E Q U I V A R I A N T   K - T H E O R Y

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ABSTRACT

(For a more discursive and propagandist account of the contents I refer the reader to the introduction)

The purpose of this thesis is to present a fairly complete account of equivariant K-theory on compact spaces. Equivariant K-theory is a generalisation of K-theory, a rather well-known cohomology theory arising from consideration of the vector-bundles on a space. Equivariant K-theory, or K_G-theory, is defined not on spaces but on G-spaces, i.e. pairs (X,α), where X is a space and α is an action of a fixed group G on X, and it arises from consideration of G-vector-bundles on X, i.e. vector-bundles on whose total space G acts in a suitable way (cf 3.1). In this thesis G will always be a compact group.

But K_G-theory does not appear in the first three chapters, which are introductory. Chapter 1 consists of preliminary discussions of little relevance to the sequel, but which permit me to make a few propositions in the later chapters shorter or more elegant. It was intended to be amusing, and the reader may prefer to omit it.

Chapter 2 is devoted to the representation-theory of compact groups. When X is a point a G-vector-bundle on X is just a representation-module for G, so the representation-ring, or character-ring, R(G) plays a fundamental role in K_G-theory. In chapter 2 I investigate its algebraic structure, and in particular when G is a compact Lie group I determine completely its prime ideals. To do this I have to discuss first the space of conjugacy-classes of a compact Lie group, and outline an induced-representation construction for obtaining finite-dimensional modules for G from modules for suitable subgroups not of finite index.

Chapter 3 is a rather full collection of technical results concerning G-vector-bundles: they are all essentially well-known, but have not been

stated in the equivariant case.

Chapter 4 presents basic equivariant K-theory. I show that it can be defined in three ways: by G-vector-bundles, by complexes of G-vector-bundles, and by Fredholm complexes of infinite-dimensional G-vector-bundles. This chapter also treats the continuity of K_G with respect to inverse limits of G-spaces, the Thom homomorphism for a G-vector-bundle and the periodicity-isomorphism, and the question of extending K_G to non-compact spaces.

In chapter 5 I obtain for $K_G(X)$ a filtration and spectral sequence generalising those of [6], but without dissecting the space X. My method is based on a Cech approach: for each open covering of X I construct an auxiliary space homotopy-equivalent to X which has the natural filtration that X lacks.

Also in chapter 5 I prove the localisation-theorem (5.3), which, together with the theory of chapter 6, is one of the most important tools in applied K_G-theory. $K_G(X)$ is a module over the character-ring $R(G)$, so one can localise it at the prime ideals of $R(G)$, which I have determined in 2.5. The simplest and most important case of the localisation-theorem states that, if $\mathfrak{P}$ is the prime ideal of characters of G vanishing at a conjugacy-class γ, and if X^γ is the part of X where elements in γ have fixed-points, then the natural restriction-map $K_G(X) \to K_G(X^\gamma)$ induces an isomorphism when localised at $\mathfrak{P}$.

In chapter 6 I show how to associate to certain maps $f : X \to Y$ of G-spaces a homomorphism $f_! : K_G(X) \to K_G(Y)$. It is the analogue of the Gysin homomorphism in ordinary cohomology-theory; but it can also be regarded as a generalisation of the induced-representation construction of 2.4. In the important special case when f is a fibration whose fibre is a rational algebraic variety I prove that $f_!$ is left-inverse to the natural map $f^! : K_G(Y) \to K_G(X)$; and I apply that to obtain the general Thom isomorphism theorem.

Finally in chapter 7 I prove the theorem towards which my thesis was originally directed. Just as a G-module defines a vector-bundle on the classifying-space B_G for G (cf [1]), so a G-vector-bundle on X defines a vector-bundle on the space X_G fibred over B_G with fibre X. Thus one gets a homomorphism $\alpha : K_G(X) \to K(X_G)$. I prove that if $K_G(X)$ and $K(X_G)$ are given suitable topologies then in certain circumstances $K(X_G)$ is complete and α induces an isomorphism of the completion of $K_G(X)$ with $K(X_G)$. This generalizes the theorem of Atiyah-Hirzebruch that $R(G)^\wedge \cong K(B_G)$.

CONTENTS

INTRODUCTION

The purpose of this thesis is to develop equivariant K-theory. K-theory is the art of constructing a cohomology-theory out of the vector-bundles on a space: I think it will need no introduction to the reader ([6], [5], [7], etc.). Equivariant K-theory is a generalisation obtained as follows. One fixes a topological group G, and considers G-spaces X, i.e. spaces X together with a given continuous action of G on X. A G-vector-bundle E on the G-space X is an ordinary vector-bundle $E \xrightarrow{p} X$ together with an action of G on E such that p is equivariant and such that the map induced by $g \in G$ from the vector-space fibre $E_x = p^{-1}(x)$ at x in X to the fibre $E_{g,x}$ is linear for all g and x. G-vector-bundles on X can be added fibrewise: $(E \oplus F)_x = E_x \oplus F_x$; and their isomorphism-classes form an abelian semigroup whose associated abelian group is $K_G(X)$, the basis group of equivariant K-theory. One can form a tensor-product of G-vector-bundles, and it induces a structure of ring on $K_G(X)$.

By way of propaganda for equivariant K-theory I shall make here just two remarks. The first is that G-vector-bundles are very common in nature. For instance, if the base-space X is a point, a G-vector-space is the same thing as a representation-module for G, and $K_G(X)$ is isomorphic to the "character-ring" $R(G)$ of G; so G-vector-bundles are a generalisation of group-representations. On the other hand they generalise ordinary vector-bundles (the case $G = \{1\}$); and whenever, say, a group acts on a differentiable manifold, all the vector-bundles naturally associated to the manifold, such as the tangent-bundle, are G-vector-bundles. Finally, there is the "homogeneous" vector-bundle: if H is a subgroup of G, and M a module for H, one can form on the quotient-space G/H a G-vector-bundle $E = (G \times M)/H$ – H acts on $G \times M$ by $(h; g, m) \mapsto (gh^{-1}, hm)$ – which is important in connection with group-representations as its space of sections is what is usually called the "representation of G induced from M".

My second remark is that equivariant K-theory is often useful for obtaining, or for making perspicuous, results in ordinary K-theory. On the simplest level this is because one can often regard a space as the quotient of a more convenient space by a group-action. It is hard to produce a profound example without going too far afield, but I can mention the case of "cohomology-operations in K-theory" [3a]. An obvious operation on vector-bundles is the k^{th} tensor-power $E \mapsto E \otimes \ldots \otimes E$ (k factors). It induces a map $K(X) \to K_G(X)$, where G is the symmetric group on k symbols, and is made to act trivially on X. Because G acts trivially on X $K_G(X) \cong R(G) \otimes K(X)$ (cf 4.1 below), so for each homomorphism $R(G) \to \mathbb{Z}$ one gets a natural operation $K(X) \to K(X)$; and in fact all such operations can be obtained in this way. On the other hand if one had not observed that the tensor-power had a G-action one would have obtained only the uninteresting operation $\xi \mapsto \xi^k$.

But from a more cynical point of view one can say that equivariant K-theory is interesting and non-trivial because the Bott periodicity of ordinary K-theory, in the form of [5], happens to extend – without change – to equivariant K-theory.

To turn to the structure of the present work: anyone writing about

K-theory is embarrassed by the fact that although there is no proper exposition of the subject in the literature there have been a large number of bites at the apple, so that most results appear somewhere, but usually only with ad hoc generality. I have decided without any conviction to present here a fairly complete and general account in what seems to have emerged as the canonical presentation; because most of the material is essentially well-known my exposition is compressed, and will appeal at best only to the converted.

Chapter 1 consists of preliminary discussions which permit me to make a few propositions in the later chapters shorter or more elegant. It was intended to be amusing, but the reader may prefer to omit it.

Chapter 2 is devoted to the representation-theory of compact groups; I describe in detail the structure of the character-ring $R(G)$. This chapter is preliminary to the body of the thesis, and it goes into more detail than is subsequently required, so I refer the reader to the summary on page 2.1.

Chapter 3 is a rather full collection of technical results on G-vector-bundles: they are all essentially well-known, but have not been stated in the equivariant case.

Chapter 4 presents basic equivariant K-theory. I show that it can be defined in three ways: by G-vector-bundles, by complexes of G-vector-bundles (cf [7] Part II), and by Fredholm complexes of infinite-dimensional G-vector-bundles. All three methods are from time to time useful. Only my discussion of Fredholm complexes is original, though even there the idea has been well-known for some time. The chapter also treats the continuity of K_G, the Thom homomorphism for a G-vector-bundle and the periodicity isomorphism, and the question of extending K_G to non-compact spaces.

In chapter 5 I obtain for $K_G(X)$ a filtration and spectral sequence generalising those of [6], but without dissecting the space X. To restrict oneself in classical K-theory to the category of triangulable spaces is, however ugly, not really disabling; but to restrict oneself to G-spaces with some kind of equivariant decomposition would be much more embarrassing. The device I use to avoid dissecting the space is probably, I think, the most interesting novelty in this thesis. It is based on a Cech approach: for each open covering of X I construct an auxiliary space homotopy-equivalent to X which has the natural filtration that X lacks.

Also in chapter 5 I prove the localisation theorem (5.3), which, together with the theory of chapter 6, is one of the most important tools in applied K_G-theory. One should think of it as an attempt to describe elements of $K_G(X)$ in the way that representations of G are described by their characters, functions on G. $K_G(X)$ is a module over the character-ring R(G), so one can localise it at any prime ideal of R(G). Now prime ideals of R(G) correspond very roughly to conjugacy-classes of G (the proper statement is in 2.5), and looking at the image of an element of $K_G(X)$ in the localized modules can be regarded as the analogue of looking at the value of the character of a G-module at the conjugacy-classes of G. The simplest and most important case of the localisation-theorem states that, if $\wp$ is the prime ideal consisting of characters of G which vanish at a conjugacy-class γ, then an element of $K_G(X)_\wp$ depends only on its restriction to X^γ, the part of X where elements in γ have fixed-points, in fact

$K_G(X)_\wp \xrightarrow{\tilde{=}} K_G(X^\gamma)_\wp$.

The contents of chapter 6 is also very important: most of the more significant applications of equivariant K-theory depend on it. I show that one can associate to certain maps $f : X \to Y$ of G-spaces a homomorphism $f_! : K_G(X) \to K_G(Y)$, not of rings but of $K_G(Y)$-modules. It corresponds to the Gysin homomorphism in ordinary cohomology-theory; but it is more enlightening to regard it as a generalisation of the "induced representation" construction of the theory of group-representations (of 2.4). In the important special case when f is a fibration whose fibre is a rational algebraic variety I prove that $f_!$ is left-inverse to the natural map $f^! : K_G(Y) \to K_G(X)$; and I apply that to obtain the general Thom isomorphism theorem.

For a preliminary account of some the the applications which have been made of the theory of chapters 5 and 6 the reader can refer to [8].

Finally in chapter 7 I prove the theorem towards which my thesis was originally directed. Just as a G-module defines a vector-bundle on the classifying-space B_G for G (of [1]), so a G-vector-bundle on X defines a vector-bundle on the space X_G fibred over B_G with fibre X. Thus one gets a homomorphism $\alpha : K_G(X) \to K(X_G)$. If these two groups are given suitable topologies then α is continuous, and I show that in suitable circumstances $K(X_G)$ is complete and α induces an isomorphism $K_G(X)^\wedge \tilde{=} K(X_G)$. This generalises the theorem of Atiyah-Hirzebruch that $R(G)^\wedge \tilde{=} K(B_G)$; but I know of no applications, except the following possibly empty one: if $f : X \to Y$ is a G-map of compact G-spaces which is a homotopy-equivalence, though not necessarily a G-homotopy-equivalence, then in certain circumstances

$f^!$ induces an isomorphism of the completions $K_G(Y)^\wedge \to K_G(X)^\wedge$.

I should mention in conclusion that I have used (from chapter 3 on) a definition of G-vector-bundle more general than that above: I have assumed that G is equipped with a homomorphism α into the Galois group of $\mathbb{C}$ over $\mathbb{R}$, and that G acts on $\mathbb{C}$ whenever it occurs by means of α. That is to say, I consider "antilinear" actions of G on complex vector-spaces. By this device I have included without extra labour the "real K-theory" of [3]; but as my subject is complex K-theory I have refrained from pursuing any question peculiar to the real case.

ACKNOWLEDGEMENTS

I have tried as far as possible to indicate by references the sources of ideas I have used which are not my own. But my indebtedness to my supervisor, Professor M. F. Atiyah, F.R.S., is so great that it was not possible to acknowledge it specifically throughout, and it must suffice for me to say here that almost every section either was suggested by him or has profited by his advice.

I am also grateful to very many other people with whom I have discussed my subject-matter, and whose ideas I have consciously or unconsciously used: I would like to mention especially Professors R. Bott and C. T. C. Wall.

NOTATIONS

I shall use the following standard notations.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote respectively: the natural numbers $0,1,2,\ldots$, the integers, the rational numbers, the real numbers, and the complex numbers.

D^q denotes the standard q-dimensional disc in $\mathbb{R}^q$, and S^{q-1} is its boundary $(q-1)$-sphere.

Δ^q is the standard q-simplex in $\mathbb{R}^{q+1}$, i.e.

$$\Delta^q = \{(\lambda_0, \ldots, \lambda_q) : \lambda_i \geq 0, \Sigma \lambda_i = 1\}.$$

T^1 is the group of complex numbers of modulus one.

I use the sign $\coprod$ for the sum in any category (and Π or $\times$ for the product).

Chapter One

PRELIMINARIES

1.1 Realization functors

In this section I shall review briefly an abstract construction whose usefulness will be shown, I hope, by the examples below.

Let F be a category, and $i : F \to T$ a functor into a category with direct limits. $\hat{F}$ denotes the category of presheaves of sets on F, i.e. of functors $F^{o} \to \underline{\mathrm{Ens}}$.

i induces two functors, $i_* : \hat{F} \to T$ and $i^* : T \to \hat{F}$, as follows:

if $X \in T$, i^*X is the presheaf $S \mapsto \mathrm{Mor}(iS; X)$;

if $A \in \hat{F}$ is a presheaf, then i_*A, called the realization of A, is $\varinjlim_{F/A} i$, where F/A is the category whose objects are pairs (S,a) with $S \in F$ and $a \in A(S)$, and whose morphisms from (S,a) to (S',a') are the morphisms $\phi : S \to S'$ of F such that $\phi^*(a') = a$; i is regarded as a functor on F/A by $i(S,a) = i(S)$.

i_* and i^* are adjoint: $\mathrm{Mor}(A; i^*X) \cong \mathrm{Mor}(i_*A; X)$. For the left-hand-side is simply $\varprojlim_{F/A} i^*X = \varprojlim_{(S,a)\in F/A} \mathrm{Mor}(iS; X) \cong \mathrm{Mor}(\varinjlim_{(S,a)} iS; X) = \mathrm{Mor}(i_*A; X)$

Examples

(i) F = Fin = category of finite sets (resp. F = Ord = category of finite totally ordered sets)

T = Top = category of topological spaces

$i = \Delta$: Fin $\to$ Top assigns to $S \in F$ the standard simplex Δ_S with S as its vertices.

Then $\hat{F}$ is the category of simplicial (resp. semi-simplicial) sets - in the sense of Grothendieck [19] - and Δ^* : Top $\to \hat{F}$ assigns to a space its "singular complex", while Δ_* assigns to a "complex" its geometrical realisation - as defined by Milnor [23] in the semi-simplicial case.

(ii) F = category of open subsets of a space X

T = Top/X = category of spaces over X

$i : F \to T$ is the inclusion.

Then $\hat{F}$ is the category of presheaves on X. i^* assigns to a space over X its sheaf of sections, and i_* assigns to a presheaf the associated space étalé over X constructed by Cartan ([18] page 110).

(iii) F = Cpt = category of compact spaces

T = Top; i : Cpt $\to$ Top is the inclusion.

One might call $\underline{\mathrm{Cpt}}^\wedge$ the category of pre-quasi-spaces. Then i^* assigns to a space the associated quasi-space as defined by Spanier [30], and i_* assigns to a pre-quasi-space its natural geometrical realisation.

1.2 A generalisation

I have explained how to extend a functor $i : F \to T$ to the category of presheaves of sets on F. This is not quite general enough for my purpose: if there is given another category C and a pairing $C \times T \to T$ then one can define a "realisation" $i_* : \hat{F}_C \to T$ of the category of presheaves on F with values in C.

To see this observe that the definition $i_*A = \varinjlim_{F/A} i$ of the realisation of a set-valued presheaf can be rewritten more explicitly

$$i_*A = \varinjlim_{(S_0 \xrightarrow{t} S_1) \in F/F^\circ} i(S_0) \otimes A(S_1),$$

where F/F° is the category whose objects are morphisms $S_0 \xrightarrow{t} S_1$ of F and whose morphisms from t to t' are pairs of morphisms $\phi_0 : S_0 \to S_0'$, $\phi_1 : S_1' \to S_1$ such that $\phi_1 t' \phi_0 = t$. (If E is a set and X is an object of some category then $E \otimes X$ means the sum "of E copies of X"; it exists here because I assume T has direct limits. Similarly $\mathrm{Map}(E;X)$ means the product "of E copies of X".)

The advantage of the new formulation is that it is meaningful for a presheaf A with values in C when one interprets $\otimes$ as the given pairing.

Notice that if $X \in T$ then $i_*i^*X \cong \varinjlim_{(S_0 \to S_1)} i(S_0) \otimes \mathrm{Mor}(i(S_1); X)$, which puts in evidence the adjuction-morphism $i_*i^*X \to X$.

<u>Examples</u>

(1) Let G be a finite group, k a field of characteristic prime to the order of G, and $k[G]$ the group-ring of G over k.

Let T be the category of $k[G]$-modules, F the full subcategory of simple $k[G]$-modules, $i : F \to T$ the inclusion, and C the category of vector-spaces over k. Then an object E of T defines a presheaf i^*E on F with values in C by $M \mapsto \mathrm{Hom}_{k[G]}(iM; E)$. Conversely a presheaf $A : F^\circ \to C$ has a realisation i_*A in T. Because morphisms in F are either zero or isomorphisms it is easy to see that $i_*A \cong \coprod_{M \in F} M \otimes_{D_M} A(M)$, where D_M is the skew-field $\mathrm{End}(M)$. It is well-known that i_* and i^* are inverse equivalences. (In particular if one takes $E = k[G]^*$ then $\mathrm{Hom}_{k[G]}(M;k[G]^*) \cong \mathrm{Hom}_{k[G]}(k[G];M^*) \cong M^*$, so one obtai

$\coprod_{M\in F} \mathrm{End}_{D_M}(M) \to k[G]^*$, and dually $k[G] \to \prod_{M\in F} \mathrm{End}_{D_M}(M)$, the decomposition of the group ring into "simple matrix algebras".)

(ii) Let G be a topological group. One can define a semi-simplicial G-space G_* by $G_*(S) = \mathrm{Map}(S; G)$, S being a finite ordered set. (It would seem more natural to regard G_* as a simplicial G-space, but that turns out to be inexpedient.) If $\Delta : \underline{\mathrm{Ord}} \to \underline{\mathrm{Top}}$ is as in 1.1 Ex.(i) then

$\Delta_*(G_*) = \varinjlim_{(S_0 \to S_1)} \Delta_{S_0} \times \mathrm{Map}(S_1; G)$ is the universal G-space EG defined by Milnor [24]. To be precise, it is not quite Milnor's $G * G * G * \ldots$, but is the space obtained from that by collapsing degenerate simplexes - i.e. those in which two consecutive vertices are the same element of G. In most cases - e.g. if G is compact - it has the same homotopy-type as $G * G * G * \ldots$. I shall return to this point later.

(iii) Let K be a category. I shall define a topological space BK such that if K is a group, regarded as a category with one object, then BK is a classifying-space for K.

Let $i : \underline{\mathrm{Ord}} \to \underline{\mathrm{Cat}}$, the category of categories, be the inclusion. It induces $i^* : \underline{\mathrm{Cat}} \to \underline{\mathrm{Ord}}^*$ - $i^*K(S) = \mathrm{Funct}\,(S; K)$ - i^*K can be thought of, according to taste, as the "singular complex" or as the "nerve" of K. Define $BK = \Delta_* i^* K = \varinjlim_{\underline{\mathrm{Ord}}/K} \Delta \cong \varinjlim_{(S_0 \to S_1)} \Delta_{S_0} \times \mathrm{Funct}\,(S_1; K)$. Thus B is a functor: <u>Cat</u> $\to$ <u>Top</u>.

<u>Remarks</u>

(a) If K "is" a group G, and EG is defined as in example (ii), then the maps Map $(S; G) \to$ Funct $(S; G)$ defined by $f \mapsto \{(\alpha\to\beta) \mapsto f(\alpha)^{-1}f(\beta)\}$ induce a map EG $\to$ BG which identifies BG with EG/G.

(b) In the next section I shall extend this definition so that it will include, among other things, the case of a topological group.

(iv) Let X be a space, and $\{U_\alpha\}_{\alpha\in\Sigma}$ a covering of X. If $\sigma \subset \Sigma$ define $U_\sigma = \bigcap_{\alpha\in\sigma} U_\alpha$. The spaces U_σ for σ finite, together with their inclusion maps, form a full subcategory R of <u>Top</u>/X. i*R (notation as above) is the barycentric subdivision of the nerve, in the usual sense, of the covering; so BR is the usual geometric nerve. It will appear in chapter 5 that it is reasonable to define also a "universal space" ER over BR.

1.3 Topological categories

A topological category is a category C in the category of topological spaces (cf [19]) i.e. it is an ordinary category but with a topology on the sets ob(C) and mor(C) such that the structural maps $ob(C) \rightleftarrows mor(C)$ and the composition $mor(C) \times_{ob(C)} mor(C) \to mor(C)$ are continuous.

We have seen that an ordinary category defines a semi-simplicial set, and correspondingly a topological category defines a semi-simplicial space C_* :

$$o, \quad m, \quad m \times_o m, \quad m \times_o m \times_o m, \quad \ldots \quad ,$$

where $o = ob(C)$, $m = mor(C)$. As before I shall write BC for the realisation $\Delta_*(C_*)$ of this semi-simplicial space, and call it the classifying-space for C.

I have pointed out already that the classifying-spaces so defined are a little squashed. The space defined by Milnor is spread out over the infinite simplex Δ^∞ – the realization of any semi-simplicial space $S \mapsto A(S)$ can be so spread out, if desired, by replacing A by $S \mapsto \coprod_{t: S \to \mathbb{N}} A(\text{coimage } t)$. The virtue of the squashed construction is that it commutes with products: $B(C \times C') \xrightarrow{\cong} BC \times BC'$; to be truthful, it commutes with products when one works in the category of compactly generated Hausdorff spaces. (The proof is the same as the classical proof that the realization of a semi-simplicial set commutes with products.)

One property of the functor B is:

Proposition 1.3.1 If F_0, F_1 : $C \to C'$ are two functors and there is a morphism of functors $F : F_0 \to F_1$ then BF_0 and BF_1 are homotopic.

Proof: F can be regarded as a functor $C \times J \to C'$, where J is the category $(\cdot \longrightarrow \cdot)$. But B commutes with products, and BJ = I, the unit interval, so $BF : BC \times I \to BC'$ is a homotopy.

Remarks: Thus C can be replaced by an equivalent category without changing the homotopy-type of BC. Another example is: an inner automorphism of a topological group induces a map homotopic to the identity in the classifying-space. But I think the following example is the most interesting. The universal space for a group G can be written BC, where C is a category with ob(C) = G, mor(C) = G × G (1.2 Ex.(ii)). This category is plainly equivalent to C' with ob(C') = point, mor(C') = point. Hence EG is contractible.

I conclude with another example. The classifying-space for a group G is BG where G is the category with ob(G) = point, mor(G) = G. A locally trivial principal G-fibration over a space X is determined by a functor

$f : \bar{R} \to G$, where $\bar{R}$ is a category with $ob(\bar{R}) = \coprod U_\alpha$ and $mor(\bar{R}) = \coprod U_{\alpha\beta}$, $\{U_\alpha\}$ being a covering of X with respect to which the fibration is trivial. (Any equivalence-relation on a space Y can be regarded as a topological category C with ob(C) = Y.) The fibration determines f up to equivalence. So it determines up to homotopy a map $B\bar{R} \to BG$. If the covering is numerable then $B\bar{R}$ – the universal space mentioned in 1.2 Ex.(iv) – is homotopy-equivalent to X (see 5.1), so one has a map $X \to BG$ determined up to homotopy.

<u>Appendix</u> Of course most everyday categories are so large that it is embarrassing to put a topology on their set of objects. This embarrassment can be relieved by remarking that among all categories equivalent to a category C there is a last one $\tilde{C}$, whose set of objects is <u>isom</u> C, the set of isomorphism-classes of objects of C. Frequently there is a sensible topology on $\tilde{C}$. Then C should be topologised by calling a set of ob(C) or mor(C) open if it is the inverse-image of an open set by the functor $C \to \tilde{C}$. Even then not enough categories are topological categories. For example one can give the set of isomorphism-classes of topological spaces the discrete topology, but there is no suitable way of topologising Map(X; Y) for two general spaces X, Y. The problem disappears if one replaces the category of topological spaces by the category of compactly

generated Hausdorff spaces, or by the category of quasi-spaces of Spanier.

1.4 Objects with group-action

If G is a topological group and C is a topological category I shall write $G - C$ for the category $\mathrm{Funct}(G; C)$, where G is regarded as a category in the usual way. $G - C$ is called the category of "objects of C with G-action".

To be explicit: an object of $G - C$ is a pair (X,a), where X is an object of C, and $a : G \to \mathrm{End}_C(X)$ is a homomorphism of topological semigroups; a morphism $f : (X,a) \to (X',a')$ in G-C is a morphism $f : X \to X'$ in C such that $f \circ a(g) = a'(g) \circ f'$ for all $g \in G$.

$G \to \{1\}$ induces a functor $T : C \to G - C$. The left- and right-adjoints of T, when they exist, are written $X \mapsto X/G$ and $X \mapsto \Gamma^G X$ respectively. (But sometimes I write X^G for $\Gamma^G X$.)

If (X,a), (X',a') are objects of $G - C$ then $\mathrm{Mor}(X; X')$ will always denote $\mathrm{Mor}_C(X; X')$. It should be regarded as a space with G-action. $\mathrm{Mor}_{G-C}((X,a); (X',a'))$ will always be denoted by $\mathrm{Mor}^G(X; X')$. Evidently $\mathrm{Mor}^G(X; X') = \Gamma^G \mathrm{Mor}(X; X')$.

The sum and the product $C \times C \to C$, if they exist, induce the sum and the product $(G - C) \times (G - C) \to (G - C)$. This means that many properties of C carry over to $G - C$. For example: if C is additive, or abelian, then so is $G - C$.

Bus I shall make the following convention. When I am concerned with categories of topological spaces for which there is no sensible topology on $\mathrm{End}(X)$, I shall regard a map $G \to \mathrm{End}(X)$ as continuous when the adjoint map $G \times X \to X$ is continuous.

1.5 G-spaces and slices

I shall be concerned throughout this thesis with spaces with G-action or G-spaces. But the only general theorem about them I shall need to use is the existence of slices, which I shall now explain.

If X is a G-space and $x \in X$ and G_x will always denote the stabiliser of X, i.e. $G_x = \{g \in G : g.x = x\}$. If $\{x\}$ is closed in X then G_x is a closed subgroup.

Definition 1.5.1 If X is a G-space and $x \in X$, a slice at x is a G_x-stable subspace S of X containing x such that

(i) $G.S$ is a neighbourhood of x, and

(ii) the G-action $G \times S \to X$ induces an isomorphism of G-spaces: $(G \times S)/G_x \to G.S$.

Thus if there is a slice at x there is an equivariant retraction of the neighbourhood $G.S$ of the orbit G/G_x of x. If X is locally compact the existence of such a retraction is easily seen to be equivalent to the existence of a slice.

Proposition 1.5.2 If X is a completely regular G-space there is a slice at every point of X.

This theorem is due to Mostow; a proof can be found in chapter 7 of [9].

1.6 Homotopy theory and K-theory

There is a functor T : Ens $\to$ Top which assigns to a set the discrete topology. One would define π_0 : Top $\to$ Ens as the left-adjoint of T, but unfortunately T is not left-exact (it doesn't commute with products), so it cannot have a left-adjoint. The best one can do is to define π_0 as the left-adjoint of the inclusion of the category of 0-dimensional spaces in

Top. (In Top the 0-dimensional spaces - those with a basis consisting of open and closed sets - form the smallest full subcategory containing the discrete spaces and closed under inverse limits.) π_0 will of course be right-exact.

π_0 acquires much more interest as a functor on the category of topological groups (with values in the category of totally disconnected groups). $\pi_0(G)$ is then G/G^0, where G is the identity-component of G. If $1 \to G' \to G \to G'' \to 1$ is a short exact sequence of topological groups then $\pi_0 G' \to \pi_0 G \to \pi_0 G'' \to 1$ is exact, and the kernel of the first map is abelian. The important thing is that one can define left-derived functors π_i ($i \in \mathbb{N}$) characterised by a universal long exact sequence

$$\ldots \to \pi_2(G'') \xrightarrow{d} \pi_1(G') \to \pi_1(G) \to \pi_1(G'') \xrightarrow{d} \pi_0(G') \to \pi_0(G) \to \pi_0(G'') \to 1.$$

The category of topological groups is a full subcategory of the category of spaces-with-composition or H-spaces, so one can extend π_0 to a functor on the latter, with values still in the category of totally disconnected groups. In this form it is the subject of classical homotopy-theory. (One speaks there usually of the composite of the π_i with the trivial exact functor which associates to a space its H-space of loops.)

The category of H-spaces is a full subcategory of the category of the category of topological categories-with-composition: a space X should be identified with the category $X \rightrightarrows X$, its trivial prolongation as a semi-simplicial space. So π_0 can be extended to a functor on the category of topological categories with composition, with values still in the category of totally disconnected group. In this guise I shall write it K instead of π_0, and call it the Grothendieck group. Thus if A is a category K(A) is the group $\pi_0(\underline{\text{ob}}\ A/R)$, where R is the equivalence relation $\underline{\text{mor}}\ A \rightrightarrows \underline{\text{ob}}\ A$.

I shall be concerned only with the case of topological additive categorie with the direct sum as composition. Then to get a non-zero K(A) one must agree to discard all the morphisms except isomorphisms. Thus $K(A) = \pi_0(\underline{\text{isom}}\ A)$.

If A is a topological additive category I shall write C(A) for the category of bounded complexes $(E^\bullet, d^\bullet)$ of A, with chain-homotopy classes of morphisms; i.e. E^q is an object of A for $q \in \mathbb{Z}$, with $E^q = 0$ for $|q|$ large; $d^q : E^q \to E^{q+1}$ is such that $d^{q+1} \circ d^q = 0$; morphisms are families $f^q : E_1^q \to E_2^q$ commuting with $d_1^\bullet$, $d_2^\bullet$, but $f_0^\bullet$ and $f_1^\bullet$ are identified if there are $h^q : E_1^q \to E_2^{q-1}$ $(q \in \mathbb{Z})$ such that $f_0^q - f_1^q = d^{q-1} \circ h^q + h^{q+1} \circ d^q$ (cf [31]). C(A) is a topological additive category, and there is a canonical inclusion $i : A \to C(A)$.

<u>Proposition 1.6.1</u> If A is a real topological additive category (i.e. mor(A) is a real vector-space in the category of spaces over ob A × ob A), then $i : A \to C(A)$ induces an isomorphism of Grothendieck groups.

<u>Proof</u>: Define a map $\text{ob}\, C(A) \to \pi_0(\text{ob}(A))$ by $(E^\bullet, d^\bullet) \mapsto \sum_{q\in} (-1)^q [E^q]$. It evidently induces $\chi : K(C(A)) \to K(A)$ such that $\chi \circ K(i) = \text{id}$. But $(E^\bullet, d^\bullet)$ is joined by a path in ob C(A) to $(E^\bullet, 0)$, whose image in $\pi_0(\text{ob}\, C(A))$ differs from $\Sigma(-1)^q[E^q]$ by addition and subtraction of complexes homotopic to elementary complexes $\ldots \to 0 \to E \xrightarrow{\text{id}} E \to 0 \to \ldots$ which are isomorphic to zero in C(A). So also $K(i) \circ \chi = \text{id}$.

There are two reasons for considering C(A) in preference to A. The first is that the set of path-components of the set of isomorphism classes of C(A) has an inverse operation for $\oplus$: the translation T of a complex by one degree. In fact $E^\bullet \oplus TE^\bullet$ is the mapping-cone of $0 : E^\bullet \to E^\bullet$, and is connected by a path to the mapping-cone of $\text{id} : E^\bullet \to E^\bullet$, which is isomorphic to zero in C(A). This means that $\text{ob}\, C(A) \to K(C(A))$ is set-theoretically surjective, which is convenient.

The second reason is that one may have a class of preferred short exact sequences in A, and would like to identify [E] with [E'] + [E"] in K(A) when there is a preferred short exact sequence $0 \to E' \to E \to E'' \to 0$

in A. That can be done very simply by replacing C(A) by D(A), its derived category with respect to the subcategory of complexes which are acyclic with respect to the given notion of exactness. (See [31]) This will be important in 4.5.

What I have said in this section for topological categories can be said for categories in any category F. Instead of totally-disconnected spaces one should use, I suppose, the subcategory of F obtained by adjoining projective limits to the subcategory of discrete objects. (An object is discrete if it is a sum of copies of the final object.) But I don't know any interesting applications.

The functor K, as I have defined it, is right-exact, i.e. $\varinjlim K(A_\alpha) \xrightarrow{\cong} K(\varinjlim A_\alpha)$. One would like to say that it has left-derived functors $\{K^{-i}\}_{i\in\mathbb{N}}$ characterised by a universal "Mayer-Vietoris" sequence:

$$\ldots \to K^{-q}(B \times_A B) \to K^{-q}(B) \oplus K^{-q}(B) \to K^{-q}(A) \xrightarrow{d} K^{-q+1}(B \times_A B) \to$$
$$\ldots \to K(A) \to 1$$

whenever A is a quotient of B. I don't know how to construct these in general. But if ob(A) and mor(A) are locally trivial spaces then one can reasonably define $K^{-q}(A)$ by a "singular" process analogous to the definition of the homotopy groups for a locally trivial H-space X:

$$\pi_q(X) = \ker\colon \pi_0 \operatorname{Map}(S^q; X) \to \pi_0 X,$$

or better: $\pi_q(X)$ is the q^{th} homology group of the complex

$$\ldots \to \operatorname{Map}(\Delta^q; X) \to \operatorname{Map}(\Delta^{q-1}; X) \to \ldots \to \operatorname{Map}(\Delta^1; X) \to X.$$

(A space X is locally trivial if whenever B is a closed subspace of a compact space C, and there is given $f : B \to X$, then f can be extended to a neighbourhood of B in C. If X is locally trivial and compact it is locally contractible, as one sees by extending $pr_1 : X \times I \to X$ to a neighbourhood of $X \times I$ in $X * X$.)

1.7 Remarks about topological vector-spaces

The purpose of this section is to establish some conventions for the sequel.

In this thesis "vector-space" will mean "Hausdorff locally convex vector-space over $\mathbb{C}$". The category of vector-spaces is a topological category: I shall always give $\operatorname{Hom}(E; F)$ the topology of uniform convergence on compact sets.

$\mathrm{Hom}(E;\mathbb{C})$ is called the <u>dual</u> of E, and denoted by E'. Evidently E' is complete if E is compactly generated. A morphism $f : E \to F$ induces a morphism ${}^t f : F' \to E'$. It follows from the Hahn-Banach theorem that f is injective (resp. surjective) if and only if ${}^t f$ is surjective (resp. injective): notice that a surjection in the category is a map whose image is dense in the target-space.

I shall call a bilinear map $f : E \times E' \to F$ <u>hypocontinuous</u> if $f|C \times E'$ and $f|E \times C'$ are continuous for all compact $C \subset E$, $C' \subset E'$. Hypocontinuous maps $E \times E' \to F$ correspond precisely to morphisms $E \to \mathrm{Hom}(E'; F)$ if E' is compactly generated.

<u>Proposition 1.7.1</u> If X is a compact space there is a complete vector-space M_X containing X as a subspace and characterised by the fact that the inclusion $X \to M_X$ has the universal property for maps of X into complete vector-spaces. Furthermore

(i) $M_X \cong C_X'$, where C_X is the Banach space $\mathrm{Map}(X;\mathbb{C})$

(ii) C_X, and therefore M_X, is reflexive

(iii) if $M_X^a = \{\mu \in M_X : |\langle\mu,f\rangle| \leq a\|f\| \text{ for all } f \in C_X\}$ for $a \in \mathbb{R}_+$ then M_X^a is compact

(iv) if E is a complete vector-space then $X \times \mathrm{Map}(X; E) \to E$ is induced by a hypocontinuous map $M_X \times \mathrm{Map}(X; E) \to E$.

(v) if Y is another compact space there is a unique hypocontinuous map $M_X \times M_Y \to M_{X\times Y}$ extending the inclusion $X \times Y \to M_{X\times Y}$.

Proof: Let us define $M_X = C_X'$. First observe that because M_X^a is an equicontinuous subset of M_X the topology of compact convergence coincides on it with the weak topology (Bourbaki, Esp. vect. top. III §3 no.5 prop.5). And M_X^a a compact for the weak topology. Furthermore any compact set S of M_X is contained in some M_X^a, otherwise one could choose for each integer k some $f_k \in C_X$ such that $||f_k|| < k^{-\frac{1}{2}}$ and $|\langle \mu, f_k\rangle| > k^{\frac{1}{2}}$ for some $\mu \in S$, and that is impossible because $\{0, f_1, f_2, \ldots\}$ is a compact set in C_X.

