Geometric insights from supergravity in the AdS/CFT correspondence

Juan Manuel Pérez Ipiña



Mathematical Institute

University of Oxford

New College

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Statement of originality

The content of this thesis is based on several works written in collaboration. It has been reduced from the published versions in some places due to space constraints, but also expanded upon in others for the sake of clarity.

In particular, Chapter 2 is based on

[1] P. Benetti Genolini, J. M. Perez Ipiña, and J. Sparks, "Localization of the action in AdS/CFT," *JHEP* **10** (2019) 252, arXiv:1906.11249 [hep-th] ,

Chapter 3 on

[2] P. Ferrero, J. P. Gauntlett, J. M. P. Ipiña, D. Martelli, and J. Sparks, "Accelerating black holes and spinning spindles," *Phys. Rev. D* **104** no. 4, (2021) 046007, arXiv:2012.08530 [hep-th] ,

and finally, Chapter 4 is adapted from

[3] P. Ferrero, J. P. Gauntlett, J. M. Pérez Ipiña, D. Martelli, and J. Sparks, "D3-Branes Wrapped on a Spindle," *Phys. Rev. Lett.* **126** no. 11, (2021) 111601, arXiv:2011.10579 [hep-th],

[4] A. Boido, J. M. P. Ipiña, and J. Sparks, "Twisted D3-brane and M5-brane compactifications from multi-charge spindles," *JHEP* **07** (2021) 222, arXiv:2104.13287 [hep-th].

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Abstract

The aim of this thesis is to explore new constructions in supergravity to gain insights into the $\mathrm{AdS}_4/\mathrm{CFT}_3$ and $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondences.

We start by deriving a simple formula for the action of any supersymmetric solution to minimal gauged supergravity in the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. We find that the holographically renormalized action localizes to the fixed points of a supersymmetric Killing vector and may then be expressed entirely in terms of topological data. In this sense, the gravitational partition function localizes in the bulk. Our general formula reproduces all known examples in the literature and makes predictions for new solutions, assuming they exist.

Also in $D = 4$, $\mathcal{N} = 2$ minimal gauged supergravity, we sudy a class of accelerating, rotating and dyonic black holes in AdS_4. These $D = 4$ black hole metrics have conical singularities but we find a new way to desingularize them by uplifting to solutions of $D = 11$ supergravity using a regular Sasaki-Einstein SE_7 manifold. These $D = 11$ solutions are associated to M2-branes wrapped around a two-dimensional weighted projective space, $\mathbb{WCP}^1_{[n_-,n_+]}$, also known as a spindle, and we find that supersymmetry is not preserved through a usual topological twist. We also discuss the supersymmetric and extremal limits of these solutions and compute the entropy of these black holes, arguing that it should be possible to derive it from certain $\mathcal{N} = 2$, $d = 3$ quiver gauge theories compactified on a spinning spindle with magnetic flux.

In the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence we continue exploring these ideas by constructing $\mathrm{AdS}_3 \times \mathbb{WCP}^1_{[n_-,n_+]}$ solutions of different $D = 5$ supergravity theories. Again we find that these solutions can be uplifted to find smooth solutions in $D = 10/11$ and that supersymmetry is preserved in a different way than through the topological twist. Our construction suggests a path to the dual $d = 2, \mathcal{N} = (0, 2)$ SCFTs in the IR, associated to D3 or M5 branes wrapped on a spindle. We make a precision test of this conjecture by computing the central charge and R-symmetry of the field theories and find agreement with the gravity computations in all cases.

Contents

1 Introduction and background **1**

1.1 The AdS/CFT correspondence 1

1.2 From branes to SCFTs 4

1.3 The field theory perspective 6

1.4 Outline 9

2 A localization principle in the bulk **11**

2.1 $D = 4$ gauged supergravity and reduction 14

2.1.1 Action and equations of motion 14

2.1.2 Reduction to base 15

2.1.3 Reduction of the bulk action 18

2.2 Supersymmetric solutions 21

2.2.1 Supersymmetric equations 21

2.2.2 Local form of supersymmetric solutions 24

2.3 Contributions from the conformal boundary 27

2.4 Contributions from the fixed points of the isometry 30

2.4.1 Contribution from a nut 31

2.4.2 Contribution from a bolt 34

2.5 Examples 38

2.5.1 AdS_4 39

2.5.2 Self-dual solutions 40

2.5.3 Non-self-dual 1/4 BPS 43

2.5.4 Non-self-dual 1/2 BPS 47

2.5.5 More general examples . . . 50
2.5.6 Supersymmetric black holes . . . 53
2.6 Discussion . . . 56

3 Accelerating black holes and their spindle horizons 61
3.1 General setting . . . 66
3.2 AdS_4 PD black holes in $D = 4$ . . . 69
3.3 Uplifting and desingularizing . . . 73
3.3.1 Metric . . . 73
3.3.2 Flux quantization . . . 81
3.4 Supersymmetric and extremal limits . . . 83
3.4.1 Near horizon limit: $AdS_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$ . . . 85
3.4.2 Killing spinors for $AdS_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$ . . . 90
3.4.3 R-symmetry Killing vector . . . 95
3.5 Conformal boundary for non-rotating solutions . . . 97
3.5.1 Supersymmetric and extremal solutions . . . 98
3.5.2 Supersymmetric and non-extremal solutions . . . 104
3.5.3 A one-parameter family of non-supersymmetric and non-extremal solutions . . . 110
3.6 Discussion . . . 114

4 Five dimensional theories on spindles 119
4.1 Minimal gauged theory . . . 124
4.1.1 $D = 5$ solutions . . . 124
4.1.2 Uplift to IIB string theory . . . 127
4.1.3 $d = 4$ SCFTS on Σ . . . 131
4.2 Multi-charge spindles . . . 133
4.2.1 Local form of the solutions . . . 134
4.2.2 Global analysis and general solution . . . 136
4.2.3 Central charge and R-symmetry . . . 140
4.2.4 Special cases . . . 143
4.2.5 M5 branes for two charge solution . . . 143

4.2.6 Field theory 147
4.3 Discussion 152

A Details on accelerating black holes **157**

A.1 Circle fibrations over spindles 157
A.2 Non-accelerating case: $\alpha = 0$ 159
A.3 Near horizon limit 161
A.4 The non-rotating $AdS_2 \times Y_9$ solutions 165
A.5 Angular momentum 168
A.6 Killing spinor equations 174

Bibliography **177**

Chapter 1

Introduction and background

1.1 The AdS/CFT correspondence

The Anti-de Sitter/Conformal Field Theory correspondence (AdS/CFT) establishes a duality between a quantum theory of fields and a higher dimensional gravitational theory. A direct realization of the Holographic principle, by relating a theory of quantum gravity to a non-gravitational theory on its boundary it provides valuable insight into some of the most ambitious questions in modern theoretical physics.

In practice, it is best understood as a duality in string theory. The first example of the correspondence was originally formulated in [5], as an equivalence between $\mathcal{N} = 4$ Super Yang-Mills (SYM) in $3 + 1$ dimensions and type IIB string theory on $\mathrm{AdS}_5 \times S^5$. At its strongest form it is a relation between the fully quantum string theory and the generic field theory, with a matching prescription for the free parameters of both sides. Because of the intrinsic difficulties in working with these theories at any value of the parameters, it is common and most useful to take limits on both sides of the duality.

The supergravity limit: By taking the string coupling to zero we remain at first order in string perturbation theory, at tree level in a string genus expansion. Classical string theory on the gravity side matches to a field theory in the large N limit (N being the rank of the gauge group) but arbitrary value for the 't Hooft coupling λ. Taking λ to be large as well we focus on strongly coupled field theories and this corresponds on the gravity side to the point-particle low energy limit of the string theory (string length much smaller than the

radius of curvature), given by a supergravity theory. An example of a weak/strong duality, this formulation of AdS/CFT allows us to learn about observables in the large N limit of the strongly interacting field theories, through the study of observables in the calculable limit of supergravity.

Shortly after the duality was formulated, an equivalence was proposed between the generating functionals of both theories, written informally as $Z_{\rm CFT}[\phi_0] = Z_{\rm string}[\phi]|_{\phi\to\phi_0}$ for the strongest form of the correspondence [6, 7]. The holographic dictionary states that at the conformal boundary of AdS the bulk fields ϕ must take the values ϕ_0, which correspond to the sources of the CFT. In the approximation of classical supergravity the right hand side of this equation is much simpler and so we find, in Euclidean signature,

$$Z_{\rm CFT}[\phi_0] \simeq e^{-I_{\rm SUGRA}[\tilde{\phi}]}\Big|_{\tilde{\phi}\to\phi_0}\,, \tag{1.1.1}$$

where $I_{\rm SUGRA}[\tilde{\phi}]$ is the supergravity action evaluated on a classical solution $\tilde{\phi}$ to the equations of motion (that satisfies the corresponding boundary conditions). One may have to consider quantum or string theory corrections to this action if this saddle point approximation is not appropriate. The action may also present divergences that need regularizing, often near the conformal boundary, which correspond to the UV divergences of the field theory. To deal with these divergences, one usually follows the procedure of holographic renormalization [8, 9].

AlAdS: We can further generalize (1.1.1), to see that in the large N limit the AdS/CFT correspondence states [6, 10]

$$Z_{\rm CFT}(M_d) = \sum e^{-I[M_{d+1}]}\,. \tag{1.1.2}$$

On the left hand side we have the partition function of the conformal field theory defined on a background M_d and on the other side we have the holographically renormalized gravity action $I[M_{d+1}]$ evaluated on an asymptotically locally (Euclidean) AdS solution M_{d+1}. This solution has a boundary M_d where the gravity fields take values that correspond to the CFT background fields. Finally we ought to sum over all such Al(E)AdS solutions that can be embedded in string theory, the possible bulk coverings of the boundary M_d.

The AdS vacuum is dual to a CFT in flat space, but moreover, asymptotically locally

AdS solutions[1] are equivalent to placing those field theories on curved backgrounds, as was shown for instance in [13, 14]. In these papers the authors study the CFT deformations this gravity structure implies. For theories with an R-symmetry, for instance, they show how preservation of supersymmetry leads at the boundary manifold to a charged conformal Killing spinor equation [13–15]. The study of placing supersymmetric field theories on curved spaces is quite a rich subject so we will briefly explain the systematic approach from [16] in section 1.3.

The role of branes: The extended objects known as Dp-branes are at the core of the duality; their dual open/closed strings interpretation was the catalyst for Maldacena's insight. From the open string perspective D-branes are seen as higher-dimensional objects where open strings can end; while in the closed string perspective D-branes are solitonic objects of the supergravity theory. The configuration in [5] describes a stack of N D3-branes in flat spacetime, and the application of both perspectives motivates the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. That is, from the former perspective, in the low energy limit the open and closed strings decouple and we find a supersymmetric gauge theory, $\mathcal{N} = 4$ SYM, on the worldvolume of the branes and type IIB supergravity theory on $\mathbb{R}^{9,1}$. From the latter perspective, we find the low energy decoupling of type IIB supergravity in a near horizon $\mathrm{AdS}_5 \times S^5$ region and type IIB supergravity on $\mathbb{R}^{9,1}$. Both perspectives must describe the same physics and this lead Maldacena to conjecture that $\mathcal{N} = 4$ SYM in four dimensions must be equivalent to type IIB supergravity on $\mathrm{AdS}_5 \times S^5$.

For the duality to work, it is essential that the global symmetries of the two theories match. The six-dimensional space perpendicular to the branes has a rotational symmetry group $SO(6)$. This group corresponds to the isometries of the S^5 on the gravity side, and to the $SU(4) \sim SO(6)$ R-symmetry group of the dual field theory. Correspondingly, the isometry group of AdS_5, $SO(4, 2)$, matches the conformal group in four dimensions. Moreover, as charged objects in string theory (or supergravity) branes must also satisfy Dirac quantization conditions. For the D3-branes, the flux through the S^5 counts the number N of coincident branes and is therefore appropriately quantized. These characteristics of the brane picture have therefore immediate implications in the resulting supergravity theories, and the world-

[1] In this thesis we will concern ourselves with solutions that present an asymptotic Anti-de Sitter geometry, and not on alternative holographic formulations on flat space [11] or de Sitter space [12], for instance.

volume duals. In the next section we will expand on these ideas and analyze different brane constructions that have been used in the context of the AdS/CFT correspondence.

1.2 From branes to SCFTs

Beyond the epitome case of $\mathcal{N} = 4$ SYM, many new examples of gauge/gravity dualities have been proposed. In this section we will review the most relevant ideas that serve as background to the work presented in this thesis. To this purpose we focus on extensions to the brane picture from the previous section, that open the way to new dualities between supergravity theories and superconformal field theories (SCFTs).

Other dimensions: Since the first work of [5] different dualities were proposed for supergravity theories associated to p-branes of superstring theory or M-theory (in dimensions $D = 10, 11$, respectively). The near horizon geometry of the supergravity brane solution takes the form $\mathrm{AdS}_{p+2} \times S^d$ ($d = D - p - 2$), and the duality arises with the $p+1$-dimensional conformal field theory living in the worldvolume of the brane. In M-theory, or its low-energy limit eleven-dimensional supergravity, there are dualities associated with both M2 and M5-branes. In the case of the M2-branes in flat space, we find a near horizon limit of the form $\mathrm{AdS}_4 \times S^7$, thus giving one of the classic examples of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. It is considerably difficult to work out the dual field theories in this correspondence, but some successes have been made by placing the branes on singularities (see below), for instance in the case of the famous ABJM theory [17]. The M5-branes in flat space give an example of the $\mathrm{AdS}_7/\mathrm{CFT}_6$ correspondence, with the six-dimensional field theories being for example $\mathcal{N} = (0, 2)$ SCFTs [5]. The degrees of freedom also scale differently with the number N of stacked branes in all these theories: as N^2 in D3-branes, $N^{3/2}$ for M2-branes and N^3 in the case of M5-branes. See the review [18] for other examples like the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality arising from a system of D1/D5 branes, or dualities arising from Dp-branes that do not have AdS factors and are not associated to conformal field theories.

Branes at singularities: We have seen solutions of the form $\mathrm{AdS}_{p+2} \times S^d$ but we can also explore products, possibly warped, with other compact spaces, that may preserve less than maximal supersymmetry. That is, geometries of the form $\mathrm{AdS}_{p+2} \times X_d$, for an Einstein

d-dimensional manifold X_d (with $d = D - p - 2$). In the brane picture this can be obtained by replacing the flat space transverse to the brane for a manifold with special holonomy. If one takes this special holonomy manifold to be a cone metric over the base X_d, we see that we get a singularity at $r = 0$ unless $X_d = S^d$. The physical picture then is a stack of N p-branes sitting at the conical singularity, and so the near horizon limit of this metric is of the form $\mathrm{AdS}_{p+2} \times X_d$ [19].

One famous example [20–22] is taking the cone metric to be a Calabi-Yau (CY) manifold. By definition, the base of a Calabi-Yau cone is a Sasaki-Einstein manifold SE_d. This way one obtains dualities as for example, $d = 4$ $\mathcal{N} = 1$ SCFTs corresponding to $\mathrm{AdS}_5 \times SE_5$ solutions to type IIB supergravity, or $d = 3$ $\mathcal{N} = 2$ SCFTs dual to $\mathrm{AdS}_4 \times SE_7$ solutions to eleven-dimensional supergravity. Furthermore it has been shown that there exist consistent reductions on these internal manifolds to the lower four or five dimensional theories [23, 24]. Other famous examples include placing branes on orbifold singularities, orientifolds, conifolds, etc. (for an early review, see [18]). In particular we note the aforementioned ABJM theory [17], associated to N M2-branes located at the apex of a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. This leads to supergravity theories on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, that are dual to Chern-Simons $U(N) \times U(N)$ gauge theories.

Branes wrapping cycles: In the constructions above the field theory lives on the world-volume of the branes, which corresponds to flat Minkowski space $\mathbb{R}^{p,1}$. Since the seminal work of [25] there have been many examples constructed out of wrapping the branes on some non-trivial cycle, and therefore having the superconformal field theory living in a curved space. For instance, in [25] the authors study the large N limit of branes wrapping a Riemann surface Σ_g of genus g. They describe how the starting d-dimensional field theory ($d = p + 1$), defined on $\mathbb{R}^{d-2} \times \Sigma_g$, reduces to a $(d-2)$-dimensional theory at low energies (length scales larger than the size of Σ_g). This "RG flow across dimensions" leads to the discovery of new SCFTs with reduced supersymmetry (since they are conformal then we will have an AdS_{d-1} geometry in the IR).

Within the AdS/CFT correspondence therefore this establishes a clear picture to follow. In $D = 10/11$ supergravity, we start with solutions describing branes wrapped on a compact n-dimensional manifold Σ. These solutions have boundaries of the form $\mathrm{AdS}_{d+1} \times M$

which describe the UV of a d-dimensional SCFT, and near horizon geometries of the form $\mathrm{AdS}_{d+1-n} \times \Sigma \times M$, which describe the SCFT in $d-n$ dimensions arising in the IR. After compactifying on M these $d+1$ solutions can be seen as black branes in AdS_{d+1} with horizon Σ. The dual field theories live in the worldvolume of the wrapped branes and are therefore like the previous paragraph (where $n=2$). There have been many generalizations including wrapping manifolds of higher dimension, relaxing the constant curvature condition, allowing for punctures, etc. (e.g. [26–28]).

A critical question to this procedure however, is how does the field theory preserve supersymmetry in this curved space. From the brane picture, it has been shown in [29] how branes wrapping cycles can preserve some supersymmetry through the so-called "topological twist"[2]. The directions transverse to the cycle form a non-trivial normal bundle, with connection A. To preserve supersymmetry this connection has to be equal to the spin connection ω in the worldvolume geometry of the cycle, schematically $A=\omega$. To understand what this means at the field theory level we need to first explain how to place supersymmetric field theories in curved spaces.

1.3 The field theory perspective

Supersymmetry in curved spaces: The study of placing supersymmetric field theories on non-trivial manifolds $\mathcal{M}$ was started by Witten in [30, 31]. In [16], the authors systematically analyze this problem and propose a novel approach. It consists of coupling the theory to some off shell formulation of supergravity and then taking the rigid limit where the excitations of the gravity fields become frozen (taking the Planck mass to infinity, $M_p \to \infty$). These rigid supersymmetry theories are then defined on $\mathcal{M}$ under some fixed background values for the metric and the auxiliary bosonic fields, which are found by demanding that supersymmetry is preserved without any consideration to their equations of motion.

We can then analyze the supersymmetry condition, through the vanishing of the gravitino variation. If the theory has a $U(1)$ R-symmetry then we find an equation of the schematic form $(\partial_\mu + \omega_\mu - A_\mu)\epsilon = 0$ (excluding other possible background fields for simplicity), where A_μ is a background gauge field for the global R-symmetry. This is called the (generalized) Killing

[2] This is a well known construction in the area of topological field theories [30], where the name "topological twist" comes from.

spinor equation, and we will have as many supercharges as solutions ϵ to this equation. Note that without the background fields the problem would reduce to finding covariantly constant spinors on the curved manifold $\mathcal{M}$, $(\partial_\mu + \omega_\mu)\epsilon = 0$, which is a very restrictive equation. We can then schematically choose the value of $A_\mu = \omega_\mu$ to cancel the spin connection and obtain $\partial_\mu \epsilon = 0$, which is solved by a constant spinor. This is the same "topological twist" as seen in the branes wrapping cycles picture. Notice that the spinors are then sections of the spin bundle tensored with the R-symmetry bundle (we say the twisting changes the spin of the fields). One can also have a "partial topological twist" where the background might be a product of manifolds and one twists on only some of these factors.

Supersymmetric localization: Computing the path integral of a field theory defined on a curved manifold $\mathcal{M}$ is a daunting task. But, if the theory is supersymmetric, the technique of supersymmetric localization [32] can provide a way to exactly compute the partition function of the theory. This was started in [33] where the authors computed the exact partition function and expectation values for Wilson loops in $\mathcal{N} = 2$ supersymmetric theories on S^4. By deforming the action by a Q-invariant term (Q is a fermionic symmetry of the theory) of the form tQV, they observe that the new partition function $Z(t) = \int D\Phi \exp^{-S[\Phi]-tQV}$ is independent of the parameter t. This means that we obtain our original integral $Z(0)$ by evaluating for instance at $t \to \infty$. For a suitable choice of V, in this limit then the integral localizes to the Q-invariant subset of V. In favorable circunstances the path integral can then be reduced to a finite dimensional integral for a matrix model. This insight opened the way for a multiple of exact computations in supersymmetric theories (see the exhaustive review [32] and references therein).

Supersymmetric localization has its origins in the localization formulas and theorems from differential geometry. In particular we note the results of Duistermaat and Heckman [34] and later of Berline and Vergne [35] and Atiyah and Bott [36]. There they show how the integrals of equivariantly closed forms can be localized to the fixed points of a $U(1)$ action. The extension of these ideas to infinite dimensional path integrals (in the context of supersymmetric quantum field theories) is covered by the techniques of supersymmetric localization. But it is interesting to note how these old localization formulas may be coming back in the context of AdS/CFT. As we have described, the holographic conjecture establishes

a dictionary between the partition functions at both sides of the duality. If the partition function in the field theory is localized, then the same should be true also for the gravitational partition function. In this thesis we will present an answer to this problem and find a bulk localization principle that very much resembles the fixed point theorems from differential geometry.

In AdS/CFT there have been many applications of supersymmetric localization, for instance for non-perturbative checks of the correspondence. The classical gravity approximation is linked to the strongly coupled limit of the field theory, which can be hard to access. Any exact result, as the ones obtained from supersymmetric localization, can be evaluated at strong coupling and holographically checked with the gravitational prediction. For a review of applications of these techniques in AdS/CFT see [37], within the larger [32]. Below we will briefly explain some results of particular relevance for the work presented in this thesis.

Supersymmetric extremization principles: In particular, localization has been critical in the AdS_4/CFT_3 correspondence, since the results for the three sphere in [38–41]. The full path integral on the three sphere localizes to a matrix model and can then be solved in the large N limit. From the partition function one can obtain the free energy on the sphere through $F = -\log|Z_{S^3}|$ to match with the supergravity computations, as was done in [39] to reproduce the $N^{3/2}$ dependence of the famous ABJM theory. Interest in the free energy F arose as it was found to be a good measure of the degrees of freedom of the theory, and that it decreases along the RG flow (that is, if there is an RG flow between two CFTs then $F_{UV} > F_{IR}$). This was dubbed the "F-theorem" [41]. Moreover, in [40, 41] the process of F-maximization was proposed as a way of computing the exact superconformal R-symmetry of a 3d $\mathcal{N} = 2$ theory. The R-symmetry that appears in the superconformal algebra of an IR SCFT can be obtained as a linear mixing between the UV R-symmetry and the possible flavor symmetries preserved along the RG flow (as long as there are no accidental symmetries in the IR). To obtain this precise linear combination, the S^3 free energy F must be maximized over the space of possible R-symmetries.

The ideas behind F-maximization and the F-theorem already existed in other dimensions, namely the work of [42] in $d = 4$ and [43, 44] in $d = 2$. In [42] the exact superconformal R-symmetry $R_\star$ of any 4d $\mathcal{N} = 1$ SCFT was obtained through the maximization of a

trial function $a_{trial}(R)$, and moreover this gives $a_{trial}(R_\star) = a$, the four dimensional central charge (that which multiplies the Euler density in the trace anomaly). In two dimensions, the authors of [43, 44] found a similar principle applied, which determines the exact superconformal R-symmetry of a two-dimensional SCFT with $\mathcal{N} = (0, 2)$ supersymmetry. This "c-extremization" principle gives the exact right-moving central charge c_R, through the use of anomaly polynomials. Both these constructions rely on the computation of anomaly polynomials, since supersymmetry relates the conformal anomaly (which determines the central charge) to the R-symmetry 't Hooft anomaly. The central charges can then be also computed from the gravity side [45] to make a precision test of the correspondence. The 2d central charge c and the 4d a are also associated to the number of degrees of freedom and decrease monotonically along the RG flow, as was originally shown in [46] in two dimensions and later proposed for the four-dimensional case in [47–49].

We have seen this analysis repeat itself in four, three and two dimensions[3]; but what about in one-dimensional theories? A proposal for such an extremization principle was described in [55]. In the AdS_4/CFT_3 correspondence, the authors reproduced the entropy of AdS_4 black holes through the extremization of a certain Witten index. The gravitational solution interpolates between AdS_4 and $AdS_2 \times \Sigma_g$, leading to an RG flow across dimensions, of a CFT_3 to a CFT_1. Using localization they obtain this twisted index from the partition function of the dual three-dimensional ABJM theory on $\Sigma_g \times S^1$, topologically twisted along the Σ_g. The extremization of this index leads to the exact superconformal R-symmetry and taking the logarithm we can exactly reproduce the black hole entropy (computed from the gravity side through the Bekenstein Hawking formula [56, 57]).

1.4 Outline

Now that we have covered the necessary background for the work presented in this dissertation, we can outline how it is presented and how it builds on the topics from the Introduction.

In chapter 2 we lean on this question of what the gravity analog to supersymmetric localization may be. Working on the AdS_4/CFT_3 correspondence we consider asymptotically

[3] A general statement can be made across dimensions, see e.g. [50], where the authors study the partition function $F = -\log Z_{S^d}$ of a CFT compactified on S^d. Moreover, the AdS/CFT correspondence implies that there must exist gravitational duals to these extremization principles, and in fact such geometric constructions were found, for instance, in [51–54].

locally AdS solutions to $\mathcal{N} = 2$ minimal gauged supergravity in four dimensions. We study the holographically renormalized on-shell action of these solutions and find that it localizes to the fixed points of a supersymmetric Killing vector. This localization principle greatly resembles the old localization formulas from differential geometry and gives an expression for the action of any such solution purely in terms of topological data. This formula reproduces the action of all known solutions to these four-dimensional theories and matches their corresponding three-dimensional field theory partition functions. Moreover, it provides a general treatment to find the action of solutions assuming they exist, generalizing and making predictions for new solutions and their corresponding unexplored dual field theories.

Also in the context of four-dimensional $\mathcal{N} = 2$ minimal gauged supergravity, starts the work of chapter 3. We study accelerating black hole solutions to this theory, which are known to be singular. However, we propose a novel way of desingularizing these solutions by uplifting them in a precise way to obtain solutions of M-theory. These eleven-dimensional solutions are associated to M2-branes wrapped around non-trivial cycles, but with the particularity that in this case supersymmetry is not preserved through the topological twist! These constructions describe a new way of preserving supersymmetry in this context, which has opened the way to many new and interesting applications in the field. Moreover, we compute the entropy of these solutions and propose how it could be derived from a field theory computation, involving supersymmetric localization on orbifolds. We expand on some details of these computations in appendix A at the end of this dissertation.

Chapter 4 further extends these ideas into five-dimensional supergravity, first in the minimal case and later in more general theories with matter couplings. In the five-dimensional version of the previous $D = 4$ solutions we do not possess the full black string solution but can reproduce an analog story working from the near horizon metric. We again find that the solutions can be regularized by an uplift, and that supersymmetry is preserved in a different way than through the topological twist. This construction suggests a way of finding the dual IR SCFTs and we test this by computing the central charge of the theory using c-extremization, finding agreement with the gravity computation. In this chapter we analyze solutions associated to D3-branes and M5-branes wrapping orbifolds, further opening the way to new and interesting constructions that have followed this work.

Chapter 2

A localization principle in the bulk

As we have seen in the Introduction, supersymmetric localization has played a key role in recent developments in AdS/CFT, allowing for the exact computation of certain field theory observables at strong coupling. These computations may in turn be compared to classical, or semi-classical, gravity. Now we want to show that localization also plays a role in the dual classical gravity computation. We describe how the holographically renormalized on-shell action I of any Euclidean supersymmetric solution to four dimensional minimal gauged supergravity localizes to the fixed points of a supersymmetric Killing vector. As can be seen in Eq. (1.1.1), in a saddle point approximation $\exp(-I)$ is identified with the gravitational partition function; so it is in this sense that the gravitational partition function localizes in the bulk. This is to be distinguished from the attempts at defining localization in the supergravity path integral - see, for example [58–61].

One of the motivations for the work presented in this chapter was to try to understand what are the fundamental structures underpinning certain supersymmetric observables in AdS/CFT. Rather than regard the agreement of dual computations of a particular observable, in a particular theory, as "miraculous", one would like to go further and try to understand at a more fundamental level why the computations are equivalent. To this end, we here focus on Euclidean solutions to $\mathcal{N} = 2$ minimal gauged supergravity in four dimensions [62]. The bosonic sector of this theory is Einstein–Maxwell theory with a negative cosmological constant. There are a number of reasons for focusing on this case. Two points worth mentioning here are that, firstly, because these solutions may be uplifted on different internal 7-manifolds Y_7 to obtain solutions of M-theory [24]. Secondly, because the supergravity theory is sim-

ple and many explicit solutions are known, whose gravitational partition functions have been matched to the large N limit of the dual field theories [63–69]. The field theory results include some cases described in section 1.3 such as the partition function of $d = 3$ $\mathcal{N} = 2$ SCFTs on the round S^3 or the topologically twisted index of $\Sigma_g \times S^1$. But also, more general manifolds $\mathcal{M}_{g,p}$, $U(1)$ bundles of Chern degree $p \in \mathbb{Z}$ over a Riemann surface Σ_g [69, 70], which include the latter examples as subcases. The classification of supersymmetric asymptotically locally AdS_4 solutions whose conformal boundaries are given by $\mathcal{M}_{g,p}$ manifolds was started in the cited papers above, and will be covered in higher detail below.

Let us summarize the main result from this chapter. As already mentioned, we consider Euclidean supersymmetric solutions to minimal $\mathcal{N} = 2$ gauged supergravity that are asymptotically locally AdS. We denote the bulk 4-manifold as M, with the conformal boundary 3-manifold $M_3 = \partial M$. Every supersymmetric solution is equipped with a canonical Killing vector field ξ on M, defined as a bilinear in the Killing spinor. We take this to be nowhere zero on the boundary M_3, so that the induced rigid supersymmetric geometry on M_3 is that in [71].[1] As usual, the on-shell action I for such a solution is divergent, but may be regularized by adding boundary counterterms [9, 72]. We show that this holographically renormalized action may be written as

$$I = \left(\sum_{\text{nuts}_\mp} \pm \frac{(b_1 \pm b_2)^2}{4 b_1 b_2} + \sum_{\text{bolts } \Sigma_\pm} \int_{\Sigma_\pm} \left(\tfrac{1}{2} c_1(T\Sigma_\pm) \mp \tfrac{1}{4} c_1(N\Sigma_\pm) \right) \right) \frac{\pi}{2G_4} \,. \tag{2.0.1}$$

The overall factor of $\pi/2G_4$, where G_4 denotes the four-dimensional Newton constant, is simply the action of Euclidean AdS_4. The terminology in the sums is taken from [73]: the fixed point set of ξ lies in the interior of M, with connected components being either fixed points, called *nuts*, or fixed two-dimensional submanifolds, called *bolts*. As we show later in the chapter, at such a zero of ξ the bulk Dirac Killing spinor necessarily becomes chiral, and the $\pm$ signs in the sums over $\text{nuts}_\pm \in M$ and bolts $\Sigma_\pm \subset M$ in (2.0.1) denote chirality. At a nut isolated fixed point we may write the vector field ξ as

$$\xi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} \,. \tag{2.0.2}$$

[1] This is not strictly necessary for the gravity computation, but it *is* necessary to compare with field theory localization results that have been developed to date.

Here the tangent space at the fixed point is $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$, with φ_1, φ_2 being standard polar angle coordinates on each copy of $\mathbb{R}^2 \cong \mathbb{C}$, respectively. As we also explain later, due to the lack of a canonical choice of certain signs, and indeed any canonical normalization for ξ, in fact only the ratio $b_1/b_2 \in \mathbb{R} \setminus \{0\}$ is well-defined. The action (2.0.1) is then a function of this ratio. On the other hand, a bolt is a fixed closed two-manifold $\Sigma_\pm \subset M$, with the $\pm$ signs again denoting chirality of the spinor over each connected component. Here $\int_{\Sigma_\pm} c_1(T\Sigma_\pm) = 2 - 2g$ is the Chern number of the tangent bundle of the Riemann surface $\Sigma_\pm$, of genus g, while $c_1(N\Sigma_\pm)$ denotes the first Chern class of the normal bundle of $\Sigma_\pm$ in M. The bolt contribution in (2.0.1) is then a topological invariant.

Using (2.0.1) we may immediately reproduce all known results in the literature. Various families of explicit solutions on $M \cong \mathbb{R}^4$, with $\partial M = M_3 \cong S^3$, were constructed in [63,64,66], and in [67] the formula (2.0.1) was proven for a general class of such "self-dual" solutions with this topology, where there is a single isolated fixed point at the origin of $M \cong \mathbb{R}^4$. On the other hand, in [65, 69] a class of 1/4 BPS "bolt" solutions was constructed. There are two branches of solutions, referred to as bolt$_\mp$, where $M \cong \mathcal{O}(-p) \to \Sigma_g$, so that $\partial M = M_3$ is the total space of a degree p circle bundle over a Riemann surface Σ_g of genus g. Here the supersymmetric Killing vector ξ fixes the zero section $\Sigma_\pm = \Sigma_g \subset M$, and we may immediately read off $\int_{\Sigma_\pm} c_1(N\Sigma_\pm) = -p$ as the degree of the line bundle $\mathcal{O}(-p)$. The formula (2.0.1) then agrees with the results of [65,69], taking upper and lower signs for bolt$_\mp$, respectively, where in [69] this was also obtained from a dual field theory computation. When $p = 0$ we have $M \cong \mathbb{R}^2 \times \Sigma_g$, and the action (2.0.1) reproduces minus the entropy of the black hole solution in [68], related to the so-called universal twist (see section 2.5.6), which again has been reproduced in field theory [68]. We also show that (2.0.1) correctly reproduces the action of the 1/2 BPS family of "bolt$_\pm$" solutions in [65], with topology $M \cong \mathcal{O}(-p) \to S^2$, where in the terminology of the present chapter the supersymmetric Killing vector has two isolated fixed points at the north and south poles of the zero section $S^2 \subset M$, which are a nut$_\pm$ and a nut$_\mp$, respectively.

In addition to reproducing all known results, the formula (2.0.1) may also be used to compute the action of solutions *assuming they exist*. Without a general existence theorem for supersymmetric solutions to minimal $\mathcal{N} = 2$ gauged supergravity, this sort of application is at present somewhat formal. However, we show that the right hand side of (2.0.1) may be

computed very readily for "toric" four-manifolds M, which by definition admit a T^2 action that contains the isometry generated by the supersymmetric Killing vector ξ. This includes all of the explicit examples mentioned in the previous paragraph (with genus $g = 0$, when relevant) as special cases, but significantly generalizes them.

We conclude this summary by noting that the form of (2.0.1) is very suggestive that there is a localization/fixed point theorem directly underlying this formula. The action depends only on topological fixed point data: the weights b_1, b_2 of the supersymmetric Killing vector ξ at its isolated fixed points, Chern numbers at the fixed submanifolds, and where certain signs are determined by chirality data at the fixed points. Its form very much resembles an equivariant index theorem, or the Berline–Vergne/Duistermaat–Heckman fixed point theorems. On the other hand, the localization here is in the bulk of a 4-manifold M that fills the boundary 3-manifold $M_3 = \partial M$ on which the dual field theory is defined. AdS/CFT states that $\exp(-I)$ should agree with the strong coupling (large N) partition function of this theory on M_3, where more precisely one should take the bulk solution of least action. We shall return to some of these comments in the discussion section 2.6.

Let us outline the structure of this chapter. In section 2.1, we introduce the four-dimensional supergravity theory of interest, and outline the process of reduction to a three-dimensional base space for solutions with at least a $U(1)$ isometry. The resulting equations of motion can be used to show that the on-shell action is naturally exact, as in [73]. In section 2.2 we study the supersymmetric nature of these solutions. Later we describe our analysis of the on-shell action for these solutions, computing the conformal boundary contributions in section 2.3 and from the Killing vector fixed points in 2.4. A number of explicit examples of supersymmetric solutions are presented in section 2.5. Finally we discuss the results in section 2.6, and give a brief overview of some of the extensions to this work.

2.1 $D = 4$ gauged supergravity and reduction

2.1.1 Action and equations of motion

The gravitational theory we consider is Einstein–Maxwell theory with a cosmological constant: the fields are the metric $g_{\mu\nu}$ and an Abelian gauge field A with field strength $F = dA$,

and we set the cosmological constant to $\Lambda = -3$. The resulting bulk Euclidean action is

$$S = -\frac{1}{16\pi G_4}\int \left(R_g + 6 - F^2\right) \mathrm{vol}_g \,, \tag{2.1.1}$$

where R_g denotes the Ricci scalar of the metric g, and $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$. The equations of motion obtained from the action are

$$0 = (E_g)_{\mu\nu} \equiv (\mathrm{Ric}_g)_{\mu\nu} + 3g_{\mu\nu} - 2\left(F_{\mu\rho}F_\nu{}^\rho - \tfrac{1}{4}F^2 g_{\mu\nu}\right) \,, \tag{2.1.2}$$

$$0 = d *_g F \,, \tag{2.1.3}$$

where (2.1.3) is the Maxwell equation for the Abelian gauge field.

The action (2.1.1) describes the bosonic sector of $\mathcal{N} = 2$ gauged supergravity [62], and a solution to the above equations is supersymmetric if there exists a non-identically zero Dirac spinor ϵ satisfying the (generalized) Killing spinor equation coming from the vanishing of the supersymmetric gravitino variation

$$\left(\nabla_\mu - \mathrm{i}A_\mu + \frac{1}{2}\Gamma_\mu + \frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\epsilon = 0 \,. \tag{2.1.4}$$

Here, the Hermitian Γ matrices generate the Clifford algebra $\mathrm{Cliff}(4,0)$, so $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$. Finally, we note that the supersymmetry equation (2.1.4) is compatible with the equations of motion (2.1.2), (2.1.3) and with the Bianchi identity $dF = 0$ for the gauge field. Specifically, the integrability condition contracted with Γ^ν leads to

$$0 = (E_g)_{\mu\nu}\Gamma^\nu\epsilon + \mathrm{i}\left[\frac{1}{2}dF_{\mu\nu\rho}\Gamma^{\nu\rho} - (*_g\, d *_g F)^\nu\Gamma_{\mu\nu} + (*_g\, dF)_\mu\Gamma_* + (*_g\, d *_g F)_\mu\right]\epsilon \,, \tag{2.1.5}$$

where E_g is defined in the Einstein equation (2.1.2), and we have defined the volume element $\Gamma_* \equiv \Gamma_{1234}$.

2.1.2 Reduction to base

Now we assume we are given a solution to the equations of motion (2.1.2), (2.1.3) that is equipped with a $U(1)$ symmetry generated by a vector field ξ. This is taken to preserve both the metric and gauge field curvature, so $\mathcal{L}_\xi g = 0 = \mathcal{L}_\xi F$. The aim will be to reduce

the various geometric quantities to a three-dimensional base space of orbits of the $U(1)$ symmetry, and also obtain an expression for the bulk action (2.1.1), evaluated on-shell. This straightforwardly generalizes the similar analysis in [73] for pure gravity to the case with an Abelian gauge field, although we will be careful to keep track of how various quantities transform under gauge transformations. We shall see in section 2.2 that supersymmetric solutions are always equipped with a canonical Killing vector ξ, and moreover there is a natural gauge choice for the Abelian gauge field A.

The assumption of a $U(1)$ symmetry acting on the spacetime manifold M immediately leads to a circle fibration

$$\pi : M \setminus M_0 \to B\,, \tag{2.1.6}$$

where

$$M_0 \equiv \{\xi = 0\} \subset M \tag{2.1.7}$$

is the subset of M where the Killing vector is zero (the fixed point set of the $U(1)$ symmetry), and B is a three-dimensional base of non-trivial orbits. In general B will be an orbifold, with orbifold points being images under π of points in M with non-trivial finite isotropy subgroups of the $U(1)$ action. We may also remove small tubular neighbourhoods of radius $\varepsilon > 0$ around each connected component of M_0 to obtain $M_\varepsilon \subset M$, so that $B_\varepsilon \equiv (M \setminus M_\varepsilon)/U(1) \subset B$ is an orbifold with boundary. We may then recover B as the $\varepsilon \to 0$ limit.

