Supplementary Information: Dissipation-induced non-equilibrium phases with temporal and spatial order

In this Supplementary Information (SI), we provide a detailed discussion of the calculation in the main text. Supplementary Notes 1–3 primarily explain the details of the section "Driven Bose-Einstein condensate (BEC) in a dissipative cavity" in the main text. In Supplementary Note 1, we derive the system's equations of motion, namely Eq. (3) and Eq. (4) in the main text, from the fundamental principles of light-atom interaction. In Supplementary Note 2, we specify the experimental parameters and discuss several subtle points on experimental realization of our model. In Supplementary Note 3, we justify the validity of the mean-field approximation based on the atom-only master equation.

Supplementary Notes 4–6 ofprimarily correspond to the content of the section "Accelerating spatial lattice in the red-detuned regime" in the main text, focusing on the system's dynamics in the red-detuned regime. In Supplementary Note 4, we establish the connection between the cavity field and the net acceleration of the atoms. This result aligns well with physical intuition, as it can be interpreted as a manifestation of momentum conservation. Furthermore, it serves as a key prerequisite for calculating the atomic acceleration in the red-detuned regime (Eq. (7) in the main text). In Supplementary Note 5, we compute in detail the evolution of the Bloch wavefunction ansatz (Eq. (5) in the main text) under the equations of motion, ultimately deriving the result that atoms accelerate at a constant value (Eq. (7) in the main text) in this regime. In Supplementary Note 6, we discuss how the Bloch wavefunction ansatz breaks down as the magnitude of the cavity detuning Δ_c crosses zero, resulting in a phase transition, which corresponds to the content of the section "Transitions between phases with different sign of Δ_c" in the main text. In these Supplementary Notes, we mainly work in real space.

Supplementary Notes 7–9 dive into the calculation details of the 1-photon temporal order in the blue-detuned regime, as described in the section "Dissipative spatio-temporal lattice in the blue-detuned regime" of the main text. In Supplementary Note 7, we use a truncated three-level model to demonstrate that when an atom is initially prepared in a specific momentum state $|q\rangle$, it preferentially transitions to the $|q+k\rangle$ state rather than the $|q-k\rangle$ state during time evolution under dissipation. In Supplementary Note 8, we employ a simple truncated two-level model to illustrate the system's intra-period dynamics, showing how an atom coherently transferred from the $|q\rangle$ momentum state to the $|q+k\rangle$ momentum state. Building on this, we combine the conclusions from Supplementary Note 7 to explain the mechanism underlying the system's temporal order. Finally, in Supplementary Note 9, we analytically compute how the picture presented in Supplementary Note 8 breaks down when the pumping strength becomes sufficiently strong. This analytical result is fully consistent with the numerical results. In these Supplementary Notes, we mainly work using discrete momentum states.

SUPPLEMENTARY NOTE 1: DERIVATION OF THE EQUATIONS OF MOTION

We consider a system where an ultracold Bosonic gas is placed at the mode center of an optical cavity, with the cavity axis aligned along the y-direction. A laser beam propagating along the x-direction drives the transition between the atomic ground state $|g\rangle$ and the excited state $|e\rangle$. The Heisenberg equations of motion for the system are [1]:

$$
\begin{aligned}
\partial_t \Psi_{\mathrm{g}}(\mathbf{r}) &= i[\frac{\hbar}{2m}\nabla^2 - \frac{V_{\mathrm{g}}(\mathbf{r})}{\hbar} - \frac{U_{\mathrm{g}}}{\hbar}\Psi_{\mathrm{g}}^\dagger(\mathbf{r})\Psi_{\mathrm{g}}(\mathbf{r})]\Psi_{\mathrm{g}}(\mathbf{r}) + [g(\mathbf{r})a^\dagger + h(\mathbf{r})]\Psi_{\mathrm{e}}(\mathbf{r}) \\
\partial_t \Psi_{\mathrm{e}}(\mathbf{r}) &= i[\frac{\hbar}{2m}\nabla^2 - \frac{V_{\mathrm{e}}(\mathbf{r})}{\hbar} - \frac{U_{\mathrm{e}}}{\hbar}\Psi_{\mathrm{e}}^\dagger(\mathbf{r})\Psi_{\mathrm{e}}(\mathbf{r})]\Psi_{\mathrm{e}}(\mathbf{r}) + \Delta_{\mathrm{a}}\Psi_{\mathrm{e}}(\mathbf{r}) + [g(\mathbf{r})a + h^*(\mathbf{r})]\Psi_{\mathrm{g}}(\mathbf{r}) \\
\partial_t a &= i\Delta_{\mathrm{c}} a + \int d^3\mathbf{r}\, g(\mathbf{r})\Psi_{\mathrm{g}}^\dagger(\mathbf{r})\Psi_{\mathrm{e}}(\mathbf{r})
\end{aligned}
\tag{S1}
$$

where a is the annihilation operator for photons in the optical cavity mode, and $\Psi_{\mathrm{g}}(\mathbf{r})$ ($\Psi_{\mathrm{e}}(\mathbf{r})$) is the annihilation operator for ground (excited) atoms at position r. $V_{\mathrm{g}}(\mathbf{r})$ ($V_{\mathrm{e}}(\mathbf{r})$) represents the external trapping potential confining the ground (excited) state atoms. U_{g} (U_{e}) describes the collisional interaction between atoms in ground (excited) state. $h(\mathbf{r})$ is the complex position-dependent coupling strength between the driving light field and the atoms, and $g(\mathbf{r})$ is the real coupling strength between the cavity light field and the atoms. $\Delta_{\mathrm{c}} = \omega_{\mathrm{p}} - \omega_{\mathrm{c}}$ and $\Delta_{\mathrm{a}} = \omega_{\mathrm{p}} - \omega_a$ are the laser detunings from cavity resonance and atomic transition, respectively. Assuming that both detunings are much larger than the linewidth of the atomic transition, we can perform an adiabatic elimination of the excited state of the atoms, as described in [1]:

$$
\Psi_{\mathrm{e}}(\mathbf{r}) = -\frac{i}{\Delta_{\mathrm{a}}}[h^*(\mathbf{r}) + g(\mathbf{r})a]\Psi_{\mathrm{g}}(\mathbf{r}) \tag{S2}
$$

Neglecting the interaction between atoms, $U_\mathrm{g} = U_\mathrm{e} = 0$, we arrive at the Heisenberg equations of motion for the atomic ground state and the optical field:

$$\begin{aligned}\partial_t \Psi(\mathbf{r}) &= i\left[\frac{\hbar}{2m}\nabla^2 - \frac{V(\mathbf{r})}{\hbar} - \frac{|h(\mathbf{r})|^2}{\Delta_\mathrm{a}} - \frac{g^2(\mathbf{r})}{\Delta_\mathrm{a}}a^\dagger a - \frac{g(\mathbf{r})}{\Delta_\mathrm{a}}(h^*(\mathbf{r})a + h(\mathbf{r})a^\dagger)\right]\Psi(\mathbf{r}) \\ \partial_t a &= i[\Delta_\mathrm{c} - \frac{1}{\Delta_\mathrm{a}}\int d^3\mathbf{r} g^2(\mathbf{r})\Psi(\mathbf{r})^\dagger\Psi(\mathbf{r})]a - \frac{i}{\Delta_\mathrm{a}}\int d^3\mathbf{r} g(\mathbf{r})h^*(\mathbf{r})\Psi(\mathbf{r})^\dagger\Psi(\mathbf{r})\end{aligned} \tag{S3}$$

