# Decoupling Configurations from Moduli Spaces of Manifolds




Luciana Basualdo Bonatto

Green Templeton College

University of Oxford

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*To my parents*
*for all the love and support*
*and in memory of*
*my dearest friend Patricia*

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# Abstract

We consider moduli spaces of $d$-dimensional manifolds with embedded particles and discs. In such a moduli space, the location of the particles and discs is constrained by the $d$-dimensional manifold. We will compare this moduli space with the moduli space of $d$-dimensional manifolds in which the location of such decorations is no longer constrained, i.e. the decorations are decoupled. We generalise work by Bödigheimer–Tillmann for oriented surfaces and obtain new results for surfaces with different tangential structures as well as to higher dimensional manifolds. We also provide a generalisation of this result to moduli spaces with more general submanifold decorations and specialise in the case of decorations being unparametrised unlinked circles.

The moduli space of oriented surfaces with one boundary component can be made into a topological monoid with the operation of gluing surfaces along a section of their boundary. This operation can be extended to the spaces of configurations on surfaces with labels. We show that the group completion of this monoid splits as a product of the group completion of the moduli space of oriented surfaces and labelled configuration spaces of points in $\mathbb{R}^\infty$. We further generalise this result for the configuration spaces on surfaces with partially summable labels, where particles are allowed to collide whenever their labels are summable.

The final project consists of endowing normalized cacti with the structure of an $\infty$-operad by showing that its existing composition laws are associative up to all higher homotopies. The higher homotopies are encoded by a new topological operad of bracketed trees which we relate both to an enrichment of the dendroidal category $\Omega$ and to the Boardman-Vogt $W$-construction on the operad of operads.

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# Chapter 1

# Introduction

In broad terms, this work is on the classifying space of manifolds with marked points. From a geometric viewpoint, this involves abstract manifolds in $\mathbb{R}^\infty$ with particles that are constrained to the manifold. On the one hand, this relates to configuration spaces of manifolds, and monoids of configurations on manifolds. On the other hand, this can be seen as a classifying space for a manifold with punctures. The relation between these two viewpoints lies in the centre of this work. Crucially, we use one of these approaches to derive a decoupling result for such spaces: from the perspective of the stable homology, the condition that the particles be restrained to the underlying manifold can be lifted, and this splits as the space of submanifolds and the space of labelled points in $\mathbb{R}^\infty$. This is what we refer to as the decoupling result (for an illustration of this decoupling see Figure 1.1).

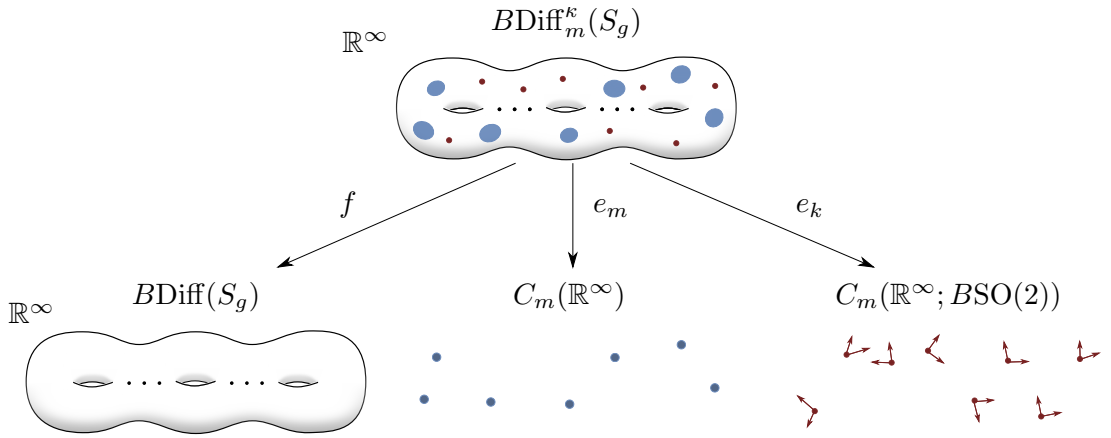



Figure 1.1: Geometric representation of the decoupling map for the oriented moduli space of a surface $S_g$ as the product of three maps: forget the decorations, record the centre of the $m$ marked discs, and record the $k$ marked points and their oriented tangent spaces. For more details on the geometric interpretation see Section 1.1.1.

We start by looking at this space from the perspective of being a classifying space for a manifold with punctures or, as we refer to it during this dissertation, a manifold with decorations. The diffeomorphism group of a smooth manifold and its classifying space are fundamental objects in topology. In particular, for a closed smooth manifold $W$, the space $B\mathrm{Diff}(W)$ classifies the smooth fibre bundles with fibre $W$. When $W$ is a manifold with non-empty boundary, we consider $\mathrm{Diff}(W)$ to be the group of those diffeomorphisms which are the identity near $\partial W$. The classifying spaces of such groups are also extremely important, as they are crucial for instance to the construction of topologically enriched categories of bordisms. To completely understand the classifying space of a diffeomorphism group is known to be extremely difficult and such a description is only available for very few manifolds. One key strategy when studying $B\mathrm{Diff}(W)$ is to understand how its homology behaves when changing the manifold $W$ by operations such as the connected sum or gluing of cobordisms. In this paper, we use this strategy to study the stable homology of the decorated diffeomorphism group.

A $d$-dimensional smooth manifold $W$ is said to be decorated if it is equipped marked points $p_1, \dots, p_k$ and disjoint parametrized discs $\phi_1, \dots, \phi_m : D^d \hookrightarrow W$. The decorated diffeomorphism group of $W$, denoted $\mathrm{Diff}^k_m(W)$, consists of those maps $\psi : W \to W$ which preserve the marked points and parametrized discs up to permutation. This notation is justified by the fact that the isomorphism type of $\mathrm{Diff}^k_m(W)$ does not depend on the choice of the $k$ points and $m$ embedded discs that comprise the decorations (see Lemma 3.1.2). The classifying space $B\mathrm{Diff}^k_m(W)$ has been studied from many different perspectives, for instance, considering the behaviour after increasing the number of marked points or discs (see [88]).

When $W = F_{g,b}$ is an oriented surface of genus $g$ with $b$ boundary components, the diffeomorphism group $\mathrm{Diff}(F_{g,b})$ has contractible path components, and therefore $B\mathrm{Diff}(F_{g,b})$ is weakly equivalent to the classifying space of the *mapping class group* $\Gamma_{g,b} = \pi_0\mathrm{Diff}(F_{g,b})$. This group has been vastly studied from several perspectives and some of the most famous results include Harer's stability theorems [37] and the Madsen-Weiss theorem [56].

Bödigheimer and Tillmann [12] studied the comparison between $B\mathrm{Diff}^k_m(F_{g,b})$ and $B\mathrm{Diff}(F_{g,b})$. They used a decoupling map

$$d\colon\ B\mathrm{Diff}^{+,k}_m(F_{g,b}) \xrightarrow{f\times e_m\times e^k} B\mathrm{Diff}^{+}(F_{g,b}) \times B\Sigma_m \times B(\Sigma_k \wr \mathrm{SO}(2)) \qquad (1.0.1)$$

where the map $f$ is induced by the inclusion $\mathrm{Diff}^k_m(F_{g,b}) \to \mathrm{Diff}(F_{g,b})$, the map $e_m$

is induced by $\mathrm{Diff}^k_m(F_{g,b}) \to \Sigma_m$ recording the permutation of the marked discs, and $e^k$ is induced by the map $\mathrm{Diff}^k_m(F_{g,b}) \to \Sigma_k \wr \mathrm{SO}(2)$ recording the permutation of the marked points together with the induced map on their tangent space (see Figure 1.1 for a geometric representation of the decoupling map). Bödigheimer and Tillmann showed that $d$ induces a homology isomorphism in degrees $\leq \frac{g}{3}$, therefore, in this range, we say that the decorations, which were bound to the manifold, get decoupled.

One of the main goals of this thesis is to address the following:

**Question 1.** How does the decoupling theorem of Bödigheimer and Tillmann generalises to other manifolds?

One of the ways $B\mathrm{Diff}^k(W)$ relates to the classifying space $B\mathrm{Diff}(W)$ is via the following fibre sequence (which follows from Palais' theorem [68, 54])

$$C_k(W) \longrightarrow B\mathrm{Diff}^k(W) \longrightarrow B\mathrm{Diff}(W) \tag{1.0.2}$$

where $C_k(W)$ is the configuration space of $k$ points in $W$. Recall that the ordered configuration space of $k$ points in $W$ is defined as

$$\widetilde{C}_k(W) \coloneqq \{(x_1, \dots, x_k) \in W^k \mid x_i \neq x_j \text{ if } i \neq j\}$$

and $C_k(W)$ is the quotient $\widetilde{C}_k(W)/\Sigma_k$ where $\Sigma_k$ acts by permutation.

Configuration spaces are important constructions in topology and can be considered in a more general setting: given a pointed space $X$, the space of configurations in $W$ with labels in $X$ is the quotient

$$C(W; X) \coloneqq \left( \coprod_{k \geq 0} \widetilde{C}_k(W) \times_{\Sigma_k} X^k \right) \Big/ \sim$$

under the relation $(w_1, \dots, w_k; x_1, \dots, x_k) \sim (w_1, \dots, w_{k-1}; x_1, \dots, x_{k-1})$ if $x_k$ is the basepoint of $X$. We can interpret this space geometrically by considering it as the space of particles in $W$ where each particle is labelled by an element of $X$, and a particle is allowed to disappear if labelled by the basepoint. This space has been of interest since the 70's, appearing on the works of May [59] and Segal [80] taking $W = \mathbb{R}^n$. They constructed what today is called a *scanning map* $C(W; X) \to \Omega^n \Sigma^n X$ and showed that for path-connected $X$ this is a weak-homotopy equivalence, while in general it is the group completion of $C(W; X)$ (with multiplication given roughly by the disjoint union). This type of result has been generalised in many directions: Bödigheimer [11] showed a similar result for general manifolds $W$, and more general

spaces of labels with more structure, such as (partial) multiplications, were considered in [63, 82, 34, 47, 76].

More recently this space has been expanded to sophisticated constructions in factorization homology [1] on the one hand and, on the other, in the form of configuration spaces of manifolds, has been used to compute the stable homology of the moduli spaces of Riemann surfaces [56] and higher dimensional manifolds [31, 30].

The spaces $C_k(W)$ admit an action of the group $\mathrm{Diff}(W)$ given by

$$\phi \cdot (p_1, \ldots, p_k) = (\phi(p_1), \ldots, \phi(p_k))$$

and we look at the Borel constructions (homotopy quotients) $C_k(W)/\!\!/\mathrm{Diff}(W)$. This action of $\mathrm{Diff}(W)$ can be further extended to an action on configuration spaces with labels $C(W; X)$. The fibre sequence (1.0.2) implies a weak homotopy equivalence

$$C_k(W)/\!\!/\mathrm{Diff}(W) \simeq B\mathrm{Diff}^k(W).$$

This gives a bridge between results about the classifying space of decorated diffeomorphism groups of $W$ and homotopy quotients of configurations on $W$.

We previously mentioned that the space $C(\mathbb{R}^n; X)$ can be given the structure of an ($A_\infty$-)monoid by taking multiplication to be roughly given by stacking configurations side by side (see [80] for details of this construction). More generally, given a multiplication rule on a collection of manifolds, we can look at the induced multiplication on the homotopy quotients of their configuration spaces. We now discuss this induced multiplication and, for simplicity, restrict to the case of oriented surfaces $F_{g,b}$ (for details see Chapter 8).

It was shown in [86] that the operation of gluing surfaces of the form $F_{g,1}$ along a section of their boundary induces a multiplication

$$\mu : B\mathrm{Diff}(F_{g,1}) \times B\mathrm{Diff}(F_{h,1}) \to B\mathrm{Diff}(F_{g+h,1})$$

making $\coprod_g B\mathrm{Diff}(F_{g,1})$ into a topological monoid. This multiplication induces a map

$$\mu : C(F_{g,1}; X)/\!\!/\mathrm{Diff}(F_{g,1}) \times C(F_{h,1}; X)/\!\!/\mathrm{Diff}(F_{h,1}) \to C(F_{g+h,1}; X)/\!\!/\mathrm{Diff}(F_{g+h,1})$$

making $\coprod_g C(F_{g,1}; X)/\!\!/\mathrm{Diff}(F_{g,1})$ into a topological monoid. This monoid is automatically an algebra over Tillmann's surface operad, so its group completion is an infinite loop space.

**Question 2.** Does the decoupling result of Bödigheimer and Tillmann imply a splitting for the infinite loop space $\Omega B\left(\coprod_g C(F_{g,1};X)/\!/\mathrm{Diff}(F_{g,1})\right)$?

Questions 1 and 2 are the core of the thesis. Chapters 2-7 focus on the perspective of classifying spaces of decorated diffeomorphisms groups and Question 1, while the last chapters (8 and 9) explore monoids of configurations on surfaces and Question 2. We now discuss in more detail the main results related to these topics.

## 1.1 Statement of results

### 1.1.1 Decorated diffeomorphism groups

For manifolds in high dimensions, Harer's stability theorems [37] and the Madsen-Weiss theorem [56] have been generalised to statements about the homology of the moduli spaces of manifolds with tangential structures [31, 30]. It is for these spaces that we provide a generalisation of the decoupling theorem of Bödigheimer and Tillmann.

Orientation, framings, spin structures and maps to a background space are all examples of a more general type of structure which can be described just from data on the tangent bundle of a manifold. A tangential structure is a topological space $\Theta$ equipped with a continuous $\mathrm{GL}_d$ action. A $\Theta$-structure on a $d$-dimensional manifold $W$ is a $\mathrm{GL}_d$-equivariant map $\rho_W : \mathrm{Fr}(TW) \to \Theta$, where $\mathrm{Fr}(TW)$ denotes the space of framings of the manifold $W$. A canonical example is $\Theta^{or} = \{\pm 1\}$ with action given by multiplication with the sign of the determinant, and it is simple to see that a $\Theta^{or}$-structure on a manifold is a choice of orientation.

The space of all $\Theta$-structures has an action of $\mathrm{Diff}(W)$ given by precomposition with the differential. Given a closed compact connected smooth manifold $W$ equipped with a $\Theta$-structure $\rho_W$, the moduli space $\mathcal{M}^{\Theta}(W,\rho_W)$ of $W$ with $\Theta$-structures concordant to $\rho_W$ is defined as the path component of $\rho_W$ in the Borel construction

$$\{\mathrm{GL}_d\text{-equivariant maps } \rho : \mathrm{Fr}(TW) \to \Theta\}/\!/\mathrm{Diff}(W).$$

Examples of this construction are the classifying spaces $B\mathrm{Diff}(W)$ and, when $W$ is orientable, $B\mathrm{Diff}^+(W)$.

Analogously, the decorated moduli space of $(W,\rho_W)$, denoted $\mathcal{M}^{\Theta,k}_m(W,\rho_W)$, is

defined as the path component of $\rho_W$ in the Borel construction

$$\{\mathrm{GL}_d\text{-equivariant maps } \rho : \mathrm{Fr}(TW) \to \Theta\} /\!\!/ \mathrm{Diff}^k_m(W).$$

If $W$ is a manifold with non-empty boundary, the moduli spaces $\mathcal{M}^\Theta(W, \rho_W)$ and $\mathcal{M}^{\Theta,k}_m(W, \rho_W)$ are defined analogously but only considering the $\mathrm{GL}_d$-equivariant maps $\rho : \mathrm{Fr}(TW) \to \Theta$ which agree with $\rho_W$ on $\mathrm{Fr}(TW_{|\partial W})$.

We start by noting that many results obtained for classifying spaces of decorated diffeomorphism groups of surfaces can be adapted to results on decorated moduli spaces of manifolds with tangential structures. For instance, the following is a generalisation of Tillmann's result in [88].

**Theorem A.** *For any compact connected smooth manifold $W$, we have maps*

$$\mathcal{M}^{\Theta,k}(W) \to \mathcal{M}^{\Theta,k+1}(W) \qquad\qquad \mathcal{M}^\Theta_m(W) \to \mathcal{M}^\Theta_{m+1}(W)$$

*which induce homology isomorphisms on degrees $* \leq k/2$ and $m/2$, respectively.*

However, our focus is on constructing a decoupling map analogous to (1.0.1). For a $d$-dimensional oriented manifold $W$ with non-empty boundary, this map is given by

$$D : \ \mathcal{M}^{\Theta,k}_m(W, \rho_W) \xrightarrow{F \times E_m \times E^k} \mathcal{M}^\Theta(W, \rho_W) \times \Theta^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}^+_d)^k /\!\!/ \Sigma_k.$$

In particular, taking $W$ to be a surface and $\Theta$ the orientation structure, $D$ is precisely the map $d$ of equation (1.0.1).

The image of $D$ is a path-component of the codomain, which we denote

$$\mathcal{M}^\Theta(W, \rho_W) \times \Theta^m_0 /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}^+_d)^k_0 /\!\!/ \Sigma_k$$

and we show that, when a stabilisation condition is satisfied, the decoupling induces a homology isomorphism onto its image, in a range depending on the genus of $W$.

The decoupling theorem relies on the comparison between the moduli space of a manifold $W$ and the moduli space of $W \setminus \coprod_k \mathrm{int}(D^d)$. For $W$ a compact connected $d$-dimensional manifold, and $\rho_W$ a fixed $\Theta$-structure on $W$, the manifold $W \setminus \coprod_k \mathrm{int}(D^d)$ is naturally endowed with a $\Theta$-structure $\rho'_W$ given by the restriction of $\rho_W$. Then there is a natural map

$$\mathcal{M}^\Theta(W \setminus \coprod_k \mathrm{int}(D^d), \rho'_W) \to \mathcal{M}^\Theta(W, \rho_W) \tag{1.1.1}$$

induced by extending a $\Theta$-structure by $\rho_W|_{D^d}$ and a diffeomorphism by the identity.

**Theorem B.** *Let $W$ be a compact, connected, orientable manifold with non-empty boundary and $\rho_W$ be a fixed $\Theta$-structure on $W$. If the map* (1.1.1) *induces a homology isomorphism in degrees $i \leq \alpha$ then the decoupling map*

$$D : \mathcal{M}_m^{\Theta,k}(W, \rho_W) \longrightarrow \mathcal{M}^{\Theta}(W, \rho_W) \times \Theta_0^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}_d^+)_0^k /\!\!/ \Sigma_k$$

*induces homology isomorphisms in degrees $i \leq \alpha$.*

A more general version of Theorem B is given in Theorem 3.3.2 for manifolds $W$ with empty boundary and also for non-orientable manifolds.

In particular, the theorem above gives us new results on moduli spaces of surfaces with tangential structures, generalising the decoupling result of Bödigheimer and Tillmann to surfaces with, amongst others, spin structures, maps to a background space, and framings.

The main corollary of Theorem B is obtained in the context of manifolds of even dimension $d = 2n \geq 6$, where the assumption on the map (1.1.1) has been shown to hold whenever $\rho_W$ is $n$-connected [30].

**Corollary C.** *For $W$ compact, connected, orientable manifold of dimension $d = 2n \geq 6$ with non-empty boundary, and $\rho_W$ an $n$-connected $\Theta$-structure on $W$,*

$$D : \mathcal{M}_m^{\Theta,k}(W, \rho_W) \longrightarrow \mathcal{M}^{\Theta}(W, \rho_W) \times \Theta_0^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}_d^+)_0^k /\!\!/ \Sigma_k$$

*induces homology isomorphisms in degrees $i \leq \frac{g-4}{3}$, where $g$ is the stable genus of $W$.*

We also show that this condition is necessary, namely that there exists a manifold $W$ and a tangential structure $\rho_W$ which is not $n$-connected, for which the decoupling theorem does not hold (see Chapter 6). Although this connectivity assumption is quite restrictive, we still obtain further results for more general tangential structures using the techniques of [30, Section 9]. In particular, we prove the following:

**Theorem D.** *Let $W_{g,1} = \#_g S^n \times S^n \setminus D^{2n}$, for $2n \geq 6$. Then for all $i \leq \frac{g-4}{3}$*

$$H_i(B\mathrm{Diff}_m^{+,k}(W_{g,1})) \cong H_i(B\mathrm{Diff}^+(W_{g,1}) \times \mathrm{SO}[0, n-1]^m /\!\!/ \Sigma_m \times B\mathrm{SO}(2n)\langle n \rangle^k /\!\!/ \Sigma_k)$$

*where $\mathrm{SO}[0, n-1]$ is the $n$-truncation of $\mathrm{SO}$ and $B\mathrm{SO}(2n)\langle n \rangle$ is the $n$-connected cover of $B\mathrm{SO}(2n)$.*

We also provide a computation of the cohomology of $B\mathrm{Diff}^{+,k}_m(W_{g,1})$ using Theorem D.

Finally, we provide a generalisation of the decoupling result for more types of decorations. When studying surfaces, it is natural to look at marked points and discs, however for high dimensional manifolds, one is allowed to explore more general types of decorations. This has been studied in recent work such as [69, 50, 70, 71]. We generalise the definition of the decorated moduli space of a manifold with more general marked submanifolds, we define a decoupling map and show that, under a homology stability hypothesis, it induces a homology isomorphism in a certain range. We analyse more closely the case that the decorations are unlinked circles, because of its relation to the literature and also its relevance for string theory. The space $\mathcal{M}^{\Theta}_{kS^1}(W, \rho_W)$ is defined to be the moduli space of manifolds diffeomorphic to $W$, equipped with a $\Theta$-structure and $k$ marked unparametrised circles. As in Theorem B, we also get a splitting result, now in terms of the space of configurations of circles in $\mathbb{R}^\infty$ with labels on a space $\mathcal{L}$ (see Definition 5.4.10), which we denote $\mathrm{C}_{kS^1}(\mathbb{R}^\infty, \mathcal{L})$.

**Theorem E.** *Let $W$ be a simply-connected spin manifold of dimension $2n \geq 6$ with non-empty boundary, equipped with $k$ marked unparametrised unlinked circles and with a $\Theta$-structure $\rho_W : \mathrm{Fr}(TW) \to \Theta$ which is $n$-connected and such that $\Theta /\!\!/ \mathrm{GL}_d$ is simply-connected. Then for all $i \leq \frac{g-4}{3}$*

$$H_i(\mathcal{M}^{\Theta}_{kS^1}(W, \rho_W)) \cong H_i(\mathcal{M}^{\Theta}(W, \rho_W) \times \mathrm{C}_{kS^1}(\mathbb{R}^\infty; (L\Theta /\!\!/ L_{null}\mathrm{GL}^+_{d-1})_0))$$

*where $L-$ is the free loop space, $L_{null}$ is the subspace of nullhomotopic loops, and $(-)_0$ indicates a path-component that is specified in the proof.*

**Geometric Interpretation** The spaces and maps used in the decoupling result all have a very concrete geometrical interpretation, which we briefly introduce.

Let $\Theta^* = *$ be the point with the trivial $\mathrm{GL}_d$-action. Then any manifold $W$ admits a unique $\Theta^*$-structure $\rho_W$. By definition, the moduli space $\mathcal{M}^{\Theta^*}(W, \rho_W)$ is a model for $B\mathrm{Diff}(W)$. A geometric interpretation for this moduli space can be obtained from the specific model of $B\mathrm{Diff}(W)$ given by the quotient $\mathrm{Emb}(W, \mathbb{R}^\infty)/\mathrm{Diff}(W)$. With this model, $\mathcal{M}^{\Theta^*}(W)$ can be seen as the subspace of all submanifolds of $\mathbb{R}^\infty$ that are abstractly diffeomorphic to $W$. Analogously, fixing an arbitrary $\Theta$-structure, $\rho_W$, on a manifold $W$ (for instance a choice of orientation), the moduli space $\mathcal{M}^{\Theta}(W, \rho_W)$ has a model as the space of all submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to $W$ together with a choice of $\Theta$-structure concordant to $\rho_W$. A detailed description of this model

can be found in sections 6 and 7 of [31].

From this perspective, the decorated moduli space $\mathcal{M}_m^{\Theta,k}(W, \rho_W)$ is the space of all submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to $W$ together with $k$ marked points, $m$ marked parametrized discs, and a choice of $\Theta$-structure concordant to $\rho_W$.

Moreover, note that the spaces $\Theta_0^m /\!\!/ \Sigma_m$ and $(\Theta /\!\!/ \mathrm{GL}_d)_0^k /\!\!/ \Sigma_k$ also have geometric models in terms of unordered configuration spaces with labels, which we denote respectively by $C_m(\mathbb{R}^\infty, \Theta_0)$ and $C_k(\mathbb{R}^\infty, (\Theta /\!\!/ \mathrm{GL}_d)_0)$.

The decoupling map in Theorem B, is the product of three maps. The first

$$\mathcal{M}_m^{\Theta,k}(W, \rho_W) \to \mathcal{M}^\Theta(W, \rho_W),$$

is the map that simply forgets the decorations; the map

$$\mathcal{M}_m^{\Theta,k}(W, \rho_W) \to C_m(\mathbb{R}^\infty, \Theta_0)$$

records the centre of the marked discs together with local tangential structure information; and finally

$$\mathcal{M}_m^{\Theta,k}(W, \rho_W) \to C_k(\mathbb{R}^\infty, (\Theta /\!\!/ \mathrm{GL}_d)_0)$$

is the map that records the positions of the marked points in $\mathbb{R}^\infty$ together with their tangent spaces and tangential structure information. See Figure 1.1 for an illustration.

With this geometric interpretation, the decoupling result tells us that the homology of the space of decorated submanifolds of $\mathbb{R}^\infty$ of a fixed diffeomorphism type, in which the points and discs are constrained to our manifolds, can be understood in terms of the homology of a space where these points and discs are not constrained anymore, i.e. they are decoupled.

### 1.1.2 Monoids of configurations on surfaces

As discussed previously, the gluing of surfaces of the form $F_{g,1}$ along their boundary induces a multiplication

$$\mu : C(F_{g,1}; X) /\!\!/ \mathrm{Diff}(F_{g,1}) \times C(F_{h,1}; X) /\!\!/ \mathrm{Diff}(F_{h,1}) \to C(F_{g+h,1}; X) /\!\!/ \mathrm{Diff}(F_{g+h,1})$$

making $\coprod_g C(F_{g,1}; X) /\!\!/ \mathrm{Diff}(F_{g,1})$ into a topological monoid.

This construction can be extended further by allowing the diffeomorphism group to act non-trivially on the space of labels. Namely, if $Z$ is a pointed $\mathrm{GL}_2$-space, we

can define an action of $\mathrm{Diff}(F_{g,1})$ by

$$\phi \cdot (p_1, \dots, p_k; z_1, \dots, z_k) = (\phi(p_1), \dots, \phi(p_k); D_{p_1}\phi \cdot z_1, \dots, D_{p_k}\phi \cdot z_k).$$

As before, gluing of surfaces of the form $F_{g,1}$ along their boundary induces a multiplication

$$\mu : C(F_{g,1}; Z)/\!\!/\mathrm{Diff}(F_{g,1}) \times C(F_{h,1}; Z)/\!\!/\mathrm{Diff}(F_{h,1}) \to C(F_{g+h,1}; Z)/\!\!/\mathrm{Diff}(F_{g+h,1})$$

making $\coprod_g C(F_{g,1}; Z)/\!\!/\mathrm{Diff}(F_{g,1})$ into a topological monoid.

**Theorem F.** *For any path-connected pointed* $\mathrm{GL}_2^+$*-space* $Z$*, there is a homotopy equivalence*

$$\tau \times \delta : \Omega B(MC(Z)) \xrightarrow{\simeq} \mathbb{Z} \times B\mathrm{Diff}(F_\infty)^+ \times Q((E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} Z).$$

In the above, $Q(-)$ denotes the functor $\Omega^\infty\Sigma^\infty(-)$. On the one hand, $\mathbb{Z} \times B\mathrm{Diff}(F_\infty)^+$ was shown in [87] to be the group completion of the monoid $\coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1})$, while Segal showed in [80] that the space $Q(X)$ is the group completion of the monoid $C(\mathbb{R}^\infty; X)$. Therefore the above theorem can be seen as a decoupling result in the context of monoids of configurations on surfaces.

### 1.1.3 Configuration spaces with partially summable labels

We finish by considering the more general construction of configuration spaces with labels in a framed partial 2-monoid, where particles are allowed to collide if their labels are summable. This is an important construction which has been developed in [63, 82, 34, 47, 76]. A partial 2-monoid $P$ is, in essence, a space with a multiplication that makes it similar to an $E_2$-algebra, but with the restriction that this multiplication does not need to be defined between every pair of elements in $P$. Whenever $P$ is equipped with a compatible action of $\mathrm{GL}_2$, we call it a framed partial monoid, and we can define the space of configurations in $F_{g,1}$ with partially summable labels in $P$, denoted $C_\Sigma(F_{g,1}; P)$. See Chapter 9 for details.

The definition $C_\Sigma(F_{g,1}; P)$ requires much more machinery then the case for non-summable labels, such as the Fulton-MacPherson configuration spaces and operad, and yields more complicated models for configuration spaces. To prove the decoupling theorem for the space $C_\Sigma(F_{g,1}; P)$, we develop a different model for this space in section 9.2, which we refer to as the *disc model for configurations*, denoted $|D_\Sigma(F_{g,1}; P)_\bullet|$.

In the decoupling context, we naturally encounter another space of discs with configurations, which we refer to as the 2-*discs model for configurations* in $\mathbb{R}^\infty$, denoted $|D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet|$.

Denote by $F_\infty$ the *surface of infinite genus*, i.e. the limit of the surfaces $F_{g,1}$. The main result for configurations with partially summable labels is the following:

**Theorem G.** *The decoupling map induces a homology isomorphism*

$$C_\Sigma(F_\infty; P)/\!/\mathrm{Diff}(F_\infty) \simeq |D_\Sigma(F_\infty; P)_\bullet|/\!/\mathrm{Diff}(F_\infty) \to B\mathrm{Diff}(F_\infty) \times |D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet|.$$

Denoting by $MC_\Sigma(P)$ the monoid for configurations in surfaces with partially summable labels in $P$, the above result implies the following generalisation of Theorem F:

**Corollary H.** *For any path-connected framed partial 2-monoid with unit $P$, there is a homotopy equivalence*

$$\Omega B(MC_\Sigma(P)) \simeq \mathbb{Z} \times B\mathrm{Diff}(F_\infty)^+ \times |D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet|.$$

## 1.2 Structure of the thesis

Chapters 2-7 relate to the decorated moduli spaces of manifolds with tangential structures, while Chapters 8 and 9 focus on monoids of configurations on manifolds. For a diagram of the chapters and how they relate to one another, see Figure 1.2.

Chapter 2 recalls the basic concepts and results needed to construct the decorated moduli spaces with tangential structures. We start by defining and giving examples of tangential structures and the topological moduli space of manifolds. Further, we prove auxiliary results on fibre sequences of Borel constructions and a spectral sequence argument which will be needed throughout the thesis.

We define the decorated moduli space of manifolds and the decoupling map, and we prove Theorem B in Chapter 3. Subsequently, in Chapter 4, we prove the corollaries of Theorem B which provide decoupling results for surfaces with many tangential structures, as well as Corollary C, for manifolds of dimension $2n \geq 6$.

In Chapter 5 we define the generalisation of the decorated moduli space for more general types of submanifold decorations, define the decoupling map and prove the decoupling theorem in this case. We look closer at the case when decorations are embedded unlinked circles and prove Theorem E.

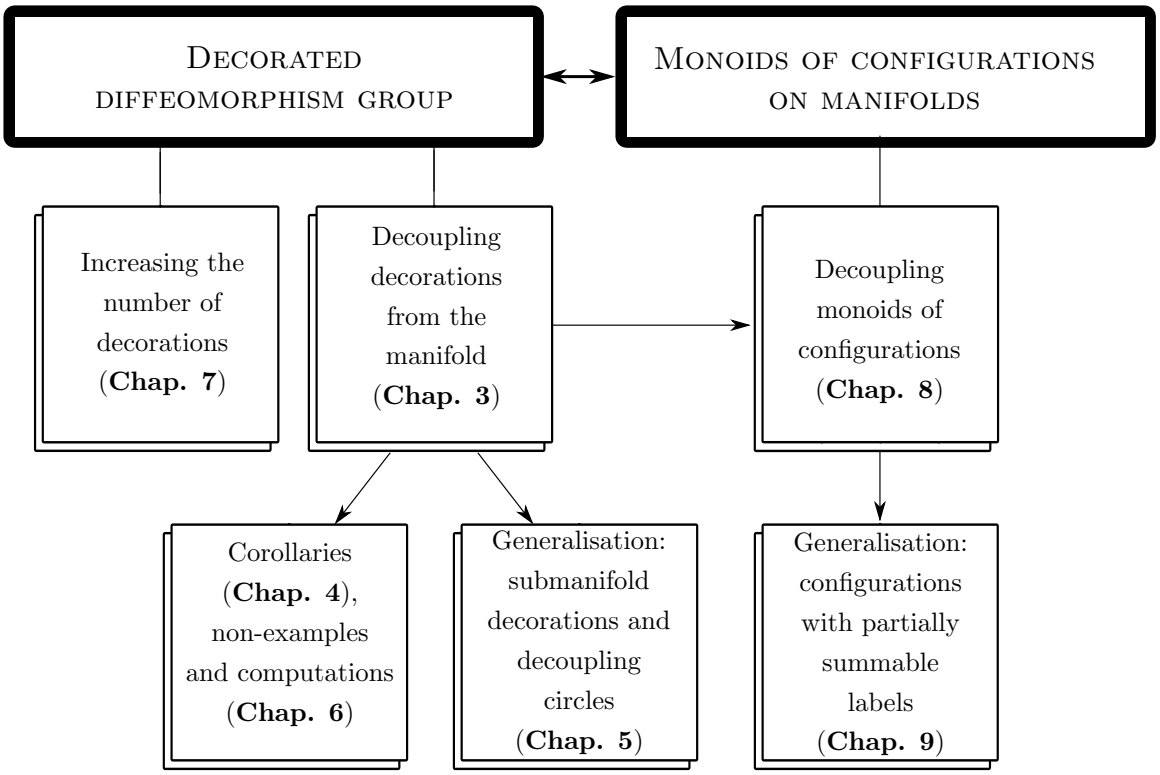



Figure 1.2: Thesis Structure and Chapter Dependencies.

Chapter 6 looks at non-examples of the decoupling theorem, i.e. high dimensional manifolds with tangential structures that fail to satisfy the hypothesis of Theorem B. We show that there are manifolds with tangential structures for which the decoupling does not hold, and discuss the techniques to understand the homology of the decorated moduli spaces in these cases. In particular, by applying this result to the manifolds $W_{g,1}$, we prove Theorem D.

Excluding Section 3.4, Chapters 2-6 effectively constitute the paper [2] on arXiv.

Chapter 7 is the last one which focuses on decorated moduli spaces of manifolds and it contains the proof of Theorem A, on increasing the number of decorations on the moduli spaces.

We finish by discussing the monoids of configurations on surfaces. In Chapter 8 we introduce the monoid for non-colliding labels and prove Theorem F.

Finally, in Chapter 9 we discuss the case of configurations with summable labels. We start by recalling the definitions of framed partial $d$-monoids and configuration spaces with partially summable labels. We develop different models for this space of configurations and use them to prove Theorem G and Corollary H.

## 1.3 Appendix: The $\infty$-operad of normalised cacti

A second project included in this dissertation relates to the algebraic structures that are determined by graphical models for moduli spaces of oriented surfaces. Since the results of this project have already been peer reviewed and accepted for publication at the *Journal of Topology and its Applications*, we defer it to the appendix. This project started as part of the Women in Topology Workshop and it is joint work with Safia Chettih, Abigail Linton, Sophie Raynor, Marcy Robertson and Nathalie Wahl.

The operadic structure defined by oriented surfaces and cobordisms has been studied previously in [86]. There, it was shown that the moduli space of oriented surfaces with at least one boundary component can be modified to define an operad. However, without the modification, the composition fails to be associative and therefore does not define an operad. This project shows that graphical models of moduli spaces of genus zero oriented surfaces with the operation of gluing along cobordism naturally define an $\infty$-operad.

On the one hand, understanding the algebraic structure determined by gluing along cobordisms is quite central to the study of Topological Field Theories. From a different perspective, these moduli spaces would provide a rare example of an $\infty$-operad that does not arise via the application of a nerve construction to a known operad. Only a few such examples exist in the literature, as the weak operad of configuration spaces [35, Corollary 5], the configuration categories of [10], or examples that arise as a result of completion as in [8, Proposition 5.1].

In our work [3], we study normalised cacti, a graphical model for the moduli space of genus 0 surfaces with at least one boundary component [92, Remark 2.8]. This model can be given a composition that corresponds to gluing along cobordisms, and this is known not to be associative [48, Remark 2.3.19]. We show that normalised cacti with this operation have the structure of an $\infty$-operad by showing that its existing composition law is associative up to all higher homotopies.

To show that normalised cacti produce such an example, we develop a new technique to express and identify $\infty$-operads. We do this by defining an *operad of bracketed trees* $B\mathcal{O}$ and relate it to two well-known models for $\infty$-operads: dendroidal Segal spaces and $W\mathcal{O}$-algebras, using the Boardman-Vogt $W$-construction on the operad of operads. In particular, we have the following:

**Theorem 1.3.1.** *There is a canonical operad map $W\mathcal{O} \to B\mathcal{O}$. In particular, any $B\mathcal{O}$-algebra is also a $W\mathcal{O}$-algebra, and therefore an $\infty$-operad.*

The algebras over $B\mathcal{O}$ are precisely $\infty$-operads with a strict composition, unit and symmetric action. With this new technique, we showed that:

**Theorem 1.3.2.** *Normalised cacti with the cobordism composition extend to a $B\mathcal{O}$-algebra, and therefore are part of the structure of an $\infty$-operad.*

My participation in this project included major contributions to: the definition of the operad $B\mathcal{O}$ of bracketed trees (Definition A.3.9), the relation between the operads $B\mathcal{O}$ and $W\mathcal{O}$ (Theorem A.6.4), the relation between $B\mathcal{O}$-algebras and dendroidal spaces (Theorem A.4.8), and on the relation between cacti and the $MS$-operad (Section A.5.1).

# Chapter 2

# Preliminaries

In this section we recall some crucial definitions such as tangential structures and the moduli space of manifolds with tangential structures. We will also give examples that will be used in Section 4. Finally, we recall basic results about descending fibre sequences to homotopy quotients as well as a spectral sequence argument that will be used throughout the thesis.

## 2.1 Tangential structures

Throughout, we consider $W$ to be a smooth compact connected $d$-dimensional manifold, possibly with non-empty boundary. If $W$ is a closed manifold, we denote by $\mathrm{Diff}(W)$ the group of diffeomorphisms of $W$ with Whitney $C^\infty$ topology. If $W$ has non-empty boundary, we assume it to be equipped with a collar neighbourhood of $\partial W$ and we denote by $\mathrm{Diff}(W)$ the group of diffeomorphisms of $W$ which restrict to the identity on this collar. If moreover $W$ is an orientable manifold, we denote by $\mathrm{Diff}^+(W)$ the subgroup of orientation preserving diffeomorphisms. Note that if $W$ is orientable and has a non-empty boundary, any element of $\mathrm{Diff}(W)$ is automatically orientation preserving.

Given a vector bundle $p : E \to B$, the frame bundle of $E$ over $B$ will be denote by $\mathrm{Fr}(E)$. Recall that the fiber of $\mathrm{Fr}(E) \to B$ over a fixed $b$ is the space of ordered bases of $p^{-1}(b)$, and this forms a $\mathrm{GL}_d$-principal bundle, with the action $A \cdot (v_1, \dots, v_n) = (A(v_1), \dots, A(v_n))$, for $A \in \mathrm{GL}_d$ and $(v_1, \dots, v_n)$ an ordered basis of $p^{-1}(b)$. Throughout this paper, we denote by $TW$ the tangent bundle of the manifold $W$, and by $\varepsilon^n \to B$ the trivial $n$-dimensional vector bundle over some base space $B$. In this thesis, we will consider only real vector bundles and denote by $\mathrm{GL}_d$ the general linear group $\mathrm{GL}_d(\mathbb{R})$.

We now define tangential structures following the terminology established by Galatius and Randal-Williams in [32].

**Definition 2.1.1.** A *tangential structure* for $d$-dimensional manifolds is a space $\Theta$ with a continuous action of $\mathrm{GL}_d$. A $\Theta$-structure on a $d$-manifold $W$ is a $\mathrm{GL}_d$-equivariant map $\rho : \mathrm{Fr}(TW) \to \Theta$.

Manifolds are usually equipped with data that can be described using tangential structures:

**Example 2.1.2.** Let $W$ be a connected manifold.

(a) An orientation consists of a coherent choice of which oriented bases of the tangent spaces are considered positive. Namely, this is the data of a $\mathrm{GL}_d$-equivariant map $\mathrm{Fr}(TW) \to \{\pm 1\}$, where the action on $\Theta^{or} := \{\pm 1\}$ is given by multiplication by the sign of the determinant. Therefore, a $\Theta^{or}$-structure on a manifold is equivalent to a choice of orientation.

(b) If we want to consider all manifolds with no extra data, we can use the trivial tangential structure $\Theta^* = \{*\}$ with the trivial action. Any manifold admits a unique $\Theta^*$-structure, and therefore it encodes no extra data.

(c) Framings on a manifold also are a tangential structure described by $\Theta^{\mathrm{fr}} = \mathrm{GL}_d$, since the data of a framing is precisely a continuous choice of basis for the tangent space at each point, which can be expressed as a $\mathrm{GL}_d$-equivariant map $\mathrm{Fr}(TW) \to \mathrm{GL}_d$.

(d) Given a space $X$, we define the tangential structure of maps to $X$, by taking $\Theta_X = X$ with the trivial action of $\mathrm{GL}_d$. Then a $\Theta_X$-structure on a manifold $W$ is the data of a continuous a map $W \to X$.

*Remark* 2.1.3. Many authors approach tangential structures in a different way, namely by defining it as a fibration $\theta : B \to B\mathrm{O}(d)$, and by setting a $\theta$-structure on a manifold $W$ to be a map $W \to B$ lifting the map $W \to B\mathrm{O}(d)$ which classifies $TW$. There is a clear way of exchanging the two approaches using the correspondence between spaces with a $\mathrm{GL}_d$ action and spaces over $B\mathrm{GL}_d \simeq B\mathrm{O}(d)$, made through the principal $\mathrm{GL}_d$-bundle $E\mathrm{GL}_d \to B\mathrm{GL}_d$. Both the spaces $\Theta$ and $B$ associated to a given tangential structure will come into the decoupling result, so it is worth making precise the relation between them: given a fibration $\theta$, the pullback space

$$\Theta := E\mathrm{GL}_d \underset{B\mathrm{GL}_d}{\times} B$$

is naturally equipped with a $\mathrm{GL}_d$ action. On the other hand, given a $\mathrm{GL}_d$-space $\Theta$, we can define $B$ as the Borel construction $\Theta /\!\!/ \mathrm{GL}_d$ (ie. the quotient of $E\mathrm{GL}_d \times \Theta$ by the diagonal action of $\mathrm{GL}_d$). Then $E\mathrm{GL}_d \times \Theta \to B$ is a principal $\mathrm{GL}_d$-bundle, which means $B$ comes equipped with a map $\theta : B \to B\mathrm{GL}_d$. Since $E\mathrm{GL}_d$ is contractible, these processes are inverse up to equivariant fibre-wise weak equivalence.

**Example 2.1.4.** Spin structures on an $d$-dimensional manifold are known to be classified by lifts along the fibration $\theta_{\mathrm{Spin}} : B\mathrm{Spin} \to B\mathrm{O}(d) \simeq B\mathrm{GL}_d$. So the corresponding $\Theta^{\mathrm{Spin}}$ is the pullback space

$$E\mathrm{GL}_d \underset{B\mathrm{GL}_d}{\times} B\mathrm{Spin}$$

and it fits into the following diagram of fibre sequences

$$\begin{array}{ccccc} \{\pm 1\} \times B\mathbb{Z}/2 & \longrightarrow & \Theta^{\mathrm{Spin}} & \longrightarrow & E\mathrm{GL}_d \\ \| & & \downarrow & \lrcorner & \downarrow \\ \{\pm 1\} \times B\mathbb{Z}/2 & \longrightarrow & B\mathrm{Spin} & \longrightarrow & B\mathrm{GL}_d \end{array}$$

which implies that the space $\Theta^{\mathrm{Spin}}$ is homotopy equivalent to $\{\pm 1\} \times B\mathbb{Z}/2$.

**Definition 2.1.5.** Let $W$ be a closed manifold and $\Theta$ a fixed tangential structure. We define the *space of $\Theta$-structures on $W$*, denoted $\mathrm{Bun}^{\Theta}(W)$, to be the space of all $\mathrm{GL}_d$-equivariant maps $\mathrm{Fr}(TW) \to \Theta$ equipped with the compact-open topology.

Let $W$ be a manifold with non-empty boundary and a collar together with a $\mathrm{GL}_d$-equivariant map $\rho_\partial : \mathrm{Fr}(T\partial W \oplus \varepsilon) \to \Theta$. We define the *space of $\Theta$-structures on $W$ restricting to $\rho_\partial$*, denoted $\mathrm{Bun}^{\Theta}_{\rho_\partial}(W)$, to be the space of all $\mathrm{GL}_d$-equivariant maps $\mathrm{Fr}(TW) \to \Theta$ that restrict to $\rho_\partial$ on $\partial W$.

Given a manifold $W$ with non-empty boundary together with a $\mathrm{GL}_d$-equivariant map $\rho_\partial : \mathrm{Fr}(T\partial W \oplus \varepsilon) \to \Theta$, it is possible that the space $\mathrm{Bun}^{\Theta}_{\rho_\partial}(W)$ is empty, in the case where the chosen map $\rho_\partial$ cannot be extended to $\mathrm{Fr}(TW)$. For instance if $W$ is an orientable manifold with disconnected boundary and $\rho_\partial$ assigns non-compatible orientations for the different components of $\partial W$. These are not the cases we are interested in, and therefore throughout the paper, whenever $W$ is a manifold non-empty boundary we assume it comes equipped with a map $\rho_\partial$ which is the restriction of a $\Theta$-structure in $W$.

**Example 2.1.6.** Let $W$ be a manifold.

(a) If $\Theta^*$ is the trivial tangential structure of Example 2.1.2(b), then there is only one $\Theta^*$-structure for any manifold, so $\mathrm{Bun}^{\Theta^*}(W)$ is a single point.

(b) Consider the tangential structure $\Theta^{or}$ for orientation described in Example 2.1.2(a). If $W$ is a closed connected orientable manifold, it admits two $\Theta^{or}$-structures which implies that $\mathrm{Bun}^{\Theta^{or}}(W)$ consists of two points. On the other hand, if $W$ has a non-empty boundary and $\rho_\partial$ is a fixed $\Theta$ structure on $\partial W$, then $\mathrm{Bun}^{\Theta^{or}}_{\rho_\partial}(W)$ consists only of those $\mathrm{GL}_d$-equivariant maps $\mathrm{Fr}(TW) \to \Theta$ which restrict to $\rho_\partial$, and therefore consists of a single point.

(c) Given a space $X$, consider the tangential structure $\Theta_X$ defined in Example 2.1.2(d). As discussed before, a $\Theta_X$ structure on a manifold $W$ is just a continuous map $W \to X$, and therefore, if $W$ is closed, $\mathrm{Bun}^{\Theta_X}(W)$ is the space of continuous maps from $W$ to $X$.

## 2.2 Moduli spaces of manifolds

The action of the diffeomorphism group of $W$ on the tangent bundle $TW$ induces an action on the space $\mathrm{Bun}^{\Theta}(W)$ for any tangential structure $\Theta$. Explicitly, given $\phi \in \mathrm{Diff}(W)$ and $\rho \in \mathrm{Bun}^{\Theta}(W)$,

$$\phi \cdot \rho = \rho \circ D\phi^{-1}$$

where $D\phi : \mathrm{Fr}(TW) \to \mathrm{Fr}(TW)$ is the map induced by the differential of $\phi$.

**Definition 2.2.1.** Let $W$ be a closed manifold and fix $\rho_W$ a $\Theta$-structure on $W$, we define $\mathrm{Bun}^{\Theta}(W, \rho_W)$ to be the orbit of the path-component of $\rho_W$ in $\mathrm{Bun}^{\Theta}(W)$ under the action of the diffeomorphism group $\mathrm{Diff}(W)$. If $W$ has non-empty boundary $\mathrm{Bun}^{\Theta}(W, \rho_W)$ is defined to be the orbit of the path-component of $\rho_W$ in $\mathrm{Bun}^{\Theta}_{\rho_\partial}(W)$, where $\rho_\partial$ is the restriction of $\rho_W$ to the boundary.

We define the *moduli space of $W$ with $\Theta$-structures concordant to $\rho_W$* to be the Borel construction (ie. homotopy orbit space)

$$\mathcal{M}^{\Theta}(W, \rho_W) := \mathrm{Bun}^{\Theta}(W, \rho_W) /\!\!/ \mathrm{Diff}(W).$$

In other words, the moduli space $\mathcal{M}^{\Theta}(W, \rho_W)$ is the path-component of the Borel construction $\mathrm{Bun}^{\Theta}(W) /\!\!/ \mathrm{Diff}(W)$ that contains the classes represented by $\rho_W$.

*Remark* 2.2.2. In the above definition, when $W$ is a manifold with boundary and $\rho_W$ a fixed $\Theta$-structure, we have omitted the symbol $\rho_\partial$ from the notation for the

space $\mathrm{Bun}^{\Theta}(W, \rho_W)$. However, it should always be understood that there is a fixed boundary condition which is determined by the restriction of the fixed $\rho_W$ to the boundary.

The most important examples of these moduli spaces come from the simplest tangential structures: for the trivial tangential structure $\Theta^*$, the space $\mathrm{Bun}^{\Theta^*}(W)$ consists of a single point for any $W$ and therefore $\mathcal{M}^{\Theta^*}(W, \rho_W)$ will be simply the classifying space $B\mathrm{Diff}(W)$. On the other hand, if $W$ is an orientable manifold $\mathrm{Bun}^{\Theta^{or}}(W, \rho_W)$ consists of either one or two points depending on whether $\mathrm{Diff}(W)$ has an element that reverses the orientation of $W$. In either case, the moduli space $\mathcal{M}^{\Theta^{or}}(W, \rho_W)$ is homotopy equivalent to $B\mathrm{Diff}^+(W)$.

## 2.3 A lemma on fibre sequences and homotopy quotients

In this section we prove a lemma that will be used throughout the paper to construct fibre sequences of moduli spaces from equivariant fibre sequences of diffeomorphism groups and spaces of $\Theta$-structures.

**Lemma 2.3.1.** *Given a commutative diagram*

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle h} & \searrow^{f} & \\ Y & \xrightarrow{g} & Z \end{array}$$

*such that $f$ and $h$ are Serre fibrations and $h$ is surjective, then $g$ is also a Serre fibration.*

*Proof.* We will show that $g$ has the homotopy lifting property with respect to any inclusion $D^i \times \{0\} \hookrightarrow D^i \times I$ using that both $f$ and $h$ have this property.

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle h} \\ D^i \times \{0\} & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow{\scriptstyle g} \\ D^i \times I & \longrightarrow & Z \end{array} \qquad f$$

Given a lift of $D^i \times \{0\} \to Y$ to $X$, we can construct a lift $\ell : D^i \times I \to X$ using

that $f$ is a Serre fibration. Then $h \circ \ell : D^i \times I \to Y$ is a lift with respect to $g$. It only remains to see that any map $D^i \times \{0\} \to Y$ admits a lift to $X$, which we can prove by induction on $i$: for $i = 0$, this is precisely the condition that $h$ is surjective, for $i > 0$, this lift can be obtained using the identification $D^i \simeq D^{i-1} \times I$ and the fact that $h$ is a Serre fibration. □

**Lemma 2.3.2.** *Let $G_i$ be a topological group and $p_i : M_i \to M_i/G_i$ be a $G_i$-principal bundle, for $i = 1, 2, 3$.*

*(a) If $\phi : G_2 \to G_3$ is a continuous homomorphism and $f : M_2 \to M_3$ is a $\phi$-equivariant fibration, then the induced map*

$$\psi : M_2/G_2 \to M_3/G_3$$

*is a fibration.*

*(b) Given a short exact sequence*

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

*and a fibre sequence of equivariant maps $M_1 \to M_2 \to M_3$, the induced maps on quotients form a fibre sequence*

$$M_1/G_1 \to M_2/G_2 \to M_3/G_3$$

*Proof.* (a) By assumption, the map $p_2$ is a surjective fibration and the composition $\psi \circ p_2$ is equals the composition of fibrations $p_3 \circ f$. Therefore, by Lemma 2.3.1, $\psi$ is a fibration.

(b) Diagrammatically, we want to show that given the diagram of fibre sequences below, there exists a fibre sequence fitting into the bottom row:

$$\begin{array}{ccccc}
G_1 & \xrightarrow{\iota} & G_2 & \xrightarrow{\phi} & G_3 \\
\circlearrowright & & \circlearrowright & & \circlearrowright \\
M_1 & \xrightarrow{i} & M_2 & \xrightarrow{f} & M_3 \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} \\
M_1/G_1 & \dashrightarrow & M_2/G_2 & \overset{\psi}{\dashrightarrow} & M_3/G_3
\end{array}$$

By item (a), the map $\psi$ is a fibration, so all that remains is to identify its fibres. The composition $p_3 \circ f$ is a fibration with fibre $G_2 \cdot i(M_1) \subset M_2$. Then the fibre of $\psi$ is

$p_2(G_2 \cdot i(M_1)) = p_2(i(M_1))$. Since the action of $G_3$ on $M_3$ is free, we know that for any $g \in G_2$ which is not in the kernel of $\phi$, the intersection $(g \cdot i(M_1)) \cap i(M_1)$ is empty. So $p_2(i(M_1))$ is simply the quotient of $i(M_1)$ by the action of $\ker \phi = \iota(G_1)$. Then the map $M_1/G_1 \to M_2/G_2$ is precisely the inclusion of the fibre of $\psi$. □

**Corollary 2.3.3.** *Let $G_i$ be a topological group and $S_i$ be a $G_i$-space, for $i = 1, 2, 3$.*

*(a) If $\phi : G_2 \to G_3$ is a continuous homomorphism and $f : S_2 \to S_3$ is a $\phi$-equivariant fibration, then we can choose a model for the Borel constructions such that the induced map*

$$\psi : S_2 /\!\!/ G_2 \to S_3 /\!\!/ G_3$$

*is a fibration.*

*(b) Given a short exact sequence*

$$0 \to G_1 \to G_2 \xrightarrow{\phi} G_3 \to 0$$

*such that $\phi$ is a principal bundle, and a fibre sequence of equivariant maps $S_1 \to S_2 \to S_3$, the induced maps on quotients form a homotopy fibre sequence*

$$S_1 /\!\!/ G_1 \to S_2 /\!\!/ G_2 \to S_3 /\!\!/ G_3$$

*Proof.* Both statements follow from Lemma 2.3.2 and the following observations: fix $EG_2$, then the inclusion $\iota : G_1 \to G_2$ induces a $G_1$ action on $EG_2$ and since $\phi$ is a principal $G_1$-bundle, then so is $EG_2 \to EG_2/G_1$. Therefore the space $EG_2$ is a model for $EG_1$ as well. Also, any model for the space $EG_3$ carries an action of $G_2$ via the map $\phi : G_2 \to G_3$, and in particular, the quotient of $EG_2 \times EG_3$ by $G_2$ is still a $G_2$-principal bundle. Then the proof follows directly from applying Lemma 2.3.2 to the diagram

$$\begin{array}{ccccc}
G_1 & \xrightarrow{\quad\iota\quad} & G_2 & \xrightarrow{\qquad\phi\qquad} & G_3 \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
S_1 \times EG_2 & \longrightarrow & S_2 \times EG_2 \times EG_3 & \longrightarrow & S_3 \times EG_3 \\
\downarrow & & \downarrow & & \downarrow \\
S_1 /\!\!/ G_1 & \dashrightarrow & S_2 /\!\!/ G_2 & \dashrightarrow & S_3 /\!\!/ G_3
\end{array}$$

where the middle row is the product of the fibre sequence $S_1 \to S_2 \to S_3$ with the

trivial fibre sequence

$$EG_2 \longrightarrow EG_2 \times EG_3 \longrightarrow EG_3$$

and the action of the groups is the diagonal action. The commutativity of the diagram follows from the fact that the action of $G_1$ on $EG_3$ induced by $\phi \circ \iota$ is trivial. □

In particular, applying the above corollary to the trivial fibration $* \to *$, gives us the well-known result that a short exact sequence of groups $G_1 \to G_2 \to G_3$ induces a fibre sequence on classifying spaces

$$BG_1 \longrightarrow BG_2 \longrightarrow BG_3$$

## 2.4 The spectral sequence argument

The decoupling result will be deduced from the comparison of the homology spectral sequence associated to fibre sequences of moduli spaces. The result then follows from a well-known spectral sequence result which we recall (for a proof see [94]):

**Lemma 2.4.1** (*Spectral Sequence Argument*)**.** *Let $f : E^{\bullet}_{p,q} \to \tilde{E}^{\bullet}_{p,q}$ be a map of homological first quadrant spectral sequences. Assume that*

$$f : E^2_{p,q} \xrightarrow{\cong} \tilde{E}^2_{p,q} \qquad \text{for } 0 \leq p < \infty \text{ and } 0 \leq q \leq l.$$

*Then $f$ induces an isomorphism on the abutments in degrees $* \leq l$.*

Using the result above, we obtain the result which is in the centre of the decoupling:

**Proposition 2.4.2.** *Given a homotopy fibre sequence $F \xrightarrow{i} E \xrightarrow{p} B$ and a map $f : E \to M$ such that a the composite $f \circ i : F \to M$ induces a homology isomorphism in degrees $\leq \alpha$. Then the map $f \times p : E \to M \times B$ induces a homology isomorphism in degrees $i \leq \alpha$.*

*Proof.* The following is a map of fibre sequences

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ {\scriptstyle f\circ i}\downarrow & & \downarrow{\scriptstyle f\times p} & & \| \\ M & \longrightarrow & M \times B & \longrightarrow & B. \end{array}$$

This induces a map of the respective Serre spectral sequences $f : E^{\bullet}_{p,q} \to \tilde{E}^{\bullet}_{p,q}$, and since $f \circ i$ is a homology isomorphism in degrees $\leq \alpha$, the map between the $E^2$ pages

$$E^2_{p,q} = H_p\left(B\ ;\ H_q(F)\right) \longrightarrow H_p\left(B\ ;\ H_q(M)\right) = \tilde{E}^2_{p,q}$$

is an isomorphism for all $q \leq \alpha$. Then by the Spectral Sequence Argument (Lemma 2.4.1), $f \times p$ induces an isomorphism $H_k(E) \xrightarrow{\cong} H_k(M \times B)$ for all $k \leq \alpha$. $\square$

# Chapter 3

# The Decoupling Theorem

In this section we introduce decorated manifolds, the decorated moduli space, and the maps that are in the centre of the decoupling theorem: the forgetful map and evaluation map. We end by defining the decoupling map and proving the decoupling theorem.

For now, we focus on decorations being points and discs. These are the extreme cases: the simplest embedded manifolds of lowest and highest possible dimension. In section 5 we show how this can be extended to more general submanifold decorations, focusing on the case of manifolds decorated with embedded unlinked circles.

## 3.1 The decorated moduli space and the forgetful map

Throughout this section, let $W$ be a compact connected smooth manifold. We will study manifolds equipped with decorations:

**Definition 3.1.1.** A $d$-dimensional *manifold with decorations* consists of a manifold $W$ together with a set of distinct marked points in its interior $p_1, \dots, p_k \in W \setminus \partial W$ and disjoint embeddings $\phi_1, \dots, \phi_m : D^d \hookrightarrow W \setminus (\partial W \cup \{p_1, \dots, p_k\})$, with $k, m \in \mathbb{N}$. If $W$ is orientable, we require all embeddings to be oriented in the same way. We refer to these choices as *decorations* on our manifold.

Given a manifold $W$ with decorations, we define the *decorated diffeomorphism group* $\mathrm{Diff}^k_m(W)$ to be the subgroup of $\mathrm{Diff}(W)$ of the diffeomorphisms $\psi$ such that

$$\psi \circ \phi_j = \phi_{\alpha(j)} \qquad\qquad \psi(p_i) = p_{\beta(i)}$$

for some $\alpha \in \Sigma_m$ and $\beta \in \Sigma_k$.

In other words, we are looking at the diffeomorphisms that preserve the marked points and parametrized discs up to permutations. Note that the notation $\mathrm{Diff}^k_m(W)$ does not record which points and embedded discs comprise the decorations. The following lemma justifies this notation.