Next we may introduce coordinates so that the Killing vector $\xi = \partial_\psi$, where on a generic orbit ψ is a local periodic coordinate with $\psi \sim \psi + \beta$. On $M \setminus M_0$ we may then write the line element for the spacetime metric g as

$$ds^2 = V(d\psi + \phi)^2 + V^{-1}\gamma_{ij}dx^i dx^j\,. \tag{2.1.8}$$

Here ϕ is a local one-form satisfying $\xi \lrcorner \phi = 0$ and $\mathcal{L}_\xi \phi = 0$, $V \equiv \langle \xi, \xi \rangle$ is the square norm of the Killing vector, and γ is a metric on B. Notice that since V is invariant under ξ, it descends to a strictly positive function on B, so that V^{-1} is well-defined on B. We will denote

$$\eta \equiv V^{-1}\xi^\flat = d\psi + \phi \tag{2.1.9}$$

where $\xi^\flat$ is the one-form dual to ξ using the metric. The one-form η is globally defined on

$M \setminus M_0$, being proportional to a global angular form for the circle bundle over B in (2.1.6), while the second expression in (2.1.9) is valid only locally. Note that under a redefinition $\psi \mapsto \psi + \lambda$, the local one-form ϕ transforms as $\phi \mapsto \phi - d\lambda$. In addition we may define the *twist* one-form by

$$H \equiv *_\gamma d\eta \,, \tag{2.1.10}$$

which is clearly conserved on B by construction

$$\overline{\nabla}^i H_i = *_\gamma \, d \, *_\gamma \, (*_\gamma d\eta) = 0 \,, \tag{2.1.11}$$

where $\overline{\nabla}$ is the Levi-Civita connection associated to γ on the base.

Ultimately our aim in this section is to reduce the bulk action (2.1.1) to an expression on B, showing that it is naturally exact. To this end, we thus begin by reducing the Ricci tensor of the spacetime metric g along the circle fibre of (2.1.6). In terms of the above quantities we find

$$(\mathrm{Ric}_g)_{\psi\psi} = \tfrac{1}{2} V^4 \langle H, H \rangle_\gamma - \tfrac{1}{2} V^2 \overline{\nabla}^2 \log V \tag{2.1.12}$$

$$(\mathrm{Ric}_g)_{\psi i} = \tfrac{1}{2} \left(*_\gamma d(V^2 H) \right)_i \,, \tag{2.1.13}$$

$$\begin{aligned} (\mathrm{Ric}_g)_{ij} = {} & (\mathrm{Ric}_\gamma)_{ij} - \tfrac{1}{2} V^{-2} \overline{\nabla}_i V \overline{\nabla}_j V + \tfrac{1}{2} \overline{\nabla}^2 \log V \, \gamma_{ij} + \tfrac{1}{2} V^2 H_i H_j \\ & - \tfrac{1}{2} V^2 \langle H, H \rangle_\gamma \gamma_{ij} \,, \end{aligned} \tag{2.1.14}$$

so that the scalar curvature takes the form

$$R_g = V R_\gamma + V \overline{\nabla}^2 \log V - \tfrac{1}{2} V^3 \langle H, H \rangle_\gamma - \tfrac{1}{2} V^{-1} \langle dV, dV \rangle_\gamma \,. \tag{2.1.15}$$

Here we have denoted the pointwise inner product of two p-forms μ, ν on B as $\langle \mu, \nu \rangle_\gamma \equiv \mu_{i_1 \cdots i_p} \nu^{i_1 \cdots i_p}$, where indices are raised using the metric γ_{ij}.

Locally on $M \setminus M_0$ we may write the Abelian gauge field as

$$A = \varphi \eta + a \,, \tag{2.1.16}$$

where we have simply decomposed this local one-form into a component along the circle fibre direction, and a transverse one-form a, with $\xi \lrcorner a = 0$. From the assumption that the gauge

field curvature F is invariant under the Killing vector, one can verify that it is possible to locally choose a gauge in which both φ and a are invariant under ξ, so that both descend to the base space B. In fact for supersymmetric solutions we shall see in section 2.2 that there is a natural gauge choice of the form (2.1.16) in which φ is a *global* function on B, determined by the Killing spinor ϵ, and a is a local gauge one-form on B, so we henceforth assume this to be the case. This leads to the following expression for the gauge field strength

$$F = d\varphi \wedge \eta + \varphi\, d\eta + f\,, \tag{2.1.17}$$

where $f = da$. Notice that we may identify $\varphi = \xi \lrcorner A$, but that a constant gauge transformation along the $d\psi$ direction leads to the changes

$$A \mapsto A + c\, d\psi \qquad \Rightarrow \qquad \varphi \mapsto \varphi + c, \qquad f \mapsto f - c\, d\eta\,, \tag{2.1.18}$$

where c is a constant. We shall keep track of how various quantities transform under this gauge transformation in what follows. Choosing the orientation $\mathrm{vol}_g = V^{-1}\eta \wedge \mathrm{vol}_\gamma$, we find that

$$*_g F = -V^{-1} *_\gamma d\varphi + \varphi V \eta \wedge *_\gamma d\eta + V\eta \wedge *_\gamma f\,, \tag{2.1.19}$$

which is indeed gauge invariant under (2.1.18).

2.1.3 Reduction of the bulk action

Upon substitution of (2.1.15) and (2.1.19) into the bulk action (2.1.1), and further integration along the circle fibre of the $U(1)$ isometry, we find an expression for the action on the base B. We then add to this a Lagrange multiplier term imposing the constraint that H is conserved (2.1.11):

$$\begin{aligned} S = -\frac{\beta}{16\pi G_4}\int_B \Big[& R_\gamma + \overline{\nabla}^2 \log V - \tfrac{1}{2}V^2\langle H, H\rangle_\gamma - \tfrac{1}{2}V^{-2}\langle dV, dV\rangle_\gamma + 6V^{-1} \\ & - 2V^{-1}\langle d\varphi, d\varphi\rangle_\gamma - 2\varphi^2 V\langle H, H\rangle_\gamma - 2\varphi V\langle *_\gamma H, f\rangle_\gamma - V\langle f, f\rangle_\gamma \\ & - \sigma\, \overline{\nabla}^i H_i \Big] \mathrm{vol}_\gamma\,. \end{aligned} \tag{2.1.20}$$

This expression extends the analogue for pure Einstein gravity found in [73]. The equation of motion for the Lagrange multiplier σ is obviously just the constraint (2.1.11) on H, whereas the equation of motion for a, where recall $f = da$, is more interesting:

$$d(\varphi V H + V *_\gamma f) = 0\,. \tag{2.1.21}$$

This equation implies that locally on B there is an electromagnetic potential function ν such that

$$d\nu = \varphi V H + V *_\gamma f \; = \xi \lrcorner *_g F\,, \tag{2.1.22}$$

which notice is gauge invariant under (2.1.18). Here the second equality follows from comparing with (2.1.19). In terms of the electromagnetic potential, the equation of motion for φ becomes

$$d *_\gamma \left(V^{-1} d\varphi - \nu H\right) = 0\,, \tag{2.1.23}$$

which in turn tells us that we may locally write

$$*_\gamma (V^{-1} d\varphi - \nu\, H) = d\varpi \tag{2.1.24}$$

for some one-form ϖ.

The Lagrange multiplier σ is known as the *nut potential* [73]. It also appears in the equation of motion for H, which may be expressed in terms of the electromagnetic potential ν as

$$d\sigma = V^2 H + 4\varphi\, d\nu\,. \tag{2.1.25}$$

The origin of the name nut potential is clearer in the pure gravity case, where the last term in (2.1.25) vanishes. With the introduction of an Abelian gauge field, the nut potential is not gauge invariant under (2.1.18): since ν, H and V are left untouched by the gauge transformation along $d\psi$, we find that σ changes as

$$\sigma \mapsto \sigma + 4c\nu\,. \tag{2.1.26}$$

Finally, we note the equations of motion for the scalar V, and the Einstein equation for the

metric γ on the base:

$$0 = \overline{\nabla}^2 \log V - V^2 \langle H, H\rangle_\gamma - 6V^{-1} + 2V^{-1}\left(\langle d\varphi, d\varphi\rangle_\gamma - \langle d\nu, d\nu\rangle_\gamma\right) , \tag{2.1.27}$$

$$\begin{aligned} 0 = {} & (\mathrm{Ric}_\gamma)_{ij} - \tfrac{1}{2}V^{-2}\overline{\nabla}_i V \overline{\nabla}_j V + 6V^{-1}\gamma_{ij} + \tfrac{1}{2}V^2 H_i H_j \\ & - 2V^{-1}\big(\overline{\nabla}_i\varphi\overline{\nabla}_j\varphi - \overline{\nabla}_i\nu\overline{\nabla}_j\nu\big) . \end{aligned} \tag{2.1.28}$$

Notice that all the equations of motion are written in terms of quantities that are invariant under (2.1.18), (2.1.26).

Substituting these equations of motion inside the expression for the bulk Euclidean action (2.1.20), we readily find an expression for the bulk *on-shell* action, which we refer to as I_{bulk}, expressed as an integral over the base:

$$I_{\text{bulk}} = \frac{\beta}{16\pi G_4}\int_B d *_\gamma \left(d\log V - \sigma H + 4V^{-1}\varphi\, d\varphi\right) . \tag{2.1.29}$$

This is not immediately gauge invariant, but can be written in a more invariant way using ϖ

$$I_{\text{bulk}} = \frac{\beta}{16\pi G_4}\int_B d\left[4\varphi\, d\varpi + *_\gamma\left(d\log V + (4\varphi\nu - \sigma)\, H\right)\right] . \tag{2.1.30}$$

The bracketed term contains the gauge invariant combination $4\varphi\nu - \sigma$, and ϖ is invariant, so the overall variation of the integrand under (2.1.18) is $d(4cd\varpi) = 0$. Indeed, the variation of the integrand of (2.1.29) is precisely the equation of motion for φ, (2.1.23).

The expression (2.1.29) shows that the on-shell action is naturally exact on the base B. Importantly, we shall see in section 2.2 that for supersymmetric solutions there is a natural gauge in which both φ and σ are *global* functions on B. Since also $V > 0$ is a global function, and H a global one-form on B, it follows that we may integrate the action (2.1.29) by parts using Stokes' theorem, reducing to an integral over the boundary of B. For an asymptotically locally Euclidean AdS spacetime, which is the case we are interested in, the latter consists of two types of boundary term: the conformal boundary at infinity, and the boundaries of the tubular neighbourhoods surrounding the fixed point loci of the isometry, where the fibration of spacetime (2.1.6) degenerates. It follows that (2.1.29) then reduces to a contribution from the conformal boundary, together with a sum over contributions from each connected component of the fixed point set of ξ. We shall evaluate this more explicitly

for supersymmetric solutions in sections 2.3 and 2.4 .

2.2 Supersymmetric solutions

2.2.1 Supersymmetric equations

In this section we shall assume we are given a supersymmetric solution to the equations of motion (2.1.2), (2.1.3), meaning there is a non-identically zero Dirac spinor ϵ satisfying the (generalized) Killing spinor equation (2.1.4).

It will be convenient for what follows to introduce the charge conjugation matrix $\mathcal{C}$ satisfying

$$\mathcal{C} = \mathcal{C}^* = -\mathcal{C}^T\,, \qquad \mathcal{C}^2 = -1\,, \qquad \Gamma_\mu^T = \mathcal{C}^{-1}\Gamma_\mu\mathcal{C}\,. \tag{2.2.1}$$

It was shown in [74] that there are no solutions to (2.1.4) in which the spinor ϵ is chiral. Thus at a generic point on M the spinor ϵ is a non-chiral Dirac spinor, which in four dimensions generates an identity structure. Effectively, this corresponds to an orthonormal frame $\{\mathrm{E}^1, \mathrm{E}^2, \mathrm{E}^3, \mathrm{E}^4\}$ constructed in terms of the normalized chiral projections of the spinor

$$\eta_\pm \equiv \frac{\epsilon_\pm}{\sqrt{S_\pm}}\,, \qquad \text{where} \qquad \epsilon_\pm \equiv \frac{1}{2}(1 \pm \Gamma_*)\epsilon\,, \quad S_\pm \equiv \overline{\epsilon_\pm}\epsilon_\pm\,, \tag{2.2.2}$$

as

$$\mathrm{iE}^3 - \mathrm{E}^4 \equiv \overline{\eta_-}\Gamma_{(1)}\eta_+\,, \qquad \mathrm{iE}^1 - \mathrm{E}^2 \equiv \overline{\eta_-^c}\Gamma_{(1)}\eta_+\,, \tag{2.2.3}$$

where $\epsilon^c \equiv \mathcal{C}\epsilon^*$ is the charge conjugate of a spinor. Furthermore, a Dirac spinor defines two functions S and θ, where S is the square norm of the spinor. These are related to the square norms $S_\pm$ of the two chiral projections via

$$S \equiv \bar{\epsilon}\epsilon\,, \qquad \cos^2\frac{\theta}{2} \equiv \frac{S_+}{S}\,, \qquad \sin^2\frac{\theta}{2} \equiv \frac{S_-}{S}\,. \tag{2.2.4}$$

Notice that the frame degenerates where the spinor becomes chiral, $S_\pm = 0$, that is, at $\theta = \pi, 0$, respectively, and also potentially where the spinor vanishes, $S = 0$. We may then express the standard supersymmetric bilinears in terms of the orthonormal frame and the two scalars constructed above. In particular, the following real bilinears will be relevant for

us

$$\begin{aligned} P &\equiv \bar{\epsilon}\Gamma_*\epsilon = S\cos\theta\,, \\ K &\equiv \bar{\epsilon}\Gamma_{(1)}\epsilon = -S\sin\theta\,\mathrm{E}^4\,, \\ \xi^\flat &\equiv -\mathrm{i}\bar{\epsilon}\Gamma_{(1)}\Gamma_*\epsilon = S\sin\theta\,\mathrm{E}^3\,, \\ U &\equiv \mathrm{i}\bar{\epsilon}\Gamma_{(2)}\epsilon = -S(\mathrm{E}^{12} + \cos\theta\,\mathrm{E}^{34})\,, \end{aligned} \tag{2.2.5}$$

where note that the volume element corresponds to the orientation given by E^{4123}, not E^{1234}.

As is usual, from the supersymmetry equation (2.1.4) we can find a number of differential equations satisfied by the bilinears. In particular, we find that the vector field ξ, dual to $\xi^\flat$, is a Killing vector, and that

$$\xi \lrcorner *_g F = K + dS\,, \tag{2.2.6}$$

$$\xi \lrcorner F = -dP\,, \tag{2.2.7}$$

$$dK = 0\,, \tag{2.2.8}$$

$$d\xi^\flat = -2\,(*_g U + S *_g F + PF)\ . \tag{2.2.9}$$

Since by construction $F = dA$ automatically satisfies the Bianchi identity $dF = 0$, then it follows from the equations above that ξ generates a symmetry of all the bosonic fields in the solution

$$\mathcal{L}_\xi S = \mathcal{L}_\xi \theta = 0\,, \qquad \mathcal{L}_\xi F = 0\,, \qquad \mathcal{L}_\xi g = 0\,. \tag{2.2.10}$$

Moreover, since K is closed, we find that a component of Maxwell's equation (2.1.3) is immediately implied

$$\xi \lrcorner d *_g F = 0\,. \tag{2.2.11}$$

We conclude this subsection by mentioning a few additional points that will be relevant to our subsequent analysis. Firstly, it is not necessary for the four-dimensional spacetime M to be a spin manifold. From (2.1.4) we see that the Killing spinor ϵ has unit charge under the Abelian gauge field A, and so generically can be a section of a spin^c bundle over M. In fact, this is precisely what happens in some of the examples we consider later: A is more precisely a connection on a virtual square root line bundle $\mathcal{L}^{1/2}$, meaning that the periods

of the globally defined curvature two-form $F = dA$ are in general half-integer multiples of 2π, rather than the integer multiples for a standard $U(1)$ gauge field.[2] In general we shall therefore assume that our Euclidean supergravity solutions are equipped with a global spin^c Dirac spinor ϵ, which we have shown above defines a canonical frame on the subspace of M where it is non-chiral (and non-zero). Notice also from (2.2.3) that since $\mathrm{iE}^1 - \mathrm{E}^2$ has charge two under the Abelian gauge field, it transforms as a section of the gauge line bundle $(\mathcal{L}^{1/2})^2 \cong \mathcal{L}$, so more precisely this is a twisted frame.

Next, since equation (2.1.4) is linear in ϵ, it is possible to rescale the spinor by an arbitrary non-zero complex number, which implies a rescaling of the Killing vector ξ by an arbitrary positive real number. Moreover, taking the gauge field A to be real implies that the charge conjugate spinor ϵ^c satisfies the same equation (2.1.4), but with $A \mapsto -A$. The latter is a symmetry of the action (2.1.1) and equations of motion (2.1.2), (2.1.3). Constructing the Killing vector instead out of this charge conjugated spinor, one finds that $\xi \mapsto -\xi$. For a given real supersymmetric solution, there is no canonical way to choose between using ϵ or ϵ^c for its supersymmetric structure. From these comments, one thus expects any final result depending on the Killing vector ξ to be independent of multiplying it by an arbitrary real number.

Finally, we note that charge conjugation is also related to the choice of orientations. On the spacetime manifold M we choose the orientation naturally provided by the supersymmetric frame as $\mathrm{vol}_g = \mathrm{E}^{4123}$. If instead we construct the same frame as (2.2.3) with $\epsilon \mapsto \epsilon^c$, we find that E^1 and E^3 have changed sign (consistently with what was said above about the Killing vector ξ, given that E^3 is proportional to the dual one-form $\xi^\flat$ (2.2.5)). Therefore, whilst the overall orientation of M is the same, the individual orientations of the two tangent planes spanned by $\{\mathrm{E}^1, \mathrm{E}^2\}$ and $\{\mathrm{E}^3, \mathrm{E}^4\}$ have changed. Identifying both of these tangent planes with the complex plane $\mathbb{C}$, this amounts to a complex conjugation of both, while note that $A \mapsto -A$ also complex conjugates the Hermitian line bundle $\mathcal{L}$ on which $2A$ is a connection.

[2] For further details in the current context, the interested reader is referred to Appendix D of [65].

2.2.2 Local form of supersymmetric solutions

In this subsection we proceed to further analyse the spinor bilinear equations presented in the previous subsection. This will allow us to determine the local form of any real supersymmetric solution. An equivalent analysis appears in [74], but we have found it more convenient to use the bilinear formalism developed in the present chapter. In particular, as we shall see, certain bilinear equations play a key role in evaluating the on-shell action.

As already noticed, ξ is a Killing vector, so as in section 2.1.2 we may introduce coordinates so that $\xi = \partial_\psi$. In section 2.1.2 we assumed that the orbits of ξ all close, so that it generates a $U(1)$ isometry of M. More generally this need not be the case, but provided the closure of the orbits of ξ in the isometry group of M is a compact group, then M will in fact have a torus isometry, i.e. at least $U(1)^2$. In this case we may approximate ξ by a sequence of Killing vectors, each of which generates a $U(1)$ isometry. We shall return to this point again later. From (2.2.5) we may write down

$$\mathrm{E}^3 = S\sin\theta(d\psi + \phi)\,, \tag{2.2.12}$$

where ϕ, as in the previous section, is a local basic one-form. We have also seen in section 2.2.1 that ξ generates a full symmetry of the solution, so all functions will be independent of ψ. Additionally, from (2.2.5) and (2.2.8), since K is closed we may locally introduce a function y satisfying

$$\mathrm{E}^4 = \frac{1}{S\sin\theta} d\left(\frac{1}{y}\right)\,. \tag{2.2.13}$$

Following [74], via an appropriate frame rotation we may choose a gauge where $\partial_y \lrcorner A = 0$. In such gauge, we may then introduce a local complex coordinate z and a real function W to write

$$\mathrm{E}^1 + \mathrm{i}\mathrm{E}^2 = \frac{2\,\mathrm{e}^{W/2}}{y^2 S\sin\theta} dz\,. \tag{2.2.14}$$

From the supersymmetry equation (2.1.4), we find that

$$d(S\sin\theta(\mathrm{E}^1 + \mathrm{i}\mathrm{E}^2)) - 2\mathrm{i}A\wedge(S\sin\theta(\mathrm{E}^1 + \mathrm{i}\mathrm{E}^2)) = 2(\mathrm{E}^4 - \mathrm{i}\cos\theta\,\mathrm{E}^3)\wedge S(\mathrm{E}^1 + \mathrm{i}\mathrm{E}^2)\,, \tag{2.2.15}$$

which will be useful below.

As for the gauge field, by writing it in the form (2.1.16) and comparing its curvature with (2.2.7), we immediately find that

$$\varphi = S\cos\theta + c_\varphi = P + c_\varphi \ , \tag{2.2.16}$$

for some real constant c_φ. Recall here that in terms of the spinor bilinears in (2.2.5) we have $P \equiv \bar{\epsilon}\Gamma_*\epsilon = S\cos\theta$, so that P and hence also φ are then manifestly global functions on the spacetime four-manifold M. Moreover, in the gauge where $\partial_y \lrcorner A = 0$, the components of (2.2.15) fix a and impose a constraint between the functions W, S, θ:

$$\frac{y}{4}\partial_y W = 1 - \frac{1}{yS\sin^2\theta}\,, \tag{2.2.17}$$

$$a_{\bar{z}} = -\frac{\mathrm{i}}{4}\partial_{\bar{z}} W\,. \tag{2.2.18}$$

It follows that the metric and gauge field on the spacetime respectively take the form

$$ds^2 = S^2\sin^2\theta(d\psi+\phi)^2 + \frac{1}{y^4 S^2\sin^2\theta}\left(dy^2 + 4\mathrm{e}^W dz d\bar{z}\right)\ , \tag{2.2.19}$$

$$A = (S\cos\theta + c_\varphi)\,(d\psi+\phi) + \frac{\mathrm{i}}{4}\left(\partial_z W\, dz - \partial_{\bar{z}} W\, d\bar{z}\right)\ . \tag{2.2.20}$$

In order to apply the formulae derived in section 2.1.2 to this class of supersymmetric solutions, we should identify the terms in (2.2.19), (2.2.20). Clearly, the square norm of the Killing vector field V and the metric on the base γ are

$$V = S^2\sin^2\theta\,, \qquad \gamma = \frac{1}{y^4}\left(dy^2 + 4\mathrm{e}^W dz d\bar{z}\right)\ . \tag{2.2.21}$$

Notice that the subset $M_0 \equiv \{\xi = 0\}$ of fixed points of the isometry is also $M_0 = \{V = 0\}$, which is precisely where the canonical frame degenerates, which in turn is where the spinor becomes chiral. Moreover, (2.2.6) fixes the electromagnetic potential

$$\nu = S - \frac{1}{y} + c_\nu\,. \tag{2.2.22}$$

Note that ν, just like the coordinate y, is *a priori* only defined locally. Recall that the gauge transformation (2.1.18) in particular shifts $\varphi \mapsto \varphi + c$, where c is a constant. Thus via an

appropriate gauge transformation we may take $c_\varphi = 0$, obtaining

$$\varphi = S\cos\theta\,, \qquad a = \frac{\mathrm{i}}{4}\left(\partial_z W\, dz - \partial_{\bar z} W\, d\bar z\right)\,, \tag{2.2.23}$$

which we refer to as the *supersymmetric gauge*.

Next, in order to determine nut potential and twist, we should find $d\eta$ from its definition and (2.2.9):

$$d\eta = V^{-2}\xi \lrcorner (d\xi^\flat \wedge \xi^\flat) = 2V^{-3/2} *_\gamma \left[2\cot\theta\, d\left(\frac{1}{y}\right) - S\, d\theta\right]\,. \tag{2.2.24}$$

In deriving this, we have compared the orientation given by supersymmetry $\mathrm{vol}_g = \mathrm{E}^{4123}$, and that chosen in the reduction to the base of the fibration $\mathrm{vol}_g = V^{-1}\eta \wedge \mathrm{vol}_\gamma$, to find $\mathrm{vol}_\gamma = -V^{3/2}\mathrm{E}^{124} = \frac{2\mathrm{i}\mathrm{e}^W}{y^6} dy\wedge dz\wedge d\bar z$. The expression for $d\eta$ then immediately gives $H = *_\gamma d\eta$ and σ via (2.1.25)

$$\sigma = 2S\varphi + c_\sigma = 2S^2\cos\theta + c_\sigma = 2SP + c_\sigma\,, \tag{2.2.25}$$

again involving a constant c_σ which we are going to fix in the next section by looking at the contribution to the on-shell action from the conformal boundary. As for φ, we observe that the expression for the nut potential only involves the spinor bilinears $\sigma = 2SP + c_\sigma$, so it is manifestly globally defined on M.

Finally, the differential equations arising from higher rank differential form spinor bilinears provide two additional constraints between the functions appearing in the metric and gauge field:

$$\partial^2_{z\bar z} W = -\mathrm{e}^W\left[\partial^2_{yy} W + \frac{1}{4}(\partial_y W)^2 + \frac{12\cos^2\theta}{y^4 S^2\sin^4\theta}\right]\,, \tag{2.2.26}$$

$$\begin{aligned}\partial^2_{z\bar z} f + \frac{\mathrm{e}^W}{y^2}\Big[f\left(f^2+2\right) - y\left(2\partial_y f + \frac{3}{2} f\partial_y W\right) \; + \\ + y^2\left(\partial^2_{yy} f + \frac{3}{2}\partial_y W\partial_y f + \frac{3}{2} f\partial^2_{yy} W + \frac{3}{4} f(\partial_y W)^2\right)\Big] \;=\; 0\,.\end{aligned} \tag{2.2.27}$$

In the latter equation, which is introduced for completeness but not needed for the purposes of this chapter, the function f is defined by

$$f \equiv -\frac{2\cos\theta}{yS\sin^2\theta}\,. \tag{2.2.28}$$

2.3 Contributions from the conformal boundary

In this subsection we begin our analysis of the on-shell action for supersymmetric solutions that are asymptotically locally Euclidean AdS. The main result of section 2.1 was the formula (2.1.29) for the bulk contribution to the on-shell action. As usual, to obtain the full on-shell action we should add to this the Gibbons–Hawking–York boundary term, together with the standard local boundary counterterms that holographically renormalize the action (see e.g. [9,72]). The latter are both terms at the conformal boundary (or more precisely near the conformal boundary, using a cut-off δ that is then removed by taking the $\delta \to 0$ limit). The bulk on-shell action (2.1.29) is naturally exact, and the main result of this subsection will be to show that by choosing the integration constant $c_\sigma = 0$ in the nut potential in (2.2.25), all the contributions at the conformal boundary in fact cancel. The only remaining contributions to the on-shell action then come from the fixed points of the supersymmetric Killing vector ξ, which we analyse in the next subsection. Of course, the on-shell action is independent of the choice of c_σ, so we may regard $c_\sigma = 0$ simply as a natural and convenient gauge choice.

Following [10,67], we take the conformal boundary to be at $\{y = 0\} = \partial M = M_3$, where the function $1/y$ then provides a natural radial coordinate near to this conformal boundary. We furthermore assume that the various terms appearing in the local structure of the solution described in the previous subsection have an analytic expansion in y in a neighbourhood of the conformal boundary at $\{y = 0\}$. Essentially the same analysis appears in [10], using a different set of variables, so here we will be brief. In a neighbourhood of the conformal boundary we thus write

$$\theta(y,z,\overline{z}) = \theta_{(0)}(z,\overline{z}) + y\,\theta_{(1)}(z,\overline{z}) + \frac{y^2}{2}\theta_{(2)}(z,\overline{z}) + \mathcal{O}(y^3)\,, \tag{2.3.1}$$

$$W(y,z,\overline{z}) = W_{(0)}(z,\overline{z}) + y\,W_{(1)}(z,\overline{z}) + \frac{y^2}{2}W_{(2)} + \mathcal{O}(y^3)\,, \tag{2.3.2}$$

$$S(y,z,\overline{z}) = \frac{1}{y}S_{(-1)}(z,\overline{z}) + S_{(0)}(z,\overline{z}) + y\,S_{(1)}(z,\overline{z}) + \frac{y^2}{2}S_{(2)}(z,\overline{z}) + \mathcal{O}(y^3)\,, \tag{2.3.3}$$

$$\phi(y,z,\overline{z}) = \phi_{(0)}(z,\overline{z}) + y\phi_{(1)}(z,\overline{z}) + \frac{y^2}{2}\phi_{(2)}(z,\overline{z}) + \mathcal{O}(y^3)\,, \tag{2.3.4}$$

and by imposing the constraints (2.2.17), (2.2.26), (2.2.24), we find

$$\theta = \frac{\pi}{2} + y\,\theta_{(1)} + \frac{y^2}{2}\theta_{(2)} + \mathcal{O}(y^3)\,, \tag{2.3.5}$$

$$W = W_{(0)} + y\,W_{(1)} + \frac{y^2}{2}\left(-\mathrm{e}^{-W_{(0)}}\partial^2_{z,\bar{z}}W_{(0)} - 12\theta^2_{(1)} - \frac{1}{4}W^2_{(1)}\right) + \mathcal{O}(y^3)\,, \tag{2.3.6}$$

$$\begin{aligned} S = \frac{1}{y} + \frac{1}{4}W_{(1)} + y\,\left(-\frac{1}{4}\mathrm{e}^{-W_{(0)}}\partial_{z,\bar{z}}W_{(0)} - 2\theta^2_{(1)}\right) + \frac{y^2}{2}\left[\frac{1}{8}e^{-W_{(0)}}\left(\partial^2_{z,\bar{z}}W_{(0)}W_{(1)}\right.\right. \\ \left.\left. - 2\partial^2_{z,\bar{z}}W_{(1)}\right) + \frac{1}{2}W_{(1)}\theta^2_{(1)} - \theta_{(1)}\theta_{(2)}\right] + \mathcal{O}(y^3)\,, \end{aligned} \tag{2.3.7}$$

$$\phi = \phi_{(0)} + y^2\,\mathrm{i}(\partial_{\bar{z}}\theta_{(1)}\,d\bar{z} - \partial_z\theta_{(1)}\,dz) + \mathcal{O}(y^3)\,, \tag{2.3.8}$$

where $\phi_{(0)}$ is constrained to satisfy

$$d\phi_{(0)} = 4\mathrm{i}\mathrm{e}^{W_{(0)}}\theta_{(1)}\,dz\wedge d\bar{z}\,. \tag{2.3.9}$$

To leading order, the metric of the four-dimensional spacetime then takes the form

$$ds^2 = [1+\mathcal{O}(y)]\,\frac{dy^2}{y^2} + \frac{1}{y^2}\left[(d\psi+\phi_{(0)})^2 + 4\mathrm{e}^{W_{(0)}}dzd\bar{z} + \mathcal{O}(y)\right]\,, \tag{2.3.10}$$

confirming that it is asymptotically locally Euclidean AdS, with defining function y. We may then choose a natural representative for the metric on the conformal boundary

$$ds^2_3 = (d\psi+\phi_{(0)})^2 + 4\mathrm{e}^{W_{(0)}}dzd\bar{z}\,. \tag{2.3.11}$$

Furthermore, the bulk Abelian gauge field A has leading order term $A_{(0)}$ given by

$$A_{(0)} \equiv A|_{y=0} = -\theta_{(1)}(d\psi + a_{(0)}) + \frac{\mathrm{i}}{4}\left(\partial_z W_{(0)}\,dz - \partial_{\bar{z}}W_{(0)}\,d\bar{z}\right)\,. \tag{2.3.12}$$

Proceeding to higher orders in the expansion of the supersymmetry equations leads to additional relations between the terms in the expansion of the functions, as described in detail in [10]. The conclusion is that all the higher order terms in the series solutions are determined in terms of the boundary data, characterized by $W_{(0)}, \theta_{(1)}$ (with $\phi_{(0)}$ constrained to satisfy (2.3.9)), and the free bulk functions $W_{(1)}, \theta_{(2)}$.

The geometric structure on $M_3 = \{y = 0\}$ induced from the bulk supersymmetry con-

ditions is the same as that for rigid supersymmetric backgrounds in three-dimensional new minimal supergravity [71]. In a little more detail, ∂_ψ restricted to the boundary coincides with the canonical vector field for the almost contact structure with global one-form $d\psi+\phi_{(0)}$, and the Abelian gauge field restricted to the boundary is identified with the non-dynamical gauge field that couples to the R-symmetry current of the field theory. Since we are assuming that the orbits of ξ all close, the boundary manifold $M_3 = \partial M$ is a Seifert three-manifold, being the total space of an orbifold circle bundle over an orbifold Riemann surface Σ.

Having determined the expansion of the fields near the conformal boundary, we may now evaluate the corresponding contribution to the bulk on-shell action (2.1.29), after integrating by parts. More precisely, we consider a cut-off spacetime M_δ, where the radial coordinate y only extends to $y = \delta > 0$, rather than to the conformal boundary located at $y = 0$. We denote the resulting boundary by $\partial M_\delta \equiv M \cap \{y = \delta\} \cong M_3$, with base $\partial B_\delta = \partial M_\delta / U(1)$. Applying Stokes' theorem to (2.1.29), we find that the conformal boundary contribution to the bulk on-shell action is

$$I^{\rm UV}_{\rm bulk} = -\frac{\beta}{16\pi G_4}\int_{\partial B_\delta} *_\gamma \left(d\log V - \sigma H + 4V^{-1}\varphi d\varphi\right) \tag{2.3.13}$$

The sign here is due to the direction of the normal, together with compatibility of the orientation choices associated to supersymmetry and the reduction along the circle fibre. As already pointed out, the volume element on (B,γ) is naturally $\mathrm{vol}_\gamma = 2\mathrm{i}\mathrm{e}^W\, dy\wedge dz\wedge d\bar z/y^6$. The outward-pointing unit normal to ∂B_δ is $n = -dy/y^2$, so in order to appeal to Stokes' theorem with a positive sign we should use the orientation on ∂B_δ given by $n^\sharp \lrcorner \mathrm{vol}_\gamma = -2\mathrm{i}\mathrm{e}^W\, dz\wedge d\bar z/y^4$, but this is opposite to the natural orientation on that surface, hence the sign. We may then lift the integral in (2.3.13) back up to an integral on ∂M_δ:

$$I^{\rm UV}_{\rm bulk} = -\frac{1}{16\pi G_4}\int_{\partial M_\delta} \eta\wedge *_\gamma \left(d\log V - \sigma H + 4V^{-1}\varphi d\varphi\right)\,. \tag{2.3.14}$$

Next we add to the bulk action the usual Gibbons–Hawking–York term, computed using the induced metric h on the cut-off hypersurface ∂M_δ

$$I_{\rm GHY} = -\frac{1}{8\pi G_4}\int_{\partial M_\delta} K\,\mathrm{vol}_h\,. \tag{2.3.15}$$

The same induced metric is used to construct the counterterm action [72] cancelling the divergences present in the sum $I^{\rm UV}_{\rm bulk}+I_{\rm GHY}$:

$$I_{\rm ct}=\frac{1}{8\pi G_4}\int_{\partial M_\delta}\left(2+\frac{1}{2}R_h\right)\mathrm{vol}_h\,. \tag{2.3.16}$$

The conformal boundary contribution $I^{\rm UV}$ to the full on-shell action is then given by the sum $I^{\rm UV}_{\rm bulk}+I_{\rm GHY}+I_{\rm ct}$, in the limit of vanishing cutoff, $\delta\to 0$. Using the expansion of the fields determined earlier, it is a straightforward computation to see that this sum reduces to

$$I^{\rm UV}=\lim_{\delta\to 0}\left(I^{\rm UV}_{\rm bulk}+I_{\rm GHY}+I_{\rm ct}\right)=c_\sigma\frac{1}{8\pi G_4}\int_{\partial M}\theta_{(1)}\,\eta_{(0)}\wedge\mathrm{vol}_2\,, \tag{2.3.17}$$

where $\mathrm{vol}_2=2\mathrm{i}\mathrm{e}^{W_{(0)}}\,dz\wedge d\bar z$. Remarkably, (2.3.17) is proportional to the constant c_σ that appears in the nut potential in (2.2.25). On the other hand, the *total* on-shell action is independent of the choice of this constant. However, clearly a natural choice is to now set

$$c_\sigma=0\qquad\Rightarrow\qquad I^{\rm UV}=0\,. \tag{2.3.18}$$

With this gauge choice for the nut potential, the only contribution to the on-shell action thus comes from the fixed point set of the supersymmetric Killing vector ξ, and it is this contribution to which we now turn.

2.4 Contributions from the fixed points of the isometry

In this subsection we would like to evaluate the contribution to the bulk on-shell action (2.1.29) from the fixed points of the isometry generated by the supersymmetric Killing vector ξ. Recall this means that we remove a small tubular neighbourhood M_ε of radius $\varepsilon>0$ around (each connected component of) the fixed point set $M_0=\{\xi=0\}$. The image in the base space is then $B_\varepsilon=(M\setminus M_\varepsilon)/U(1)\subset B$, which is in general an orbifold with boundary. We then wish to apply Stokes' theorem to (2.1.29), and evaluate the boundary contributions around the fixed point set, in the limit $\varepsilon\to 0$.

The square norm of the Killing vector ξ is $V=S^2\sin^2\theta$. This vanishes where the spinor is chiral and non-zero ($\theta=0,\pi$), and also potentially where the spinor is zero, where the

latter is equivalent to $S = 0$. It follows that the fixed points of the supersymmetric isometry precisely correspond to the subspaces where the orthonormal frame associated to the identity structure degenerates. The connected components of the fixed point set of a $U(1)$ isometry must have even codimension, so in four dimensions there may only be zero-dimensional nuts, and two-dimensional bolts [73]. At this point we notice that in fact S can never be zero anywhere, unless the isometry acts trivially on M: S is only zero where ϵ is zero, but then (2.2.9) implies that $d\xi^\flat$ is also zero at such a point, which in turn means that the vector field is identically zero on M (assuming M is connected).

It follows that on a given connected component of the fixed point set the spinor ϵ has either positive or negative chirality, corresponding to $\theta = 0$, $\theta = \pi$, respectively, and we may thus correspondingly label isolated fixed points as $\text{nut}_\pm$, or connected two-dimensional fixed point sets as $\Sigma_\pm$, if the spinor has positive or negative chirality there, respectively. At such points $P = \pm S$, and in the $\varepsilon \to 0$ limit the equation for $d\xi^\flat$ (2.2.9) greatly simplifies:

$$d\xi^\flat|_\pm \equiv -2 \lim_{\varepsilon\to 0} S \left(\mathrm{E}^{34} \pm \mathrm{E}^{12} \pm F + *_g F\right) . \tag{2.4.1}$$

By taking the (anti-)self-dual parts at subspaces with spinors with definite chirality we find

$$\left(d\xi^\flat|_\pm\right)^\pm = \mp 4 S F^\pm , \qquad \left(d\xi^\flat|_\pm\right)^\mp = \mp 2 \lim_{\varepsilon\to 0} S \left(\mathrm{E}^{12} \pm \mathrm{E}^{34}\right) . \tag{2.4.2}$$

Here the $\pm$ superscripts denote self-dual and anti-self-dual parts of the two-forms, respectively. The second equation above is particularly useful, as its square norm gives $S^2|_\pm$:

$$\text{At a fixed point of chirality } \pm: \qquad S^2\big|_\pm = \frac{1}{16} \left\langle (d\xi^\flat|_\pm)^\mp, (d\xi^\flat|_\pm)^\mp \right\rangle_g . \tag{2.4.3}$$

2.4.1 Contribution from a nut

A nut is an isolated fixed point of ξ, and we may introduce a radial geodesic distance function ρ from the nut, so that the nut is at $\rho = 0$. To leading order near the nut, the metric is simply the flat space metric

$$ds^2 = d\rho_1^2 + \rho_1^2 d\varphi_1^2 + d\rho_2^2 + \rho_2^2 d\varphi_2^2 , \tag{2.4.4}$$

where $\rho^2 = \rho_1^2 + \rho_2^2$. For $\rho = \varepsilon > 0$ small, the induced metric on $\{\rho = \varepsilon\}$ is then approximately the round metric on the three-sphere of radius ε, S^3_ε. Moreover, the supersymmetric Killing vector near the nut is then

$$\xi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} \,. \tag{2.4.5}$$

Here we have identified the tangent space to the nut as $\mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C}$, where φ_1, φ_2 are standard polar coordinates on each copy of the complex plane $\mathbb{C}$ with periodicity 2π, and b_1, b_2 are the weights of the rotations in the two two-planes. Notice that the orientations here are not unique: reversing the orientations of both two-planes is equivalent to simultaneous complex conjugation of both copies of $\mathbb{C}$, and gives the same orientation on $\mathbb{R}^4$, and likewise swapping the two two-planes also leaves the overall orientation invariant. These act as $(b_1, b_2) \mapsto (-b_1, -b_2)$ and $(b_1, b_2) \mapsto (b_2, b_1)$ on the weights, respectively. Without loss of generality, we may then assume that $b_1 > 0$, and b_2 may then either be positive or negative. Requiring that the orbits of ξ close means that we can write $b_1/b_2 = p/q$, with p, q coprime integers of the same signs as b_1, b_2, respectively, so the period of a generic orbit is $\beta = 2\pi p/b_1 = 2\pi q/b_2 > 0$. The connected component of the boundary of the tubular neighbourhood ∂B_ε in the base space B is then the complex weighted projective space $\Sigma_2 = \mathbb{WCP}^1_{[p,|q|]}$.