Here, subscripts g are omitted. The corresponding many-body Hamiltonian for this equation of motion is:

$$\mathcal{H}/\hbar = \int d^3r\Psi^\dagger(\mathbf{r})\left\{-\frac{\hbar}{2m}\nabla^2 + V(\mathbf{r}) + \frac{1}{\Delta_\mathrm{a}}[|h(\mathbf{r})|^2 + g^2(\mathbf{r})a^\dagger a + g(\mathbf{r})(h^*(\mathbf{r})a + h(\mathbf{r})a^\dagger)]\right\}\Psi(\mathbf{r}) - \Delta_\mathrm{c}a^\dagger a \tag{S4}$$

Substituting the explicit form of the coupling strengths for the atoms and the light into the Hamiltonian, the cavity field forms a standing wave, described by $g(\mathbf{r}) = g_0\cos(ky)$, and the driving laser forms a running wave, described by $h(\mathbf{r}) = h_0 e^{ikx}$. By defining the driving strength as $\eta_\mathrm{p} = h_0 g_0/\Delta_\mathrm{a}$, we obtain the many-body Hamiltonian of our system as:

$$\mathcal{H}/\hbar = \int d^3r\Psi^\dagger(\mathbf{r})\left\{-\frac{\hbar}{2m}\nabla^2 + V(\mathbf{r}) + \frac{g_0^2}{\Delta_\mathrm{a}}\cos^2(ky)a^\dagger a + \cos(ky)\eta_\mathrm{p}(a^\dagger e^{ikx} + ae^{-ikx})\right\}\Psi(\mathbf{r}) - \Delta_\mathrm{c}a^\dagger a\,, \tag{S5}$$

where we neglected the constant term $|h(\mathbf{r})|^2/\Delta_\mathrm{a} = h_0^2/\Delta_\mathrm{a}$. In this Hamiltonian, the term $\frac{g_0^2}{\Delta_\mathrm{a}}\cos^2(ky)a^\dagger a$ is the optical potential acting on the atoms from the cavity field, while term $\cos(ky)\eta_\mathrm{p}(a^\dagger e^{ikx} + ae^{-ikx})$ can be viewed as the AC-Stark shift potential from an interference between the driving laser field and the cavity field. We would like to point out that the Doppler shift the atoms experience due to the acceleration discussed in this work is negligible since it scales with the ratio of the recoil frequency ω_r and the atomic detuning Δ_a: $\omega_r/\Delta_\mathrm{a} \sim 10^{-6}$.

We restrict our analysis for simplicity to the x-direction, and assume the atoms to be confined at $y = 0$ and $z = 0$, with no additional confinement along the x-direction (see Supplementary Note 2 for experimental relevance of this restriction). Under this assumption, we can neglect the now trivial terms $V(\mathbf{r})$ and $\frac{g_0^2}{\Delta_\mathrm{a}}\cos^2(ky)a^\dagger a$ and obtain the Hamiltonian of the system as:

$$H/\hbar = \int dx\Psi^\dagger(x)\left\{-\frac{\hbar}{2m}\nabla^2 + \eta_\mathrm{p}(a^\dagger e^{ikx} + ae^{-ikx})\right\}\Psi(x) - \Delta_\mathrm{c}a^\dagger a \tag{S6}$$

By moving into the momentum basis, we reach at Hamiltonian in Eq. (2) in the main text:

$$H/\hbar = \sum_q \frac{\hbar q^2}{2m}c_q^\dagger c_q - \Delta_\mathrm{c}a^\dagger a + \eta_\mathrm{p}[a^\dagger(\sum_q c_{q+k}^\dagger c_q) + a(\sum_q c_q^\dagger c_{q+k})], \tag{S7}$$

where $c_q = \int dx e^{-iqx}\Psi(x)/\sqrt{L}$ is the annihilation operator for an atom in momentum state $|q\rangle$ and L is the quantization length. For convenience in future discussions, we define the operator $O = \sum_q c_{q+k}^\dagger c_q = \int dx\Psi^\dagger(x)e^{ikx}\Psi(x)$. Real and imaginary parts of the expectation value of this operator serve as the order parameters for the Dicke phase transition in cQED systems in previous study [1]. From the Hamiltonian in Eq. (S6) and Eq. (S7), we derive the Heisenberg equations of motion in real space:

$$\begin{aligned}i\partial_t\Psi(x) &= [-\frac{\hbar}{2m}\partial_x^2 + \eta_\mathrm{p}(a^\dagger e^{ikx} + ae^{-ikx})]\Psi(x) \\ i\partial_t a &= -(\Delta_\mathrm{c} + i\kappa)a + \eta_\mathrm{p}\int d^3\mathbf{r}\Psi(x)^\dagger e^{ikx}\Psi(x)\end{aligned} \tag{S8}$$

and momentum space (Eq. (3) in the main text):

$$\begin{aligned}i\partial_t c_q &= \frac{\hbar q^2}{2m}c_q + \eta_\mathrm{p}(a^\dagger c_{q-k} + ac_{q+k}), \\ i\partial_t a &= -(\Delta_\mathrm{c} + i\kappa)a + \eta_\mathrm{p}\sum_q c_{q+k}^\dagger c_q.\end{aligned} \tag{S9}$$

In these equations, an additional term $-i\kappa a$ which is not included in Hamitonian is added, describing the loss of photons due to cavity field dissipation with rate κ.

The interacting quantum many-body models in Eq. (S8) or Eq. (S9) are generically difficult to study due to the exponential number of degrees of freedom in terms of the number of particles. To perform an accessible computational study on the model, we apply the mean field approximation by replacing the cavity photon operator with the field amplitude and the bosonic atom operator with a macroscopic wave function,

$$\langle a\rangle \to \alpha, \quad \langle \Psi(\mathbf{r})\rangle \to \sqrt{N}\psi(\mathbf{r}), \quad \langle c_q\rangle \to \sqrt{N}\mathcal{C}_q\,. \tag{S10}$$

In the mean field approximation we neglect quantum fluctuations,

$$\langle a\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\rangle \to \langle a\rangle\langle \Psi^\dagger(\mathbf{r})\rangle\langle\Psi(\mathbf{r})\rangle = N\alpha\psi^*(\mathbf{r})\psi(\mathbf{r}), \quad \text{etc}, \tag{S11}$$

and obtain the mean field equations of motion for cavity field and atoms in real space

$$\begin{aligned} i\partial_t\psi(x) &= [-\frac{\hbar}{2m}\partial_x^2 + \eta_{\mathrm{p}}(\alpha^* e^{ikx} + \alpha e^{-ikx})]\psi(x) \\ i\partial_t\alpha &= -(\Delta_{\mathrm{c}} + i\kappa)\alpha + \eta_{\mathrm{p}} N\int d^3\mathbf{r}\psi(\mathbf{r})^\dagger e^{ikx}\psi(\mathbf{r}) \end{aligned} \tag{S12}$$

and in momentum space

$$\begin{aligned} i\partial_t\mathcal{C}_q &= \frac{\hbar q^2}{2m}\mathcal{C}_q + \eta_{\mathrm{p}}(\alpha^*\mathcal{C}_{q-k} + \alpha\mathcal{C}_{q+k}), \\ i\partial_t\alpha &= -(\Delta_{\mathrm{c}} + i\kappa)\alpha + \eta_{\mathrm{p}}N\sum_q \mathcal{C}^\dagger_{q+k}\mathcal{C}_q. \end{aligned} \tag{S13}$$

All the numerical simulation shown in the work are based on solving equation Eq. (S12). The analytical calculations are based on atomic equations of motion in Eq. (S12) and Eq. (S13), together with an adiabatic elimination of cavity field (valid since the cavity field evolves much faster than the atomic state),

$$\alpha = \frac{N\eta_{\mathrm{p}}}{\Delta_{\mathrm{c}} + i\kappa}\int d^3\mathbf{r}\psi(\mathbf{r})^\dagger e^{ikx}\psi(\mathbf{r}) = \frac{N\eta_{\mathrm{p}}}{\Delta_{\mathrm{c}} + i\kappa}\sum_q \mathcal{C}^\dagger_{q+k}\mathcal{C}_q. \tag{S14}$$

The validity of mean-field approximation is discussed in Supplementary Note 3. In particular, we would like to emphasize that this adiabatic elimination works in the leading order of $1/\kappa$ even when $\Delta_{\mathrm{c}} \approx 0$, because physically the leading order does not depend on Δ_{c} and the cavity is close to the vacuum. More intuitively, the frequency shift of the light field scattered into the cavity by accelerating the atoms is much smaller than the cavity dissipation. Thus, even for $\Delta_{\mathrm{c}} \to 0$, this effective Doppler shift is not dominating the dynamics.