In general, compact sets of C_X are those closed bounded sets A with the property that for each $\epsilon > 0$ there is a finite partition of unity $\{\phi_i\}$ on X, and points $\{x_i\}$ such that $||f - \Sigma f(x_i)\phi_i|| < \epsilon$ for all $f \in A$. Reciprocally one finds that elements of M_X^1 can be approximated by measures $\Sigma \lambda_i x_i$ with $\Sigma|\lambda_i| \leqslant 1$.

I shall show that for any complete vector-space E the restriction $i^* : \mathrm{Hom}(M_X; E) \to \mathrm{Map}(X; E)$ is an isomorphism. That will in one step prove the universal property, (ii) and (iv). Let the topology of E be defined by seminorms $\tilde{p}$ such that $\tilde{p}(f) = \sup_{x \in X} p(f(x))$, and that of $\mathrm{Hom}(M_X; E)$ by seminorms $\bar{p}$ where $\bar{p}(f) = \sup_{\mu \in M_X'} p(f(\mu))$. By the approximation property just mentioned $\bar{p}(f) = \tilde{p}(i^*f)$. So i^* is an embedding. But the image contains all $f : X \to E$ which factorise through finite-dimensional spaces, and these are dense in $\mathrm{Map}(X; E)$; and $\mathrm{Hom}(M_X; E)$ is complete.

To prove (v): $X \times Y \to M_{X \times Y}$ induces $X \to \mathrm{Map}(Y; M_{X\times Y})$, so $M_X \to \mathrm{Map}(Y; M_{X\times Y}) \cong \mathrm{Map}(M_Y; M_{X\times Y})$. And this induces $M_X \times M_Y \to M_{X\times Y}$. It is unique because X and Y generate M_X and M_Y.

Chapter Two

C O M P A C T   G R O U P S

This chapter is devoted to the representation theory of compact groups, and in particular to the determination of the structure of the representation-ring R(G). As this is of interest apart from its applications to equivariant K-theory I shall treat it more fully than is necessary for the sequel.

The contents of the chapter are as follows.

§ 2.1 is a review of the classical theory of compact groups. There are only two important theorems - the existence of Haar measure, and the Peter-Weyl theorem - and I formulate them in a way convenient for my purposes.

§ 2.2 contains the definition and the most elementary properties of the representation-ring R(G), all essentially well-known.

In § 2.3 I determine the conjugacy-classes of a compact Lie group by obtaining generalisations of the well-know propositions about maximal tori in connected groups, i.e.

(i) each group-element is contained in a maximal torus (cf. 2.3.3 below),

(ii) any two maximal tori are conjugate (cf. 2.3.6)

(iii) two elements of a maximal torus are conjugate in the group if and only if they are conjugate in the normaliser of the torus (cf. 2.3.8).

Some of the results of this section have been obtained by Siebenthal [29], but by completely different methods.

§ 2.4 is concerned with generalising the "induced representation" homomorphism $R(H) \to R(G)$ which is familiar when H is a subgroup of finite index in G. The method is rather profound, involving elliptic differential operators and the Lefschetz formula of Atiyah and Bott. I have followed the lines of a paper of Bott [10].

§ 2.5 contains the real matter of the chapter, the study of the structure of $R(G)$. I determine completely the set of prime ideals, and give various applications, such as a generalisation of Brauer's theorem. My methods are a direct generalisation of those which have been used when G is finite in [27], [11], and [1].

Henceforth all groups will be compact unless the contrary is mentioned or obvious. In particular, "subgroup" will mean "closed subgroup", and "cyclic group" will mean "compact group containing an element whose powers are dense".

2.1 Let G be a compact group.

M_G will denote the space of measures on G (cf. 1.7), dual of the commutative ring $C_G = \mathrm{Map}(G;\mathbb{C})$. The composition $G \times G \to G$ induces a composition (hypocontinuous) $M_G \times M_G \to M_G$, making M_G an algebra over . It is also a C_G-module.

The homomorphism $G \to 1$ induces a morphism of rings $\varepsilon : M_G \to \mathbb{C}$. In fact $\varepsilon(\mu) = \langle\mu, 1\rangle$. The first main theorem of the subject states that ε splits, i.e. $M_G \cong \mathbb{C} \times \tilde{M}_G$ as ring. The central idempotent μ_G generating $\mathbb{C} \subset M_G$ is called the <u>Haar measure</u> of G. It is clearly unique, and characterised by $\mu.\mu_G = \langle\mu.1\rangle\mu_G$ for all $\mu \in M_G$. (e.g. $g.\mu_G = \mu_G$).

Because M_G is a module over C_G one can define a map $C_G \to M_G$ by $f \mapsto f.\mu_G$. ($f.\mu_G$ is defined by $\langle f.\mu_G, f'\rangle = \langle\mu_G, ff'\rangle$.) It is injective because the support of μ_G, being G-stable, must be the whole of G. And the image is dense, for M_G is generated by its subset G, and $g \in G \subset M_G$ can be approximated by measures $f.\mu_G$. The map $C_G \to M_G$ is equivalent to a quadratic form $C_G \times C_G \to \mathbb{C}$ invariant under $G \times G$. There is also an involution in C_G induced by inversion in G.

Let E be a G-vector-space (subject to the convention of 1.7). The map $G \times E \to E$ defining the G-action extends to a map (hypocontinuous)

$M_G \times E \to E$ which makes E an M_G-module. (For $G \times E \to E$ induces $E \to \mathrm{Map}(G;E)$, hence $M_G \times E \to M_G \times \mathrm{Map}(G;E)$, which maps hypocontinuously into E.) And conversely a hypocontinuous action $M_G \times E \to E$ induces a G-action $G \times E \to E$ because G is embedded in M_G. Henceforth I shall extend G-actions to M_G-actions without comment, so the terms "G-vector-space" and "M_G-module" will be interchangeable; but I shall reserve <u>G-module</u> for a <u>finite-dimensional</u> G-vector-space. The abelian category of G-modules will be denoted by G-<u>Vect</u>.

If M is a G-module, the defining map $G \to \mathrm{End}(M)$ extends to a homomorphism of rings: $M_G \to \mathrm{End}(M)$. Let Σ_G be a set of representatives of the isomorphism-classes of simple G-modules. The second fundamental theorem concerning compact groups is:

<u>Theorem 2.1.1</u> (Peter-Weyl) The natural homomorphism of rings $M_G \to \prod_{M\in\Sigma_G} \mathrm{End}(M)$ is a bijection of Hausdorff vector-spaces, i.e. it is injective and the image is dense.

This can be reformulated in various ways. For instance one can dualise: $\coprod_{M\in\Sigma_G} \mathrm{End}(M^*) \to C_G$ is a bijection of $(G \times G)$-vector-spaces. This is the form in which the theorem is usually proved: injectivity comes at once from the fact that the modules $\mathrm{End}(M^*)$ are non-isomorphic simple $(G \times G)$-modules (2.2.5).

End(M^*) can be rewritten as $\varinjlim_{(M_2 \to M_1)} \mathrm{Hom}(M_1;M_2)$, where $(M_2 \to M_1)$ runs through the category G-$\underline{\mathrm{Vect}}/G$-$\underline{\mathrm{Vect}}^\circ$ (cf. 1.2). In particular this shows that it has a ring-structure induced by $\mathrm{Hom}(M_1;M_2) \times \mathrm{Hom}(N_1;N_2) \to \mathrm{Hom}(M_1 \otimes N_1; M_2 \otimes N_2)$ which is compatible with its injection into C_G. It is called the "ring of representative functions" of G; I shall denote it by A_G. Yet another way of writing it is:

$$A_G \cong \coprod_{M \in \Sigma_G} M \otimes \mathrm{Hom}^G(M;C_G),$$

for $\mathrm{Hom}^G(M;C_G) \cong \mathrm{Hom}^G(M_G;M^*) \cong M^*$.

By the Stone-Weierstrass theorem, to prove 2.1.1 amounts to showing that the functions of A_G separate the points of G.

In view of the injection $C_G \to M_G$ one can state a stronger form of 2.1.1: define a map $\coprod \mathrm{End}(M) \to M_G$ so as to make the diagram

$$\begin{array}{ccc} \coprod \mathrm{End}(M^*) & \longrightarrow & C_G \\ \cong \big\downarrow \text{transpose} & & \big\downarrow \\ \coprod \mathrm{End}(M) & \longrightarrow & M_G \end{array}$$

commutative. It turns out to be a morphism of rings, and a partial inverse to the canonical map $M_G \to \prod \mathrm{End}(M)$; the proof amounts to showing that the quadratic form on C_G induces the standard quadratic form on $\mathrm{End}(M^*)$, which is easy. So one can say:

<u>Corollary 2.1.2</u> There are canonical bijections of rings

$$\coprod_{M\in\Sigma_G} \mathrm{End}(M) \to M_G \to \prod_{M\in\Sigma_G} \mathrm{End}(M),$$

whose composite is the inclusion.

<u>Note</u>: Here and elsewhere $\coprod$, when used in the category of rings, denotes <u>not</u> the sum in that category, but the sum in the category of abelian groups, endowed with an obvious ring-structure.

It follows from the corollary that any simple M_G-module is finite-dimensional, for any simple module for $\coprod \mathrm{End}(M)$ is finite-dimensional. More generally

<u>Proposition 2.1.3</u> For any G-vector-space E the canonical map

$$\alpha : \coprod_{M\in\Sigma_G} M \otimes \mathrm{Hom}^G(M;E) \to E \quad \text{is a bijection.}$$

The proof will be preceded by a lemma.

<u>Lemma 2.1.4</u> If M is finite-dimensional then any G-surjection $p : E \to M$,

or G-injection $i : M \to E$, splits.

<u>Proof</u>: Given a surjection $p \in \mathrm{Hom}^G(E;M)$, choose any splitting $j \in \mathrm{Hom}(M;E)$, not necessarily a G-map. $\mathrm{Hom}(M;E)$ is an M_G-module, so one can form $M_G . j$. This is the required splitting for $p \circ (\mu_G . j) = \mu_G . (p \circ j) = \mu_G . 1 = 1$.

Conversely, given an injection $i \in \mathrm{Hom}^G(M;E)$, choose a splitting $q \in \mathrm{Hom}(E;M) \cong \mathrm{Hom}(E \otimes M^*; \mathbb{C})$. It is not clear, without some mild restriction on E, that this space is an M_G-module, so define $\tilde{q} \in \mathrm{Hom}^G(E \otimes M^*; \mathbb{C})$ by $\tilde{q}(\xi) = q(\mu_G . \xi)$. If $p \in \mathrm{Hom}^G(E;M)$ corresponds to $\tilde{q}$ then $p \circ i = 1$.

<u>Proof of 2.1.3</u> If M_0 is a simple G-module then

$$\mathrm{Hom}^G(M_0; \coprod M \otimes \mathrm{Hom}^G(M;E)) \cong \coprod \mathrm{Hom}^G(M_0; M \otimes \mathrm{Hom}^G(M;E))$$

$$\cong \coprod \mathrm{Hom}^G(M_0;M) \otimes \mathrm{Hom}^G(M;E) \xrightarrow{\cong} \mathrm{Hom}^G(M_0;E).$$

This shows that α is injective, for if it had a non-zero kernel there would be a non-zero map from some simple M_0 into the kernel. On the other hand, $\mathrm{Hom}^G(\coprod M \otimes \mathrm{Hom}^G(M;E); M_0) \cong \Pi\, \mathrm{Hom}^G(M \otimes \mathrm{Hom}^G(M;E); M_0)$

$$\cong \Pi\, \mathrm{Hom}\,(\mathrm{Hom}^G(M;E); \mathrm{Hom}^G(M;M_0))$$

$$\cong \mathrm{Hom}^G(E;M_0) \qquad \text{which implies that } \alpha$$

is surjective, for if it had a non-zero cokernel then there would be by 2.1.4 a non-zero map from E to some simple M_0 which vanished on the image of α.

Proposition 2.1.5 The following conditions on a compact group G are equivalent:

(a) the subgroups of G satisfy the descending chain condition

(b) there exists a faithful G-module M_0

(c) A_G is a finitely-generated algebra over $\mathbb{C}$.

Proof: (a) $\Rightarrow$ (b): One has injections $G \to M_G \to \prod_{M \in \Sigma_G} \mathrm{End}(M)$. But (a) implies that one can find a finite set $\sigma \subset \Sigma_G$ such that $G \to \prod_{M \in \sigma} \mathrm{End}(M)$ is injective. Then $M_0 = \prod_{M \in \sigma} M$ is faithful.

(b) $\Rightarrow$ (c): G injects into the vector-space $\mathrm{End}(M_0)$, so by the Stone-Weierstrass theorem $S(\mathrm{End}(M_G^*)) \to C_G$ is an epimorphism of topological vector-spaces. (S denotes the symmetric algebra.) But it factorizes through the injection $A_G = \coprod_{M \in \Sigma_G} \mathrm{End}(M^*) \to C_G$, and the image of $S(\mathrm{End}(M_0^*))$ in A_G is a $(G \times G)$-subspace whose image in $\prod_{M \in \Sigma_G} \mathrm{End}(M^*)$ is dense, so it must be the whole of A_G. So A_G is finitely-generated.

(c) $\Rightarrow$ (a): A_G is noetherian, and to each subgroup H of G there corresponds the ideal of A_G vanishing on H. So one has only to show that whenever $H_1 < H_2$ strictly there exists $f \in A_G$ such that $f|H_1 = 0$ but $f|H_2 \neq 0$.

Write μ_1, μ_2 for the images in M_G of $\mu_{H_1} \in M_{H_1}$, $\mu_{H_2} \in M_{H_2}$. Because $H_1 \neq H_2$ one can find $f'' \in C_G$ such that $\langle \mu_1, f'' \rangle \neq \langle \mu_2, f'' \rangle$. And because A_G is dense in C_G one can find $f' \in A_G$ such that $\langle \mu_1, f' \rangle = 0 \neq \langle \mu_2, f' \rangle$. Let $f = \mu_1 . f' \in A_G$. Then for any $h \in H_1$ one has $f(h) = \langle h, \mu_1, f' \rangle = \langle \mu_1 . h, f' \rangle = \langle \mu_1, f' \rangle = 0$. But f does not vanish on H_2, otherwise $\langle \mu_2, f' \rangle = \langle \mu_1 \mu_2, f' \rangle = \langle \mu_2, f \rangle = 0$.

<u>Definition 2.1.6</u> A compact group satisfying the equivalent conditions of 2.1.5 is called a <u>compact Lie group</u>.

<u>Remarks</u> (i) It is clear that subgroups and quotient groups of compact Lie groups are again compact Lie groups.

(ii) The complex affine algebraic scheme Spec A_G becomes, by virtue of the co-ring structure of A_G dual to the ring-structure of Π End(M), an algebraic group defined over $\mathbb{R}$. The inclusion $A_G \to C_G$ induces an embedding $G \to$ Specm A_G. G is precisely the real part of Specm A_G. (See [13] p.199.) So G is a real algebraic manifold, necessarily non-singular.

(iii) If G is a compact group and G^0 is the component containing the identity then G/G^0, called the group of components of G, is profinite, and hence finite if G is a compact Lie group. ([26] pp.39, 54.)

2.2 The representation-ring

If G is a compact group, the Grothendieck group of the category G-**Vect** of (finite-dimensional) G-modules with respect to direct sum will be denoted by $R(G)$. The tensor-product in G-**Vect** induces a structure of commutative ring in $R(G)$, and the exterior powers $M \mapsto \lambda^k M$ induce operations $\lambda^k : R(G) \to R(G)$ making it into a λ-ring [1]. $R(G)$ is called the *representation-ring* of G; it is easy to see it is a contravariant functor from compact groups to λ-rings.

The following proposition is an immediate consequence of 2.1.3.

Proposition 2.2.1 $R(G)$ can be identified canonically with the free abelian group generated by the set Σ_G of isomorphism-classes of simple G-modules.

Example If G is abelian then $R(G)$ is (as a ring) the group-ring over

$\mathbb{Z}$ of the dual $\hat{G} = \mathrm{Hom}(G; \mathbb{C}^*)$ of G.

There is an integral quadratic form on $R(G)$ induced by $(M,M') \mapsto \langle M,M' \rangle = \dim_{\mathbb{C}} \mathrm{Hom}^G(M;M')$. It satisfies the relations: $\langle \xi,\eta \rangle = \langle \eta^*,\xi^* \rangle = \langle \eta,\xi \rangle = \langle \xi^*,\eta^* \rangle$, where $*$ is the involution in $R(G)$ induced by $M \mapsto M^*$ in G-<u>Vect</u>. It is positive-definite, and the set Σ_G of 2.2.1 is an orthonormal basis of $R(G)$ with respect to it. One has also:
$\langle \xi.\eta,\zeta \rangle = \langle \zeta,\eta.\zeta \rangle$.

To a G-module M is associated its character χ_M in C_G: χ_M is the image of 1_{M^*} under the $(G \times G)$-map : $\mathrm{End}(M^*) \to C_G$ dual to the structural map $G \to \mathrm{End}(M)$. 1^*_M is invariant under the diagonal action of $G \subset G \times G$, so χ_M is invariant under the diagonal action ("conjugation") of G on C_G. Thus χ_M can be regarded as a function on $[G]$, the space of conjugacy classes of G. One can check that $M \mapsto \chi_M$ induces a morphism of rings $\chi : R(G) \to C_G$ (it factorises, of course, through $A_G \to C_G$). And χ is natural in G.

<u>Proposition 2.2.2</u> $\chi : R(G) \to C_G$ is compatible with the involutions and quadratic forms on $R(G)$ and C_G.

<u>Proof</u>: (i) $\chi_{M^*}(g) = \chi_M(g^{-1})$ is obvious.

(ii) $\langle \chi_{M_1}, \chi_{M_2} \rangle = \langle \mu_G, \chi_{M_1^* \times M_2} \rangle$, so it suffices to show that $\langle \mu_G, \chi_M \rangle = \dim M^G$, and hence to show that $\langle \mu_G, \chi_M \rangle = 0$ when M is non-trivial and simple. But $\langle \mu_G, \chi_M \rangle$ is the image of $1_{M^*} \in \mathrm{End}(M^*)$ under a $(G \times G)$-map into $\mathbb{C}$, and any such map is zero as it corresponds to an element of $\mathrm{End}(M)$ fixed under $G \times G$.

<u>Corollary 2.2.3</u> χ is injective.

<u>Corollary 2.2.4</u> If $G \to G'$ is surjective then $R(G') \to R(G)$ is injective.

<u>Proposition 2.2.5</u> $R(G_1) \otimes R(G_2) \to R(G_1 \times G_2)$ is an isomorphism of rings.

<u>Proof</u>: It suffices to show that if M is a simple G_1-module and N a simple G_2-module then $M \otimes N$ (i.e. $pr_1^* M \otimes pr_2^* N$) is a simple $(G_1 \times G_2)$-module, and that any simple $(G_1 \times G_2)$-module is of this form. But $\mathrm{Hom}^{G_1 \times G_2}(M \otimes N; M \otimes N) \cong \mathrm{Hom}^{G_1 \times G_2}(M; \mathrm{Hom}(N; M \otimes N)) \cong \mathrm{Hom}^{G_1}(M; \mathrm{Hom}^{G_2}(N; M \otimes N)) \cong \mathrm{Hom}^{G_1}(M; \mathrm{Hom}^{G_2}(N;N) \otimes M) \cong \mathrm{Hom}^{G_1}(M;M) \otimes \mathrm{Hom}^{G_2}(N;N) \cong \mathbb{C}$, so $M \otimes N$ is simple.

On the other hand if P is a $(G_1 \times G_2)$-module one has

$$\coprod_{M \in L_{G_1}} M \otimes \mathrm{Hom}^{G_1}(M; P) \xrightarrow{\cong} P,$$ and $\mathrm{Hom}^{G_1}(M; P)$ is a G_2-module.

<u>Proposition 2.2.6</u> $\varinjlim_N R(G/N) \xrightarrow{\cong} R(G)$, where N runs through the category of normal subgroups of G such that G/N is Lie.

Proof: Injectivity comes from 2.2.4. Surjectivity is obvious.

Proposition 2.2.7 If $\{G_\alpha\}_{\alpha\in A}$ is any family of compact groups, then $\varprojlim_S R(\prod_{\alpha\in S} G_\alpha) \xrightarrow{\cong} R(\prod_{\alpha\in A} G_\alpha)$, where, on the left, S runs through the category of finite subsets of A.

Proof: Injectivity comes again from 2.2.4.

On the other hand any element of the right-hand-side comes from some $R(G/N)$, where N is a normal subgroup of $G = \prod_{\alpha\in A} G_\alpha$, and G/N is Lie. Set $K_S = \ker : G \to \prod_{\alpha\in S} G_\alpha$. $\{K_S\}$ is a family of normal subgroups of G whose intersection is 1; so $K_S/H \cap K_S \cong K_S H/H$ is a family of subgroups of the Lie group G/H which also has intersection 1 because $\varprojlim$ is exact for compact spaces. So $K_S \subset H$ for some finite S, and $G \to G/H$ factorises through $G \to \prod_{\alpha\in S} G_\alpha$, and the proposition is proved.

In Chapter 4 I shall prove by more sophisticated methods that $\varprojlim_\alpha R(G_\alpha) \xrightarrow{\cong} R(\varprojlim_\alpha G_\alpha)$ for any directed inverse system $\{G_\alpha\}$ of compact groups. (I don't know a direct proof.) In view of 2.2.5 this might lead one to conjecture that the functor R is right-exact in general. But that is not true: if T is a subgroup of G such that $R(G) \to R(T)$ is not surjective then $T \times_G T \cong T$, but the natural map $R(T) \otimes_{R(G)} R(T) \to R(T)$

is not an isomorphism. But R is right-exact for abelian groups (2.5.4).

2.2.6 shows that it is in one sense sufficient to study R(G) when G is a compact Lie group. For the rest of the chapter I shall make that assumption.

2.3 Conjugacy in compact Lie groups

In this section G is a compact Lie group, G^0 is its identity-component, and $\Gamma = G/G^0$ is its group of components.

Definition 2.3.1 A subgroup S of G is called a Cartan subgroup if it is cyclic and of finite index in its normaliser N(S). The finite group N(S)/S is called the Weyl group of S, and denoted by W_S.

Remark: Cartan subgroup is not a very good name, and it conflicts with the definition of Chevalley [15]; but the groups seem to me to correspond to Cartan subalgebras in the theory of Lie algebras.

Proposition 2.3.2 Each cyclic subgroup C of Γ is covered by a Cartan subgroup S of G.

Proof: Choose $y \in G$ such that yG^0 generates C. Let T be a maximal torus of the centraliser $Z(y)$ of y (I shall not presuppose any properties of maximal tori, however), and let S be the subgroup generated by T and y.

The identity-component S^0 of S contains T; but if it contained any 1-parameter subgroup L not in T then L and T would generate a torus of $Z(y)$ strictly containing T. S/S^0 is cyclic, generated by $y.S^0$, so S, which is isomorphic to $(S/S^0) \times S^0$ is also cyclic. Finally $(N(S) : S) = (N(S) : Z(S))(Z(S) : S)$; but $N(S)/Z(S)$ is a compact subgroup of the discrete group of automorphisms of S, so it is finite; and $Z(S)/S$ is finite because the two groups have the same identity-component T. So $(N(S) : S)$ is finite, and S is a Cartan subgroup.

Proposition 2.3.3 Each element g of G is contained in a Cartan subgroup.

The proof will be based on the following topological lemma.

Lemma 2.3.4 If X is a compact metric space, and $f : X \to X$ a map which is homotopic to an isometry of X with a finite number of fixed-points, then f has a fixed point.

I shall not prove this lemma here, though it is quite elementary, for in the case I need X is a differentiable manifold and the result can be deduced from the more profound Lefschetz fixed-point formula.

Proof of 2.3.3: Let gG^0 generate C in Γ, and let S be a Cartan subgroup of G covering C. I shall show that g is contained in a conjugate ySy^{-1} of S.

Consider the map f_g of the compact metric space G/S defined by $f_g(yS) = gyS$. If yS is a fixed-point then $y^{-1}gy \in S$, and $g \in y^{-1}Sy$ as required. But if x is a generator of S contained in gG^0 then f_g is homotopic to f_x whose fixed-point set is $N(S)/S$, which is finite. And one can certainly suppose that f_x is an isometry, so one is in the situation of 2.3.4.

Proposition 2.3.3 can be sharpened slightly, as follows.

Proposition 2.3.5 If S is a Cartan subgroup of G generated by x, then any $g \in xG^0$ is conjugate by an element of G^0 to an element of xS^0.

(Notice that there might be several components of S in xG^0.)

Proof: Proceed as in 2.3.3 but consider the map $h_g : G^0/S^0 \to G^0/S^0$ defined by $h_g(yS^0) = gyx^{-1}S^0$. A fixed-point of h_g is a coset yS^0 with $y \in G^0$ such that $y^{-1}gy \in xS^0$.

Proposition 2.3.6 The projection {Cartan subgroups of G} $\to$ {cyclic subgroups of Γ} induces a bijection of conjugacy-classes.

Proof: One half of this is 2.3.2, the other half, that two Cartan subgroups are conjugate if they have the same projection into Γ, follows from the proof of 2.3.3. (For a Cartan subgroup cannot be conjugate to a proper subgroup of itself.)

2.3.5 has the following consequence, which I include for future reference.

Proposition 2.3.7 If S is a Cartan subgroup of G projecting into C in Γ, then the projection $N(S) \to N(C)$ is surjective.

It remains to decide when two elements of a Cartan subgroup S are conjugate. I don't know whether it is true that two elements of S are conjugate in G if and only if they are conjugate in $N(S)$; but the following result suffices for all practical purposes.

Proposition 2.3.8 If S is a Cartan subgroup of G generated by x, then two elements of xS^0 are conjugate in G if and only if they are conjugate in $N(S)$.

Proof: If y and gyg^{-1} are both in xS^0 then S and $g^{-1}Sg$ are two Cartan subgroups of $Z(y)$, both containing y. They both have the same projection into the group of components of $Z(y)$ - it is the cyclic group generated by $y.Z(y)^0$ - so they are conjugate in $Z(y)$, i.e. there is a $z \in Z(y)$ such that $zg^{-1}Sgz^{-1} = S$. Then $zg^{-1} \in N(S)$, and $(zg^{-1})^{-1}y(zg^{-1}) = gyg^{-1}$, as required.

Corollary 2.3.9 If $[G]$ is the space of conjugacy-classes of G, and $[\Gamma]$ is that of Γ, then in the projection $[G] \to [\Gamma]$ the inverse image of a conjugacy class $[g.G^0] \in [\Gamma]$ is isomorphic to $(g.S^0)/W_S^1$, where S is a Cartan subgroup of G with a generator in $g.G^0$, and W_S^1 is the subgroup of W_S which normalises $g.S^0$.

Definition 2.3.10 An element of G is called regular if it generates a Cartan subgroup, and singular otherwise.

Clearly regular elements are dense in G.

Proposition 2.3.11 If g is a regular element of G, and H is a subgroup of G, then g acts with only a finite number of fixed-points on G/H.

Proof: Let S be the Cartan subgroup generated by g. The coset $yH \in G/H$ is fixed by g if and only if $y^{-1}Sy < H$. So there are no fixed-points at all unless S is conjugate to a subgroup of H, in which case one may as well assume $S < H$. Because there are only a finite number of conjugacy-classes of Cartan subgroups of H one can choose a finite subset A of G such that

$y^{-1}Sy < H \Rightarrow y^{-1}Sy$ is conjugate in H to $a^{-1}Sa$ for some $a \in A$. The fixed-point set F of the action of g can be decomposed correspondingly into a finite disjoint union $F = \coprod_{a\in A} F_a$. But $y^{-1}Sy = ha^{-1}Sah^{-1}$ with $h \in H \Leftrightarrow yh \in N(S).a$, so $F_a \cong N(S).a.H/H \cong N(S)/aHa^{-1} \cap N(S)$, which is a quotient of the finite set $N(S)/S$. So F is finite.

I conclude this section with the following remarks. The reader should not assume when S is a Cartan subgroup of G covering C in Γ that $(S : S^0) = (C : 1)$: a counterexample is given by the Cartan subgroup C_4 of $(T^1 \tilde{\times} C_4)/C_2$, where $T^1 \tilde{\times} C_4$ is the only non-trivial semi-direct product of the circle-group T^1 and the cyclic group C_4 of order 4, and C_2 (cyclic of order 2) is embedded diagonally. But one does have $(C : 1)|(S : S^0)|(C : 1)^2$, so no "new" primes appear in $(S : S^0)$. This emerges from the following discussion, which has also some independent interest.

S/S^0 acts on the centraliser $Z(S^0)$ of S^0 in G, and whenever a cyclic group acts on a compact group it leaves fixed a maximal torus T ([22] Exp.24). But maximal tori of $Z(S^0)$ are the same as maximal tori of G containing S^0. So T contains S^0 and is normalised by S. Also $S \cap G^0 < T$, as it commutes with T. Thus $\tilde{T} = T.S$ is an extension of T by $\tilde{C}$ covering C : a "fattening" of S; and $(S \cap G^0)/S^0$ is a cyclic subgroup of the group of components of $H^0(C; T)$. But this group of components is annihilated by $(C : 1)$: the reader can easily check that when a finite group F acts on a torus T the group of components of T^F is $H^1(F; \mathrm{Hom}(T^1; T))$.

2.4 Induced representations

Let $i : H \to G$ be an inclusion of compact groups. There is a restriction-functor $i^* : G\text{-}\underline{\mathrm{Vect}} \to H\text{-}\underline{\mathrm{Vect}}$ inducing a homomorphism of rings $i^* : R(G) \to R(H)$, so that $R(H)$ can be regarded as an $R(G)$-module.

There is a functor i_*, left-adjoint to i^*, which assigns to an H-module M a G-vector-space i_*M with the property

$$\mathrm{Hom}^G(i_*M; N) \cong \mathrm{Hom}^H(M; i^*N) \qquad \text{for all } G\text{-modules } N.$$

It follows from 2.1.3 that $i_*M \cong \coprod_{P \in \Sigma_G} P \otimes \mathrm{Hom}^H(P; M)$ canonically, where the sum is over the set Σ_G of simple G-modules. i_* does not define a map $R(H) \to R(G)$, for i_*M has infinite dimension unless $(G : H)$ is finite. But if $R'(G)$ denotes the group of formal infinite sums $\sum_P n_P[P]$ $(n_P \in \mathbb{Z})$, i.e. $R'(G) = \mathrm{Map}(\Sigma_G; \mathbb{Z})$, then i_*M defines an element of $R'(G)$, and $i_* : R(H) \to R'(G)$ is a homomorphism of $R(G)$-modules. It follows at once from the adjointness property that if $j : G \to F$ is another inclusion, and $m \in R(H)$ is such that $i_*m \in R(G) \subset R'(G)$, then $(ji)_*m = j_*i_*m \in R'(F)$.

It is my object in this section to define for certain inclusions $i : H \to G$ of compact Lie groups an element $\Delta_{G/H} \in R(H)$ with the property that $i_*(m.\Delta_{G/H}) \in R(G) \subset R'(G)$ for all $m \in R(H)$. When $(G : H)$ is finite $\Delta_{G/H}$ will be 1. Then I shall define $i_!(m) = i_*(m.\Delta_{G/H})$; $i_! : R(H) \to R(G)$ will be a homomorphism of $R(G)$-modules which will take over the role played by i_* in the analysis of $R(G)$ when G is finite. I shall give a formula for computing $i_!(m)$.

Let T_G, T_H, $T_{G/H}$ be the tangent-spaces to G, H, G/H at their identity-elements. Choose once for all a G-invariant positive-definite quadratic form on T_G. $T_G \otimes_{\mathbb{R}} \mathbb{C}$ is a G-module, the "adjoint representation" of G. $T_H \otimes_{\mathbb{R}} \mathbb{C}$ and $T_{G/H} \otimes_{\mathbb{R}} \mathbb{C}$ are H-modules. I shall abbreviate $T_{G/H} = T$, and $T_{G/H} \otimes_{\mathbb{R}} \mathbb{C} = T_{\mathbb{C}}$.

$\Delta_{G/H}$ will be defined only when the relative tangent-space T admits a "Pin^c-structure" [7]. A real H-module T with a quadratic form admits a Pin^c-structure if there is a homomorphism $\phi : H \to C(T_{\mathbb{C}})$ of H into the multiplicative group of the complex Clifford algebra of $T_{\mathbb{C}}$ such that $h.\xi = (-1)^{\deg \phi(h)}.\phi(h).\xi.\phi(h)^{-1}$ for all $h \in H$ and $\xi \in T_{\mathbb{C}}$. (The multiplications on the right are in $C(T_{\mathbb{C}})$; recall that the algebra $C(T_{\mathbb{C}})$ is graded module 2.) If T has a complex-structure then it has a Pin^c-structure, though that is not obvious.

I must distinguish two cases.

Case 1: the dimension of G/H is even. Then one must require also that T is orientable, i.e. T admits a Spin^c-structure. Define $\Delta_{G/H} = [\Delta^0] - [\Delta^1] \in R(H)$, where (Δ^0, Δ^1) is a simple graded module for $C(T_{\mathbb{C}})$, and Δ^0, Δ^1 are regarded as H-modules by means of ϕ. There are two such simple graded modules [7], but they differ only by interchange of the grading, and that affects only the sign of $\Delta_{G/H}$.

If $\phi : H \to C^0(T_{\mathbb{C}})$ exists at all it is unique up to multiplication by a homomorphism $\theta : H \to \mathbb{C}^* \subset C^0(T_{\mathbb{C}})$. Changing ϕ by θ replaces Δ^i by $\Delta^i \otimes L_\theta$, where L_θ is the one-dimensional H-module defined by θ. $[L_\theta]$ is a unit in $R(H)$.

Case 2: the dimension of G/H is odd. Orientability is no longer required. There is a unique simple graded module (Δ^0, Δ^1) for $C(T_{\mathbb{C}})$; but Δ^0 and Δ^1, which are isomorphic as $C^0(T_{\mathbb{C}})$-modules, can be made into separate ungraded $C(T_{\mathbb{C}})$-modules by observing that $C(T_{\mathbb{C}}) \cong C^0(T_{\mathbb{C}})[X]/(X^2 - 1)$

and letting X act as 1 on Δ^0 and as -1 on Δ^1. Then Δ^0 and Δ^1 become H-modules, and one can define $\Delta_{G/H} = [\Delta^0] - [\Delta^1] \in R(H)$, as before. There is the same ambiguous factor $\pm [L_\theta]$ as before.

To understand these definitions and to prove the results I need, one must change one's viewpoint. If M is an H-module one can form a G-vector-bundle (see chapter 3) $(G \times M)/H$ over the space G/H. Let $i_d M$ be the G-vector-space of smooth sections, i.e. of differentiable maps $s : G/H \to (G \times M)/H$ such that $p \circ s = id$, where $p : (G \times M)/H \to G/H$ is the projection. $i_d M \cong \mathrm{Map}^H(G; M)$ canonically, and by 2.1.3 one has a bijection $i_* M \to i_d M$, for $i_* M \cong \coprod_{P \in \Sigma_G} P \otimes \mathrm{Hom}^G(P; i_d M)$.

Now it happens that there is an equivariant elliptic differential operator, the Dirac operator, a linear G-map $d_M : i_d(M \otimes \Delta^0) \to i_d(M \otimes \Delta^1)$, for any H-module M. Its definition can be found, for example, in [10]. Because it is elliptic its kernel and cokernel have finite-dimension. Applying the functor $E \mapsto \coprod P \otimes \mathrm{Hom}^G(P; E)$, which is exact, one finds that d_M induces a map $i_*(M \otimes \Delta^0) \to i_*(M \otimes \Delta^1)$ which has the same kernel and cokernel as d_M. So $i_*([M].\Delta_{G/H}) = [i_*(M \otimes \Delta^0)] - [i_*(M \otimes \Delta^1)]$ $= [\ker d_M] - [\operatorname{coker} d_M] \in R(H)$ is finite, as desired.

The character $\chi_{i_! M}$ of $i_! M$ can be computed by the Lefschetz formula of Atiyah and Bott [4]. It suffices to determine it for regular elements $g \in G$, and then

$$\chi_{i_!M}(g) = \sum_{x \in F} \frac{\chi_{M.\Delta_{G/H}}(x^{-1}gx)}{\det(1 - \mathrm{ad}_{G/H}(x^{-1}gx))},$$

where F is the finite set of fixed-points of the translation of G/H induced by g (thus $x \in F \Leftrightarrow x^{-1}gx \in H$, where $x^{-1}gx$ is defined up to conjugation), and $\mathrm{ad}_{G/H}(y)$ for $y \in H$ means the action of y on T. To see that this is the Lefschetz one has only to remark that $\chi_{M.\Delta^{i}_{G/H}}(x^{-1}gx)$ is the trace of the map induced by g in the fibre of $(G \times M)/H$ over the fixed-point $x \in G/H$, and that $\mathrm{ad}_{G/H}(x^{-1}gx)$ is isomorphic to the action of g in the tangent-space to G/H at x. The formula reduces when $(G : H)$ is finite (and $\Delta_{G/H} = 1$) to the classical formula for χ_{i_*M}; and when G is connected we shall see that it is the Weyl character-formula.