Using the form of the metric (2.4.4) shows that the square norm $V = O(\rho^2)$ as $\rho \to 0$, and it is straightforward to see that the first term in (2.1.29) does not contribute to the action in the case of a nut (see also [73]). On the other hand, the last term in the action, namely the integral of $*_\gamma 4\beta V^{-1} \varphi \, d\varphi$ over the boundary of B, can also be written using (2.1.19) as the integral of $-4\varphi\eta \wedge *F$ over the corresponding boundary in M, which is a three-sphere of radius ε. Now for any smooth two-form, such as $*F$, the components tangent to the three-sphere must vanish at least as fast as $O(\rho^2)$ as $\rho = \varepsilon \to 0$. This term hence also gives zero contribution as $\varepsilon \to 0$. It follows that the entire contribution of a zero-dimensional fixed point to the action arises from the term involving $d\eta = *_\gamma H$. On the other hand, as in [73] (or [10]) we may easily compute

$$\int_{\Sigma_2} d\eta = \beta \int_{\mathbb{WCP}^1_{[p,|q|]}} c_1(\mathcal{L}) = -\frac{\beta}{pq} \,, \tag{2.4.6}$$

where $\mathcal{L}$ is the orbifold line bundle over B associated to the $U(1)$ fibration (2.1.6) (for further discussion on weighed projective spaces see the following chapters, and specially appendix

A.1). The contribution to the on-shell action of isolated nuts is hence

$$I_{\text{nuts}} = -\sum_{\text{nuts}} \frac{\beta^2}{16\pi G_4}\sigma|_{\text{nut}}\frac{1}{pq} = -\frac{\pi}{2G_4}\sum_{\text{nuts}}\sigma|_{\text{nut}}\frac{1}{2b_1b_2}\,, \tag{2.4.7}$$

where the sign is due to the normal being $d\rho$. This shows that in order to evaluate the on-shell action contribution from a nut, we now only need an expression for the nut potential. Here supersymmetry helps, because for supersymmetric solutions the nut potential is fixed by (2.2.25), and the choice $c_\sigma = 0$ taken in the previous subsection (2.3.18) in order to have vanishing contribution to the action from conformal infinity gives

$$\sigma = 2S\varphi\,. \tag{2.4.8}$$

Since also $S\big|_\pm = \pm P = \pm\varphi$, using (2.4.3) we may hence write

$$I_{\text{nuts}} = -\frac{\pi}{2G_4}\sum_{\text{nuts}_\pm} \pm\big\langle (d\xi^\flat|_\pm)^\mp, (d\xi^\flat|_\pm)^\mp\big\rangle_g \frac{1}{16b_1b_2}\,. \tag{2.4.9}$$

Finally, in the orthonormal frame compatible with (2.4.4), (2.4.5) we have

$$d\xi^\flat\big|_{\text{nut}} = 2\begin{pmatrix} 0 & b_1 & 0 & 0\\ -b_1 & 0 & 0 & 0\\ 0 & 0 & 0 & b_2\\ 0 & 0 & -b_2 & 0\end{pmatrix}\,, \tag{2.4.10}$$

so

$$\big\langle (d\xi^\flat)^\mp, (d\xi^\flat)^\mp\big\rangle_g\big|_{\text{nut}} = 4(b_1 \mp b_2)^2\,, \tag{2.4.11}$$

and we reach the final expression

$$I_{\text{nuts}} = \frac{\pi}{2G_4}\sum_{\text{nuts}_\pm} \mp\frac{(b_1\mp b_2)^2}{4b_1b_2} = \frac{\pi}{2G_4}\sum_{\text{nuts}_\mp} \pm\frac{(b_1\pm b_2)^2}{4b_1b_2}\,. \tag{2.4.12}$$

Of course here the values of the weights (b_1, b_2) at each nut will in general be different, although we have suppressed this in the notation. Notice that for each nut, the expression in the sum is indeed invariant under both $(b_1, b_2) \mapsto (-b_1, -b_2)$ and $(b_1, b_2) \mapsto (b_2, b_1)$, as it

must be since these are simply different choices of bases for the same vector field action near the nut. Moreover, it is also invariant under scaling $(b_1, b_2) \mapsto \lambda(b_1, b_2)$, which for $\lambda > 0$ is equivalent to rescaling the spinor by a non-zero complex number. Since the overall spinor normalization is arbitrary, it again follows that the formula for the action had to be invariant under such a rescaling.

2.4.2 Contribution from a bolt

A bolt is a two-dimensional surface $\Sigma \subset M$. This means that the image of the boundary of a small ξ-invariant tubular neighbourhood around Σ in the base B is also a copy of Σ. We denote this as $T_\varepsilon \cong \Sigma$, which is the connected component of ∂B_ε around the bolt.

We next look at equation (2.4.1). Notice that we may always decompose a differential form Ψ uniquely as $\Psi = \eta \wedge \Psi_\xi + \Psi_T$, where we identify $\Psi_\xi \equiv \xi \lrcorner \Psi$. It then follows that the transverse part of a form Ψ with $\mathcal{L}_\xi \Psi = 0$, namely $\Psi_T = \Psi - \eta \wedge \xi \lrcorner \Psi$, may be interpreted as a two-form on our base B. Moreover $\Psi = 0$ if and only if both $\Psi_\xi = 0$ and $\Psi_T = 0$. For a bolt, the transverse part of $d\xi^\flat$ is zero at the bolt, since this is the rotation matrix for the linear action of ξ, which only rotates the normal directions. On the other hand, manifestly $(\mathrm{E}^{34})_T = 0$, $(\mathrm{E}^{12})_T = \mathrm{E}^{12}$, while from (2.1.17) and (2.1.19) we have

$$F_T = \varphi\, d\eta + f\,, \qquad (*_g F)_T = -V^{-1} *_\gamma d\varphi\,. \tag{2.4.13}$$

Integrating the transverse part of (2.4.1) over T_ε hence gives the equation

$$\lim_{\varepsilon\to 0} \int_{T_\varepsilon} V^{-1} *_\gamma d\varphi = \pm \lim_{\varepsilon\to 0} \int_{T_\varepsilon} (\mathrm{E}^{12} + F_T)\,. \tag{2.4.14}$$

Moreover, from the spinor bilinear equations in sections 2.2.1 and 2.2.2 we deduce the equation

$$*_\gamma d \log V = -2S\mathrm{E}^{12} - 2V^{-1}\varphi *_\gamma d\varphi + 2S(\varphi\, d\eta + f)\,. \tag{2.4.15}$$

Using these preliminary results, we may now go back to the evaluation of the on-shell

action (2.1.29) for a bolt. In particular, using (2.4.14) and (2.4.15) we compute

$$\begin{aligned}\frac{\beta}{16\pi G_4}\lim_{\varepsilon\to 0}\int_{B_\varepsilon} d*_\gamma\left(d\log V+4V^{-1}\varphi\, d\varphi\right)\Big|_{\text{bolt}} &= -\frac{\beta}{16\pi G_4}\lim_{\varepsilon\to 0}\int_{T_\varepsilon} *_\gamma(d\log V+4V^{-1}\varphi\, d\varphi)\\ &= -\frac{\beta}{8\pi G_4}\lim_{\varepsilon\to 0}\int_{T_\varepsilon}\left(SF_T+S\left(\varphi\, d\eta+f\right)\right)\\ &= -\frac{\beta}{4\pi G_4}\lim_{\varepsilon\to 0}\int_{T_\varepsilon} SF_T\\ &= -\frac{\beta}{2G_4}S|_{\Sigma_\pm}\int_{\Sigma_\pm} c_1(F)\,. \end{aligned} \tag{2.4.16}$$

Here we have used the first equation in (2.4.13) in the third line, while in the last line $c_1(F)$ is the first Chern class of the Abelian gauge bundle, and we have used that S is constant along the bolt. To see the latter, notice from (2.1.17) that

$$d\varphi=-F_\xi\equiv-\xi\lrcorner F\,. \tag{2.4.17}$$

We may restrict this equation to the bolt, and then dot with a tangent vector $\mathtt{t}$ to the bolt. Since by assumption F is smooth everywhere, notice that $F_\xi(\mathtt{t})$ must then tend to zero at the bolt (where η is singular). But then (2.4.17) implies that φ is constant over a bolt, and since $S\big|_\pm=\pm\varphi$ at a fixed point of positive/negative chirality, we deduce that also S is constant over a bolt. Notice that an expansion in geodesic normal coordinates would give, consistently with [73], that the $*_\gamma\, d\log V$ term in (2.4.16) results in a contribution proportional to the area of the bolt. However, for supersymmetric solutions this area term is then cancelled by the $4V^{-1}\varphi\, d\varphi$ term, leaving the topological contribution on the last line of (2.4.16).

Finally, we turn to the contribution of the middle term in (2.1.29). Again, the calculation of this is essentially the same as that in [73], and is proportional to the self-intersection number of the bolt:

$$\frac{\beta}{16\pi G_4}\lim_{\varepsilon\to 0}\int_{B_\varepsilon} d*_\gamma\left(-\sigma H\right)\Big|_{\text{bolt}}=\frac{\beta^2}{16\pi G_4}\sigma|_{\Sigma_\pm}\int_{\Sigma_\pm} c_1(N\Sigma_\pm)\,. \tag{2.4.18}$$

Since S is constant on the bolt, we may evaluate it at any point using (2.4.3). At a bolt, $d\xi^\flat$ is a skew-symmetric operator with rank 2, so there exists an orthonormal frame in which it

has the form

$$d\xi^\flat\big|_{\text{bolt}} = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa \\ 0 & 0 & -\kappa & 0 \end{pmatrix}, \tag{2.4.19}$$

and the "surface gravity" κ is related to the period β of a generic orbit of ξ as $\kappa = 2\pi/\beta$. Thus, applying (2.4.3) gives

$$S|_{\Sigma_\pm} = \frac{\pi}{\beta}\,. \tag{2.4.20}$$

Notice that S is proportional to the surface gravity, consistently with the fact that it is constant on the bolt. Finally, adding the two contributions in (2.4.16), (2.4.18), and summing over all bolts, we obtain

$$I_{\text{bolts}} = \frac{\pi}{2G_4}\sum_{\text{bolts}\ \Sigma_\pm}\int_{\Sigma_\pm}\left(-c_1(F) \pm \tfrac{1}{4}c_1(N\Sigma_\pm)\right)\,. \tag{2.4.21}$$

The expression (2.4.21) depends on the first Chern class $c_1(F)$ of the Abelian gauge bundle, but this may in turn be related to certain topological invariants at a bolt Σ by further considering the topology of $\Sigma \subset M$. Since a bolt is a two-dimensional orientable Riemannian manifold, it can be given a complex structure and viewed as a Riemann surface $\Sigma = \Sigma_g$ of genus g. Notice this involves a choice of orientation. Complex line bundles over a compact Riemann surface are in one-to-one correspondence with $H^2(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}$, and the (group) isomorphism is given by the first Chern number of the bundle. Thus, we may use unambiguously the notation $\mathcal{O}(n)$ for the line bundle over Σ_g with degree n. In particular, having fixed a choice of orientation for the bolt, in order to agree with the given orientation on M this also fixes an orientation for the normal bundle $N\Sigma$ of Σ in M. We may then write $N\Sigma = \mathcal{O}(Y)$, where the integer $Y = \int_\Sigma c_1(N\Sigma) \in \mathbb{Z}$ is the self-intersection number of Σ in M. Similarly, the tangent bundle of the bolt is $T\Sigma = \mathcal{O}(2-2g)$, where $\Sigma = \Sigma_g$ has genus g.

From the above discussion it follows that topologically $TM|_\Sigma \cong \mathcal{O}(2-2g) \oplus \mathcal{O}(Y)$ may be written as a direct sum of two complex line bundles, and from this we may then compute

the chiral spin bundles of M restricted to the bolt, which we denote as $\mathcal{S}_\pm$. One finds

$$\begin{aligned}\mathcal{S}_+ &= \mathcal{O}\left(-(1-g)+\frac{Y}{2}\right)\oplus\mathcal{O}\left((1-g)-\frac{Y}{2}\right)\,,\\ \mathcal{S}_- &= \mathcal{O}\left(-(1-g)-\frac{Y}{2}\right)\oplus\mathcal{O}\left((1-g)+\frac{Y}{2}\right)\,.\end{aligned} \tag{2.4.22}$$

Of course, as spin bundles here we should also keep track of the inequivalent spin structures, classified by $H^1(M;\mathbb{Z}_2)$. This amounts to the choice of a $\mathbb{Z}_2$ principal bundle, or equivalently a flat complex line bundle $\mathscr{S}$ with $\mathbb{Z}_2$-valued holonomy, that arise since both $\pm 1 \in \mathrm{Spin}(4)$ map to the identity in $SO(4)$. More precisely, we should then tensor $\mathcal{S}_\pm$ in (2.4.22) with $\mathscr{S}$, although as we shall see below these different choices of spin structure play no role in the following argument. Independently of this, the bundles in (2.4.22) in general only exist as genuine vector bundles when M is a spin manifold. However, as briefly mentioned in section 2.2.1, generally M is spin^c. This means that $2A$ is a connection on a genuine complex line bundle $\mathcal{L}$ over M, with $c_1(\mathcal{L}) \bmod 2 = w_2(M) \in H^2(M;\mathbb{Z}_2)$ reducing mod 2 to the second Stiefel–Whitney class of M. Indeed, we see from (2.1.4) that the connection acting on the spinor is twisted by A. This ensures that a corresponding twisting by $\mathcal{L}^{1/2}$ leads to well-defined spin^c spinor bundles:

$$\begin{aligned}\mathcal{S}_+\otimes\mathcal{L}^{1/2} &= \mathcal{O}\left(-(1-g)+\frac{m+Y}{2}\right)\oplus\mathcal{O}\left((1-g)+\frac{m-Y}{2}\right)\,,\\ \mathcal{S}_-\otimes\mathcal{L}^{1/2} &= \mathcal{O}\left(-(1-g)+\frac{m-Y}{2}\right)\oplus\mathcal{O}\left((1-g)+\frac{m+Y}{2}\right)\,.\end{aligned} \tag{2.4.23}$$

where we have denoted $\mathcal{L}|_\Sigma = \mathcal{O}(m)$. These correspond to the $2+2=4$ components of a spin^c Dirac spinor.

As we have shown, the spinor is necessarily chiral on (a connected component of) the fixed point set, but with non-zero constant norm. This means that there is a nowhere-zero section of one of the two pairs of line bundles in (2.4.23), which means that one of the complex line bundles is just the trivial bundle at the bolt. In order to find out which one it is, we can refer to the form of U in terms of the supersymmetric vierbein in (2.2.5), which near a bolt$_\pm$, reduces to

$$U|_{\Sigma_\pm} = -\lim_{\varepsilon\to 0} S(\mathrm{E}^{12}\pm\mathrm{E}^{34})\,, \tag{2.4.24}$$

which we can verify to be consistent with the following projection conditions[3] on the spinor

$$\mathrm{i}\Gamma_{12}\epsilon = \epsilon\,, \quad \mathrm{i}\Gamma_{34}\epsilon = \mp\epsilon\,. \tag{2.4.25}$$

With our choice of conventions, the only complex line bundles in (2.4.23) that could admit a non-vanishing section satisfying the projection conditions near the bolt are the second summands in each line, so in the limit $\varepsilon \to 0$, near a bolt$_\pm$, the non-zero spinor component is a section of

$$\mathcal{O}\left((1-g) + \frac{m \mp Y}{2}\right) \overset{!}{=} \mathcal{O}(0)\,. \tag{2.4.26}$$

This in turn then constrains the gauge bundle near a bolt in terms of the local topology

$$\mathcal{L}|_{\Sigma_\pm} = \mathcal{O}(m) = \mathcal{O}(\pm Y) \oplus \mathcal{O}(-2(1-g)) = N\Sigma_\pm^{\pm 1} \oplus T\Sigma_\pm^{-1}\,, \tag{2.4.27}$$

or, equivalently,

$$\int_{\Sigma_\pm} c_1(F) = \int_{\Sigma_\pm} \frac{\pm c_1(N\Sigma_\pm) - c_1(T\Sigma_\pm)}{2}\,. \tag{2.4.28}$$

With this result, we can go back to substitute the first Chern class of the gauge bundle through the bolt in (2.4.21), whence

$$I_{\text{bolts}_\pm} = \frac{\pi}{2G_4} \sum_{\text{bolts}\ \Sigma_\pm} \int_{\Sigma_\pm} \left(\tfrac{1}{2}c_1(T\Sigma_\pm) \mp \tfrac{1}{4}c_1(N\Sigma_\pm)\right)\,. \tag{2.4.29}$$

Summing (2.4.12) and (2.4.29) leads to the expression for the on-shell action already quoted in the Introduction, equation (2.0.1).

2.5 Examples

As already remarked in the Introduction, there are a number of explicitly known supersymmetric solutions to $\mathcal{N} = 2$ gauged supergravity. In this section, we illustrate some features of the general formulae given above by considering concrete examples.

[3] Recall here that the supersymmetric vierbein is not that used to define the Γ matrices, as remarked below (2.2.5).

2.5.1 AdS_4

The simplest example we consider is AdS_4, realised as a hyperbolic ball foliated by three-spheres

$$ds^2 = \frac{dr^2}{r^2+1} + r^2\left(d\vartheta^2 + \cos^2\vartheta\, d\varphi_1^2 + \sin^2\vartheta\, d\varphi_2^2\right) , \tag{2.5.1}$$

with a trivial Abelian gauge field, $A = 0$. The radial coordinate is $r \in [0, \infty)$, while the S^3 is viewed as a torus fibration over the interval, with $\vartheta \in [0, \frac{\pi}{2}]$ parametrizing the interval, and $\varphi_1, \varphi_2 \in [0, 2\pi)$ parametrizing the torus.[4] This is a maximally supersymmetric solution to the equations of motion of our theory and can be given a number of supersymmetric structures. Indeed solving the (generalized) Killing spinor equation on this background leads to a spinor that depends on four complex numbers. By appropriately choosing them, we may construct supersymmetric Killing vectors as in (2.2.5) that have the form

$$\xi^{\pm} = \partial_{\varphi_1} \pm \partial_{\varphi_2} . \tag{2.5.3}$$

Such vectors generate an isometric torus action on AdS_4 and have a nut at $r = 0$ (in the nomenclature of [73], respectively a nut or an anti-nut). The corresponding spinors, in the vierbein where $e^1 = r\cos\vartheta\, d\varphi_1$, $e^2 = r\sin\vartheta\, d\varphi_2$, $e^3 = r\, d\vartheta$, $e^4 = dr/\sqrt{r^2+1}$ are

$$\epsilon^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sinh\frac{\operatorname{arcsinh} r}{2}\, e^{i(-\vartheta+\varphi_1+\varphi_2)/2} \\ \sinh\frac{\operatorname{arcsinh} r}{2}\, e^{i(\vartheta+\varphi_1+\varphi_2)/2} \\ -i\cosh\frac{\operatorname{arcsinh} r}{2}\, e^{i(-\vartheta+\varphi_1+\varphi_2)/2} \\ i\cosh\frac{\operatorname{arcsinh} r}{2}\, e^{i(\vartheta+\varphi_1+\varphi_2)/2} \end{pmatrix} , \quad \epsilon^- = \frac{1}{\sqrt{2}} \begin{pmatrix} \cosh\frac{\operatorname{arcsinh} r}{2}\, e^{i(\vartheta-\varphi_1+\varphi_2)/2} \\ -\cosh\frac{\operatorname{arcsinh} r}{2}\, e^{i(-\vartheta-\varphi_1+\varphi_2)/2} \\ i\sinh\frac{\operatorname{arcsinh} r}{2}\, e^{i(\vartheta-\varphi_1+\varphi_2)/2} \\ -i\sinh\frac{\operatorname{arcsinh} r}{2}\, e^{i(-\vartheta-\varphi_1+\varphi_2)/2} \end{pmatrix} . \tag{2.5.4}$$

Here we have chosen the following form of the Γ matrices generating $\mathrm{Cliff}(4,0)$

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} , \qquad \Gamma_4 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ -i\mathbb{1}_2 & 0 \end{pmatrix} \qquad \Gamma_* \equiv \Gamma_{1234} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} , \tag{2.5.5}$$

[4] The change of variable $r^2 = 4y^2/(1-y^2)^2$ takes (2.5.1) to the standard metric on the Poincaré ball

$$ds^2 = \frac{4}{(1-y^2)^2}\left(dy^2 + y^2 d\Omega_3^2\right) . \tag{2.5.2}$$

where σ_i are the Pauli matrices, and the charge conjugation matrix is

$$\mathcal{C} = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix} . \tag{2.5.6}$$

It is immediately clear from the above form of the spinors that near the nut the spinors $\epsilon^{\pm}$ have negative/positive chirality.[5] Therefore, corresponding to the relative signs of the two circle actions in the torus we have either a nut$_-$ or a nut$_+$ (the nut is a nut$_-$ and the anti-nut is a nut$_+$). It is then immediate to see that applying the formula (2.0.1) leads in both cases to the action of AdS_4

$$I_{\mathrm{AdS}_4} = \frac{\pi}{2G_4} . \tag{2.5.7}$$

However, AdS_4 admits a number of different supersymmetric structures with corresponding Killing vectors constructed out of the generalized Killing spinors. For instance, a different supersymmetric structure was found in [63] and reviewed in [67]: this corresponds to a non-trivial Abelian instanton gauge field and a Killing spinor leading to the generic circle action

$$\xi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} , \tag{2.5.8}$$

which subsumes the above case, corresponding to $b_1/b_2 = \pm 1$. As shown in [67], the resulting solution is in fact regular only if $b_1/b_2 > 0$ or if $b_1/b_2 = -1$, and the chirality analysis can be shown to carry through: if $b_1/b_2 > 0$, we have a nut$_-$. Thus, application of (2.0.1) leads to the same result as [67].

2.5.2 Self-dual solutions

As we have just reviewed, starting with the standard hyperbolic metric (2.5.1) on AdS_4, one can turn on an instanton gauge field that modifies the standard Killing spinors on AdS_4, and hence also the supersymmetric Killing vector. In fact requiring the gauge field to be anti-self-dual is natural in four dimensions, and leads to interesting classes of solutions related to integrable systems, described in general in [75]. Under the assumption that the topology of the spacetime is that of a four-ball, with a nut at the origin (as for the explicit solutions

[5] One should bear in mind that a rotation of the S^3 frame is needed to make it regular at the origin, so that the spinors are then manifestly regular at the origin in such a frame.

in section 2.5.1 above), the value of the on-shell action has been computed in terms of the weights of a torus action at the nut in [67], and further analysis has been performed in [10], so here we will be succinct.

Requiring the gauge field to be anti-self-dual implies that

$$d(\varphi - \nu) = 0 \quad \Rightarrow \quad \varphi = \nu + k\,, \tag{2.5.9}$$

where $k \in \mathbb{R}$ is a constant, which we choose to be zero (assuming that spacetime is path-connected). For supersymmetric solutions, for which the electromagnetic potential ν and φ are fixed by (2.2.22) and (2.2.23), this also means that

$$\cos\theta = 1 - \frac{1}{Sy}\,. \tag{2.5.10}$$

Therefore, the square norm of the Killing vector is $V = (2Sy-1)/y^2$. In order to more easily compare with [67], we introduce the function

$$\mathcal{V} \equiv \frac{1}{2Sy-1}\,. \tag{2.5.11}$$

Using this notation, it is immediate to show that the metric has the form

$$ds^2 = \frac{1}{y^2}\left[\mathcal{V}^{-1}\eta^2 + \mathcal{V}(dy^2 + 4\mathrm{e}^W\, dzd\bar{z})\right]\,, \tag{2.5.12}$$

whereas (2.2.17) and (2.2.24) reduce to

$$\begin{aligned} \mathcal{V} &= 1 - \frac{1}{2}y\,\partial_y W\,, \\ d\phi &= -\mathrm{i}dy \wedge (\partial_z\mathcal{V}\,dz - \partial_{\bar{z}}\mathcal{V}\,d\bar{z}) - 2\mathrm{i}\,\partial_y\left(\mathcal{V}\mathrm{e}^W\right)\, dz\wedge d\bar{z}\,. \end{aligned} \tag{2.5.13}$$

and (2.2.26) becomes the $SU(\infty)$ Toda equation

$$\partial^2_{z\bar{z}}W + \partial^2_y \mathrm{e}^W = 0\,. \tag{2.5.14}$$

This matches precisely the structure described in [67, 74].[6] Thus, the entire solution is de-

[6]There is one caveat: the supersymmetric Killing vector field there has the opposite sign to ours, so $\eta^{\text{here}} = -\eta^{\text{there}}$. This is consistent with both works considering anti-self-dual gauge fields.

scribed by the solution to a single partial differential equation. More geometrically, requiring the gauge field to be anti-self-dual forces the $U(1)$ stress-energy tensor in (2.1.2) to vanish (one uses the Schouten identity), so the metric is Einstein, and the Weyl tensor of (2.5.12) is anti-self-dual as well. For reference, four-dimensional self-dual Einstein manifolds are considered the four-dimensional analogues of higher-dimensional quaternionic Kähler manifolds, and hence are sometimes referred to as such.

In terms of the metric function $\mathcal{V}$, we may rewrite

$$\cos\theta = \frac{1-\mathcal{V}}{1+\mathcal{V}}\,, \tag{2.5.15}$$

which we use to determine whether at the fixed point of the isometry the spinor has positive or negative chirality. As in [10,67], we restrict to supersymmetric solutions where the spacetime has the topology of $\mathbb{R}^4$ with a nut at the origin. As proved in the original papers, given a generic toric Killing vector field

$$\xi = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}\,, \tag{2.5.16}$$

the metric is regular everywhere outside the nut provided $b_1/b_2 > 0$ (in which case the nut is at a finite radial coordinate $y = y_{\rm NUT} = 1/(b_1+b_2)$), or $b_1 = -b_2$ (in which case the radial coordinate diverges at the location of the nut). In the first case, if ρ^2 denotes the geodesic distance from the nut, near the nut we have $\mathcal{V}\sim\rho^{-2}$ (as required in order to have a smooth metric), so equation (2.5.15) tells us that the fixed point is a nut$_-$. The value of the on-shell action given by our formula (2.0.1) matches precisely that computed in [67, eq. (1.2)]

$$I = \frac{\pi}{2G_4}\frac{(b_1+b_2)^2}{4b_1b_2}\,. \tag{2.5.17}$$

If, instead, $b_1 = -b_2$, in order to have a smooth metric near the nut we need $\mathcal{V}\sim\rho^2$, so (2.5.15) gives that the spinor at the fixed point has positive chirality. The on-shell action in this case is given by

$$I = \frac{\pi}{2G_4}\,, \tag{2.5.18}$$

which again matches the result of [67].

2.5.3 Non-self-dual 1/4 BPS

Non-self-dual solutions that are 1/4 BPS have been constructed in [65] and generalized in [69]. The spacetime has the topology of a complex line bundle over a Riemann surface of genus g, so that we may write $M = \mathcal{O}(-p) \to \Sigma_g$, where $p \in \mathbb{Z}_{>0}$. The metric and gauge field are given by

$$\begin{aligned} ds^2 &= \frac{r^2 - s^2}{\Omega(r)} dr^2 + (r^2 - s^2)(\sigma_1^2 + \sigma_2^2) + 4s^2 \frac{\Omega(r)}{r^2 - s^2} \sigma_3^2 \\ A &= -\frac{(4s^2 - \kappa)(r^2 + s^2) + 4Qrs}{2(r^2 - s^2)} \sigma_3 \,, \end{aligned} \tag{2.5.19}$$

where $\kappa = +1, 0, -1$ denotes the curvature of Σ_g and the one-forms σ_i are

- for $\kappa = +1$, $g = 0$ and $\Sigma_0 = S^2$

$$\sigma_1 = \cos\tau \, d\vartheta + \sin\tau \sin\vartheta \, d\phi \,, \quad \sigma_2 = -\sin\tau \, d\vartheta + \cos\tau \sin\vartheta \, d\phi \,, \quad \sigma_3 = d\tau + \cos\vartheta \, d\phi \,, \tag{2.5.20}$$

$$\sigma_1^2 + \sigma_2^2 = d\vartheta^2 + \sin^2\vartheta \, d\phi^2 \,, \tag{2.5.21}$$

- for $\kappa = 0$, $g = 1$ and $\Sigma_1 = T^2$

$$\sigma_1 = \sin\tau \, d\vartheta + \cos\tau \, d\phi \,, \quad \sigma_2 = -\cos\tau \, d\vartheta + \sin\tau \, d\phi \,, \quad \sigma_3 = d\tau - \vartheta \, d\phi \,, \tag{2.5.22}$$

$$\sigma_1^2 + \sigma_2^2 = d\vartheta^2 + d\phi^2 \,, \tag{2.5.23}$$

- for $\kappa = -1$, $g > 1$ and Σ_g a Riemann surface obtained by compactifying $\mathbb{H}^2$

$$\sigma_1 = \sin\tau \, d\vartheta + \cos\tau \sinh\vartheta \, d\phi \,, \quad \sigma_2 = -\cos\tau \, d\vartheta + \sin\tau \sinh\vartheta \, d\phi \,, \quad \sigma_3 = d\tau - \cosh\vartheta \, d\phi \,, \tag{2.5.24}$$

$$\sigma_1^2 + \sigma_2^2 = d\vartheta^2 + \sinh^2\vartheta \, d\phi^2 \,. \tag{2.5.25}$$

The function $\Omega(r)$ has the form

$$\Omega(r) = (r^2 - s^2)^2 + (\kappa - 4s^2)(r^2 - s^2) - 4sQ\, r + \frac{1}{4}(4s^2 - \kappa)^2 - Q^2\,, \tag{2.5.26}$$

and is required to be positive.

In order to write down the expression for the Killing spinor, we introduce a frame

$$\begin{aligned} &\mathrm{e}^1 = \sqrt{r^2 - s^2}\,\sigma_1\,, \quad \mathrm{e}^2 = \sqrt{r^2 - s^2}\,\sigma_2\,, \\ &\mathrm{e}^3 = 2s\sqrt{\frac{\Omega(r)}{r^2 - s^2}}\,\sigma_3\,, \quad \mathrm{e}^4 = \sqrt{\frac{r^2 - s^2}{\Omega(r)}}\,dr\,. \end{aligned} \tag{2.5.27}$$

The Killing spinor satisfying (2.1.4) with our choice of Γ matrices (2.5.5) and in our frame is

$$\epsilon = \begin{pmatrix} 0 \\ \sqrt{\frac{(r-r_3)(r-r_4)}{r-s}}\,\chi_{(0)} \\ 0 \\ \mathrm{i}\sqrt{\frac{(r-r_1)(r-r_2)}{r+s}}\,\chi_{(0)} \end{pmatrix}, \tag{2.5.28}$$

where $r_{1,2,3,4}$ are the four roots of Ω and $\chi_{(0)}$ is a complex constant which we set to $\sqrt{s}$ to simplify expressions (recall that we may always renormalize the Killing spinor by a complex constant without affecting the result). Since the frame (2.5.27) introduced above is only locally defined, so is the expression (2.5.28) for the spinor: a more detailed discussion of the global regularity of the solution can be found in the appendices to [65, 69].

The supersymmetric Killing vector obtained as a bilinear from (2.5.28) is $\xi = \partial_\tau$, and it is clear by looking at the form of the metric (2.5.19) that the norm of the supersymmetric Killing vector vanishes at the roots of Ω. We have a conformal boundary as $r \to \infty$ and the metric is complete if it collapses smoothly at the largest root of Ω. If Q is such that Ω has a double root, that is $Q = \pm\frac{1}{2}(4s^2 - \kappa)$, then necessarily this is at $r = s$, we are only allowed to consider the $g = 0$ case, and we reduce to the self-dual Taub–NUT–AdS solution, which is covered by the discussion in the previous subsection [76]. However, generically the solution is not self-dual, and considering a neighbourhood of the simple largest root r_0 shows that the fixed point locus of ξ is a two-dimensional bolt Σ_g. In order to have a smooth collapse of the

plane perpendicular to the bolt, we need to impose

$$\frac{r_0^2 - s^2}{|\Omega'(r_0)|} = \frac{2s}{\mathbf{p}}\,, \tag{2.5.29}$$

where

$$\mathbf{p} = \frac{p}{|g-1|} \quad \text{for } g \neq 1\,, \qquad \mathbf{p} = p \quad \text{for } g = 1\,. \tag{2.5.30}$$

This shows concretely that the spacetime has the topology of an $\mathcal{O}(-p)$ fibration over the bolt Σ_g. In particular, the coordinate τ parametrizes the polar angle direction in the complex line fibre, having period $4\pi|g-1|/p$ when $g \neq 1$, and period $4\pi/p$ when $g = 1$ [65, 69].

Imposing the regularity condition (2.5.29), together with some algebra, implies that there are two branches of solutions with different largest root of Ω and different value for Q; we refer to them as positive and negative branches and label the bolts as $\Sigma_g^{\pm}$ (note that here $\pm$ does not refer to the chirality of the spinor at the bolt, although see the discussion further below). Moreover, for different values of the integer p, each branch will only exist for certain ranges of the deformation parameter s. The moduli space of solutions is quite intricate and we would not do justice to it here, so we refer the interested reader to [65] and [69] for a more extensive discussion.

In the following, we move to the local analysis of the $U(1)$ fibration by the supersymmetric Killing vector orbits. Knowing the Killing vector and its norm allows us to find the metric on the base

$$\gamma = 4s^2\left(dr^2 + \Omega(r)(\sigma_1^2 + \sigma_2^2)\right)\,, \tag{2.5.31}$$

and the twist vector

$$H = -\frac{1}{2s\Omega(r)}dr\,. \tag{2.5.32}$$

Comparing the expression (2.5.19) with (2.1.16) gives

$$\varphi = -\frac{(4s^2 - \kappa)(r^2 + s^2) + 4Qrs}{2(r^2 - s^2)} + c_\varphi\,, \tag{2.5.33}$$

and because of the gauge choice in c_φ, (2.1.18) imposes that $f = -c_\varphi\, d\eta$. Because of the form of the metric, ν and σ may be simply found by integrating along the radial coordinate.

From (2.1.22) we find

$$\nu = -s\frac{(4s^2-\kappa)r+2sQ}{r^2-s^2}+c_\nu\,, \tag{2.5.34}$$

and from (2.1.25), we find

$$\begin{aligned}\sigma = \frac{2s}{(r^2-s^2)^2}\Big[&4s^2\left(Q^2r+2Qs\left(r^2+s^2\right)-r\left(r^2-3s^2\right)\left(r^2+s^2\right)\right)\\&-4\kappa rs\left(Qr+s^3+r^2s\right)+\kappa^2r^3\Big]+4c_\varphi\nu+c_\sigma\,,\end{aligned} \tag{2.5.35}$$

where we notice the natural appearance of the $4c_\varphi\nu$ term, consistently with (2.1.26).

At this point, we should fix the gauge choices c_φ, c_σ: to do so, we additionally require that the functions satisfy the relations (2.2.23) and (2.4.8), characteristic of supersymmetric solutions in the supersymmetric gauge. This fixes

$$c_\varphi = -\frac{\kappa}{2}\,,\qquad c_\sigma = 2(\kappa c_\nu - 4Qs^2)\,. \tag{2.5.36}$$

By performing standard holographic renormalization, we readily see that the contribution from conformal infinity to the on-shell action vanishes for the choice (2.5.36), consistently with the fact that in the supersymmetric gauge the contribution from conformal infinity is zero (2.3.18).

Now we can finally check the validity of our computation near the fixed point locus: near the bolt in the positive/negative branch, the spinor (2.5.28) has negative/positive chirality, so the bolt $\Sigma_g^\pm$ is a bolt$_\mp$. Computing the flux of the gauge field through the bolt, this confirms the relation between gauge bundle and geometry (2.4.28), since

$$\int_{\Sigma_g^\pm} c_1(F) = \frac{\pm p-2(1-g)}{2} = \int_{\Sigma_g^\pm}\frac{\mp c_1(N\Sigma_g^\pm)-c_1(T\Sigma_g^\pm)}{2}\,, \tag{2.5.37}$$

where $T\Sigma_g \cong \mathcal{O}(2-2g)$ and $N\Sigma_g \cong \mathcal{O}(-p)$ (recall that in the notation of (2.4.28), $\Sigma_g^\pm$ is a bolt $\Sigma_\mp$!). Moreover, this concretely shows that the gauge field is generically a connection on a virtual unitary line bundle, making the spinor a spin^c spinor – only for even p is A a connection on an honest $U(1)$ bundle.

It is then easy to find that the only non-vanishing skew-eigenvalue of $d\xi^\flat$ is the surface gravity of the bolt $\kappa = -\frac{\mathfrak{p}}{2}$ (in the normalization of (2.4.19)), and that consistently with

(2.4.20), at leading order we have $S = |\kappa|/2$. We can then apply (2.0.1) to obtain the value of the on-shell action

$$I_{\pm} = \frac{\pi}{2G_4}\left(\mp\frac{p}{4} + (1-g)\right) , \tag{2.5.38}$$

consistently with the results obtained previously in the literature [69, eqs (3.73) and (3.74)]. Notice that the analysis of the supersymmetric structure performed here clarifies further the appearance of the branches of solutions, which appeared in the original literature out of considering the smoothness of the metric, and justifies their different on-shell action.

2.5.4 Non-self-dual 1/2 BPS

In addition to the solutions considered in the previous section that preserve 1/4 of the supersymmetries, solutions preserving 1/2 of the supersymmetries have been found in [65]. The ansatz has $SU(2) \times U(1)$ isometry, and the local form of the metric and gauge field is very similar to (2.5.19)

$$\begin{aligned} ds^2 &= \frac{r^2-s^2}{\Omega(r)}dr^2 + (r^2-s^2)(\sigma_1^2+\sigma_2^2) + 4s^2\frac{\Omega(r)}{r^2-s^2}\sigma_3^2 \\ A &= -\frac{s\sqrt{4s^2-1}(r^2+s^2)+2Qrs}{(r^2-s^2)}\sigma_3 , \end{aligned} \tag{2.5.39}$$

but now σ_i are the $SU(2)$ left-invariant one-forms

$$\sigma_1 + \mathrm{i}\sigma_2 = \mathrm{e}^{-\mathrm{i}\tau}\left(d\vartheta + \mathrm{i}\sin\vartheta\, d\phi\right) , \qquad \sigma_3 = d\tau + \cos\vartheta\, d\phi , \tag{2.5.40}$$

and the function Ω reads

$$\Omega(r) = (r^2-s^2)^2 + (1-4s^2)(r^2-s^2) - 2Q\sqrt{4s^2-1}\, r + s^2(4s^2-1) - Q^2 . \tag{2.5.41}$$

Because the solution has the same local form (at least formally, Ω being different), we choose the same vierbein as (2.5.27). In that frame, the spinor satisfying (2.1.4) has the form

$$\epsilon = \begin{pmatrix} \sqrt{\frac{(r-r_3)(r-r_4)}{r-s}}\chi^{(+)} \\ \sqrt{\frac{(r-r_1)(r-r_2)}{r-s}}\chi^{(-)} \\ \mathrm{i}\sqrt{\frac{(r-r_1)(r-r_2)}{r+s}}\chi^{(+)} \\ \mathrm{i}\sqrt{\frac{(r-r_3)(r-r_4)}{r+s}}\chi^{(-)} \end{pmatrix}, \tag{2.5.42}$$

where r_i are the roots of Ω, and

$$\begin{pmatrix} \chi^{(+)} \\ \chi^{(-)} \end{pmatrix} = \begin{pmatrix} \cos\frac{\vartheta}{2}\,\mathrm{e}^{\mathrm{i}(\tau+\phi)/2} & -\sin\frac{\vartheta}{2}\,\mathrm{e}^{\mathrm{i}(\tau-\phi)/2} \\ \gamma\sin\frac{\vartheta}{2}\,\mathrm{e}^{-\mathrm{i}(\tau-\phi)/2} & \gamma\cos\frac{\vartheta}{2}\,\mathrm{e}^{-\mathrm{i}(\tau+\phi)/2} \end{pmatrix} \begin{pmatrix} \mathtt{p} \\ \mathtt{q} \end{pmatrix}, \tag{2.5.43}$$

where $\gamma \equiv \mathrm{i}(2s+\sqrt{4s^2-1})$ and $(\mathtt{p},\mathtt{q}) \in \mathbb{C}^2 \setminus \{0\}$. The supersymmetric Killing vector field constructed with this spinor according to (2.2.5) is

$$\begin{aligned} \xi \equiv \partial_\psi = -2\Big\{&(2s+\sqrt{4s^2-1})\Big[2\Im\left(\mathtt{p}\bar{\mathtt{q}}\mathrm{e}^{\mathrm{i}\phi}\right)\partial_\vartheta + \left(|\mathtt{p}|^2-|\mathtt{q}|^2+2\Re\left(\mathtt{p}\bar{\mathtt{q}}\mathrm{e}^{\mathrm{i}\phi}\right)\cos\vartheta\right)\partial_\phi\Big] \\ &+\left[\left(\frac{1}{2s}-2s-\sqrt{4s^2-1}\right)\left(|\mathtt{p}|^2+|\mathtt{q}|^2\right)-2\Re\left(\mathtt{p}\bar{\mathtt{q}}\mathrm{e}^{\mathrm{i}\phi}\right)\left(2s+\sqrt{4s^2-1}\right)\csc\vartheta\right]\partial_\tau\Big\}. \end{aligned} \tag{2.5.44}$$

Apart from an irrelevant overall normalization factor, this agrees with [67, eq. (5.23)]. Notice that the supersymmetric Killing vector does not simply correspond to the Killing vector ∂_τ generating the Hopf fibration. Since the form of the metric is the same as in the previous subsection, the solution has spherical bolts for ∂_τ at the largest root of Ω, as was the case in the 1/4 BPS solutions. These bolts correspond to the base spaces of the fibration that determines the topology of spacetime: for the self-dual case, the largest root of Ω is $r = s$ and the solution is just Taub–NUT–AdS, whereas in general the topology is that of a bundle $\mathcal{O}(-p) \to S^2$. In this sense, these solutions are referred to as "bolt" solutions in the literature. However, we are interested in the fixed points of the isometry generated by the supersymmetric Killing vector, so in the nomenclature of this work they should be referred to as "nut" solutions. Indeed, ξ vanishes at the two poles of the S^2 when one of the two parameters $\mathtt{p},\mathtt{q}$ is chosen

to be zero. In other words, for $(\mathfrak{p},\mathfrak{q}) = (1,0)$ or $(0,1)$, the supersymmetric Killing vector has nuts at the two poles of the S^2 bolt of the Killing vector ∂_τ.

