Topologizing Group Actions

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Abstract

This thesis is centered on the following question: Given an abstract group action by an Abelian group G on a set X, when is there a compact Hausdorff topology on X such that the group action is continuous? If such a topology exists, we call the group action compact-realizable.

We show that if G is a locally-compact group, a necessary condition for a G-action to be compact-realizable, is that the image of X under the stabilizer map must be a compact subspace of the collection of closed subgroups of G equipped with the co-compact topology. We apply this result to give a complete characterization for the case when G is a compact Abelian group in terms of the existence of continuous compact Hausdorff pre-images of a certain topological space associated with the group action. If G is not compact, we will show that the necessary condition is not sufficient. Together with various examples, we then present a general two-stage method of construction for compact Hausdorff topologies for $\mathbb{R}$-actions.

For discrete groups, the necessary condition above turns out to be not very strong. In the case of $G = \mathbb{Z}^2$ we will see that the two cases $|X| < \mathfrak{c}$ and $|X| \geq \mathfrak{c}$ must be treated very differently. We derive necessary conditions for a group action with $|X| < \mathfrak{c}$ to be compact-realizable by constructing particularly nice open partitions of the space X. We then use symbolic dynamics together with some generic constructions to obtain a partial converse in this case. If $|X| \geq \mathfrak{c}$ we give further constructions of compact Hausdorff topologies for which the group action is continuous.

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Chapter 1

Introduction

Although continuity is one of the fundamental concepts in general topology, its implications for the structure of functions and the underlying space have only been given scarce attention. The easily stated question what self-maps on a set X can possibly be continuous if X is equipped with a sensible topology seems so general that a satisfying answer appears to be out of reach. In spite of this in this thesis we obtain far-reaching results which can, for the most part, be stated using little more mathematical language than an undergraduate would know. In the proofs, however, a wide variety of mathematical topics is used, touching amongst others on the lattice of topologies, non-Hausdorff spaces, and convergence spaces. The use of these techniques gives another striking argument why they should move from their current status as fringe interests into the heart of mainstream mathematics.

Already in the 1950s there was considerable interest in the consequences of continuity. Ellis [5] was the first to explicitly ask

> [...] how may one construct a topology in S [the set], in general non-trivial, so that in this topology γ [the map] will be continuous?

Powderly and Tong [11] solved this problem, but only de Groot and de Vries [1] construct non-discrete T_1 topologies for all cases in which S is infinite. They are the first to show more generally that the topology can always be chosen to be metrizable and non-discrete (provided the set is infinite).

We can rephrase these questions as problems about continuous semi-group actions, in this particular case $\mathbb{Z}^+$, or, provided the map is bijective, group actions. The phase spaces modern day dynamicists are mostly interested in are compact metric spaces. So given a group G, for example $\mathbb{Z}$ or $\mathbb{R}$, we would like to characterize all continuous G-actions for which the phase space is compact metric. However, already in 1957 de Vries [2] had shown that even if the phase space has size ω or $\mathfrak{c}$, it is in general impossible (in ZFC) to achieve this. In fact, he proved that the Continuum Hypothesis is equivalent to the assertion that for every set X of size $\mathfrak{c}$ and every bijection f on X there is a compact metrizable topology on X such that f is continuous, i.e. a homeomorphism.

Therefore, when the question was picked up again by Iwanik [8], he chose to focus on compact Hausdorff topologies. He gave a characterization of those $\mathbb{Z}$-actions for which a compact Hausdorff topology on the phase space exists, such that the action is continuous. Independently and using an entirely different approach, Good et al [7] give necessary and sufficient conditions for the phase space of a $\mathbb{Z}^+$-action to have a compact Hausdorff topology under which the action is continuous.

It then becomes natural to ask about other groups or classes of groups leading to the main question around which the results of this thesis revolve:

Question. Given a topological group G and an abstract G-action on a set X, when is there a compact Hausdorff topology on X such that the action is continuous?

Why have we chosen compact Hausdorff as the required property on the phase space? From the early studies mentioned above, it seems reasonable that a form of separation axiom, T_1, T_2 or even metrizability, should be used to exclude the indiscrete topology.

It is harder to exclude the discrete topology by some general topological property. De Groot and de Vries [1] note that "in general, *it is impossible to render (the infinite) S dense in itself so that γ becomes continuous.*" (their emphasis; S is the phase space, γ the self-map in question).

However, it becomes clear that a characterization might be possible if we use Hausdorffness instead of metrizability to exclude the indiscrete topology and compactness to exclude the discrete topology. The combination of compactness and Hausdorffness is also interesting for the reason that these topologies are minimal among the Hausdorff topologies and maximal among the compact topologies. In this sense they occupy a well-defined position in the lattice of topologies on the set X.

Given the abstract setting of the problem, a group action on a set without any additional structure, the characterizations have to be given in terms of the periodicity of the orbits, i.e. the stabilizers. In chapter 2, we introduce the concept of the (weighted) orbit spectrum to capture this abstract structure of a group action. The (weighted) orbit spectrum provides a natural ordering of the actions of a fixed group, which can be viewed as measuring their complexity.

We will then prove in chapter 3 that for an Abelian group the orbit spectrum can be equipped with a suitable topology such that it is the image of the stabilizer map, $x \mapsto \mathrm{stab}(x)$, and that this map is continuous. Factoring out the orbit equivalence relation leaves us with a continuous map from the orbit space of a group action into the set of closed subgroups of the group under consideration. This gives us the Continuity Theorem (3.11), namely that (under fairly general assumptions) the orbit spectrum must be a compact space in the co-compact topology on the collection of closed subgroups of G. In this chapter we will also establish the Continuum Theorem (3.13) which states that if G is a σ-compact Abelian group acting continuously on a compact Hausdorff space X such that all orbits are non-compact, then there are at least $\mathfrak{c}$ many orbits.

In chapter 4 we show how the order on the weighted orbit spectra introduced in chapter 2 gives rise to a notion of reducibility of group actions. One application is the rather nice Theorem 4.4: If G is an Abelian, locally compact group and an abstract group action has a fixed point, then there will exist a compact Hausdorff topology making the group action continuous. We will also give other methods of modifying a group action for which a compact Hausdorff topology making it continuous exists.

The next three chapters are devoted to the study of particular groups. We start in chapter 5 with a precise statement of the characterization given in [7] and [8] of abstract $\mathbb{Z}$-actions for which the phase space can be equipped with a compact Hausdorff topology. This result indicates that small $\mathbb{Z}$-actions (i.e. actions where the phase space has cardinality less than $\mathfrak{c}$) and large $\mathbb{Z}$-actions (the cardinality of the phase space is at least $\mathfrak{c}$) require a fundamentally different treatment.

This difference generalizes to $\mathbb{Z}^2$-actions where we discuss actions with small and large phase spaces separately. For small phase spaces we present two approaches for deriving necessary conditions that a $\mathbb{Z}^2$-action can be continuous on a compact Hausdorff space. A major difference compared to $\mathbb{Z}$-actions turns out to be the two-dimensional nature of $\mathbb{Z}^2$. While all $\mathbb{Z}$-actions which have periodic and non-periodic orbits can be realized as compact, continuous $\mathbb{Z}$-actions, we can show that this is no longer the case for $\mathbb{Z}^2$-actions. We then give a variety of constructions for large classes of $\mathbb{Z}^2$-actions. Unfortunately we were unable to construct all $\mathbb{Z}^2$-actions which are not explicitly forbidden by the necessary conditions derived earlier. However, I feel that substantial progress has been made towards a complete characterization of $\mathbb{Z}^2$-actions which can be realized as compact continuous actions. The situation for large phase spaces is much more satisfying. Considering the differences between $\mathbb{Z}$ and $\mathbb{Z}^2$-actions in the realm of small phase spaces, it is surprising that for large phase spaces the results are very similar. We manage to construct all $\mathbb{Z}^2$-actions with large phase space which do not have essential forced-compact subsets. These are the fundamental building blocks of $\mathbb{Z}^2$-actions with large phase spaces.

Chapter 6 focuses on compact Abelian groups and shows that for these that the conditions in the Corollary 3.12 to the Continuity Theorem are not only necessary but in fact sufficient. For compact Abelian Lie groups this amounts to the simple theorem that there will exist a compact Hausdorff topology on the phase space making an abstract group action continuous if and only if the orbit spectrum of the action is compact in the co-compact topology.

Chapter 7 concerns itself with the group of additive reals, perhaps the most important group after $\mathbb{Z}$ in this context. A general two-stage method of construction for $\mathbb{R}$-actions is provided with which appropriate compact Hausdorff topologies can be constructed for a large class of abstract $\mathbb{R}$-actions. Examples showing the limitation of this construction method are given as well.

I would like to thank the countless number of mathematicians who have either indirectly, through their published papers, or directly, through talks and conversations, nudged me in the right direction and influenced my style of doing mathematics. In particular I would like to thank my supervisor Dr R W Knight for his excellent support and friendship, his ideas and his critical questions during my time in Oxford. I would also like to thank Dr P J Collins for making it possible for me to study at the University of Oxford and to arrange meetings with numerous mathematicians in a relaxed atmosphere. This thesis would not have seen the light of day, were it not for the moral and financial support of my parents and Brasenose College. My deepest love and thanks go to Anna Hoerder who shared my happiness during the ups and consoled me during the downs in the production of this work.

Chapter 2

Basic Concepts

In this chapter we will introduce the definitions and notations which will be used throughout the thesis. Apart from standard terminology, we introduce the fundamental notion of an order relation on group actions for some fixed Abelian group G. This order relation approximately describes how complicated a group action is, when seen from an abstract point of view. Whereas standard notation and definitions are given in the running text, we will number the new concepts defined in this thesis.

We also remark that we assume that every group is an Abelian Hausdorff topological group.

2.1 Abstract Group Actions

Given a group G and a set X, an abstract G-action on X is a map

$$\rho\colon G \times X \to X;\ \rho(g,x) = gx$$

such that $g(hx) = (gh)x$ for all $g, h \in G, x \in X$ and $ex = x$ for all $x \in X$ where e is the identity of G. We will frequently omit explicit mention of the map ρ and simply write (X, G) for the G-action. If G is clear from the context, we will refer to the abstract group action by its phase space X.

For sets $F \subset G$ and $A \subset X$ we write $FA = \{gx\colon g \in F, x \in A\} \subset X$ and shorten $\{g\}\, A$ and $F\, \{x\}$ to gA and Fx if no ambiguity can arise. With this notation the orbit of $x \in X$ is the set Gx. The relation $x \sim y \iff y \in Gx$

defines an equivalence relation on X (the orbit equivalence relation) and we denote the collection of equivalence classes by $\mathcal{O}(X) = \{Gx \colon x \in X\}$.

The stabilizer of a point $x \in X$ is the subgroup $\operatorname{stab}(x) = \{g \in G \colon gx = x\}$. This defines a map from X into the collection of subgroups of G. Note that if $gx = y$ then $\operatorname{stab}(x) = g^{-1} \operatorname{stab}(y) g$. Therefore, as G is Abelian, stab is constant on orbits and thus induces a map, also denoted by stab, from $\mathcal{O}(X)$ into the collection of subgroups of G. The image of an orbit under this induced map is also called the type of this orbit.

Definition 2.1. For an abstract group action by an Abelian group G we define the weighted orbit spectrum as the family $(\kappa_H)_{H \leq G}$ where

$$\kappa_H = \left|\{Gx \colon \operatorname{stab}(Gx) = H\}\right|.$$

The orbit spectrum on the other hand is the set

$$\{H \leq G \colon \exists x \in X.\ \operatorname{stab}(x) = H\}.$$

Given an abstract action defined by the weighted orbit spectrum $(\kappa_H)_{H \leq G}$, the orbit spectrum $\{H \leq G \colon \kappa_H > 0\}$ of the action is called the induced orbit spectrum.

Note that two abstract G-actions (G Abelian) with the same weighted orbit spectrum are conjugate, i.e. if $\rho \colon G \times X \to X, \rho' \colon G \times X' \to X'$ have the same weighted orbit spectrum then there is a bijection $f \colon X \to X'$ such that $f(\rho(g, x)) = \rho'(g, f(x))$ for all $g \in G, x \in X$.

Definition 2.2. The canonical representation of a weighted orbit spectrum $w\mathcal{O}$, denoted by $C(w\mathcal{O})$, is the space

$$\bigcup_{H \leq G} \{H\} \times \kappa_H \times G/H$$

with action $\rho(g, (H, \alpha, hH)) = (H, \alpha, ghH)$ for $h, g \in G, H \leq G, \alpha \in \kappa_H$.

Definition 2.3. Given two weighted orbit spectra $w\mathcal{O} = (\kappa_H)_{H \leq G}$ and $w\mathcal{O}' = (\kappa'_H)_{H \leq G}$ we define $w\mathcal{O} \leq_{wos} w\mathcal{O}'$ if and only if for each $H \leq G$

(i) $\kappa_H \leq \kappa'_H$;

(ii) if $\kappa'_H \neq 0$ then there is $H' \geq H$ with $\kappa_{H'} \neq 0$;

Similarly for two orbit spectra $\mathcal{O}, \mathcal{O}'$ we define $\mathcal{O} \leq_{os} \mathcal{O}'$ if and only if $\mathcal{O} \subset \mathcal{O}'$ and for each $H \in \mathcal{O}'$ there is an $H' \geq H$ with $H' \in \mathcal{O}$.

If $w\mathcal{O} \leq_{wos} w\mathcal{O}'$ then the condition that $\kappa_H \leq \kappa'_H$ for each $H \leq G$ implies that $C(w\mathcal{O}) \subset C(w\mathcal{O}')$. The second condition then gives rise to a retraction of the canonical representation of $w\mathcal{O}'$ onto the canonical representation of $w\mathcal{O}$ as follows.

Lemma 2.4. *Let G be an Abelian group, $w\mathcal{O}, w\mathcal{O}'$ be weighted orbit spectra of abstract G-actions with $w\mathcal{O} \leq_{wos} w\mathcal{O}'$. Then there is a retraction $R\colon C(w\mathcal{O}') \to C(w\mathcal{O})$ such that* $\operatorname{stab}(x) \leq \operatorname{stab}(R(x))$ *for each $x \in C(w\mathcal{O}')$.*

Proof. For each orbit $O_1 \subset C(w\mathcal{O}') \setminus C(w\mathcal{O})$ choose an orbit $O_2 \subset C(w\mathcal{O})$ with $\operatorname{stab}(O_1) \leq \operatorname{stab}(O_2)$. Such an orbit O_2 always exists since by the second condition there exists $H' \geq \operatorname{stab}(O_1)$ with $\kappa_{H'} \neq 0$. Choose $x' \in O_1 = Gx'$ and $x \in O_2 = Gx$ and let $R(gx') = gx$ for every $g \in G$. Since $\operatorname{stab}(x')$ is a subgroup of $\operatorname{stab}(x)$ this is well defined. On $C(w\mathcal{O})$ we define R to be the identity.

Clearly R thus defined is a retraction from $C(w\mathcal{O}')$ to $C(w\mathcal{O})$ with $\operatorname{stab}(x) \leq \operatorname{stab}(R(x))$ for each $x \in C(w\mathcal{O}')$. □

It is convenient to also define an order relation between two orbit types along the same lines. Given two orbit types $H_1, H_2 \leq G$ we say that $H_1 \leq_{ot} H_2$ and say that H_1 covers H_2 if and only if $H_1 \supset H_2$. $\leq_{ot}$ can be seen as an information order: intuitively we can say that the smaller the orbit type the more restricted its appearance is.

With regards to our question of when an abstract group action is compact-realizable, the rule of thumb is that orbits with smaller orbit type are harder to include, but once taken care of help in adding further orbits. For a precise statement we refer the reader to Chapter 4.

2.2 Continuous Group Actions

If G is a topological group and X a topological space we call a G-action continuous if and only if ρ is a continuous map. In this case $\mathcal{O}(X)$ will be equipped with the quotient topology and referred to as the orbit space.

A compact group action will denote a continuous group action on a compact Hausdorff space. Note that a compact group action whose orbits are all compact has a Hausdorff orbit space.

Definition 2.5. A weighted orbit spectrum $w\mathcal{O}$ of an abstract G-action (G Abelian) is called compact-realizable if and only if there is a compact G-action which has weighted orbit spectrum $w\mathcal{O}$. An orbit spectrum $\mathcal{O}$ is called compactifiable if and only if there is an compact G-action which has orbit spectrum $\mathcal{O}$ and for which $Gx \neq Gy \implies \mathrm{stab}(Gx) \neq \mathrm{stab}(Gy)$, i.e. each orbit type occurs at most once. Since the weighted orbit spectrum determines an abstract group action (up to conjugacy) we call an action compact-realizable if and only if its weighted orbit spectrum is.

Trivially, every element of the orbit spectrum of a continuous group action is a closed subgroup.

Lemma 2.6. *If $\mathcal{O}$ is the orbit spectrum of a continuous, Hausdorff G-action (G Abelian) then its elements are closed subgroups of G.*

Proof. Suppose that (X, G) is a continuous group action, that $x \in X$ and that $g \in \overline{\mathrm{stab}(x)}$. If U is an open set containing gx then by continuity there are open sets $V \ni x$ and $F \ni g$ such that $FV \subset U$. Since F is a neighbourhood of g there must be $h \in F \cap \mathrm{stab}(x)$ and thus $hx = x \in U$. Thus, every open set containing gx also contains x which, by Hausdorffness of X implies $gx = x$, i.e. $g \in \mathrm{stab}(x)$. □

Since $\mathrm{stab}(x) \neq \emptyset$ for all $x \in X$, stab is a map into the collection of closed subgroups of G, which we will denote by $G^{\leq}$.

In view of the last lemma, we can restrict our attention to group actions which only have closed subgroups as stabilizers.

Definition 2.7. Let (X, G) be an abstract group action where G is an Abelian topological group. We say that (X, G) is an admissable group action if and only if for all $x \in X$, $\text{stab}(x)$ is a closed subgroup of G.

Chapter 3

Restrictions on the Orbit Spectrum

Even under very general conditions on the group, there are, surprisingly, a number of restrictions on the structure of a compact group action and on the orbit spectrum. In this chapter we will explore these restrictions at a general level. For applications to specific groups we point the reader to the appropriate chapters, namely chapter 5 for $\mathbb{Z}$ and $\mathbb{Z}^2$, chapter 6 for compact groups and in particular compact Lie groups, and chapter 7 for the group of additive reals.

The most important result of this chapter consists of finding a topology, namely the co-compact topology, on the collection of closed subgroups of an Abelian group G such that the stabilizer is a continuous map from X into $G^{\leq}$. The order on the orbit types which was given in the previous chapter is intimately tied up with this topology. In fact, this order turns out to be the specialization order of the co-compact topology.

Note that the co-compact topology is a very non-Hausdorff topology. In fact, for some groups (e.g. $\mathbb{Z}$) this topology is anti-Hausdorff. In the authors opinion, the natural occurrence of a non-Hausdorff topology in this setting emphasizes the need to further the study of these topologies along the lines that are started in e.g. [9]. At the moment non-Hausdorff topologies seem to be internal topological problems or come up in the study of problems inspired from a computer science background. Our discoveries point to wider applicability in particular in the area of function spaces.

We will also observe the curious behaviour of the category of topological spaces when confronted with function spaces. Although this could be overcome by switching to a suitable Cartesian closed category, specifically the category of convergence spaces, a lot of the sharpness and appeal would be lost. The category of topological spaces seems to be right on the edge between too broad categories (e.g. convergence spaces) and too narrow categories (e.g. metric spaces). It is this tension which leads to a better understanding of the subtleties and nuances of the problem at hand.

3.1 The Structure of Orbits

There is only little that we can say about the topology of an individual orbit. However, the following lemma is of great importance.

Lemma 3.1. *If (X, G) is a compact group action with G Abelian, $\mathcal{O}$ is an orbit of type H and G/H is compact Hausdorff, then the restriction of the group action to $\mathcal{O}$ is topologically conjugate to the quotient group action $(G/H, G)$ given by $G \times G/H \to G/H; (g, hH) \mapsto ghH$.*

Proof. Fix $x \in \mathcal{O}$ and note that the map $G/H \to \mathcal{O}; gH \mapsto gx$ is a continuous bijection from a compact to a Hausdorff space and thus a homeomorphism. Trivially it respects the group action, i.e. is a conjugacy. □

No such theorem can be given about orbits of type H where G/H is non-compact. The most we can say generally is that an orbit of type H is the continuous bijective image of G/H. However in certain situations we can show that if G/H is non-compact then the orbit of type H cannot be compact. Surprisingly, the following theorem appears to be unpublished.

Theorem 3.2 (Non-compactness Theorem)**.** *Suppose G is a locally compact, Lindelöf, non-compact Abelian Hausdorff topological group. If (X, G) is a compact group action with $X = Gx$ then* $\operatorname{stab}(x) \neq \{e\}$.

Proof. Write σ for the topology on G and suppose $\operatorname{stab}(x) = \{e\}$. We may then identify X with G by picking some $x_e \in X$ and identify $y = gx_e$ with

$g \in G$. We will write τ for the induced compact Hausdorff topology on G and G_τ for the space (G, τ), whereas (G, σ) will be denoted by G_σ. For subsets A of G we will write A_σ and A_τ for the subspaces A of G_σ and G_τ respectively.

If U is τ-open and V the inverse image of U under the group action then $V \cap G_\sigma \times \{x_e\} = U \times \{x_e\}$ must be open in $G_\sigma \times \{x_e\}$ and thus $\tau \subset \sigma$.

Since G_σ is locally compact we can find $U \in \sigma$ with $e \in U$ and $\overline{U}^\sigma$ compact. Since $\tau \subset \sigma$ the identity map $\mathrm{id}\colon G_\sigma \to G_\tau$ is continuous and thus by compactness $\overline{U}^\tau = \overline{U}^\sigma = C$. We claim that C is nowhere dense in G_τ.

Let V be the τ-interior of $\overline{C} = C$ and assume that $h \in V$. We will show that $\mathrm{id}\colon G_\sigma \to G_\tau$ must then be open and therefore a homeomorphism which is a contradiction to G_σ being non-compact and G_τ being compact. Thus $V = \emptyset$ showing that C is nowhere dense in G_τ as claimed.

Since $\mathrm{id}\colon G_\sigma \to G_\tau$ is a continuous bijection and C is compact Hausdorff $\mathrm{id}\colon C_\sigma \to C_\tau$ is a homeomorphism and thus $\mathrm{id}\colon V_\sigma \to V_\tau$ is a homeomorphism. If $W \in \sigma$ we can write $W = \bigcup_{g \in W} gh^{-1}V_g$ for some σ-open $V_g \subset V$, giving $\mathrm{id}(W) = \bigcup_{g \in W} gh^{-1}\,\mathrm{id}(V_g)$. For every $g \in W$, V_g is σ-open in V and id is a homeomorphism from V_σ to V_τ. Therefore $\mathrm{id}(V_g)$ will be a τ-open subset of V and thus τ-open. Hence $\mathrm{id}(W)$ is τ-open, giving the desired result.

Observe that $U \subset C$ so that if $\{gU : g \in H\}$ covers G for some $H \subset G$ then so does $\{gC : g \in H\}$. Since G_σ is Lindelöf and $\{gU : g \in G\}$ covers G there is a countable $H \subset G$ with $\{gU : g \in H\}$ covering G.

Lastly note that since C is nowhere dense in G_τ and the group action is continuous, in particular $h \mapsto gh$ is an autohomeomorphism of G_τ for every $g \in G$, gC is nowhere dense in G_τ for every $g \in G$. But then $G_\tau = \bigcup_{g \in H} gC$ is a countable union of nowhere dense sets contradicting the Baire Category theorem, which must hold for the compact space G_τ. □

Note that if H is a closed subgroup of G, then G/H is a Hausdorff topological group. Moreover, if G acts continuously on a space X with $H \subset \mathrm{stab}(x)$ for all $x \in X$, then $G/H \times X \to X; (gH, x) \mapsto gx$ is a continuous group action as well. Thus we have the following corollary.

Corollary 3.3. *If (X, G) is a compact group action, where G is Abelian, with precisely one orbit, say of type H, and G/H is locally compact and Lindelöf, then G/H is in fact compact.*

Let us show that the conditions 'locally compact and Lindelöf' are strict. The first example shows that we cannot omit Lindelöfness.

Example 3.4. Let $G_\sigma = 2^\omega$ with the box topology and $G_\tau = 2^\omega$ with the usual Tychonoff topology. Clearly $\tau \subset \sigma$ and G_σ is non-compact whereas G_τ is compact. Observe that G_σ is a locally compact topological group (in fact it has the discrete topology).

Note that the natural action $G_\sigma \times G_\tau \to G_\tau; (g, h) \mapsto gh$ is continuous as G_τ is a topological group, i.e. $G_\tau \times G_\tau \to G_\tau; (g, h) \mapsto gh$ is continuous and $\tau \subset \sigma$. Then G_τ is a compact orbit with trivial stabilizer under the action of G_σ.

The second example show that local compactness cannot be omitted either.

Example 3.5. Let $G_\tau = \mathbb{T}$ with its usual topology τ and $G_\sigma = \mathbb{T}$ with the topology σ where a neighbourhood basis at x is given by sets of the form $U \setminus S$ where $U \ni x$ is at τ-open set and $S \not\ni x$ is a countable τ-discrete set with unique τ-limit point x. G_σ is clearly Lindelöf, non-compact and finer than τ. However, it is not locally compact.

Note that as above the natural action of G_σ on G_τ is continuous so that G_τ is a compact orbit with trivial stabilizer under the action of G_σ.

3.2 Forced-compact Sets

We have an easy lemma which forces some sets to be compact invariant subsets of X.

Lemma 3.6. *Suppose (X, G) is a compact group action with G Abelian. For every subset H of G the set $\mathrm{Fix}_H = \{x \in X : \forall h \in H.\ hx = x\}$ is a closed, compact, G-invariant subset of X.*

Proof. Since the map $x \mapsto gx$ is continuous for every $g \in G$ all the sets $\mathrm{Fix}_g = \{x \in X : gx = x\}$ are closed and invariant under G. Thus an intersection of these sets, $\mathrm{Fix}_H = \bigcap_{h \in H} \mathrm{Fix}_h$, is closed and invariant under G. □

The sets Fix_H for $H \subset G$ will be called forced-compact subsets of X. It turns out that conceptually there are two types of forced-compact sets. Those which have only finitely many $\leq_{ot}$-minimal orbit types are typically easy to handle and either make compactifiability outright impossible (e.g. they consist of precisely one orbit which cannot be compact for example due to the Non-compactness Theorem 3.2) or will not pose a problem. The other type has infinitely many $\leq_{ot}$-minimal orbit types and therefore presents a non-trivial challenge. The latter will be called 'essential' forced-compact sets. Since the group actions without proper essential forced-compact sets can be viewed as the building blocks of more general group actions we will call them 'fundamental' group actions.

We will see that these forced-compact sets are in some sense a glimpse of the following section. However, their ease and simplicity make them worth mentioning as a separate phenomenon.

3.3 Continuity of stab

In this section we will equip $G^{\leq}$ with a topology such that the stabilizer map $\mathrm{stab}\colon X \to G^{\leq}$ and therefore the induced map on the orbit space are both continuous. Note that the orbit spectrum is in fact the image of X under stab. Thus, if stab is continuous, then the orbit spectrum is a compact subspace of $G^{\leq}$!

We begin by shifting our attention from the topological point of view to a convergence-theoretic point of view. To this end, we give a very brief definition of convergence structures and how they are related to topological spaces.

In the same way that topologies can be seen as generalized metric spaces, we can see convergences as generalized topological spaces. One motivation for their invention is that their category-theoretic behaviour is much better than that of topological spaces. We will not concern ourselves with these considerations but merely remark that the category of convergences is Cartesian closed and that the category of topological spaces forms a full subcategory.

3.3.1 Definition of Convergence Theoretic Structures

A convergence is a relation $\to$ between the filters on a set X and its elements. It satisfies the following axioms:

(i) $\{A \subset X : x \in A\} \to x$;

(ii) if $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_1 \to x$ then $\mathcal{F}_2 \to x$;

If $\mathcal{F}$ is a filter on X we write $\lim \mathcal{F} = \{x \in X : \mathcal{F} \to x\}$.

The concept of continuity is generalized as expected:

A map f between to convergence spaces X and Y is said to be (convergence)-continuous if and only if $\mathcal{F} \to_X x$ implies $f(\mathcal{F}) \to_Y f(x)$ for each filter $\mathcal{F}$ of X and each point $x \in X$, where $f(\mathcal{F})$ denotes the filter generated by $\{f(F) : F \in \mathcal{F}\}$.

Finally we briefly mention the relation between topologies and convergences. Every topology τ on X gives rise to a convergence $\to_\tau$ on X, namely that $\mathcal{F} \to_\tau x$ if and only if $\mathcal{F}$ contains all neighbourhoods of x. Note that a continuous map between two topological spaces is convergence-continuous between the two induced convergence spaces.

Conversely every convergence $\to$ on X gives rise to a topology $\tau_\to$ on X defined via the closure operator:

$$\overline{A} = \{x \in X : \exists \mathcal{F} \to x.\ A \in \mathcal{F}\}$$

Note however that these operations are not inverses of each other as in general $\to_{\tau_\to} \neq \to$. If however $\to_{\tau_\to} = \to$ then we say that $\to$ is topological. If X, Y are two topological convergence spaces and f a convergence-continuous map from X to Y then f is continuous with respect to the induced topologies.

A more detailed introduction to convergence spaces can be found in [4] and [3].

3.3.2 A Topology on $G^{\leq}$

We can now proceed to use the above machinery to deal with our specific problem of finding a topology on $G^{\leq}$ such that stab is continuous. The convergence of interest to us is the upper Kuratowski convergence.