SUPPLEMENTARY NOTE 2: EXPERIMENTAL IMPLEMENTATION

For concreteness, we consider in this work the example of a Bose-Einstein condensate with $N = 10^5$ ^{87}Rb atoms and the excited state $5P_{3/2}$. This transition has a natural linewidth of $\gamma = 2\pi \times 6.07$MHz. When a ^{87}Rb atom absorbs a photon of this wavelength, the corresponding recoil frequency is $\omega_r = 2\pi \times 3.7$kHz. In the experiment, the driving laser is red-detuned from the atomic transition, with a detuning frequency Δ_{a} of several to tens of GHz. The optical cavity parameters are based on [1]: the cavity mode waist is approximately $w = 25\mu$m, corresponding to a single-photon atom-coupling coefficient $g_0 = 2\pi \times 12$MHz, and the cavity linewidth due to photon leakage is $\kappa = 2\pi \times 1$ MHz. Under these conditions, the single-atom cooperativity of the cavity is $C_0 = 20$. With commonly used laser powers, the Rabi frequency induced by the driving laser h_0 does not exceed 10 MHz, resulting in pump strength parameter η_{p} on the order of a few kHz.

We first examine the effect of incoherent scattering of photons by atoms. When the driving laser illuminates the atoms, the incoherent scattering rate of a single atom under far-detuned conditions is given by $Nh_0^2\gamma/(\Delta_{\mathrm{a}}^2 + \gamma^2)$. Consequently, the scattering force experienced by all the atoms is $F_{\mathrm{sc}} = \hbar kNh_0^2\gamma/(\Delta_{\mathrm{a}}^2 + \gamma^2)$. In comparison, the coherent scattering force exerted when atoms scatter the driving photons into the cavity mode, which will be discussed in detail in Supplementary Note 5, is $F_{\mathrm{coh}} = \frac{\hbar\kappa k}{2}\frac{(N\eta_{\mathrm{p}})^2}{|\Delta_{\mathrm{c}}|^2+\kappa^2}$. The ratio of the incoherent scattering force to the coherent scattering force is

$$\frac{F_{\mathrm{sc}}}{F_{\mathrm{coh}}} = \frac{2h_0^2\gamma/(\Delta_{\mathrm{a}}^2 + \gamma^2)}{N\kappa\eta_{\mathrm{p}}^2/(\Delta_{\mathrm{c}}^2 + \kappa^2)} = \frac{2}{NC_0}\frac{1 + (\Delta_{\mathrm{c}}/\kappa)^2}{1 + (\gamma/\Delta_{\mathrm{a}})^2}. \tag{S15}$$

For the parameters we have chosen, this ratio is 1/2500 making the incoherent scattering force negligible.

For simplicity we reduced the model to a 1D description along the pump direction, perpendicular to the cavity axis. We do not expect the observed physics to be different in an extended sample as long as (1) the atoms at a specific x-position experience the same phase of the driving field, and (2) all atoms couple to the same phase of the cavity field. The first condition is trivially fulfilled if the driving field is a plane wave. The second condition can be fulfilled if the atoms are confined in a 1D lattice with λ-spacing along the cavity axis. This can be achieved by driving the cavity with a laser at twice the wave length of the transverse drive that acts as a lattice potential.

In our model, we assume the coupling coefficient η_{p} to remain constant in space. However, as atoms continue to accelerate in the x-direction, they will eventually move out of the cavity mode volume, causing the coupling to shut off. When the waist radius is 25μm, atoms can uniformly accelerate in the red-detuned regime for approximately 0.5 ms within this 50μm range or gain momentum k three times in the blue-detuned regime. Thus, this limited waist size is sufficient to observe different phases discussed in this work. As shown in Figure 4 in the main text, especially the light field is suited to distinguish the different regimes which differ in (1) the evolution times by more than an order of magnitude, (2) the maximum light field amplitudes, (3) the temporal evolution of the light field amplitudes (approximately constant, pulsing, or noise dominated). If a longer evolution time is desired, the optical mode's width must be increased. This would increase the cavity mode volume and lead to a reduction in the coupling coefficient. Potential approaches to address this issue include: (1) Balancing the trade-off between evolution time and coupling coefficient; (2) Increasing the atom number to achieve higher collective cooperativity and thus compensate for the reduction in the single-atom coupling coefficient.

SUPPLEMENTARY NOTE 3: ATOM-ONLY MASTER EQUATION AND VALIDITY OF MEAN-FIELD APPROXIMATION

We prove the validity of the mean field approximation by deriving the atom-only master equation of this system. To start with, we divide the operator $O = \sum_q c^\dagger_{q+k} c_q$ into two parts, its expectation value and quantum fluctuation: $O = \langle O \rangle + \delta O$. Then, the Lindblad master equation of the atom and cavity system is

$$\begin{aligned} \frac{d}{dt}\rho =& - i[-\Delta_{\mathrm{c}} a^\dagger a + \eta_{\mathrm{p}}(\langle O \rangle a^\dagger + \langle O \rangle^* a), \rho] + \kappa(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \\ & - i[\sum_q \frac{\hbar\omega_q}{2m} c^\dagger_q c_q + \eta_{\mathrm{p}}(\delta O a^\dagger + \delta O^\dagger a), \rho]. \end{aligned} \tag{S16}$$

The first line describes the dissipation-driven dynamics of the cavity field. In the time scale of the cavity field dynamics, the change in $\langle O \rangle$ is negligible. Thus, $\langle O \rangle$ acts as a quasi-static classical driving source on the cavity field. The second line describes the dynamics of the atoms and the interactions between the cavity field and the fluctuations of the atom operator δO.

Since the atoms are weakly coupled to a cavity with strong dissipation and large detuning ($\Delta_{\mathrm{c}}, \kappa \gg \eta_{\mathrm{p}}$), the cavity field can be adiabatically eliminated and we get the atom-only master equation [2],

$$\frac{d}{dt}\rho_A = - i[\sum_q \frac{\hbar\omega_q}{2m} c^\dagger_q c_q + \eta_{\mathrm{p}}(\alpha_0 O^\dagger + \alpha_0^* O), \rho_A] + \frac{\eta_{\mathrm{p}}^2}{\kappa}(2\delta O \rho_A \delta O^\dagger - \delta O^\dagger \delta O \rho_A - \rho_A \delta O^\dagger \delta O), \tag{S17}$$

together with the relation between steady state cavity field and atomic order, $\alpha_0 = \langle a \rangle = \frac{\eta_{\mathrm{p}}}{\Delta_{\mathrm{c}} + i\kappa}\langle O \rangle$. The first term in the atom-only master equation describes the coherent evolution of atoms driven by a self-generated classical light field. The second term describes the dissipative dynamics governed by the quantum fluctuations δO at rate $\eta_{\mathrm{p}}^2/\kappa$. If $\eta_{\mathrm{p}}^2/\kappa$ is much smaller than all coupling strengths in the equations of motion (S12) that govern the system's dynamics, which is the case for the parameters provided in Supplementary Note 2, the effect of quantum fluctuations is negligible compared to the dominating mean field dynamics.