Regularity of the metric near the largest root of Ω again imposes a constraint on Q analogous to (2.5.29). As in the previous case, there are two branches of solutions, labelled positive and negative, and an intricate moduli space of their existence depending on the value of the deformation parameter s and the self-intersection number p. Moreover, the moduli spaces of the 1/4 BPS and 1/2 BPS supersymmetric solutions intersect non-trivially, making the problem of matching the value of the on-shell action particularly interesting. As before, we refer the interested readers to the original paper [65] for more details.

Examining the Killing spinor near the poles, we see that for the choice $(\mathfrak{p},\mathfrak{q}) = (1,0)$ we have a nut of $\pm$ type near the north/south pole for the positive/negative branch of solutions, and for the choice $(\mathfrak{p},\mathfrak{q}) = (0,1)$ we have a nut of $\mp$ type near the north/south pole for the positive/negative branch of solutions, but the value of the on-shell action does not depend on the choice of spinor (within a branch).

In the $(\mathfrak{p},\mathfrak{q}) = (1,0)$ case, the weights of the Killing vector near the north pole are

$$b_1 = -4s - 2\sqrt{4s^2-1}\,, \qquad b_2 = \frac{p}{2s} \tag{2.5.45}$$

for both positive and negative branch of solutions, and near the south pole

$$b_1 = 4s + 2\sqrt{4s^2-1}\,, \qquad b_2 = \frac{p}{2}\left(\frac{1}{s} - 8s - 4\sqrt{4s^2-1}\right)\,, \tag{2.5.46}$$

again for both branches. Applying (2.0.1) then leads to

$$I_\pm = \frac{\pi}{2G_4}\left[1 \pm \frac{2\sqrt{4s^2-1}}{sp}\left(s^2 - \frac{p^2}{16}\right)\right]\,, \tag{2.5.47}$$

which matches [65, eq. (4.29)]. Similarly, in the $(\mathfrak{p},\mathfrak{q}) = (0,1)$ case, the weights of the Killing vector near the north pole are

$$b_1 = 4s + 2\sqrt{4s^2-1}\,, \qquad b_2 = \frac{p}{2}\left(\frac{1}{s} - 8s - 4\sqrt{4s^2-1}\right) \tag{2.5.48}$$

and near the south pole

$$b_1 = -4s - 2\sqrt{4s^2 - 1}\,, \qquad b_2 = \frac{p}{2s}\,. \tag{2.5.49}$$

Again, applying (2.0.1) leads to (2.5.47) (the weights are exchanged between the two poles, but so is the chirality of the spinor near the nuts).

2.5.5 More general examples

The two solutions considered in the previous two subsections share the same topology $M = \mathcal{O}(-p) \to S^2$, where $p \in \mathbb{Z}_{>0}$ and with both admitting a $U(1)^2$ torus action which contains the isometry generated by the supersymmetric Killing vector field ξ. On the other hand, our main formula (2.0.1) for the action of a solution only requires knowledge of the action of ξ on M, together with the chirality data (determining certain signs) at the fixed points of ξ. We can in fact compute the action for *any* supersymmetric solution on $M = \mathcal{O}(-p) \to S^2$, *assuming the solution exists*, for which the 1/4 BPS and 1/2 BPS results in sections 2.5.3 and 2.5.4 arise as special cases. As well as recovering known results very simply, we are then also able to give the actions of solutions that have not (yet) been found in closed form, but of course we do need to assume the solutions actually exist. We discuss this further in section 2.6. Note that we have summarized the results from this section for clarity and because of space constraints; the details and background of these computations can be found in the original paper [1].

Using the notation of sections 2.5.3 and 2.5.4, we let ∂_ϕ be the lift of the vector field that rotates the S^2 zero section, fixing the north and south poles at $\vartheta = 0,\, \pi$, and $\frac{2}{p}\partial_\tau$ be the vector field that rotates the complex line fibre with weight one. We then introduce the following basis for the $U(1)^2$ action:

$$\partial_{\psi_1} = \partial_\phi + (1 - \frac{p}{2})\frac{2}{p}\partial_\tau\,, \qquad \partial_{\psi_2} = \frac{2}{p}\partial_\tau\,. \tag{2.5.50}$$

The vector fields ∂_{ψ_1}, ∂_{ψ_2} generate an effective action of the torus on $M = \mathcal{O}(-p) \to S^2$, for any (non-zero) p. We can then write down a general toric Killing vector as

$$\xi = a_1\partial_{\psi_1} + a_2\partial_{\psi_2} = (a_1, a_2)\ , \tag{2.5.51}$$

where $a_1, a_2 \in \mathbb{R}$ (not both zero) determine the choice of supersymmetric Killing vector field on M.

Assuming that such a supersymmetric solution exists on $M = \mathcal{O}(-p) \to S^2$, with supersymmetric Killing vector given by (2.5.51), the action of such a solution given by (2.0.1) also depends on the chiralities associated to the two vertices (fixed points of the torus action). We denote these as $\kappa_1, \kappa_2 \in \{\pm 1\}$, respectively. Then for generic $a_1, a_2 \in \mathbb{R}$ we may write down the action (see [1] for details)

$$I = I_p(\kappa_1, \kappa_2; a_1, a_2) = \left[\frac{\mathcal{Q}_p(\kappa_1, \kappa_2; a_1, a_2)}{4a_1(a_1+a_2)(a_1(p-1)-a_2)}\right]\frac{\pi}{2G_4}\,, \tag{2.5.52}$$

where we have defined the homogeneous cubic polynomial

$$\begin{aligned}\mathcal{Q}_p(\kappa_1, \kappa_2; a_1, a_2) = \Big\{ & a_1^3\left[\kappa_1\left(p^2-2p+2\right)+2(\kappa_2+2)(p-1)\right]+a_2^3(\kappa_1-\kappa_2) \\ & + a_1^2 a_2(p-2)[\kappa_1(p-2)+2(\kappa_2+2)] + a_1 a_2^2[\kappa_1(3-2p)+\kappa_2(p-3)-4]\Big\}\,.\end{aligned} \tag{2.5.53}$$

The action $I_p(\kappa_1, \kappa_2; a_1, a_2)$ in (2.5.52) depends on the choice of 4-manifold $M = \mathcal{O}(-p) \to S^2$, via the integer $p \in \mathbb{Z}_{\geq 0}$, the assignment of chiralities $\kappa_i \in \{\pm 1\}$ to each of the two vertices, and also on the choice of supersymmetric Killing vector (2.5.51), via the coefficients $a_1, a_2 \in \mathbb{R}$.

From this general result, we may immediately recover both the 1/4 BPS and 1/2 BPS results in the previous two subsections. Notice first that the 1/4 BPS solution is a degenerate case of the above analysis, where the entire S^2 zero section of $M = \mathcal{O}(-p) \to S^2$ is fixed by ξ. This immediately requires us to take $\kappa_1 = \kappa_2$, since these are precisely the chiralities at the two poles of the S^2 zero section, which must be the same since the spinor is now chiral over the entire S^2. Setting $\kappa_1 = \kappa_2 = \kappa$, the action simplifies to

$$I_p(\kappa, \kappa; a_1, a_2) = \left[\frac{a_1^2\left(\kappa p^2+4p-4\right)+a_1 a_2(p-2)(\kappa p+4)-a_2^2(\kappa p+4)}{4(a_1+a_2)(a_1(p-1)-a_2)}\right]\frac{\pi}{2G_4}\,. \tag{2.5.54}$$

On the other hand, the 1/4 BPS Killing vector precisely rotates the complex line fibre of $M = \mathcal{O}(-p) \to S^2$, we see this means setting $a_1 = 0$, so that $\xi \propto \partial_\tau$. This gives

$$I = I_p(\kappa, \kappa; 0, a_2 \neq 0) = \left(\frac{\kappa p}{4}+1\right)\frac{\pi}{2G_4}\,. \tag{2.5.55}$$

This correctly reproduces the 1/4 BPS bolt action $I_\pm$ in (2.5.38), where recall we should set $\kappa = \mp 1$ and the genus $g = 0$ for S^2. Notice we have recovered this formula as a limit of the general nut fixed point in (2.5.52), in the limit where the Killing vector ξ develops an S^2 bolt, rather than two nuts. Conversely, we may regard (2.5.54) as the 1/4 BPS action with *generic* choice of toric Killing vector, specified by a_1/a_2 and the choice of chirality κ, while the explicitly known solution has $a_1 = 0$. To date such a solution is not known explicitly, or even known to exist, but assuming it *does* exist, its action is given by (2.5.54). Notice that such a solution will necessarily have only $U(1)^2$ as isometry, rather than the $SU(2) \times U(1)$ isometry of the 1/4 BPS solution with $a_1 = 0$.

Next we may recover the 1/2 BPS solution result in section 2.5.4. Recall this had a nut$_-$ and a nut$_+$, so we now set $\kappa_1 = -\kappa_2 = \kappa$. Using the general form of the Killing vector ξ in that section, with the choice $(\mathfrak{p}, \mathfrak{q}) = (1, 0)$, it is straightforward to read off the weights a_1, a_2 in (2.5.51), where recall the basis is defined by (2.5.50). One finds

$$a_1 = -2(2s + \sqrt{4s^2 - 1})\,, \qquad a_2 = -2\left(\frac{p}{4s} - 2s - \sqrt{4s^2 - 1}\right)\,. \tag{2.5.56}$$

From (2.5.52) we then compute

$$\begin{aligned} I &= I_p\left(\kappa, -\kappa; -2(2s + \sqrt{4s^2 - 1}), -2\left(\frac{p}{4s} - 2s - \sqrt{4s^2 - 1}\right)\right) \\ &= \left[1 - \frac{2\kappa\sqrt{4s^2 - 1}}{sp}\left(s^2 - \frac{p^2}{16}\right)\right]\frac{\pi}{2G_4}\,. \end{aligned} \tag{2.5.57}$$

This precisely agrees with the 1/2 BPS action $I_\pm$ in (2.5.47) on setting $\kappa = \mp 1$ for the two branches, respectively.

We can generalize these expressions even further. For the $M = \mathcal{O}(-p) \to S^2$ examples above, the conformal boundary is a Lens space $M_3 = \partial M = L(p, 1)$. We can instead consider fillings of different Lens spaces

$$L(p, q) = S^3/\mathbb{Z}_p\,, \qquad (z_1, z_2) \mapsto (\omega_p z_1, \omega_p^q z_2)\,, \tag{2.5.58}$$

where p and q are coprime integers with $p > q > 0$, and we identify S^3 as the unit sphere in $\mathbb{C}^2$ with coordinates (z_1, z_2). We can compute the action of supersymmetric solutions on these four-manifolds, provided they exist; refer to the original paper [1] for the details of this

computation. We will discuss these results further in section 2.6.

2.5.6 Supersymmetric black holes

The explicit solutions considered in sections 2.5.3 and 2.5.4 involve a non-trivial fibration over a base surface, which in some cases (for example, choosing $a_1 = 0$ in (2.5.51)) is a bolt for the supersymmetric Killing vector. However, there are also known solutions where the fibration is trivial and the topology is that of $M = \mathbb{R}^2 \times \Sigma_g$ (with a warped metric). One simple class of such solutions that preserve 1/4 of the supersymmetry can be obtained by Wick rotation of the dyonic static solutions studied in [77–79]

$$ds^2 = V(r)\, d\tau^2 + \frac{dr^2}{V(r)} + r^2(d\vartheta^2 + \sinh^2\vartheta\, d\phi^2)\,, \tag{2.5.59}$$

with

$$V(r) = -1 + \frac{\frac{1}{4} - Q^2}{r^2} + r^2\,. \tag{2.5.60}$$

The two-dimensional metric in round brackets in (2.5.59) is that on H^2, and quotienting by discrete subgroups of $SO(1,2)$ we may find Riemann surfaces with any genus $g > 1$. The gauge field and its curvature are given by

$$A = \frac{Q}{r} d\tau + \frac{1}{2}\cosh\vartheta\, d\phi\,, \qquad F = \frac{Q}{r^2} d\tau \wedge dr + \frac{1}{2}\sinh\vartheta\, d\vartheta \wedge d\phi\,, \tag{2.5.61}$$

so we identify the electric charge with Q and the magnetic charge as

$$\int_{\Sigma_g} \frac{F}{2\pi} = \frac{1}{4\pi}\int_{\Sigma_g} \mathrm{vol}_{\Sigma_g} = g - 1 \in \mathbb{Z}\,. \tag{2.5.62}$$

This last condition ensures that the gauge field is a connection on an honest $U(1)$ gauge bundle over the spacetime, for any Riemann surface.

Choosing the vierbein

$$\mathrm{e}^1 = r\, d\vartheta\,, \quad \mathrm{e}^2 = r\sinh\vartheta\, d\phi\,, \quad \mathrm{e}^3 = \sqrt{V}\, d\tau\,, \quad \mathrm{e}^4 = \frac{dr}{\sqrt{V}}\,, \tag{2.5.63}$$

and using the same Γ matrices as in (2.5.5), we find that the following spinor satisfies the

(generalized) Killing spinor equation (2.1.4)

$$\epsilon = \begin{pmatrix} 0 \\ \sqrt{\frac{2(r-r_-)(r+r_-)}{r}}\chi_{(0)} \\ 0 \\ \mathrm{i}\sqrt{\frac{2(r-r_+)(r+r_+)}{r}}\chi_{(0)} \end{pmatrix}, \tag{2.5.64}$$

where $\chi_{(0)}$ is a complex constant which we set to $1/2$ to simplify some expressions, and $r_\pm$ are the following roots of $V(r)$

$$r_\pm = \sqrt{\frac{1}{2} \pm Q}\,. \tag{2.5.65}$$

The solution is regular as long as $V(r) > 0$. If we denote the largest root of V by r_0 and assume that $Q \neq 0$, then a local analysis near r_0 shows that, to avoid conical singularities in the space orthogonal to the bolt, the period of τ should satisfy the constraint

$$\frac{|V'(r_0)|}{2}\Delta\tau = 2\pi\,. \tag{2.5.66}$$

If $Q = 0$, instead, $V'(r_0) = 0$, so near $r = 1/\sqrt{2}$ we introduce the coordinate $\rho = r - 1/\sqrt{2}$ and we see that the metric approaches that of $H^2 \times \Sigma_g$

$$ds^2 \sim 4\rho^2 d\tau^2 + \frac{d\rho^2}{4\rho^2} + \frac{1}{2}ds^2_{\Sigma_g}\,, \tag{2.5.67}$$

with an infinite throat as $\rho \to 0$. Therefore, we have a family of solutions depending on a parameter Q: as long as $Q \neq 0$, we have a bolt at $r = r_0 > 1/\sqrt{2}$ for ∂_τ at finite distance, whereas in the case $Q = 0$ the "bolt" has effectively receded to infinite distance. As we shall see, this limit is especially interesting in Lorentzian signature, and we will come back to it after computing the action of these solutions.

Using the spinor (2.5.64), we can immediately see that the supersymmetric Killing vector is $\xi = \partial_\tau$. We may then compute the functions appearing the three-dimensional reduction of section 2.1: in the supersymmetric gauge, where φ and σ satisfy (2.2.23) and (2.4.8), we have

$$\varphi = \frac{Q}{r}\,, \qquad \sigma = -\frac{Q}{r^2} + 2Q\,, \qquad \cos\theta = \frac{2Q}{2r^2 - 1}\,. \tag{2.5.68}$$

One can perform the holographic renormalization according to the procedure outlined in subsection 2.3 and see that indeed in this gauge, the contribution to the on-shell action from the conformal infinity is vanishing, and the value is determined by the bolt contribution. Turning instead to our main formula (2.0.1), at the fixed point set $\cos\theta = \mathrm{sgn}(Q)$, so the distinction $\mathrm{bolt}_{\pm}$ only depends on the charge Q. However, this does not affect the value of the on-shell action, as the chirality label in (2.0.1) only appears in front of the self-intersection number of the bolt, which is zero for the trivial fibration as in this case. Therefore, we conclude from (2.0.1) that the on-shell action for the entire family of solutions (2.5.59), (2.5.61) is simply proportional to the Euler characteristic of the bolt

$$I = \frac{\pi}{2G_4}(1-g)\,. \tag{2.5.69}$$

Notice that this corresponds to the $p = 0$ case of (2.5.38), since the on-shell action only depends on the topological data of the solution. Indeed, the action is manifestly independent of Q, as it had to be. The fact that (2.5.69) agrees with the purely magnetically charged $Q = 0$ black hole action in [68] then really follows only because the $Q \neq 0$ deformation with a bolt exists: as soon as one knows this deformation exists, the action is given by the topological formula (2.0.1), which is independent of the deformation parameter. In this way, one can compute the action of solutions with an infinite throat, provided one knows an appropriate deformation of it exists that is closed in the interior. Notice that a similar regularization of the action of a solution with an infinite throat, by deforming away from extremality but preserving supersymmetry, was used in various dimensions in [80, 81]. Notice also that the relation between gauge field and geometry fixed by (2.4.28) is satisfied: the magnetic charge is in this case proportional to the Euler characteristic of the surface, since the normal bundle to the bolt is trivial.

In Lorentzian signature (obtained via $t = \mathrm{i}\tau$ and $q_e = -\mathrm{i}Q$), the solutions with $Q \neq 0$ describe dyonic naked singularities, whereas the case $Q = 0$ is an extremal magnetically charged black hole with horizon homeomorphic to a Riemann surface. As observed in [68], the Bekenstein–Hawking entropy of the black hole is directly reproduced by the Euclidean

on-shell action (note that indeed the entropy is positive for genus $g > 1$)[7]

$$I = -S_{\rm BH}\,, \tag{2.5.70}$$

and thus, via the AdS/CFT dictionary, the entropy of the black hole can be connected to the large N limit of the partition function of the boundary SCFT, leading to the relation originally advocated in [55]

$$S_{\rm BH} = \log Z\,. \tag{2.5.71}$$

Moreover, the fact that the formula for the on-shell action factorizes as the action for AdS_4, multiplied by the Euler number of the surface, can be interpreted as the manifestation of the boundary universal twist [68, 82, 83].

2.6 Discussion

Inspired by the analysis in [73], in this chapter we presented a formula (2.0.1) for the on-shell action of supersymmetric asymptotically locally Euclidean AdS solutions to four-dimensional minimal gauged supergravity in terms of contributions only from the fixed point locus of a canonical supersymmetric Killing vector. The formula is such that it may be evaluated knowing only the topology of the four-manifold M and the action generated by the vector field, together with certain signs that are determined by chirality data. We have shown that (2.0.1) straightforwardly reproduces the actions of explicitly known solutions in the literature, but also that it may be used to simply write down the actions of solutions, *assuming they exist*.

Recall that the standard holographic dictionary connects the on-shell action to the partition function of a dual field theory defined the boundary manifold $M_3 = \partial M$, at least in an appropriate strong coupling limit. We may make this more precise in the current setting by embedding the construction into string/M-theory, where recall that minimal $\mathcal{N} = 2$ gauged supergravity is a consistent truncation of 11-dimensional supergravity on any Sasaki–Einstein 7-manifold Y_7 [24]. Thus, at least locally, any bulk solution on a four-manifold M uplifts to

[7] Note the difference with the deformation of the extremal case considered in [68]: there, the authors consider non-supersymmetric magnetically charged solutions, whereas we have to preserve supersymmetry in order to apply our methods, so we consider dyons.

an 11-dimensional solution that is the total space of a fibration of $Y_7 \to M$. As discussed in [65, 69], globally there are some restrictions on which Y_7 may be fibred, depending on the topology of M and the spin^c gauge bundle defined by the Abelian gauge field. However, there are by now large classes of Sasaki–Einstein Y_7 for which the dual three-dimensional superconformal field theories on $M_3 = \partial M$ are known explicitly, starting with the seminal work of [17] for $Y_7 = S^7/\mathbb{Z}_k$. These are typically Chern–Simons-matter theories, with the supergravity saddle point limit corresponding to a limit of large rank N of the gauge group. As mentioned already, many of the expressions for the on-shell action reviewed in this chapter have already been matched to such corresponding field theory calculations, showing agreement.

However, thanks to (2.0.1) we now have much more general expressions. For example, (2.5.54) gives the action of solutions with topology $M = \mathcal{O}(-p) \to S^2$ with a *general* choice of supersymmetric Killing vector, which are continuously connected to the explicit 1/4 BPS solution of [65] with this topology, which has a particular fixed supersymmetric Killing vector. The formula (2.5.54) is hence a prediction for the large N limit of appropriate classes of Chern–Simons-matter theories on the Lens space $L(p,1) = S^3/\mathbb{Z}_p$, as discussed in section 2.5.5. The field theory computation required to check this prediction is a generalization of that appearing in [69], which reproduces the 1/4 BPS result (2.5.38). However, one can go much further. One could start with any Lens space $M_3 = L(p,q)$ with any choice of toric supersymmetric Killing vector (2.5.51). This defines a rigid supersymmetric three-manifold background [71], and moreover there are now techniques to compute the partition functions of supersymmetric Chern–Simons-matter theories on *any* such three-manifold [70, 84] (for rational $a_1/a_2 \in \mathbb{Q}$). One can then take the appropriate large N limit to compare to the supergravity saddle point result (2.0.1).

In the absence of explicit solutions, application of (2.0.1) also requires us to assume such a supergravity solution actually exists, which leads naturally to the question of existence and uniqueness of solutions. We note this is similar to the situation in [51], where the volumes of (toric) Sasaki–Einstein manifolds could be computed explicitly, again assuming that the Einstein equation actually has a solution. In this setting, reference [85] subsequently proved that such solutions do indeed always exist. This is in general a problem in geometric analysis. However, we also note that one might more simply address this existence problem for self-dual solutions, as briefly summarized in section (2.5.2), since the PDE (2.5.14) in this case

is integrable. Indeed, formally infinite families of solutions to this equation may be written down, where M has a T^2 isometry, as summarized in section 5.4 of [67], following [86]. Given that local solutions satisfying the supergravity equations are trivial to construct within this ansatz, this is likely the best place to begin to answer these (global) existence questions.

As already remarked in the Introduction, supersymmetric localization for field theories on curved backgrounds has allowed for spectacular improvements in our understanding of quantum field theories at strong coupling, and the duality relations that appear in that regime. Here we have presented a formula (2.0.1) that suggests that a localization similar in spirit already happens in classical supergravity. It is interesting to speculate whether there is a more precise connection between the boundary and bulk computations, and indeed whether the formula (2.0.1) can be understood directly from a large N field theory computation on the boundary three-manifold $M_3 = \partial M$.

Of course, in the context of holographic approaches to quantum gravity, it would also be very interesting to consider the subleading corrections in the rank of the gauge group of the boundary theory, as done for instance in [87–90]. In particular, we notice that in the latter reference the corrections have been computed for the minimal gauged supergravity considered here, for the black hole solution (2.5.59), even though there it was seen in the context of a reduction from 7-dimensional supergravity with a view to holography and the 3d/3d correspondence. Since the publication of the work described in this chapter, several papers have appeared with development in these directions. Namely the work of [91], where the authors consider higher derivative corrections to the supergravity action, making a conjecture for the extension of our formula 2.0.1 with the first subleading corrections in N (for the case of M2-branes constructions, this corresponds to the $N^{1/2}$ correction to the $N^{3/2}$ leading term). Later in [92] the authors prove this conjecture by extending the formalism explained in this chapter to the four-derivative action from [91]. More recently, in [93], the authors present an extension of the ideas in this chapter by proposing a general formula for the log-corrections to the supergravity on-shell action, also linking to the aforementioned results in localizing the supergravity partition function. Moreover, they also add arbitrary matter supermultiplets in their analysis so it extends beyond the minimal theory as well.

A natural generalization of this work would then be to consider different, less simple, supergravity theories. This is particularly motivated by holographic computations of the en-

tropy of black holes. For instance, in agreement with [68], we found in section 2.5.6 a relation between the on-shell action and the Bekenstein–Hawking entropy of the Lorentzian black hole. *A priori*, the standard gauge/gravity dictionary relates the bulk on-shell action to the boundary partition function. However, in models where gravity is coupled to vector multiplets (namely the STU model), a relation, involving a Legendre transform, between the black hole entropy and the supersymmetric partition function computed via localization has been originally advocated in [82], and then shown to be a consequence of the BPS relation [94,95]. It would be interesting to see if there is a generalization of the structure underlying supersymmetric solutions that we have found here, see e.g. [68,82,96–104]. From the gravity side, the attractor mechanism [105–108] also appears as a holographic counterpart to the microscopic counting of black hole entropy. In these papers the authors study how the scalar fields in extremal black hole solutions exhibit a gradient flow towards fixed points at the horizon; you can obtain the entropy of the black hole by extremizing an entropy functional with respect to these horizon modes. Recently, in [109], a new proposal was described, with the intention of unifying the entropy functionals of spherical black hole solutions with arbitrary rotation and electric and magnetic charges into a single formalism, which reduces to the attractor mechanism in particular cases. They describe some similarities with the formalism in this chapter and state that their results should agree with a non-minimal generalization of the action localization approach presented here.

Black hole entropy is quite a vast and rich field of study in the world of Holography, as we have started to see above. In the next chapter we will explore further this complex landscape of AdS_4 black holes, in the context of $D = 4$, $\mathcal{N} = 2$ minimal gauged supergravity. In particular, we find an interesting class of accelerating, rotating and dyonically charged black holes with novel characteristics that have remarkable implications for supersymmetry and the dual holographic computations.

Chapter 3

Accelerating black holes and their spindle horizons

In this chapter we continue the exploration of $D = 4$ minimal gauged supergravity, focusing on a class of accelerating, rotating and dyonically charged black holes in AdS_4. It is well known that when the acceleration is non-vanishing the $D = 4$ black hole metrics have conical singularities. By uplifting the solutions to $D = 11$ supergravity using a regular Sasaki-Einstein 7-manifold, SE_7, we show how the free parameters can be chosen to eliminate the conical singularities. Topologically, the $D = 11$ solutions incorporate an SE_7 fibration over a two-dimensional weighted projective space, $\mathbb{WCP}^1_{[n_-,n_+]}$, also known as a "spindle", which is labelled by two integers that determine the conical singularities of the $D = 4$ metrics. The $D = 11$ solutions are associated to a stack of M2-branes wrapped around the spindle, but we find that in this case supersymmetry is not preserved by a topological twist, as was the case for the constructions in section 1.3. We explain how these novel ideas work in different limits of the solutions, while also computing the entropy for these accelerating black holes and making predictions for the dual field theories.

In the literature of accelerating black holes, the appearance of the conical singularities is well known. The C-metrics of $D = 4$ Einstein-Maxwell theory [110] describe two charged black holes undergoing uniform acceleration. The force for the acceleration is provided by a conical angle excess (a strut) between the two black holes, a conical angle deficit (a string) attached to the black holes and extending out to infinity, or a combination of the two. By

extending the model to include additional matter fields that allow for cosmic strings, the conical singularity can be smoothed out by having two cosmic strings pull the black holes apart [111, 112]. Another approach for removing the conical singularities, in the case of electrically or magnetically charged black holes, is provided by the Ernst metric in which the black holes are being accelerated by a background electric or magnetic field, respectively [113]. Generalizations of the Ernst metric were found for Einstein-Maxwell-dilaton gravity in [114]. For the particular case associated with $D = 5$ Kaluza-Klein theory it was shown in [115] that the accelerating $D = 4$ extremal, magnetically charged black hole solutions can be obtained by a dimensional reduction of a double Wick rotation of the $D = 5$ generalization of the Kerr solution [116]. Recall the extremal magnetically charged black holes in Kaluza-Klein theory are Kaluza-Klein monopoles [117, 118], which are in fact naked singularities from the four-dimensional point of view. A key ingredient in the construction of [115] is that the Kaluza-Klein circle action associated with the reduction has two fixed points which precisely correspond to a Kaluza-Klein monopole, anti-monopole pair.

As we have already introduced, in this chapter we provide a new way of desingularizing the conical deficits for a specific class of accelerating, rotating and dyonically charged black holes in AdS_4, by embedding them in $D = 11$ supergravity. We study the Plebański–Demiański (PD) solutions [119] of $D = 4$ Einstein-Maxwell theory, generalizing the C-metrics of [110], which in particular allow for a cosmological constant, which we take to be negative. Such accelerating AdS_4 black holes have been extensively studied (e.g. [120–122]) and here we will only be interested in the case of small acceleration for which there is just a single black hole. From a $D = 4$ perspective we will consider a single accelerating black hole with a two-dimensional horizon given by the aforementioned spindle, $\mathbb{WCP}^1_{[n_-,n_+]}$, which is topologically a sphere with conical deficits at both poles specified by $n_\pm \in \mathbb{N}$, which stretch out to the AdS_4 boundary. The net acceleration of the black hole is due to the mismatch of the conical deficits on either side of the black hole. Interestingly, the same mismatch also leads to a non-zero magnetic flux though the horizon, given by $G_{(4)}Q_m = \frac{n_- - n_+}{4n_+n_-}$.

As for the solutions of the previous chapter we can embed these solutions in $D = 11$ supergravity, locally, using an arbitrary seven-dimensional Sasaki-Einstein manifold SE_7. In our construction it is only the *regular class* of SE_7 that plays a role in obtaining smooth $D = 11$ solutions. These have the property that the SE_7 manifold is a circle bundle over KE_6,

a six-dimensional Kähler-Einstein manifold with positive curvature. It will also be important to recall that this includes the generic class where the circle bundle is the canonical bundle of the Kähler-Einstein manifold, but it is also possible, depending on the choice of KE_6, to enlarge the period of the circle to get other Sasaki-Einstein manifolds. The simplest example is for the case of $\mathbb{CP}^3$: for this case $S^7/\mathbb{Z}_4$ is the canonical circle bundle over $\mathbb{CP}^3$ and is a smooth Sasaki-Einstein manifold, but so too are $S^7/\mathbb{Z}_2$ and S^7.

After uplifting the PD black hole solutions with horizon $\mathbb{WCP}^1_{[n_-,n_+]}$ on a regular SE_7 we will show, somewhat miraculously, that by choosing the parameters in the PD metric appropriately, the $D = 11$ solution is free from any conical deficit singularities.[1] Importantly, precisely which regular Sasaki-Einstein manifold one can uplift upon depends on the integers $n_\pm \in \mathbb{N}$. For example, for the case of $\mathbb{CP}^3$, for given $n_\pm \in \mathbb{N}$ we can uplift on precisely one of the three cases S^7, $S^7/\mathbb{Z}_2$ or $S^7/\mathbb{Z}_4$. It is also important to note that for the Kaluza-Klein reduction on the SE_7 the fibred circle action does not have any fixed points and hence there are no Kaluza-Klein monopoles as in the construction of [115]. On the other hand, the circle action is not free and this leads to the conical deficits of the spindle $\mathbb{WCP}^1_{[n_-,n_+]}$.

Of particular interest is that our construction also includes accelerating black holes that preserve supersymmetry, as considered in a general context in [123], and, moreover, have regular extremal horizons. From the $D = 4$ perspective, the near horizon limit of these supersymmetric, extremal, rotating black holes are of the form $\mathrm{AdS}_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$, with non-vanishing magnetic flux through the horizon $\mathbb{WCP}^1_{[n_-,n_+]}$. The way that supersymmetry is preserved for these AdS_2 solutions is non-standard: the magnetic flux is not describing a topological twist [29,30], and the Killing spinors are then correspondingly not simply constant, even when the rotation is turned off; indeed, they are sections of non-trivial bundles over the spindle $\mathbb{WCP}^1_{[n_-,n_+]}$. The uplifted supersymmetric $D = 11$ solutions have a near horizon limit of the form $\mathrm{AdS}_2 \times Y_9$. In the special case that we set the rotation parameter of the PD metrics to zero, we find an $\mathrm{AdS}_2 \times Y_9$ geometry in the general class of [124], where Y_9 is a nine-dimensional GK geometry [125]. In fact, remarkably, for vanishing rotation parameter we find exactly the same class of supersymmetric $\mathrm{AdS}_2 \times Y_9$ solutions first constructed in [126] from a quite different perspective. When the rotation parameter of the PD metrics is non-zero we find a new class of supersymmetric $\mathrm{AdS}_2 \times Y_9$ solutions that lie outside the class

[1] Of course the black hole curvature singularity remains, behind the black hole horizon.

considered in [124], and generalizing those of [126] with an extra rotation parameter.[2]

We calculate the entropy S_{BH} for the PD black hole solutions. In particular, for the "supersymmetric spinning spindles" that in addition have extremal horizons, we find

$$S_{BH} = \frac{\pi}{G_{(4)}}\frac{J}{Q_e} = \frac{\pi}{G_{(4)}}\frac{\sqrt{2}\sqrt{n_-^2+n_+^2+8n_-^2n_+^2(G_{(4)}Q_e)^2}-(n_-+n_+)}{4n_-n_+}\,, \tag{3.0.1}$$

where J is the angular momentum of the black hole, and Q_e is its electric charge. Notice this gives a relation between J and Q_e for extremal solutions. We also note that with non-zero acceleration, we can still have supersymmetric extremal black holes with $J = Q_e = 0$; for these black holes the second expression in (3.0.1) is the valid expression. We also recall that $\frac{\pi}{2G_{(4)}} = \mathcal{F}_{S^3}$ is the free energy of the $d = 3$ SCFT dual to the $\mathrm{AdS}_4 \times SE_7$ solution.

We define the angular momentum J of the black hole, appearing in (3.0.1), to be a conserved quantity that can be equally evaluated either at the conformal boundary or at the black hole horizon (see e.g. [128]). As a consequence it is a type of Page charge that depends on the gauge used for the gauge field. However, there is a different natural angular momentum defined in the near horizon limit which has a gauge field that is invariant under the AdS_2 symmetries, that we denote J_{AdS_2}. We find

$$J_{\mathrm{AdS}_2} - J = \frac{Q_e}{4}\chi(\Sigma)\,, \tag{3.0.2}$$

where $\chi(\Sigma) = \frac{n_-+n_+}{n_+n_-}$ is the Euler number of the spindle horizon $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$.

The family of black holes that we consider includes the well-known Kerr-Newman-AdS spacetime [129] in the non-accelerating case, whose BPS and extremal limits have been extensively discussed both from the gravitational, as well as from the dual field theory points of view, see e.g. [81, 130–132] and [133, 134], respectively. Interestingly, the entropy of the BPS and extremal Kerr-Newman-AdS black hole can be immediately recovered from (3.0.1) by setting $n_+ = n_- = 1$. In addition, the expression (3.0.2) is also valid for the Kerr-Newman-AdS black holes, a point that seems to have been overlooked in the literature.

These results leads to the challenge of recovering the entropy of the rotating and accelerating black hole by evaluating a suitable index (as the principle of I-extremization in the case

[2] These should arise within the recent classification of rotating $D = 11$ AdS_2 solutions in [127], and it would be interesting to investigate this further.

of the topological twist, covered in section 1.3). This index will be an appropriately defined localized partition function of the dual $d = 3$ SCFT compactified on $S^1 \times \mathbb{WCP}^1_{[n_-,n_+]}$, with electric flux and magnetic flux through $\mathbb{WCP}^1_{[n_-,n_+]}$, and with additional rotation. There are a number of subtleties related to this computation, which will need to be further developed in future work. Here we will make a number of particularly interesting observations concerning the BPS extremal black holes with acceleration but without rotation or electric charge i.e. $J = Q_e = 0$. One feature is that these solutions have an acceleration horizon that intersects the conformal boundary, effectively dividing the spindle in half. Moreover, we find that the Killing spinor on this conformal boundary is given by a topological twist, so that the spinor is constant, but it is a *different* constant spinor on each half of the space! We also explain how these unusual features are "regulated" by keeping, for example, supersymmetry but relaxing the extremality condition (although this class with $J = 0$ then has a naked singularity in the bulk).

The plan of the rest of the chapter is as follows. In section 3.1 we briefly review $D = 4$ minimal gauged supergravity, and its uplift to $D = 11$ supergravity on Sasaki-Einstein seven-manifolds. Section 3.2 introduces the class of Plebański–Demiański solutions of interest, showing that the parameters can be chosen so that the black hole horizon is topologically a spindle, $\mathbb{WCP}^1_{[n_-,n_+]}$. In section 3.3 we show that on uplifting to $D = 11$ supergravity the PD solutions become completely smooth, with properly quantized flux. Supersymmetric and extremal solutions are studied in section 3.4, where we focus on the near horizon AdS_2 geometries and associated Killing spinors. In section 3.5 we discuss some global properties of the accelerating but non-rotating extremal supersymmetric black hole, discussing the conformal boundary and the acceleration horizon. We conclude with a discussion of the results and the open problems in section 3.6.

We have also included a number of appendices to this chapter at the end of this dissertation. Appendix A.1 has some general comments on circle bundles over spindles. Appendix A.2 briefly discusses some aspects of the well-known non-accelerating class of Kerr-Newman-AdS black holes. Appendix A.3 contains some technical details concerning the near horizon limit, while appendix A.4 shows that the resulting supersymmetric $\mathrm{AdS}_2 \times Y_9$ solutions generalize those of [126] by the addition of an angular momentum parameter. In appendix A.5 we discuss how we define the angular momentum of the black holes, clarify various aspects of the

gauge dependence and also derive (3.0.2). Finally, appendix A.6 discusses the conventions used for computing the Killing spinors.

3.1 General setting

As in chapter 2 we will again be interested in solutions of $D = 4$ Einstein-Maxwell theory, this time in Lorentzian signature. For completeness, the action is then given by

$$S = \frac{1}{16\pi G_{(4)}} \int d^4x \sqrt{-g} \big(R + 6 - F^2\big) \,, \tag{3.1.1}$$

with $F = dA$ (notice the sign difference with (2.1.1), because of the different signature). This is the action for the bosonic fields of $D = 4$, $\mathcal{N} = 2$ minimal gauged supergravity. A solution to the equations of motion will preserve supersymmetry provided that it admits appropriate Killing spinors which we define later. Note that we have again performed a scaling to set the cosmological constant as $\Lambda = -3$ so that a unit radius AdS_4 vacuum solution, with vanishing gauge field, solves the equations of motion. This vacuum solution preserves all of the $D = 4$ supersymmetry.