Definition 3.7. Let X be a topological space. The upper Kuratowski convergence on the set of closed, non-empty subsets of X, $\mathcal{C}(X)$, is defined by $\mathcal{F} \to C$ if and only if

$$\bigcap_{F \in \mathcal{F}} \overline{\bigcup F} \subset C.$$

The proof of the next lemma, namely that stab is a convergence-continuous map, formalizes the idea that if x_α is a net in X converging to x and h_α a net in G converging to h such that $h_\alpha x_\alpha = x_\alpha$ for each α then $hx = x$. Note however that due to diagonalization issues, a naïve approach does not work. The notation regarding uniform spaces is taken from [6], where a uniformity is defined as a family $\mathcal{U}$ of entourages of the diagonal (i.e. a family of symmetric subsets of X^2 containing $\mathrm{id}_X = \{(x, x)\colon x \in X\}$) which is closed under taking supersets and finite intersections and furthermore satisfies

(i) $\forall V \in \mathcal{U}.\ \exists W \in \mathcal{U}.\ 2W \subset V$,

(ii) $\bigcap \mathcal{U} = \Delta$.

For $x, y \in X$ and $U \in \mathcal{U}$ we say that $|x - y| < U$ if and only if $(x, y) \in U$ and define the set $U(x) = \{z \in X\colon\ |x - z| < U\}$.

Lemma 3.8. *If (X, G) is a completely regular continuous group action where G is an Abelian Hausdorff topological group, then* $\mathrm{stab}\colon X \to G^{\leq}$ *is convergence-continuous from X with the convergence induced by the topology, to $G^{\leq}$ with the upper Kuratowski convergence.*

Proof. First, equip X with a uniformity $\mathcal{U}$ inducing its topology, which is possible as X is completely regular.

Let $\mathcal{F}$ be a filter on X and $x \in \lim \mathcal{F}$. To show that stab is convergence-continuous, we need to prove that $\mathrm{stab}(x) \in \lim \mathrm{stab}(\mathcal{F})$, or using the definition of the upper Kuratowski convergence that $\bigcap_{F \in \mathcal{F}} \overline{\bigcup_{y \in F} \mathrm{stab}(y)} \subset \mathrm{stab}(x)$. So let h be an element of the left hand side.

We will show that for each $U \in \mathcal{U}$ we have $|hx - x| < U$. T_1-ness of X then implies that $x = hx$ as required. To do this let $V \in \mathcal{U}$ with $2V \subset U$. If $|hx - gy| < V$ and $|gy - x| < V$ then we must have $|hx - x| < U$.

Continuity of the group action implies that there is a neighbourhood A of h and some $W \in \mathcal{U}$ such that $AW(x) \subset V(hx)$ and $W \subset V$. As $\mathcal{F} \to x$ we have $W(x) \in \mathcal{F}$ and thus $h \in \overline{\bigcup_{y \in W(x)} \operatorname{stab}(y)}$, i.e. there exists some $g \in A \cap \operatorname{stab}(y)$ for some $y \in W(x)$. But with these choices $gy = y$ giving $|gy - x| = |y - x| < V$ and $|hx - gy| < V$. Hence g and y witness that $|hx - x| < U$ as required. □

So far we used the convenient framework of convergences to obtain our result. We will now translate back into the better known and more widely used topological spaces.

The appropriate topology on $G^{\leq}$ turns out to be the co-compact topology.

Definition 3.9. The co-compact topology, $\tau_{\mathbf{Co}}$, on the set $\mathcal{C}(G)$ of closed, non-empty subsets of G is the topology generated by the base

$$\mathcal{B} = \{\{C \in \mathcal{C}(G) \colon C \cap K = \emptyset\} : K \text{ compact} \subset G\}.$$

In other words, a typical basic open set is the collection of closed subsets of G which miss a fixed compact subset of G.

As G will always be an Abelian topological group in our applications and we are only interested in the subspace $G^{\leq}$ of closed subgroups of $\mathcal{C}(G)$, we will denote $\left\{H \in G^{\leq} \colon H \subset U\right\}$ by $\hat{U}$ for every subset U of G.

In [4] the close connection between the upper Kuratowski convergence and the co-compact topology has been demonstrated.

Theorem 3.10. *If G is a locally compact, completely regular space then the upper Kuratowski convergence on $\mathcal{C}(G)$ is topological and induces a topology which coincides with the co-compact topology on $\mathcal{C}(G)$.*

Note that in the same paper other premises replacing Čech-completeness are given. The above is sufficient for us, as all specific groups considered in the sequel are indeed Čech-complete.

Collecting the above theorems and lemmas together we arrive at the following rather neat result.

Theorem 3.11 (Continuity Theorem). *If G is a locally compact Abelian Hausdorff group acting continuously on a Tychonoff space X then the map*

$$\begin{aligned} \mathrm{stab}\colon X &\to (G^{\leq}, \tau_{\mathbf{Co}}); \\ x &\mapsto \mathrm{stab}(x) \end{aligned}$$

is continuous. In particular, this theorem holds if X is compact Hausdorff.

Theorem 3.11 gives us a strong restriction on the image of X under stab, i.e. the orbit spectrum.

Corollary 3.12. *The orbit spectrum of a continuous action of a locally compact Abelian Hausdorff group on a compact Hausdorff space is the continuous image of a compact Hausdorff space, thus in particular compact. Moreover, if every orbit type of this action occurs at most once and every orbit is compact then the orbit spectrum has a finer compact Hausdorff topology.*

Note that not only compactness, but also connectedness is preserved by continuous maps. The orbit spectrum of a continuous action therefore gives insight into the dynamical structure of the action.

Finally we address the comment made in the section on forced-compact sets. If $g \in G$ and stab is continuous into $G^{\leq}$, e.g. G is locally compact and X is compact Hausdorff, we may consider the co-compact subset $U = G \setminus \{g\}$ of G. The associated open set $\hat{U}$ in the co-compact topology contains precisely those elements of $G^{\leq}$ that do not contain g. Its complement, i.e. all $H \in G^{\leq}$ with $g \in H$ is closed and thus $\mathrm{stab}^{-1}(\{H \in G^{\leq}\colon g \in H\})$ is a closed subset of X, hence compact. Noting that $\mathrm{stab}^{-1}(\{H \in G^{\leq}\colon g \in H\}) = \mathrm{Fix}_{\{g\}}$ we see that the forced-compact sets are inverse images of closed sets under stab.

3.4 Non-compact Orbits

Our considerations above give only restrictions on the orbit spectrum and only indirectly on the weighted orbit spectrum. In particular they do not give any information about the size of κ_H for $H \leq G$ but only distinguish between $\kappa_H = 0$ and $\kappa_H > 0$. In view of Lemma 4.1 there cannot be a stronger general theorem.

However, if all orbits are non-compact, it is possible to say a lot more. Dr Knight (private communication) proved that continuous $\mathbb{R}$-actions on compact Hausdorff spaces with only non-compact orbits must have $\mathfrak{c}$ many orbits.

We can generalize this to the following powerful theorem which holds for a wide variety of groups.

Theorem 3.13 (Continuum Theorem)**.** *If G is a σ-compact Abelian Hausdorff group acting continuously on a compact Hausdorff space X such that every orbit is non-compact, then there are at least $\mathfrak{c}$ many orbits.*

We will prove this theorem in two steps.

Lemma 3.14. *Suppose (X, ρ, G) is a compact group action such that every orbit is non-compact. Then there exists a non-empty closed invariant subset X' of X, such that no orbit in $(X', \rho|_{X'}, G)$ is open. We will call this subset the core of the group action.*

Proof. We will define X' by recursion. Let $X_0 = X$ and define

$$X_{\alpha+1} = X_\alpha \setminus \bigcup \{Gx \colon Gx \text{ open in } X_\alpha\}.$$

For limit ordinals $\gamma > 0$ we let $X_\gamma = \bigcap_{\beta<\gamma} X_\beta$.

As $\bigcup \{Gx \colon Gx \text{ open in } X_\alpha\}$ is a union of X_α open sets and therefore open in X_α, each $X_{\alpha+1}$ will be closed, provided X_α is. Since the intersection of closed sets is closed, X_γ will be closed for every limit ordinal γ. Hence by recursion every X_α will be closed.

Note also that if $\gamma > 0$ is a limit ordinal and $X_\gamma = \emptyset$ then compactness of X implies that $X_\beta = \emptyset$ for some $\beta < \gamma$. However if $X_{\alpha+1} = \emptyset$ but $X_\alpha \neq \emptyset$ then X_α is a closed, therefore compact subset of X which is the topological sum of non-empty non-compact sets, a contradiction. Thus by recursion $X_\alpha \neq \emptyset$ for all α provided $X_0 = X \neq \emptyset$.

Let α_0 be the least ordinal such that $X_{\alpha_0} = X_{\alpha_0+1}$. Then no orbit of X_{α_0} is open in X_{α_0} and X_{α_0} is a non-empty closed invariant subsystem of X as required. □

Next we will construct a set of size $\geq \mathfrak{c}$ in this core that meets every orbit at most once. An adaptation of the standard bisection construction of a Cantor subset of $\mathbb{I}$ easily gives a Cantor subset of X. We will further modify this construction to ensure our condition on the intersections with the orbits.

Lemma 3.15. *Let (X, G) be a compact group action without open orbits. If G is σ-compact, Abelian and Hausdorff there is a subset of size $\mathfrak{c}$ of X which meets each orbit at most once.*

Proof. Since G is σ-compact we may write $G = \bigcup_{n\in\omega} G_n$ where for each $n \in \omega$ G_n is compact, $e \in G_n \subset G_{n+1}$ and $G_n^{-1} = G_n$.

Since X has no open orbits, every open subset of X meets at least two orbits. For if U is open and $U = Gx \cap U$ then $GU = Gx$ is an open subset of X. In particular if $n \in \omega$ and $x_1, \ldots, x_k \in X$ then $U \setminus \bigcup_{l=1}^{k} G_n x_l$ is a non-empty open set.

By induction on n we will construct open sets U_f for each $f \in 2^n$ such that

- if g extends f then $U_f \supset \overline{U_g}$;
- if $f, g \in 2^n$ are distinct then $G_n U_f \cap G_n U_g = \emptyset$.

For $f = \emptyset$ we let $U_f = X$. Suppose we have constructed U_f as claimed for all $f \in 2^n$. For $f \in 2^n$ we write f^i for the element g of 2^{n+1} such that $g(n) = i$ and $g|_{2^n} = f$. Enumerate all elements of 2^n as $\{f_1, \ldots, f_{2^n}\}$. A simple recursion on k provides $x_k^0, x_k^1 \in U_{f_k}$ for $k = 1, \ldots, 2^n$ such that the $G_{n+1} x_k^i$ are all disjoint. The recursion works since $U_k^0 = U_{f_k} \setminus \bigcup_{l \leq k, i=1,2} G_{2(n+1)} x_k^i$ is non-empty, so choosing $x_{k+1}^0 \in U_k^0$ and $x_{k+1}^1 \in U_k^0 \setminus G_{2(n+1)} x_{k+1}^0$ works. Since G_{n+1} is compact, the $G_{n+1} x_k^i$ are closed and thus can be separated by disjoint open sets U_k^i. Continuity of the group action, $G_{n+1}^{-1} = G_{n+1}$ and compactness of the latter provides a $U'_{f_k^i} \ni x_k^i$ with $G_{n+1} U'_{f_k^i} \subset U_k^i$. Now regularity gives a

$U_{f^i_k}$ with $x^i_k \in U_{f^i_k} \subset \overline{U_{f^i_k}} \subset U'_{f^i_k} \cap U_f$. It is clear that the $U_{f^i_k}$ satisfy the two conditions.

Finally note that for each $f \in 2^\omega$ we have $C_f = \bigcap_{n\in\omega} U_{f|n} = \bigcap_{n\in\omega} \overline{U_{f|n}}$ and the latter is non-empty as X is compact. Hence we may choose $x_f \in C_f$.

We want to show that $\{x_f \colon f \in 2^\omega\}$ meets every orbit at most once. So suppose that $x \in Gx_f \cap Gx_h$ for $f \neq h \in 2^\omega$. Write $x = g_f x_f = g_h x_h$ with $g_f, g_h \in G$ and choose n such that $g_f, g_h \in G_n$ and $f|_n \neq h|_n$. Then $g_f x_f \in G_n U_{f|n} = V_f$, $g_h x_h \in G_n U_{h|n} = V_h$ and $V_f \cap V_h = \emptyset$ by construction. However this contradicts $g_f x_f = g_h x_h$.

Therefore $\{x_f \colon f \in 2^\omega\}$ is a set of size $\mathfrak{c}$ which meets every orbit at most once. □

Proof of theorem. Use Lemma 3.14 to obtain the core X' of X to which we may apply Lemma 3.15 to see that X' and hence X contains at least $\mathfrak{c}$ many different orbits. □

Since every σ-compact topological space is also Lindelöf we can combine Theorem 3.13 with the Non-compactness Theorem (Theorem 3.2). We therefore obtain the corollary:

Corollary 3.16. *Suppose G is a locally compact, σ-compact, Abelian, Hausdorff topological group acting continuously on a compact Hausdorff space X. If $G/\operatorname{stab}(x)$ is locally compact, Lindelöf and non-compact for each $x \in X$, then there are at least $\mathfrak{c}$ many orbits.*

Chapter 4

General Constructions

We will now give some generic constructions which allow us to modify the orbit spectrum of a continuous compact group action under very general assumptions. These will be helpful when we show that certain abstract group actions are compact-realizable.

Although all of these results serve mostly as a means to an end, there is some interest in how these constructions work. Essentially we consider the orbit space of a compact group action and try to assign levels, as one would if one had a scattered space. We then have the options of

- adding more isolated orbits: in the language of scattered spaces that amounts to adding new isolated points and possibly turning some of the formally isolated points into limit points.

- changing isolated orbits, in particular enlarging them: this can be seen as replacing one point of an isolated orbit by multiple copies.

- shrinking the orbits in the top layer: this amounts to a taking a quotient under a suitable closed relation.

Since the orbit space is neither necessarily Hausdorff, nor in general scattered, this cannot be done formally. Nevertheless it serves as a good image to have in mind.

Specific mention should be made of Theorem 4.4 which exploits the order relation on G-actions to show that abstract actions with a fixed point are compact-realizable provided G is locally compact, Abelian Hausdorff.

4.1 Extending Orbit Spectra

In Lemma 2.4 we have seen that the order relation between (weighted) orbit spectra can be interpreted as denoting extendibility. This carries through to the continuous case and further justifies our choice of the order.

Lemma 4.1 (Adding Lemma)**.** *Let G be a locally compact, Abelian, Hausdorff topological group and let $w\mathcal{O}, w\mathcal{O}'$ be weighted orbit spectra of admissable G-actions. If $w\mathcal{O}$ is compact-realizable and $w\mathcal{O} \leq_{wos} w\mathcal{O}'$ then $w\mathcal{O}'$ is compact-realizable.*

Proof. Let τ be a compact Hausdorff topology on $C(w\mathcal{O})$ such that the canonical group action is continuous. We will extend this topology to one on $C(w\mathcal{O}')$.

Equip $M = C(w\mathcal{O}') \setminus C(w\mathcal{O})$ with the topology τ_M in which every orbit Gx is clopen and has topology $G/\operatorname{stab}(Gx)$.

The topology τ' on $C(w\mathcal{O}')$ is then generated by

$$\tau_M \cup \left\{R^{-1}(U) \setminus C \colon U \in \tau, C \text{ compact} \subset M\right\},$$

where R is the retraction defined in Lemma 2.4.

Clearly τ' is a compact Hausdorff topology on $C(w\mathcal{O}')$.

If $x \in M$ then ρ' is continuous at each (g, x) for $g \in G$, since G acts continuously on $G/\operatorname{stab}(Gx) \approx Gx \subset M \subset C(w\mathcal{O}')$. So let $x \in C(w\mathcal{O})$, $g \in G$ and consider $R^{-1}(U) \setminus C$ for $x \in U \in \tau$, C a compact subset of M. Continuity of ρ implies that there are open $F \subset G, V \in \tau$ such that $g \in F, x \in V, FV \subset U$. Since G is locally compact, we may assume that $\overline{F}$ is compact. Now $y \in R^{-1}(V)$, $f \in F$ implies $R(fy) = fR(y) \in U$ so $FR^{-1}(V) \subset R^{-1}(U)$. Note that $\overline{F}^{-1}C$ is a compact subset of M and if $fy \in C$ with $f \in F$ then $y \in F^{-1}C$. Thus $F(R^{-1}(V) \setminus \overline{F}^{-1}C) \subset R^{-1}(U) \setminus C$ and $R^{-1}(V) \setminus \overline{F}^{-1}C$ is an open set containing x. This shows continuity of ρ' at (g, x). $\square$

The restriction to locally compact groups is essential as the next example shows.

Example 4.2. Let $G = \mathbb{Q}$ and consider an abstract group action $(X, \mathbb{Q})$ consisting of $\mathfrak{c}$ many orbits of type $\mathbb{Z}$ (i.e. of period 1) and one orbit $\mathcal{O}_H$ of type $H \subsetneq \mathbb{Z}$, e.g. a non-periodic orbit.

We claim that this group action is not compact-realizable. For suppose τ were a compact Hausdorff topology on X making the group action continuous. Considering $\mathrm{Fix}_{\mathbb{Z}}$ we see that the orbit of type H must be open, therefore locally compact (an open subset of a compact space is locally compact). Take an open subset U of $\mathcal{O}_H$ with $\overline{U} \subset \mathcal{O}_H$ and $\overline{U}$ compact (which exists by local compactness and regularity of (X, τ)). Then $\overline{U}$ is a countable compact set, and therefore must contain an isolated point x_0, say. However, x_0 then is also isolated in U and thus in X. Since every map $X \to X; x \mapsto qx$ for $q \in \mathbb{Q}$ is a homeomorphism, that implies that every point of $\mathbb{Q}x_0 = \mathcal{O}_H$ is isolated. This contradicts that the map $\mathbb{Q} \to X; q \mapsto qx_0$ is continuous, i.e. that $\mathbb{Q}x$ has a coarser topology than $\mathbb{Q}$.

Note on the other hand that an abstract $\mathbb{Q}$-action that consists of $\mathfrak{c}$ many orbits of type $\mathbb{Z}$ is compact-realizable. Thus the Adding Lemma cannot be applied to groups which are not locally compact.

From Lemma 4.1 we immediately get the following corollary.

Corollary 4.3. *Let G be a locally compact, Abelian, Hausdorff topological group and $\mathcal{O}, \mathcal{O}'$ orbit spectra of admissable G-actions. If $\mathcal{O} \leq_{os} \mathcal{O}'$ and $\mathcal{O}$ is compactifiable then so is $\mathcal{O}'$.*

Proof. We observe that if $\mathcal{O} \leq \mathcal{O}'$ then there are G-actions ρ, ρ' with orbit spectra $\mathcal{O}, \mathcal{O}'$ respectively such that their weighted orbit spectra, $w\mathcal{O}, w\mathcal{O}'$, also satisfy $w\mathcal{O} \leq_{wos} w\mathcal{O}'$. Moreover if $w\mathcal{O}$ is chosen such that $\kappa_H \leq 1$ for every $H \leq G$ then we may choose $w\mathcal{O}'$ to satisfy the same condition. □

We will begin to see the importance of these constructions in the next chapters. For now, let us note that they give the following theorem.

Theorem 4.4 (Fixed-point Compactification Theorem)**.** *If G is a locally compact, Abelian, Hausdorff topological group, and ρ an admissable G-action with a fixed point, then ρ is compact-realizable.*

Proof. Note that the weighted orbit spectrum with $\kappa_G = 1, \kappa_H = 0$ for $H \neq G$ is $\leq_{wos}$-minimal and trivially compact-realizable. Now apply Lemma 4.1. □

4.2 Modifying Orbit Types

In the previous section we have shown how to add orbits to a compact group action. Now we concern ourselves with modifying orbits of a compact group action. There are three basic operations which may be carried out. Since they modify already existing orbits (instead of adding new ones) the topology of the modified orbits has to be taken into account. The first two operations concern themselves with compact orbits while the last considers isolated orbits. Note that the two former change only finitely many orbits at a time, whereas the last one allows to change infinitely many orbits at once.

Lemma 4.5 (Identification Lemma)**.** *Suppose that (X, G) is a compact group action with G Abelian Hausdorff. If two orbits $\mathcal{O}_1, \mathcal{O}_2$ are compact and have the same orbit type, then we can identify them and obtain a continuous compact G-action.*

Proof. Let $x_i \in \mathcal{O}_i$ and consider the relation

$$R = \mathrm{id} \cup \{(gx_0, gx_1) \colon g \in G\} \cup \{(gx_1, gx_0) \colon g \in G\}$$

on X^2. R is invariant under G (since the orbit types of $\mathcal{O}_1$ and $\mathcal{O}_2$ are the same) so the result follows provided R is closed. To that end note that $(\mathcal{O}_1, G), (\mathcal{O}_2, G)$ are conjugate by Lemma 3.1 and hence $\{(gx_0, gx_1) \colon g \in G\}$ is homeomorphic to $\mathrm{id}_{\mathcal{O}_1 \times \mathcal{O}_2}$ which is compact, hence closed. □

Lemma 4.6 (Shrinking Lemma)**.** *Suppose that (X, G) is a compact group action with G Abelian Hausdorff. If $\mathcal{O}$ is a compact orbit and* $\mathrm{stab}(\mathcal{O}) \subset H \in G^{\leq}$ *then $\mathcal{O}$ may be replaced by an orbit of type H.*

Proof. Since $\mathcal{O}$ is compact it is conjugate to $G/\mathrm{stab}(\mathcal{O})$ so that the relation

$$\mathrm{id} \cup \{(x, hx) \colon x \in \mathcal{O}, h \in H\}$$

is closed in X^2 and invariant under G. □

Lemma 4.7 (Expanding Lemma)**.** *Suppose that (X, G) is a compact group action where G is locally compact, Abelian, Hausdorff. If each $\mathcal{O}_\alpha, \alpha \in I$ with* $\operatorname{stab}(\mathcal{O}_\alpha) = H_\alpha$ *is open in X and G/H_α is compact, then we may replace each $\mathcal{O}_\alpha$ with $\mathcal{O}'_\alpha$ with* $\operatorname{stab}(\mathcal{O}'_\alpha) = H'_\alpha \in G^{\leq}$ *provided $H'_\alpha \subset H_\alpha$ and each G/H'_α is compact Hausdorff.*

Proof. Let τ be the compact Hausdorff topology on X. Define

$$Y = (X \setminus \bigcup_{\alpha \in I} \mathcal{O}_\alpha) \cup \bigcup_{\alpha \in I} \mathcal{O}'_\alpha$$

and define a topology τ' on Y as follows: choose $x_\alpha \in \mathcal{O}_\alpha$ and $x'_\alpha \in \mathcal{O}'_\alpha$ $(\alpha \in I)$.

- each orbit $\mathcal{O}'_\alpha$ $(\alpha \in I)$ is open and equipped with the topology G/H'_α;
- if $U \in \tau$ then

  $$U' = (U \setminus \bigcup_{\alpha \in I} \mathcal{O}_\alpha) \cup \{gx'_\alpha \colon g \in G, gx_\alpha \in U, \alpha \in I\}$$

  is open in τ'.

τ' is Hausdorff: Suppose $U, V \in \tau$ are disjoint and that $y \in U' \cap V'$. Clearly $y \in \mathcal{O}'_\alpha$ for some α. Hence there must be $g_U, g_V \in G$ such that $y = g_U x'_\alpha = g_V x'_\alpha$ with $g_U x_\alpha \in U, g_V x_\alpha \in V$. But then $g_U = g_V h$ for some $h \in H'_\alpha \subset H_\alpha$ so $g_U x_\alpha = g_V h x_\alpha = g_V x_\alpha$ contradicting disjointness of U and V. Hence Hausdorffness of X implies that any two points from $Y \setminus \bigcup_\alpha \mathcal{O}'_\alpha$ can be separated by open sets. Since each orbit $\mathcal{O}'_\alpha$ is closed (compact) and open in Y and since G/H'_α is Hausdorff this is sufficient to guarantee Hausdorffness of τ'.

τ' is compact: Since for every every τ-open set U containing $\mathcal{O}_\alpha$, U' contains $\mathcal{O}'_\alpha$ and since all the $\mathcal{O}_\alpha$ are open and all the $\mathcal{O}'_\alpha$ compact, compactness of X implies compactness of Y.

The group action on Y is continuous: This follows directly from the definition and the fact that the group action on X is continuous. □

Chapter 5

Discrete Groups

In this chapter we will consider countable discrete (Abelian) groups, in particular $\mathbb{Z}$ and $\mathbb{Z}^2$. We note however that the techniques presented will work for $\mathbb{Z}^n$ as well. We will find that a major factor in compactifiability is the size of the set X. In particular, there is a fundamental difference between sets with $|X| \geq \mathfrak{c}$ and sets with $|X| < \mathfrak{c}$ with the latter being far more restricted than the former. As will be explained below this is due to the fact that a compact Hausdorff space of size less than $\mathfrak{c}$ is a zero-dimensional, scattered space.

Note that since all discrete groups are locally compact (in their discrete topology), the results from chapters 2, 3 and 4 apply to the abstract group actions we will be investigating in this chapter.

As remarked in the introduction the case for $\mathbb{Z}$-actions has been solved independently in [8] and [7]. We may summarize their results as follows.

Theorem 5.1. *An abstract group action $(X, \mathbb{Z})$ is compact-realizable if and only if*

1. *all orbits are finite $\implies$ there are finitely many minimal orbit types or $|X| \geq \mathfrak{c}$;*

2. *all orbits are infinite $\implies$ $|X| \geq \mathfrak{c}$.*

This theorem provides a good starting point to explore the situation in $\mathbb{Z}^2$.

We note that the second implication is reflected in the corollary to the Continuum Theorem (Corollary 3.16) and still holds for $\mathbb{Z}^2$. As there are a variety of different infinite orbits, it does not tell the whole story. There might be essential forced-compact subsets and we need to apply this result to each of those.

A similar observation can be made about the first implication. Here the Adding Lemma (Lemma 4.1), which is also applicable to $G = \mathbb{Z}$, partially explains the condition. Again we have to be careful to apply it to all the essential forced-compact subsets. However, even if there are no essential forced-compact subsets and we have at least $\mathfrak{c}$ many orbits, there is a distinctly different flavour in the construction of an appropriate topology. This comes from the potential interaction between two generators of $\mathbb{Z}^2$, i.e. the existence of orbits whose stabilizer contains $S^l T^m$ but neither S^l nor T^m.

The most striking difference between $G = \mathbb{Z}$ and $G = \mathbb{Z}^2$ however lies in the case which is not explicitly mentioned in the above theorem, namely the mixture of finite and infinite orbits in a countable phase space. For $G = \mathbb{Z}$ a clever construction (in fact [7] and [8] use two entirely different constructions) ensures that in this case we always have a compactifiable group action. As we will see when examining the same case for $G = \mathbb{Z}^2$, these constructions rely on the fact that for $G = \mathbb{Z}$ and any two distinct points x, y in the same infinite orbit, there is essentially only one path between the two, i.e. one of $x, Sx, S^2x, \ldots, S^nx, y$ or $x, S^{-1}x, S^{-2}x, \ldots, y$ where S is a generator of $\mathbb{Z}$. This clearly does not hold in $\mathbb{Z}^2$ any more and the multiplicity of paths between such two points allows the deduction of additional necessary conditions. In particular, we show that the existence of a finite orbit and an orbit with trivial stabilizer will, in general, not be sufficient to guarantee that an abstract group action $(X, \mathbb{Z}^2)$ is compactifiable.

5.1 The Group $\mathbb{Z}^2$

Let us note a corollary of Lemma 4.1 which is similar in spirit to Theorem 4.4.

Lemma 5.2. *If an abstract group action $(X, \mathbb{Z}^2)$ has only finitely many minimal orbit types and each of the minimal orbit types is finite, then the group action is compact-realizable.*

It will also be convenient to fix a set of generators S, T for $\mathbb{Z}^2$. Given a subset $\mathcal{S}$ of a group G we will write $\langle \mathcal{S} \rangle$ for the smallest subgroup of G containing $\mathcal{S}$, i.e. the group generated by $\mathcal{S}$.

5.1.1 Forced Compact Subsystems

Note that for $G = \mathbb{Z}$ there are no essential forced-compact subsets. This is different for $G = \mathbb{Z}^2$ as can be easily seen from an abstract group action composed of orbits of type $\left\langle S^{l_0}, T^{n_i} \right\rangle \cup \left\langle S^{l_i}, ST^{n_i} \right\rangle$ where n_i does not divide n_j and l_i does not divide l_j for $i \neq j$.

We will therefore focus on fundamental group actions, i.e. group actions which do not contain an essential forced-compact proper subset. Note that finitely many compact-realizable fundamental group actions can be pieced together by simply taking the topological sum. Problems only occur when infinitely many, possibly intersecting essential forced-compact subsets exist.

5.1.2 The Subgroups of $\mathbb{Z}^2$

Since all orbit types of abstract group actions of $\mathbb{Z}^2$ will be subgroups of $\mathbb{Z}^2$ we now give a listing of all of these, using the generators S and T.

Lemma 5.3. *Every subgroup H of $\mathbb{Z}^2 = \langle S, T \rangle$ is one of the following:*

1. $\left\langle S^k, S^l T^m \right\rangle = \left\langle S^j T^n, T^{mk/j} \right\rangle$ *where* $0 \leq |l| < k,\ 0 < m,\ j = \gcd(k, l)$, *$n$ is chosen such that* $j \equiv nl \pmod{k}$ *and* $0 \leq n < mk/j$; *in this case the corresponding orbit is finite.*

2. $\left\langle S^l T^m \right\rangle$ *where* $l, m \in \mathbb{Z}$; *in this case the corresponding orbit is infinite.*

Proof. We write $(\alpha, \beta)\, \mathbb{Z}$ for $\{(n\alpha, n\beta) : n \in \mathbb{Z}\}$.