Because $i_!$ is defined only up to multiplication with a factor $\pm [L_\rho]$ in $R(H)$ one must state its transitivity-property cautiously. If $i : H \to G$ and $j : G \to F$ are inclusions and $\Delta_{G/H}$ and $\Delta_{F/G}$ are defined then $\Delta_{G/H}.i^*\Delta_{F/G}$ is a possible choice for $\Delta_{F/H}$, and with that choice $j_! \circ i_! = (j \circ i)_!$. The proof amounts to observing that $T_{F/H} \cong T_{F/G} \oplus T_{G/H}$, and that if Δ_1, Δ_2 are simple graded modules for $C(T_1)$, $C(T_2)$, then $\Delta_1 \otimes \Delta_2$ is a simple graded module for $C(T_1 \oplus T_2) \cong C(T_1) \otimes C(T_2)$.

Examples

(i) If H does not contain a Cartan subgroup of G then $i_!$ is zero whenever it is defined. For then any regular element of G acts without fixed-points on G/H.

(ii) $\Delta_{G/H}$ is always defined if H is cyclic, provided only that the relative adjoint-action $\alpha : H \to \mathrm{Aut}(T_{G/H})$ is orientation-preserving if dim(G/H) is even. For the image of H lies in a Cartan subgroup of O(T) covered by a Cartan subgroup of $\mathrm{Pin}^c(T_{\mathbb{C}})$, and the projection of Cartan subgroups is split (it has kernel T^1), so α lifts. Furthermore the orientability-condition is always satisfied if H is a Cartan subgroup, for its generator acts with isolated fixed-points on G/H and so induces an orthogonal transformation of T which does not have the eigenvalue + 1: and any such transformation of an even-dimensional space has determinant + 1.

Of course any even-dimensional real orientable module for a cyclic group actually has a complex-structure.

(iii) If $N \to H \to P$ is an extension by a group N of a p-group P, and $p \neq 2$, then $\Delta_{G/H}$ is defined if and only if $\Delta_{G/N}$ is defined.

<u>Proof</u>: Orientability presents no problem here. Let $ad : H \to O(T)$ be the adjoint-action of H; one must show that ad lifts to $Pin^c(T_c)$ if and only if $ad|N$ does. Now $Hom(H; O(T))$ can be identified with $H^1(BH; O(T))$, where BH is the classifying-space for H. Consider the diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}/2 & \to & Pin(T) & \to & O(T) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & T^1 & \to & Pin^c(T_c) & \to & O(T) & \to & 1. \end{array}$$

$ad \in H^1(BH; O(T))$ lifts to $H^1(BH; Pin^c(T_c))$ if and only if its image $d(ad) \in H^2(BH; T^1)$ is zero. But $2.d(ad) = 0$ because $d(ad)$ is in the image of $H^2(BH; \mathbb{Z}/2)$, so $d(ad)$ lies in the 2-primary part $H^2(BH; T^1)_{(2)}$ of $H^2(BH; T^1)$. On the other hand the restriction $H^2(BH; T^1)_{(2)} \to H^2(BN; T^1)_{(2)}$ is an isomorphism because $(H : N)$ is prime to 2, so one finds that ad lifts if and only if $ad|N$ lifts.

2.5 The structure of R(G)

In this section all groups will be compact Lie groups.

I shall begin with some examples.

(i) If $G = \mathbb{Z}/n$, $R(G) \cong \mathbb{Z}[X]/(X^n-1)$.

(ii) If $G = T^1$, $R(G) \cong \mathbb{Z}[X,X^{-1}]$, i.e. $\mathbb{Z}[X,Y]/(XY-1)$.

(iii) If $G = T^k = T^1 \times \ldots \times T^1$ (k factors), $R(G) \cong \mathbb{Z}[X_1,X_1^{-1};\ldots;X_k,X_k^{-1}]$.

(iv) If G is the unitary group $U(n)$, and $T = T^n$ is a maximal torus of $U(n)$ - say the diagonal matrices - then the restriction $R(G) \to R(T)$ is injective, for each element of G is conjugate to an element of T. So $R(G)$ can be identified with a subring of $R(T) \cong \mathbb{Z}[X_1,X_1^{-1};\ldots;X_n,X_n^{-1}]$. But the entries of a diagonal matrix can be permuted by conjugation in $U(n)$, so $R(G)$ consists of symmetric expressions in $X_1, \ldots, X_n$. In fact

<u>Proposition 2.5.1</u> $R(U(n)) \cong \mathbb{Z}[s_1,\ldots,s_n,s_n^{-1}]$, where s_k is the k^{th} symmetric function in $X_1, \ldots, X_n \in R(T)$. In particular $R(U(n))$ is a noetherian ring.

<u>Proof</u>: s_k is in $R(U(n))$, for it is the character of the exterior power $\Lambda^k M$, where $M = \mathbb{C}^n$ is the standard $U(n)$-module. And s_n^{-1} is the character

of $(\Lambda^n M)^*$. On the other hand $\mathbb{Z}[s_1,\ldots,s_n,s_n^{-1}]$ contains all the symmetric functions in $\mathbb{Z}[X_1,\ldots,X_n^{-1}]$, so $R(U(n))$ cannot be any bigger. It is well-known that $s_1, \ldots, s_n$ are algebraically independent.

The following theorem of Atiyah is crucial for the sequel.

<u>Proposition 2.5.2</u> If H is a subgroup of G then the restriction $R(G) \to R(H)$ makes $R(H)$ a finite $R(G)$-module.

<u>Proof</u>: Embed G in a unitary group $U = U(n)$. It suffices to show that $R(H)$ is finite over $R(U)$, i.e. one can assume $G = U$. Furthermore the restriction $R(H) \to \Pi\, R(S)$, where S runs through the finite set of conjugacy classes of Cartan subgroups of H, is injective, so - because $R(U)$ is noetherian - it suffices to show each $R(S)$ is finite over $R(U)$.

For given S one can choose a maximal torus T of U so that $S < T < U$. $R(T) \cong \mathbb{Z}[X_1,\ldots,X_n^{-1}]$ is finite over $R(U)$, for each X_i satisfies the integral equation $\sum_{k=0}^{n} (-1)^k X_i^k s_{n-k} = 0$ over $R(U)$, and each X_i^{-1} a similar one. Finally, $R(S)$ is finite over $R(T)$, indeed $R(T) \to R(S)$ is surjective: for $R(T) \cong \mathbb{Z}[\hat{T}]$, $R(S) \cong \mathbb{Z}[\hat{S}]$, and $S \to T$ induces a surjection $\hat{T} \to \hat{S}$

<u>Corollary 2.5.3</u> $R(G)$ is noetherian for any compact Lie group G.

For it is noetherian even as an $R(U)$-module.

It is easy to see that 2.5.3 characterises the compact Lie groups among the compact groups.

Now I can begin to discuss the prime ideas of $R(G)$.

Consider the restriction $R(G) \to \Pi\, R(S)$, where S runs through the (conjugacy classes of) Cartan subgroups of G. It is injective, and $\Pi\, R(S)$ is finite over $R(G)$, so by the theorem of Cohen-Seidenberg every prime ideal $\mathfrak{p}$ of $R(G)$ is of the form $\{X : X|S \in \mathfrak{p}_0\}$ for some S and some prime $\mathfrak{p}_0$ of $R(S)$. I shall prove that for each $\mathfrak{p}$ there is a minimum cyclic subgroup $S_{\mathfrak{p}}$ of G, unique up to conjugacy, such that $\mathfrak{p}$ is induced by a prime $\mathfrak{p}_0$ of $R(S_{\mathfrak{p}})$. $S_{\mathfrak{p}}$ will be called the <u>support</u> of $\mathfrak{p}$.

<u>Lemma 2.5.4</u> If S is abelian, and S_1, S_2 are subgroups, then $R(S_1) \otimes_{R(S)} R(S_2) \xrightarrow{\cong} R(S_1 \cap S_2)$ by the natural map.

<u>Proof</u>: For any subgroup A of S write $J_A = \{X \in R(S) : X|A = 0\}$. $R(A) = R(S)/J_A$, for $R(S) \to R(A)$ is surjective. J_A is, as ideal in $R(S)$, generated by the elements $\alpha - 1$ for all homomorphisms $\alpha : S \to \mathbb{C}^*$ such that $\alpha|A = 1$. In fact if $\Sigma\, n_i\alpha_i \in J_A$, with $\alpha_i : S \to \mathbb{C}^*$, and the α_i all have the same restrictions to A, then $\Sigma\, n_i\alpha_i = \Sigma\, n_i\alpha_1^{-1}(\alpha_i\alpha_1^{-1} - 1)$, which

is of the required form. But any element of J_Λ is a sum of elements of that type.

Now the left-hand-side of 2.5.4 is $R(S)/(J_{S_1} + J_{S_2})$, so one must show that $J_{S_1} + J_{S_2} = J_{S_1 \cap S_2}$. But if $\alpha : S \to \mathbb{C}^*$ is a homomorphism such that $\alpha | S_1 \cap S_2 = 1$, one can choose $\tilde{\alpha} : S \to \mathbb{C}^*$ so that $\tilde{\alpha} | S_1 = \alpha | S_1$ and $\tilde{\alpha} | S_2 = 1$, for $(S/S_2)^\wedge \to (S_1/S_1 \cap S_2)^\wedge$ is surjective. Then
$\alpha - 1 = (\alpha - \tilde{\alpha}) + (\tilde{\alpha} - 1) \in J_{S_1} + J_{S_2}$.

Remark 2.5.5 From 2.5.4 and 2.2.5 it follows that if $S_1 \to S$, $S_2 \to S$ are any homomorphisms of abelian groups then $R(S_1) \otimes_{R(S)} R(S_2) \xrightarrow{\cong} R(S_1 \times_S S_2)$.

The reader will remember that the set of prime ideals of a ring R can be made into a topological space Spec R which depends contravariantly on R. ([20] Chap.1.) Furthermore, if $R \to R_1$, $R \to R_2$ are morphisms of rings, the induced map $\mathrm{Spec}(R_1 \otimes_R R_2) \to (\mathrm{Spec}\ R_1) \times_{(\mathrm{Spec}\ R)} (\mathrm{Spec}\ R_2)$ is surjective. ([20] I 3.4.7) Thus

Corollary 2.5.6

$\{\text{image} : \mathrm{Spec}\ R(S_1)\} \cap \{\text{image} : \mathrm{Spec}\ R(S_2)\} = \{\text{image} : \mathrm{Spec}\ R(S_1 \cap S_2)\}$
where "image" in each case refers to the natural map into Spec $R(S)$.

Notice that if S_1 and S_2 are cyclic it may be that $S_1 \cap S_2$ is not cyclic. But any prime which comes from S_1 and S_2 comes from $S_1 \cap S_2$, and hence from some cyclic subgroup of $S_1 \cap S_2$.

Any prime of $R(G)$ comes from a minimal cyclic subgroup of G because compact Lie groups obey the descending chain condition. To prove that each prime comes from a well-defined minimum cyclic subgroup of G it would now suffice to show that each prime comes from a well-defined Cartan subgroup. But I shall prove first a more precise result, and return to the prime ideals afterwards.

The inclusion $\mathbb{Z} \subset R(G)$ induces π : Spec $R(S) \to$ Spec $\mathbb{Z}$, i.e. if $\wp$ is a prime of $R(G)$ then $p = \wp \cap \mathbb{Z}$ is a prime $\pi(\wp)$ of $\mathbb{Z}$ (an ordinary prime or zero). It is expedient to consider the fibres of π separately. Let us fix p; then $\pi^{-1}(p)$ can be identified with Spec $R_p(G)$, where $R_p(G) = R(G) \otimes_{\mathbb{Z}} k_p$, and k_p is the prime field of characteristic p.

If $\wp$ is a prime of $R(G)$ above p, and S is a minimal cyclic subgroup from which it arises, then $(S : S^0)$ is not divisible by p. For otherwise one can write $S = T \times (\mathbb{Z}/p^k\ )$, so that $R(S) \cong R(T)[X]/(X^{p^k} - 1)$, and $R_p(S) \cong R_p(T)[X]/(X - 1)^{p^k}$. Then $R_p(S) \to R_p(T)$, given by $X \mapsto 1$, has a nilpotent kernel, and induces a homeomorphism of spectra, so $\wp$ comes from T. I shall call a cyclic group S <u>p-regular</u> if p does not divide

$(S : S^0)$.

If S is a cyclic group generated by g I shall write $\wp_S$ for the prime ideal of characters of S which vanish at g. $\wp_S$ does not depend on the generator chosen: if $(S : S^0) = n$ it is the principal ideal $(\phi_n(X))$, where ϕ_n is the n^{th} cyclotomic polynomial and $X : S \to \mathbb{C}^*$ is a homomorphism with kernel S^0. (Any such X gives the same $\phi_n(X)$.) If S is a torus, $\wp_S = 0$. I shall write $\tilde{R}(S) = R(S)/\wp_S$, an integral domain; the primes of $\tilde{R}(S)$ are precisely the primes of $R(S)$ which do not come from subgroups of finite index. I put also $\tilde{R}_p(S) = \tilde{R}(S) \otimes_{\mathbb{Z}} k_p \cong R_p(S)/(\phi_n(X))$ The Weyl group W_S of S acts naturally on $\tilde{R}_p(S)$.

After these preliminaries I can state

__Proposition 2.5.7__ Consider the restriction $R_p(G) \to \Pi\, \tilde{R}_p(S)^{W_S}$, where S runs through the p-regular Cartan subgroups.

Its kernel is the nilradical of $R_p(G)$.

If G is finite or if $p = 0$ it is surjective. In general, given $y \in \Pi\, \tilde{R}_p(S)^{W_S}$ one can find k such that y^{p^k} comes from $R_p(G)$.

In any case the map $\coprod (\operatorname{Spec} \tilde{R}_p(S))/W_S \to \operatorname{Spec} R_p(G)$ is a homeomorphism. ($\coprod$ denotes topological sum.)

<u>Remarks</u> 1) The reader can check that $\tilde{R}_p(S)^{W_S} \cong \tilde{R}(S)^{W_S} \otimes k_p \cong R(S)^{W_S}/(\phi_n(X)) \otimes k_p$ if S is p-regular.

2) The proof to follow is much simpler when $p = 0$, but one must then sometimes interpret p as 1, in a familiar way.

The proof will be preceded by a lemma.

<u>Lemma 2.5.8</u> Let $S \to H \to P$ be an extension of p-group P of order $q = p^k$ by a p-regular cyclic group S. Then the kernel of $r : R_p(H) \to R_p(S)^P$ is the nilradical of $R_p(H)$; if the extension is split r is surjective; in general, if $x \in R_p(S)^P$ then x^q comes from $R_p(H)$.

<u>Proof</u>: We know that any prime of $R_p(H)$ comes from a p-regular cyclic subgroup S_1. But any such S_1 lies in S, so each prime of $R_p(H)$ comes from $R_p(S)^P$. This implies that the kernel consists of nilpotent elements. On the other hand there are no nilpotent elements in $R_p(S)$.

Now I shall show that $R(H) \to R(S)^P$ is surjective if the extension is split. $R(S)^P$ can be identified with the free abelian group generated by the orbits of the set $\hat{S} = \hom(S; \mathbb{C}^*)$ under P. Given $a \in \hat{S}$, let $K < H$ be its stabiliser, i.e. $K = \{h \in H : a(hsh^{-1}) = a(s) \text{ for all } s \in S\}$. Then $a : S \to \mathbb{C}^*$ extends to $\bar{a} : K \to \mathbb{C}^*$ in $R(K)$ (because the extension is is split), and the character of H induced from K restricts to $\sum_{h \in H/K} h.a$, a typical basis element of $R(S)^P$.

In general the extension $S \to H \to P$ corresponds to an element $e \in H^2(P;S)$; and $q.e = 0$. Set $S_0 = \{s \in S : q.s = 1\}$, a finite p-group. From the short exact sequence $S_0 \to S \xrightarrow{\times q} S = S/S_0$ one finds that the induced extension $S/S_0 \to H/S_0 \to P$ is split. Now consider the diagram

$$\begin{array}{ccc} R_p(H/S_0) & \to & R_p(S/S_0)^P \\ \downarrow & & \downarrow \\ R_p(H) & \to & R_p(S)^P \end{array} .$$

The top line is surjective. And if $a \in \hat{S}$ then $a^q \in (S/S_0)^\wedge$. But $(\Sigma a_i)^q = \Sigma a_i^q \bmod p$, so the q^{th} power of an element of $R_p(S)^P$ comes from $R_p(H/S_0)$, and a fortiori from $R_p(H)$.

Proof of 2.5.7 We know that the map induces a surjection of spectra, so its kernel is nilpotent. And again there are no nilpotent elements on the right.

Now suppose given $x \in R(S)^{W_S}$. One must find $z \in R(G)$ which has the same image as x^{p^k} in $\tilde{R}_p(S)$ and has image 0 in the other $\tilde{R}_p(S')$. Let P be a Sylow p-subgroup of $W_S = N(S)/S$, and H its inverse image in $N(S)$.

I shall show that a suitable z can be induced from H.

Let us suppose first that $p \neq 2$ and $\dim(G/H)$ is even. Then one can define an induction-homomorphism $i_! : R(H) \to R(G)$ (§ 2.4 Ex.(iii)). In fact $i_* x$ lies in $R(G)$ if x is divisible by $\Delta_{G/H}$ in $R(H)$. Recall that if $g \in G$ is regular then $\chi_{i_* x}(g) = \sum_{\substack{y \in G/H \\ y^{-1} gy \in H}} \chi_x(y^{-1}gy)/\lambda_{G/H}(y^{-1}gy)$, where $\lambda_{G/H}(h) = \det(1 - \mathrm{ad}_{G/H}(h)) = \sum_k (-1)^k \operatorname{trace} \Lambda^k \mathrm{ad}_{G/H}(h)$. Now one has $\Delta_{G/H} \cdot \bar{\Delta}_{G/H} = \lambda_{G/H}$ (see e.g. [14] p.96), so $i_*(x.\lambda_{G/H}) \in R(G)$ for any $x \in R(H)$. Choose a character e of S which vanishes on its subgroups of finite index and elsewhere takes a single integral value m not divisible by p. That is possible because $R_p(S) = \prod_{S'} \tilde{R}_p(S')$ when S' runs through the subgroups of finite index in S (for $R_p(S) = R_p(S^0)[X]/(X^n - 1)$, with $n = (S : S^0)$, and $X^n - 1$ is separable over k_p). If $x \in R(S)^{W_S}$ then so is $e.x$, and $(e.x)^q$ can be pulled back to $R(H)$ $(q = (H : S))$. Consider $z = i_*((e.x)^q \lambda_{G/H}) \in R(G)$. If g generates a p-regular Cartan subgroup of G not conjugate to S then $z(g) = 0$, for any p-regular subgroup of H is contained in S. But if g generates S then $\chi_z(g) = (N(S) : H) m^q \chi_x(g)^q$, which is good enough for our purpose, as $(N(S) : H)m^q$ is invertible in k_p.

If $p \neq 2$ and $\dim(G/H)$ is odd the argument is the same: one sets $z = i_*((e.x)^q \Delta_{G/H} \bar{\Delta}_{G/H})$ and observes that $\Delta_{G/H} \bar{\Delta}_{G/H} = 2\lambda_{G/H}$ in $\tilde{R}(H)$ – in fact the characters coincide on the odd component of $O(T_{G/H})$. 2 is invertible in k_p.

When $p = 2$ I am reduced to the following. I shall produce a central double covering $\tilde{G}$ of G for which I can prove the proposition. But if $x \in R_2(\tilde{G})$ then x^2 comes from $R_2(G)$. For if $x = \Sigma \pm [M_i]$ with M_i simple then $x^2 = \Sigma\,[M_i \otimes M_i]$ (mod 2), and $M_i \otimes M_i$ comes from G because the kernel of $\tilde{G} \to G$ acts as ± 1 on M_i and hence trivially on $M_i \otimes M_i$. Now consider the diagram

$$\begin{tikzcd} R_2(G) \arrow[r, "r"] \arrow[d, "u"'] & \Pi\, \tilde{R}_2(S)^{W_S} \arrow[d, "v"] \\ R_2(\tilde{G}) \arrow[r, "\tilde{r}"] & \Pi\, \tilde{R}_2(\tilde{S})^{W_S} \end{tikzcd} .$$

v is injective. If $y \in \Pi\, \tilde{R}_2(S)^{W_S}$ one can find $\tilde{x} \in R_2(\tilde{G})$ such that $\tilde{r}(\tilde{x}) = v(y)^{2^k}$. But $\tilde{x}^2 = u(x)$ with $x \in R_2(G)$. So $vr(x) = ru(x) = v(y^{2^{k+1}})$, and $y^{2^{k+1}} = r(x)$, as required.

It remains to produce $\tilde{G}$ and prove the proposition for it. Define $\tilde{G}$ as the fibred product $G \times_{\mathrm{Aut}(T_G)} \mathrm{Pin}(T_G)$, where $G \to \mathrm{Aut}(T_G)$ is the adjoint-action. I shall write from now on G for $\tilde{G}$. Then all possible $T_{G/H}$ have Pin-structures, for a submodule of a module with a Pin-structure has a Pin-structure. But one is in trouble about orientability: $\dim(G/H)$ is always even (say $2r$) in this case. Let H_1 be the part of H which preserves the orientation of $T_{G/H}$. If $H_1 = H$ the argument above applies. Otherwise one must define $z = j_*((s.x)^q \Delta_{G/H} \bar{\Delta}^0{}_{G/H})$, where $j : H_1 \to G$. $\Delta^0_{G/H}$ is not invariant under W_S. In fact $\Delta_{G/H} \bar{\Delta}^0{}_{G/H} = \sum_{k=0}^{r-1} (-1)^{r-k} \Lambda^k T_{\mathbb{C}} + \Lambda^r_+ T_{\mathbb{C}}$, and the orientation-reversing elements of W_S change it into $\sum_{k=0}^{r-1} (-1)^{r-k} \Lambda^k T_{\mathbb{C}} + \Lambda^r_- T_{\mathbb{C}}$, where $\Lambda^r_{\pm} T_{\mathbb{C}}$ are the two modules into which $\Lambda^r T_{\mathbb{C}}$ splits because T is orientable. (Notice that $\Lambda^k T_{\mathbb{C}} = \Lambda^{2r-k} T_{\mathbb{C}}$; cf. [14], p.96.) All in all, if g generates S, $\chi_z(g) = (N(S) : H) m^q\, \chi_x(g)^q$, as before. This completes the proof of the bulk of 2.5.7.

When G is finite the groups H are always split, and there is no problem of Spin^c-structures, so the morphism of 2.5.7 is actually surjective.

The last statement, about spectra, is a consequence of the earlier ones. The only non-trivial point is that $(\operatorname{Spec} R)/W \xrightarrow{\cong} \operatorname{Spec} R^W$, when W is a finite group. But that map is surjective because R is finite over R^W. And if $\mathfrak{p}_1, \mathfrak{p}_2$ are primes of R such that $\mathfrak{p}_1 \cap R^W = \mathfrak{p}_2 \cap R^W$, but $\mathfrak{p}_1 \neq s.\mathfrak{p}_2$ for any $s \in W$, then choose $x_s \in s.\mathfrak{p}_2$, $x_s \notin \mathfrak{p}_1$ for each

$s \in W$, so that Πx_s is both in and not in $\mathcal{P}_1$.

From Proposition 2.5.7 one obtains the following statements about the primes of $R(G)$.

<u>Proposition 2.5.9</u> (i) The primes of $R(G)$ can be represented $(p,S,\wp)$, where p is a prime of $\mathbb{Z}$, S is a p-regular cyclic subgroup, and $\wp$ is a prime of $\tilde{R}_p(S)$.

$(p,S,\wp) = \rho^{-1}\wp$, where $\rho : R(G) \to \tilde{R}_p(S)$ is the natural map.

$(p,S,\wp) = (p',S',\wp')$ if and only if $p = p'$ and S is conjugate to S' by a conjugacy which takes $\wp$ to $\wp'$.

(ii) $(p,S,\wp)$ is maximal if and only if $p \neq 0$ and S is finite.

It is minimal if and only if $p = \wp = 0$ and S is a Cartan subgroup.

(iii) If H is a subgroup of G the following are equivalent:

(a) $(p,S,\wp)$ comes from $R(H)$

(b) S is conjugate to a subgroup of H

(c) the localised module $R(H)_{(p,S,\wp)}$ is non-zero.

<u>Proof</u> (i) is already established.

(ii): If $(p,S,\wp)$ is maximal then p is maximal, for $\mathbb{Z}$ is a direct factor of $R(G)$; so $p \neq 0$. Then $\wp$ is the kernel of a morphism of $R_p(S) = k_p[\hat{S}]$

into a finite field K. Such a morphism is induced by a homomorphism of $\hat{S}$ onto a finite cyclic subgroup $\hat{S}_0$ of K^*. Then $R_p(S) \to K$ factorises through $R_p(S) \to R_p(S_0)$, where S_0 is a finite cyclic subgroup of S.

Conversely if S is finite cyclic then $\tilde{R}_p(S)$ is a sum of fields, so $(p,S,\wp)$ is maximal.

(iii) (a) $\Leftrightarrow$ (b) follows from (i)

(a) $\Leftrightarrow$ (c) because $R(H)_{(p,S,\wp)} = 0 \Leftrightarrow \exists x \notin (p,S,\wp)$ such that $x|H = 0$, while $(p,S,\wp)$ comes from $R(H)$ if and only if it contains the kernel of $R(G) \to R(H)$.

<u>Example</u>

The prime ideal I_G consisting of characters vanishing at the identity-element is called the <u>augmentation-ideal</u>. It will be important in chapter 7. In the notation of 2.5.9, $I_G = (0,1,0)$. Because $R(G)/I_G \cong \mathbb{Z}$, I_G is not maximal, but any prime ideal containing it is maximal, and so of the form $(p,S,\wp)$ with $p \neq 0$ and S finite and p-regular. Clearly this contains I_G if and only if $S = 1$ (and $\wp = 0$). One can read off now the generalisations of two propositions of [1].

<u>Proposition 2.5.10</u> (cf. [1] 6.7) If H is a subgroup of G then the primes

of $R(H)$ containing I_H are the same as those containing the image of I_G. Hence the I_G-adic topology of $R(H)$ when it is regarded as an $R(G)$-module coincides with its I_H-adic topology.

<u>Proof</u>: Let $r : R(G) \to R(H)$ be the restriction. $(p,S,\wp)$ in $R(H)$ contains $r(I_G)$ if and only if $(p,S,\wp)$ in $R(G)$ contains I_G, i.e. if and only if $S = 1$. But then $(p,S,\wp) \supset I_H$.

<u>Proposition 2.5.11</u> (cf. [1] 6.9) $\bigcap_{n \in \mathbb{N}} I_G^n$ consists of the elements whose characters vanish on all components of prime-power order in G/G^0.

<u>Proof</u>: $\bigcap I_G^n$ is the intersection of those minimal primes $(0,S,0)$ of $R(G)$ which are not coprime to I_G, i.e. are such that $I_G + (0,S,0) \neq R(G)$, which means that $(0,S,0)$ is contained in $(p,1,0)$ for some p. But that is true if and only if $(p,S',0)$ is contained in $(p,1,0)$, where S' is the p-regular part of S, and so if and only if S' is connected, i.e. $(S : S^0) = p^k$ for some k. And finally we know that the generators of Cartan subgroups with $(S : S^0)$ a prime-power form a dense subset of the components of prime-power order in G/G^0.

<u>Brauer's theorem</u> In the theory of the representations of a finite group G one has the following three results, any or all of which I shall

call Brauer's theorem.

(B1) $R(G)$ is generated as abelian group by the modules induced from modules for elementary subgroups of G.

(B2) $R(G)$ is generated as abelian group by the modules induced from one-dimensional modules for subgroups of G.

(B3) a class-function on G is a character if its restriction to each elementary subgroup is a character.

I shall now generalise these statements to compact Lie groups. Let us for the present define an <u>elementary group</u> as an extension of a p-group by a p-regular cyclic group - the usual definition prescribes a <u>central</u> extension. Then one has

<u>Proposition 2.5.12</u> If G is a group whose adjoint-action admits a Pin^c-structure, then (B1) and (B2) are true.

<u>Proposition 2.5.13</u> If G is any compact Lie group, then (B3) is true.

<u>Proof of 2.5.12</u> (B1): The modules induced from elementary subgroups generate an ideal in $R(G)$. The substance of the proof of 2.5.7 was to shew that this ideal is not contained in any prime ideal of $R(G)$.

(B1) implies (B2) in view of the following proposition, which was pointed out to me by Professor J. Tits.

<u>Proposition 2.5.14</u> If $S \to H \to P$ is an extension of a nilpotent group P by a cyclic group S then any simple H-module is monomial, i.e. induced from a one-dimensional module of a subgroup of finite index.

<u>Remark</u>: This result is well-known, and I shall quote it, when H is nilpotent. ([22] Exposé 24)

<u>Proof</u>: Let M be a simple H-module. Suppose $M = M_1 \oplus \ldots \oplus M_k$ is its decomposition into isotypical S-modules.

Let $K = \{h \in H : hM_1 \subset M_1\}$. If $j : K \hookrightarrow H$ then $j_* M_1 \xrightarrow{\cong} M$, so it suffices to show M_1 is monomial.

Let $\bar{K}$ be the image of K in $\mathrm{Aut}\,(M_1)$. $\bar{K}$ is a <u>central</u> extension of a nilpotent group by a cyclic group, so it is nilpotent. Therefore M_1 is monomial as a $\bar{K}$-module, say $M_1 \cong \mathbb{C}[\bar{K}] \otimes_{\mathbb{C}[\bar{K}_0]} N$. But that is the same as $\mathbb{C}[K] \otimes_{\mathbb{C}[K_0]} N$, so M_1 is monomial also as K-module.

<u>Proof of 2.5.13</u>: First suppose that the adjoint-action of G admits a Pin^c-structure. Let $\tilde{R}$ be the ring of class-functions on G which restrict to characters of all elementary subgroups. $R(G)$ is a subring of $\tilde{R}$: it

suffices to show that it is an ideal. But the characters induced from elementary subgroups generate an ideal in $\tilde{R}$, as one sees by looking at the formula for the induced character, and on the other hand by 2.5.12 they generate $R(G)$.

<u>Remarks</u>

The theory I have developed in this section is very crude. For example it does not yield, when G is connected and T a maximal torus, that $R(G) \xrightarrow{\cong} R(T)^{W_T}$ (though one can deduce from 2.5.7 that any element of $R(T)^{W_T}$ has a power in $R(G)$). This is because my "elementary subgroups" are much too large. By choosing a more sensible e in the proof of 2.5.7 one can make H a bit smaller so that H/H^0 is a classical elementary subgroup, i.e. $H/H^0 \cong P \times (S/S^0)$, and then my Brauer's theorem includes the classical one; but that is only scratching the problem. The real point is that I have used - say when G is connected - only the fact that G/T has an almost-complex structure, whereas it is actually a complex manifold, indeed a rational algebraic variety. To put it another way: I ordered the roots of G randomly, ignoring their remarkable properties of coherence. This meant that I threw away the cancelling-power of the factor $\Delta_{G/H}/\lambda_{G/H} = 1/\bar{\Delta}_{G/H}$ in the formula for the induced character, and so had to use a very large H. For example when G is connected there is an identity, due to Hermann Weyl,

$$i_! 1 = \sum_{s \in W_T} 1/s.\bar{\Delta}_{G/T} = 1,$$

so one never needs to take H larger than T. The identity depends on the fact that G/T is a rational algebraic variety.

By changing one's point of view, and looking at the complex algebraic group $G_{\mathbb{C}}$ of which G is the real part, one can do better. One can find subgroups H of G which are extensions of ordinary elementary subgroups of G/G^0 by maximal tori of G^0 - thus they are in general not extensions of p-groups by cyclic groups, and not nilpotent, but by 2.5.14 their simple modules are monomial - such that G/H is a complete algebraic variety. (<u>Proof</u>: Let B be a Borel subgroup of $G_{\mathbb{C}}$ [15]. It can be shown that $N(B)/B \cong G_{\mathbb{C}}/G^0_{\mathbb{C}} \cong G/G^0$. Let $H_{\mathbb{C}} < N(B)$ be the inverse image of the elementary subgroup of G/G^0; and $H = H_{\mathbb{C}} \cap G$. Then $G/H \cong G_{\mathbb{C}}/H_{\mathbb{C}}$, so G/H is a complete variety.)

Using these groups as elementary subgroups one can prove Brauer's theorem without restriction on G: indeed the theorem reduces to the finite case when one has shown that the diagram

$$\begin{tikzcd} R(H) \arrow[r, "i_!"] & R(G) \\ R(H/H^0) \arrow[u, "p^*"] \arrow[r, "i_!"'] & R(G/G^0) \arrow[u, "p^*"'] \end{tikzcd}$$

(Note that $H^0 = H \cap G^0$)

is commutative. This commutativity is the natural generalisation of Weyl's identity above.

One can try to do better still. Inside H one can choose E which is an extension by a Cartan subgroup S of a p-subgroup of G/G^0 centralising $S/S \cap G^0$.

(<u>Proof</u>: $H/H^0 = C \times P$. Choose a Cartan subgroup S (of H and of G) covering C. $N_H(S)$ covers $N_{H/H^0}(C)$ by 2.3.7. Let $E < N_H(S)$ be the inverse image of a Sylow p-subgroup of $N_H(S)/S$: it can be shown that $E/S \cong P$.)

If $\Delta_{H/E}$ is defined, and $j : E \to H$ is the inclusion, the induction $x \to j_*(x.\Delta_{H/E})$ multiplies $\chi_x(g)$, where g generates S, by $(N_H(S) : E)$, which is not divisible by p. So if $\Delta_{H/E}$ were always defined then Brauer's theorem would hold using groups like E - extensions by a Cartan subgroup S of a Sylow p-subgroup of the centraliser of $S/S \cap G^0$ in the group of components G/G^0. We know that $\Delta_{H/E}$ exists when $p \neq 2$, but it need not when $p = 2$. So we are back to Proposition 2.5.12 above, but with much smaller elementary subgroups.

This is still not very satisfactory. My latest and best elementary subgroups are not characterised intrinsically, as groups, but only as subgroups. Also one would like elementary subgroups to be nilpotent, as in the classical theory. For the statement (B1) that is not possible. Eg. if G is the orthogonal group $O(2)$ the nilpotent subgroups are $O(1)$ and the subgroups of $SO(2)$, and any character induced from them has an even value at the identity-element. For the statement (B3) it may well be possible - e.g. a class-function on $O(2)$ is a character if it restricts to characters of $O(1)$ and $SO(2)$ - but I haven't proved it.

It may be worth mentioning another matter. One can read off from 2.5.7 (the case $p = 0$) the structure of $\operatorname{Spec} R(G) \otimes_{\mathbb{Z}} \mathbb{C}$. It is a sum of quotients of algebraic tori $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$ by Weyl groups. It can be shown that this is the space of conjugacy-classes of semi-simple elements of the complex algebraic group $G_{\mathbb{C}}$. (An element of a group is semi-simple if it is represented by a semi-simple transformation in any representation. Thus any element of a compact group is semi-simple.)

I shall end with an amusing example. Let $G^0 = U(2n)$, and let G be the semi-direct product $G^0 \tilde{\times} (\mathbb{Z}/2)$, where $\mathbb{Z}/2 = \{1, s\}$ and s acts on G^0 by complex-conjugation. Then $G^0_{\mathbb{C}} = Gl(2n, \mathbb{C})$, and s acts on it by

$A \mapsto ({}^tA)^{-1}$. The centraliser of z is $O(2n) \times (\mathbb{Z}/2)$, so a Cartan subgroup is $S = SO(2) \times \ldots \times SO(2) \times (\mathbb{Z}/2)$. Writing down the condition that an element of $G^0.z$ is conjugate by an element of G^0 to an element of $S^0.z$ one finds: for any unitary matrix A there is a unitary matrix B such that ${}^tB.A.B$ is of the form $\bigoplus \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$. The analogue for the complex group gives: for any non-singular complex matrix A such that ${}^tA^{-1}.A$ is semi-simple there is a matrix B such that ${}^tB.A.B$ is of the form $\bigoplus \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$. I have not come across this result in the literature, though it can presumably be proved by non-transcendental methods.

Chapter Three

G - V E C T O R - B U N D L E S

From now on "group" will mean "compact group over $\mathbb{Z}/2$ ", i.e. each group G will be equipped with a homomorphism $\alpha : G \to \mathbb{Z}/2$. $\mathbb{Z}/2$ is the Galois group of $\mathbb{C}$ over $\mathbb{R}$, and wherever the field $\mathbb{C}$ occurs I shall assume that G acts on it by α.

3.1 A complex vector-space E in the category of G-spaces over a fixed G-space X will be called a G-vector-bundle on X if

(i) the fibre E_x is finite dimensional for all $x \in X$, and

(ii) E is locally trivial in the sense that, for some numerable covering $\mathcal{R}$ of X, $Y \in \mathcal{R} \Rightarrow E \times_X Y \cong \mathbb{C}^n \times Y$ as vector-space over Y (neglecting the G-action), where n can depend on Y.

I emphasise that "complex vector-space" means that the structural-map $\mathbb{C} \times E \to E$ is a G-map when G acts on both $\mathbb{C}$ and E. I shall usually only be interested in the case of a compact base-space X.

G-Vect (X) will denote the category of G-vector-bundles on X. G-Vect (point) will be identified with G-Vect.