Any (supersymmetric) solution of this theory automatically uplifts to a (supersymmetric) solution of $D = 11$ supergravity on an arbitrary Sasaki-Einstein 7-manifold, SE_7 [24]. The general uplifting ansatz is[3]

$$\begin{aligned} ds^2_{11} &= L^2 \left[\tfrac{1}{4} ds^2_4 + \big(\eta + \tfrac{1}{2} A\big)^2 + ds^2_T \right] , \\ G &= L^3 \left[\tfrac{3}{8} \mathrm{vol}_4 - \tfrac{1}{2} *_4 F \wedge J \right] , \end{aligned} \tag{3.1.2}$$

where $L > 0$ is a constant which is eventually fixed by flux quantization. Here η is the contact one-form of the Sasaki-Einstein manifold, with transverse Kähler two-form[4] J and associated transverse metric ds^2_T, with $d\eta = 2J$. The vacuum AdS_4 solution, with $\mathrm{d}s^2_4$ a unit radius AdS_4 and $A = 0$, uplifts to the supersymmetric $\mathrm{AdS}_4 \times SE_7$ solution, dual to an $\mathcal{N} = 2$

[3] Note that in [24] $A^{\text{there}} = 2A^{\text{here}}$.

[4] We have used the letter J to denote both the Kähler two-form and the angular momentum of the black holes; the context should make it clear which one we are referring to.

SCFT in $d = 3$. For later use we note that

$$*_{11}G = L^6\Big[J^3 \wedge (\eta + \tfrac{1}{2}A) + \frac{1}{2^4}F \wedge J^2 \wedge (\eta + \tfrac{1}{2}A)\Big]\,. \tag{3.1.3}$$

In this chapter we will be interested in the case that the Sasaki-Einstein manifold is in the *regular class*, for which $ds_T^2 = ds_{KE_6}^2$ is a metric on KE_6, a Kähler-Einstein manifold. Necessarily, KE_6 has positive curvature and ds_T^2 is normalized so that $Ric(KE_6) = 8g(KE_6)$ and hence $\rho = 8J$, where ρ is the Ricci form. Introducing local coordinates we can write

$$\eta = \tfrac{1}{4}d\psi + \sigma\,, \qquad d\sigma = 2J\,. \tag{3.1.4}$$

The Sasaki-Einstein manifold admits a Killing spinor which has charge $1/2$ under ∂_ψ (see appendix A.6). That is, in a frame invariant under ∂_ψ there is an explicit phase $\mathrm{e}^{\mathrm{i}\psi/2}$ in the spinor. This gives rise to a phase $\mathrm{e}^{\mathrm{i}\psi}$ in the holomorphic $(4,0)$-form on the cone over the SE_7, which is a bilinear in the Killing spinor.

In the sequel it will be important to recall that the KE_6 has a *Fano index* $I \in \mathbb{N}$. If $\mathcal{L}$ is the canonical line bundle over the KE_6 then I is the largest integer for which there is a line bundle $\mathcal{N}$ with $\mathcal{L} = \mathcal{N}^I$, i.e. we are able to take the Ith root of the canonical bundle. In general we may then take the period of ψ to be

$$\Delta\psi = \frac{2\pi I}{k}\,, \tag{3.1.5}$$

where k is a positive integer that divides I. The fundamental group is then $\pi_1(SE_7) \cong \mathbb{Z}_k$, so that in particular when $k = 1$ the resulting manifold is simply-connected. For example, if $KE_6 = \mathbb{CP}^3$, then $I = 4$ and the associated SE_7 manifolds with $k = 1, 2$ and 4 are respectively S^7, $S^7/\mathbb{Z}_2$ and $S^7/\mathbb{Z}_4$. Note that $I/k \in \mathbb{N}$ ensures that the Killing spinor and holomorphic $(4,0)$-form mentioned above are well-defined. Some other well-known examples of regular SE_7 are summarized in Table 3.1, with the associated values of I (see e.g. Theorem 3.1 of [135].)

Another integer quantity associated with the KE_6 that will frequently appear is defined

by

$$M \equiv \int_{KE_6} \left(\frac{\rho}{2\pi}\right)^3 = \int_{KE_6} c_1^3 \,, \tag{3.1.6}$$

where ρ is the Ricci form for the KE_6. Values of M for examples of SE_7 can also be found in Table 3.1. We note that the volume of the KE_6 and the SE_7 can be expressed in terms of M as follows

$$\mathrm{vol}(KE_6) = \frac{\pi^3 M}{3\times 2^7} \,, \qquad \mathrm{vol}(SE_7) = \frac{\pi^4 M I}{3\times 2^8 k} \,. \tag{3.1.7}$$

KE_6	I	M	SE_7 (with $k=1$)
$\mathbb{CP}^3$	4	2^6	S^7
$\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1$	2	$2^4\times 3$	$Q^{1,1,1}$
$\mathbb{CP}^2\times\mathbb{CP}^1$	1	2×3^3	$M^{3,2}$
$SU(3)/T^2$	2	$2^4\times 3$	$N^{1,1}$
$Gr_{5,2}$	3	2×3^3	$V_{5,2}$
$dP_n\times\mathbb{CP}^1$	1	$6\times(9-n)$	$\star$

Table 3.1: Some examples of simply-connected ($k=1$) SE_7 manifolds obtained as circle bundles over a KE_6 manifold. The integer I is the Fano index for the KE_6 and M is defined in (3.1.6). Note that dP_n are del-Pezzo surfaces with $n=3,\ldots,8$ and the associated SE_7 do not have a name.

Our conventions for $D=11$ supergravity are as in e.g. [136, 137]. We define $N_{SE}>0$ to be the quantized flux through the SE_7 manifold:

$$N_{SE} \equiv \frac{1}{(2\pi\ell_p)^6}\int_{SE_7} *_{11} G = \frac{3L^6\mathrm{vol}(SE_7)}{\pi^6 2^5 \ell_p^6} \,. \tag{3.1.8}$$

Here ℓ_p is the $D=11$ Planck length with the $D=11$ Newton's constant defined by $1/(16\pi G_{(11)}) = 1/(2\pi)^8\ell_p^9$. Carrying out the dimensional reduction to $D=4$ we can now express the $D=4$ Newton constant in the following useful form

$$\frac{1}{G_{(4)}} = \frac{2^{3/2}\pi^2}{3^{3/2}\mathrm{vol}(SE_7)^{1/2}} N_{SE_7}^{3/2} = \frac{2^{11/2}k^{1/2}}{3M^{1/2}I^{1/2}} N_{SE_7}^{3/2} \,. \tag{3.1.9}$$

We also recall that the free energy of the $d=3$ SCFT dual to the $\mathrm{AdS}_4\times SE_7$ solution is

given by

$$\mathcal{F}_{S^3} = \frac{\pi}{2G_{(4)}}\,. \tag{3.1.10}$$

3.2 AdS_4 PD black holes in $D = 4$

We start with a sub-class of the class of Plebański–Demiański (PD) solutions [119] of Einstein-Maxwell theory, as presented in [138]. The metric is given by

$$\begin{aligned} ds^2 = \frac{1}{H^2}\Big\{ &-\frac{Q}{\Sigma}\big[dt - a\sin^2\theta\, d\phi\big]^2 + \frac{\Sigma}{Q}\,dr^2 + \frac{\Sigma}{P}d\theta^2 \\ &+ \frac{P}{\Sigma}\sin^2\theta\big[adt - (r^2+a^2)d\phi\big]^2\Big\}, \end{aligned} \tag{3.2.1}$$

where

$$\begin{aligned} H &= 1 - \alpha\, r\cos\theta\,, \\ \Sigma &= r^2 + a^2\cos^2\theta\,, \\ P &= 1 - 2\alpha m\cos\theta + \Big(\alpha^2(a^2+e^2+g^2) - a^2\Big)\cos^2\theta\,, \\ Q &= \Big(r^2 - 2mr + a^2 + e^2 + g^2\Big)(1-\alpha^2 r^2) + (a^2+r^2)r^2\,, \end{aligned} \tag{3.2.2}$$

and we note that $P = P(\theta)$ and $Q = Q(r)$ while H and Σ depend on both r and θ. The gauge field is given by

$$A = -e\frac{r}{\Sigma}(dt - a\sin^2\theta d\phi) + g\frac{\cos\theta}{\Sigma}(adt - (r^2+a^2)d\phi)\,. \tag{3.2.3}$$

The solution depends on five free parameters m, e, g, a and α, which we can loosely associate with mass, electric charge, magnetic charge, rotation and acceleration, respectively. We note that we have set a possible NUT parameter in the PD metrics to zero, since we want to avoid closed timelike curves. We have fixed the cosmological constant to be $\Lambda = -3$, as in (3.1.1), and finally we note that we have changed the sign of g compared with [138].

We will assume $m > 0$. Note that there are various coordinate changes $\theta \to \pi - \theta$, $t \to -t$, $\phi \to -\phi$ as well as $A \to -A$, which change the signs of the pairs (α, g), (a, e), (a, g) and (e, g), respectively. If $\alpha = 0$ we can thus choose $e, g, a \geq 0$ without loss of generality. If $\alpha \neq 0$

we can choose $m, \alpha > 0$ and $e, g \geq 0$ with either $a \geq 0$ or $a \leq 0$. However, in order to get supersymmetric, extremal black holes one should take $a \geq 0$. Thus, in the sequel we focus on

$$m > 0\,, \qquad \alpha, e, g, a \geq 0\,. \tag{3.2.4}$$

The range of the θ coordinate is taken to be $0 \leq \theta \leq \pi$. We will assume that $P(\theta) > 0$ in this range.[5] It will be convenient to define

$$\theta_- = 0\,, \qquad \text{and} \qquad \theta_+ = \pi\ . \tag{3.2.5}$$

On slices of constant t, r, we can examine the behaviour of the metric as we approach $\theta = \theta_\pm$ and we find

$$ds^2_{\theta,\phi} \approx \left[\frac{\Sigma}{PH^2}\right]_{\theta=\theta_\pm} \left[d\theta^2 + P_\pm^2(\theta - \theta_\pm)^2 d\phi^2\right]\ , \tag{3.2.6}$$

where

$$P_\pm \equiv P(\theta_\pm) = 1 \pm 2\alpha m + \alpha^2(a^2 + e^2 + g^2) - a^2\,, \tag{3.2.7}$$

are constants. With $\alpha m \neq 0$ it is not possible to choose a period for ϕ so that we obtain a smooth metric on a round two sphere as there will always be conical deficits at one or both of the poles. Thus, as is well known, the mismatch of these conical deficits at the two poles is directly connected with the non-vanishing of the acceleration parameter.

A simple observation, which will turn out to be important in obtaining regular solutions in $D = 11$, is that we can suitably constrain the parameters in the metric, and choose a corresponding period for ϕ, so that the conical defects give rise to an orbifold known as a spindle, or equivalently a weighted projective space. We can demand that the ratio P_+/P_- is a rational number, which we write as

$$\frac{P_+}{P_-} = \frac{n_-}{n_+}\,, \tag{3.2.8}$$

[5] As in [122], this can be achieved if $m\alpha < \frac{1}{2}\Xi$ for $\Xi \in (0, 2]$ and $m\alpha < \frac{1}{2}(\Xi - 1)^{1/2}$ for $\Xi > 2$, where $\Xi \equiv 1 + \alpha^2(a^2 + e^2 + g^2) - a^2$.

with $n_\pm \in \mathbb{N}$, and then choose the period of ϕ to be

$$\Delta\phi = \frac{2\pi}{n_+ P_+} = \frac{2\pi}{n_- P_-}\,, \tag{3.2.9}$$

and hence from (3.2.7) we can also write $\Delta\phi = \frac{\pi}{2m\alpha}\frac{n_- - n_+}{n_- n_+}$. The two-dimensional space parametrized by θ and ϕ is then the weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$ (for some further discussion on $\mathbb{WCP}^1_{[n_-,n_+]}$, see appendix A.1). We will see that we are always able to desingularize the singularities of this spindle after uplifting to $D = 11$. As we will see, our procedure will require that we impose the condition $m\alpha = g$; we will also find that $m\alpha = g$ is implied by supersymmetry. It will also be useful in the sequel to introduce a coordinate φ on the spindle that has period 2π:

$$\varphi \equiv \left(\frac{2\pi}{\Delta\phi}\right)\phi\,. \tag{3.2.10}$$

In order to have a spacetime with a black hole horizon, located at $r = r_+$, we demand that $Q(r_+) = 0$ and the region exterior to the black hole has $r \geq r_+$. We also require $H > 0$ and hence the radial coordinate is constrained via

$$\alpha r\cos\theta < 1\,, \tag{3.2.11}$$

and so, in particular, $\alpha r_+ < 1$. The conformal boundary is approached when $\alpha r\cos\theta \to 1$ and one finds that the conical defects are still present when $\alpha \neq 0$. There can also be an acceleration horizon. We discuss this in section 3.5, where we study the global structure of the non-rotating black holes in more detail. In particular, although Q has no other roots for $r > r_+$, one can continue the radial coordinate r past $r = \infty$, and there can effectively be another root beyond this, corresponding to an acceleration horizon. A detailed analysis of the various cases and their Penrose diagrams may be found in [121]. Some further discussion of these black holes may also be found in [122] which calculates various thermodynamic quantities (when $g = 0$). Here we record that the entropy of the black holes is given by

$$S_{BH} = \frac{1}{4G_{(4)}}A = \frac{(r_+^2 + a^2)\Delta\phi}{2G_{(4)}(1-\alpha^2 r_+^2)}\,. \tag{3.2.12}$$

In the next section we show how the PD metrics can be desingularized by uplifting on certain SE_7 manifolds described in the previous section. This analysis will fix $\Delta\phi$, and furthermore the entropy can then be expressed in terms of either N_{SE_7} using (3.1.9), or the free energy of the $d = 3$ SCFT using (3.1.10). For the special class of supersymmetric extremal black holes we then obtain the expression (3.0.1).

We also record that the magnetic flux through the horizon, Q_m, is defined by

$$G_{(4)}Q_m \equiv \frac{1}{4\pi}\int_{r=r_+} F = g\,\frac{\Delta\phi}{2\pi} \tag{3.2.13}$$

and the electric flux, Q_e, is defined by

$$G_{(4)}Q_e \equiv \frac{1}{4\pi}\int_{r=r_+} *F = e\,\frac{\Delta\phi}{2\pi}\,. \tag{3.2.14}$$

Finally, we can introduce the angular momentum of the black hole. Rather than a simple Komar integral, as often used, we define the angular momentum as a conserved quantity that can be equally evaluated either at the conformal boundary or at the black hole horizon (see e.g. [128]). Associated with the Killing vector $k = \partial_\varphi$, where $\varphi \in [0, 2\pi)$ was introduced in (3.2.10), we introduce the two-form

$$Z = dk + 4(A\cdot k)F\,, \tag{3.2.15}$$

and then define the angular momentum via

$$J(A) = \frac{1}{16\pi G_{(4)}}\int_{r=r_+} *Z\,. \tag{3.2.16}$$

The angular momentum is then a kind of Page charge that depends on the gauge.[6] We explore this gauge dependence in appendix A.5, where we also highlight a subtle difference with a natural definition of the angular momentum of the near horizon $AdS_2 \times Y_9$ limit for

[6]Note that when $\alpha = 0$ one can explicitly check that the expression for the angular momentum, in the gauge we are using, agrees with a Komar integral (A.5.1) evaluated at the conformal boundary, which is an expression that is often used to define the angular momentum. We have not verified whether or not this is also the case when $\alpha \neq 0$.

the supersymmetric and extremal black holes. Using the gauge as in (3.2.3) we obtain

$$G_{(4)}J = m\,a\left(\frac{\Delta\phi}{2\pi}\right)^2\,, \tag{3.2.17}$$

which agrees with formulas in the literature for the non-accelerating limit.

3.3 Uplifting and desingularizing

We now consider the PD metrics uplifted to $D = 11$ on a Sasaki-Einstein manifold as in (3.1.2). In this section we will assume that we have non-zero acceleration in $D = 4$:

$$\alpha \neq 0\,. \tag{3.3.1}$$

In section 3.3.1 we will first analyse the regularity of the metric; as this is somewhat involved, we have included a summary section at the end. We then analyse flux quantization in section 3.3.2.

3.3.1 Metric

In analysing the regularity of the $D = 11$ supergravity solution, it is convenient to first perform[7] a local gauge transformation on the gauge field of the form $A \to A + c\,d\phi$, where c is a constant which will be fixed shortly. In the associated $D = 11$ metric, we can focus on the metric on a constant r, t slice which is given by

$$ds^2_{\theta,\phi,KE_6,\psi} = \frac{1}{4}\left[g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2\right] + (\tfrac{1}{4}d\psi + \sigma + \tfrac{1}{2}\tilde{A}_\phi d\phi)^2 + ds^2(KE_6)\,, \tag{3.3.2}$$

where

$$\begin{aligned}
\tilde{A}_\phi &= \frac{1}{\Sigma}\left[era\sin^2\theta - g\cos\theta(r^2+a^2)\right] + c\,,\\
g_{\phi\phi} &= \frac{\sin^2\theta}{H^2\Sigma}\left[-Qa^2\sin^2\theta + P(r^2+a^2)^2\right]\,,\\
g_{\theta\theta} &= \frac{\Sigma}{H^2P}\,,
\end{aligned} \tag{3.3.3}$$

[7] Notice that, effectively, the same thing is achieved by shifting the ψ coordinate via $\psi \to \psi + 2c\phi$.

with $\tilde{A}_\phi = A_\phi + c$. The most general construction will show that the (θ, ϕ, ψ) part of the metric can be taken to parametrize a smooth S^3, or more generally a Lens space $S^3/\mathbb{Z}_\mathfrak{q}$, that is then fibred over the KE_6.

This S^3 can be embedded as $S^3 \subset \mathbb{C}^2$, and we would like to find the $U(1)$ generators that rotate the two copies of $\mathbb{C}$. That is, we introduce $\ell_\pm = \partial_{\alpha_\pm}$ where $\alpha_\pm$ are (2π)-period polar coordinates for each copy of $\mathbb{C}$. By definition, then $\|\ell_+\|^2(\theta_+) = 0$ and $\|\ell_-\|^2(\theta_-) = 0$, and moreover the surface gravity for $\ell_\pm$ satisfies $\kappa_\pm^2 = 1$ at $\theta = \theta_\pm$. Choosing the constant c appearing in the gauge field to be

$$c = \frac{g}{4m\alpha}(P_+ + P_-) \ , \tag{3.3.4}$$

implies that $\ell_\pm$ have the same coefficient of ∂_ψ, with

$$\partial_{\alpha_\pm} = \ell_\pm = \frac{g}{m\alpha}\partial_\psi - \frac{1}{P_\pm}\partial_\phi \ . \tag{3.3.5}$$

Equivalently

$$\psi = \frac{g}{m\alpha}\left(\alpha_+ + \alpha_-\right) \ , \qquad \phi = -\frac{1}{P_+}\alpha_+ - \frac{1}{P_-}\alpha_- \ . \tag{3.3.6}$$

Using the comments in section 3.1, we then conclude that the $D = 11$ Killing spinor will have charge $g/2m\alpha$ under both $\partial_{\alpha_\pm}$. Furthermore, later we will see that one of the BPS equations which ensures that the $D = 4$ or $D = 11$ solutions preserve supersymmetry is precisely $g/m\alpha = 1$.

To proceed we next define the functions

$$\begin{aligned} A_\pm &= \frac{1}{4P_\pm^2}g_{\phi\phi} + \Big(\frac{g}{4m\alpha} - \frac{1}{2P_\pm}\tilde{A}_\phi\Big)^2 \ , \\ B &= \frac{1}{4P_+P_-}g_{\phi\phi} + \Big(\frac{g}{4m\alpha} - \frac{1}{2P_+}\tilde{A}_\phi\Big)\Big(\frac{g}{4m\alpha} - \frac{1}{2P_-}\tilde{A}_\phi\Big) \ , \end{aligned} \tag{3.3.7}$$

and the connection one-forms

$$D\alpha_\pm = d\alpha_\pm \pm \frac{P_\pm}{g}\sigma \ , \tag{3.3.8}$$

to find that the metric can be written

$$ds^2_{\theta,\phi,KE_6,\psi} = \frac{g_{\theta\theta}}{4}d\theta^2 + A_+(D\alpha_+)^2 + A_-(D\alpha_-)^2 + 2B(D\alpha_+)(D\alpha_-) + ds^2(KE_6)\ . \quad (3.3.9)$$

Here $A_+(\theta_+) = 0 = A_-(\theta_-)$, and $B(\theta_\pm) = 0$.

It is next convenient to introduce the coordinates

$$\gamma = \alpha_+ + \alpha_-\ , \qquad \beta = \alpha_+ - \alpha_-\ , \quad (3.3.10)$$

so that $\partial_\gamma = \frac{1}{2}(\partial_{\alpha+} + \partial_{\alpha-})$, $\partial_\beta = \frac{1}{2}(\partial_{\alpha+} - \partial_{\alpha-})$. We then make the periodic identifications

$$\Delta\gamma = 2\pi\,, \qquad \Delta\beta = 4\pi/\mathtt{q}\,. \quad (3.3.11)$$

This leads to a smooth S^3 fibre when $\mathtt{q} = 1$, and more generally a Lens space $S^3/\mathbb{Z}_\mathtt{q}$. Here we are using the fact that a Lens space $S^3/\mathbb{Z}_\mathtt{q}$ may be viewed as the total space of a circle fibration, here with circle fibre coordinate β, over a two-sphere. Specifically, in the construction above the two-sphere has standard spherical polar coordinates (θ, γ). Notice from our comments above that the spinors are charged under ∂_γ, but not under ∂_β, so we may quotient the period of β by $\mathtt{q}$ and preserve supersymmetry. For bosonic solutions more generally we could in principle take further/different quotients but we will not investigate this further here.

Having established that the fibre is a Lens space, we now examine the fibration over the KE_6. We will do this in two steps, firstly discussing an S^2 bundle over the KE_6, with the S^2 parametrized by (θ, γ), and then discussing the circle bundle over this, with the circle parametrized by β. After writing $D\alpha_\pm \equiv \frac{1}{2}D\gamma \pm \frac{1}{2}D\beta$, the connection forms are given by

$$D\gamma = d\gamma + \frac{1}{g}(P_+ - P_-)\sigma\ , \qquad D\beta = d\beta + \frac{1}{g}(P_+ + P_-)\sigma\ . \quad (3.3.12)$$

Focusing on $D\gamma$ we recall from (3.1.4) that $d\sigma = \rho/4$, where ρ is the Ricci form for the KE_6. We can also compute

$$\frac{1}{4g}(P_+ - P-) = \frac{m\alpha}{g}\ . \quad (3.3.13)$$

Thus, since γ has period 2π, we will obtain an S^2 bundle over the KE_6 with fibre parametrized by (θ, γ) being the Riemann sphere compactification of a well-defined line bundle, provided that $m\alpha/g = n/I$, where I is the Fano index of the KE_6 and n is an integer. The line bundle is then $\mathcal{L}^{n/I}$. However, recall we also commented earlier that $g/2m\alpha$ is precisely the charge of the $D = 11$ Killing spinors under $\partial_{\alpha_\pm}$, and hence ∂_γ, and we shall later find that one of the BPS equations is precisely $g/m\alpha = 1$, so these charges are all $1/2$. Thus, we will impose

$$\frac{1}{4g}(P_+ - P_-) = \frac{m\alpha}{g} = 1\,, \tag{3.3.14}$$

and this implies that the (θ, γ) S^2 bundle over the KE_6 is necessarily that associated to the canonical bundle. For the bosonic solutions more generally we could instead take powers of the canonical bundle, with different n above, but here we do not.

It remains to ensure that we have a well-defined circle bundle, with circle fibre coordinate β, over the above S^2 bundle over the KE_6. As described in [139], a necessary and sufficient condition for this is to verify that the corresponding curvature two-form has appropriately quantized periods over a basis of two-cycles. One such two-cycle is a copy of the fibre S^2, at a fixed point on the KE_6, and this has already fixed the period $\Delta\beta = 4\pi/\mathtt{q}$ in (3.3.11), to obtain a Lens space fibre $S^3/\mathbb{Z}_\mathtt{q}$ over the KE_6. The remaining two-cycles may be taken to be two-cycles in the copy of the KE_6 base at either $\theta = \theta_+$ or $\theta = \theta_-$. For example, in the former case the corresponding circle bundle over the KE_6 has connection term $D\alpha_-$, and quantizing the periods of the associated curvature two-form leads to setting

$$\frac{P_-}{4g} = \frac{\mathtt{p}}{\mathtt{q}I}\,, \tag{3.3.15}$$

where $\mathtt{p} \in \mathbb{N}$. Specifically, setting $\theta = \theta_+ = \pi$ then gives a circle, parametrized by α_-, inside the Lens space fibre. Since on this circle α_- has period $2\pi/\mathtt{q}$, the choice (3.3.15) implies that we obtain the circle bundle associated to $\mathcal{L}^{\mathtt{p}/I}$. It then follows that

$$\frac{P_+}{4g} = 1 + \frac{\mathtt{p}}{\mathtt{q}I} = \frac{\mathtt{q}I + \mathtt{p}}{\mathtt{q}I}\,, \tag{3.3.16}$$

so that the α_+ circle bundle at $\theta = \theta_- = 0$ is that associated to $\mathcal{L}^{(\mathtt{q}I+\mathtt{p})/I}$. We also deduce

that

$$\frac{P_+}{P_-} = \frac{\mathtt{q}I + \mathtt{p}}{\mathtt{p}} \ . \tag{3.3.17}$$

With these choices we have thus constructed a regular Lens space $S^3/\mathbb{Z}_{\mathtt{q}}$ fibration over the KE_6, where the positive integer parameter $\mathtt{p}$ determines the twisting. The total space Y_9 of this fibration is simply-connected if we further require $\mathrm{hcf}(\mathtt{p},\mathtt{q}) = 1$. We shall henceforth also assume this condition, but note that more generally the topology is simply a free $\mathbb{Z}_{\mathrm{hcf}(\mathtt{p},\mathtt{q})}$ quotient of the solution with parameters $(\mathtt{p},\mathtt{q})/\mathrm{hcf}(\mathtt{p},\mathtt{q})$, so there is essentially no loss of generality.

Recall that as originally presented in (3.3.2), the space Y_9 is a fibration of a SE_7 over the two-dimensional space parametrized by (θ,ϕ). It will be important to understand this fibration structure also. In particular, the Reeb vector ∂_ψ that rotates the SE_7 $U(1)$ fibre over the KE_6 may be computed to be

$$\partial_\psi = \partial_{\alpha_+} + \frac{\mathtt{p}}{I}\cdot\frac{1}{\mathtt{q}}\left(\partial_{\alpha_+} - \partial_{\alpha_-}\right) \ . \tag{3.3.18}$$

Note that moving 2π along the orbits of both ∂_{α_+} and $\frac{1}{\mathtt{q}}\left(\partial_{\alpha_+} - \partial_{\alpha_-}\right)$ returns to the same point on the Lens space fibre, in the latter case precisely because we took a $\mathbb{Z}_{\mathtt{q}}$ quotient of S^3 along the $U(1)$ generated by $\partial_{\alpha_+} - \partial_{\alpha_-}$. To determine the period of ψ it is useful to rewrite (3.3.18) as

$$\frac{I}{k}\partial_\psi = \frac{I}{k}\partial_+ + \frac{\mathtt{p}}{k}\cdot\frac{1}{\mathtt{q}}\left(\partial_+ - \partial_-\right) \ , \tag{3.3.19}$$

where we have defined

$$k = \mathrm{hcf}(I,\mathtt{p}) \ . \tag{3.3.20}$$

This ensures that moving 2π along the orbit of the vector field on the right hand side of (3.3.19) closes, and moreover this is the minimal period. But this shows that on the Lens

space ψ has period $2\pi I/k$,

$$\Delta\psi = \frac{2\pi I}{k}\,, \tag{3.3.21}$$

precisely as in (3.1.5), where specifically k is fixed via (3.3.20).

Recalling that

$$\psi = \alpha_+ + \alpha_-\,, \qquad \phi = -\frac{1}{P_+}\alpha_+ - \frac{1}{P_-}\alpha_-\,, \tag{3.3.22}$$

and that the torus made up of α_+, α_- has volume $(2\pi)^2/\mathtt{q}$, we deduce from the Jacobian of this transformation that ϕ has period

$$\Delta\phi = \left(\frac{1}{P_-} - \frac{1}{P_+}\right)\frac{2\pi}{\mathtt{q}}\frac{k}{I}\,. \tag{3.3.23}$$

From the discussion below (3.2.6) and specifically (3.2.9), we see that (θ,ϕ) on the base of the ψ fibration at fixed point on the KE_6 will be a spindle/weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$, where we may identify

$$n_- = \frac{\mathtt{q}I + \mathtt{p}}{k}\,, \qquad n_+ = \frac{\mathtt{p}}{k}\,. \tag{3.3.24}$$

Notice that these are indeed integers, due to the definition (3.3.20) of k, and moreover since $\mathrm{hcf}(\mathtt{p},\mathtt{q}) = 1$ we also note that $\mathrm{hcf}(n_+,n_-) = 1$.

To complete the viewpoint of the SE_7 as the fibre, we may also look directly at the ψ circle fibration over the weighted projective space. Recalling that ψ has period $2\pi I/k$ and using $A_\phi(\theta_\pm) = \pm g$, we calculate the Chern number of this fibration as

$$\frac{1}{\pi I/2k}\int_{\mathbb{WCP}^1}\frac{1}{2}d(\tilde{A}_\phi \mathrm{d}\phi) = \frac{k^2\mathtt{q}}{\mathtt{p}^2 + \mathtt{pq}I} = \frac{k}{I}\frac{n_- - n_+}{n_+ n_-} = \frac{\mathtt{q}}{n_+ n_-}\,. \tag{3.3.25}$$

Here we have used the fact that

$$\mathtt{q} = \frac{k}{I}(n_- - n_+)\,. \tag{3.3.26}$$

Since $\mathtt{q}$ is an integer, notice that the difference of the weights $n_- - n_+$ is necessarily divisible

by the integer I/k. The orbifold line bundle over $\mathbb{WCP}^1_{[n_-,n_+]}$ with Chern number (3.3.25), denoted $O(\mathfrak{q})$, is discussed in appendix A.1. In that appendix it is shown that the total space of the associated circle bundle is indeed a Lens space $S^3/\mathbb{Z}_{\mathfrak{q}}$, completing the circle of arguments.

Summary: We can summarize the results of this section as follows. The $D = 4$ PD metrics of interest depend on five parameters m, a, e, g, α. We uplift to $D = 11$ using a SE_7 manifold in the regular class, which is a circle bundle over a KE_6 manifold. We then obtain a regular $D = 11$ solution after imposing the following two constraints on the five parameters:

$$\frac{P_+}{4g} = 1 + \frac{\mathfrak{p}}{I\mathfrak{q}}\,, \qquad \frac{P_-}{4g} = \frac{\mathfrak{p}}{I\mathfrak{q}}\,, \tag{3.3.27}$$

where $\mathfrak{q} \in \mathbb{N}$ and $\mathfrak{p} \in \mathbb{N}$, and we take $\mathrm{hcf}(\mathfrak{p}, \mathfrak{q}) = 1$ so that the total space is simply-connected. Here I is the Fano index for the KE_6 associated with the regular SE_7 and $P_\pm$ are given in (3.2.7). Defining

$$k = \mathrm{hcf}(I, \mathfrak{p})\ , \tag{3.3.28}$$

we then choose the periods of ψ and ϕ to be

$$\Delta\psi = \frac{2\pi I}{k}\,, \qquad \Delta\phi = \left(\frac{1}{P_-} - \frac{1}{P_+}\right)\frac{2\pi}{\mathfrak{q}}\frac{k}{I}\,. \tag{3.3.29}$$

The nine-dimensional manifold Y_9 at fixed t, r is then a Lens space $S^3/\mathbb{Z}_{\mathfrak{q}}$ bundle over the KE_6, while the SE_7 at fixed point in $D = 4$ spacetime has fundamental group $\mathbb{Z}_k$, and is the circle bundle over the KE_6 associated to the line bundle $\mathcal{L}^{k/I}$, where $\mathcal{L}$ is the canonical bundle over the KE_6. Moreover, the base space of the SE_7 fibration, at fixed t, r, is parametrized by θ, ϕ of the $D = 4$ metric, which is topologically a spindle, a weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$. Here

$$n_- = \frac{\mathfrak{q}I + \mathfrak{p}}{k}\ , \qquad n_+ = \frac{\mathfrak{p}}{k}\,, \tag{3.3.30}$$

are relatively prime integers, where $\theta = \theta_- = 0$ is an $\mathbb{R}^2/\mathbb{Z}_{n_-}$ orbifold singularity, while

$\theta = \theta_+ = \pi$ is an $\mathbb{R}^2/\mathbb{Z}_{n_+}$ orbifold singularity. The magnetic flux in (3.2.13) through the spindle horizon in the $D = 4$ spacetime is given by the rational number

$$G_{(4)}Q_m = \frac{n_- - n_+}{4n_+n_-}\,. \tag{3.3.31}$$

Conversely, we can begin with a weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$, with arbitrary coprime integers $n_- > n_+$. We then set

$$\mathtt{q} = \frac{n_- - n_+}{I/k}\,, \qquad \mathtt{p} = kn_+\,. \tag{3.3.32}$$

Here we choose the integer

$$k = \frac{I}{\mathrm{hcf}(I, n_- - n_+)}\,. \tag{3.3.33}$$

With this definition of k we have that I/k is an integer that divides $n_- - n_+$, which ensures that $\mathtt{p}$ and $\mathtt{q}$ in (3.3.32) are integer. Moreover, note that we also have $k = \mathrm{hcf}(I, \mathtt{p})$, as in (3.3.28).[8] The above construction then leads to a Lens space fibration over the KE_6.

Since we have imposed two conditions (3.3.27), we are left with a three-parameter family of non-singular, rotating and accelerating black hole solutions with spindle horizon. The three parameters correspond to the three independent physical conserved quantities, namely mass, electric charge Q_e and angular momentum J. The above conditions are consistent[9] with preservation of supersymmetry as we discuss in the next section. The entropy of the black holes is given by (3.2.12), after using (3.1.9). In particular, from (3.2.7), the conditions (3.3.27) imply

$$\frac{m\alpha}{g} = 1\,. \tag{3.3.34}$$

Finally, we further illustrate with a concrete example. We take $KE_6 = \mathbb{CP}^3$ and there are then three cases. Firstly, we have an S^7 fibration over $\mathbb{WCP}^1_{[n_-,n_+]}$ for $\mathtt{p} = 1, 3, 5, \ldots$ and relatively prime $\mathtt{q}$, with $n_+ = \mathtt{p}$ and $n_- = 4\mathtt{q} + \mathtt{p}$. Secondly, we have an $S^7/\mathbb{Z}_2$ fibration for

[8] To see this, write $\mathtt{f} = \mathrm{hcf}(I, n_- - n_+)$, so that $k = I/\mathtt{f}$. Then compute $\mathrm{hcf}(I, \mathtt{p}) = \mathrm{hcf}\left((I/\mathtt{f})\mathtt{f}, (I/\mathtt{f})n_+\right) = (I/\mathtt{f})\mathrm{hcf}(\mathtt{f}, n_+) = I/\mathtt{f} = k$, where in the penultimate step we have used $\mathrm{hcf}(n_- - n_+, n_+) = \mathrm{hcf}(n_-, n_+) = 1$.

[9] As we noted just below (3.3.14), if one is just interested in a purely bosonic solution, then one can relax the conditions a little and still maintain regularity.

$\mathtt{p} = 2, 6, 10, \ldots$ and relatively prime $\mathtt{q}$, with $n_+ = \mathtt{p}/2$ and $n_- = 2\mathtt{q}+\mathtt{p}/2$. Finally, we have an $S^7/\mathbb{Z}_4$ fibration for $\mathtt{p} = 4, 8, 12, \ldots$ and relatively prime $\mathtt{q}$, with $n_+ = \mathtt{p}/4$ and $n_- = \mathtt{q}+\mathtt{p}/4$.

3.3.2 Flux quantization

We can also quantize the flux in the $D = 11$ solution (3.1.2). There is no quantization condition on the four-form G as there are no non-trivial four-cycles. We therefore consider the dual seven-form $*_{11}G$ as given in (3.1.3). We have already seen in (3.1.8) that the flux through the SE_7 fibre over a point in the $D = 4$ spacetime gives N_{SE}. In this section we present a general analysis for ensuring the fluxes through an appropriate basis of seven-cycles are quantized, by determining the constant L in (3.1.2).

Note first that as for the previous subsection we may restrict to a constant r, t slice, since these directions span $\mathbb{R}^2$ and hence don't contribute to any non-trivial cycles. The resulting nine-manifold Y_9 is a Lens space $S^3/\mathbb{Z}_{\mathtt{q}}$ fibred over the KE_6. This is the same topology as the solutions discussed in appendix D.2 of [140], and indeed later in the chapter we shall see that those solutions are precisely the near horizon limit of the black holes we are discussing, when the rotation parameter is set to zero.

Setting $\theta = \theta_\pm$ gives rise to two seven-cycles that we call $D_\pm$.[10] There are also seven-cycles D_a that arise as the Lens space fibred over four-cycles $\Sigma_a \in H_4(KE_6, \mathbb{Z})$, where by definition these form a basis for the free part of the latter homology group. We may then write

$$c_1 = c_1(KE_6) = I s_a \Sigma_a \ , \tag{3.3.35}$$

where recall that I is the Fano index and the s_a are then coprime integers, and we have identified $H^2(KE_6, \mathbb{Z}) \cong H_4(KE_6, \mathbb{Z})$ using Poincaré duality. As discussed in [140], we then have the homology relation

$$D_- = D_+ - I s_a D_a \ . \tag{3.3.36}$$

[10] These were called $\tilde{D}_0$ and D_0 in [140], respectively.

Writing

$$N(D) \equiv \frac{1}{(2\pi\ell_p)^6}\int_D *_{11}G\,, \tag{3.3.37}$$

as the flux through the seven-cycle D, from (3.1.3) we compute

$$N(D_\pm) = \frac{L^6}{(2\pi\ell_p)^6}6\mathrm{vol}(KE_6)\int_{S^1_{\theta=\theta_\pm}}\eta\ , \tag{3.3.38}$$

where recall that η is the contact one-form of the Sasaki-Einstein manifold. A short computation shows that

$$\eta\,|_{\theta=\theta_\pm} = \frac{1}{4}d\psi + \frac{1}{2}\tilde{A}_\phi(\theta_\pm)d\phi = \mp\frac{I\mathtt{q}}{4kn_\pm}d\alpha_\mp\ , \tag{3.3.39}$$

where $\alpha_\pm$ are the coordinates introduced in the previous subsection in (3.3.6), and recall that $n_\pm$ are defined in terms of $\mathtt{q}$, $\mathtt{p}$, I and k via (3.3.30). Since $\alpha_\pm$ have period $2\pi/\mathtt{q}$ through their respective circles $S^1 \subset S^3/\mathbb{Z}_\mathtt{q}$, we deduce that (choosing orientations to give a positive flux)

$$N(D_\pm) = \frac{L^6}{(2\pi\ell_p)^6}\frac{IM\pi^4}{128kn_\pm}\ , \tag{3.3.40}$$

where we used (3.1.7). Similarly, we compute

$$N(D_a) = \frac{L^6}{(2\pi\ell_p)^6}\frac{1}{4}g\Delta\phi\Delta\psi\frac{1}{2!}\int_{\Sigma_a}J^2\ . \tag{3.3.41}$$

Using

$$\frac{1}{2!}\int_{\Sigma_a}J^2 = \frac{\pi^2}{2^5}I^2n_a\ , \tag{3.3.42}$$

where $n_a \equiv \int_{\Sigma_a}(c_1/I)^2$ are coprime integers, and inserting the periods $\Delta\phi = 2\pi/(n_+P_+)$, $\Delta\psi = 2\pi I/k$ from the previous section, we find

$$N(D_a) = \frac{L^6}{(2\pi\ell_p)^6}\frac{I^4\mathtt{q}\pi^4}{128k^2n_+n_-}n_a\ . \tag{3.3.43}$$

Using $s_a n_a = M/I^3$, one can check that the fluxes (3.3.40), (3.3.43) satisfy the homology relation (3.3.36).

Since the cycles we have introduced form a basis of seven-cycles, we can now introduce a minimal flux number, that we call N, such that all fluxes are integer multiples of N. Specifically, this fixes L via[11]

$$\frac{L^6}{(2\pi\ell_p)^6} = \frac{128k^2 n_+ n_-}{I^4 h \pi^4} N \ , \tag{3.3.44}$$

where we have introduced $h = \text{hcf}(M/I^3, \mathtt{q})$, where recall that $\mathtt{q} = (k/I)(n_- - n_+)$ is an integer. We then find

$$N(D_\pm) = \frac{M}{I^3 h} k n_\mp N \ , \qquad N(D_a) = \frac{\mathtt{q}}{h} n_a N \ , \tag{3.3.45}$$

where the factors are all integers. Moreover, the expression for N_{SE} given in (3.1.8) can then be written as

$$N_{SE} = n_- N(D_-) = n_+ N(D_+) = \frac{M}{I^3 h} k n_+ n_- N \ . \tag{3.3.46}$$

Notice that this is indeed an integer.