First observe that for any $m_1, n_1, m_2, n_2, \alpha \in \mathbb{Z}$ we have

$$(m_1, n_1)\, \mathbb{Z} + (m_2, n_2)\, \mathbb{Z} = (m_1, n_1)\, \mathbb{Z} + (m_2 + \alpha m_1, n_2 + \alpha n_1)\, \mathbb{Z}.$$

Hence, using the technique from the Euclidean algorithm, for each (m_1, n_1), $(m_2, n_2) \in \mathbb{Z}^2$ where the m_i are non-zero, there are $(m, 0), (m', n') \in \mathbb{Z}^2$ such that

$$(m_1, n_1)\,\mathbb{Z} + (m_2, n_2)\,\mathbb{Z} = (m, 0)\,\mathbb{Z} + (m', n')\,\mathbb{Z}. \tag{5.1}$$

Thus if $\{S^{m_1}T^{n_1}, S^{m_2}T^{n_2}\} \cup F$ (m_i non-zero) is a set of generators of $H \leq G$, then so is $\left\{S^m, S^{m'}T^{n'}\right\} \cup F$ for some $m, m', n' \in \mathbb{Z}$. We may then further simplify F so that if $S^pT^q \in F$ then $|p| < |m|, |q| < |n'|$.

Thus F is finite and we may apply equation 5.1 repeatedly, to obtain a set of generators consisting of at most one element of type S^lT^m with l, m non-zero and elements of type S^{k_i} and T^{m_i}. Taking the greatest common divisor for both the families $\{k_i\}$ and $\{m_i\}$, we arrive at a generating set of at most $\left\{S^lT^m, S^{k_0}, T^{m_0}\right\}$ with $k_0, m_0, l, m \in \mathbb{Z}$.

If this set actually contains less than three elements we stop. Otherwise, note that

$$(l, m)\,\mathbb{Z} + (0, m_0)\,\mathbb{Z} = \left(l\frac{m_0}{j}, 0\right)\mathbb{Z} + (l', j)\,\mathbb{Z}$$

for $j = \gcd(m, m_0) = \alpha m + \beta m_0$ and $l' = \alpha l$, to obtain a set of generators of the form $\left\{S^lT^m, S^{k_0}, S^{k_1}\right\}$. If k then is the greatest common divisor of k_0 and k_1 a generating set $\left\{S^k, S^lT^m\right\}$ with two elements is found. □

Above we have fixed an arbitrary set of generators $\{S, T\}$ for $\mathbb{Z}^2$. It can be advantageous however, to choose some specific generators. It is therefore important to know that given any $g = S^LT^M \neq e$, we can find generators S', T' of $\mathbb{Z}^2$ such that $g = S'^{L'}$ for some $L' \in \mathbb{Z}$.

Lemma 5.4. *Suppose $\{S, T\}$ generate $\mathbb{Z}^2$ and that $g = S^LT^M \neq e$ with $\gcd(L, M) = 1$. Then there is $h \in \mathbb{Z}^2$ such that $\{g, h\}$ generate $\mathbb{Z}^2$.*

Proof. Since $\gcd(L, M) = 1$ there are $P, Q \in \mathbb{Z}$ such that $PL + QM = 1$. Consider $h = S^QT^{-P}$ and note that $g^Ph^M = S^{PL+QM}T^{PM-MP} = S$ whereas $g^Qh^{-L} = S^{QL-LQ}T^{QM+LP} = T$. □

5.2 Small Phase Spaces

We will first consider group actions on sets of cardinality less than $\mathfrak{c}$. We will see that these are fundamentally different from group actions on sets of cardinality greater or equal to $\mathfrak{c}$, although the latter may contain closed G-invariant subsets of size $< \mathfrak{c}$.

5.2.1 Necessary Conditions using Scattered Spaces

The main result of this section will be the following theorem. Just as important as the theorem, however, are the techniques used in the proof. In fact, these techniques allow a detailed study of the structure of the group action.

Theorem 5.5. *Suppose that $(X, \mathbb{Z}^2)$ is a continuous compact Hausdorff group action with $|X| < \mathfrak{c}$.*

- *If $\mathbb{Z}^2 x$ is an infinite orbit, then there is $y \in X$ with finite orbit satisfying* $\mathrm{stab}(x) \subset \mathrm{stab}(y)$.
- *If there is no orbit with trivial stabilizer, then there are finitely many $g_1, \ldots, g_n \in \mathbb{Z}^2$ such that there are orbits of type $\langle g_i \rangle$, $i = 1, \ldots, n$ and all but finitely many minimal orbits have stabilizer $\left\langle g_i^{k_i}, h_i \right\rangle$, $k_i \neq 0$, $h_i \in \mathbb{Z}^2$.*
- *If there is no orbit with stabilizer $\langle g \rangle$ for $g \neq e$ then there are only finitely many minimal orbit types.*

5.2.1.1 Zero-dimensional and scattered spaces

We need two facts about compact Hausdorff spaces X with $|X| < \mathfrak{c}$.

Theorem 5.6. *Every compact Hausdorff space X with $|X| < \mathfrak{c}$ is zero-dimensional, i.e. has a basis of closed-and-open sets.*

Recall that a scattered space is a topological space in which every non-empty subspace contains an isolated point. In a scattered space we can define a tower of subsets, the derived sets, by recursion. For any subspace A of X,

A' denotes the set of limit points of A in X. With this notation define by recursion

$$\begin{aligned} X^{(0)} &= X \\ X^{(\alpha+1)} &= \left(X^{(\alpha)}\right)' \\ X^{(\gamma)} &= \bigcap_{\beta<\gamma} X^{(\beta)} \text{ for } \gamma \text{ a limit ordinal} \end{aligned}$$

The height of X, $\mathrm{ht}(X)$, is the least ordinal α such that $X^{(\alpha)} = \emptyset$ if such an α exists. The height of a point $x \in X$, $\mathrm{ht}(x)$, is the unique ordinal α such that $x \in I_\alpha(X) = X^{(\alpha)} \setminus X^{(\alpha+1)}$, the α^{th} level of X. Thus $I_\alpha(X)$ is the collection of isolated points in $X^{(\alpha)}$. We may totally order the points of X by $x \leq_s y$ if and only if $\mathrm{ht}(x) \leq \mathrm{ht}(y)$ with $x =_s y$ if and only if $\mathrm{ht}(x) = \mathrm{ht}(y)$. The top level of the scattered space is then the collection of all maximal points with respect to $\leq_s$.

Using this terminology we can state the following well known theorem.

Theorem 5.7. *Every compact Hausdorff space X with $|X| < \mathfrak{c}$ is scattered with height a successor ordinal $< \mathfrak{c}$ and finite top-level.*

5.2.1.2 Analysing scattered zero-dimensional compact Hausdorff phase spaces

In the following we identify conditions on the orbit spectrum of (X, G) which must hold under this assumption. Although our analysis is mainly concerned with the case $G = \mathbb{Z}^2$, the techniques can be used to study the cases $G = \mathbb{Z}^n$ for any $n \in \mathbb{N}$.

We start with a two simple observations.

Lemma 5.8. *If (X, G) is a compact group action with $|X| < \mathfrak{c}$ then every level $I_\alpha(X)$ and every derived set $X^{(\alpha)}$ is invariant under the group action.*

Proof. For each $x \in X$, $\mathrm{ht}(x)$ is preserved under homeomorphisms. □

Corollary 5.9. *If (X, G) is a compact group action with $|X| < \mathfrak{c}$ then there is a finite orbit.*

Proof. The top-level of X is finite, non-empty and invariant under the group action, so it must consist of finite orbits. ☐

We now proceed with an intricate analysis of various cases, depending on the existence or non-existence of certain infinite orbits.

Our approach will be made in two steps. First, we show that if we have a finite open partition satisfying certain properties, then we can make deductions about the relationship of the stabilizers of various points of the phase space. Secondly, we show that under certain conditions on the orbit spectrum of (X, G), a particularly nice finite open partition exists.

Let us start with a definition.

Definition 5.10. Suppose $\mathcal{U}$ is a finite open partition of the scattered space X.

We say that $\mathcal{U}$ satisfies the maximality condition if and only if for each $U \in \mathcal{U}$ there is a unique $\leq_s$-maximal point in U, say z_U. For $x \in X$ we write U_x for the unique $U \in \mathcal{U}$ with $x \in U$ and z_x for the $\leq_s$-maximal element of U_x. Write $M_{\mathcal{U}} = \{z_U : U \in \mathcal{U}\}$, the set of $\leq_s$-maximal points with respect to $\mathcal{U}$. An element z_x of $M_{\mathcal{U}}$ will be called $\mathcal{U}$-maximal or simply maximal if $\mathcal{U}$ is clear from the context.

Now assume that G acts on X. If $\mathcal{U}$ satisfies the maximality condition we say that the pair $(x, h) \in X \times G$ is obedient with respect to $\mathcal{U}$ if and only if

$$\begin{aligned} z_x \leq_s z_{hx} &\implies hx \in U_{hz_x} \\ \text{and } z_x \geq_s z_{hx} &\implies x \in U_{h^{-1}z_{hx}}. \end{aligned}$$

For a fixed $H \subset G$ invariant under taking inverses, we say that x is obedient with respect to $\mathcal{U}$ and H if and only (x, h) is obedient with respect to $\mathcal{U}$ for all $h \in H$. If H or $\mathcal{U}$ are clear from the context we omit the references to them. We use the term rebellious to indicate that a pair $(x, h) \in X \times G$ or a point $x \in X$ is not obedient.

We denote the set of all $\mathcal{U}, H$-obedient points by $G^H_{\mathcal{U}}$ or simply $G_{\mathcal{U}}$ if H is clear from the context.

We call a set $M \subset X$ periodically closed with respect to H if and only if for each $x \in M$ and each $W = S^L T^M \in \mathbb{Z}^2$ such that x is W-periodic, there is a finite sequence $f_0 = e, f_1, \ldots, f_{k(x,W)} = W$ of elements of G (depending on both x and W) such that $h_i = f_{i+1}f_i^{-1} \in H$ for $i = 0, \ldots, k-1$, $i \neq j \implies h_i \neq h_j^{-1}$ and

$$\forall i \in \{0, \ldots, k(x,W)-1\}, l \in \mathbb{Z}.\ f_i W^l x \in M.$$

Again, if H is clear from the context, we will omit the reference to it.

We say that $\mathcal{U}$ satisfies the periodic closure condition (pcc) in $X^{(\beta)}$ if the set $X^{(\beta)} \cap \{z_U : U \in \mathcal{U}\}$ is periodically closed. If $\mathcal{U}$ satisfies the pcc in $X^{(0)}$ then we simply say that $\mathcal{U}$ satisfies the pcc.

The definition and a lot of what is to follow makes sense for H not closed under taking inverses. However, since we want to maintain a certain symmetry, we will always assume in the following that in fact $H^{-1} = H$.

We also remark that (x,h) being obedient roughly means that x follows z_x under h up to the open partition $\mathcal{U}$. That a set M is periodically closed means that for each W-periodic point x of M there is a path in the set M through $x, Wx, W^2x, W^3x, \ldots$ with each step in the path being an element of H. It can therefore be thought of as a form of (discrete) path-connectedness.

From now on we will always assume that H is a finite, fixed subset of G and that $\langle H \rangle = G$. If $G = \mathbb{Z}^2$ we will always work with $H = \{S^{\pm 1}, T^{\pm 1}\}$ in the remainder of this section unless explicitly stated otherwise.

Let us note a few facts about the various concepts defined here.

Lemma 5.11. *Suppose that (X,G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$ and that $\mathcal{U}$ is a finite open partition satisfying the maximality condition.*

Then the following holds for all $x \in X, g \in G$.

(i) (x,g) is obedient if and only if

$$z_x \leq_s z_{gx} \implies z_{gx} = z_{gz_x}$$
$$\text{and } z_{gx} \leq_s z_x \implies z_x = z_{g^{-1}z_{gx}}.$$

(ii) If $gz_x = z_{gx}$ then (x,g) is obedient.

(iii) *If* (x, g) *is obedient and* $z_x =_s z_{gx}$ *then* $gz_x = z_{gx}$.

(iv) *If* $z_x = x$ *then* (x, g) *is obedient.*

(v) *If* $y \in U_x$ *then* $x \in U_y$ *and* $U_x = U_y$.

(vi) *If* $z_y \in U_x$ *then* $z_x = z_y$.

(vii) *If* (x, g) *is obedient then so is* (gx, g^{-1}).

Proof. This follows directly from the definitions and the fact that

$$w \in U_y \iff y \in U_w.$$

□

These facts will be used in the following without mentioning them explicitly.

Our interest in open partitions satisfying the maximality condition and the pcc stems from the next theorem. It shows that a lot of information about the structure of a dynamical system is contained in these open partition.

Theorem 5.12. *Suppose that* (X, G) *is a compact group action (G Abelian) with* $|X| < \mathfrak{c}$ *and that* $\mathcal{U}$ *is a finite open partition satisfying the maximality condition and the pcc such that all points of* X *are obedient.*

Suppose that x *is a point of* X *such that* z_x *is* $\leq_s$*-minimal among the* z_y *for* $y \in Gx$. *If* x *is* W*-periodic* ($W \in G$) *with period* q *and* z_x *is* W*-periodic with period* p *then* $q \in p\mathbb{Z}$.

Proof. Let $f_0(z_x, W) = e, f_1(z_x, W), \ldots, f_{k(z_x,W)}(z_x, W)$ be the sequence provided by periodic closedness of $\mathcal{U}$. Then $f_i W^l z_x =_s z_x$ is maximal for all i, l and hence all of them are $\leq_s$-minimal among the $z_y, y \in Gx$.

The obedience of $f_i W^l x$ $(i = 0, \ldots, k(z_x, W) - 1; l \in \mathbb{Z})$ then implies by induction that $z_{f_i W^l x} = f_i W^l z_x$ for $i = 0, \ldots, k_W - 1; l \in \mathbb{Z}$. But $W^q x = x$ so $z_x = z_{W^q x} = W^q z_x$. If p is the W-period of z_x then we must clearly have $p|q$, i.e. $q \in p\mathbb{Z}$. □

It is clear that using the last theorem with different partitions of X will give different information about the group action. If, for example, a partition contains a point with trivial stabilizer as a $\mathcal{U}$-maximal point, then the last theorem does not tell us very much unless we have a more detailed knowledge of the partition. Our next task is therefore to construct particularly nice open partitions, for example ones where every $\mathcal{U}$-maximal point has non-trivial stabilizer. However, first we give a lemma that extends the information we can extract using the above theorem.

Lemma 5.13. *Suppose that (X, G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$ and that $\mathcal{U}$ is a finite open partition satisfying the maximality condition and the pcc such that all points of X are obedient.*

If $x \in X$, z_x is $\leq_s$-minimal among the $z_y, y \in Gx$ and z_x is g-periodic for all $g \in G$, then $z_{gx} = gz_x$ for all $g \in G$ and $\operatorname{stab}(x) \subset \operatorname{stab}(z_x)$*.*

Proof. Suppose that $z_x \neq x$ and let $W \in H$. Since $z_x \leq_s z_{Wx}$ we have $Wx \in U_{Wz_x}$. However, z_x being W-periodic implies that Wz_x is $\mathcal{U}$-maximal. Thus $z_{Wx} = Wz_x$ and z_{Wx} is also $\leq_s$-minimal among the $z_y, y \in Gx$. Since H was assumed to generate G, we have by induction that in fact $z_{gx} = gz_x$ for all $g \in G$. If now $g \in \operatorname{stab}(x)$ then $gz_x = z_{gx} = z_x$ so that $g \in \operatorname{stab}(z_x)$ and thus $\operatorname{stab}(x) \subset \operatorname{stab}(z_x)$ as claimed. □

The scatteredness of the space X suggest to start with an open partition approximating the structure of the group action rather crudely and refining this further and further until all points are obedient. We will go down through the layers of the scattered space, since for each point $x \in X$ we can find a closed-and-open neighbourhood of it in which x is the $\leq_s$-maximal point.

Our starting point, the partition approximating the structure of (X, G) in a rather crude way, is provided by the following.

Lemma 5.14. *Suppose that (X, G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$.*

Then there is a finite open partition satisfying the maximality condition such that the top-level of X consists of obedient points only.

Proof. Since the top-level is finite, we can enumerate it as $T = \{x_1, \dots, x_n\}$, say. For $i = 1, \dots, n-1$ choose closed-and-open sets U_i such that $U_i \cap T = \{x_i\}$, $i = 1, \dots, n-1$ and that the U_i are pairwise disjoint. This is possible as X is Hausdorff. Now set $U_n = X \setminus \bigcup_{i=1}^{n-1} U_i$ and note that $\mathcal{U} = \{U_1, \dots, U_n\}$ is as desired. □

The important observation which allows us the inductive refinement of the open partitions consist of the following two lemmas.

Lemma 5.15. *Suppose that (X, G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$ and that $\mathcal{U}$ is a finite open partition satisfying the maximality condition.*

If H is is a finite subset of G then the collection of obedient points, $G^H_{\mathcal{U}}$, is closed-and-open in X.

Proof. First assume that $x \in X$ is obedient. Then (x, h) is obedient for every $h \in H$. Consider the open neighbourhood $V_h = U_x \cap h^{-1}(U_{hx})$ of x. If $y \in V_h$ then $z_y = z_x$, $z_{hy} = z_{hx}$ and $hz_y = hz_x$. Therefore if $z_y \leq_s z_{hy}$ then $z_x \leq_s z_{hx}$ and therefore $hy \in U_{hx} = U_{hz_x} = U_{hz_y}$ where the first equality follows from obedience of (x, h). If on the other hand $z_y \geq_s z_{hy}$ then similarly $z_x \geq_s z_{hx}$ and therefore $y \in U_x = U_{h^{-1}z_{hx}} = U_{h^{-1}z_{hy}}$. Thus (y, h) is obedient. Now consider the open neighbourhood $\bigcap_{h \in H} V_h$ of x and note that by the foregoing every point of it is obedient. Therefore $G^H_{\mathcal{U}}$ is open.

Conversely suppose that x is rebellious. Then there is some $h \in H$ such that (x, h) is rebellious. Take any point $y \in V_h$ as defined above. If $z_y \leq_s z_{hy}$ then $hy \in U_{hx} \neq U_{hz_x} = U_{hz_y}$ where the inequality expresses the fact that (x, h) is rebellious. If on the other hand $z_y \geq_s z_{hy}$ we may deduce that $y \in U_x \neq U_{h^{-1}z_{hx}} = U_{h^{-1}z_{hy}}$. Thus (y, h) is rebellious for every $y \in V_h$. Therefore $G^H_{\mathcal{U}}$ is closed. □

The last lemma implies that the set of rebellious points of X is a closed hence compact subset and as such a scattered space itself.

Lemma 5.16. *Suppose that (X, G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$ and that $\mathcal{U}$ is a finite open partition satisfying the maximality condition.*

The least ordinal α such that $X^{(\alpha)} \subset G_{\mathcal{U}}$ is a successor ordinal $\beta_{\mathcal{U}} + 1$ or 0 in which case we write $\beta_{\mathcal{U}} = -1$. In the case that $\beta_{\mathcal{U}} \neq -1$ the set of rebellious points in level $\beta_{\mathcal{U}}$, $I_{\beta_{\mathcal{U}}}(X) \setminus G_{\mathcal{U}}$, is non-empty and finite.

Proof. This is a direct consequence of Lemma 5.14, Theorem 5.7 and the compactness of the set of rebellious points which follows from Lemma 5.15. □

In order to describe how to change a finite open partition $\mathcal{U}$ satisfying the maximality condition, it is convenient to have some descriptive language which hides the technical complexities. The following definition provides two basic operations which we will use over and over in the process of refinement.

Definition 5.17. Suppose that (X, G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$. Let $\mathcal{U}$ be a finite open partition satisfying the maximality condition. When saying that we move $x \in X$ to $U \in \mathcal{U}$ we mean that we choose a closed-and-open set $U' \subset U_x$ such that x is the unique $\leq_s$-maximal point in U' and replace $\mathcal{U}$ by the finite open partition

$$(\mathcal{U} \setminus \{U, U_x\}) \cup \{U_x \setminus U', U \cup U'\}\,.$$

When saying that we isolate $x \in X$ we mean that we choose a closed-and-open set $U' \subset U_x$ such that x is the unique $\leq_s$-maximal point in U' and replace $\mathcal{U}$ by the finite open partition

$$(\mathcal{U} \setminus \{U_x\}) \cup \{U_x \setminus U', U'\}\,.$$

It is important to know how the obedience status of pairs (x, g) of $X \times G$ change when moving or isolating a point. This is what the next lemma describes.

Lemma 5.18. *Suppose that (X, G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$. Let $\mathcal{U}$ be a finite open partition satisfying the maximality condition.*

If we move a point $x \in X$ to $U \in \mathcal{U}$ then the obedience status of all (y, g) with $y \in X^{(\mathrm{ht}(x))}$ and $y, gy \neq x$ is unchanged ($g \in G$).

If we isolate a point $x \in X$ then the obedience status of all (y, g) with $y \in X^{(\mathrm{ht}(x))}$ and $y, gy \neq x$ is unchanged ($g \in G$), whereas (x, g) will be obedient for any $g \in G$.

Proof. Suppose that $y, gy \neq x$ and $y \in X^{(\mathrm{ht}(x))}$. Then by moving x to $U \in \mathcal{U}$ we have not changed $z_y, z_{gy}, gz_y, g^{-1}z_{gy}, z_{gz_y}, z_{g^{-1}z_{hy}}$. Since obedience of (y, g) depends only on the latter, it has not changed. By isolating x on the other hand we have not changed z_y, z_{gy}. If $y \neq z_y$ then $\mathrm{ht}(z_y) > \mathrm{ht}(x)$ so none of hz_y, z_{gz_y} has changed. If on the other hand $y = z_y$ then $gz_y = gy, z_{gz_y} = z_{gy}$ and these have not been changed. Similar considerations apply to $g^{-1}z_{gy}$ and $z_{g^{-1}z_y}$. Finally to see that (x, g) is now obedient just note that by definition $z_x = x$ after the isolation. □

In the process of isolating points in $I_\alpha(X)$, we might arrive at an open partition that does not satisfy the pcc in $X^{(\alpha)}$ although the original partition did. The next lemma ensures that further isolation of at most finitely many points gives an open partition which has the pcc in $X^{(\alpha)}$ without worsening the obedience status of points in $X^{(\alpha)}$.

Lemma 5.19. *Suppose that $(X, \mathbb{Z}^2)$ is a compact group action with $|X| < \mathfrak{c}$ and that $\mathcal{U}$ is a finite open partition satisfying the maximality condition.*

If all points of $X^{(\beta)}$ are obedient and all $\mathcal{U}$-maximal points have height at least β, i.e. $M_{\mathcal{U}} \subset X^{(\beta)}$, and $\mathcal{U}$ satisfies the pcc in $X^{(\beta+1)}$ then there exists a finite open partition $\mathcal{U}'$ satisfying the maximality condition such that all points of $X^{(\beta)}$ are $\mathcal{U}'$-obedient and $\mathcal{U}'$ satisfies the pcc in $X^{(\beta)}$. Moreover the sets $\{z_x \colon x \in X, \mathrm{stab}(z_x) = \{e\}\}$ are identical with respect to $\mathcal{U}, \mathcal{U}'$.

Proof. We will isolate finitely many points in $I_\beta(X)$. By Lemma 5.18 this will give a finite open partition $\mathcal{U}'$ satisfying the maximality condition such that all points of $X^{(\beta)}$ remain obedient.

Let

$$S = \{x \in I_\beta(X) \colon z_x = x\}$$

and assume $x \in S$. If x is periodic, then isolate all points in Gx, noting that the latter is finite. We then certainly have that for any $W \in \mathbb{Z}^2$ any $y \in Gx$ and any sequence $f_0 = e, \ldots, f_{k(y,W)} = W \in \mathbb{Z}^2$ with $f_{i+1}f_i^{-1} \in H$

that $z_{f_i W^l y} = f_i W^l z_y$. If on the other hand x has stabilizer $\langle W^n \rangle$, where $W = S^L T^M \in \mathbb{Z}^2$ with $\gcd(L, M) = 1$ we let

$$g_0 = e, g_1 = S^{\operatorname{sgn}(L)1}, \dots, g_{|L|} = S^L,$$
$$g_{|L|+1} = S^L T^{\operatorname{sgn}(M)1}, \dots, g_{|L|+|M|} = S^L T^M)$$

and $h_{i+|L|+|M|} = h_i = g_{i+1} g_i^{-1} \in H$ for $i = 0, \dots, |L| + |M| - 1$. Then isolate all points $y_{i,m} = g_i W^m x$ for $i = 0, \dots, |L| + |M| - 1; m = 0, \dots, n-1$ and define

$$f_k(y_{i,m}, W) = \prod_{j=1}^{k} h_{j+i-1}, \quad k = 0, \dots, |L| + |M|$$

(we redefine $f_k(y_{i,m}, W)$ in this way if necessary). As every element W' under which $y_{i,m}$ is periodic can be generated by W we can define the sequence $f_k(y_{i,m}, W')$ by concatenation and possible reversal from $f_k(y_{i,m}, W)$.

Informally, what we do is choose a band (without backtracking) through x around the cylinder representing Gx and isolate all its points. Then we choose a sequence for each point in the band so that its images under elements from the sequence are in the band again.

Doing this for every $x \in S$ gives a finite open partition $\mathcal{U}'$ as required.

As no points with stabilizer $\{e\}$ were isolated the sets

$$\{z_x \colon x \in X, \operatorname{stab}(z_x) = \{e\}\}$$

are indeed identical for $\mathcal{U}, \mathcal{U}'$. □

Recall that we would like to arrive eventually at a finite open partition with respect to which there are no maximal points with trivial stabilizer. As this is our only aim, we can deal with other points in a crude way.

Lemma 5.20. *Suppose that (X, G) is a compact group action (G Abelian) with $|X| < \mathfrak{c}$ and that $\mathcal{U}$ is a finite open partition satisfying the maximality condition such that all points of $X^{(\beta+1)}$ are obedient and $\mathcal{U}$ satisfies the pcc in $X^{(\beta+1)}$.*

If all points $x \in I_\beta(X)$ with $\operatorname{stab}(x) = \{e\}$ are $\mathcal{U}$-obedient, then there is a finite open partition $\mathcal{U}'$ satisfying the maximality condition such that all *points of* $X^{(\beta)}$ *are* $\mathcal{U}'$*-obedient, the sets* $\{z_x\colon \operatorname{stab}(z_x) = \{e\}\}$ *are identical for* $\mathcal{U}$ *and* $\mathcal{U}'$*, and* $\mathcal{U}'$ *satisfies the pcc in* $X^{(\beta)}$*.*

Proof. Isolate all $\mathcal{U}$-rebellious points in the β^{th} level of X all of which have non-trivial stabilizer by assumption. By Lemma 5.15 there are only finitely many of them and by Lemma 5.18 all points of $X^{(\beta)}$ will be obedient in the new partition. Finally apply Lemma 5.19 to the new partition in order to obtain $\mathcal{U}'$ as desired. □

Note that quite often we could probably do much better, in that we do not really need to isolate as many points as we describe above. For our purposes, however, the previous lemma is sufficient.

We are therefore left with the problem that some points (in the level under consideration) with trivial stabilizer are rebellious. What we need to do is to move them, so that they will be obedient in the new partition. We will make essential use of the two-dimensional nature of Gx in case $\operatorname{stab}(x) = \{e\}$.

An auxiliary lemma will be shown first. This involves some form of 'diagram chasing'. The conditions we list in the lemma can be read as 'all points in higher layers behave very well' and referred to later when actually using these lemmas.

Lemma 5.21. *Suppose that* (X, G) *is a compact group action (*G *Abelian) with* $|X| < \mathfrak{c}$ *and that* $\mathcal{U}$ *is a finite open partition satisfying the maximality condition.*

Suppose we have a diagram as shown

$$\begin{array}{ccc} a & \xrightarrow{W} & x \\ {\scriptstyle V}\big\downarrow & & \big\downarrow{\scriptstyle V} \\ b & \xrightarrow[W]{} & y \end{array}$$

where $W, V \in H$*,* $(Vz_a, W), (V^{-1}z_b, W)$ *are obedient and*

$$Wz_a = z_x, \quad Wz_{VW^{-1}z_x} = z_{WVW^{-1}z_x},$$
$$Wz_b = z_y, \quad Wz_{V^{-1}W^{-1}z_y} = z_{WV^{-1}W^{-1}z_y}$$

$(a, b, x, y \in X; W, V \in G)$*.*

Then (x, V) *is obedient.*

Proof. Suppose first that $z_x \leq_s z_y$. $z_a = W^{-1}z_x \leq_s W^{-1}z_y = z_b$ and obedience of (a, V) means $b = Va \in U_{Vz_a} = U_{VW^{-1}z_x}$. Therefore $VW^{-1}z_x \in U_b$ giving $z_{VW^{-1}z_x} = z_b = W^{-1}z_y$. Hence $z_{Vz_x} = z_{WVW^{-1}z_x} = Wz_{VW^{-1}z_x} = z_y$ and therefore $Vz_x \in U_{z_y} = U_y$. Thus $Vx = y \in U_{Vz_x}$ as required. If on the other hand $z_x \geq_s z_y$ then the same argument with a, b, x, y, V replaced by b,a,y,x,V^{-1} respectively shows that $x \in U_{V^{-1}z_{Vx}}$. We have therefore shown that (x, V) is obedient as claimed. □

We have assembled all the necessary ingredients to be able to deal with rebellious points with trivial stabilizer.