For any X there is a functor from G- <u>Vect</u> to G-<u>Vect</u> (X) which assigns to the complex G-module M the "trivial" G-vector-bundle $M \times X$ on X. I shall abbreviate $M \times X$ to $\underline{M}$ when the base-space is clear from the context.

If $f : (H,Y) \to (G,X)$ is a morphism of spaces with group-action and E is a G-vector-bundle on X then $f^*E = E \times_X Y$ is an H-vector-bundle on Y. In fact f^* is a functor from G-<u>Vect</u> (X) to H-<u>Vect</u> (Y). Sometimes it is convenient to combine all the G-<u>Vect</u> (X) for variable (G,X) into a single category $*$-<u>Vect</u>$(*)$. Its objects are triples (G,X,E), and a morphism from (G,X,E) to (G',X',E') is a pair (f,ϕ) where $f : (G,X) \to (G',X')$ and $\phi : E \to f^*E'$ is a morphism in G-<u>Vect</u> (X).

If E is a G-vector-bundle on X then $\mathrm{Map}_X(X;E)$ is by definition a G-vector-space. It is called the space of sections of E, and will be written $\Gamma_X E$. The G-invariant subspace $\Gamma^G \Gamma_X E$ is called the space of "equivariant sections", and written $\Gamma_X^G E$.

If $i : A \to X$ is the inclusion of a subspace, and $E \in$ G-<u>Vect</u> (X), then i^*E will be written $E|A$.

3.2 Descent or clutching

Suppose the G-space X is a quotient of the H-space Y by an open equivalence relation : $(H \times_G H, Y \times_X Y) \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} (H,Y) \xrightarrow{f} (G,X)$ with f, pr_1, pr_2 open. (In fact: f open $\Leftrightarrow$ pr_1 open $\Leftrightarrow$ pr_2 open.)

Let E be an H-vector-bundle on Y, and suppose given an isomorphism $\alpha : pr_1^* E \to pr_2^* E$ satisfying the conditions

(i) (reflexivity) $\Delta^*(\alpha)$ = identity, where $\Delta : Y \to Y \times_X Y$ is the diagonal

(ii) (symmetry) $t^*(\alpha) = \alpha^{-1}$, where $t : Y \times_X Y \to Y \times_X Y$ permutes the factors

(iii) (transitivity) $pr_{13}^*(\alpha) = pr_{23}^*(\alpha) \circ pr_{12}^*(\alpha)$, where $pr_{ij} : Y \times_X Y \times_X Y \to Y \times_X Y$ is projection on to the i^{th} and j^{th} factors.

Then α defines an equivalence relation $R : E' \rightrightarrows E$ on E. R is open, because open maps are universally open. Therefore $(H \times E)/(R \times R) \xrightarrow{\cong} G \times (E/R)$ and $(E \times_Y E)/(R \times_Y R) \xrightarrow{\cong} (E/R) \times_X (E/R)$ (Top.gén. I.5), and E/R is a G-vector-space on X. Furthermore the map $E \to f^*(E/R) = (E/R) \times_X Y$, which is clearly bijective, is an isomoprhism because both spaces are quotients of $E \times_X Y$. If f is in addition locally split - not necessarily respecting the group-action - then E/R is locally trivial on X. So in this case there is equivalence between G-Vect (X) and the category of pairs (E,α) as above, and one says that E/R, which I shall normally write $(E,\alpha)/f$, or simply E/f, is obtained by "descent" from E with "descent-datum" α, or by "clutching E along $Y \times_X Y$ by means of α". One says also that f is a "descent-morphism" for $*$-Vect $(*)$.

If one tries to carry through the above reasoning when f is a closed map one finds that it fails unless f is also universally closed, i.e. proper. For proper maps the argument is as above.

When $H = G$ the following formulation is often convenient. Let X be given in the form $\varinjlim_{\alpha\in S} X_\alpha$, where $\{X_\alpha\}_{\alpha\in S}$ are G-subspaces of X - i.e. X is a quotient of $Y = \coprod_{\alpha\in S} X_\alpha$. Suppose given $E_\alpha \in G$-Vect (X_α) for all $\alpha \in S$, and also coherent isomorphisms $\phi_t : E_\alpha \xrightarrow{\cong} t^*E_\beta$ for each arrow $t : \alpha \to \beta$ in S. ("Coherent" means $\phi_{id} = id$ and $\phi_{st} = t^*(\phi_s) \circ \phi_t$.) Then if the morphism $Y \to X$ satisfies suitable conditions one can form a vector-bundle $\varinjlim_{\alpha\in S} E_\alpha$ on X by descent, and one can say

$$G - \underline{\text{Vect}}\,(X) \;=\; \varprojlim_{\alpha\in S}\; G - \underline{\text{Vect}}\,(X_\alpha).$$

Examples

(i) $\{X_\alpha\}_{\alpha\in S}$ is a numerable covering of X by G-stable open sets. Then

$X = \varinjlim_{\sigma\in\Sigma} X_\sigma$, where Σ is the category of finite subsets of S, and

$X_\sigma = \bigcap_{\alpha\in\sigma} X_\alpha$. $f : \coprod X_\sigma \to X$ is open, and by definition locally split,

so G-Vect $(X) = \varprojlim_{\sigma\in\Sigma} G$-Vect (X_σ).

(ii) $\{X_\alpha\}_{\alpha\in S}$ is a locally finite covering of X by closed G-stable sets. Then $X = \varinjlim X_\sigma$ as before. In this case $f : \coprod X_\sigma \to X$ is proper, but not locally split. I shall nevertheless show (3.5.3) that if X is compactly generated then G-Vect $(X) = \varprojlim G$-Vect (X_σ)

(iii) Let G be a compact Lie group, and $X = Y/N$, where N is a normal subgroup of G which acts freely on X. Then $f : (G,Y) \to (G/N,X)$ is both open and proper. If Y is paracompact then f is locally split (by the existence of slices for the action on N (1.5)). There is a unique possible descent datum, and $E/f = E/N$. This proves part of

Proposition 3.2.1 If G is a compact group and Y is paracompact and compactly generated, and $N \lhd G$ acts freely on Y, then

$$f^* : (G/N)\text{-}\underline{\text{Vect}}\,(Y/N) \xrightarrow{\cong} G\text{-}\underline{\text{Vect}}\,(Y).$$

The proof when G is not a Lie group will be explained in 3.3.

(iv) X is obtained from the G-space Y by shrinking to a point a compact G-subspace A. (I shall write $X = Y/A$.) Then $f : Y \to X$ is proper. It is not hard to see that a descent-datum for f is an isomorphism $A \times M \xrightarrow{\cong} E|A$, where M is a complex G-module. In fact

<u>Proposition 3.2.2</u> If A is a compact G-stable subspace of the compactly generated G-space Y, and $X = Y/A$, then G-<u>Vect</u> (X) is equivalent to the category of triples (E,M,α), where $E \in$ G-<u>Vect</u> (Y), M is a G-module, and $\alpha : M \xrightarrow{\cong} E|A$.

The proof will be given in 3.5. (It remains only to show that E/f is locally trivial.)

3.3 <u>Tensor products, etc.</u>

G-<u>Vect</u> (X) is a "multiadditive" category: as well as abelian groups of morphisms $\mathrm{Hom}_X^G(E; E')$ from E to E' there are abelian groups $\mathrm{Hom}_X^G(E_1,E_2;E')$ of "multilinear" morphisms $E_1 \times_X E_2 \to E'$. (Of course this is so in the category of vector-spaces in <u>any</u> category.) So it is reasonable to consider the existence of functors $(E,E') \mapsto E \otimes E'$,

$(E',E'') \mapsto \underline{\mathrm{Hom}}_X(E';E'')$ defined by

$$\mathrm{Hom}_X^G(E \otimes E';\ E'') \cong \mathrm{Hom}_X^G(E,E';E'') \cong \mathrm{Hom}_X^G(E;\ \underline{\mathrm{Hom}}_X(E';E'')).$$

The functors exist, but I shall not give the proof here. The fibre of $E \otimes E'$ at $x \in X$ is $E_x \otimes E'_x$; that of $\underline{\mathrm{Hom}}_X(E;E')$ is $\mathrm{Hom}(E_x;E'_x)$. Similarly the functors $E \mapsto \otimes^q E$, $\Lambda^q E$, $S^q E$ exist.

Notice that $\mathrm{Hom}_X^G(\underline{\mathbb{C}};E) = \Gamma_X^G E$ for any G-vector-bundle E, and therefore $\Gamma_X^G\, \underline{\mathrm{Hom}}_X(E;E') = \mathrm{Hom}_X^G(E;E')$.

Let $\underline{\mathrm{Isom}}_X(E;E')$ be the open subspace of $\underline{\mathrm{Hom}}_X(E;E')$ whose fibre over $x \in X$ is $\mathrm{Isom}\,(E_x;E'_x)$. It is locally trivial over X and has the property: $\Gamma_X^G\, \underline{\mathrm{Isom}}_X(E;E') = \mathrm{Isom}_G^G(E;E')$. If E_x has constant dimension n for $x \in X$, then $P = \underline{\mathrm{Isom}}_X(\underline{\mathbb{C}}^n;E)$ is called the <u>principal bundle</u> associated to E. It is a $(G \times A)$-space, where $A = \mathrm{Aut}\,(\mathbb{C}^n)$; and $X = P/A$, $E = (P \times \mathbb{C}^n)/A$. When E has a hermitian structure (3.5.5) it has a subbundle P_0 (also locally trivial because the map $\underline{\mathrm{Isom}}_X(\underline{\mathbb{C}}^n;E) \to \underline{\mathrm{Isom}}_X(\underline{\mathbb{C}}^n;E)$ given by orthonormalisation is continuous) which is a $(G \times U(n))$-space such that $P = (P_0 \times A)/U(n)$, $E = (P_0 \times \mathbb{C}^n)/U(n)$.

With the aid of this construction one can complete the proof of 3.2.1: one has a G-vector-bundle E on Y, and $N \triangleleft G$ acts freely on Y; one wants to show that E/N is locally trivial on $X = Y/N$. But

$E/N \cong ((P_0 \times \mathbb{C}^n)/U(n))/N \cong ((P_0/N) \times \mathbb{C}^n)/U(n)$, so it suffices to show that P_0/N is locally trivial on $(P_0/N)/U(n) \cong X$. But $U(n)$ is by construction a compact Lie group, and P_0/N is paracompact. (For Y paracompact $\Rightarrow P_0$ paracompact $\Rightarrow P_0/N$ paracompact.)

The functorial properties of $\otimes$, $\underline{\mathrm{Hom}}_X$, etc. are continually needed:

Proposition 3.3.1 If $f : Y \to X$ is a G-map, and $E, E' \in$ G-Vect (X), then $f^*(E \otimes E') \cong f^*E \otimes f^*E'$,

$f^*\underline{\mathrm{Hom}}_X(E;E') \cong \underline{\mathrm{Hom}}_Y(f^*E;f^*E')$; $f^*\underline{\mathrm{Isom}}_X(E;E') \cong \underline{\mathrm{Isom}}_Y(f^*E;f^*E')$;

and similarly for the other functors mentioned above.

Proposition 3.3.2 If $f : Y \to X$ is a G-map, then

$$\mathrm{Map}_X^G(Y; \underline{\mathrm{Hom}}_X(E;E')) \cong \mathrm{Hom}_Y^G(f^*E; f^*E').$$

(This generalizes, and in view of 3.3.1 is implied by,

$$\Gamma_X^G \, \underline{\mathrm{Hom}}_X(E;E') \cong \mathrm{Hom}_X^G(E;E') \text{ .)}$$

I shall omit the proofs of 3.3.1 and 3.3.2.

The G-vector-bundle $\underline{\mathrm{Hom}}_X(E; \underline{\mathbb{C}})$ is called the dual of E, and written E^*. The fundamental property of duality in G-Vect (X) is

<u>Proposition 3.3.3</u> $\underline{\mathrm{Hom}}_X(E_1;E_2) \xrightarrow{\cong} \underline{\mathrm{Hom}}_X(E_2^*;E_1^*)$.

One can deduce from this such facts as:

$E \xrightarrow{\cong} (E^*)^*$; $\underline{\mathrm{Hom}}_X(E_1;E_2) \cong E_1^* \otimes E_2$; $(E_1 \otimes E_2)^* \cong E_1^* \otimes E_2^*$; $\underline{\mathrm{Hom}}_X(E_1;E_2)^* \cong \underline{\mathrm{Hom}}_X(E_1^*;E_2^*)$.

Finally, I shall mention the most important practical consequence of the constructions in this section. Because the inverse map $\mathrm{Isom}(M;M') \to \mathrm{Isom}(M';M)$ is continuous there is a continuous inverse $\underline{\mathrm{Isom}}_X(E;E') \to \underline{\mathrm{Isom}}_X(E';E)$ over X. Applying Γ_X^G one discovers

<u>Proposition 3.3.4</u> A morphism in G-<u>Vect</u>(X) is an isomorphism if its restriction to each fibre is an isomorphism.

From now on I shall commit continually the following abuse of language. I shall call a morphism in G-<u>Vect</u>(X) an injection (resp. surjection, bijection) if its restriction to each fibre is an injection (resp. ...). This is clearly different from an injection (resp. ...) in the category G-<u>Vect</u>(X): in fact it is the same as a <u>strict</u> injection (resp. ...) in the category. I shall use the words "exact sequence" similarly. With this abuse I can write $\mathrm{Monom}_X^G(E;E') = \Gamma_X^G\, \underline{\mathrm{Monom}}_X(E;E')$, $\mathrm{Epim}_X^G(E;E') = \Gamma_X^G\, \underline{\mathrm{Epim}}_X(E;E')$, where $\underline{\mathrm{Monom}}_X(E;E')$, $\underline{\mathrm{Epim}}_X(E;E')$ are certain

subspaces of $\underline{\mathrm{Hom}}_X(E;E')$ whose definition will be clear. I shall introduce also another subspace $\underline{\mathrm{Proj}}_X(E)$ of $\underline{\mathrm{Hom}}_X(E;E) = \underline{\mathrm{End}}_X(E)$: an element of the fibre of $\underline{\mathrm{Proj}}_X(E)$ over $x \in X$ is a projection in E_x. Thus
$\Gamma_X^G\, \underline{\mathrm{Proj}}_X(E) = \mathrm{Proj}_X^G(E)$.

3.4 Extension of group

We have seen that a homomorphism of compact groups $\phi : G \to G'$ induces adjoint functors $\phi_* : \{G\text{-spaces}\} \to \{G'\text{-spaces}\}$ and $\phi^* : \{G'\text{-spaces}\} \to \{G\text{-spaces}\}$. The restriction ϕ^* obviously induces a functor
$\phi^* : G\text{-}\underline{\mathrm{Vect}}\,(Y) \to G\text{-}\underline{\mathrm{Vect}}\,(\phi^*Y)$ for a G'-space Y; but the same is not true in general for ϕ_*. One has, however,

<u>Proposition 3.4.1</u> If X is a paracompact G-space and $\phi : G \to G'$ is an inclusion - or, more generally, if $\ker \phi$ acts freely on X - then ϕ_* induces a functor $\phi_* : G\text{-}\underline{\mathrm{Vect}}\,(X) \to G'\text{-}\underline{\mathrm{Vect}}\,(\phi_*X)$ which is an equivalence of categories inverse to the restriction i^*, where $i : (G,X) \to (G',\phi_*X)$ is $X = G \times_G X \to G' \times_G X = \phi_*X$.

<u>Proof</u>: ϕ_* is the composite of two functors:

(i) pr_2^* : G-Vect $(X) \to (G' \times G)$-Vect $(G' \times X)$. Because G' acts freely on $G' \times X$ 3.2.(iii) shows that pr_2^* is inverse to the descent from $(G' \times G, G' \times X)$ to (G,X), which is always split.

(ii) descent from $(G' \times G)$ - Vect $(G' \times X)$ to G' - Vect $(G' \times_G X)$, also an equivalence by 3.2.(iii) under the conditions stated.

Remark: In Chapter 6 I shall define, under much stronger conditions on ϕ, a more subtle functor $\phi_!$: G - Vect $(\phi^*Y) \to G'$ - Vect (Y) for a G'-space Y.

3.5 Extension of Isomorphisms

If E is a G-vector-bundle on a locally compact G-space X, the G-vector-space $\Gamma_X E = \mathrm{Map}_X(X; E)$ has a topology defined by $\mathrm{Map}^G(Y; \Gamma_X E) = \Gamma^G_{X \times Y}(E \times Y)$ for any G-space Y. If X is compactly generated $\Gamma_X E$ "exists" only in the category of compactly generated G-spaces. The topology is of course in either case that of uniform convergence on compact sets.

<u>Proposition 3.5.1</u> If X is compactly generated, $\Gamma_X E$ is complete. If X is compact it is normable.

<u>Proof</u>: X can be written $\varinjlim_{\alpha \in S} X_\alpha$, where $\{X_\alpha\}$ are compact subspaces and $E|X_\alpha$ is trivial (say $= X_\alpha \times M_\alpha$). (One can forget about G.) Then $\mathrm{Map}_X(X; E) = \varprojlim \mathrm{Map}_X(X_\alpha; E) = \varprojlim \mathrm{Map}\,(X_\alpha; M_\alpha)$. But each $\mathrm{Map}\,(X_\alpha; M_\alpha)$ is complete and normable, and an inverse limit of complete spaces is complete. If X is compact one can choose S finite and then the inverse limit will be normable.

Thus the action of G on $\Gamma_X E$ can be extended to an action of M_G (the measures on G). In particular the Haar measure μ_G induces a projection of $\Gamma_X E$ on to $\Gamma_X^G E$, which is therefore a closed subspace.

<u>Convention</u> For the remainder of this chapter X will be a compactly generated paracompact space - though the paracompactness assumption could if necessary be avoided.

<u>Proposition 3.5.2</u> If A is a closed G-subspace of X then an equivariant section of $E|A$ can be extended to an equivariant section of E.

<u>Proof</u>: Let $\{X_\alpha\}_{\alpha \in S}$ be closed numerable covering of X with $E|X_\alpha$ trivial

for each α. By Tietze's theorem $\Gamma_{X_\alpha}(E|X_\alpha) \to \Gamma_{X_\alpha \cap A}(E|X_\alpha \cap A)$ is onto (set-theoretically). A partition of unity for $\{X_\alpha\}$ is a splitting of the restriction $\Gamma_X(E) \to \Pi\, \Gamma_{X_\alpha}(E|X_\alpha)$, and it induces a partition for $\{X_\alpha \cap A\}$. The proposition follows from the commutative diagram:

$$\begin{tikzcd}
\Gamma_X^G E \arrow[r, shift left, leftarrow, "\mu_G"] \arrow[r, shift right] \arrow[d, "r_1"] & \Gamma_X E \arrow[r, shift left] \arrow[r, shift right, leftarrow] \arrow[d, "r_2"] & \Pi\, \Gamma_{X_\alpha}(E|X_\alpha) \arrow[d, "r_3"] \\
\Gamma_A^G (E|A) \arrow[r, shift left, leftarrow] \arrow[r, shift right] & \Gamma_A(E|A) \arrow[r, shift left] \arrow[r, shift right, leftarrow] & \Pi\, \Gamma_{X_\alpha \cap A}(E|X_\alpha \cap A)
\end{tikzcd}$$

r_3 onto $\Rightarrow r_2$ onto $\Rightarrow r_1$ onto.

<u>Corollary 1</u> If E, E' are G-vector-bundles on X, and A is a closed G-subspace of X, and $f : E|A \to E'|A$ is a morphism, then f extends to a morphism $\tilde{f} : E \to E'$.

<u>Proof</u>: $\mathrm{Hom}_X^G(E; E') \cong \Gamma_X^G\, \mathrm{Hom}_X(E; E')$, etc.

<u>Corollary 2</u> If, in the same circumstances, $f : E|A \to E'|A$ is an isomorphis then there is a G-neighbourhood U of A in X such that f extends to an isomorphism $f' : E|U \xrightarrow{\cong} E'|U$.

Furthermore, if f'_0, f'_1 are two extensions of f then

$t \mapsto (1-t).f'_0 + t.f'_1$ $(t \in I)$ defines a homotopy from f'_0 to f'_1 through isomorphisms over some G-neighbourhood U' of A contained in U.

Proof: Choose some extension $\tilde{f}$ of f. Regard $\tilde{f}$ as a section of $\underline{\mathrm{Hom}}_X(E; E')$, and define $U = \tilde{f}^{-1}\, \underline{\mathrm{Isom}}_X(E; E')$: this will do. The last statement is proved by applying the same reasoning to the subspace $(U \times 0) \cup (A \times I) \cup (U \times 1)$ of $X \times I$.

I can now prove two statements I made without proof in 3.2.

Proposition 3.5.3 If $f : \coprod_{\alpha \in S} X_\alpha \to X$ is a locally finite closed covering of X by G-stable sets, and E is a G-vector-bundle on $\coprod_{\alpha \in S} X_\alpha$ with descent-data for f, then E/f is a G-vector-bundle on X.

Proof: G is irrelevant. One can assume that S is finite, and in fact that it has two elements; also that E is trivial. Then $X = X_1 \cup X_2$, and one has an isomorphism between $\underline{M}_1 | X_1 \cap X_2$ and $\underline{M}_2 | X_1 \cap X_2$. Extend it to an isomorphism over $U \supset X_1 \cap X_2$ (U open). Set $\tilde{X}_i = X_i \cup U$ $(i = 1,2)$. Then E/f is also defined by clutching $M_1 \times \tilde{X}_1$ and $M_2 \times \tilde{X}_2$ along $U = \tilde{X}_1 \cap \tilde{X}_2$, because $X_1 \amalg X_2 \to X$ factorises through $\tilde{X}_1 \amalg \tilde{X}_2 \to X$. But the latter is locally split, so E/f is locally trivial.

Proof of Proposition 3.2.2

A is a compact G-subspace of X. E is a G-vector-bundle on X; $\alpha : \underline{M} \xrightarrow{\cong} E|A$ is a G-trivialisation. $f : X \to X/A$.

I shall show that E/f is locally trivial.

Extend α to $\tilde{\alpha} : \underline{M} \to E|U$, where U is open and contains A. Then $(E/f)|fU \cong fU \times M$, because descent is unique. And fU is a neighbourhood of f(x) in X/A. So E/f is trivial near f(x). But elsewhere in X/A f is locally split, and there is nothing to prove.

Proposition 3.5.5 A G-vector-bundle E on X can be given a G-invariant Hermitian metric - i.e. there is a morphism $\alpha : \bar{E} \otimes E \to \underline{\mathbb{C}}$ such that $\alpha(\bar{x},y) = \overline{\alpha(\bar{y},x)}$, and $\alpha(\bar{x},x) > 0$ whenever $x \neq 0$.

Proof: The hermitian maps $\bar{E} \otimes E \to \underline{\mathbb{C}}$ correspond to real sections of the G-vector-bundle $F = \underline{\mathrm{Hom}}_X(\bar{E} \otimes E; \underline{\mathbb{C}})$. Choose a closed numerable covering $\{X_\alpha\}$ of X such that $E|X_\alpha$ is trivial. $F|X_\alpha$ is then also trivial, and has real positive sections. But the splittings used earlier

$$\Gamma_X^G F \overset{\mu_G}{\rightleftarrows} \Gamma_X F \overset{\{\phi_\alpha\}}{\rightleftarrows} \Pi\, \Gamma_{X_\alpha}(F|X_\alpha)$$

are real and positive, so one obtains a suitable section of F.

Corollary: $\overline{E} \cong E^*$ as G-vector bundle.

Proposition 3.5.4 If $\alpha : E' \to E$ is a monomorphism of G-vector-bundles then the cokernel $\beta : E \to E''$ is a G-vector-bundle. Dually, if $\beta : E \to E''$ is an epimorphism of G-vector-bundles then the kernel $\alpha : E' \to E$ is a G-vector-bundle.

Proof: G is irrelevant, and one can suppose $E' = \underline{\mathbb{C}}^n$, $E = \underline{\mathbb{C}}^N$. If $x \in X$ one can find a subspace M of $\mathbb{C}^N$ such that $\alpha(E'_x) \oplus M = \mathbb{C}^N$. Consider the morphism $E' \oplus \underline{M} \to E$. It is an isomorphism at x, and so also in a neighbourhood of x. Therefore coker $\alpha \cong \underline{M}$ near x, as required. The argument for kernels is precisely dual.

Proposition 3.5.6 A G-vector-bundle E on X is semi-simple, i.e. if E' is a sub-G-vector-bundle of E there is complementary sub-G-vector-bundle E'' such that $E \cong E' \oplus E''$. Equivalently, in view of 3.5.4, every injection and surjection of G-vector-bundles splits.

First proof: A hermitian structure on E induces one on E'. $E' \to E$ induces $\overline{E}^* \to \overline{E}'^*$, and the diagram

$$\begin{array}{ccc} E' & \to & E \\ \cong\downarrow & & \downarrow\cong \\ \overline{E}'^* & \leftarrow & \overline{E}^* \end{array}$$

commutes. But the vertical maps are isomorphisms, so we have a splitting.

<u>Second proof</u>: Choose, as one can by the proof of 3.5.4, a closed numerable covering $\{X_\alpha\}$ of X and local splittings $f_\alpha \in \mathrm{Hom}_X(E|X_\alpha; E'|X_\alpha)$. The usual pull-backs $\mathrm{Hom}_X^G(E; E') \xleftarrow{\mu_G} \mathrm{Hom}_X(E; E') \xleftarrow{\{\phi_\alpha\}} \Pi\, \mathrm{Hom}_{X_\alpha}(E|X_\alpha; E'|X_\alpha)$ provide a global splitting.

(I mention this proof because of its application to infinite-dimensional bundles (cf. § 4.5), when one doesn't have a Hermitian structure.)

<u>Proposition 3.5.7</u> A G-vector-bundle E on a <u>compact</u> G-space X is a quotient - hence also a sub-bundle - of a trivial G-vector-bundle $\underline{\underline{M}}$.

<u>Proof</u>: By 3.5.2 there is a surjection $\Gamma_X E \times X \to E$; one must find a finite-dimensional sub-G-module M of $\Gamma_X E$ such that the induced map $\underline{\underline{M}} \to E$ is surjective. But by 2.1.3 there is a bijection $\coprod M_\alpha \to \Gamma_X E$ with all M_α finite-dimensional. So $\coprod \underline{\underline{M}}_\alpha \to E$ is surjective: one is reduced to showing that G-vector-bundles are "noetherian". Certainly for each point $x \in X$ one can find a finite set S_x such that $\coprod_{\alpha \in S_x} M_\alpha \to E_x$ is surjective. But then $\coprod_{\alpha\in S_x} M_\alpha \to E_y$ is surjective for all y in some neighbourhood U_x of x, for $\underline{\mathrm{Epim}}_X(\coprod_{\alpha\in S_x} \underline{\underline{M}}_\alpha; E)$ is open in $\underline{\mathrm{Hom}}_X(\coprod_{\alpha\in S_x} \underline{\underline{M}}_\alpha; E)$. Choose a finite set of points x such that the corresponding U_x cover X, and let S be the union

of the corresponding S_x. Then $\coprod_{\alpha\in S} \underline{M}_{m_\alpha} \to E$ is surjective.

__Remarks__: If M is a G-module I shall write $V(n,M)$ for the G-manifold Monom $(\mathbb{C}^n; M)$, and call it a G-Stiefel-manifold. Aut $(\mathbb{C}^n)$ acts on $V(n,M)$, commuting with the action of G. The quotient $V(n,M)/\text{Aut}\,(\mathbb{C}^n)$ I call the G-Grassmannian $Gr(n,M)$. There is a canonical G-vector-bundle $E(n,M) = (V(n,M) \times \mathbb{C}^n)/\text{Aut}\,(\mathbb{C}^n)$ on $Gr(n,M)$. If E is a G-vector-bundle on X with n-dimensional fibres then to a monomorphism $E \to \underline{M}$ corresponds a map $P \cong \underline{\text{Isom}}_X(\underline{\mathbb{C}}^n; E) \to \text{Monom}\,(\mathbb{C}^n; M) = V(n,M)$. But $P/\text{Aut}\,(\mathbb{C}^n) \cong X$, so this induces $f : X \to Gr(n,M)$. And $f^* E(n,M) \cong (P \times \mathbb{C}^n)/\text{Aut}\,(\mathbb{C}^n) \cong E$, i.e. "any G-vectorbundle can be induced by a G-map into a G-grassmannian". Furthermore if one has two injections $E \to \underline{M}_0$, $E \to \underline{M}_1$ one obtains two maps $E \to \underline{M_0 \oplus M_1}$ which have a common splitting, and so are linearly homotopic through monomorphisms, and induce G-homotopic maps $X \to Gr(n, M_0 \oplus M_1)$.

__Proposition 3.5.8__ If $p : E \to E$ is a projection (i.e. $p \circ p = p$) in G-__Vect__ (X), then ker p, coker p, im p all exist in G-__Vect__ (X).

__Proof__: As usual only local triviality is in question: one can disregard G and assume $E = \underline{\mathbb{C}}^n$. p corresponds to $\tilde{p} : X \to \text{Proj}\,(\mathbb{C}^n)$. Now the dimension of the image is lower-semi-continuous on End $(\mathbb{C}^n)$, and therefore the dimension of the kernel is upper-semi-continuous. But on $\mathrm{Proj}\,(\mathbb{C}^n)$ im $q = \ker(1-q)$, so the dimension of the image is continuous, i.e. locally constant.

Let $x \in X$, $M = \ker \tilde{p}_x$, and let $M \to \mathbb{C}^n$ be the inclusion. $(1-p) \circ i : \underline{M} \to \underline{\mathbb{C}}^n$ is annihilated by p. It is injective at x, and hence in a neighbourhood of x. But dim $(\ker \tilde{p}_x)$ is locally constant, so $0 \to \underline{M} \xrightarrow{(1-p)\circ i} \underline{\mathbb{C}}^n \xrightarrow{p} \underline{\mathbb{C}}^n$ is exact near x. That is, ker p is locally trivial near x; and with its coker α and im α are locally trivial by 3.5.4.

<u>Remark</u>: The same argument shows that any acyclic complex (not necessarily bounded) of G-vector-bundles splits.

<u>Proposition 3.5.9</u> If E is a G-vector-bundle on X, and A is a closed G-subspace of X, and $p : E|A \to E|A$ is a projection, then there is a G-stable neighbourhood U of A such that p can be extended to a projection $q : E|U \to E|U$.

<u>Proof</u>: Let V be the subspace of $\underline{\mathrm{End}}_X(E)$ consisting of endomorphisms with no eigenvalues of modulus $\frac{1}{2}$. It is open, and it contains $\underline{\mathrm{Proj}}_X(E)$. Define a map $\pi : V \to \underline{\mathrm{End}}_X(E)$ over X by $y \mapsto \frac{1}{2\pi i} \int_{|z| = \frac{1}{2}} (z.1 - y)^{-1} dz$

(To be precise one should write $\pi = \frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}(z.d - y)^{-1}dz$, regarding π, d, y as elements of the topological ring $\Gamma_V^G(pr^* \underline{\mathrm{End}}_X(E))$, where $pr : V \to X$ is the projection. $d : V \to V \times_X \underline{\mathrm{End}}_X(E)$ is the diagonal map. The ring is complete because V is compactly generated.) π is an equivariant retraction of V on to $\underline{\mathrm{Proj}}_X(E)$: one can look at each End (E_x) separately, and decompose $E_x = E'_x \oplus E''_x$ so that E'_x and E''_x are stable under y, and $y|E'_x$ (resp. $y|E''_x$) has no eigenvalues of modulus $\leq \frac{1}{2}$ (resp. $\geq \frac{1}{2}$). Then $\pi(y) = 0 \oplus \mathrm{id}$.

Now the given projection p corresponds to a section $\tilde{p}$ of $\underline{\mathrm{Proj}}_X(E)|A$. Extend $\tilde{p}$ to $\tilde{r} \in \Gamma_X^G \underline{\mathrm{End}}_X(E)$ and define $U = \tilde{r}^{-1}V$. Then $\tilde{q} = \pi \circ (\tilde{r}|U)$: $U \to \underline{\mathrm{Proj}}_X(E)$ extends $\tilde{p}$ and defines the required q.

<u>Proposition 3.5.10</u> If A is a compact G-subspace of X, and E is a G-vector-bundle on A, then there is a G-neighbourhood U of A and a G-vector-bundle $\tilde{E}$ on U such that $\tilde{E}|A \cong E$.

<u>Proof</u>: Express E as a summand of some $\underline{M}$; extend $\underline{M}$ over X; extend the projection $\underline{M} \to E$ over U; define $\tilde{E}$ as its image.

It is reasonable to define a topology on the set <u>isom</u> G-<u>Vect</u> (X) of isomorphism-classes of G-vector-bundles as follows: a subset B is open if and only if, for any space Y (with trivial G-action) and any G-vector-bundle E on $X \times Y$, the set $\{y \in Y : [E_y] \in B\}$ is open in Y. Here E_y means $i_y^* E$, where $i_y : X \to X \times Y$ is $x \mapsto (x,y)$. Then one has

<u>Proposition 3.5.11</u> If X is compact, <u>isom</u> G-<u>Vect</u> (X) is discrete.

<u>Proof</u>: Given Y and E as above, and $y_0 \in Y$, one must show there is a neighbourhood U of y_0 such that $E_y \cong E_{y_0}$ for all $y \in U$. Consider the G-vector-bundle $pr^* E_{y_0}$. $(pr : X \times Y \to X)$. One has $E|X \times y_0 \cong (pr^* E_{y_0})|X \times y_0$, and can extend this to an isomorphism $E|X \times U \cong (pr^* E_{y_0})|X \times U$. But then $i_y^* E \cong i_y^* \, pr^* \, E_{y_0} = E_{y_0}$, as required.

<u>Proposition 3.5.12</u> If X is compact, and $f_0 \simeq f_1 : X \to Y$ as G-maps, and E is a G-vector-bundle on Y, then $f_0^* E \cong f_1^* E$, by an isomorphism which is determined up to homotopy by the homotopy $f : X \times I \to Y$ between f_0 and f_1.

<u>Remarks</u>: (i) Two <u>isomorphisms</u> $J_0, J_1 : F \to F'$ of G-<u>Vect</u>(X) are said to be homotopic if there is an isomorphism $J : pr^* F \to pr^* F'$ $(pr : X \times I \to X)$ such that $J_0 = i_0^*(J)$, $J_1 = i_1^*(J)$. ($i_t : X \to X \times I$ is $x \mapsto (x,t)$.)

(ii) The proposition is actually true for any X. [16]

<u>Proof</u>: The first statement follows at once from 3.5.11. But

$f = f_0 \circ pr$, so also $f^* E \cong pr^* f_0^* E$, and one can choose the isomorphism $J : f_0^* E \xrightarrow{\cong} f_1^* E$ of the form $i_1^*(K)$, where $K : pr^* f_0^* E \xrightarrow{\cong} f^* E$ and $i_0^*(K) = id$. If J' is another isomorphism satisfying this condition then $(J')^{-1} \circ J$ is homotopic to the identity, so $J \simeq J'$.

<u>Proposition 3.5.13</u> ("Homotopy extension") If A is a closed G-subspace of X, E and E' are G-vector-bundles on X, and $f_0 : E \to E'$, $h_1 : E|A \to E'|A$ are isomorphisms such that $h_0 = f_0|A \simeq h_1$, then h_1 can be extended to an isomorphism $f_1 : E \to E'$ such that $f_0 \simeq f_1$.

<u>Proof</u>: One has an isomorphism $Q : pr^* E|B \to pr^* E'|B$, where $B = (X \times 0) \cup (A \times I) \subset X \times I$. Extend Q to an isomorphism R over $\tilde{B} = (X \times 0) \cup (U \times I)$ for some open $U \supset A$. Choose $\rho : X \to I$ with support in U such that $\rho(x) = 1$ for $x \in A$. Define $r : X \times I \to \tilde{B}$ by $r(x,t) = (x, t\rho(x))$. Define $f = r^*(R) : pr^*E \to pr^*E'$. Then $f_1 = i_1^*(f)$ as required.

<u>Scholium</u>: The homotopy between f_0 and f_1 is determined up to homotopy by that between h_0 and h_1.

3.6 Trivial G-spaces

X is said to be a trivial G-space if G acts trivially on X and - i.e. on the sheaf of complex functions on X.

In this case (and not otherwise) there is a functor $T : \underline{\text{Vect}}(X) \to G\text{-}\underline{\text{Vect}}(X)$ which assigns to a vector-bundle the same bundle with trivial G-action. So it is reasonable to investigate the existence of adjoint functors Γ^G and $/G$ from $G\text{-}\underline{\text{Vect}}(X)$ to $\underline{\text{Vect}}(X)$ such that

$$\mathrm{Hom}_X^G(TE;\ E') \cong \mathrm{Hom}_X(E;\ \Gamma^G E')$$

$$\mathrm{Hom}_X^G(E;\ TE') \cong \mathrm{Hom}_X(E/G;\ E').$$

Proposition 3.6.1 If E is a G-vector-bundle on a trivial G-space X, then the G-invariant subspace is a vector-bundle on X which is both $\Gamma^G E$ and E/G.

Proof: The G-action $G \to \mathrm{End}_X(E)$ extends to a homomorphism of rings $M_G \to \mathrm{End}_X(E)$. The image of the central idempotent $\mu_G \in M_G$ is a G-projection $\mu_G : E \to E$. Its image is by 3.5.8 a G-vector-bundle which is both a subbundle and a quotient-bundle of E, and clearly satisfies the universal properties for $\Gamma^G E$ and E/G. It contains all the invariant points of E because through any such point there passes an equivariant section, i.e.

an element of $\mathrm{Hom}_X^G(\underline{\underline{\mathbb{C}}};\, E) \;\cong\; \mathrm{Hom}_X(\underline{\underline{\mathbb{C}}};\, \Gamma^G E)$.