We have thus shown that the $D = 4$ PD black hole metrics of interest uplift to smooth $D = 11$ solutions with properly quantized flux, provided that we impose (3.3.27)–(3.3.30) and fix L via (3.3.44).

3.4 Supersymmetric and extremal limits

In this section we analyse the additional conditions for supersymmetry, as well as the conditions required to have an extremal black hole horizon with vanishing surface gravity. In general, the extremality condition is not implied by supersymmetry. We also derive the black hole entropy formulae (3.0.1) for the most general case of the solution.

By definition the supersymmetric (BPS) limit occurs when the solutions admit solutions

[11] Note that this is consistent with (D.18) of [140] after taking into account a difference of 32/3 in the L^2 between here and there, as a result of this factor in (A.4.4).

to the $D = 4$ Killing spinor equation of minimal gauged supergravity, again given by

$$\nabla_\mu \epsilon = \left(\mathrm{i} A_\mu - \tfrac{1}{2}\gamma_\mu - \tfrac{\mathrm{i}}{4} F_{\alpha\beta}\, \gamma^{\alpha\beta}\, \gamma_\mu \right) \epsilon \,, \tag{3.4.1}$$

where ϵ is a $D = 4$ Dirac spinor (see appendix A.6). The conditions for the PD solutions to admit Killing spinors were determined in [123]. Our primary interest in this chapter is when $\alpha \neq 0$, and we will continue with this, but for reference, in appendix A.2 we briefly discuss the PD black holes when $\alpha = 0$ where the supersymmetry analysis is different. The non-rotating solutions with $a = 0$ are discussed in more detail in section 3.5.1.

By examining the integrability conditions for (3.4.1), as in [123, 131] we find that supersymmetry implies that the five parameters are constrained by the following two conditions

$$\begin{aligned} 0 &= 2aeg\alpha + g^4\alpha^4 + g^2(-1 + \alpha^2 + (a^2 + 2e^2)\alpha^4) + e^2\alpha^2(1 + e^2\alpha^2 + a^2(-1 + \alpha^2)) \,, \\ 0 &= 2a^2g^2\alpha + 2e^4\alpha^3 + 2g^4\alpha^3 - 2aeg(-1 + a^2\alpha^2) - 2e^2\alpha(a^2 - 2g^2\alpha^2) \\ &\quad - \alpha(m + a^2 m\alpha^2)^2 - (e^2 + g^2)\alpha(1 + a^2\alpha^2)(-1 + e^2\alpha^2 + g^2\alpha^2 + a^2(-1 + \alpha^2)) \,. \end{aligned} \tag{3.4.2}$$

With $\alpha \neq 0$, and $m > 0$ we must have $g > 0$. With the signs of the parameters as in (3.2.4), we can solve these conditions to obtain

$$\begin{aligned} m &= \frac{g}{\alpha} \,, \\ a &= \frac{-eg\alpha + (e^2 + g^2)\ \alpha^2 \sqrt{1 + (-1 + e^2)\alpha^2 - (e^2 + g^2)\ \alpha^4}}{g^2\alpha^4 + e^2\alpha^2(-1 + \alpha^2)} \,. \end{aligned} \tag{3.4.3}$$

Recall that we imposed the first of these two conditions in our construction of regular uplifted $D = 11$ solutions (it is implied by (3.3.27)).

We now consider the additional conditions imposed by extremality when $r = r_+$ becomes a double root of $Q(r)$. Assuming that both equations in (3.4.2) are satisfied we find

$$a^3\alpha^2 eg^3 + a^2\alpha g^2(e - g)(e + g) - aeg^3 + \alpha^3 e^2 \left(e^2 + g^2\right)^2 = 0 \,. \tag{3.4.4}$$

In principle, one could solve these three constraints in terms of two independent parameters, and then solve (3.2.8) to find the relation between these and $n_\pm$. However, this is a little cumbersome to do in practice, and it is clearer to keep all the parameters in our expressions,

where it is then understood that they must solve the constraints given by (3.4.2) and (3.4.4). For example, a simple expression for the horizon radius of the supersymmetric and extremal black hole is given by

$$r_+ = \sqrt{\frac{a\,(m - a\,e)}{e + a\,m\,\alpha^2}}\,, \tag{3.4.5}$$

which would not be so simple if one were to use the explicit solutions of the constraints above. However, for the special case when we set the rotation parameter $a = 0$, we will find simple and explicit expressions as we discuss in section 3.5.1. For supersymmetric and extremal black holes with $a = 0$ we also have $e = 0$, and the expression (3.4.5) doesn't apply directly.

Notice that substituting (3.4.5) into the general black hole entropy formula (3.2.12) immediately gives

$$S_{BH} = \frac{am}{2G_{(4)}e}\Delta\phi\,, \tag{3.4.6}$$

as long as the parameter $e \neq 0$. Further using the expressions for the black hole electric charge Q_e in (3.2.14) and angular momentum J in (3.2.17) then leads to

$$S_{BH} = \frac{J}{Q_e}\frac{\pi}{G_{(4)}}\,, \tag{3.4.7}$$

which is the first expression in (3.0.1). This formula for the entropy holds for the subfamily of supersymmetric extremal Kerr-Newman black holes, as discussed in appendix A.2 (see equation (A.2.12)). We have shown that, remarkably, exactly the same formula holds also when we turn on acceleration. On the other hand, as we discuss in section 3.5, for the supersymmetric extremal black holes with $J = Q_e = 0$, or equivalently $a = e = 0$, neither (3.4.6) nor (3.4.7) apply directly, although we shall see later in section 3.5 that the second expression in (3.0.1) is valid in this limit.

3.4.1 Near horizon limit: $\mathbf{AdS}_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$

We now elucidate the near horizon limit of these supersymmetric extremal black holes. The result is a new class of supersymmetric $\mathrm{AdS}_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$ solutions of $D = 4$ gauged supergravity. These uplift to regular $\mathrm{AdS}_2 \times Y_9$ solutions, which generalize those of [126] by

an extra rotation parameter.

A convenient way to find the near horizon solution is to implement the following coordinate transformation,

$$r \to r_+ + \lambda\, s\, \rho\,, \quad t \to \lambda^{-1}\, s\, \tau\,, \quad \phi \to \phi' + \lambda^{-1}\, s\, W \frac{\Delta\phi}{2\pi}\, \tau\,, \tag{3.4.8}$$

where s is a constant, and then take the $\lambda \to 0$ limit. Here W is given by

$$W = \frac{a}{r_+^2 + a^2} \frac{2\pi}{\Delta\phi}\,, \tag{3.4.9}$$

with $\partial_t + W \frac{\Delta\phi}{2\pi} \partial_\phi = \partial_t + W \partial_\varphi$ a null generator of the horizon.

Some details of the limiting procedure are given in appendix A.3. After carrying out various coordinate and gauge transformations, as well as redefining the parameters, we eventually end up with the following class of $\mathrm{AdS}_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$ solutions:

$$\begin{aligned} \mathrm{d}s^2 &= \tfrac{1}{4}(y^2 + \mathtt{j}^2) \left(-\rho^2\, d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + \frac{y^2 + \mathtt{j}^2}{q(y)} dy^2 + \frac{q(y)}{4(y^2 + \mathtt{j}^2)} (dz + \mathtt{j}\, \rho\, d\tau)^2\,, \\ A &= h(y)\, (dz + \mathtt{j}\, \rho\, d\tau)\,, \end{aligned} \tag{3.4.10}$$

where we have defined

$$\begin{aligned} q(y) &= \left(y^2 + \mathtt{j}^2\right)^2 - 4\left(1 - \mathtt{j}^2\right) y^2 + 4\mathtt{a}\sqrt{1 - \mathtt{j}^2}\, y - \mathtt{a}^2\,, \\ h(y) &= \frac{\sqrt{1 - \mathtt{j}^2}}{2} - \frac{1}{2(y^2 + \mathtt{j}^2)} (\mathtt{a} y + 2\mathtt{j}^2 \sqrt{1 - \mathtt{j}^2})\,. \end{aligned} \tag{3.4.11}$$

The solutions depend on two free parameters $\mathtt{j}$, $\mathtt{a}$, which are functions of the original m, α, e, g, a, given implicitly in appendix A.3. Indeed, it is remarkable how simple the solution is in the above parametrization. The parameter $\mathtt{j}$ is a rotation parameter and $\mathtt{j} = 0$ is the non-rotating limit. In fact if we set $\mathtt{j} = 0$ then we precisely recover the $\mathrm{AdS}_2 \times Y_9$ solutions of [126], after dimensional reduction on a SE_7, as we explain in appendix A.4.

Continuing[12] with $\mathtt{j} \geq 0$, we need $\mathtt{j} \in [0, 1]$ in order to get a real solution. We next analyse the roots of $q(y)$, which are given by $y = \sqrt{1 - \mathtt{j}^2} \pm \sqrt{1 - \mathtt{a} - 2\mathtt{j}^2}$ and $y = -\sqrt{1 - \mathtt{j}^2} \pm \sqrt{1 + \mathtt{a} - 2\mathtt{j}^2}$. Given that $q(y)$ is a quartic in y with positive coefficient of y^4, in order for

[12] Note that we can change the sign of $\mathtt{j}$ by changing the sign of τ, z and also the gauge field A.

$q(y) \geq 0$ we need to choose y to lie in between the middle two roots of the quartic, with all roots real, which fixes $y \in [y_2, y_3]$ with

$$y_2 = -\sqrt{1-\mathtt{j}^2} + \sqrt{1+\mathtt{a}-2\mathtt{j}^2}\,, \qquad y_3 = \sqrt{1-\mathtt{j}^2} - \sqrt{1-\mathtt{a}-2\mathtt{j}^2}\,. \tag{3.4.12}$$

For these to be real we need to take

$$\mathtt{j} \in [0, \tfrac{1}{\sqrt{2}})\,. \tag{3.4.13}$$

By analysing how the metric behaves at the roots, we demand that the (y,z) part of the metric becomes $\mathbb{WCP}^1_{[n_-,n_+]}$, which fixes

$$\sqrt{1+\mathtt{a}-2\mathtt{j}^2}\cdot\Delta z = \frac{2\pi}{n_+}\,, \qquad \sqrt{1-\mathtt{a}-2\mathtt{j}^2}\cdot\Delta z = \frac{2\pi}{n_-}\,, \tag{3.4.14}$$

where we have identified $y = y_2, y_3$ with θ_+, θ_- of previous sections, respectively, which has solution

$$\mathtt{a} = \frac{(1-2\mathtt{j}^2)(n_-^2-n_+^2)}{n_+^2+n_-^2}\,, \qquad \Delta z = \frac{\sqrt{2}\sqrt{n_+^2+n_-^2}}{n_+ n_- \sqrt{1-2\mathtt{j}^2}}\pi\,. \tag{3.4.15}$$

We may now compute the magnetic flux (3.2.13) of the gauge field and find

$$G_{(4)}Q_m = \frac{1}{4\pi}\int_\Sigma dA = \frac{n_- - n_+}{4n_+n_-}\,, \tag{3.4.16}$$

where $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ is the spindle horizon, precisely as we had for the general PD black holes. In particular notice this result is independent of the continuous rotation parameter $\mathtt{j}$. We can also derive a very useful expression for the electric charge Q_e (3.2.14) by calculating it directly in the near horizon solution. Doing so we find

$$G_{(4)}Q_e = \frac{1}{4\pi}\int_\Sigma *F = \mathtt{j}\frac{\Delta z}{4\pi}\,. \tag{3.4.17}$$

Given the expression (3.4.15) for Δz, we may now solve for $\mathtt{j}$ in terms of the physical black

hole parameter Q_e:

$$\mathtt{j} = \frac{2\sqrt{2}n_-n_+(G_{(4)}Q_e)}{\sqrt{16n_-^2n_+^2(G_{(4)}Q_e)^2+n_-^2+n_+^2}}\,. \tag{3.4.18}$$

The area of the horizon is then: Area $= \frac{1}{2}(y_3-y_2)\Delta z$. Substituting for $\mathtt{j}$ in terms of Q_e using (3.4.18), we find the entropy is

$$S_{BH} = \frac{1}{4G_{(4)}}\text{Area} = \frac{\sqrt{2}\sqrt{8n_-^2n_+^2(G_{(4)}Q_e)^2+n_-^2+n_+^2}-(n_-+n_+)}{n_-n_+}\frac{\pi}{4G_{(4)}}\,. \tag{3.4.19}$$

This is precisely the second expression in (3.0.1). Notice that setting $\mathtt{j}=0$ is equivalent to $Q_e=0$, which gives the non-rotating limit. The expression (3.4.19) with $Q_e=0$ gives the entropy of the non-rotating but accelerating extremal supersymmetric black holes, studied in more detail in section 3.5. We also note that the angular momentum J and electric charge Q_e are related by

$$J = Q_e\frac{\sqrt{2}\sqrt{8n_-^2n_+^2(G_{(4)}Q_e)^2+n_-^2+n_+^2}-(n_-+n_+)}{4n_-n_+}\,, \tag{3.4.20}$$

for these extremal solutions. Formally setting $n_-=n_+=1$ into the relation (3.4.20) gives the corresponding relation for the supersymmetric extremal Kerr-Newman-AdS black holes, discussed in appendix A.2 (see equation (A.2.11)).

We may also rewrite (3.4.19) using equations (3.1.9), (3.3.24), and (3.3.46), which gives

$$S_{BH} = \frac{2^{7/2}\pi M\sqrt{\mathtt{p}(I\mathtt{q}+\mathtt{p})}\left[\sqrt{\frac{(1-\mathtt{j}^2)}{(1-2\mathtt{j}^2)}}\sqrt{(I\mathtt{q}+2\mathtt{p})^2+I^2\mathtt{q}^2}-(I\mathtt{q}+2\mathtt{p})\right]}{3I^5h^{3/2}}N^{3/2}\,. \tag{3.4.21}$$

Notice that all quantities appearing, except for $\mathtt{j}$, are integers. We also note that if we set $\mathtt{j}=0$ then we precisely recover the expression for the entropy of the $\text{AdS}_2\times Y_9$ solutions as given in (D.21) of [140].

We note that we can express the black hole entropy in yet another way, namely as

$$S_{BH} = \left(\frac{J_{\text{AdS}_2}}{Q_e}-\frac{1}{4}\chi(\Sigma)\right)\frac{\pi}{G_{(4)}}\,. \tag{3.4.22}$$

Here $\chi(\Sigma)$ is the Euler number of the spindle horizon $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ given by

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R_2 \,\mathrm{vol}_2 = \frac{n_- + n_+}{n_+ n_-}\,, \tag{3.4.23}$$

where R_2 denotes the Ricci scalar of the spindle, and J_{AdS_2} is the angular momentum that is defined naturally for the near horizon AdS_2 solutions described in this subsection. Specifically, J_{AdS_2} is invariant under the AdS_2 symmetries. We refer the reader to appendix A.5 for further details, as well as a derivation of the formula we gave in (3.0.2):

$$J_{\mathrm{AdS}_2} - J = \frac{Q_e}{4}\chi(\Sigma)\,. \tag{3.4.24}$$

This formula as well as (3.4.22) are also valid for non-accelerating Kerr-Newman-AdS black holes upon setting $n_- = n_+ = 1$.

We can also express the entropy in another form[13], by replacing the orbifold parameters $n_\pm$ with the magnetic flux Q_m, given in (3.4.16), and the Euler number (3.4.23):

$$S_{BH} = \frac{\pi}{4G_{(4)}}\left(\sqrt{\chi^2 + 16\left[(G_{(4)}Q_e)^2 + (G_{(4)}Q_m)^2\right]} - \chi\right)\,. \tag{3.4.25}$$

Correspondingly, this implies that the near horizon angular momentum can be written in the form

$$J_{\mathrm{AdS}_2} = \frac{Q_e}{4}\sqrt{\chi^2 + 16\left[(G_{(4)}Q_e)^2 + (G_{(4)}Q_m)^2\right]}\,. \tag{3.4.26}$$

Finally, we point out that the local metric appearing in (3.4.10) was also used in [141] in a completely different context of constructing supersymmetric wormholes in AdS_4. To do this, the authors used ranges of the parameters and the coordinates so that, in particular, $q(y) > 0$, in contrast to what we have done here.

[13] It is interesting to wonder if this formula can also be used for supersymmetric black holes with no acceleration, electric charge or rotation. The answer is no. However, we note that for the so called universal twist black holes, with horizon consisting of Riemman surface with genus $g > 1$, one obtains the correct entropy formula (e.g as in [68]), after setting $Q_e = 0$ and formally taking $G_{(4)}Q_m \to \mathrm{i}\chi/4$ with $\chi = 2(1-g)$.

3.4.2 Killing spinors for $\mathrm{AdS}_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$

We now construct the $D = 4$ Killing spinors associated with the rotating, magnetically charged $\mathrm{AdS}_2 \times \mathbb{WCP}^1_{[n_-,n_+]}$ solutions given in (3.4.10) with (3.4.15) and $\mathtt{j} \in [0, \frac{1}{\sqrt{2}})$. The fact that these solutions describe M2-branes wrapped on a surface $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, with a magnetic flux (3.4.16) through the surface, looks similar to a topological twist. However, in the latter case one instead needs the flux to be proportional to the Euler number $\chi(\Sigma)$ of the spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ given in (3.4.23). The flux (3.4.16) instead leads to spinors that are sections of non-trivial line bundles over $\mathbb{WCP}^1_{[n_-,n_+]}$, which we shall describe explicitly, rather than the constant spinor solutions one obtains for the topological twist.

We first introduce the following orthonormal frame for the near horizon metric (3.4.10):

$$\begin{aligned} e^0 &= \tfrac{1}{2}\sqrt{y^2+\mathtt{j}^2}\,\rho\,d\tau\,, \qquad & e^1 &= \tfrac{1}{2}\sqrt{y^2+\mathtt{j}^2}\,\frac{d\rho}{\rho}\,,\\ e^2 &= \sqrt{\frac{y^2+\mathtt{j}^2}{q(y)}}\,dy\,, & e^3 &= \tfrac{1}{2}\sqrt{\frac{q(y)}{y^2+\mathtt{j}^2}}\,(dz+\mathtt{j}\,\rho\,d\tau)\,. \end{aligned} \tag{3.4.27}$$

For this frame, we then take the four-dimensional gamma matrices[14] to be

$$\begin{aligned} &\gamma_a = \beta_a \otimes 1_2, && a = 0,1\,,\\ &\gamma_2 = \beta_3 \otimes \sigma^1\,, \quad \gamma_3 = \beta_3 \otimes \sigma^2\,, \end{aligned} \tag{3.4.28}$$

where the two-dimensional gamma-matrices β_a are defined by

$$\beta_0 = \mathrm{i}\,\sigma^2\,, \quad \beta_1 = \sigma^1\,, \quad \beta_3 \equiv \beta_0\,\beta_1 = \sigma^3\,, \tag{3.4.29}$$

and the σ^i are Pauli matrices.

We next recall that the Killing spinor equation for AdS_2 is

$$\nabla_a \theta = \frac{\mathrm{i}}{2} n\,\beta_a\,\beta_3\,\theta\,, \tag{3.4.30}$$

with $n = \pm 1$. This is solved by Majorana spinors that can be decomposed as $\theta_{1,2} = \theta^{(+)}_{1,2} + \theta^{(-)}_{1,2}$,

[14] Explicitly, $\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}$, $\gamma_3 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$.

with the Majorana-Weyl spinors $\theta^{(\pm)}_{1,2}$ of chirality $\beta_3\theta^{(\pm)}_{1,2} = \pm\theta^{(\pm)}_{1,2}$, given by

$$\theta^{(+)}_1 = \begin{pmatrix} \sqrt{\rho} \\ 0 \end{pmatrix}, \qquad \theta^{(-)}_1 = \begin{pmatrix} 0 \\ \mathrm{i}\, n\sqrt{\rho} \end{pmatrix},$$
$$\theta^{(+)}_2 = \begin{pmatrix} \sqrt{\rho}\tau - \frac{1}{\sqrt{\rho}} \\ 0 \end{pmatrix}, \quad \theta^{(-)}_2 = \begin{pmatrix} 0 \\ \mathrm{i}\, n\left(\sqrt{\rho}\tau + \frac{1}{\sqrt{\rho}}\right) \end{pmatrix}. \tag{3.4.31}$$

After a lengthy calculation we find that the $D = 4$ Killing spinors for the near horizon limit of the supersymmetric, extremal PD black hole, satisfying (3.4.1), can be expressed in the remarkably simple form

$$\epsilon_1 = \theta^{(+)}_1 \otimes \chi_1 + \theta^{(-)}_1 \otimes \chi_2\,,$$
$$\epsilon_2 = \theta^{(+)}_2 \otimes \chi_1 + \theta^{(-)}_2 \otimes \chi_2\,, \tag{3.4.32}$$

where $\chi_{1,2}$ are two two-dimensional spinors, given by

$$\chi_1 = \begin{pmatrix} \frac{\mathcal{Y}_1^{1/2}(y)}{\sqrt{y-\mathrm{i}\,\mathtt{j}}} \\ -\frac{\mathcal{Y}_2^{1/2}(y)}{\sqrt{y+\mathrm{i}\,\mathtt{j}}} \end{pmatrix}, \quad \chi_2 = -n\,\mathrm{e}^{\mathrm{i}\,\xi} \begin{pmatrix} \frac{\mathcal{Y}_1^{1/2}(y)}{\sqrt{y+\mathrm{i}\,\mathtt{j}}} \\ \frac{\mathcal{Y}_2^{1/2}(y)}{\sqrt{y-\mathrm{i}\,\mathtt{j}}} \end{pmatrix}.$$

Here

$$\mathcal{Y}_1(y) = y^2 - 2\sqrt{1-\mathtt{j}^2}y + \mathtt{j}^2 + \mathtt{a}\,,$$
$$\mathcal{Y}_2(y) = y^2 + 2\sqrt{1-\mathtt{j}^2}y + \mathtt{j}^2 - \mathtt{a}\,, \tag{3.4.33}$$

which satisfy $\mathcal{Y}_1(y)\,\mathcal{Y}_2(y) = q(y)$, and the phase ξ appearing in the Killing spinor is given by

$$\xi = \arccos \mathtt{j}\,. \tag{3.4.34}$$

Let us look more carefully at the global structure of the Killing spinors ϵ_i in (3.4.32). The spinors $\theta^{(\pm)}_i$ are simply the standard Killing spinors on AdS_2, where $i = 1, 2$, so our focus will be on the two-dimensional spinors χ_i (3.4.33) on $\mathbb{WCP}^1_{[n_-,n_+]}$. Note first that the two components of χ_i have chiralities ± 1 under σ_3, which is the two-dimensional chirality

operator on $\mathbb{WCP}^1_{[n_-,n_+]}$. Thus we can write $\chi_i = \chi_i^{(+)} + \chi_i^{(-)}$, $i = 1, 2$, where

$$\chi_1^{(+)} = \begin{pmatrix} \frac{\mathcal{Y}_1^{1/2}(y)}{\sqrt{y-\mathrm{i}\,\mathfrak{j}}} \\ 0 \end{pmatrix}, \qquad \chi_1^{(-)} = \begin{pmatrix} 0 \\ -\frac{\mathcal{Y}_2^{1/2}(y)}{\sqrt{y+\mathrm{i}\,\mathfrak{j}}} \end{pmatrix},$$
$$\chi_2^{(+)} = -n\,\mathrm{e}^{\mathrm{i}\,\xi} \begin{pmatrix} \frac{\mathcal{Y}_1^{1/2}(y)}{\sqrt{y+\mathrm{i}\,\mathfrak{j}}} \\ 0 \end{pmatrix}, \qquad \chi_2^{(-)} = -n\,\mathrm{e}^{\mathrm{i}\,\xi} \begin{pmatrix} 0 \\ \frac{\mathcal{Y}_2^{1/2}(y)}{\sqrt{y-\mathrm{i}\,\mathfrak{j}}} \end{pmatrix}. \tag{3.4.35}$$

We also note that $\mathcal{Y}_1(y_3) = 0 = \mathcal{Y}_2(y_2)$, where y_2, y_3 are the roots (3.4.12), so that the positive chirality components $\chi_i^{(+)}$ are zero at $y = y_3$, while the negative chirality components $\chi_i^{(-)}$ are zero at $y = y_2$.

Both the frame (3.4.27) and the R-symmetry $U(1)$ gauge field A in (3.4.10) are singular at the roots $y = y_2$, y_3. Let us first look at the gauge field. The four-dimensional Killing spinors ϵ_i have charge $+1$ under A, as we see from the Killing spinor equation (3.4.1). A gauge transformation $A \to A + d\gamma$ then leads to a $U(1)$ rotation $\epsilon_i \to \mathrm{e}^{\mathrm{i}\gamma}\epsilon_i$. The magnetic flux of this gauge field through $\mathbb{WCP}^1_{[n_-,n_+]}$ is given by (3.4.16). As explained further in appendix A.1, this identifies $2A$ as a connection on the complex line bundle $O(n_- - n_+)$. When $n_- - n_+$ is not divisible by 2, this is a spin^c gauge field on the weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$. We shall comment on this further below when we describe the spin structure more explicitly.

Note that we may write the gauge field in (3.4.10), restricted to $\mathbb{WCP}^1_{[n_-,n_+]}$, as

$$A\,|_{\mathbb{WCP}^1_{[n_-,n_+]}} = h(y)\frac{\sqrt{n_+^2 + n_-^2}}{\sqrt{2}n_+n_-\sqrt{1-2\mathfrak{j}^2}}d\varphi\,, \tag{3.4.36}$$

where we have defined

$$\varphi = \frac{\sqrt{2}n_+n_-\sqrt{1-2\mathfrak{j}^2}}{\sqrt{n_+^2+n_-^2}}z\,. \tag{3.4.37}$$

Here φ is the same as the coordinate introduced in (3.2.10), and has the canonical period of 2π. However, $d\varphi$ is not defined at the roots $y = y_3$, $y = y_2$, so that the gauge field (3.4.36) is singular at the roots. We may then introduce open sets U_- and U_+ on $\mathbb{WCP}^1_{[n_-,n_+]}$ that cover hemispheres containing the roots $y = y_3$ and $y = y_2$, respectively. Here $y = y_3$ is a

$\mathbb{C}/\mathbb{Z}_{n_-}$ orbifold singularity, while $y = y_2$ is a $\mathbb{C}/\mathbb{Z}_{n_+}$ orbifold singularity. One can then check that to obtain a well-defined connection in each patch, we need to make the local gauge transformations

$$\begin{aligned} U_-: \qquad & A \to A + \frac{1}{2n_-} d\varphi \equiv A_- \,, \\ U_+: \qquad & A \to A + \frac{1}{2n_+} d\varphi \equiv A_+ \,. \end{aligned} \tag{3.4.38}$$

The gauge fields $A_\pm$ are now smooth one-forms in their respective patches $U_\pm$, and on the overlap $U_- \cap U_+$ they are related by

$$A_+ = A_- + \frac{n_- - n_+}{2n_+ n_-} d\varphi \,. \tag{3.4.39}$$

This again identifies the complex line bundle on which $2A$ is a connection as $O(n_- - n_+)$, where by definition the transition function defining this line bundle over $\mathbb{WCP}^1_{[n_-,n_+]}$ is given by the gauge transformation in (3.4.39), cf. (3.4.16). We discuss this further towards the end of this subsection.

Next let us look at the two-dimensional frame $\{e^2, e^3\}$ for $\mathbb{WCP}^1_{[n_-,n_+]}$ in (3.4.27), which is again singular at the roots. Specifically,

$$\begin{aligned} U_-, \text{ near } y = y_3: \qquad & e^2 \sim -d\varrho_- \qquad e^3 \sim \varrho_- \frac{d\varphi}{n_-} \,, \\ U_+, \text{ near } y = y_2: \qquad & e^2 \sim +d\varrho_+ \qquad e^3 \sim \varrho_+ \frac{d\varphi}{n_+} \,, \end{aligned} \tag{3.4.40}$$

where $\varrho_\pm$ is geodesic distance measured from each root, to leading order. Note here that y is increasing as one approaches $y = y_3 > y_2$, while ϱ_- is decreasing, hence the minus sign. We may then introduce a complex coordinate $z_- \equiv -\varrho_- \mathrm{e}^{-\mathrm{i}\varphi/n_-}$ in the patch U_-, which defines a smooth one-form dz_- on the orbifold; that is, dz_- is a smooth one-form on the covering space $\mathbb{C}$ in which φ has period $2\pi n_-$. We may then write $z_- \equiv x_- + \mathrm{i}y_- = -\varrho_- \cos\frac{\varphi}{n_-} + \mathrm{i}\varrho_- \sin\frac{\varphi}{n_-}$, and rotate the frame

$$\begin{pmatrix} e^2 \\ e^3 \end{pmatrix} \to \begin{pmatrix} \cos\frac{\varphi}{n_-} & \sin\frac{\varphi}{n_-} \\ -\sin\frac{\varphi}{n_-} & \cos\frac{\varphi}{n_-} \end{pmatrix} \begin{pmatrix} e^2 \\ e^3 \end{pmatrix} \sim \begin{pmatrix} dx_- \\ dy_- \end{pmatrix} . \tag{3.4.41}$$

This is an $SO(2) \cong U(1)$ rotation of the frame on the patch U_-, which leads to a corresponding spinor $U(1)$ rotation $\exp(\pm \mathrm{i}\varphi/2n_-)$, where the sign is correlated with two-dimensional chirality of the spinor. This follows from exponentiating the spinor representation of the infinitesimal version of the above $SO(2)$ rotation, namely

$$\exp\left(-\frac{1}{2}\sigma_2\sigma_1\varphi/n_-\right) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi/2n_-} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi/2n_-} \end{pmatrix} . \tag{3.4.42}$$

We may then rotate the spinors χ_i, $i = 1, 2$, in the patch U_-, noting there is both a spinor rotation and an R-symmetry rotation (3.4.38):

$$\begin{aligned} U_-, \text{ near } y_3 : \qquad \chi_i = \chi_i^{(+)} + \chi_i^{(-)} &\to \left(\mathrm{e}^{\mathrm{i}\varphi/2n_-}\chi_i^{(+)} + \mathrm{e}^{-\mathrm{i}\varphi/2n_-}\chi_i^{(-)}\right)\mathrm{e}^{\mathrm{i}\varphi/2n_-} \\ &= \mathrm{e}^{\mathrm{i}\varphi/n_-}\chi_i^{(+)} + \chi_i^{(-)} . \end{aligned} \tag{3.4.43}$$

The coordinate φ is not defined at the root $y = y_3$, but on the other hand $\chi_i^{(+)}(y_3) = 0$. Thus the above spinor is smooth and well-defined near to $y = y_3$, in the patch U_-. Note in particular that the spinor rotation and R-symmetry rotation cancel each other for the non-vanishing negative chirality component $\chi_i^{(-)}$.

A similar calculation goes through at the other root $y = y_2$, in the patch U_+. We introduce a coordinate $z_+ \equiv \varrho_+ \mathrm{e}^{\mathrm{i}\varphi/n_+} = x_+ + \mathrm{i}y_+ = \varrho_+ \cos\frac{\varphi}{n_+} + \mathrm{i}\varrho_+ \sin\frac{\varphi}{n_+}$. The rotation is now in the opposite direction

$$\begin{pmatrix} e^2 \\ e^3 \end{pmatrix} \to \begin{pmatrix} \cos\frac{\varphi}{n_+} & -\sin\frac{\varphi}{n_+} \\ \sin\frac{\varphi}{n_+} & \cos\frac{\varphi}{n_+} \end{pmatrix} \begin{pmatrix} e^2 \\ e^3 \end{pmatrix} \sim \begin{pmatrix} dx_+ \\ dy_+ \end{pmatrix} . \tag{3.4.44}$$

The corresponding spinor rotation and R-symmetry rotation is thus

$$\begin{aligned} U_+, \text{ near } y_2 : \qquad \chi_i = \chi_i^{(+)} + \chi_i^{(-)} &\to \left(\mathrm{e}^{-\mathrm{i}\varphi/2n_+}\chi_i^{(+)} + \mathrm{e}^{\mathrm{i}\varphi/2n_+}\chi_i^{(-)}\right)\mathrm{e}^{\mathrm{i}\varphi/2n_+} \\ &= \chi_i^{(+)} + \mathrm{e}^{\mathrm{i}\varphi/n_+}\chi_i^{(-)} . \end{aligned} \tag{3.4.45}$$

Now $\chi_i^{(-)}(y_2) = 0$, and we see the spinor is smooth and well-defined near the root.

The above analysis shows that the two-dimensional spinors χ_i on $\mathbb{WCP}^1_{[n_-,n_+]}$ are smooth

and well-defined, in the appropriate orbifold sense. Notice that the above computations show that the spinor transition function in going from the patch U_- to the patch U_+ is, similarly to (3.4.39), given by

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi/2n_+} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi/2n_+} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi/2n_-} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi/2n_-} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi\frac{n_++n_-}{2n_+n_-}} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi\frac{n_++n_-}{2n_+n_-}} \end{pmatrix} . \quad (3.4.46)$$

Here the original spinor rotation (3.4.42) is inverted, since we begin with the smooth spinor in the patch U_-. This identifies the positive and negative chirality spin bundles $\mathcal{S}^{(\pm)}$ on $\mathbb{WCP}^1_{[n_-,n_+]}$ as $O(\mp(n_+ + n_-)/2)$. Notice these are well-defined as line bundles when $n_+ + n_-$ is divisible by 2, which is the case if and only if $n_+ - n_-$ is divisible by 2, when the gauge field A is a connection on a well-defined line bundle. We may understand this more abstractly as follows. On any oriented two-manifold (or orbifold) the spinor bundles are

$$\mathcal{S}^{(+)} = \mathcal{K}^{1/2} , \qquad \mathcal{S}^{(-)} = \mathcal{K}^{-1/2} = \Lambda^{(0,1)} \otimes \mathcal{K}^{1/2} , \quad (3.4.47)$$

where $\mathcal{K} \cong \Lambda^{(1,0)}$ is the canonical bundle, namely the cotangent bundle, and $\Lambda^{(0,1)}$ denote $(0,1)$-forms with respect to the canonical complex structure. Again, the above computations explicitly show that the cotangent bundle is $\mathcal{K} = O(-(n_+ + n_-))$. Instead, our spinors χ_i are spin^c spinors that are also charged under the gauge field A. Denoting the line bundle on which $2A$ is a connection as $\mathcal{L}$, the spinors $\chi_i^{(\pm)}$ are hence sections of

$$\begin{aligned} \chi_i^{(+)} : \quad & \mathcal{S}^{(+)} \otimes \mathcal{L}^{1/2} = O(-\tfrac{1}{2}(n_+ + n_-)) \otimes O(\tfrac{1}{2}(n_- - n_+)) = O(-n_+) , \\ \chi_i^{(-)} : \quad & \mathcal{S}^{(-)} \otimes \mathcal{L}^{1/2} = O(\tfrac{1}{2}(n_+ + n_-)) \otimes O(\tfrac{1}{2}(n_- - n_+)) = O(n_-) . \end{aligned} \quad (3.4.48)$$

Again, these also follow directly from composing the transition functions we worked out explicitly above. Notice that these chiral spin^c bundles $O(-n_+)$ and $O(n_-)$ are always well-defined as line bundles, irrespective of whether $n_+ \pm n_-$ is divisible by two.

3.4.3 R-symmetry Killing vector

The supersymmetric $\mathrm{AdS}_2 \times Y_9$ solutions we have constructed should have a holographic dual description in terms of a $d = 1$ superconformal quantum mechanics (SCQM). This has an

Abelian R-symmetry, that is realized in the supergravity solution as a canonically defined Killing vector field under which the Killing spinors are charged. As usual, this R-symmetry Killing vector R may be constructed as a bilinear in the Killing spinors as we explain below. For the solutions in this chapter we find

$$R = R_{3d} + 2\sqrt{1-\mathtt{j}^2}\partial_z = R_{3d} + 2\sqrt{2}\frac{n_+n_-\sqrt{8n_-^2n_+^2(G_{(4)}Q_e)^2+n_-^2+n_+^2}}{\sqrt{16n_-^2n_+^2(G_{(4)}Q_e)^2+n_-^2+n_+^2}}\partial_\varphi\,, \tag{3.4.49}$$

where in the second expression we have used (3.4.18), and we have also defined $R_{3d} = 2\partial_\psi$. The latter is precisely the R-symmetry Killing vector for the corresponding $\mathrm{AdS}_4 \times SE_7$ solutions, normalized so that the Killing spinor on the SE_7 has unit charge under R_{3d} (see appendix A.6). This is the geometric counterpart to the superconformal R-symmetry of the dual $d = 3$, $\mathcal{N} = 2$ SCFT. We note that (3.4.49) reduces to (A.4.12) on setting $\mathtt{j} = 0$, and that ∂_φ generates the $U(1)$ isometry of the spindle $\mathbb{WCP}^1_{[n_-,n_+]}$, normalized so that φ has period 2π (see (3.2.10)). We discuss the physical interpretation of (3.4.49) in the discussion section 3.6.

One way to identify the R-symmetry Killing vector is to construct bilinears of the Killing spinors on Y_9, as was essentially done in [126] for the case of $\mathtt{j} = 0$. Instead here we will construct bilinears of the $D = 11$ Killing spinors for the $\mathrm{AdS}_2 \times Y_9$ solution. With the conventions of appendix A.6, we can obtain the $D = 11$ Killing spinors as a tensor product of the $D = 4$ Killing spinors (3.4.32) with the Killing spinor χ on SE_7. The solution preserves four $D = 11$ Majorana spinors which we can package into two complex $D = 11$ spinors via

$$\varepsilon_1 = \epsilon_1 \otimes \chi\,, \quad \varepsilon_2 = \epsilon_2 \otimes \chi\,. \tag{3.4.50}$$

We then obtain the following bilinears

$$\begin{aligned}
&\mathrm{i}\,\bar{\varepsilon}_1\,\Gamma_M\,\varepsilon_1 = P_M\,,\\
&\mathrm{i}\,\bar{\varepsilon}_2\,\Gamma_M\,\varepsilon_2 = -K_M\,,\\
&\mathrm{i}\,\bar{\varepsilon}_1\,\Gamma_M\,\varepsilon_2 = D_M + \tfrac{\mathrm{i}}{2}\,R_M\,,
\end{aligned} \tag{3.4.51}$$

where

$$
\begin{aligned}
P &= \partial_\tau \,, \\
D &= \tau\, \partial_\tau - \rho\, \partial_\rho \,, \\
K &= -(\tau^2 + \rho^{-2})\, \partial_\tau + 2\, \tau\, \rho\, \partial_\rho + 2\, \mathtt{j}\, \rho^{-1}\, \partial_z \,, \\
R &= 2\, \partial_\psi + 2\, \sqrt{1 - \mathtt{j}^2}\, \partial_z \,.
\end{aligned} \tag{3.4.52}
$$

The Killing vectors P, D and K generate the $\mathfrak{sl}(2)$ symmetry algebra of AdS_2:

$$
[D,\, P] = -P \,, \quad [D,\, K] = K \,, \quad [P,\, K] = -2\, D \,, \tag{3.4.53}
$$

and hence we can identify the Killing vector R to be the R-symmetry Killing vector, as claimed above.

3.5 Conformal boundary for non-rotating solutions

An ultimate goal for holography would be to reproduce the black hole entropy (3.0.1) for the extremal supersymmetric solutions via a dual field theory computation. The dual theory lives on the conformal boundary three-manifold of the full black hole solution. In this section we therefore turn to looking at this conformal boundary and, for simplicity, we will now set the rotation parameter $a = 0$. We will see that for the supersymmetric extremal black holes we must have $e = 0$, hence $J = Q_e = 0$, and the near horizon solutions can be obtained by setting $\mathtt{j} = 0$ in the near horizon metric of section 3.4.1.

We shall see that the global black hole geometry of the solutions with $a = 0$ have some interesting features, including an acceleration horizon beyond $r = \infty$. For the supersymmetric and extremal solution this acceleration horizon intersects the conformal boundary, effectively dividing the latter in half. Moreover, we shall find that the Killing spinor on this conformal boundary is given by a topological twist, so that the spinor is constant, but it is a *different* constant spinor on each half of the space.

These exotic features will make a dual field theory calculation more challenging, but in the remaining subsections we show that the features arise as a limit of more well-behaved solutions, still with $a = 0$. In particular in section 3.5.2 we relax the extremality condition,

while preserving supersymmetry. The boundary three-manifold is now a smooth product of the time direction with a spindle, with a single component, and has a smooth boundary Killing spinor. It is therefore natural to perform any field theory localization calculation in this setting. One could then take the extremal limit. However, in the bulk of this solution there is a naked curvature singularity. In section 3.5.3 we discuss a simple way of regulating this feature by also relaxing the requirement of supersymmetry, to obtain completely regular accelerating black holes with a smooth conformal boundary.