Lemma 5.22. *Suppose that $(X, \mathbb{Z}^2)$ is a compact group action with $|X| < \mathfrak{c}$. Assume that $\mathcal{U}$ is a finite open partition of X satisfying the maximality property, that all points of $X^{(\beta+1)}$ are obedient, all $\mathcal{U}$-maximal points are contained in $X^{(\beta+1)}$ and $\mathcal{U}$ satisfies the pcc in $X^{(\beta+1)}$.*

If $\mathcal{O}$ is an orbit of type $\{e\}$ such that all points $\{z_y : y \in \mathcal{O}\}$ are periodic, then there is a finite open partition $\mathcal{U}'$ of X satisfying the maximality condition such that all $\mathcal{U}$-obedient points of $X^{(\beta)}$ are $\mathcal{U}'$-obedient, all points of $\mathcal{O}$ are $\mathcal{U}'$-obedient, all $\mathcal{U}'$-maximal points are contained in $X^{(\beta+1)}$ and $\mathcal{U}'$ satisfies the pcc in $X^{(\beta+1)}$.

Proof. If no point of $\mathcal{O}$ is rebellious we are done. So assume that there are rebellious points in $\mathcal{O}$. By Lemma 5.15 there are only finitely many rebellious points in $\mathcal{O}$. Thus there is a rectangle $R = (0, K) \times (0, L) \subset \mathbb{Z}^2$ with (K, L) minimal in the partial product order on $\mathbb{N} \times \mathbb{N}$ such that there is $x \in \mathcal{O}$ with all rebellious points contained in W^kV^lx for $(k, l) \in R$. In the following diagram of the situation the arrows labelled 1 are obedient by assumption.

$$
\begin{array}{ccccccccc}
x & \xrightarrow[1]{W} & Wx & \xrightarrow[1]{W} & W^2x & \cdots & W^{K-1}x & \xrightarrow[1]{W} & W^Kx \\
V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 \\
Vx & \xrightarrow[1]{W} & WVx & \xrightarrow{W} & W^2Vx & \cdots & W^{K-1}Vx & \xrightarrow[1]{W} & W^KVx \\
V\Big\downarrow 1 & & V\Big\downarrow 2 & & V\Big\downarrow & & V\Big\downarrow & & V\Big\downarrow 1 \\
V^2x & \xrightarrow[1]{W} & WV^2x & \xrightarrow{W} & W^2V^2x & \cdots & WK-1V^2x & \xrightarrow[1]{W} & W^KV^2x \\
V\Big\downarrow 1 & & V\Big\downarrow 2 & & V\Big\downarrow & & V\Big\downarrow & & V\Big\downarrow 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
V\Big\downarrow 1 & & V\Big\downarrow 2 & & V\Big\downarrow & & V\Big\downarrow & & V\Big\downarrow 1 \\
V^{L-1}x & \xrightarrow[1]{W} & WV^{L-1}x & \xrightarrow{W} & W^2V^{L-1}x & \cdot\cdot & W^{K-1}V^{L-1}x & \xrightarrow[1]{W} & W^KV^{L-1}x \\
V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 \\
V^Lx & \xrightarrow[1]{W} & WV^Lx & \xrightarrow[1]{W} & W^2V^Lx & \cdots & W^{K-1}V^Lx & \xrightarrow[1]{W} & W^KV^Lx
\end{array}
$$

Our plan is to move points in Rx to shrink the rectangle until all points of $\mathcal{O}$ are obedient.

Since $\mathcal{U}$ satisfies the pcc in $X^{(\beta+1)}$ we have $Wz_u = z_{Wu}$ and $Vz_u = z_{Vu}$ for every $z_u \in X^{(\beta+1)}$. From Lemma 5.21 we can deduce that (WV^lx, V) (the arrows labelled 2) are obedient for $l = 1, \ldots, L-1$. If $K \geq 4$, we move the points W^2V^lx to $U_{Wz_{WV^lx}}$ for $l = 1, \ldots, L-1$. From Lemma 5.18 and $K \geq 4$ this only changes the obedience status of points of R and possibly V^2x, W^KV^2x. By the choice of the moves, however, we have made sure that (WV^lx, W) is obedient. Therefore we have shrunk the width, K, of R and possibly increased its height, L. Repeat this until $K < 4$. If $K \leq 2$ we use Lemma 5.21 to see that in fact $K = 0$, i.e. there are no more rebellious points in $\mathcal{O}$. Thus we may assume that $K = 3$. The picture now looks as follows, where points labelled $*_g, *_a, *_b$ and arrows labelled 1 are obedient by assumption, arrows labelled 2 are obedient by Lemma 5.21.

$$
\begin{array}{ccccccc}
*_g & \xrightarrow[1]{W} & *_a & \xrightarrow[1]{W} & *_b & \xrightarrow[1]{W} & *_g \\
V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 & & V\Big\downarrow 1 \\
*_g & \xrightarrow[1]{W} & *_u & \xrightarrow[4]{W} & *_y & \xrightarrow[1]{W} & *_g \\
V\Big\downarrow 1 & & V\Big\downarrow 2 & & V\Big\downarrow 2 & & V\Big\downarrow 1 \\
*_g & \xrightarrow[1]{W} & * & \xrightarrow{W} & * & \xrightarrow[1]{W} & * \\
V\Big\downarrow 1 & & V\Big\downarrow 2 & & V\Big\downarrow 2 & & V\Big\downarrow 1 \\
\vdots & & \vdots & & \vdots & & \vdots \\
V\Big\downarrow 1 & & V\Big\downarrow 2 & & V\Big\downarrow 2 & & V\Big\downarrow 1 \\
*_g & \xrightarrow[1]{W} & * & \xrightarrow{W} & * & \xrightarrow[1]{W} & *_g \\
V\Big\downarrow 1 & & V\Big\downarrow 2 & & V\Big\downarrow 2 & & V\Big\downarrow 1 \\
*_g & \xrightarrow[1]{W} & *_g & \xrightarrow[1]{W} & *_g & \xrightarrow[1]{W} & *_g
\end{array}
$$

Using Lemma 5.21 with W and V interchanged shows that in fact the arrow labelled 4 is obedient, and thus, by induction, all arrows are obedient. $\square$

Let us collect the lemmas together to construct certain open partitions to which we can then apply Theorem 5.12.

First among these is the case in which no orbits of type $\langle g \rangle, g \neq e$ exist. Note that this includes the possibility that all orbits are in fact finite. The idea is that when combined with Theorem 5.12, we can conclude that there are only finitely many minimal orbit types.

Theorem 5.23. *Suppose that $(X, \mathbb{Z}^2)$ is a compact group action with $|X| < \mathfrak{c}$. If there are no orbit types of the form $\langle g \rangle, g \neq e$, then there is a finite open partition $\mathcal{U}$ of X satisfying the maximality property and the pcc, such that* $\mathrm{stab}(z_x) \neq \{e\}$ *for every $x \in X$ and with respect to which every point is obedient. Hence X contains only finitely many minimal orbits.*

Proof. We claim that for every α there is an open partition $\mathcal{U}$ of X with $\{z_x : x \in X\} \subset X^{(\alpha)}$ satisfying the maximality property and the pcc in $X^{(\alpha)}$ such that all points in $X^{(\alpha)}$ are obedient and $\mathrm{stab}(z_x) \neq \{e\}$ for every $x \in X$.

By Lemma 5.14 this is true for every α with $\alpha + 1 \geq \mathrm{ht}(X)$. Let α_0 be the minimal α for which the above is true. If $\alpha_0 = 0$ we are done. Otherwise let $\mathcal{U}$ witness that α_0 satisfies the above. By Lemma 5.16 we may assume that $\alpha_0 = \beta + 1$ and that the set N of rebellious points in $I_\beta(X)$ is non-empty and finite. If N does not contain points with stabilizer $\{e\}$ we can apply Lemma 5.20 and obtain a contradiction to minimality of α_0. Otherwise apply Lemma 5.22 and again Lemma 5.20 to obtain a contradiction to minimality of α_0.

To see that this implies that there are only finitely many minimal orbit types, enumerate $G_{\mathcal{U}}$ as $z_1, \ldots, z_n$. If $x \in X$ has finite orbit, let $y_x \in Gx$ be such that z_{y_x} is minimal among the $z_y, y \in Gx$. Since z_{y_x} is periodic under every element of $\mathbb{Z}^2$ (it has a finite orbit) we can apply Lemma 5.13 to see that $\mathrm{stab}(y_x) \subset \mathrm{stab}(z_{y_x})$. Hence only those orbits containing points from $G_{\mathcal{U}}$ have minimal orbit type and there are only finitely many of these. □

Secondly, we consider the case if there are no orbits with trivial stabilizer. This can be proven by the same technique as the previous.

Theorem 5.24. *Suppose $(X, \mathbb{Z}^2)$ is a compact group action with $|X| < \mathfrak{c}$. If there are no orbits with trivial stabilizer, then there is a finite open partition $\mathcal{U}$ of X satisfying the maximality condition and the pcc such that every point of X is obedient. Thus, there are $g_1 = S^{L_1}T^{M_1}, \ldots, g_n = S^{L_n}T^{M_n} \in \mathbb{Z}^2$ such that $\gcd(L_i, M_i) = 1$, there exists an orbit of type $\langle g_i^{k_i} \rangle$ for every i and for all but finitely many minimal orbit types there is i (depending on the orbit) such that the g_i-period of it is a multiple of k_i.*

Proof. By Lemma 5.13 every orbit Gx with minimal type must either contain a $\mathcal{U}$-maximal point (and therefore consists entirely of $\mathcal{U}$-maximal points) or z_x must be in an infinite orbit and $\leq_s$-minimal among the $z_y, y \in Gx$. Write $\mathrm{stab}(z_x) = \langle (S^L T^M)^k \rangle$ with L, M co-prime and observe that by Theorem 5.12 the $S^L T^M$-period of x is a multiple of k.

As there are only finitely many $\mathcal{U}$-maximal points with respect to $\mathcal{U}$ the result follows. □

We can now prove the Theorem 5.5.

Proof of Theorem 5.5. We prove the three statements in turn.

If Gx is an infinite orbit, then there is $y \in X$ with finite orbit satisfying $\mathrm{stab}(x) \subset \mathrm{stab}(y)$**.** Write $\mathrm{stab}(x) = \langle g \rangle$ for $g \in \mathbb{Z}^2$ and consider Fix_g. This is an invariant closed subset of X and as such must contain a finite orbit, all of whose elements are necessarily fixed under g.

If there is no orbit with trivial stabilizer, then there are finitely many $g_1, \dots, g_n \in \mathbb{Z}^2$ such that there are orbits of type $\langle g_i \rangle, i = 1, \dots, n$ and all but finitely many minimal orbits have stabilizer $\langle g_i^k, h \rangle$, $k \neq 0$, $h \in \mathbb{Z}^2$. This follows directly from Theorem 5.24.

If there is no orbit with stabilizer $\langle g \rangle$ for $g \neq e$ then there are only finitely many minimal orbit types. This follows directly from Theorem 5.23. □

5.2.2 Necessary Conditions using Even Continuity

There is an alternative approach to derive some of the results from the section 5.2.1. This has been used in [8] to obtain necessary conditions for the case $G = \mathbb{Z}$. It centres around the concept of even continuity which originates in the study of function spaces.

5.2.2.1 Even Continuity and the Ascoli Theorem

This material can be found in great detail in [6]. We only give the definition of a family of evenly continuous functions and the Ascoli Theorem.

Suppose a F is a family of continuous functions from X to Y, both being topological spaces. Fixing some $x \in X$, we say that F is evenly continuous at x provided for every $y \in Y$ and every open set $V \ni y$ there is an open set $U \ni x$ and an open set $W \ni y$ such that for all $f \in F$ $f(x) \in W \implies f(U) \subset V$.

We can explain the concept roughly as follows: Given an x and some open set V continuity of a map f guarantees an open neighbourhood U of

x with $f(U) \subset V$ provided of course that $f(x) \in V$. Even continuity of a collection F of maps guarantees that the same U works for every element of F. If, for example f is a continuous self-map on a space X fixing a point x then continuity of f says that points sufficiently close to x will be mapped to points which are also close to x. Even continuity of $\{f^n : n \in \mathbb{N}\}$ means that in fact the whole itinerary of points sufficiently close to x will stay close to x.

Trivially every finite collection of continuous maps is evenly continuous. In this context we can see even continuity as a kind of compactness for a collection of maps. The precise statement of this is the Ascoli Theorem.

Theorem 5.25 (Ascoli Theorem)**.** *If X is a k-space (i.e. the continuous Hausdorff image of a locally compact space) and Y is a regular space, then a closed subset F of the collection of continuous functions from X to Y, Y^X, with the compact-open topology is compact if and only if F is an evenly continuous family of mappings and the set $\{f(x) : f \in F\} \subset Y$ has compact closure for every $x \in X$.*

In the case of interest to us we have that $X = Y$ is a compact Hausdorff space and the collection F is a subset of the group G, i.e. consists entirely of homeomorphisms. Compactness of X ensures that the last condition of the theorem is fulfilled so that we may paraphrase it in this particular situation as "F is evenly continuous if and only if it is a compact subset of X^X with the compact-open topology".

5.2.2.2 Showing even continuity of group actions

When proving that a certain collection of functions is evenly continuous at a point we make use of the following observations. The first of these concerns even continuity at fixed points.

Lemma 5.26. *Suppose that X is a Hausdorff space. If x is fixed under every element of F and F is not evenly continuous at x then we can find a basic neighbourhood V of x, a net $x_\alpha \to x$ and $f_\alpha \in F$ such that $f_\alpha(x_\alpha) \notin V$ for all α.*

Proof. Suppose that F is not evenly continuous at x and that $f(x) = x$ for every $f \in F$. By the definition of even continuity, we can find a $y \in X$ and a neighbourhood V of y such that for every neighbourhood U of x and every neighbourhood W of y there is some $f_U \in F$ with $f_U(x) \in W$ and $f_U(U) \not\subset V$.

Note that we must have $x = y$. For, if $x \neq y$ then by Hausdorffness we can find a neighbourhood W of y not containing x and then $f_U(x) = x \notin W$, a contradiction. On the other hand, if $x = y$ then the condition $f_U(x) \in W$ is trivially fulfilled.

If we let the U range through a neighbourhood basis of x then choosing an $x_U \in U$ with $f_U(x_U) \notin V$ results in a net x_U converging to x and $f_U \in F$ such that $f_U(x_U) \notin V$. Clearly we may shrink V to a basic neighbourhood V' of x. □

We can now extend the previous Lemma from fixed points (points whose orbit is has size 1) to points with finite orbit under F. This is expected since in view of the Ascoli theorem, even continuity can be seen as a compactness-like property.

Lemma 5.27. *Suppose X is a Hausdorff space. If F is an Abelian group of homeomorphisms and the orbit of x under F is finite, then F is evenly continuous at x if and only if $\operatorname{stab}(x)$ is evenly continuous at x.*

Proof. If F is evenly continuous at x, then so is $\operatorname{stab}(x)$. Conversely, suppose $\operatorname{stab}(x)$ is evenly continuous at x. Take any $y \in X$ and any neighbourhood V of y.

If y is not in the finite orbit of x then choose a neighbourhood W of y disjoint from the orbit of x under F. Clearly $f(x) \notin W$ for every $f \in F$, so we may choose any neighbourhood U of x.

If on the other hand $y = f(x)$ for some $f \in F$, then $f^{-1}(V)$ is a neighbourhood of x. Thus by even continuity of $\operatorname{stab}(x)$ at x, there is a neighbourhood U of x such that $h(U) \subset f^{-1}(V)$ for every $h \in \operatorname{stab}(x)$ and thus $fh(U) \subset V$ for every $h \in \operatorname{stab}(x)$. Choose W such that $W \cap Fx = \{y\}$.

With these choices of U and W we have that if $g \in F$ then either $g(x) = y$ hence $f^{-1}g \in \operatorname{stab}(x)$ and so $g(U) = f(f^{-1}g(U)) \subset V$ or $g(x) \neq y$ and then $g(x) \notin W$. $\square$

The previous two Lemmas suggest that provided that all orbits are finite, a continuous group action is in fact evenly continuous. That this is indeed the case for finitely generated groups is shown in the next Lemma.

Lemma 5.28. *Suppose X is a compact Hausdorff zero-dimensional space and that G is a finitely generated Abelian group of auto-homeomorphisms of X such that (X, G) has only finite orbits. Then G is evenly continuous on X.*

Proof. Suppose not. Then G is not evenly continuous at some $x \in X$. By Lemma 5.27 we may assume that in fact x is fixed under G. Then by Lemma 5.26 and zero-dimensionality of X, we can find a closed-and-open neighbourhood U of x, a net $x_\alpha \to x$ and $f_\alpha \in G$ such that $f_\alpha(x) \notin U$.

Let $f_1, \ldots, f_n$ be a minimal set of generators for G and write $f_\alpha = f_1^{m_1^\alpha} \cdots f_n^{m_n^\alpha}$ for $(m_1^\alpha, \ldots, m_n^\alpha) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n |m_i^\alpha|$ is minimal.

Choose the f_α such that $s_\alpha = \sum |m_i^\alpha|$ is minimal. We must have s_α unbounded for otherwise only finitely many different f_α were involved. Hence without loss of generality m_1^α is unbounded. By passing to a subnet and possibly replacing f_1 by f_1^{-1} we may assume that $m_1^\alpha \to \infty$ and that $m_1^\alpha > 0$.

Consider the points $y_\alpha = f_1^{-1} f_\alpha x_\alpha$. By minimality of s_α we have $y_\alpha \in U$. By compactness, we may assume, by taking an appropriate subnet, that y_α converges to some $y \in \overline{U} = U$. By continuity of f_1 we know that $f_1 y_\alpha \to f_1 y \in X \setminus U$. Since y has a finite orbit under G, there is a $p \in \mathbb{N}$ such that $f_1^p y = y$ and hence that $f_1^{1-p} y = f_1 y \notin U$. As $n_1^\alpha \to \infty$ eventually $n_\alpha - p > 0$ and thus by minimality of s_α we then have

$$f_1^{1-p} y_\alpha = f_1^{-p} f_\alpha x_\alpha \in U.$$

But then considering the convergence of the left hand side, we find

$$f_1^{1-p} y = f_1 y \in \overline{U} = U,$$

a contradiction. □

5.2.2.3 Using even continuity of group actions

Having established that for certain group actions continuity implies even continuity, we can use this stronger property to construct open sets which interact nicely with the stab-map. These open sets are analogous to the open sets in the obedient partitions constructed earlier.

Lemma 5.29. *Suppose X is a compact Hausdorff space, (X, G) (G Abelian) has only finite orbits and G is evenly continuous on X.*

Then $U(x) = \{y \in X\colon \operatorname{stab}(y) \leq \operatorname{stab}(x)\}$ is open for every $x \in X$.

Proof. Note that it is sufficient to show that x is in the interior of $U(x)$, since if $y \in U(x)$ then $U(y) \subset U(x)$.

As the orbit of x under G is finite, we can enumerate its elements as $x_0, \ldots, x_n$ with $x = x_0$. Choose pairwise disjoint open sets $U_0, \ldots, U_n$ such that $x_i \in U_i$ for $i = 0, \ldots, n$.

By even continuity for each i there is an open neighbourhood V_i of x such that $\forall g \in G.\ g(x) = x_i \implies g(V_i) \subset U_i$. Let $V = U_0 \cap \bigcap_{i=0}^{n} V_i$.

Then for every $g \in G$, there is precisely one j such that $g(V) \cap U_j \neq \emptyset$. This follows since $g(x) = x_j$ for a unique j and then $g(V) \subset g(V_j) \subset U_j$ and the U_j are all disjoint.

If $y \in V \subset U_0$ and $g \in \operatorname{stab}(y)$ then $y \in g(V) \cap U_0 \neq \emptyset$, so $g(V) \subset U_0$. But then necessarily $g(x) = x_0 = x$ as $Gx \cap U_0 = \{x\}$, so $g \in \operatorname{stab}(x)$.

Thus for every $y \in V$ $\operatorname{stab}(y) \leq \operatorname{stab}(x)$. Hence $V \subset U(x)$ and since $x \in V$ and V is the finite intersection of open sets, thus open, the result follows. □

We collect the results together to obtain a strong necessary condition for compact group actions by a finitely generated group with countable, compact, Hausdorff phase space in which all orbits are finite. Note that in the special case $G = \mathbb{Z}^2$ this is a strictly weaker version of Theorem 5.5. The ease of its proof, the greater generality and the different point of view, however, make the approach using even continuity worthwhile. A careful analysis of even continuity and its connection to the Ascoli theorem might show that in fact the two approaches lead to equivalent results.

Theorem 5.30. *Suppose the group action (X, G) with G finitely generated and Abelian has only finite orbits.*

If it is compact-realizable, then either one orbit type occurs at least $\mathfrak{c}$ often or there are only finitely many minimal orbit types.

Proof. Suppose that no orbit type occurs at least $\mathfrak{c}$ often. Since G is finitely generated, G must be countable and hence $|X| < \mathfrak{c}$. Thus if (X, τ) is a compact Hausdorff space for some topology τ, it must be zero-dimensional. Hence if the elements of G are auto-homeomorphisms of (X, τ), then G is evenly continuous and so the sets $U(x)$ are open for every $x \in X$. Since the $U(x)$ cover the compact space X there is a finite subcover $U(x_1), \ldots, U(x_n)$. But then for any $y \in X$, we have $\operatorname{stab}(y) \leq \operatorname{stab}(x_i)$ for some $i = 1, \ldots, n$. Hence the only minimal orbit types can be $\operatorname{stab} Gx_1, \ldots, \operatorname{stab} Gx_n$. □

5.2.3 Constructions

One of the most useful methods of constructing compact group actions with small phase spaces uses symbolic dynamical systems. Classically a symbolic dynamical system is a closed subset of 2^ω with shift map σ defined by $\sigma((x_n)_{n\in\omega}) = (x_{n+1})_{n\in\omega}$. σ is then a (non-injective) continuous self-map of the compact Hausdorff space 2^ω. This can be easily extended to $2^\mathbb{Z}$. In fact, if G is any discrete group then 2^G is a compact Hausdorff space on which G acts continuously by $g(x_h)_{h\in G} = (x_{gh})_{h\in G}$. Of course the 'base' space 2 can be replaced by any other compact Hausdorff space F, although typically F is finite. In fact, if F is finite then it can be shown that G acting on F^G is conjugate to a G-action on a compact, shift-invariant subset of 2^G. Note that it is important that G is discrete. If G were not discrete, then the continuity of the G-action could no longer be guaranteed, although every map $x \mapsto gx$ for fixed $g \in G$ would still be continuous.

We will be interested in symbolic dynamical systems $F^{\mathbb{Z}^2}$ where the $\mathbb{Z}^2$ action is represented by the left-shift $S : (x_{n,m}) \mapsto (x_{n-1,m})$ and the up-shift $T : (x_{n,m}) \mapsto (x_{n,m-1})$.

We have, unfortunately, not managed to construct all the fundamental group actions not forbidden by the results from Theorem 5.5. In particular, it appears necessary in various cases that there are two orbits with types of the form $\langle g^K \rangle \neq \{e\}$ which are linearly independent. We therefore present a selection of constructions which we feel to be particularly instructive.

The first construction is a direct generalization of the construction in [8] for $G = \mathbb{Z}$ with finite and infinite orbits. One might be tempted to assume that it should generalize easily when the finite orbits of types $\langle S^{k_i}, T^{m_i} \rangle$ are replaced by more general finite orbits of types $\langle S^{k_i}, S^{l_i} T^{m_i} \rangle$. However, this seems not to be the case unless $\gcd(l_i, k_i) \to \infty$ as $i \to \infty$ or we also introduce more infinite orbits. We will consider this case below.

We use the notation $a_i \to \infty$ to mean that a_i diverges, i.e. that for every $N \in \mathbb{N}$ there is $I \in \mathbb{N}$ such that $i > I \implies a_i > N$.

Lemma 5.31. *Suppose the group action $(X, \mathbb{Z}^2)$ has orbit spectrum*

$$\left\{\{e\}, \langle S^l \rangle, \langle T^n \rangle, \langle S^k, T^m \rangle\right\} \cup \left\{\langle S^{k_i}, T^{m_i} \rangle : i \in \mathbb{N}\right\}$$

where l, n, k, m, k_i, m_i are positive integers, $k|l$, $m|n$, $k_i \to \infty$ and $m_i \to \infty$.

Then (X, G) is compactifiable.

Proof. Let $j = l/k$, $j' = n/m$. $F = \{0, 1, 2, 3\}$ and let A be a $k \times m$-block

$$A = \begin{matrix} 0 & 1 & \cdots & 1 \\ 1 & & & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{matrix},$$

i.e. A consists of 1s except for a 0 in the top left corner.

For each i write $k_i = \alpha_i j k + \beta_i$, $m_i = \alpha'_i j' m + \beta'_i$ for $0 < \beta_i \leq m$, $0 < \beta'_i \leq k$. Write $\alpha_i = \gamma_i \beta_i + \delta_i$, $\alpha'_i = \gamma'_i \beta'_i + \delta'_i$ with $0 < \delta_i \leq \beta_i$ and $0 < \delta'_i \leq \beta'_i$. Choose N sufficiently large that $\gamma_i < \beta_i$, $\gamma'_i > \beta'_i$ for all $i \geq N$. For $i \geq N$ consider the block

$$
B_i = \begin{array}{ccccccccccccc}
 & & & & & & & & & \downarrow & & & \\
A & \cdots & A & Y_1 & A & \cdots & A & Y_1 & \cdots & Y_1 & A & \cdots & A \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\
A & \cdots & A & Y_{j'\gamma'_i} & A & \cdots & A & Y_{j'\gamma'_i} & \cdots & Y_{j'\gamma'_i} & A & \cdots & A \\
X_1 & \cdots & X_{j\gamma_i} & Z & X_1 & \cdots & X_{j\gamma_i} & Z & \cdots & Z & X_1 & \cdots & X_{j\delta_i} \\
A & \cdots & A & Y_1 & A & \cdots & A & Y_1 & \cdots & Y_1 & A & \cdots & A \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\
A & \cdots & A & Y_{j'\gamma'_i} & A & \cdots & A & Y_{j'\gamma'_i} & \cdots & Y_{j'\gamma'_i} & A & \cdots & A \\
X_1 & \cdots & X_{j\gamma_i} & Z & X_1 & \cdots & X_{j\gamma_i} & Z & \cdots & Z & X_1 & \cdots & X_{j\delta_i} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X_1 & \cdots & X_{j\gamma_i} & Z & X_1 & \cdots & X_{j\gamma_i} & Z & \cdots & Z & X_1 & \cdots & X_{j\delta_i} \\
A & \cdots & A & Y_1 & A & \cdots & A & Y_1 & \cdots & Y_1 & A & \cdots & A \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\
A & \cdots & A & Y_{j'\delta'_i} & A & \cdots & A & Y_{j'\delta'_i} & \cdots & Y_{j'\delta'_i} & A & \cdots & A
\end{array}
\begin{array}{c}
\\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \longleftarrow \\ \\ \\ \\
\end{array}
$$

Each $\begin{smallmatrix} A & \cdots & A \\ \vdots & & \vdots \\ A & \cdots & A \end{smallmatrix}$ except for the ones furthest to the right and those furthest to the bottom consists of $\gamma_i j \times \gamma'_i j'$ blocks of A. There are $\beta_i \times \beta'_i$ of such blocks (separated by columns and rows of Ys and Xs). The ones right of the rightmost column of Ys (marked with $\downarrow$) but not below the lowest row of Xs (marked with $\longleftarrow$) consist of $\delta_i \times \gamma'_i$ blocks of A, those below the lowest row of Xs but not right of the rightmost column of Ys consist of $\gamma_i \times \delta'_i$ blocks of A and the one in the bottom right corner consists of $\delta_i \times \delta'_i$ blocks of A.

The Y_is represent a column (height m) of 2s when i is not divisible by j' and a column (height m) of 3s if it is. The X_is similarly represent a row (length k) of 2s when i is not divisible by j and a row (length k) of 3s if it is. The Zs are just single 2s.

This turns B_i into a

$$\beta_i\left(\gamma_i jk + 1\right) + \delta_i jk \times \beta'_i\left(\gamma'_i j'm + 1\right) + \delta'_i j'm = k_i \times m_i$$

block of symbols from F.

Since $\gamma_i > \delta_i > 0$ and $\gamma_i' > \delta_i' > 0$ it is clear that the element of $F^{\mathbb{Z}^2}$,

$$C_i = \begin{matrix} & \vdots & \vdots & \vdots & \\ \cdots & B_i & B_i & B_i & \cdots \\ \cdots & B_i & B_i & B_i & \cdots \\ \cdots & B_i & B_i & B_i & \cdots \\ & \vdots & \vdots & \vdots & \end{matrix},$$

has stabilizer $\left\langle S^{k_i}, T^{m_i} \right\rangle$.

Consider now the closure of $\bigcup_{i\in\omega} \langle S, T\rangle C_i$. Clearly, since $k_i, m_i \to \infty$, we also have $\gamma_i, \gamma_i' \to \infty$. Hence, apart from $\bigcup_{i\in\omega} \langle S, T\rangle C_i$ the closure also contains the following elements and their orbits under G:

- $$D_0 = \begin{matrix} & \vdots & \vdots & \vdots & \\ \cdots & A & A & A & \cdots \\ \cdots & A & A & A & \cdots \\ \cdots & A & A & A & \cdots \\ & \vdots & \vdots & \vdots & \end{matrix}$$

  (stay unboundedly far away from any X and any Y) with stabilizer $\left\langle S^k, T^m \right\rangle$,

- $$D_1 = \begin{matrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A & A & Y & A & A & \cdots \\ \cdots & A & A & Y & A & A & \cdots \\ \cdots & A & A & Y & A & A & \cdots \\ \cdots & A & A & Y & A & A & \cdots \\ \cdots & A & A & Y & A & A & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix},$$

  where each Y is a column of 2s except that every j'^{th} Y is a column of 3s (stay close to a column of Ys but unboundedly far from a row of Xs), having stabilizer $\left\langle T^{j'm} \right\rangle = \langle T^n \rangle$,

- a similar element D_2 for with a row of Xs instead of a column of Ys with stabilizer $\left\langle S^{jk} \right\rangle = \left\langle S^k \right\rangle$,

- and

$$D_3 = \begin{matrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A & A & Y_{-2} & A & A & \cdots \\ \cdots & A & A & Y_{-1} & A & A & \cdots \\ \cdots & X_{-2} & X_{-1} & 2 & X_1 & X_2 & \cdots \\ \cdots & A & A & Y_1 & A & A & \cdots \\ \cdots & A & A & Y_2 & A & A & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix},$$

  where X_{-1-nj} and X_{nj} are rows of 3s for $n \in \mathbb{N}$ whereas the other X_i are rows of 2s, and $Y_{-1-nj'}$ and $Y_{nj'}$ are columns of 3s for $n \in \mathbb{N}$ whereas the other Y_i are columns of 2s. Clearly D_3 has trivial stabilizer.