<u>Corollary 3.6.2</u> If $E, E' \in G$-<u>Vect</u> (X) there is a vector-bundle $\underline{\mathrm{Hom}}_X^G(E;\, E')$ on X with the property $\Gamma_X\, \underline{\mathrm{Hom}}_X^G(E;\, E') \;\cong\; \mathrm{Hom}_X^G(E;\, E')$.

<u>Proposition 3.6.3</u> If G acts trivially on X then any E in G-<u>Vect</u> (X) can be decomposed: $\coprod_{M\in\Sigma_G} \underline{\underline{M}} \otimes \underline{\mathrm{Hom}}_X^G(\underline{\underline{M}};\, E) \xrightarrow{\cong} E$, where Σ_G is the set of isomorphism classes of simple G-modules.

<u>Proof</u>: The definition of the map is clear. And its restriction to each fibre is bijective by 2.1.3. So it is an isomorphism.

Chapter Four

EQUIVARIANT K-THEORY

From now until section 4.6 all G-spaces will be supposed compact.

4.1 Definition of K_G

Definition 4.1.1 If X is a compact G-space, $K_G(X)$ is the Grothendieck group (1.6) of the topological category G-Vect (X) with respect to the composition $\oplus$.

In fact the isomorphism-classes of G-Vect (X) form an abelian semi-group under $\oplus$, and $K_G(X)$ is the associated abelian group: in the framework I set up in 1.6 this is not quite trivial, but follows from 3.5.11 which says that the set of isomorphism-classes of G-Vect (X) has the discrete topology.

If $G = \{1\}$ one writes $K(X)$ for $K_G(X)$.

Example If X is a point then G-Vect (X) $\cong$ G-Vect, so $K_G(X) \cong R(G)$.

A G-map $f : Y \to X$ induces a functor $f^* :$ G-Vect (X) $\to$ G-Vect (Y) which is additive and so induces a homomorphism $f^! : K_G(X) \to K_G(Y)$.

By defining $K_G(f) = f^!$ one obtains a functor $K_G : (G - \underline{Cpt})^0 \to \underline{Ab}$.

More generally, a morphism $(\phi,f) : (H,Y) \to (G,X)$ in the category of spaces-with-group-action induces a homomorphism $K_G(X) \to K_H(Y)$.

The tensor-product $(E,E') \mapsto E \otimes E'$ is a functor from G-$\underline{Vect}$ $(X) \times G$-$\underline{Vect}$ (X) to G-$\underline{Vect}$ (X) which is additive in both variables. So it induces a homomorphism $K_G(X) \otimes K_G(X) \to K_G(X)$; $K_G(X)$ acquires thereby a structure of commutative ring. The formula $f^*(E \otimes E') \cong f^*E \otimes f^*E'$ implies that $f^! : K_G(X) \to K_G(Y)$ is a homomorphism of rings. That is, K_G is a functor into the category of rings. In particular, because any G-space has a unique map on to a point, $K_G(X)$ has always a natural structure of $R(G)$-module. More generally, if X, X' are two G-spaces there is a natural morphism of rings:

$$K_G(X) \otimes K_G(X') \to K_G(X \times X').$$

Several of the propositions of Chapter 3 can be translated immediately into statements about K_G.

If $\phi : G \to G'$ is a homomorphism of groups there is a natural homomorphism of rings $\phi^* : K_{G'}(X) \to K_G(\phi^*X)$ for a G'-space X.

<u>Proposition 4.1.2</u> If $\phi : G \to G'$ is a homomorphism, and $N = \ker \phi$ acts

freely on the G-space X, then there is a natural isomorphism of rings and of R(G)-modules

$$\phi_* : K_G(X) \xrightarrow{\cong} K_{G'}(\phi_* X). \qquad \text{(cf 3.2.1, 3.4.1).}$$

<u>Examples</u>: (i) $K_G(G/H) \cong R(H)$ as R(G)-module

(ii) $K_G(X) \cong K(X/G)$ if G acts freely on X.

<u>Proposition 4.1.3</u> If $f_0 \simeq f_1 : Y \to X$ as G-maps, then

$$f_0^! = f_1^! : K_G(X) \to K_G(Y).$$

This follows from 3.5.12 because the Grothendieck group is formed from the isomorphism-classes of G-<u>Vect</u> (X).

<u>Proposition 4.1.4</u> If A is a closed G-contractible G-subspace of X then $p : X \to X/A$ induces an isomorphism $p^! : K_G(X/A) \to K_G(X)$.

<u>Proof</u>: By 3.2.2 G-Vect (X/A) is equivalent to the category of triples (E,M,α) with $E \in$ G-<u>Vect</u> (X), M a G-module, and $\alpha : E|A \xrightarrow{\cong} \underline{M}$; and with this identification p^* is $(E,M,\alpha) \mapsto E$. But if A is contractible every E forms part of a triple (E,M,α) by 3.5.12. So $p^!$ is surjective. But 3.5.13 implies that any two triples with the same E are isomorphic: given (E,M,α), (E,N,β), $\beta^{-1} \circ \alpha : E|A \xrightarrow{\cong} E|A$ is homotopic to the identity

because A is contractible, so it can be extended over X. Hence p^* induces an isomorphism of semigroups: $\underline{\text{isom}}\ G\text{-}\underline{\text{Vect}}\,(X/A) \xrightarrow{\cong} \underline{\text{isom}}\ G\text{-}\underline{\text{Vect}}\,(X)$.

If X is a trivial G-space there is an additive functor $T : \underline{\text{Vect}}\,(X) \to G\text{-}\underline{\text{Vect}}\,(X)$ which was discussed in 3.6. It induces a homomorphism of rings $K(X) \to K_G(X)$. So in this case there is a natural morphism of rings $\mu : R(G) \otimes K(X) \to K_G(X)$.

<u>Proposition 4.1.5</u> If X is a trivial G-space then $\mu : R(G) \otimes K(X) \to K_G(X)$ is an isomorphism of rings.

<u>Proof</u>: If M is a G-module the functor $E \mapsto \underline{\text{Hom}}_X^G(\underline{M}; E)$ from $G\text{-}\underline{\text{Vect}}\,(X)$ to $\underline{\text{Vect}}\,(X)$ induces a morphism $\pi_M : K_G(X) \to K(X)$. $R(G)$ is the free abelian group generated by the set Σ_G of isomorphism classes of simple G-modules, so one can define a morphism $\nu : K_G(X) \to R(G) \otimes K(X)$ by $x \mapsto \sum_{M \in \Sigma_G} [M] \otimes \pi_M(x)$. It follows from 3.6.3 that ν is the required inverse of μ.

The existence of complementary bundles (3.5.7) gives one a very manageable description of elements of $K_G(X)$.

<u>Proposition 4.1.6</u> An element of $K_G(X)$ can be written $[E] - [\underline{M}]$, where E

is a G-vector-bundle on X, M is a G-module, and [E], [$\underline{\underline{M}}$] are their images in $K_G(X)$. And $[E_1] - [\underline{\underline{M}}_1] = [E_2] - [\underline{\underline{M}}_2]$ if and only if there exists a G-module M such that $E_1 \oplus \underline{\underline{M}}_2 \oplus \underline{\underline{M}} \cong E_2 \oplus M_1 \oplus \underline{\underline{M}}$.

4.2 K_G^{-q} and the exact sequence

Proposition 4.2.1 If $\{X_\alpha\}_{\alpha\in S}$ is a finite family of G-spaces, then

$$K_G(\coprod_{\alpha\in S} X_\alpha) \xrightarrow{\cong} \prod_{\alpha\in S} K_G(X_\alpha).$$

This is trivial.

Proposition 4.2.2 If $X \to X_1$, $X \to X_2$ are inclusions, then

$$K_G(X_1 \amalg_X X_2) \to K_G(X_1) \oplus K_G(X_2) \rightrightarrows K_G(X) \text{ is exact.}$$

Proof: One half of the exactness just expresses the functoriality of K_G. On the other hand, an element of the middle group can be written $([E_1] - [\underline{\underline{M}}], [E_2] - [\underline{\underline{M}}])$, and if its two images in $K_G(X)$ coincide then $(E_1|X) \oplus \underline{\underline{N}} \cong (E_2|X) \oplus \underline{\underline{N}}$ for some G-module N. Construct E on $X_1 \amalg_X X_2$ by clutching $E_1 \oplus \underline{\underline{N}}$ and $E_2 \oplus \underline{\underline{N}}$ along X: it is clear that the image of $[E] - [\underline{M}] - [\underline{N}]$ in $K_G(X_1) \oplus K_G(X_2)$ is $([E_1] - [\underline{M}], [E_2] - [\underline{M}])$, as required.

The homomorphism $K_G(X_1 \amalg_X X_2) \to K_G(X_1) \oplus K_G(X_2)$ will not, normally, be injective; i.e. K_G is not left-exact. One would like to extend the sequence of 4.2.2 to the left to obtain a long exact sequence

$$(*) \quad \ldots \to K_G^{-q}(X_1 \amalg_X X_2) \to K_G^{-q}(X_1) \oplus K_G^{-q}(X_2) \rightrightarrows K_G^{-q}(X) \xrightarrow{d} K_G^{-q+1}(X_1 \amalg_X X_2)$$

$$\to \ldots \to K_G^{-1}(X) \xrightarrow{d} K_G(X_1 \amalg_X X_2) \to K_G(X_1) \oplus K_G(X_2) \rightrightarrows K_G(X),$$

where $\{K_G^{-q}\}_{q \in \mathbb{N}}$ are in some sense "left-derived functors" of K_G. This can be done, but I don't know a satisfying way of doing it, and I shall simply follow the classical discussion.

Let G-$\underline{\text{Vect}}^{-q}(X)$ be the additive category whose objects are pairs (E,α), where $E \in G$-$\underline{\text{Vect}}(X)$ and $\alpha : S^{q-1}$, point $\to \mathrm{Aut}_X^G(E)$, 1 is a map. (One can use this "singular" definition because the spaces in question are locally trivial (1.6).) There is an evident continuous functor G-$\underline{\text{Vect}}^{-q}(X) \to G$-$\underline{\text{Vect}}(S^q \times X)$ obtained by regarding $S^q \times X$ as $(D^q \times X) \amalg_{S^{q-1} \times X} (D^q \times X)$. Two objects (E,α_0), (E,α_1) define isomorphic objects of G-$\underline{\text{Vect}}(S^q \times X)$ if and only if $\alpha_0 \simeq \alpha_1$: "if" because $\underline{\text{isom}}$ G-$\underline{\text{Vect}}(S^q \times X)$ is discrete; "only if" because if $(E,\alpha_i) \mapsto \tilde{E}_i$ then an

isomorphism $\psi : \tilde{E}_0 \to \tilde{E}_1$ consists of two automorphisms ψ_+, ψ_- of $E \times D^q$ such that $(\psi_+|X \times S^{q-1}) \circ \alpha_0 = \alpha_1 \circ (\psi_-|X \times S^{q-1})$ i.e.
$\alpha_0 \circ \alpha_1^{-1} = (\psi_+|X \times S^{q-1})^{-1} \circ \alpha_1 \circ (\psi_-|X \times S^{q-1}) \circ \alpha_1^{-1}$ and the right-hand-side of this last is null-homotopic.

So $K(G\text{-}\underline{\mathrm{Vect}}^{-q}(X)) \xrightarrow{\cong} K_G(S^q \times X)$. Now $K_G(X)$ is a canonical direct summand of $K(G\text{-}\underline{\mathrm{Vect}}^{-q}(X))$. I define $K_G^{-q}(X)$ as the complementary direct summand. That means

$$K_G^{-q}(X) \;=\; \mathrm{coker} : K_G(X) \;\to\; K_G(X \times S^q).$$

Alternatively one can say : $K_G^{-q}(X) \xrightarrow{\cong} \varinjlim_{E} \; \pi_{q-1} \mathrm{Aut}_X^G(E)$, where E runs through the category of G-vector-bundles on X and inclusions.

I shall define a coboundary homomorphism $d : K_G^{-q}(X) \to K_G^{-q+1}(X_1 \amalg_X X_2)$. Let (E,α) represent an element of $G\text{-}\underline{\mathrm{Vect}}^{-q}(X)$. One can suppose that E is trivial: say $E = \underline{M}$.

$(X_1 \amalg_X X_2) \times S^{q-1} \cong (X_1 \times S^{q-1}) \amalg_{X \times S^{q-1}} (X_2 \times S^{q-1})$. Clutch two copies of $\underline{M}$ along $X \times S^{q-1}$ by α to form a G-vector-bundle $\tilde{E}$ on $S^{q-1} \times (X_1 \amalg_X X_2)$. Define $d(\underline{M},\alpha) = [\tilde{E}] - [\underline{M}] \in K_G^{-q+1}(X_1 \amalg_X X_2)$. The exactness of $K_G^{-q}(X) \xrightarrow{d} K_G^{-q+1}(X_1 \amalg_X X_2) \to K_G^{-q+1}(X_1) \oplus K_G^{-q+1}(X_2)$ is immediate: a bundle on $(X_1 \amalg_X X_2) \times S^{q-1}$ which is trivial on $X_1 \times S^{q-1}$ and $X_2 \times S^{q-1}$ is of the form of $\tilde{E}$ above.

The exactness of $K_G^{-q}(X_1 \amalg_X X_2) \to K_G^{-q}(X_1) \oplus K_G^{-q}(X_2) \rightrightarrows K_G^{-q}(X)$ follows from 4.2.2.

Finally, consider $K_G^{-q}(X_1) \oplus K_G^{-q}(X_2) \rightrightarrows K_G^{-q}(X) \xrightarrow{d} K_G^{-q+1}(X_1 \amalg_X X_2)$. If $\alpha : S^{q-1} \to \mathrm{Aut}_X^G(\underline{M})$ extends over either X_1 or X_2 then $\widetilde{E}$ is trivial: this implies one half of the exactness. On the other hand if $\widetilde{E} \cong \underline{M}$ one has isomorphisms $\alpha_1 : S^{q-1} \to \mathrm{Aut}_{X_1}^G(\underline{M})$, $\alpha_2 : S^{q-1} \to \mathrm{Aut}_{X_2}^G(\underline{M})$ such that $(\alpha_2|X) \circ (\alpha_1|X)^{-1} = \alpha$; and the rest follows from the lemma - which is also of independent interest:

Lemma 4.2.3 If (E,α), (E,β) define two elements of $K_G^{-q}(X)$, their sum is defined by $(E, \beta\circ\alpha)$.

Proof: It suffices to show that the two maps $S^{q-1} \to \mathrm{Aut}_X^G(E \oplus E)$ represented diagrammatically by $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ and $\begin{pmatrix}\beta\alpha & 0\\ 0 & 1\end{pmatrix}$ are canonically homotopic. So it suffices to show that $\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}$ is canonically null-homotopic. But it is linearly homotopic to $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, for $\begin{pmatrix}t\beta & (1-t)1\\ -(1-t)1 & t\beta^{-1}\end{pmatrix} = B_t$ cannot be singular for $t \in \mathbb{R}$. (Its inverse is $k\begin{pmatrix}t\beta^{-1} & -(1-t)\\ 1-t & t\beta\end{pmatrix}$ with $k \neq 0$.); and $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is connected by an obvious path to 1.

This completes the construction of the desired long exact sequence (*).

4.3 Relative groups

Let A be a closed G-subspace of X. $K_G(X)$ is a canonical direct summand of $K_G(X \amalg_A X)$: the complementary direct summand is called $K_G(X,A)$.

Example: $K_G(X \times D^q, X \times S^{q-1}) \cong K_G^{-q}(X)$ because $D^q \amalg_{S^{q-1}} D^q \cong S^q$.

Similarly one defines: $K_G^{-q}(X \amalg_A X) \cong K_G^{-q}(X) \oplus K_G^{-q}(X,A)$ for any $q \in \mathbb{N}$.

The exact sequence for $A \rightrightarrows X \amalg X \to X \amalg_A X$ can be written

$$\ldots \xrightarrow{d} K_G^{-q}(X,A) \to K_G^{-q}(X) \to K_G^{-q}(A) \xrightarrow{d} K_G^{-q+1}(X,A) \to \ldots$$

$$\ldots \to K_G^{-1}(A) \xrightarrow{d} K_G(X,A) \to K_G(X) \to K_G(A).$$

There is a "strong excision" isomorphism $K_G^{-q}(X \amalg_A Y, Y) \xrightarrow{\cong} K_G^{-q}(X,A)$ when $A \to X$, $A \to Y$ are inclusions. For $(X \amalg_A Y) \amalg_Y (X \amalg_A Y) \cong (X \amalg_A X) \amalg_X (X \amalg_A Y)$ and the exact sequence breaks up:

$$0 \to K_G^{-q}((X \amalg_A X) \amalg_X (X \amalg_A Y)) \to K_G^{-q}(X \amalg_A X) \oplus K_G^{-q}(X \amalg_A Y) \to$$

$$K_G^{-q}(X) \to 0$$

because of the splitting $X \amalg_A X \to X$.

But an even stronger statement can be made:

<u>Proposition 4.3.1</u> If $f : X,A \to Y,B$ is a relative homeomorphism (by which I mean: $f|(X-A) : X - A \xrightarrow{\cong} Y - B$) then

$$f^! : K_G^{-q}(Y,B) \xrightarrow{\cong} K_G^{-q}(X,A) \quad \text{for all } q \in \mathbb{N} .$$

<u>Proof</u>: It is sufficient to consider the case of the projection $f : X,A \to X/A$, point. That can be factorised into the inclusion $X,A \to X \amalg_A CA, CA$ (CA is the cone on A), which induces an isomorphism by the discussion above, and the projection $X \amalg_A CA, CA \to (X \amalg_A CA)/CA$, point, which induces an isomorphism by the 5-lemma and 4.1.4.

4.4 <u>Complexes of G-vector-bundles</u>

In this section I shall describe a new way of defining $K_G(X)$ and $K_G(X,A)$ which is for many purposes more convenient than the one I have been using. My discussion is an elaboration of Part 2 of [7].

Let $C_G(X)$ be the category whose objects are bounded complexes $E^\bullet = (E^\bullet, d^\bullet)$ in G-<u>Vect</u> (X) (i.e. $\{E^q\}_{q \in \mathbb{Z}}$ are G-vector-bundles with

$E^q = 0$ for $|q|$ large; $d^q : E^q \to E^{q+1}$, and $d^{q+1} \circ d^q = 0$) and whose morphisms $f^\bullet : E_1^\bullet \to E_2^\bullet$ are maps $f^q : E_1^q \to E_2^q$ such that $d_2^q \circ f^q = f^{q+1} \circ d_1^q$, but modulo the equivalence-relation ("chain-homotopy") defined by

$$f_1^\bullet \sim f_2^\bullet \Leftrightarrow \exists\, h^q : E_1^q \to E_2^{q-1} \ (q \in \mathbb{Z}) \text{ such that } f_1^q - f_2^q = d^{q-1} \circ h^q + h^{q+1} \circ d^q.$$

We saw in 3.5 that any complex which is pointwise acyclic splits, i.e. is isomorphic to a sum of "elementary" complexes $\ldots \to 0 \to E \xrightarrow{id} E \to 0 \to \ldots$, and so is homotopically trivial, i.e. isomorphic to zero in $C_G(X)$. This has the important consequence that any morphism $f^\bullet : E^\bullet \to F^\bullet$ which induces, pointwise, isomorphisms of the homology of the complexes is in fact an isomorphism in $C_G(X)$: one has only to write down the condition that the mapping-cone of $f^\bullet$ is homotopically trivial.

The functor $X \mapsto C_G(X)$ has a clutching-property like that of $X \mapsto G$-<u>Vect</u> (X): it resembles, though not precisely, a sheaf of categories.

<u>Proposition 4.4.1</u> If a G-space is the union of compact G-subspaces X and Y, and $A = X \cap Y$, and there are given $E^\bullet \in C_G(X)$, $F^\bullet \in C_G(Y)$ and an isomorphism $\alpha : E^\bullet|A \to F^\bullet|A$ in $C_G(A)$, then there is an object $H^\bullet$ of $C_G(X \amalg_A Y)$ and isomorphisms $i^\bullet : E^\bullet \to H^\bullet|X$, $j^\bullet : F^\bullet \to H^\bullet|Y$ in $C_G(X)$ (resp. $C_G(Y)$) such that $(j^\bullet|A)^{-1} \circ (i^\bullet|A) = \alpha$. Furthermore $H^\bullet$ is

unique up to canonical isomorphism in $C_G(X \amalg_A Y)$.

I shall establish a couple of other results before proving 4.4.1.

<u>Proposition 4.4.2</u> If $f^\bullet : E_1^\bullet \xrightarrow{\simeq} E_2^\bullet$ in $C_G(X)$ then

$$\sum_{q \in \mathbb{Z}} (-1)^q [E_1^q] = \sum_{q \in \mathbb{Z}} (-1)^q [E_2^q] \text{ in } K_G(X).$$

<u>Proof</u>: The mapping-cone $C_f^\bullet$ of $f^\bullet$ is an acyclic complex. But $C_f^q \cong E_1^{q+1} \oplus E_2^q$. So $\Sigma(-1)^q[E_2^q] + \Sigma(-1)^q[E_1^{q+1}] = \Sigma(-1)^q[C_f^q] = 0$.

<u>Proposition 4.4.3</u> If $f^\bullet : E_1^\bullet \xrightarrow{\simeq} E_2^\bullet$ in $C_G(X)$, and $f^q : E_1^q \to E_2^q$ is surjective for all $q \in \mathbb{Z}$, then $E_1^\bullet$ is the mapping-cone of a map $\phi^\bullet : E_2^\bullet \to \ker f^\bullet$.

<u>Note</u>: "is the mapping-cone" does not mean merely "up to chain-homotopy-equivalence" in this case.

<u>Proof</u>: Once one has observed that ker $f^\bullet$ exists, and that any short exact sequence in G-<u>Vect</u> (X) splits, this is a formal triviality (see e.g. [31] § 2.4).

<u>Proof of 4.4.1</u> One can write $\alpha = \alpha_1^{-1} \circ \alpha_0$, where $\alpha_0^\bullet : Q^\bullet \to E^\bullet|A$, $\alpha_1^\bullet : Q^\bullet \to F^\bullet|A$ are chain-homotopy-equivalences. By adding to $Q^\bullet$ the

mapping-cones of $\mathrm{id}_{E^\cdot|A}$ and $\mathrm{id}_{F^\cdot|A}$ one can suppose α_0^q, α_1^q surjective for all q. And, by adding another acyclic complex, that Q^q is trivial for $q \neq 0$. Then Q^0 must extend over both X and Y (perhaps after adding another pair of trivial bundles to $Q^\cdot$) for $\Sigma(-)^q[Q^q] = \Sigma(-)^q[E^q|A] = \Sigma(-)^q[F^q|A]$ in $K_G(A)$, so is in the image of both $K_G(X)$ and $K_G(Y)$. Thus one can assume that all the bundles Q^q extend ovér X and Y. Define $U^\cdot = \ker : Q^\cdot \to E^\cdot|A$, $V^\cdot = \ker : Q^\cdot \to F^\cdot|A$. $U^\cdot$ (resp. $V^\cdot$) is acyclic, and its bundles extend over X (resp. Y) - perhaps after adding some more trivial bundles to $Q^\cdot$. $U^\cdot$ is split, i.e. $U^q \cong Z^q(U^\cdot) \oplus B^{q+1}(U^\cdot)$, and one sees by induction on q that $Z^q(U^\cdot)$ and $B^{q+1}(U^\cdot)$ extend over X (perhaps ...). Suppose then that $U^\cdot$ has been extended to an acyclic complex on X. $Q^\cdot \to E^\cdot|A$ is the mapping-cone of a map $u^\cdot : E^\cdot|A \to U^\cdot|A$ determined by the choice of an isomorphism $Q^\cdot \cong E^\cdot|A \mathbin{\tilde{\oplus}} U^\cdot|A$. Because $U^\cdot$ is split one can extend $u^\cdot$ over X, and it will then determine a chain-homotopy-equivalence $\tilde{\alpha}_0 : C^\cdot_{u^\cdot} \to E^\cdot$ extending α_0; similarly one can extend α_1 to $\tilde{\alpha}_1 : C^\cdot_{v^\cdot} \to F^\cdot$. Define $H^\cdot$ on $X \amalg_A Y$ by clutching $C^\cdot_{u^\cdot}$ and $C^\cdot_{v^\cdot}$, which both coincide with $Q^\cdot$ on A.

As to the uniqueness of $H^\cdot$, the point is to show that if $H_1^\cdot$, $H_2^\cdot \in C_G(X \amalg_A Y)$ and if $f_X^\cdot : H_1^\cdot|X \xrightarrow{\cong} H_2^\cdot|X$, $f_Y^\cdot : H_1^\cdot|Y \xrightarrow{\cong} H_2^\cdot|Y$ are such that $f_X^\cdot|A = f_Y^\cdot|A$ in $C_G(A)$ then there is a well-defined $f^\cdot : H_1^\cdot \xrightarrow{\cong} H_2^\cdot$ such that $f^\bullet|X = f^\bullet_X$, $f^\bullet|Y = f^\bullet_Y$. Now there is a chain-homotopy $h^q : H_1^q|A \to H_0^{q-1}|A$ such that $f^\bullet_Y|A = f^\bullet_X|A + d^\bullet h^\bullet + h^\bullet d^\bullet$. But $h^\bullet$ can be extended over Y, so one can define $f^\bullet : H_1^\bullet \to H_0^\bullet$ by $f^\bullet|X = f^\bullet_X$, $f^\bullet|Y = f^\bullet_Y + d^\bullet h^\bullet + h^\bullet d^\bullet$.

Recall (from 1.6) that the Grothendieck group of a topological triangulated category C can be defined by $K(C) = \pi_0$ (isom C). The main result of this section is

Proposition 4.4.4 There are natural isomorphisms

(i) $K_G(X) \xrightarrow{\cong} K(C_G(X))$

(ii) $K_G(X,A) \xrightarrow{\cong} K(C_G(X,A))$

(i) is a formal triviality explained already in 1.6. The proof of (ii) will be preceded by a lemma.

Lemma 4.4.5 If A is a closed G-subspace of X and there are given bundles E^q ($q \in \mathbb{Z}$) on X such that $E^q = 0$ for $|q|$ large, and morphisms $d^q : E^q|A \to E^{q+1}|A$ making $E^\bullet|A$ into an acyclic complex, then the d^q can be extended over X so that $(E^\bullet, d^\bullet)$ is a complex.

Proof: Let $F^q = \ker(d^q|A)$. Extend F^q to a bundle F_1^q on a neighbourhood U of A independent of q. Choose $\phi : X, A, X - U \to I, 0, 1$, and choose extensions d_i^q of $d^q : (E^q/F_i^q)|A \to F_i^{q+1}|A$ over U. Then define $d^q : E^q \to E^{q+1}$ as zero outside U and as $\phi(x)d_{i,x}^q$ for $x \in U$. This is a suitable extension.

<u>Proof of 4.4.4 (ii)</u>: There is an additive functor $e : C_G(X,A) \to C_G(X \amalg_A X)$ which takes $E^\cdot$ to $\tilde{E}^\cdot$ obtained by clutching $E^\cdot$ and 0 by the unique isomorphism $0 \to E^\cdot|A$; and there is another functor $\delta^* : C_G(X) \to C_G(X \amalg_A X)$ induced by the codiagonal. It suffices to show that $e \oplus \delta^* : C_G(X) \times C_G(X,A) \to C_G(X \amalg_A X)$ induces isomorphism of Grothendieck groups. But there is a map $\underline{ob}\ C_G(X \amalg_A X) \to \underline{ob}\ C_G(X) \times \pi_0\ \underline{isom}\ C_G(X,A)$ whose first component is restriction to the second factor and whose second component is $E^\cdot \mapsto (F^\cdot, d^\cdot)$ where $F^q = i_1^* E^q \oplus i_2^* E^{q+1}$ (i_1, $i_2 : X \to X \amalg_A X$ the inclusions) and $d^\cdot$ is an extension of the differential of the mapping-cone of the identity: $i_1^* E^\cdot|A \to i_2^* E^\cdot|A$. I leave it to the reader to verify that this induces a map of Grothendieck groups inverse to $(e \oplus \delta^*)^!$.

The only application I shall make here of the new definition of $K_G(X,A)$ is to define its multiplicative structure. The tensor-product of complexes induces a biadditive functor $C_G(X,A) \times C_G(Y,B) \to$

$C_G(X \times Y,\ (X \times B) \cup (A \times Y))$, which induces $K_G(X,A) \otimes K_G(Y,B) \to K_G(X \times Y,\ (X \times B) \cup (A \times Y))$. Furthermore, because $K_G^{-q}(X,A) \cong K_G(X \times D^q,\ (X \times S^{q-1}) \cup (A \times D^q))$, this leads at once to $K_G^{-q}(X,A) \otimes K_G^{-r}(Y,B) \to K_G^{-q-r}(X \times Y,\ (X \times B) \cup (A \times Y))$. By composing this exterior product with the diagonal $X \to X \times X$ one obtains a structure of graded ring on $\{K_G^{-q}(X)\}_{q \in \mathbb{N}}$. It is anticommutative: to see that amounts to showing that a map $f : D^q, S^{q-1} \to D^q, S^{q-1}$ induces multiplication by its degree in $K_G(X \times D^q,\ X \times S^{q-1})$, and hence to showing that the map $\pi_q(D^q, S^{q-1}) \to \operatorname{End} K_G(X \times D^q,\ X \times S^{q-1})$ is a homomorphism. But the group-law in π_q is induced by the cogroup-structure $c : D^q, S^{q-1} \to D^q \amalg_{S^{q-1}} D^q,\ S^{q-1}$, so one need only check the commutativity of the diagram:

$$
\begin{tikzcd}
 & K_G((D \amalg_S D)X,\ SX) \arrow[r, "(f_1 \amalg f_2)^!"] \arrow[dd, "i_1^! \oplus i_2^!"] & K_G((D \amalg_S D)X,\ SX) \arrow[dd, "i_1^! \oplus i_2^!"'] \arrow[dr, "c^!"] & \\
K_G(DX,\ SX) \arrow[ur, "(id \amalg id)^!"] \arrow[dr, "id^! \oplus id^!"'] & & & K_G(DX,\ SX) \\
 & K_G(DX,\ SX) \oplus K_G(DX,\ SX) \arrow[r, "f_1^! \oplus f_2^!"'] & K_G(DX,\ SX) \oplus K_G(DX,\ SX) \arrow[ur, "id^! \oplus id^!"'] &
\end{tikzcd}
$$

(Here $DX = D^q \times X$, $SX = S^{q-1} \times X$, etc.) But everything except the right-hand triangle commutes by functoriality: and the right-hand triangle commutes because of the homotopy-properties of the cogroup-law - consider the diagram

$$\begin{tikzcd}
 & & K_G((D \amalg_S D)X, D_-X) \oplus K_G((D \amalg_S D)X, D_+X) \arrow[d, "\cong"] \arrow[ddll, "i_1^! \oplus i_2^!"', "\cong"] \arrow[ddrr, "i_1^! \oplus i_2^!"] & & \\
 & & K_G((D \amalg_S D)X, SX) \arrow[dll, "i_1^! \oplus i_2^!"] \arrow[drr, "c^!"'] & & \\
K_G(DX,SX) \oplus K_G(DX,SX) \arrow[rrrr, "\mathrm{id}^! \oplus \mathrm{id}^!"'] & & & & K_G(DX,SX)
\end{tikzcd}$$

(Everything else commutes, so the bottom triangle commutes.)

Similarly $\{K_G^{-q}(X,A)\}_{q\in \mathbb{N}}$ is a graded ring-without-unit, and a module over $\{K_G^{-q}(X)\}$: I forbear the details.

4.5 Fredholm complexes

In this section I shall present a third way of defining K_G for a compact space X. I shall change (in this section only) the meaning of "G-vector-bundle" as follows:

<u>Definition 4.5.1</u> A complex vector-space E in the category of G-spaces over a fixed G-space X will be called a <u>G-vector-bundle</u> if

(i) it is locally trivial when one neglects the G-action

(ii) the fibres E_x $(x \in X)$ are locally convex, hausdorff, complete, and have the property: there is a neighbourhood U_x of 0 in E_x such that $\xi \in E_x$, $\mathbb{C} . \xi \subset U_x \Rightarrow \xi = 0$.

<u>Remark</u>: The last condition is equivalent to the existence of a continuous norm on E_x: thus it is satisfied for $C_Y = \mathrm{Map}(Y ; \mathbb{C})$ when Y is compact, for $D_Y = \{\text{smooth maps}: Y \to \mathbb{C}\}$ when Y is a compact differentiable manifold; but also by $M_Y = C_Y'$ when Y is compact and metrisable, and by the distributions D_Y' on a compact differentiable manifold.

I shall write $G\text{-}\underline{\mathrm{Vect}}_\infty(X)$ for the category of such G-vector-bundles on X; a G-vector-bundle in the sense of 3.1 will now be called a "<u>finite</u> G-vector-bundle".

<u>Definition 4.5.2</u> A morphism $f : E \to E'$ in $G\text{-}\underline{\mathrm{Vect}}_\infty(X)$ is <u>compact</u> if there is a neighbourhood U of the zero-section of E and a subspace K of E' proper

over X such that $f(U) \subset K$.

<u>Definition 4.5.3</u> (i) A bounded complex $(E^{\cdot},d^{\cdot})$ in G-<u>Vect</u>$_{\infty}(X)$ is called a <u>G-Fredholm-complex</u> if there exist morphisms $h^i : E^i \to E^{i-1}$ $(i \in \mathbb{Z})$ such that $d^{i-1} h^i + h^{i+1} d^i = 1 - k^i : E^i \to E^i$, where $k^i : E^i \to E^i$ is compact; that is, if the identity $E^{\cdot} \to E^{\cdot}$ is chain-homotopic to a compact morphism.

(ii) $(E^{\cdot},d^{\cdot})$ is <u>of excess $\leq n$</u> if for some $m \in \mathbb{Z}$ E^i is finite for $i \notin [m,m+n]$.

(iii) $(E^{\cdot},d^{\cdot})$ is <u>acyclic</u> if $(E^{\cdot}_x, d^{\cdot}_x)$ is an exact sequence of vector-spaces for each $x \in X$.

The category of G-Fredholm complexes on X and chain-homotopy-classes of morphisms is denoted by G-<u>Fred</u>(X). It is an additive category. But in contrast to the finite case it is not true that any acyclic complex splits, i.e. is isomorphic to zero in G-<u>Fred</u>(X). To cope with this it is convenient to adjoin more morphisms to G-<u>Fred</u>(X), forming a new category called its <u>derived category</u>, so that acyclic complexes are isomorphic to zero. Let us call a morphism of G-<u>Fred</u>(X) a quasi-isomorphism if it induces, pointwise, isomorphisms of the homology of the complexes - or, equivalently, if its mapping-cone is acyclic. Then one quite simply adjoins to G-<u>Fred</u>(X) inverses of all quasi-isomorphisms. For the details the reader can refer to [31].

__Proposition 4.5.4__ If X is compact, the inclusion $C_G(X) \to G\text{-}\underline{\mathrm{Fred}}(X)$ induces an equivalence of derived categories. Thus $K_G(X)$ is the Grothendieck group of the derived category of $G\text{-}\underline{\mathrm{Fred}}(X)$.

The proof depends on the following proposition, whose proof I shall postpone.

__Proposition 4.5.5__ If E is a G-vector-bundle on a compact space X, and $d : E \to E$ is a morphism of the form $1 - k$ with k compact, then there is a G-module M and morphisms $\phi : E \to \underline{\underline{M}}$ and $\psi : E \oplus \underline{\underline{M}} \to \underline{\underline{M}}$ such that

$$0 \to E \xrightarrow{d\oplus\phi} E \oplus \underline{\underline{M}} \xrightarrow{\psi} \underline{\underline{M}} \to 0$$

is a split short exact sequence.

__Proof of 4.5.4:__ If suffices to show how to associate to a complex $(E^\cdot, d^\cdot)$ in $G\text{-}\underline{\mathrm{Fred}}(X)$ of excess $n \geq 1$ a complex of excess $n - 1$ to which it is __canonically__ quasi-isomorphic in $G\text{-}\underline{\mathrm{Fred}}(X)$.

One can suppose that E^i is finite for $i < 0$ and that E^0 is infinite. Then one can choose the h^i of 4.5.3 so that $h^i = 0$ for $i < 0$, and then

$h^1 d^0 : E^0 \to E^0$ is of the form $1 - k^0$. Choose a G-module M and $\phi : E^0 \to \underline{\underline{M}}$ as in 4.5.5. The complex

$$(*) \quad \ldots \xrightarrow[d^{-2}]{} E^{-1} \xrightarrow[d^{-1}\oplus(-\phi d^{-1})]{} E^0 \oplus \underline{\underline{M}} \xrightarrow[\begin{pmatrix} d^0 & 0 \\ \phi & 1 \end{pmatrix}]{} E^1 \oplus \underline{\underline{M}} \xrightarrow[d^1 \oplus 0]{} E^2 \xrightarrow[d^2]{} \ldots$$

is the mapping-cone of a map of $E^\cdot$ into $\underline{\underline{M}} \xrightarrow{id} \underline{\underline{M}}$, so it is canonically quasi-isomorphic to $E^\cdot$.