SUPPLEMENTARY NOTE 4: ACCELERATION AND CONSERVATION OF MOMENTUM

In this part, we connect the acceleration of the center-of-mass of the atoms with the cavity photon occupation. The total momentum of the atomic system is $P = \hbar \sum_q q c^\dagger_q c_q$. To calculate its evolution, we start with the equation of motion of the atomic operators from the main text,

$$i\partial_t c_q = \frac{\hbar q^2}{2m} c_q + \eta_{\mathrm{p}}(a^\dagger c_{q-k} + a c_{q+k}). \tag{S18}$$

Then, the time evolution of the operator P is governed by

$$
\begin{aligned}
i\partial_t P = i\hbar \sum_q q(\partial_t c_q^\dagger \cdot c_q + c_q^\dagger \cdot \partial_t c_q) &= \sum_q q\{[-\frac{\hbar q^2}{2m} c_q^\dagger - \eta_\mathrm{p}(a c_{q-k}^\dagger + a^\dagger c_{q+k}^\dagger)]c_q + c_q^\dagger[\frac{\hbar q^2}{2m} c_q + \eta_\mathrm{p}(a^\dagger c_{q-k} + a c_{q+k})]\} \\
&= \hbar\eta_\mathrm{p} \sum_q q[a^\dagger(c_q^\dagger c_{q-k} - c_{q+k}^\dagger c_q) + a(c_q^\dagger c_{q+k} - c_{q-k}^\dagger c_q)] = \hbar\eta_\mathrm{p} k(a^\dagger O - a O^\dagger),
\end{aligned}
\tag{S19}
$$

where $O = \sum_q c_{q+k}^\dagger c_q$. Under mean field approximation, the expectation value for the acceleration is

$$
\partial_t \langle P \rangle \approx -i\eta_\mathrm{p}\hbar k(\alpha^* \langle O \rangle + \alpha \langle O^\dagger \rangle). \tag{S20}
$$

Finally, we adiabatically eliminate the cavity field, $\alpha \approx \eta_\mathrm{p}\langle O \rangle/(\Delta_\mathrm{c} + i\kappa)$, and get a compact relation between acceleration and cavity field,

$$
\partial_t \langle P \rangle \approx \frac{(2\kappa)\eta_\mathrm{p}^2 k}{\Delta_\mathrm{c}^2 + \kappa^2} |\langle O \rangle|^2 \approx (2\kappa) k |\alpha|^2. \tag{S21}
$$

In this equation, $|\alpha|^2$ is the number of photon in the cavity, while $2\kappa|\alpha|^2$ is the leaking rate of photons from the cavity. Since the relaxation of the cavity field is much faster then the atom dynamics, the number of photons leaking from the cavity is the same as the number of photons scattered into the cavity in the time scale of the atom dynamics. So, $(2\kappa)k|\alpha|^2$ is the momentum of the photons scattered by the atoms per unit time, which equals the momentum gained by the atoms per unit time according to the conservation of momentum. Based on this result, we will calculate the time-independent acceleration of atom-lattice system in the red-detuned regime in next section.

SUPPLEMENTARY NOTE 5: BLOCH WAVEFUNCTION ANSATZ IN THE RED-DETUNED REGIME

In this section, we work with Eq. (S12) in the red-detuned regime, deriving the result Eq. (7) in the main text. Note that in the equation of motion of atom, term $\eta_\mathrm{p}(\alpha^* e^{ikx} + \alpha e^{-ikx}) = V_\mathrm{o}(x)$ can be understood as the optical potential acting on the mean-field atomic wavefunction $\psi(x)$. This potential is a sinusoidal lattice with depth $2\eta_\mathrm{p}|\alpha|$:

$$
V_\mathrm{o}(x) = 2\eta_\mathrm{p}|\alpha| \cos(kx - \arg(\alpha)) \tag{S22}
$$

The potential itself depends on atomic wavefunction, since α depends on atomic wave function (according to Eq. (S14), if cavity field is adiabatically eliminated). To solve this problem, we use a Bloch wavefunction in solid state system as an ansatz"

$$
\psi(x) = f(x - x_0) e^{iqx} u_q(x - x_0), \tag{S23}
$$

Here, x_0 is the center of the wave packet, q is the quasi-momentum, while $u_q(x)$ is an even periodic function that satisfies $u_q(x) = u_q(x + 2\pi/k)$ and narrowly peaks at $x = 2\pi s\lambda$, s are integer. The slow-varying envelope function $f(x)$ satisfies $|(\partial_x f(x))/f(x)| \ll k$, and we assume that the maximum of $u_q(x)$ is located at $x = 0$. Substitute Eq. (S23) into Eq. (S14), we obtain the cavity optical amplitude under this ansatz:

$$
\begin{aligned}
\alpha &= \frac{N\eta_\mathrm{p}}{\Delta_\mathrm{c} + i\kappa} \int dx \psi^*(x) e^{ikx} \psi(x) = \frac{N\eta_\mathrm{p}}{\Delta_\mathrm{c} + i\kappa}\ e^{ikx_0} \int |f(x)|^2 e^{ikx} |u_q(x)|^2 dx \\
&\approx \frac{N\eta_\mathrm{p}}{\Delta_\mathrm{c} + i\kappa}\ e^{ikx_0} \sum_{s=-\infty}^{\infty} |f(\frac{2\pi s}{k})|^2 \int_{\frac{\pi}{k}(2s-1)}^{\frac{\pi}{k}(2s+1)} e^{ikx} |u_q(x)|^2 dx = \frac{N\eta_\mathrm{p}}{\Delta_\mathrm{c} + i\kappa}\ e^{ikx_0} \cdot \frac{k}{2\pi} \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} \cos(kx) |u_q(x)|^2 dx.
\end{aligned}
\tag{S24}
$$

From the first line to the second line in equations above, we applied the assumption that $f(x)$ is slow-varying spatially while wavepacket $u(x)$ is narrow around its peak in each period, so that we can replace f(x) in integral by its value at the center of each u(x) peak. Since the integral in the last line is real, the phase of optical field is

$$
\arg(\alpha) = \arg(\frac{N\eta_\mathrm{p}}{\Delta_\mathrm{c} + i\kappa}\ e^{ikx_0}) = kx_0 + \arg(\frac{1}{\Delta_\mathrm{c} + i\kappa}) \tag{S25}
$$

Thus, the optical potential is

$$
V_\mathrm{o}(x) = \eta_\mathrm{p}|\alpha_0| \cos[k(x - x_0) - \arg(\frac{1}{\Delta_\mathrm{c} + i\kappa})]. \tag{S26}
$$

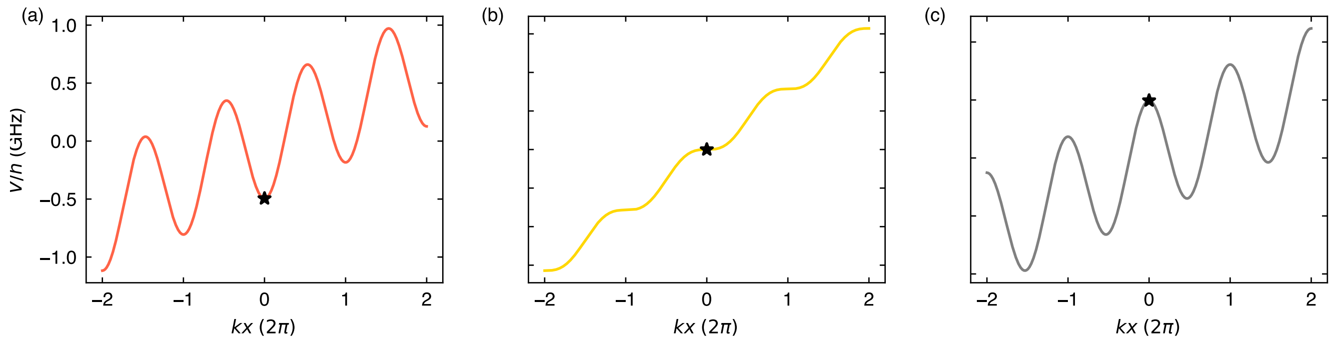

FIG. S1. Effective potential for atoms in (a) red-detuned regime ($\Delta_c < 0$, spatial-order phase); (b) phase boundary ($\Delta_c = 0$); (c) blue-detuned regime ($\Delta_c > 0$, dephasing regime). The positions of the atoms in the potential (marked with asterisks) are (a) stable ($\partial^2 V/\partial x^2 > 0$); (b) critical ($\partial^2 V/\partial x^2 = 0$); (c) unstable ($\partial^2 V/\partial x^2 < 0$).