On this compact Hausdorff space the maps S and T act continuously with the desired orbit spectrum. □

We can apply the results from chapter 4 to the last construction to obtain the following.

Theorem 5.32. *Suppose that $(X, \mathbb{Z}^2)$ is a fundamental abstract group action with orbit spectrum*

$$\left\{\{e\}, \left\langle S^l \right\rangle, \langle T^n \rangle, H\right\} \cup \{H_i \colon i \in \mathbb{N}\}$$

where $\mathbb{Z}^2/H$ is finite, $S^l, T^n \in H$ and $H_i = \left\langle S^{k_i}, S^{l_i} T^{m_i} \right\rangle$ with $\gcd(k_i, l_i) \to \infty$, $m_i \to \infty$.

Then (X, G) is compactifiable.

In the above it is important (and implied by the non-existence of essential forced-compact proper subsets) that $m_i \to \infty$. If this is not the case, a different construction is possible.

Lemma 5.33. *Suppose that $(X, \mathbb{Z}^2)$ is a fundamental abstract group action with orbit spectrum*

$$\left\{\left\langle S^{k_0} \right\rangle, \{e\}\right\} \cup \{H_i \colon i \in \mathbb{N}\}$$

where $H_i = \left\langle S^{k_i}, S^{l_i} T^m \right\rangle$ with $0 \leq l_i < k_i, 0 < m$ are minimal orbit types

If $S^{l_i} T^m \in H_0$ for each i then $(X, \mathbb{Z}^2)$ is compactifiable.

Proof. Let A be the $k_0 \times m$ block

$$A = \begin{matrix} 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & & \cdots & 1 \end{matrix}.$$

Write $k_i = \alpha_i k_0 + \beta_i, \alpha_i = \gamma_i \beta_i + \delta_i$ where $0 < \beta_i \leq k_0, 0 < \delta_i \leq \beta_i$.

Let B_i be the element

$$\begin{matrix} & & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ S^{-2l_i} & (\cdots & A & \cdots & A & X & A & \cdots & A & X & \cdots & \cdots & X & A & \cdots & A\cdots) \\ S^{-l_i} & (\cdots & A & \cdots & A & X & A & \cdots & A & X & \cdots & \cdots & X & A & \cdots & A\cdots) \\ S^{0} & (\cdots & A & \cdots & A & X & A & \cdots & A & X & \cdots & \cdots & X & A & \cdots & A\cdots) \\ S^{l_i} & (\cdots & A & \cdots & A & X & A & \cdots & A & X & \cdots & \cdots & X & A & \cdots & A\cdots) \\ S^{2l_i} & (\cdots & A & \cdots & A & X & A & \cdots & A & X & \cdots & \cdots & X & A & \cdots & A\cdots) \\ & & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & & \vdots \end{matrix}$$

of $\{0,1,2\}^{\mathbb{Z}^2}$, where X is a $1 \times m$ block of 2s and each line is the repetition of β_i copies of $A \cdots AX$ followed by δ_i copies of A. Clearly B_i has stabilizer H_i. We now consider the limit points of $\bigcup_{i\in\omega} \langle S, T \rangle B_i$. Here and in the following we write $S^r(\cdots)$ to mean that this row is shifted by r symbols to the right.

Note that since the group action does not contain essential forced-compact proper subsets we must have $k_i \to \infty$ and $l_i \to \infty$. Also, for every sequence $\eta_i \in \mathbb{Z}$ and every $\xi \in \mathbb{Z}$ we must have $\eta_i k_i + \xi l_i \to \infty$, for if $\eta_i k_i + \xi l_i = L \in \mathbb{Z}$ infinitely often, then $S^L T^{\xi m}$ will be in infinitely many H_i. However, this means that the distance between any to Xs in B_i tends to ∞ as $i \to \infty$. Also, note that $l_i \equiv l_0(\pmod k)_0)$ as $S^{l_i} T^m \in H_0$. Thus, we only get the following limit points:

- $$\begin{matrix} & & \vdots & \vdots & \vdots & \\ S^{-2l} & (\cdots & A & A & A & \cdots) \\ S^{-l} & (\cdots & A & A & A & \cdots) \\ S^{0} & (\cdots & A & A & A & \cdots) \\ S^{l} & (\cdots & A & A & A & \cdots) \\ S^{2l} & (\cdots & A & A & A & \cdots) \\ & & \vdots & \vdots & \vdots & \end{matrix}$$

  with stabilizer H_0;

- $$\begin{array}{ll} & \quad\;\; \vdots \quad \vdots \quad \vdots \\ S^{-2l+1} & (\cdots \; A \; A \; A \; \cdots) \\ S^{-l+1} & (\cdots \; A \; A \; A \; \cdots) \\ S^{0} & (\cdots \; A \; A \; A \; \cdots) \\ S^{l} & (\cdots \; A \; A \; A \; \cdots) \\ S^{2l} & (\cdots \; A \; A \; A \; \cdots) \\ & \quad\;\; \vdots \quad \vdots \quad \vdots \end{array}$$

  with stabilizer $\left\langle S^{k_0} \right\rangle$;

- $$\begin{array}{ll} & \quad\;\; \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ S^{-2l+1} & (\cdots \; A \; A \; A \; A \; A \; \cdots) \\ S^{-l+1} & (\cdots \; A \; A \; A \; A \; A \; \cdots) \\ S^{0} & (\cdots \; A \; A \; X \; A \; A \; \cdots) \\ S^{l} & (\cdots \; A \; A \; A \; A \; A \; \cdots) \\ S^{2l} & (\cdots \; A \; A \; A \; A \; A \; \cdots) \\ & \quad\;\; \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}$$

  with stabilizer $\{e\}$.

□

Finally we give a construction which picks up on the second claim of Theorem 5.5.

Lemma 5.34. *Suppose $(X, \mathbb{Z}^2)$ is a fundamental abstract group action with orbits of type $H_i = \left\langle S^{k_i}, T^{m_i} \right\rangle, k_i, m_i > 0$ and one orbit of type $H_\infty = \left\langle S^K \right\rangle$ such that $k_0 | K$ and $K | k_i$ for $i > 0$.*

Then $(X, \mathbb{Z}^2)$ is compactifiable.

Proof. Again, let A be the $k_0 \times m_0$ block

$$\begin{array}{cccc} 0 & 1 & \cdots & 1 \\ 1 & & \cdots & 1 \\ \vdots & & \vdots & \\ 1 & & \cdots & 1 \end{array}$$

and write $m_i = \alpha_i m_0 + \beta_i$ with $0 < \beta_i \leq m_0$, $\alpha_i = \gamma_i \beta_i + \delta_i$ with $0 < \delta_i \leq \beta_i$.

Then define a $k_i \times m_i$-block

$$B_i = \begin{matrix} A & A & \cdots & A \\ \vdots & \vdots & & \vdots \\ A & A & \cdots & A \\ X_1 & X_2 & \cdots & X_{k_i} \\ A & A & \cdots & A \\ \vdots & \vdots & & \vdots \\ A & A & \cdots & A \\ X_1 & X_2 & \cdots & X_{k_i} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ X_1 & X_2 & \cdots & X_{k_i} \\ A & A & \cdots & A \\ \vdots & \vdots & & \vdots \\ A & A & \cdots & A \end{matrix}$$

where all but the last $\begin{smallmatrix} A & A & \cdots & A \\ \vdots & \vdots & & \vdots \\ A & A & \cdots & A \end{smallmatrix}$ block has $k_i/k_0 \times \gamma_i$ As and the last block has $k_i/k_0 \times \delta_i$ As and X_i is a $k_0 \times 1$ block consisting of 2s if $i \in K\mathbb{Z}$ and 3s otherwise.

C_i is then the element of $\{0,1,2,3\}^{\mathbb{Z}^2}$ that consists entirely of Bs, i.e.

$$C_i = \begin{matrix} & \vdots & \vdots & \vdots & \\ \cdots & B_i & B_i & B_i & \cdots \\ \cdots & B_i & B_i & B_i & \cdots \\ \cdots & B_i & B_i & B_i & \cdots \\ & \vdots & \vdots & \vdots & \end{matrix}.$$

The limit points of $\bigcup_{i>0} \langle S, T \rangle C_i$ are then

- in the orbit corresponding to H_0, i.e.

$$\begin{matrix} & \vdots & \vdots & \vdots & \\ \cdots & A & A & A & \cdots \\ \cdots & A & A & A & \cdots \\ \cdots & A & A & A & \cdots \\ & \vdots & \vdots & \vdots & \end{matrix};$$

- in the orbit corresponding to H_∞, i.e.

$$\begin{array}{ccccc} & \vdots & \vdots & \vdots & \\ \cdots & A & A & A & \cdots \\ \cdots & X & X & X & \cdots, \\ \cdots & A & A & A & \cdots \\ & \vdots & \vdots & \vdots & \end{array}$$

  where every Kth X consists of 2s and all other Xs of 3s;

Note, however, that the stabilizer of each $C_i, i > 0$ is in fact $\left\langle S^K, T^{m_i} \right\rangle \supset H_i$ so that we use Lemma 4.7 to replace them with orbits of type H_i. □

5.3 Large Phase Spaces

Having examined $\mathbb{Z}^2$-actions for which the phase space has cardinality less than $\mathfrak{c}$, we will now turn to actions where the phase space is large, i.e. its cardinality is at least $\mathfrak{c}$. Whereas before we had a variety of restrictions regarding the number and type of minimal orbits, we will show that in the case of fundamental group actions with large orbit spaces, none of these restrictions apply.

5.3.1 Constructions

When the phase space is large, we have a new method of construction at our disposal, namely the torus. For small phase spaces, the orbit in the top layer of the scattered space could be regarded as a discrete torus. Most of the other orbits had to wind about this and because of the discrete nature, we were able to separate its elements cleanly. The constructions which follow have a similar feel to them, since we also have a 'base torus' about which the other orbits wind. However, since this time the base torus is solid, we can 'slip' from one orbit in it to another while converging to it. This makes constructions a lot easier. We outline the general method of construction by the observation below.

Theorem 5.35. *Suppose that Y is a compact Hausdorff space and that $\alpha\colon Y \to \mathbb{R}^n$ is a continuous map. Let $X = \mathbb{T}^n \times Y$. Then the map from X to itself given by*

$$(\theta_1, \dots, \theta_n, x) \mapsto (\theta_1 + \alpha_1(x), \dots, \theta_n + \alpha_n(x), x)$$

is continuous on X.

Typically Y will be homeomorphic to a convergent sequence, $\omega + 1$. In this case we may think of this construction as a sequence of tori shrinking to the base torus at ω and the map restricted to each $\mathbb{T}^n \times \{\beta\}$ as a rotation of this torus. Continuity means that these rotations approach the rotation on the base torus, as the tori shrink to it. Also note that if X_β is a closed subset of $\mathbb{T}^n \times \{\beta\}$ for $\beta < \omega$ then $\mathbb{T}^n \times \{\omega\} \cup \bigcup_{\beta > 0} X_\beta$ is a closed subset of X.

5.3.1.1 Technical results

Showing that the conditions in Theorem 5.35 can be satisfied in certain situations is rather technical. In this section we provide the lemmas required. The reader is encouraged to think about the η_i, ζ_i, ξ_i mentioned below as k_i, l_i, m_i that occur during compactification of a $\mathbb{Z}^2$-action with orbits of type $H_i = \left\langle S^{k_i}, S^{l_i} T^{m_i} \right\rangle$.

The first lemma will be applied in situations without essential forced-compact sets with non-trivial stabilizer, where most of the orbits are finite.

Lemma 5.36. *Suppose $(\eta_i, \zeta_i, \xi_i) \in \mathbb{Z}^3$ with $0 \le \zeta_i < \eta_i$ and $\xi_i \neq 0$ for all $i \in \omega$ are all distinct. Suppose also that $r, r' \in \mathbb{I}$.*

If for all $\Theta, \Theta' \in \mathbb{Z}$ the set

$$\{i \in \omega \colon \exists \alpha_i, \beta_i \in \mathbb{Z} \text{ s.t. } \alpha_i \eta_i + \beta_i \zeta_i = \Theta \text{ and } \beta_i \xi_i = \Theta'\} \tag{5.2}$$

is finite, then there are $\phi_i, \psi_i \in \mathbb{Z}$ such that

$$\begin{aligned} \frac{\phi_i}{\eta_i} &\to r \\ \frac{\psi_i \eta_i - \phi_i \zeta_i}{\xi_i \eta_i} &\to r'. \end{aligned}$$

It is easy to see that such choices can be made if both $|\eta_i|$ and $|\xi_i|$ converge to ∞. Since the assumption implies that $|\eta_i| \to \infty$ the problematic situation is that ξ_i takes some value infinitely often. We solve this special case first via an auxiliary lemma.

Lemma 5.37. *Suppose* $(\eta_i, \zeta_i, \xi_i) \in \mathbb{Z}^3$ *for* $i \in N \subset \mathbb{N}$ *are distinct where* N *is infinite,* η_i *diverging to* ∞*,* $0 \leq \zeta_i < \eta_i, \xi_i \neq 0$ *and* $\xi_i = \xi$ *for all* $i \in N$*.*

Suppose that for every $q \in \mathbb{Q}$ *and every* $M \in \omega$ *there is* $\delta > 0$ *such that for all but finitely many* $i \in N$*,* $|\zeta_i - q\eta_i| > M$*.*

Given $r, r' \in \mathbb{I}$ *and* $\epsilon > 0$ *there are* $\phi_i, \psi_i \in \mathbb{Z}$ *with*

$$\left|\frac{\phi_i}{\eta_i} - r\right| < \epsilon$$

and

$$\left|\frac{\psi_i \eta_i - \phi_i \zeta_i}{\xi_i \eta_i} - r'\right| < \epsilon$$

for all but finitely many $i \in N$*.*

The strategy in this proof and the one of Lemma 5.39 is the same. We guess a potential ϕ_i and a corresponding ψ_i. We then add a correction term $\Delta\phi_i$ which is small compared to η_i for large i. With a corresponding correction term $\Delta\psi_i$ both inequalities can then be satisfied simultaneously, since $\eta_i \to \infty$.

Proof. Without loss of generality we may assume that $N = \mathbb{N}$ by choosing a bijection between them.

Write

$$\delta_i(\phi) = \left|\frac{\phi}{\eta_i} - r\right| = \left|\frac{\phi - r\eta_i}{\eta_i}\right|$$

$$\epsilon_i(\phi, \psi) = \left|\frac{\psi\eta_i - \phi\zeta_i}{\xi_i\eta_i} - r'\right| = \left|\frac{\psi - \frac{\phi\zeta_i}{\eta_i} - r'\xi_i}{\xi_i}\right|$$

Let

$$A = \{x \in \mathbb{R}\colon \forall\delta > 0 \ \ |\{i \in \mathbb{N}\colon \zeta_i/\eta_i \in B_\delta(x)\}| = \aleph_0\}.$$

For each element α of A we will find an $\epsilon_\alpha > 0$ and an $N_\alpha \in \mathbb{N}$ such that for all $i > N_\alpha$ with $\zeta_i/\eta_i \in B_{\epsilon_\alpha}(\alpha)$, there are $\phi_i, \psi_i \in Z$ such that we have $\delta_i(\phi_i), \epsilon_i(\phi_i, \psi_i) < \epsilon$. Since A is compact, there are then finitely many $\alpha_1, \ldots, \alpha_n \in A$ such that $\bigcup_{i=1}^n B_{\epsilon_{\alpha_i}}(\alpha_i)$ covers A. For all but finitely many i we then have $\zeta_i/\eta_i \in \bigcup_{i=1}^n B_{\epsilon_{\alpha_i}}(\alpha_i)$ so there is $N' \in \mathbb{N}$ such that

$$i > N' \implies \zeta_i/\eta_i \in \bigcup_{i=1}^n B_{\epsilon_{\alpha_i}}(\alpha_i).$$

By considering $N_\epsilon = \max\{N', N_{\alpha_1}, \ldots, N_{\alpha_n}\}$ the result then follows.

Case $\alpha \in A \setminus \mathbb{Q}$: Suppose that $\alpha \in A \setminus \mathbb{Q}$. Since $\alpha, 2\alpha, 3\alpha, \ldots$ is dense in $\mathbb{T}$, there is a $K \in \mathbb{N}$ such that

$$\alpha, 2\alpha, \ldots, K\alpha$$

is $\epsilon/4$-dense in $\mathbb{T}$.

Let $\nu = \min\{i\alpha, 1 - i\alpha\colon i = 1, \ldots, K\}$ (all arithmetic taking place in $\mathbb{T}$, i.e. mod 1). Note that since α is irrational we have $\nu > 0$. Let $\epsilon_\alpha = \frac{\min\{\epsilon/4,\nu\}}{K}$. If $\zeta_i/\eta_i \in B_{\epsilon_\alpha}(\alpha)$ then

$$\zeta_i/\eta_i, \ldots, K\zeta_i/\eta_i$$

is $\epsilon/2$-dense in $\mathbb{T}$ (by the choice of $\epsilon_\alpha < \nu/K$ it is guaranteed that in $\mathbb{R}$ $k\zeta_i/\eta_i$ is not further from $k\alpha$ than ν for $k = 1, \ldots, K$, so that for some $n \in \mathbb{Z}$, both $k\zeta_i/\eta_i$ and $k\alpha$ are in $[n, n+1]$).

Let $\phi_i^0 \in \mathbb{Z}$ be such that $\delta_i(\phi_i^0)$ is minimal. Choose $\psi_i^0 \in \mathbb{Z}$ such that $\epsilon_i(\phi_i^0, \psi_i^0)$ is minimal.

Now choose $\Delta\phi_i \in \{0, \ldots, K\}$, $\Delta\psi_i \in \mathbb{Z}$ such that $\epsilon(\phi_i^0 + \Delta\phi_i, \psi_i^0 + \Delta\psi_i)$ is minimal. Since $\zeta_i/\eta_i, \ldots, K\zeta_i/\eta_i$ is $\epsilon/2$-dense we have

$$|\xi_i|\,\epsilon\left(\phi_i^0 + \Delta\phi_i, \psi_i^0 + \Delta\psi_i\right) < \epsilon$$

as long as $\zeta_i/\eta_i \in B_{\epsilon_\alpha}(\alpha)$.

On the other hand, since $\eta_i \to \infty$ we can find $N_\alpha \in \mathbb{N}$ such that if $i > N_\alpha$ then $\left|(1+K)/\eta_i\right| < \epsilon$. Noting that

$$\begin{aligned} \delta_i\left(\phi_i^0 + \Delta\phi_i\right) &\leq \delta_i\left(\phi_i^0\right) + K/\eta_i \\ &\leq (1+K)/\eta_i \\ &< \epsilon \end{aligned}$$

we may conclude that $i > N_\alpha$ and $\zeta_i/\eta_i \in B_{\epsilon_\alpha}(\alpha)$ imply that there are $\phi_i = \phi_i^0 + \Delta\phi_i$, $\psi_i = \psi_i^0 + \Delta\psi_i$ such that $\delta_i(\phi_i) < \epsilon$ and $\epsilon_i(\phi_i, \psi_i) < \epsilon$.

Case $\alpha \in A \cap \mathbb{Q}$: Now suppose that $\alpha \in A \cap \mathbb{Q}$. Write $\alpha = p/q$ in lowest terms with $p, q \in \mathbb{N}, 0 \leq p < q$. For every $i > 0$ write $\zeta_i/\eta_i = \alpha + \mu_i/\eta_i$ for some $\mu_i \in \mathbb{Q}$. By the assumption that $|\zeta_i - \alpha\eta_i| \to \infty$ as $i \to \infty$, we must have $|\mu_i| \to \infty$ as well. Choose $N'_\alpha \in \mathbb{N}$ such that $i > N'_\alpha \implies |\mu_i| > 4/\epsilon$ and assume $i > N'_\alpha$ in the following. Then

$$q\zeta_i/\eta_i, \ldots, \lfloor \eta_i/\mu_i \rfloor q\zeta_i/\eta_i$$

is just

$$q\mu_i/\eta_i, \ldots \lfloor \eta_i/\mu_i \rfloor q\mu_i/\eta_i$$

in $\mathbb{T}$ (i.e. $\mod 1$) and hence $|q\mu_i/\eta_i|$-dense in $\mathbb{T}$. Choose $\epsilon_\alpha > 0$ such that if $\zeta_i/\eta_i \in B_{\epsilon_\alpha}(\alpha)$ then $|q\mu_i/\eta_i| < \epsilon/2$.

Choose $\phi_i^0 \in \mathbb{Z}$ such that $\delta_i(\phi_i^0)$ is minimal and $\psi_i^0 \in \mathbb{Z}$ such that $\epsilon_i(\phi_i^0, \psi_i^0)$ is minimal. We can then find $\Delta\phi_i, \Delta\psi_i \in \mathbb{Z}$ with $0 \leq \Delta\phi_i \leq \eta_i/\mu_i$ such that $\epsilon_i(\phi_i^0 + \Delta\phi_i, \psi_i^0 + \Delta\psi_i) < \epsilon$. Since $\eta_i \to \infty$ we can choose $N''_\alpha \in \mathbb{N}$ such that if $i > N''_\alpha$ then $\delta_i(\phi_i^0) < \epsilon/4$.

By the choice of N'_α, if $i > N'_\alpha$ then $|1/\mu_i| < \epsilon/4$. Moreover we have $|\Delta\phi_i/\eta_i| \leq |1/\mu_i| < \epsilon/4$ and hence $\delta_i(\phi_i^0 + \Delta\phi_i) < \epsilon$.

Let $N_\alpha = \max\{N'_\alpha, N''_\alpha\}$. If $i > N_\alpha$ and $\zeta_i/\eta_i \in B_{\epsilon_\alpha}(\alpha)$ then for $\phi_i = \phi_i^0 + \Delta\phi_i$, $\psi_i = \psi_i^0 + \Delta\psi_i$ we have $\delta_i(\phi_i) < \epsilon$ and $\epsilon_i(\phi_i, \psi_i) < \epsilon$ as desired. □

Proof of Lemma 5.36. Note that the assumptions imply that $\eta_i \to \infty$. In fact, if η_i takes the value η infinitely often, then for $\Theta = \eta$ and $\Theta' = 0$ the assumption 5.2 is violated.

We will show that for every $\epsilon > 0$ there are $\phi_i, \psi_i \in \mathbb{Z}$ such that for all but finitely many $i \in \omega$ we have

$$\begin{aligned} \left|\frac{\phi_i}{\eta_i} - r\right| &< \epsilon \\ \left|\frac{\psi_i\eta_i - \phi_i\zeta_i}{\xi_i\eta_i} - r'\right| &< \epsilon. \end{aligned} \tag{5.3}$$

Clearly this implies the lemma.

Fix $\epsilon > 0$. Choose $\Theta, \Xi \in \mathbb{N}$ with $1/\Theta < \epsilon$ and $1/\Xi < \epsilon$.

If $|\eta_i| \geq \Theta$ and $|\xi_i| \geq \Xi$ then with

$$\phi_i = \lfloor \eta_i r \rfloor \text{ and } \psi_i = \lfloor \frac{\phi_i\zeta_i}{\eta_i} + \xi_i r' \rfloor$$

the conditions 5.3 are satisfied.

For each ξ with $|\xi| \leq \Xi$ let $N_\xi = \{i \in \omega \colon \xi_i = \xi\}$. For every $q \in \mathbb{Q}$, $M \in \omega$ there are only finitely many $i \in N_\xi$ with $|\zeta_i - q\eta_i| \leq M$. Otherwise, writing $q = \alpha/\beta$ in lowest terms with $\alpha, \beta \in \mathbb{Z}$ we would have $|\beta\zeta_i - \alpha\eta_i| \leq |\beta| M$ for infinitely many $i \in \omega$ contradicting 5.2 by the pigeon-hole principle.

If N_ξ is infinite we may hence apply Lemma 5.37 with $N = N_\xi$, to see that for all but finitely many $i \in N_\xi$ we can choose ϕ_i and ψ_i satisfying condition 5.3. □

We now prove a similar lemma for the case when there is an essential forced-compact set with non-trivial stabilizer.

Lemma 5.38. *Suppose $(\eta_i, \zeta_i, \xi_i) \in \mathbb{Z}^3$ with $0 \leq \zeta_i < \eta_i$ and $\xi_i \neq 0$ for all $i \in \omega$ are all distinct.*

Suppose also that $r, r' \in \mathbb{I}$ and $\Theta, \Theta' \in \mathbb{Z}$ with Θ, Θ' not both zero and $\Theta r + \Theta' r' \in \mathbb{Z}$ are given.

If

1. *for all $i \in \omega$ there exist $\alpha_i, \beta_i \in \mathbb{Z}$ such that*

$$\begin{aligned} \alpha_i\eta_i + \quad \beta_i\zeta_i &= \Theta \\ \beta_i\xi_i &= \Theta'; \end{aligned}$$

2. *for infinitely many $i \in \omega$ there exist $\alpha_i, \beta_i \in \mathbb{Z}$ such that*

$$\begin{aligned} \alpha_i\eta_i + \beta_i\zeta_i &= \Theta_2 \\ \beta_i\xi_i &= \Theta_2' \end{aligned}$$

implies that there is $P \in \mathbb{Z}$ with $P\Theta = \Theta_2$ and $P\Theta' = \Theta_2'$;

then there are $\phi_i, \psi_i \in \mathbb{Z}$ such that

$$\begin{aligned} \frac{\phi_i}{\eta_i} &\to r \\ \frac{\psi_i\eta_i - \phi_i\zeta_i}{\xi_i\eta_i} &\to r'. \end{aligned}$$

As before an auxiliary lemma for the special case where ξ_i is fixed needs to be proven first.

Lemma 5.39. *Suppose $(\eta_i, \zeta_i, \xi_i) \in \mathbb{Z}^3$ with $i \in N \subset \mathbb{N}$ are distinct, where η_i diverging to ∞, $0 \le \zeta_i < \eta_i, \xi_i \neq 0$ and $\xi_i = \xi$ for $i \in N$.*

($\star$) Suppose further that there are Θ, Θ' such that for each $i \in N$ there are $\alpha_i, \beta_i \in \mathbb{Z}$ with $\alpha_i\eta_i + \beta_i\zeta_i = \Theta,\ \beta_i\xi_i = \Theta'$.

($\dagger$) Suppose also that for any infinite $N' \subset N$ if there are $\Theta_2, \Theta_2' \in \mathbb{Z}$ such that for each $i \in N'$ there are $\alpha_i', \beta_i' \in \mathbb{Z}$ with $\alpha_i'\eta_i + \beta_i'\zeta_i = \Theta_2$, $\beta_i'\xi_i = \Theta_2'$ then there is $P \in \mathbb{Z}$ with $P\Theta = \Theta_2, P\Theta' = \Theta_2'$.

Given $r, r' \in \mathbb{I}$ with $\Theta r + \Theta' r' \in \mathbb{Z}$ and $\epsilon > 0$ there are $\phi_i, \psi_i \in \mathbb{Z}$ with

$$\begin{aligned} |\phi_i/\eta_i - r| &< \epsilon \\ |(\psi_i\eta_i - \phi_i\zeta_i) / (\xi_i\eta_i)| &< \epsilon \end{aligned}$$

for all but finitely many $i \in N$.

Proof. Without loss of generality we may assume that $N = \mathbb{N}$.

Note that $\beta_i = \beta = \Theta'/\xi$ is constant for $i > 0$.

Also, if $\gcd(\alpha_i, \beta) \neq 1$ for infinitely many $i > 0$ then by the pigeonhole principle there is some infinite $N_1 \subset N$ such that for $i \in N_1$ we have $\gcd(\alpha_i, \beta) = \gamma > 1$. But for this N_1 we then have $(\alpha_i/\gamma)\eta_i + (\beta/\gamma)\zeta_i = \Theta/\gamma$ a contradiction to the assumptions. Thus we may assume that there is some $N' \in \mathbb{N}$ such that for $i > N'$ $\gcd(\alpha_i, \beta) = 1$. In the following we assume $i > N'$.

Let $A = \{x \in \mathbb{R} \colon \forall \delta > 0 \, |\{i \in \mathbb{N} \colon \zeta_i/\eta_i \in B_\delta(x)\}| = \aleph_0\}$. Clearly A is closed and bounded, hence compact. Now the argument is similar as in Lemma 5.37. We will show that for every $x \in A$ there is an $\epsilon_x > 0$ and an $N_x \in \mathbb{N}$ such that for all $i > N_x$ we have that $\zeta_i/\eta_i \in B_{\epsilon_x}(x)$ implies the existence of $\phi_i, \psi_i \in \mathbb{Z}$ satisfying the required conditions. By compactness of A the conclusion of the theorem follows as in Lemma 5.37.

For $x \in A$ let $\delta < \min\{\beta x - \lfloor \beta x \rfloor, \lceil \beta x \rceil - \beta x\}$ and write $\alpha_i = \Theta/\eta_i - \beta\zeta_i/\eta_i$ $(i > 0)$, so that if $\Theta/\eta_i < \delta/4$ and $\zeta_i/\eta_i \in B_{\delta/4\beta}(x)$ then $|\alpha_i + \beta x| < \delta/2$ and hence for all such i, α_i must have the same integer value, α_x say. By the condition $x \in A$ and $\eta_i \to \infty$, there are in fact infinitely many such i. But since δ can be arbitrarily small we actually have $x = -\alpha_x/\beta \in \mathbb{Q}$.