3.5.1 Supersymmetric and extremal solutions

In this section we focus on the solutions in (3.2.1) with $a = 0$, which depend on four free parameters m, e, g and α. Explicitly, we then have

$$ds^2 = \frac{1}{H^2}\left[-\frac{Q}{r^2}dt^2 + \frac{r^2}{Q}\,dr^2 + \frac{r^2}{P}d\theta^2 + Pr^2\sin^2\theta d\phi^2\right], \tag{3.5.1}$$

with

$$\begin{aligned} H &= 1-\alpha\, r\cos\theta\,,\\ P &= 1-2\alpha m\cos\theta+\alpha^2(e^2+g^2)\cos^2\theta\,,\\ Q &= (r^2-2mr+e^2+g^2)(1-\alpha^2r^2)+r^4\,. \end{aligned} \tag{3.5.2}$$

The gauge field is given by

$$A = -\frac{e}{r}dt - g\cos\theta d\phi\,. \tag{3.5.3}$$

By directly examining the integrability conditions for the Killing spinor equations when $a = 0$, we find that supersymmetry requires

$$\begin{aligned} 0 &= m^2-(e^2+g^2)(1+(e^2+g^2)\alpha^2)\,,\\ 0 &= m^4-m^2(e^2+g^2)-g^2(e^2+g^2)^2\,. \end{aligned} \tag{3.5.4}$$

In fact we get the same system of equations from (3.4.2) after setting $a = 0$ and assuming

$\alpha \neq 0$, as well as one of e, g non-vanishing. These can be solved to give

$$m^2 = \frac{1-\alpha^2-2\alpha^4e^2+\sqrt{(1-\alpha^2)^2-4\alpha^4e^2}}{2\alpha^6}\,,$$
$$g^2 = \frac{1-\alpha^2-2\alpha^4e^2+\sqrt{(1-\alpha^2)^2-4\alpha^4e^2}}{2\alpha^4}\,, \tag{3.5.5}$$

where we have taken the positive square roots in order to continuously connect with the extremal solution below. Note that (3.5.5) implies that $m = g/\alpha$.

The extremal limit is given by setting $e = 0$. Everything may then be expressed in terms of one parameter, for example α, via

$$e = 0\,, \quad g = \frac{\sqrt{1-\alpha^2}}{\alpha^2}\,, \quad m = \frac{\sqrt{1-\alpha^2}}{\alpha^3}\,. \tag{3.5.6}$$

The black hole horizon radius is given by the largest double root $r_+ > 0$ of $Q(r)$, where

$$r_\pm = \frac{-1 \pm \sqrt{5-4\alpha^2}}{2\alpha\sqrt{1-\alpha^2}}\,. \tag{3.5.7}$$

We require the function $P \geq 0$ and so we should restrict the range of α to be

$$\frac{\sqrt{3}}{2} < \alpha < 1\,, \tag{3.5.8}$$

and we also observe that in this range r_+ is positive and r_- is negative. Recall that regularity of the uplifted solution requires the conditions (3.3.27) are also imposed, and this fixes the parameter α in terms of the integers n_-, n_+, so that there are no remaining free parameters. In particular, we note that in terms of the integers $n_- > n_+$ specifying the orbifold singularities at the poles $\theta = 0$, $\theta = \pi$, we have

$$\alpha = \frac{\sqrt{(3n_- + n_+)(3n_+ + n_-)}}{2(n_- + n_+)}\,. \tag{3.5.9}$$

Thus the lower limit $\sqrt{3}/2$ for α in (3.5.8) is the limit $n_- \to \infty$, holding n_+ fixed, while the upper limit of 1 corresponds[15] to $n_- - n_+ \to 0$.

Next we look in more detail at the global structure of this extremal solution. We take

[15]Note that in this limit we have $\alpha = 1$, $g = m = 0$ and hence, in particular, vanishing gauge field, and the metric is locally that of AdS_4.

$r \geq r_+$, where recall that the conformal boundary is at $\alpha r \cos\theta = 1$. Notice immediately that for $\theta > \pi/2$ this requires r *negative*. Globally r is not a good coordinate, and we instead put $y = 1/r$. The black hole metric (3.5.1) now reads

$$ds^2 = \frac{1}{(y - \alpha\cos\theta)^2}\Big[- Y dt^2 + \frac{dy^2}{Y} + \frac{1}{P} d\theta^2 + P \sin^2\theta d\phi^2 \Big] , \tag{3.5.10}$$

where for the extremal supersymmetric solution we have introduced

$$Y = Y(y) = (1-\alpha^2)(1 - r_- y)^2 (1 - r_+ y)^2 . \tag{3.5.11}$$

For the original r coordinate we have $r > r_+ > 0$, which implies

$$y \leq \frac{1}{r_+} \equiv y_+ , \tag{3.5.12}$$

and we may then continue y past zero to negative values (effectively extending beyond $r = \infty$). The y coordinate decreases as one moves away from the horizon at $y = y_+$, eventually hitting the conformal boundary at $y = \alpha\cos\theta$, with $y - \alpha\cos\theta > 0$ in the interior of the spacetime. However, although $1 - r_+ y > 0$ for $y < y_+$, for negative y one can reach the double root of the metric function Y at $y = 1/r_-$, where we recall that $r_- < 0$ given (3.5.8). One can show that this is an *acceleration horizon*. To emphasize this we write $r_- \equiv r_A$, and then the acceleration horizon is located at

$$y = \frac{1}{r_A} \equiv y_A , \qquad r_A \equiv r_- . \tag{3.5.13}$$

One can ask when the acceleration horizon at $y = y_A$ intersects the conformal boundary. This is determined by the equation

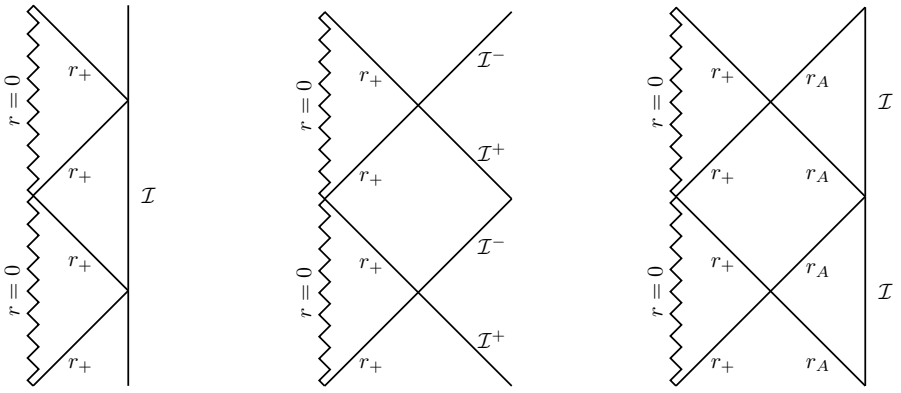
$$y_A = \frac{1}{r_A} = \alpha\cos\theta , \tag{3.5.14}$$

which can be solved to give

$$\theta = \theta_0 \equiv \arccos\left(\frac{1 - \sqrt{5 - 4\alpha^2}}{2\sqrt{1-\alpha^2}} \right) = \arccos \frac{n_- + n_+ - \sqrt{2n_-^2 + 2n_+^2}}{n_- - n_+} . \tag{3.5.15}$$

On the other hand, for given α and fixed $\theta > \theta_0$, note $y_A > \alpha\cos\theta$, which means that as one approaches from the black hole one hits the acceleration horizon before the conformal boundary. On the conformal boundary itself the lower half of the spindle, with $\theta \in (\theta_0, \pi]$, then effectively lies behind the acceleration horizon. Interestingly, the acceleration horizon is also extremal, and there is an asymptotic AdS_2 region as one approaches $y \to y_A$ from above.

An extensive analysis of the causal structure of the AdS C-metrics is presented in [121], for general values of the parameters. The Penrose diagram for the extremal supersymmetric black hole solution is shown in Figure 3.1. In particular we note that the lower half of the spindle boundary, with $\theta > \theta_0$, lies behind the acceleration horizon $r = r_A$.



(a) $\theta < \theta_0$ (b) $\theta = \theta_0$ (c) $\theta > \theta_0$

Figure 3.1: Penrose diagram for the non-rotating supersymmetric extremal black hole, for different slices of constant θ. The black hole horizon is denoted by r_+. The left panel is associated with a conformal boundary, consisting of the product of the time direction with half a spindle with $\theta < \theta_0$. In the right panel one reaches the other half of the spindle with $\theta > \theta_0$ at the conformal boundary after passing through the extremal acceleration horizon, denoted by r_A. For $\theta = \theta_0$, in the middle panel, a null infinity appears.

Next let us look more closely at the conformal boundary itself. Starting from the general non-rotating black hole metric (3.5.1), the conformal boundary is located at $H = 0$. Choosing the conformal factor so that the timelike Killing vector ∂_t has unit norm on the boundary, we find that the general form of the conformal boundary metric is

$$ds^2_{3d} = -dt^2 + ds^2_\Sigma = -dt^2 + \frac{d\theta^2}{P\,(1-\alpha^2\,\sin^2\theta\,P)^2} + \frac{P\,\sin^2\theta}{1-\alpha^2\,\sin^2\theta\,P}\,d\phi^2\,. \tag{3.5.16}$$

That is, the conformal boundary is a static product metric, where the induced metric on a constant time slice is ds^2_Σ.

We have seen that for the extremal supersymmetric black hole the acceleration horizon

intersects the conformal boundary at $\theta = \theta_0$, and in fact the boundary actually splits in half along this slice. Indeed, although $P = P(\theta) > 0$ for all $\theta \in [0, \pi]$, we find that for the extremal solution $1 - \alpha^2 \sin^2\theta P(\theta) \geq 0$, with equality if and only if $\theta = \theta_0$. The metric on Σ in (3.5.16) is then singular for $\theta = \theta_0$. Introducing the new coordinate

$$\rho = \frac{\alpha\sqrt{1+\sqrt{5-4\alpha^2}}}{\sqrt{2}\,(5-4\alpha^2)}\frac{1}{\theta-\theta_0}\,, \tag{3.5.17}$$

we find that near to $\theta = \theta_0$ (which is $\rho = \infty$), the metric on Σ takes the form

$$\mathrm{d}s^2_\Sigma \simeq d\rho^2 + \frac{5-4\alpha^2}{\alpha^4}\rho^2 d\phi^2\,. \tag{3.5.18}$$

Thus each side of $\theta = \theta_0$ opens out into a non-compact asymptotically locally Euclidean end, with each of the poles $\theta = 0$ and $\theta = \pi$ being an infinite distance from $\theta = \theta_0$. The conformal boundary thus effectively has two halves, with the lower half $\theta \in (\theta_0, \pi]$ lying behind the acceleration horizon.

We can now discuss the behaviour of the Killing spinor on the conformal boundary. The bulk Killing spinor equation for minimal $D = 4$, $\mathcal{N} = 2$ gauged supergravity induces the following conformal Killing spinor equation (CKSE) on the conformal boundary [14]:

$$\nabla_a\zeta = \frac{1}{3}\gamma\nabla\zeta\,, \tag{3.5.19}$$

where we have introduced the covariant derivative $\nabla_a = \partial_a + \frac{1}{4}\omega_a^{bc}\gamma_{bc} - \mathrm{i}A_a$. Here $a = 0, 1, 2$ is a tangent space index, and we may take the gamma matrices to be $\gamma_0 = \mathrm{i}\sigma^1$, $\gamma_1 = \sigma^2$, $\gamma_2 = \sigma^3$, in terms of Pauli matrices. To solve this equation we begin by introducing the obvious orthonormal frame

$$e^0 = dt\,, \qquad e^1 = \frac{d\theta}{\sqrt{P}(1-\alpha^2\sin^2\theta P)}\,, \qquad e^2 = \sqrt{\frac{P}{1-\alpha^2\sin^2\theta P}}\sin\theta d\phi\,. \tag{3.5.20}$$

For the extremal solution with $e = 0$ the gauge field is

$$A = -g\cos\theta d\phi = -\frac{\sqrt{1-\alpha^2}}{\alpha^2}\cos\theta d\phi\,. \tag{3.5.21}$$

It is convenient to define the gauge-equivalent gauge field

$$A_0 \equiv A + \frac{1}{2\alpha^2} d\phi \,. \tag{3.5.22}$$

Then remarkably we find that this gauge field is equal to plus or minus $\frac{1}{2}\omega^{12}$, where ω^{12} is the non-zero spin connection component for Σ, with the sign depending on which half of Σ the expressions are compared in:

$$\tfrac{1}{2}\omega^{12} = \begin{cases} -A_0 & \theta \in [0, \theta_0) \,, \\ +A_0 & \theta \in (\theta_0, \pi] \,. \end{cases} \tag{3.5.23}$$

We find that the solution to (3.5.19) is in fact covariantly constant, so $\nabla_a \zeta = 0$, and moreover due to (3.5.23) in the gauge (3.5.22) the solution for ζ is in fact constant:

$$\zeta = \begin{cases} \begin{pmatrix} -\mathrm{i} \\ \\ \mathrm{i} \end{pmatrix} & \theta \in [0, \theta_0) \,, \\ \\ \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} & \theta \in (\theta_0, \pi] \,. \end{cases} \tag{3.5.24}$$

There is thus a topological twist on each half of the conformal boundary, with the gauge field effectively cancelling the spin connection and leading to a constant spinor, but with a discontinuity in the spinor as one moves across the slice $\theta = \theta_0$ that intersects the acceleration horizon of the bulk black hole. Of course we may multiply each spinor in (3.5.24) by any constant, and this will still be a solution. The reason for normalizing the spinors in the way that we have, in particular taking a purely imaginary spinor in $\theta \in [0, \theta_0)$, will be come apparent in the next subsection.

Finally in this subsection we note that it is possible to derive an explicit expression for the entropy of the non-rotating extremal supersymmetric solution directly from (3.2.12). In terms of $n_\pm$ we find

$$g = \frac{2(n_-^2 - n_+^2)}{(3n_- + n_+)(n_- + 3n_+)} \,, \qquad \Delta\phi = \frac{(3n_- + n_+)(n_- + 3n_+)\pi}{4n_- n_+ (n_- + n_+)} \,. \tag{3.5.25}$$

The horizon radii $r = r_\pm$ are given by

$$r_\pm = \frac{2\,(n_- + n_+)\left(\pm\sqrt{2}\sqrt{n_-^2 + n_+^2} - (n_- + n_+)\right)}{(n_- - n_+)\sqrt{(3n_- + n_+)\,(n_- + 3n_+)}}\,. \tag{3.5.26}$$

Using (3.2.12) we then find the entropy is given by

$$\begin{aligned} S_{BH} &= \frac{\sqrt{2}\sqrt{n_-^2 + n_+^2} - (n_- + n_+)}{n_- n_+}\frac{\pi}{4G_{(4)}} \\ &= \frac{2^{7/2}\pi M\sqrt{\mathtt{p}(I\mathtt{q} + \mathtt{p})}\left[\sqrt{(I\mathtt{q} + 2\mathtt{p})^2 + I^2\mathtt{q}^2} - (I\mathtt{q} + 2\mathtt{p})\right]}{3I^5 h^{3/2}} N^{3/2}\,. \end{aligned} \tag{3.5.27}$$

The first expression agrees with (3.4.19) after setting $Q_e = 0$, where recall that (3.4.19) was instead computed using the near horizon metric for the general supersymmetric extremal solution. The second expression agrees with (3.4.21) after correspondingly setting $\mathtt{j} = 0$, and is the same as the expression for the entropy of the $\mathrm{AdS}_2 \times Y_9$ solutions as given in equation (D.21) of [140].

3.5.2 Supersymmetric and non-extremal solutions

The non-rotating, extremal supersymmetric black hole has some slightly exotic features, especially the behaviour of the conformal boundary and its Killing spinor. In this section we relax the extremality condition $e = 0$, instead imposing the BPS relations (3.5.5) on the conformal boundary geometry with $e \neq 0$. We shall find that the conformal boundary has the same form as (3.5.16), but now with a completely regular metric on the spindle Σ, apart from the usual orbifold singularities at $\theta = 0$ and $\theta = \pi$, so that $\Sigma \cong \mathbb{WCP}^1_{[n_-,n_+]}$ has the same topology as the black hole horizon in the extremal limit. The circumference of Σ near to $\theta = \theta_0$ grows as $e \to 0$, as does the distance between the poles and $\theta = \theta_0$, with the spindle effectively completely splitting in half in the extremal limit $e = 0$. There is correspondingly a smooth solution ζ to the $d = 3$ Killing spinor equation for $e \neq 0$, that approaches the piecewise constant solution (3.5.24) in the extremal limit. As we discuss, the non-rotating BPS and non-extremal solutions no longer have a smooth black hole horizon but a naked singularity.

We first note that with $a = 0$, the BPS conditions (3.5.5) and the regularity conditions

(3.2.8) imply that

$$m = \frac{g}{\alpha}, \qquad \alpha = \frac{\sqrt{(3n_- + n_+)(3n_+ + n_-)}}{2(n_- + n_+)\sqrt{1 + \frac{e^2(3n_- + n_+)^2(3n_+ + n_-)^2}{4(n_-^2 - n_+^2)^2}}},$$
$$g = \frac{2(n_-^2 - n_+^2)}{(3n_- + n_+)(n_- + 3n_+)}, \qquad \Delta\phi = \frac{(3n_- + n_+)(n_- + 3n_+)}{4n_- n_+(n_- + n_+)}\pi . \tag{3.5.28}$$

These expressions can be obtained by solving the regularity condition (3.2.8) for e and then substiuting this into the first line of (3.4.2) to derive the expression for g. Then substituting this expression for g into (3.2.8) we get the expression for α. We also note that if we set $e = 0$ then we recover the same conditions for the BPS and extremal solutions that we considered in the previous subsection.

To analyse the conformal boundary and the Killing spinors both on the boundary and in the bulk, it is convenient to change to PD-type coordinates via

$$t = \alpha\tau\,, \qquad \cos\theta = p\,, \qquad \phi = \alpha^2\sigma\,, \qquad r = -1/(\alpha q)\,, \tag{3.5.29}$$

where $p \in [-1, 1]$. We also change parameters by introducing

$$g = \frac{\mathsf{P}}{\alpha^2}\,, \qquad e = \frac{\mathsf{Q}}{\alpha^2}\,, \qquad \mathsf{C} = \mathsf{P}^2 + \mathsf{Q}^2\,. \tag{3.5.30}$$

The BPS conditions (3.5.5) imply $m = g/\alpha$, as usual, and $\alpha = (\mathsf{P}^2/\mathsf{C} - \mathsf{C})^{1/2}$. Using these relations, we can then write the full non-rotating solution (3.5.1) in these coordinates:

$$\begin{aligned} ds^2 &= \frac{1}{(p+q)^2}\left(-\mathcal{Q}(q)\,d\tau^2 + \frac{dq^2}{\mathcal{Q}(q)} + \frac{dp^2}{\mathcal{P}(p)} + \mathcal{P}(p)\,d\sigma^2\right), \\ A &= \mathsf{Q}\,q\,d\tau - \mathsf{P}\,p\,d\sigma\,, \end{aligned} \tag{3.5.31}$$

where the metric functions are

$$\mathcal{P}(p) = \mathsf{C}^{-1}\,\mathcal{P}_1(p)\,\mathcal{P}_2(p)\,, \quad \mathcal{Q}(q) = \mathsf{C}^{-1}\,\mathcal{Q}_1(q)\,\mathcal{Q}_2(q)\,, \tag{3.5.32}$$

and we have introduced

$$\begin{aligned}\mathcal{P}_1(p) &= -(1-p)(\mathsf{C}p+\mathsf{C}-\mathsf{P})\,, & \mathcal{Q}_1(q) &= \mathsf{C}q^2-\mathsf{C}+\mathsf{P}q+\mathrm{i}\mathsf{Q}\,,\\ \mathcal{P}_2(p) &= -(1+p)(\mathsf{C}p-\mathsf{C}-\mathsf{P})\,, & \mathcal{Q}_2(q) &= \mathsf{C}q^2-\mathsf{C}+\mathsf{P}q-\mathrm{i}\mathsf{Q}\,.\end{aligned} \tag{3.5.33}$$

The regularity conditions on the metric imply that

$$\mathsf{P}=\frac{n_- - n_+ + \sqrt{(n_- - n_+)^2 - 16(n_- + n_+)^2\mathsf{Q}^2}}{4(n_- + n_+)}\,, \tag{3.5.34}$$

with $\mathsf{P} < \frac{1}{2}$, while the period of σ can be expressed as

$$\Delta\sigma = 2\pi\frac{n_-^2 - n_+^2}{n_- n_+(n_- - n_+ + \sqrt{(n_- - n_+)^2 - 16(n_- + n_+)^2\mathsf{Q}^2})}\,. \tag{3.5.35}$$

Notice that we can parametrize this class of solutions in terms of $n_\pm$ and Q, with the extremal limit obtained when $\mathsf{Q}\to 0$. The reality of P in (3.5.34) requires that we impose

$$\mathsf{Q}\leq\frac{n_- - n_+}{4(n_- + n_+)}\,, \tag{3.5.36}$$

The conformal boundary metric (3.5.16), obtained at $p=-q$, is then (after a rescaling by the constant α^2) given by

$$ds^2_{3d} = -d\tau^2 + ds^2_\Sigma = -d\tau^2 + \frac{dp^2}{\mathcal{P}(p)(1-\mathcal{P}(p))^2} + \frac{\mathcal{P}(p)}{1-\mathcal{P}(p)}d\sigma^2\,. \tag{3.5.37}$$

Notice that $\mathcal{P}(p) > 0$ for $p\in[-1,1]$ is implied by (3.5.36). The circumference $\mathcal{C}$ of the spindle, at fixed $p\in[-1,1]$, is given by the function

$$\mathcal{C}=\mathcal{C}(p)=\sqrt{\frac{\mathcal{P}(p)}{1-\mathcal{P}(p)}}\Delta\sigma\,. \tag{3.5.38}$$

We have plotted this in Figure 3.2 for the spindle $\Sigma=\mathbb{WCP}^1_{[3,1]}$, with progressively smaller values of Q, tending to the extremal solution with $\mathsf{Q}=0$. The circumference at $p=p_0$ is

infinite for $\mathsf{Q}=0$, where $p_0=\cos\theta_0$, with θ_0 given by (3.5.15):

$$p_0=\frac{n_-+n_+-\sqrt{2n_-^2+2n_+^2}}{n_--n_+}\,. \tag{3.5.39}$$

In the $\mathsf{Q}=0$ limit the spindle has then effectively split in half.

At this point one might picture the geometry as breaking up into two "pancakes" as $\mathsf{Q}\to 0$. However, this is not correct and is clarified by calculating the proper distance from $p=\pm 1$ to $p=p_0$, both of which diverge as $\mathsf{Q}\to 0$. It is illuminating to present the geometry of the spindle as an embedding in three-dimensional Euclidean space, as in Figure 3.3. As $\mathsf{Q}\to 0$, while the circumference at p_0 is diverging, so too is the height of the figures.

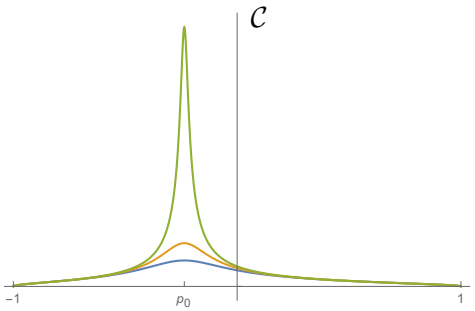


Figure 3.2: Circumference $\mathcal{C}$ of the metric on the spindle $\Sigma=\mathbb{WCP}^1_{[3,1]}$ on the conformal boundary as a function of $p\in[-1,1]$, for the supersymmetric non-extremal black holes. The blue, orange, and green curves have progressively smaller values of Q, namely $\mathsf{Q}=0.04$, $\mathsf{Q}=0.01$ and $\mathsf{Q}=0.0002$, respectively, tending to the extremal solution with $\mathsf{Q}=0$. The same values of Q are plotted also in Figures 3.4 and 3.5.

Introducing the orthonormal frame[16] for the conformal boundary metric (3.5.37)

$$e^0=d\tau\,,\qquad e^1=-\frac{dp}{\sqrt{\mathcal{P}(p)}(1-\mathcal{P}(p))}\,,\qquad e^2=\sqrt{\frac{\mathcal{P}(p)}{1-\mathcal{P}(p)}}d\sigma\,, \tag{3.5.40}$$

we may solve the conformal Killing spinor equation (3.5.19) using the same basis of gamma matrices as the previous subsection (see below (3.5.19)). We find that

$$\zeta=\mathrm{e}^{-\mathrm{i}(\kappa_1\tau+\kappa_2\sigma)}\begin{pmatrix}\zeta_1(p)\\ \zeta_2(p)\end{pmatrix}\,, \tag{3.5.41}$$

[16]Notice the overall minus sign in e^1 is so as to match the orientation in the corresponding frame (3.5.20), where $dp=-\sin\theta d\theta$.

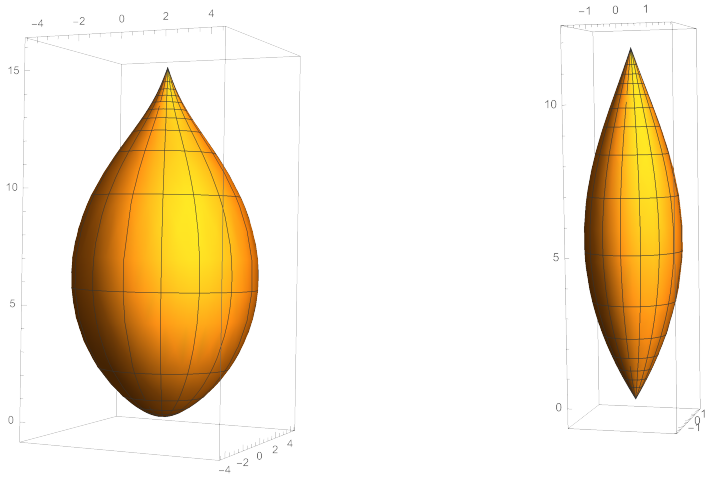


Figure 3.3: Embedding of two spindles on the conformal boundary for the supersymmetric non-extremal black holes in three dimensional Euclidean space. The left plot is for $\Sigma = \mathbb{WCP}^1_{[3,1]}$ and $\mathsf{Q} = 0.04$ and the right plot is for $\Sigma = \mathbb{WCP}^1_{[3,2]}$ and $\mathsf{Q} = 0.025$.

where we have introduced the constants

$$\kappa_1 = \frac{\mathsf{PQ}}{2(\mathsf{P}^2+\mathsf{Q}^2)}\,, \qquad \kappa_2 = \frac{\mathsf{P}^2}{2(\mathsf{P}^2+\mathsf{Q}^2)}\,. \tag{3.5.42}$$

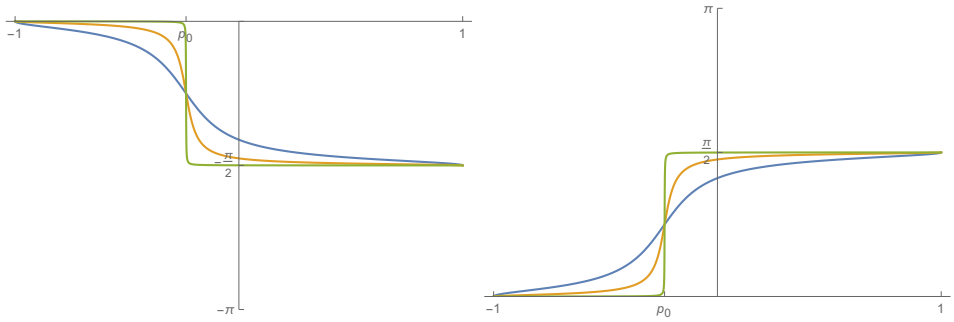
Notice that in the extremal limit $\mathsf{Q} = 0$ the phase in (3.5.41) is $\mathrm{e}^{-\mathrm{i}\sigma/2} = \mathrm{e}^{-\mathrm{i}\phi/2\alpha^2}$, which was compensated for in the previous subsection by making the gauge transformation (3.5.22). The components $\zeta_1(p)$, $\zeta_2(p)$ satisfy the equations

$$\begin{aligned}\zeta_1' &= \frac{\mathrm{i}(p\mathsf{Q}-\kappa_1)}{\sqrt{\mathcal{P}(p)}(1-\mathcal{P}(p))}\zeta_1\,,\\ \zeta_2 &= \frac{4\mathrm{i}(p\mathsf{Q}-\kappa_1)\mathcal{P}(p)-\sqrt{\mathcal{P}(p)}\mathcal{P}'(p)}{4(p\mathsf{P}-\kappa_2)\sqrt{\mathcal{P}(p)(1-\mathcal{P}(p))}}\zeta_1\,.\end{aligned} \tag{3.5.43}$$

After some effort, one finds the solution

$$\begin{aligned}\zeta_1(p) &= \sqrt{\frac{\mathcal{P}_1(p)-\mathcal{P}_2(p)-2\mathrm{i}\mathsf{Q}\sqrt{\mathcal{P}(p)}}{2\mathsf{P}\sqrt{1-\mathcal{P}(p)}}}\,,\\ \zeta_2(p) &= \sqrt{\frac{\mathcal{P}_1(p)-\mathcal{P}_2(p)+2\mathrm{i}\mathsf{Q}\sqrt{\mathcal{P}(p)}}{2\mathsf{P}\sqrt{1-\mathcal{P}(p)}}}\,.\end{aligned} \tag{3.5.44}$$

Notice that $\zeta_1^* = \zeta_2$ and $|\zeta_1| = |\zeta_2|$. In fact we have chosen the overall normalization constant of the spinor so that the components lie on the unit circle in the complex plane, namely $|\zeta_1| = |\zeta_2| = 1$.



(a) $\arg\zeta_1(p)$ (b) $\arg\zeta_2(p)$

Figure 3.4: Arguments of $\zeta_1(p)$, $\zeta_2(p)$, appearing in the conformal Killing spinors on the conformal boundary for the supersymmetric non-extremal black holes, as a function of $p \in [-1,1]$ and plotted for the spindle $\Sigma = \mathbb{WCP}^1_{[3,1]}$ The blue, orange, and green curves have progressively smaller values of Q, as given in Figure 3.2, tending to the extremal solution with $\mathsf{Q} = 0$. For $\mathsf{Q} \neq 0$ we have a smooth conformal Killing spinor on the spindle that approaches two different constant values on each half of the spindle in the extremal limit $\mathsf{Q} \to 0$.

In Figure 3.4 we have plotted the arguments $\arg\zeta_1(p)$, $\arg\zeta_2(p)$, for the spindle $\Sigma = \mathbb{WCP}^1_{[3,1]}$ with progressively smaller values of Q, tending to the extremal solution with $\mathsf{Q} = 0$. In the latter case notice that in the $\mathsf{Q} \to 0$ limit we have

$$\text{for } \mathsf{Q} = 0: \quad \begin{cases} \arg\zeta_2(p) = -\arg\zeta_1(p) = \frac{\pi}{2}\,, & p \in (p_0, 1]\,, \\ \arg\zeta_2(p) = \arg\zeta_1(p) = 0\,, & p \in [-1, p_0)\,. \end{cases} \tag{3.5.45}$$

This precisely corresponds to the extremal solution (3.5.24).

We can also display the way in which the two different topological twists for the extremal case arise in the limit that $\mathsf{Q} \to 0$. In Figure 3.5 we have plotted the spin connection and gauge field for the conformal boundary geometries corresponding to Figure 3.4. More precisely, the non-trivial spin connection component is ω^{12}, and we define its holonomy around a circle in the spindle Σ at constant p, parametrized by σ, via

$$\omega(p) = \frac{1}{2\pi}\int_{S^1} \omega^{12}\,. \tag{3.5.46}$$

The solid curves in Figure 3.5 are then $\frac{1}{2}\omega(p)$, while the dashed lines are $\pm$ the corresponding holonomy of the gauge field $A + \kappa_2 d\sigma$, *cf.* equation (3.5.23).

Although the conformal boundary of this non-extremal supersymmetric solution is perfectly regular, in the bulk the black hole horizon has disappeared and there is a naked curva-

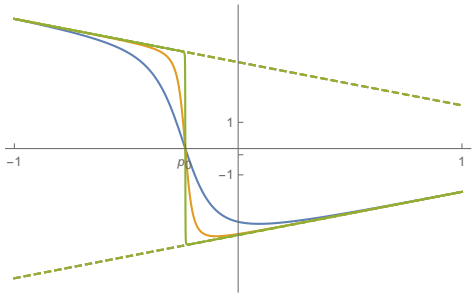


Figure 3.5: Spin connection and gauge field holonomies around a circle in the spindle Σ for the supersymmetric non-extremal black holes, as a function of $p \in [-1,1]$ and plotted for the spindle $\Sigma = \mathbb{WCP}^1_{[3,1]}$. The dashed lines are $\pm$ the holonomy of the gauge field, while the solid blue, orange, and green lines are the corresponding holonomy of the spin connection for progressively smaller values of Q, as given in Figure 3.2, tending to the extremal solution with $\mathsf{Q} = 0$. For $\mathsf{Q} \neq 0$ we do not have a topological twist, but in the extremal limit $\mathsf{Q} \to 0$ we get a different topological twist on each half of the spindle.

ture singularity. To see this, we return to the black hole metric given in (3.5.1) in PD-type coordinates and observe that the function $\mathcal{Q}(q)$ has no real roots for $\mathsf{Q} \neq 0$. There is then a naked curvature singularity at $q = \infty$, with Penrose diagram given by Figure 3.6.

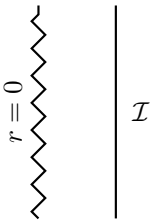


Figure 3.6: Penrose diagram for the non-extremal supersymmetric black holes, with $e \neq 0$. The black hole horizon has disappeared leaving a naked singularity. There is now a smooth conformal boundary which consists of the product of the time direction with a spindle.

3.5.3 A one-parameter family of non-supersymmetric and non-extremal solutions

In the previous two subsections we have discussed special cases that lie inside the more general class of non-rotating PD black holes. While interesting because they preserve supersymmetry, as discussed they both have some pathologies: either an acceleration horizon that cuts the conformal boundary, or a naked singularity. These pathologies arise because of the specific restrictions we have imposed on the parameters in those cases. According to the number and value of the roots of the functions Q and P there are many other possibilities. A detailed

analysis of the causal structure in various cases can be found in [121]. In this subsection we will consider another special case that, while allowing some degree of analytic control over the roots of the metric functions, gives a black hole with completely regular conformal boundary, and two ordinary horizons. This configuration is also smoothly connected with the extremal and BPS black hole, thus providing a kind of "regulator" of the latter solution, while staying within[17] the non-rotating family of solutions.

We start again with the general metric (3.2.1) with $a = 0$, and we further restrict the parameters to satisfy

$$m = \frac{g}{\alpha}\,, \quad e = 0\,, \tag{3.5.47}$$

and hence, in particular, we have $J = Q_e = 0$. The first of the conditions in (3.5.47) is satisfied when *both* BPS conditions (3.5.4) are met, so it amounts to imposing only one of the two conditions. In addition, we recall that we imposed this condition in constructing regular uplifted solutions, as we discussed in section 3.3. The second is the extremality condition in the BPS case. It follows that the black hole that we obtain with these restrictions is neither BPS nor extremal, but it is continuously connected with the case discussed in section 3.5.1 by taking the limit

$$g \to g_{BPS} = \frac{\sqrt{1-\alpha^2}}{\alpha^2}\,. \tag{3.5.48}$$

Since we are taking $a = e = 0$ in (3.2.3), we note that the gauge field is simply

$$A = -g\,\cos\theta\, d\phi\,. \tag{3.5.49}$$

We continue to take $\sqrt{3}/2 < \alpha < 1$ as in section 3.5.1, or equivalently $0 < \mathsf{P} < 1/2$ as in section 3.5.2. For fixed α, the roots of $Q(r)$ depend on g in a very simple way. In particular, we find:

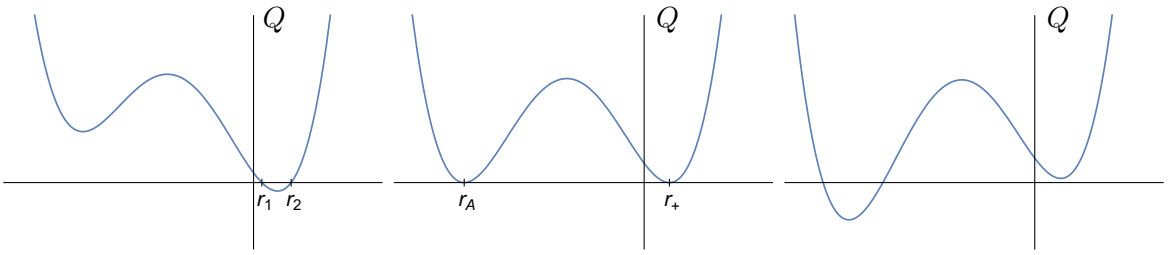
- When $g > g_{BPS}$, $Q(r)$ has two real roots for *negative* r which correspond to acceleration horizons, and no black hole horizon. As in section 3.5.1, the former intersect the

[17] We can also regulate the solutions in sections 3.5.1 and 3.5.2 by turning on the rotation parameter a as considered in sections 3.3, 3.4. In particular, the supersymmetric and extremal rotating black holes have no acceleration horizons.

conformal boundary, which causes pathologies: this can be seen from the fact that in this case the combination $1-\alpha^2\sin^2\theta\,P(\theta)$ appearing in the boundary metric (3.5.16) has two real roots, and is negative between the two.

- When $g < g_{BPS}$, $Q(r)$ has two real roots for *positive* r, which give two black hole horizons (an inner and an outer horizon) with no acceleration horizons. For this case $1-\alpha^2\sin^2\theta\,P(\theta)$ has no roots for $0\leq\theta\leq\pi$, which means the conformal boundary is a smooth spindle as in section 3.5.2.

- Finally, when $g = g_{BPS}$ then $Q(r)$ has two pairs of coincident real roots, which is the case discussed in section 3.5.1.



(a) $g < g_{BPS}$ (b) $g = g_{BPS}$ (c) $g > g_{BPS}$

Figure 3.7: The function $Q(r)$ for three different cases in the class of non-supersymmetric and non-extremal black holes with $m = g/\alpha$ and $a = e = 0$. We focus on the cases $g < g_{BPS}$ and $g = g_{BPS}$. When $g < g_{BPS}$ the two positive roots, r_1, r_2, correspond to an inner and outer black hole horizon. When $g = g_{BPS}$ these two horizons coalesce to give an extremal horizon, r_+, and, in addition, an acceleration horizon, r_A, also appears. For $g > g_{BPS}$ there are no positive roots of Q, and hence no black hole horizon, but instead two negative roots which correspond to two acceleration horizons.

Since the main point of this section is showing that we can have an ordinary black hole with no acceleration horizons, we shall focus on the case

$$g < g_{BPS}\,. \tag{3.5.50}$$

Furthermore, note that while so far we have focused on the roots of $Q(r)$, the restrictions we have put on α and g are also such as to guarantee $P(\theta) > 0$ for $0\leq\theta\leq\pi$. Hence this case indeed corresponds to a completely regular black hole, whose Penrose diagram is given in Figure 3.8, where we have denoted by $r_{1,2}$ the two positive roots of $Q(r)$, with $r_1 < r_2$. In the supersymmetric and extremal limit $g = g_{BPS}$ we have $r_1 = r_2 = r_+$.

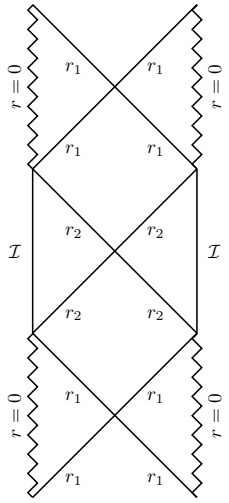


Figure 3.8: Penrose diagram for the non-supersymmetric and non-extremal black holes with $m = g/\alpha$ and $a = e = 0$ and $g < g_{BPS}$. In addition to the black hole event horizon, denoted by r_2, there is also an inner horizon, denoted by r_1. There is a smooth conformal boundary which consists of the product of the time direction with a spindle.