**Lemma 3.1.2.** *If $d \geq 2$, the isomorphism type of $\mathrm{Diff}^k_m(W)$ does not depend on the choice of the $k$ points and $m$ embedded discs that comprise the decorations.*

*Proof.* For any two collections of decorations in $W$ denoted

$$(p_1, \ldots, p_k, \phi_1, \ldots, \phi_m) \text{ and } (p'_1, \ldots, p'_k, \phi'_1, \ldots, \phi'_m),$$

there exists a diffeomorphism $\psi$ of $W$ such that

$$\psi(p_i) = p'_i \qquad\qquad \psi \circ \phi_i = \phi'_i$$

which can be constructed recursively by extending isotopies of the points and discs to diffeotopies of $W$ as described in [67, Theorem B] and [23, Proposition 6.2.4]. Then conjugation with $\psi$ defines an isomorphism between the group of diffeomorphisms preserving $(p_1, \ldots, p_k, \phi_1, \ldots, \phi_m)$ and the one preserving $(p'_1, \ldots, p'_k, \phi'_1, \ldots, \phi'_m)$. □

We are now ready to define the analogue of the moduli space including the decorations:

**Definition 3.1.3.** Given a manifold $W$ with a $\Theta$-structure $\rho_W$, we define the *decorated moduli space* of $W$ with $k$ points and $m$ discs to be

$$\mathcal{M}^{\Theta,k}_m(W, \rho_W) := \mathrm{Bun}^{\Theta}(W, \rho_W) /\!\!/ \mathrm{Diff}^k_m(W).$$

Recall that, if $W$ has non-empty boundary, then $\mathrm{Diff}(W)$ consists only of those diffeomorphisms fixing a collar of the boundary and the elements of $\mathrm{Bun}^{\Theta}(W, \rho_W)$ agree with $\rho_W$ on $\partial W$.

We define the forgetful map

$$F : \mathcal{M}^{\Theta,k}_m(W, \rho_W) \to \mathcal{M}^{\Theta}(W, \rho_W) \tag{3.1.1}$$

to be the one induced by the identity map on $\mathrm{Bun}^{\Theta}(W)$ and the subgroup inclusion $\mathrm{Diff}^k_m(W) \to \mathrm{Diff}(W)$.

## 3.2 The evaluation map

Let $W$ be a decorated manifold with $k$ marked points and $m$ marked discs. For each marked point $p_i$, we choose once and for all a frame of $T_{p_i}W$, and if $W$ is oriented, we ask that these frames have the same orientation. We also fix throughout this section $N \subset W$ which is the union of a tubular neighbourhood of the marked points and the interiors of the parametrized discs. We denote by $W_{m+k}$ the manifold $W \setminus N$. The decoupling result follows from understanding the difference between the decorated moduli space of $W$ and the moduli space of $W_{m+k}$.

For instance, assume $k = 0$ and $m = 1$, then there is a group isomorphism

$$\mathrm{Diff}(W_1) \to \mathrm{Diff}_1(W)$$

given by extending a diffeomorphism on $W_1$ by the identity on the marked disc. More generally, if $W$ is a manifold with $m$ embedded discs, the map

$$e_m : \mathrm{Diff}_m(W) \to \Sigma_m$$

taking a diffeomorphism $\phi$ to the $\alpha \in \Sigma_m$ recording the permutation induced on the discs by $\phi$, is a surjective homomorphism with kernel $\mathrm{Diff}(W_m)$, where, as above, $W_m$ is the manifold obtained from $W$ by removing the interior of the $m$ embedded discs.

Assume now $W$ has $k$ marked points $\{p_1, \ldots, p_k\}$ and no marked discs. We still get a homomorphism

$$\mathrm{Diff}(W_k) \to \mathrm{Diff}^k(W)$$

by extending a diffeomorphism on $W_k$ by the identity on the removed neighbourhood of the points, but this is not an isomorphism, since the elements of $\mathrm{Diff}^k(W)$ are not required to fix the entire neighbourhood of the marked points. A way to understand the diffeomorphisms around these is by looking at the differential map on the chosen frames at the marked points. So we define a map to the wreath product

$$e^k : \mathrm{Diff}^k(W) \longrightarrow \Sigma_k \wr \mathrm{GL}_d \tag{3.2.1}$$

$$\phi \longmapsto (D_{p_1}\phi, \ldots, D_{p_k}\phi, \beta) \tag{3.2.2}$$

where $\beta \in \Sigma_k$ is the permutation induced on the marked points by $\phi$. The image of $e^k$ depends the manifold $W$.

**Definition 3.2.1.** An orientable decorated manifold $W$ with $k$ marked points and $m$ marked discs is called *decorated-chiral* if every $\phi \in \mathrm{Diff}^k_m(W)$ preserves the orientation.

*Remark* 3.2.2. If the manifold $W$ is decorated by $m > 0$ discs or $\partial W \neq \emptyset$ then it is immediately decorated-chiral. There are also many manifolds for which any diffeomorphisms (not necessarily decorated) are orientation preserving, these are called *chiral* manifolds. A classical example is $\mathbb{C}P^2$, which can be deduced by analysing the automorphisms of its cohomology ring. Trivially, any chiral manifold is always decorated-chiral.

It follows from [88, Lemmas 2.3 and 2.4] that:

**Lemma 3.2.3** ([88])**.** *Let $W$ be a compact connected decorated manifold, then:*

*(a) the map*

$$e: \ \mathrm{Diff}^k_m(W) \xrightarrow{e_m \times e^k} \Sigma_m \times (\Sigma_k \wr \mathrm{GL}^\dagger_d)$$

*is a surjective principal bundle, where the group $\mathrm{GL}^\dagger_d$ is $\mathrm{GL}^+_d$ if $W$ is decorated-chiral, and $\mathrm{GL}_d$ otherwise.*

*(b) $\mathrm{Diff}(W_{m+k})$ is the homotopy fibre of $e$.*

*Remark* 3.2.4. Identifying the image of $e$ is important because we want to use a Serre spectral sequence to compare the homology of the total spaces of two fibre sequences. Therefore it is important to identify precisely the images of the fibrations we define.

A generalisation of the above lemma provides a fibre sequence on moduli spaces with tangential structures which is the key to the proof of the decoupling.

**Proposition 3.2.5.** *Let $W$ be a compact connected decorated manifold and $\rho_W$ a fixed $\Theta$-structure on $W$, then:*

*(a) the homomorphism $e$ induces an evaluation map*

$$E: \ \mathcal{M}^{\Theta,k}_m(W, \rho_W) \longrightarrow \Theta^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}^\dagger_d)^k /\!\!/ \Sigma_k$$

*which is a Serre fibration onto the path component which it hits, where the group $\mathrm{GL}^\dagger_d$ is $\mathrm{GL}^+_d$ if $W$ is decorated-chiral, and $\mathrm{GL}_d$ otherwise.*

*(b) let $W_{m+k}$ be equipped with the $\Theta$-structure $\rho_{W_{m+k}}$ given by the restriction of $\rho_W$. Then*

$$\mathcal{M}^\Theta(W_{m+k}, \rho_{W_{m+k}})$$

*is the homotopy fibre of $E$ over its image.*

To prove the Proposition, we will need the following lemma:

**Lemma 3.2.6.** *Let $W$ be a connected manifold and $S$ a smooth submanifold, then the restriction map*

$$r_S : \ \mathrm{Bun}^{\Theta}(W) \to \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_S), \Theta)$$

*is a Serre fibration.*

*Proof.* For any $i \geq 0$, a lift for the diagram

$$\begin{array}{ccc} D^i \times \{0\} & \longrightarrow & \mathrm{Bun}^{\Theta}(W) \\ \downarrow & \nearrow & \downarrow r_S \\ D^i \times I & \longrightarrow & \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_S), \Theta) \end{array}$$

is equivalent to a $\mathrm{GL}_d$-equivariant extension of the following

$$\begin{array}{ccc} (D^i \times \{0\} \times \mathrm{Fr}(TW)) \cup (D^i \times I \times \mathrm{Fr}(TW|_S)) & \xrightarrow{\rho} & \Theta \\ \downarrow & \nearrow & \\ D^i \times I \times \mathrm{Fr}(TW) & & \end{array} \tag{3.2.3}$$

Since the inclusion $S \hookrightarrow W$ is an embedding, there exists a strong deformation retract

$$r : D^i \times I \times W \longrightarrow (D^i \times \{0\} \times W) \cup (D^i \times I \times S).$$

If $i$ denotes the inclusion of $(D^i \times \{0\} \times W) \cup (D^i \times I \times S)$ into $D^i \times I \times W$, we have an isomorphism

$$f : D^i \times I \times \mathrm{Fr}(TW) \xrightarrow{\cong} r^* i^* (D^i \times I \times \mathrm{Fr}(TW))$$

which is the identity on $(D^i \times \{0\} \times \mathrm{Fr}(TW)) \cup (D^i \times I \times \mathrm{Fr}(TW|_S))$.

Therefore the composite

$$D^i \times I \times \mathrm{Fr}(TW) \xrightarrow{f} r^* i^* (D^i \times I \times \mathrm{Fr}(TW)) \xrightarrow{r^*} i^* (D^i \times I \times \mathrm{Fr}(TW)) \xrightarrow{\rho} \Theta$$

gives a lift to diagram 3.2.3. This implies that the map $\mathrm{Bun}^{\Theta}(W) \to \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_S), \Theta)$ is a Serre fibration. □

*Proof of Proposition 3.2.5.* (a) Let $P \subset W$ be the union of the $k$ marked points and

the centres of the $m$ marked discs. By Lemma 3.2.6, the restriction map

$$r_P : \mathrm{Bun}^{\Theta}(W, \rho_W) \to \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_P), \Theta)$$

is a Serre fibration. For each marked point, we chose a frame of its tangent space. Each point in the centre of a marked disc, comes with a preferred frame induced by the parametrization of the disc. So every point in $P$ is equipped with a frame of its tangent space, and this gives us a diffeomorphism $\mathrm{Fr}(TW|_P) \cong \mathrm{GL}_d \times P$. Therefore the space of $\mathrm{GL}_d$-equivariant maps $\mathrm{Fr}(TW|_P) \to \Theta$ can be identified with the space of continuous maps $P \to \Theta$, which is just $\Theta^m \times \Theta^k$. The result then follows by applying Corollary 2.3.3 to combine the fibration $r_P$ with the homomorphism of Lemma 3.2.3

$$\mathrm{Diff}^k_m(W) \xrightarrow{e} \Sigma_m \times (\Sigma_k \wr \mathrm{GL}^\dagger_d). \tag{3.2.4}$$

We apply Corollary 2.3.3(a) by taking $G_2 = \mathrm{Diff}^k_m(W)$ and $S_2 = \mathrm{Bun}^{\Theta}(W, \rho_W)$, with the usual action by precomposition with the differential. On the other hand, we take $G_3 = \Sigma_m \times (\Sigma_k \wr \mathrm{GL}^\dagger_d)$, and $S_3 = \Theta^m \times \Theta^k$ with the following action: the space $\Theta^m \times \Theta^k$ can be spit into the $m$ factors corresponding to the marked discs and $k$ factors corresponding to the marked points. Then we have an action of $\Sigma_m \times (\Sigma_k \wr \mathrm{GL}^\dagger_d)$ on $\Theta^m \times \Theta^k$ induced by the actions

$$\Sigma_m \curvearrowright \Theta^m \qquad \Sigma_k \wr \mathrm{GL}^\dagger_d \curvearrowright \Theta^k.$$

Then the fibration $\mathrm{Bun}^{\Theta}(W, \rho_W) \to \Theta^m \times \Theta^k$ is $e$-equivariant and therefore, by Corollary 2.3.3(a), we have a fibration

$$E: \ \mathcal{M}^{\Theta,k}_m(W, \rho_W) \longrightarrow \Theta^m /\!\!/ \Sigma_m \times \Theta^k /\!\!/ (\Sigma_k \wr \mathrm{GL}^\dagger_d).$$

Since $E\Sigma_k \times (E\mathrm{GL}_d)^k$ is a model for $E(\Sigma_k \wr \mathrm{GL}^\dagger_d)$, then

$$(\Theta /\!\!/ \mathrm{GL}^\dagger_d)^k /\!\!/ \Sigma_k$$

is a model for $\Theta^k /\!\!/ (\Sigma_k \wr \mathrm{GL}^\dagger_d)$, and the result follows.

(b) Recall that $W_{m+k}$ is defined as the submanifold $W \setminus N$, where $N$ is the union of a tubular neighbourhood of the marked points and the interiors of the marked discs. The restriction $\rho_{W_{m+k}}$ of $\rho_W$ is a $\Theta$-structure on $W_{m+k}$. In the remainder of the proof, we will show that $\mathcal{M}^{\Theta}(W_{m+k}, \rho_{W_{m+k}})$ is the homotopy fibre of $E$.

A description of the fibre of $E$ can be obtained using Corollary 2.3.3(b) with the short exact sequence of groups being

$$\ker e \to \mathrm{Diff}^k_m(W) \xrightarrow{e} \Sigma_m \times (\Sigma_k \wr \mathrm{GL}^\dagger_d) \tag{3.2.5}$$

and the fibre sequence of $S_1 \to S_2 \to S_3$ being the one associated to the fibration $r_P$ of item (a). The fibre of $r_P$ over $r_P(\rho_W)$ is the subspace of all elements of $\mathrm{Bun}^\Theta(W, \rho_W)$ which restrict to $r_P(\rho_W)$ over $P$, which we here denote $\mathrm{Bun}^\Theta_P(W, \rho_W)$. This space carries an action of $\ker e$ by precomposition with the differential, and it is simple to check that this fibre sequence is equivariant with respect to (3.2.5). Then by Corollary 2.3.3(b), the fibre of the evaluation map $E$ is given by

$$\mathrm{Bun}^\Theta_P(W, \rho_W) /\!\!/ \ker e.$$

Applying Lemma 3.2.6 to both submanifolds $P$ and $N$, we obtain two fibrations fitting into the following commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}^\Theta(W, \rho_W) & \xrightarrow{r_N} & \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_N), \Theta) \\ \downarrow{=} & & \downarrow{i^*} \\ \mathrm{Bun}^\Theta(W, \rho_W) & \xrightarrow{r_P} & \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_P), \Theta) \cong \Theta^m \times \Theta^k \end{array}$$

where the right-hand vertical map is induced by the inclusion $i : P \hookrightarrow N$. Since $\mathrm{Fr}(TD^d)$ is isomorphic to $\mathrm{GL}_d \times D^d$ as $\mathrm{GL}_d$-bundles, and the spaces $\mathrm{GL}_d \times D^d$ and $\mathrm{GL}_d \times \{*\}$ are homotopy equivalent as $\mathrm{GL}_d$-spaces, the map $i^*$ is a homotopy equivalence. In particular, this implies that the map from the fibre of $r_N$ to $\mathrm{Bun}^\Theta_P(W, \rho_W)$ is a homotopy equivalence.

The fibre of $r_{_N}$ over $r_{_N}(\rho_W)$ is by definition the space of all $\Theta$ structures on $W$ which agree with $\rho_W$ on $N$. We claim that this space is homeomorphic to $\mathrm{Bun}^\Theta(W_{m+k}, \rho_{W_{m+k}})$, since the restriction map $r_{_{W_{m+k}}}$ takes the fibre of $r_{_N}$ bijectively to $\mathrm{Bun}^\Theta(W_{m+k}, \rho_{W_{m+k}})$ and it has an inverse given by extending an element by $r_{_N}(\rho_W)$.

So we have a commutative diagram of principal fibre bundles

$$\begin{array}{ccc}
\mathrm{Diff}(W_{m+k}) & \xrightarrow{\quad\simeq\quad} & \ker e \\
\downarrow & & \downarrow \\
\mathrm{Bun}^{\Theta}(W_{m+k}, \rho_{W_{m+k}}) \times E\mathrm{Diff}(W) & \xrightarrow{\simeq} & \mathrm{Bun}^{\Theta}_{P}(W, \rho_W) \times E\mathrm{Diff}(W) \\
\downarrow & & \downarrow \\
\mathcal{M}^{\Theta}(W_{m+k}, \rho_{W_{m+k}}) & \longrightarrow & \mathrm{Bun}^{\Theta}_{P}(W, \rho_W) /\!\!/ \ker e
\end{array}$$

where the top horizontal map is a homotopy equivalence by Lemma 3.2.3 and the middle map is a homotopy equivalence by the discussion above. Therefore the map

$$\mathcal{M}^{\Theta}(W_{m+k}, \rho_{W_{m+k}}) \to \mathrm{Bun}^{\Theta}_{P}(W, \rho_W) /\!\!/ \ker e$$

is also a homotopy equivalence, as required. □

## 3.3 Proof of the Decoupling

In this section we prove the decoupling result by comparing the homotopy fibration sequence from Proposition 3.2.5 to the product fibre sequence via the aforementioned spectral sequence argument.

**Definition 3.3.1.** The *decoupling map*

$$D: \ \mathcal{M}^{\Theta,k}_{m}(W, \rho_W) \xrightarrow{F \times E} \mathcal{M}^{\Theta}(W, \rho_W) \times \Theta^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}^{\dagger}_{d})^k /\!\!/ \Sigma_k$$

is the product of the forgetful map (3.1.1) and the evaluation map $E$ defined in Proposition 3.2.5.

We now restate the decoupling theorem:

**Theorem 3.3.2.** *Let $W$ be a smooth connected compact manifold equipped with a $\Theta$-structure $\rho_W$. If the map*

$$\tau : H_i(\mathcal{M}^{\Theta}(W_{m+k}, \rho_{W_{m+k}})) \to H_i(\mathcal{M}^{\Theta}(W, \rho_W))$$

*induces a homology isomorphism in degrees $i \leq \alpha$, then for all such $i$ the decoupling map $D$ induces an isomorphism*

$$H_i(\mathcal{M}^{\Theta,k}_{m}(W, \rho_W)) \cong H_i(\mathcal{M}^{\Theta}(W, \rho_W) \times \Theta^m_0 /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}^{\dagger}_{d})^k_0 /\!\!/ \Sigma_k)$$

*where* $(-)_0$ *denotes a path component of the image of* $\rho_W$*, and the group* $\mathrm{GL}_d^\dagger$ *is* $\mathrm{GL}_d^+$ *if* $W$ *is orientable and decorated-chiral, and* $\mathrm{GL}_d$ *otherwise.*

*Proof.* By Proposition 3.2.5, $E$ is a fibration. Since $\mathcal{M}_m^{\Theta,k}(W, \rho_W)$ is connected by definition, we know that the image of $E$ is precisely the path component

$$\Theta_0^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}_d^\dagger)_0^k /\!\!/ \Sigma_k.$$

So we have a homotopy fibre sequence

$$\mathcal{M}^\Theta(W_{m+k}, \rho_{W_{m+k}}) \longrightarrow \mathcal{M}_m^{\Theta,k}(W, \rho_W) \xrightarrow{E} \Theta_0^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}_d^\dagger)_0^k /\!\!/ \Sigma_k.$$

By assumption, the composition

$$\mathcal{M}^\Theta(W_{m+k}, \rho_{W_{m+k}}) \to \mathcal{M}_m^{\Theta,k}(W, \rho_W) \xrightarrow{F} \mathcal{M}^\Theta(W, \rho_W)$$

induces a homology isomorphism in degrees $i \leq \alpha$. Therefore, by Proposition 2.4.2, the decoupling map $D$ induces a homology isomorphism in the same range of degrees. □

We now discuss how the decoupling result can be re-stated with a geometric interpretation. As discussed in Section 1.1.1, the space $\mathrm{Emb}(W, \mathbb{R}^\infty)$ is a model for $E\mathrm{Diff}(W)$, and therefore it is also a model for $E\mathrm{Diff}_m^k(W)$. With this model, the elements of $\mathcal{M}_m^{\Theta,k}(W, \rho_W)$ are decorated submanifolds of $\mathbb{R}^\infty$ diffeomorphic to $W$ with $k$ marked points and $m$ disjoint embedded discs, with a choice of $\Theta$-structure concordant (ie. equivariantly homotopic) to $\rho_W$. With this model, the forgetful map

$$F : \mathcal{M}_m^{\Theta,k}(W, \rho_W) \to \mathcal{M}^\Theta(W, \rho_W) \tag{3.3.1}$$

simply forgets the marked points and discs.

To interpret the evaluation map $E$ with this model, we need also a geometric model for $E\Sigma_s$. Recall that the *configuration space of* $s$ *points* in a manifold $M$ is defined as

$$\mathrm{C}_s(M) := \mathrm{Emb}(\{1, \ldots, s\}, M)/\Sigma_s$$

where the action of $\Sigma_s$ is given by permutation of the points in $\{1, \ldots, s\}$. In other words, $\mathrm{C}_s(M)$ is the space of unordered collections of $s$ distinct points in $M$. More generally, given a space $X$, the *configuration space of* $s$ *points in* $M$ *with labels in* $X$

is defined as

$$\mathrm{C}_s(M;X) := (\mathrm{Emb}(\{1,\dots,s\},M)\times X^s)/\Sigma_s$$

where $\Sigma_s$ acts by permuting the factors of $X^s$, and acts on the product diagonally. In other words, $\mathrm{C}_s(M)$ is the space of unordered collections of $s$ distinct points in $M$, where each point is labelled by a point in $X$.

Since the space $\mathrm{Emb}(\{1,\dots,m\},\mathbb{R}^\infty)$ is weakly contractible, it is a model for the total space $E\Sigma_m$. Therefore a model for $\Theta^m /\!\!/ \Sigma_m$ is precisely the space of unordered configurations of $m$ points in $\mathbb{R}^\infty$ with labels in $\Theta$. Analogously, a model for the $(\Theta /\!\!/ \mathrm{GL}_d^\dagger)^k /\!\!/ \Sigma_k$ is given by the space of unordered configurations of $k$ points in $\mathbb{R}^\infty$ with labels in $(\Theta /\!\!/ \mathrm{GL}_d^\dagger)$.

Then the evaluation map $E$ takes a decorated submanifold $S$ in $\mathbb{R}^\infty$ together with a tangential structure $\rho$ to the configurations given by the centres of the $m$ marked points, and the $k$ marked discs. The labels of such configurations are determined by the tangential structure $\rho$: let $p$ be the centre point of a marked disc and let $V_p$ be the canonical frame of the tangent space of $W$ at $p$ induced by the parametrization of the disc. Then the label of the point corresponding to $p$ in the configuration space $\mathrm{C}_m(\mathbb{R}^\infty;\Theta)$ is given by $\rho(V_p)\in\Theta$. Analogously, if $p$ is a marked point and $V_p$ is our chosen frame of its tangent space, then the label of the point corresponding to $p$ in $\mathrm{C}_k(\mathbb{R}^\infty;\Theta /\!\!/ \mathrm{GL}_d^\dagger)$ is simply the class of $\rho(V_p)$ in the Borel construction $\Theta /\!\!/ \mathrm{GL}_d^\dagger$.

With these models, the decoupling map can be interpreted geometrically (see Figure 1.1), and the decoupling theorem can be re-stated as:

**Corollary 3.3.3.** *Let $W$ be a smooth connected compact manifold equipped with a $\Theta$-structure $\rho_W$. If the map*

$$\tau : H_i(\mathcal{M}^\Theta(W_{m+k},\rho_{W_{m+k}})) \to H_i(\mathcal{M}^\Theta(W,\rho_W))$$

*induces a homology isomorphism in degrees $i\leq\alpha$, then for all such $i$ the decoupling map $D$ induces an isomorphism*

$$H_i(\mathcal{M}_m^{\Theta,k}(W,\rho_W)) \cong H_i(\mathcal{M}^\Theta(W,\rho_W)\times \mathrm{C}_m(\mathbb{R}^\infty;\Theta_0)\times \mathrm{C}_k(\mathbb{R}^\infty;(\Theta /\!\!/ \mathrm{GL}_d^\dagger)_0))$$

*where $(-)_0$ denotes a path component of the image of $\rho_W$, and the group $\mathrm{GL}_d^\dagger$ is $\mathrm{GL}_d^+$ if $W$ is orientable and decorated-chiral, and $\mathrm{GL}_d$ otherwise.*

## 3.4 Stable decoupling with non-trivial coefficients

We finish this chapter discussing the implications of the decoupling theorem for the colimit of a manifold after stabilising by increasing the genus. For surfaces this gives a result on the homology of the $B\mathrm{Diff}^k(F_\infty)$, where $F_\infty$ is the surface of infinite genus. This result will be central in chapter 8.

In this section we assume $W$ to be an $d = 2n$-dimensional smooth manifold with non-empty boundary decorated with $k$ points $p_1, \dots, p_k$. As before, we fix trivializations of the tangent spaces of the marked points, positively oriented if $W$ is orientable, and fix a tubular neighbourhood $N$ of the marked points. As before, we denote by $\mathrm{Diff}^k(W)$ the subgroup of diffeomorphisms fixing a collar of the boundary, and fixing the set of marked points but allowing permutations.

To make the stabilisation precise, define $S$ to be the connected sum of $[0,1] \times \partial W$ with $S^n \times S^n$. We define stabilisation homomorphisms

$$\sigma : \mathrm{Diff}(W) \to \mathrm{Diff}(W \cup_{\partial W} S), \qquad \sigma : \mathrm{Diff}^k(W) \to \mathrm{Diff}^k(W \cup_{\partial W} S)$$

by extending a diffeomorphism by the identity on $S$. We define the stable diffeomorphism group $\mathrm{Diff}(W_\infty)$ as the colimit

$$\mathrm{Diff}(W_\infty) \coloneqq \mathrm{co\,lim}(\mathrm{Diff}(W) \xrightarrow{\sigma} \mathrm{Diff}(W \cup_{\partial W} S) \xrightarrow{\sigma} \dots)$$

And define analogously the subgroup $\mathrm{Diff}^k(W_\infty)$ as the respective colimit. This stabilisation is compatible with the forgetful map and evaluation maps of the previous section and also with the map

$$\mathrm{Diff}(W \setminus \mathring{D}^d) \to \mathrm{Diff}(W)$$

induced by extending diffeomorphisms by the identity. Similarly, let $\rho_W$ be a $\Theta$-structure on $W$ and fix $\rho_S$ a $\Theta$-structure on $S$ which agrees with $\rho_W$ on the boundary. We define

$$\mathrm{Bun}^\Theta(W_\infty, \rho_\infty) \coloneqq \mathrm{co\,lim}(\mathrm{Bun}^\Theta(W, \rho_W) \xrightarrow{\sigma} \mathrm{Bun}^\Theta(W \cup_{\partial W} S, \rho_{W \cup S}) \xrightarrow{\sigma} \dots)$$

This definition only makes sense when such a $\Theta$-structure in $S$ exists. From now we will assume this is the case. The space $\mathrm{Bun}^\Theta(W_\infty, \rho_\infty)$ carries an action of the group $\mathrm{Diff}(W_\infty)$ and we can define the stable moduli space $\mathcal{M}^\Theta(W_\infty, \rho_\infty)$ as the Borel

construction

$$\mathrm{Bun}^{\Theta}(W_\infty, \rho_\infty) /\!\!/ \mathrm{Diff}(W_\infty).$$

In this section we discuss the decoupling for these colimits and we also allow some extra information encoded by a $\Sigma_k \wr \mathrm{GL}_d$-space $X$. Any such $X$ can be endowed with an action of $\mathrm{Diff}^k(W)$: using the fixed trivializations of the tangent space of the marked points and restricting a diffeomorphism to the action on the marked points, we get the evaluation homomorphism

$$e^k : \mathrm{Diff}^k(W) \to \Sigma_k \wr \mathrm{GL}_d^\dagger$$

as discussed in (3.2.1). This map commutes with the stabilisation map $\sigma$, thus any $\Sigma_k \wr \mathrm{GL}_d^\dagger$-space $X$ is via $e$ also a $\mathrm{Diff}^k(W)$-space. Moreover $\sigma$ induces a map

$$X /\!\!/ \mathrm{Diff}^k(W) \to X /\!\!/ \mathrm{Diff}^k(W \cup_{\partial W} S)$$

and we can consider the (homotopy limit space) $X /\!\!/ \mathrm{Diff}^k(W_\infty)$.

**Theorem 3.4.1.** *Assume the map*

$$\tau : H_i(\mathcal{M}^{\Theta}(W \setminus N, \rho_{W\setminus N})) \to H_i(\mathcal{M}^{\Theta}(W, \rho_W))$$

*induces a homology isomorphism in degrees increasing with the genus of $W$. Then for any $\Sigma_k \wr \mathrm{GL}_d^\dagger$ space $X$, the decoupling map $D$ induces a homology isomorphism*

$$(\mathrm{Bun}^{\Theta}(W_\infty, \rho_\infty) \times X) /\!\!/ \mathrm{Diff}^k(W_\infty) \to \mathcal{M}^{\Theta}(W_\infty) \times (\Theta^k \times X) /\!\!/ (\Sigma_k \wr \mathrm{GL}_d^\dagger)$$

*onto the path component which it hits. Here the group $\mathrm{GL}_d^\dagger$ is $\mathrm{GL}_d^+$ if $W$ is orientable and decorated-chiral, and $\mathrm{GL}_d$ otherwise.*

*Remark* 3.4.2. This stable result can also be generalised to include marked discs and the proof is essentially the same. We choose to just discuss the points case for simplicity and because this is the only case we will need in the future sections.

*Proof.* By Proposition 3.2.5 and Corollary 2.3.3, the evaluation map induces

$$(\mathrm{Bun}^{\Theta}(W_\infty, \rho_\infty) \times X) /\!\!/ \mathrm{Diff}^k(W_\infty) \xrightarrow{E} (\Theta^k \times X) /\!\!/ (\Sigma_k \wr \mathrm{GL}_d^\dagger)$$

which is a Serre fibration onto the path component which it hits. We also know that its fiber is of the homotopy type $\mathcal{M}^{\Theta}(W_\infty \setminus N, \rho_{\infty\setminus N})$ where $\rho_{\infty\setminus N}$ is the restriction

of $\rho_\infty$ to $W_\infty \setminus N$. The composition with the forgetful map $F$

$$F \circ i : \ \mathcal{M}^\Theta(W_\infty \setminus N, \rho_{\infty\setminus N}) \longrightarrow (\mathrm{Bun}^\Theta(W_\infty, \rho_\infty) \times X) /\!\!/ \mathrm{Diff}^k(W_\infty) \longrightarrow \mathcal{M}^\Theta(W_\infty, \rho_\infty)$$

is precisely the map $\tau$ and therefore it induces a homology isomorphism, by assumption. Therefore, by Proposition 2.4.2, the map

$$(\mathrm{Bun}^\Theta(W_\infty, \rho_W) \times X) /\!\!/ \mathrm{Diff}^k(W_\infty) \xrightarrow{F \times E} \mathcal{M}^\Theta(W_\infty) \times (\Theta^k \times X) /\!\!/ (\Sigma_k \wr \mathrm{GL}_d^\dagger)$$

is a homology isomorphism. □

In the next chapter we will discuss applications and examples of this decoupling theorem. This result will also be central in Chapter 8, when studying the group completion of monoids of configuration spaces on manifolds.

We finish the chapter by exploring further this result when $W$ is an oriented surface and the tangential structure is just orientation, which will be an important case used in chapter 8. We denote the surface of genus $g$, $b$ boundary components and $k$ punctures by $F_{g,b}^k$. As before, we define

$$\mathrm{Diff}(F_\infty) \coloneqq \mathrm{co\,lim}(\mathrm{Diff}(F_{g,1}) \xrightarrow{\sigma} \mathrm{Diff}(F_{g,1} \cup_{\partial F_{g,1}} S) \xrightarrow{\sigma} \dots)$$

where $S$ is the torus with the interior of two discs removed. In this case, the theorem above can be upgraded to a stronger statement.

Let $P(G)$ denote the maximal perfect subgroup of a group $G$. It is well-known that the maximal perfect subgroup of the symmetric group is the alternating group, $P(\Sigma_k) = A_k$, for $k \geq 5$, while the mapping class groups $\Gamma_{g,b}^k = \pi_0 \mathrm{Diff}(F_{g,b}^k)$ are perfect when $g \geq 3$ (for a nice exposition see [49, Section 5]). Then the maximal perfect subgroup of $\pi_0 \mathrm{Diff}^k(F_{g,b})$ is the group extension

$$\Gamma_{g,b}^k \to P(\pi_0 \mathrm{Diff}^k(F_{g,b})) \to A_k.$$

For any space $Z$, let $Z^+$ denote the *Quillen plus construction* with respect to the maximal perfect subgroup of the fundamental group $\pi_1 Z$.

**Corollary 3.4.3.** *For any $\mathrm{GL}_2^+$-space $X$, the decoupling map induces a homotopy equivalence*

$$(X /\!\!/ \mathrm{Diff}^k(F_\infty))^+ \xrightarrow{\simeq} (B\mathrm{Diff}(F_\infty))^+ \times (X /\!\!/ (\Sigma_k \wr S^1))^+$$

*Proof.* As discussed in [12, Section 2] the plus construction has the following property: if $F \to E \xrightarrow{p} B$ is a fibration of connected spaces then so is $F^+ \to E^+ \to B^+$ provided $p$ is quasi-nilpotent and $F^+$ is nilpotent. Quasi-nilpotent here means that the fundamental group $\pi_1 B$ of the base space acts nilpotently on the homology $H_* F$ of the fibre. We want to apply this to the fibre sequence

$$B\mathrm{Diff}(F_{\infty,k}) \xrightarrow{\ i\ } X /\!\!/ \mathrm{Diff}^k(F_\infty) \xrightarrow{\ E\ } X /\!\!/ (\Sigma_k \wr \mathrm{GL}_2^+) \tag{3.4.1}$$

which is a specialisation of the fibre sequence in the proof of theorem 3.4.1 to the case of surfaces with orientation. The fundamental group of the fibre is simply the stable mapping class group $\Gamma_{\infty,k}$ which is perfect, so the plus construction of the fibre has trivial fundamental group by and is in particular nilpotent. The action of the fundamental group of the basespace on the homology of the fibre factors through the surjection

$$\pi_1(X /\!\!/ (\Sigma_k \wr S^1)) \to \Sigma_k.$$

However, by [12, Lemma 3.3], $\Sigma_k$ acts trivially on the homology of the fibre. Therefore, after application of the plus construction on (3.4.1) we still get a fibre sequence fitting into the following diagram

$$\begin{array}{ccccc}
(B\mathrm{Diff}(F_{\infty,k}))^+ & \xrightarrow{\quad i \quad} & (X /\!\!/ \mathrm{Diff}^k(F_\infty))^+ & \xrightarrow{\quad E \quad} & (X /\!\!/ (\Sigma_k \wr \mathrm{GL}_2^+))^+ \\
\downarrow \simeq & & \downarrow (F\times E)_* & & \| \\
(B\mathrm{Diff}(F_\infty))^+ & \longrightarrow & (B\mathrm{Diff}(F_\infty))^+ \times (X /\!\!/ (\Sigma_k \wr \mathrm{GL}_2^+))^+ & \longrightarrow & (X /\!\!/ (\Sigma_k \wr \mathrm{GL}_2^+))^+
\end{array}$$

The map between the fibres is a homology isomorphism by Harer's theorem [38]. Therefore the maps between the base and the fibre are both homotopy equivalences, hence so is the map between the total spaces. □

The following is an example of the above Corollary for specific $X$ which will be relevant in Chapter 8.

**Example 3.4.4.** If $Y$ is an $\mathrm{GL}_2^+$-space then the $k$-fold Cartesian product $X = Y^k$ is naturally a $\Sigma_k \wr \mathrm{GL}_2^+$-space. The right hand term in the theorem can be written as

$$B\mathrm{Diff}(F_\infty) \times (Y /\!\!/ \mathrm{GL}_2^+)^k /\!\!/ \Sigma_k.$$

If $Y$ is a pointed $\mathrm{GL}_2^+$-space, then the $k$-fold wedge product $X = Y^{\wedge k}$ is an $\Sigma_k \wr \mathrm{GL}_2^+$-

space and the right hand term in the theorem can be written as

$$B\mathrm{Diff}(F_\infty) \times ((E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} Y)^{\wedge k} /\!\!/ \Sigma_k.$$

Here $Z_+ = Z \cup \{+\}$ denotes the space with a disjoint basepoint.

# Chapter 4

# Applications of the Decoupling Theorem

The main hypothesis of the decoupling theorem is that the map

$$\tau : H_i(\mathcal{M}^\Theta(W_{m+k}, r_{W_{m+k}}\rho_W)) \to H_i(\mathcal{M}^\Theta(W, \rho_W))$$

induces an isomorphism in a certain range $i \leq \alpha$. We know this condition holds in many circumstances for a variety of manifolds dimensions and tangential structures. In this section we recall some of these cases and discuss the applications of the decoupling theorem, focusing on the results for dimension 2 and for even dimensions greater than 4.

We remark that, for odd higher dimensions, many stability results on the homology of the moduli space have also been proven, but it is not yet known whether the map $\tau$ needed for the decoupling induces isomorphisms in a stable range.

## 4.1 Applications for surfaces

We now consider the 2 dimensional case, where the stability holds in many circumstances.

### 4.1.1 Orientation

For orientations, the classical stability result of Harer on the homology of mapping class groups of surfaces shows that the hypothesis of the decoupling theorem is satisfied for every oriented surface of genus $g$ and $b$ boundary components, $S_{g,b}$.

The range in which the isomorphism holds has been improved throughout the years [38, 44, 45, 46, 13, 75]. The most recent bound, by Randal-Williams in [75], implies that the map $H_i(B\mathrm{Diff}^+(S_{g,b+1})) \to H_i(B\mathrm{Diff}^+(S_{g,b}))$ is an isomorphism for all $3i \leq 2g$. Then applying the decoupling theorem, we recover the result of Bödigheimer and Tillmann, now with an improved isomorphism range:

**Corollary 4.1.1** ([12])**.** *For all* $3i \leq 2g$

$$H_i(B\mathrm{Diff}^{+,k}_m(S_{g,b})) \cong H_i(B\mathrm{Diff}^+(S_{g,b}) \times B\Sigma_m \times B(\Sigma_k \wr \mathrm{SO}(2))).$$

*Proof.* This is a direct application of the decoupling theorem. In this case, $\Theta^{or} = \{\pm 1\}$ and therefore $\Theta^{or}_0 = *$. Moreover, $\Theta^{or} /\!\!/ \mathrm{GL}_2^+$ is the disjoint union of two copies of $B\mathrm{GL}_2^+ \simeq B\mathrm{SO}(2)$, so $(\Theta^{or} /\!\!/ \mathrm{GL}_2^+)_0 \simeq B\mathrm{SO}(2)$, and therefore

$$(\Theta^{or} /\!\!/ \mathrm{GL}_2^+)^k_0 /\!\!/ \Sigma^k \simeq B(\Sigma_k \wr \mathrm{SO}(2))$$

as required. □

### 4.1.2 Non-orientable surfaces

Let $\mathcal{N}_{g,b}$ be the decorated non-orientable surface $\#_g \mathbb{R}P^\infty \setminus \coprod_b D^2$. Wahl showed in [91] that the map $H_i(B\mathrm{Diff}(\mathcal{N}_{g,b+1})) \to H_i(B\mathrm{Diff}(\mathcal{N}_{g,b}))$ is an isomorphism for all $4i \leq g - 3$. Applying the decoupling theorem, we recover the result of Hanbury in [36]:

**Corollary 4.1.2** ([36])**.** *For all* $4i \leq g - 5$

$$H_i(B\mathrm{Diff}^k_m(\mathcal{N}_{g,b})) \cong H_i(B\mathrm{Diff}(\mathcal{N}_{g,b}) \times B\Sigma_m \times B(\Sigma_k \wr \mathrm{O}(2))).$$

*Proof.* The result follows from applying the decoupling theorem for $\Theta^* = *$. Then $\Theta^* /\!\!/ \mathrm{GL}_2$ is homotopy equivalent to $B\mathrm{GL}_2 \simeq B\mathrm{O}(2)$, and the result follows. □

### 4.1.3 Framings

In [74], Randal-Williams showed that for the oriented surface $S_{g,b}$ with a framing $\rho$, the map $H_i(\mathcal{M}^{\mathrm{fr}}(S_{g,b+1}, \rho_{S_{g,b+1}})) \to H_i(\mathcal{M}^{\mathrm{fr}}(S_{g,b}, \rho))$ is an isomorphism for all $6i \leq 2g - 8$.

**Corollary 4.1.3.** *Let $\rho$ be a framing on $S_{g,b}$, then for all $6i \leq 2g-8$*

$$H_i(\mathcal{M}_m^{\mathrm{fr},k}(S_{g,b},\rho)) \cong H_i(\mathcal{M}^{\mathrm{fr}}(S_{g,b},\rho) \times \mathrm{SO}(2)^m /\!\!/ \Sigma_m \times B\Sigma_k).$$

*Proof.* The result is a direct application of the decoupling theorem, together with the fact that any path-component of $\Theta^{\mathrm{fr}} = \mathrm{GL}_2$ is homeomorphic to $\mathrm{GL}_2^+ \simeq \mathrm{SO}(2)$ and that $\Theta^{\mathrm{fr}} /\!\!/ \mathrm{GL}_2^+$ is equivalent to two points. □

### 4.1.4 Spin structures

In [40, 4, 74] it was shown that for the oriented surface $S_{g,b}$ with spin structure $\rho$, the map $H_i(\mathcal{M}^{\mathrm{Spin}}(S_{g,b+1},\rho_{S_{g,b+1}})) \to H_i(\mathcal{M}^{\mathrm{Spin}}(S_{g,b},\rho))$ is an isomorphism for all $5i \leq 2g-7$.

**Corollary 4.1.4.** *Let $\rho$ be a spin structure on $S_{g,b}$. For all $4i \leq g-2$, the group $H_i(\mathcal{M}_m^{\mathrm{Spin},k}(S_{g,b},\rho))$ is isomorphic to*

$$H_i\left(\mathcal{M}^{\mathrm{Spin}}(S_{g,b},\rho) \times B(\Sigma_m \wr \mathbb{Z}/2) \times B(\Sigma_k \wr \mathrm{Spin}(2))\right).$$

*Proof.* From Example 2.1.4, we know that $\Theta^{\mathrm{Spin}}$ is weakly equivalent to $\{\pm 1\} \times B\mathbb{Z}/2$, and therefore $(\Theta^{\mathrm{Spin}})_0$ is weakly equivalent to $B\mathbb{Z}/2$. On the other hand, $\Theta^{\mathrm{Spin}} /\!\!/ \mathrm{GL}_2^+$ is the disjoint union of two copies of $B\mathrm{Spin}(2)$ and therefore

$$(\Theta^{\mathrm{Spin}} /\!\!/ \mathrm{GL}_2^+)_0^k /\!\!/ \Sigma_k \simeq B\mathrm{Spin}(2)^k /\!\!/ \Sigma_k \simeq B(\Sigma_k \wr \mathrm{Spin}(2)).$$

□

### 4.1.5 Maps to a background space

Consider the tangential structure given by maps to a simply-connected background space $X$. It was shown in [21, 22, 75] that if $X$ is a simply-connected space, then the map $H_i(\mathcal{M}^{\mathrm{X}}(S_{g,b+1},\rho_{S_{g,b+1}})) \to H_i(\mathcal{M}^{\mathrm{X}}(S_{g,b},\rho))$ is an isomorphism for all $3i \leq 2g$, and all $\Theta_X$-structure $\rho$. Applying the decoupling theorem in this case, we obtain a generalisation of [22]*Theorem 8:

**Corollary 4.1.5.** *Let $\rho : S_{g,b} \to X$ be a continuous map and let $X_0$ be the path component containing the image of $\rho$. Then for all $3i \leq 2g$,*

$$H_i(\mathcal{M}_m^{\mathrm{X},k}(S_{g,b},\rho)) \cong H_i\left(\mathcal{M}^{\mathrm{X}}(S_{g,b},\rho) \times (X_0)^m /\!\!/ \Sigma_m \times (X_0)^k /\!\!/ (\Sigma_k \wr \mathrm{SO}(2))\right).$$

*Proof.* The result follows from applying the decoupling theorem together with the fact that $\Theta_X = X$ with the trivial action and $\Theta_X /\!\!/ \mathrm{GL}_2^+ \simeq X \times B\mathrm{GL}_2^+ \simeq X \times B\mathrm{SO}(2)$. □

### 4.1.6 $\mathrm{Spin}^r$ structures

The tangential structure called $\mathrm{Spin}^r$ is a generalisation of Spin which was thoroughly studied in [74]*Section 2. In this paper, Randal-Williams showed that for the oriented surface $S_{g,b}$ with a $\mathrm{Spin}^r$ structure $\rho$, the map

$$H_i(\mathcal{M}^{\mathrm{Spin}^r}(S_{g,b+1}, \rho_{S_{g,b+1}})) \to H_i(\mathcal{M}^{\mathrm{Spin}^r}(S_{g,b}, \rho))$$

is an isomorphism for all $6i \leq 2g - 8$.

**Corollary 4.1.6.** *Let $\rho$ be a $\mathrm{Spin}^r$ structure on $S_{g,b}$. For all $6i \leq 2g - 8$, the group $H_i(\mathcal{M}_m^{\mathrm{Spin}^r,k}(S_{g,b}, \rho))$ is isomorphic to*

$$H_i\left(\mathcal{M}^{\mathrm{Spin}^r}(S_{g,b}, \rho) \times B(\Sigma_m \wr \mathbb{Z}/r) \times B(\Sigma_k \wr \mathrm{Spin}^r(2))\right).$$

*Proof.* Using the fibre sequence

$$\{\pm 1\} \times B\mathbb{Z}/r \longrightarrow B\mathrm{Spin}^r(2) \longrightarrow B\mathrm{GL}_2$$

we can apply the procedure described in Example 2.1.4 to deduce that a path component of $\Theta^{\mathrm{Spin}^r}$ is weakly equivalent to the Lens space $B\mathbb{Z}/r$ and $\Theta^{\mathrm{Spin}^r} /\!\!/ \mathrm{GL}_2^+$ is the disjoint union of two copies of $B\mathrm{Spin}^r(2)$. □

### 4.1.7 $\mathrm{Pin}^{\pm}$ structures

The tangential structures called $\mathrm{Pin}^+$ and $\mathrm{Pin}^-$ are generalisations of Spin for non-orientable manifolds, and they were thoroughly studied in [74]*Section 4. In this paper, Randal-Williams showed that for the non-orientable surface $\mathcal{N}_{g,b} = \#_g \mathbb{R}P^\infty \setminus \coprod_b D^2$ with a $\mathrm{Pin}^+$-structure, the map

$$H_i(\mathcal{M}^{\mathrm{Pin}^+}(\mathcal{N}_{g,b+1}, \rho_{\mathcal{N}_{g,b+1}})) \to H_i(\mathcal{M}^{\mathrm{Pin}^+}(\mathcal{N}_{g,b}, \rho))$$

is an isomorphism for all $4i \leq g - 6$, and $\mathrm{Pin}^+$-structure $\rho$.

It was also shown in [74] that the map

$$H_i(\mathcal{M}^{\mathrm{Pin}^-}(\mathcal{N}_{g,b+1}, \rho_{\mathcal{N}_{g,b+1}})) \to H_i(\mathcal{M}^{\mathrm{Pin}^-}(\mathcal{N}_{g,b}, \rho))$$

is an isomorphism for all $5i \leq g-8$, and $\mathrm{Pin}^-$-structure $\rho$.

**Corollary 4.1.7.** *Let $\rho$ be a $\mathrm{Pin}^+$ structure on $\mathcal{N}_{g,b}$, then for all $4i \leq g-6$*

$$H_i(\mathcal{M}_m^{\mathrm{Pin}^+,k}(\mathcal{N}_{g,b},\rho)) \cong H_i(\mathcal{M}^{\mathrm{Pin}^+}(\mathcal{N}_{g,b},\rho) \times B(\Sigma_m \wr \mathbb{Z}/2) \times B(\Sigma_k \wr \mathrm{Pin}^+(2))).$$

**Corollary 4.1.8.** *Let $\rho$ be a $\mathrm{Pin}^-$ structure on $\mathcal{N}_{g,b}$, then for all $5i \leq g-8$*

$$H_i(\mathcal{M}_m^{\mathrm{Pin}^-,k}(\mathcal{N}_{g,b},\rho)) \cong H_i(\mathcal{M}^{\mathrm{Pin}^-}(\mathcal{N}_{g,b},\rho) \times B(\Sigma_m \wr \mathbb{Z}/2) \times B(\Sigma_k \wr \mathrm{Pin}^-(2))).$$

*Proof of Corollaries 4.1.7 and 4.1.8.* Using the fibre sequence

$$B\mathbb{Z}/2 \longrightarrow B\mathrm{Pin}^{\pm}(d) \longrightarrow B\mathrm{GL}_d$$

we can apply the procedure described in Example 2.1.4 to deduce that a path component of $\Theta^{\mathrm{Pin}^{\pm}}$ is weakly equivalent to $B\mathbb{Z}/2$, and $\Theta^{\mathrm{Pin}^{\pm}}/\!\!/\mathrm{GL}_2$ is the space $B\mathrm{Pin}^{\pm}(2)$. □

## 4.2 Applications for high dimensional manifolds

One of the most interesting applications of the decoupling result appears when looking at higher dimensional manifolds. In [30], it was shown that the hypothesis of the decoupling theorem holds for many manifolds $W$ of even dimension greater or equal to 6, and many tangential structures. The range of the homology isomorphism is given in terms of the stable genus of $W$, which we now recall, following the notation of [32].

Analogously to the surface case, the genus will be measured by disjoint embeddings of the space $(S^n \times S^n) \setminus \{*\}$, but now taking into account the tangential structure as well. Namely, Galatius and Randal-Williams define what it means for a $\Theta$-structure on $(S^n \times S^n) \setminus \{*\}$ to be *admissible* (see [32]*Section 3.2) and define the *genus* of a manifold $W$ with $\Theta$-structure $\rho_W$ to be

$$g(W,\rho_W) = \max\left\{ g \in \mathbb{N} \;\middle|\; \begin{array}{c} \text{there are } g \text{ disjoint embeddings } j : (S^n \times S^n) \setminus \{*\} \hookrightarrow W \\ \text{such that } j^*\rho_W \text{ is admissible} \end{array} \right\}.$$

The *stable genus* of $(W,\rho_W)$ is defined to be

$$\overline{g}(W,\rho_W) = \max\left\{ g\left(W \# W_{k,1}, \rho_W^{(k)}\right) - k \,|\, k \in \mathbb{N} \right\}$$

where $W\#W_{k,1}$ is obtained from $W$ by removing $k$ discs and attaching $k$ copies of

$(S^n \times S^n) \setminus \mathrm{int}(D^{2n})$ along the new boundary. The $\Theta$-structure $\rho_W^{(k)}$ is obtained by extending the restriction of $\rho_W$ by any admissible structure on $(S^n \times S^n) \setminus \mathrm{int}(D^{2n})$.

**Lemma 4.2.1.** *Let $W$ be a smooth compact manifold of dimension $2n \geq 2$, $L \subset \mathrm{int}(W)$ a closed submanifold of dimension $\leq n-1$, and $N$ a tubular neighbourhood of $L$. Then the genus of $W \setminus N$ is equal to the genus of $W$.*

*Proof.* Sard's theorem implies that for any submanifold $L' \subset (S^n \times S^n) \setminus \{*\}$ with $dim(L') \leq n-1$ there is an embedding $(S^n \times S^n) \setminus \{*\} \hookrightarrow (S^n \times S^n) \setminus \{*\}$ that avoids $L'$ and is isotopic to the identity. In particular, this implies that for any $\phi : \coprod_g (S^n \times S^n) \setminus \{*\} \hookrightarrow W$, there is an embedding

$$\phi' : \coprod_g (S^n \times S^n) \setminus \{*\} \hookrightarrow \coprod_g (S^n \times S^n) \setminus \{*\} \xhookrightarrow{\phi} W$$

that avoids $L$ and is isotopic to $\phi$. Since $W \setminus L$ is diffeomorphic to $\mathrm{int}(W \setminus N)$ via a diffeomorphism fixing everything but a collar of $L$, the result follows. □

Let $W$ be a manifold with non-empty boundary $P$, and $\rho_W$ a $\Theta$-structure on $W$. Given $M$ a cobordism from $P$ to $Q$ together with a $\Theta$-structure $\rho_M$ on $M$ which restricts to $\rho_W$ over $P$, there is an induced map

$$- \cup_P (M, \rho_M) : \mathcal{M}^\Theta(W, \rho_W) \longrightarrow \mathcal{M}^\Theta(W \cup_P M, \rho_W \cup \rho_M) \tag{4.2.1}$$

which is induced by the $\mathrm{Diff}(W)$-equivariant map $\mathrm{Bun}^\Theta(W, \rho_W) \to \mathrm{Bun}^\Theta(W \cup_P M, \rho_W \cup \rho_M)$ given by extending a map by $\rho_M$, and the homomorphism $\mathrm{Diff}(W) \to \mathrm{Diff}(W \cup_P M)$ given by extending a map by the identity on $M$.

**Theorem 4.2.2** ([30], Corollary 1.7)**.** *Assume $d = 2n \geq 6$, and $\Theta$ is such that $\Theta /\!\!/ \mathrm{GL}_d$ is simply-connected. Let $\rho_W$ be an $n$-connected $\Theta$-structure on $W$ and let $g = \overline{g}(W, \rho_W)$. Given a cobordism $(M, \rho_M)$ as above such that $(M, P)$ is $(n-1)$-connected, the map*

$$(- \cup_P (M, \rho_M))_* : H_i(\mathcal{M}^\Theta(W, \rho_W)) \longrightarrow H_i(\mathcal{M}^\Theta(W \cup_P M, \rho_{W \cup_P M}))$$

*is an isomorphism for all $3i \leq g-4$.*

We recall that a map is called $n$-connected if the map induced on homotopy groups $\pi_i$ is an isomorphism for $i < n$ and a surjection for $i = n$.

*Remark* 4.2.3. In [30]*Corollary 1.7, the result above is given in much more generality, allowing arbitrary coefficient systems and providing a better stability range depending on the coefficient system and the tangential structure. We restrict ourselves to the case above, for simplicity, but remark that such generalisations can also be immediately carried out in the decoupling theorem.

**Corollary 4.2.4.** *Assume $d = 2n \geq 6$, and $\Theta$ is such that $\Theta /\!\!/ \mathrm{GL}_d$ is simply-connected. Let $\rho_W$ be an $n$-connected $\Theta$-structure on $W$ and let $g = \overline{g}(W, \rho_W)$. Then for all $i \leq \frac{g-4}{3}$ we have an isomorphism*

$$H_i(\mathcal{M}_m^{\Theta,k}(W, \rho_W)) \cong H_i(\mathcal{M}^{\Theta}(W, \rho_W) \times \Theta_0^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}_{2n}^{\dagger})_0^k /\!\!/ \Sigma_k)$$

*where $\mathrm{GL}_{2n}^{\dagger}$ equals to $\mathrm{GL}_{2n}^{+}$ if $W$ is decorated-chiral, and is $\mathrm{GL}_{2n}$ otherwise.*

*Proof.* First notice that since the map $\rho_W$ is $n$-connected and the pair $(W, W_{m+k})$ is $(2n-1)$-connected, then the restriction $\rho_{W_{m+k}}$ is still an $n$-connected $\Theta$-structure.

The map $\tau : H_i(\mathcal{M}^{\Theta}(W_{m+k}, \rho_{W_{m+k}})) \to H_i(\mathcal{M}^{\Theta}(W, \rho_W))$ in the hypothesis of the decoupling theorem, is induced by attaching $\coprod_{m+k} D^{2n}$ along the $m+k$ boundary sphere components of $W_{m+k}$. Since

$$(M, P) = (\coprod_{m+k} D^{2n}, \partial \coprod_{m+k} D^{2n})$$

is $(n-1)$-connected, the hypotheses of Theorem 4.2.2 are satisfied, which implies that $\tau$ induces a homology isomorphism in degrees $3i \leq g - 4$. Applying Theorem 3.3.2, the result follows. □

**Example 4.2.5.** Let $W_{g,1} = (S^n \times S^n) \# D^{2n}$. Since $TW_{g,1}$ is trivialisable, we know $W_{g,1}$ admits a framing $\rho_{W_{g,1}} : \mathrm{Fr}(TW_{g,1}) \to \mathrm{GL}_{2n}$ fitting into the following pullback diagram:

$$\begin{array}{ccc} \mathrm{Fr}(TW_{g,1}) & \xrightarrow{\rho_{W_{g,1}}} & \mathrm{GL}_{2n} \\ \downarrow & \lrcorner & \downarrow \\ W_{g,1} & \longrightarrow & E\mathrm{GL}_{2n} \end{array}$$

The bottom arrow is necessarily $n$-connected because $W_{g,1}$ is $(n-1)$-connected and $E\mathrm{GL}_{2n}$ is weakly contractible. Therefore, $\rho$ is $n$-connected as well.

Let $\overline{g}$ denote the stable genus $\overline{g}(W_{g,1}, \rho_{W_{g,1}})$. By Corollary 4.2.4, for all $i \leq \frac{\overline{g}-4}{3}$, the group $H_i(\mathcal{M}_m^{\mathrm{fr},k}(W_{g,1}, \rho_{W_{g,1}}))$ is isomorphic to

$$H_i\left(\mathcal{M}^{\mathrm{fr}}(W_{g,1}, \rho_{W_{g,1}}) \times \mathrm{SO}(2n)^m /\!\!/ \Sigma_m \times B\Sigma_k\right).$$

**Example 4.2.6.** Let $V_d \subset \mathbb{C}P^4$ be a smooth hypersurface determined by a homogeneous complex polynomial of degree $d$. This is an orientable chiral 6-dimensional manifold whose diffeomorphism type depends only on the degree $d$. In section 5.3 of [32], Galatius and Randal-Williams show that, if $d$ is even, there exists a 3-connected $\mathrm{Spin}^c$-structure $\rho_{V_d}$ on $V_d$. They also compute an expression for the stable genus $\overline{g}(V_d, \rho_{V_d})$ in terms of $d$.

Applying the procedure of Example 2.1.4 to the fibre sequence

$$\{\pm 1\} \times B\mathrm{U}(1) \longrightarrow B\mathrm{Spin}^c(d) \longrightarrow B\mathrm{GL}_d$$

we get that $\Theta^{\mathrm{Spin}^c} \simeq \{\pm 1\} \times B\mathrm{U}(1)$, and $\Theta^{\mathrm{Spin}^c} /\!\!/ \mathrm{GL}_6^+ \simeq \{\pm 1\} \times B\mathrm{Spin}^c(6)$. Therefore, by Corollary 4.2.4, for all $i \leq \frac{d^4-5d^3+10d^2-10d+4}{4}$, the group $H_i(\mathcal{M}_m^{\mathrm{Spin^c},k}(V_d, \rho_{V_d}))$ is isomorphic to

$$H_i\left(\mathcal{M}^{\mathrm{Spin^c}}(V_d, \rho_{V_d}) \times B(\Sigma_m \wr \mathrm{U}(1)) \times B(\Sigma_k \wr \mathrm{Spin}^c(6))\right)$$

The conditions on the tangential structure in Theorem 4.2.2 are quite restrictive, for instance the trivial tangential structure $\Theta^*$ does not satisfy the hypothesis because $B\mathrm{GL}_d$ is not simply connected for any $d$. Moreover, the condition that we start with an $n$-connected $\Theta$-structure $\rho_W$ excludes many of the cases we are interested in. For instance, it implies that the manifold $W_{g,1}$ with an orientation does not satisfy the hypothesis of Theorem 4.2.2. However, in [30]*Section 9, Galatius and Randal-Williams provided a generalisation of this result for general tangential structures. In Section 6, we use their techniques to prove a generalisation of Theorem B for high dimensional manifolds with any tangential structure.

# Chapter 5

# Decoupling Submanifolds

In Section 3, we proved a decoupling result for the decorated moduli space of a manifold with marked points and discs, following the works of [12, 36, 22]. Recently, in [69, 70, 71] Palmer has studied manifolds equipped with more general decorations, allowed to be any embedded closed manifold $P$. In this section, we show that there is a decoupling result for these generalised decorations. As a specific example, we focus on the case where the decorations are unlinked circles, which have also been closely studied in dimension 3 by Kupers in [50].

## 5.1 The $L$-decorated moduli space

In this section, we generalise the definition of a decorated manifold to allow more general submanifolds as decorations. Throughout, let $W$ be a smooth connected compact $d$-dimensional manifold.

**Definition 5.1.1.** A $d$-dimensional *$L$-decorated manifold* is a pair $(W, L)$ of a manifold $W$ together with a closed submanifold $L \subset W$.

Given a $L$-decorated manifold $(W, L)$, we define the *decorated diffeomorphism group* $\mathrm{Diff}_L(W)$ to be the subgroup of $\mathrm{Diff}(W)$ of the diffeomorphisms $\psi$ such that

$$\psi(L) = L.$$

In other words, we are looking at the diffeomorphisms preserving the marked submanifold, but not necessarily pointwise.

**Definition 5.1.2.** Given a closed manifold $W$ and a $\Theta$-structure $\rho_W$ on $W$, we define

the *L-decorated moduli space of* $W$ to be

$$\mathcal{M}_L^\Theta(W, \rho_W) := \mathrm{Bun}^\Theta(W, \rho_W) /\!\!/ \mathrm{Diff}_L(W).$$

The inclusion of groups $\mathrm{Diff}_L(W) \to \mathrm{Diff}(W)$ induces a map

$$F_L : \mathcal{M}_L^\Theta(W, \rho_W) \to \mathcal{M}^\Theta(W \rho_W) \tag{5.1.1}$$

which we call the *forgetful map*.

## 5.2 The evaluation map $E_L$

Let $(W, L)$ be an $L$-decorated manifold and let $\nu_L \coloneqq (TW_{|L})/TL$ be the normal bundle of $L$ in $W$. Let $N$ be the tubular neighbourhood of the decoration identified as the image of an embedding $\Phi : \nu_L \to W$, and denote by $W_N$ the manifold $W \setminus N$. The theorem for decoupling submanifolds relies on understanding the difference between the $L$-decorated moduli space of $W$ and the moduli space of $W_N$.