In the red cavity detuned regime ($\Delta_c < 0$), if the dissipation κ is nonzero, there is a phase shift $\delta = \tan^{-1}(\kappa/|\Delta_c|)$ between atom wave function and the potential it generates. This deviation of the atom wavefunction from the minima of the potential causes an effective force on the atoms and leads to transport in $+x$ direction. The expectation of the force on atoms is in first order approximation (by neglecting the shape of $u(x)$)

$$\left\langle \frac{dP}{dt} \right\rangle = -N \left\langle \frac{\partial \hbar V_o}{\partial x} \right\rangle \approx -N\hbar \frac{\partial V_o}{\partial x} |_{x=x_0} = \eta_p N\hbar k|\alpha| \sin(\delta) = \eta_p N\hbar k|\alpha| \frac{\kappa}{\sqrt{|\Delta_c|^2 + \kappa^2}}. \tag{S27}$$

Comparing the equation above and with Eq. (S21), we can solve both the force and the steady state cavity field and effective force on the atom (also the acceleration), as functions of the experimentally tunable parameters η_p and Δ_c, in the small phase shift ($\Delta_c \ll -\kappa < 0$) regime:

$$|\alpha_0| \approx \frac{N\eta_p}{2\sqrt{|\Delta_c|^2 + \kappa^2}}, \quad \left\langle \frac{dP}{dt} \right\rangle \approx \frac{\hbar\kappa k}{2} \frac{(N\eta_p)^2}{|\Delta_c|^2 + \kappa^2}. \tag{S28}$$

The picture described in this section is only valid when the cavity detuning is red ($\Delta_c < 0$), as well as its absolute value is larger the dissipation $|\Delta_c| \gg \kappa$. When the cavity detuning crosses $\Delta_c = 0$ from negative to positive values, this picture breaks down because ansatz Eq. (S23) is not stable anymore. This is discussed in the next section.

SUPPLEMENTARY NOTE 6: PHASE TRANSITION BETWEEN SPATIAL ORDER TO DEPHASING REGIME

The phase transition between spatial order in red-detuned regime ($\Delta_c < 0$) can be explained by the effective potential in the accelerating frame. When the atoms are in a co-accelerating lattice, the total potential includes the optical potential together with the linear potential from the acceleration:

$$V(x) = V_o(x) + \left\langle \frac{dP}{dt} \right\rangle x = \frac{\hbar (N\eta_p)^2}{2\sqrt{|\Delta_c|^2 + \kappa^2}} \cos[k(x - x_0) - \arg(\frac{1}{\Delta_c + i\kappa})] + \frac{\kappa\hbar k}{2} \frac{(N\eta_p)^2}{|\Delta_c|^2 + \kappa^2} x. \tag{S29}$$

The extremal point of this potential is x_0: $\frac{\partial V}{\partial x}|_{x=x_0} = 0$. In the red-detuned regime ($\Delta_c < 0$), this point is stable since

$$\frac{\partial^2 V}{\partial x^2}|_{x=x_0} = -\frac{\hbar (N\eta_p)^2 k^2}{2\sqrt{|\Delta_c|^2 + \kappa^2}} \cos[\arg(\frac{1}{\Delta_c + i\kappa})] > 0. \tag{S30}$$

Therefore, the atom wavepacket can stay at this minimum point and the Bloch wavefunction ansatz works. When the detuning is blue ($\Delta_c > 0$), the extremum becomes the maximum point of the potential energy: $\frac{\partial^2 V}{\partial x^2}|_{x=x_0} < 0$. Therefore, the Bloch wavefunction ansatz is not stable and the spatial order breaks down. The effective potentials in the red and the blue-detuned regimes, as well as the potential on the boundary line, are showed in Fig. S1.

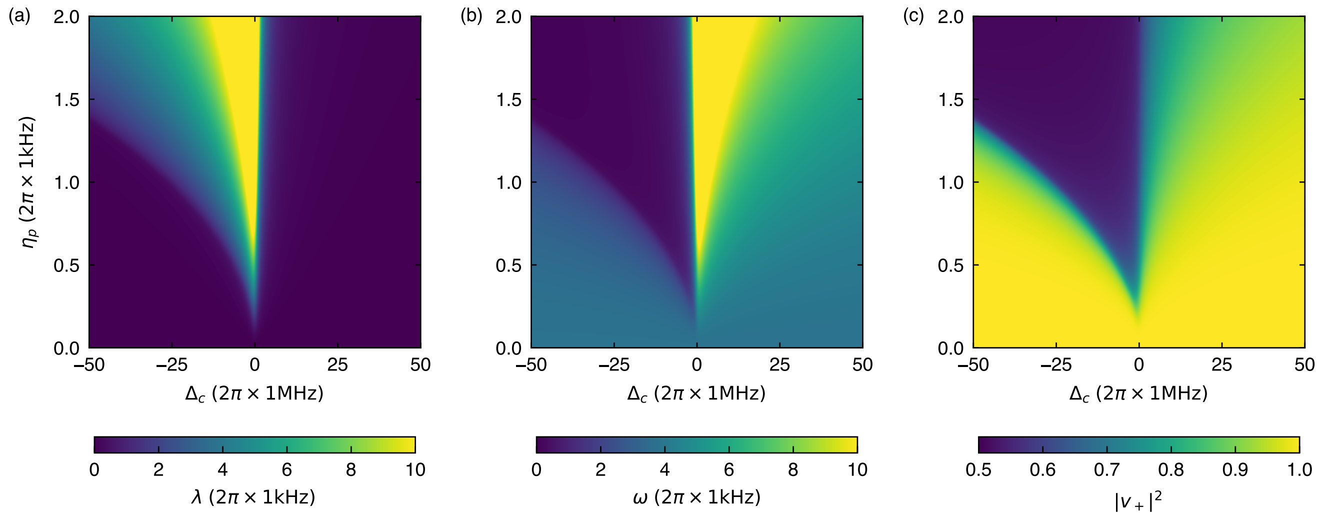

FIG. S2. The eigenvalues and eigenvectors of the matrix M corresponding to the dynamics. (a) The real part of the eigenvalue which is the exponential growth rate $\lambda > 0$ of the eigenmode. (b) The imaginary part of the eigenvalue which is the oscillation frequency ω of the corresponding eigenmode. (c) The weight $|v_+|^2$ of the $q+k$ momentum component in the eigenvector.

SUPPLEMENTARY NOTE 7: SHORT-TIME DYNAMICS OF ATOMS FROM A PURE MOMENTUM STATE

In the blue-detuned regime, numerical results reveal that the cavity field exhibits periodic oscillations over time. Simultaneously, with each oscillation of the cavity field, the atom coherently absorbs a photon momentum, transitioning from its current momentum state $|q\rangle$ to the state $|q+k\rangle$. To understand this phenomenon, we first study the short-time dynamics of the system in this regime in this section. Specifically, assuming the atom is initially prepared in the momentum state $|q\rangle$ and the cavity field is in the vacuum state $|0\rangle$, we will demonstrate that, due to dissipation, the atom transitions to the state $|q+k\rangle$ rather than $|q-k\rangle$. This can be understood from the sign of the corresponding eigenvalues.