By 4.5.5 one can find $\theta = \theta_0 \oplus \theta_1 : \underline{\underline{M}} \to E^0 \oplus \underline{\underline{M}}$ such that the map whose matrix is $\begin{pmatrix} h^1 d^0 & \theta_0 \\ \phi & \theta \end{pmatrix}$ an automorphism of $E^0 \oplus \underline{\underline{M}}$. Let its inverse be $\begin{pmatrix} B_0 & B_1 \\ C_0 & C_1 \end{pmatrix}$.

I must now distinguish two cases. First suppose $E^\cdot$ has excess ≥ 2, i.e. E^2 is infinite.

Consider the morphism $f^\cdot$ of complexes (it is injective):

$$\begin{array}{ccccccccccccc}
\ldots & \to & 0 & \longrightarrow & E^0 & \xrightarrow{id} & E^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow f^{-1} & & \downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & & \downarrow f^3 & & \\
\ldots & \to & E^{-1} & \longrightarrow & E^0 \oplus \underline{\underline{M}} & \longrightarrow & E^1 \oplus \underline{\underline{M}} \oplus E^0 & \longrightarrow & E^2 \oplus E^0 & \longrightarrow & E^3 & \longrightarrow & \ldots
\end{array}$$

where the second complex is the direct sum of $(*)$ and $(\ldots \to 0 \to E^0 \xrightarrow{id} E^0 \to 0 \to \ldots)$, and $f^0 = id \oplus 0$, $f^1 = d^0 \oplus \phi \oplus 0$. There is an automorphism α of $E^1 \oplus \underline{\underline{M}} \oplus E^0$ such that $\alpha i_3 = f^1$, where $i_3 : E^0 \to E^1 \oplus \underline{\underline{M}} \oplus E^0$ is the injection of the third factor. In fact α is given by

$$\begin{pmatrix} 1 & 0 & d^0 \\ 0 & \theta_1 & \phi \\ -h^1 & \theta_0 & 0 \end{pmatrix}, \quad \text{which has the inverse} \quad \begin{pmatrix} 1-d^0B_0h^1 & -d^0B_1 & -d^0B_0 \\ C_0h^1 & C_1 & C_0 \\ B_0h^1 & B_1 & B_0 \end{pmatrix}.$$

Because of the existence of α one knows that $f^{\cdot}$ has a cokernel isomorphic in G-*Fred*(X) to

$$\ldots \to E^{-1} \to \underline{\underline{M}} \to E^1 \oplus \underline{\underline{M}} \to E^2 \oplus E^0 \to E^3 \to \ldots,$$

which has excess smaller by one than that of $E^{\cdot}$. But the cokernel is canonically quasi-isomorphic to (*), and hence to $E^{\cdot}$.

The other case, in which $E^{\cdot}$ has excess one, is treated in exactly the same way except that one does not add $(\ldots \to 0 \to E^0 \to E^0 \to 0 \to \ldots)$ to the target of $f^{\cdot}$; and one uses 4.5.9 below instead of α.

I shall now prove a sequence of lemmas leading to 4.5.5.

Let $d : E \to E$ be a morphism of the form $1 - k$, where k is compact. Let U be a closed convex neighbourhood of the zero-section of E such that $k(U) \subset K$ which is proper over X.

Lemma 4.5.6 If d is injective then it is a closed map.

Proof: $d|U$ is closed, for it is the composite

$U \xrightarrow{1\times k} E \times_X K \xrightarrow{\cong} E \times_X K \xrightarrow{pr} E$, where the middle map is $(x,y) \mapsto (x-y,y)$.

It follows that $d(2U - \mathring{U})$ is a closed set of E not containing 0. Choose a neighbourhood V of 0 such that $d^{-1}V \subset U$, e.g. $V = \{x : \lambda x \notin d(2U - \mathring{U})$ for $0 \leq \lambda \leq 1\}$. Let F be closed in E. dF is closed if and only if $dF \cap n.dU$ is closed for all integers n. But $dF \cap n.dU = n.d(\frac{1}{n}F \cap U)$, and the latter is closed.

Lemma 4.5.7 If d_x is injective for some $x \in X$ then d_y is injective for all y near x.

Proof: One can suppose that U has the property: $\mathbb{C}.\xi \subset U \Rightarrow \xi = 0$. Now $d^{-1}(0) \cap (2K - \mathring{U})$ is closed in $2K$, so $C = pr\{d^{-1}(0) \cap (2K - \mathring{U})\}$ is closed in X $(pr : E \to X)$. Furthermore x is not in C, and d is injective over the complement of C, for if d_y is not injective there is some ξ in $2U - \mathring{U}$ over y such that $d(\xi) = 0$. Then $\xi = k\xi$, so $\xi \in d^{-1}(0) \cap (2K - \mathring{U})$, and $y \in C$.

Lemma 4.5.8 If X is a point then the image of d is closed and the kernel and cokernel have the same (finite) dimension.

This is well-known; see [20a] Théorème 5 p.272. The kernel and cokernel are finite because k induces the identity on them.

<u>Lemma 4.5.9</u> Let $d : E \to F$ be an injection such that there exists $h : F \to E$ with $hd = 1 - k$, k compact. Then d is split; and if also $dh = 1 - k'$ with k' compact then $\operatorname{coker}(d)$ is a finite G-vector-bundle and $F \cong E \oplus \operatorname{coker}(d)$.

<u>Proof</u>: It suffices to show that d is locally split (cf 3.5), so it suffices to find for each $x \in X$ a morphism $h^x : F \to E$ such that $h^x d$ is an isomorphism near x.

Let $N_x = \ker(h^x_x\, d_x) \cong \operatorname{coker}(h^x_x\, d_x) = N''_x$. Define $N'_x = d_x(N_x) \cong N_x$, and choose $\phi : F \to \underline{\underline{N}}'_x$ which is the identity on N'_x. Choose also $\psi : \underline{\underline{N}}''_x \to E$ which is a splitting of $h^x d$ at x. Then, if $h^x = h + \psi\phi$, $h^x d$ is of the form 1-(compact) and is injective at x, hence isomorphic near x.

If also dh is of the form $1 - k'$ then $C_x = \operatorname{coker} d_x$ is finite. Choose $\theta : \underline{\underline{C}}_x \to F$ which is a splitting at x. $d \oplus \theta : E \oplus \underline{\underline{C}}_x \to F$ is then an isomorphism near x: for its composite, in both senses, with $h^x \oplus \chi$ (where $\chi\theta = 1$ locally) is of the form 1-(compact) and is isomorphic at x.

<u>Proof of 4.5.5</u>: Given $d = 1 - k : E \to E$ it will now be clear how to

choose $\phi : E \to \underline{\underline{M}}$ so that $d \oplus \phi$ is injective. It remains only to show that $\operatorname{coker}(d \oplus \phi) \cong \underline{\underline{M}}$ (for suitable M, ϕ).

Let $pr : X \times I \to X$ be the projection, and define $\tilde{d} : pr^*E \to pr^*E$ by $\tilde{d}_{(x,t)} = 1 - tk_x$. Choose M and $\tilde{\phi} : pr^*E \to \underline{\underline{M}}$ such that $\tilde{d} \oplus \tilde{\phi}$ is injective. Then $\tilde{d} \oplus t\tilde{\phi}$ (in evident notation) is also injective, so its cokernel is a G-vector-bundle which is $\underline{\underline{M}}$ at one end of $X \times I$ and $\operatorname{coker}(d \oplus \phi)$ at the other: so $\operatorname{coker}(d \oplus \phi) \cong \underline{\underline{M}}$.

4.6 K_G for general spaces

If one were to define, as might seem at first sight reasonable, $K_G(X)$ for a general space X as the Grothendieck group of G-<u>Vect</u>(X), then because bundles no longer have complementary bundles and one can no longer add to a bundle on a subspace A of X another bundle so that the sum can be extended over X, one would find that the sequence

$$K_G(X \amalg_A Y) \to K_G(X) \oplus K_G(Y) \rightrightarrows K_G(A)$$

was in general not exact, and K_G would not be a cohomology theory.

Now there is actually no reasonable doubt or confusion as to what $K_G(X)$

should be (for, say, paracompact X). But there are at least three ways of defining it, all of which one would like to think unnecessarily cumbersome. They are:

(i) by the method of "stable bundles" presented below;

(ii) by the use of Fredholm complexes - but I haven't carried this through;

(iii) by considering G-homotopy-classes of G-maps of X into a suitable infinite "grassmannian" - one obvious choice is the space of Fredholm operators in the Hilbert sum of the Hilbert spaces $M \otimes H$, where H is a fixed infinite separable Hilbert space, and M runs through the set of simple G-modules. The virtue of this method is that for paracompact X it leads at once (by the arguments of Puppe) to a cohomology theory.

Let X be a G-space; C a numerable covering of X by open G-sets; and R_C the category of all the open G-sets subordinate to C and their inclusions.

Let $C_G(R_C)$ be the category whose objects are families $\{E^{\bullet}_U\}_{U \in R_C}$ of objects of $C_G(U)$ together with isomorphisms $\phi_f : E^{\bullet}_U \to f^* E^{\bullet}_{U'}$ in $C_G(U)$ for all $f : U \to U'$ in R_C satisfying the evident naturality condition.

Define $G\text{-}\underline{\mathrm{Stab}}(X) = \varinjlim_{C} C_G(R_C)$.

<u>Proposition 4.6.1</u> If C is a numerable open covering of X and there is given for each $U \in R_C$ an object $\{E^\bullet_{U,V}\}$ of G-<u>Stab</u>(U) defined with respect to some covering C_U of U, and for each $f : U \to U'$ in R_C an isomorphism $\{E^\bullet_{U,V}\} \to \{f^* E^\bullet_{U',V'}\}$ in G-<u>Stab</u>(U), then there is an object of G-<u>Stab</u>(X), unique up to isomorphism, whose restriction to each U is canonically isomorphic to $\{E^\bullet_{U,V}\}$.

<u>Proof</u>: Let $\tilde{C}$ be the numerable covering of X obtained by putting together all the C_U $(U \in C)$. If $W \in R_{\tilde{C}}$ then $W \in R_{C_U}$ for some U, and one has an object $E^\bullet_{U,W}$ of $C_G(U)$. The point is to show that if $W \in R_{C_U} \cap R_{C_V}$ then $E^\bullet_{U,W}$ is canonically isomorphic to $E^\bullet_{V,W}$. But for some numerable covering $\{W_\alpha\}$ of W one has isomorphisms $\theta^\bullet_\alpha : E^\bullet_{U,W}|W_\alpha \to E^\bullet_{U,W}|W_\alpha$ in $C_G(W_\alpha)$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{W_\alpha\}$. Then $\sum_\alpha \rho_\alpha \theta_\alpha$ is the required isomorphism: because $\theta^\bullet_\alpha|W_\alpha \cap W_\beta$ is chain-homotopic to $\theta^\bullet_\beta|W_\alpha \cap W_\beta$ one knows that $\sum_\beta \rho_\beta \theta^\bullet_\beta$ is chain-homotopic to $\theta^\bullet_\alpha$ in W_α.

An analogous argument proves the uniqueness.

An object $\{E^\bullet_U\}$ of G-<u>Stab</u>(X) has a <u>support</u> which is a closed G-subset of X: it is the complement of the union of all the open G-subsets U in R_C such that $E^\bullet_U \cong 0$.

If A is a G-subspace of X let G-<u>Stab</u>(X,A) be the full subcategory of G-<u>Stab</u>(X) with objects whose supports do not meet A. More generally, if Φ is a family of supports in X, let G-<u>Stab</u>$_\Phi(X)$ be the full subcategory with objects whose supports are in Φ. It is $\varinjlim_{C \in \Phi}$ G-<u>Stab</u>$(X,X-C)$. Finally, G-<u>Stab</u>$_\Phi(X,A)$ consists of objects whose supports are in Φ but don't meet A.

I shall only make use of the two cases $\Phi = \{$all closed G-subsets of $X\}$ and $\Phi = \text{cpt} = \{$all compact G-subsets$\}$.

G-<u>Stab</u>$_\Phi(X,A)$ is an additive category.

<u>Definition 4.6.2</u> $K_{G,\Phi}(X,A)$ is the Grothendieck group of G-<u>Stab</u>$_\Phi(X,A)$, which I interpret to mean $K_{G,\Phi}(X,A) = \underline{\text{isom}}$ G-<u>Stab</u>$_\Phi(X,A)/\approx$, where $E_0^\bullet \approx E_1^\bullet$ if there is $E^\bullet \in$ G-<u>Stab</u>$_\Phi(X \times I, A \times I)$ such that $E_0^\bullet \cong i_0^* E^\bullet$, $E_1^\bullet \cong i_1^* E^\bullet$. $(i_0, i_1 : X \to X \times I)$.

(I have adopted this streamlined interpretation - looking, so to speak, at the path-components instead of the proper π_0 - in order not to have to invoke the extensibility of complexes.)

One must show that the groups so defined constitute in some sense a cohomology theory: that is, have properties of homotopy-invariance, exactness, excision, and additivity.

The homotopy-axiom is immediate: it was built into the definition. And additivity is obvious because $G\text{-}\underline{\mathrm{Stab}}(\coprod X_\alpha) \cong \prod G\text{-}\underline{\mathrm{Stab}}(X_\alpha)$. Excision is easy - I shall state two results:

<u>Proposition 4.6.3</u> If $A \to X$, $A \to Y$ are closed embeddings then $G\text{-}\underline{\mathrm{Stab}}(X \amalg_A Y, Y) \xrightarrow{\cong} G\text{-}\underline{\mathrm{Stab}}(X,A)$, so that

$$K_G(X \amalg_A Y, Y) \xrightarrow{\cong} K_G(X,A).$$

<u>Proposition 4.6.4</u> If X is compact and A is closed then $G\text{-}\underline{\mathrm{Stab}}(X,A) \xrightarrow{\approx} G\text{-}\underline{\mathrm{Stab}}_{\mathrm{cpt}}(X-A)$ so that $K_G(X,A) \xrightarrow{\cong} K_{G,\mathrm{cpt}}(X-A)$. In particular, if X is locally compact and X^+ is its one-point compactification then $K_{G,\mathrm{cpt}}(X) \cong K_G(X^+, \text{point})$.

(One should notice in this connection that if $U \to X$ is an open embedding there is a natural functor "extension by zero" $K_{G,\mathrm{cpt}}(U) \to K_{G,\mathrm{cpt}}(X)$.)

Finally, the question of exactness:

<u>Proposition 4.6.5</u> If A is a closed G-subset of X then $K_G(X \amalg_A CA, \text{point}) \to K_G(X) \to K_G(A)$ is exact. (CA means the cone on A.)

<u>Proof:</u> One half is trivial. For the other, let $E^{\cdot} \in G\text{-}\underline{\mathrm{Stab}}(X)$ represent

an element of $K_G(X)$ whose restriction in $K_G(A)$ vanishes. Then there is $F^\cdot \in G$-Stab$(A \times [\frac{2}{5}, \frac{3}{5}])$ whose restrictions to $A \times \frac{2}{5}$ and $A \times \frac{3}{5}$ are isomorphic to $E^\cdot|A$ and 0 respectively. Let $\rho_1 : X \amalg_A (A \times [0, \frac{2}{5}[) \to X$ be the projection, and define $\rho_2 : A \times]\frac{1}{5}, \frac{4}{5}[\to A \times [\frac{2}{5}, \frac{3}{5}]$ by $(a,t) \mapsto (a, \inf(\sup(\frac{2}{5},t), \frac{3}{5}))$. Define $\tilde{E}^\cdot$ in G-Stab$(X \amalg_A CA$, point) by clutching $\rho_1^* E^\cdot$, $\rho_2^* F^\cdot$, and 0 on $A \times]\frac{3}{5}, 1] \subset CA$. Then $\tilde{E}$ restricts to $E^\cdot$.

It is now classical (cf [2]) that one can define K_G^{-q} to obtain a long exact sequence in which the role of the relative group $K_G(X,A)$ is played by $K_G(X \amalg_A CA$, point). $K_G^{-q}(X)$ $(q > 0)$ can be defined equally well as $K_G(\Sigma^q X$, point) or $K_G(X \times D^q, X \times S^{q-1})$ or $K_{G,\mathrm{prop}}(X \times \mathbb{R}^q)$, where Σ^q denotes the q-fold suspension, and prop denotes the family of closed subsets of $X \times \mathbb{R}^q$ which are proper over X. But one has a choice: either one can change the definition of the relative groups so that exactness holds for any pair but not excision; or one can adopt the definitions above, as I shall do, and forgo exactness except when one knows that $K_G(X,A) \xrightarrow{\tilde{\cong}} K_G(X \amalg_A CA$, point) - which is true, for example, when A is a G-neighbourhood-deformation-retract in X, for then $X \amalg_A CA$, point $\xrightarrow{\cong} X \amalg_A CA, CA$, while $K_G(X \amalg_A CA, CA) \xrightarrow{\tilde{\cong}} K_G(X,A)$ by excision.

So far I haven't mentioned multiplicative properties, but they present no difficulty. $\{K_G^{-q}(X)\}$ is a graded ring; $\{K_{G,cpt}^{-q}(X)\}$ is a graded ring-without-unit, which has a unit if and only if X is compact, and a graded module over $\{K_G^{-q}(X)\}$. And there are pairings

$K_G^{-q}(X,A) \otimes K_G^{-r}(Y,B) \to K_G^{-q-r}(X \times Y,\ X \times B \cup Y \times A)$, etc.

4.7 Continuity

In this section I shall prove that the functor K_G from the category of compact G-spaces to the category of abelian groups is continuous in the sense that if $\{X_\alpha\}_{\alpha\in\Sigma}$ is an inverse system of compact G-spaces over the directed set Σ then the natural map $\varinjlim_{\alpha\in\Sigma} K_G(X_\alpha) \to K_G(\varprojlim_{\alpha\in\Sigma} X_\alpha)$ is an isomorphism.

Lemma 4.7.1 To prove K_G continuous it suffices to prove 4.7.2 and 4.7.3.

Lemma 4.7.2 For any family $\{X_\alpha\}_{\alpha\in\Sigma}$ of compact G-spaces

$$\varinjlim_{\sigma} K_G(\prod_{\alpha\in\sigma} X_\alpha) \xrightarrow{\cong} K_G(\prod_{\alpha\in\Sigma} X_\alpha)$$

where σ runs through the category of finite subsets of Σ.

Lemma 4.7.3 For any inverse system $\{X_\alpha\}_{\alpha\in\Sigma}$ of closed G-subspaces of a compact G-space X, over a directed set Σ,

$$\varinjlim_{\alpha\in\Sigma} K_G(X_\alpha) \xrightarrow{\cong} K_G(\bigcap_{\alpha\in\Sigma} X_\alpha).$$

<u>Proof of 4.7.1</u> Given a directed inverse system $\{X_\alpha\}_{\alpha\in\Sigma}$ write $\Pi = \prod_{\alpha\in\Sigma} X_\alpha$; $\Sigma_\alpha = \{\beta \in \Sigma : \beta < \alpha\}$; $\Pi_\alpha = \prod_{\beta\notin\Sigma_\alpha} X_\beta$. Define $\Pi_\alpha \to \Pi = \prod_{\beta\in\Sigma_\alpha} X_\beta \times \Pi_\alpha$ as the identity on to Π_α and the composite $\Pi_\alpha \to X_\alpha \to \varprojlim_{\beta\in\Sigma_\alpha} X_\beta$ on to the other factor. Π_α is thereby identified with a closed subspace of Π, and $\Pi_{\alpha'} \subset \Pi_\alpha$ if $\alpha < \alpha'$. Furthermore $\bigcap_{\alpha\in\Sigma} \Pi_\alpha = \varprojlim_{\alpha\in\Sigma} X_\alpha$. So by 4.7.3 $\varinjlim K_G(\Pi_\alpha) \xrightarrow{\cong} K_G(\varprojlim X_\alpha)$.

Now consider the direct set $\tilde{\Sigma}$ of pairs (α,σ) with σ a finite subset of Σ and α a minimal element of σ, ordered by $(\alpha,\sigma) < (\alpha',\sigma') \Longleftrightarrow \alpha < \alpha'$ and $\sigma - \Sigma_{\alpha'} \subset \sigma'$. $(\alpha,\sigma) \mapsto \prod_{\beta\in\sigma} X_\beta$ is an inverse system in an evident way over $\tilde{\Sigma}$. And

$$\begin{aligned}
\varinjlim_{(\alpha,\sigma)\in\tilde{\Sigma}} K_G(\prod_{\beta\in\sigma} X_\beta) &\cong \varinjlim_{\alpha\in\Sigma} \varinjlim_{\sigma\subset\Sigma-\Sigma_\alpha} K_G(\prod_{\beta\in\sigma} X_\beta)\\
&\cong \varinjlim_{\alpha\in\Sigma} K_G(\Pi_\alpha) \quad \text{by 4.7.2}\\
&\cong K_G(\varprojlim X_\alpha) \quad \text{as above.}
\end{aligned}$$

On the other hand there is a morphism $\Sigma \to \tilde{\Sigma}$ defined by $\alpha \mapsto (\alpha, \{\alpha\})$. And $(\alpha,\sigma) \leq (\gamma,\{\gamma\})$ if $\beta < \gamma$ for all $\beta \in \sigma$, so the morphism is cofinal, and $\varinjlim_{\alpha\in\Sigma} K_G(X_\alpha) \xrightarrow{\cong} \varinjlim_{(\alpha,\sigma)\in\tilde{\Sigma}} K_G(\prod_{\beta\in\sigma} X_\beta) \xrightarrow{\cong} K_G(\varprojlim X_\alpha)$, as required.

<u>Proof of 4.7.2</u>: Let $X = \prod_{\alpha\in\Sigma} X_\alpha$, $X_\sigma = \prod_{\alpha\in\sigma} X_\alpha$, and $p_\sigma : X \to X_\sigma$, $p_{\tau\sigma} : X_\tau \to X_\sigma$ be the projection ($\sigma \subset \tau$). In view of the existence of complementary bundles it suffices to prove that any bundle E on X is of the form $p_\sigma^* E_\sigma$ for some finite σ, and that if $p_\sigma^* E_\sigma \cong p_{\tau\sigma}^* E'_\sigma$ then $p_{\tau\sigma}^* E_\sigma \cong p_{\tau\sigma}^* E'_\sigma$ for some finite τ containing σ.

Choose $\xi_\alpha \in X_\alpha$ for each $\alpha \in \Sigma$. Define $f_\sigma : X \to X$ by $(f_\sigma x)_\alpha = x_\alpha$ if $\alpha \in \sigma$ and $= \xi_\alpha$ if $\alpha \notin \sigma$. (f_σ is not a G-map.) Let $i : X \to M_X$ be the embedding of X in its space of measures. The system of maps $\{i \circ f_\sigma\}$ converges uniformly to i. So the system of G-maps $\{\phi_\sigma = \mu_G.(i \circ f_\sigma)\}$ obtained by averaging over G also converges uniformly to i. But the bundle E on X can be extended over a G-neighbourhood U of X in M_X, and $\phi_\sigma(X) \subset U$ for large σ, and is furthermore linearly G-homotopic to i for large σ. As ϕ_σ factorizes through p_σ this proves the first half of the result. If now $E = p_\sigma^* E_\sigma \overset{\alpha}{\cong} p_\sigma^* E'_\sigma = E'$ one can extend E, E' and α over a G-neighbourhood of X in M_X. Then for large τ $\phi_\tau^*(\alpha)$ is an isomorphism

$E \to E'$ homotopic to α induced from X_τ. So $p^*_{\tau\sigma} E_\sigma \cong p^*_{\tau\sigma} E'_\sigma$, as required.

(This would have been easier if $G = \{1\}$, for then the injectivity is trivial.)

Lemma 4.7.4 If A is a closed G-subspace of the compact G-space X then $\varinjlim_U K_G(U) \xrightarrow{\cong} K_G(A)$, where U runs through the G-neighbourhoods of A in X.

This is immediate.

Proof of 4.7.3 It suffices to show that if $\{X_\alpha\}_{\alpha\in\Sigma}$ is a family of closed G-subspaces of a compact G-space Y, and $X_\sigma = \bigcap_{\alpha\in\sigma} X_\alpha$, $X = X_\Sigma$, then

$$\varinjlim_{\sigma \text{ finite}} K_G(X_\sigma) \xrightarrow{\cong} K_G(X).$$

That follows from 4.7.4, the only point being to observe that if U is open in Y and contains X then it contains X_σ for large finite σ: otherwise $\{X_\alpha - U\}$ would be a family of compact sets with the finite-intersection property but with empty intersection.

Proposition 4.7.5 If $\{(X_\alpha, A_\alpha)\}_{\alpha\in\Sigma}$ is an inverse system of pairs of compact G-spaces, over a directed set Σ, then

$$\varinjlim K_G^{-q}(X_\alpha, A_\alpha) \xrightarrow{\cong} K_G^{-q}(\varprojlim X_\alpha, \varprojlim A_\alpha). \quad (q \in \mathbb{N}).$$

Proof: $\varinjlim K_G(X_\alpha) \xrightarrow{\cong} K_G(\varprojlim X_\alpha)$, so $\varinjlim K_G(S^q \times X_\alpha) \xrightarrow{\cong} K_G(S^q \times \varprojlim X_\alpha)$, so $\varinjlim K_G^{-q}(X_\alpha) \xrightarrow{\cong} K_G^{-q}(\varprojlim X_\alpha)$. Similarly $\varinjlim K_G^{-q}(A_\alpha) \xrightarrow{\cong} K_G^{-q}(\varprojlim A_\alpha)$, so $\varinjlim K_G^{-q}(X_\alpha, A_\alpha) \xrightarrow{\cong} K_G^{-q}(\varprojlim X_\alpha, \varprojlim A_\alpha)$.

<u>Proposition 4.7.6</u> If $\{G_\alpha\}_{\alpha\in\Sigma}$ is an inverse system of compact groups over a directed set Σ, then $\varinjlim R(G_\alpha) \xrightarrow{\cong} R(\varprojlim G_\alpha)$.

<u>Proof</u>: Write $G = \varprojlim_{\alpha\in\Sigma} G_\alpha < \Pi = \prod_{\alpha\in\Sigma} G_\alpha$. Also $G_\alpha = \varprojlim_{\beta\leq\alpha} G_\beta$ can be regarded as a subgroup of $\Pi_\alpha = \prod_{\beta\leq\alpha} G_\beta$ so that $\Pi/G \to \Pi_\alpha/G_\alpha$ is a map of Π-spaces. Clearly $\Pi/G \xrightarrow{\cong} \varprojlim \Pi_\alpha/G_\alpha$. Applying K_Π to both sides gives $R(G) \xleftarrow{\cong} \varinjlim R(G_\alpha)$.

I don't know a more direct way of proving this.

4.8 The Thom homomorphism and periodicity

Let X be a G-space and E a G-vector-bundle on X. I shall define an element λ_E, called the Thom element, in $K_{G,prop}(E)$, where 'prop' means the family of closed sets of E which are proper over X. $K_{G,prop}(E)$ is a module over $K_G(E) \cong K_G(X)$, so one can define a homomorphism of $K_G(X)$-modules $\phi : K_G(X) \to K_{G,prop}(E)$ by $x \mapsto \lambda_E . x$. ϕ is called the Thom homomorphism.

λ_E is defined by the following complex on E:

$$\dots \to 0 \to \mathbb{C} \to pr^*E \to \Lambda^2 pr^*E \to \Lambda^3 pr^*E \to \dots ,$$

where $pr : E \to X$, and all the differentials are exterior multiplication by δ, the diagonal map $E \to E \times_X E$ regarded as a section of pr^*E.

I have listed properties of the Thom homomorphism in [8], and I shall not repeat them here.

The periodicity-theorem of Bott can be generalized and stated in the following form.

Proposition 4.8.1 If L is a G-line-bundle on a compact G-space X then $\phi : K_G(X) \to K_{G,cpt}(L)$ is an isomorphism.

<u>Proof</u>: $K_{G,\mathrm{cpt}}(L) \cong K_G(P(L \oplus \underline{\mathbb{C}}), X)$, where P denotes the associated projective bundle. The complex λ_E extends to $P(L \oplus \underline{\mathbb{C}})$ as $\ldots \to \underline{\mathbb{C}} \to H \otimes pr^* L \to 0 \to \ldots$, where H is the Hopf bundle (see [8] chap. 4). There is a split short exact sequence

$0 \to K_G(P(L \oplus \underline{\mathbb{C}}), X) \to K_G(P(L \oplus \underline{\mathbb{C}})) \to K_G(X) \to 0$, so it is plain that 4.8.1 will be implied by

<u>Proposition 4.8.2</u> $K_G(P(L \oplus \underline{\mathbb{C}}))$ is generated as an algebra over $K_G(X)$ by $y = [H]$, the class of the Hopf bundle, subject to the sole relation $(1 - y)(1 - [L]y) = 0$.

<u>Proof</u>: $P(L \oplus \underline{\mathbb{C}})$ is a fibre-bundle over X with fibre S^2. It can be written $(P \times S^2)/T^1$, where P is a principal T^1-bundle, i.e. a $(G \times T^1)$-space, and the circle-group T^1 acts on S^2 keeping the poles fixed. Now $K_G((P \times S^2)/T^1) \cong K_{G\times T^1}(P \times S^2)$, and one can check that 4.8.2 is implied by

<u>Proposition 4.8.3</u> If G acts on S^2 by a homomorphism $t : G \to T^1$, then $K_G(X \times S^2)$ is generated as algebra over $K_G(X)$ by an element y subject to the sole relation $(1 - y)(1 - ty) = 0$, where $t = [\underline{M}]$, and M is the G-module defined by $t : G \to T^1$.

Having performed the reduction as far as this one can now turn to the proof of Bott's theorem given in [5]. One finds that the proof there goes through without change in the presence of a group-action. One doesn't even need to assume that G acts trivially on the field $\mathbb{C}$.

I have shown in [8] how one gets from 4.8.1 to the following proposition.

<u>Proposition 4.8.3</u> If X is a compact G-space and $E = L_1 \oplus \ldots \oplus L_k$ is a is a G-vector-bundle on X which is a sum of G-line-bundles, then the Thom homomorphism $\phi : K_G(X) \to K_{G,cpt}(E)$ is an isomorphism of $K_G(X)$-modules.

This proposition holds even without the assumption that E is a sum of line-bundles, as I shall prove in 6.7.

The most important consequence of 4.8.1 or 4.8.3 is

<u>Proposition 4.8.4</u> If X is a compact G-space and G acts trivially on $\mathbb{C}$ then $K_G^{-q}(X)$ is naturally isomorphic to $K_G^{-q-2}(X)$ as $K_G(X)$-module for any q in $\mathbb{N}$.

<u>Proof</u>: $K_G^{-2}(X) = K_{G,cpt}(X \times \mathbb{R}^2) \cong K_{G,cpt}(X \times \mathbb{C})$, so the case $q = 0$ is immediate; the general case follows mechanically.

Because of 4.8.4 it is reasonable to define $K_G^q(X)$ for positive q as $K_G(X)$ or $K_G^{-1}(X)$ according as q is even or odd. One will then have long exact sequences of the type

$$\cdots \to K_G^q(X,A) \to K_G^q(X) \to K_G^q(A) \to K_G^{q+1}(X,A) \to \cdots$$

extending for all $q \in \mathbb{Z}$. And because the periodicity-isomorphism is compatible with multiplication $K_G^*(X)$ as so defined will be a graded ring. Alternatively one can regard it as graded modulo two.

The criterion of a satisfactory definition of $K_G(X)$ for non-compact spaces X is that, besides giving rise to a cohomology-theory, it has the periodicity-property 4.8.4. I have not verified this for my definition given in 4.6, and therefore I have essentially confined myself to compact spaces in this thesis. But I should emphasise that when $G = \{1\}$ it is possible satisfactorily to define $K(X)$ as $[X;B]$, the group of homotopy-classes of maps of X into a suitable triangulable H-space B: for then $K^{-2}(X) \cong [X;B']$ for some other triangulable H-space B', and the existence of a natural isomorphism $[X;B] \cong [X;B']$ for all compact X implies that $B \simeq B'$. (This argument appears to apply also to G-spaces when G is finite.) But of course I haven't proved that my definition of K is representable in this way.

In the next chapter I shall construct a number of spectral sequences involving K_G. The methods used have nothing to do with compactness, and they would apply quite generally if K_G^q were defined for all $q \in \mathbb{Z}$ (with the properties verified in 4.6). And my results are new even in the case $G = \{1\}$, when there is no difficulty defining K. So I have stated the results of 5.1 and 5.2 in illegitimate generality: the reader should either assume that $G = \{1\}$, or else observe throughout that the constructions I make do not lead out of the category of compact spaces.

Chapter Five

SPECTRAL SEQUENCE and FILTRATION

5.1 The spectral sequence for a covering

In Chapter 1 I defined a functor B from categories to topological spaces. For a category R, BR was the realization of the semi-simplicial set R_* : S $\mapsto$ Funct (S; R), which can be thought of as the "singular complex" or the "nerve" of R, of better still can be identified with R.

The case I shall be concerned with here is when R is a subcategory of G-Top/X for some G-space X - for example a category of G-stable subsets of X. I shall write R_σ for $\sigma(0_S)$ when $\sigma : S \to R$ is a simplex of R_* and 0_S is the initial object of the finite ordered set S. One can define a "universal object" ER in G-Top (X) as the realisation of the semi-simplicial G-space $S \mapsto \coprod_{\sigma : S \to R} R_\sigma$.

To be quite explicit: ER is the quotient by a certain equivalence relation of the G-space $\coprod_{(S,\sigma)} \Delta_S \times R_\sigma$, where σ runs through all functors $\sigma : S \to R$ for all $S \in$ Ord. So a point of ER can be represented (S,σ,ξ,x) with $\xi \in \Delta_S$ and $x \in R_S$; the equivalence relation is generated by the identifications: $(S,\sigma,\xi,x) = (S',\sigma',\xi', x')$ if there is a map $\theta : S \to S'$ of ordered sets such that $\sigma = \sigma' \circ \theta$, $\xi' = \theta_*\xi$, $x = \theta^*x'$. Similarly a point of BR can be represented (S,σ,ξ). The projection $ER \to BR$ is evident.

ER and BR have natural increasing filtrations $\{E^pR\}$, $\{B^pR\}$: E^pR is the image in ER of the $\Delta_S \times R_\sigma$ with card $(S) \leq p + 1$; and similarly for B^pR.

One can be more explicit still. ER is constructed as follows: E^0R is the disjoint union of the objects of R. To form E^1R one attaches, for each morphism $Y_0 \to Y_1$ in R, a cylinder $Y_0 \times I$ by its ends to Y_0 and Y_1. But each cylinder corresponding to an identity-morphism $Y_0 \to Y_0$ is to be collapsed and can be omitted. Then to obtain E^2R one attaches, for each diagram $Y_0 \to Y_1 \to Y_2$ in R, a prism $Y_0 \times \Delta^2$, by its boundary $Y_0 \times \dot{\Delta}^2$, to the cylinders corresponding to $Y_1 \to Y_2$, $Y_0 \to Y_2$, $Y_0 \to Y_1$. But again, diagrams containing an identity can be ignored. All in all one has:

<u>Lemma 5.1.1</u> The map $\coprod_\sigma (\Delta^p \times R_\sigma, \dot{\Delta}^p \times R_\sigma) \to (E^pR, E^{p-1}R)$, where σ runs through the non-degenerate p-simplexes $\sigma : [0,p] \to R$, is a relative homeomorphism.

<u>Remarks</u>: (i) $\sigma : S \to R$ is called "degenerate", if it factorises $\sigma = \sigma' \circ \theta$ with $\theta : S \to S'$ not injective.

(ii) By "relative homeomorphism" I mean only that the map induces a homeomorphism of the complements of the subspaces. However the map induces an isomorphism of K_G because the two subspaces are compatible neighbourhood-deformation-retracts, as one sees by a very classical argument.

<u>Proof</u>: Surjectivity is trivial. Injectivity is "well-known" - one can use the argument of Milnor ([23] Theorem 1). Finally, the map is open: for an open set of $\overset{\circ}{\Delta}{}^p \times R_{\sigma_0}$ is the intersection of $\overset{\circ}{\Delta}{}^p \times R_{\sigma_0}$ with an open set of $\coprod_{\sigma} \Delta^p \times R_\sigma$ saturated for the equivalence relation $\sim$ above.

<u>Proposition 5.1.2</u> There is a spectral sequence $H(R)$ depending functorially on R with

$$H_2^{pq}(R) \cong H^p(R_*; K_G^q) \quad \text{and} \quad H_\infty^{pq}(R) \cong gr^p K_G^{p+q}(ER). \text{ (*)}$$

<u>Proof</u>: One has only to apply the method of Cartan-Eilenberg ([12] p.333) with, in their notation, $H(p,q) = K_G(E^{q-1}R, E^{p-1}R)$, and to observe that $K_G^{p+q}(\Delta^q \times R_\sigma, \dot{\Delta}^q \times R_\sigma) \cong K_G^p(R_\sigma)$ Of course $H^p(R_*; K_G^q)$ means the p^{th} cohomology group of the cochain-complex into which the functor K_G^q transforms

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(*) See the remarks on page 4.41. This spectral sequence can be regarded as defined only when $G = \{1\}$ or when ER is compact (cf. case 2 below).

the semi-simplicial complex R_*.

I shall now show that in certain circumstances the projection $\pi : ER \to X$ induces an isomorphism $K_G^*(X) \to K_G^*(ER)$, so that the spectral sequence of 5.1.2 terminates with $K_G^*(X)$.