We start by constructing an equivariant fibre sequence relating the decorated diffeomorphism groups $\mathrm{Diff}_L(W)$ and $\mathrm{Diff}(W_N)$. Recall that $\mathrm{Diff}(W_N)$ consists only of those diffeomorphisms fixing a collar neighbourhood of the boundary of $W_N$, including the newly formed boundary obtained by removing $N$. Extending a diffeomorphism by the identity on $N$, gives us a homomorphism

$$\mathrm{Diff}(W_N) \to \mathrm{Diff}_L(W).$$

On the other hand, since any diffeomorphism $\phi \in \mathrm{Diff}_L(W)$ fixes $L$, the differential of $\phi$ induces an isomorphism of the tangent bundle $TW_{|L}$ fixing $TL$ (not necessarily pointwise). This gives a map:

$$e_L : \mathrm{Diff}_L(W) \longrightarrow \mathrm{Iso}(TW_{|L}, TL) \tag{5.2.1}$$

$$\phi \longmapsto D\phi|_L \tag{5.2.2}$$

where $D\phi|_L$ denotes the isomorphism of $TW_{|L}$ induced by the differential of $\phi$, and $\mathrm{Iso}(TW_{|L}, TL)$ denotes the group of bundle isomorphisms of $TW_{|L}$ fitting into the

following diagram:

$$\begin{array}{ccc} TL & \xrightarrow{Df} & TL \\ \downarrow & & \downarrow \\ TW_{|L} & \xrightarrow{\overline{f}} & TW_{|L} \\ \downarrow & & \downarrow \\ L & \xrightarrow{f} & L. \end{array}$$

**Definition 5.2.1.** For any subgroup $G \subset \operatorname{Im} e_L$, we define $\mathrm{Diff}_G(W)$ to be the subgroup $e_L^{-1}(G)$. Given a closed manifold $W$ and a $\Theta$-structure $\rho_W$ on $W$, we define

$$\mathcal{M}_G^\Theta(W, \rho_W) := \mathrm{Bun}^\Theta(W, \rho_W) /\!\!/ \mathrm{Diff}_G(W).$$

Note that taking $G = \operatorname{Im} e_L$, one recovers precisely the definition of $\mathcal{M}_L^\Theta(W, \rho_W)$. The kernel of $e_L$ consists precisely of those elements of $\mathrm{Diff}_L(W)$ which fix the submanifold $L$ pointwise and whose differential $D_p\phi$ is the identity on every point of the submanifold $L$. We denote the kernel of $e_L$ by $\mathrm{Diff}(W, TW|_L)$.

**Lemma 5.2.2.** *Let $(W, L)$ be a compact connected $L$-decorated manifold, then*

*(a) the homomorphism*

$$e_L : \ \mathrm{Diff}_L(W) \longrightarrow \mathrm{Iso}(TW_{|L}, TL)$$

*is a principal bundle;*

*(b) the map*

$$i : \ \mathrm{Diff}(W_N) \longrightarrow \mathrm{Diff}(W, TW|_L)$$

*is a homotopy equivalence.*

*Proof.* Throughout this proof, we will use a generalisation of Palais' theorem in [68] proved by Lima in [54], which gives us a principal bundle

$$\mathrm{Diff}(W_N) \longrightarrow \mathrm{Diff}(W) \longrightarrow \mathrm{Emb}(N, W).$$

Let $\mathrm{Emb}_L(N, W)$ be the subspace of embeddings $f : N \hookrightarrow W$ such that the core of $N$ is taken to our marked submanifold $L$ in $W$. Then taking the pullback along the inclusion $\mathrm{Emb}_L(N, W) \hookrightarrow \mathrm{Emb}(N, W)$ gives as the principal bundle:

$$\mathrm{Diff}(W_N) \longrightarrow \mathrm{Diff}_L(W) \xrightarrow{r} \mathrm{Emb}_L(N, W) \tag{5.2.3}$$

*Part (a):* Write $N$ as the image of an embedding $\exp \circ \Phi : \nu_L \hookrightarrow W$, where $\Phi : \nu_L \to TW_{|L}$. Consider the forgetful map

$$d : \mathrm{Emb}_L(N, W) \to \mathrm{Iso}(TW_{|L}, TL)$$

taking an embedding to the map induced on the normal bundle of the zero section $L \subset N$. Then $e_L = d \circ r$. We will show $d$ is a fibre bundle, which implies $e_L$ is a principal bundle. It is enough to exhibit a local section of $d$ at a neighbourhood of the identity (for details see [84, Part I, Section 7.4]).

Given $\overline{f} : TW_{|L} \to TW_{|L}$ in $\mathrm{Iso}(TW_{|L}, TL)$ we can define

$$s_f : \ \nu_L \xrightarrow{\Phi} TW_{|L} \xrightarrow{\overline{f}} TW_{|L} \xrightarrow{\exp} W.$$

Since the assignment $f \mapsto s_f$ is continuous and $\mathrm{Emb}(\nu_L, W)$ is an open subset of $\mathcal{C}^\infty(\nu_L, W)$, then the space of maps $\overline{f} \in \mathrm{Iso}(TW_L, TL)$ such that $s_f$ is an embedding, is an open neighbourhood $U$ of the identity. Therefore, the map

$$\begin{aligned} U &\longrightarrow \mathrm{Emb}(\nu_L, W) \\ \overline{f} &\longmapsto s_f \end{aligned}$$

is a local section for $d$ at the identity.

*Part (b):* We will show the map $i$ is a homotopy equivalence, from the fact that it fits into the following commutative diagram of fibre sequences:

$$\begin{array}{ccccc} \mathrm{Diff}(W_N) & \longrightarrow & \mathrm{Diff}_L(W) & \longrightarrow & \mathrm{Emb}_L(N, W) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle d} \\ \mathrm{Diff}(W, TW|_L) & \longrightarrow & \mathrm{Diff}_L(W) & \xrightarrow{e_L} & \mathrm{Iso}(TW_{|L}, TL) \end{array}$$

The fiber of the forgetful map $d$ over the identity is simply the space of tubular neighbourhoods of $L$ in $W$, which is contractible. This implies $d$ is a homotopy equivalence and therefore so is $i$. □

**Proposition 5.2.3.** *Let $W$ be a compact connected manifold and $\rho_W$ a fixed $\Theta$-structure on $W$, then:*

*(a) the homomorphism $e_L$, induces an evaluation map*

$$E_L : \ \mathcal{M}_G^\Theta(W, \rho_W) \longrightarrow \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta) /\!\!/ G$$

*which is a Serre fibration onto the path component which it hits.*

*(b) Let $W_N$ be equipped with the $\Theta$-structure $\rho_{W_N}$ given by the restriction of $\rho_W$. Then*

$$\mathcal{M}^{\Theta}(W_N, \rho_{W_N})$$

*is the homotopy fibre of $E_L$ over its image.*

*Proof.* (a) By Lemma 3.2.6, the restriction map

$$r_L : \mathrm{Bun}^{\Theta}(W, \rho_W) \to \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)$$

is a Serre fibration. Then the result follows by applying Corollary 2.3.3 to combine the fibration $r_L$ with the homomorphism

$$e_L : \mathrm{Diff}_G(W) \to G. \tag{5.2.4}$$

We apply Corollary 2.3.3(a) by taking $G_2 = \mathrm{Diff}_G(W)$ and $S_2 = \mathrm{Bun}^{\Theta}(W, \rho_W)$, with the usual action by precomposition with the differential. On the other hand, we take $G_3 = G$, and $S_3 = \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)$ with the action induced by

$$\mathrm{Iso}(TW_{|L}, TL) \curvearrowright \mathrm{Fr}(TW|_L).$$

Then the fibration $\mathrm{Bun}^{\Theta}(W, \rho_W) \to \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)$ is $e_L$-equivariant and therefore, by Corollary 2.3.3(a), we have a fibration

$$\mathcal{M}^{\Theta}_G(W, \rho_W) \xrightarrow{E_L} \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta) /\!\!/ G$$

onto the path components which it hits.

(b) Recall that $W_N$ is defined as the submanifold $W \setminus N$, where $N$ is a tubular neighbourhood of the submanifold $L$. Then the restriction $\rho_{W_N}$ is a $\Theta$-structure on $W_N$. In the remainder of the proof, we will show that $\mathcal{M}^{\Theta}(W_N, \rho_{W_N})$ is the homotopy fibre of $E_L$.

A description of the fibre of $E_L$ can be obtained using Corollary 2.3.3(b) with the short exact sequence of groups being

$$\ker e_L \to \mathrm{Diff}_G(W) \xrightarrow{e_L} G \tag{5.2.5}$$

and the fibre sequence of $S_1 \to S_2 \to S_3$ being the one associated to the fibration $r_L$.

The fibre of $r_L$ over $r_L(\rho_W)$ is the subspace of all elements of $\mathrm{Bun}^\Theta(W, \rho_W)$ which restrict to $r_L(\rho_W)$ over $L$, which we here denote $\mathrm{Bun}^\Theta_L(W, \rho_W)$. This space carries an action of $\ker e_L$ by precomposition with the differential, and it is simple to check that this fibre sequence is equivariant with respect to (5.2.5). Then by Corollary 2.3.3(b), the fibre of the evaluation map $E_L$ is given by

$$\mathrm{Bun}^\Theta_L(W, \rho_W) /\!\!/ \ker e_L.$$

Applying Lemma 3.2.6 to both submanifolds $L$ and $N$, we obtain two fibrations fitting into the following commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}^\Theta(W, \rho_W) & \xrightarrow{r_N} & \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_N), \Theta) \\ \downarrow{=} & & \downarrow{i^*} \\ \mathrm{Bun}^\Theta(W, \rho_W) & \xrightarrow{r_L} & \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta) \end{array}$$

where the right-hand vertical map is induced by the inclusion $i : L \hookrightarrow N$. Since $i$ is a strong deformation retract, the map $i^*$ is a homotopy equivalence. In particular, this implies that the map from the fibre of $r_N$ to $\mathrm{Bun}^\Theta_L(W, \rho_W)$ is a homotopy equivalence.

The fibre of $r_{_N}$ over $r_{_N}(\rho_W)$ is by definition the space of all $\Theta$ structures on $W$ which agree with $\rho_W$ on $N$. We claim that this space is homeomorphic to $\mathrm{Bun}^\Theta(W_N, \rho_{W_N})$, since the restriction map $r_{_{W_N}}$ takes the fibre of $r_{_N}$ bijectively to $\mathrm{Bun}^\Theta(W_N, \rho_{W_N})$ and it has an inverse given by extending an element by $r_{_N}\rho_W$.

So we have a commutative diagram of principal fibre bundles

$$\begin{array}{ccc} \mathrm{Diff}(W_N) & \xrightarrow{\quad\simeq\quad} & \ker e_L \\ \downarrow & & \downarrow \\ \mathrm{Bun}^\Theta(W_N, \rho_{W_N}) \times E\mathrm{Diff}(W) & \xrightarrow{\simeq} & \mathrm{Bun}^\Theta_L(W, \rho_W) \times E\mathrm{Diff}(W) \\ \downarrow & & \downarrow \\ \mathcal{M}^\Theta(W_N, \rho_{W_N}) & \longrightarrow & \mathrm{Bun}^\Theta_L(W, \rho_W) /\!\!/ \ker e_L \end{array}$$

where the top horizontal map is a homotopy equivalence by Lemma 5.2.2 and the middle map is a homotopy equivalence by the discussion above. Therefore the map

$$\mathcal{M}^\Theta(W_N, \rho_{W_N}) \to \mathrm{Bun}^\Theta_L(W, \rho_W) /\!\!/ \ker e_L$$

is also a homotopy equivalence, as required. □

Proposition 3.2.5 can be recovered as a special case of Proposition 5.2.3: let $L$ be comprised of $m+k$ points (which are the $k$ marked points and the centres of the $m$ marked discs), then $TL$ is a zero-dimensional bundle and $TW_{|L}$ is a trivial bundle of dimension $d$. Defining $G \subset \mathrm{Iso}(TW_{|L}, TL) \cong \mathrm{Iso}(\coprod_{m+k} \mathbb{R}^d)$ to be the subgroup $(\Sigma_k \wr \mathrm{GL}_d) \times \Sigma_m$, we recover precisely the case analysed in Proposition 3.2.5. Note that an element of $\mathrm{Diff}_G(W)$ can permute the $k$ marked points with no restrictions on the map induced on their tangent bundle, on the other hand, the $m$ points are allowed to be permuted, but the map induced on their tangent spaces has to be the identity.

## 5.3 Decoupling $L$-decorations

In this section we prove the decoupling result by comparing a homotopy fibre sequence constructed in Proposition 5.2.3 to the product fibre sequence via the aforementioned spectral sequence argument.

**Definition 5.3.1.** The *decoupling map*

$$D_L : \mathcal{M}^{\Theta}_G(W, \rho_W) \xrightarrow{F_L \times E_L} \mathcal{M}^{\Theta}(W, \rho_W) \times \mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L, \Theta)) /\!\!/ G$$

is the product of the forgetful map 5.1.1 and the evaluation map $E_L$ defined in Proposition 5.2.3.

We now state the decoupling theorem:

**Theorem 5.3.2.** *Let $(W, L)$ be an $L$-decorated manifold, with $W$ a connected compact manifold equipped with a $\Theta$-structure $\rho_W$, and $G \subset \mathrm{Im}\, e_L$. If $\tau : H_i(\mathcal{M}^{\Theta}(W_N, \rho_{W_N})) \to H_i(\mathcal{M}^{\Theta}(W, \rho_W))$ is an isomorphism in degrees $i \leq \alpha$, then for all such $i$ the decoupling map $D_L$ induces an isomorphism*

$$H_i(\mathcal{M}^{\Theta}_G(W, \rho_W)) \cong H_i(\mathcal{M}^{\Theta}(W, \rho_W) \times (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta) /\!\!/ G)_0)$$

*where $(-)_0$ denotes a path component of $E_L(\rho_W)$.*

*Proof.* By Proposition 3.2.5, $E_L$ is a fibration onto the path-components which it hits, therefore the restriction of $E_L$ to the subspace $\mathcal{M}^{\Theta}_G(W, \rho_W)$ is a fibration onto the path-component of $\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta) /\!\!/ G$ which it hits. We denote it

$$(\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta) /\!\!/ G)_0.$$

Therefore, we have a homotopy fibre sequence

$$\mathcal{M}^{\Theta}(W_N, \rho_{W_N}) \longrightarrow \mathcal{M}^{\Theta}_G(W, \rho_W) \xrightarrow{E_L} (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0$$

The proof of the theorem follows from the comparison between this homotopy fibre sequence and the one associated to the trivial fibration

$$\mathcal{M}^{\Theta}(W, \rho_W) \times (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0 \longrightarrow (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0$$

By definition of the maps in Proposition 5.2.3, the following is a commutative diagram of homotopy fibre sequences

$$\begin{array}{ccccc}
\mathcal{M}^{\Theta}(W_N, \rho_{W_N}) & \longrightarrow & \mathcal{M}^{\Theta}_G(W, \rho_W) & \xrightarrow{E_L} & (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0 \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle D_L} & & \| \\
\mathcal{M}^{\Theta}(W, \rho_W) & \longrightarrow & \mathcal{M}^{\Theta}(W, \rho_W) \times (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0 & \to & (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0
\end{array}$$

where the middle vertical map is the decoupling map. This induces a map of the respective Serre spectral sequences $f : E^{\bullet}_{p,q} \to \tilde{E}^{\bullet}_{p,q}$, and since $\tau$ is a homology isomorphism in degrees $i \leq \alpha$, the map between the $E^2$ pages

$$\begin{array}{c}
E^2_{**} = H_*\left((\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0 \,;\, H_*(\mathcal{M}^{\Theta}(W_N, \rho_{W_N}))\right) \\
\downarrow \\
\tilde{E}^2_{**} = H_*\left((\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0 \,;\, H_*(\mathcal{M}^{\Theta}(W, \rho_W))\right)
\end{array}$$

is an isomorphism for all $q \leq \alpha$. Then by the Spectral Sequence Argument recalled in Lemma 2.4.1, $D$ induces an isomorphism

$$H_i(\mathcal{M}^{\Theta}_G(W, \rho_W)) \xrightarrow{\cong} H_i(\mathcal{M}^{\Theta}(W, \rho_W) \times (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta)/\!/G)_0$$

for all $i \leq \alpha$. □

We give an application of this theorem for high even-dimensional manifolds using [30, Corollary 1.7], which we recalled in Theorem 4.2.2.

**Corollary 5.3.3.** *Let $(W, L)$ be an $L$-decorated manifold, with $W$ a compact simply-connected manifold of dimension $2n \geq 6$, and $L$ of dimension less than $n$. Let $\rho_W$ be an $n$-connected $\Theta$-structure on $W$, and denote by $g$ the stable genus of $W$. Then for*

*all* $i \leq \frac{g-3}{2}$ *and* $G \subset \operatorname{Im} e_L$*, the decoupling map* $D_L$ *induces an isomorphism*

$$H_i(\mathcal{M}_G^\Theta(W, \rho_W)) \cong H_i\left(\mathcal{M}^\Theta(W, \rho_W) \times (\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_L), \Theta) /\!\!/ G)_0\right)$$

*where* $(-)_0$ *is the path component of the image of* $\rho_W$*.*

*Proof.* We know that $N$ is homotopy equivalent to $L$, and that the boundary of $N$ is a sphere bundle over $L$ with fibre $S^{c-1}$, where $c$ is the codimension of $L$ and $W$. The dimension assumption on $L$ implies that $c \leq n+1$, and therefore the pair $(N, \partial N)$ is $(n-1)$-connected. Moreover, by Lemma 4.2.1, the stable genus of $W_N$ is equal to $g$. Hence we are under the hypothesis of Theorem 4.2.2 and

$$\tau : H_i(\mathcal{M}^\Theta(W_N, \rho_{W_N})) \to H_i(\mathcal{M}^\Theta(W, \rho_W))$$

is an isomorphism for all $i \leq \frac{g-3}{2}$. By Theorem 5.3.2, the result follows. □

## 5.4 Decoupling unlinked circles

In this section, we apply Theorem 5.3.2 to the specific case where $L$ is a collection of $k$ unlinked circles. Therefore, throughout this section, we assume $W$ to be a compact simply-connected manifold of dimension $2n \geq 6$, to satisfy the hypothesis of Corollary 5.3.3. Note that, since $W$ is simply-connected, it is always orientable.

**Definition 5.4.1.** An embedding $f : \coprod_k S^1 \to W \setminus \partial W$ is said to be *unlinked* if it extends to an embedding $\overline{f} : \coprod_k D^2 \to W \setminus \partial W$. If $W$ is oriented and 2-dimensional we also assume that the embedding $\overline{f}$ is orientation preserving.

**Notation 5.4.2.** Throughout this section, we let $kS^1$ denote the space $\coprod_k S^1$, and $kD^2$ denote the space $\coprod_k D^2$.

In this section, we will repeatedly use the following result, which follows from [42, Chapter 8, Theorems 3.1, 3.2].

**Lemma 5.4.3.** *Let* $W$ *be a connected* $d$*-manifold and and* $f, g : kD^2 \hookrightarrow W$ *embeddings of* $k$ *disjoint discs into* $W$*. If* $d = 2$ *and* $W$ *is oriented, assume also that* $f$ *and* $g$ *both preserve, or both reverse, orientation. Then there is a diffeomorphism* $\phi$ *of* $W$ *which is diffeotopic to the identity, such that* $\phi \circ f = g$*.*

An immediate consequence of the result above is the following

**Lemma 5.4.4.** *The isomorphism type of* $\mathrm{Diff}_{f(kS^1)}(W)$ *does not depend on the choice of the unlinked embedding* $f : kS^1 \hookrightarrow W$.

*Proof.* For any two unlinked embeddings $f, g : kS^1 \hookrightarrow W$, there are embeddings $\overline{f}, \overline{g} : kD^2 \hookrightarrow W$ extending $f, g$. By Lemma 5.4.3, there is a diffeomorphism $\phi$ of $W$ with $\phi \circ f = g$. Then conjugation with $\phi$ defines an isomorphism between the group of diffeomorphism preserving $f(kS^1)$ and the one preserving $g(kS^1)$. □

From here on, we denote by $\mathrm{Diff}_{kS^1}(W)$ the isomorphism type of $\mathrm{Diff}_{f(kS^1)}(W)$ for any embedding $f : kS^1 \hookrightarrow W$, which is well-defined by Lemma 5.4.4.

We want to use Theorem 5.3.2 for the case where the submanifold $L$ is an collection of unlinked circles, but instead of choosing a subgroup of $G$, we will take $G = \mathrm{Im}\, e_{kS^1}$, so we start by analysing what this image is. Fix an unlinked embedding of $kS^1$ in $W$ (we will refer to it as $kS^1 \subset W$), since $W$ is orientable and by fixing a Riemmannian metric we get an explicit isomorphism

$$TW_{|kS^1} \cong TkS^1 \oplus \nu_{kS^1}$$

**Lemma 5.4.5.** *There is a quotient map* $q : \mathrm{Iso}(TW_{|kS^1}, TkS^1) \to \mathrm{Iso}(\nu_{kS^1})$ *which is a homomorphism, a Serre fibration and a homotopy equivalence.*

*Proof.* Any isomorphism $\overline{f} \in \mathrm{Iso}(TW_{|kS^1}, TkS^1)$ satisfies $\overline{f}(TkS^1) = TkS^1$. Therefore, it induces a map on the quotient bundle $[\overline{f}] : TW_{|kS^1}/TkS^1 = \nu_{kS^1} \to \nu_{kS^1}$. Using the inclusion $\nu_{kS^1} \to TW_{|kS^1}$ induced by the choice of a Riemannian metric, it is simple to check that this map satisfies the homotopy lifting property of Serre fibrations. Moreover, using the identification $TW_{|kS^1} \cong TkS^1 \oplus \nu_{kS^1}$, we can verify easily that the fibre of $q$ over the identity is the space of sections of the vector bundle $\mathrm{Hom}(\nu_{kS^1}, TkS^1) \to kS^1$ which is contractible. □

Since the normal bundle of the marked circles is also orientable and any orientable vector bundle over a circle is trivial, we know there is a bundle isomorphism $\nu_{kS^1} \cong kS^1 \times \mathbb{R}^{d-1}$ giving a short exact sequence

$$C^\infty(S^1, \mathrm{GL}_{d-1})^k \longrightarrow \mathrm{Iso}(\nu kS^1) \xrightarrow{f} \mathrm{Diff}(kS^1)$$

where $f$ takes an isomorphism of $\nu_{kS^1}$ to the underlying diffeomorphism of the base $kS^1$. The map $\mathrm{Diff}(kS^1) \to \mathrm{Iso}(\nu kS^1)$ defined by taking $\phi$ to the isomorphism $\phi \times \mathrm{Id}$ is a section for $f$, and therefore

$$\mathrm{Iso}(\nu_{kS^1}) \cong C^\infty(S^1, \mathrm{GL}_{d-1})^k \rtimes \mathrm{Diff}(kS^1). \tag{5.4.1}$$

Fixing such isomorphism, the evaluation map (5.2.1) together with the quotient map of Lemma 5.4.5 induce a homomorphism

$$\overline{e_{kS^1}} : \ \mathrm{Diff}_{kS^1}(W) \longrightarrow C^\infty(S^1, \mathrm{GL}_{d-1})^k \rtimes \mathrm{Diff}(kS^1).$$

We want to determine the image of the map $\overline{e_{kS^1}}$, which is equivalent to identifying the isomorphisms of the normal bundle of the circles that can actually be realised by a diffeomorphism of $W$.

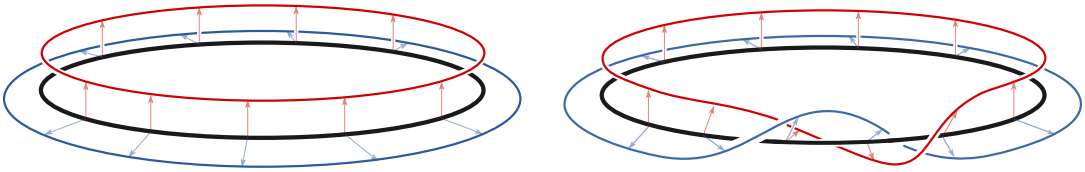


Figure 5.1: A non trivial isomorphism of the normal bundle of $S^1$ in a 3-dimensional manifold.

For simplicity, we first look at the surface case:

**Lemma 5.4.6.** *Let Let $S_{g,b}$ be the oriented surface of genus $g$ and $b \geq 1$ boundary components. Then the image of $\overline{e_{kS^1}}$ is*

$$C^\infty(S^1, \mathrm{GL}_1^+) \rtimes \mathrm{Diff}^+(kS^1).$$

*Proof.* Fix once and for all an embedding $f : kD^2 \hookrightarrow S_{g,b}$, and consider $f(\partial kD^2)$ to be the $kS^1$ decoration in $S_{g,b}$. We start by showing that any diffeomorphism of $\mathrm{Diff}^+(kS^1)$ can be realised by an element in $\mathrm{Diff}_{kS^1}(S_{g,b})$. Any diffeomorphism $\phi \in \mathrm{Diff}^+(kS^1)$ can be extended to $\overline{\phi} \in \mathrm{Diff}^+(kD^2)$ [42, Chapter 8, Theorem 3.3], so it is sufficient to find a diffeomorphism $\psi$ of $W$ such that $\psi \circ f = f \circ \overline{\phi}$. But this can always be done, by Lemma 5.4.3.

This implies that the coset $\overline{e_{kS^1}}(Id) \cdot \mathrm{Diff}^+(kS^1)$ is contained in the image of $\overline{e_{kS^1}}$, and since this map is surjective on path components, we know that $C^\infty(S^1, GL_1^+)^k \rtimes \mathrm{Diff}^+(kS^1)$ is contained in the image of $\overline{e_{kS^1}}$. Moreover, since we assume $b \geq 1$, then for any $k$, all elements of $\mathrm{Diff}_{kS^1}(S_{g,b})$ are orientation preserving and fix $f(kD^2)$ as a set, which means that $\mathrm{Im}\,\overline{e_{kS^1}}$ is contained in $C^\infty(S^1, GL_{d-1}^+)^k \rtimes \mathrm{Diff}^+(kS^1)$, as required. □

We now analyse $\mathrm{Im}\,\overline{e_{kS^1}}$ for a manifold $W$ of higher dimension. In particular, we need to understand when there exists a diffeomorphism of $W$ that induces a loop with

non-trivial homotopy class in $\pi_1(\mathrm{GL}_{d-1})$ as depicted in Figure 5.1. It is clear that determining this image does not only depend on the orientablity of $W$, as it was for the case of points and discs, but it will also depend on the spinnability of $W$.

**Lemma 5.4.7.** *Let $W$ be a simply-connected manifold of dimension $d \geq 5$. Then the image of $\overline{e_{kS^1}}$ is*

$$C^\infty_\diamond(S^1, \mathrm{GL}_{d-1})^k \rtimes \mathrm{Diff}(kS^1)$$

*where $C^\infty_\diamond(S^1, -)$ is equal to the subspace $C^\infty_{null}(S^1, -)$ of nullhomotopic loops if $W$ is spinnable, and is equal to $C^\infty(S^1, -)$ otherwise.*

Here we are not thinking of the spaces as pointed, so we consider a nullhomotopic loop to be one that is homotopic to a constant loop, not necessarily at a base point.

*Proof.* Fix once and for all an embedding $f : kD^2 \hookrightarrow W$, and consider $f(\partial kD^2)$ to be the $kS^1$ decoration in $W$. By the same argument as in the proof of Lemma 5.4.6, we know that any diffeomorphism of $\mathrm{Diff}(kS^1)$ can be realised by an element in $\mathrm{Diff}_{kS^1}(W)$. Without loss of generality, we assume from now on that $k = 1$.

Since $\overline{e_{S^1}}$ is surjective on path components, we know that $C^\infty_{null}(S^1, GL^+_{d-1}) \rtimes \mathrm{Diff}(S^1)$ is contained in the image of $\overline{e_{S^1}}$ because it is the path component of $\overline{e_{S^1}}(Id)$.

By Lemma 5.4.3 there exists an orientation preserving diffeomorphism $\phi_c \in \mathrm{Diff}_{S^1}(W)$ that restricts to complex conjugation along the marked circles. This implies that $\phi_c$ induces an orientation reversing diffeomorphism on the normal bundle of $S^1$. Since the marked circle bounds an embedded 2-disc, we can define such a $\phi_c$ by taking an embedded disc $D^d$ in $W$ containing the marked circle in its equator, and applying a rotation that flips the circle. By the Isotopy Extension Theorem, such a rotation can be extended to an isotopy in $W$. Then $\overline{e_{S^1}}(\phi_c)$ is contained in $C^\infty_{null}(S^1, GL_{d-1}) \rtimes \mathrm{Diff}(S^1)$. Since $\overline{e_{S^1}}$ is surjective on path-components, we conclude that

$$C^\infty_{null}(S^1, GL_{d-1}) \rtimes \mathrm{Diff}(S^1) \subset \mathrm{Im}\, \overline{e_{S^1}}.$$

We now show that a smooth curve $\gamma \notin C^\infty_{null}(S^1, \mathrm{GL}_{d-1})$ is in the image of $\overline{e_{S^1}}$ if, and only if, $W$ is not spin.

First, assume $W$ is spin, and choose $\phi \in \mathrm{Diff}_{S^1}(W)$. We know that any diffeomorphism of the circle is isotopic either to the identity or to complex conjugation, so without loss of generality, we can assume that $\phi$ restricts to one of these two maps on the marked circle. Start by assuming that $\phi$ restricts to the identity on the marked circle. Then using the embedding $f : D^2 \hookrightarrow W$ bounding the marked circle, we can define a continuous function $g : S^2 \to W$ by sending the bottom hemisphere $D^2_-$ to

$f(D^2)$, and the top hemisphere $D^2_+$ to $\phi \circ f(D^2)$. Since we assume $W$ to be spin, we know that $w_2(g^*(TW)) = g^*(w_2(TW)) = 0$. Since $d \geq 5$, the class $w_2$ detects the only obstruction to lifting to to $E\mathrm{SO}(d)$ the map $S^2 \to B\mathrm{SO}(d)$ classifying the bundle $g^*(TW)$. Since $w_2(g^*(TW)) = 0$, this implies that $g^*(TW)$ is a trivial bundle, and in particular, its clutching function $S^1 \to \mathrm{GL}_d^+$ is nullhomotopic. But note that $D\phi$ along the marked circle is a clutching function of $g^*(TW)$, and therefore $\overline{e_{S^1}}(\phi)$ is contained in $C^\infty_{null}(S^1, \mathrm{GL}^+_{d-1}) \rtimes \mathrm{Diff}(S^1)$.

On the other hand, if $\phi$ restricts to complex conjugation on the marked circle, then composing with the map $\phi_c$ constructed above, we get a map restricting to the identity. By the same arguments as above, we can conclude that $\overline{e_{S^1}}(\phi) \in C^\infty_{null}(S^1, \mathrm{GL}_{d-1}) \rtimes \mathrm{Diff}(S^1)$.

Now assume $W$ is not spin. Since $W$ is simply-connected, by Hurewicz theorem, all its second homology classes are represented by maps $S^2 \to W$ and by [85, Theorem II.27] we can always pick a representative given by an embedding. Since $W$ is not spin, there exists an embedding $h : S^2 \to W$ such that $w_2(h^*(TW)) \neq 0$, and since $d \geq 5$, we can pick one such $h$ not intersecting $f(D^2)$ by the Transversality Theorem (see [51, Corollary 12.2.7]). By Lemma 5.4.3, we know there exists a diffeomorphism $\phi_h$ of $W$ taking $\phi_h \circ h|_{D^2_-}$ to $h|_{D^2_+}$, and which restricts to the identity on the image of $f(D^2)$. By definition, $D\phi_h|_{h(S^1)}$ is a clutching function for $h^*(TW)$ and therefore is not nullhomotopic as $h^*(TW)$ is non-trivial.

Let $\psi$ be a diffeomorphism taking $f(D^2)$ to $h|_{D^2_-}$. Then $\psi^{-1} \circ \phi_h \circ \psi$ is a diffeomorphism of $W$ whose image through $\overline{e_{S^1}}$ is not in $C^\infty_{null}(S^1, \mathrm{GL}^+_{d-1}) \rtimes \mathrm{Diff}(S^1)$. Since $\overline{e_{S^1}}$ is surjective on path components, we conclude that the image of $\overline{e_{S^1}}$ contains $C^\infty(S^1, \mathrm{GL}^+_{d-1}) \rtimes \mathrm{Diff}(S^1)$. Analogously, looking at the composition $\psi^{-1} \circ \phi_f \circ \psi \circ \phi_c$, we conclude that the image of $\overline{e_{S^1}}$ is $C^\infty(S^1, \mathrm{GL}_{d-1}) \rtimes \mathrm{Diff}(S^1)$. □

*Remark* 5.4.8. Lemma 5.4.7 can be generalised for dimension 4 assuming the manifold is spin, using exactly the same argument as above.

Now that we have analysed the image of $\overline{e_{kS^1}}$ we apply Theorem 5.3.2. As before, we start by looking at the surface case.

**Corollary 5.4.9.** *Let $S_{g,b}$ be the oriented surface of genus $g$ and $b \geq 1$ boundary components. Then for all $3i \leq 2g$*

$$H_i(B\mathrm{Diff}_{kS^1}(S_{g,b})) \cong H_i(B\mathrm{Diff}(S_{g,b}) \times B(\Sigma_k \wr \mathrm{SO}(2))).$$

*Proof.* Start by fixing an unlinked embedding $f : kS^1 \hookrightarrow S_{g,b}$ and $\overline{f} : kD^2 \hookrightarrow S_{g,b}$ extending $f$. The proof follows from applying Theorem 5.3.2 taking $L$ to be the

embedded circles. To do this, we start by verifying that the hypothesis of the Theorem are satisfied, ie. that the map defined by extending diffeomorphisms by the identity

$$\varepsilon_N : \mathrm{Diff}(S_{g,b} \setminus N) \to \mathrm{Diff}(S_{g,b})$$

induces isomorphisms on the homology groups of the classifying spaces in the range $3i \leq 2g$.

Since the embedded circles are unlinked, we know they are nullhomotopic and therefore $S_{g,b} \setminus N$ is diffeomorphic to $S_{g,b+k} \cup \coprod_k D^2$. Then

$$\mathrm{Diff}(S_{g,b} \setminus N) \cong \mathrm{Diff}(S_{g,b+k}) \times \mathrm{Diff}(D^2)^k.$$

The inclusion $S_{g,b} \setminus N \to S_{g,b}$ induces maps

$$\varepsilon_S : \mathrm{Diff}(S_{g,b+k}) \to \mathrm{Diff}(S_{g,b})$$

$$\varepsilon_D : \mathrm{Diff}(D^2)^k \to \mathrm{Diff}(S_{g,b})$$

given by extending the the diffeomorphims by the identity, and it is simple to check that the following diagram commutes

$$\begin{array}{ccc} & & \mathrm{Diff}(S_{g,b}) \times \mathrm{Diff}(D^2)^k \\ & \overset{\varepsilon_S \times id}{\nearrow} & \qquad\qquad \overset{c}{\searrow} \\ \mathrm{Diff}(S_{g,b+k}) \times \mathrm{Diff}(D^2)^k & & \xrightarrow{\qquad\varepsilon_N\qquad} \mathrm{Diff}(S_{g,b}) \end{array}$$

where $c(\phi, \psi) := \phi \circ \varepsilon_D(\psi)$.

By [75, Theorem 7.1], we know that the map $\varepsilon_S$ induces a homology isomorphism in the range $3i \leq 2g$. Moreover, the map $c$ is a homotopy equivalence since the space $\mathrm{Diff}(D^2)$ is contractible. Therefore, $\varepsilon_N$ induces a homology isomorphism in the range $3i \leq 2g$.

Hence we are under the hypothesis of Theorem 5.3.2. Moreover, we know that $TW|_{kS^1}$ is trivial so

$$\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TW|_{kS^1}), \Theta^{or}) \simeq \mathrm{Map}(S^1, \{\pm 1\})^k$$

is simply a disjoint union of points. And, by Lemma 5.4.6,

$$\mathrm{Im}\, e_{kS^1} \simeq \mathrm{Im}\, \overline{e_{kS^1}} \cong C^\infty(S^1, \mathrm{GL}_1^+)^k \rtimes \mathrm{Diff}^+(kS^1) \simeq \mathrm{Diff}^+(kS^1) \simeq \Sigma_k \wr \mathrm{SO}(2).$$

Applying Theorem 5.3.2, the result follows. □

Corollary 5.4.9 can be re-stated with a geometric interpretation. As discussed in Section 1.1.1, the space $\mathrm{Emb}(S_{g,b}, \mathbb{R}^\infty)$ is a model for $E\mathrm{Diff}(S_{g,b})$, and therefore it is also a model for $E\mathrm{Diff}_{kS^1}(S_{g,b})$. With this model, the elements of $B\mathrm{Diff}_{kS^1}(S_{g,b})$ are oriented submanifolds of $\mathbb{R}^\infty$ diffeomorphic to $S_{g,b}$ with $k$ marked unlinked circles. With this model, the forgetful map

$$F_{kS^1} : B\mathrm{Diff}_{kS^1}(S_{g,b}) \to B\mathrm{Diff}(S_{g,b}) \tag{5.4.2}$$

simply forgets the marked circles.

To interpret the evaluation map $E_{kS^1}$ in this model, we recall a definition that will also be useful for the interpretation of the evaluation map for the moduli space in higher dimensions with general tangential structures.

**Definition 5.4.10.** Let $W$ be a manifold and $X$ be a space with an action of $\mathrm{Diff}(S^1)$, the *space of $k$-unlinked circles in $W$* with labels in $X$ is defined to be

$$\mathrm{C}_{kS^1}(W; X) := \mathrm{Emb}^{\mathrm{unl}}(kS^1, W) \times X^k / \mathrm{Diff}(kS^1)$$

where $\mathrm{Emb}^{\mathrm{unl}}$ denotes the space of unlinked embeddings.

Note that if $W$ is a simply connected manifold of dimension $d \geq 5$, all embeddings of $kS^1$ into $W$ are unlinked.

Then a model for $B\mathrm{Diff}^+(kS^1)$ is precisely the configuration space $\mathrm{C}_{kS^1}(\mathbb{R}^\infty; \{\pm 1\})$ of $k$ circles in $\mathbb{R}^\infty$ with labels in $\{\pm 1\}$, and the evaluation map $E_{kS^1}$ simply takes an oriented decorated submanifold $S$ in $\mathbb{R}^\infty$, to the configurations given by the marked $k$ oriented circles.

**Corollary 5.4.11.** *Let $S_{g,b}$ be the oriented surface of genus $g$ and $b \geq 1$ boundary components. Then for all $3i \leq 2g$*

$$H_i(B\mathrm{Diff}_{kS^1}(S_{g,b})) \cong H_i(B\mathrm{Diff}(S_{g,b}) \times \mathrm{C}_{kS^1}(\mathbb{R}^\infty; \{\pm 1\})).$$

Analogous results for other tangential structures can be obtained by the same arguments. We now look at how the result above generalises for higher dimensions. From now on, we let $L(-) \coloneqq \mathrm{Map}(S^1, -)$ denote the free loop space.

**Corollary 5.4.12.** *Let $W$ be a compact simply-connected manifold of dimension $2n \geq 6$, $\rho_W$ an $n$-connected $\Theta$-structure on $W$, and denote by $g$ the stable genus of $W$. Then*

*for all* $i \leq \frac{g-3}{2}$*, the decoupling map* $D_{kS^1}$ *induces an isomorphism*

$$H_i(\mathcal{M}^{\Theta}_{kS^1}(W, \rho_W)) \cong H_i\left(\mathcal{M}^{\Theta}(W, \rho_W) \times \mathrm{C}_{kS^1}(\mathbb{R}^\infty; (L(\Theta) /\!\!/ L_\diamond(\mathrm{GL}_{d-1}))_0)\right)$$

*where* $(-)_0$ *is the path component of the image of* $\rho_W$*,* $L_\diamond(-)$ *is equal to the subspace* $L_{null}(-)$ *of nullhomotopic loops if* $W$ *is spinnable, and is equal to* $L(-)$ *otherwise.*

*Proof.* The result follows from applying Corollary 5.3.3, taking $G = \operatorname{Im} e_{kS^1} \simeq \operatorname{Im} \overline{e_{kS^1}}$, which was identified in Lemma 5.4.7. Moreover, since $TkS^1$ is orientable, it is a trivial bundle and therefore the space $\mathrm{Map}_{\mathrm{GL}_d}(\mathrm{Fr}(TkS^1), \Theta)$ is equivalent to the space of continuous maps $kS^1 \to \Theta$, which is precisely $L(\Theta)^k$.

Then, by Corollary 5.3.3, for all $i \leq \frac{g-3}{2}$, the decoupling map $D_{kS^1}$ induces an isomorphism

$$H_i(\mathcal{M}^{\Theta}_{kS^1}(W, \rho_W)) \cong H_i\left(\mathcal{M}^{\Theta}(W, \rho_W) \times (L(\Theta)^k /\!\!/ C^\infty_\diamond(S^1, \mathrm{GL}_{d-1})^k \rtimes \mathrm{Diff}(kS^1))_0\right)$$

Moreover, the space $(L(\Theta)^k /\!\!/ C^\infty_\diamond(S^1, \mathrm{GL}_{d-1})^k \rtimes \mathrm{Diff}(kS^1))_0$ is homotopy equivalent to

$$(L(\Theta) /\!\!/ C^\infty_\diamond(S^1, \mathrm{GL}_{d-1}))^k /\!\!/ \mathrm{Diff}(kS^1))_0 \tag{5.4.3}$$

Taking $\mathrm{Emb}(kS^1, \mathbb{R}^\infty)$ as the model for $E\mathrm{Diff}(kS^1)$, we get a model for the space in 5.4.3, which is precisely the configuration space of $k$ circles in $\mathbb{R}^\infty$ with labels in $(L(\Theta) /\!\!/ C^\infty_\diamond(S^1, \mathrm{GL}_{d-1}))_0$, as required. Since the space of smooth loops is homotopy equivalent to the free loop space, the result follows. □

# Chapter 6

# Decoupling for general tangential structures in higher dimensions

In this section we show how Corollaries 4.2.4 and 5.3.3 can be generalised for other tangential structures, based on the techniques used by Galatius and Randal-Williams in [30, Section 9]. Recall that the decoupling theorems (3.3.2 and 5.3.2) relied on the hypothesis that the map

$$\mathcal{M}^{\Theta}(W_N, \rho_{W_N}) \to \mathcal{M}^{\Theta}(W, \rho_W)$$

induces a homology isomorphism in a range. In even dimensions at least 6, this assumption was shown to hold in several cases in [31]*Corollary 1.7 as recalled in 4.2.2, but only when the $\Theta$-structure $\rho_W : \mathrm{Fr}(TW) \to \Theta$ is $n$-connected (ie. the induced map $\pi_i(\mathrm{Fr}(TW)) \to \pi_i(\Theta)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$). In [31, Section 9] Galatius and Randal-Williams provide a generalisation of the result to general tangential structures. In this section we introduce the tools used to construct this generalisation and show how they also provide an extension of the decoupling result in higher dimensions for general tangential structures.

One could hope that for any manifold $W$ and any $\Theta$-structure $\rho_W$, the decoupling map would still induce a homology isomorphism, but this is not the case, as it is shown by the following example.

**Example 6.0.1.** Consider the manifold $W_g = \#_g S^n \times S^n$ with one embedded disc as a decoration. Let $W_{g,1} = \#_g(S^n \times S^n) \setminus \mathrm{int}(D^{2n})$ and recall there is an isomorphism

$$\mathrm{Diff}^+(W_{g,1}) \xrightarrow{\cong} \mathrm{Diff}^+_1(W_g)$$

given by extending the diffeomorphism of $W_{g,1}$ by the identity on the marked disc (see Lemma 3.2.3).

Therefore, the decorated moduli space $\mathcal{M}_1^{or}(W_g, \rho_W) \simeq B\mathrm{Diff}_1^+(W_g)$ is weakly equivalent to $\mathcal{M}^{or}(W_{g,1}, \rho_W) \simeq B\mathrm{Diff}^+(W_{g,1})$. In this case, the decoupling map

$$\begin{array}{ccc} \mathcal{M}_1^{or}(W_g, \rho_{W_g}) & \xrightarrow{D} & \mathcal{M}^{or}(W_g, \rho_{W_g}) \times \Theta_0^{or} \\ \wr & & \wr \\ \mathcal{M}^{or}(W_{g,1}, \rho_{W_{g,1}}) & & \mathcal{M}^{or}(W_g, \rho_{W_g}) \end{array} \tag{6.0.1}$$

does not induce a homology isomorphism on integral coefficients in a stable range as was shown in [32, Sections 5.1 and 5.2]. This implies that the decoupling as stated in Corollary 4.2.4 is not true for general tangential structures.

Let $W$ be a $2n$-dimensional manifold, $2n \geq 6$, with possibly non-empty boundary, and $\lambda_W$ a $\Lambda$-structure on $W$. If the map $\lambda_W : \mathrm{Fr}(TW) \to \Lambda$ is not $n$-connected we will use an "intermediate" tangential structure $\Theta$ which is better behaved. Precisely, let the following be the Moore-Postnikov $n$-stage of $\lambda_W$,

$$\begin{array}{ccccc} & & \Theta & & \\ & \overset{\rho_W}{\nearrow} & & \overset{u}{\searrow} & \\ \mathrm{Fr}(TW) & & \xrightarrow{\lambda_W} & & \Lambda. \end{array} \tag{6.0.2}$$

This means that $\Theta$ is a $\mathrm{GL}_d$-space, $u$ is an $n$-co-connected (ie. the induced map $\pi_i(\mathrm{Fr}(TW)) \to \pi_i(\Theta)$ is an isomorphism for $i > n$ and a monomorphism for $i = n$) equivariant fibration and $\rho_W$ an $n$-connected equivariant cofibration. Such a factorization always exist and it is unique up to homotopy equivalence.

Denote by $\rho_\partial$ and $\lambda_\partial$ the restriction of $\rho_W$ and $\lambda_W$ respectively to $\mathrm{Fr}(TW)|_{\partial W}$. Any $\Theta$-structure on $W$ induces a $\Lambda$-structure by postcomposition with $u$, giving us a map

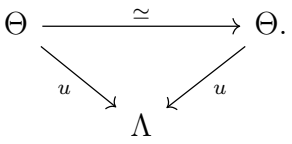
$$\mathrm{Bun}^\Theta(W, \rho_W) \to \mathrm{Bun}^\Lambda(W, \lambda_W).$$

**Lemma 6.0.2** ([30], Lemma 9.4)**.** *If $W$ is a manifold equipped with a $\Lambda$-structure $\lambda_W$ and $\mathrm{Fr}(TW) \xrightarrow{\rho_W} \Theta \xrightarrow{u} \Lambda$ is a Moore-Postnikov $n$-stage of $\lambda_W$, then the stable genus $\overline{g}(W, \rho_W)$ is equal to $\overline{g}(W, \lambda_W)$.*

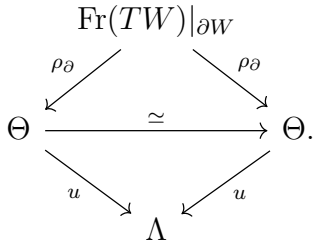
We now define a topological monoid that is crucial to the comparison between the moduli spaces $\mathcal{M}^\Lambda(W, \lambda_W)$ and $\mathcal{M}^\Theta(W, \rho_W)$.

**Definition 6.0.3.** If $W$ is a closed manifold, denote by $\mathrm{hAut}(u)$ the group-like topological monoid consisting of equivariant weak equivalences $\Theta \to \Theta$ over $u$, ie. $\mathrm{GL}_d$-

equivariant maps $\Theta \to \Theta$ fitting into the following commutative diagram




If $W$ has non-empty boundary, let $\rho_\partial$ be the restriction of $\rho_W$ to $\partial W$. Denote by $\mathrm{hAut}(u, \rho_\partial)$ the group-like topological monoid consisting of equivariant weak equivalences $\Theta \to \Theta$ over $u$ and under $\rho_\partial$




The monoid $\mathrm{hAut}(u, \rho_\partial)$ acts on the space of $\Theta$-structures on $W$ by post-composition, and the following result shows that this action encodes precisely the relation between $\Theta$ and $\Lambda$-structures on $W$.

**Lemma 6.0.4** ([30], Lemma 9.2)**.** *In the context defined above, the map induced by postcomposition with $u$*

$$\mathrm{Bun}^{\Theta}_{\rho_\partial}(W) /\!\!/ \mathrm{hAut}(u, \rho_\partial) \to \mathrm{Bun}^{\Lambda}_{\lambda_\partial}(W)$$

*is a homotopy equivalence onto the path components which it hits.*

Let $\mathrm{hAut}(u, \rho_\partial)_{[W,\rho_W]}$ denote the components of $\mathrm{hAut}(u, \rho_\partial)$ that map $\mathrm{Bun}^{\Theta}(W, \rho_W)$ to itself. By the orbit-stabiliser theorem

$$\mathrm{Bun}^{\Theta}(W, \rho_W) /\!\!/ \mathrm{hAut}(u, \rho_\partial)_{[W,\rho_W]} \to \mathrm{Bun}^{\Lambda}(W, \lambda_W)$$

is also a homotopy equivalence onto the path components which it hits. Taking a further Borel construction with the groups $\mathrm{Diff}(W)$, $\mathrm{Diff}^k_m(W)$, $\mathrm{Diff}_L(W)$, we get that the induced maps

$$\mathcal{M}^{\Theta}(W, \rho_W) /\!\!/ \mathrm{hAut}(u, \rho_\partial)_{[W,\rho_W]} \to \mathcal{M}^{\Lambda}(W, \lambda_W) \tag{6.0.3}$$

$$\mathcal{M}^{\Theta,k}_m(W, \rho_W) /\!\!/ \mathrm{hAut}(u, \rho_\partial)_{[W,\rho_W]} \to \mathcal{M}^{\Lambda,k}_m(W, \lambda_W) \tag{6.0.4}$$

$$\mathcal{M}^{\Theta}_L(W, \rho_W) /\!\!/ \mathrm{hAut}(u, \rho_\partial)_{[W,\rho_W]} \to \mathcal{M}^{\Lambda}_L(W, \lambda_W) \tag{6.0.5}$$

are weak homotopy equivalences. Therefore, analysing $\mathrm{hAut}(u, \rho_\partial)_{[W,\rho_W]}$ and applying Corollaries 4.2.4 and 5.3.3 we get decoupling results for general $\Lambda$.

**Lemma 6.0.5** ([32])**.** *If $(W, \partial W)$ is $c$-connected for some $c \leq n-1$, then the monoid $\mathrm{hAut}(u, \rho_\partial)$ is a non-empty $(n-c-2)$-type. In particular, it is contractible if $(W, \partial W)$ is $(n-1)$-connected.*

We now focus on applying these techniques to the manifold $W_{g,1} = \#_g S^n \times S^n \setminus D^{2n}$, for $n \geq 3$.

**Proposition 6.0.6.** *Let $W_{g,1} = \#_g S^n \times S^n \setminus D^{2n}$, for $n \geq 3$, and $\lambda_W$ a $\Lambda$-structure on $W$. Let $g$ denote the stable genus $\overline{g}(W, \lambda_W)$. For all $i \leq \frac{g-4}{3}$, the group $H_i(\mathcal{M}_m^{\Lambda,k}(W_{g,1}, \lambda_W))$ is isomorphic to*

$$H_i(\mathcal{M}^\Lambda(W_{g,1}, \lambda_W) \times \Theta_0^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}_{2n}^+)_0^k /\!\!/ \Sigma_k)$$

*where $(-)_0$ denotes the path-component of $E(\rho_W)$, for $\rho_W$ as in* (6.0.2)*.*

Note that in the above proposition, the decorations on $\mathcal{M}_m^{\Lambda,k}(W)$ get decoupled into components depending on $\Theta$, the tangential structure that appeared in the Moore-Postnikov $n$-stage factorisation of $\lambda_W$. This is quite different than what was obtained in Corollary 4.2.4 as well as in the other decoupling theorems of sections 3 and 4, where the decoupled components corresponding to the marked points and discs depended on the original chosen tangential structure $\Lambda$.

*Proof.* By Lemma 6.0.5, we know that $\mathrm{hAut}(u, \rho_\partial)$ is contractible, and therefore

$$\mathcal{M}^\Theta(W_{g,1}, \rho_W) \simeq \mathcal{M}^\Lambda(W_{g,1}, \lambda_W)$$
$$\mathcal{M}_m^{\Theta,k}(W_{g,1}, \rho_W) \simeq \mathcal{M}_m^{\Lambda,k}(W_{g,1}, \lambda_W)$$

are weak homotopy equivalences. Since $\rho_W$ is $n$-connected, we can apply Corollary 4.2.4 to understand the homology of $\mathcal{M}_m^{\Theta,k}(W_{g,1}, \rho_W)$. Putting this together with the above identifications we get that the group $H_i(\mathcal{M}_m^{\Lambda,k}(W_{g,1}, \lambda_W))$ is isomorphic to $i$th homology group of

$$\mathcal{M}^\Lambda(W_{g,1}, \lambda_W) \times \Theta_0^m /\!\!/ \Sigma_m \times (\Theta /\!\!/ \mathrm{GL}_{2n}^+)_0^k /\!\!/ \Sigma_k.$$

□

We now look at the case where $\Lambda$ is the tangential structure for orientations: a $\mathrm{GL}_d$-equivariant map $\lambda_W : \mathrm{Fr}(TW_{g,1}) \to \{\pm 1\}$ determines, up to a contractible choice, a map $\ell'_W : W_{g,1} \to B\mathrm{SO}(2n)$ fitting into the following homotopy pullback square

$$\begin{array}{ccc} \mathrm{Fr}(TW_{g,1}) & \xrightarrow{\lambda_W} & \{\pm 1\} \\ \downarrow & \lrcorner & \downarrow \\ W_{g,1} & \xrightarrow[\ell'_W]{} & B\mathrm{SO}(2n). \end{array}$$

Then an equivariant Moore-Postnikov factorization of $\lambda_W$ can be obtained from a Moore-Postnikov factorization of $\ell'_W$. Since $W_{g,1}$ is $(n-1)$-connected and parallelizable, we know that the $n$-stage of this factorization is given by maps

$$W_{g,1} \xrightarrow{\ell_W} B\mathrm{O}(2n)\langle n\rangle \xrightarrow{u} B\mathrm{SO}(2n)$$

where $B\mathrm{O}(2n)\langle n\rangle$ is the $n$-connected cover of $B\mathrm{O}(2n)$. Taking the pullback of $\{\pm 1\} \to B\mathrm{SO}(2n)$ along these maps, we get

$$\begin{array}{ccccc} \mathrm{Fr}(TW_{g,1}) & \longrightarrow & \mathrm{O}[0,n-1] & \longrightarrow & \{\pm 1\} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ W_{g,1} & \xrightarrow[\ell_W]{} & B\mathrm{O}(2n)\langle n\rangle & \xrightarrow{u} & B\mathrm{SO}(2n) \end{array}$$

where $\mathrm{O}[0,n-1]$ is the $(n-1)$-truncation of O. Note that a path-component of $\mathrm{O}[0,n-1]$ is homotopy equivalent to $\mathrm{SO}[0,n-1]$, the $(n-1)$-truncation of SO.

**Corollary 6.0.7.** *Let $W_{g,1} = \#_g S^n \times S^n \setminus D^{2n}$, for $n \geq 3$. Then for all $i \leq \frac{g-4}{3}$, the group $H_i(B\mathrm{Diff}^{+,k}_m(W_{g,1}))$ is isomorphic to*

$$H_i(B\mathrm{Diff}^+(W_{g,1}) \times \mathrm{SO}[0,n-1]^m /\!\!/ \Sigma_m \times B\mathrm{O}(2n)\langle n\rangle^k /\!\!/ \Sigma_k).$$

The proof is a direct application of Proposition 6.0.6 using the factorization described above, and the fact that for an orientation $\rho_{W_{g,1}} : \mathrm{Fr}(TW_{g,1}) \to \{\pm 1\}$, the stable genus $\overline{g}(W_{g,1}, \rho_{W_{g,1}})$ is equal to $g$ (see [32]*Section 3.2).

We end by using the result above to explicitly compute the cohomology of $B\mathrm{Diff}^{+,k}_m(W_{g,1})$ with rational coefficients, in the stable range. As an immediate consequence of Corollary 6.0.7 and Kunneth Theorem, the elements of $H^*(B\mathrm{Diff}^{+,k}_m(W_{g,1}); \mathbb{Q})$ of degree $i \leq \frac{g-4}{3}$, are given by the elements of such degrees in the tensor product of the cohomology rings of $B\mathrm{Diff}^+(W_{g,1})$, $\mathrm{SO}[0,n-1]^m /\!\!/ \Sigma_m$ and $B\mathrm{O}(2n)\langle n\rangle^k /\!\!/ \Sigma_k$.

By [31]*Corollary 1.8, in degrees $i \leq \frac{g-3}{2}$, the ring $H^*(B\mathrm{Diff}^+(W_{g,1}); \mathbb{Q})$ is isomorphic to

$$\mathbb{Q}[\kappa_c | c \in \mathcal{B}, |c| > 2n]$$

where $\mathcal{B}$ denotes the set of monomials in the classes $e$, $p_{n-1}$, $p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$ of $H^*(B\mathrm{SO}(2n))$ and $|\kappa_c| = |c| - 2n$.

By the Cartan-Leray spectral sequence, we also know that

$$H^*(B\mathrm{O}(2n)\langle n \rangle^k /\!\!/ \Sigma_k; \mathbb{Q}) \cong H^*(B\mathrm{O}(2n)\langle n \rangle^k; \mathbb{Q})^{\Sigma_k}$$

the fixed points by the action of $\Sigma_k$ which permutes the factors of $(B\mathrm{O}(2n)\langle n \rangle)^k$. We know that $H^*(B\mathrm{O}(2n)\langle n \rangle; \mathbb{Q})$ is simply the subalgebra of $H^*(B\mathrm{O}(2n); \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_{n-1}, e]$ with no generators of degrees $\leq n$. So

$$H^*(B\mathrm{O}(2n)\langle n \rangle; \mathbb{Q}) \cong \mathbb{Q}[e, p_{\lceil \frac{n+1}{4} \rceil}, \ldots, p_{n-1}].$$

Then $H^*(B\mathrm{O}(2n)\langle n \rangle^k; \mathbb{Q})^{\Sigma_k}$ is isomorphic to

$$\left( \bigotimes_k \mathbb{Q}[e, p_{\lceil \frac{n+1}{4} \rceil}, \ldots, p_{n-1}] \right)^{\Sigma_k}$$

the fixed points by the action of $\Sigma_k$ which permutes the factors of the $k$-fold tensor product.

Analogously,

$$H^*(\mathrm{SO}[0, n-1]^m /\!\!/ \Sigma_m; \mathbb{Q}) \cong H^*(\mathrm{SO}[0, n-1]^m; \mathbb{Q})^{\Sigma_m}.$$

Using the fibre sequence

$$\mathrm{SO}(2n)\langle n-1 \rangle \longrightarrow \mathrm{SO}(2n) \longrightarrow \mathrm{SO}[0, n-1]$$

then by the Leray-Hirsch Theorem, we have a $\mathbb{Q}$-module isomorphism

$$H^*(\mathrm{SO}(2n); \mathbb{Q}) \cong H^*(\mathrm{SO}[0, n-1]; \mathbb{Q}) \otimes H^*(\mathrm{SO}(2n)\langle n-1 \rangle; \mathbb{Q}).$$

Together with the fact that we have a canonical ring monomorphism

$$H^*(\mathrm{SO}(2n)\langle n-1 \rangle; \mathbb{Q}) \longrightarrow H^*(\mathrm{SO}(2n); \mathbb{Q})$$

we conclude that

$$H^*(\mathrm{SO}[0,n-1];\mathbb{Q}) \cong \bigwedge[y_1,\dots,y_{\lfloor\frac{n-1}{4}\rfloor}]$$

with $|y_i| = 4i-1$.

Then $H^*(\mathrm{SO}(2n)[0,n-1]^m;\mathbb{Q})^{\Sigma_m}$ is isomorphic to

$$\left(\bigotimes_m \bigwedge[y_1,\dots,y_{\lfloor\frac{n-1}{4}\rfloor}]\right)^{\Sigma_m}$$

the fixed points by the action of $\Sigma_m$ which permutes the factors of the $m$-fold tensor product.

Therefore in degrees $i \leq \frac{g-4}{3}$, the ring $H^*(B\mathrm{Diff}_m^{+,k}(W_{g,1}))$ is isomorphic to the graded commutative algebra

$$\mathbb{Q}[\kappa_c | c \in \mathcal{B}, |c| > 2n] \otimes \left(\bigotimes_m \bigwedge[y_1,\dots,y_{\lfloor\frac{n-1}{4}\rfloor}]\right)^{\Sigma_m} \otimes \left(\bigotimes_k \mathbb{Q}[e, p_{\lceil\frac{n+1}{4}\rceil},\dots,p_{n-1}]\right)^{\Sigma_k}.$$

# Chapter 7

# Increasing the number of decorations

In the previous chapters, we discussed the behaviour of the homology of the moduli space of manifolds when stabilising by increasing the genus. For completeness, we show in this section that the decorated moduli spaces of manifolds with tangential structures also have homology stability with respect to increasing the number of decorations. This is a direct generalisation of Tillmann's result in [88].

Throughout this section, we assume $W$ to be a smooth, compact, path-connected manifold of dimension $d \geq 2$ with non-empty boundary. Unlike in previous chapters, here we can work with the moduli spaces with tangential structures without restricting to path components. Namely, let $\mathrm{Bun}^{\Theta}(W)$ be the space of $\Theta$-structures on $W$ as defined in Definition 2.1.5. We denote by $\mathcal{M}_m^{\Theta,k}(W)$ the Borel construction

$$\mathrm{Bun}^{\Theta}(W) /\!\!/ \mathrm{Diff}_m^k(W).$$

The space $\mathcal{M}_m^{\Theta,k}(W, \rho_W)$ used previously is the collection of path components of $\mathcal{M}_m^{\Theta,k}(W)$ containing the orbit of $\rho_W$ via the $\mathrm{Diff}(W)$ action.