During a very short time ($t \ll (\frac{N\eta_{\mathrm{p}}^2}{|\Delta_{\mathrm{c}}-i\kappa|})^{-1}$) after the coupling is turned on, the atoms are only pumped into the neighboring states $|q-k\rangle$ and $|q+k\rangle$, since only these two momentum states are directly coupled to the $|q\rangle$ state according to equations of motion (S13). Moreover, we neglect the depletion of the q state and assume its amplitude is 1: $\mathcal{C}_q = 1$, since the evolution time is short. With these two approximations, as well as with the adiabatic elimination of the cavity mode, we obtain equations of motion with only two modes, $|q-k\rangle$ and $|q+k\rangle$:

$$\partial_t \begin{pmatrix} \mathcal{C}_{q+k} \\ \mathcal{C}_{q-k} \end{pmatrix} = M \begin{pmatrix} \mathcal{C}_{q+k} \\ \mathcal{C}_{q-k} \end{pmatrix}, \quad M = -i \begin{pmatrix} (\omega_{q+k}-\omega_q) + \frac{N\eta_{\mathrm{p}}^2}{\Delta_{\mathrm{c}}-i\kappa} & \frac{N\eta_{\mathrm{p}}^2}{\Delta_{\mathrm{c}}-i\kappa} \\ -\frac{N\eta_{\mathrm{p}}^2}{\Delta_{\mathrm{c}}-i\kappa} & -(\omega_{q-k}-\omega_q) - \frac{N\eta_{\mathrm{p}}^2}{\Delta_{\mathrm{c}}-i\kappa} \end{pmatrix}. \tag{S31}$$

If the atoms are initially trapped in a static potential ($q=0$), $\omega_{q+k}-\omega_q = \omega_{q-k}-\omega_q = \omega_r$.

The short-time dynamics of atoms in the $q-k$ and $q+k$ modes is governed by the eigenvalues and eigenvectors of matrix M in Eqn. (S31) and can be calculated as:

$$M \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = (\lambda + i\omega) \begin{pmatrix} v_+ \\ v_- \end{pmatrix}. \tag{S32}$$

The eigenvector $[v_+, v_-]^T$ describes how the $q+k$ and $q-k$ states hybridize into the short-time eigenmodes. The imaginary part ω is the angular frequency of the eigenmode oscillation, while the real part λ is the rate of exponential growth ($\lambda > 0$) or decay ($\lambda < 0$) of the eigenmode. Since we are not interested in the eigenmode with $\lambda < 0$ that decays away, we focus on the eigenmode with positive real part eigenvalue ($\lambda > 0$). In this eigenvector, the $q+k$ mode always dominates: $|v_+| > |v_-|$. Experimentally, the atoms are usually loaded into a static trap. We thus focus on the initial state $q=0$: The real and imaginary part of the eigenvalue, and the weight of the k momentum mode in the eigenvector $|v_+|^2$ are plotted in the (Δ_c, η_p) parameter plane (see Fig. S2). In the 1-photon temporal phase, the growing eigenmode is almost $|q+k\rangle$ since $|v_+|^2$ is almost 1. This can explain why atoms only transition from $|q\rangle$ to $|q+k\rangle$ state: the eigenvector dominated by $|q+k\rangle$ state growths exponentially, while the eigenvector dominated by $|q-k\rangle$ state decays exponentially.

SUPPLEMENTARY NOTE 8: SELF-INDUCED OSCILLATIONS IN THE BLUE-DETUNED REGIME

In the previous section, we demonstrated that, in the blue-detuned one-photon temporal ordered regime, an atom in the momentum state $|q\rangle$ only transitions to the state $|q+k\rangle$ within a short period. In this section, we focus exclusively on these two momentum states and use a truncated two-level model to explain the entire process of the atom's transition from $|q\rangle$ to $|q+k\rangle$. The coupling driving the transition from $|q\rangle$ to $|q+k\rangle$ originates from the cavity field formed by the system's self-organization, and is proportional to the product of the amplitudes of the two momentum states $\mathcal{C}^*_{q+k}\mathcal{C}_q$. Thus, we refer to this process as self-induced oscillation. This gives a picture of the transition of atomic population from $|q\rangle$ to $|q+k\rangle$ and describes the system's dynamics within one period of a 1-photon temporal order. Moreover, we will explain why, despite the presence of a cavity field, the transition from $|q+k\rangle$ to $|q+2k\rangle$ is suppressed. This suppression justifies truncating the system to include only the $|q\rangle$ and $|q+k\rangle$ states under certain conditions. We will provide an analytical calculation of this process in the next section. Based on that results, how the system transitions from one period to the next, and whether the above picture remains valid, will be discussed in the next section. While also momentum states outside the discrete values $q = 0 \mod k$ might be occupied due to localization or finite temperature, the coupling for these states to states shifted by k will be negligible, since the coupling strength is proportional to the mode occupations. In addition, due to the nature of the coupling, no mixing of states outside the discrete ladder can appear.

Assuming that only $|q\rangle$ and $|q+k\rangle$ states are occupied, then the truncated equation of motion from Eq. (S13) is

$$i\partial_t \begin{pmatrix} \mathcal{C}_q \\ \mathcal{C}_{q+k} \end{pmatrix} = \begin{pmatrix} \omega_q & \eta_p\alpha \\ \eta_p\alpha^* & \omega_{q+k} \end{pmatrix} \begin{pmatrix} \mathcal{C}_q \\ \mathcal{C}_{q+k} \end{pmatrix}. \tag{S33}$$

The field amplitude is $\alpha = \frac{N\eta_p}{\Delta_c + i\kappa}\mathcal{C}^*_{q+k}\mathcal{C}_q$ using adiabatic elimination. If the coupling $\eta_p\alpha$ is weak, the amplitude $\mathcal{C}_q$ and $\mathcal{C}_{q+k}$ can be separated into a slow-varying term and a free oscillation term: $\mathcal{C}_q = \tilde{c}_q(t)e^{-i\omega_q t}$, $\mathcal{C}_{q+k} = \tilde{c}_{q+k}(t)e^{-i\omega_{q+k}t}$. Then, the cavity field is

$$\alpha(t) = \frac{N\eta_p}{\Delta_c + i\kappa}\tilde{c}^*_{q+k}(t)\tilde{c}_q(t)e^{i(\omega_{q+k}-\omega_q)t}, \tag{S34}$$

which can be viewed as a coupling oscillating with frequency $\omega_{q+k} - \omega_q$, on resonant with energy difference between $|q\rangle$ and $|q+k\rangle$ state, and amplitude-modulated with pulse shape $\tilde{c}^*_{q+k}(t)\tilde{c}_q(t)$. This coupling drives oscillations between $|q\rangle$ and $|q+k\rangle$ state. In the rotating frame, the equation of motion is

$$i\partial_t \begin{pmatrix} \tilde{c}_q \\ \tilde{c}_{q+k} \end{pmatrix} = \begin{pmatrix} 0 & \frac{N\eta_p^2}{\Delta_c + i\kappa}\tilde{c}^*_{q+k}\tilde{c}_q \\ \frac{N\eta_p^2}{\Delta_c - i\kappa}\tilde{c}^*_q\tilde{c}_{q+k} & 0 \end{pmatrix} \begin{pmatrix} \tilde{c}_q \\ \tilde{c}_{q+k} \end{pmatrix}. \tag{S35}$$

Once all atoms are coherently driven into the $|q+k\rangle$ mode from the $|q\rangle$ mode, the coupling $\tilde{c}^*_{q+k}(t)\tilde{c}_q(t)$ is automatically turned off. Then, the $|q+k\rangle$ state is occupied by all atoms and a new cycle of transition from $|q+k\rangle$ to $|q+2k\rangle$ mode starts. Fig. S3 is a illustration of this process where atoms climb the momentum ladder and accelerate in a step-like fashion.