We may conclude that for every $x \in A$ we have $x = -\alpha_x/\beta$ for some $\alpha_x \in \mathbb{Z}$ coprime to β and that there is an $\epsilon'_x > 0$ and an $N'_x \in \mathbb{N}$ such that for all $i > N'_x$ (namely if η_i is sufficiently large), $\zeta_i/\eta_i \in B_{\epsilon'_x}(x) \implies \alpha_i = \alpha_x$

Write

$$\begin{aligned}\delta_i(\phi) &= \left|\frac{\phi}{\eta_i} - r\right| = \left|\frac{\phi - \eta_i r}{\eta_i}\right| \\ \epsilon_i(\phi,\psi) &= \left|\frac{\psi\eta_i - \phi\zeta_i}{\xi_i\eta_i} - r'\right| = \left|\frac{\psi - \frac{\phi\zeta_i}{\eta_i} - r'\xi_i}{\xi_i}\right|\end{aligned}$$

Fix $x \in A$. For all the i in this paragraph we will assume that $i > N'_x$ and $\alpha_x\eta_i + \beta\zeta_i = \Theta$ (i.e. that ζ_i/η_i is close enough to x and η_i is large enough). Choose $\phi_i^0 \in \mathbb{Z}$ minimizing $\delta_i(\phi_i^0)$ and $\psi_i^0 \in \mathbb{Z}$ minimizing $\epsilon_i(\phi_i^0, \psi_i^0)$. From the equations $\alpha_x\eta_i + \beta\zeta_i = \Theta$, $\beta\xi_i = \Xi$ and $\Theta r + \Theta' r' = \Theta'' \in \mathbb{Z}$ we can rewrite

$$|\xi_i|\,\epsilon_i(\phi,\psi) = \left|\psi - \frac{\phi\Theta}{\beta\eta_i} + \frac{\phi\alpha_x}{\beta} - \xi_i r'\right| = \left|\psi - \frac{\Theta}{\beta}\left(\frac{\phi}{\eta_i} - r\right) + \frac{\alpha_x\phi - \Theta''}{\beta}\right|.$$

Since β and α_x are coprime, there is $\Delta\phi_i \in \{0, \ldots, \beta - 1\}$ such that

$$\frac{\alpha_x(\phi_i^0 + \Delta\phi_i) - \Theta''}{\beta} \in \mathbb{Z}.$$

For some $\Delta\psi_i \in \mathbb{Z}$ we then have

$$\begin{aligned} |\xi_i| \, \epsilon_i \left(\phi_i^0 + \Delta\phi_i, \psi_i^0 + \Delta\psi_i\right) &= \left| \left(\frac{\phi_i^0 + \Delta\phi_i}{\eta_i} - r \right) \frac{\Theta}{\beta} \right| \\ &= \delta_i \left(\phi_i^0 + \Delta\phi_i\right) \left| \frac{\Theta}{\beta} \right| . \end{aligned}$$

Since $\Delta\phi_i$ is bounded by β we can conclude that if η_i is large enough and thus since $\eta_i \to \infty$ if $i > N''_x$ for some $N''_x \in \mathbb{N}$ (in particular we need $(1+\beta)\,\Theta/\beta\eta_i \leq 2\Theta/\eta_i < \epsilon$), then there are $\phi_i = \phi_i^0 + \Delta\phi_i, \psi_i = \psi_i^0 + \Delta\psi_i \in \mathbb{Z}$ such that $\delta_i(\phi_i) < \epsilon, \epsilon_i(\psi_i, n_i) < \epsilon$ for $i > N_x = \max\{N'_x, N''_x\}$. As x was arbitrary in A this concludes the proof. □

Proof of Lemma 5.38. As before, we will show that for every $\epsilon > 0$ there are $\phi_i, \psi_i \in \mathbb{Z}$ such that for all but finitely many $i \in \omega$ we have

$$\begin{aligned} \left| \frac{\phi_i}{\eta_i} - r \right| &< \epsilon \\ \left| \frac{\psi_i\eta_i - \phi_i\zeta_i}{\xi_i\eta_i} - r' \right| &< \epsilon. \end{aligned} \tag{5.4}$$

Fix $\epsilon > 0$. There are two cases to consider.

Case $\Theta' \neq 0$**:** Note that $\Theta' \neq 0$ implies that $\xi_i \leq \Theta'$.

For every $\xi \leq \Theta'$ let $N_\xi = \{i \in \omega \colon \xi_i = \xi\}$. Note that on every infinite N_ξ we must have $\eta_i \to \infty$ since the (η_i, ζ_i, ξ_i) are all distinct and $0 \leq \zeta_i < \eta_i$. Apply Lemma 5.39 with $N = N_\xi$ to see that for all but finitely many $i \in N_\xi$ a choice of ϕ_i, ψ_i satisfying 5.4 is possible.

Case $\Theta' = 0$**:** Since $\xi_i \neq 0$ we must have $\beta_i = 0$ for all $i \in \omega$. By the assumptions we then have $\eta_i = \Theta$ for all but finitely many $i \in \omega$. Exclude all those for which $\eta_i \neq \Theta$ from consideration. $\Theta r = \Theta r + \Theta' r' \in \mathbb{Z}$ implies that for all i we may choose $\phi_i \in \mathbb{Z}$ such that actually we have $|\phi_i/\eta_i - r| = 0$.

Choose $\psi_i = \lfloor \phi_i\zeta_i/\eta_i + \xi_i r' \rfloor$. The conditions 5.4 are then met for sufficiently large $|\xi_i|$. But since (η_i, ζ_i, ξ_i) are all distinct and η_i and hence ζ_i are bounded, we must actually have $|\xi_i| \to \infty$. Thus 5.4 can be met for all but finitely many $i \in \omega$. □

Finally we prove a similar lemma about choices of irrational numbers. This will be applied when considering group actions which consist of infinite orbits.

Lemma 5.40. *Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and $k, m \in \mathbb{Z}$ coprime.*
Let $N = \max\{|k|, |m|\}$. There exist $\alpha', \beta' \in \mathbb{R} \setminus \mathbb{Q}$ such that $|\alpha - \alpha'| < 1/N$, $|\beta - \beta'| < 1/N$ and $k\alpha' + m\beta' \in \mathbb{Z}$.

Furthermore if $N = |m|$ then $k'\alpha' + mm'\beta' \in \mathbb{Z} \implies k' = m'k$ for $k', m' \in \mathbb{Z}$. If $N = |k| > |m|$ then $k'k\alpha' + m'\beta' \in \mathbb{Z} \implies m' = k'm$ for $k', m' \in \mathbb{Z}$.

Proof. Suppose without loss of generality that $N = |m|$. Since k and m are coprime the set $\{r - k(\alpha + s)/m \colon s, r \in \mathbb{Z}\}$ is $|1/m|$-dense in $\mathbb{R}$. Choose $s, r \in \mathbb{Z}$ such that

$$\left|\beta - \left(r - \frac{k(\alpha + s)}{m}\right)\right| < \left|\frac{1}{m}\right| = \frac{1}{N}.$$

Let

$$\alpha' = \alpha, \ \beta' = r - \frac{k(\alpha + s)}{m}.$$

Clearly $|\alpha - \alpha'| < 1/N$ and $|\beta - \beta'| < 1/N$. Further

$$k\alpha' + m\beta' = k\alpha - k\alpha - ks + rm = rm - ks \in \mathbb{Z}$$

as required. Note as well that if $k'\alpha' + m'm\beta' \in \mathbb{Z}$ for $m' \in \mathbb{Z}$ then $k' = m'k$ since by irrationality of α' we must have $k' - kmm'/m = 0$ □

5.3.1.2 Concrete Constructions

Using the lemmas from the previous section we now give concrete constructions for fundamental abstract group actions with large phase spaces. Note that since there are only countably many different orbit types and every orbit is countable, $|X| \geq \mathfrak{c}$ implies that at least one orbit type must occur at least $\mathfrak{c}$ often.

The most fundamental constructions consists just of a single torus consisting of $\mathfrak{c}$ many orbits with the same stabilizer.

Lemma 5.41. *Suppose $(X, \mathbb{Z}^2)$ is an abstract group action in which all orbits have type $H = \langle S^K, S^L T^M \rangle$ and that there are $\mathfrak{c}$ many of them. Then $(X, \mathbb{Z}^2)$ is compact-realizable.*

Proof. There are three cases to consider.

Firstly $\mathbb{Z}^2/H$ could be finite. Then we have without loss of generality $K > 0, 0 \leq L < K$ and $M > 0$. We therefore choose $r_1 = 1/K, r_2 = 0$, $r_1' = -L/(KM), r_2' = 1/M$.

Secondly we could have $H = \{e\}$, i.e. $K = S = M = 0$. We then let $r_1, r_2' \in \mathbb{I} \setminus \mathbb{Q}$ and $r_2 = r_1' = 0$.

Thirdly we might have $H = \langle S^L T^M \rangle \neq \{e\}$. We can then assume by Lemma 5.4 that $M = 0, L > 0$ and choose $r_1 = 1/L, r_2 = r_1' = 0$ and $r_2' \in \mathbb{I} \setminus \mathbb{Q}$.

In any case we consider the set $\mathbb{T}^2$ with maps

$$S(\theta, \theta') = (\theta + r_1, \theta' + r_2) \tag{5.5}$$

$$T(\theta, \theta') = (\theta + r_1', \theta' + r_2'), \tag{5.6}$$

observe that these are continuous and that every point in $\mathbb{T}^2$ has indeed stabilizer H. □

Since we only consider fundamental group action for the moment we can assume that the intersection of infinitely many different minimal orbit types will be constant.

Theorem 5.42. *Let $(X, \mathbb{Z}^2)$ be a fundamental abstract group action with orbits of type $H_i = \langle S^{k_i}, S^{l_i} T^{m_i} \rangle$ $(0 \leq l_i < k_i,\ m_i \neq 0)$ for $i \in \omega$, all of which are $\leq_{ot}$-minimal. Assume further that there are $\mathfrak{c}$ many orbits of type $H_\infty = \langle S^K, S^L T^M \rangle$.*

Then $(X, \mathbb{Z}^2)$ is compact-realizable.

Proof. Since the group action is fundamental, there is $H \subset H_\infty$ with $\bigcap_{i \in N} H_i = H$ for every infinite $N \subset \omega$.

If there are only finitely many different H_i then $(X, \mathbb{Z}^2)$ is compactifiable by Lemma 5.2 and Lemma 5.41. We may thus assume in the following that all the H_i are different.

We will use Theorem 5.35 with $n = 2$ and $Y = \{1/i \colon i > 0\} \cup \{0\}$. Our proceeding will vary depending on both H_∞ and H.

Case $H_\infty = \{e\}$**:** Let $r_1 = r'_2 \in \mathbb{I} \setminus \mathbb{Q}$ and $r_2 = r'_1 = 0$.

Case $H_\infty = \langle S^K \rangle$ **for some** $K > 0$**:** Let $r_1 = 1/K, r_2 = 0, r'_1 = 0$ and $r'_2 \in \mathbb{I} \setminus \mathbb{Q}$.

Case $H_\infty = \langle S^L T^M \rangle$ **with** $M \neq 0$**:** Adjust S, T.

Case $H_\infty = \langle S^K, S^L T^M \rangle$ **for** $0 \leq L < K, 0 < M$**:** Let $r_1 = 1/K$, $r_2 = 0$, $r'_1 = -L/(KM)$, $r'_2 = 1/M$.

Write $\langle S^{k_i}, S^{l_i} T^{m_i} \rangle = \langle S^{m'_i} T^{l'_i}, T^{k'_i} \rangle$ for $m'_i = \gcd(k_i, l_i) = p_i k_i + q_i l_i$, $k'_i = m_i k_i / m'_i$ and $l'_i \equiv q_i m_i \pmod{k'_i}$. We then define

$$S(\theta, \theta', 0) = (\theta + r_1, \theta' + r_2, 0) \tag{5.7}$$

$$T(\theta, \theta', 0) = (\theta + r'_1, \theta' + r'_2, 0) \tag{5.8}$$

and

$$S\left(\theta, \theta', \frac{1}{i}\right) = \left(\theta + \frac{s_i}{k_i}, \theta' + \frac{n'_i k'_i - s'_i l'_i}{m'_i k'_i}, \frac{1}{i}\right) \tag{5.9}$$

$$T\left(\theta, \theta', \frac{1}{i}\right) = \left(\theta + \frac{n_i k_i - s_i l_i}{m_i k_i}, \theta' + \frac{s'_i}{k'_i}, \frac{1}{i}\right) \tag{5.10}$$

for $i < \omega$ where s_i, s'_i, n_i, n'_i will be chosen below such that both S and T are continuous.

Case $H = \{e\}$**:** Apply Lemma 5.36 once with $r = r_1$, $r' = r'_1$, $\eta_i = k_i$, $\zeta_i = l_i$, $\xi_i = m_i$ to obtain $s_i = \phi_i$ and $n_i = \psi_i$ and a second time with $r = r_2$, $r' = r'_2$, $\eta_i = k'_i$, $\zeta_i = l'_i$, $\xi_i = m'_i$ to obtain $s'_i = \phi_i$ and $n'_i = \psi_i$. Since for every infinite set $N \subset \omega$ we have $\bigcap_{i \in \omega} H_i = \{e\}$ the conditions for Lemma 5.36 are satisfied.

Case $H = \langle S^A T^B \rangle$ **for** A, B **not both zero:** Apply Lemma 5.38 with $r = r_1$, $r' = r'_1$, $\eta_i = k_i$, $\zeta_i = l_i$, $\xi_i = m_i$, $\Theta = A$, $\Theta' = B$ to obtain $s_i = \phi_i$ and $n_i = \psi_i$. The condition $H_\infty \cap \bigcap_{i \in N} H_i = \langle S^K T^M \rangle$ for all infinite $N \subset \omega$ ensures that the conditions of Lemma 5.38 are satisfied.

Similarly, apply Lemma 5.38 a second time, this time with $r = r_2$, $r' = r'_2$, $\eta_i = k'_i$, $\zeta_i = l'_i$, $\xi_i = m'_i$, $\Theta = B$, $\Theta' = A$ to obtain $s'_i = \phi_i$ and $n'_i = \psi_i$.

In either case, Theorem 5.35 with these choices shows that both S and T are continuous. For each $i > 0$ we only consider the closed subset $\langle S, T \rangle (0, 0, 1/i)$ of the torus $\mathbb{T}^2 \times \{1/i\}$.

Note that for each $i > 0$ we indeed have $H_i \subset \text{stab}((0, 0, 1/i))$ and since these orbits are isolated we can apply Lemma 4.7 to insure that they have in fact stabilizer H_i.

The continuum many orbits in $\mathbb{T}^2 \times \{0\}$ on the other hand have stabilizer precisely H_0. For suppose $x \in \mathbb{T}^2 \times \{0\}$ and $S^m T^n x = x$. Then

$$mr_1 + nr_1' = \frac{m}{k_0} - \frac{nl_0}{m_0 k_0} \in \mathbb{Z}$$

and $mr_2 + nr_2' = n/m_0 \in \mathbb{Z}$. Write $beta = n/m_0 \in \mathbb{Z}$ and observe that this implies $m - \beta l_0 = \alpha k_0$ for some $\alpha \in \mathbb{Z}$. But then $S^m T^n = S^{\alpha k_0 + \beta l_0} T^{\beta m_0} \in H_0$ so that $\text{stab}(x) \subset H_0$. Conversely it is easily seen that x is fixed by each element of H_0 proving that $\text{stab}(x) = H_0$ as desired. □

Previously the orbits whose types occur $\mathfrak{c}$ often were finite. We now prove a similar theorem where these orbits are infinite. There are three cases to consider.

Theorem 5.43. *Suppose the orbits of the fundamental group action $(X, \mathbb{Z}^2)$ have stabilizers $\langle S^{k_i} \rangle$ for some $k_i \in \mathbb{Z}^+$, all of which are minimal. Suppose further that each such orbit type occurs $\mathfrak{c}$ often.*

Then $(X, \mathbb{Z}^2)$ is compact-realizable.

Proof. Suppose first that there are infinitely many different minimal orbit types. Choose some irrational α and $s_i \in \omega, i > 0$ such that $|s_i/k_i - 1/k_0|$ is minimal. Let $j_i = \gcd(s_i, k_i) \neq 0$. For $i > 0$ define

$$S_i \left(\theta, \theta', \frac{1}{i} \right) = \left(\theta + \frac{s_i}{k_i}, \theta' + \frac{1}{2^i j_i}, \frac{1}{i} \right) \tag{5.11}$$

$$T_i \left(\theta, \theta', \frac{1}{i} \right) = \left(\theta + \alpha, \theta', \frac{1}{i} \right) \tag{5.12}$$

on $X_i = \mathbb{T} \times \mathbb{R}/2^{-i}\mathbb{Z} \times \{1/i\}$ (i.e. the radius of the second circle converges to 0 as $i \to \infty$). For $i = 0$ define

$$S_0 (\theta, 0, 0) = (\theta + 1/k_0, 0, 1/i)$$

and

$$T_0(\theta, 0, 0) = (\theta + \alpha, 0, 0)$$

on $X_0 = \mathbb{T} \times \{0\} \times \{0\}$. Let $S = \bigcup_{i\in\omega} S_i$ and $T = \bigcup_{i\in\omega} T_i$. By Theorem 5.35, S and T are auto-homeomorphisms on $\bigcup_{i\in\omega} X_i$ when seen as a subspace of Euclidean n-space.

If there are only finitely many different orbit types, say $i = 1, \ldots, n$ then define $S_i(\theta, \theta', 1/i) = (\theta + 1/k_i, \theta', 1/i)$ and $T(\theta, \theta', 1/i) = (\theta, \theta' + \alpha, 1/i)$ on $\mathbb{T} \times \mathbb{T} \times \{1, \ldots, n\}$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. This clearly realizes (X, G) as a compact Hausdorff group action. □

As Iwanik did in his paper, one could define the two rotations S and T on $\mathbb{T} \times \{0\} \cup \{1/i\colon i > 0\}$ by carefully choosing the s_i to be coprime to the k_i. This is possible due to the prime number theorem: There are approximately $2\epsilon\lambda n/\log(\lambda n)$ primes between $(1-\epsilon)\lambda n$ and $(1+\epsilon)\lambda n$ for $0 < \epsilon, \lambda < 1$. However, there are at most $\log_2 n$ prime divisors of n, so for large enough n we can find a prime very close (compared to n) to n/k_0 which does not divide n. Choosing prime p_i for $n = k_i$ we then have $p_i/k_i \to 1/k_0$ and p_i is coprime to k_i. So using p_i instead of our 'ideal' s_i immediately gives orbits of the correct type.

Theorem 5.44. *Suppose the orbits of the fundamental group action $(X, \mathbb{Z}^2)$ have stabilizer $\left\langle S^{j_i k_i} T^{j_i m_i} \right\rangle$ with k_i, m_i coprime and j_i non-zero. Suppose further there are $\mathfrak{c}$ many orbits of each such type and these are $\leq_{ot}$-minimal.*

Then $(X, \mathbb{Z}^2)$ is compact-realizable.

Here [8] ignores the possibility that $j_i \neq 1$. As in the previous case, we will take care of this by working on $\mathbb{T} \times \mathbb{T} \times \mathbb{I}$ instead of $\mathbb{T} \times \mathbb{I}$.

Proof. Suppose that there are infinitely many different orbit types. Choose some $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$\beta_0 = \frac{1}{m_0 j_0} - \frac{k_0}{m_0}\alpha_0.$$

Clearly $j_0 k_0 \alpha_0 + j_0 m_0 \beta_0 \in \mathbb{Z}$. Further, if

$$k\alpha_0 + m\beta_0 = \frac{m}{m_0 j_0} + \left(k - \frac{m k_0}{m_0}\right)\alpha_0 \in \mathbb{Z}$$

for $k, m \in \mathbb{Z}$ then $k - mk_0/m_0 = 0$ since α_0 is irrational. Thus $m/(m_0 j_0) \in \mathbb{Z}$ giving $m = rm_0 j_0$ for some $r \in \mathbb{Z}$. Hence $k = rk_0 j_0$, i.e. (k, m) is an integer multiple of $(j_0 k_0, j_0 m_0)$.

For $i > 0$ use Lemma 5.40 with $k = k_i, m = m_i$ and $\alpha = \alpha_0, \beta = \beta_0$ to obtain a $\alpha_i, \beta_i \in \mathbb{R} \setminus \mathbb{Q}$. If $|k_i| > |m_i|$ let $q_i = 1/(j_i k_i)$, $r_i = 0$, otherwise let $q_i = 0$, $r_i = 1/(j_i m_i)$. Define S_i, T_i on $\mathbb{T} \times \mathbb{T} \times \mathbb{T} \times \{1/i\}$ by

$$S_i\left(\theta, \theta', \theta'', \frac{1}{i}\right) = \left(\theta + \alpha_i, \theta' + q_i, \theta'', \frac{1}{i}\right) \tag{5.13}$$

$$T_i\left(\theta, \theta', \theta'', \frac{1}{i}\right) = \left(\theta + \beta_i, \theta', \theta'' + r_i, \frac{1}{i}\right) \tag{5.14}$$

and $S_0(\theta, \theta', \theta'', 0) = (\theta + \alpha_0, \theta', \theta'', 0)$, $T_0(\theta, \theta', \theta'', 0) = (\theta + \beta_0, \theta', \theta'', 0)$.

Note that since $\max\{|k_i|, |m_i|\} \to \infty$ as $i \to \infty$, we have $\alpha_i \to \alpha$, $\beta_i \to \beta$, $q_i \to 0$ and $r_i \to 0$ as $i \to \infty$.

Thus $S = \bigcup_{i\in\omega} S_i$, $T = \bigcup_{i\in\omega} T_i$ are continuous as explained in Theorem 5.35.

Note as well that the orbit spectrum of the thus constructed group action is correct.

If there are only finitely many minimal orbit types, say $i = 1, \ldots, n$ then define

$$S_i(\theta, \theta', 1/i) = (\theta + \alpha, \theta' + 1/(j_i k_i), 1/i)$$

and

$$T_i(\theta, \theta', 1/i) = (\theta - k_i\alpha/m_i, \theta', 1/i)$$

for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. on $\mathbb{T} \times \mathbb{T} \times \{1/i\colon i = 1, \ldots, n\}$. □

Theorem 5.45. *Suppose all orbits of the abstract group action $(X, \mathbb{Z}^2)$ have trivial stabilizer and that there are $\mathfrak{c}$ many of them. Then $(X, \mathbb{Z}^2)$ is compact-realizable.*

Proof. Choose any two (possibly equal) irrationals α, β. On $\mathbb{T} \times \mathbb{T}$ define $S(\theta, \theta') = (\theta + \alpha, \theta')$, $T(\theta, \theta') = (\theta, \theta' + \beta)$. Clearly S, T are auto-homeomorphisms of the compact Hausdorff space $\mathbb{T} \times \mathbb{T}$. Since α, β are irrational each orbit is of the required type. □

Note that the less intuitive definition on $\mathbb{T}$ with $S(\theta) = \theta + \alpha$, $T(\theta) = \theta + \beta$ would have worked as well, if one requires α and β to be independent over $\mathbb{Q}$.

5.4 Summary

In this section we have given some necessary conditions and some sufficient conditions on the (weighted) orbit spectrum of compact $\mathbb{Z}^2$-action. The techniques for deriving the necessary conditions can give further insight into specific actions. Similarly, the techniques for constructing group actions can sometimes be modified for (weighted) orbit spectra not explicitly covered by the theorems and lemmas.

Especially when there is little or no 'interaction'between the two generating maps S and T (i.e. most finite orbits have type $\langle S^k, T^m \rangle$) the sufficient conditions are close to the necessary conditions. In fact, the only significant gap remaining there is how to piece together infinitely many, possibly overlapping fundamental group actions. This seems possible by ad hoc methods in most cases, but a general formulation in the form of a theorem still eludes the author.

In the more general case with plenty of interaction between the generating maps, the constructions for small phase spaces become decidedly more difficult. It seems that the method of construction using symbolic dynamics reaches its limitations in these cases. Whether this is an artefact of this particular method, or whether there are further genuine necessary conditions in this case is unclear.

Chapter 6

Compact Groups

Before looking at the supremely important group $\mathbb{R}$ in the next chapter, it is useful to sharpen our understanding of the techniques involved in the easier setting of compact groups. This is especially true since we may consider an $\mathbb{R}$-action under which every point is fixed by the same element of $\mathbb{R}$, as a $\mathbb{T}$-action instead. We will see below that a $\mathbb{T}$-action is compact-realizable if and only if there are finitely many minimal orbit types, implying that either there is a fixed point, or there are only finitely many orbit types.

One of the chief difficulty compared to discrete groups is that we not only have to ensure that every map $x \mapsto gx$ is continuous for $g \in G$, but that joint continuity $G \times X \to X$ occurs. This joint continuity finds its expression already in Corollary 3.12. We will see that at least for compact Abelian Lie groups compactness of the orbit spectrum in the co-compact topology is sufficient to guarantee compactifiability. The case for general compact Abelian groups is more difficult but turns out to be covered by Theorem 3.11.

6.1 A characterization of compact-realizable group actions by compact Abelian groups

The main result of this chapter can be summarized as follows:

Theorem 6.1. *Suppose G is a compact Abelian Hausdorff group, and that (X, G) is an admissable G-action. Then (X, G) is compact-realizable if and* *only if the stabilizer of every point is a closed subgroup of G and there is a compact Hausdorff topology on the weighted orbit spectrum regarded as $\bigcup_{H\in G^{\leq}} \kappa_H \times \{H\}$ such that the map $(\beta, H) \mapsto H \in G^{\leq}$ is continuous.*

The proof of this theorem comes in two steps. We have already assembled all the ingredients for the necessity in chapter 3.

Proof of necessity. From Lemma 2.6 we have that the stabilizer of every point must be a closed subgroup of G.

Let us note that if G is a compact Hausdorff group and (X, G) a compact group action, then every orbit Gx is compact, as it is the continuous image of the compact set $G \times \{x\}$. Thus the orbit relation on X is closed and the orbit space is in fact a compact Hausdorff space. By Theorem 3.11 and the fact that stab is constant on orbits, we can deduce that the map $\mathrm{stab}\colon \mathcal{O}(X) \to G^{\leq}$ is continuous. Identifying $\bigcup_{H\in G^{\leq}} \kappa_H \times \{H\}$ with the orbit space in the obvious manner (i.e. an orbit with stabilizer H is identified with a point $(\beta, H), \beta < \kappa_H$) we see that the condition is in fact necessary. □

To show that the condition is in fact sufficient we have to work slightly harder.

Theorem 6.2. *If G is a compact Abelian Hausdorff group, Y a compact Hausdorff space and $S\colon Y \to G^{\leq}$ a continuous function then there is a compact group action (X, G) such that $\mathcal{O}(X) = Y$ and S is induced by the* stab*-map.*

Proof. Let $X' = Y \times G$ with $\rho'\colon G \times X' \to X'$ given by $\rho'(g, (y, h)) = (y, gh)$. Then (X', G, ρ') is clearly a compact group action. Let

$$X = \bigcup_{y\in Y} \{y\} \times G/S(y)$$

be the quotient space of X' under the equivalence relation $(y, g) \sim (y', h)$ if and only if $y = y'$ and $gh^{-1} \in S(y)$. X is clearly compact and the map ρ' factors through the equivalence relation to the continuous map

$$\rho(g, (y, hS(y))) = (y, ghS(y)).$$

It is easy to see that indeed $Y = X/\sim_G$ and S is induced by stab. To complete the proof we will show that X is Hausdorff in the quotient topology.

If $y \neq y'$ then Hausdorffness of Y immediately implies that $(y, gS(y))$ and $(y', g'S(y'))$ can be separated by open sets for any $g, g' \in G$.

On the other hand, if $(y, gS(y)) \neq (y, g'S(y))$ then $gS(y) \cap g'S(y) = \emptyset$. Both $gS(y)$ and $g'S(y)$ are closed subsets of the normal space G so that they may be separated by disjoint open sets U', V'. Then $U = g^{-1}U' \cap g'^{-1}V' \supset S(y)$ is open and $gU \cap g'U = \emptyset$.

Now consider the two sets

$$U_1 = \bigcup_{z \in S^{-1}\hat{U}} \{(z, hS(z)) \colon hS(z) \subset gU\}$$

$$U_2 = \bigcup_{z \in S^{-1}\hat{U}} \{(z, hS(z)) \colon hS(z) \subset g'U\}.$$

Clearly U_1 and U_2 are disjoint since gU and $g'U$ are. Further $(y, gS(y)) \in U_1$ and $(y, g'S(y)) \in U_2$. We claim that the U_i are open. To see this, let π be the quotient map and consider $(z, h) \in \pi^{-1}U_1$. Then $hS(z) \subset gU$, so continuity of the multiplication and compactness of $S(z)$ gives open $V' \ni h, V \supset S(z)$ such that $V'V \subset gU$. If $(z', h') \in S^{-1}\hat{V} \times V'$ then $h'S(z') \subset V'V \subset gU$. Hence $(z, h) \in S^{-1}\hat{V} \times V' \subset \pi^{-1}U_1$ and continuity of S implies that $S^{-1}\hat{V} \times V'$ is open in X'. As (z, h) was an arbitrary element of $\pi^{-1}U_1$ this shows that the latter is open. Therefore by the definition of the quotient topology U_1 is open. Similarly U_2 is open which completes the proof. □

This finishes the proof of Theorem 6.1.