A collection of $k$ marked points in the interior of a manifold $W$ is precisely an element of the configuration space $C_k(W)$, and similarly, a collection of $m$ marked discs in $W$ can be identified with a configuration of $m$ points in $W$ with labels. Therefore any stability of the homology of the moduli space with respect to increasing the number of decorations is closely related to the stability of the maps between configuration spaces in $W$ with labels.

Such a stabilisation map $C_k(W) \to C_{k+1}(W)$ can be intuitively described as adding a marked point via the boundary of $W$ in a prescribed way. Precisely, fix a col-

lar of $\partial W$, a marked point $p_0 \in \partial W \times (0,1)$, and an isotopy $e$ of $W^+ \coloneqq W \cup_{\partial W} (\partial W \times I)$ onto $W$ mapping $\partial W \times I$ into the collar of $\partial W$ while leaving the complement of the collar in $W$ fixed. Then we get a stabilisation map

$$\sigma^k : C_k(W) \to C_{k+1}(W^+) \to C_{k+1}(W)$$

where the first map adds the particle $p_0$ to the configuration and the second map is the one induced by $e$. This map was shown to induce an isomorphism on homology in degrees $* \leq k/2$ (see [80, 63, 82, 73, 20]). The same procedure gives a stabilisation map $\sigma_m$ for increasing the number of marked discs, and this was also shown to induce an isomorphism on homology in degrees $* \leq m/2$ (see [16, 52]).

In [88], Tillmann used these maps between configurations to construct stabilisation maps for any $k, m$

$$\mathrm{Diff}^k(W) \to \mathrm{Diff}^{k+1}(W) \qquad\qquad \mathrm{Diff}_m(W) \to \mathrm{Diff}_{m+1}(W).$$

These maps are defined by first extending diffeomorphisms by the identity to get a map to the decorated diffeomorphism of $W^+$ and using the fixed isotopy to obtain a diffeomorphism of $W$. Since we look only at diffeomorphisms fixing the collar of $\partial W$, this construction produces diffeomorphisms that indeed preserve the marked points. Tillmann showed in [88, Theorems 1.2 and 1.3] that these maps induce homology isomorphisms on classifying spaces in degrees $* \leq k/2$ and $* \leq m/2$, respectively. We show that these maps also induce homology isomorphisms of decorated moduli spaces with tangential structures.

**Theorem 7.0.1.** *The maps*

$$S^k : \mathcal{M}^{\Theta,k}(W) \to \mathcal{M}^{\Theta,k+1}(W) \qquad\qquad S_m : \mathcal{M}^{\Theta}_m(W) \to \mathcal{M}^{\Theta}_{m+1}(W)$$

*induced by $\sigma^k$ and $\sigma_m$, respectively, induce homology isomorphisms on degrees $* \leq k/2$ and $m/2$, respectively.*

*Proof.* The proof is essentially the same for increasing the number of points or discs, so we discuss only the points case, for simplicity.

Fix once and for all a model for $E\mathrm{Diff}(W)$. We start by constructing, for each $k$ a fibre sequence: let $p_1, \ldots, p_k$ be $k$ marked points in $W$. The diagram below denotes the application of Lemma 2.3.2(a) with $G_2 = \mathrm{Diff}^k(W)$, $G_3 = \mathrm{Diff}(W)$ and $f$ the

identity on $\mathrm{Bun}^\Theta(W) \times E\mathrm{Diff}(W)$

$$\begin{array}{ccc}
\mathrm{Diff}^k(W) & \xrightarrow{\quad i \quad} & \mathrm{Diff}(W) \\
\curvearrowright & & \curvearrowright \\
\mathrm{Bun}^\Theta(W) \times E\mathrm{Diff}(W) & = & \mathrm{Bun}^\Theta(W) \times E\mathrm{Diff}(W) \\
\downarrow & & \downarrow \\
\mathrm{Bun}^\Theta(W) \times_{\mathrm{Diff}^k(W)} E\mathrm{Diff}(W) & \xrightarrow{Q_k} & \mathrm{Bun}^\Theta(W) \times_{\mathrm{Diff}(W)} E\mathrm{Diff}(W)
\end{array}$$

where $i$ is the subgroup inclusion. By Lemma 2.3.2(a), the map $Q_k$ is a fibration for every $k$.

The projections onto the $E\mathrm{Diff}(W)$ induce a commutative square of fibrations

$$\begin{array}{ccc}
\mathrm{Bun}^\Theta(W) \times_{\mathrm{Diff}^k(W)} E\mathrm{Diff}(W) & \xrightarrow{Q_k} & \mathrm{Bun}^\Theta(W) \times_{\mathrm{Diff}(W)} E\mathrm{Diff}(W) \\
\downarrow & & \downarrow \\
B\mathrm{Diff}^k W & \longrightarrow & B\mathrm{Diff}(W).
\end{array} \tag{7.0.1}$$

Since the vertical fibers are both $\mathrm{Bun}^\Theta(W)$ and the map between them is the identity, this is a homotopy pullback square, which implies the horizontal fibers are also equivalent.

Palais' fibre sequence [68, 54]

$$\mathrm{Diff}(W, \{p_1, \dots, p_k\}) \longrightarrow \mathrm{Diff}(W) \longrightarrow \mathrm{Emb}(\coprod_k *, W).$$

implies that the bottom map of diagram (7.0.1) fits into the fibration sequence

$$C_k(W) \longrightarrow B\mathrm{Diff}^k(W) \longrightarrow B\mathrm{Diff}(W).$$

Therefore, for each $k$, the map $Q_k$ is a fibration with homotopy fibre $C_k(W)$ and the map $\sigma^k$ induces a map of fibre sequences

$$\begin{array}{ccccc}
C_k(W) & \longrightarrow & \mathcal{M}^{\Theta,k}(W) & \xrightarrow{Q_k} & \mathcal{M}^\Theta(W) \\
\downarrow{\scriptstyle \sigma^k} & & \downarrow{\scriptstyle S^k} & & \| \\
C_{k+1}(W) & \longrightarrow & \mathcal{M}^{\Theta,k+1}(W) & \xrightarrow{Q_{k+1}} & \mathcal{M}^\Theta(W)
\end{array}$$

The left-hand map induces a homology isomorphism in degrees $* \leq k/2$ (see for instance [73, Theorem A]) and therefore by Zeeman's comparison theorem [94], so does the middle map. □

A generalisation of the study of homology stability for configuration spaces has been recently carried out for configurations of more general submanifolds in $W$ as seen in [69, 70, 71, 50]. For instance, it was shown that the stabilisation map between configuration spaces of unlinked circles

$$\sigma_{kS^1} : \mathrm{C}_{kS^1}(W) \to \mathrm{C}_{(k+1)S^1}(W)$$

induces a homology isomorphism in degrees $* \geq k/2$ if the dimension of $W$ is $\geq 5$ [69, Main Theorem, Chapter 5], and in degrees $* \geq \lfloor \frac{k-2}{2} \rfloor$ if the dimension is 3 [50, Theorem 1.3].

**Proposition 7.0.2.** *The map $S_{kS^1} : \mathcal{M}^{\Theta}_{kS^1}(W) \to \mathcal{M}^{\Theta}_{(k+1)S^1}(W)$ induced by $\sigma_{kS^1}$ gives a homology isomorphism on degrees $* \geq k/2$ if the dimension of $W$ is $\geq 5$, and in degrees $* \geq \lfloor \frac{k-2}{2} \rfloor$ if the dimension is 3.*

The proof is analogous to the proof of Theorem 7.0.1.

# Chapter 8

# Decoupling Configuration Spaces

Let $Z$ be a pointed $\mathrm{GL}_2^+$-space with a good point denoted by 0. The configuration space $C(F_{g,1}; Z)$ is the quotient

$$\coprod_{n\geq 0} \widetilde{C}_n(M) \times_{\Sigma_n} Z^n / \sim$$

where $\widetilde{C}_n(M)$ denotes the ordered configuration space on $M$, and

$$(m_1, \ldots, m_n; z_1, \ldots, z_n) \sim (m_1, \ldots, m_{n-1}; z_1, \ldots, z_{n-1})$$

whenever $z_n = 0$. Since $Z$ is a pointed $\mathrm{GL}_2^+$-space, $C(F_{g,1}; Z)$ carries a natural action by the diffeomorphism group of $F_{g,1}$: for $\phi \in \mathrm{Diff}(F_{g,1})$

$$\phi \cdot (m_1, \ldots, m_n; z_1, \ldots, z_n) \coloneqq (\phi(m_1), \ldots, \phi(m_n); D_{m_1}\phi(z_1), \ldots, D_{m_1}\phi(z_n)).$$

The basepoint relation is preserved by this action as $Z$ is a pointed $\mathrm{GL}_2^+$-space. Denote the associated Borel construction by

$$MC(Z)_g = C(F_{g,1}; Z) /\!\!/ \mathrm{Diff}(F_{g,1}).$$

Gluing the boundary circles of surfaces $F_{g,1}$ and $F_{h,1}$, to two of the three boundary circles of $F_{0,3}$ defines a map of topological groups

$$\mathrm{Diff}(F_{g,1}) \times \mathrm{Diff}(F_{h,1}) \to \mathrm{Diff}(F_{g+h,1}).$$

Up to homotopy, the above gluing process is equivalent to gluing the two surfaces

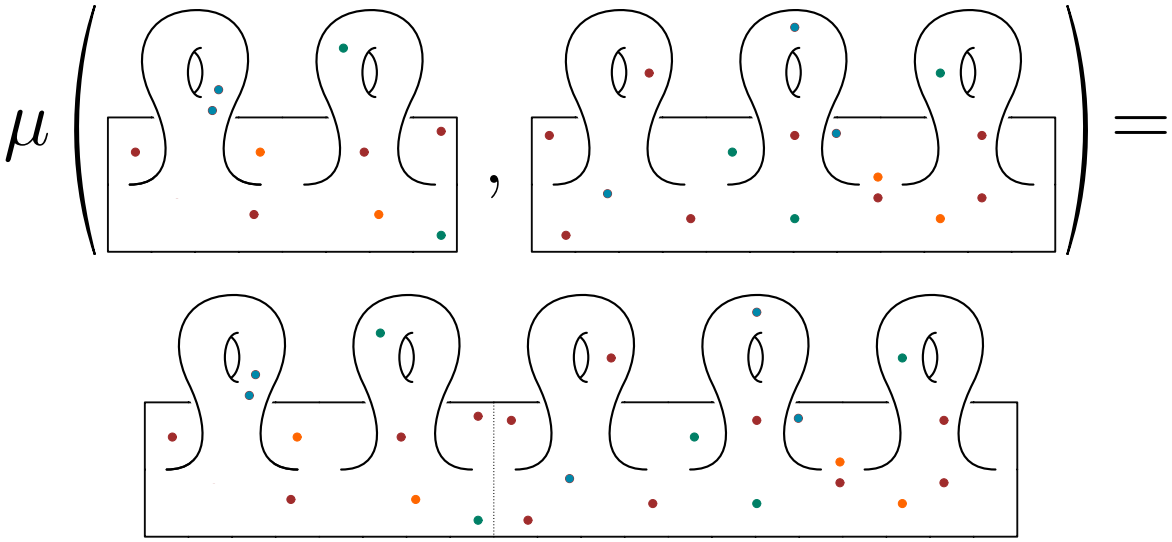


Figure 8.1: Example of the map $\mu : MC(\mathcal{C})_2 \times MC(\mathcal{C})_3 \to MC(\mathcal{C})_5$ where $\mathcal{C}$ is the space of colours, with white as the basepoint.

$F_{g,1}$ and $F_{h,1}$ along part of their boundary. This way the product can be made strictly associative by fixing embeddings $F_{g,1} \hookrightarrow \mathbb{R}^{\infty-1} \times [0,g] \subset \mathbb{R}^\infty = \operatorname{colim} \mathbb{R}^n$ that commute with the stabilisation map $\mathrm{Diff}(F_{g,b}) \to \mathrm{Diff}(F_{g,b+1})$. An example of such embeddings and multiplication is depicted in Figure 8.1. This product extends naturally to a map on the Borel constructions

$$\mu : MC(Z)_g \times MC(Z)_h \to MC(Z)_{g+h}$$

and makes the disjoint union $MC(Z) = \coprod_{g \geq 0} MC(Z)_g$ into a topological monoid, which we refer to as the *monoid of configurations labelled by* $Z$. Consider its group completion $\Omega BMC(Z)$.

**Theorem 8.0.1.** *For any path-connected pointed* $\mathrm{GL}_2^+$*-space* $Z$*, there is a homotopy equivalence*

$$\tau \times \delta : \Omega B(MC(Z)) \xrightarrow{\simeq} \mathbb{Z} \times B\mathrm{Diff}(F_\infty)^+ \times Q((E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} Z).$$

In the above, $Q(-)$ denotes the functor $\Omega^\infty\Sigma^\infty(-)$. Segal showed in [80] that the space $Q(X)$ is the group completion of the monoid $C(\mathbb{R}^\infty; X)$ where the operation is roughly given by the disjoint union. The space $\mathbb{Z} \times B\mathrm{Diff}(F_\infty)^+$ was shown by Tillmann in [87] to be the group completion of the monoid $\coprod_{g \geq 0} B\mathrm{Diff}(F_{g,1})$. The proof of the above theorem will consist of creating a monoidal map between $MC(Z)$ and the monoids of Segal and Tillmann, and to show that this induces a homotopy equivalence

on group completions.

## 8.1 Construction of the map

Let $X$ be any $\Sigma_n \wr \mathrm{GL}_2^+$-space. The space $\widetilde{C}_n(F_{g,1}) \times X$ comes equipped with two actions: $\Sigma_n$ acts diagonally by permuting the points in the configuration and by the action on $X$, and $\mathrm{Diff}(F_{g,1})$ acts by

$$\phi \cdot (m_1, \dots, m_n; x) = (\phi(m_1), \dots, \phi(m_n); (d\phi_{m_1}, \dots, d\phi_{m_n})(x)).$$

These two actions commute. Hence,

$$(\widetilde{C}_n(F_{g,1}) \times_{\Sigma_n} X) /\!\!/ \mathrm{Diff}(F_{g,1}) = [(\widetilde{C}_n(F_{g,1}) \times X) /\!\!/ \mathrm{Diff}(F_{g,1})]/\Sigma_n.$$

In Theorem 8.0.1 we take $X = Z^n$, but it will be convenient for the proof to construct the map for more general $X$. To define the map of monoids, we will use a specific model for the Borel construction

$$(\widetilde{C}_n(F_{g,1}) \times X) /\!\!/ \mathrm{Diff}(F_{g,1}) = E\mathrm{Diff}(F_{g,1}) \times_{\mathrm{Diff}(F_{g,1})} (\widetilde{C}_n(F_{g,1}) \times X). \tag{8.1.1}$$

Namely, we will take the model for $E\mathrm{Diff}(F_{g,1})$ given by the geometric realisation of the nerve of the groupoid with objects $\mathrm{Diff}(F_{g,1})$ and a unique morphism between any two objects. Similarly we will take the model for $B\mathrm{Diff}(F_{g,1})$ given by the geometric realisation of the nerve of the groupoid with a unique object and automorphism group $\mathrm{Diff}(F_{g,1})$.

The Borel construction in (8.1.1) can then be described as the geometric realisation of the simplicial space with $k$-simplices uniquely written as $(\phi_1, \dots, \phi_k; (m_i)_i, x)$ where $\phi_j \in \mathrm{Diff}(F_{g,1})$, $m_1, \dots, m_n$ are $n$ distinct points of $F_{g,1}$, and $x \in X$. Note that $\Sigma_n$ acts freely by permutations of $(m_i)_i$ and the action on $X$.

We first define a map $(\widetilde{C}_n(F_{g,1}) \times X) /\!\!/ \mathrm{Diff}(F_{g,1}) \to B\mathrm{Diff}(F_{g,1})$, as the realisation of the simplicial map given on the $k$-th skeleton by

$$(\phi_1, \dots, \phi_k; (m_i)_i, x) \mapsto (\phi_1, \dots, \phi_k).$$

This map is trivially $\Sigma_n$-equivariant and for $X = Z^n$, it induces

$$\tau : C(F_{g,1}; Z) /\!\!/ \mathrm{Diff}(F_{g,1}) \to B\mathrm{Diff}(F_{g,1}) \tag{8.1.2}$$

We refer to $\tau$ as a *forgetful map*, since it simply forgets the extra data of the labelled configurations. This gives a map from $MC(Z)$ to the monoid of oriented surfaces with one boundary component.

To construct the other monoidal map we start by constructing for any $(\Sigma_n \wr \mathrm{GL}_2^+)$-space $X$, a $\Sigma_n$-equivariant map

$$\widetilde{\delta} : (\widetilde{C}_n(F_{g,1}) \times X) /\!\!/ \mathrm{Diff}(F_{g,1}) \to \widetilde{C}_n(\mathbb{R}^\infty) \times (E\mathrm{GL}_2^+)^n \times_{(\mathrm{GL}_2^+)^n} X. \tag{8.1.3}$$

Recall that we have fixed embeddings $F_{g,1} \hookrightarrow \mathbb{R}^{\infty-1} \times [0,g] \subset \mathbb{R}^\infty = \mathrm{colim}\, \mathbb{R}^n$ that commute with the stabilisation map $\sigma$ and the monoidal product structure. These induce a $\Sigma_n$-equivariant inclusion map $\iota : \widetilde{C}_n(F_{g,1}) \to \widetilde{C}_n(\mathbb{R}^\infty)$. Analogous to what we did before before, we use a model for the Borel construction $(E\mathrm{GL}_2^+)^n \times_{(\mathrm{GL}_2^+)^n} X$ given by the geometric realisation of a simplicial space with $k$-simplices uniquely written as $(A_1, \dots, A_n, x)$ where $A_j \in (\mathrm{GL}_2^+)^k$, and $x \in X$.

The map (8.1.3) is the geometric realisation of the simplicial map given on $k$-simplices by

$$(\phi_1, \dots, \phi_k; (m_i)_i, x) \mapsto (\iota(m_i)_i, (d\phi_1|_{\phi_2 \dots \phi_k(m_i)}, \dots, d\phi_k|_{m_i})_i, x).$$

The map $\widetilde{\delta}$ is clearly $\Sigma_n$-equivariant and for $X = Z^n$, $\widetilde{\delta}$ respects the basepoint. Then $\widetilde{\delta}$ induces a map

$$\delta : C(F_{g,1}; Z) /\!\!/ \mathrm{Diff}(F_{g,1}) \to C(\mathbb{R}^\infty; (E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} Z). \tag{8.1.4}$$

## 8.2 Relation to the decoupling map

Consider the surface $F_{g,b}$ equipped with $n$ marked points $p_1, \dots, p_n$. As in the previous chapters, we consider the decorated diffeomorphism group $\mathrm{Diff}^n(F_{g,b})$ to be the subgroup of those diffeomorphisms which fix the set of marked points, but allows them to be permuted. Crucial to the relation between the maps above and the decoupling map is the well-known weak equivalence

$$B\mathrm{Diff}^n(F_{g,b}) \simeq C_n(F_{g,b}) /\!\!/ \mathrm{Diff}(F_{g,b}). \tag{8.2.1}$$

This can be derived from Palais' fibre bundle [67, 54]

$$\mathrm{Diff}(F_{g,b}, \{p_1, \dots, p_n\}) \longrightarrow \mathrm{Diff}(F_{g,b}) \longrightarrow \mathrm{Emb}(\{p_1, \dots, p_n\}, F_{g,b})$$

which induces the homotopy fibre sequence

$$\mathrm{Diff}(F_{g,b}) \longrightarrow C_n(F_{g,b}) \longrightarrow B\mathrm{Diff}^n(F_{g,b})$$

The projection $E\mathrm{Diff}(F_{g,b}) \times C_n(F_{g,b}) \to C_n(F_{g,b})$ is $\mathrm{Diff}(F_{g,b})$-equivariant and therefore it induces a map of fibre sequences

$$\begin{array}{ccc} \mathrm{Diff}(F_{g,b}) & = & \mathrm{Diff}(F_{g,b}) \\ \downarrow & & \downarrow \\ E\mathrm{Diff}(F_{g,b}) \times C_n(F_{g,b}) & \xrightarrow{\simeq} & C_n(F_{g,b}) \\ \downarrow & & \downarrow \\ C_n(F_{g,b}) /\!\!/ \mathrm{Diff}(F_{g,b}) & \longrightarrow & B\mathrm{Diff}^n(F_{g,b}) \end{array}$$

Since the map between fibers and total spaces are weak equivalences, so is the bottom map, as required.

Recall that fixing trivialisations of the tangent bundle of $F_{g,b}$, the action of an element of $\mathrm{Diff}^n(F_{g,b})$ on the tangent space of the marked points induces a homomorphism on the wreath product

$$e : \mathrm{Diff}^n(F_{g,b}) \to \Sigma_n \wr \mathrm{GL}_2^+$$

which surjects onto $\Sigma_n \wr \mathrm{GL}_2^+$, as discussed in Lemma 3.2.3. Thus any $\Sigma_n \wr \mathrm{GL}_2^+$-space $X$ is via $e$ also a $\mathrm{Diff}^n(F_{g,b})$-space. As in section 3.4, we consider the (homotopy) limit space $X /\!\!/ \mathrm{Diff}^n(F_\infty)$ obtained via the map

$$X /\!\!/ \mathrm{Diff}^n(F_{g,b}) \to X /\!\!/ \mathrm{Diff}^n(F_{g+1,b})$$

induced by the stabilisation map $\sigma : \mathrm{Diff}^n(F_{g,b}) \to \mathrm{Diff}^n(F_{g+1,b})$.

By comparing fibrations over the homotopy equivalence (8.2.1) with fibre a $\Sigma_n \wr \mathrm{GL}_2^+$-space $X$, one proves

$$X /\!\!/ \mathrm{Diff}^n(F_{g,1}) \xrightarrow{\simeq} (\widetilde{C}_n(F_{g,1}) \times_{\Sigma_n} X) /\!\!/ \mathrm{Diff}(F_{g,1}).$$

Under this homotopy equivalence, the map $\delta$ can be identified up to homotopy with the evaluation map $E$ of section 3.4.

Similarly, the forgetful map $F : B\mathrm{Diff}^n(F_{g,1}) \to B\mathrm{Diff}(F_{g,1})$ corresponds to map $\tau : C_n(F_{g,1})/\!/\mathrm{Diff}(F_{g,1}) \to B\mathrm{Diff}(F_{g,1})$. Summarising, we restate Corollary 3.4.3 as

**Corollary 8.2.1.** *For any $\Sigma_n \wr \mathrm{GL}_2^+$-space $X$, the maps $\tau$ and $\delta$ induce a homology isomorphism*

$$(\widetilde{C}_n(F_\infty) \times_{\Sigma_n} X)/\!/\mathrm{Diff}(F_\infty) \xrightarrow{\tau\times\delta} B\mathrm{Diff}(F_\infty) \times (\widetilde{C}_n(\mathbb{R}^\infty) \times (E\mathrm{GL}_2^+)^n) \times_{\Sigma_n\wr\mathrm{GL}_2^+} X.$$

*Remark* 8.2.2. This result may be interpreted in physical terms: in the high genus limit, the constraints in a two-dimensional sigma model for the particles to stay on the underlying surface is lifted when the theory is coupled to gravity; the particles are now free.

## 8.3 A group completion argument

By [87] the space $\mathbb{Z}\times B\mathrm{Diff}(F_\infty)^+$ is the group completion of the monoid $\coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1})$. Moreover, in [80], Segal showed that the space $Q(X)$ is the group completion of the monoid $C(\mathbb{R}^\infty; X)$ where the operation is roughly given by the disjoint union. We show that the maps defined above are compatible with these structures.

Some care has to be taken into making the configuration spaces into actual topological monoids (see [80]). Namely, instead of $C_n(\mathbb{R}^k; X)$, we use the homotopy equivalent space

$$C'_n(\mathbb{R}^k, X) = \{(c,t) \in C_n(\mathbb{R}^k, X) \times \mathbb{R} : t \geq 0, c \subset (0,t) \times \mathbb{R}^{k-1}\}.$$

The monoidal structure is given by *juxtaposition*, ie. $(c,t),(c',t') \mapsto (c \cup T_t(c'), t+t')$ where $T_t(-)$ is the map that translates a configuration by $t$ on the first direction.

The monoidal structure of $\coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1})$ is constructed very similar to the one of $C_\Sigma(P)$. When constructing $C_\Sigma(P)$, we fixed representatives of the surfaces $F_{g,1}$ embedded in $[0,g]\times\mathbb{R}^{\infty-1}$ compatible with the stabilisation maps $F_{g,1}\times F_{h,1} \to F_{g+h,1}$. These representatives induce maps between the respective classifying spaces which are associative and make $\coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1})$ into a topological monoid.

**Lemma 8.3.1.** *The map $\delta$ and the forgetful map $\tau$ are compatible with these monoidal structures.*

*Proof.* The map $\delta$ takes an element in $MC(Z)_g$ to a pair $(c, g) \in C'_n(\mathbb{R}^\infty)$, for some $n$, and that the definition of the monoidal structure in $C_\Sigma(P)$ is compatible with the monoid structure of juxtaposing configurations as defined in above. Moreover, on the underlying surfaces, our definition of the monoidal structure in $MC(Z)_g$ was defined using the same representatives and composition rule as $\coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1})$ above, making the forgetful map compatible with these monoidal structures. □

To prove Theorem 8.0.1 is thus enough to show that the map of monoids induces a homotopy equivalence on group completions. As the group completions are loop spaces, they are in particular simple and, by the Whitehead theorem for simple spaces, it suffices to show that it induces a homology equivalence on the group completions. Both monoids are homotopy commutative, hence the group completion theorem [62] can be applied. It implies that it is enough to prove that the induced map on the limit spaces (defined in section 3.4)

$$\tau \times \delta : C(F_\infty; Z) /\!\!/ \mathrm{Diff}(F_\infty) \to B\mathrm{Diff}(F_\infty) \times C(\mathbb{R}^\infty; (E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} Z) \qquad (8.3.1)$$

is a homology equivalence. We will show this in the next subsection.

## 8.4 Homological argument

In this section, we will show that:

**Proposition 8.4.1.** *The map* (8.3.1) *is a homology equivalence.*

The limit spaces in (8.3.1) consist of configuration spaces with arbitrary number of points. These they have natural filtrations

$$\begin{aligned} X_n &\coloneqq C^{\leq n}(F_\infty; Z) /\!\!/ \mathrm{Diff}(F_\infty) \\ Y_n &\coloneqq B\mathrm{Diff}(F_\infty) \times C^{\leq n}(\mathbb{R}^\infty; (E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} Z) \end{aligned}$$

with respect to the number of points in the configuration. These filtrations are preserved under the map $\tau \times \delta$. We will inductively show the restrictions $X_n \to Y_n$ are homology isomorphisms for all $n$.

*Proof.* Since the basepoint of $Z$ is a good point, the inclusions $X_{n-1} \hookrightarrow X_n$ and $Y_{n-1} \hookrightarrow Y_n$ are cofibrations. Their subquotients are

$$X_n / X_{n-1} = (E\mathrm{Diff}(F_\infty))_+ \wedge_{\mathrm{Diff}(F_\infty)} \widetilde{C}_n(F_\infty)_+ \wedge_{\Sigma_n} Z^{\wedge n}$$

and

$$Y_n/Y_{n-1} = (B\mathrm{Diff}(F_\infty))_+ \wedge \widetilde{C}_n(\mathbb{R}^\infty)_+ \wedge_{\Sigma_n} ((E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} Z)^{\wedge n}.$$

Comparing the spectral sequences associated to these filtrations, it is enough to show that the induced map on these subquotients is a homology isomorphism. Consider the map of cofibrations:

$$\begin{array}{ccccc}
E\mathrm{Diff}(F_\infty) \underset{\mathrm{Diff}(F_\infty)}{\times} C_n(F_\infty) & \longrightarrow & E\mathrm{Diff}(F_\infty) \underset{\mathrm{Diff}(F_\infty)}{\times} (\widetilde{C}_n(F_\infty) \underset{\Sigma_n}{\times} Z^{\wedge n}) & \longrightarrow & X_n/X_{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
B\mathrm{Diff}(F_\infty) \times C_n(\mathbb{R}^\infty, B\mathrm{GL}_2^+) & \longrightarrow & B\mathrm{Diff}(F_\infty) \times (\widetilde{C}_n(\mathbb{R}^\infty) \underset{\Sigma_n}{\times} ((E\mathrm{GL}_2^+)_+ \underset{\mathrm{GL}_2^+}{\wedge} Z)^{\wedge n}) & \longrightarrow & Y_n/Y_{n-1}
\end{array}$$

By corollary 8.2.1 with $X = *$ the left-hand map is a homology isomorphism, and by the same result with $X = Z^{\wedge n}$, so is the middle map. Then by the five lemma, the right-hand map is also a homology isomorphism, as required. □

# Chapter 9

# Decoupling Configuration Spaces with Partially Summable Labels

In this chapter we discuss how to extend the result on monoids of configuration spaces developed in the previous chapter to the case where the points are given partially summable labels and the particles are allowed to collide whenever their labels can be added. The space $C_\Sigma(W;P)$ of configurations in $W$ with partially summable labels in $P$ has been defined in [76] and its definition requires more sophisticated tools such as the Fulton-MacPherson configuration spaces and operad. In section 9.1 we recall these definitions and the concept of a partial $d$-monoids.

To prove the decoupling theorem for the space $C_\Sigma(W;P)$, we develop a different model for this space in section 9.2, which we refer to as the *disc model for configurations*, denoted $|D_\Sigma(W;P)_\bullet|$. In Proposition 9.2.2 we show that indeed this space is weakly equivalent to $C_\Sigma(W;P)$. In the decoupling context, we naturally encounter another space of discs with configurations, which we refer to as the *d-discs model for configurations* in $\mathbb{R}^\infty$, denoted $|D^d_\Sigma(\mathbb{R}^\infty;P)_\bullet|$, which is constructed in definition 9.2.5.

All the constructions above are carried out in any dimension. When proving the decoupling in this context in section 9.3, we specialise to $d = 2$ for clarity of exposition. The main theorems are:

**Theorem 9.0.1.** *The decoupling map induces a homology isomorphism*

$$C_\Sigma(F_\infty;P)/\!/\mathrm{Diff}(F_\infty) \simeq |D_\Sigma(F_\infty;P)_\bullet|/\!/\mathrm{Diff}(F_\infty) \to B\mathrm{Diff}(F_\infty) \times |D^2_\Sigma(\mathbb{R}^\infty;P)_\bullet|.$$

Denoting by $MC_\Sigma(P)$ the monoid for configurations in surfaces with partially summable labels in $P$, the above result implies the following:

**Corollary 9.0.2.** *For any path-connected framed partial 2-monoid with unit $P$, there*

*is a homotopy equivalence*

$$\Omega B(MC_\Sigma(P)) \simeq \mathbb{Z} \times B\mathrm{Diff}(F_\infty)^+ \times |D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet|.$$

## 9.1 Fulton-MacPherson configuration space and partially summable labels

In this section we recall the concept of a configuration space with summable labels in a partial monoid as described in [76].

A partial monoid $P$ consists of a space with a continuous operation that is not defined on all collections of elements, but only on a subset of *composable elements*. The data defining such a partial monoid consists of the subset of composable elements, together with an operation on this subset which is associative. We are interested in partial monoids that moreover have the structure of an $E_d$-algebra. We make this precise in this section, using the concepts developed by Salvatore in [76].

Crucial to our construction is $\mathcal{F}_d$, the Fulton-MacPherson operad in dimension $d$ (for a detailed exposition see [83]). This is a cofibrant replacement for the little $d$-discs operad and therefore will be used to make precise the notion of a $E_d$-partial monoid. Crucially for us, the Fulton-MacPherson operad is defined in terms of configuration spaces of points, in which the particles in the configuration are allowed to collide, but keeping track of the relative position of the particles before they collided and the order in which collided.

We follow the definition of the Fulton-MacPherson configuration space $\overline{C}_k(W)$ of a manifold $W$ as described in [83]. To make it precise, we consider $W$ as embedded in some $\mathbb{R}^m$ and we denote by $\underline{n}$ the set $\{1, \ldots, n\}$. We record the directions of particles in a collision, by defining for $(i,j) \in \widetilde{C}_2(\underline{n})$, a map $\pi_{i,j} : \widetilde{C}_n(\mathbb{R}^m) \to S^{m-1}$ sending a configuration $\xi = (x_i)$ to the unit vector in the direction $x_i - x_j$. The order of collision is recorded by defining for $(i,j,k) \in \widetilde{C}_3(\underline{n})$ the map $s_{i,j,k} : \widetilde{C}_n(\mathbb{R}^m) \to [0,\infty]$ sending $\xi = (x_i)$ to $|x_i - x_j|/|x_i - x_k|$.

**Definition 9.1.1** ([83], Definition 1.3)**.** The *Fulton-MacPherson configuration space* $\overline{C}_n(W)$ of the manifold $W$ is the closure of the image of the map

$$i \times (\pi_{i,j}|_{\widetilde{C}_n(W)}) \times (s_{i,j,k}|_{\widetilde{C}_n(W)}) : \widetilde{C}_n(W) \longrightarrow W^n \times (S^{m-1})^{n(p-1)} \times [0,\infty]^{n(p-1)(n-2)}.$$

The space $\overline{C}_n(W)$ is homotopy equivalent to the ordered configuration space $\widetilde{C}_n(W)$ [83, Corollary 4.5] and whenever $W$ is compact, $C_n(W)$ is a compactifica-

tion of $\widetilde{C}_n(W)$. Moreover, this construction is functorial with respect to embeddings [83, Corollary 4.8], i.e. any embedding $f : W \hookrightarrow N$ induces an embedding $f_* : \overline{C}_n(W) \to \overline{C}_n(N)$.

The following result gives a convenient way to represent elements of $\overline{C}_n(W)$, which will be used throughout the chapter.

**Proposition 9.1.2** ([83], Theorem 3.8)**.** *Each element in $\overline{C}_n(W)$ is uniquely determined by:*

1. *A configuration of points $P_1, \ldots, P_l$ in the interior of $W$, with $1 \leq l \leq n$ (we refer to these as the infinitesimal configuration),*

2. *For each $1 \leq i \leq l$, a tree $T_i$ with $f_i$ leaves (twigs), no bivalent vertices, so that $\sum_{i=1}^{l} f_i = n$, and for each vertex in $T_i$ of valence $m$ an element in $C_m(T_{P_i}W)/G(d)$, where $G(d)$ is the group of affine transformations of $\mathbb{R}^d$ generated by translations and positive dilations.*

3. *A global ordering of the $k$ leaves of the trees.*

This interpretation of the elements of $\overline{C}_n(W)$ also provides a way of describing the map $f_* : \overline{C}_n(W) \to \overline{C}_n(N)$ induced by an embedding $f : W \hookrightarrow N$ into a $d'$-manifold $N$: it takes a point with infinitesimal configurations $P_1, \ldots, P_l \in W$ to one with infinitesimal configurations $f(P_1), \ldots, f(P_l) \in N$, it preserves the trees and ordering of the leaves, but changes the labels of the vertices of the trees by taking a label $\xi \in C_m(T_{P_i}W)/G(d)$ to the label $D_{P_i}f(\xi) \in C_m(T_{f(P_i)}N)/G(d')$.

The Fulton-MacPherson operad $\mathcal{F}_d$ uses a specific subspace of this construction. Intuitively, for each $k \geq 0$ the space $F_d(k)$ is the subspace of the ordered configurations of $k$ points in $\mathbb{R}^d$, in which the points have collided at the origin.

**Definition 9.1.3.** The *Fulton-MacPherson operad* in dimension $d$, denoted $\mathcal{F}_d$, is defined by taking $\mathcal{F}_d(k)$ to be the pullback

$$\begin{array}{ccc} \mathcal{F}_d(k) & \longrightarrow & \overline{C}_k(\mathbb{R}^d) \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & (\mathbb{R}^d)^k. \end{array}$$

In other words, it is the subspace of $\overline{C}_k(\mathbb{R}^d)$ with infinitesimal configuration given by a single point at the origin.

With the description of this space as in Proposition 9.1.2, the composition of this operad is given by grafting of trees.

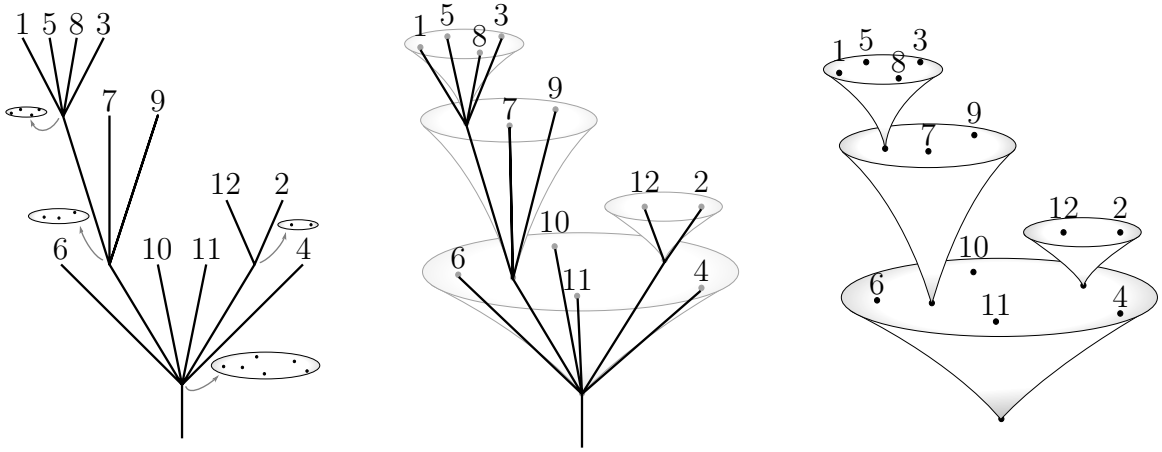



Figure 9.1: Representation of an element in the space $\mathcal{F}_2(12)$ in terms of a tree of infinitesimal configurations.

Pictorially, we will often represent elements of $\mathcal{F}_d(k)$ as trees of configurations such as in the rightmost picture of Figure 9.1. The composition is given simply by grafting (ie. stacking a tree on top of another along the specified leaf).

As shown in [76], there exists a model structure on the category of topological operads in which $\mathcal{F}_d$ is a cofibrant replacement of the little $d$-discs operad. An algebra over $\mathcal{F}_d$, called by Salvatore a *$d$-monoid*, consists of a space $A$ together with $\Sigma_k$-equivariant maps

$$\mathcal{F}_d(k) \times A^k \to A$$

which commute with the structure maps of $\mathcal{F}_d$.

**Definition 9.1.4** ([76], Definition 1.7 and 2.6)**.** A *partial $d$-monoid* is a space $P$ together with monomorphisms of $\Sigma_k$-spaces

$$i : Comp_k \hookrightarrow \mathcal{F}_d(k) \times_{\Sigma_k} P^k.$$

and composition maps $\rho : Comp_k \to P$ such that

1. The unit $\eta \times P : P \to \mathcal{F}_d(1) \times P$ factors uniquely through $\widetilde{\eta} : P \to Comp_1$ and the composition $\rho(\widetilde{\eta}) = \mathrm{id}_P$ is the identity;

2. For $f \in \mathcal{F}_d(k)$ and $[f_j; \overline{p}_j] \subset Comp_{m_j}$, for $j = 1, \ldots, k$, the element

$$[f; \rho(f_1; \overline{p}_1), \ldots, \rho(f_k; \overline{p}_k)] \in \mathcal{F}_d(k) \times_{\Sigma_k} P^k$$

belongs to $Comp_k$ if and only if

$$[\mu(f; f_1, \dots, f_k); \overline{p}_1, \dots, \overline{p}_k]$$

belongs to $Comp_{m_1+\cdots+m_k}$. Moreover, if that is the case, then their image by $\rho$ coincide.

A pointed space $(P, 0)$ is a *partial $d$-monoid with unit* if in addition it satisfies

3. For every $k$ there is an inclusion $u : \mathcal{F}_d(k) \times_{\Sigma_k} \bigvee_k P \hookrightarrow Comp_k$ such that the composition with $i \circ u$ is the subspace inclusion, and $\rho \circ i : \mathcal{F}_d(k) \times_{\Sigma_k} \bigvee_k P \to P$ is the projection onto the $P$ coordinate.

The definition above is better understood when in comparison to $d$-monoids. Given a $d$-monoid $M$ and a infinitesimal configuration $f \in \mathcal{F}_d(k)$, we can always compose any $k$ elements of $M$ via the composition rule described by $f$. In a partial $d$-monoid $P$, that is not the case. We instead are given a subset of the $k$-tuples of $P$ which are composable via the operation described by a given $f \in \mathcal{F}_d(k)$. The space $Comp_k$ should be then thought of as the pairs of possible composition rules (given by $\mathcal{F}_d$) together with the tuples of elements of $P$ which can actually be composable with this composition rule. The map $\rho$ is the evaluation map that actually computes the results of compositions, whenever they are defined.

**Example 9.1.5.** (a) Any space $X$ admits the structure of a trivial partial $d$-monoid, by defining $Comp_1 = \{[1, x] : x \in X\}$ and $Comp_k = \emptyset$, for all $k \neq 1$. In this case, the only composition rule that can be performed is the identity and no other collection of points in $X$ is composable in any way.

(b) When $X$ is a space equipped with a basepoint $*$, we can define a unital partial $d$-monoid by setting $Comp_k = \mathcal{F}_d(k) \times \vee_k X$ and $\rho(f, \overline{x}) = x_i$, where $x_i$ is the unique non-basepoint coordinate, or $*$ otherwise. In this case, the basepoint acts as a unit and compositions are only defined when done with the unit.

(c) In [80], Segal defines a partial monoid of ordered configuration spaces with composition given by the union of particles. Precisely, $P \coloneqq \coprod_{k \geq 0} \widetilde{C}_k(\mathbb{R}^n)$, and two configurations $\overline{p} = (p_1, \dots, p_n)$, $\overline{q} = (q_1, \dots, q_m)$ are composable if they are disjoint. In this case, their multiplication $\overline{p} \cdot \overline{q}$ is defined as the union $(p_1, \dots, p_n, q_1, \dots, q_m)$. This is a partial 1-monoid, where $[f, \overline{p}_1 \dots, \overline{p}_k] \in Comp_k$ if $\overline{p}_1, \dots, \overline{p}_k$ are disjoint, and composition is given by the union of all configurations, with the new ordering determined by $f$.

(d) Every $d$-monoid is trivially a partial $d$-monoid. In particular, every $\Omega^d$-space is a partial $d$-monoid.

(e) Any partial $m$-monoid is also a partial $d$-monoid for any $d < m$. The canonical inclusion $\mathbb{R}^d \hookrightarrow \mathbb{R}^m$, for $m > d$, induces an injective map of operads $\mathcal{F}_d \to \mathcal{F}_m$. If $P$ is a partial $m$-monoid, the pullback of the diagram

$$\begin{array}{ccc} & & Comp_k \\ & & \downarrow \\ \mathcal{F}_d(k) \times_{\Sigma_k} P^k & \longrightarrow & \mathcal{F}_m(k) \times_{\Sigma_k} P^k \end{array}$$

is a space $\overline{Comp_k}$ with a monomorphism to $\mathcal{F}_d(k) \times_{\Sigma_k} P^k$ and a composition map

$$\overline{Comp_k} \to Comp_k \xrightarrow{\rho} P.$$

It is simple to check that this gives $P$ the structure of a partial $d$-monoid. Intuitively, this is as if we have forgotten how to perform some of the compositions in $P$, the ones that are not given by composition rules of configurations in $\mathbb{R}^d \hookrightarrow \mathbb{R}^m$.

(f) The canonical inclusion $i_{d,n} : \mathbb{R}^d \hookrightarrow \mathbb{R}^{d+n}$ also allows us to construct see a partial $d$-monoid $P$ as a partial $(d+n)$-monoid with a trivial $n$ composition directions. We denote this trivial upgrade of a partial $d$-monoid $P$ by $T_{d,n}P$. The underlying space of $T_{d,n}P$ is $P$, and we define $Comp_k^{T_{d,n}P}$ to be the image of

$$Comp_k^P \hookrightarrow \mathcal{F}_d(k) \times_{\Sigma_k} P^k \xrightarrow{(i_{n,k})_*} \mathcal{F}_{d+n}(k) \times_{\Sigma_k} P^k = \mathcal{F}_{d+n}(k) \times_{\Sigma_k} (T_{d,n}P)^k.$$

In [76], it was always assumed that the partial $d$-monoids were *good*, in the sense described below.

**Definition 9.1.6.** A partial $n$-monoid $P$ is *good* if for every $k$ the inclusion $Comp_k^P \hookrightarrow \mathcal{F}_d(k) \times_{\Sigma_k} P^k$ is a cofibration.

From now on, we always assume partial $n$-monoids to be good and to have a unit. For future applications, we are further interested in partial $d$-monoids with compatible actions of $\mathrm{GL}_d$, so we introduce this concept here and perform our constructions in this more general setting.

Recall from Proposition 9.1.2 that an element of $\mathcal{F}_d(k)$ is described by a tree with $k$ ordered leaves and a decoration of the vertices by elements of $\widetilde{C}_{|v|}(\mathbb{R}^d)/G(d)$. The

action of $\mathrm{GL}_d$ on $\mathbb{R}^d$ induces an action on $\widetilde{C}_{|v|}(\mathbb{R}^d)/G(d)$. This gives an action of this group on $\mathcal{F}_d(k)$ for every $k$, and it is simple to check that the operad maps $\mu$ are all $\mathrm{GL}_d$ equivariant.

**Definition 9.1.7** ([76], Definition 4.3)**.** The *framed Fulton-MacPherson operad* denoted $f\mathcal{F}_d$, is the operad defined by $f\mathcal{F}_d(k) = \mathcal{F}_d(k) \times \mathrm{GL}_d^k$, with structure map

$$\begin{aligned}\widetilde{\mu}((x, g_1, \ldots, g_k);(x_1, g_1^1, \ldots, g_1^{m_1}), \ldots, (x_k, g_k^1, \ldots, g_k^{m_k})) \\ = (\mu(x; g_1x_1, \ldots, g_kx_k), g_1g_1^1, \ldots, g_kg_k^{m_k}).\end{aligned}$$

The construction above is an instance of the construction $\mathcal{A} \rtimes G$, the semi-direct product of an operad $\mathcal{A}$ and group $G$. This construction, and the proof that the above indeed defines an operad can be found in [78, Definition 2.1].

**Definition 9.1.8** ([76], Definition 4.8)**.** A *framed partial $d$-monoid* with unit is a pointed $\mathrm{GL}_d$-space $P$ together with monomorphisms of $\Sigma_k \wr \mathrm{GL}_d$-spaces

$$i : fComp_k \hookrightarrow f\mathcal{F}_d(k) \times_{\Sigma_k} P^k.$$

and $\mathrm{GL}_d$-equivariant composition maps $\rho : fComp_k \to P$ satisfying properties 1-3 of Definition 9.1.4.

The $\mathrm{GL}_d$-bundle of frames on $W$ induces a $(\mathrm{GL}_d)^k$-bundle $f\overline{C}_k(W)$ on $C_k(W)$, acted on by $\Sigma_k$. This is called the *framed configuration space* of $k$ points in $W$. Using the description of Proposition 9.1.2, an element of the space $f\overline{C}_k(W)$ can be uniquely determined by an infinitesimal configuration in $W$ with labelled trees, together with additional $k$ frames of the tangent planes associated to the $k$ leaves of the trees.

Then the space of framed configurations $f\overline{C}(W) = \coprod_k f\overline{C}_k(W)$ is a right $f\mathcal{F}_d$-module, with $\mathrm{GL}_d$-equivariant multiplication maps

$$m : f\overline{C}_k(W) \times_{\Sigma_k} (f\mathcal{F}_d(n_1) \times \cdots \times f\mathcal{F}_d(n_k)) \to f\overline{C}_{n_1+\cdots+n_k}(W).$$

defined by grafting the element of $f\mathcal{F}_d(n_i)$ on the $i$-th leaf of the element of $f\overline{C}_k(W)$, for all $i = 1, \ldots, k$ and using the frame on the leaves to identify $C_m(\mathbb{R}^d)/G(d)$ with a configuration on the tangent space of $W$ ([76, Proposition 4.5]). It is simple to verify that any co-dimension zero embedding $e : W \hookrightarrow N$ induces a right $f\mathcal{F}_d$-homomorphism $e_* : f\overline{C}(W) \hookrightarrow f\overline{C}(N)$.

**Definition 9.1.9** ([76], Definition 4.14)**.** Let $P$ be a framed partial $d$-monoid, and let $W$ be a manifold of dimension $d$. Then the space of configurations in $W$ with

partially summable labels in $P$, denoted $C_\Sigma(W;P)$ is defined as the co-equalizer of the following

$$\coprod_k \left[ f\overline{C}_k(W) \times_{\Sigma_k} \left( \coprod_{\pi \in \mathrm{Map}(n,k)} \prod_{i=1}^{k} fComp^P_{\pi^{-1}(i)} \right) \right] \underset{(m\times \mathrm{id})\circ(\mathrm{id}\times i)}{\overset{\mathrm{id}\times\rho^k}{\rightrightarrows}} \coprod_k f\overline{C}_k(W) \times_{\Sigma_k} P^k$$

An element of $C_\Sigma(W;P)$ is then an equivalence class of elements in $\coprod_k f\overline{C}_k(W) \times_{\Sigma_k} P^k$. From the description of $\overline{C}_n(W)$ above, we can see that an element in $C_\Sigma(W;P)$ can be represented by an infinitesimal configuration $P_1, \ldots, P_l$ of $l < k$ points in $W$ together with trees $T_i$, for $i = 1, \ldots, l$, where the vertices of $T_i$ are labelled by elements in $f\mathcal{F}_d$, and the leaves of the trees are labelled by elements of $P$. The equivalence relation defining $C_\Sigma(W;P)$ implies that if some leaves labelled by $a_1, \ldots, a_k$ are departing from a vertex labelled by $c \in f\mathcal{F}_d(k)$ and the composition $\rho(c; a_1, \ldots, a_k)$ is defined, then we identify this configuration with the one in which such leaves are removed and their vertex is replaces by a leaf labelled by $\rho(c; a_1, \ldots, a_k)$.

With this description, it is simple to verify that for a framed partial $d$-monoid $P$ and a co-dimension zero embedding $f : W \hookrightarrow N$, the induced map

$$f_* : \coprod_k f\overline{C}_k(W) \times_{\Sigma_k} P^k \to \coprod_k f\overline{C}_k(N) \times_{\Sigma_k} P^k$$

preserves the equivalence classes described above. Therefore any such co-dimension zero embedding induces a map

$$f_* : C_\Sigma(W;P) \to C_\Sigma(N;P).$$

Seeing the group $\mathrm{Diff}(W)$ as a subspace of $\mathrm{Emb}(W,W)$, we can see that the above construction shows that $C_\Sigma(W;P)$ admits an action of $\mathrm{Diff}(W)$. We will be interested in a decoupling theorem for the space $C_\Sigma(W;P)/\!/\mathrm{Diff}(W)$.

## 9.2 Disc models for configuration spaces with partially summable labels

Let $W$ be a smooth compact $d$-manifold, possibly with boundary, and $P$ a framed partial $d$-monoid. The group $(\Sigma_k \wr \mathrm{GL}_d)$ has a canonical inclusion into $\mathrm{Diff}(\coprod_k \mathbb{R}^d)$. We consider $\mathrm{Emb}(\coprod_k \mathbb{R}^d, W)$ to be a $(\Sigma_k \wr \mathrm{GL}_d)$-space with action given by pre-composition by the inverse, and $C_\Sigma(\coprod_k \mathbb{R}^d; P)$ to be a $(\Sigma_k \wr \mathrm{GL}_d)$-space with action induced by the

action of $\mathrm{Diff}(\coprod_k \mathbb{R}^d)$, as described in the previous section.

**Definition 9.2.1.** The *disc model for configurations* in $W$ with summable labels in $P$, denoted $D_\Sigma(W;P)$ , is the quotient

$$\left(\coprod_{k\geq 0} \mathrm{Emb}(\coprod_k \mathbb{R}^d, W) \times_{(\Sigma_k \wr \mathrm{GL}_d)} C_\Sigma(\coprod_k \mathbb{R}^d; P)\right) \Big/ \sim$$

where $(e : \coprod_m \mathbb{R}^d \to W, \xi) \sim (e' : \coprod_n \mathbb{R}^d \to W, \xi')$ if there are injections $\underline{k} \hookrightarrow \underline{m}$ and $\underline{k} \hookrightarrow \underline{n}$ such that the induced inclusions $i_1 : \coprod_k \mathbb{R}^d \to \coprod_m \mathbb{R}^d$ and $i_2 : \coprod_k \mathbb{R}^d \to \coprod_n \mathbb{R}^d$ satisfy

$$\xi \subset Im\, i_1,\ \xi' \subset Im\, i_2 \text{ and } e \circ i_1 = e' \circ i_2.$$

Since the Fulton-MacPherson configuration spaces are functorial for co-dimension zero embeddings, we get a map

$$p : D_\Sigma(W;P) \to C_\Sigma(W;P)$$

taking a class $(e, \xi)$ to the configuration $e_*(\xi)$. It is simple to verify that this map does not depend on the choice of representative $(e, \xi)$. We equip the space $D_\Sigma(W;P)$ with a partial ordering by declaring $(e, \xi) < (e', \xi')$ if, for some representative of the classes, $e(\xi) = e'(\xi')$ and $Im\, e \supset Im\, e'$.

The space we will use in the decoupling theorem is the geometric realisation of the semi-simplicial nerve of the poset $D_\Sigma(W;P)$. We quickly recall the definitions of augmented semi-simplicial spaces and their realisation, for a detailed exposition see [27].

A *semi-simplicial space* is a functor $\Delta_{inj}^{op} \to \mathrm{Top}$, where $\Delta_{inj}$ is the category with object the linearly ordered sets $[p] = \{0 < \cdots < p\}$ and morphisms the injective monotone maps. We denote such a functor by $X_\bullet$ and write $X_p = X_\bullet(\{1, \ldots, p\})$. The datum of a semi-simplicial space is equivalent to the collection of spaces $X_p$, $p \geq 0$, together with *face maps* $d_i : X_p \to X_{p-1}$ for $i = 0, \ldots, p$, satisfying $d_i d_j = d_{j-1} d_i$ if $i < j$.

Denote by $\Delta^p$ the *standard p-simplex*

$$\Delta^p = \left\{ (t_0, \ldots, t_p) \in \mathbb{R}^{p+1} | \sum_{i=1}^{p} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

To each morphism $\phi : [p] \to [q]$ in $\Delta_{inj}$, there is a continuous map $\phi_* : \Delta^p \to \Delta^q$ such

that $\phi_*(t_0, \dots, t_p) = (s_0, \dots, s_q)$ with $s_j = \sum_{i \in \phi^{-1}(j)} t_i$. The *geometric realisation* of a semi-simplicial space $X_\bullet$ is the quotient space

$$|X_\bullet| \coloneqq \left( \coprod_p X_p \times \Delta^p \right) \Big/ \sim$$

where $(x, \phi_* t) \sim (\phi^* x, t)$, and $\phi$ is a morphism of $\Delta_{inj}$.

An *augmented semi-simplicial space* is a triple $(X_\bullet, X_{-1}, \varepsilon_\bullet)$, where $X_\bullet$ is a semi-simplicial space, $X_{-1}$ is a space and $\varepsilon_\bullet$ is a collection of continuous maps $\varepsilon_p : X_p \to X_{-1}$ satisfying $d_i \varepsilon_p = \varepsilon_{p-1}$ for all $p \geq 0$ and all face maps $d_i$. We also say that $\varepsilon_\bullet : X_\bullet \to X_{-1}$ is an *augmentation* for $X_\bullet$. It is simple to verify that an augmentation induces a continuous map $|\varepsilon_\bullet| : |X_\bullet| \to X_{-1}$.

Any topological poset $Q$ defines a semi-simplicial space $Q_\bullet$ by setting $Q_p$ to be the subspace of tuples $(q_0 < \cdots < q_p) \in Q^{p+1}$, and face maps

$$\begin{aligned} d_i : Q_p &\longrightarrow Q_{p-1} && \text{for } 0 \leq i \leq p \\ (q_0 < \cdots < q_p) &\longmapsto (q_0 < \cdots < \widehat{q_i} < \cdots < q_p). \end{aligned}$$

We refer to $Q_\bullet$ as the *semi-simplicial nerve* of the poset $Q$.

Let $D_\Sigma(W; P)_\bullet$ be the semi-simplicial nerve of the poset $D_\Sigma(W; P)$. By the definition of the partial order, the map $p$ induces an augmentation $D_\Sigma(W; P)_\bullet \to C_\Sigma(W; P)$. It might be helpful to keep in mind the following visualisation for elements in the space $|D_\Sigma(W; P)_\bullet|$: any point can be expressed by a tuple

$$((e_0, \xi_0) < \cdots < (e_k, \xi_k); t_0, \dots, t_k) \qquad\qquad t_0 \cdots + t_k = 1.$$

The poset construction implies that for such a tuple $e_0(\xi_0) = \cdots = e_k(\xi_k)$ and $Im\, e_0 \supset \cdots \supset Im\, e_k$. We can visualise it as a collection of descending embedded discs around the configuration $e_i(\xi_i)$ in $W$. Each collection of embedded discs has weights adding up to 1 and when the weight associated to a collection of embeddings goes to zero, that collection disappears. The equivalence relation guarantees that we can always get a representative for the collection of discs such that each component has at least one particle of the configuration inside it. See Figure 9.2

**Proposition 9.2.2.** *The map $p : D_\Sigma(W; P) \to C_\Sigma(W; P)$ induces a weak homotopy equivalence*

$$|D_\Sigma(W; P)_\bullet| \to C_\Sigma(W; P).$$

The proof of the above will be a direct application of [29, Theorem 6.2]. For

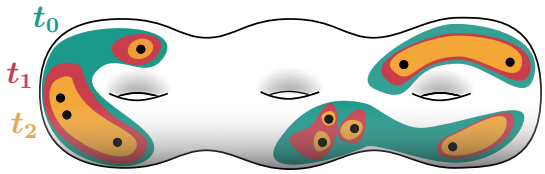



Figure 9.2: Element in a 2-simplex of $|D_\Sigma(F_3)|$.

completeness, we quickly recall the statement of this result to clarify the conditions that will be checked in the proof of Proposition 9.2.2. We follow the notation and definitions of [29].

An *augmented topological flag complex* [29, Definition 6.1] is an augmented semi-simplicial space $\varepsilon : X_\bullet \to X_{-1}$ such that

(i) The map $X_n \to X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$ taking an $n$-simplex to its $(n+1)$ vertices is a homeomorphism onto its image, which is an open subset.

(ii) A tuple $(v_0, \ldots, v_n) \in X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$ lies in $X_n$ if and only if $(v_i, v_j) \in X_1$ for all $i < j$.

The result below is a criterion to determine when an augmented topological flag complex $X_\bullet \to X_{-1}$ induces a weak equivalence $|X_\bullet| \to X_{-1}$.

**Theorem 9.2.3** ([29], Theorem 6.2)**.** *Let $X_\bullet \to X_{-1}$ be an augmented topological flag complex. Suppose that*

*(i) The map $\varepsilon : X_0 \to X_{-1}$ has local sections (in the strong sense that given any $x \in X_0$, there is a neighbourhood $U \subset X_{-1}$ of $\varepsilon(x)$ and a section $s : U \to X_0$ with $s(\varepsilon(x)) = x$.*

*(ii) $\varepsilon : X_0 \to X_{-1}$ is surjective.*

*(iii) For any $p \in X_{-1}$ and any non-empty finite set $\{v_1, \ldots, v_n\} \in \varepsilon^{-1}(p)$ there exists a $v \in \varepsilon^{-1}(p)$ with $(v_1, v) \in X_{-1}$ for all $i$.*

*Then $|X_\bullet| \to X_{-1}$ is a weak homotopy equivalence.*

The proof of Proposition 9.2.2 consists of showing that the augmented semi-simplicial space $p : D_\Sigma(W; P)_\bullet \to C_\Sigma(W; P)$ satisfies all the hypothesis above.

*Proof of Proposition 9.2.2.* It is simple to verify that $D_\Sigma(W; P)_\bullet \to C_\Sigma(W; P)$ is an augmented topological flag complex. We need to verify that the hypotheses of Theorem 9.2.3 are satisfied:

(i) To see that $p : D_\Sigma(W;P) \to C_\Sigma(W;P)$ has local sections, fix $(e,\xi) \in D_\Sigma(W;P)$ and let $U \subset C_\Sigma(W;P)$ be the subset of configurations which are contained in the image of the embedding $e$. Then $s : U \to D_\Sigma(W;P)$ defined by $\chi \mapsto (e, e^{-1}(\chi))$ is a local section.

(ii) $p : D_\Sigma(W;P) \to C_\Sigma(W;P)$ is surjective, since any configuration $\xi$ admits a tubular neighbourhood $e$. Then $(e, e^{-1}(\xi))$ is an element of $D_\Sigma(W;P)$ in the pre-image of $\xi$.

(iii) For a configuration $\xi \in C_\Sigma(W;P)$ and a non-empty finite subset $\{(e_1,\xi_1),\dots,(e_k,\xi_k)\}$ in its pre-image, we can always find $e$ an embedding of $\mathbb{R}^d$'s containing $e_i(\xi_i)$ and contained in all $e_i$. For instance by taking $e$ to small open discs around the points in the configuration.