During the oscillation between the $|q\rangle$ and the $|q+k\rangle$ mode, both modes are occupied. Thus, also transitions from the $|q+k\rangle$ to the $|q+2k\rangle$ state may simultaneously happen. Note that according to Eq. (S34), the oscillation frequency of coupling is $\omega_{\text{drive}} = \omega_{q+k} - \omega_q$, while the transition frequency between unwanted transition $|q+k\rangle \to |q+2k\rangle$ is $\omega_{q+2k} - \omega_{q+k}$. The detuning between the frequency of the driving and transition frequency is:

$$\delta_q = |\omega_{\text{drive}} - (\omega_{q+2k} - \omega_{q+k})| = |[(\frac{q}{k}+1)^2 - (\frac{q}{k})^2] - [(\frac{q}{k}+2)^2 - (\frac{q}{k}+1)^2]|\omega_r = 2\omega_r. \tag{S36}$$

If the driving is much smaller than the level detuning δ_q:

$$|\eta_p\alpha| < \frac{N\eta_p^2}{4|\Delta_c + i\kappa|} \ll \delta_q = 2\omega_r, \tag{S37}$$

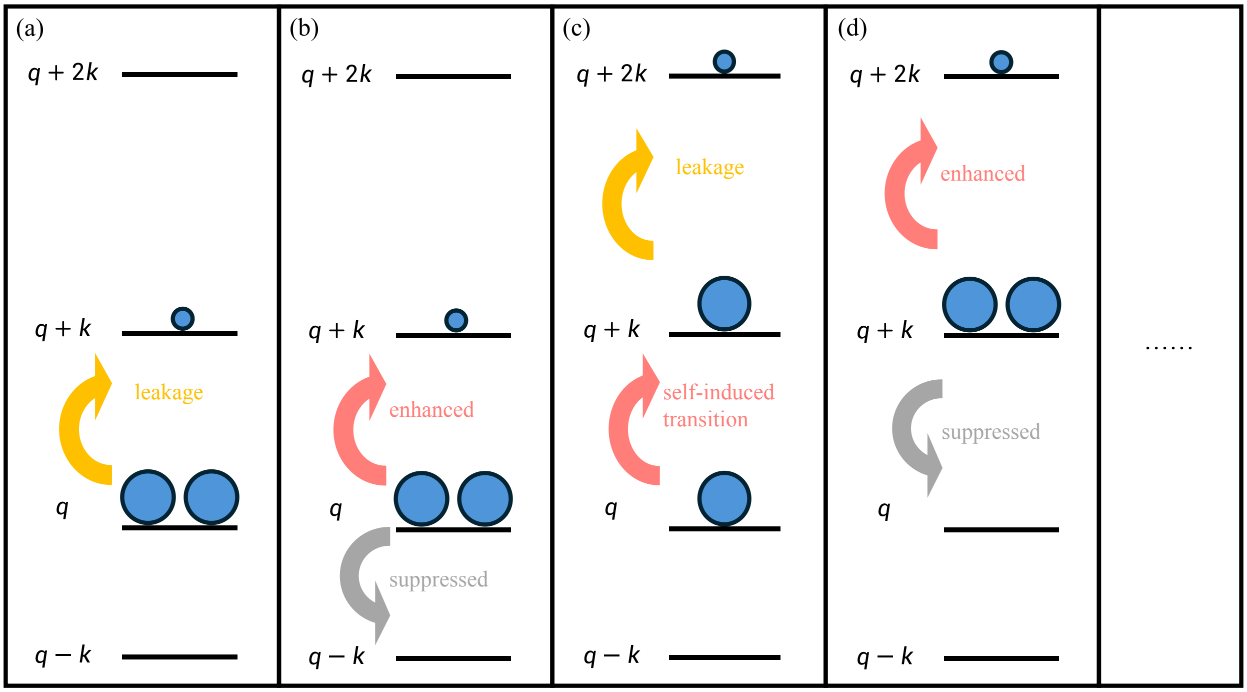

FIG. S3. Step-like acceleration of atoms in the blue-detuned regime. (a) Atoms initially populate the momentum state $|q\rangle$, and a small amount leaks into the state $|q+k\rangle$ due to incoherent scattering, quantum fluctuation, and finite wave package size. (b) The transition from $|q\rangle$ to $|q+k\rangle$ is enhanced, while transition from $|q\rangle$ to $|q-k\rangle$ is suppressed, see Supplementary Note 7. For low driving strengths, the process to populate $q+k$ is initially slow, leading to the slow evolution shown in figure 4b in the main text. (c) When the population in $|q+k\rangle$ is large enough, the dynamics is governed by the discussion in Supplementary Note 8. A small amount of atoms leaks from $|q+k\rangle$ to $|q+2k\rangle$ during the transition, see Supplementary Note 9. However, due to to the small gain in the one-photon regime, it again takes significant time until this growth takes off. (d) The process repeats starting from state $|q+k\rangle$ to higher momentum states.

the leakage is negligible. If the pumping η_p is however strong enough or the cavity detuning Δ_c is too small, transitions to momentum states higher than $q+k$ become non-negligible. Once many momentum modes are sufficiently occupied, components of the cavity field oscillate at all frequency differences between any two modes. These different frequency components cause the dephasing of the cavity field and the picture of self-induced oscillation of two levels breaks down.

I. SUPPLEMENTARY NOTE 9: BREAKDOWN OF 1-PHOTON TEMPORAL ORDER

In the previous section, we described the effective 2-level dynamics during a single period in the 1-photon temporal order, and how this picture breaks down. In this section, we formalize the above two points through analytical calculations. Starting from Eq. (S35), we impose the condition $|\tilde{c}_q|^2+|\tilde{c}_{q+k}|^2=1$ to decouple the equations of motion of $\tilde{c}_q$ and $\tilde{c}_{q+k}$,

$$\partial_t\tilde{c}_q=-iz(1-|\tilde{c}_q|^2)\tilde{c}_q,\quad \partial_t\tilde{c}_{q+k}=-iz^*(1-|\tilde{c}_{q+k}|^2)\tilde{c}_{q+k}. \tag{S38}$$

where $z=\frac{N\eta_\mathrm{p}^2}{\Delta_\mathrm{c}+i\kappa}=a-bi$ and

$$a=\frac{N\eta_\mathrm{p}^2\Delta_\mathrm{c}}{\Delta_\mathrm{c}^2+\kappa^2},\quad b=\frac{N\eta_\mathrm{p}^2\kappa}{\Delta_\mathrm{c}^2+\kappa^2}.$$