6.2 Further investigation

Having achieved our initial aim of giving a characterization of the compact-realizable group actions of a compact Abelian group, we will examine the meaning of this characterization a bit more closely. In the previous chapters we were mainly concerned with compactifiability. In this setting we obtain the following beautiful result.

Theorem 6.3. *If G is a compact Abelian Hausdorff group then every compact subset of $G^{\leq}$ which has a finer compact Hausdorff topology is the orbit of some compact group action (X, G) such that every orbit type occurs at most once.*

Proof. This follows immediately from the previous theorem, since we may take Y to be the orbit spectrum with the finer compact Hausdorff topology and S the identity. □

A slight uneasiness in the last two theorems is the reliance on an 'external' characterization of compact group actions which are compact-realizable (resp. compactifiable). We refer to an unknown compact Hausdorff space and a continuous map (resp. a 'finer compact Hausdorff' topology). Really we would like to look at any orbit spectrum of a compact group and decide whether it is compactifiable without too much effort.

Therefore we would now like a characterization of those compact subsets of $G^{\leq}$ that have a finer compact Hausdorff topology. Also, it would be interesting to have an internal characterization of all those compact spaces which are images of compact Hausdorff spaces. Unfortunately these seem to be a hard open problems as shown in the next section. However, for Lie groups we may apply a theorem from [10] which clarifies what the co-compact topology on $G^{\leq}$ looks like.

Theorem 6.4. *If G is a compact Abelian Lie group and F is a closed subgroup of G, then there is an open set $U \supset F$ such that if H is a closed subgroup of G contained in U then H is conjugate to a subgroup of F.*

This last theorem allows us to transform our previous result into an internal characterization.

Theorem 6.5. *If G is a compact Abelian Lie group, then every compact subset of $G^{\leq}$ is the orbit spectrum of some compact group action (X, G) such that every orbit type occurs at most once.*

Proof. Let C be the orbit spectrum we are trying to realize. Note that the previous theorem implies that the co-compact topology on $G^{\leq}$ is generated by sets of the form $\left\{H \in G^{\leq} \colon H \leq H_0\right\}$ where $H_0 \in G^{\leq}$. Thus compact subsets are of $G^{\leq}$ are precisely those which have finitely many maximal elements under the $\subset$-relation. Therefore we may partition C into finitely many sets $C_1, \ldots, C_n$ each of which has a unique maximal element $H_1, \ldots, H_n$. Define a topology τ_i on C_i by isolating all non-maximal points and letting the neighbourhoods of H_i be the co-finite subsets of C_i. Each τ_i is compact Hausdorff and since there are only finitely many C_i, the direct sum, τ, is a compact Hausdorff topology on C. Clearly τ is finer than $\tau_{\mathbf{Co}}$. Thus by Theorem 6.3 the result follows. □

Using all of the above we may now present a simple, complete characterization of those abstract group actions of compact Abelian Lie groups which are realizable as compact Hausdorff dynamical systems.

Theorem 6.6. *Given an admissable group action (X, G) of a compact Abelian Lie group, there is a compact Hausdorff topology on X making the group action continuous if and only if the orbit spectrum is a compact subset of the co-compact topology on $G^{\leq}$, i.e. the orbit spectrum has finitely many maximal elements under the $\subset$-relation.*

For compact Abelian Lie groups we have presented a complete, easy to use solution to our original question above. This depended crucially on the fact that the co-compact topology on compact Lie groups is particularly simple. In general compact Abelian topological groups the situation is less clear. In fact, we will show that every T_0 compact space of weight α is a subspace of $\mathbb{T}^{\alpha\leq}$. Thus the problem of giving an internal characterization of the compactifiable orbit spectra for compact Abelian group is as least as hard as characterizing all compact spaces which have a finer compact Hausdorff topology.

Recall that the Sierpinski space S is the set $S = \{0, 1\}$ with topology $\{\emptyset, \{0\}, S\}$. Also recall that every T_0 space of weight α can be embedded in S^α. We will now embed S^α in $\mathbb{T}^{\alpha\leq}$.

Lemma 6.7. *For every cardinal α, S^α can be embedded into $\mathbb{T}^{\alpha\leq}$ with the co-compact topology. Hence every compact space of weight at most α can be embedded into $\mathbb{T}^{\alpha\leq}$.*

Proof. Let $G_0 = \{e\} \subset \mathbb{T}, G_1 = \mathbb{T}$ and define $e\colon S^\alpha \to \mathbb{T}^{\alpha\leq}$ by

$$f((x_\beta)_{\beta<\alpha}) = \prod_{\beta<\alpha} G_{x_\beta}.$$

Clearly f is injective and thus a bijection onto its image. We need to show that f is continuous and open onto its image.

First consider a basic open set $U = \prod_{\beta<\alpha} U_\beta \subset \mathbb{T}^\alpha$ with $U_\beta = \mathbb{T}$ for cofinitely many β, say $\beta \notin \{\beta_0, \ldots, \beta_n\}$. Then

$$\begin{aligned} f^{-1}\hat{U} &= \left\{x \in S^\alpha \colon \forall \beta < \alpha.\ G_{x_\beta} \subset U_\beta\right\} \\ &= \{x \in S^\alpha \colon \forall i = 0, \ldots, n.\ x_\beta = 0\} \\ &= \bigcap_{i=0}^{n} \pi_{\beta_i}^{-1} \{0\} \end{aligned}$$

and the latter is open. Hence, if V is any set containing U then

$$f^{-1}\hat{V} = \bigcap_{i\in F} \pi_{\beta_i}^{-1} \{0\}$$

for some $F \subset \{0, \ldots, n\}$ and therefore open. But every non-empty open set contains some basic open set. Thus f is continuous.

Now consider a non-empty basic open set $U = \bigcap_{\beta\in F} \pi_\beta^{-1} \{0\} \subset S^\alpha$ where F is a finite subset of α. For each $\beta \in F$ let $V_\beta \neq T$ be a neighbourhood of $e \in \mathbb{T}$ and let $V_\beta = \mathbb{T}$ if $\beta \notin F$. Then $f(U) = f(S^\alpha) \cap \prod_{\beta<\alpha} \hat{V}_\beta$ and hence $f(U)$ is open in $f(S^\alpha)$ as required. ☐

This last lemma enables us to give compact subsets of $G^{\leq}$ which cannot occur as orbit spectra of a compact G-action for some compact group G.

Corollary 6.8. *There is a compact Hausdorff group G and a compact subset of $G^{\leq}$ which is not the orbit spectrum of any compact group action (X, G) such that every orbit type occurs at most once.*

Proof. It is enough to show the existence of a compact space without finer compact Hausdorff topology. In [12] a second countable compact topological space without finer compact Hausdorff topology is constructed. ☐

As remarked above, an internal solution to our question for compact Hausdorff Abelian groups could be given, once the following two questions are answered.

Question. Which compact spaces have a finer compact Hausdorff topology?

Question. Which compact spaces are continuous images of compact Hausdorff spaces?

However both these questions seem to defy a solution. In fact, until recently the related, but easier problem of showing that all compact spaces have a finer, maximally compact topology was open as well.

Chapter 7

The Additive Reals

The results from the previous chapter gave a first idea of what kind of techniques can be used with non-discrete groups. The most important of these is undoubtedly the group of additive reals. Its non-compact nature however, makes constructions of compact group actions far more difficult then simply finding a finer compact Hausdorff topology on the orbit spectrum (or a continuous compact Hausdorff pre-image which acts as the orbit space).

The main result of this chapter is a proof that what can naturally be called 'bounded, Euclidean nowhere dense' orbit spectra of abstract $\mathbb{R}$-actions are compactifiable. The technique of the concrete construction presented, as well as examples why a more general, easy-to-understand sufficient condition is impossible, give a thorough understanding of the issues in these group actions. Once these are understood, some further important classes of orbit spectra are investigated.

We need to introduce some notation specific to this section. There are at least two notions of the boundary of a subset of a topological space. Engelking [6] defines the boundary to be its closure minus its interior. Sometimes the boundary is taken to be the set itself minus the interior. We are interested in the non-interior limit points of a set. Therefore we define the following. Given a subset A of $\mathbb{R}$ in this section we will write $\overline{A}$ for the Euclidean closure of A. We will also write $\operatorname{ess}(A) = \overline{A} \setminus A$ as these point turn out to be the essential obstacles to finding a finer compact Hausdorff topology.

7.1 The co-compact Topology

From Theorem 3.11 and Corollary 3.12 we can see that the co-compact topology is of major importance in the understanding of the orbit spectra of compact $\mathbb{R}$-actions. For $\mathbb{R}$ there is a particularly nice representation of $(\mathbb{R}^{\leq}, \tau_{\mathbf{Co}})$ which we shall exhibit now. Recall that the closed subgroups of $\mathbb{R}$ are precisely $\{0\}, \mathbb{R}$ and $t\mathbb{Z}$ for $t \in \mathbb{R}^+$.

Theorem 7.1. *Under the identification*

$$\mathbb{R}^{\leq} \to \mathbb{R}_0^+ \cup \{\infty\}\,; H \mapsto \begin{cases} \inf(H \cap \mathbb{R}^+); & \text{if } H \neq \{0\} \\ \infty; & \text{if } H = \{0\} \end{cases}$$

the co-compact topology on $\mathbb{R}^{\leq}$ maps to the topology consisting of sets of the form $\mathbb{Z}U \cup \{\infty\}$ where U is an open subset of $\mathbb{R}$ and $\emptyset$.

Moreover, under this identification a subset T of $\mathbb{R}^{\leq}$ is compact if and only if $\overline{T \cap \mathbb{R}^+} \subset \mathbb{N}(T \cap \mathbb{R}^+)$ or $0 \in T$.

Proof. First note that for every Euclidean open set U there is a $u \in \mathbb{R}^+$ such that $(-\infty, -u) \cup (u, \infty) \subset \mathbb{Z}U$. For suppose $(a, b) \subset U$ with $0 < a < b$. Then there is $n \in \mathbb{N}$ with $(n+1)a < nb < (n+1)b$ and therefore $(m+1)a < mb < (m+1)b$ for $m \geq n$. Hence

$$\mathbb{Z}U \supset \mathbb{Z}(a, b) \supset \bigcup_{m \geq n} (ma, mb) \supset (na, \infty).$$

Similarly $\mathbb{Z}U \supset (-\infty, -ma)$ for some $m \in \mathbb{N}$ as required. If no such a, b exist then, as U is Euclidean open we have $(a, b) \subset U$ with $a < b \leq 0$ and an analogous argument gives the claim.

As $\mathbb{Z}U$ is also Euclidean open it is a co-compact subset of $\mathbb{R}$. Finally note that $H \subset \mathbb{Z}U$ implies that $\inf H \cap \mathbb{R}^+ \in \mathbb{Z}U$ for a closed, non-trivial subgroup H of $\mathbb{R}$. Thus $\mathbb{Z}U \cup \{\infty\}$ corresponds to an open set under the identification.

Conversely suppose U is a co-compact proper subset of $\mathbb{R}$. Consider the set $U' = \bigcup \left\{H \in \mathbb{R}^{\leq} \colon H \subset U\right\} \setminus \{0\}$. If $r \in \mathbb{Z}U'$ then $r \in \mathbb{Z}H = H$ for some $H \in \mathbb{R}^{\leq}$ so $r\mathbb{Z} \subset H \subset U$ and thus $r\mathbb{Z}$ is in the open set $\hat{U} \subset \mathbb{R}^{\leq}$. Finally

U' is an open subset of $\mathbb{R}$. For suppose $r \in U'$, i.e. $r\mathbb{Z} \subset U$. Since U is co-compact, there is $n \in \mathbb{N}$ with $(-\infty, -nr) \cup (nr, \infty) \subset U$. Then

$$B = (nr/(n+1), \infty) \cap \bigcap_{k=-n}^{n} U/n$$

is a Euclidean neighbourhood of r and thus if $s \in B$ then $s\mathbb{Z} \subset U$.

Hence the open subset $\hat{U}$ of $\mathbb{R}^{\leq}$ is either empty or corresponds to the set $\mathbb{Z}U' \cup \{\infty\}$ under the identification.

For the characterization of the compact subsets of $\mathbb{R}^{\leq}$ under the identification, first note that the only open set containing 0 is the whole space $\mathbb{R}_0^+ \cup \{\infty\}$. Thus any set containing 0 will be compact.

Next, consider a compact subset $T \not\ni 0$ of $\mathbb{R}_0^+ \cup \{\infty\}$. If $\infty \neq r \in \overline{T}$ then we can choose a sequence r_n from T which converges to r in the Euclidean sense. As $\{0\} \cup r/\mathbb{N}$ is a Euclidean closed subset of $\mathbb{R}$ for any $t \in \mathbb{R}^+ \setminus r/\mathbb{N}$ there is a Euclidean open $U \subset \mathbb{R}^+$ containing t which is disjoint from $\{0\} \cup r/\mathbb{N}$. Hence $\mathbb{Z}U$ does not contain r but is a (co-compact) open neighbourhood of t. Thus r_n cannot have t as a limit point in the co-compact topology. However, since compactness of T implies that r_n has a limit point, $\{0\} \cup r/\mathbb{N}$ must meet T. As $0 \notin T$ that implies that $r \in \mathbb{N}T$ as required.

Conversely suppose $T \subset \mathbb{R}_0^+ \cup \{\infty\}$ satisfies $\overline{T \cap \mathbb{R}} \subset \mathbb{N}(T \cap \mathbb{R})$. If $T \subset \{\infty\}$ then T is clearly compact. Otherwise let $\mathcal{U}$ be a (co-compact) open cover of T and choose some $t_0 \in T \setminus \{0\}$ and some $U_0 \in \mathcal{U}$ with $t_0 \in U_0$. Note that U_0 contains a set of the form $(a, \infty) \cap T$. On the other hand since every $U \in \mathcal{U}$ is invariant under multiplication by $\mathbb{N}$, $\mathcal{U}$ also covers $\mathbb{N}(T \cap \mathbb{R})$ and therefore in particular $\overline{T \cap \mathbb{R}}$. Since $\overline{T \cap \mathbb{R}} \setminus U_0$ is a Euclidean closed and bounded subset of $\mathbb{R}$ it is Euclidean compact. As the co-compact topology on $\mathbb{R}_0^+$ is a coarser topology than the Euclidean one, it is therefore compact in the co-compact topology. Thus there is a finite subcover $\mathcal{V} \subset \mathcal{U}$ of $\overline{T \cap \mathbb{R}}$. Hence $\{U_0\} \cup \mathcal{V}$ is a finite subcover of T, i.e. T is compact. $\square$

7.2 A Method of Construction

Now that we have a good description of the co-compact topology on $\mathbb{R}^{\leq}$ the next obvious step is to ask which of the compact sets does have a finer compact Hausdorff topology, and which of them is the continuous image of a compact Hausdorff space. We will focus on the first problem, as it seems the more fundamental as an independent question in general topology.

Let us first give an informal outline of our strategy constructing a finer compact Hausdorff topology. We will then give examples of the two typical problems we have to overcome.

It is helpful to recast the general problem of when a compact topology on a set can be refined to a compact Hausdorff topology in the context of convergence structures. A convergence is compact if every ultrafilter converges somewhere and Hausdorff if every filter converges to at most one point. Note that if $\to$ is a relation between the ultrafilters of X and points of X such that for every ultrafilter $\mathcal{U}$ there is a unique point $x \in X$ with $\mathcal{U} \to x$ we can extend $\to$ to a compact Hausdorff convergence on X. In fact there is a unique maximal (with respect to inclusion) extension of $\to$ to a convergence on X given by $\mathcal{U} \to x$ if and only if every ultrafilter finer than $\mathcal{U}$ converges to x. Note also that every topological convergence $\to$ has the property that $\mathcal{U} \to x$ if and only if every ultrafilter finer than $\mathcal{U}$ converges to x.

Thus, given a compact topological convergence $\to$ on X, for each ultrafilter $\mathcal{U}$ on X we choose some $x_{\mathcal{U}}$ with $\mathcal{U} \to x_{\mathcal{U}}$ and set $\mathcal{U} \to' x \iff x = x_{\mathcal{U}}$. Then $\to'$ finer than $\to$ on the ultrafilters and by the above $\to'$ extends to a compact Hausdorff convergence on X. It is now easy to check that the extension of $\to'$ is finer than $\to$. Thus every compact topological convergence has a finer compact Hausdorff convergence.

However, although $\to'$ is a perfectly good compact Hausdorff convergence, it is not necessarily topological. In particular, the topology that is induced by $\to'$ is not necessarily Hausdorff, although it will always be compact. The problem is that in a topological space, as opposed to a convergence space, the convergence of two ultrafilters is in general not independent. We can illustrate this with the following example.

Example 7.2. The compact subset

$$\begin{aligned} T = \{2,3\} &\cup \{2+1/(3n) \colon n \text{ even}\} \\ &\cup \{3+1/(2n) \colon n \text{ odd}\} \\ &\cup (6,7] \setminus \{6+1/n \colon n \in \mathbb{N}\} \end{aligned}$$

of $R^{\leq}$ (under the identification) has no finer compact Hausdorff topology. However, it is the continuous image of a compact Hausdorff space.

Proof. Clearly T is compact in the co-compact topology. First note that

$$\{2+1/(3n), 3+1/(2(n+1))\} \cup (6+1/(n+1), 6+1/n)$$

for n even, and

$$\{2+1/(3(n+1)), 2+1/(3n)\} \cup (6+1/(n+1), 6+1/n)$$

for n odd are both compact Hausdorff subsets of T.

Suppose now that τ is a finer compact Hausdorff topology. Clearly every $U \in \tau$ that contains 2 (3 resp.) must contain all but finitely many $2+1/(3n), n$ odd $(3+1/(2n), n$ even resp.). So let U_2, U_3 be disjoint elements of τ containing 2 and 3 respectively and let $N \in \mathbb{N}$ such that $n \geq N \implies 2+1/(3n) \in U_2$ (n even), $3+1/(2n) \in U_3$ (n odd). By the first remark $(6+1/(n+1), 6+1/n) \setminus U_2 \cup U_3$ must be non-empty for $n \geq N$ and will have the usual Euclidean topology as a subspace. Let

$$V_n = (6+1/(2n+1), 6+1/(2n)),$$

a co-compact and hence τ-open set. Let

$$\begin{aligned} V = &\{2+1/(3n) \colon n < N, n \text{ even}\} \\ &\cup \{3+1/(2n) \colon n < N, n \text{ odd}\} \\ &\cup (6+1/N, 7] \cap T, \end{aligned}$$

again co-compact open and hence τ-open. Then $\{U_2, U_3, V\} \cup \{V_n \colon n \geq N\}$ is a τ-open cover of T which has no finite subcover, since each V_n contains an element x_n not contained in any other element of the cover. Thus no finer compact Hausdorff topology can exists. $\square$

This counterexample exhibits one type of problem that may occur in a compact subset of $R^{\leq}$. We may say that we cannot decide which of the points 2 and 3 should represent or 'cover' the 'missing' $6 \in \overline{T}$. The impossibility of this decision can be traced back to the connectedness properties of T. In fact, below we show that provided T is Euclidean nowhere dense this type of problem cannot occur. Intuitively, we can then pull T apart and deal with the individual parts separately.

Theorem 7.3. *If T is a compact subset of $\mathbb{R}^{\leq} \setminus \{\infty\}$, T is Euclidean nowhere dense and bounded, then there is a finer compact Hausdorff topology on T.*

We will prove this theorem in a series of lemmas. However, since the proof is fairly intricate, we will first describe it informally. First, let us observe that if T has arbitrarily small elements, then it must contain 0 and we let 0 be the one point at infinity in the compactification of $T \setminus \{0\}$ with the discrete topology, an application of Theorem 4.4. Also, by using Lemma 4.1 we may restrict our attention to the a T_1 subset $T' = T \setminus \bigcup_{2<k} kT$. We have argued above that the problematic points are those of $\operatorname{ess}(T')$. We will show that we can choose one such point x, together with a Euclidean neighbourhood U of x such that $\overline{U \cap T'} \subset T' \cup n\overline{T'}$ for some $n > 1$, i.e. that all problematic points in a small neighbourhood around x can be covered with the same factor n. Provided we find a finer compact Hausdorff topology on $T' \setminus U$ we can then add points of $T' \cap U$ later by taking the disjoint union of $T' \setminus U$ and $\overline{T' \cap U}$ and identify the point $x \in \operatorname{ess}(T' \cap U)$ with the point representing x/n. Note that x/n itself may not be in $T' \setminus U$ but only in $\overline{T' \setminus U}$, so the identification requires some care. If $U \subset (\sup T'/2, \infty)$ then $T' \setminus U$ will still be compact (in the co-compact topology), so we will do this construction recursively. To succeed, however, it is important that at each step of the recursion we will decrease the number of problematic points, i.e. that $\operatorname{ess}(T' \setminus U) \subsetneq \operatorname{ess}(T')$.

Here the fact that T' is Euclidean nowhere dense allows us to choose U such that $\operatorname{ess}(U) \cap \overline{T'} = \emptyset$, i.e. that taking away U does not add any new problematic points.

As we will see now, some care and technical detail is needed to execute the above plan. The first step in the proof of the theorem has to be the identification of a point x and a Euclidean open neighbourhood U of x which we may remove now and add back in at a later stage. Since the remainder of T will be split into parts, we also have to make sure that U respects this split, which is guaranteed by the second part of the Lemma.

Lemma 7.4. *Suppose T satisfies the conditions of the theorem and that it is also T_1 and bounded away from 0.*

If $\operatorname{ess}(T) \neq \emptyset$, $y \in \operatorname{ess}(T)$ and V a Euclidean open set containing y then there is $x \in \operatorname{ess}(T) \cap V$, a Euclidean open $U \ni x$, and an integer $n > 1$ such that $\operatorname{ess}(U) \cap \overline{T} = \emptyset$, $\operatorname{ess}(T \cap U) \subset n\overline{T}$ and $x/n \in T$.

Moreover, if $U_1, \ldots, U_N$ is a finite partition of T, we may choose x, U and n such that $\operatorname{ess}(T \cap U) \subset n\overline{U_i}$.

Proof. First note that $\sup T / \inf T < N$ for some $N \in \mathbb{N}$, so that $\overline{T} \subset \bigcup_{i \leq N} iT$.

Now suppose that the Lemma is not true for T. Choose witnesses $y \in \operatorname{ess}(T)$, $V \ni y$ for this.

We will define $x_k \in U_k, 1 < n_k \leq N$ such that for every k,

(i) $x_k \in \operatorname{ess}(T \cap U_k)$,

(ii) $\overline{U_k} \subset U_k \subset V$,

(iii) $\operatorname{ess}(U_k) \cap \overline{T} = \emptyset$,

(iv) $\operatorname{ess}(T \cap U_{k+1}) \cap n_k\overline{T} = \emptyset$,

(v) $n_k \neq n_i$ for $i < k$.

Let $x_1 = y$. Since $\overline{T} \subset \bigcup_{i \leq N} iT$ there is $1 < n_1 \leq N$ with $x_1/n_1 \in T$. As T is Euclidean nowhere dense, we can find Euclidean open $U_1 \ni x_1$ such that $U_1 \subset V$ and $\operatorname{ess}(U_1) \cap \overline{T} = \emptyset$.

Having defined x_k, U_k, n_k for $k \leq K$ consider $T \cap U_K$. Since $x_K \in V$, $x_K/n_K \in T$, $\operatorname{ess}(U_K) \cap \overline{T} = \emptyset$ and y, V witness the failure of the Lemma, there must be $x_{K+1} \in \operatorname{ess}(T \cap U_K) \setminus n_K\overline{T}$. As $\mathbb{R}$ is regular, $n_K\overline{T}$ Euclidean closed, and T Euclidean nowhere dense, we can find Euclidean open U_{K+1} with $\operatorname{ess}(U_{K+1})$ disjoint from $\overline{T}$ such that $\operatorname{ess}(T \cap U_{K+1}) \cap n_K\overline{T} = \emptyset$ and $x_{K+1} \in U_{K+1} \subset \overline{U_{K+1}} \subset U_K$. Finally note that since

$$x_{K+1} \in \operatorname{ess}(T \cap U_{K+1}) \subset \operatorname{ess}(T \cap U_k)$$

for $k \leq K$ we have $x_{K+1} \notin n_k\overline{T}$ for $k \leq K$. But $x_{K+1} \in \bigcup_{i \leq N} iT$ so there is $n_{K+1} \neq n_k$ for $k \leq K$ with $x_{K+1}/n_{K+1} \in T, 1 < n_{K+1} \leq N$.

However, there are only finitely many possible n_k, namely $2, \ldots, N$, so eventually we will arrive at a contradiction.

The proof of the second part works similarly. Suppose it were not true. We choose witnesses y, V and apply the first part. We then have $x \in U$, $\operatorname{ess}(U) \subset n\overline{T} = \bigcup_{i \leq N} n\overline{U_i}$, $x/n \in U_i$ for some i, so there must be $x' \in \operatorname{ess}(U)$ with $x' \notin n\overline{U_i}$. The latter is closed so we can separate x' from $n\overline{U_i}$ and proceed by induction as above. Again only finitely many U_is have to be considered so we will eventually arrive at a contradiction. □

As described before, we may now iterate this construction.

Lemma 7.5. *Suppose T satisfies the conditions of the theorem and that it is also T_1 and bounded away from 0.*

Then there is a finite partition U_k of T, integers n_k, p_k $(k = 0, \ldots, N)$ such that

(i) U_0 *is Euclidean closed in* $\mathbb{R}$,

(ii) $n_i > 1$ *and* $p_i < i$ *for* $i > 0$

(iii) $\operatorname{ess}(U_i) \subset n_i\overline{U_{p_i}}$ *for* $i \geq 0$.

Proof. Let $t_0 = \inf T > 0$ and note that compactness of T implies that $T \cap [0, 3t_0/2]$ is Euclidean closed. As T is Euclidean nowhere dense, there is $t > t_0$, $t \notin \overline{T}$ such that a partition as required exists for $T \cap [0, t]$.

We will now show that if such a partition exists for $T \cap [0,t]$ with $t \notin \overline{T}$ then there is $t' > 5/4t, t' \notin \overline{T}$ such that a partition as required exists for $T \cap [0,t']$. Boundedness of T then implies that a partition as required exists for T.

So let $U_0, \ldots, U_N, n_0, \ldots, n_N, p_0, \ldots, p_n$ be as described for $T \cap [0,t]$. Since T is Euclidean nowhere dense, we may choose $t' \in [5t/4, 3t/2] \setminus \overline{T}$. Note that if $V \subset (t, \infty)$ is Euclidean open then and $T' \subset T \cap [0,t']$ is compact (in the co-compact topology), then $T' \setminus V$ is compact (in the co-compact topology) as well.

Fix a basis $\mathcal{B}$ of $\mathbb{R}$ such that for every $\epsilon > 0$ all but finitely many basic open sets have diameter less than ϵ with $\mathrm{ess}\,(U) \cap T = \emptyset$ for every $U \in \mathcal{B}$.

By induction we will define $T_\alpha \subset T \cap [0,t']$, Euclidean open $V_\alpha \subset (t, \infty)$ with $V_\alpha \in \mathcal{B}$, integers $n_\alpha > 1, i_\alpha \leq N$

(i) T_α is compact in the co-compact topology,

(ii) $V_\alpha \cap \mathrm{ess}\,(T_\alpha) \neq \emptyset$,

(iii) $\beta < \alpha \implies \mathrm{ess}\,(T_\beta) \subsetneq \mathrm{ess}\,(T_\alpha)$,

(iv) $\mathrm{ess}\,(V_\alpha \cap T_\alpha) \subset n_\alpha \overline{U_{i_\alpha}}$.

Let $T_\alpha = T \cap [0,t'] \setminus \bigcup_{\beta < \alpha} V_\beta$. Since the V_α are Euclidean open and meet $\mathrm{ess}\,(T_\alpha)$ the first two conditions are satisfied.

If there is $y \in \mathrm{ess}\,(T_\alpha \cap (t, \infty)) \neq \emptyset$ apply Lemma 7.4 with $T_\alpha \cap (t, \infty)$, y and $V = (t, t')$ to obtain U, x, n. Shrink U to an element V_α of $\mathcal{B}$ still containing x and note that the conclusions of Lemma 7.4 still hold.

If on the other hand $\mathrm{ess}\,(T_\alpha \cap (t, \infty)) = \emptyset$, we set $\alpha_0 = \alpha$ and stop.

Since $\mathrm{ess}\,(T_\alpha)$ is strictly decreasing as α increases the process must eventually stop. We then define

$$U_{n,p} = \bigcup_{\alpha : n_\alpha = n, i_\alpha = p} V_\alpha \cap T_\alpha$$

and claim that $\mathrm{ess}\,(U_{n,p}) \subset n\overline{U_p}$. For suppose $x \in \mathrm{ess}\,(U_{n,p})$ and let x_m be a sequence in $U_{n,p}$ converging to x. If there is a subsequence of x_m which is contained in some $V_\alpha \cap T_\alpha$ for some α then $x \in \operatorname{ess}(V_\alpha \cap T_\alpha) \subset n\overline{U_p}$ as claimed. Otherwise we may pick a subsequence such that each $V_\alpha \cap T_\alpha$ contains at most one x_m. If $x_m \in V_\alpha \cap T_\alpha$ then by the construction there is some $y_m \in V_\alpha$ such that $y_m/n \in U_p$. By the choice of the basis the diameter of the V_α converges to 0 so y_m must converge to x as well. But then $x/n \in \overline{U_p}$ so $x \in n\overline{U_p}$ as desired.

Finally note that $T_{\alpha_0} \cap (t, \infty)$ is closed, so that if we relabel $U_{n,p}, T_{\alpha_0} \cap (t, \infty)$ by $U_{N+1}, \ldots, U_M$ and set the n_k, p_k accordingly (i.e. if $U_{n,p} = U_k$ then $n_k = n$, $p_k = p$ and otherwise arbitrarily subject to $n_k > 1, p_k < k$ if $T_{\alpha_0} \cap (t, \infty) = U_k$) we have in fact extended the partition to $T \cap [0, t']$ as required. □

The above lemma essentially constructs a tree where the nodes U_k are disjoint subsets of T and the edge between U_k and U_{p_k} is labelled by n_k. Starting from the root, U_0 of this tree, we may now construct a finer compact Hausdorff topology on T.