Then by [29, Theorem 6.2], the map $|D_\Sigma(W;P)_\bullet| \to C_\Sigma(W;P)$ is a weak homotopy equivalence. □

The space $D_\Sigma(W;P)$ is equipped with an action of $\mathrm{Diff}(W)$: for $\psi \in \mathrm{Diff}(W)$, an embedding $e : \coprod_k \mathbb{R}^d \hookrightarrow W$, and $\xi = (x_1,\dots,x_n;p_1,\dots,p_n)$ a configuration of points in $\coprod_k \mathbb{R}^d$ with labels in $P$, we define

$$\phi \cdot [e,\xi] = [e \circ \phi, x_1,\dots,x_n; D_{e(x_1)}\phi \cdot p_1,\dots,D_{e(x_n)}\phi \cdot p_n]$$

It is simple to verify this action is well-defined and preserves the partial order in $D_\Sigma(W;P)$. Moreover it follows directly from the definition that the augmentation map $p : D_\Sigma(W;P) \to C_\Sigma(W;P)$ is $\mathrm{Diff}(W)$-equivariant.

Since the partial order in $D_\Sigma(W;P)$ is compatible with the $\mathrm{Diff}(W)$-action, it induces a fibrewise partial order on $D_\Sigma(W;P) \times_{\mathrm{Diff}(W)} \mathrm{Emb}(W,\mathbb{R}^\infty)$ over $B\mathrm{Diff}(W)$. Then the semi-simplicial nerve of the poset $D_\Sigma(W;P) \times_{\mathrm{Diff}(W)} \mathrm{Emb}(W,\mathbb{R}^\infty)$ is simply is the fibrewise semi-simplicial nerve $D_\Sigma(W;P)_\bullet \times_{\mathrm{Diff}(W)} \mathrm{Emb}(W,\mathbb{R}^\infty)$.

**Corollary 9.2.4.** *The map $D_\Sigma(W;P) \to C_\Sigma(W;P)$ induces a weak homotopy equivalence*

$$|D_\Sigma(W;P)_\bullet \times_{\mathrm{Diff}(W)} \mathrm{Emb}(W,\mathbb{R}^\infty)| \longrightarrow C_\Sigma(W;P)/\!\!/\mathrm{Diff}(W).$$

The above follows directly from the definition of the definition of the partial order on $D_\Sigma(W;P) \times_{\mathrm{Diff}(W)} \mathrm{Emb}(W,\mathbb{R}^\infty)$ and Proposition 9.2.2.

Corollary 9.2.4 allows us to use a disc model for the space of configurations with partially summable labels when proving the decoupling in this setting. We finish this

section by introducing another space of discs and configurations which will be used in the decoupling.

**Definition 9.2.5.** Let $M$ be a smooth compact manifold of dimension $n > d$. The *$d$-discs model for configurations* in $M$ with summable labels in $P$, denoted $D^d_\Sigma(M;P)$, is the quotient

$$\left(\coprod_{k\geq 0} \mathrm{Emb}(\coprod_k \mathbb{R}^d, M) \times_{(\Sigma_k \wr \mathrm{GL}^+_d)} C_\Sigma(\coprod_k \mathbb{R}^d; P)\right) \Big/ \sim$$

where $(e : \coprod_m \mathbb{R}^d \to M, \xi) \sim (e' : \coprod_n \mathbb{R}^d \to M, \xi')$ if there are injections $\underline{k} \hookrightarrow \underline{m}$ and $\underline{k} \hookrightarrow \underline{n}$ such that the induced inclusions $i_1 : \coprod_k \mathbb{R}^d \to \coprod_m \mathbb{R}^d$ and $i_2 : \coprod_k \mathbb{R}^d \to \coprod_n \mathbb{R}^d$ satisfy

$$\xi \subset Im\, i_1,\ \xi' \subset Im\, i_2 \text{ and } e \circ i_1 = e' \circ i_2.$$

We equip this space with a partial ordering by declaring $(e,\xi) < (e',\xi')$ if, for some representative of the classes, $e(\xi) = e'(\xi')$ and $Im\, e \supset Im\, e'$. We denote the semi-simpicial nerve of this poset by $D^d_\Sigma(M;P)_\bullet$.

## 9.3 The decoupling theorem

In this section we prove the main theorem of this chapter, which is a decoupling result for $C_\Sigma(F_\infty;P)/\!\!/\mathrm{Diff}(F_\infty)$.

For every $g \geq 0$, we have a splitting map

$$\begin{aligned} D_\Sigma(F_{g,1};P) \times \mathrm{Emb}(F_{g,1},\mathbb{R}^\infty) &\longrightarrow \mathrm{Emb}(F_{g,1},\mathbb{R}^\infty) \times D^2_\Sigma(\mathbb{R}^\infty;P) \\ ((e : \coprod_k \mathbb{R}^2 \hookrightarrow F_\infty, \xi), f : F_\infty \hookrightarrow \mathbb{R}^\infty) &\longmapsto (f, (f \circ e, \xi)). \end{aligned}$$

This map is $\mathrm{Diff}(F_{g,1})$-equivariant with respect to the diagonal action on the left and the action on $\mathrm{Emb}(F_{g,1},\mathbb{R}^\infty)$ on the right, and it preserves the poset structures. This map is compatible with the stabilisation $F_{g,1} \to F_{g+1,1}$, so it induces a map $D$ between the limit spaces fitting into the following diagram

$$\begin{array}{ccc} & |D_\Sigma(F_\infty;P)_\bullet \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty,\mathbb{R}^\infty)| & \\ \overset{\simeq}{\swarrow} & & \overset{D}{\searrow} \\ C_\Sigma(F_\infty;P) \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty,\mathbb{R}^\infty) & & |B\mathrm{Diff}(F_\infty) \times D^2_\Sigma(\mathbb{R}^\infty;P)_\bullet| \end{array}$$

**Theorem 9.3.1.** *The map*

$$D : |D_\Sigma(F_\infty; P)_\bullet \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty, \mathbb{R}^\infty)| \to |B\mathrm{Diff}(F_\infty) \times D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet|$$

*is a homology isomorphism.*

The proof will consist on showing that $D$ is a level-wise homology isomorphism of semi-simplicial spaces and then show that this implies a homology isomorphism of geometric realisations, as done in [30, Section 4]. We quickly recall below the spectral sequence defined in [79, Proposition 5.1] associated to a semi-simplicial space, which is the key for the homology argument used in the proof of Theorem 9.3.1 (see [27, Section 1.4] for a detailed discussion).

For any semi-simplicial space $X_\bullet$, the geometric realisation $|X_\bullet|$ admits a filtration by its skeleta, with $|X_\bullet|^{(0)} = X_0$ and

$$|X_\bullet|^{(q)} = |X_\bullet|^{(q-1)} \cup_{X_q \times \partial\Delta^q} X_q \times \Delta^q.$$

This filtration yields a spectral sequence

$$E^1_{p,q} = H_{p+q}(|X_\bullet|^{(q)}, |X_\bullet|^{(q-1)}) \Longrightarrow H_{p+q}(|X_\bullet|)$$

and by excision and the Kunneth isomorphism, the left-hand term can be re-written to give a spectral sequence with

$$E^1_{p,q} \cong H_p(X_q) \Longrightarrow H_{p+q}(|X_\bullet|).$$

Therefore a map of semi-simplicial spaces inducing a level-wise homology isomorphism gives an isomorphism of the first pages of the respective spectral sequences, and therefore a homology isomorphism between the geometric realisations.

The proof of Theorem 9.3.1 will use the decoupling result proved in Chapter 8 for $C(F_\infty; L)$, the space of non-colliding configurations with labels in $L$, where particles are allowed to vanish when labelled by the basepoint (Proposition 8.4.1). We will further use that the space $C(F_\infty; L)$ is $\mathrm{Diff}(F_\infty)$-homotopy equivalent to the space of *tubular configurations* in $F_\infty$ labelled by $L$ [57], denoted $E(F_\infty; L)$. The space $E(F_\infty; L)$ is defined as follows: for a configuration $\xi$ of points in $F_\infty$, let $Tub(\xi)$ denote the space of tubular neighbourhoods of $\xi$. Then we define

$$E_k(F_\infty; L) = \coprod_{\xi \in C_k(F_\infty)} Tub(\xi) \times_{\Sigma_k} L^k$$

and

$$E(F_\infty; L) = \left( \coprod_{k \geq 0} E_k(F_\infty; L) \right) \Big/ \approx \tag{9.3.1}$$

where $(f_1, \ldots, f_k; s_1, \ldots, s_k) \approx (f_1, \ldots, f_{k-1}; s_1, \ldots, s_{k-1})$ if $s_k$ is the basepoint of $L$. In [57, Proposition 2.7, 3.6], it was shown that the map $Tub(\xi) \mapsto \xi$ induces a $\mathrm{Diff}(F_\infty)$-equivariant homotopy equivalence

$$E(F_\infty; L) \xrightarrow{\simeq} C(F_\infty; L)$$

which will be essential in the proof below.

Throughout the proof, it will be helpful to keep in mind the following picture

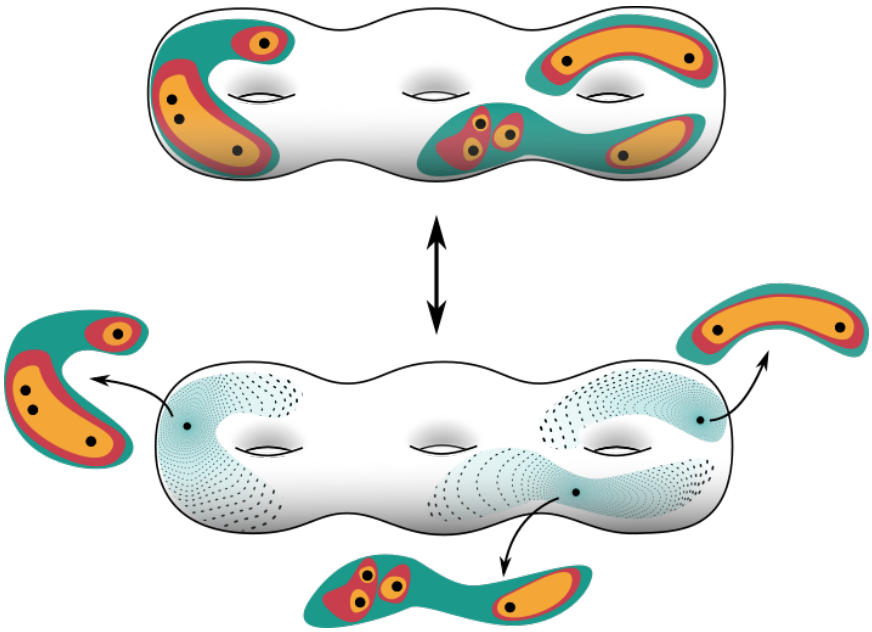


Figure 9.3: Correspondence of (9.3.2) between an element of $D_\Sigma(F_3; P)_2$ (top) and $E(F_3, \mathcal{D}_2)$ (bottom). The dotted regions represent the tubular neighbourhoods and the arrows indicate their labels.

*Proof of Theorem 9.3.1.* We start by showing that

$$D_\Sigma(F_\infty; P)_\bullet \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty, \mathbb{R}^\infty) \to B\mathrm{Diff}(F_\infty) \times D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet$$

is a level-wise homology isomorphism.

As before, we can fix for every $g \geq 0$ compatible trivialisation of the tangent bundle $TF_{g,1}$. This gives a fixed trivialisation of the tangent bundle of $F_\infty$. Then

every class $(e : \mathbb{R}^2 \hookrightarrow F_\infty, \xi) \in D_\Sigma(F_\infty; P)$ admits a unique representative $(e', \xi')$ such that

$$T_{e'(0)}F_\infty \xrightarrow{\cong} \mathbb{R}^2 \cong T_0\mathbb{R}^2 \xrightarrow{D_0 e'} T_{e(0)}F_\infty$$

is the identity map. Similarly, for every class $(e : \coprod_k \mathbb{R}^2 \hookrightarrow F_\infty, \xi) \in D_\Sigma(F_\infty; P)$, there is only one representative $(e', \xi')$, up to permutation of the path components of $\coprod_k \mathbb{R}^2$, such that

$$\coprod_{i=1}^{k} T_{e'(0_i)}F_\infty \xrightarrow{\cong} \coprod_k \mathbb{R}^2 \cong \coprod_k T_0\mathbb{R}^2 \xrightarrow{D_0 e'} \coprod_{i=1}^{k} T_{e(0_i)}F_\infty$$

is the identity map. We call such a pair $(e', \xi')$ a tubular representative of the class.

Let $\mathcal{D}_p$ be the subspace of $D_\Sigma(\mathbb{R}^2; P)_p$ consisting of those tuples $\overline{e} = ((e_0, \xi_0), \ldots, (e_p, \xi_p))$ with $e_0 = \mathrm{id}$. This space is pointed by the unique class on the empty configuration. We show there is a homeomorphism

$$D_\Sigma(F_\infty; P)_p \cong E(F_\infty; \mathcal{D}_p) \tag{9.3.2}$$

where $E(F_\infty; \mathcal{D}_p)$ is the space of tubular configurations with labels in $\mathcal{D}_p$ (see (9.3.1)), where a tubular neighbourhood is allowed to disappear if it is labelled by the basepoint of $\mathcal{D}_p$, the empty configuration. To describe the homeomorphism, we denote by $i_j : \mathbb{R}^2 \hookrightarrow \coprod_k \mathbb{R}^2$ the map induced by the inclusion $\{j\} \hookrightarrow \{1, \ldots, k\}$, for all $1 \leq j \leq k$. The map $D_\Sigma(F_\infty; P)_p \to E(F_\infty; \mathcal{D}_p)$ takes a sequence $((e_0, \xi_0), \ldots, (e_p, \xi_p))$, where $(e_0 : \coprod_k \mathbb{R}^2 \hookrightarrow F_\infty, \xi_0)$ is a tubular representative, to the class represented by the tubular neighbourhood

$$\coprod_{i=1}^{k} T_{e_0(0_i)}F_\infty \xrightarrow{\cong} \coprod_k \mathbb{R}^2 \xrightarrow{e_0} F_\infty$$

and the label associated to the $j$th component $T_{e_0(0_j)}F_\infty \subset \coprod_{i=1}^{k} T_{e_0(0_i)}F_\infty$ given by

$$((id, \xi_0), ((e_0 \circ i_j)^{-1} \circ e_1, \xi_1) \ldots, ((e_0 \circ i_j)^{-1} \circ e_p, \xi_p)) \in \mathcal{D}_p$$

For intuition behind this homeomorphism, see Figure 9.3. It is simple to explicitly construct an inverse for this map and check this is a $\mathrm{Diff}(F_\infty)$-equivariant homeomorphism.

By [57, Proposition 2.7, 3.6], the map taking a tubular neighbourhood to the image of its zero section defines a $\mathrm{Diff}(F_\infty)$-equivariant map to the space $C(F_\infty; \mathcal{D}_p)$ which is a weak-equivalence of $\mathrm{Diff}(F_\infty)$-spaces. Together with the homeomorphism

(9.3.2) this implies that

$$D_\Sigma(F_\infty;P)_p \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty,\mathbb{R}^\infty) \xrightarrow{\simeq} C(F_\infty;\mathcal{D}_p) \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty,\mathbb{R}^\infty) \quad (9.3.3)$$

Similarly, we now show that

$$D^2_\Sigma(\mathbb{R}^\infty;P)_p \xrightarrow{\simeq} C(\mathbb{R}^\infty;(E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} \mathcal{D}_p). \quad (9.3.4)$$

Using the same idea as above, we can see there is a homeomorphism

$$D^2_\Sigma(\mathbb{R}^\infty;P)_p \cong \left( \coprod_{k\geq 0} \mathrm{Emb}(\coprod_k \mathbb{R}^2, \mathbb{R}^\infty) \times_{(\Sigma_k \wr \mathrm{GL}_2^+)} (\mathcal{D}_p)^k \right) \Big/ \sim$$

where the relation $\sim$ allows embeddings labelled by the empty configuration to disappear (this is very similar to the construction of the space of tubular configurations, with the difference that the embedded submanifolds are not of the same dimension as the ambient manifold). Sending an embedding to the image of the origin and the standard frame gives a homotopy equivalence

$$\mathrm{Emb}(\coprod_k \mathbb{R}^2, \mathbb{R}^\infty) \simeq \widetilde{C}_k(\mathbb{R}^\infty) \times (E\mathrm{GL}_2^+)^k$$

which implies the homotopy equivalence (9.3.4).

Then we have a commutative diagram

$$\begin{array}{ccc} D_\Sigma(F_\infty;P)_p \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty,\mathbb{R}^\infty) & \xrightarrow{\simeq} & C(F_\infty;\mathcal{D}_p) \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty,\mathbb{R}^\infty) \\ \downarrow{\scriptstyle D_p} & & \downarrow{\scriptstyle H_*\cong} \\ B\mathrm{Diff}(F_\infty) \times D^2_\Sigma(\mathbb{R}^\infty;P)_p & \xrightarrow{\simeq} & B\mathrm{Diff}(F_\infty) \times C(\mathbb{R}^\infty;(E\mathrm{GL}_2^+)_+ \wedge_{\mathrm{GL}_2^+} \mathcal{D}_p) \end{array}$$

where the top and bottom maps are weak equivalences by (9.3.3) and (9.3.4), respectively. The right-hand map induces homology isomorphisms by Proposition 8.4.1 with $Z = \mathcal{D}_p$. Therefore $D_p$ also induces isomorphisms in homology.

This implies that the map between the spectral sequences associated to the semi-simplicial spaces $X_\bullet = D_\Sigma(F_\infty;P)_\bullet \times_{\mathrm{Diff}(F_\infty)} \mathrm{Emb}(F_\infty,\mathbb{R}^\infty)$, and $Y_\bullet = B\mathrm{Diff}(F_\infty) \times D^2_\Sigma(\mathbb{R}^\infty;P)_\bullet$

$$\begin{array}{ccc} E^1_{p,q} = H_q(X_p) & \Longrightarrow & H_{p+q}(|X_\bullet|) \\ \downarrow & & \downarrow{\scriptstyle D} \\ E'^1_{p,q} = H_q(Y_p) & \Longrightarrow & H_{p+q}(|Y_\bullet|). \end{array}$$

induces an isomorphism of $E^1$-pages, and therefore the right-hand map is also an isomorphism. □

## 9.4 Monoids of configurations on surfaces with partially summable labels

In this section, we show that Theorem 9.3.1 implies a splitting for the group completion of the monoid of configuration on surfaces with partially summable labels, analogous to Theorem 8.0.1.

As described in Section 9.1, for $(P, 0)$ a framed partial 2-monoid with unit, the configuration space $C_\Sigma(F_{g,1}; P)$ carries a natural action of the diffeomorphism group $\mathrm{Diff}(F_{g,1})$. Analogous to the construction of Chapter 8, we denote the associated Borel construction by

$$MC_\Sigma(P)_g = C_\Sigma(F_{g,1}; P) /\!\!/ \mathrm{Diff}(F_{g,1}).$$

As in the previous chapter, fixing embeddings $F_{g,1} \hookrightarrow \mathbb{R}^{\infty-1} \times [0, g] \subset \mathbb{R}^\infty = \operatorname{colim} \mathbb{R}^n$ that commute with the stabilisation map $\mathrm{Diff}(F_{g,b}) \to \mathrm{Diff}(F_{g,b+1})$ we can define a product

$$\mu : MC_\Sigma(P)_g \times MC_\Sigma(P)_h \to MC_\Sigma(P)_{g+h}$$

and makes the disjoint union $MC_\Sigma(P) = \coprod_{g\geq 0} MC_\Sigma(P)_g$ into a topological monoid, which we refer to as the *monoid of configurations with summable labels* in $P$.

**Corollary 9.4.1.** *For any path-connected framed partial* 2*-monoid with unit* $P$*, there is a homotopy equivalence*

$$\Omega B(MC_\Sigma(P)) \simeq \mathbb{Z} \times B\mathrm{Diff}(F_\infty)^+ \times |D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet|.$$

The proof of the above result consists of constructing a zig-zag of monoids

$$\begin{array}{ccc} \coprod_{g\geq 0} |D_\Sigma(F_{g,1}; P)_\bullet /\!\!/ \mathrm{Diff}(F_{g,1})| & & \\ \swarrow^{\simeq} & & \searrow^{D} \\ MC_\Sigma(P) & & \coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1}) \times |D^2_\Sigma(\mathbb{R}^\infty; P)_\bullet|. \end{array} \tag{9.4.1}$$

The left-hand arrow is induced by Corollary 9.2.4 while the right-hand arrow is in-

duced by the decoupling map. We describe the monoidal structures using a general construction for posets.

Given a space $Q$ with a partial order $<$, the space $Q \times Q$ is equipped with a partial order where

$$(q_1, q_2) < (q_1', q_2')$$

if $q_i < q_i'$ and $q_j \leq q_j'$, for $\{i, j\} = \{1, 2\}$. With say that a pointed such $Q$ is a *partially ordered topological monoid* if it is equipped with a multiplication

$$\mu : Q \times Q \to Q$$

which is strictly associative, unital and order preserving. The geometric realisation of the semi-simplicial nerve $Q_\bullet$ is naturally endowed with a multiplication $|\mu|$ defined by

$$\begin{aligned} |\mu|\big((q_0 < \cdots < q_m; t_0, \ldots, t_m), (\overline{q_0} < \cdots < \overline{q_n}; \bar{t}_0, \ldots, \bar{t}_n)\big) = \\ = (q_0 \cdot \overline{q} < \cdots < q_m \cdot \overline{q}; t_0 \cdot \bar{t}, \ldots, t_m \cdot \bar{t}) \end{aligned}$$

where $q_i \cdot \overline{q} = \mu(q_i, \overline{q_0}) < \cdots < \mu(q_i, \overline{q_n})$ and $t_i \cdot \bar{t} = t_i\overline{t_0}, \ldots, t_i\overline{t_n}$, for all $i = 0, \ldots, m$. It is straightforward to verify that this is well-defined.

**Lemma 9.4.2.** *For $(Q, <, \mu)$ a partially ordered topological monoid, $(|Q_\bullet|, |\mu|)$ is a topological monoid. Moreover, any map of partially ordered topological monoids $f : Q \to Q'$ induces a map of topological monoids*

$$f_* : |Q_\bullet| \to |Q'_\bullet|$$

The proof is a straightforward computation and follows directly from the definition of the semi-simplicial nerve of a poset, $|\mu|$ and the properties of a partially ordered topological monoid. We omit the proof for simplicity.

*Proof of Corollary 9.4.1.* The space $\coprod_{g \geq 0} |D_\Sigma(F_{g,1}; P)_\bullet /\!/ \mathrm{Diff}(F_{g,1})|$ is homotopy equivalent to the geometric realisation of the semi-simplicial nerve of the poset

$$\coprod_{g \geq 0} D_\Sigma(F_{g,1}; P) /\!/ \mathrm{Diff}(F_{g,1})$$

with partial order defined component-wise. The fixed the embeddings $F_{g,1} \hookrightarrow \mathbb{R}^{\infty-1} \times$

$[0,g] \subset \mathbb{R}^\infty = \operatorname{co\,lim} \mathbb{R}^n$ defined before allow us to define a product

$$D_\Sigma(F_{g,1};P)/\!/\mathrm{Diff}(F_{g,1}) \times D_\Sigma(F_{h,1};P)/\!/\mathrm{Diff}(F_{h,1}) \to D_\Sigma(F_{g+h,1};P)/\!/\mathrm{Diff}(F_{g+h,1})$$

which makes $\coprod_{g\geq 0} D_\Sigma(F_{g,1};P)/\!/\mathrm{Diff}(F_{g,1})$ into a partially ordered topological monoid. Moreover, one can verify that the augmentations $D_\Sigma(F_{g,1};P)_\bullet \to C_\Sigma(F_{g,1};P)$ induce a map of monoids. Together with Lemma 9.4.2, this gives a map of monoids

$$\coprod_{g\geq 0} |D_\Sigma(F_{g,1};P)_\bullet/\!/\mathrm{Diff}(F_{g,1})| \to MC_\Sigma(P). \tag{9.4.2}$$

On the other hand, the poset $D^2_\Sigma(\mathbb{R}^\infty;P)$ can be modified to be a partially ordered topological monoid, using the same strategy Segal used to define a topological monoid equivalent to $C(\mathbb{R}^\infty,X)$, as we recalled in section 8.3. One can then verify that the decoupling map gives a map of topological monoids

$$\coprod_{g\geq 0} |D_\Sigma(F_{g,1};P)_\bullet/\!/\mathrm{Diff}(F_{g,1})| \to \coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1}) \times |D^2_\Sigma(\mathbb{R}^\infty;P)_\bullet| \tag{9.4.3}$$

just as in Lemma 8.3.1.

All that remains is to verify that the maps (9.4.2) and (9.4.3) induce homotopy equivalences on group completions. As the group completions are loop spaces, they are simple and, by Whitehead's theorem, it suffices to show the maps induce homology equivalences on the group completions. All monoids are homotopy commutative, hence the group completion theorem [62] can be applied. It implies that it is enough to show that the induced maps on limit spaces

$$|D_\Sigma(F_\infty;P)_\bullet/\!/\mathrm{Diff}(F_\infty)| \to C_\Sigma(F_\infty;P)/\!/\mathrm{Diff}(F_\infty) \tag{9.4.4}$$

$$|D_\Sigma(F_\infty;P)_\bullet/\!/\mathrm{Diff}(F_\infty)| \to B\mathrm{Diff}(F_\infty) \times |D^2_\Sigma(\mathbb{R}^\infty;P)_\bullet| \tag{9.4.5}$$

are homology equivalences. The first map is a weak equivalence by Corollary 9.2.4, while the second map is a homology equivalence by Theorem 9.3.1. □

# Appendix A

# Infinity operads and normalized cacti

This chapter consists effectively of the paper [3], which is to be published in the *Journal of Topology and its Applications*. This project started as part of the Women in Topology Workshop and it is joint work with Safia Chettih, Abigail Linton, Sophie Raynor, Marcy Robertson and Nathalie Wahl.

## A.1 Introduction

Gluing surfaces along their boundary allows us to define composition laws that have been used to define cobordism categories, as well as operads and props associated to surfaces. These have played an important role in recent years, for example in constructing topological field theory or computing the homology of the moduli space of Riemann surfaces. Of particular interest is the cobordism category whose morphism spaces are moduli spaces of Riemann surfaces. It has long been known that such moduli spaces admit a graph model: they have the homotopy type of spaces of metric fat graphs [14, 39, 72]. The composition of moduli spaces induced by the gluing of surfaces was modeled using graphs in [28, Construction 3.29]. Though the resulting composition is associative on the associated cellular chain complex, it is not associative on the space level, and, at present, it is not known how to make it associative, or even coherently homotopy associative [28, Remark 3.31]. In genus 0, this graph model of the cobordism category includes normalized cacti (eg. [92, Remark 2.8]), whose composition was also known not to be associative [48, Remark 2.3.19]. The goal of our paper is to show that the composition of normalized cacti is associative up to all higher homotopies, and hence normailzed cacti form an $\infty$-operad in the

way detailed below. We expect that the technique presented here can be extended to likewise show that the composition in the graph model of the cobordism category is also associative up to all higher homotopies.

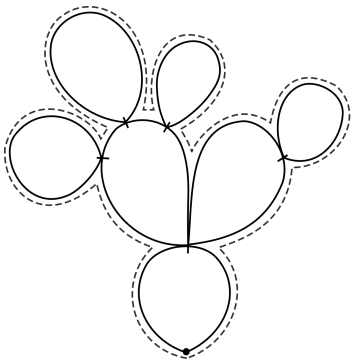


Figure A.1: Spineless cactus with 7 lobes, with its outside the dotted line.

A cactus is a treelike configuration of circles (Figure A.1). The cactus operad, originally introduced by Voronov [89, Section 2.7], and its spineless version, introduced by Kaufmann [48, Section 2.3], are models for the framed and unframed little disc operads respectively [48, Section 3.2.1]. Operadic composition is by insertion: identifying the outside contour of one cactus with the lobe of another cactus and scaling the inserted cactus appropriately. Here we work with the spineless version for simplicity.

A cactus is *normalized* if each circle in the cactus has circumference of length one. The space of all normalized cacti with $k$-lobes is denoted by $\mathsf{cact}^1(k)$ and these spaces assemble into the symmetric sequence $\mathsf{cact}^1 = \{\mathsf{cact}^1(k)\}_{k\geq 0}$, with each $\mathsf{cact}^1(k) \subset \mathsf{cact}(k)$ a homotopy equivalent subspace, for $\mathsf{cact}(k)$ the space of all cacti with $k$ lobes. (See [48, Section 2.3].) Composition of normalized cacti is defined by insertion as for the cactus operad, but instead of scaling the inserted cactus to the size of the lobe it is inserted in, one scales the lobe to the size of the inserted cactus. Surprisingly, as illustrated in Figure A.16, this new composition is not associative ([48, Remark 2.3.19]). So, normalized cacti do *not* form an operad. This non-associative composition is, however, the one relevant to the graph model of the cobordism category, as we explain in Remark A.5.1.

Operads can be described as algebras over the operad of operads $\mathcal{O}$, an operad whose elements can be represented by certain trees (Definition A.2.8). In Section A.3, we define a bracketing of a tree and use it to construct a new topological operad $B\mathcal{O}$ (Definition A.3.9) whose algebras are homotopy associative versions of operads: Any

$B\mathcal{O}$–algebra has an underlying symmetric sequence and a preferred composition, but the composition is only associative up to coherent homotopy. The operad $B\mathcal{O}$ is the realization of an operad whose operations lie in the poset of bracketings of the trees in $\mathcal{O}$.

Given a composition on a symmetric sequence, an action of the operad $B\mathcal{O}$ gives a hands-on way to keep track of the homotopies required to show that it is coherently homotopy associative. We illustrate how to construct a $B\mathcal{O}$–algebra in practice by showing:

**Theorem A.1.1** (Theorem A.5.12)**.** *The symmetric sequence* $\{\mathsf{cact}^1(k)\}_{k\geq 0}$ *of normalized cacti, together with the* $\mathsf{cact}^1$ *composition described above, extends to a* $B\mathcal{O}$*–algebra structure.*

In this paper, we show that this hands-on notion of an operad up to homotopy is related to more well-known notions of $\infty$-operads. There exists several models of $\infty$-operads in the literature: lax operads, obtained by resolving the operad of operads [15], dendroidal sets or spaces satisfying an inner Kan or Segal condition [18, Proposition 6.3, Theorem 8.15] and [18, Definition 8.1], Lurie's $\infty$-operads [41, Section 2.5], [17, Corollary 1.2] and Barwick's complete Segal operads [17, Theorem 1.1]. These models are all Quillen equivalent, and also Quillen equivalent to topological operads themselves [19, Theorem 8.15]. We compare our construction with two of these notions of $\infty$-operads. In Section A.4, we show that any $B\mathcal{O}$–algebra defines a dendroidal space satisfying a weak Segal condition, and in Section A.6, we show that any $B\mathcal{O}$-algebra is also a lax operad in the sense of [15].

To associate a dendroidal space to a $B\mathcal{O}$-algebra, we first construct a topological enrichment $\widetilde{\Omega}_0$ of the dendroidal category $\Omega$. The objects of $\widetilde{\Omega}_0$ are trees, as for $\Omega$, but morphisms are the realisation of certain posets of bracketings in trees, defined in a similar fashion to the operad $B\mathcal{O}$. Diagrams over this thickened dendroidal category $\widetilde{\Omega}_0$ are types of homotopy coherent dendroidal spaces. By defining a nerve functor that takes a $B\mathcal{O}$-algebra to the category of reduced $\widetilde{\Omega}_0$-diagrams that satisfy a strict Segal condition, we prove the following:

**Theorem A.1.2** (Theorem A.4.8)**.** *There is an isomorphism of categories between* $B\mathcal{O}$*-algebras and the category of reduced* $\widetilde{\Omega}_0$*-diagrams that satisfy a strict Segal condition.*

In Proposition A.4.10, we show that a $\widetilde{\Omega}_0$–diagram can be rectified to an equivalent $\Omega$–diagram, i.e. an actual dendroidal space. Moreover, we show that this rectified

dendroidal space satisfies a weak version of the Segal condition if, and only if, the original $\widetilde{\Omega}_0$–diagram satisfies such a condition. We describe this rectification explicitly in the case of normalized cacti in Corollary A.5.13.

The other notion of $\infty$-operads we consider are a more classical notion of an operad up to homotopy obtained by resolving the operad of operads $\mathcal{O}$ via the Boardman-Vogt $W$-construction. A $W\mathcal{O}$–algebra is an operad up to homotopy, where the symmetric group action, the unit and associativity relation are all assumed to hold only up to coherent homotopy. For this reason, $W\mathcal{O}$-algebras are called lax operads in the Ph.D. thesis [15]. Algebras over the operad $W\mathcal{O}$ are a model for $\infty$-operads: there exists a zig-zag of Quillen equivalences between the category of $W\mathcal{O}$-algebras and reduced dendroidal Segal spaces. (For example this can be seen by combining Theorem 4.1 of [5] with either Theorem 1.1 of [7] or Theorem 8.15 of [19].) However, the operad $W\mathcal{O}$ is not easy to work with directly. Indeed, its elements are trees (from the $W$–construction) whose vertices are themselves decorated by trees (from the operad $\mathcal{O}$), where the first trees compose by grafting and the second trees compose by vertex substitution. In Section A.6, we show that the operad $B\mathcal{O}$ is actually isomorphic to a quotient $W_0\mathcal{O}$ of $W\mathcal{O}$:

**Theorem A.1.3.** *[Theorem A.6.4] There exists an isomorphism of topological operads $W_0\mathcal{O} \cong B\mathcal{O}$, for $W_0\mathcal{O}$ an explicit quotient of the operad $W\mathcal{O}$. In particular, any $B\mathcal{O}$–algebra is also a $W\mathcal{O}$-algebra.*

A $W_0\mathcal{O}$–algebra (or equivalently $B\mathcal{O}$–algebra) is a homotopy operad where the composition is still only homotopy associative, but where the symmetric group action and unit are strict. We show in Corollary A.6.5 that every $B\mathcal{O}$-algebra is homotopy equivalent to a strict operad.

The relationship between the operad $B\mathcal{O} \cong W_0\mathcal{O}$ and the dendroidal category $\Omega$ is detailed in Theorem A.1.2, in which we show a "bracketed version" of the equivalence between $O$–algebras and appropriate $\Omega$–diagrams. As the operad $W\mathcal{O}$ is a more complete resolution of $\mathcal{O}$, it is natural to ask if an even larger resolution of the dendroidal category $\Omega$ extends this relationship. For a category $\mathcal{K}$, there exists a resolution similar to the $W$–construction, namely the "explosion" $\widetilde{\mathcal{K}}$ of the category, as studied by Segal [81, Appendix B] and Leitch [53]. This "explosion" has the property that $\widetilde{\mathcal{K}}$–diagrams are coherently homotopy $\mathcal{K}$–diagrams. Applying this construction to the category $\Omega$, one could expect that $W\mathcal{O}$–algebras are related to $\widetilde{\Omega}$–diagrams in the same way that $B\mathcal{O} = W_0\mathcal{O}$–algebras are related to $\widetilde{\Omega}_0$–diagrams. We show in Theorem A.7.6 that this does not quite hold, proving instead that there is an

embedding of the category of $W\mathcal{O}$-algebras as a full subcategory of the category of $\widetilde{\Omega}$–diagrams satisfying a strict Segal condition.

By combining Theorem A.1.1 and Theorem A.1.2 or Theorem A.1.3, normalized cacti are a rare example of an $\infty$-operad that does not arise via the application of a nerve construction to a known (discrete or topological) operad. Indeed, to our knowledge, only a few such examples exist in the literature; see the weak operad of configuration spaces [35, Corollary 5], the configuration categories of [**?**] or examples that arise as a result of completion as in [8, Proposition 5.1].

The input of the construction of our explicit infinity operad structure on normalized cacti is a pre-given composition that we show to be associative up to coherent homotopy by using the operad $B\mathcal{O} = W\mathcal{O}/\sim$. The homotopies are constructed using the contractible space of basepoint preserving monotone reparametrizations of the circle (see the proof of Theorem A.5.12). To extend the results to the cobordism category of graphs described above, one would need to replace $\mathcal{O}$ by the operad $P\mathcal{O}$, whose algebras are all symmetric properads [93, Section 14.1.2], define a resolution "$BP\mathcal{O}$", as the appropriate quotient of the $W$-construction applied to $P\mathcal{O}$. Our expectation is that these same reparametrisations of the circle will likewise provide all the necessary homotopies to provide an infinity composition in the graph model of the cobordism category.

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## A.2 Preliminaries on Operads

A *symmetric sequence* in a symmetric monoidal category $\mathcal{S}$ is a collection $\mathcal{P} = \{\mathcal{P}(k)\}_{k\geq 0}$ of objects in $\mathcal{S}$ in which each $\mathcal{P}(k)$ comes equipped with an action of the symmetric group $\Sigma_k$. In this paper, our symmetric monoidal category $\mathcal{S}$ will either be the discrete category of sets, the category of simplicial sets, or the category of topological spaces with their standard Cartesian products.

An *operad* in $\mathcal{S}$ is a symmetric sequence $\mathcal{P} = \{\mathcal{P}(k)\}_{k\geq 0}$ together with a distinguished element $\iota \in \mathcal{P}(1)$, called the *unit*, and a collection of composition maps

$$\circ_i : \mathcal{P}(k) \times \mathcal{P}(j) \longrightarrow \mathcal{P}(k+j-1),$$

$1 \leq i \leq k$, which are associative, unital, and equivariant. For more complete details see, for example, [58, Definition 11]. Given an operad $\mathcal{P}$, a symmetric sequence $\mathcal{Q} = \{\mathcal{Q}(k) \subseteq \mathcal{P}(k)\}_{k\geq 0}$ is a *suboperad* of $\mathcal{P}$ if the restriction of the composition maps in $\mathcal{P}$ induce an operad structure on $\mathcal{Q}$. A *morphism of operads* $f : \mathcal{P} \to \mathcal{Q}$ is a family of equivariant maps

$$\{f(k) : \mathcal{P}(k) \to \mathcal{Q}(k)\}_{k\geq 0}$$

that are compatible with composition and units.

*Remark* A.2.1. It is equivalent to work with individual compositions

$$\circ_i \colon \mathcal{P}(k) \times \mathcal{P}(j_i) \to \mathcal{P}(k + j_i - 1)$$

or with all $\circ_i$-compositions simultaneously. In the latter case, the simultaneous compositions are denoted by a map

$$\gamma_{\mathcal{P}} \colon \mathcal{P}(k) \times \mathcal{P}(j_1) \times \ldots \times \mathcal{P}(j_k) \to \mathcal{P}(\Sigma_{i=1}^{k} j_i).$$

(eg:[58, Proposition 13]).

More generally, we will use *colored* operads. For any non-empty set $\mathfrak{C}$, a $\mathfrak{C}$-*colored symmetric sequence* is a family of objects $\mathcal{P} := \{\mathcal{P}(c; c_1, \ldots, c_k)\}_{k\geq 0}$ in $\mathcal{S}$, where $(c; c_1, \ldots, c_k)$ ranges over every list of colors in $\mathfrak{C}$ together with a map $\sigma^\star : \mathcal{P}(c; c_1, \ldots, c_k) \to \mathcal{P}(c; c_{\sigma(1)}, \ldots, c_{\sigma(k)})$ for each $\sigma \in \Sigma_k$. A $\mathfrak{C}$-*colored operad* is a $\mathfrak{C}$-colored symmetric sequence $\mathcal{P}$ together with a family of partial composition maps

$$\circ_i \colon \mathcal{P}(c; c_1, \ldots, c_k) \times \mathcal{P}(d; d_1, \ldots, d_j) \to \mathcal{P}(c; c_1, \ldots, c_{i-1}, d_1, \ldots, d_j, c_{i+1} \ldots, c_k)$$

defined only when $c_i = d$, together with an element $\iota_c \in \mathcal{P}(c;c)$ for each $c \in \mathfrak{C}$, which satisfies unit, equivariance and associativity conditions. For more details see, for example, [5, Definition 1.1]. When the color set is $\mathfrak{C} = \{*\}$, a $\mathfrak{C}$–colored operad is a one-colored operad. In this paper we will refer to both operads and colored operads as "operads", only mentioning the color set when necessary.

An *algebra* over a ($\mathfrak{C}$–colored) operad $\mathcal{P}$ is a collection of objects $\{X(c)\}_{c\in\mathfrak{C}}$ in $\mathcal{S}$ together with evaluation maps

$$\alpha\colon \mathcal{P}(c; c_1, \dots, c_k) \times X(c_1) \times \cdots \times X(c_k) \longrightarrow X(c)$$

satisfying appropriate associativity, unit and equivariance conditions, see e.g. [5, Definition 1.2]. The category of $\mathcal{P}$-algebras in $\mathcal{S}$ is denoted $\mathcal{P}-\mathrm{Alg}_{\mathcal{S}}$.

Our main example of a colored operad will be the $\mathbb{N}$-colored operad $\mathcal{O}$, whose algebras are the (non-colored) operads, see Definition A.2.8. In Section A.5, we will also make use of the following operad:

**Example A.2.2.** Let $X$ be a fixed space in $\mathcal{S}$. The *coendomorphism operad* of $X$, $\mathrm{CoEnd}(X)$, has an underlying symmetric sequence with arity $k$ spaces

$$\mathrm{CoEnd}(n)(X) := \mathrm{Map}(X, X^{\times k}).$$

The symmetric groups act by permuting the factors of $f = (f_1, \dots, f_k) \in \mathrm{CoEnd}(k)$. If $f = (f_1, \dots, f_k) \in \mathrm{CoEnd}(k)(X)$ and $g = (g_1, \dots, g_j) \in \mathrm{CoEnd}(j)(X)$ the partial compositions

$$\circ_i : \mathrm{CoEnd}(k)(X) \times \mathrm{CoEnd}(j)(X) \longrightarrow \mathrm{CoEnd}(k+j-1)(X)$$

are given by

$$f \circ_i g = (f_1\,,\,\dots\,,\,f_{i-1}\,,\,g_1 \circ f_i\,,\,\dots\,,\,g_j \circ f_i\,,\,f_{i+1}\,,\,\dots\,,\,f_k).$$

### A.2.1 Trees

Throughout this paper, we use trees to model operad compositions and as the basis of our main constructions. A graph $G$ is a tuple $(V(G), H(G), s, i)$ where $V(G)$ is a set of vertices, $H(G)$ a set of half-edges, $s : H(G) \to V(G)$ is the source map and $i : H(G) \to H(G)$ is an involution. Orbits of the involution $i$ are called *edges* of $G$

and the set of edges is denoted by $E(G)$. An edge represented by a pair $\{h, i(h)\}$ with $i(h) \neq h$ is called an *internal edge*, and the set of internal edges is denoted $iE(G)$. Edges corresponding to orbits of fixed points of the involution are *external*.

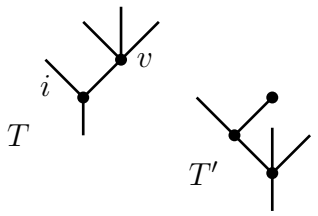
A tree is a simply connected graph. All our trees will be *rooted*, i.e. they come with a distinguished "outgoing" external edge called the *root*. All other external edges are "incoming" and called *leaves*. The set of leaves is denoted $L(T)$. The *arity* of $T$ is the number of leaves $|L(T)|$. The root of the tree is denoted $R(T)$.

Note that a rooted tree can be canonically made into a directed graph by setting all the edges to point towards the root. Then note that the set of edges incident to a vertex always has a unique outgoing edge, the one closest to the root, and all other edges are *incoming edges*. The number of incoming edges of a vertex $v$ is called the *arity* of the vertex and denoted by $|v|$, with $|v| \geq 0$ any natural number.

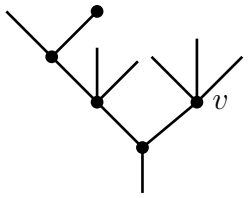
We allow the special tree $\eta = |$, with no vertices and a single edge. The trees with a single vertex and $n$ leaves are called *$n$-corollas* and denoted $C_n$. A rooted tree $S$ is a *subtree* of $T$ if $V(S) \subseteq V(T)$ preserving the arity, $H(S) \subseteq H(T)$, and the structure maps for $S$ are restrictions of the structure maps for $T$, defining $i(h) = h$ in $S$ if $i(h) = h'$ in $T$ with $h' \notin H(S)$. A *planar tree* is a rooted tree together with a preferred isotopy class of embedding into the plane. Note that this is equivalent to defining a planar tree as a tree with a canonical ordering of the incoming edges at each vertex.

We use planar trees to model operad compositions via an operation called grafting. Given trees $T$ and $T'$, of arity $n$ and $m$ respectively, and a leaf $i \in L(T)$, the grafting of $T'$ onto $T$ along the leaf $i$ is defined to be the tree $T \circ_i T'$ obtained by attaching the root of $T'$ to the leaf $i$ of $T$ so that they form a new internal edge in the grafted tree (Figure A.2). Grafting of trees is also used to model the free operad generated by a symmetric sequence, as we will explain now. To avoid confusion later, when we will have to decorate vertices of trees by other trees, we will use blackboard fonts for the trees in the free operad construction (and later the associated $W$-construction in Section A.6.1), as we will soon apply this construction to a symmetric sequence of trees, which will give (blackboard) trees of (plain) trees.

**Definition A.2.3.** Let $\mathcal{P} = \{\mathcal{P}(c; c_1, \ldots, c_k)\}_{c_i, c \in \mathfrak{C}}$ be a $\mathfrak{C}$-colored symmetric sequence in $\mathcal{S}$. A planar tree $\mathbb{T}$ is $\mathfrak{C}$-colored if it is equipped with a map $f : E(\mathbb{T}) \to \mathfrak{C}$, we refer to $f(e)$ as the color of the edge $e$. A $\mathfrak{C}$-colored planar tree $\mathbb{T}$ is *decorated* by $\mathcal{P}$ if each vertex $v \in V(\mathbb{T})$ is labeled by an operation in $p_v \in \mathcal{P}(out(v); in(v))$, where $out(v)$ is the color of the outgoing edge of $v$, and $in(v)$ is the list of colors of the incoming edges of $v$, ordered by the planar structure. The *free operad* $F(\mathcal{P})$ on $\mathcal{P}$ is




(a) Example trees $T$ and $T'$




(b) The grafting $T \circ_i T'$

Figure A.2: Grafting of trees.

the $\mathfrak{C}$-colored operad whose $k$-ary operations are the $\mathfrak{C}$-colored, $\mathcal{P}$ decorated, planar trees $\mathbb{T}$ of arity $k$ with leaves labeled by a bijection $\lambda : \{1, \dots, k\} \to L(\mathbb{T})$.

Explicitly, for each $c, c_1, \dots, c_k \in \mathfrak{C}$,

$$F(\mathcal{P})(c; c_1, \dots, c_k) := \Big( \coprod_{(\mathbb{T}, f, \lambda)} \prod_{v \in V(\mathbb{T})} \mathcal{P}(out(v); in(v)) \Big) / \sim,$$

where $(\mathbb{T}, f, \lambda)$ runs over all isomorphism classes of leaf-labeled $\mathfrak{C}$-colored planar trees with $k$ leaves such that $f(\lambda(i)) = c_i$, $f(R(\mathbb{T})) = c$, and where the equivalence relation is generated by the following:

(∗) two labeled trees $(\mathbb{T}, f, \lambda, (p_v)_{v \in V(\mathbb{T})})$ and $(\mathbb{T}', f', \lambda', (p'_w)_{w \in V(\mathbb{T}')})$ are equivalent if there exists a non-planar isomorphism $\alpha : \mathbb{T} \to \mathbb{T}'$ such that $f \circ \alpha = f'$, $\alpha \circ \lambda = \lambda'$, and $\sigma_v(\alpha) p_v = p_{\alpha(v)}$, for $\sigma_v(\alpha)$ the permutation on $in(v)$ induced by $\alpha$.

The symmetric group acts on $F(\mathcal{P})$ by permuting the labels of the leaves, acting on $\lambda$, and composition in $F(\mathcal{P})$ is given by grafting of trees, with $\circ_i$ grafting at the leaf $\lambda(i)$. For full details see, for example, the construction under Corollary 3.3 [5].

We now employ the free operad construction to define a class of free operads $\Omega(T)$ generated by a planar tree $T$. This will play a fundamental role in the definition of the dendroidal category (Section A.2.2), which describes a model for $\infty$-operads.

**Example A.2.4.** A planar tree $T$ generates a free colored operad $\Omega(T)$ as follows. The set of colors of $\Omega(T)$ is the set of edges $\mathfrak{C} = E(T)$. We define a discrete $E(T)$-coloured symmetric sequence $X(T)$ by

$$X(T)(e; e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \begin{cases} \{\sigma v\} & \text{if } (e; e_1, \dots, e_n) = (\mathrm{out}(v); \mathrm{in}(v)), \text{ for } v \in V(T), \\ \emptyset & \text{otherwise,} \end{cases}$$

with its built in free symmetric group action. Then $\Omega(T) := FX(T)$ is the free operad on the collection $X(T)$. Explicitly, $\Omega(T)$ is an $E(T)$-coloured operad with

$$\Omega(T)(e; e_{\sigma(1)}, \ldots, e_{\sigma(n)}) = \begin{cases} \sigma S & \text{if } (e; e_1, \ldots, e_n) = (R(S); L(S)),\ S \subset T, \\ \emptyset & \text{otherwise.} \end{cases}$$

for $S \subset T$ a subtree of $T$. Composition, as in the free operad, is given by grafting of subtrees. For further details, see Section 2.2 and just above Definition 2.3.1 in [64].

### A.2.2 The dendroidal category $\Omega$

The model we use for $\infty$-operads is that of dendroidal Segal spaces that satisfy the *weak Segal condition*. Dendroidal spaces are diagrams of the dendroidal category.

The dendroidal category $\Omega$ is the full subcategory of colored operads whose objects are the free operads $\Omega(T)$ generated by trees (as in Example A.2.4). In other words, objects of $\Omega$ are planar isomorphism classes of planar rooted trees and morphisms in $\Omega$ are defined to be operad maps

$$\mathrm{Hom}_{\Omega}(S, T) = \mathrm{Hom}_{\mathsf{Op}}(\Omega(S), \Omega(T)).$$

Morphisms in $\Omega$ can be described as a composition of four types of elementary morphisms: isomorphisms, degeneracies, inner and outer face maps. In terms of trees, isomorphisms are non-planar tree isomorphisms, *inner face maps* are of the form $\partial_e : T/e \to T$, where $T/e$ is the tree obtained from $T$ by contracting an inner edge $e \in iE(T)$. If $v$ is a vertex of $T$ with only one inner edge attached to it then $T/v$ is the tree obtained from $T$ by chopping off the vertex $v$ and the inclusion $\partial_v : T/v \to T$ is an *outer face map*. A *degeneracy* is a map $s_v : T/v \to T$ where $T/v$ is obtained from $T$ by deleting a vertex $v$, with $|v| = 1$, in $T$.

In the opposite category $\Omega^{op}$, outer face maps correspond to restriction to certain allowed subtrees, while inner face maps correspond to edge collapses. For more details and plenty of examples see [65, 64].

*Remark* A.2.5. Our definition of $\Omega$ differs slightly from the usual definition in that we have chosen our objects to be planar trees. Technically, what we have described here is the equivalent category $\Omega'$ from [64, 2.3.2].

**Definition A.2.6.** A *dendroidal space* $X$ is an $\Omega$-diagram $X : \Omega^{op} \to \mathcal{S}$, where $\mathcal{S}$ is either the category of simplicial sets or topological spaces.

The evaluation of $X$ at a tree $T$ is denoted $X(T)$. A dendroidal space is called *reduced* if $X(\eta) = *$, where $\eta = |$. We will write $\mathcal{S}^{\Omega^{op}}$ for the category of dendroidal spaces.

For any vertex $v$ in a tree $T \in \Omega$, we have an associated outer face map in $\Omega$

$$C_v \longrightarrow T$$

taking the unique vertex of the corolla to $v \in V(T)$, where $C_v$ is the corolla with $|v|$ leaves. Likewise, for any internal edge between vertices $u$ and $v$ in $T$, there is a commuting diagram in $\Omega$

$$\begin{array}{ccc} \eta & \longrightarrow & C_u \\ \downarrow & & \downarrow \\ C_v & \longrightarrow & T. \end{array}$$

Let $\mathrm{Sk}_1(T)$ be the category whose objects are the edges and vertices of $T$, thought of copies of $\eta$ and corollas $C_v$, and whose morphisms are associated to edge inclusions in $T$, as in the top left corner of the above diagram.

For a dendroidal space $X$, the *Segal map* is the unique map from $X(T)$ to the limit $\lim_{\mathrm{Sk}_1(T)^{op}} X$ induced by the corolla inclusions. When $X(\eta) = *$, this limit becomes a product over the value of $X$ at the corollas, and the Segal map becomes the map

$$\chi \colon X(T) \longrightarrow \prod_{v \in V(T)} X(C_v)$$

with components the restriction to the value of $X$ at each corolla.

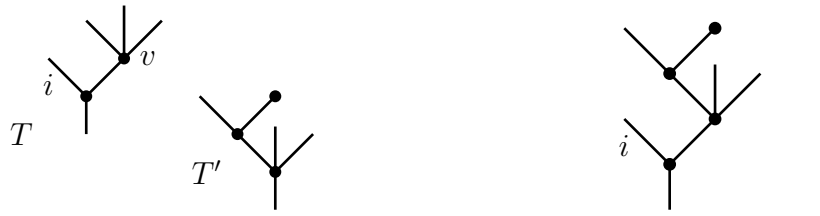
The category of $\Omega$-diagrams admits two Quillen model category structures: the Reedy model structure and the projective model structure which are Quillen equivalent (see e.g. [9, Remark 2.5]). Throughout, we take the *projective model structure* in which a morphism of $\Omega$-diagrams is a weak equivalence or fibration if it is entrywise a weak equivalence or fibration.

**Definition A.2.7.** A dendroidal space $X \in \mathcal{S}^{\Omega^{op}}$ satisfies a *strict Segal condition* if the Segal map is an isomorphism for each $\eta \neq T \in \Omega$.

### A.2.3 The operad of operads

One of the main constructions in this paper is the operad $B\mathcal{O}$. This operad builds on an $\mathbb{N}$-colored operad $\mathcal{O}$ called the *operad of operads*, whose algebras are one-colored operads.

Let $T$ be a planar tree. For a vertex $v \in V(T)$ with arity $|v| = m$ and a planar tree $T'$ with $m$ leaves, the *substitution* $T \bullet_v T'$ is obtained by removing the vertex $v$ from $T$ and identifying the incoming and outgoing edges of $v$ with the leaves and root of $T'$, respectively. An example is shown in Figure A.3.




(a) Example trees $T$ and $T'$

(b) The substitution $T \bullet_v T'$

Figure A.3: Tree substitution (Compare with grafting in Figure A.2).

A *labelled* planar tree is a triple $(T, \sigma, \tau)$, consisting of a planar tree $T$ equipped with bijections $\sigma : |V(T)| \to V(T)$ and $\tau : |L(T)| \to L(T)$. Two such triples $(T, \sigma, \tau)$ and $(T', \sigma', \tau')$ are isomorphic if there is a planar tree isomorphism $T \to T'$ that respects the labelling $\sigma, \tau$. We represent a labelled planar tree $(T, \sigma, \tau)$ by writing above each leaf $\ell \in L(T)$ the number $\tau^{-1}(\ell)$, and writing by each vertex $v \in V(T)$ the number $\sigma^{-1}(v)$, as depicted in Figure A.4.

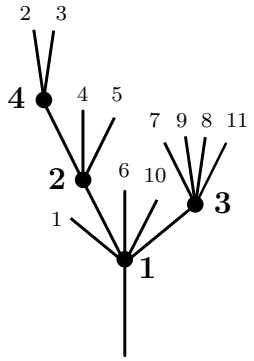



Figure A.4: Example of a labelled planar tree in $\mathcal{O}(11; 5, 3, 4, 2)$.

We also define a tree substitution that is *compatible with the labellings of the leaves*. Let $(T, \sigma, \tau)$ and $(T', \sigma', \tau')$ be two planar labelled trees with $|V(T)| = k$, $|V(T')| = l$ and $|L(T')| = |\sigma(i)| = m_i$. The map $\tau'$ encodes a permutation in the symmetric group with $m_i$ elements. We obtain a new planar tree $(\tau'_{\sigma(i)})T$ by applying the permutation $\tau'$ on the $m_i$ incoming edges of the vertex $\sigma(i) \in V(T)$. We then define

$$T \bullet_{\sigma(i),\tau'} T' = (\tau'_{\sigma i})T \bullet_{\sigma(i)} T'. \tag{A.2.1}$$

In particular, $V(T \bullet_{\sigma(i),\tau'} T') = \{V(T) - \sigma(i)\} \amalg V(T')$. The labelling on the vertices of $T \bullet_{\sigma(i),\tau'} T'$ is given by the map $\sigma \circ_i \sigma'$, which is the induced bijection $\{1, \dots, k+l-1\} \to V(T \bullet_{\sigma(i),\tau'} T')$

$$j \mapsto \begin{cases} \sigma(j) & 1 \leq j < i \\ \sigma'(j-i+1) & i \leq j \leq i+l \\ \sigma(j-l+1) & i+l < j \leq k+l-1. \end{cases}$$

An example is shown in Figure A.5. In the case where the order induced by $\tau'$ on the

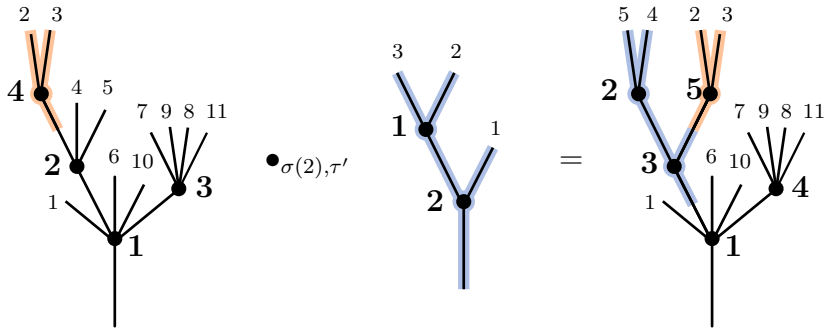



Figure A.5: Example of composition in $\mathcal{O}$ where $\tau'$ is different to the planar order of $in(\sigma(2))$.

$m_i$ incoming edges of $\sigma(i)$ is the same as the order induced by the planar structure, then $T \bullet_{\sigma(i),\tau'} T' = T \bullet_{\sigma(i)} T'$.

**Definition A.2.8.** The *operad of operads* $\mathcal{O}$ is the $\mathbb{N}$–colored operad, for which

$$\mathcal{O}(n; m_1, \dots, m_k)$$

is the discrete space whose elements are isomorphism classes of *labelled* planar rooted trees $(T, \sigma, \tau)$ where $T$ is a planar tree with $k$ vertices and $n$ leaves, with bijections $\sigma\colon |V(T)| \to V(T)$, $\tau\colon |L(T)| \to L(T)$, such that the vertex $\sigma(i)$ has arity $m_i$ for each $1 \leq i \leq k$. The composition operation

$$\mathcal{O}(n; m_1, \dots, m_k) \times \mathcal{O}(m_i; b_1, \dots, b_l) \xrightarrow{\circ_i} \mathcal{O}(n; m_1, \dots, b_1, \dots, b_l, \dots, m_k)$$
$$((T, \sigma, \tau), (T', \sigma', \tau')) \longmapsto (T, \sigma, \tau) \circ_i (T', \sigma', \tau')$$

is induced by tree substitution that is compatible with the labelling as in (A.2.1), where

$$(T, \sigma, \tau) \circ_i (T', \sigma', \tau') = (T \bullet_{\sigma(i),\tau'} T', \sigma \circ_i \sigma', \tau).$$

The unit for this composition, for the color $n$, is the element of $\mathcal{O}(n;n)$ represented by the corolla $C_n$ equipped with the canonical left-right labelling. The symmetric group $\Sigma_k$ acts on $(T,\sigma,\tau) \in \mathcal{O}(n;m_1,\ldots,m_k)$ by precomposition on the labelling $\sigma$ of the vertices $V(T)$.

We further observe that, for each $m,n \in \mathbb{N}$,

$$\mathcal{O}(m;n) \cong \begin{cases} \Sigma_n & \text{for } m=n, \\ \emptyset & \text{when } m \neq n. \end{cases}$$

The isomorphism $\mathcal{O}(n;n) \cong \Sigma_n$ corresponds to labelling the leaves of a corolla $C_n$ in all possible ways. The unique arity 0 operation in $\mathcal{O}$ is represented by the special tree $\eta \in \mathcal{O}(1;\emptyset)$. An $\mathcal{O}$-algebra, $\mathcal{P}$, is precisely a one-colored operad. That is to say, $\mathcal{P}$ has an underlying $\mathbb{N}$-graded object $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ in $\mathcal{S}$. Moreover, $\mathcal{P}$ admits actions $\mathcal{O}(n;n) \times \mathcal{P}(n) \to \mathcal{P}(n)$ for all $n$ and thus $\mathcal{P}$ has an underlying symmetric sequence. By definition, we have

$$\mathcal{O}(n;m_1,\ldots,m_k) \times \mathcal{P}(m_1) \times \ldots \times \mathcal{P}(m_k) \subset F\mathcal{P}(n),$$

where $F\mathcal{P}$ is the free operad on the symmetric sequence $\mathcal{P}$, and

$$F\mathcal{P}(n) = \coprod_{k\in\mathbb{N}} \left( \coprod_{(m_1,\ldots,m_k)\in\mathbb{N}^k} \mathcal{O}(n;m_1,\ldots,m_k) \times \mathcal{P}(m_1) \times \ldots \times \mathcal{P}(m_k) \right)_{\Sigma_k}$$

so the action maps

$$\alpha : \mathcal{O}(n;m_1,\ldots,m_k) \times \mathcal{P}(m_1) \times \ldots \times \mathcal{P}(m_k) \to \mathcal{P}(n)$$

induce maps $F\mathcal{P}(n) \to \mathcal{P}(n)$ for all $n$, and by the algebra axioms, this is precisely the data of a symmetric operad in $\mathcal{S}$ (See [5, Example 1.5.6]). Note that, in particular, the $\circ_i$-compositions of an operad $\mathcal{P}$ are governed by the trees with one internal edge in $\mathcal{O}(n;m_1,m_2)$, where $n = m_1 + m_2 - 1$.