Let $\{X_\alpha\}_{\alpha\in\Sigma}$ be a covering of X by G-subspaces. For any subset τ of Σ, X_τ denotes $\bigcap_{\alpha\in\tau} X_\alpha$. Let R be the full subcategory of G-Top/X whose objects are the X_τ for all finite subsets τ of Σ. In this case ER maps injectively into $BR \times X$ which is ER' for the category R' isomorphic to R but with all the objects X_τ replaced by X. And ER can be written $\cup\, \Delta_\tau \times X_\tau$, where τ runs through the finite subsets of Σ.

<u>Case 1</u> The covering $\{X_\alpha\}_{\alpha\in\Sigma}$ is <u>numerable</u>.

One can choose a G-invariant subordinate partition of unity $\{\phi_\alpha\}$, which should be regarded as a map $\phi : X \to BR$. If $\psi = \phi \times id : X \to BR \times X$, it is immediate that $\psi(X) \subset ER$, and $\pi \circ \psi = id_X$. But $\psi \circ \pi : ER \to ER$ can be joined to the identity by a linear homotopy, so $\pi : ER \to X$ is a G-homotopy-equivalence, and $\pi^!$ is an isomorphism.

<u>Case 2</u> X and all the X_α are compact, and Σ is finite.

I shall introduce a new filtration of ER. First filter X as follows:

$X = X_0 \supset X_1 \supset X_2 \supset \ldots$, where X_r consists of all points of X which are in at least $r + 1$ of the sets of the covering. Then define

$E_p R = \pi^{-1} X_p \subset ER$.

<u>Lemma 5.1.3</u> $\quad \pi^! : K_G^*(X_p, X_{p+1}) \xrightarrow{\cong} K_G^*(E_p R, E_{p+1} R)$.

<u>Proof</u>: $K_G^*(X_p, X_{p+1}) \cong K_G^*(X_{p+1} \cup \bigcup_\sigma X_\sigma, X_{p+1}) \cong \prod_\sigma K_G^*(X_\sigma, X_\sigma \cap X_{p+1})$

where σ runs through the subsets of Σ such that $\mathrm{card}(\sigma) = p + 1$. But similarly:

$$\begin{aligned} K_G^*(E_p R, E_{p+1} R) &\cong K_G^*(E_{p+1} R \cup \bigcup_\sigma (\Delta^p \times X_\sigma), E_{p+1} R) \\ &\cong \prod_\sigma K_G^*(\Delta^p \times X_\sigma, (\Delta^p \times X_\sigma) \cap E_{p+1} R) \\ &\cong \prod_\sigma K_G^*(\Delta^p \times X_\sigma, \Delta^p \times (X_\sigma \cap X_{p+1})) \cong \prod_\sigma K_G^*(X_\sigma, X_\sigma \cap X_{p+1}). \end{aligned}$$

<u>Corollary 5.1.4</u> $\quad \pi^! : K_G^*(X) \xrightarrow{\cong} K_G^*(ER)$.

Proof by induction.

<u>Case 3</u> $\quad X$ is paracompact, and $\{X_\alpha\}_{\alpha \in \Sigma}$ is locally finite.

I have shown elsewhere [28] that this case follows from Case 1 by a limiting argument using a continuity-property of K_G^*, at least when $G = \{1\}$.

<u>Remark</u>: I have now established what is usually called the Leray spectral

sequence for the covering $\{X_\alpha\}_{\alpha\in\Sigma}$. It is to be regarded as a generalisation of the Mayer-Vietoris sequence, to which it reduces when Σ has two elements. The argument of Case 2 above then affords a proof of the exactness of the Mayer-Vietoris sequence which some people find more perspicuous than the argument of Eilenberg and Streenrod. But, unlike Eilenberg and Steenrod, I have used the homotopy axiom for K_G^* in a rather essential way. I think it likely that one can derive the Leray spectral sequence (say, for Σ finite) by a generalisation of the Eilenberg-Steenrod method, independently of the homotopy-axiom, but I have not succeeded in doing so.

5.2 <u>The spectral sequence for a map</u>

In this section I shall prove

<u>Proposition 5.2.1</u> Let $f : X \to Y$ be a proper G-map, where Y is a paracompact and compactly generated trivial G-space. Then there is a spectral sequence $H(f)$ with $H_2^{pq}(f) = H^p(Y; K_G^q f)$ and $H_\infty^{pq}(f) = gr^p\, K_G^{p+q}(X)$, where $K_G^q f$ is the sheaf on Y associated to the presheaf $U \mapsto K_G^q(\overline{f^{-1}U})$.

The stalk of $K_G^q f$ at $y \in Y$ is $K_G^q(X_y)$, where $X_y = f^{-1}(y)$. (*)

(Notice that X is paracompact and compactly generated if and only if Y is.)

<u>Proof</u>: For each open covering $\mathcal{C}$ of Y let $R_{\mathcal{C}}$ be the subcategory of G-<u>Top</u>/X associated to the covering $\{\overline{f^{-1}V}\}_{V\in\mathcal{C}}$ of X. If $\mathcal{C}$ refines $\mathcal{C}'$ there are functors $t : R_{\mathcal{C}} \to R_{\mathcal{C}'}$ such that $\tilde{V} \subset t(\tilde{V})$ for each $\tilde{V} \in R_{\mathcal{C}}$; and if t_0, t_1 are two such functors there is a third t (say $t(\tilde{V}) = t_0(\tilde{V}) \cap t_1(\tilde{V})$) with morphisms $t \to t_0$, $t \to t_1$. This amounts to a functor $\tilde{t} : R_{\mathcal{C}} \times J \to R_{\mathcal{C}}$ where J is the category $(\overset{0}{\cdot} \leftarrow \overset{(01)}{\cdot} \to \overset{1}{\cdot})$. Then $B(R_{\mathcal{C}} \times J) \cong BR_{\mathcal{C}} \times BJ \cong BR_{\mathcal{C}} \times I$, and correspondingly $E(R_{\mathcal{C}} \times J) \cong ER_{\mathcal{C}} \times I$, so that $E\tilde{t}$ is a homotopy between Et_0 and Et_1. Furthermore, because $B\tilde{t}$ is filtration-preserving and J has dimension one, one has $Et_\lambda(ER^p_{\mathcal{C}}) \subset ER^{p+1}_{\mathcal{C}'}$ for all $p \in \mathbb{N}$, $\lambda \in I$. So there is a morphism of spectral sequences $\{H_r(R_{\mathcal{C}'})\} \to \{H_r(R_{\mathcal{C}})\}$ well-defined (i.e. independent of t) for $r \geq 2$; and one obtains a system of spectral sequences over the directed set of open coverings of Y. Each of them terminates with $K_G^*(X)$ by Case 1 above. Passing to the direct limit gives the spectral sequence desired. (It was necessary to prove that the functor $\mathcal{C} \mapsto H(R_{\mathcal{C}})$ factorised through a directed set, for only then is $\varinjlim$ an exact functor.)

(*) Again one should assume $G = \{1\}$ or else X and Y compact.

One knows that the stalk of $K_G^* f$ at $y \in Y$ is $\varinjlim_U K_G^*(\overline{f^{-1}U})$ where U runs through the neighbourhoods of y in Y. But that is the same as $\varinjlim_V K_G^*(\bar{V})$ where V runs through the G-neighbourhoods of X_y in X. (One must show that each V contains some $f^{-1}U$; let V be open, then $U = Y - f(X - V)$ is a neighbourhood of y because f is closed and $f^{-1}U \subset V$.) And $\varinjlim_V K_G^*(V) \xrightarrow{\cong} K_G^*(X_y)$ by 4.7.4.

<u>Corollary 5.2.2</u> If X is a G-space there is a spectral sequence $H^p(X/G; R^q) \Rightarrow K_G^*(X)$, where $R^q = 0$ for q odd, while for q even it is a sheaf on X/G whose stalk at Gx is isomorphic to $R(G_x)$, where G_x is the stabiliser of x.

Of course one cannot make any deductions from this statement unless one knows that the spectral sequence converges. It certainly converges if X/G has finite covering dimension, and that is the only condition I know. But I don't know, for example, if X/G has finite covering dimension whenever X does. If G is a compact Lie group and X is a smooth compact G-manifold on which G acts smoothly then X/G has finite covering dimension because it is triangulable [32], or better, because it is a finite union of smooth manifolds.

5.3 Localisation

In this section G is a compact Lie group.

The most immediate application of the spectral sequence 5.2.2 is to suggest the localization-theorem I shall now state; but it can be proved better without the spectral sequence, as will appear.

If H is a subgroup of G, and X is a G-space, I shall write $X^{(H)}$ for the subspace of X consisting of points x for which H is conjugate to a subgroup of G_x. It is a G-subspace.

Proposition 5.3.1 Let X be a compact G-space, and γ a prime ideal of $R(G)$ whose support (see 2.5.9) is S, a cyclic subgroup of G. Then the restriction

$$K_G^*(X)_\gamma \to K_G^*(X^{(S)})_\gamma \quad \text{is an isomorphism.}$$

(If M is an $R(G)$-module, M_γ denotes its localisation at γ.)

Proof: If X/G has finite covering dimension one can proceed as follows. Consider the morphism of spectral sequences induced by restriction:

$$\begin{array}{ccc} H^*(X/G;\, R^*) & \to & H^*(X^{(S)}/G,\, R^*) \\ \Downarrow & & \Downarrow \\ K_G^*(X) & \to & K_G^*(X^{(S)}) \end{array}$$

(I use R^* to denote also the restriction of this sheaf to $X^{(S)}/G$.) Because localisation is an exact functor one can localise both spectral sequences at γ to obtain

$$\begin{array}{ccc} H^*(X/G;\, R^*_\gamma) & \rightarrow & H^*(X^{(S)}/G;\, R^*_\gamma) \\ \Downarrow & & \Downarrow \\ K^*_G(X)_\gamma & \rightarrow & K^*_G(X^{(S)})_\gamma , \end{array}$$

where R^*_γ is a sheaf whose stalk at xG is $R(G_x)_\gamma$. But by 2.5.10, $R(G_x)_\gamma = 0$ if $x \notin X^{(S)}$; and this implies the theorem provided that the spectral sequences are convergent.

A better proof, not requiring finite-dimensionality, is as follows. Let $\mathcal{C}$ denote the collection of compact G-subspaces Y of X with the property that for any compact G-subspace Z containing Y the restriction $K^*_G(X)_\gamma \rightarrow K^*_G(Z)_\gamma$ is an isomorphism. If Y_1 and Y_2 are in $\mathcal{C}$ then so is $Y_1 \cap Y_2$, by the Mayer-Vietoris sequence. So, by the continuity of K^*_G, $Y_0 = \bigcap_{Y \in \mathcal{C}} Y$ is in $\mathcal{C}$. One must show $Y_0 \subset X^{(S)}$. But otherwise there is some $x \in Y_0$ which is not in $X^{(S)}$. Let A be a compact slice at x, i.e. $G.A$ is a G-neighbourhood of x isomorphic to $G \times_{G_x} A$ as G-space. And let U be an open G-neighbourhood of x contained in $G.A$. I shall derive the contradiction that $Y_0 - U \in \mathcal{C}$. It suffices to show that $K^*_G(Y_0, Y_0 - V)_\gamma = 0$

when V is open and contained in U. But $K_G^*(Y_0, Y_0 - V) \cong K_G^*(G.A, G.A - V)$ by excision. Now the structural map $R(G) \to K_G^*(G \times_{G_x} A)$ comes from the map $G \times_{G_x} A \to \text{Point}$, which factorises through G/G_x, so $R(G)_\gamma \to K_G^*(G.A)_\gamma$, factorises through $R(G_x)_\gamma$, which is zero. Therefore $1 = 0$ in the ring $K_G^*(G.A)_\gamma$; and hence $K_G^*(G.A, G.A - V)_\gamma = 0$, as it is a unitary module over $K_G^*(G.A)_\gamma$.

5.4 The filtration of $K_G(X)$

I shall define a decreasing filtration $\{K_{G,p}(X)\}_{p\in\mathbb{N}}$ of $K_G(X)$ so that it becomes a filtered ring.

In the case $G = 1$ this can be done by defining $x \in K_p(X)$ if x is in the kernel of $K(X) \to K(\Xi^{p-1}R)$ for some R corresponding as in 5.1 to a numerable covering of X. Thus $K_1(X)$ consists of those elements whose restriction to each set of some covering of X is zero. But in the general case it is necessary to consider more subtle refinements than coverings by subsets, and I think it is simplest to proceed as follows.

If $Y \to X$ is a G-space over X one can form a semi-simplicial (actually simplicial) G-space Y_* defined by $Y_*(S) = Y \times_X Y \times_X \cdots \times_X Y$ (S copies).

The realisation of Y_* I shall denote by EY; it has a natural filtration $\{E^q Y\}$.

<u>Example</u> If X is a compact G-space and $Y = X \times G$ then $EY \cong X \times EG$.

<u>Definition 5.4.1</u> $x \in K_{G,p}(X)$ if x is in the kernel of $K_G(X) \to K_G(E^{p-1}Y)$ for some G-space Y which is locally split - but not necessarily locally G-split - over X. (This is to be interpreted so that $K_{G,0}(X) = K_G(X)$.)

$K_{G,p}(X)$ is an ideal in $K_G(X)$.

<u>Proposition 5.4.2</u> If $g : (G, X) \to (G', X')$ is a morphism, then $f^! K_{G',p}(X') \subset K_{G,p}(X)$ for all p.

The proof is clear.

<u>Proposition 5.4.3</u> $K_{G,p}(X).K_{G,q}(X) \subset K_{G,p+q}(X)$.

<u>Proof</u>: If $x \in K_{G,p}(X)$, $y \in K_{G,q}(X)$ one can suppose they vanish in $K_G(E^{p-1}Y)$, $K_G(E^{q-1}Y)$ for the same Y. I shall abbreviate E^kY to E^k. If $x \mapsto \bar{x} \in K_G(E)$ then $\bar{x}$ comes from some $\bar{\bar{x}} \in K_G(E,E^{p-1})$ - in view of the neighbourhood deformation property explained in 5.1. Similarly $y \mapsto \bar{y} \leftarrow \bar{\bar{y}} \in K_G(E,E^{q-1})$. Now $\bar{x}.\bar{y} = d^!(\bar{x} \times \bar{y})$, where

$d = id \times id : E \to E \times E$, so comes from $\bar{\bar{x}} \times \bar{\bar{y}} \in K_G(E \times E, E^{p-1} \times E \cup E \times E^{q-1})$. One must show that the composite $K_G(E \times E, E^{p-1} \times E \cup E \times E^{q-1}) \to K_G(E \times E) \xrightarrow{d^!} K_G(E^{p+q-1})$ is zero. That follows from:

<u>Lemma 5.4.4</u> If A is a semi-simplicial G-space, and ρ^n is the functor taking a semi-simplicial G-space to the "n-skeleton" of its realization, then the diagonal: $\rho A \to \rho A \times \rho A$ is homotopic to a map taking $\rho^n A$ into $\bigcup_{p+q=n} \rho^p A \times \rho^q A$.

<u>Proof</u>: If S is a finite ordered set I shall describe points $\xi \in \Delta_S$ not by the barycentric coordinates $x^S_\alpha : \Delta_S \to I (\alpha \in S)$ but by "cumulative" coordinates $y^S_\alpha : \Delta_S \to I$, where $y_\alpha(\xi) = \sum_{\beta \leq \alpha} x_\alpha(\xi)$. The diagonal $d^S : \Delta_S \to \Delta_S \times \Delta_S$ is linearly homotopic to the map $h^S = h^S_1 \times h^S_2 :$ $\Delta_S \to \Delta_S \times \Delta_S$ defined by

$$y^S_\alpha \, h^S_2 \, (\xi) = \sup(2y^S_\alpha(\xi) - 1, 0); \quad y^S_\alpha \, h^S_1(\xi) = \inf(2y^S_\alpha, 1).$$

I leave it to the reader to check that h^S is functorial in S. Now $h^S(\Delta_S) \subset \bigcup_{\alpha \in S} \Delta_{S_\alpha} \times \Delta_{T_\alpha}$, where $S_\alpha = \{\beta \in S : \beta \leq \alpha\}$ and $T_\alpha = \{\beta \in S : \beta \geq \alpha\}$ - for $h^S(\xi) \in \Delta_{S_\alpha} \times \Delta_{T_\alpha}$ whenever $y^S_\alpha(\xi) \geq \frac{1}{2}$ and $y^S_{\alpha-1}(\xi) \leq \frac{1}{2}$.

And

$$\rho A = \lim_{(S\to T)} \Delta_S \times A(T) \xrightarrow{d'} \lim_{(S\to T)} \Delta_S \times \Delta_S \times A(T) \times A(T) \to \rho A \times \rho A.$$

The diagonal d factorises through d' induced by the $\{d^S\}$; and d' is linearly homotopic to h' induced by the $\{h^S\}$. So d is homotopic to $h : \rho A \to \rho A \times \rho A$. But $h(\rho^n A) \subset \bigcup_{p+q=n} \rho^p A \times \rho^q A$.

The following property of the filtration is rather important.

<u>Proposition 5.4.5</u> If N is a normal subgroup of G which acts freely on the compact G-space X, then $p^! : K_{G/N}(X/N) \to K_G(X)$ is an isomorphism of filtered rings.

<u>Proof</u>: It suffices to show that $p^! x \in K_{G,q}(X) \Longrightarrow x \in K_{G/N,q}(X/N)$. Let $Y \to X$ be such that $p^! x \in \ker : K_G(X) \to K_G(E^{q-1}Y)$. The induced map $Y/N \to X/N$ leads to a semi-simplicial G-space $(Y/N)_*$, and as N acts freely on X one has $(Y/N)_*(S) = (Y/N) \times_{(X/N)} \cdots \times_{(X/N)} (Y/N) \cong Y_*(S)/N$. Therefore $(EY)/N \xrightarrow{\tilde{\cong}} E(Y/N)$, for direct-limit formations commute among themselves. The result now follows from the diagram

$$\begin{array}{ccc} K_{G/N}(X/N) & \xrightarrow{\cong} & K_G(X) \\ \downarrow & & \downarrow \\ K_{G/N}(E(Y/N)) & \xrightarrow{\tilde{\cong}} & K_G(E^{q-1}Y) \end{array}$$

Corollary 5.4.6 If $\phi : G \to G'$ and $\ker \phi$ acts freely on the compact G-space X, then $\phi_* : K_G(X) \to K_{G'}(\phi_* X)$ is an isomorphism of filtered rings.

Proof: This follows from 5.4.5 in view of:

$$K_G(X) \xrightarrow{\cong} K_{G'\times G}(G' \times X) \xleftarrow{\cong} K_{G'}(G' \times_G X).$$

Remark: It is plain that $K_{G,1}$ (point) is the augmentation-ideal I_G in $R(G)$. So by 5.4.3, $I_G^q . K_{G,p}(X) \subset K_{G,p+q}(X)$. I shall show in Chapter 8 that if X is compact and $K_G(X)$ is finite over $R(G)$ then the filtration-topology of $K_G(X)$ coincides with its I_G-adic topology.

5.5 Finiteness

It is often desirable to know that $K_G^*(X)$, by which I mean here $K_G(X) \oplus K_G^{-1}(X)$, is a finite module over $R(G)$. The best result that I have to offer in this direction is the following, which includes the case of a compact smooth manifold on which a compact Lie group acts smoothly.

Proposition 5.5.1 If G is a compact Lie group and X is a compact G-space which is locally G-contractible and such that X/G has finite covering-dimension, then $K_G^*(X)$ is finite over $R(G)$.

<u>Locally G-contractible</u> means that each point x of X has arbitrarily small G_x-stable neighboruhoods which are G_x-contractible in themselves to x; alternatively one can say: each point x of X has arbitrarily small G-stable neighbourhoods of which the orbit Gx is a G-deformation-retract.

<u>Proof</u>: We have seen in 5.2 that when X/G has finite covering-dimension there is a convergent spectral sequence $H^*(X/G;\ R^*) \Longrightarrow K_G^*(X)$, where R^0 is a sheaf whose stalk at an orbit Gx is $R(G_x)$, and $R^1 = 0$. So it suffices to show that $H^*(X/G;\ R^*)$ is finite over $R(G)$ (because $R(G)$ is noetherian).

Now by a theorem of Grothendieck (Bull. Soc. Math. France, <u>84</u> (1956), 1-7) if A is a sheaf on a compact space Y there are arbitrarily fine open coverings U of Y such that $H^*(U;\ A) \to H^*(Y;\ A)$ is surjective, provided that the sheaf A is <u>calculable</u>, which means that for each neighbourhood V of a point y of Y there is a smaller neighbourhood W such that the image of $H^q(V;\ A)$ in $H^q(W;\ A)$ is zero for $q > 0$. The sheaf R^* on X/G that we are concerned with is calculable: given V, choose W of the form $\tilde{W}/G$ where $\tilde{W}$ is G-contractible to a single orbit; then $(W,\ R^*)$ is contractible <u>as sheaved-space</u>, so $H^q(W;\ R^*) = 0$ for $q > 0$.

Now let U' be a shrinkage of the finite open covering U: i.e. the sets of U' are relatively compact in those of U. The map $H^*(U; R^*) \to H^*(X/G; R^*)$ factorises through $H^*(U'; R^*)$, so it suffices to show that the image of $H^*(U; R^*)$ in $H^*(U'; R^*)$ is finite over $R(G)$. Finally then, it suffices to prove

<u>Lemma 5.5.2</u> If V, W are open sets of X/G such that $\bar{W} \subset V$ then the image of $R^*(V)$ in $R^*(W)$ is finite over $R(G)$.

This follows simply from a second lemma:

<u>Lemma 5.5.3</u> Each point Gx of X/G has arbitrarily small neighbourhoods N such that $R^*(N) \cong R(G_x)$.

<u>Proof that (5.5.3)⇒(5.5.2)</u>: For each point Gx of V choose a neighbourhood N_x contained in V such that $R^*(N_x) \cong R(G_x)$. A finite number, say $N_{x_1}, \ldots, N_{x_n}$, of these cover W. $R^*(V) \to R^*(W)$ factorises through $R^*(\cup N_{x_i})$, which is a submodule of $\Pi\, R^*(N_{x_i}) \cong \Pi\, R(G_{x_i})$, which is finite over $R(G)$.

<u>Proof of 5.5.3</u>: R^* is the sheaf associated to the presheaf P defined by $P(V) = K_G(\overline{p^{-1}V})$, where $p : X \to X/G$ is the projection. Define another presheaf Q on X/G by prescribing $Q(V) = K_G(\overline{p^{-1}V})$ when $\overline{p^{-1}V}$ is G-contractible

to Gx.

One must show (cf. [20] p.26): if U is a G-subset of X G-contractible to Gx, and if U is covered by G-contractible sets U_α, and one has $\xi_\alpha \in K_G(U_\alpha)$ for each α, and each pair ξ_α, ξ_β agrees on G-contractible subsets of $U_\alpha \cap U_\beta$, then there is a unique ξ in $K_G(U)$ which restricts to each ξ_α. x is in some U_α, say U_0, and $K_G(U) \cong K_G(U_0)$, so there is a unique candidate for ξ. It remains to see that ξ restricts to each ξ_α. Suppose U_α contracts to Gy. Without loss of generality one can suppose that y is connected by a path pointwise invariant under G_y to x. By comparing the restrictions of ξ and of the various ξ_β to $R(G_y)$ as one moves along the path from x to y one finds that ξ must restrict to ξ_α. This completes all the proofs in motion.

Chapter Six

DIRECT-IMAGE MAPS

6.1 The finite case

Suppose that the G-map $f : X \to Y$ is a finite covering (i.e. f is locally trivial with finite fibres). Then the functor f^* has a right-adjoint $f_* : G\text{-}\underline{\text{Vect}}\,(X) \to G\text{-}\underline{\text{Vect}}\,(Y)$,

i.e. $$\text{Hom}_X^G\,(f^*E';\, E) \;\cong\; \text{Hom}_Y^G\,(E';\, f_*E).$$

The existence of f_* is very easy to see: the fibre of f_*E at $y \in Y$ will be $\coprod_{x \in f^{-1}y} E_x$; I shall not give the details.

It happens that f_* is also left-adjoint to f^*, in consequence of the isomorphism $\coprod_{x \in f^{-1}y} E_x \xrightarrow{\cong} \prod_{x \in f^{-1}y} E_x$. This implies the following proposition.

Proposition 6.1.1 If $E \in G\text{-}\underline{\text{Vect}}\,(X)$, $E' \in G\text{-}\underline{\text{Vect}}\,(Y)$, then

$$f_*(E \otimes f^*E') \;\cong\; (f_*E) \otimes E'.$$

Proof: For any F in $G\text{-}\underline{\text{Vect}}\,(Y)$ one has

$$\begin{aligned}\mathrm{Hom}^G_Y(f_*(E\otimes f^*E');F) &\cong \mathrm{Hom}^G_X(E\otimes f^*E';f^*F)\\ &\cong \mathrm{Hom}^G_X(E;f^*\underline{\mathrm{Hom}}_Y(E';F))\\ &\cong \mathrm{Hom}^G_Y(f_*E;\underline{\mathrm{Hom}}_Y(E';F))\\ &\cong \mathrm{Hom}^G_Y((f_*E)\otimes E';F).\end{aligned}$$

f_* is an additive functor - it suffices to consider the matter pointwise - so it induces a homomorphism $f_! : K^*_G(X) \to K^*_G(Y)$ of abelian groups. Proposition 6.1.1 shows that $f_!(x.f^!y) = f_!x.y$, which means that $f_!$, while not a homomorphism of rings, is a homomorphism of $K^*_G(Y)$-modules.

6.2 Group extensions of finite index

The most important application of f_* is to the finite covering $f : \phi_*\phi^*X \cong (G/H)\times X \to X$, where $\phi : H \to G$ is the inclusion of a subgroup of finite index, and X is a G-space. In 3.4 I defined a functor $\phi_* : H$-$\underline{\mathrm{Vect}}$ $(\phi^*X) \to G$-$\underline{\mathrm{Vect}}$ $(\phi_*\phi^*X)$; the composite $f_*\circ\phi_* : H\text{-}\mathrm{Vect}(\phi^*X) \to$ G-$\underline{\mathrm{Vect}}$ (X) will be called $\phi_!$. It induces $\phi_! : K^*_H(\phi^*X) \to K^*_G(X)$.

It is easy to verify

<u>Proposition 6.2.1</u> If $E \in G$-<u>Vect</u> (X), $E' \in H$-<u>Vect</u> (ϕ^*X), then

$$\phi_!(E' \otimes \phi^*E) \cong (\phi_!E') \otimes E.$$

In particular $\phi_!\phi^*E \cong (\phi_!\underline{\mathbb{C}}) \otimes E$.

The next task is to examine $\psi^*\phi_!E$. In fact I shall give a formula for $\psi^*\phi_!E$, where $\phi : H_1 \to G$, $\psi : H_2 \to G$ are two inclusions of finite index. I shall write $H_g = g H_1 g^{-1} \cap H_2$ for any $g \in G$; but it depends only on the double coset H_2gH_1 in $H_2 \backslash G/H_1$. H_g has evident embeddings:

$$\begin{tikzcd}
 & H_1 \arrow[dr, "\phi"] & \\
H_g \arrow[ur, "\phi_g"] \arrow[dr, "\phi_g"'] & & H \\
 & H_2 \arrow[ur, "\phi"'] &
\end{tikzcd}.$$

$\phi_!E = f_*\phi_*E$, where $f : G \times_{H_1} X \to X$. Restriction of the group commutes with f_*, so $\psi^*\phi_! = f_*\psi^*\phi_*E$. As an H_2-space,

$G \times_{H_1} X \cong \coprod_{g \in H_2\backslash G/H_1} (H_2gH_1) \times_{H_1} X$, and correspondingly $f_* = \Sigma\, f_{g*}\, i_g^*$,

where $i_g : (H_2gH_1) \times_{H_1} X \to G \times_{H_1} X$ and $f_g : (H_2gH_1) \times_{H_1} X \to X$. Now

$i_g^*\psi^*\phi_*E = (H_2gH_1) \times_{H_1} E \xrightarrow{\cong} H_2 \times_{H_g} \phi_g^*E = \phi_{g*}\,\phi_g^*E$; the isomorphism being $(h,\xi) \longmapsto (hg,\xi)$. Thus

<u>Proposition 6.2.2</u> If $\phi : H_1 \to G$, $\psi : H_2 \to G$ are inclusions of finite index, X is a G-space, and $E \in H_1$-<u>Vect</u> (X), then

$$\psi^*\phi_! E \cong \coprod_{g \in H_2 \backslash G/H_1} \phi_{g!}\, \psi_g^* E, \quad \text{with } \phi_g, \psi_g \text{ as above.}$$

<u>Corollary 6.2.3</u> If $\phi : N \to G$ is the inclusion of a normal subgroup of finite index in G, X is a G-space, and $E \in N$-<u>Vect</u> (X), then

$$\phi^*\phi_! E \cong \coprod_{\sigma \in G/N} \sigma^* E, \quad \text{where } \sigma^* \text{ is induces by conjugation } \sigma : (N,X) \to (N,X).$$

6.3 <u>Brauer's theoremfor finite groups</u>

In this section G is a finite group, X is a compact G-space, and Σ is the set of elementary subgroups of G. (See 2.5.)

<u>Proposition 6.3.1</u> $K_G(X) \to \prod_{H \in \Sigma} K_H(X)$ is injective.

<u>Proof</u>: This follows from the injectivity of $K_G(X) \to \prod_{\gamma \in \operatorname{Spec} R(G)} K_G(X)_\gamma$ and the following lemma.

<u>Lemma 6.3.2</u> If $\gamma \in \operatorname{Spec} R(G)$ has support S (a cyclic subgroup of G) and lies over $p \in \operatorname{Spec} \mathbb{Z}$, and P is a Sylow p-subgroup of the centraliser of S,

then the restriction $K_G(X)_{\gamma} \to K_{S\times P}(X)_{\gamma}$ is injective.

Proof: If $\phi : S \times P \to G$, $x \in K_G(X)_{\gamma}$, and $y \in R(S \times P)_{\gamma}$, then $\phi_!(y.\phi^* x) = (\phi_! y).x$. But in 2.5 I showed that one could find $y \in R(S \times P)$ such that $\phi_! y \notin \gamma$, so $\phi_! y$ is invertible in $K_G(X)_{\gamma}$, and $(\phi_! y)^{-1}\phi_!(y.\phi^* x) = x$, i.e. ϕ^* is injective.

Proposition 6.3.3 (Brauer's Theorem)

(a) $\coprod_{H\in\Sigma} K_H(X) \to K_G(X)$ is surjective, and

(b) $K_G(X) \xrightarrow{\cong} \varprojlim_{H\in\Sigma} K_H(X)$;

where the map (a) is the sum of the direct-image maps; and the map (b) is restriction, and Σ is regarded as a category whose morphisms are all the homomorphisms $\phi : H \to H'$ of the form $\phi(h) = ghg^{-1}$, with $g \in G$.

Proof: (a) is clear, for the map is a homomorphism of $K_G(X)$-modules whose image contains 1: for the image $\coprod R(H) \to R(G)$ already contains 1, being an ideal not contained in any prime ideal.

To prove (b) surjective (it is injective by 6.3.1) it suffices to show that the image is an ideal in $\varprojlim K_H(X)$. But by (a) the image is the group generated by $\{\phi_H^* \phi_{F!}\, x\}_{H\in\Sigma}$ in $\prod K_H(X)$, where $\phi_H : H \to G$,

$\phi_F : F \to G$, and F runs through Σ. It certainly suffices to show that $y_H\, \phi_H^* \phi_{F!}\, x = \phi^*_H\, \phi_{F!}\,(y_F x)$ for every $\{y_H\}_{H\in\Sigma} \in \varprojlim K_H(X)$. Now

$$y_H\, \phi_H^*\, \phi_{F!}\, x = \sum_{g\in H\backslash G/F} y_H\, \phi_{g!}\, \psi_g^*\, x \text{ by 6.2.2, i.e. } \sum \phi_{g!}(\phi_g^*\, y_H \cdot \psi_g^*\, x).$$

But $\phi_g^*\, y_H = \psi_g^*\, y_F$ because $\{y_H\} \in \varprojlim K_H(X)$, so the last expression is

$$\sum \phi_{g!}\, \psi_g^*\,(y_F \cdot x) = \phi_H^*\, \phi_{F!}\,(y_F \cdot x), \text{ as required.}$$

6.5 The general case

The object of the remainder of this chapter is to generalise the direct-image homomorphism $f_!$ which was associated in 6.1 to a finite covering $f : X \to S$. I shall only deal with fibrations with a compact smooth manifold as fibre. Then there is a tangent-bundle-along-the-fibres $T_{X/S}$, a real G-vector-bundle on X, and it turns out that there is a natural homomorphism $f_a : K_{G,\mathrm{prop}}(T^*_{X/S}) \to K_G(S)$, where 'prop' means the family of closed sets of $T^*_{X/S}$ proper over X. (When X is finite over S $T_{X/S}$ is just X, and f_a is $f_!$ as already defined.) When the fibration satisfies suitable conditions, in particular when the fibre is a complex manifold, one can compose f_a with the Thom homomorphism $\phi : K_G(X) \to K_{G,prop}(T^*_{X/S})$ to obtain $f_! : K_G(X) \to K_G(S)$.

Broadly speaking f_a is defined as follows. An element of $K_{G,prop}(T_{X/S})$ is interpreted as the symbol of a family (determined up to homotopy) of elliptic pseudo-differential operators on the fibres of f. This family of operators on X defines a Fredholm complex on S, and so an element of $K_G(S)$ by 4.5. But quite a lot of technical discussion is necessary, and the next section is devoted to that.

6.6 Relative manifolds and pseudo-differential operators

A relative manifold over the G-space S is a G-space $X \xrightarrow{f} S$ over S together with a G-sheaf D_X of complex-valued functions on X called smooth functions, and such that (X, D_X) is a ringed-space over S locally isomorphic to $(S \times \mathbb{R}^n, D_{S\times\mathbb{R}^n})$, where $D_{S\times\mathbb{R}^n}$ is the sheaf on $S \times \mathbb{R}^n$ defined on basic open sets by $D_{S\times\mathbb{R}^n}(U \times V) = Map(U; D(V))$, $D(V)$ being the space of smooth functions on $V \subset \mathbb{R}^n$, with its usual topology.

For example, if X is a fibre-bundle over S with a smooth manifold M as fibre, and the structure-group is a Lie group L which acts smoothly on

M, then X is a relative manifold. One has only to verify that a smooth map $h : A \times B \to C$, where A, B, C are open sets of euclidean space, induces a continuous map $A \times D(C) \to D(B)$.

I think it will be clear what I mean by a smooth G-vector-bundle on a relative manifold. The relative tangent-bundle $T_{X/S}$, or tangent-bundle-along-the-fibres, is an example of a smooth real G-vector-bundle: a relative tangent-vector is defined as an operator on the smooth functions as usual, but one requires it to annihilate functions induced from S. A real smooth function g on X defines a function dg on $T_{X/S}$; dg will usually be regarded as a section of the dual bundle $T^*_{X/S}$.

Now I must introduce pseudo-differential operators (cf [21], [4]). If E and F are smooth vector-bundles on X then a pseudo-differential operator P or order n from E to F is a continuous linear map $P : \Gamma_{cpt}E \to \Gamma F$, where Γ indicates smooth sections and Γ_{cpt} those with compact support, such that

(i) $P(us) = uP(s)$ if u is a function induces from S, and

(ii) there is an asymptotic expansion

$$e^{-i\lambda g}P(e^{i\lambda g}s) \sim \sum_{k\leq n} P_k(s,g)\lambda^k$$

as $\lambda \to \infty$ in $\mathbb{R}_+$ for any real smooth function g for which $dg \neq 0$ on the support of s, uniform on compact sets of suitable g's.

I shall write $Ps^n(E; F)$ for the vector-space of such operators. One has $Ps^n(E; F) \supset Ps^{n-1}(E; F) \supset \ldots \supset Ps^{-\infty}(E; F)$; an element of $Ps^{-\infty}(E; F)$ is called a 'smoothing operator'.

The first important property of pseudo-differential operators is

<u>Proposition 6.6.1</u> $\qquad Ps^n(E; F)/Ps^{n-1}(E; F) \;\tilde{=}\; \mathrm{Map}^n_X(T^*_X - 0; \underline{\mathrm{Hom}}_X(E; F))$, canonically, where $T^*_X - 0$ means the complement of the zero-section, and Map^n_X means the maps which are positively homogeneous of degree n.

The isomorphism is given by assigning to $P \in Ps^n(E; F)$ the map σ_P derived from the first term of the hypothesized asymptotic expansion, defined by $\sigma_P(\xi)(e) = P_n(s_e, g_\xi)(x)$, where $x = pr(\xi)$, s_e in $\Gamma_{opt}E$ has the value e at x, and g_ξ is a function which in a local identification at x of X with $S \times T_{X/S,x}$ is given by $g_\xi(s,y) = \langle y,\xi\rangle$. σ_P is called the <u>symbol</u> of P. It is not obvious that it is well-defined, or that one can find P for any prescribed σ_P, as asserted in 6.6.1, but I shall not prove these facts here. A proof in the local non-relative case can be found in [21]; the generalization needed is routine but tedious.

Now let us suppose that X and S are compact. Then pseudo-differential operators P in $Ps^n(E; F)$ and Q in $Ps^m(F; G)$ can be composed, and it is fairly easy to see that QP is in $Ps^{n+m}(E; G)$. Furthermore $\sigma_{QP} = \sigma_Q \sigma_P$, where the last composition refers to the natural pairing $\underline{\mathrm{Hom}}_X(E; F) \times \underline{\mathrm{Hom}}_X(F ; G) \to \underline{\mathrm{Hom}}_X(E; G)$; this is immediate from the definition provided one knows that the symbols are well-defined.