Lemma 7.6. *Suppose T satisfies the conditions of the theorem and that it is also T_1 and bounded away from 0.*

Then there is a finer compact Hausdorff topology on T.

Proof. First construct a partition $U_k, k = 0, \ldots, N$ by using Lemma 7.5.

Let $Y = \sum_{k \le N} \overline{U_k}$ be the disjoint union of the U_k each of which is equipped with its Euclidean topology. Let R be the reflexive, symmetric, transitive closure of $x \sim y \iff y \in \overline{U_k}, x = y/n_k \in \overline{U_{p_k}}$. Note that

$$R = \bigcup_{n \in \omega} (\mathrm{id} \cup \sim \cup \sim^{-1})^n.$$

Also observe that since T is bounded and there are only finitely many U_k there must in fact be some $N \in \omega$ with $R = \bigcup_{n \le N} (\mathrm{id} \cup \sim \cup \sim^{-1})^n$. Since $\sim$ is closed by the conditions on the U_k and T_1-ness of T we can conclude that R is a closed relation. Thus $X = Y/R$ is a compact Hausdorff space. We will show that the identification $[x] \in X \to \min[x] \in T$ is a continuous bijection, thereby completing the proof.

First observe that since T is bounded away from 0 every equivalence class $[x]$ under R has a least element. Thus the map $S\colon Y \to T; y \to \min[y]$ is well defined and constant on equivalence classes. Since the U_k partition T the map S is surjective and $S/R = ([x] \mapsto \min[x])$ is injective as T is T_1. It remains to show that S is continuous. So let I be an open interval in $\mathbb{R}$ and consider a point $y \in S^{-1}(\mathbb{N}I \cap T)$ with $y \in \overline{U_k}, k \leq N$. Since $y/\min[y] \in \mathbb{N}$ for each $y \in Y$ and $y \in \overline{U_k} \subset \overline{T}$, this means that $y \in \mathbb{N}I \cap \overline{T}$. But then $\mathbb{N}I \cap \overline{U_k}$ is a $\overline{U_k}$-open neighbourhood of y and $S(\mathbb{N}I \cap \overline{U_k}) \subset \mathbb{N}I$. Therefore $S^{-1}(\mathbb{N}I \cap T)$ is indeed open and since the $\mathbb{N}I \cap T$ form a basis for T, S will be continuous. □

Proof of Theorem 7.3. If $0 \in \overline{T}$ then $0 \in T$ and we isolate every point of $T \setminus \{0\}$ and let 0 be the point at infinity of the one-point compactification of $T \setminus 0$.

Otherwise consider the T_1-space $T' = T \setminus \bigcup_{k \geq 2} kT$. We can apply Lemma 7.6 to T' and obtain a finer compact Hausdorff topology τ. Similarly to Lemma 4.1 we may now add the points of $T \setminus T'$. More precisely, for each $x \in T \setminus T'$ we choose $R(x) \in T'$ such that $x/R(x) \in \mathbb{N}$ and extend R to T by letting $R|_{T'} = \mathrm{id}$. The topology on T is then generated by $\{x\}$ for $x \notin T'$ and $R^{-1}(U) \setminus F$ where $U \in \tau$ and F is a finite subset of $T \setminus T'$. It is easily checked that this topology is in fact a finer compact Hausdorff topology on T. □

We have shown that the construction of a finer compact Hausdorff convergence can be modified to yield a construction of a finer compact Hausdorff topology, provided the original orbit spectrum T is bounded and Euclidean nowhere dense. The reason for the nowhere dense requirement has already been exhibited. If we turn our attention to unbounded subspaces T of $\mathbb{R}^{\leq} \setminus \{\infty\}$ we can first note that unless T is nowhere dense, no new problems arise. For, if $\overline{T}$ contains some interval, then the T_1-core, i.e. the set $T \setminus \bigcup_{2<k} kT$, will be bounded. If T however is nowhere dense, and hence no obstacles to a finer compact Hausdorff topology will occur in any bounded part of T, we encounter a new problem in the unbounded part.

This arises from the fact that every co-compact open set in $\mathbb{R}_0^+$ contains a tail of $\mathbb{R}_0^+$. Thus if we consider any ultrafilter $\mathcal{U}$ finer than $\left\{(r,\infty) \cap T \colon r \in \mathbb{R}_0^+\right\}$ then $\mathcal{U}$ will converge to every point of T. Now, one might hope that we can just choose any $x \in T$ and simply define all these ultrafilters to converge to this particular x, as one would in the context of convergence spaces. Unfortunately, the following example shows that this is, in general, impossible.

Example 7.7. Let $X_0 = \{1\}$. Inductively construct well ordered sets X_n such that $\overline{X_{n+1}} \setminus X_{n+1} = 2\overline{X_n}$ and each X_{n+1} is isolated in $\mathbb{R}$.

Having constructed X_n we will build X_{n+1} as follows. For each $x \in X_n$ let $x' = \min\{y \in X_n \colon x < y\}$ if this exists and 2^{n+1} otherwise. Let y_n^x be a decreasing sequence in $(2x, 2x')$ that converges to $2x$. Now set

$$X_{n+1} = \bigcup_{x \in X_n} \{y_n^x \colon n \in \mathbb{N}\}.$$

Finally let $X = \bigcup_{n\in\mathbb{N}} X_n$ and observe that X is a compact subset of $\mathbb{R}^{\leq}$ but is Euclidean discrete. Note that every bounded part of X has a finer compact Hausdorff topology. X on the other hand cannot have a finer compact Hausdorff topology.

Proof. X is countable, so if X had a finer compact Hausdorff topology it would be a scattered space, thus there would be isolated points. But isolating any point $x \in X$ clearly leads to non-compactness and is therefore impossible. □

What we can prove is that if T is eventually Euclidean closed then the unboundedness does not introduce any new problems.

Theorem 7.8. *Suppose T is a compact, Euclidean nowhere dense subset of $\mathbb{R}^{\leq}$. If there is $r \in \mathbb{R}$ with $T \cap [r,\infty)$ Euclidean closed and $T \cap [0,r]$ has a finer compact Hausdorff topology, then T has a finer compact Hausdorff topology.*

Proof. Let τ be the finer topology on $T_r = T \cap [0,r]$. If $T \subset [0,r]$ we are done. Otherwise equip $\overline{T \cap (r,\infty)}$ with its Euclidean topology and let $T^\star$ be its one point compactification. Identify the point at infinity of $T^\star$ with any $x \in T^\star \setminus \{\infty\}$ to obtain T_∞, a compact Hausdorff space. Thus the disjoint sum of (T_r,τ) and T_∞ is a compact Hausdorff space. Finally, identify $r \in T_r$ with $r \in T_\infty$ if both exist. □

We now return to our original aim, namely to construct the actual $\mathbb{R}$-action. The finer compact Hausdorff topologies constructed above will serve as the orbit space. To obtain an $\mathbb{R}$-action from an orbit space we apply the following theorem. Apart from the induced stabilizer from the orbit space to $\mathbb{R}^{\leq}$, we also need a 'winding' number n. Given two orbits represented by elements x, y of the orbit space, $n(x, y)$ describes how often the orbit corresponding to y is wound around the orbit corresponding to x.

Theorem 7.9. *Suppose T is a compact Hausdorff space, $S\colon T \to \mathbb{R}^+$ a continuous map (where $\mathbb{R}^+$ is regarded as a subspace of $\mathbb{R}^{\leq}$) and there is a function $n\colon T^2 \to \mathbb{N}$ such that*

(i) given $\epsilon > 0, x \in T$ there is a τ-open $U \ni x$ such that

$$y \in U \implies \left| \frac{S(y)}{n(x,y)S(x)} - 1 \right| < \epsilon;$$

(ii) given $x \in T$ there is a τ-open $V_x \ni x$ such that for every $y \in V_x$ there is a τ-open $W_{x,y} \ni y$ with $W_{x,y} \subset V_x$ and

$$z \in W_{x,y} \implies n(x,z) = n(x,y)n(y,z).$$

Then there is a compact Hausdorff space X with a continuous $\mathbb{R}$ action such that T is the orbit space and S the induced stab*-map.*

Proof. In the proof we will use various quotients of $\mathbb{R}$. Elements of $\mathbb{T}_x = \mathbb{R}/S(x)\mathbb{Z}$ are subsets of $\mathbb{R}$ of the form $r + S(x)\mathbb{Z}$. If U is an open subset of $\S_x$ and π_x the projection map from $\mathbb{R}$ to $\mathbb{T}_x$ then $\pi_x^{-1}(U)$ is an open subset of $\mathbb{R}$ such that $\pi_x^{-1}(U) + S(x)\mathbb{Z} = \pi_x^{-1}(U)$. Conversely any such open subset U of $\mathbb{R}$ such that $U + S(x)\mathbb{Z}$ is the inverse image of an open subset of $\mathbb{T}_x$ under π_x. We will therefore identify open subsets of $\mathbb{T}_x$ with those open subsets U of $\mathbb{R}$ such that $U + S(x)\mathbb{Z} = U$. Note then that if U is an open subset of $\mathbb{T}_x$ and $n \in \mathbb{N}$ then $\frac{US(y)}{S(x)n}$ is an open subset of $\mathbb{T}_y$ since it is invariant under adding $S(y)\mathbb{Z}$.

Let $X = \bigcup_{x \in T} \{x\} \times \mathbb{T}_x$. A point $(x, t + S(x)\mathbb{Z})$ has a neighbourhood basis consisting of sets of the form

$$W_{x,U,V} = \bigcup_{y \in U} \{y\} \times \frac{VS(y)}{S(x)n(x,y)}$$

where $U \subset V_x$ is an open neighbourhood of $x \in T$, V is an subset of $\mathbb{R}$ such that $t \in V$ and $V + S(x)\mathbb{Z} = V$. Note that if $y \in U$ then

$$W_{y, U \cap W_{x,y} \cap V_y, \frac{VS(y)}{S(x)n(x,y)}} \subset W_{x,U,V}$$

using the fact that $z \in W_{x,y}, y \in V_x \implies n(x,y)n(y,z) = n(x,z)$. Hence every $W_{x,U,V}$ is in fact open as claimed.

Since $rV \cap rV' = \emptyset \iff V \cap V' = \emptyset$ for subsets V of $\mathbb{R}$ and $r \in \mathbb{R}$ and since both T and $\mathbb{T}_x$ for each $x \in T$ are Hausdorff, X is Hausdorff.

Similarly if the $U_i, i \in I$ cover $\mathbb{T}_x$ then $\frac{U_i S(y)}{S(x)n(x,y)}, i \in I$ cover $\mathbb{T}_y$. Thus if W_{x,U_i,V_i} covers $\{x\} \times \mathbb{T}_x$ then in fact it covers $\bigcup_{y \in \bigcap V_i} \{y\} \times \mathbb{T}_y$. Therefore compactness of T implies compactness of X.

Finally we need to check continuity of the action. Let $t_0 \in \mathbb{R}$, $(x_0, s_0 + S(x_0)\mathbb{Z}) \in X$ and let $W(x_0, U_0, V_0) \ni t(x_0, s_0 + S(x_0)\mathbb{Z})$. Choose open $V \ni s_0 + S(x_0)$ and $\delta > 0$ such that $t_0 + B_\delta(V) \subset V_0$. Next choose $U \subset U_0$ such that

$$x, y \in U \implies |S(x)n(x,y)/S(y) - 1| < \delta/(2|t_0| + \delta)$$

and observe that for $t \in B_{\delta/2}(t_0)$ we have

$$\begin{aligned} B_{\delta/2}(t_0)W(x_0,U,V) &= \bigcup_{y \in U} \{y\} \times \frac{VS(y)}{S(x_0)n(x_0,y)} + B_{\delta/2}(t_0) \\ &= \bigcup_{y \in U} \{y\} \times \left(V + \frac{B_{\delta/2}(t_0)S(x_0)n(x_0,y)}{S(y)}\right) \frac{S(y)}{S(x_0)n(x_0,y)} \\ &= \bigcup_{y \in U} \{y\} \times \frac{(V + B_{\delta/2}(t_0) + \epsilon(x_0,y)t)S(y)}{S(x_0)n(x_0,y)} \end{aligned}$$

where $|\epsilon(x_0,y)| < \delta/(2|t_0| + \delta)$. Thus $V + B_{\delta/2}(t_0) + \epsilon(x,y)t \subset t_0 + B_\delta(V) \subset V_0$ for every $y \in U$. Therefore $B_{\delta/2}(t_0)W(x_0,U,V) \subset W_(x_0,U_0,V_0)$, $t_0 \in B_{\delta/2}(t_0)$ and $(x_0, s_0 + S(x_0)\mathbb{Z}) \in W(x_0,U,V)$ giving continuity of the action. $\square$

To use this theorem, we not only need the orbit space, but also the winding numbers $n(x,y)$. Out constructions of finer compact Hausdorff topologies already gave an idea of the winding numbers n. However, provided $S(T)$ is bounded, we can reconstruct appropriate winding numbers.

Lemma 7.10. *If T is a compact Hausdorff space and $S\colon T \to R^{\leq}$ continuous such that $S(T) \subset \mathbb{R}_0^+$ is bounded and bounded away from 0, then there is a map $n\colon T^2 \to \mathbb{N}$ as required in Theorem 7.9.*

Proof. Continuity of S implies that a function $n\colon T^2 \to \mathbb{N}$ exists which satisfies the first condition, namely as y tends to x, $S(y)/n(x,y)S(x)$ tends to 1. If $S(T)$ is bounded this automatically gives the second condition: write $S(y)/S(x)n(x,y) = 1 + \epsilon_{x,y}$ and note that

$$\begin{aligned} |\epsilon_{x,z}| = \left|\frac{S(z)}{S(x)n(x,z)} - 1\right| &= \left|\frac{S(z)}{S(y)n(y,z)}\frac{S(y)}{S(x)n(x,y)}\frac{n(x,y)n(y,z)}{n(x,z} - 1\right| \\ &= \left|(1+\epsilon_{y,z})(1+(\epsilon_{x,y})\frac{n(x,y)n(y,z)}{n(x,z)} - 1\right| \end{aligned}$$

Thus as z tends to y and y tends to x all of $|\epsilon_{x,z}|, |\epsilon_{x,y}|, |\epsilon_{y,z}|$ become small, so that $n(x,y)n(y,z)/n(x,z)$ tends to 1, noting that n will be bounded by the bounds on $S(T)$. Thus $n(x,y)n(y,z)$ must tend to $n(x,z)$ (again since n is bounded) and therefore eventually they must be equal, as required. □

Theorem 7.11. *If T is an orbit spectrum of an abstract $\mathbb{R}$-action which has a bounded, Euclidean nowhere dense T_1-core, then T is compactifiable.*

Proof. This follows from Theorem 7.3, Theorem 7.9 and Lemma 7.10. □

7.2.1 Examples

We will illustrate the results from above by example constructions.

Example 7.12. The most basic case is where the orbit spectrum (in $\mathbb{R}_0^+ \cup \{\infty\}$) is a closed interval, i.e. $[r_1, r_2]$ with $0 < r_1 < r_2 < 2r_1$. In this case, we can see that the identity from $[r_1, r_2]$ with the Euclidean topology into the co-compact topology is continuous. We let $T = [r_1, r_2]$ with the Euclidean topology, $S = \mathrm{id}$ in Lemma 7.10 to obtain the winding numbers $n(x, y)$. The only possible choice of n is simply $n(x, y) = 1$ (due to connectedness of T). We can then apply Theorem 7.9 to obtain the required compact group action.

Geometrically, the phase space consists of an annulus in the plane. The group action is a rotation of this which has the same speed everywhere.

Prof Haydon calls this construction the horse-racing track. Each 'horse' has to stay on its lane and races around the circle and each horse has the same speed (so the horses on the outer circle take longer).

Example 7.13. Let us modify the previous example slightly by replacing r_1 from the orbit spectrum by $r_1/2$, i.e. the orbit spectrum will be $\{r_1/2\} \cup (r_1, r_2]$ again with $0 < r_1 < r_2 < 2r_1$. This time, the continuous map will have range $[r_1, r_2]$ with the Euclidean topology and map r_1 to $r_1/2$ and r to itself for $r \neq r_1$. Note that the induced finer topology on the orbit spectrum is not the usual Euclidean topology! We can then use the same process to obtain a compact group action. This time, $n(r_1, y)$ will have to be 2 if y is very close to r_1.

Geometrically we again obtain an annulus where the speed of rotation is proportional to the distance from the center, but where opposite points on the innermost circle are identified, illustrating the concept of the winding number: all other circles wind twice around the circle at r_1.

In the pictorial terminology of the horse-racing track, we can picture this as two indistinguishable horses racing on the innermost track on opposite sides. To the observer the horses on the innermost track seem to take only half as long to run once around the circle.

Example 7.14. Let us modify the example again by adding the interval $(3r_1/2, r_3]$, so that we have an orbit spectrum $\{r_1/2\} \cup (r_1, r_2] \cup (3r_1/2, r_3]$ where $0 < r_1 < r_2 < 3r_1/2 < r_3 < 2r_1$. The orbit space will be the union of the two intervals $[r_1, r_2]$ and $[3r_1/2, r_3]$ with r_1 and $2r_1$ identified. The continuous map is the identity except that the equivalence class $\{r_1, 2_r1\}$ is mapped to $r_1/2$. We obtain winding numbers $n(x, y)$ where $n(\{r_1, 2r_1\}, y) = 2$ when y is close to r_1 and $n(\{r_1, 2r_1\}, z) = 3$ when z is close to $2r_1$.

Geometrically, the orbits corresponding to $y \in (r_1, r_2]$ wind twice around the orbit corresponding to $\{r_1, 2r_1\}$ whereay the orbits corresponding to $z \in (3r_1/2, r_3]$ wind three times around it.

If we want to give the horse-racing track analogy, we now have a middle circle which has either two indistinguishable horses if looking from the outside, or three indistinguishable horses if looking from the inside.

It is straightforward to modify the above examples so that instead of intervals with open endpoints r_1 and $2r_1$, we had converging sequences with limit r_1 and $2r_1$ respectively. The winding number is then still uniquely determined when the two arguments are sufficiently close, but not anymore when the arguments are far apart. We can also provide an example where we have various choices of the winding number which lead to different group actions:

Example 7.15. Consider the orbit spectrum $\{r_1/2, r_1/3\} \cup (r_1, r_2] \cup (3r_1/2, r_3]$ with $0 < r_1 < r_2 < 3r_1/2 < r_3 < 2r_1$. Now the following choices of orbit space and map S are all possible:

- $[r_1, r_2] \cup [3r_1/2, r_3]$ (Euclidean topology) with $S(r) = r$ for $r \neq r_1, 2r_2$ and $S(r_1) = r_1/2$, $S(3r_1/2) = r_1/3$. The resulting group action is the disjoint sum of two annuli as given in example 7.13.

- $[3r_1/2 - r_3, r_2 - r_1] \cup \{p\}$ (Euclidean topology, $p > r_2 - r_1$) with

$$S(r) = \begin{cases} r + r_1 & r > 0 \\ r_1/2 & r = 0 \\ r_1/3 & r = p \\ -r + 3r_1/2 & r < 0 \end{cases}.$$

  We obtain the group action of example 7.14 plus a single circle (far away at p) representing the orbit of type $r_1/3$.

Finally we look at unbounded orbit spectra.

Example 7.16. Let $S^n = \{n + 1/m \colon m \in \mathbb{N}\}$ for $n \in \mathbb{N}$, i.e. a sequence converging to n. Consider the orbit spectrum $T = \{1\} \cup \bigcup_{n \in \mathbb{N}} S^n$. One way to construct a compact $\mathbb{R}$-action with this orbit spectrum is as follows: let $T^n = \{\alpha_n : \alpha < \omega + 1\}$ be homeomorphic to $\omega + 1$ and consider the quotient space T' of $\bigcup_{n\in\mathbb{N}} T^n$ where all the limit points $\omega_n, n \in \mathbb{N}$ are identified to a single point $\star$. Let $S : T' \to T$ be defined by $S(\alpha_n) = n + 1/\alpha$ for $\alpha \neq \omega$ and $S(\star) = 1$. Clearly S is continuous into the orbit spectrum. We can define the winding numbers $n(\star, \alpha_n) = n$ for $\alpha \neq \omega$, $n(x, x) = 1$ for $x \in T'$ and extend n arbitrarily to $T' \times T'$ and apply Theorem 7.9.

Note however, that this is not the only way of constructing an appropriate $\mathbb{R}$-action. An alternative can look as follows: Use the same notation as above but this time, let T' be the quotient of $\bigcup_{n\in\mathbb{N}} T^n$ where all ω_n with $n > 1$ will be identified to the point $\star$. Define $S(\alpha_n) = n + 1/(\alpha + 1)$ for $\alpha \neq \omega, n > 1$. Let $S(\star) = 1$ and let $S(\alpha_1) = \alpha - 1 + 1/2$ for $\alpha \neq \omega$ and define $S(\omega_1)$ arbitrarily. Now the winding numbers are defined by $n(\star, \alpha_n) = n$ for $n > 1, \alpha \neq \omega$ and $n(\omega_1, \alpha_n) = \text{round}(S(\alpha_n)/S(\omega_1))$. Again $n(x, x) = 1$ for $x \in T'$ and every other value of n is arbitrary. Finally apply Theorem 7.9 to obtain a very different $\mathbb{R}$-action from the one above.

These example show that the winding number can usually be defined in a natural way. However, the author is unable to prove the existence of the winding number satisfying the condition in Theorem 7.9 from the assumption that a compact $\mathbb{R}$-action with the appropriate orbit space and induced map from the orbit space to the orbit spectrum exists. On the other hand, no example of a compact subset of $\mathbb{R}^{\leq}$ which has a finer compact Hausdorff topology but for which no winding numbers can be defined is known.

7.3 Non-periodic Orbits

In the previous section we have concerned ourselves mainly with $\mathbb{R}$-actions where all orbits are periodic, i.e. compact. In this section we take a closer look at $\mathbb{R}$-actions without periodic orbits. Fortunately, most of the work has already been done in previous chapters.

Theorem 7.17. *Every non-periodic orbit of a continuous $\mathbb{R}$-actionis non-compact.*

Proof. This follows directly from Theorem 3.2 since $\mathbb{R}$ is a locally compact, Lindelöf, non-compact Hausdorff group and every non-periodic $\mathbb{R}$-orbit must have stabilizer $\{e\}$. □

Together with Theorem 3.13 we now have a clear picture of what happens for continuous $\mathbb{R}$-actions on compact Hausdorff spaces that do not have periodic orbits.

Corollary 7.18. *If $\mathbb{R}$ acts continuously on a compact Hausdorff space X such that no orbit is periodic, then there are at least $\mathfrak{c}$ many orbits.*

Theorem 7.19. *An abstract $\mathbb{R}$-action consisting of κ many non-periodic orbits is compact realizable if and only if $\kappa \geq \mathfrak{c}$.*

Proof. Necessity follows from the previous Corollary. For sufficiency of the condition we give an explicit construction of a compact $\mathbb{R}$-action with $\mathfrak{c}$ many non-periodic orbits. Using the Adding Lemma (4.1) this gives the result.

So let $X = \mathbb{T} \times \mathbb{T}$ and choose an irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then define the map $\mathbb{R} \times X \to X$ by $(t, (\phi, \psi)) \mapsto (\phi + t, \psi + \alpha t)$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Clearly this is a continuous $\mathbb{R}$-action on the compact Hausdorff space X. Furthermore if $(t, (\phi, \psi)) \mapsto (\phi, \psi)$ then we must have $t, \alpha t \in \mathbb{Z}$ which implies $t = 0$ as $\alpha \notin \mathbb{Q}$. Thus every point has stabilizer $\{e\}$ as required. □

Chapter 8

Conclusion

We have shown that the apparently simple demand of continuity of a group action on a compact Hausdorff space gives rise to surprising restrictions on its structure as an abstract group action. Using a wide variety of techniques we were able to prove these results in a geometrically meaningful way. On the other hand, for large classes of abstract group actions we gave constructions of compact Hausdorff topologies making them continuous. For compact Abelian groups the necessary and sufficient conditions for the existence of these compact Hausdorff topologies coincide. Thus a complete characterization of the compact group actions of these groups was obtained. Furthermore, if the groups in question are Lie groups, then a particularly simple statement of the characterization is possible. Unfortunately both for $\mathbb{Z}^2$-actions and for $\mathbb{R}$-actions some open questions remain. Where possible we have tried to describe these questions by giving concrete examples, exhibiting the limitations of our current results. Below we summarize the main open questions and briefly speculate about possible approaches to solve these.

8.1 Open questions

8.1.1 $\mathbb{Z}^n$-actions

We have seen that a straight generalization from $\mathbb{Z}$-actions to $\mathbb{Z}^2$-actions, let alone general $\mathbb{Z}^n$-actions, is impossible. This is primarily due to the multi-dimensional nature of the orbits of most $\mathbb{Z}^n$-actions for $n > 1$.

A refinement of the techniques using scattered spaces presented in section 5.2.1, in particular the construction of even better open partitions, seems possible in principle, although complicated to execute in practice. In the same vein, the applications of these techniques to $\mathbb{Z}^n$ actions should produce necessary conditions for continuity of the actions.

The different approach used in section 5.2.2 on the other hand is comparatively simple, but does not produce as strong results. However, we have already noted that the two approaches might in fact be equivalent or could at least be generalized to produce identical results. If this were the case, then the connection of even continuity to the Ascoli theorem and thereby to the theory of function spaces, could be a large advantage. This seems a promising field for further investigations.

When comparing the results for $\mathbb{Z}$-actions and for $\mathbb{Z}^2$-actions there are two striking differences. Firstly in $\mathbb{Z}$-actions, a combination of finite and infinite orbits is always compact-realizable which does definitely not hold for $\mathbb{Z}^2$-actions, although the existence of infinite orbits definitely helps. Secondly, it just so happens that the precise number of orbits of a particular type is not important for $\mathbb{Z}$-actions - only the distinction between less than $\mathfrak{c}$ and at least $\mathfrak{c}$ many orbits is important. However, this is just a corollary from the theorem characterizing all compact-realizable $\mathbb{Z}$-actions and is not explained by the proof of this result. Preliminary results indicate however, that in $\mathbb{Z}^2$-actions the precise number of orbits of a particular type is important in deciding whether the action is compact-realizable. Thirdly and lastly we have observed that $\mathbb{Z}^2$-actions can have infinitely many non-trivial forced-compact subsystems. Even if each of these subsytems is compact-realizable it remains unclear how to piece them together.

8.1.2 Finer compact Hausdorff topologies

Our discoveries have shown that compact non-Hausdorff topologies are crucial in determining whether an abstract group action is compact-realizable. In particular, the question of whether these topologies have finer compact

Hausdorff topologies or more generally continuous compact Hausdorff preimages is of great importance. In fact, at least for compact Abelian groups we were able to show that the existence of certain the continuous preimages of compact topologies is equivalent to the question of their compact-realizability.

8.1.3 $\mathbb{R}$-actions

Apart from the question of the existence of finer compact Hausdorff topologies mentioned in the previous section, we encounter the problem of 'winding' numbers when asking whether an abstract $\mathbb{R}$-action is compact-realizable. The existence of compatible winding numbers could turn out to be an extra necessary condition for an $\mathbb{R}$-action to be compact-realizable. On the other hand, no compact subset of $\mathbb{R}^{\leq}$ with finer compact Hausdorff topology is known that is not the orbit spectrum of a compact $\mathbb{R}$-action.

When discussing the existence of finer compact Hausdorff topologies, we gave two examples of compact subsets of $\mathbb{R}^{\leq}$ that do have such a finer compact Hausdorff topology. One of these examples was 'bounded' whereas the other was unbounded and Euclidean nowhere-dense. Since we could not find fundamentally different examples of compact subsets of $\mathbb{R}^{\leq}$ without finer compact Hausdorff topologies, we conjecture that these two exhibit all the problems one might possibly encounter when trying to find finer compact Hausdorff topologies to compact subsets of $\mathbb{R}^{\leq}$.

8.1.4 Outlook

Apart from the fairly specific questions mentioned above, a wide field of more broad research directions remains.

We have seen that a diverse range of categories, all related to the category of topological spaces, was used in the proofs in this thesis. In some of these, for example convergence spaces, the problem seems to become easier (every compact convergence structure has a finer compact Hausdorff convergence structure). In others, for example metric spaces, our question becomes harder and in fact undecidable in ZFC as mentioned in the introduction. I am convinced, however, that investigating this or similar questions for these categories would produce interesting results. Particularly if one starts to consider the problem on topological or differentiable manifolds, direct applications in the field of theoretical physics might be possible.

Closely related, sometimes indistinguishable, is the variation on the properties we require the phase spaces to have. We argued in the introduction why compact Hausdorff seems a good choice, but other properties are certainly interesting as well. In view of Theorem 3.11 properties which are invariants of continuous maps suggest themselves. Among these are weaker forms of compactness (e.g. Lindelöfness) but also completely different properties like connectedness. Combinations, for example requiring the phase space to be a continuum, i.e. a compact, connected Hausdorff space, are further possibilities.

Finally, it is possible to not only impose further restrictions on the phase space, but also on the group action. Apart from requiring it to be uniformly continuous when working in the category of uniform spaces or differentiable when working on differentiable manifolds, we can impose restrictions from the theory of dynamical systems on the action. Examples of such restrictions are the existence or absence of attractors, whether the action is topologically mixing, and so on. This would require the development of new and different techniques, among them a generalization the results to semi-group actions (e.g. $\mathbb{R}^+$-actions) which brings with it completely new problems.

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