### A.2.4 The relationship between operads and dendroidal spaces

Dendroidal spaces with $X(\eta) = *$ that satisfy a *strict* Segal condition are closely related to one-colored operads. Explicitly, every operad $\mathcal{P}$ can be viewed as a den-

droidal space via the dendroidal *nerve* construction that defines a functor

$$N^d(\mathcal{P})(T) = \mathrm{Hom}_{\mathsf{Op}}(\Omega(T), \mathcal{P})$$

as $T$ ranges over $\Omega$. The nerve of the free operad $\Omega(T)$ is just the representable dendroidal space $\Omega[T] := \mathrm{Hom}_{\Omega}(-, T)$. A dendroidal space $X$ is the *nerve* of an operad if, and only if, the Segal map of Definition A.2.7 is an isomorphism for all $T$ [19, Lemma 6.4; Proposition 6.5]. To put this altogether, there is an isomorphism of categories

$$\mathcal{O}\text{−Alg}_{\mathcal{S}} \cong (\mathcal{S}^{\Omega^{op}})_{strict}$$

where $\mathcal{O}$ is the colored operad whose algebras are one-colored operads (Definition A.2.8, below) and $(\mathcal{S}^{\Omega^{op}})_{strict}$ denotes the category of reduced dendroidal spaces satisfying the strict Segal condition. We will prove similar statements for "thickened" versions of $\Omega$ in Theorem A.4.8 and Theorem A.7.6.

## A.3 The operad of brackets $\mathcal{BO}$

In this section we introduce a new topological operad called the *operad of bracketed trees*. In short, the operad $B\mathcal{O}$ captures a weak notion of an operad in the sense that a $B\mathcal{O}$-algebra is a symmetric sequence with $\circ_i$-operations that are only associative up to higher homotopy. The construction of the operad $B\mathcal{O}$ allows one to check with relative ease whether a symmetric sequence with compositions assembles into an $\infty$-operad. In Theorem A.5.12, we use this to show that normalized cacti admit such a structure. Moreover, we expect that this construction provides a general method that one can use to construct other examples of $\infty$-operads.

One could instead use the classical Boardman-Vogt $W$-construction on the operad $\mathcal{O}$ to obtain an operad $W\mathcal{O}$ whose algebras are homotopy operads (lax operads in the language of [15]). It is known to experts that bracketings in trees are related to this operad $W\mathcal{O}$, but the precise details are difficult to find in the literature. (However, see [66, Section 2.3], in particular Theorem 4, together with Remark A.3.8 below, for an algebraic version of this in the case of non-symmetric operads.) In Section A.6 we will show that $B\mathcal{O}$ identifies with a quotient of the operad $W\mathcal{O}$. Bracketings in trees have also appeared elsewhere, see eg. [24, 25], and the parenthesizations of [83, 2.6].

### A.3.1 Bracketings of trees

We define in this section the poset of bracketings of a tree, starting with the definition of a bracketing:

**Definition A.3.1.** A tree is called *large* if it has at least two vertices (or equivalently, at least one internal edge). A set $\{S_j\}_{j\in J}$ of subtrees of a tree $T$ is *nested* if, for any $i, j \in J$, the set of common vertices $V(S_i) \cap V(S_j)$ is either $V(S_i)$, $V(S_j)$ or empty. A *bracketing* $B$ of a tree $T$ is a (possibly empty) collection $B = \{S_j\}_{j\in J}$ of nested large proper subtrees of $T$.

Recall from Section A.2.1 that a subtree of $T$ is a tree $S$ whose vertices are a subset of the vertices of $T$, and whose half-edges are all the half-edges in $T$ attached to such vertices. Therefore, a subtree is completely determined by its vertices. With this in mind, we will represent bracketings as in Figure A.6.

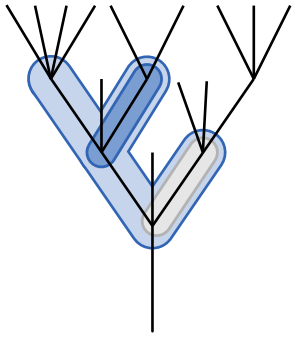


Figure A.6: Example of a tree bracketing with 3 nested subtrees.

**Definition A.3.2.** Bracketings of a tree $T$ form a poset of bracketings $\mathcal{B}(T)$ with the relation $B' \leq B$ if $B' \subseteq B$.

We denote the geometric realisation of the nerve of the poset $\mathcal{B}(T)$ by $|\mathcal{B}(T)|$. A point in

$$|\mathcal{B}(T)| = \coprod_{r\geq 0} N_r\mathcal{B}(T) \times \Delta^r / \sim$$

is a pair $(B, t)$ with $B = B_0 \subset \cdots \subset B_r$ a sequence of bracketings and $t \in \Delta^r$. Such a pair $(B, t)$ can be interpreted as a weighted bracketing with underlying set of brackets $B_r = \cup_{i=0}^r B_i$ and weights given by

$$t = (1, t_1, \ldots, t_r) \in \Delta^r = \{1 = t_0 \geq t_1 \geq \cdots \geq t_r \geq 0\}$$

where we assign the weight $t_0 = 1$ to all brackets in $B_0$, and for each $1 \leq i \leq r$, the weight $t_i$ to all brackets in $B_i \backslash B_{i-1}$. In particular, a weighted bracketing with all brackets having weight 1 corresponds to a vertex $B = B_0$ in the nerve of the poset. Also, the equivalence relation on the realization implies that a bracket of weight 0 can be discarded. (See also Section A.6 and in particular the proof of Lemma A.6.8 where this point of view is used to relate $B\mathcal{O}$ to the operad $W\mathcal{O}$.)

**Example A.3.3.** If $T = C_n$ then $T$ does not admit any large subtree, therefore $\mathcal{B}(T) = \{\emptyset\}$ only has the empty (or trivial) bracketing.

**Example A.3.4.** Let $T$ be the tree , then the space $|\mathcal{B}(T)|$ is depicted in Figure A.7 (left). Note that the initial object in the poset is the empty bracket, in the centre of the pentagon (the 4th associahedron).

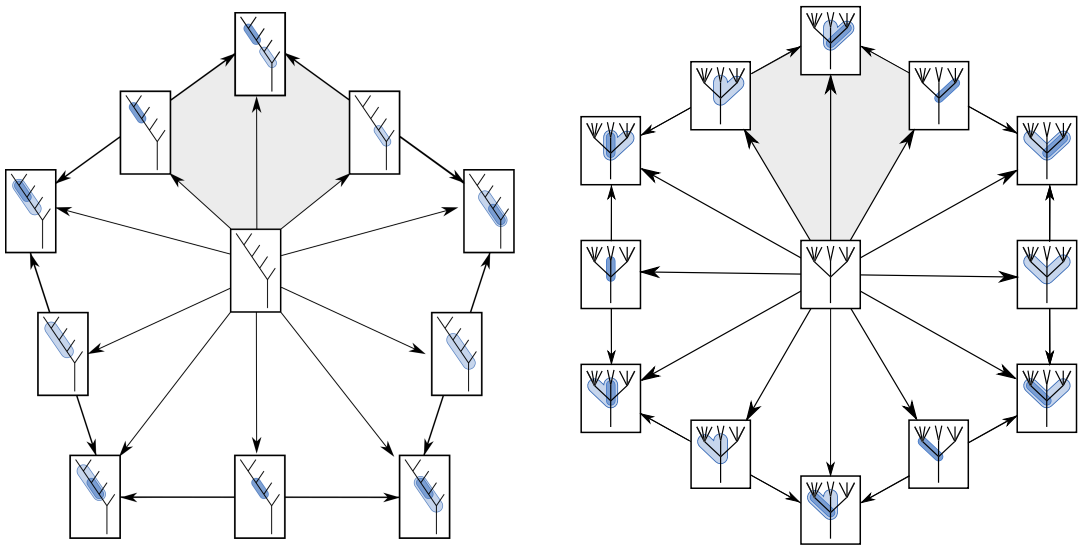


Figure A.7: Geometric realization of the poset $\mathcal{B}(T)$ of Examples A.3.4 and A.3.5.

More generally, let $T_n$ be a tree with $n$ vertices such that no vertex is connected to more than two inner edges. For such trees, the set of vertices can always be given a total ordering, for instance by constructing a list starting with a vertex $v$ connected to only one internal edge, and defining the next element of the list to be the vertex sharing an edge with $v$ that has not yet been listed. Then a bracket of $T_n$ can be immediately identified with a meaningful placement of parentheses on a word with $n$ letters where the word is represented by the ordered set of vertices. Therefore, $|\mathcal{B}(T_n)|$ can always be identified with the $n$-th associahedron (see also Remark A.3.8 for another approach to this statement).

**Example A.3.5.** Consider a tree $T$ with three inner edges all meeting at a single vertex. Note that the poset of bracketings depends only on the relative positions of the vertices (or analogously, the inner edges) of the tree $T$, and is independent of the number of leaves at each vertex. Therefore, the realization poset of bracketings of $T$ is the hexagon (two-dimensional permutahedron) depicted in the Figure A.7 (right), using as an example the tree $T =$ .

**Example A.3.6.** Figure A.8 depicts the realisation of the poset of bracketings of a tree $T$ with four inner edges meeting at a single vertex. Note that by fixing a large subtree $S$ of $T$, the realisation of the subposet of bracketings of $T$ containing $S$ will correspond to a subspace of the boundary of $|\mathcal{B}(\mathcal{T})|$. Each boundary face of top dimension is then associated to a subtree $S$ of $T$, and two such faces $S_1, S_2$ share a subface if $\{S_1, S_2\}$ is nested. In this case, $|\mathcal{B}(T)|$ is the three-dimensional permutahedron.

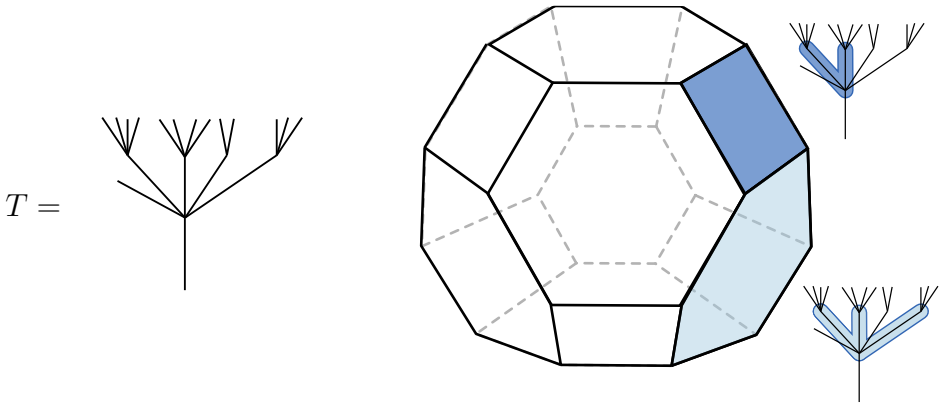



Figure A.8: Tree satisfying the conditions of Example A.3.6 together with the geometric realisation of its poset of bracketings.

**Lemma A.3.7.** *For any tree $T$, the space $|\mathcal{B}(T)|$, is contractible.*

*Proof.* The contractibility of the space $|\mathcal{B}(T)|$ follows directly from the fact that the poset $\mathcal{B}(T)$ has a minimal element, namely the empty bracketing. □

*Remark* A.3.8. The spaces $|\mathcal{B}(T)|$ are related to the family of nestohedra that were first described by Feichtner and Sturmfels [**?**] and Postnikov [**?**]. Examples of nestohedra include simplices, permutahedra, Stasheff's associahedra and more generally Carr and Devadoss' graph associahedra [**?**]. Hypergraph polytopes are an interpretation of nestohedra by Došen and Petrić [**?**], with the advantage that they have a convenient tree notation (called *constructs*) to label the faces of the polytope and that encode

the face inclusions. The space $|\mathcal{B}(T)|$ can be identified with a hypergraph polytope as follows. Using the definition and notation by Obradović [66, Section 2.2.1], the edge-graph $\mathbf{H}_T$ of $T$ is the graph whose vertices correspond to the inner edges of $T$, and two vertices of $\mathbf{H}_T$ share an edge if the corresponding inner edges of $T$ have a common vertex. A subgraph $\mathbf{S}$ of $\mathbf{H}_T$ uniquely defines a subforest $\langle \mathbf{S} \rangle$ of $T$ whose internal edges correspond precisely with the vertices of $\mathbf{S}$, and each tree in this forest is necessarily large because it has an inner edge (see [66, Section 2.2.1, Lemma 3]). Then we have an order reversing bijection $b$ between the hypergraph polytope of the edge-graph $\mathbf{H}_T$ and $\mathcal{B}(T)$ that can be recursively defined as follows: we take the construct $V(\mathbf{H}_T)$ given by the vertex set of $\mathbf{H}_T$ to the empty bracketing, and if $\mathbf{H}_T \setminus Y$ for $Y \subset V(\mathbf{H}_T)$ decomposes into $n$ connected components $\mathbf{H}_{T_1}, \ldots, \mathbf{H}_{T_n}$, then for constructs $C_i$ of $\mathbf{H}_{T_i}$ we take the construct $Y\{C_1, \ldots, C_n\}$ to the bracketing $\{\langle \mathbf{H}_{T_1} \rangle, \ldots, \langle \mathbf{H}_{T_n} \rangle, b(C_1), \ldots, b(C_n)\}$. The definition of constructs guarantees that these sets are nested and therefore define a bracketing, and it is simple to check that this is an order reversing bijection.

### A.3.2 An operad of bracketings

We'll use the bracketings $\mathcal{B}(T)$ to construct a topological operad. Let the collection

$$B\mathcal{O}(n; m_1, \ldots, m_k) = \coprod_{(T,\sigma,\tau) \in \mathcal{O}(n; m_1, \ldots, m_k)} |\mathcal{B}(T)|$$

define the $\mathbb{N}$-coloured symmetric sequence $B\mathcal{O}$. So, elements of $B\mathcal{O}(n; m_1, \ldots, m_k)$ are tuples $(T, \sigma, \tau, B, t)$ where $(T, \sigma, \tau)$ is an element of $\mathcal{O}(n; m_1, \ldots, m_k)$ (Definition A.2.8) and $(B, t)$ is a weighted bracketing of $T$ (ie. a point in $|\mathcal{B}(T)|$).

To define operadic composition in $B\mathcal{O}$, we use the composition of trees in $\mathcal{O}$ and induce a bracketing of the resulting tree. Let $(T, \sigma, \tau, B)$ and $(T', \sigma', \tau', B')$ be labeled trees with bracketings. The composition in $\mathcal{O}$ (Definition A.2.8) is given by the substitution of $T'$ into the vertex $\sigma(i) \in V(T)$,

$$(T, \sigma, \tau) \circ_i (T', \sigma', \tau') = (T \bullet_{\sigma(i), \tau'} T', \sigma \circ_i \sigma', \tau).$$

Since $T'$ is canonically a subtree of $T \bullet_{\sigma(i),\tau'} T'$, the bracketing $B'$ on $T'$ defines a nested collection of subtrees of $T \bullet_{\sigma(i),\tau'} T'$. We also construct a nested collection of subtrees $\tilde{B} = \{\tilde{S}_j\}_{j \in J}$ on $T \bullet_{\sigma(i),\tau'} T'$ that is induced by the bracketing $B = \{S_j\}_{j \in J}$

on $T$. If $T' \neq \eta$, then $\tilde{B} \cong B$ is given by

$$\tilde{S}_j = \begin{cases} S_j & \text{if } \sigma(i) \notin V(S_j), \\ S_j \bullet_{\sigma(i),\tau'} T' & \text{if } \sigma(i) \in V(S_j). \end{cases} \tag{A.3.1}$$

If $T' = \eta$, then $\tilde{B} = \{\tilde{S}_j\}_{j \in J}$ is defined in the same way, unless $\sigma(i) \in V(S_j)$ and $S_j$ has two vertices, in which case $S_j \bullet_{\sigma(i),\tau'} \eta$ is a corolla and is discarded as it is not large. That is, we replace $J$ with another indexing set $J' \subset J$, which is the subset of indices $j$ such that $S_j$ is large.

We define a bracketing of the tree $T \bullet_{\sigma(i),\tau'} T'$ by

$$B'' = \begin{cases} \tilde{B} \cup B' \cup \{T'\} & \text{if } T' \text{ is large} \\ \tilde{B} & \text{else.} \end{cases} \tag{A.3.2}$$

See Figure A.9. This defines a composition of bracketings of trees. This composition is associative as follows. Suppose $S_j \subset T$ is a bracket with only two vertices $v$ and $w$, and $T'$ is a tree with at least two vertices. If we first compose $\eta$ in $v$ and then $T'$ in $w$, the bracket $S_j$ is discarded during the first composition, and then replaced by a new bracket $T'$. Reversing the order of these two compositions yields the same result because first composing $T'$ in $w$ will create a new bracket $T'$, and $S_j$ will not be discarded, but composing further $\eta$ in $v$ will equate $S_j$ and $T'$. Otherwise, the associativity of the composition follows from the associativity on the composition in $\mathcal{O}$.

The composition also respects inclusions and thus is a poset map

$$\mathcal{B}(T) \times \mathcal{B}(T') \longrightarrow \mathcal{B}(T \bullet_{\sigma(i),\tau'} T'). \tag{A.3.3}$$

The realization of the poset map (A.3.3) induces a map between the geometric realisations of the nerve of the posets.

Also recall that the unary elements of $\mathcal{O}$, i.e. the elements of $\mathcal{O}(n;n)$ for some $n$, are given by labeled corollas. Since there are no non-trivial bracketings of corollas, unary elements of $B\mathcal{O}$ have the form $(C_n, \sigma, *, \emptyset, \emptyset) \in B\mathcal{O}(n;n)$ with $\sigma \in \Sigma_n$. In particular, the $n$-coloured identity for the composition $\circ$ in $B\mathcal{O}$ is given by $(C_n, id_n, *, \emptyset, \emptyset) \in B\mathcal{O}(n;n)$. Therefore $B\mathcal{O}$ is an operad.

**Definition A.3.9.** The *operad of bracketed trees* $B\mathcal{O}$ is the $\mathbb{N}$-coloured topological

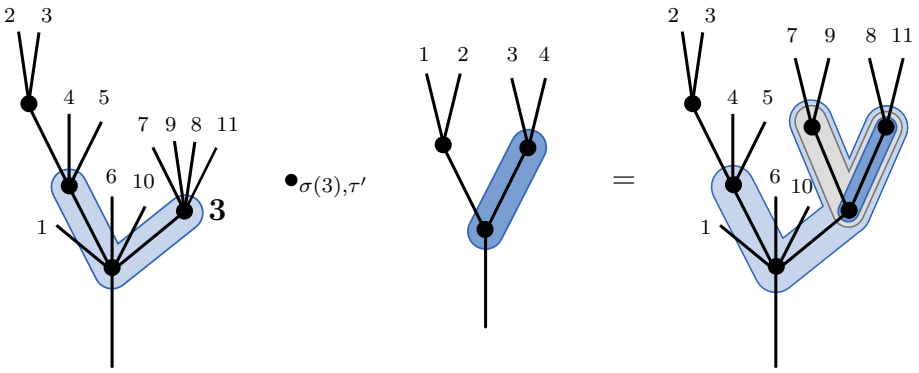



Figure A.9: Example of composition in $B\mathcal{O}$ with labelling of the vertices omitted for simplicity.

operad with underlying symmetric sequence

$$B\mathcal{O}(n; m_1, \dots, m_k) = \coprod_{(T,\sigma,\tau)\in\mathcal{O}(n;m_1,\dots,m_k)} |\mathcal{B}(T)|$$

and composition given by combining the composition in $\mathcal{O}$ with the map (A.3.3) described above.

*Remark* A.3.10.

1. Note that there is an isomorphism of operads $\mathcal{O} \cong \pi_0 B\mathcal{O}$ and, by Lemma A.3.7, the projection $B\mathcal{O} \to \mathcal{O}$ is a homotopy equivalence of operads.

2. The topological operad $B\mathcal{O}$ is the realization of an operad in posets. Indeed, the space $B\mathcal{O}(n; m_1, \dots, m_k)$ is the realization of the poset of elements $(T, \sigma, \tau)$ of $\mathcal{O}(n; m_1, \dots, m_k)$ together with a bracketing of $T$, where two elements are comparable only if they have the same underlying element of $\mathcal{O}$. Likewise, the operad structure is defined as the realization of a map on the level of posets.

### A.3.3 $B\mathcal{O}$-algebras

A *$B\mathcal{O}$-algebra* is an operad whose $\circ_i$-compositions are associative up to all higher homotopies. In particular, a $B\mathcal{O}$-algebra $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ has an underlying *symmetric sequence*. To see this, we note that the labelling of the leaves of a corolla $(C_n, \tau, *, \emptyset, \emptyset) \in B\mathcal{O}(n; n)$ identifies with elements of the symmetric group and we have isomorphisms

$$B\mathcal{O}(n; n) \cong \mathcal{O}(n; n) \cong \Sigma_n.$$

The action

$$B\mathcal{O}(n; n) \times \mathcal{P}(n) \longrightarrow \mathcal{P}(n)$$

makes $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ into a symmetric sequence.

$B\mathcal{O}$-algebras also have a notion of operadic $\circ_i$ composition. To see this, recall that such compositions are encoded in the operad $\mathcal{O}$ by the trees with exactly two vertices, one attached to the $i$th incoming edge of the other. As such trees admit no large, proper subtrees, they admit no non-trivial bracketing and we have isomorphisms for any $n, m \geq 0$

$$B\mathcal{O}(m+n-1; m, n)|_{V(T)\leq 2} \cong \mathcal{O}(m+n-1; m, n)|_{V(T)\leq 2}$$

between the components of the tuples $(T, \sigma, \tau, \emptyset, 0)$ (resp. $(T, \sigma, \tau)$) with $T$ having at most two vertices. It follows then that $\mathcal{P}$ is equipped with operadic $\circ_i$-compositions.

A $B\mathcal{O}$-algebra is not in general an operad, however. The brackets that arise in trees with more than two vertices capture the different choices one has in iterated compositions of $\circ_i$ operations. More explicitly, if $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ is a $B\mathcal{O}$–algebra, then for any collection of elements $x_i \in \mathcal{P}(m_i)$ that decorate the vertices of a tree $(T, \sigma, \tau) \in \mathcal{O}(n; m_1, \ldots, m_k)$, we have a chosen composition of those elements, namely the one determined by $(T, \sigma, \tau, \emptyset, \emptyset) \in B\mathcal{O}(n; m_1, \ldots, m_k)$. This "unbracketed" tree sits in the middle of a polytope of all possible elements $(T, \sigma, \tau, B, s)$ for any bracketing $B$, as in Figure A.7. The corners of this polytope correspond to the possible maximal bracketings of $T$ (the maximal elements of $\mathcal{B}(T)$). Just like the corners of the Stasheff polytopes give all the possible ways to bracket a $k$–fold multiplication, these maximal bracketings correspond precisely to the possible ways to bracket the composition of $\circ_i$ operations, which are those defined using trees with exactly two vertices. The polytopes arising from the posets of bracketing in trees can be thought of as an operadic analogue of the Stasheff polytopes.

*Remark* A.3.11. In [48, Definition 1.1.1], a *quasi-operad* is a symmetric sequence $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ together with operadic $\circ_i$-compositions and no further structure. In this way, a $B\mathcal{O}$-algebra is an extension of a quasi-operad. The operad $B\mathcal{O}$ is closely related to the $W$-construction of $\mathcal{O}$, whose algebras go under the name lax operads, see Section A.6, where we show that $B\mathcal{O}$-algebras can be described as strictly symmetric lax operads.

## A.4 Thickening the category $\Omega$

We have seen that operads are $\mathcal{O}$-algebras. Also recall from Section A.2.4 that operads can be described as strict Segal dendroidal spaces. The dendroidal category

$\Omega$ is defined as a full subcategory of the coloured operads generated by trees. To obtain a similar description of $B\mathcal{O}$–algebras as certain "homotopy dendroidal Segal spaces," we construct a topological category $\widetilde{\Omega}_0$ that is a category with the same objects as $\Omega$ but its spaces of morphisms are built using posets similar to the posets used to define $B\mathcal{O}$. Theorem A.4.8 establishes that this category, $\widetilde{\Omega}_0$, has the desired property that reduced Segal $\widetilde{\Omega}_0^{op}$–spaces are precisely $B\mathcal{O}$-algebras. In Section A.4.3, we then show how rectification of diagrams can be used to produce an actual Segal dendroidal space from such a homotopy version of a dendroidal space.

Given any category $\mathcal{K}$ with a discrete set of objects, Leitch [53] constructed a new category $\widetilde{\mathcal{K}}$ with the property that $\widetilde{\mathcal{K}}$–diagrams are homotopy coherent $\mathcal{K}$–diagrams. A similar enrichment (the *explosion category*) was also used by Segal [81, Appendix B] to relate his $\Gamma$–space approach to infinite loop spaces to the operadic approach of Boardman-Vogt and May. Because $\widetilde{\Omega}_0$–diagrams are homotopy coherent $\Omega_0$–diagrams, one can expect that the category $\widetilde{\Omega}_0$ is related to this construction of Leitch applied to $\Omega$. In Section A.7, we construct an equivalence between these two categories, and show that strict Segal $\widetilde{\Omega}$–spaces are closely related to $W\mathcal{O}$–algebras.

### A.4.1 Bracketing $\Omega$ and the category $\widetilde{\Omega}_0$

Recall from Section A.2.2 that the objects of $\Omega$ are planar isomorphism classes of planar rooted trees. Morphisms in $\Omega$ are compositions of inner and outer face maps, degeneracies and isomorphisms of trees. Inner face maps $\partial_e : T/e \to T$ create inner edges and correspond to operadic composition, while outer face maps are subtree inclusions and are associated to projection maps. A degeneracy creates a vertex that is adjacent to exactly two edges. The category $\widetilde{\Omega}_0$ is a version of $\Omega$ with the same set of objects, but with the realization of a poset of bracketings over each composition of inner face maps.

We define the morphism spaces of $\widetilde{\Omega}_0$ as follows. Let $g : S \to T$ be a morphism in $\Omega$. For each vertex $v \in V(S)$, let $C_v \subset S$ denote the corolla of the vertex $v$ that is, $C_v = i_v(C_{|in(v)|})$ where $i_v : C_{|in(v)|} \to S$ is the composition of outer faces in $\Omega$ sending the vertex of the corolla $C_{|in(v)|}$ to $v$. Since $g$ is alternatively considered as a map of operads between $\Omega(S)$ and $\Omega(T)$, the image in $S$ of $C_v$ under $g$ is a subtree in $T$, which we denote

$$g(C_v) \ \subset \ T.$$

Note that the trees $g(C_v)$ are precisely the subtrees of $T$ that correspond to expansion of vertices into subtrees, going from $S$ to $T$, or collapsed by $g^{op} : T \to S$ in the opposite

category $\Omega^{op}$. These subtrees correspond to the part of $g$ made out of inner face maps.

For a vertex $v \in V(S)$, let $B_v^g$ be a bracketing of $g(C_v)$ as defined in Definition A.3.1. We define a poset $\mathcal{L}_g$ whose objects are tuples $(B_v^g)_{v \in V(S)}$ of bracketings of the trees $g(C_v)$. The poset relation is componentwise inclusion. Taking the realization of these posets, for each morphism $g$ we associate the space

$$L_g := \prod_{v \in V(S)} |\mathcal{B}(g(C_v))|$$

where $\mathcal{B}(g(C_v))$ is the poset of bracketings of the tree $g(C_v)$ as defined in Definition A.3.2. Note also that $|\mathcal{B}(g(C_v))| = *$ if $g(C_v)$ admits only the trivial bracketing.

**Example A.4.1.** Consider the morphism $f \in \mathrm{Hom}_\Omega(R, S)$ of Figure A.10. Since the image of each corolla under $f$ only admits a trivial bracketing,

$$L_f = \left|\mathcal{B}\left(\begin{array}{c}\includegraphics[height=0.6em]{}\end{array}\right)\right| \times \left|\mathcal{B}\left(\begin{array}{c}\includegraphics[height=0.6em]{}\end{array}\right)\right| \times \left|\mathcal{B}\left(\begin{array}{c}\includegraphics[height=0.6em]{}\end{array}\right)\right| \times \left|\mathcal{B}\left(\begin{array}{c}\includegraphics[height=0.6em]{}\end{array}\right)\right| = *.$$

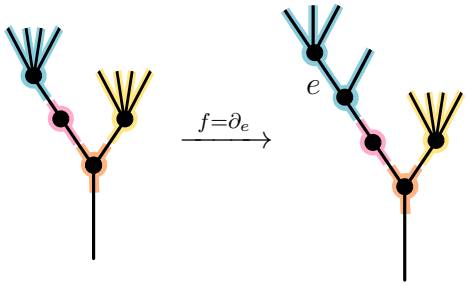
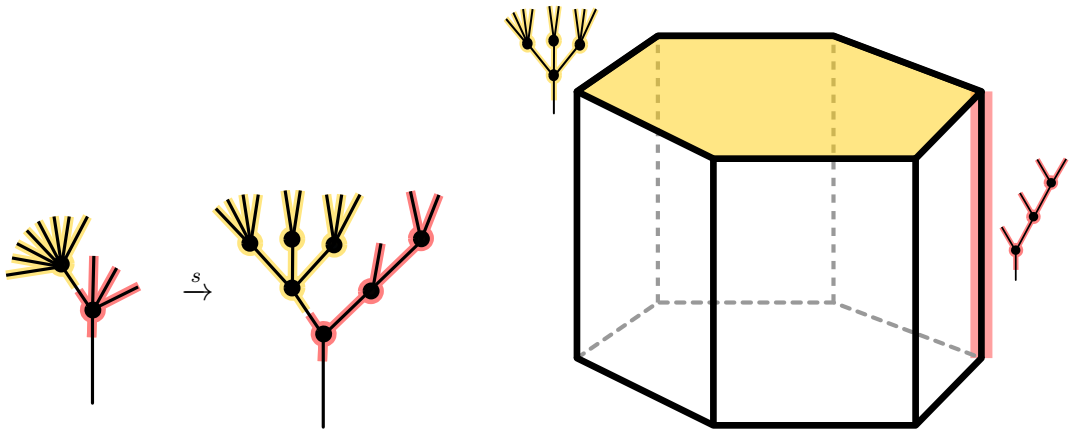



Figure A.10: Example of map $f$ in $\Omega$ and the subtrees $f(C_v)$.

**Example A.4.2.** Let $s$ be the morphism in Figure A.11(a). By Example A.3.4, if $s(C_v)$ has 3 vertices such that no vertex is connected to more than two inner edges, then $|\mathcal{B}(s(C_v))|$ is the 3rd associahedron, which is an interval. As in Example A.3.5, when $s(C_v)$ is a tree whose three internal edges meet at a single vertex, the realization poset $|\mathcal{B}(s(C_v))|$ corresponds to a hexagon. Thus $L_s$ is identified with the hexagonal prism of Figure A.11(b).

The space of morphisms between any two objects in $\widetilde{\Omega}_0$ is

$$\mathrm{Hom}_{\widetilde{\Omega}_0}(S, T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} L_g.$$




(a) Map $s$ in $\Omega$ and the subtrees $s(C_v)$. (b) Space $L_s$ for the map $s$ of Figure A.11(a)

Figure A.11: A map $s$ in $\Omega$ and its corresponding space $L_s$.

It remains to define composition in $\widetilde{\Omega}_0$. To do this, we first define a map of posets

$$\mathcal{L}_g \times \mathcal{L}_f \longrightarrow \mathcal{L}_{g\circ f} \tag{A.4.1}$$

for any two morphisms $f : R \to S$ and $g : S \to T$ in $\Omega$, then we take the realization of this composition map to get a composition of spaces $L_g$. Let $(B^g_v)_{v\in V(S)} \in \mathcal{L}_g$ and $(B^f_w)_{w\in V(R)} \in \mathcal{L}_f$ be two collections of bracketings. So for each $v \in V(S)$, $B^g_v$ is a bracketing of $g(C_v) \subset T$ and for each $w \in V(R)$, $B^f_w$ is a bracketing of the tree $f(C_w) \subset S$. To define the image of (A.4.1), we construct a bracketing of the tree $(g \circ f)(C_w)$ from the bracketings of $f$ and $g$.

Fix a vertex $w \in V(R)$. For each $v \in f(C_w) \subset S$, there is the subtree $g(C_v) \subset (g\circ f)(C_w)$, as well as a bracketing $B^g_v$ of $g(C_v)$. Also, for each bracket in $S_i \in B^f_w$, the image $g(S_i)$ is a subtree of $(g \circ f)(C_w)$. Therefore we have the following collections of subtrees in $(g \circ f)(C_w)$:

$$\begin{aligned}
\tilde{B}^g_{f(w)} &= \bigcup_{v\in f(C_w)} B^g_v \;=\; \{S_j :\; S_j \in B^g_v \text{ and } v \in f(C_w)\} \\
\tilde{B}^{g\circ f}_w &= \;\{g(C_v) :\; v \in f(C_w) \text{ and } g(C_v) \subsetneq (g \circ f)(C_w) \text{ is large}\} \\
\tilde{B}^f_w &= \;\{g(S_i) :\; S_i \in B^f_w \text{ and } g(S_y) \subsetneq (g \circ f)(C_w) \text{ is large}\}.
\end{aligned}$$

All of these are collections of proper large subtrees of $(g \circ f)(C_w)$. We set the bracketing $\tilde{B}_w$ of $(g \circ f)(C_w)$ to be the union

$$\tilde{B}_w := \tilde{B}^g_{f(w)} \cup \tilde{B}^{g\circ f}_w \cup \tilde{B}^f_w.$$

To see that $\tilde{B}_w$ is a bracketing of $(g \circ f)(C_w)$, it remains to verify that this collection is comprised of nested subtrees. First, each $B_v^g \subset \tilde{B}_{f(w)}^g$ is a bracketing of $g(C_v) \subset (g \circ f)(C_w)$, so it is nested. Moreover, the subtrees $g(C_v)$ are all disjoint and each tree of $\tilde{B}_{f(w)}^g$ is contained in a tree of $\tilde{B}_w^{g\circ f}$, so the union $\tilde{B}_{f(w)}^g \cup \tilde{B}_w^{g\circ f}$ is nested too. The $\tilde{B}_w^{g\circ f} \cup \tilde{B}_w^f$ is also nested, since each bracket $g(C_v)$ in the first set is included in each $g(S_y)$ of the second set whenever $v \in S_y$ and otherwise is disjoint from it. Hence $\tilde{B}_{f(w)}^g \cup \tilde{B}_w^f$ is also nested, and thus $\tilde{B}_w$ is nested.

Define the composition $(B_v^f)_{v\in V(S)} \circ (B_w^g)_{w\in V(R)}$ to be the collection

$$(\tilde{B}_w)_{w\in V(R)} \in \mathcal{L}_{g\circ f}.$$

Associativity of this composition is analogous to the associativity of the $B\mathcal{O}$ composition in Section A.3.2. In most cases, the composition is associative because vertex substitution is associative. In a composition with a degeneracy, a vertex is removed and so a bracket may be discarded if it is no longer large. Any discarded bracket is recreated in a subsequent composition if it should not have been discarded in the total composition.

Furthermore, this composition definition respects componentwise inclusion and thus defines the poset map (A.4.1). The realization of this poset map induces a map

$$L_g \times L_f \longrightarrow L_{g\circ f}. \tag{A.4.2}$$

This defines a composition on the morphism spaces of $\tilde{\Omega}_0$.

**Example A.4.3.** Let $f\colon R \to S$ and $g\colon S \to T$ be the morphisms in Figure A.12. Then $R$ is a corolla $C_w = C_9$, and $f(C_w) \subset S$ is the proper subtree of $S$ whose vertices are $v_1, v_2, v_3$. The images $g(C_{v_1}), g(C_{v_3}), g(C_{v_4}) \subset T$ are the corollas $C_{u_1}, C_{u_5}, C_{u_6}$ respectively and $g(C_{v_2})$ is the subtree with vertices $u_2, u_3, u_4$. The only images of corollas that admit a non-trivial bracketing are $f(C_w)$ and $g(C_{v_2})$. If the bracketing of $f(C_w)$ consists of the bracket $B_1$ in Figure A.13(a) and the bracketing of $g(C_{v_2})$ consists of $B_2$ in Figure A.13(b), then

$$\tilde{B}_{f(w)}^g = \{B_2\}, \quad \tilde{B}_w^{g\circ f} = \{g(C_{v_2})\}, \quad \tilde{B}_w^f = \{g(B_1)\}.$$

The bracketing $\tilde{B}_w \in \mathcal{B}((g \circ f)(C_w))$ is illustrated in Figure A.13(c).

By Example A.3.4, if $T_n$ is a tree with $n$ vertices such that no vertex is connected to more than two inner edges, then $|\mathcal{B}(T_n)|$ is the $n$th associahedron. The 3rd

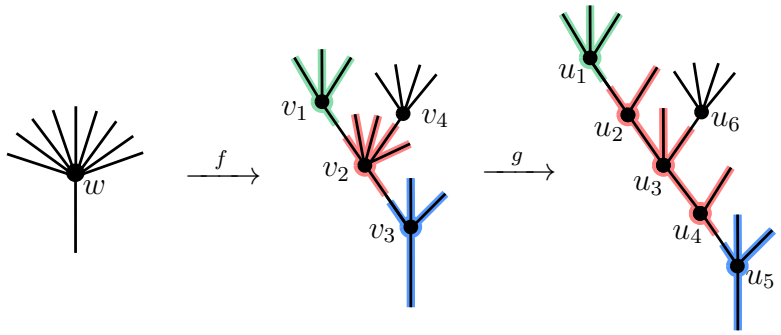
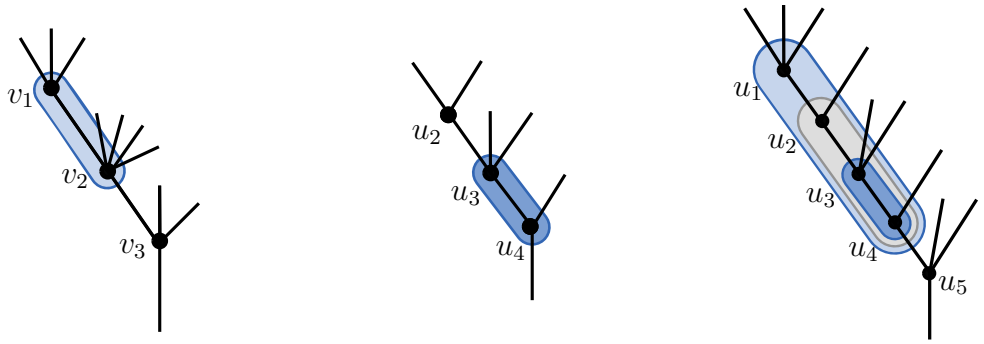



Figure A.12: Morphisms $f$ and $g$ in $\Omega$.




(a) Bracketing $B_1$ of $f(C_w)$.

(b) Bracketing $B_2$ of $g(C_{v_2})$.

(c) Bracketing $\tilde{B}_w$ of $g \circ f(C_w)$.

Figure A.13: Example of a bracketing induced by $f$ and $g$ in Figure A.12.

associahedron is an interval. Thus,

$$L_f = |\mathcal{B}(f(C_w))| = \left|\mathcal{B}\left(\begin{smallmatrix}\diagup\!\diagdown\\ \diagup\end{smallmatrix}\right)\right| \cong [0,1]$$
$$L_g = |\mathcal{B}(g(C_{v_1}))| \times \left|\mathcal{B}\left(\begin{smallmatrix}\diagup\!\diagdown\\ \diagup\end{smallmatrix}\right)\right| \times |\mathcal{B}(g(C_{v_3}))| \cong [0,1].$$

Again by Example A.3.4 and since $(g \circ f)(C_w)$ is a tree on five vertices, $L_{g\circ f} = |\mathcal{B}((g \circ f)(C_w))|$ is the 5th associahedron, which is a three dimensional polytope called an enneahedron.

**Definition A.4.4.** The category $\widetilde{\Omega}_0$ has the same objects as $\Omega$. Morphism spaces in $\widetilde{\Omega}_0$ are

$$\mathrm{Hom}_{\widetilde{\Omega}_0}(S,T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} L_g = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} \prod_{v \in V(S)} |\mathcal{B}(g(C_v))|$$

with composition (A.4.2) as described above.

**Example A.4.5.** Suppose $T_n$ is a planar tree with $(n+1)$ leaves and $n$ vertices, each of which is connected to at most two inner edges. Let the inner edges of $T_n$ be named $e_1, \ldots, e_{n-1}$. Morphisms $g \in \mathrm{Hom}_\Omega(C_{n+1}, T_n)$ are compositions of inner

face maps $\partial_{e_1}, \ldots, \partial_{e_{n-1}}$ but since the order of the composition does not affect the total composition, there is only one such morphism $g$. Hence $\mathrm{Hom}_{\widetilde{\Omega}_0}(C_{n+1}, T_n) = L_g = |\mathcal{B}(T_n)|$. Thus $\mathrm{Hom}_{\widetilde{\Omega}_0}(C_{n+1}, T_n)$ is the $n$th associahedron by Example A.3.4; the centre point of the polytope is defined by the empty bracket, which is the initial object in the poset $\mathcal{B}(T_n)$.

Lemma A.3.7 tells us that each bracketing space $|\mathcal{B}(g(C_v))|$ is contractible, which implies that each $L_g$ is contractible. Let $p : \widetilde{\Omega}_0 \to \Omega$ be the functor that is the identity on objects and projects each morphism space $L_g$ to $g$. By considering $\Omega$ as a discrete topological category, we have the following proposition.

**Proposition A.4.6.** *The functor $p : \widetilde{\Omega}_0 \to \Omega$ induces a homotopy equivalence on morphism spaces.* □

This proposition will allow us to associate an actual dendroidal space to any homotopy dendroidal space in Section A.4.3.

### A.4.2 Homotopy dendroidal spaces

In Section A.2.2, we defined a Segal condition for dendroidal spaces $X : \Omega^{op} \to \mathcal{S}$ using the Segal map

$$\chi \colon X(T) \longrightarrow \lim_{\mathrm{Sk}_1(T)^{op}} X.$$

We recall that the category $\mathrm{Sk}_1(T)$ has the vertices and edges of $T$ as objects, with morphisms given by edge inclusions $\iota_e : \eta \to C_v$ into the corollas of adjacent vertices. The Segal map $\chi$ is the unique map to the limit induced by the edge and corolla inclusions

$$\iota_e : \eta \to T \quad \text{and} \quad \iota_v : C_{|v|} \to T.$$

Note that the spaces $L_{\iota_e}$ and $L_{\iota_v}$ in $\widetilde{\Omega}_0^{op}$ which lie above the morphisms $\iota_e$ and $\iota_v$ are always just a single point, so the Segal map exists unchanged for functors $X : \widetilde{\Omega}_0^{op} \to \mathcal{S}$. This allows us to make the following definition:

**Definition A.4.7.** A *homotopy dendroidal space* $X$ is a diagram $X : \widetilde{\Omega}_0^{op} \to \mathcal{S}$. A homotopy dendroidal space is *reduced* if $X(\eta) = *$ and satisfies the *strict* Segal condition if the Segal map is an *isomorphism* for each $\eta \neq T \in \Omega$.

Recall from Section A.2.4 that one-colored operads are identified with reduced dendroidal Segal spaces via the dendroidal nerve

$$N^d : \mathcal{O}\text{-Alg} \to \mathcal{S}^{\Omega^{op}}.$$

The following theorem is a version of this nerve theorem for homotopy dendroidal spaces. We construct a functor

$$\Phi : B\mathcal{O}-\mathrm{Alg}_{\mathcal{S}} \longrightarrow \mathcal{S}^{\widetilde{\Omega}_0^{op}}$$

and show that a reduced homotopy dendroidal space $X \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$ is strictly Segal if, and only if, $X \cong \Phi(\mathcal{P})$ for some $B\mathcal{O}$-algebra $\mathcal{P}$.

Write $(\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}$ for the full subcategory of $\widetilde{\Omega}_0$-diagrams whose objects are reduced homotopy dendroidal spaces satisfying the strict Segal condition. Then we have the following result:

**Theorem A.4.8.** *There exists an isomorphism of categories*

$$\Phi : B\mathcal{O}-\mathrm{Alg}_{\mathcal{S}} \xrightarrow{\cong} (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}.$$

*Proof.* Given a $B\mathcal{O}$–algebra $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ with structure maps

$$\alpha_{\mathcal{P}} : B\mathcal{O}(n; m_1, \ldots, m_k) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_k) \longrightarrow \mathcal{P}(n)$$

we will define

$$\Phi(\mathcal{P}) = \Phi(\mathcal{P}, \alpha_{\mathcal{P}})\colon \widetilde{\Omega}_0^{op} \longrightarrow \mathcal{S}$$

as follows. We set $\Phi(\mathcal{P})(\eta) = *$. On objects $T \neq \eta$ of $\widetilde{\Omega}_0$, we set

$$\Phi(\mathcal{P})(T) = \prod_{w\in V(T)} \mathcal{P}(|w|).$$

Given a morphism $g : S \to T$ in $\Omega$, we need to define maps

$$\Phi(\mathcal{P})(g) : L_g \times \prod_{w\in V(T)} \mathcal{P}(|w|) \longrightarrow \prod_{v\in V(S)} \mathcal{P}(|v|).$$

We proceed one vertex $v$ at a time. As $L_g = \prod_{v\in V(S)} |\mathcal{B}(g(C_v))|$, at each $v \in V(S)$ we have projection maps

$$\pi_v : L_g \times \prod_{w\in V(T)} \mathcal{P}(|w|) \longrightarrow |\mathcal{B}(g(C_v))| \times \prod_{w\in V(g(C_v))} \mathcal{P}(|w|).$$

An application of the structure map $\alpha_{\mathcal{P}}$ defines a map

$$\alpha_v : |\mathcal{B}(g(C_v))| \times \prod_{w \in V(g(C_v))} \mathcal{P}(|w|) \longrightarrow \mathcal{P}(|v|). \qquad (*)$$

Indeed, an element of $|\mathcal{B}(g(C_v))|$ is a weighted bracketing $(B, t)$ of the subtree $g(C_v) \subset T$. Because $T$ is a planar tree, $g(C_v)$ inherits a planar structure. We consider $g(C_v)$ as an element of $\mathcal{O}$ by picking an ordering $\sigma$ of its vertices $\{w_1, \ldots, w_k\}$, and labeling its leaves via the map $\tau$ ordering them according to its planar structure. This way $((g(C_v), \sigma, \tau), B, t)$ is an element of $B\mathcal{O}(|v|; |w_1|, \ldots, |w_k|)$. To define the map $(*)$, we first order the factors $\mathcal{P}(|w|)$ for $w \in V(g(C_v))$, in accordance with our chosen $\sigma$, and then apply $\alpha_{\mathcal{P}}$ noting that our choice of ordering does not affect the result by the equivariance of $\alpha_{\mathcal{P}}$. Finally we act on the resulting element of $\mathcal{P}(|v|)$ by the permutation induced by $g$ that identifies the inputs of $v$ with the leaves of $g(C_v)$, comparing the labeling $\tau$ from the planar structure of $T$ to the planar ordering of $in(v)$ (which comes from the planar structure of $S$). We now set

$$\Phi(\mathcal{P})(g) := (\alpha_v \circ \pi_v)_{v \in V(S)}.$$

The fact that $\Phi(\mathcal{P})$ commutes with composition follows from the fact that composition in $\widetilde{\Omega}_0$ is defined exactly as the operadic composition of $B\mathcal{O}$ by taking the union of the brackets from the first morphism which remain large after applying the second morphisms, the brackets from the second morphism, and new "middle brackets", the images of the middle corollas, if they are large. It follows then that $\Phi(\mathcal{P}) : \widetilde{\Omega}_0 \to \mathcal{S}$ is a functor. Since $\Phi(\mathcal{P})(\eta) = *$, the Segal map is the map

$$\Phi(\mathcal{P})(T) \longrightarrow \prod_{v \in V(T)} \Phi(\mathcal{P})(C_v)$$

induced by the inclusions of the corollas. It is an isomorphism by definition of $\Phi(\mathcal{P})$.

The data required in the definition of the homotopy dendroidal space $\Phi(\mathcal{P})$ is the underlying symmetric sequence $\mathcal{P} = \{\mathcal{P}(m)\}$, the $B\mathcal{O}$-algebra structure maps $\alpha_{\mathcal{P}}$ and the projection maps $\pi_v$, all of which are natural under maps of $B\mathcal{O}$-algebras. Thus, the assignment $\mathcal{P} \mapsto \Phi(\mathcal{P})$ defines a functor

$$\Phi : B\mathcal{O}\text{−Alg}_{\mathcal{S}} \longrightarrow (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}.$$

It remains to show that the functor $\Phi$ is an isomorphism of categories. Given

two $B\mathcal{O}$-algebras $\mathcal{P}$ and $\mathcal{Q}$ with $\Phi(\mathcal{P}) = \Phi(\mathcal{Q})$, the underlying symmetric sequences $\{\mathcal{P}(n)\}_{n\geq 0}$ and $\{\mathcal{Q}(n)\}_{n\geq 0}$ are necessarily equal, being the value at the corollas $C_n$ and the corolla isomorphisms $\mathrm{Hom}_{\widetilde{\Omega}_0}(C_n, C_n) \cong \mathrm{Hom}_{\Omega}(C_n, C_n) \cong \Sigma_n$. Moreover, the structure maps $\alpha_{\mathcal{P}}$ and $\alpha_{\mathcal{Q}}$ likewise must agree as they agree with the evaluation of $\Phi(\mathcal{P}) = \Phi(\mathcal{Q})$ at corresponding morphisms in $\widetilde{\Omega}_0^{op}$. It follows that $\Phi$ is injective.

On the other hand, given any $X \in (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}$, we can construct a $B\mathcal{O}$-algebra $\mathcal{P}_X$ by setting $\mathcal{P}_X(n) = X(C_n)$ with a symmetric group action induced by the image under $X$ of the isomorphisms of $C_n$ in $\widetilde{\Omega}_0$. The $B\mathcal{O}$-algebra structure maps of $\mathcal{P}_X$ are defined using the above identification of the spaces $B\mathcal{O}(n; m_1, \ldots, m_k)$ with morphism spaces in $\widetilde{\Omega}_0$. The fact that $X$ is a functor will then give that $\mathcal{P}_X$ is a $B\mathcal{O}$–algebra. Thus the functor $\Phi$ is surjective. □

### A.4.3 Rectifying homotopy dendroidal spaces

We have just seen that $B\mathcal{O}$-algebras correspond to homotopy dendroidal spaces satisfying the strict Segal condition. In this section we will show how to produce, from a $B\mathcal{O}$-algebra, an actual dendroidal space satisfying a weak version of the Segal condition.

A commutative diagram in a topological category $\mathcal{S}$ is a functor from a discrete category $\mathcal{K}$ to $\mathcal{S}$. A homotopy commutative diagram can be similarly described as a functor from a topological category $\widetilde{\mathcal{K}}$ to $\mathcal{S}$, with the homotopies encoded as paths in the spaces of morphisms. In this language, a homotopy commutative diagram $X : \widetilde{\mathcal{K}} \to \mathcal{S}$ can be rectified, or strictified, to a functor $X' : \mathcal{K} \to \mathcal{S}$ precisely when there is an equivalence $p : \widetilde{\mathcal{K}} \to \mathcal{K}$. We briefly recall this rectification of diagrams, which was used by Segal in [81], and treated in great generality by Dwyer and Kan [26]; see also [90, Sec 2] for a detailed account of what we will use here. Our examples will be $\mathcal{K} = \Omega$ with $\widetilde{\mathcal{K}} = \widetilde{\Omega}_0$.

Let $p : \widetilde{\mathcal{K}} \to \mathcal{K}$ be a functor between categories enriched over topological spaces. There is an induced functor

$$p^* : \mathcal{S}^{\mathcal{K}} \longrightarrow \mathcal{S}^{\widetilde{\mathcal{K}}}$$

defined by precomposition with $p$. The homotopy left Kan extension defines also a functor

$$p_! : \mathcal{S}^{\widetilde{\mathcal{K}}} \longrightarrow \mathcal{S}^{\mathcal{K}}$$

that can be explicitly given as follows: given a diagram $Y \in \mathcal{S}^{\widetilde{\mathcal{K}}}$, its evaluation at an

object $d$ of $\mathcal{K}$ is the realization of a simplicial space with space of $k$–simplices

$$(p_! Y(d))_k = \coprod_{c_0,\dots,c_k \in ob(\widetilde{\mathcal{K}})} Y(c_0) \times \mathrm{Hom}_{\widetilde{\mathcal{K}}}(c_0, c_1) \times \cdots \times \mathrm{Hom}_{\widetilde{\mathcal{K}}}(c_{k-1}, c_k) \times \mathrm{Hom}_{\mathcal{K}}(c_k, d).$$

**Lemma A.4.9.** *[90, Proposition 2.1] Let $p\colon \widetilde{\mathcal{K}} \to \mathcal{K}$ be a functor inducing a homotopy equivalence of morphism spaces, and let $Y\colon \widetilde{\mathcal{K}} \to \mathcal{S}$ be a diagram, with $p_!Y\colon \mathcal{K} \to \mathcal{S}$ its rectification as defined above. Then there exists a zig-zag of natural transformations $p^*p_!Y \leftarrow \overline{p^*p_!Y} \to Y$, which induces a homotopy equivalence on objects: $p^*p_!Y(d) \simeq \overline{p^*p_!Y}(d) \simeq Y(d)$.*

In the statement, $\overline{p^*p_!Y}$ is an explicit functor from $\widetilde{\mathcal{K}}$ to $\mathcal{S}$ associated to $Y$ given by a two-sided bar construction (details in the proof of [90, Proposition 2.1]).

Proposition A.4.6 states that the functor $p : \widetilde{\Omega}_0 \to \Omega$ induces a homotopy equivalence of morphism spaces. Below, we apply Lemma A.4.9 to describe weak Segal dendroidal spaces that arise as the rectification of a homotopy dendroidal Segal space $Y \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$ in the following sense:

Consider the functors

$$\mathcal{S}^{\widetilde{\Omega}_0^{op}} \underset{p^*}{\overset{p_!}{\rightleftarrows}} \mathcal{S}^{\Omega^{op}}.$$

For any small category $\mathcal{K}$, the category of diagrams $\mathcal{S}^{\mathcal{K}^{op}}$ admits a projective model structure in which weak equivalences and fibrations are defined entrywise [43, 11.6.1]. In particular, when $\mathcal{S}$ is the category of topological spaces, every object is fibrant in this model structure. An application of [55, Proposition A.3.3.7] implies that $p_!$ and $p^*$ are the left and right Quillen functor of a Quillen equivalence.

**Proposition A.4.10.** *Let $\mathcal{S}$ be the category of topological spaces, and let $Y \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$ be a homotopy dendroidal space. Then the rectification $p_!Y \in \mathcal{S}^{\Omega^{op}}$ is a dendroidal space such that $p_!Y(\eta) \simeq *$, and such that the weak Segal map*

$$\tilde{\chi} : p_!Y(T) \xrightarrow{\simeq} \mathrm{holim}_{Sk_1(T)^{op}} p_!Y(C_v)$$

*is a homotopy equivalence for each $\eta \neq T \in \Omega$ if, and only if, the same two properties hold for the homotopy dendroidal space $Y$.*

*Proof.* Note first that, because $p$ is the identity on objects, we have $p^*X(T) = X(T)$ for any $X : \Omega^{op} \to \mathcal{S}$ and any $T \in Obj(\Omega) \equiv Obj(\widetilde{\Omega}_0)$. It follows that $p_!Y(\eta) \simeq *$ if and only if $Y(\eta) \simeq *$ as $p_!Y(\eta) = p^*p_!Y(\eta) \simeq Y(\eta) \simeq *$.

We are left to show that the weak Segal map $\tilde{\chi}$ is a weak equivalence for every $T \neq \eta$ for $p_!Y : \Omega^{op} \to \mathcal{S}$ if and only if it is the case for the original functor $Y : \widetilde{\Omega}_0^{op} \to \mathcal{S}$. Recall that the Segal map $\chi$ and its weak version $\tilde{\chi}$ for both $\Omega^{op}$–spaces and $\widetilde{\Omega}_0^{op}$–spaces are induced by corolla and edge inclusions $\iota_v$ and $\iota_e$ in $\widetilde{\Omega}_0$ and $\Omega$, respectively. As $p$ takes each map $\iota_v$ and $\iota_e$ in $\widetilde{\Omega}_0$, to the corresponding map in $\Omega$, and $p$ is the identity on objects, for any $X : \Omega^{op} \to \mathcal{S}$, we have

$$\begin{array}{ccc} X(T) & \xrightarrow{\tilde{\chi}} & \mathrm{holim}_{\mathrm{Sk}_1(T)^{op}} X(C_v) \\ \| & & \| \\ p^*X(T) & \xrightarrow{\tilde{\chi}} & \mathrm{holim}_{\mathrm{Sk}_1(T)^{op}} p^*X(C_v) \end{array}$$

in which the two horizontal maps describe the exact same map in $\mathcal{S}$.

The natural equivalences of functors $p^*p_!Y \leftarrow \overline{p^*p_!Y} \to Y$ of Lemma A.4.9 give us the vertical homotopy equivalences in the following commuting diagram in $\mathcal{S}$

$$\begin{array}{ccc} p^*p_!Y(T) & \xrightarrow{\tilde{\chi}} & \mathrm{holim}_{\mathrm{Sk}_1(T)^{op}} p^*p_!Y(C_v) \\ \simeq\uparrow & & \uparrow\simeq \\ \overline{p^*p_!Y}(T) & \xrightarrow{\tilde{\chi}} & \mathrm{holim}_{\mathrm{Sk}_1(T)^{op}} \overline{p^*p_!Y}(C_v) \\ \simeq\downarrow & & \downarrow\simeq \\ Y(T) & \xrightarrow{\tilde{\chi}} & \mathrm{holim}_{\mathrm{Sk}_1(T)^{op}} Y(C_v). \end{array}$$

which gives the result using the previous remark in the case $X = p_!Y$. □

*Remark* A.4.11. We used in the proof of Proposition A.4.10 that every topological $\widetilde{\Omega}_0$ diagram is projectively fibrant to obtain that the homotopy limits preserve homotopy equivalences.

As the dendroidal category is a generalized Reedy category [6, Example 1.6], there is a Reedy model structure on the category of reduced dendroidal spaces. We do not currently have an enriched generalized Reedy model structure on $\widetilde{\Omega}_0$-spaces. An advantage of the Reedy model structure is that homotopy limits of fibrant objects are weakly equivalent to the limit. One expects that, using such Reedy model structures, one could prove a version of Proposition A.4.10 with the actual Segal map $\chi$ replacing the weak Segal map $\tilde{\chi}$.

## A.5 Normalized Cacti as an infinity operad

The first goal of this section is to define an operad $MS^+$ and show that, despite not being an operad itself, normalized cacti and their composition can be described as elements and compositions inside $MS^+$. In Section A.5.2, we will use $MS^+$ to show that normalized cacti and the normalized composition extends to define a $B\mathcal{O}$-algebra structure. Using the results of Sections A.3 and A.4, this implies that we have an explicit construction of an $\infty$-operad with underlying sequence the spaces $\mathsf{cact}^1(n)$.

A cactus is a configuration of circles of various lengths attached to each other in a treelike fashion. In the original definition by Voronov [89, Section 2.7], there is a global basepoint associated to the "outside circle" of the cactus, as well as a basepoint for each circle (or *lobe*). A *spineless cactus* is a variant introduced by Kaufmann [48, Section 2.3], where the basepoint of each lobe is its closest point to the global basepoint along the outside circle. See Figure A.14 for an example. The

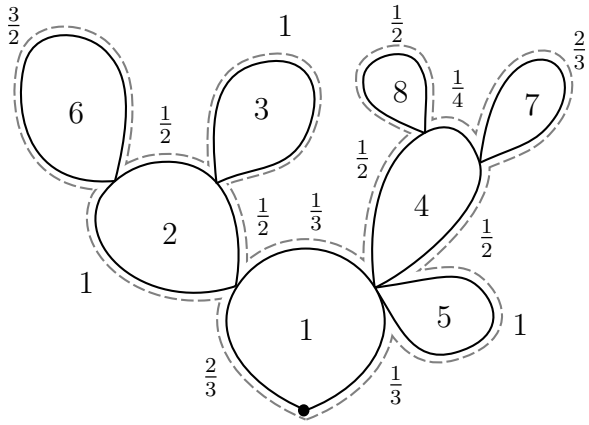



Figure A.14: Cactus with 8 lobes, its outside circle indicated by the dotted line.

space of all spineless cacti with $k$ lobes is denoted $\mathsf{cact}(k)$. The symmetric group acts on this space by permuting the labels of the lobes. The symmetric sequence $\mathsf{cact} = \{\mathsf{cact}(k)\}_{k\geq 0}$ is given a composition

$$\circ_i \colon \mathsf{cact}(k) \times \mathsf{cact}(j) \to \mathsf{cact}(k+j-1)$$

that is defined by inserting the second cactus into the $i$th lobe of the first cactus and aligning its global basepoint with the basepoint of the $i$th lobe. The insertion is done by rescaling the second cactus so that its total length is equal to the length of the $i$th lobe of the first cactus, then identifying the outside circle of the second cactus with the $i$th lobe of the first cactus. This composition makes $\mathsf{cact}$ into an operad, which

is equivalent to the little 2-discs operad [48, Section 3.2.1]. A rigorous definition of this composition requires close attention to subtleties and we refer to [48, Section 2] for precise definitions.