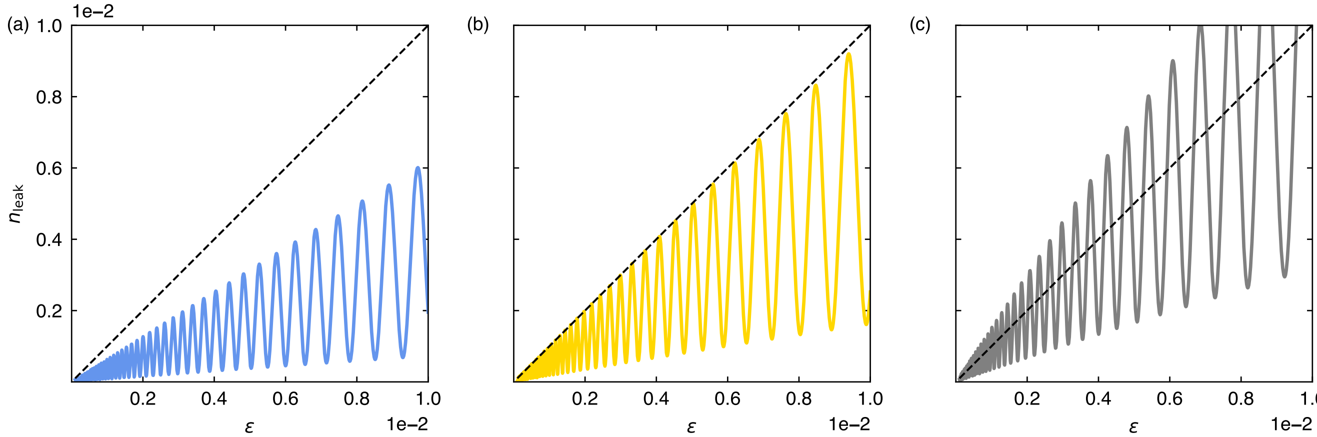

FIG. S4. Comparison between the initial occupation ϵ and occupation leakage n_{leak} after the transfer of atoms from the $q-k$ to the q mode at (a) $\Delta_c = 2\pi \times 25$MHz, $\eta_p = 2\pi \times 1$kHz (one-photon spatial-temporal order regime);(b)$\Delta_c = 2\pi \times 25$MHz, $\eta_p = 2\pi \times 1.15$kHz (phase boundary); (c)$\Delta_c = 2\pi \times 25$MHz, $\eta_p = 2\pi \times 1.3$kHz (beyond one-photon spatial-temporal order regime).

Let $\tilde{c}_q = A_q \exp(i\theta_q)$ and $\tilde{c}_{q+k} = A_{q+k} \exp(i\theta_{q+k})$, the amplitudes and phases obey the equations

$$\begin{aligned} \dot{A}_q &= -b(1 - A_q^2)A_q, \quad \dot{\theta}_q = -a(1 - A_q^2) = \frac{a}{b}\frac{\dot{A}_q}{A_q}, \\ \dot{A}_{q+k} &= b(1 - A_{q+k}^2)A_{q+k}, \quad \dot{\theta}_{q+k} = -a(1 - A_{q+k}^2) = -\frac{a}{b}\frac{\dot{A}_{q+k}}{A_{q+k}}. \end{aligned} \tag{S39}$$

We assume that the system is initially mostly populated in the $|q\rangle$ mode, while the $|q+k\rangle$ mode only has a small but nonzero population ϵ due to the finite initial size of the atomic cloud in real space (or quantum fluctuations, or incoherent scattering). We set the initial condition to $\tilde{c}_q(0) = \sqrt{1-\epsilon}\exp(i\theta_q(0))$, $\tilde{c}_{q+k}(0) = \sqrt{\epsilon}\exp(i\theta_{q+k}(0))$, and find as solutions

$$\begin{aligned} A_q(t) &= \sqrt{\frac{1-\epsilon}{(1-\epsilon) + \epsilon e^{2bt}}}, \quad \theta_q(t) = \theta_q(0) - \frac{a}{2b}\ln[(1-\epsilon) + \epsilon e^{2bt}], \\ A_{q+k}(t) &= \sqrt{\frac{\epsilon}{\epsilon + (1-\epsilon)e^{-2bt}}}, \quad \theta_{q+k}(t) = \theta_{q+k}(0) - \frac{a}{2b}\ln[\varepsilon + (1-\varepsilon)e^{-2bt}]. \end{aligned} \tag{S40}$$

The above results formalize the physical picture described in the previous section.

In the previous discussion, we considered a two-level model involving only the momentum modes $|q\rangle$ and $|q+k\rangle$, assuming that other momentum states are negligible during the transition of the atom from state $|q\rangle$ to $|q+k\rangle$. Specifically, we assumed that the transition from $|q+k\rangle$ to $|q+2k\rangle$ can be neglected. This assumption is valid only under certain conditions (Eq. (S37)). In fact, the small but non-zero transition from $|q+k\rangle$ to $|q+2k\rangle$ plays a crucial role. It determines whether the system will transition step by step to higher momentum states, as described in the picture from the previous section, or ultimately distribute its population among several momentum states, leading to cavity-field dephasing. From Eq. (S13), we get the equation of motion that $q+2k$ mode:

$$i\partial_t \mathcal{C}_{q+2k} = \omega_{q+2k}\mathcal{C}_{q+2k} + \eta_p \alpha^* \mathcal{C}_{q+k} = \omega_{q+2k}\mathcal{C}_{q+2k} + z^*[\mathcal{C}_q^* \mathcal{C}_{q+k}^2 + |\mathcal{C}_q|^2 \mathcal{C}_{q+2k}] \tag{S41}$$

Note that since we ignore the higher order leakage effect out of $|q\rangle$ to $|q+k\rangle$, here we only include 3 modes: $|q\rangle$, $|q+k\rangle$ and $|q+2k\rangle$. Applying $\tilde{c}_{q+2k} = \mathcal{C}_{q+2k}\exp(i\omega_{q+2k}t)$, we find in a rotating frame

$$\partial_t \tilde{c}_{q+2k} = -iz^*[e^{2i\omega_r t}\tilde{c}_q^* \tilde{c}_{q+k}^2 + |\tilde{c}_{q+k}|^2 \tilde{c}_{q+2k}]. \tag{S42}$$

The solution is

$$\tilde{c}_{q+2k}(t) = -iz^*[\int_0^t d\tau e^{2i\omega_r \tau}\tilde{c}_q^*(\tau)\tilde{c}_{q+k}^2(\tau)e^{-h(\tau)}]e^{h(\tau)}, \tag{S43}$$

where $h(\tau) = \frac{-iz^*}{2b} \ln[(1-\epsilon) + \epsilon e^{2b\tau}]$. At a fixed parameter point (Δ_c, η_p) in the phase diagram, the leakage to the $|q+2k\rangle$ state during an atom transfer from $|q\rangle$ to $|q+k\rangle$ depends on the initial population in $|q+k\rangle$, that is ϵ:

$$n_{\mathrm{leak}}[\epsilon, (\Delta_c, \eta_p)] = |\tilde{c}_{q+2k}(t = +\infty)|^2. \tag{S44}$$

We consider two scenarios. In the first scenario, if the leaked population is smaller than any arbitrarily small ϵ: $n_{\mathrm{leak}}[\epsilon, (\Delta_c, \eta_p)] \leq \epsilon$, the cavity field coupling the $|q\rangle$ and $|q+k\rangle$ states is automatically switched off once the population in the $|q\rangle$ state drops to zero after the atom transitions to the $|q+k\rangle$ state. Following this, the atoms in the $|q+k\rangle$ state start transitioning to the $|q+2k\rangle$ state due to the mechanism described in Supplementary Note 7, and the next cycle begins. In this case, the atoms can coherently transition step by step from one momentum state to higher momentum states, which corresponds to the dynamics under the one-photon temporal order. In the second scenario, if the leaked population exceeds the threshold: $n_{\mathrm{leak}}[\epsilon, (\Delta_c, \eta_p)] > \epsilon$, this implies that with each transition between momentum states, the population leaked to higher momentum states increases. After a few transitions, the leaked population becomes significant enough to trigger a "chain reaction," leading to the occupation of a large number of different momentum states. In this case, the picture described in Supplementary Note 8 breaks down. Atoms in multiple momentum states contribute multiple frequency components to the cavity field, placing the system in the dephasing regime of the phase diagram. For three different points (a) in the 1-photon temporal order phase; (b) on the phase boundary; (3) in the dephasing phase, we show the leakage as a function of ϵ in Fig. S4.

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