The second result about pseudo-differential operators that I shall need is:

<u>Proposition 6.6.2</u> The vector-space $Ps^n(E; F)/Ps^{-\infty}(E; F)$ is complete with respect to its filtration by the subspaces $Ps^k(E; F)/Ps^{-\infty}(E; F)$ $(k \leq n)$.

Once again the proof is essentially contained in [21], and I shall not discuss it further here.

I shall apply 6.6.1 and 6.6.2 to the following purpose. If $P \in Ps^n(E; F)$ and $\sigma_P(\xi)$ is an isomorphism for all $\xi \in T^*_{X/S} - 0$, then P is said to be <u>elliptic</u>. Then - if X and S are compact - one can find $\sigma_Q : (T^*_{X/S} - 0) \to \underline{\mathrm{Hom}}_X(F; E)$, positively homogeneous of degree $-n$, such that $\sigma_P \sigma_Q = 1$, $\sigma_Q \sigma_P = 1$. By 6.6.1 one has then Q_1 such that $Q_1 P - 1 \in Ps^{-1}(E; E)$ and $PQ_1 - 1 \in Ps^{-1}(F; F)$. Proceeding inductively, then using 6.6.2, one can find $Q \in Ps^{-n}(F; E)$ such that $QP = 1 - S$ and $PQ = 1 - S'$, with $S \in Ps^{-\infty}(E; E)$ and $S' \in Ps^{-\infty}(F; F)$.

Now suppose that X is locally trivial on S. Then to each smooth G-vector-bundle E on X one can associate f_*E, an infinite G-vector-bundle on S in the sense of 4.5; the fibre of f_*E at s in S is $\Gamma(E|f^{-1}s)$, and sections of f_*E over an open set V of S correspond precisely to smooth sections of $E|f^{-1}V$. If X and S are compact then a pseudo-differential operator P from E to F defines a morphism $P_* : f_*E \to f_*F$ of infinite vector-bundles.

<u>Proof</u>: (i) P_* is well-defined, for if e, e' are two sections of E coincident on $X_s = f^{-1}s$ one can write $e - e' = \phi.e''$, where e" is a third section and ϕ is a smooth function induced from S vanishing on X_s. Then $(P_*e - P_*e')|X_s = (\phi.P_*e'')|X_s = 0$.

(ii) P_* is continuous: the question is local, so one can assume $X = S \times M$, $E = S \times E_o$, $F = S \times F_o$. By definition one has a continuous map $\Gamma E = \mathrm{Map}(S; \Gamma E_o) \to \Gamma F = \mathrm{Map}(S; \Gamma F_o)$, and it induces $S \times \mathrm{Map}(S; \Gamma E_o) \to \Gamma F_o$. But $P_* : S \times \Gamma E_o \to \Gamma F_o$ is obtained from the last by reducing to the quotient $S \times \Gamma E_o$ of $S \times \mathrm{Map}(S; \Gamma E_o)$, so P_* is continuous.

This having been said, the third fact I need concerning pseudo-differential operators is

<u>Proposition 6.6.3</u> If X and S are compact then a smooth operator P from E to F induces a compact morphism $P_* : f_*E \to f_*F$.

<u>Proof</u>: Again the question is local, and P_* can be regarded as a map $S \times \Gamma E_0 \to \Gamma F_0$. The argument of [21] shows that it is of the form $(s,u) \longmapsto v$, where $v(x) = \int_M k(s,x,y)u(y)dy$ and k is a smooth section of the bundle $\underline{\mathrm{Hom}}_{S\times M\times M}(pr_2^* E; pr_1^* F)$. But by inspection this takes a neighbourhood of the origin in ΓE_0 into a subset of ΓF_0 bounded uniformly for s in S. Bounded subsets of ΓF_0 are compact, so P_* is compact.

6.6.3 shows that an elliptic pseudo-differential operator P from E to F induces a Fredholm-complex $f_*E \xrightarrow{P_*} f_*F$ on S, in the sense of 4.5. If E and F are smooth G-vector-bundles on X and P is equivariant then one obtains a G-Fredholm-complex on S: the homotopy-operator can be made into a G-morphism simply by averaging over the group, for f_*E is compactly generated and so $\mathrm{Hom}(f_*E; f_*F)$ is complete.

But because it is not clear that averaging pseudo-differential operators leads to a pseudo-differential operator I have had to use a devious argument to prove the following proposition.

Proposition 6.6.4 If $\sigma : (T^*_{X/S} - 0) \to \underline{\mathrm{Hom}}_X(E; F)$ is an equivariant symbol, positively homogeneous of degree n, then one can find an equivariant pseudo-differential operator P whose symbol is σ.

Proof: Let P_0 be a p.d.o. of order n whose symbol is σ, and let $g.P_0$ be its transform by $g \in G$. Then $g.P_0 - P_0$ is a p.d.o. of order $n - 1$ whose symbol $\sigma_1(g)$ depends continuously on g. (The continuity results from the continuity in the arguments (s,h) of the functions $P_k(s,h)$ which occur in the asymptotic expansion defining a p.d.o.). Furthermore one has: $\sigma_1(gg') = g.\sigma_1(g') + \sigma_1(g)$. Average $\sigma_1(g)$ over the group to obtain $\bar{\sigma}_1$; then $\bar{\sigma}_1 - g.\bar{\sigma}_1 = \sigma_1(g)$. Let P_1 be a p.d.o. with symbol $\bar{\sigma}_1$. $P_0 + P_1$ is a p.d.o. with symbol σ, and it has the property that $g.(P_0 + P_1) - (P_0 + P_1)$ is a p.d.o. of order $n - 2$ (for its symbol as an operator of order $n - 1$ is zero). Proceeding inductively, then using 6.6.2, one finds a p.d.o. P' with symbol σ such that $g.P' - P'$ is a smoothing-operator for all g in G. Finally one obtains the desired equivariant operator when one observes that smoothing-operators can, like symbols, be averaged, using the isomorphism $Ps^{-\infty}(E; F) \cong \mathrm{Hom}_{X \times_S X}(pr_1^*E; pr_2^*F)$

already remarked. One needs to know that $g.P' - P'$ depends continuously on g in the last space: it suffices to see that $g \mapsto g.P' - P'$ defines a smoothing-operator on $G \times X$ over $G \times S$, and that follows from the uniformity required in the definition of a p.d.o.

Now let us define $S_{X/S} = (T^*_{X/S} - 0)/\mathbb{R}^*_+$, a G-sphere-bundle on X, and $\hat{S}_{X/S} = ((T^*_{X/S} \oplus \underline{\mathbb{R}}) - 0)/\mathbb{R}^*_+$, a G-sphere-bundle of dimension one higher, with privileged "polar" sections, and containing $S_{X/S}$ as its "equator".

Let E and F be smooth G-vector-bundles on X, and $\sigma : pr^*E \to pr^*F$ an isomorphism, where $pr : S_{X/S} \to X$. σ can be regarded as the symbol of an equivariant elliptic p.d.o. of order 0 between E and F, and such an operator defines a G-Fredholm complex on S and hence an element of $K_G(S)$. The element does not depend on the particular equivariant p.d.o. with symbol σ selected, as two such are linearly homotopic through equivariant elliptic p.d.o.'s. Thus to the triple (E,F,σ) is associated a well-defined element $f(E,F,\sigma)$ of $K_G(S)$. On the other hand (E,F,σ) defines in an obvious way a G-vector-bundle $cl(E,F,\sigma)$ on $\hat{S}_{X/S}$. I shall show

(i) $f(E,F,\sigma)$ depends only on the G-vector-bundle $cl(E,F,\sigma)$, and

(ii) every G-vector-bundle on $\hat{S}_{X/S}$, disregarding its differentiable structure, is isomorphic to a bundle of the form $cl(E,F,\sigma)$.

Let (i) and (ii) be granted for the moment; I shall construct the homomorphism $f_! : K_{G,\mathrm{opt}}(T^*_{X/S}) \to K_G(S)$. By associating $f(E,F,\sigma)$ to $\mathrm{cl}(E,F,\sigma)$ one obtains a function from G-vector-bundles on X to $K_G(S)$ which is plainly additive. So one has a homomorphism $K_G(\hat{S}_{X/S}) \to K_G(S)$. But $K_G(\hat{S}_{X/S}) \cong K_G(X) \oplus K_G(\hat{S}_{X/S},X) \cong K_G(X) \oplus K_{G,\mathrm{opt}}(T^*_{X/S})$, where X is identified with a polar section in $\hat{S}_{X/S}$. The summand $K_G(X)$ in $K_G(\hat{S}_{X/S})$ is generated by the bundles $\mathrm{cl}(E,E,\mathrm{id})$, and $f(E,E,\mathrm{id}) = 0$ as it is represented by the Fredholm-complex $f_*E \xrightarrow{\mathrm{id}} f_*E$. So one has the required $f_! : K_{G,\mathrm{opt}}(T^*_{X/S}) \to K_G(S)$. $f_!$ is a homomorphism of $K_G(S)$-modules - for if Q is a G-vector-bundle on S a Fredholm-complex representing $f(E \otimes f^*Q,\ F \otimes f^*Q,\ \sigma \otimes \mathrm{id})$ is obtained by tensoring a complex representing $f(E,F,\sigma)$ with Q. (Of course $f_*(E \otimes f^*Q) \cong (f_*E) \otimes Q$.)

It remains to prove the statements (i) and (ii); but I shall need two lemmas.

<u>Lemma 6.6.5</u> If E is a smooth G-vector-bundle on a compact relative manifold X over S then any continuous section of E can be approximated uniformly by smooth sections, and any continuous equivariant section by a smooth equivariant section.

<u>Proof</u>: In the case $E = \underline{\underline{C}}$ the first statement follows from the Stone-Weierstrass theorem. But one can construct a smooth partition of unity subordinated to a covering by coordinate neighbourhoods, so the general case follows by patching together. The statement about equivariant sections is obtained by averaging.

<u>Lemma 6.6.6</u> (a) If $X \to S$ is a compact relative G-manifold and E is a G-vector-bundle on X then there is a smooth G-vector-bundle E' on X isomorphic to E.

(b) If E_1, E_2 are two smooth G-vector-bundles on X which are isomorphic when one disregards the differentiable structure, then they are isomorphic as smooth G-vector-bundles.

<u>Proof</u>: (a) E is the image of a projection $p : \underline{\underline{M}} \to \underline{\underline{M}}$, where M is a G-module. p corresponds to a G-map $\tilde{p} : X \to \mathrm{End}(M)$ such that $\tilde{p}(X) \subset \mathrm{Proj}(M)$. But Proj(M) is a G-retract of an open set $U \subset \mathrm{End}(M)$, and the retraction given explicitly in 3.5 is smooth. Approximate $\tilde{p}$ by a smooth G-map $X \to U$ which is G-homotopic in U to $\tilde{p}$. By applying the retraction one gets a smooth G-map $\tilde{p}' : X \to \mathrm{Proj}(M)$ homotopic in Proj(M) to $\tilde{p}$. $\tilde{p}'$ defines a projection in $\underline{\underline{M}}$ whose image E' is a smooth G-vector-

bundle isomorphic to E.

(b) I shall leave to the reader.

<u>Proof of (i) above</u>: $cl(E,F,\sigma)$ is isomorphic to $cl(E',F',\sigma')$ if and only if $E \cong E'$ and $F \cong F'$ in such a way that σ becomes continuously homotopic to σ'. As isomorphism implies smooth isomorphism one can suppose $E = E'$ and $F = F'$. However $cl(E,F,\sigma) \cong cl(E,F,\sigma')$ if and only if $\sigma \simeq \sigma'$, by an argument already used in 4.2. The homotopy between σ and σ' can be approximated by a smooth homotopy, whose ends σ_0 and σ_1 can be supposed linearly (smoothly) homotopic to σ and σ' respectively: so altogether one can suppose σ smoothly homotopic to σ'. But then the Fredholm-complexes representing $f(E,F,\sigma)$ and $f(E,F,\sigma')$ will be homotopic, so $f(E,F,\sigma) = f(E,F,\sigma')$ as required.

<u>Proof of (ii) above</u>: Any bundle on $\hat{S}_{X/S}$ can be obtained by clutching pr^*E and pr^*F, where E and F are its restrictions to the polar sections of $\hat{S}_{X/S}$. One can suppose that E and F are smooth bundles. Approximate the clutching isomorphism $\tilde{\sigma}$ by a smooth isomorphism σ. One can suppose that σ is homotopic to $\tilde{\sigma}$, so it will define the same bundle on $\hat{S}_{X/S}$.

Before I leave this section I should make the following remark. If one chooses a metric in the cotangent-bundle $T^*_{X/S}$ then one can identify the sphere-bundle $S_{X/S}$ with a subspace of $T^*_{X/S}$. If E and F are smooth G-vector-bundles on X and $\sigma : pr^*E \to pr^*F$ is an isomorphism ($pr : S_{X/S} \to X$), then σ can be extended to a symbol $\sigma : (T^*_{X/S} - 0) \to \underline{\mathrm{Isom}}_X(E;F)$ positively homogeneous of degree k for any integer k. (The case already discussed is k = 0.) Thus it happens that one can define homomorphisms $f_a^{(k)} : K_{G,\mathrm{cpt}}(T^*_{X/S}) \to K_G(S)$ for any k. I think that $f_a^{(k)}$ is independent of k, though I have no use for that fact in this thesis. At any rate $f_a^{(k)}$ depends at most on the parity of k: if $f_a^{(k)}(\xi)$ is represented by a Fredholm-complex $f_*E \xrightarrow{A} f_*F$ then $f_a^{(k+2)}(\xi)$ can be represented by $f_*E \xrightarrow{A\Delta} f_*F$, where $\Delta : f_*E \to f_*E$ is the Laplace operator; A and $A\Delta$ define the same element of $K_G(S)$ in view of the following two facts.

(a) If $U \xrightarrow{A} V$ and $V \xrightarrow{B} W$ are Fredholm-complexes defining elements [A], [B] of $K_G(S)$ then $U \xrightarrow{BA} W$ is a Fredholm-complex defining the element [A] + [B].

(b) The Laplace operator Δ has a kernel and cokernel which are G-vector-bundles and which are isomorphic because Δ is self-adjoint. Thus $[\Delta] = 0$.

6.7 An application: the Thom isomorphism

In this section I shall suppose that X and S are compact and that $f : X \to S$ is a relative G-manifold locally trivial on S with fibre a compact complex manifold. I do not imply that f is a relative complex manifold, but I do require G to act holomorphically in the sense that $g : f^{-1}(s) \to f^{-1}(Gs)$ is a holomorphic map for any g in G and s in S. Then $T^*_{X/S}$ has a natural structure of complex G-vector-bundle on X, and there is a Thom homomorphism $\phi : K_G(X) \to K_{G,\mathrm{cpt}}(T^*_{X/S})$.

We have seen that $\phi(1)$ is defined by the complex of G-vector-bundles $\underline{\mathbb{C}} \xrightarrow{d} pr^*T \xrightarrow{d} \Lambda^2 pr^*T \xrightarrow{d} \cdots$ on $T = T^*_{X/S}$, where $pr : T \to X$. Everyone knows that it can also be defined by the complex with two terms $\coprod_i \Lambda^{2i} pr^*T \to \coprod_i \Lambda^{2i+1} pr^*T$, where the map is now $d + d^*$, d^* being the adjoint of d with respect to any hermitian metrics in the bundles. An indirect proof of this is given in [7], but I shall digress to prove it directly.

<u>Proposition 6.7.1</u> If the complex $E^\cdot = (E^0 \xrightarrow{d} E^1 \xrightarrow{d} \cdots \xrightarrow{d} E^n)$ represents an element ξ of $K_G(X,A)$, then ξ is also represented by $\coprod E^{2i} \xrightarrow{d+d^*} \coprod E^{2i+1}$.

<u>Proof</u>: It suffices to show that ξ is represented by the complex

$$F^{\cdot} = (E^0 \xrightarrow{d} \cdots \xrightarrow{d} E^{n-3} \xrightarrow{d} E^{n-2} \oplus E^n \xrightarrow{d+d^*} E^{n-1}).$$

Consider the complex associated to the double complex

$$\begin{tikzcd}
E^0 \arrow[r, "d"] & \cdots \arrow[r, "d"] & E^{n-2} \arrow[r, "d"] & E^{n-1} \arrow[r, "td"] & E^n \\
 & & & E^n \arrow[u, "d^*"] \arrow[r, "t.id"] & E^n \arrow[u, "dd^*"']
\end{tikzcd}$$

for t in $\mathbb{R}$. When $t = 1$ it is isomorphic to $C_G(X,A)$ to $E^{\cdot}$. Furthermore for any t in $\mathbb{R}$ it is acyclic whenever $E^{\cdot}$ is acyclic. And when $t = 0$ it is the direct sum of $F^{\cdot}$ and the complex $(E^n \xrightarrow{dd^*} E^n)$, which represents zero as it is linearly homotopic to $(E^n \xrightarrow{id} E^n)$.

Returning to the situation in hand, one sees that $d + d^*$ amounts to a map $(T-0) \to \underline{\mathrm{Isom}}_X(\coprod \Lambda^{2i}T; \coprod \Lambda^{2i+1}T)$ which is homogeneous of degree 1, and which is in fact on each manifold $f^{-1}(s)$ the symbol of the differential operator $\bar\partial + \bar\partial^*$ familiar in the theory of complex manifolds: $\bar\partial$ is the operation of exterior differentiation in the antiholomorphic coordinates, taking forms of type $(0,q)$ to forms of type $(0,q+1)$; $\bar\partial^*$ is its adjoint. Thus it is natural to look at the homomorphism $f_!^{(1)} = f_a^{(1)}.\phi : K_G(X) \to K_G(S)$. I shall discuss it only in the following very special but extremely important case.

<u>Proposition 6.7.2</u> If, in the situation above, the fibres of $f : X \to S$

are rational algebraic varieties then $f_!^{(1)}f^! : K_G(S) \to K_G(S)$ is the identity, so $f^!$ embeds $K_G(S)$ as a direct summand in $K_G(X)$.

Proof: As $f_!^{(1)}$ is a homomorphism of $K_G(S)$-modules it is enough to show that $f_!^{(1)}f^!(1) = 1$, i.e. that $f_!^{(1)} = f_a^{(1)}\phi(1) = 1$. We have just seen that one can use the Fredholm-complex

$$\coprod f_*(\Lambda^{2i}T^*_{X/S}) \xrightarrow{\bar\partial + \bar\partial^*} \coprod f_*(\Lambda^{2i+1}T^*_{X/S})$$

to represent $f_a^{(1)}\phi(1)$; but it is well-known that if V is a rational algebraic variety the map $\bar\partial + \bar\partial : \coprod \Gamma(\Lambda^{2i}T^*_V) \to \coprod \Gamma(\Lambda^{2i+1}T^*_V)$ is surjective, and its kernel is $\mathbb{C}$, regarded as the constant sections of $\Lambda^0T^*_V$. It follows that $f_a^{(1)}\phi(1)$ is represented equally well by the subcomplex $\underline{\underline{\mathbb{C}}} \to 0$ of the Fredholm-complex, and thus $f_!^{(1)}(1) = 1$, as required.

The most important rational algebraic varieties are those of the form H/T, where H is a compact connected Lie group and T is a maximal torus of H. They occur in the following two important consequences of 6.7.2.

Proposition 6.7.3 If G is a compact connected Lie group, and T is a maximal torus, then for any compact G-space X the restriction $K_G(X) \to K_T(X)$ embeds $K_G(X)$ as a direct summand.

Proof: If one identifies $K_T(X)$ with $K_G(G \times_T X)$ then the restriction map in question is identified with $f^!$, where $f : G \times_T X \to X$. But as X is a G-space one has $G \times_T X \cong (G/T) \times X$, so one can apply 6.7.2.

Finally we have the general Thom isomorphism theorem.

Proposition 6.7.4 If E is a G-vector-bundle on a compact G-space X then the Thom homomorphism $\phi : K_G(X) \to K_{G,\mathrm{opt}}(E)$ is an isomorphism.

Proof: E can be written $(P \times \mathbb{C}^n)/U$, where U is the unitary group $U(n)$, and P is a compact $(G \times U)$-space on which U acts freely. (I assume without loss of generality that E has constant fibre-dimension.) Let T be a maximal torus of U, and let $f : P/T \to X = P/U$ be the projection. One has $f^*E \cong (P \times \mathbb{C}^n)/T$ which is evidently the sum of n G-line-bundles. So we know from 4.8 that the Thom homomorphism $\phi : K_G(P/T) \to K_{G,\mathrm{opt}}(f^*E)$ is an isomorphism. Now consider the diagram

$$\begin{tikzcd}
K_G(X) \arrow[r, "f^!"] \arrow[d, "\phi"'] & K_G(P/T) \arrow[d, "\phi", "\cong"'] \\
K_{G,\mathrm{opt}}(E) \arrow[r, "f^!"] & K_{G,\mathrm{opt}}(f^*E).
\end{tikzcd}$$

It commutes in view of the evident functoriality-property $f^!(\lambda_E) = \lambda_{f^*E}$.

$f : P/T \to X$ is a fibration with fibre U/T and the Lie group U as structure-group, so one can apply 6.7.2, and has $f_!$ such that $f_! f^! = \mathrm{id}$.
$f^! : K_{G,\mathrm{opt}}(E) \to K_{G,\mathrm{opt}}(f^*E)$ is likewise split, as it is a direct summand of $f^! : K_G((P \times S^{2n})/U) \to K_G((P \times S^{2n})/T)$. So the Thom isomorphism for f^*E implies that for E.

Chapter Seven

THE COMPLETION OF $K_G(X)$

This chapter, I am afraid, is disappointing. I shall prove, for example, the following: if $f : Y \to X$ is a G-map of finite-dimensional compact G-spaces which is a homotopy-equivalence, but not necessarily a G-homotopy-equivalence, then, provided $K_G^*(X)$ and $K_G^*(Y)$ are finite over $R(G)$, the map $f^! : K_G^*(X) \to K_G^*(Y)$ induces an isomorphism of the I_G-adic completions of the two modules (cf. 2.5). I shall show also that for such X and Y the I_G-adic topology on $K_G^*(X)$ and $K_G^*(Y)$ coincides with the filtration-topology defined in 5.4. At first this seems interesting enough; but I don't even know a non-trivial example, let alone an application.

The result just mentioned is deduced from the following, which is a little more significant. Let X be a finite-dimensional compact G-space such that $K_G^*(X)$ is finite over $R(G)$, and let EG be the universal space for G. Then the I_G-adic topology and the filtration-topology coincide on both $K_G^*(X)$ and $K_G^*(X \times EG)$, the latter is complete, and the natural map $pr^! : K_G^*(X) \to K_G^*(X \times EG)$ induces an isomorphism $K_G^*(X)^\wedge \xrightarrow{\cong} K_G^*(X \times EG)$. Because $K_G^*(X \times EG) \cong K^*(X_G)$, where $X_G = (X \times EG)/G$, this has the effect of expressing the completion of $K_G^*(X)$ as the ordinary K^* of a space. It is a generalisation of the theorem of Atiyah-Hirzebruch that $R(G)^\wedge \xrightarrow{\cong} K^*(BG)$; my result reduces to theirs when X is a point. The theorem obviously has some content, but I know of no application, and so I have not tried very hard to remove the ugly restriction, or to settle the question whether the I_G-adic topology or the filtration-topology is the "right" one on $K_G^*(X)$. (But surely it should be the latter ...)

As we are operating in the realm of pure art, I shall state my theorem in the following form, which seems to me elegant, and afterwards I shall deduce the statements above.

Let $\mathcal{C}_X$ be the category of compact free G-spaces over X.

<u>Proposition 7.1.1</u> If $K_G^*(X)$ is a finite $R(G)$-module, then

(i) $K_G^*(X)^\wedge \xrightarrow{\cong} \varprojlim_Y K_G^*(Y)^\wedge$, and

(ii) $R' \varprojlim_Y K_G^*(Y)^\wedge = 0$,

where Y runs through the category $\mathcal{C}_X$, and all completions refer to the I_G-adic topology.

Before proceeding with the proof observe that it is permissible, and will be convenient, to redefine $\mathcal{C}_X$ as the category of compact free G-spaces over X and <u>G-fibre-homotopy-classes</u> of maps over X. Recall also that a functor $i : \mathcal{C}_0 \to \mathcal{C}$ is called cofinal if

(i) each object of $\mathcal{C}$ admits a morphism into an object $i(Y)$, $Y \in \mathcal{C}_0$.

and (ii) whenever one has $f_1 : P \to i(Y_1)$ and $f_2 : P \to i(Y_2)$ one can find $g_1 : Y_1 \to Y$ and $g_2 : Y_2 \to Y$ in $\mathcal{C}_0$ such that $i(g_1) \circ f_1 = i(g_2) \circ f_2$.

It is then a trivial exercise to show that the natural map $i^* : \varprojlim_{\mathcal{C}} F \to \varprojlim_{\mathcal{C}_0} F \circ i$ is an isomorphism for any contravariant functor from $\mathcal{C}$ to abelian groups.

I shall prove 7.1.1 in several stages.

<u>Proposition 7.1.2</u> 7.1.1 is equivalent to: if $K_G^*(X)$ is finite over $R(G)$, then

(i) $$K_G^*(X)^\wedge \xrightarrow{\cong} \varprojlim_P K_G^*(P \times X)^\wedge;$$

(ii) $$R^1 \varprojlim_P K_G^*(P \times X)^\wedge = 0;$$

where P runs through the category of compact free G-spaces and GGhomotopy-classes of maps. (Completions are I_G-adic.)

<u>Proof</u>: One must show that the subcategory of projections $P \times X \to X$ is cofinal in the category $\mathcal{C}_X$. But any $Y \to X$ in $\mathcal{C}_X$ admits a map into

$E \times X \to X$. On the other hand given $Y \to P_1 \times X$, $Y \to P_2 \times X$ one can form the diagram

$$\begin{tikzcd} & P_1 \times X \arrow[dr] & \\ Y \arrow[ur] \arrow[dr] & & (P_1 * P_2) \times X, \\ & P_2 \times X \arrow[ur] & \end{tikzcd}$$

which commutes up to G-fibre-homotopy.

Now choose a fixed embedding $\phi : G \to U$ of G in a unitary group. ϕ induces $\phi_* : \{G\text{-spaces}\} \to \{U\text{-spaces}\}$ with $\phi_* X = U \times_G X$. We know $K_G^*(X) \cong K_U^*(\phi_* X)$, and the I_G-adic topology on an $R(G)$-module coincides with the I_U-adic topology (2.5), so $K_G^*(X)^\wedge \cong K_U^*(\phi_* X)^\wedge$. ϕ_* takes free G-spaces to free U-spaces, and as a functor from $\mathcal{C}_X$ to $\mathcal{C}_{\phi_* X}$ it is cofinal, as one can easily check. Finally $K_G^*(X)$ is finite over $R(G)$ if and only if $K_U^*(\phi_* X)$ is finite over $R(U)$. Hence

<u>Proposition 7.1.3</u> It suffices to prove 7.1.1 for the unitary groups $U(n)$.

<u>Proposition 7.1.4</u> It suffices to prove 7.1.1 when G is a torus $T^n = T$.

<u>Proof:</u> Recall that I defined in chapter 6 a natural homomorphism $pr_! : K_G^*(F \times X) \to K_G^*(X)$ such that $pr_! \, pr^!$ is the identity, when F is the flag-manifold $U(n)/T^n = U/T$. Apply this when $G = U(n)$, observing that

if X is a U-space then $U \times_T X \cong (U/T) \times X$. One obtains a natural homomorphism $K_T^*(X) \cong K_U^*(U \times_T X) \xrightarrow{pr_!} K_U^*(X)$ which is left-inverse to the "restriction": $K_U^*(X) \to K_T^*(X)$. Consider the diagram

$$\begin{tikzcd} K_U^*(X)^\wedge \arrow[r, "A"] \arrow[d, shift right, "pr^!"'] & \varprojlim K_U^*(Y)^\wedge \arrow[d, shift right, "pr^!"'] \\ K_T^*(X)^\wedge \arrow[r, "B"] \arrow[u, shift right, "pr_!"'] & \varprojlim K_T^*(Y)^\wedge \arrow[u, shift right, "pr_!"'] \end{tikzcd} .$$

$pr_!\, pr^! = \mathrm{id}$ on both sides; if B is an isomorphism then A must be too. Similarly $R^1 \varprojlim K_U^*(Y)^\wedge$ is a direct summand of $R^1 \varprojlim K_T^*(Y)^\wedge$, which proves the second part.

Proof of 7.1.1 when G is a torus T:

First notice that if $G = G_1 \times G_2$ and if P_1, P_2 are free G_1-, G_2-spaces respectively then $P_1 \times P_2$ is a free G-space, and that free G-spaces of this form are cofinal in the category of all free G-spaces. (Any free G-space admits the diagonal map into $P/G_1 \times P/G_2$, etc.)

Applying this to the torus T one finds by induction using 7.1.2 that when $G = T$ 7.1.1 is implied by

Proposition 7.1.5 If $K_T^*(X)$ is finite over $R(T)$, then

(i) $$K_T^*(X)^\wedge \xrightarrow{\cong} \varprojlim_P K_T^*(X \times P)^\wedge, \quad \text{and}$$

(ii) $$R^1 \varprojlim_P K_T^*(X \times P)^\wedge = 0,$$

when P runs through the category of compact free T^1-spaces and T^1-homotopy-classes of maps. (One supposes given a fixed homomorphism $t : T \to T^1$ whereby T^1-spaces become T-spaces.)

<u>Proof</u>: Regard T^1 as the complex numbers of modulus 1. It acts naturally on S^{2n-1}, the unit-sphere in $\mathbb{C}^n$. Furthermore the set of spheres S^{2n-1} with inclusion maps is cofinal in the category of free T^1-spaces. For any free T^1-space P is the unit-sphere bundle of a complex line-bundle L. L is a sub-bundle of a trivial bundle $\underline{\mathbb{C}}^n$. The resulting $\mathbb{C}^*$-map $L \to \mathbb{C}^n$ induces a T^1-map $P \to S^{2n-1}$. But any two T^1-maps $P \to S^{2n-1}$, $P \to S^{2m-1}$ become homotopic in $S^{2(n+m)-1} \cong S^{2n-1} * S^{2m-1}$. So one has to show

(i) $$K_T^*(X)^\wedge \xrightarrow{\cong} \varprojlim_n K_T^*(X \times S^{2n-1})^\wedge$$

(ii) $$R^1 \varprojlim_n K_T^*(X \times S^{2n-1})^\wedge = 0.$$

Consider the long exact sequence

$$\ldots \to K_T^*(X \times D^{2n}, X \times S^{2n-1}) \to K_T^*(X \times D^{2n}) \to K_T^*(X \times S^{2n-1}) \to \ldots \quad .$$

Using the Thom isomorphism this becomes

$$\cdots \to K_T^*(X) \xrightarrow{\alpha} K_T^*(X) \to K_T^*(X \times S^{2n-1}) \to \cdots,$$

where α is multiplication by $\lambda_{-1}[\mathbb{C}^n] = (1 - t)^n$. Because $K_T^*(X)$ is finite over $R(T)$ this sequence remains exact when completed. (This is the only place where one uses finiteness; it is conceivable that for the filtration-topology the maps of the sequence would be strict, and then one would avoid the assumption.) Thus we have short exact sequences

(i) $0 \to A/(1-t)^n A \to K_T^*(X \times S^{2n-1})^\wedge \to B_n \to 0$

(ii) $0 \to B_n \to A_n \to (1-t)^n A \to 0,$

where $A = K_T^*(X)^\wedge$, and $A_n = A$, but the map from A_{n+1} to A_n is multiplication by $(1-t)$. Now observe that $(1-t)^n A \subset I_T^n.A$, so $\varprojlim (1-t)^n A = 0$. For the same reason $\varprojlim A/(1-t)^n A = A$. And $R^1 \varprojlim A/(1-t)^n A = 0$ because the maps in the sequence are surjective. Finally recall that

$0 \to \varprojlim A_n \to \Pi A_n \xrightarrow{s} \Pi A_n \to R' \varprojlim A_n \to 0$ is exact, where

$s(\{a_i\}) = \{a_i + (1-t)a_{i+1}\}$. [25] But s is an isomorphism on the complete module ΠA_n because it has an inverse $\{a_i\} \mapsto \{\Sigma(1-t)^j a_{i+j}\}$. So

$\varprojlim A_n = R^1 \varprojlim A_n = 0.$

This completes the proof, as (i) induces

$$0 \to \varprojlim A/(1-t)^n A \to \varprojlim K_T^*(X \times S^{2n-1})^\wedge \to \varprojlim B_n \to R^1 \varprojlim A/(1-t)^n A \to$$

$$R^1 \varprojlim K_T^*(X \times S^{2n-1})^\wedge \to R^1 \varprojlim B_n \to 0,$$

while from (ii)

$$0 \to \varprojlim B_n \to \varprojlim A_n \to \varprojlim (1-t)^n A \to R^1\varprojlim B_n \to R^1\varprojlim A_n,$$

showing that $\varprojlim B_n = R^1\varprojlim B_n = 0$.

It remains for me to deduce the assertions I made at the beginning of the chapter. In the following propositions X is a finite-dimensional compact G-space with $K_G^*(X)$ finite over $R(G)$.

Proposition 7.1.6 The I_G-adic topology on $K_G^*(X)$ coincides with the filtration-topology.

Proof: If a denotes the I_G-adic topology, and f the filtration-topology, one must show that the identity-map $K_G^*(X)_a \to K_G^*(X)_f$, which is plainly continuous, is a homeomorphism. It suffices to show that $K_G^*(X)^\wedge_a \to K_G^*(X)^\wedge_f$ is a homeomorphism.

One can choose a cofinal system of compact free G-spaces which are finite-dimensional, e.g. G, $G * G$, $G * G * G$, ... , If P belongs to this system $K_G^*(P \times X)_a = K_G^*(P \times X)_f$, as both are discrete. So there is a continuous map $K_G^*(X)_f \to \varprojlim K_G^*(P \times X)_f \cong \varprojlim K_G^*(P \times X)^\wedge_a \cong K_G^*(X)^\wedge_a$. This induces the required inverse $K_G^*(X)^\wedge_f \to K_G^*(X)^\wedge_a$.

<u>Proposition 7.1.7</u> The filtration-topology on $K^*_G(X \times EG)$ can be defined by the sequence of submodules $\{m_k\}$, where $m_k = \ker : K^*_G(X \times EG) \to K^*_G(X \times E^{k-1}G)$.

<u>Proof</u>: If X has dimension x and G has dimension G then $(X \times E^{k-1}G)/G$ has dimension $x + (k-1)(g+1)$, so $K^*_{G,x-g+k(g+1)}(X \times E^{k-1}G) = 0$, and $K^*_{G,x-g+k(g+1)}(X \times EG) \subset m_k$. On the other hand $m_k \subset K^*_{G,k}(X \times EG)$ because $X \times E^{k-1}G$ is G-homotopy-equivalent to $E^{k-1}Y$, where Y is the G-space $X \times EG \times G$ over $X \times EG$.

<u>Proposition 7.1.8</u> $K^*_G(X \times EG) \xrightarrow{\cong} \varprojlim K^*_G(X \times E^kG)$; hence the I_G-adic topology and the filtration-topology on $K^*_G(X \times EG)$ coincide, and $K^*_G(X \times EG)$ is complete.

<u>Proof</u>: Consider the filtration of $X_G = (X \times EG)/G$ by the compact subspaces $X^k_G = (X \times E^kG)/G$. By a theorem of Milnor [25] one has a short exact sequence

$$(*) \quad 0 \to R^1\varprojlim K^{q-1}(X^k_G) \to K^q(X_G) \to \varprojlim K^q(X^k_G) \to 0.$$

(This is actually a special case of the spectral sequence of 5.1. The spaces $\{X^k_G\}$ do not quite satisfy Milnor's conditions, but at any rate one has a short exact sequence if the middle term is replaced by $K^q(W)$, where W is the universal space of the category $X^0_G \to X^1_G \to X^2_G \to \ldots$; and W is just

$(X \times E')/G$, where E' is a new triangulable universal space for G, so $W \simeq X_G$. But really one can prove a much more general result: I didn't include it in chapter 5 because I haven't developed enough K_G-theory for non-compact spaces.) The sequence (*) can be rewritten

$$0 \to R^1\lim K_G^{q-1}(X \times E^kG) \to K_G^q(X \times EG) \to \lim K_G^q(X \times E^kG) \to 0;$$

but the system $\{E^kG\}$ is cofinal in the category of compact free G-spaces, so 7.1.2 gives one the result desired, and also the corollary.

Corollary 7.1.9 $\quad pr^! : K_G^*(X)^\wedge \to K_G^*(X \times EG)$ is an isomorphism.

I conclude with

Proposition 7.1.10 If Y is another finite-dimensional compact G-space such that $K_G^*(Y)$ is finite over $R(G)$, and $f : Y \to X$ is a G-map which is a homotopy-equivalence when G is disregarded, then $f^!$ induces an isomorphism $f^! : K_G^*(X)^\wedge \to K_G^*(Y)^\wedge$.

Proof: It suffices to show that f induces a homotopy-equivalence $f_G : Y_G \to X_G$. But X_G and Y_G are fibred locally trivially over BG with fibres X and Y respectively, so f_G is locally a fibre-homotopy-equivalence, therefore by [16] a fibre-homotopy-equivalence globally, and in particular a homotopy-equivalence.

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