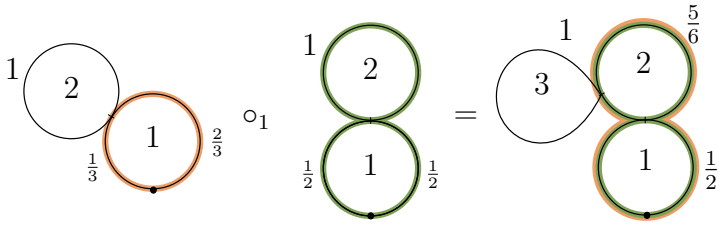



Figure A.15: A composition of normalized cacti.

The space of normalized cacti $\mathsf{cact}^1(k) \subset \mathsf{cact}(k)$ is the subspace of spineless cacti whose lobes all have length equal to 1 ([48, Definition 2.3.1]). They form a symmetric sequence $\mathsf{cact}^1 = \{\mathsf{cact}^1(k)\}_{k\geq 0}$. Composition of normalized cacti

$$\circ_i \colon \mathsf{cact}^1(k) \times \mathsf{cact}^1(j) \to \mathsf{cact}^1(k+j-1), \tag{A.5.1}$$

is defined by reparameterizing the $i$th lobe of a cactus $x \in \mathsf{cact}^1(k)$ to have length $j$, then identifying this lobe with the outer circle of the second cactus $y \in \mathsf{cact}^1(j)$ and aligning their basepoints. In contrast to $\mathsf{cact}$, the $i$th lobe of the first cactus is scaled instead of scaling the second cactus to the length of the $i$th lobe. See Figure A.15 for an example. This composition is not associative [48, Remark 2.3.19], as illustrated in Figure A.16. Thus $\mathsf{cact}^1$ is *not* an operad.

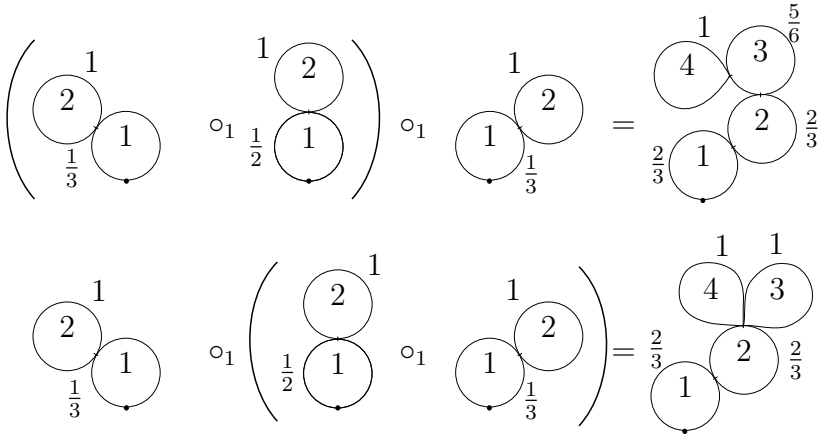



Figure A.16: Non-associativity in $\mathsf{cact}^1$

*Remark* A.5.1 (Composition in the graph cobordism category). This composition of normalized cacti is highly relevant to the graph model of the cobordism category of Riemann surfaces mentioned in the introduction of the paper. To model the gluing of

cobordisms, we use graphs to represent surfaces with potentially many incoming and outgoing boundary components. Normalized cacti are a simple case of this model, representing surfaces of genus zero with potentially many inputs but always just one output. Two surfaces are glued by attaching the incoming boundaries of the first surface to the outgoing boundaries of the second. According to [33] (see also [28, Theorem A]), we may assume that all incoming boundaries of a surface are disjoint embedded circles in the corresponding graph (like the lobes of the cactus, if they where pulled apart a little bit). Since these boundary circles are disjoint in the graph, they can be scaled independently to each match the length of an outgoing boundary in the graph of the second surface, just like scaling the $i$th lobe of the first cactus in $\mathsf{cact}^1$ composition. There is no obvious way to define a "$\mathsf{cact}$-like" composition for such more general graphs, because the outgoing circles of the second surface cannot be assumed to be disjoint, and hence cannot be scaled independently to the appropriate length. (See [28, Section 3.3] for more details about this gluing of fat graphs.)

### A.5.1 An operad $MS^+$ that contains $\mathsf{cact}^1$

In their proof of the Deligne conjecture, McClure and Smith [60, 61] introduced an operad $MS$ equivalent the little 2-discs operad.[1] Later, Salvatore [77, Section 4] used similar methods to show directly that the operad $MS$ is equivalent to the non-normalized cactus operad $\mathsf{cact}$. Here we will define a variant of $MS$ called $MS^+$, and, following [77], start by showing that it is an operad by proving that it embeds in $\mathrm{CoEnd}(S^1)$. We then show that normalized cacti are a subspace of the underlying symmetric sequence of $MS^+$ and that their composition can be written in terms of compositions in $MS^+$.

The space of operations $MS^+(k)$ is built from a space $\mathcal{F}(k)$, which we will show is homeomorphic to $\mathsf{cact}^1(k)$. In fact, we can think of an element of $\mathcal{F}(k)$ as the outer circle of a cactus.

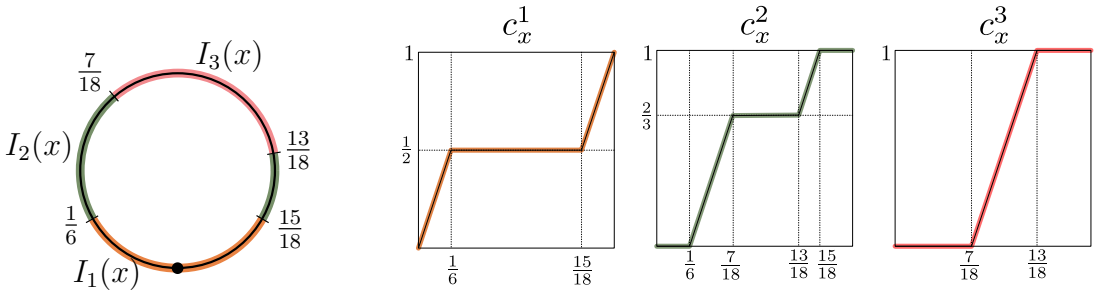
**Definition A.5.2.** [77, Definition 4.1] Let $S^1 = [0,1]/0 \sim 1$ be the circle of circumference equal to 1. Define $\mathcal{F}(k)$ as the space of partitions $x = (I_1(x), \ldots, I_k(x))$ of $S^1$ into closed 1-manifolds $I_j(x) \subset S^1$, each of which have total length $\frac{1}{k}$, with pairwise disjoint interiors, and such that

(∗) there does not exist a cyclically ordered 4-tuple $(z_1; z_2; z_3; z_4) \in S^1$ with $z_1, z_3 \in \mathring{I}_j(x)$ and $z_2, z_4 \in \mathring{I}_i(x)$, for $j \neq i$.

[1] The operad $MS$ is denoted $\mathcal{C}'$ in [60, Section 5].

For an example, see Figure A.17(a). The topology of $\mathcal{F}(k)$ is induced by the metric measuring the size of the overlap between partitions: for $x, y \in \mathcal{F}(k)$, $d(x,y) = 1 - \sum_{j=1}^{k} \ell(I_j(x) \cap I_j(y))$ for $\ell$ the length function on submanifolds of $S^1$.

The symmetric group $\Sigma_k$ acts on $\mathcal{F}(k)$ by reindexing the labels of the 1-manifolds.




(a) $x \in \mathcal{F}(3)$. (b) Maps $c_x^i$.

Figure A.17: Element of $x \in \mathcal{F}(3)$ and associated projections

**Definition A.5.3.** Given an element $x \in \mathcal{F}(k)$, we associate to each $I_j(x)$ a projection map $c_x^j : S^1 \to S^1$ that takes the quotient of $S^1$ under the identification of all the points in the same path component of $\overline{S^1 \setminus \mathring{I}_j}$ and then scales this circle by a factor of $k$. See Figure A.17(b) for an example. The *cactus map* $c_x \colon S^1 \to (S^1)^k$ is the collection of maps $c_x := (c_x^1, \ldots, c_x^k)$. Then there is a map

$$\begin{aligned} c : \mathcal{F}(k) &\longrightarrow Map(S^1, (S^1)^k) \\ x &\longmapsto c_x = (c_x^1, \ldots, c_x^k) \colon S^1 \to (S^1)^k. \end{aligned}$$

For any $x \in \mathcal{F}(k)$, we also use $x$ to denote the configuration of circles in the image of the cactus map $c_x \colon S^1 \to (S^1)^k$. Condition $(*)$ in Definition A.5.2 guarantees that this configuration is treelike, as it forces the submanifolds $I_j(x)$ to be nested. The global basepoint of $x$ is the image of the basepoint of $S^1$ and a planar structure is induced by the orientation of the source $S^1$ (see [77, Definition 4.2]). Since each part of a partition $x \in \mathcal{F}(k)$ has equal length, $x$ is a normalized cactus as shown in Figure A.18. This is the sketch of the proof for the next lemma.

**Lemma A.5.4.** *[77, Section 4] For each $k \geq 1$, the space $\mathcal{F}(k)$ is homeomorphic to* $\mathsf{cact}^1(k)$.

Recall the coendomorphism operad $\mathrm{CoEnd}(S^1)$ from Example A.2.2, whose underlying symmetric sequence is a collection of $\mathrm{CoEnd}(k)(S^1) := \mathrm{Map}(S^1, (S^1)^k)$. We

use the map

$$c : \mathsf{cact}^1(k) \cong \mathcal{F}(k) \hookrightarrow \mathrm{Map}(S^1, (S^1)^k) = \mathrm{CoEnd}(S^1)(k)$$

to define an embedding of symmetric sequences.

**Lemma A.5.5.** *The map $c : \mathcal{F}(k) \to Map(S^1, (S^1)^k)$ is a topological embedding.*

*Proof.* We first check injectivity. Given a map $c_x = (c_x^1, \ldots, c_x^k)$ in the image of $c$, we can completely determine $x \in \mathcal{F}(k)$. We know that each $c_x^j$ is a "step-map" with linear of slope $k$ over its non-constant parts, by the definition of $c$. (See Figure A.17(b).) Then $I_j(x)$ is precisely the subset of points of $S^1$ where the derivative $(c_x^j)'$ equals $k$. Continuity of $c$ follows from the fact that the topology in the mapping space can be defined using the convergence metric, using likewise the metric on $S^1$.

□

This embedding of symmetric sequences does not extend to an embedding of operads. As already mentioned, $\mathsf{cact}^1$ is not an operad and one can check that the image of $c$ is not a suboperad of $\mathrm{CoEnd}(S^1)$. Indeed, if we compose two elements in $\mathrm{CoEnd}(S^1)$ that came from elements of $\mathcal{F}$, their composition will not be in the image of any $\mathcal{F}(k)$ because all elements in the image of $\mathcal{F}(k)$ are piecewise linear graphs of slope 0 or $k$, and this property is not preserved by the composition in $\mathrm{CoEnd}(S^1)$.

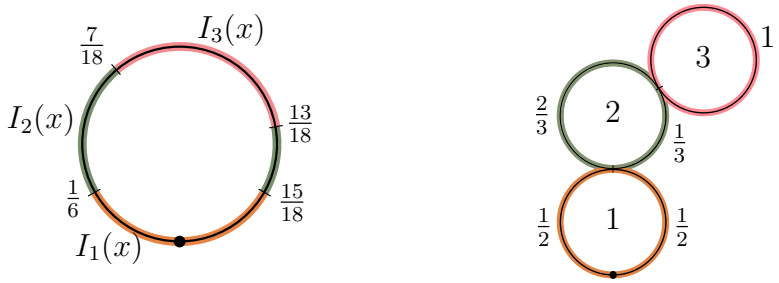



Figure A.18: An element $x$ of $\mathcal{F}(3)$ and the corresponding normalized cactus $c_x$.

Here we define the symmetric sequence $MS^+ = \{MS^+(k)\}_{k\geq 0}$, which is built from $\mathcal{F}(k)$ and a collection $Mon^+(I, \partial I)$ of scaling maps on the interval $I$. It has the important property that $\mathsf{cact}^1(k) \subset MS^+(k)$ for each $k \geq 0$.

**Definition A.5.6** ($MS^+$ as a symmetric sequence)**.** For each $k \geq 0$, we define the space $MS^+(k)$ as

$$MS^+(0) = *$$
$$MS^+(k) = \mathcal{F}(k) \times Mon^+(I, \partial I)$$

where $Mon^+(I, \partial I)$ is the space of strictly monotone self-maps of $I$ that restrict to the identity on $\partial I$. We consider $Mon^+(I, \partial I)$ as a subspace of the space of self-maps of $S^1 = I/\partial I$. For each $k$, there is an action of the symmetric group $\Sigma_k$ on $MS^+(k)$ by the reindexing of the labels of the 1-manifolds in $\mathcal{F}(k)$.

*Remark* A.5.7. The operad $MS$ that appears in [60, 61, 77] has an underlying symmetric sequence obtained by replacing $Mon^+(I, \partial I)$ by the larger space $Mon(I, \partial I)$ of weakly monotone maps. The inclusion $MS^+ \hookrightarrow MS$ is a homotopy equivalence as both $Mon(I, \partial I)$ and $Mon^+(I, \partial I)$ are contractible (in fact, they are both convex).

In order to show that $MS^+$ is an operad, we start by showing that each space of operations $MS^+(k)$ embeds in $\mathrm{CoEnd}(S^1)(k)$. We also check that the operad composition of $\mathrm{CoEnd}(S^1)$ preserves the image of $MS^+$, and hence is a suitable composition for $MS^+$, thus making $MS^+$ a suboperad of $\mathrm{CoEnd}(S^1)$.

**Proposition A.5.8.** *There is a topological embedding $\phi : MS^+(k) \to \mathrm{CoEnd}(S^1)(k)$ that sends $(x, f) \in MS^+(k)$ to the composite*

$$S^1 \xrightarrow{f} S^1 \xrightarrow{c_x} (S^1)^k$$

*where $c_x$ is the cactus map as in Definition A.5.3.*

A version of Proposition A.5.8 is stated for the operad $MS$ in [77, Section 4]. As we rely heavily on this result we give more complete details here.

*Proof.* The fact that $\phi$ is continuous follows from Lemma A.5.5, so we are left to check that $\phi$ is injective. Let $x \in \mathcal{F}(k)$. Recall that the map $c_x = (c_x^1, \ldots, c_x^k) \colon S^1 \to (S^1)^k$ is a collection of "step-maps" of linear of slope $k$ over its non-constant parts. Each map $c_x^j \colon S^1 \to S^1$ identifies points in the same path component of $\overline{S^1 \setminus \mathring{I}_j(x)}$ and linearly takes $I_j(x)$ (of length $1/k$) to a circle of circumference 1. So, these maps satisfy that

$$\frac{1}{k} \sum_{j=1}^{k} c_x^j = \mathrm{Id}_{S^1}.$$

In particular, this means that if $c_x \circ f = c_y \circ g$, then

$$f = (\frac{1}{k} \sum_{j=1}^{k} c_x^j) \circ f = \frac{1}{k} \sum_{j=1}^{k} (c_x^j \circ f) = \frac{1}{k} \sum_{j=1}^{k} (c_x^j \circ g) = (\frac{1}{k} \sum_{j=1}^{k} c_y^j) \circ g = g.$$

Moreover, as $f, g$ are strictly monotone and hence invertible, for each $j = 1, \ldots, k$,

$$c_x^j = (c_x^j \circ f) \circ f^{-1} = (c_y^j \circ g) \circ f^{-1} = (c_y^j \circ g) \circ g^{-1} = c_y^j.$$

This shows that $c_x = c_y$ and therefore the map is injective. □

Proposition A.5.8 shows that $MS^+$ is a symmetric subsequence of CoEnd and this next lemma shows that the operad structure maps of CoEnd preserve this structure.

**Lemma A.5.9.** *The operad structure maps of* CoEnd *preserve the symmetric subsequence* $MS^+$.

*Proof.* It suffices to consider the composition operations $\circ_i$ in CoEnd as defined in Example A.2.2. Given $(x, f)$ and $(y, g)$ in $MS^+$, we need to check that the composition

$$S^1 \xrightarrow{f} S^1 \xrightarrow{c_x} (S^1)^k \xrightarrow{1\times g\times 1} (S^1)^k \xrightarrow{1\times c_y\times 1} (S^1)^{j+k-1} \tag{A.5.2}$$

is in the image of $MS^+$, where $1 \times g \times 1$ denotes the map where $g$ acts only on the $i$th circle. For this, we will show two things:

(i) $(1 \times g \times 1) \circ c_x = c_{\tilde{x}} \circ \tilde{g}$, for some $\tilde{g} \in Mon^+(I, \partial I)$ and $\tilde{x} \in \mathcal{F}(k)$,

(ii) $(1 \times c_y \times 1) \circ c_x = c_z \circ h_{x,y}$ for some $h_{x,y} \in Mon^+(I, \partial I)$ and $z \in \mathcal{F}(j + k - 1)$.

For statement (i), the map $(1 \times g \times 1)$ acts only on the $i$th circle, so in the composition with $c_x$ it only affects points in $I_i(x)$. Recall that we identify $S^1$ with $I/\partial I$. Suppose $I_i(x) = J_1 \sqcup \cdots \sqcup J_r$ with each $J_s$ a subinterval of $[0, 1]$ and $I_i(x)$ of total length $\frac{1}{k}$. We obtain $\tilde{x} \in \mathcal{F}(k)$ from $x$ by replacing each subinterval $J_s$ by an interval $\tilde{J}_s$ of length $\frac{1}{k}\ell(g(c_x^i(J_s))))$ and shifting each path component of $[0, 1] \setminus \mathring{I}_i(x)$ accordingly. Then $I_i(\tilde{x}) = \tilde{J}_1 \sqcup \cdots \sqcup \tilde{J}_r$. This makes sense as, by construction, the total length of $I_i(\tilde{x})$ is again $\frac{1}{k}$. The map $\tilde{g}$ is defined as the canonical identification of $x$ with $\tilde{x}$ that maps $I_i(x)$ to $I_i(\tilde{x})$ for all $i \in \{1, \ldots, k\}$. See Figure A.19 for an example.

For statement (ii), we consider a composition $(1 \times c_y \times 1) \circ c_x : S^1 \to (S^1)^{j+k-1}$ with $c_y$ on the $i$th position. Such a composition maps the $r$th partition $I_r(x)$, for $r \neq i$, to the $r$th (if $r < i$) or $(r + k - 1)$st (if $r > i$) component in the target by a slope $k$ map, while $I_i(x)$ is mapped by slope $jk$ maps to the remaining components. Let $h_{x,y} : S^1 \to S^1$ be the rescaling map that scales each $I_r(x)$ by a factor $\frac{k}{j+k-1}$ for $r \neq i$, and $I_i(x)$ by a factor $\frac{jk}{j+k-1}$. Then the image under $h_{x,y}$ of each $I_r(x)$ will be of size $\frac{1}{j+k-1}$ for $r \neq i$, while $I_i(x)$ will have image of total size $\frac{j}{j+k-1}$. Note that this gives a well-defined map in $Mon^+(I, \partial I)$ as the sum of the length of the images

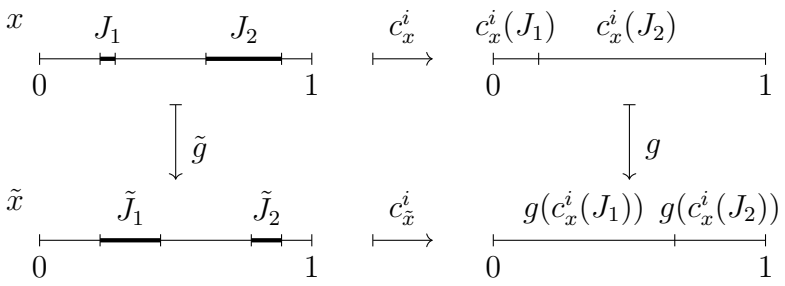



Figure A.19: An example of the commutative diagram $g \circ c^i_x = c^i_{\tilde{x}} \circ \tilde{g}$

$h_{x,y}(I_r(x))$ is $(k-1)\frac{1}{j+k-1} + \frac{j}{j+k-1} = 1$. Subdividing the image under $h_{x,y}$ of $I_i(x)$ into $j$ parts as prescribed by $y$, together with the images of the other $I_r(x)$'s, then defines $z \in \mathcal{F}(j+k-1)$. The relation $(1 \times c_y \times 1) \circ c_x = c_z \circ h_{x,y}$ holds by construction.

By putting (i) and (ii) together, the composition in (A.5.2) is given by $(h_{\tilde{x},y} \circ \tilde{g} \circ f, z) \in MS^+$. □

Therefore we have shown that $MS^+$ is a suboperad of $\mathrm{CoEnd}(S^1)$ via the embedding $\phi$ in Proposition A.5.8.

**Definition A.5.10** ($MS^+$ as an operad)**.** The symmetric sequence $MS^+ = \{MS^+(k)\}_{k\in\mathbb{N}}$ becomes an operad with composition

$$(x, f) \bullet_i (y, g) := \phi^{-1}(\phi(x, f) \circ_i \phi(y, g)) \tag{A.5.3}$$

where $\circ_i$ is the composition in $\mathrm{CoEnd}(S^1)$ defined in (A.5.2), and the pre-image exists as a consequence of Lemma A.5.9.

We will often use scaling maps in $Mon^+(I, \partial I)$ to encode the scaling of lobes in the composition of normalized cacti. Given a partition $x = (I_1(x), \ldots, I_k(x)) \in \mathcal{F}(k) \cong \mathsf{cact}^1(k)$, and natural numbers $m_1, \ldots, m_k \geq 0$, we let

$$g = g(x; m_1, \ldots, m_k) : S^1 \longrightarrow S^1 \tag{A.5.4}$$

be the element of $Mon^+(I, \partial I)$ that scales $I_j(x)$ by the factor $\frac{km_j}{m_1+\cdots+m_k}$, $1 \leq j \leq k$. Each $I_j(x)$ has total length $\frac{1}{k}$, so the image of $I_j(x)$ will have length $\frac{m_j}{m_1+\cdots+m_k}$ for each $1 \leq j \leq k$. Note that $g(x; 1, \ldots, 1) = \mathrm{id}$ is just the identity map on $S^1$.

We will now show that the $\circ_i$-compositions of normalized cacti from (A.5.1),

$$\circ_i : \mathsf{cact}^1(k) \times \mathsf{cact}^1(j) \longrightarrow \mathsf{cact}^1(k+j-1),$$

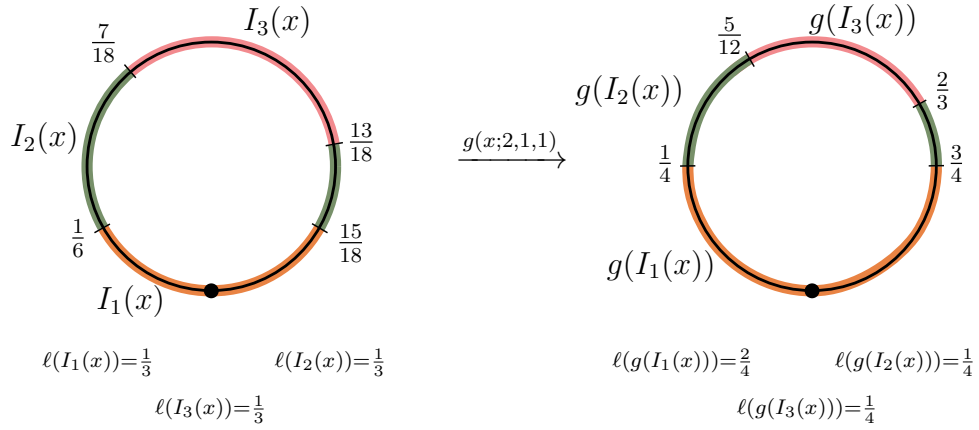



Figure A.20: Map $g(x;2,1,1)$, for $x$ from Figure A.18.

and, more generally, the $\mathsf{cact}^1$–composition maps

$$\gamma_{\mathsf{cact}^1} : \mathsf{cact}^1(k) \times \mathsf{cact}^1(m_1) \times \cdots \times \mathsf{cact}^1(m_k) \longrightarrow \mathsf{cact}^1(\Sigma_{i=1}^k m_i),$$

are restrictions of the corresponding compositions of appropriately chosen elements of $MS^+$.

For a collection of cacti $x \in \mathsf{cact}^1(k)$ and $y_j \in \mathsf{cact}^1(m_j)$, $1 \le j \le k$, the quasi-operad composition $\gamma_{\mathsf{cact}^1}(x; y_1, \ldots, y_k)$ scales each lobe of $x$ so that the $i$th lobe now has length $m_i$, and then inserts (without any further scaling) each $y_i$ in place of the $i$th scaled lobe.

Under the homeomorphism $\mathsf{cact}^1(k) \cong \mathcal{F}(k)$ in Lemma A.5.4, a normalized cactus $x \in \mathsf{cact}^1(k)$ precisely corresponds to a partition $x \in \mathcal{F}(k)$ of $[0,1]$ into $k$ submanifolds $I_j(x)$ of equal lengths $\frac{1}{k}$, satisfying the conditions of Definition A.5.2. Since the $i$th lobe of $x$ corresponds to the submanifold $I_i(x)$, there is a scaling by $\frac{1}{k}$ in the identification $\mathsf{cact}^1(k)$ to take a lobe of length 1 to $I_i(x)$. In the next lemma we will use $x \in \mathcal{F}(k)$ and $y_j \in \mathcal{F}(m_j)$ for $1 \le j \le k$ to represent a sequence of cacti in $\mathsf{cact}^1(k)$ and $\mathsf{cact}^1(m_j)$ respectively. We will still denote the composition by $\circ_i$ or $\gamma_{\mathsf{cact}^1}$.

**Lemma A.5.11.** *Let $\gamma_{MS^+}$ and $\gamma_{\mathsf{cact}^1}$ denote the (quasi-)operad compositions in $MS^+$ and $\mathsf{cact}^1$, respectively. Then for $x \in \mathcal{F}(k)$ and $y_j \in \mathcal{F}(m_j)$, with $1 \le j \le k$, we have*

$$\gamma_{MS^+}((x, g^{-1}(x; m_1, \ldots, m_k)); (y_1, id), \ldots, (y_k, id)) = (\gamma_{\mathsf{cact}^1}(x; y_1, \ldots, y_k), id)$$

*in $MS^+(\sum m_j)$. In particular,*

$$(x, g^{-1}(x; 1, \ldots, m_i, \ldots, 1)) \bullet_i (y_i, \mathrm{id}) = (x \circ_i y_i, \mathrm{id})$$

*where $\bullet_i$ denotes the composition* (A.5.3) *of $MS^+$ and $\circ_i$ represents the composition* (A.5.1) *of* $\mathsf{cact}^1$.

*Proof.* Let $\mathcal{F}_{(m_1,\ldots,m_k)}(k)$ denote the scaled version of $\mathcal{F}(k)$ where the $i$th partition, $I_i$, now has length $m_i$ instead of $\frac{1}{k}$. In particular, we have that $\mathcal{F}(k) = \mathcal{F}_{\left(\frac{1}{k},\ldots,\frac{1}{k}\right)}(k)$ and $\mathsf{cact}^1(k) = \mathcal{F}_{(1,\ldots,1)}(k)$. Thus the homeomorphism $\mathsf{cact}^1(k) \cong \mathcal{F}(k)$ implies that the composition $\gamma_{\mathsf{cact}^1}$ on $\mathsf{cact}^1$ can be interpreted as a map in $\mathcal{F}$, written as

$$\begin{aligned}
&\mathcal{F}(k)\times (\mathcal{F}(m_1) \times \cdots \times \mathcal{F}(m_k)) \\
&\qquad \xrightarrow{S} \ \mathcal{F}_{(m_1,\ldots,m_k)}(k) \times \left(\mathcal{F}_{(1,\ldots,1)}(m_1) \times \cdots \times \mathcal{F}_{(1,\ldots,1)}(m_k)\right) \\
&\qquad \xrightarrow{\gamma} \ \mathcal{F}_{(1,\ldots,1)}(\sum_i m_i) \\
&\qquad \xrightarrow{N} \ \mathcal{F}_{\frac{1}{\sum m_i},\ldots,\frac{1}{\sum m_i}}(\sum_i m_i) = \mathcal{F}(\sum_i m_i)
\end{aligned}$$

where $S$ and $N$ are scaling and normalising maps and the map labelled $\gamma$ is the insertion map.

Our task is to write the composition $\gamma_{\mathsf{cact}^1}$ in terms of the operad $MS^+$. To do this, we will use scaling maps inside $Mon^+(I, \partial I)$. More precisely, define a map

$$\mathcal{F}(k) \times (\mathcal{F}(m_1) \times \cdots \times \mathcal{F}(m_k)) \xrightarrow{G} MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k))$$

that takes $(x; y_1, \ldots, y_k)$ to $((x, g^{-1}); (y_1, \mathrm{id}), \ldots, (y_k, \mathrm{id}))$ where $g = g(x; m_1, \ldots, m_k) \in Mon^+(I, \partial I)$ is the map in equation (A.5.4). The statement we want to prove is that $(\gamma_{\mathsf{cact}^1}, \mathrm{id})$ can be written as the composition

$$\begin{aligned}
\mathcal{F}(k) \times (\mathcal{F}(m_1) \times \cdots \times \mathcal{F}(m_k)) &\xrightarrow{G} MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k)) \\
&\xrightarrow{\gamma_{MS^+}} MS^+(\sum_i m_i).
\end{aligned}$$

In particular, we claim that the resulting element of $MS^+(\sum_i m_i)$ is in the image of $\mathsf{cact}^1$, that is, of the form $(z, \mathrm{id})$.

To prove this, we start by expressing $G$ as a composition $G' \circ S$, where $S$ is the scaling map in the description $\gamma_{\mathsf{cact}^1} = N \circ \gamma \circ S$ given above and

$$\begin{aligned}
G' \colon \mathcal{F}_{(m_1,\ldots,m_k)}(k) \times (\mathcal{F}_{(1,\ldots,1)}(m_1) \times \cdots \times \mathcal{F}_{(1,\ldots,1)}(m_k)) \\
\longrightarrow MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k))
\end{aligned}$$

is the map that takes a tuple $(x; y_1, \ldots, y_k)$ to the tuple $((N(x), g^{-1}); (N(y_1), \mathrm{id}), \ldots, (N(y_k), \mathrm{id}))$, with $N$ the normalization also as above. In order to compare $\gamma_{\mathsf{cact}^1} = N \circ \gamma \circ S$ with $\gamma_{MS^+} \circ G' \circ S$, we have to show that the diagram

$$\begin{array}{ccc} \mathcal{F}_{(m_1,\ldots,m_k)}(k) \times (\mathcal{F}_{(1,\ldots,1)}(m_1) \times \cdots \times \mathcal{F}_{(1,\ldots,1)}(m_k)) & \xrightarrow{\gamma} & \mathcal{F}_{(1,\ldots,1)}(\sum_i m_i) \\ \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle (N,id)} \\ MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k)) & \xrightarrow{\gamma_{MS^+}} & MS^+(\sum_i m_i) \end{array}$$

commutes, where the right vertical map takes $z$ to $(N(z), \mathrm{id})$. To see this, let $(x; y_1, \ldots, y_k)$ be an element in the top left corner of the square. Its image $\gamma_{MS^+} \circ G'(x; y_1, \ldots, y_k)$ along the bottom composition is the element of $MS^+(m_1 + \cdots + m_k)$ given by the following composition:

$$S^1 \xrightarrow{g^{-1}} S^1 \xrightarrow{c_x} (S^1)^k \xrightarrow{c_{y_1} \times \cdots \times c_{y_k}} (S^1)^{m_1 + \cdots + m_k}$$

since we consider $MS^+(m_1 + \cdots + m_k)$ as a subspace of $\mathrm{CoEnd}(m_1 + \cdots + m_k)$ and use the composition in (A.5.3). The $j$th factor $S^1$ in the above $(S^1)^k$ is subdivided into submanifolds $I_s(y_j)$ according to $c_{y_j}$.

Their inverse image $g \circ (c_x)_j^{-1}(I_s(y_j))$ in the source $S^1$ of the composition is thus taken to the $(m_1 + \cdots + m_{j-1} + s)$th factor $S^1$ in $(S^1)^{m_1 + \cdots + m_k}$, being first scaled by a factor $\frac{\sum m_i}{k m_j}$ (using $g^{-1}$), then by a factor $k$ (via the $j$th component of $c_x$) and finally by a factor $m_j$ (via the $s$th component of $c_{y_j}$). So in total the composition takes $g \circ (c_x)_j^{-1}(I_s(y_j))$ to $S^1 = I/\partial I$ linearly by a factor $\sum m_i$, and is constant on the connected components of the complement of $g \circ (c_x)_j^{-1}(I_s(y_j))$. In particular, $g \circ (c_x)_j^{-1}(I_s(y_j))$ has length $\frac{1}{\sum m_i}$, which is independent of $j$ and $s$. Thus we see that the resulting element does indeed live in the image of $\mathsf{cact}^1$. As the scaling is always independent of $s$ and $j$, the proportion of each $g \circ (c_x)_j^{-1}(I_s(y_j))$ inside the source $S^1$ is always as dictated by $c_{y_j}$, with each $g^{-1}(I_j(x))$ having total length $\frac{m_j}{\sum_i m_i}$. Hence the composition is the same as following the other side of the square, which inserts $I_s(y_j)$ inside $I_j(x)$, scaling each $I_j(x)$ to length $m_j$, then scales it by $\frac{1}{\sum m_i}$ to be inside $MS^+$.

□

Therefore up to scaling in accordance with the homeomorphism $\mathsf{cact}^1 \cong \mathcal{F}$ in Lemma A.5.4, we have shown that both $\mathsf{cact}^1$ and its composition are contained within the operad $MS^+$, but not as a suboperad.

### A.5.2 $\mathsf{cact}^1$ is a $B\mathcal{O}$–algebra

Here we will construct an action of $B\mathcal{O}$ on normalized cacti using the fact that $\mathsf{cact}^1 \subset MS^+$, and that its composition can also be described in terms of the composition in $MS^+$. In Theorem A.5.12, we show that the quasi-operad structure on normalized cacti $\mathsf{cact}^1 = \{\mathsf{cact}^1(k)\}_{k\geq 0}$ is part of a $B\mathcal{O}$-algebra structure. In Corollary A.5.13, we conclude that $\mathsf{cact}^1$ determines a dendroidal Segal space $X \in \mathcal{S}^{\Omega^{op}}$ with $X(C_v) = \mathsf{cact}^1(|v|)$.

Recall from Section A.3.3 that a $B\mathcal{O}$–algebra is a symmetric sequence with $\circ_i$–operations that are homotopy associative up to all higher homotopies. Elements from $B\mathcal{O}$ are $(T, \sigma, \tau, B, t)$ where $T$ is a planar tree equipped with bijections $\sigma : |V(T)| \to V(T)$ and $\tau : |L(T)| \to L(T)$, and $(B, t)$ is a weighted bracketing of $T$.

Let

$$\mathcal{R}\colon MS^+ \longrightarrow \mathcal{F} \tag{A.5.5}$$

denote the projection map that forgets the $Mon^+(I, \partial I)$ component, $\mathcal{R}(x, f) = x$. This is a map of symmetric sequences. If we think of elements of $MS^+$ as cacti, the map $\mathcal{R}$ has the effect of renormalizing, that is, rescaling the lobes so that they all have the same length. Since $MS^+$ is an operad, it is an $\mathcal{O}$-algebra. The $\mathcal{O}$-action

$$\lambda_{MS^+} : \mathcal{O}(k; m_1, \ldots, m_k) \times MS^+(m_1) \times \ldots \times MS^+(m_k) \longrightarrow MS^+(\sum_i m_i)$$

takes a sequence of elements

$$((T, \sigma, \tau), (x_1, f_1), \ldots, (x_k, f_k)) \in \mathcal{O}(k; m_1, \ldots, m_k) \times MS^+(m_1) \times \ldots \times MS^+(m_k)$$

to the composition of the elements $(x_1, f_1), \ldots, (x_k, f_k)$ according to $\gamma_{MS^+}$, in the order prescribed by the labeled tree $(T, \sigma)$, acting by the permutation $\tau$ on the resulting element of $MS^+(\sum_i m_i)$. This composition can be depicted by labeling the $i$th vertex of $(T, \sigma, \tau)$ by $(x_i, f_i) \in MS^+(m_i)$. This action is compatible with the composition in $\mathcal{O}$ because $MS^+$ is an operad. We will use this existing $\mathcal{O}$-algebra structure to define the $B\mathcal{O}$-algebra structure of $\mathsf{cact}^1$ by representing the $\mathsf{cact}^1$-composition by $\mathcal{R} \circ \lambda_{MS^+}$.

**Theorem A.5.12.** *The* $\mathsf{cact}^1$*-composition* (A.5.1) *is part of a* $B\mathcal{O}$*–algebra structure.*

*Proof.* In order to construct a $B\mathcal{O}$–algebra structure on the sequence $\{\mathsf{cact}^1(n)\}_{n\geq 0}$,

we want to define a map

$$B\mathcal{O}(k; m_1, \ldots, m_k) \times \mathsf{cact}^1(m_1) \times \ldots \times \mathsf{cact}^1(m_k) \longrightarrow \mathsf{cact}^1(\sum_i m_i)$$

that restricts to the $\Sigma_n$–action on $\mathsf{cact}^1(n)$, which permutes the labels on the lobes, and its already defined $\circ_i$–compositions. Using the homeomorphism $\mathsf{cact}^1 \cong \mathcal{F}$ from Lemma A.5.4, we will equivalently construct a map

$$\lambda : B\mathcal{O}(k; m_1, \ldots, m_k) \times \mathcal{F}(m_1) \times \ldots \times \mathcal{F}(m_k) \longrightarrow \mathcal{F}(\sum_i m_i).$$

Firstly, the $\Sigma_n$-action on the $n$–space of a $B\mathcal{O}$–algebra is encoded by the labeled corollas

$$(C_n, 1, \tau, \emptyset, \emptyset) \in B\mathcal{O}(n; n) \cong \mathcal{O}(n; n) \cong \Sigma_n,$$

where $\tau$ labels the leaves of the corollas $C_n$, which are thought of as elements of the symmetric group $\Sigma_n$, and the identity corresponds to the planar ordering. This fixes the action of such elements of $B\mathcal{O}$ as we have already fixed the $\Sigma_n$–action on $\mathsf{cact}^1(n)$.

The $\mathsf{cact}^1$ $\circ_i$-composition is encoded in $B\mathcal{O}$ by the trees with exactly two vertices, one attached to the $i$th incoming edge of the other. These trees admit no non-trivial bracketings so such elements of $B\mathcal{O}$ have the form

$$(T, \sigma, \tau, \emptyset, \emptyset) \in B\mathcal{O}(m + n - 1; m, n)$$

where $\sigma$ labels the two vertices of $T$ and $\tau$ labels its $n+m-1$ leaves. The compatibility with the pre-chosen operadic composition of $\mathsf{cact}^1$ dictates the action of such elements of $B\mathcal{O}$: $(T, \sigma, \tau, \emptyset, \emptyset)$ acts on $x_1 \in \mathsf{cact}^1(m)$ and $x_2 \in \mathsf{cact}^1(n)$ by taking their $\circ_i$-composition, as dictated by the tree, and then acting by $\tau$ on the lobes of the resulting element of $\mathsf{cact}^1(m + n - 1)$. Figure A.21 illustrates an example of this action.

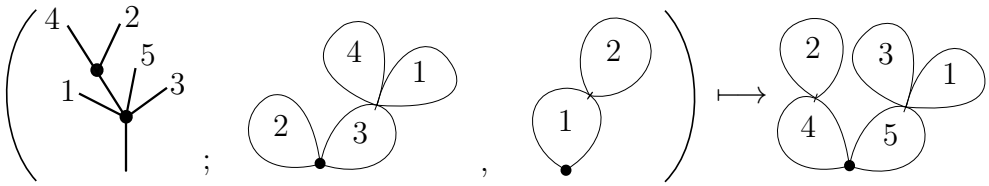



Figure A.21: Example of the $B\mathcal{O}$-action on $\mathsf{cact}^1$.

By Lemma A.5.11, this $\mathsf{cact}^1$-composition can be defined in terms of the $MS^+$ composition:

$$\mathcal{R} \circ \lambda_{MS^+}\big((T, \sigma, \tau); (x_1, g_1), (x_2, g_2)\big)$$

where $\mathcal{R}$ is the projection map (A.5.5), and $g_1 = g(x_1, 1, \dots, k_i, \dots, 1)$ and $g_2 = g(x_2, 1, \dots, l_j, \dots, 1)$ are the rescaling maps of (A.5.4), with $k_i = n$ and $l_j = 1$ if first vertex is the bottom vertex and the second is attached to its $i$th input, or $k_i = 1$ and $l_j = m$ if the second vertex is the bottom vertex with the first attached to its $j$th input. Let $(y_T, f_T) \in MS^+$ denote the element $\lambda_{MS^+}\big((T, \sigma, \tau); (x_1, g_1), (x_2, g_2)\big)$.

We will now extend this definition of the $B\mathcal{O}$–action of trees with at most two vertices to an action of the whole operad. We start by defining an explicit expression for the action of bracketings of trees $(T, \sigma, \tau, B, \underline{1})$ with brackets of weight 1, and afterwards extend this definition to the remaining elements of $B\mathcal{O}$, whose brackets have weight strictly between 0 and 1.

Let $\mathbf{T} = (T, \sigma, \tau, B, \underline{1})$ be an element of $B\mathcal{O}(n; m_1, \dots, m_k)$ with all brackets of weight 1, and let $x_i \in \mathcal{F}(m_i) \cong \mathsf{cact}^1(m_i)$ for each $1 \leq i \leq k$. We first construct scaling maps $g \in Mon^+(I, \partial I)$ as in (A.5.4). Recall from Definition A.3.1 that a bracketing $B = \{S_j\}_{j \in J}$ consists of large, nested proper subtrees of $T$. Here we allow $B$ to be empty. Recall that $\sigma$ orders the vertices of $T$. For a fixed $i \in \{1, \dots, k\}$, let $S \in B$ be the smallest bracket that contains the vertex $\sigma(i)$, allowing $S = T$ if there are no such bracket. Recall that $in(\sigma(i))$ is the set of incoming edges of $\sigma(i)$, and $L(S)$ is the set of leaves of the bracket $S$. We define a map

$$\xi : in(\sigma(i)) \longrightarrow \mathbb{N} \tag{A.5.6}$$

by setting

(i) $\xi(e) = 1$ if $e \in L(S)$;

(ii) $\xi(e) = |L(S')|$ if $e$ is the root of a bracket $S' \subset S$ in $B$, with $S' \subset S$ the largest such bracket;

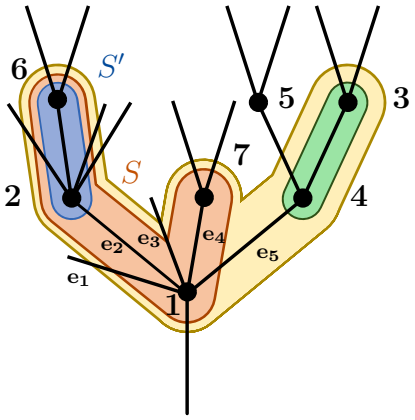
(iii) $\xi(e) = |w|$ if $e \in iE(S)$ is not the root of any $S' \in B$, where $|w|$ denotes the arity of the vertex $w \in V(S)$ for which $e$ is the outgoing edge.

Figure A.22 shows an example of the map $\xi$. We then set

$$g_i := g(x_i; \xi(e_1), \dots, \xi(e_{m_i})) \tag{A.5.7}$$

for $e_1, \dots, e_m$ the incoming edges of $\sigma(i)$ ordered by the planar ordering of $T$.

We define the action of $B\mathcal{O}$ inductively on the size of the bracketing $B$.

$T =$ 


$\xi : in(\sigma(1)) \to \mathbb{N}$

$\xi(e_1) = 1$
$\xi(e_2) = |L(S')| = 5$
$\xi(e_3) = 1$
$\xi(e_4) = |\sigma(7)| = 2$
$\xi(e_5) = 1$

Figure A.22: An example of $\xi$.

If $B$ is empty, then we define

$$\lambda(\mathbf{T}; x_1, \ldots, x_k) := \mathcal{R} \circ \lambda_{MS^+}((T, \sigma, \tau); (x_1, g_1), \ldots, (x_k, g_k))$$

and use $(y_T, f_T) \in MS^+$ to denote the image of $\lambda_{MS^+}$. Note that when $k = 1$ or 2, this is the same as the $B\mathcal{O}$-structure already defined above.

If $B$ is not empty, then we define additional scaling maps for each bracket, using the inductive hypothesis that the action has already been defined action on subtrees with fewer brackets.

Let $T'$ be the tree obtained from $T$ by adding a binary vertex at the root of each bracket $S_j \in B$. Extend the order $\sigma$ of the vertices of $T$ to an order $\sigma'$ of vertices of $T'$ by setting the $|J| = |B|$ new vertices last. An example of $T'$ is shown in Figure A.23. We will use each additional vertex of $T'$ to assign a scaling map to the associated bracket.

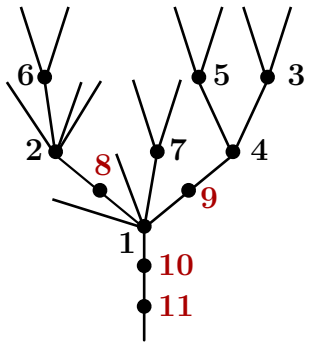
$T' =$ 


Figure A.23: An example of $T'$ for the bracketing of $T$ in Figure A.22

Let $w_j \in V(T') \setminus V(T)$ be the $j$th vertex of $T'$ not in $T$, according to the chosen order $\sigma'$. Let $S_j \in B$ be the bracket associated to $w_j$. Since the number of brackets

of $B$ that lie inside $S_j$ is less than $|B|$, we have an element

$$(y_{S_j}, f_{S_j}) \in MS^+$$

defined by the inductive assumption by restricting $\mathbf{T} = (T, \sigma, \tau, B, \underline{1})$ to the subtree $S_j$. Consider the tree $T/S_j$ in which all vertices in $S_j$ are identified and internal edges between them are collapsed. The tree $T/S_j$ has a vertex $[S_j]$ associated to the collapsed tree $S_j$. We have an induced bracketing $\tilde{B}$ of $T/S_j$ from the bracketing $B$ of $T$, and thus can define a map $\xi_j : in([S_j]) \to \mathbb{N}$ as in (A.5.6) by replacing $(T, B)$ by $(T/S_j, \tilde{B})$. Then define

$$h_j := g(y_{S_j}; \xi_j(e_1), \ldots, \xi_j(e_l)) \circ f_{S_j}^{-1} \tag{A.5.8}$$

for $e_1, \ldots, e_l$ the incoming edges of $[S_j]$ in $T/S_j$. We define the action of $B\mathcal{O}$ by setting

$$\lambda(\mathbf{T}; x_1, \ldots, x_k) := \mathcal{R} \circ \lambda_{MS^+}\big((T', \sigma', \tau); (x_1, g_1), \ldots, (x_k, g_k), (1, h_1), \ldots, (1, h_{|B|})\big) \tag{A.5.9}$$

for the rescaling maps $g_i$ and $h_j$ defined above.

We claim that the formula for the action (A.5.9) is indeed compatible with composition of bracketings of trees of weight 1. It is enough to check this for a $\circ_i$-composition in $B\mathcal{O}$, so consider $\mathbf{T_1} = (T_1, \sigma_1, \tau_1, B_1, \underline{1})$ and $\mathbf{T_2} = (T_2, \sigma_2, \tau_2, B_2, \underline{1})$ in $B\mathcal{O}$. We need to check that

$$\lambda(\mathbf{T_1}; x_1, \ldots, x_{i-1}, \lambda(\mathbf{T_2}; x_i, \ldots, x_{i+l-1}), x_{i+1}, \ldots, x_{k+l-1}) = \lambda(\mathbf{T_1} \circ_i \mathbf{T_2}; x_1, \ldots, x_{k+l-1}). \tag{A.5.10}$$

From the above definition, we have $\lambda(\mathbf{T_1}; x_1, \ldots, x_{i-1}, \lambda(\mathbf{T_2}; x_i, \ldots, x_{i+l-1}), x_{i+1}, \ldots, x_{k+l-1}) =$

$$\mathcal{R} \circ \lambda_{MS^+}\big((T_1', \sigma_1', \tau_1); (x_1, g_1), \ldots, (x_{i-1}, g_{i-1}), (y_{T_2}, g_i), (x_{i+l}, g_{i+l}), \ldots, (x_k, g_k), (1, h_1), \ldots, (1, h_{|B_1|})\big)$$

for

$$y_{T_2} = \mathcal{R} \circ \lambda_{MS^+}\big((T_2', \sigma_2', \tau_2); (x_i, g_1'), \ldots, (x_{i+l-1}, g_l'), (1, h_1'), \ldots, (1, h_{|B_2|}')\big)$$

where the maps $g_i$ and $h_i$ are those associated to $(T_1, B_1)$ and the maps $g_i'$ and $h_i'$

associated to $(T_2, B_2)$. In the above notation, we also have

$$(y_{T_2}, f_{T_2}) = \lambda_{MS^+}\big((T_2', \sigma_2', \tau_2); (x_i, g_i'), \dots, (x_{i+l-1}, g_{i+l-1}'), (1, h_1'), \dots, (1, h_{|B_2|}')\big).$$

Note that one can change the $Mon^+(I, \partial I)$ component of an element of $MS^+$ by doing a $\circ_1$–composition in the operad. In particular,

$$(y_{T_2}, g_i) = (1, g_i \circ f_{T_2}^{-1}) \circ_1 (y_{T_2}, f_{T_2})$$

in $MS^+$ ane we can rewrite the left hand side of (A.5.10) as the first component of the $MS^+$–composition

$$\begin{aligned}\lambda_{MS^+}\big((T_1', \sigma_1', \tau_1) \circ_i (\bar{T}_2', \bar{\sigma}_2', \tau_2); (x_1, g_1), \dots, (x_{i-1}, g_{i-1}),\\ (x_i, g_i'), \dots, (x_{i+l-1}, g_{i+l-1}'), (1, h_1'), \dots, (1, h_{|B_2|}'), (1, g_i \circ f_{T_2}^{-1}),\\ (x_{i+l}, g_{i+l}), \dots, (x_k, g_k), (1, h_1), \dots, (1, h_{|B_1|})\big)\end{aligned}$$

where $\bar{T}_2'$ has an extra vertex at the bottom of the tree to encode the change of $Mon^+(I, \partial I)$–component for the $T_2$ composition. If $T_2$ is large, this extra vertex corresponds exactly to the extra bracket $T_2$ arising in the $B\mathcal{O}$–composition, and one checks that the corresponding scaling map $h$ defined by the formula (A.5.8) is precisely the map $g_i \circ f_{T_2}^{-1}$. As the other labels of the vertices of the composed tree agree with those of the right hand side, we see that we recover the right hand side of (A.5.10). If $T_2$ is not large, then there is no such additional bracket in the $B\mathcal{O}$–composition, but in this case $f_{T_2} = \mathrm{id}$ and the left and right hand side agree directly.

Recall from Remark A.3.10 that we may consider $B\mathcal{O}$ as the geometric realization of the simplicial operad of bracket trees. Then the above definition of $\lambda$ on bracketings of weight 1 defines the action of the vertices of $B\mathcal{O}$. We finally extend this action to all bracketings of a tree $T$ by linear interpolation on the rescaling maps $g_i$. For a fixed tree $T$ and a point $((B_0 \subset \cdots \subset B_r), \underline{t})$ in the realization of the poset $\mathcal{B}(T)$, let $g_i(T, B_j)$ denote the definition of the rescaling map $g_i$ with respect to the bracketing $B_j$ on $T$ in (A.5.7), and likewise for the maps $f_j$ in (A.5.8). We set

$$g_i = t_0 g_i(T, B_0) + \dots + t_r g_i(T, B_r).$$

This is well-defined as $Mon^+(I, \partial I)$ is convex. Also note that this is continuous in $B\mathcal{O}$ as going to the $l$th face of the simplex $(B_0 \subset \cdots \subset B_r)$ corresponds to $t_l$ going to 0, that is, dropping the bracket $B_l$.Then we define $(y_T, f_T)$ and $\lambda(T, \sigma, \tau, B, t) :=$

$\mathcal{R}(y_T, f_T)$ as in (A.5.9) but with this definition of $g_i$ instead.

This defines the action of $B\mathcal{O}$ on $\mathsf{cact}^1$. It is compatible under composition because the composition in $B\mathcal{O}$ is the realization of the composition in the poset operad, and we have already checked the compatibility under composition there.

□

Given that normalized cacti, together with the cactus composition (A.5.1), forms a $B\mathcal{O}$-algebra we can now use the rectification results from Proposition A.4.10 to define an $\infty$-operad.

**Corollary A.5.13.** *Normalized cacti define dendroidal spaces of the following two flavors:*

*(i) There exists a reduced homotopy dendroidal space $X \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$, satisfying the strict Segal condition, such that $X(C_n) = \mathsf{cact}^1(n)$ and with value on the inner face maps $\partial_e$ given by the $\mathsf{cact}^1$–composition.*

*(ii) There exists a weakly reduced dendroidal space $Y \in \mathcal{S}^{\Omega^{op}}$, satisfying the weak Segal condition (in the sense of Proposition A.4.10), such that $Y(C_n) \simeq \mathsf{cact}^1(n)$ and with value on the inner face maps $\partial_e$ homotopic to the $\mathsf{cact}^1$–composition.*

*Proof.* Theorem A.5.12 shows that $\mathsf{cact}^1$ is a $B\mathcal{O}$-algebra. Applying the construction from Theorem A.4.8, we define a homotopy dendroidal space $X := \Phi(\mathsf{cact}^1) \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$. By construction, $\Phi(\mathsf{cact}^1)(C_n) = \mathsf{cact}^1(n)$, and by the theorem it is a reduced homotopy dendroidal space satisfying the strict Segal condition. The evaluation of $\Phi(\mathsf{cact}^1)$ on an inner edge is the $\circ_i$ composition, as encoded by the $B\mathcal{O}$-structure, which in the present case is the $\mathsf{cact}^1$–composition by Theorem A.5.12. This proves (i) in the statement.

For (ii), we set $Y := p_!X = p_!\Phi(\mathsf{cact}^1) \in \mathcal{S}^{\Omega^{op}}$ to be the rectification of $X$, as constructed in Proposition A.4.10. By Lemma A.4.9, $Y(C_w) = p_!\Phi(\mathsf{cact}^1)(C_w) \simeq \Phi(\mathsf{cact}^1)(C_w) = \mathsf{cact}^1(|w|)$ and the value of $Y$ on inner face maps is identifies under these homotopy equivalences with the value of $X$ on inner face maps, and hence identifies with the $\mathsf{cact}^1$–composition. By the proposition, $Y$ is weakly reduced and satisfies the weak Segal condition. □

## A.6 Relation between the operads $B\mathcal{O}$ and $W\mathcal{O}$

The Boardman-Vogt $W$-construction is a construction on operads with the property that, for any topological operad $\mathcal{P}$, algebras over $W\mathcal{P}$ are "up-to-homotopy"

or "weak" $\mathcal{P}$-algebras. A lax operad [15] is an algebra over the operad $W\mathcal{O}$, the Boardman-Vogt $W$–construction applied to the operad of operads $\mathcal{O}$ (Definition A.2.8), and is a notion of a "weak" or "infinity" operad. It is known that there exists a zig-zag of Quillen equivalences between the category of $W\mathcal{O}$-algebras and the category of reduced dendroidal spaces by, for example, combining Theorem 4.1 of [5] with either Theorem 1.1 of [7] or a restriction of Theorem 8.15 of [19].

Here we show how the operad $B\mathcal{O}$ can be identified with a variant $W_0$ of the $W$-construction of the operad $\mathcal{O}$ of operads (see Theorem A.6.4). From this, it will follow that $B\mathcal{O}$–algebras are lax operads that are strictly symmetric and with a strict identity (see Example A.6.2). We start by recalling the $W$–construction.

### A.6.1 The $W$-Construction

The Boardman-Vogt $W$-construction is an enlargement of the free operad construction. Given an operad $\mathcal{P}$, there are canonical morphisms of topological operads

$$F\mathcal{P} \hookrightarrow W\mathcal{P} \xrightarrow{\sim} \mathcal{P},$$

where the map $p : W\mathcal{P} \to \mathcal{P}$ is a surjective homotopy equivalence. Algebras for $W\mathcal{P}$ are up-to-homotopy $\mathcal{P}$-algebras. We briefly recall the construction here and refer the reader to [15, Section 17] or [5, Section 3] for full details.

**Definition A.6.1.** Let $\mathcal{P}$ be a $\mathfrak{C}$–colored (discrete or topological) operad. The operad $W\mathcal{P}$ is a topological operad with the same set of colors $\mathfrak{C}$, built from the free operad $F(\mathcal{P})$ (Definition A.2.3) by adding length in $[0, 1]$ to the internal edges of the trees that define the elements of $F(\mathcal{P})$. More precisely, for each list of colors $c; c_1, \ldots, c_k$ in $\mathfrak{C}$, we have

$$W\mathcal{P}(c; c_1, \ldots, c_k) = \Big( \coprod_{(\mathbb{T}, f, \lambda)} \big([0,1]^{|iE(\mathbb{T})|} \times \prod_{v \in V(\mathbb{T})} \mathcal{P}(out(v); in(v))\big)\Big) / \sim$$

where the disjoint union, as for the free operad, runs over the isomorphim classes of leaf-labeled $\mathfrak{C}$–colored planar trees

$$(\mathbb{T}, f\colon E(\mathbb{T}) \to \mathfrak{C}, \lambda\colon \{1, \ldots, k\} \to L(\mathbb{T}))$$

with $k$ leaves such that $f(\lambda(i)) = c_i$, $f(R(\mathbb{T})) = c$. The equivalence relation is generated by the relation $(*)$ in Definition A.2.3 in addition to the following additional relations that capture "weak" operadic composition and units:

1. any tree with an internal edge of length of zero is identified with the tree where that edge has been collapsed and the operations labelling its end vertices composed;

2. any tree that has a vertex with only one input and one output, both colored by $c \in \mathfrak{C}$, labeled by the identity in $\iota_c \in \mathcal{P}(c;c)$, is identified with the tree where that vertex is deleted. The resulting new edge, if internal, has length the maximum length of the two original internal edges connected to the deleted vertex.

See [15, p 75] for a pictorial version of these relations. The symmetric group acts on $W\mathcal{P}$ by relabeling the leaves, as for the free operad. Composition is by grafting, giving length 1 to the newly created internal edge.

We will denote elements of $W\mathcal{P}$ by $(\mathbb{T}, f, \lambda, s, p)$, where $\mathbb{T}$ is a planar tree, $f : E(\mathbb{T}) \to \mathfrak{C}$ is the map coloring its edges, $\lambda : \{1, \ldots, k\} \to L(\mathbb{T})$ is the bijection labeling its leaves, $s \in [0,1]^{|iE(\mathbb{T})|}$ is a collection of weights, and $p = (p_v)_{v \in V(\mathbb{T})}$ is a labeling of the vertices by operations in $\mathcal{P}$. An example is shown in Figure A.24. There is a canonical projection map $\pi : W\mathcal{P} \to \mathcal{P}$ defined by sending all the edge lengths to 0 and composing the operations of $\mathcal{P}$ as dictated by the trees.

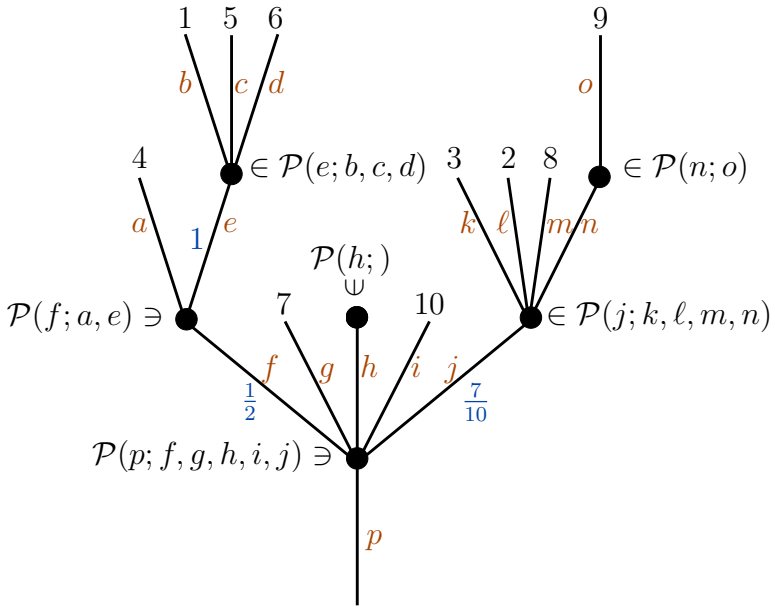



Figure A.24: Example of an element of $W\mathcal{P}(p;a,b,c,d,g,i,k,\ell,m,o)$.

### A.6.2 A variant on the $W$-construction

Given a (discrete or topological) $\mathfrak{C}$-coloured operad $\mathcal{P}$, the topological operad $W_0\mathcal{P}$ is defined as the quotient of $W\mathcal{P}$ by replacing relation (2) in Definition A.6.1 by the following stronger relations for arity one vertices, as well as a version for arity zero vertices:

(2′) any tree that has a vertex $v$ with only one input and one output both colored by $c$, adjacent to at least one other vertex $w$, with $v$ labeled by <u>any</u> element $\mathcal{P}(c;c)$, is identified with the tree where the vertex $v$ is deleted, and the label of $v$ and $w$ are composed in $\mathcal{P}$ (Figure A.25).

If the resulting new edge is internal, then its length is the maximum length of the two original (then necessarily internal) edges adjacent to $v$.

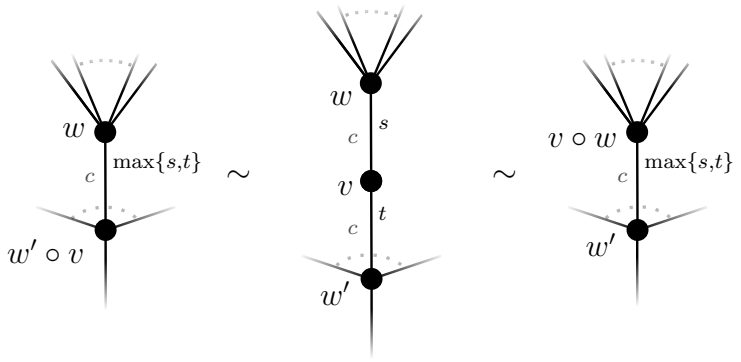



Figure A.25: Local representation of the relation (2′) on a tree.

(3') any tree that has a vertex $v$ with no input, adjacent to another vertex $w$, with $v$ labeled by <u>any</u> element of $\mathcal{P}(c;\emptyset)$, is identified with the tree where the vertex $v$ and the edge between $v$ and $w$ are deleted, and the labels of $v$ and $w$ are composed in $\mathcal{P}$ (Figure A.26).

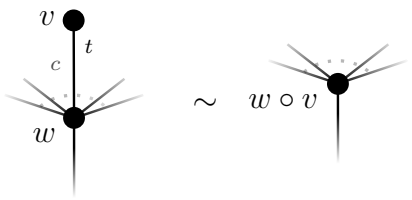



Figure A.26: Local representation of the relation (3') on a tree.

So a $W_0\mathcal{P}$–algebra is a weak $\mathcal{P}$–algebra ($W\mathcal{P}$–algebra) for which the nullary and unary operation are strict. And in particular, one has that $W_0\mathcal{P}(c;c) = \mathcal{P}(c;c)$ and $W_0\mathcal{P}(c;\emptyset) = \mathcal{P}(c;\emptyset)$ for any color $c$. Also, one can always choose representatives of elements of $W_0\mathcal{P}$ using trees with no valence 0 or 1 vertices (unless it only has 0 or 1 vertex). In a tree that defines an element of $W_0\mathcal{P}$, an arity one vertex lying in between two other vertices can be slid up or down to either of its neighboring vertices, composing its label with that of the chosen vertex, while an arity zero vertex can be "pushed down" to the vertex it is attached to.

**Example A.6.2.** The example relevant to us here is when we set $\mathcal{P} = \mathcal{O}$ is the operad of operads. In this case, $\mathfrak{C} = \mathbb{N}$ is the natural numbers and an $\mathcal{O}$-algebra is a (monochrome) operad. The nullary operations in $\mathcal{O}(1;\emptyset)$ encode the identity operation in the $\mathcal{O}$–algebra, while the unary operations in $\mathcal{O}(n;n)$ encode the action of the symmetric groups. It follows that a $W_0\mathcal{O}$–algebra is a strictly symmetric weak operad with a strict identity.

By construction, the canonical projection $p : W\mathcal{P} \to \mathcal{P}$ factors through the quotient map $q : W\mathcal{P} \to W_0\mathcal{P}$. Moreover, both $W\mathcal{P}$ and $W_0\mathcal{P}$ are homotopy equivalent to $\mathcal{P}$:

**Proposition A.6.3.** *There are operad maps $W\mathcal{P} \to W_0\mathcal{P} \to \mathcal{P}$, inducing homotopy equivalences*

$$W\mathcal{P}(c;c_1,\dots,c_n) \xrightarrow[\sim]{q} W_0\mathcal{P}(c;c_1,\dots,c_n) \xrightarrow[\sim]{p_0} \mathcal{P}(c;c_1,\dots,c_n)$$

*for each $n \geq 0$ and each $c;c_1,\dots,c_n$ in $\mathfrak{C}$.*

*Proof.* For each $n \geq 0$ and $c;c_1,\dots,c_n$ the map

$$q : W\mathcal{P}(c;c_1,\dots,c_n) \to W_0\mathcal{P}(c;c_1,\dots,c_n)$$

is the projection on to the quotient. It is an operad map because if elements of $W\mathcal{P}$ are equivalent in $W_0\mathcal{P}$ before being composed, they are necessarily also equivalent in $W_0\mathcal{P}$ after composition. The map $p_0 : W_0\mathcal{P} \to \mathcal{P}$ contracts the remaining edges in the trees of $W_0\mathcal{P}$ by sending the lengths to 0 (and composing the operations in $\mathcal{P}$). This map is well-defined, as it is compatible with the relations (2') and (3'), and respects the operad structure. These maps induce homotopy equivalences, with homotopy inverses given by including $\mathcal{P}(c_1,\dots,c_n;c)$ as labelled corollas in $W_0\mathcal{P}(c;c_1,\dots,c_n)$ or $W\mathcal{P}(c;c_1,\dots,c_n)$. □

### A.6.3 $B\mathcal{O}$-algebras are strictly symmetric lax operads

In this section we prove that there is an isomorphism of topological operads $B\mathcal{O} \cong W_0\mathcal{O}$.

**Theorem A.6.4.** *The operads $W_0\mathcal{O}$ and $B\mathcal{O}$ are isomorphic.*

The combination of Theorem A.6.4 with Example A.6.2, describes $B\mathcal{O}$–algebras as strictly symmetric lax operads with strict identity. Moreover, the isomorphism $B\mathcal{O} \cong W_0\mathcal{O}$ allows us to observe that any $B\mathcal{O}$–algebra will receive a canonical $W\mathcal{O}$–structure via the quotient map $q : W\mathcal{O} \longrightarrow W_0\mathcal{O}$. In fact, Theorem A.6.4 together with Proposition A.6.3 gives homotopy equivalences

$$W\mathcal{O} \xrightarrow{\simeq} B\mathcal{O} \xrightarrow{\simeq} \mathcal{O},$$

where $\mathcal{O} \cong \pi_0 W\mathcal{O} \cong \pi_0 B\mathcal{O}$ and the latter equivalence is the one already considered in Remark A.3.10. Each of the topological operads $W\mathcal{O}$, $B\mathcal{O}$ and $\mathcal{O}$ are admissible in the sense of [5, Theorem 2.1] and thus the natural projection maps induce Quillen adjunctions

$$W\mathcal{O}-\mathrm{Alg} \rightleftarrows B\mathcal{O}-\mathrm{Alg} \rightleftarrows \mathcal{O}-\mathrm{Alg}.$$

We expect that these adjunctions induce Quillen equivalences, though the operad $B\mathcal{O}$ is not $\Sigma$-cofibrant so we can not immediately apply [5, Theorem 4.1]. We nonetheless have the following:

**Corollary A.6.5.** *Every $B\mathcal{O}$-algebra is homotopy equivalent to a strict operad.*

*Proof.* Every $B\mathcal{O}$-algebra is canonically a $W\mathcal{O}$-algebra via the quotient map $q : W\mathcal{O} \to B\mathcal{O}$, and every $W\mathcal{O}$-algebra is homotopic to a strict operad, i.e. an $\mathcal{O}$-algebra, using e.g. Theorem 4.1 of [5] that gives a Quillen equivalence between the categories $W\mathcal{O}-\mathrm{Alg}$ and $\mathcal{O}-\mathrm{Alg}$ (using that the operads $W\mathcal{O}$ and $\mathcal{O}$ are well-pointed, $\Sigma$-cofibrant admissible operads).

Given a $B\mathcal{O}$-algebra $\mathcal{P}$, one can alternatively directly construct such a strict operad $\mathcal{P}'$ homotopic to $\mathcal{P}$ using the following double bar construction:

$$\mathcal{P} \xleftarrow{\simeq} B(B\mathcal{O}, B\mathcal{O}, \mathcal{P}) \xrightarrow{\simeq} B(\mathcal{O}, B\mathcal{O}, \mathcal{P}) =: \mathcal{P}',$$

where $\mathcal{P}'$ is an $\mathcal{O}$-algebra and hence a strict operad. In the above zig-zag, we use the notation of e.g. [**?**, Const 9.6 and Thm 9.10], identifying an operad with its associated monad. The leftmost map is an equivalence by properties of the bar construction and the rightmost because of the equivalence $B\mathcal{O} \xrightarrow{\simeq} \mathcal{O}$ of Remark A.3.10. □

Combining Theorem A.4.8 with Theorem A.6.4, we can also relate $B\mathcal{O}$-algebras with reduced, strictly Segal $\widetilde{\Omega}_0^{op}$-diagrams.

**Corollary A.6.6.** *There exist isomorphisms of categories*

$$W_0O\text{−Alg}_{\mathcal{S}} \cong (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}.$$

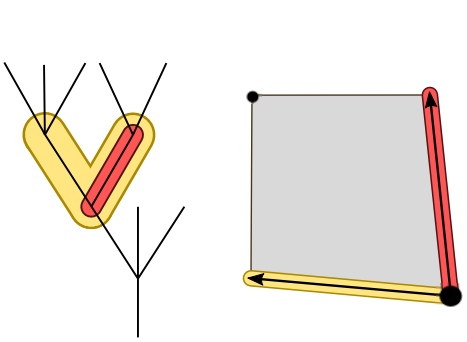
The proof of Theorem A.6.4 will be given in Section A.6.4. Though not saying this explicitly, the proof uses the natural association of a bracketing to a *clustering tree*, which is described for instance in [83, Definition 2.7].

Since the $W$-construction is built out of cubes, to prepare for the proof, we start by giving an alternative description of $B\mathcal{O}$ in terms of cubes as well.

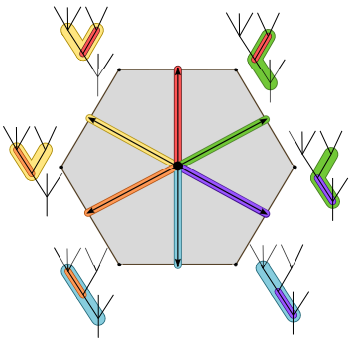
**Definition A.6.7.** We can define a *weighted bracketing* of a tree $T$ to be a pair $(B, t)$ with bracketing $B = \{S_j\}_{j\in J}$ of $T$ and $t \in [0,1]^J$. The $j$th coordinate $t_j \in t$ is the *weight* of $S_j$. The addition of weights associates to each bracketing a cube $[0,1]^{|B|}$. These cubes fit together to form a space:

$$\mathbb{B}(T) = \coprod_{B\in\mathcal{B}(T)} [0,1]^{|B|}/_{\sim}$$

where the equivalence relation is by identifying any bracketings with weights that only differ by a bracket of weight 0 (see Figure A.27(b)).



(a) A bracketing $B$ of $T$ with 2 brackets and its associated cube $[0,1]^{|B|}$ in $\mathbb{B}(T)$.



(b) All bracketings of $T$ and their associated cubes assembled to make $\mathbb{B}(T)$ a hexagon.

Figure A.27: A tree $T$ and its corresponding space of bracketings $\mathbb{B}(T)$.

Recall from Definition A.3.2 the poset $\mathcal{B}(T)$ of bracketings of a tree $T$ under the inclusion relation.

**Lemma A.6.8.** *Let $T$ be a tree. There is a homeomorphism $|\mathcal{B}(T)| \cong \mathbb{B}(T)$, between the realization of the nerve of the poset $\mathcal{B}(T)$ and the cubical space $\mathbb{B}(T)$.*

*Proof.* We consider the topological $k$–simplex as the space

$$\Delta^k = \{(s_1, \ldots, s_k) \in \mathbb{R}^k \colon 1 = s_0 \geq s_1 \geq \cdots \geq s_k \geq 0\}.$$

Fix a tree $T$ and let $\sigma$ denote a $k$–simplex $B_0 \subseteq \cdots \subseteq B_k$ of the nerve of the poset $\mathcal{B}(T)$. To each $\sigma$ we associate a map

$$\chi_\sigma : \Delta^k \longrightarrow \mathbb{B}(T)$$

where $\chi_\sigma(s_1, \ldots, s_k)$ is the weighted bracketing of $T$ in which all trees of $B_0$ have weight $1 = s_0$ and all trees of $B_i \backslash B_{i-1}$ have weight $s_i$ for $i \geq 1$.

The maps $\chi_\sigma$ assemble into a continuous map

$$\chi : |\mathcal{B}(T)| = \Big( \coprod_{k \geq 0} \mathcal{B}(T)_k \times \Delta^k \Big) /_\sim \ \longrightarrow \ \mathbb{B}(T) = \coprod_{B \in \mathcal{B}(T)} [0,1]^{|B|} /_\sim .$$

This map is a homeomorphism with inverse defined by mapping a cube $[0,1]^{|B|}$ in $\mathbb{B}(T)$ to the realization of the sub-poset $\mathcal{B}_{\leq B}$, which is a cube whose dimension is the cardinality $|B|$ of the bracketing. Explicitly, given an element $(B, t) \in \mathbb{B}(T)$ with $B = (B_0, \ldots, B_k)$, we order the coordinates of $t = (t_1, \ldots, t_k)$ so that they are in decreasing order

$$1 = t_{\sigma(1)} = \cdots = t_{\sigma(r_1)} > t_{\sigma(r_1+1)} = \cdots = t_{\sigma(r_1+r_2)} > \\ \cdots > t_{\sigma(r_1+\cdots+r_{l+1}+1)} = \cdots = t_{\sigma(r_1+\cdots+r_{l+2})} = 0.$$

This defines an $l$–simplex $\bar{B}_0 \subset \bar{B}_1 \subset \cdots \subset \bar{B}_l$ by setting

$$\bar{B}_i = B_{\sigma(1)} \cup \cdots \cup B_{\sigma(r_1+\cdots+r_{i+1})}. \qquad \square$$

## A.6.4 Proof of Theorem A.6.4

In order to prove $B\mathcal{O} \cong W_0\mathcal{O}$, we first recall some definitions. Recall that elements $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \ldots, m_k)$ are represented by a planar tree $\mathbb{T}$ with $k$ leaves ordered by the bijection $\lambda : \{1, \ldots, k\} \to L(\mathbb{T})$ and with an edge colouring $f : E(\mathbb{T}) \to \mathbb{N}$ that, in particular, colours the leaves by $m_1, \ldots, m_k$. In addition, $\mathbb{T}$ is

equipped with a collection of lengths $s \in [0,1]^{|iE(\mathbb{T})|}$, and a decoration of the vertices $p = (p_v)_{v \in V(\mathbb{T})}$ by operations by $p_v$ in $\mathcal{O}(out(v); in(v))$.

We call a representative $(\mathbb{T}, f, \lambda, s, p)$ reduced if the tree $\mathbb{T}$ has no vertices of arity zero or one, unless such a vertex cannot be removed using the equivalence relation in $W_0\mathcal{O}$, i.e. if $\mathbb{T}$ is the corolla $C_0$ or $C_1$. In particular, every element of $W_0\mathcal{O}$ has a reduced representative, which in general is not unique. It greatly simplifies the proof of Lemma A.6.9 to work with reduced representative.

For a given tree $\mathbb{T}$, and vertices $v, w \in V(\mathbb{T})$, we say that $w$ is *above* $v$ if the unique shortest path between $w$ and the root of the tree goes through $v$. In this case $v$ is *below* $w$. Every other vertex of $\mathbb{T}$ is above the *root vertex* $v_0$ whose outgoing edge is the root of $\mathbb{T}$.

**Lemma A.6.9.** *There is a map of topological operads* $\Psi : W_0\mathcal{O} \to B\mathcal{O}$.

The map $\Psi$ is illustrated in Figure A.28.

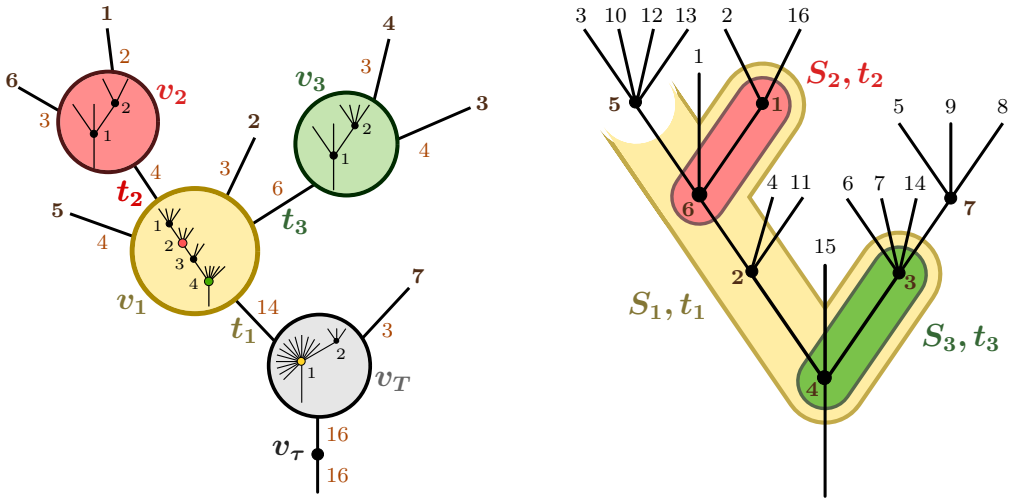



Figure A.28: Element of $W_0\mathcal{O}$ and corresponding element of $B\mathcal{O}(16; 2, 3, 4, 3, 4, 3, 3)$.

*Proof.* Given a reduced element $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \ldots, m_k)$, we construct

$$\Psi(\mathbb{T}, f, \lambda, s, p) = (T, \sigma, \tau, B, t) \in B\mathcal{O}(n; m_1, \ldots, m_k),$$

where $(B, t)$ is a weighted bracketing on the labelled tree

$$(T, \sigma, \tau) = p_0(\mathbb{T}, f, \lambda, s, p) \in \mathcal{O}(n; m_1, \ldots, m_k)$$

that is the image of $(\mathbb{T}, f, \lambda, s, p)$ under the canonical projection $p_0 \colon W_0\mathcal{O} \to \mathcal{O}$.

The bracketing $B$ is constructed from the set of vertices of $\mathbb{T}$. If $\mathbb{T}$ has at most one vertex, then set $B = \emptyset$ to be the trivial bracketing, in which case there are no weights to chose so $t$ is the empty map.

Otherwise, since $(\mathbb{T}, f, \lambda, s, p)$ is reduced, and $\mathbb{T}$ is not a corolla, all its vertices have arity $\geq 2$. Let $v_0$ be the root vertex of $\mathbb{T}$. For each $v \in V(\mathbb{T})\backslash\{v_0\}$, let

$$(S_v, \sigma_v, \tau_v) = p_0(\mathbb{T}_v, f|_{\mathbb{T}_v}, \lambda|_{\mathbb{T}_v}, p|_{\mathbb{T}_v}),$$

where $\mathbb{T}_v$ is the subtree of $\mathbb{T}$ with $v$ as its root vertex, and containing all the vertices above $v$. Observe, in particular, that, since $v \neq v_0$, the outgoing edge $e_v$ of $v$ – that is the root of $S_v$ – is internal in $T$. Since the vertices of $\mathbb{T}$ have arity at least 2, each $S_v$ is a large proper subtree of $T$, and because composition in $\mathcal{O}$ is by substitution,

$$B = \{S_v \ : \ v \in V(\mathbb{T})\backslash\{v_0\}\}$$

is a collection of nested subtrees, and hence a bracketing.

To define the weight function $t$ of $B$, we associate, to each $S_v$ the weight $t_v = s(e_v)$, the length of $e_v \in iE(\mathbb{T})$. This completes the definition of $\Psi(\mathbb{T}, f, \lambda, s, p)$.

We need to check that the defined bracketing is independent of our choice of (reduced) representative $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \ldots, m_k)$, and continuous. In particular, we must check that it is compatible with the relations (1), (2') and (3') in Definition A.6.1 and Section A.6.2.

To prove that $\Psi$ is well defined with respect to relation (1), and hence also continuous, let $s_j$ be the length of an internal edge of $\mathbb{T}$ with end vertices $v, w$, where $v$ is above $w$. Then if $s_j$ goes to 0 in $W_0\mathcal{O}$, the vertices $v, w$ are identified and their labels are composed in $\mathcal{O}$. Applying $\Psi$, this will precisely have the effect of taking the weight of the bracketing $S_v$ to 0, which is equivalent to simply forgetting the bracketing $S_v$ in $B\mathcal{O}$.

Relation (2') allows that a vertex $v$ with only 1 input in $\mathbb{T}$, labeled by a permutation $\alpha \in \mathcal{O}(n; n) \cong \Sigma_n$, to be composed to either of the vertices it shares an edge with. So suppose $\mathbb{T}$ is the reduction of a tree $\widetilde{\mathbb{T}}$ with an arity one vertex $v$ attached to two vertices $w$ and $w'$, with $w'$ below $w$. We may assume that $w$ and $w'$ both have arity at least two. We let $\mathbb{T}$ be the tree obtained from $\widetilde{\mathbb{T}}$ by collapsing the edge between $w$ and $v$ and let $\mathbb{T}'$ be the tree obtained from $\widetilde{\mathbb{T}}$ by collapsing the edge between $v$ and $w'$. We need to check that the brackets $S_w$ and $S_{w'}$ are the same if computed using the representative $(\mathbb{T}, f, \lambda, s, p)$ associated to $\mathbb{T}$ or

$(\mathbb{T}', f', \lambda, s', p')$ associated to $\mathbb{T}'$. This is immediate for the bracket $S_{w'}$ because $w'$ is below $v$ and thus $p_0(\mathbb{T}_{w'}, f|_{\mathbb{T}_{w'}}, \lambda|_{\mathbb{T}_{w'}}, p|_{\mathbb{T}_{w'}}) = p_0(\mathbb{T}'_{w'}, f'|_{\mathbb{T}'_{w'}}, \lambda|_{\mathbb{T}'_{w'}}, p'|_{\mathbb{T}'_{w'}})$. For the vertex $w$, the two representatives in general do not have the same image under $p_0$, but if $p_0(\mathbb{T}'_w, f'|_{\mathbb{T}'_w}, \lambda|_{\mathbb{T}'_w}, p'|_{\mathbb{T}'_w}) = (S'_w, \sigma'_w, \tau'_w)$, we still have that $S'_w = S_w$. In fact, only $\tau'_w$ might differ from $\tau_w$ as the vertex $v$ is a permutation $\alpha \in \mathcal{O}(n;n) = \Sigma_n$ that acts on a labeling $p \in \mathcal{O}(n; k_1, \ldots, k_l)$ by permuting the leaves of the labeled tree representing $p$.

For relation (3') in the definition of $W_0$, the relation gives a unique way to reduce a tree if an arity zero vertex is attached to another vertex, so the representative with no arity 0 vertices is unique and nothing needs to be checked.

Finally, we check that $\Psi$ is a map of operads. Consider a composition $(\mathbb{T}_1, f_1, \lambda_1, s_1, p_1) \circ_i (\mathbb{T}_2, f_2, \lambda_2, s_2, p_2)$ of reduced representatives in $W_0\mathcal{O}$, and let $\Psi(\mathbb{T}_j, f_j, \lambda_j, s_j, p_j) = (T_j, \sigma_j, \tau_j, B_j, t_j)$ for $j = 1, 2$. Composition in $W_0\mathcal{O}$ is induced by grafting a tree $\mathbb{T}_2$ onto the $i$th leaf of $\mathbb{T}_1$, creating a new internal edge of length 1. If $\mathbb{T}_2$ has at least one vertex of arity 2, this corresponds exactly under $\Psi$ to adding a new bracket $T_2$ of weight 1 in the composed tree $T_1 \bullet_{\sigma_1(i),\tau_2} T_2$ where the composition here is by insertion. If not, then, since $(\mathbb{T}_2, f_2, \lambda_2, s_2, p_2)$ is reduced, $\mathbb{T}_2$ has either no vertices or a single arity 1 vertex, so $T_2$ is either the exceptional tree $\eta$ or a corolla $C_n$. In each case, the newly added edge in the composed tree $\mathbb{T}_1 \circ_i \mathbb{T}_2$ will be collapsed when going to a reduced tree, corresponding under $\Psi$ to a composition in $B\mathcal{O}$ where no extra bracket is added. This finishes the proof. □

**Lemma A.6.10.** *For every $(n; m_1, \ldots, m_k)$ the map $\Psi : W_0\mathcal{O}(n; m_1, \ldots, m_k) \longrightarrow B\mathcal{O}(n; m_1, \ldots, m_k)$ is a bijection.*

*Proof.* We start by checking that $\Psi$ is surjective. So let $(T, \sigma, \tau, B, t)$ of $B\mathcal{O}(n; m_1, \ldots, m_k)$ with $B = \{S_j\}_{j \in J}$ and $t \in [0,1]^J$. We may always choose a representative where all brackets have non-zero weight, so we assume that $t_j \neq 0$ for any $j \in J$. We will construct an element $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \ldots, m_k)$ in the preimage of $(T, \sigma, \tau, B, t)$.

If $B = \emptyset$ is the empty bracketing then define $\mathbb{T}$ to be the corolla with $k$ leaves, with $f$ coloring its leaves $m_1, \ldots, m_k$ in the ordering given by $\lambda$, and the root by $n$ and $p$ labeling the unique vertex by $(T, \sigma, \tau)$. The weights $s$ are trivial in this case. By definition, $\Psi$ takes this element to $(T, \sigma, \tau, \emptyset, 0)$ as required.

We now assume that $B = \{S_j\}_{j \in J}$ is non-empty. To encode the leaf labelling $\tau$ on $T$, it is convenient to choose a non-reduced representative of its preimage, using a tree $\mathbb{T}$ with one valence 1 vertex at its root. We define $\mathbb{T}$ as follows: we set

$V(\mathbb{T}) = \{v_\tau, v_T\} \cup \{v_j\}_{j\in J}$, where the vertex $v_j$ corresponds to the bracket $S_j \in B$, $v_T$ corresponds to an additional "trivial bracket" $S_T := T$, and $v_\tau$ will be associated to the permutation $\tau$. To construct the tree, we set $v_i$ above $v_j$ if $S_i \subset S_j$, connecting the two vertices by an edge $e_i$ if there is no $k \in J$ such that $S_i \subsetneq S_k \subsetneq S_j$, where we allow $S_j = S_T$. This edge $e_i$ is colored by the number $|L(S_i)|$ of leaves of the smaller tree $S_i$ and we define its length by setting $s_i = t_i$ is the weight of the corresponding bracket. The nesting condition on the brackets implies that no cycles are formed this way. We also connect $v_T$ and $v_\tau$ by an edge of length 1, colored by $n = |L(T)|$, which is also the color of the root of the tree.

Finally for each vertex $v$ of $T$, we attach a leaf $l_v$ to the vertex $v_i \in V(\mathbb{T})$ if $S_i$ is the smallest tree of the bracketing containing $v$, attaching it to $v_T$ if $v$ is contained in no bracket. This leaf is colored by the arity of $v$ in $T$. This defines the tree $\mathbb{T}$, with edge lengths $s$ and edge coloring $f$.

We pick some planar structure for $\mathbb{T}$. (Recall that elements of $W_0\mathcal{O}$ are only defined up to non-planar isomorphism, which is why there is some freedom here.) Note that the leaves of $\mathbb{T}$ correspond exactly to the vertices of $T$. The ordering $\lambda : \{1, \dots, k\} \to L(\mathbb{T})$ is determined by $\sigma$ and this identification. This defines the tuple $(\mathbb{T}, f, \lambda, s)$.

All that remains is to define the decoration $p$ of the vertices of $\mathbb{T}$ by elements of $\mathcal{O}$. We need to have that $(T, \sigma, \tau)$ is given by the composition of the elements of the vertices of $\mathbb{T}$ so to determine the decorations in $\mathbb{T}$, what we need is to "undo" the compositions in $T$ marked by the bracketings.

Let $v_j \in V(\mathbb{T})$. We define $p(v_j)$ to be the element $(S_j/\sim, \sigma_j, \tau_j) \in \mathcal{O}(out(v_j); in(v_j))$ where $S_j/\sim$ is the planar tree $S_j$ with each subtree $S_i \subsetneq S_j$ collapsed to a corolla with the same set of leaves, $\sigma_j$ orders the vertices according to the above chosen planar ordering of $\mathbb{T}$, where we note that the incoming edges of $v_j$ correspond precisely to the vertices of $S_j/\sim$, and $\tau_j$ labels the leaves of $S_j/\sim$, which are also the leaves of $S_j$, in the order given by the planar embedding of $T$. (Here it is important that the chosen planar structure of $\mathbb{T}$ is compatible with the chosen order $\sigma_j$ of $V(S_j/\sim)$. On the other hand, the chosen order $\tau_j$ of $L(S_j/\sim)$ is not important, as we will fix it below using the vertex $v_\tau$.) This determines $p$ uniquely on all vertices $\{v_j\}_{j\in J} \cup \{v_T\}$. Finally, the vertex $v_\tau$ is labeled by the permutation $\tau \in \Sigma_n$, considered as an element of $\mathcal{O}(n; n)$.

This finishes the construction of $(\mathbb{T}, f, \lambda, s, p)$. To compute its image under $\Psi$, we have to pass to a reduced representative, which means collapsing the edge between $v_\tau$ and $v_T$ and composing their labeling. (The length of that edge is forgotten.) We

have that $\Psi(\mathbb{T}, f, \lambda, s, p) = (T, \sigma, \tau, B, t)$, by our choice of $p$ for the tree $T$ and its leaf-labeling $\tau$, our choice of $\lambda$ for the ordering $\sigma$ of the vertices, our choice of vertices of $\mathbb{T}$ for $B$, and with a direct correspondence between the length $s_i$ of the edge $e_i$ and the weight $t_i$ of $B_i$.

To finish the proof, we check that $\Psi$ is injective. We will check that, up to the equivalence relations defining $W_0\mathcal{O}$, there is a unique reduced $(\mathbb{T}', f', \lambda', s', p')$ in the preimage of $(T, \sigma, \tau, B, t)$. Note that the number of vertices of such a reduced representative is determined by the tree $T$ and the cardinality of $B$. We consider first the cases where $\mathbb{T}'$ has 0 or 1 vertex.

If $\mathbb{T}'$ has no vertices, then $\mathbb{T}' = \eta$ representing the identity element in $\mathcal{O}(1;1)$, $B = \emptyset$, and, up to the equivalence relations of $W_0\mathcal{O}$, there is only one possibility for $(\mathbb{T}', f', \lambda', s', p')$.

Suppose now that $\mathbb{T}' = C_k$ has exactly one vertex of arity $k$. The leaves of $\mathbb{T}'$ are in one-to-on correspondence with the vertices of $T$, with $\lambda'$ ordering its leaves, and $f'$ coloring them $m_1, \ldots, m_k, n$, with $m_i$ the color of $\lambda'(i)$. We can choose a representative of $(\mathbb{T}', f', \lambda', s', p')$ so that the planar structure of $\mathbb{T}' = C_k$ is given by the ordering $\sigma$ of the vertices of $T$. Then the labeling $p$ of the vertex is necessarily precisely $(T, \sigma, \tau)$. So there is only one possibility for $(\mathbb{T}', f', \lambda', s', p')$.

Finally, if $\mathbb{T}'$ has at least two vertices, then it must have precisely $|B| + 1$ vertices arranged in a tree according to the nested structure of the bracket, and $k$ leaves, with each leaf attached to the vertex corresponding to the appropriate bracket. The root vertex of $\mathbb{T}$ corresponds to the whole tree $T$. The coloring of the edges is determined by the arity of the vertices and brackets in $T$, and the labeling of the leaves $\lambda$ is determined by the ordering $\sigma$. The vertices are decorated by tuples $(T_j, \sigma_j, \tau_j)$, with $T_j$ determined by the bracketing $B$, $\sigma_j$ determined by the nesting of the bracketing once a planar structure for $\mathbb{T}'$ is chosen. Choosing a different planar structure will give an equivalent element of $W_0\mathcal{O}$ (in fact also of $W\mathcal{O}$). The ordering $\tau_j$ is likewise not uniquely determined by the situation, but a different choice that does yield the same tuple $(T, \sigma, \tau, B, s)$ under $\Psi$ will be equivalent in $W_0\mathcal{O}$, using relation $(2')$. This finishes the proof of injectivity.

□

We are now ready to prove our main result in this section, namely that $W_0\mathcal{O}$ and $B\mathcal{O}$ are isomorphic as topological operads.

*Proof of Theorem A.6.4.* In Lemma A.6.9 we constructed a map of topological oper-

ads

$$\Psi : W_0\mathcal{O} \longrightarrow B\mathcal{O}.$$

Combining this with Lemma A.6.10 we know that, for each tuple $(n; m_1, \ldots, m_k)$, the map

$$\Psi : W_0\mathcal{O}(n; m_1, \ldots, m_k) \longrightarrow B\mathcal{O}(n; m_1, \ldots, m_k)$$

is a continuous bijection. As the source of this map is a compact space ($\pi_0 W_0\mathcal{O}(n; m_1, \ldots, m_k) = \mathcal{O}(n; m_1, \ldots, m_k)$ is finite and there are finitely many reduced representatives $(\mathbb{T}, f, \lambda, s, p)$ defining a cube in each component), and the target is a Hausdorff space, $\Psi$ is therefore a local homeomorphism and hence an isomorphism of topological operads. □

*Remark* A.6.11. A corollary of the result we just proved is that $W_0\mathcal{O}$ is the realization of an operad in posets, namely the operad $B\mathcal{O}$. The operad $W\mathcal{O}$ can likewise be seen as the realization of an operad in posets, namely the poset of elements of the free operad $F\mathcal{O}$, with poset structure generated by edge collapses. The map of operads $q : W\mathcal{O} \to W_0\mathcal{O}$ is the realization of a map of posets. Indeed, the map $q\colon W\mathcal{O} \to W_0\mathcal{O} \cong B\mathcal{O}$ respects the poset structure because collapsing an edge in $\mathbb{T}$, which defines the poset structure underlying $W\mathcal{O}$, corresponds under the map $q$ to forgetting a bracket, which defines the poset structure underlying $W_0\mathcal{O} = B\mathcal{O}$.

## A.7 The explosion category of $\Omega$

In Section A.4.1 we introduced an enriched version of the dendroidal category $\widetilde{\Omega}_0$ which is closely related to the category of $B\mathcal{O}$-algebras. As mentioned in the introduction of Section A.4, the idea of the category $\widetilde{\Omega}_0$ is to encode homotopy coherent $\Omega$–diagram, and hence $\widetilde{\Omega}_0$ should be connected to the *explosion category* of $\Omega$, as defined by Leitch [53] and Segal [81, Appendix B].

In this appendix we describe the explosion category of $\Omega$, denoted $\widetilde{\Omega}$, and show that our topological category $\widetilde{\Omega}_0$ sits between $\widetilde{\Omega}$ and $\Omega$ in the sense that there exist equivalences of topological categories

$$\widetilde{\Omega} \xrightarrow[q]{} \widetilde{\Omega}_0 \xrightarrow[p]{} \Omega \ , \qquad \widetilde{\Omega} \xrightarrow{\tilde{p}} \Omega$$

The explosion construction and the $W$–construction are very closely related in spirit. One might thus expect a relationship between Segal $\widetilde{\Omega}$–diagrams and $W\mathcal{O}$–algebras, similar to the relationship between Segal dendroidal spaces ($\Omega$–diagrams)

and $\mathcal{O}$–algebras, and between Segal homotopy dendroidal spaces ($\widetilde{\Omega}_0$–diagrams) and $W_0\mathcal{O}$– or $B\mathcal{O}$–algebras. Theorem A.7.6 below will show that such a relationship exists, but without being as close as in the other cases: $W\mathcal{O}$–algebras identify with a full subcategory of the category of reduced strict Segal $\widetilde{\Omega}$–diagrams.

### A.7.1 The explosion of $\Omega$

For each morphism $g : S \to T$ in $\Omega$, we define a *poset of paths* $\mathrm{Path}_\Omega(S,T)_g$ whose objects are the factorizations of $g : S \to T$ in $\Omega$

$$S \xrightarrow[g_1]{} T_1 \xrightarrow[g_2]{} \dots \longrightarrow T_{n-1} \xrightarrow[g_n]{} T, \qquad (g)$$

where we identify two factorizations if they differ only by identity morphisms. In particular, each such factorisation $(g_1, \dots, g_n)$ has a unique reduced representative containing no identity morphisms unless $n = 1$ and $g$ is the identity on $S$. (Such a factorisation can be thought of as a path in the nerve of $\Omega$.) The poset structure is by refinement of factorisation: $(g_1, \dots, g_n) \leq (g'_1, \dots, g'_m)$ if $n \leq m$ and there is a monotone map $\alpha : \{0, \dots, n\} \to \{0, \dots, m\}$ such that $\alpha(0) = 0$, $\alpha(n) = (m)$, and $g_i = g'_{\alpha(i)} \circ \dots \circ g'_{\alpha(i-1)+1}$ for each $1 \leq i \leq n$.

We denote the geometric realization of this poset by

$$K_g := |\mathrm{Path}_\Omega(S,T)_g|.$$

**Definition A.7.1.** The topological category $\widetilde{\Omega}$ has the same objects as $\Omega$. Morphism spaces in $\widetilde{\Omega}$ are defined as

$$\mathrm{Hom}_{\widetilde{\Omega}}(S,T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} K_g = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} |\mathrm{Path}_\Omega(S,T)_g|.$$

Composition of morphisms of $\widetilde{\Omega}$ is given by concatenation of factorizations.

**Example A.7.2.** Fix a tree $T$ with $|L(T)| = n$ leaves and three inner edges: $e_1$, $e_2$, $e_3$. Recall that $C_n$ denotes the corolla with $n$ leaves. Let $\partial_{e_1}$, $\partial_{e_2}$, $\partial_{e_3}$ denote the inner face maps in $\Omega$ associated to each inner edge, and let $g = \partial_{e_1}\partial_{e_2}\partial_{e_3} : C_n \to T$ be their composition. Then $g$ admits a factorization

$$C_n \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T. \qquad (g)$$

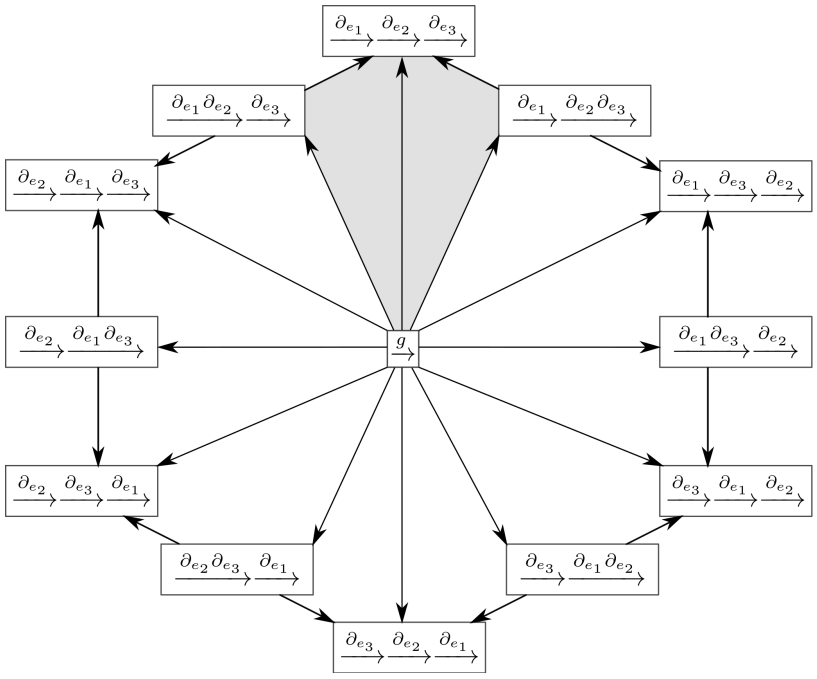
as a composition of three inner face maps for each permutation of $\{1,2,3\}$. The elements of $\mathrm{Path}_\Omega(C_n,T)_g$ that involve only these three inner face maps form a subposet with $([1],\xrightarrow{g})$ as minimum, and for each permutation $\sigma\in\Sigma_3$ the elements

$$([3],\xrightarrow{\partial_{\sigma(1)}}\xrightarrow{\partial_{\sigma(2)}}\xrightarrow{\partial_{\sigma(3)}}) \qquad ([2],\xrightarrow{\partial_{\sigma(1)}}\xrightarrow{\partial_{\sigma(2)}\partial_{\sigma(3)}}) \qquad ([2],\xrightarrow{\partial_{\sigma(1)}\partial_{\sigma(2)}}\xrightarrow{\partial_{\sigma(3)}}).$$

Each permutation $\sigma$ this way contributes to a square

$$\begin{array}{ccc}
(\xrightarrow{\partial_{\sigma(1)}\partial_{\sigma(2)}}\xrightarrow{\partial_{\sigma(3)}}) & \longrightarrow & (\xrightarrow{\partial_{\sigma(1)}}\xrightarrow{\partial_{\sigma(2)}}\xrightarrow{\partial_{\sigma(3)}}) \\
\uparrow & \nearrow & \uparrow \\
(\xrightarrow{g}) & \longrightarrow & (\xrightarrow{\partial_{\sigma(1)}}\xrightarrow{\partial_{\sigma(2)}\partial_{\sigma(3)}})
\end{array}$$

in this subposet, and the dendroidal identities tell us that these squares together form the following hexagon inside $|\mathrm{Path}_\Omega(C_n,T)_g|$:




Additional elements of $\mathrm{Path}_\Omega(C_n,T)_g$ can be obtained by inserting tree isomorphisms. This example should be compared to Examples A.3.4 and A.3.5 which can be interpreted as computing morphism spaces in the category $\widetilde{\Omega}_0$ likewise associated to trees with three internal edges, where in one case a pentagon occurs, and in the other it is a hexagon.

**Lemma A.7.3.** *For each $g \in \mathrm{Hom}_\Omega(S,T)$ the space $K_g = |Path_\Omega(S,T)_g|$ is contractible.*

*Proof.* The poset $\mathrm{Path}_\Omega(S,T)_g$ has the trivial factorisation $S \xrightarrow{g} T$ as a minimal element. □

Let $\tilde{p} : \widetilde{\Omega} \to \Omega$ be the functor that is the identity on objects and projects each morphism space $K_g$ to $g$. Considering $\Omega$ as a discrete topological category, the lemma immediately gives the following proposition.

**Proposition A.7.4.** *The functor $\tilde{p} : \widetilde{\Omega} \to \Omega$ induces a homotopy equivalence on morphism spaces.*

Note that the proposition identifies $\Omega$ with the "path component category" $\pi_0\widetilde{\Omega}$, which has the same objects as $\widetilde{\Omega}$ and $\mathrm{Hom}_{\pi_0\widetilde{\Omega}}(S,T) := \pi_0(\mathrm{Hom}_{\widetilde{\Omega}}(S,T))$.

## A.7.2 The relationship between $\widetilde{\Omega}$ and $\widetilde{\Omega}_0$

The category $\widetilde{\Omega}_0$ sits between $\widetilde{\Omega}$ and $\Omega$ in the sense of the following proposition.

**Proposition A.7.5.** *There is a functor $q : \widetilde{\Omega} \to \widetilde{\Omega}_0$, which is the identity on objects and induces a homotopy equivalence on each morphism space. Moreover, the composition $p \circ q = \tilde{p} : \widetilde{\Omega} \to \Omega$ is the projection functor of Proposition A.7.4.*

*Proof.* Fix two objects $S, T \in \widetilde{\Omega}$. Recall from Definition A.7.1 that

$$\mathrm{Hom}_{\widetilde{\Omega}}(S,T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} K_g$$

for $K_g = |\mathrm{Path}_\Omega(S,T)_g|$ is the realization of the poset of factorizations of $g$, and $K_g$ is contractible. Likewise by Definition A.4.4

$$\mathrm{Hom}_{\widetilde{\Omega}_0}(S,T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} L_g$$

and $L_g = \prod_{v \in V(S)} |\mathcal{B}(g(C_v))|$ is the realization of the poset $\mathcal{L}_g$ of bracketings of the trees $g(C_v)$, with $L_g$ likewise contractible. So to prove the proposition, it is enough to produce a functor $q$ which is the identity on objects and takes $K_g$ to $L_g$ for each $g$. We will define the functor by defining a poset map

$$q_g : \mathrm{Path}_\Omega(S,T)_g \to \mathcal{L}_g$$

and show that it is compatible with composition.

Fix a map $g : S \to T$ in $\Omega$. An object of $\mathrm{Path}_\Omega(S,T)_g$ is a factorization $(g_1, \dots, g_n)$ of $g$ and to such a factorization of $g$, for each $v \in V(S)$, we associate a bracketing of $g(C_v)$ as follows: set

$$B_v = \{S_w = g_n \circ \cdots \circ g_{i+1}(C_w)\}_{\substack{1 \le i \le n-1 \\ w \in V(g_i \circ \cdots \circ g_1(C_v)) \\ S_w \subsetneq g(C_v) \text{ large}}}$$

This is a (possibly empty) bracketing as these sets are by definition nested. We then define $q_g(g_1, \dots, g_n) = (B_v)_{v \in V(S)}$. Note that this association is a map of posets as refining a factorization will correspond under $q_g$ to an inclusion of bracketings.

We are left to check that the maps $q_g$ assemble to define a functor, i.e. that they are compatible with composition in $\widetilde{\Omega}$ and $\widetilde{\Omega}_0$. Let $f : R \to S$ be another morphism in $\Omega$. We need to check that

$$\begin{array}{ccc} \mathrm{Path}_\Omega(S,T)_g \times \mathrm{Path}_\Omega(R,S)_f & \longrightarrow & \mathrm{Path}_\Omega(R,T)_{g \circ f} \\ \downarrow{\scriptstyle q_g \times q_f} & & \downarrow{\scriptstyle q_{g \circ f}} \\ \mathcal{L}_g^{op} \times \mathcal{L}_f^{op} & \longrightarrow & \mathcal{L}_{g \circ f}^{op} \end{array}$$

commutes. Because the target is a poset, it is enough to check that it commutes on objects. Let $(g_1, \dots, g_n)$ and $(f_1, \dots, f_m)$ be objects of $\mathrm{Path}_\Omega(S,T)_g$ and $\mathrm{Path}_\Omega(R,S)_f$. By definition, their composition is $(f_1, \dots, f_m, g_1, \dots, g_n) \in \mathrm{Path}_\Omega(R,T)_{g \circ f}$. We have $q_f(f_1, \dots, f_m) = (B_x^f)_{x \in V(R)}$ and $q_g(g_1, \dots, g_n) = (B_v^g)_{v \in V(S)}$ with

$$B_x^f = \{S_y = f_m \circ \cdots \circ f_{i+1}(C_y)\}_{\substack{1 \le i \le m-1 \\ y \in f_i \circ \cdots \circ f_1(C_x) \\ S_y \subsetneq f(C_x) \text{ large}}}$$

and bracketing of $f(C_x) \subset S$, and

$$B_v^g = \{S_w = g_n \circ \cdots \circ g_{i+1}(C_w)\}_{\substack{1 \le i \le n-1 \\ w \in g_i \circ \cdots \circ g_1(C_v) \\ S_w \subsetneq g(C_v) \text{ large}}}$$

a bracketing of $g(C_v) \subset T$. By definition, $q_g(g_1, \dots, g_n) \circ q_f(f_1, \dots, f_m)$ is the collection $(\bar{B}_x)_{x \in V(R)}$ of bracketings of each tree $g \circ f(C_x) \subset T$ defined by

$$\bar{B}_x = \Big( \bigcup_{v \in f(C_x)} B_v^g \Big) \cup \Big( \bigcup_{\substack{v \in f(C_x) \\ g(C_v) \subsetneq g \circ f(C_x) \text{ large}}} \{g(C_v)\} \Big) \cup \Big( \bigcup_{\substack{S_y \in B_x^f \\ g(S_y) \text{ large}}} \{g(S_y)\} \Big),$$

where $B^g_v$ is considered as a bracketing of $g\circ f(C_x)$ via the inclusion $g(C_v)\subset g\circ f(C_x)$. Now we see that this is exactly the bracketing of $g\circ f(C_x)$ defined by the factorization $(f_1,\dots,f_m,g_1,\dots,g_m)$, which indeed is the union of the sets

$$\{g(S_y)=g\circ f_m\circ\cdots\circ f_{i+1}(C_y)\}_{\substack{1\leq i\leq m-1\\ y\in f_i\circ\cdots\circ f_1(C_x)\\ g(S_y)\subsetneq g\circ f(C_x)\text{ large}}} \cup\{S_v=g(C_v)\}_{\substack{v\in f(C_x)\\ S_v\subsetneq g\circ f(C_x)\text{ large}}} \cup\{S_w=g_n\circ\cdots\circ g_{i+1}(C_w)\}_{\substack{1\leq i\leq n-1\\ w\in g_i\circ\cdots\circ g_1\circ f(C_x)\\ S_w\subsetneq g\circ f(C_x)\text{ large}}}.$$

Hence the poset maps $q_g$ assemble to define a functor $q:\widetilde{\Omega}\to\widetilde{\Omega}_0$ as claimed. Moreover, one readily checks that the composition with the projection $p:\widetilde{\Omega}_0\to\Omega$ is the canonical projection $\tilde{p}:\widetilde{\Omega}\to\Omega$. □

### A.7.3 $W\mathcal{O}$–algebras as $\widetilde{\Omega}$–diagrams

In Section A.4.2 we showed that $B\mathcal{O}$-algebras describe dendroidal Segal spaces. For completeness, we now show how homotopy dendroidal spaces $\mathcal{S}^{\widetilde{\Omega}^{op}}$ are related to $W\mathcal{O}$-algebras.

We will only need to consider $\widetilde{\Omega}$–diagrams $X:\widetilde{\Omega}^{op}\to\mathcal{S}$ that are reduced, i.e. such that $X(\eta)=*$. Recall that in this case, for $X:\Omega\to\mathcal{S}$, the Segal map becomes the map

$$X(T)\xrightarrow{\chi}\prod_{v\in V(T)}X(C_v)$$

induced by the restriction maps $T\to C_v$ in $\Omega^{op}$. Considering these morphisms as morphisms of $\widetilde{\Omega}$, we likewise have a Segal map for $X:\widetilde{\Omega}\to\mathcal{S}$ in the reduced case.

In analogy to the case of dendroidal and homotopy dendroidal spaces, let $(\mathcal{S}^{\widetilde{\Omega}^{op}})_{strict}$ denote the full subcategory of $\mathcal{S}^{\widetilde{\Omega}^{op}}$ of $\widetilde{\Omega}$–diagrams $X:\widetilde{\Omega}^{op}\to\mathcal{S}$ such that $X(\eta)=*$ and such that the Segal map $\chi$ as above is an isomorphism for every $T\neq\eta$. We have the following:

**Theorem A.7.6.** *There exists a functor*

$$\Psi: W\mathcal{O}\text{–Alg}\to(\mathcal{S}^{\widetilde{\Omega}^{op}})_{strict}$$

*that embeds the category of $W\mathcal{O}$-algebras as a full subcategory of the category of reduced $\widetilde{\Omega}$–diagrams satisfying the strict Segal condition.*

As we will see in the proof, $\widetilde{\Omega}$–diagrams are governed by a version of $W\mathcal{O}$ where the trees $\mathbb{T}$ have an additional level structure, and $W\mathcal{O}$–algebras identify then as the

subcategory of diagrams where this level structure does not matter. If one wished to describe a category of homotopy dendroidal spaces which is isomorphic to $W\mathcal{O}$-algebras, one could use this observation to take an appropriate quotient of $\widetilde{\Omega}$. As this is particularly messy, and not the main focus of this article, we have elected not to include such a construction.

*Proof.* The proof is similar to that of Theorem A.4.8 treating the case of $B\mathcal{O}$–algebras. We start with the definition of the functor $\Psi$. Let $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ be a $W\mathcal{O}$–algebra with structure maps

$$\alpha_{\mathcal{P}} : W\mathcal{O}(n; m_1, \dots, m_k) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_k) \longrightarrow \mathcal{P}(n).$$

We associate to this data an $\widetilde{\Omega}$–diagram

$$\Psi(\mathcal{P}) = \Psi(\mathcal{P}, \alpha_{\mathcal{P}}) : \widetilde{\Omega}^{op} \to \mathcal{S}$$

as follows. Set $\Psi(\mathcal{P})(\eta) = *$ and, for $T \neq \eta$ in $\widetilde{\Omega}$, set

$$\Psi(\mathcal{P})(T) = \prod_{w \in V(T)} \mathcal{P}(|w|).$$

For every morphism $g : S \to T$ in $\Omega$, we need to define maps

$$\Psi(\mathcal{P})(g) : K_g \times \prod_{w \in V(T)} \mathcal{P}(|w|) \longrightarrow \prod_{v \in V(S)} \mathcal{P}(|v|).$$

As in Theorem A.4.8, we do this one vertex of $S$ at a time.

Recall that $K_g$ is the realisation of the poset $\mathrm{Path}_{\Omega}(S,T)_g$ of factorizations

$$S \xrightarrow{g_1} T_1 \xrightarrow{g_2} \dots \xrightarrow{g_{n-1}} T_{n-1} \xrightarrow{g_n} T$$

of $g$ in $\Omega$. For each $v \in V(S)$, we consider the restriction of these maps to $C_v \in S$:

$$C_v \xrightarrow{g_1} g_1(C_v) \xrightarrow{g_2} \dots \xrightarrow{g_{n-1}} g_{n-1} \circ \cdots \circ g_1(C_v) \xrightarrow{g_n} g(C_v) \subset T. \qquad (*)$$

Recall from Remark A.6.11 that $W\mathcal{O}$ is the realization of an operad in posets, whose elements are those of the free operad $F\mathcal{O}$ (identifying elements of $F\mathcal{O}$ with elements of $W\mathcal{O}$ in which all weights of internal edges are 1). We will now use the restriction

$(*)$ of $(g_1, \ldots, g_n)$ to $C_v$ to construct a labeled planar tree

$$(\mathbb{T}, f, \lambda, p) \in F\mathcal{O}(|v|; (|w|)_{w \in V(g(C_v))})$$

by induction on the height of the tree:

Starting at the root, we attach a vertex $\bar{v}$ of valence $|V(g_1(C_v))|$. The incoming edges of $\bar{v}$ are labelled in accordance with $(g_1(C_v), \sigma_v, \tau_v)$, where $\sigma_v$ is a chosen ordering of the vertices of the tree $g_1(C_v)$, and $\tau_v$ is induced by the planar structure of $g_1(C_v) \subset T_1$. Specifically, the incoming edges of $\bar{v}$ are labeled by the vertices of $g_1(C_v)$ and ordered via the map $\sigma_v$.

For each vertex $w \in g_1(C_v)$, which is now an incoming edge of $\bar{v}$, we can attach a vertex $\bar{w}$ of valence $|V(g_2(C_w))|$. These incoming edges are labeled with the tuple $(g_2(C_w), \sigma_w, \tau_w)$, as in the previous case.

More generally, for vertices with height $2 \leq i \leq n$, the tree $\mathbb{T}$ has a vertex $\bar{y}$ for every vertex $y$ in $(g_{i-1} \circ \cdots \circ g_1)(C_v)$, attached to the previously constructed vertex $\bar{x}$ associated to the vertex $x \in (g_{i-2} \circ \cdots \circ g_1)(C_v)$ satisfying that $y \in g_{i-1}(C_x)$. We label $\bar{y}$ by the tuple $(g_i(\bar{y}), \sigma_y, \tau_y)$ with $\tau_y$ induced by the planar structure of $T_i$, giving $C_{\bar{y}}$ the planar structure dictated by the chosen $\sigma_y$.

We now set $f : E(\mathbb{T}) \to \mathbb{N}$ to be the unique meaningful colouring which makes $\mathbb{T}$ an element of $F\mathcal{O}(|v|; |w_1|, \ldots, |w_k|)$. We set the ordering $\sigma$ of the vertices $w_1, \ldots, w_k$ of $g(C_v)$ in accordance to the resulting planar structure on $\mathbb{T}$. As the set of vertices of $g(C_v)$ is also the set of leaves of the tree $\mathbb{T}$, this also defines $\lambda$. We note that the tree constructed this way is in no way reduced and will, a priori, have many arity one vertices labelled by identities. We can use relations defining $F\mathcal{O}$, however, to remove such vertices and give an equivalent element in $F\mathcal{O}$.

This assignment of the restriction of a factorisation $(*)$ to a labelled tree $\mathbb{T}$ respects the poset structure of $\mathrm{Path}_\Omega(S,T)_g$ and $W\mathcal{O}$ as refining the factorisation corresponds to undoing the collapse of edges, namely if $(g_1, \ldots, g_n) \leq (g'_1, \ldots, g'_m)$, then the image $(\mathbb{T}, f, \lambda, p)$ of the first factorisation can be obtained from the image $(\mathbb{T}', f', \lambda', p')$ of the second by collapsing the edges corresponding to the added levels, as collapsing level in the tree correspond in this construction to composing consecutive maps $g_i$.

In this way we can apply the structure map $\alpha_{\mathcal{P}}$ one vertex at a time and define a map

$$\alpha_v : |\mathrm{Path}_\Omega(S,T)_g| \times \prod_{w_i \in V(g(C_v))} \mathcal{P}(|w_i|) \longrightarrow \mathcal{P}(|v|) \tag{A.7.1}$$

and we can define

$$\Psi(\mathcal{P})(g) = (\alpha_v)_{v \in V(S)}.$$

By construction, the action of $\Psi(P)$ on morphisms commutes with composition in $\widetilde{\Omega}$, and thus $\Psi(\mathcal{P}) : \widetilde{\Omega}^{op} \to \mathcal{S}$ defines a functor. That the Segal map for $\Psi(\mathcal{P})$ is an isomorphism for every $T \neq \eta$ follows immediately from our definition of $\Psi(\mathcal{P})$.

The assignment $\mathcal{P} \mapsto \Psi(\mathcal{P})$ requires only the data of underlying symmetric sequence of $\mathcal{P}$ and the algebra structure maps $\alpha_{\mathcal{P}}$. This data is natural under maps of $W\mathcal{O}$-algebras and thus

$$\Psi : W\mathcal{O}\text{−Alg}_{\mathcal{S}} \longrightarrow \mathcal{S}^{\widetilde{\Omega}^{op}}$$

is a functor.

It remains to check that $\Psi$ is an embedding of a full subcategory. Injectivity on objects follows from the fact that if $\mathcal{P}$ and $\mathcal{Q}$ satisfy that $\Psi(\mathcal{P}) = \Psi(\mathcal{Q})$, then we necessarily have that $\mathcal{P}(n) = \mathcal{Q}(n)$ for each $n$, as given by the value at the corolla, with agreeing symmetric group actions as given by the isomorphisms of corollas, and the structure maps $\alpha_{\mathcal{P}}$ and $\alpha_{\mathcal{Q}}$ likewise must agree as the value of the structure map on every element of $W\mathcal{O}$ is the value of the functor $\Psi(\mathcal{P}) = \Psi(\mathcal{Q})$ on an associated morphism of $\widetilde{\Omega}$ obtained by choosing a level structure on the tree and interpreting the collapse of each level of the tree as a morphism in $\Omega$. As morphisms of $W\mathcal{O}$–algebras are determined by what they do on spaces $\mathcal{P}(n)$, we see that the functor is faithful. It is also full as natural transformations between diagrams originating from $W\mathcal{O}$–algebras, will necessarily respect the $W\mathcal{O}$-algebra structure of their values at the corollas.

□

*Remark* A.7.7. The reader might be tempted to compare the functor $\Psi(-)$ from Theorem A.7.6 with the *homotopy coherent nerve* of a topological operad $\mathcal{P}$. This is a functor

$$w^* : \mathcal{O}\text{−Alg} \to \mathsf{Set}^{\Omega^{op}}$$

defined by

$$(w^*\mathcal{P})(T) = \mathrm{Hom}_{\mathsf{Op}}(W\Omega(T), \mathcal{P}),$$

where $W\Omega(T)$ denotes the Boardman-Vogt $W$-construction applied to the free operad generated by a tree $T$ (Example A.2.4). The functors $\Psi$ and $w^*$ are not equivalent on operads, though if one has a $W\mathcal{O}$-algebra $\mathcal{P}$ which happens to be an operad then one can define a dendroidal space $X_{\mathcal{P}} \in \mathcal{S}^{\Omega^{op}}/w^*\mathcal{P}$, where the later denotes the slice

category. For more on this point of view, see [65, Remark 6.2] or [9, Corollary 1.7].

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