We can then require that the topology in the θ, ϕ directions is that of a spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, by appropriately quantizing the conical deficits at $\theta = 0, \pi$. This gives

$$\alpha = \sqrt{\frac{4\,n_-}{g(n_- - n_+)} - \frac{2g+1}{g^2}}\,, \tag{3.5.51}$$

while the periodicity of ϕ is given by

$$\Delta\phi = \frac{\pi}{2\,g}\frac{n_- - n_+}{n_-\,n_+}\,. \tag{3.5.52}$$

As discussed above, in this case the black hole is completely smooth, and the spindle topology at fixed t and r persists to the conformal boundary, where the metric is again given by (3.5.16) and has the topology $\mathbb{R} \times \mathbb{WCP}^1_{[n_-,n_+]}$. This is regular for any $0 < g < g_{BPS}$, but degenerates as described in section 3.5.1 when g approaches the BPS value. The behaviour of the circumference of the spindle at the boundary and of the spin connection of the boundary metric are very similar to those given in Figures 3.2 and 3.5, respectively. Namely, when $g \to g_{BPS}$, the spindle splits in half at

$$\theta = \theta_0 \equiv \arccos\left(\frac{1-\sqrt{5-4\alpha^2}}{2\sqrt{1-\alpha^2}}\right)\,, \tag{3.5.53}$$

while the spin connection approaches $\pm$ the gauge field, up to the pure gauge term discussed in section 3.5.1.

3.6 Discussion

In this chapter we have studied a very general class of four-dimensional dyonically charged, rotating, and accelerating black holes in four-dimensional anti-de Sitter space. The acceleration leads to conical deficit singularities at the horizon which can be taken to stretch out to the conformal boundary. When these conical deficits are appropriately "quantized", so that the deficit angles are $2\pi(1-1/n_{\mp})$ with positive coprime integers $n_{\pm}$, the resulting space is known in the mathematics literature as a spindle, or equivalently a weighted projective space $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$. Remarkably, when uplifted to $D = 11$ on a regular Sasaki-Einstein seven-manifold SE_7, the solutions become completely regular, free from any conical deficit singularities whatsoever. We have also quantized the flux of these $D = 11$ solutions, thus showing that they give good M-theory backgrounds.

We have shown that there is a sub-family of both supersymmetric and extremal black hole solutions, which interpolate between AdS_4 in the UV and $\mathrm{AdS}_2 \times \Sigma$ in the near horizon IR limit. These are characterized by the integers $n_{\pm}$, that determine the spindle horizon geometry $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, and a continuous parameter, which parametrizes both the electric charge Q_e and the angular momentum J. The entropy of these black holes, which also carry magnetic charge (given in (3.3.31)), can be expressed simply in terms of $n_{\pm}$ and Q_e. We have shown that the entropy can be expressed in a number of equivalent ways, generalizing previous expressions applicable for non-accelerating supersymmetric AdS_4 black holes. In particular, (3.4.25) reduces to the entropy of the extremal Kerr-Newman-AdS black hole upon setting $Q_m = 0$ and $\chi = 2$ (equivalently setting $n_- = n_+ = 1$ in (3.0.1)). The formula (3.4.22), which applies also to the Kerr-Newman-AdS black hole, highlights the dependence of the entropy on the angular momentum computed at the horizon. When uplifted, the near horizon limit gives a new class of rotating $\mathrm{AdS}_2 \times Y_9$ solutions, where we have shown that Y_9 may be viewed as either a regular SE_7 fibration over $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, or equivalently as a Lens space $S^3/\mathbb{Z}_{\mathfrak{q}}$ fibred over the KE_6 base of the SE_7. Remarkably, setting $Q_e = 0$, which also sets $J = 0$, these reduce to a known class of supersymmetric $\mathrm{AdS}_2 \times Y_9$ solutions first

constructed in [126]. We have thus provided a new physical interpretation of those solutions: they are the near horizon limits of the accelerating (but non-rotating) black holes described in section 3.5, and we have generalized those solutions by adding angular momentum, preserving supersymmetry and extremality. It would be interesting to understand in more generality what kind of singularities in lower-dimensional supergravity theories can be uplifted to obtain regular solutions in higher dimensions. For example, it would be interesting to explore this for the $D = 4$ black holes of [142, 143] which have non-compact horizons, but with finite entropy. Many new works have been developed applying the ideas of this chapter to another class of Riemann surface with non-constant curvature, known as topological disks [144] (more on this in section 4.3).

In this chapter we have restricted our attention to solutions of minimal $D = 4$ gauged supergravity. However, it is very likely that our constructions can be generalized to more general gauged supergravity theories with various matter content. More specifically, we expect to be able to construct supersymmetric spinning spindles which would generalize the constructions of [145], for example, where it was assumed that the horizon has spherical topology. We note the similarity of our formula for the black hole entropy (3.4.25) with eq. (54) of [145]. Since the appearance of this chapter's work in the literature, some papers have emerged studying generalizations to more general supergravity theories with matter, see especially [146] and more recently [147] (also relevant are the works of [144, 148]).

We now return to the holographic interpretation of the supersymmetric extremal black holes. The $D = 11$ black hole solutions interpolate between $\mathrm{AdS}_4 \times SE_7$ in the UV and $\mathrm{AdS}_2 \times Y_9$ in the IR, where Y_9 is a SE_7 fibration over the spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$. The $\mathrm{AdS}_4 \times SE_7$ vacuum solution describes N_{SE} M2-branes at the Calabi-Yau four-fold singularity with conical metric $dr^2 + r^2 ds^2(SE_7)$, and these typically have dual field theory descriptions as Chern-Simons quiver gauge theories, with the integer N_{SE} determining the ranks of the gauge groups. Physically we are then wrapping the world-volume of the M2-branes over Σ. We have studied this conformal boundary geometry in some detail in section 3.5 for the non-rotating solutions with $Q_e = J = 0$. An important subtlety in this case is that in the UV the conformal boundary is such that the spindle is split into two components. Moreover, in this limit supersymmetry is preserved via a different topological twist on each component. However, we have also shown that this split can be regulated in a family of non-rotating

supersymmetric but non-extremal black holes (or by further relaxing the supersymmetry condition). Moreover, we do not expect the generic supersymmetric extremal rotating black holes, with $Q_e \neq 0$, to have this pathology. Indeed, later in [149] the authors carry out a detailed analysis of the thermodynamics for this general class of accelerating black holes and explicitly construct the boundary Killing spinors. They find that indeed when $Q_e \neq 0$ the conformal boundary is always regular and supersymmetry is not preserved as a result of a topological twist. On the other hand, the $Q_e = 0$ solutions studied in section 3.5 are a somewhat degenerate limit.

With the above holographic interpretation, it should be possible to reproduce the black hole entropy formulae (3.0.1) by studying the dual M2-brane field theories. Indeed, there has been considerable progress on this topic for various classes of supersymmetric AdS_4 black holes, as was briefly discussed in section 1.3. In particular, the first class of black holes for which a dual field theory interpretation has been found have just magnetic flux through a Riemann surface horizon Σ. The field theory calculation utilizes I-extremization, where the index can be identified with the localized partition function of the dual field theory on $S^1 \times \Sigma$ [54, 55, 68, 99, 100, 150–152]. More recently, following the approach put forward in [80], progress has also been made in understanding the class of electrically charged and rotating AdS_4 black holes from the dual field theory point of view [81, 132–134]. For the accelerating black hole solutions that we discussed in this chapter, it should be straightforward to compute the suitably regularized on-shell action of the corresponding Euclidean solutions and reproduce the entropy by extremizing the corresponding entropy function. This was indeed later done in the above mentioned [149], where they explain how to relate the on-shell action to the black hole entropy (3.0.1). We note that they also recover the on-shell action from the application of the fixed point formula (2.0.1), after analytic continuation, thus further showing the generality and importance of the results from chapter 2. From the field theory side we should then focus on the Euclidean version of the conformal boundary geometry of the charged, rotating and accelerating black holes, and compute a certain twisted topological index associated with the $d = 3$, $\mathcal{N} = 2$ SCFTs on the M2-branes, wrapped on the spinning spindle Σ. While some care may be required in taking the BPS and extremal limit, it seems possible that we can get agreement between these computations using localization techniques in the large N limit (for more details, see [149]).

The wrapped M2-brane theories flow to a $d = 1$ superconformal quantum mechanics in the IR, that is dual to the $\mathrm{AdS}_2 \times Y_9$ solutions that arise as the near horizon limit of the black holes. Equation (3.4.49) says that the $d = 3$ superconformal R-symmetry mixes with the $U(1)$ isometry of the internal space Σ in flowing to the $d = 1$ superconformal R-symmetry in the IR. It should similarly be possible to reproduce this formula via a dual field theory/SCQM calculation by computing and then extremizing a suitable index. Indeed, in the following chapter we will study an analogous class of supersymmetric $D = 5$ $\mathrm{AdS}_3 \times \Sigma$ solutions, with $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ again a spindle, that uplift on a regular Sasaki-Einstein five-manifold to smooth solutions of type IIB supergravity (or to M-theory in section 4.2.5). These are the holographic duals to D3-branes (M5-branes in 4.2.5) wrapping the spindle, and in this case we are able to reproduce both the central charge and the mixing of $d = 4$ and $d = 2$ superconformal R-symmetries in the supergravity solution, where in the field theory dual we make use of anomaly polynomials and c-extremization. More solutions of this form have been found for different brane pictures and we will give an overview of these new results in section 4.3.

Chapter 4

Five dimensional theories on spindles

The work described in chapter 3 presented solutions to four-dimensional supergravity with some novel characteristics. In particular we studied accelerating black holes that are asymptotically AdS_4, with a near horizon metric of the form $\mathrm{AdS}_2 \times \Sigma$, where $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ is an orbifold known as a spindle. One important result is that the quantized conical deficits of the spindle can be resolved by appropriately uplifting to $D = 11$ on a regular Sasaki-Einstein seven-manifold SE_7. From the near horizon metric not only could we compute the entropy of the solutions but also the Killing spinors on the spindle, with the surprising result that these are not constant, as is usual in these constructions. This can be further analyzed at the $D = 11$ level, where the uplifted solution interpolates between $\mathrm{AdS}_4 \times SE_7$ in the UV and $\mathrm{AdS}_2 \times Y_9$ in the IR, where Y_9 is a SE_7 fibration over Σ. The $\mathrm{AdS}_4 \times SE_7$ solution is associated to a stack of M2-branes at the apex of a Calabi-Yau cone singularity and physically we say that we are wrapping the worldvolume of the M2-branes over the spindle Σ. This leads to a field theory prediction, to reproduce the black hole entropy from the $d = 1$ superconformal quantum mechanics dual to the $\mathrm{AdS}_2 \times Y_9$ solutions, by extremizing a certain index associated with the $d = 3$ SCFTs wrapped on the spindle Σ. This dual description should also capture the mixing of the superconformal R-symmetry with the isometry of Σ in flowing to the IR, which was observed from the gravity side. For these solutions, the field theory computation was left for future work.

In this chapter we build on these ideas and present similar constructions in five dimensional supergravity theories. In five dimensions we do not possess the analog of the full interpolating solution (an AdS_5 black string), but we can work from near-horizon $AdS_3 \times \Sigma$ solutions, where again $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, a spindle. By uplifting to solutions of string theory (or M-theory) we again find a way to obtain a regular, smooth solution, free of conical singularities. Furthermore, the Killing spinors depend on the coordinates of Σ, again suggesting that supersymmetry is not being preserved by the usual topological twist. Our construction suggests a field theory computation, from $d = 4$ to $d = 2$ SCFTs, which we explore and are able to reproduce the gravity predictions. In particular we find matching for the central charge and for the mixing of the $d = 4$ and $d = 2$ superconformal R-symmetry with the isometry of the spindle. We achieve this for different supergravity theories (minimal gauged or with matter couplings), which can be uplifted to solutions associated to D3-branes and M5-branes wrapped on the spindle.

First, we consider $AdS_3 \times \Sigma$ solutions to $D = 5$ *minimal* gauged supergravity, where $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ is again the two-dimensional orbifold known as a spindle. We interpret these solutions as the near horizon limit of black brane solutions associated with D3-branes wrapped on Σ. As we introduced, we find that the conical deficits at the poles of Σ can be removed by appropriately uplifting the five-dimensional solution on a regular SE_5 to obtain a solution of type IIB supergravity. To obtain a good string theory background we also quantize the flux N through the SE_5 fibre and find restrictions on which SE_5 are allowed, given a choice of n_-, n_+. The magnetic flux through Σ is proportional to the difference $n_- - n_+$ and not to the Euler number of the spindle

$$\chi(\Sigma) \;=\; \frac{n_- + n_+}{n_- n_+}\,, \tag{4.0.1}$$

as is characteristic of a topological twist. Also in contrast to the topological twist, we find that the Killing spinors depend on the coordinates of Σ, and are in fact sections of non-trivial bundles over Σ. We also compute the central charge of this solutions and find

$$c = \frac{4(n_- - n_+)^3}{3n_-n_+(n_-^2 + n_-n_+ + n_+^2)} a_{4d}\,, \tag{4.0.2}$$

where

$$a_{4d} \equiv \frac{\pi^2 N^2}{4\mathrm{vol}(SE_5)} \,. \tag{4.0.3}$$

After uplifting, the resulting $\mathrm{AdS}_3 \times \mathcal{M}_7$ solutions are precisely those of [153]. Our construction suggests that the $d = 2, \mathcal{N} = (0,2)$ SCFTs dual to the $\mathrm{AdS}_3 \times \mathcal{M}_7$ solutions of [153] arise from compactifying $d = 4$, $\mathcal{N} = 1$ SCFTs on a spindle, where the $d = 4$ theories are dual to $\mathrm{AdS}_5 \times SE_5$. This includes the case of $\mathcal{N} = 4$ SYM, where $SE_5 = S^5$. We make a precision test of this interpretation: we compute the central charge and superconformal R-symmetry of the $d = 2$ field theories using anomaly polynomials and c-extremization [43] and find exact agreement with the gravity result (4.0.2).

Later, we extend this work by going beyond the minimal gauged theory. The five-dimensional solutions described above can be uplifted on an S^5 when $n_- - n_+$ is divisible by 3, leading to $\mathrm{AdS}_3 \times Y_7$ solutions of type IIB supergravity in which Y_7 takes the fibred form

$$S^5 \;\hookrightarrow\; Y_7 \;\to\; \Sigma \;=\; \mathbb{WCP}^1_{[n_-,n_+]} \,. \tag{4.0.4}$$

We can generalize this construction by allowing for more general fibrations in 4.0.4. More precisely, the $D = 5$ minimal gauged supergravity solution is a special case of a more general class of (local) AdS_3 solutions to $D = 5$, $U(1)^3$ gauged supergravity with two additional parameters, first constructed in [154]. These also uplift on S^5, with the three gauge fields of $U(1)^3$ determining the fibration of this internal space over the $D = 5$ spacetime. The resulting supersymmetric $\mathrm{AdS}_3 \times Y_7$ solutions were in fact constructed before reference [154], in [126]. The approach we take to the global analysis of these supergravity solutions is different to that in [154], [126], and instead follows as in the minimal gauged case. Specifically, we first construct appropriately regular $\mathrm{AdS}_3 \times \Sigma$ solutions of the $D = 5$, $U(1)^3$ gauged supergravity theory, where $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ is a spindle specified by coprime positive integers $n_- > n_+$. We then appropriately quantize the three Abelian R-symmetry gauge field fluxes through Σ, in such a way that the resulting fibration (4.0.4) is well-defined, with a smooth total space

Y_7. More precisely, defining the fluxes

$$Q_I \equiv \frac{1}{2\pi}\int_\Sigma F^I\,, \tag{4.0.5}$$

where F^I, $I=1,2,3$, are the three Abelian R-symmetry gauge field curvatures, we construct solutions with

$$Q_1 = \frac{q}{n_-n_+}\,, \qquad Q_2 = \frac{p}{n_-n_+}\,, \qquad Q_3 = \frac{n_- - n_+ - q - p}{n_-n_+}\,. \tag{4.0.6}$$

Here $q,p>0$ are positive integers, constrained so that $q+p<n_- - n_+$, and where q and p have no common factor with either of n_-, n_+. Notice that the charges (4.0.6) satisfy

$$Q_1+Q_2+Q_3 = \frac{n_- - n_+}{n_-n_+}\,. \tag{4.0.7}$$

This should be again contrasted with a topological twist, where instead on the right hand side of (4.0.7) one would have the Euler number of the spindle (4.0.1). The minimal gauged solutions are recovered by setting $Q_1=Q_2=Q_3$, or equivalently $p=q=\frac{1}{3}(n_- - n_+)$, which then requires $n_- - n_+$ to be divisible by 3.

The central charge of these solutions is given by

$$c = \frac{3pq(n_- - n_+ - q - p)}{n_-n_+[(n_- - n_+)(p+q)+n_-n_+ - p^2 - pq - q^2]}N^2\,. \tag{4.0.8}$$

The solution is thus specified by five positive integers: n_-, n_+ determine the topology of the wrapped spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, q,p determine the R-symmetry fluxes (4.0.6) and hence twisting in the fibration (4.0.4), and finally N is the number of wrapped D3-branes. As in the minimal case, the description of these solutions suggests that the dual $d=2$, $(0,2)$ SCFTs arise from a twisted compactification of $\mathcal{N}=4$ SYM on Σ, where there are background R-symmetry fluxes given by (4.0.6). We make a check on this conjecture by computing both the central charge and superconformal R-symmetry of the $d=2$ field theories, making use of the anomaly polynomial of $\mathcal{N}=4$ SYM together with c-extremization [43]. We find precise agreement with the corresponding supergravity quantities.

When $Q_1=Q_2$, or equivalently $p=q$, the solutions to $D=5$, $U(1)^3$ supergravity are

also solutions to Romans $\mathcal{N}=4$, $SU(2)\times U(1)$ supergravity [155]. The latter is a consistent truncation of $D=11$ supergravity on the Maldacena-Núñez [25] solution [156]. We thus also obtain $\mathrm{AdS}_3\times Y_8$ solutions of $D=11$ supergravity, where Y_8 takes the fibred form

$$S^4 \hookrightarrow Y_8 \rightarrow \Sigma_g\times\Sigma\,. \tag{4.0.9}$$

Here Σ_g is a constant curvature Riemann surface of genus $g>1$. The original Maldacena-Núñez solution corresponds to the AdS_5 vacuum of the Romans theory, and is dual to the $d=4$, $\mathcal{N}=2$ SCFT living on N M5-branes wrapped on the Riemann surface Σ_g, which is holomorphically embedded inside a Calabi-Yau two-fold. The uplift to $D=11$ of our $Q_1=Q_2$ solution, with parameter q, then describes the further twisted compactification of this four-dimensional theory on the spindle Σ, with $0<q<\frac{1}{2}(n_--n_+)$ determining the twisting. The central charge of these gravity solutions is

$$c=\frac{4q^2(n_--n_+-2q)}{n_-n_+[n_-(n_++2q)-q(2n_++3q)]}(g-1)N^3\,. \tag{4.0.10}$$

We again reproduce this result from a dual field theory calculation, this time utilizing the anomaly polynomial of the theory on N M5-branes, doubly wrapped on $\Sigma_g\times\Sigma$.

The outline for this chapter is as follows. We start by studying the minimal gauged case in section 4.1, beginning with a description of the five-dimensional $\mathrm{AdS}_3\times\Sigma$ solutions in section 4.1.1. We uplift these solutions on specific SE_5 manifolds to obtain completely smooth solutions of type IIB supergravity in 4.1.2. In 4.1.3 we compute the central charge and superconformal R-symmetry of the dual $d=2$ field theories using anomaly polynomials and c-extremization, and find exact agreement with the gravity result. Then we consider the extensions to multi-charged spindles in the $U(1)^3$ gauged supergravity in section 4.2. We construct the $\mathrm{AdS}_3\times Y_7$ solutions of interest by first constructing $D=5$ $\mathrm{AdS}_3\times\Sigma$ solutions in $U(1)^3$ gauged supergravity, and then imposing appropriate quantization conditions on the gauge field fluxes through the spindle $\Sigma=\mathbb{WCP}^1_{[n_-,n_+]}$ so that the uplift on S^5 leads to a smooth seven-manifold Y_7. We also compute the central charge and R-symmetry gauge field for these gravity solutions. In section 4.2.5 we similarly construct $\mathrm{AdS}_3\times Y_8$ solutions to M-theory by uplifting the same $D=5$ solutions, but now necessarily with $Q_1=Q_2$, and

compute the central charge. In section 4.2.6 we compute the $d = 2$ anomaly polynomial for the twisted compactification of $\mathcal{N} = 4$ SYM on Σ, and using c-extremization then compute the exact central charge and superconformal R-symmetry in field theory, finding agreement with the gravity duals. We perform a similar computation for the theory on N M5-branes wrapped on $\Sigma_g \times \Sigma$, again finding agreement with the central charge computed from the supergravity solution. Finally we conclude this chapter in section 4.3 with a discussion of these results and an overview of the new spindle solutions that have emerged in the literature since the publication of this work.

4.1 Minimal gauged theory

4.1.1 $D = 5$ solutions

The equations of motion for $D = 5$ minimal gauged supergravity [157] are given by

$$\begin{aligned} R_{\mu\nu} &= -4g_{\mu\nu} + \tfrac{2}{3}F_{\mu\rho}F_{\nu}{}^{\rho} - \tfrac{1}{9}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,, \\ d*F &= -\tfrac{2}{3}F\wedge F\,, \end{aligned} \tag{4.1.1}$$

where $F = dA$, with A the Abelian R-symmetry gauge field. A solution is supersymmetric if it admits a Killing spinor satisfying

$$\left[\nabla_\mu - \tfrac{\mathrm{i}}{12}(\Gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\Gamma^\rho)F_{\nu\rho} - \tfrac{1}{2}\Gamma_\mu - \mathrm{i}A_\mu\right]\epsilon = 0\,, \tag{4.1.2}$$

where ϵ is a Dirac spinor and $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$.

The supersymmetric solution of interest is given by

$$ds_5^2 = \frac{4y}{9}ds^2_{\mathrm{AdS}_3} + ds^2_\Sigma\,, \qquad A = \frac{1}{4}\left(1 - \frac{a}{y}\right)dz\,. \tag{4.1.3}$$

Here $ds^2_{\mathrm{AdS}_3}$ is a unit radius metric on AdS_3, while

$$ds^2_\Sigma = \frac{y}{q(y)}dy^2 + \frac{q(y)}{36y^2}dz^2\,, \tag{4.1.4}$$

is the metric on the horizon, Σ, and

$$q(y) = 4y^3 - 9y^2 + 6ay - a^2\,, \tag{4.1.5}$$

with a a constant.

Assuming $a \in (0,1)$ the three roots y_i of $q(y)$ are all real and positive. Defining $y_1 < y_2 < y_3$, we then take $y \in [y_1, y_2]$ to obtain a positive definite metric (4.1.4) on Σ. We can then examine the behaviour of the metric as y approaches y_1 and y_2. Defining $\rho_1 = 2(y-y_1)^{1/2}$ and $\rho_2 = 2(y_2-y)^{1/2}$, we find that as $y \to y_i$, the y, z part of the metric approaches

$$ds^2_{y,z} \approx \frac{y_i}{q'(y_i)}\left[d\rho_i^2 + c_i^2\rho_i^2 dz^2\right], \qquad c_i = \frac{|q'(y_i)|}{12y_i^{3/2}}\,, \qquad i = 1,2. \tag{4.1.6}$$

As for the solutions of the previous chapter, it is not possible to remove both conical singularities by a single choice of period Δz for z, to obtain a smooth two-sphere. Instead, we will demand that y, z parametrize a weighted projective space, $\mathbb{WCP}_{[n_+,n_-]}$ by demanding that

$$\Delta z = \frac{2\pi}{c_1 n_+} = \frac{2\pi}{c_2 n_-}\,. \tag{4.1.7}$$

where $n_\pm \in \mathbb{N}$ and we will shortly see that $n_- > n_+$.

To further analyze these conditions it is helpful to introduce another function

$$g = \frac{a-y}{2(y^2-2y+a)}\,, \tag{4.1.8}$$

which naturally appears in the uplifted solutions. We note that $q'^2 = -36q + 144y^2\frac{(a-y)}{2g}$. We then define $g_i = g(y_i)$, the value of g at the two roots. Using (14) of [153] we find

$$\frac{a}{y_1} = \frac{(g_1+2g_2)^2}{g_1^2+g_1g_2+g_2^2}\,, \qquad \frac{a}{y_2} = \frac{(2g_1+g_2)^2}{g_1^2+g_1g_2+g_2^2}\,. \tag{4.1.9}$$

Using (13) of [153] we then find that a can be expressed in terms of the g_i as

$$a = \frac{(g_2-g_1)^2(2g_1+g_2)^2(g_1+2g_2)^2}{4(g_1^2+g_1g_2+g_2^2)^3}\,. \tag{4.1.10}$$

From these relations we conclude that

$$y_1 = \frac{(g_1 - g_2)^2(2g_1 + g_2)^2}{4(g_1^2 + g_1 g_2 + g_2^2)^2}, \qquad y_2 = \frac{(g_1 - g_2)^2(g_1 + 2g_2)^2}{4(g_1^2 + g_1 g_2 + g_2^2)^2}. \tag{4.1.11}$$

Notice that a, and hence y_i, only depend on the ratio g_2/g_1. Substituting into the expression for g we then deduce the useful result

$$g_1 = \frac{2(g_1^2 + g_1 g_2 + g_2^2)}{3g_2(g_1 + g_2)}\,. \tag{4.1.12}$$

With these results in hand we can conclude that

$$c_1 = \frac{1}{g_1}\,, \qquad c_2 = \frac{1}{g_2}\,, \tag{4.1.13}$$

and hence

$$\frac{g_2}{g_1} = \frac{n_-}{n_+}\,, \qquad \Delta z = \frac{2\pi g_1}{n_+}\,, \tag{4.1.14}$$

results that can be used to express all quantities in terms of $n_\pm$.

Therefore we conclude that if

$$\begin{aligned} a &= \frac{(n_- - n_+)^2(2n_- + n_+)^2(n_- + 2n_+)^2}{4(n_-^2 + n_- n_+ + n_+^2)^3}\,, \\ \Delta z &= \frac{2(n_-^2 + n_- n_+ + n_+^2)}{3n_- n_+(n_- + n_+)} 2\pi\,, \end{aligned} \tag{4.1.15}$$

then ds_Σ^2 is a smooth metric on the orbifold $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$. Specifically, there are conical deficit angles $2\pi(1-1/n_\mp)$ at $y = y_1, y_2$, respectively, where $n_\pm$ are arbitrary coprime positive integers with $n_- > n_+$.

Note that there is magnetic flux through Σ:

$$\frac{1}{2\pi}\int_\Sigma F = \frac{n_- - n_+}{2n_- n_+}\,. \tag{4.1.16}$$

This may be contrasted with the Euler number:

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R_\Sigma \mathrm{vol}_\Sigma = \frac{n_- + n_+}{n_- n_+}\,, \tag{4.1.17}$$

where R_Σ is the Ricci scalar of Σ, and vol_Σ is its volume form[1].

To solve (4.1.2) we write $\Gamma^a = \gamma^a \otimes \sigma^3$, for $a = 0, 1, 2$ with $\gamma^0 = -\mathrm{i}\sigma^2$, $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^3$ and $\Gamma^3 = 1 \otimes \sigma^2$, $\Gamma^4 = 1 \otimes \sigma^1$, where σ^i are Pauli matrices. We then write $\epsilon = \vartheta \otimes \chi$ with ϑ a Killing spinor for AdS_3 satisfying $\nabla_a \vartheta = \frac{1}{2}\gamma_a \vartheta$. The two-component spinor χ on the spindle is given by

$$\chi = \left(\frac{\sqrt{q_1(y)}}{\sqrt{y}}, \mathrm{i}\frac{\sqrt{q_2(y)}}{\sqrt{y}}\right)\,, \tag{4.1.18}$$

where

$$q_1(y) = -a + 2y^{3/2} + 3y\,,\ q_2(y) = a + 2y^{3/2} - 3y\,, \tag{4.1.19}$$

which satisfy $q(y) = q_1(y)q_2(y)$. In contrast to the topological twist, this spinor depends on the coordinates of Σ. An analog analysis to section 3.4.2 follows, to show that the 2d spinors on Σ are smooth and well defined, in the appropriate orbifold sense, and that they are sections of a non-trivial bundle over Σ. Note that the gauge choice used in (4.1.3) has been fixed by requiring ϵ to be independent of z.

4.1.2 Uplift to IIB string theory

Any supersymmetric solution to (4.1.1) uplifts (locally) to type IIB supergravity via [23]:

$$\begin{aligned} ds_{10}^2 &= L^2\left[ds_5^2 + (\tfrac{1}{3}d\psi + \sigma + \tfrac{2}{3}A)^2 + ds_{KE_4}^2\right] \\ g_s F_5 &= L^4\Big[4\mathrm{vol}_5 - \tfrac{2}{3}*_5 F\wedge J \\ &\qquad + (2J\wedge J - \tfrac{2}{3}F\wedge J)\wedge(\tfrac{1}{3}d\psi + \sigma + \tfrac{2}{3}A)\Big]\,. \end{aligned} \tag{4.1.20}$$

[1] In general the integral of the curvature for a $U(1)$ gauge field on $\mathbb{WCP}^1_{[n_-,n_+]}$ is necessarily $2\pi/(n_- n_+)$ times an integer, as was seen in the previous chapter (see also appendix A.1). This makes the gauge field A, with flux given by (4.1.16), again a spin^c gauge field: $n_- - n_+$ is even/odd precisely when $n_- + n_+$ is even/odd, so that spinor fields with unit charge under A are always globally well-defined.

Here F_5 is the self-dual five-form, g_s is the string coupling constant, and $L > 0$ is a length scale that is fixed by flux quantization. KE_4 is an arbitrary positively curved Kähler-Einstein four-manifold with Kähler form J, normalized so that the Ricci form is $\mathcal{R} = 6J$, and σ is a local one-form with $d\sigma = 2J$.

Substituting (4.1.3) into (4.1.20) we find that the $D = 10$ metric may be written as

$$ds_{10}^2 = \frac{4}{9}L^2 y \left[ds_{\mathrm{AdS}_3}^2 + ds_{\mathcal{M}_7}^2\right] \,, \tag{4.1.21}$$

where $\mathcal{M}_7$ is a compact seven-manifold. This is the same solution of type IIB supergravity given in [153].

Analogously to the work in section 3.3, it was shown in [153] that $\mathcal{M}_7$ is the total space of a Lens space $S^3/\mathbb{Z}_q$ fibration over the KE_4, where the twisting is parametrized by another positive integer p. The Lens space fibre has coordinates y, z, ψ. In terms of our parameters $n_\pm$ we identify

$$p = kn_+ \,, \quad q = \frac{k}{I}(n_- - n_+) \,, \tag{4.1.22}$$

where $p, q \in \mathbb{N}$ are coprime. The Fano index, I, is the largest positive integer for which $\int_S c_1/I \in \mathbb{Z}$, for all two-cycles S in the KE_4, where $c_1 = [\mathcal{R}/2\pi] \in H^2(KE_4, \mathbb{Z})$. We have also defined

$$k = \mathrm{hcf}(I, p) \,, \tag{4.1.23}$$

and identify ψ with period $\Delta\psi$ given by [2]

$$\Delta\psi = \frac{2\pi I}{k} \,. \tag{4.1.24}$$

At a fixed point in the $D = 5$ spacetime, the internal five-dimensional metric $ds_{SE_5}^2 = (\frac{1}{3}d\psi + \sigma)^2 + ds_{KE_4}^2$ in (4.1.20) is then a *regular* Sasaki-Einstein manifold, which is simply-connected when $k = 1$.

To obtain a string theory background one must also quantize the five-form flux F_5 through

[2] The torus parametrized by (z, ψ) is the same as [153]. We have made identifications on z and ψ in the opposite order to [153], so (4.1.24) is not the same but $\Delta\psi\Delta z$ is.

all five-cycles in $\mathcal{M}_7$. This was carried out in [153]. We define the integers

$$M = \int_{KE_4} c_1 \wedge c_1 = \frac{1}{4\pi^2} \int_{KE_4} \mathcal{R} \wedge \mathcal{R} \,, \tag{4.1.25}$$

and $h = \text{hcf}(M/I^2, q)$. Then if we choose L to satisfy

$$\frac{L^4}{g_s \ell_s^4} = \frac{108\pi}{I^3 h} k^2 n_+ n_- n \,, \tag{4.1.26}$$

where ℓ_s is the string length and $n \in \mathbb{N}$, then one finds that $1/(2\pi\ell_s)^4 \int_D F_5 \in \mathbb{Z}$, for all five-cycles D in $\mathcal{M}_7$.

There is a finite set of choices for the positively curved KE_4. If $KE_4 = \mathbb{CP}^2$ then $I = 3$, $M = 9$. For this case, $k = 1$ gives $SE_5 = S^5$ as the internal space while $k = 3$ gives $S^5/\mathbb{Z}_3$. If $KE_4 = S^2 \times S^2$ we have $I = 2, M = 8$. Now $k = 1$ gives $SE_5 = T^{1,1}$, while $k = 2$ gives $SE_5 = T^{1,1}/\mathbb{Z}_2$. Finally, for $KE_4 = dP_m$, $3 \leq m \leq 8$, where dP_m is a del Pezzo surface, we have $I = 1$, $M = 9 - m$.

Like in the previous chapter, a key observation is that $\mathcal{M}_7$ in the AdS_3 solutions of [153] may also be viewed as SE_5 fibrations over $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$. We can begin with any weighted projective space, with weights $n_- > n_+$, and then define $p, q \in \mathbb{Z}$ via (4.1.22), where we also define

$$k = \frac{I}{\text{hcf}(I, n_- - n_+)} \,. \tag{4.1.27}$$

With this definition, p and q are manifestly coprime, and one can check that (4.1.23) is equivalent to (4.1.27). With this perspective, we can calculate the flux of F_5 through the SE_5 fibre:

$$N \equiv \frac{1}{(2\pi\ell_s)^4} \int_{SE_5} F_5 = \frac{M}{I^2 h} k n_+ n_- n \in \mathbb{N} \,. \tag{4.1.28}$$

Notice that for a given spindle, specified by $n_\pm$, and a given choice of KE_4 we only get a smooth type IIB solution for k as in (4.1.27) and hence a specific SE_5. E.g. if $KE_4 = \mathbb{CP}^2$ and $n_+ = 2$, then for $n_- = 3, 7, \ldots$ and $n_- = 5, 9, \ldots$ we can uplift on $S^5/\mathbb{Z}_3$ and S^5, respectively.

The central charge is given by $c = 3L/2G_{(3)}$, where $G_{(3)}$ is the Newton constant obtained

by compactifying type IIB supergravity on $\mathcal{M}_7$ [45]. We can rewrite the result of [153] as

$$c = \frac{4(n_- - n_+)^3}{3n_-n_+(n_-^2 + n_-n_+ + n_+^2)} a_{4d}\,, \tag{4.1.29}$$

where

$$a_{4d} \equiv \frac{\pi^2 N^2}{4\mathrm{vol}(SE_5)}\,. \tag{4.1.30}$$

In [153] the dual $d = 2$, $\mathcal{N} = (0,2)$ SCFTs were not identified, but our $D = 5$ construction of the solutions, together with the flux condition (4.1.28), leads to a conjecture. Begin with the $d = 4$ SCFT dual to $\mathrm{AdS}_5 \times SE_5$, describing N D3-branes at the Calabi-Yau three-fold singularity with conical metric $dr^2 + r^2 ds^2_{SE_5}$. The large N a-central charge of this theory is precisely given by a_{4d} in (4.1.30) [158]. One then compactifies that theory on $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, with a background R-symmetry gauge field with magnetic flux (4.1.16). The solutions we have described suggest the theory flows to a $d = 2$, $\mathcal{N} = (0,2)$ SCFT in the IR, and we will give evidence for this below by computing the central charge via a field theory calculation.

The $U(1)_R$ symmetry of the $(0,2)$ theory dual to the $\mathrm{AdS}_3 \times \mathcal{M}_7$ solutions is realized by a Killing vector, R_{2d}, on $\mathcal{M}_7$. Using the results of [126, 159] we deduce

$$R_{2d} = 2\partial_\psi + \frac{3n_-n_+(n_- + n_+)}{n_-^2 + n_-n_+ + n_+^2}\partial_\varphi\,. \tag{4.1.31}$$

Here we have defined $\varphi = \frac{2\pi z}{\Delta z}$ so that $\Delta\varphi = 2\pi$, and ∂_φ generates the $U(1)$ isometry of the weighted projective space Σ, which we shall refer to as $U(1)_J$. Note that the Killing spinor on the SE_5 has unit charge under $R_{4d} = 2\partial_\psi$, which may therefore be identified with the superconformal $U(1)_R$ symmetry of the $d = 4$ SCFT before compactification on Σ. In other words, equation (4.1.31) states that the $d = 4$ R-symmetry mixes with $U(1)_J$ in flowing to the $d = 2$ R-symmetry in the IR. We shall also recover (4.1.31) from a field theory calculation in the next section.

4.1.3 $d = 4$ SCFTS on Σ

We begin with a general $d = 4$ SCFT with anomaly polynomial given by the 6-form

$$\mathcal{A}_{4d} = \frac{\mathrm{tr}\, R^3}{6} c_1(R_{4d})^3 - \frac{\mathrm{tr}\, R}{24} c_1(R_{4d}) p_1(TZ_6)\,. \tag{4.1.32}$$

As is standard, in (even) dimension d the anomaly polynomial is a $(d+2)$-form on an abstract $(d+2)$-dimensional space, here called Z_6. $c_1(R_{4d})$ denotes the first Chern class of the $d = 4$ superconformal $U(1)_R$ symmetry bundle over Z_6, and p_1 denotes the first Pontryagin class. The trace is over Weyl fermions when the theory has a Lagrangian description, and in any case we may always write

$$\mathrm{tr}\, R^3 = \frac{16}{9}(5a_{4d} - 3c_{4d})\,, \quad \mathrm{tr}\, R = 16(a_{4d} - c_{4d})\,, \tag{4.1.33}$$

in terms of the central charges a_{4d}, c_{4d}. We focus on the large N limit in which $a_{4d} = c_{4d}$ to leading order, and hence

$$\mathcal{A}_{4d} = \frac{16 a_{4d}}{27} c_1(R_{4d})^3 \quad (\text{at large } N)\,. \tag{4.1.34}$$

We now compactify the $d = 4$ theory on $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, with magnetic flux (4.1.16) for the $d = 4$ R-symmetry gauge field. The resulting $d = 2$ anomaly polynomial will then capture the right-moving central charge c_r, but crucially we need to include the $U(1)_J$ global symmetry in $d = 2$ that comes from the isometry of Σ. Geometrically, this involves taking Z_6 to be the total space of a Σ fibration over a four-manifold Z_4 [160]. More precisely, we let J be a $U(1)$ bundle over Z_4, with connection corresponding to a background gauge field A_J for the $d = 2$ $U(1)_J$ global symmetry, and then fibre Σ over Z_4 using the $U(1)_J$ action and connection A_J. In practice this amounts to the replacement $d\varphi \mapsto d\varphi + A_J$.

Incorporating the magnetic flux (4.1.16) into this construction amounts to "gauging" the $U(1)$ gauge field A in (4.1.3), as just described. Thus, we define the following connection one-form on Z_6:

$$\mathscr{A} = \frac{1}{4}\left(1 - \frac{a}{y}\right)\frac{\Delta z}{2\pi}(d\varphi + A_J) \equiv \rho(y)(d\varphi + A_J)\,, \tag{4.1.35}$$

where recall that the gauge choice we made is such that the Killing spinors are uncharged under the $U(1)_J$ symmetry generated by ∂_φ. This is necessary for the twisting to make sense. $\mathscr{A}$ defined by (4.1.35) is a gauge field on Z_6, which restricts to the supergravity gauge field A in (4.1.3) on each Σ fibre. We compute the curvature

$$\mathscr{F} = d\mathscr{A} = \rho'(y)dy \wedge (d\varphi + A_J) + \rho(y)F_J\,, \tag{4.1.36}$$

where $F_J = dA_J$. The one-form $d\varphi + A_J$ is precisely the global angular form for the $U(1)$ bundle, and so is globally defined on Z_6 away from the poles of $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ at $y = y_1, y_2$. Moreover, one can verify that $\rho'(y)dy \wedge d\varphi$ vanishes smoothly at the poles, where the angular coordinate φ is not defined, implying that $\mathscr{F}$ is a globally defined closed two-form on Z_6. By construction the integral of $\mathscr{F}$ over a fibre Σ of Z_6 satisfies (4.1.16). More generally, the integrals of wedge products over the fibres are given, for $s \in \mathbb{N}$, by

$$\int_\Sigma \left(\frac{\mathscr{F}}{2\pi}\right)^s = \frac{1}{2^s}\left(\frac{1}{n_+^s} - \frac{1}{n_-^s}\right)\left(-\frac{F_J}{2\pi}\right)^{s-1}\,. \tag{4.1.37}$$

The curvature form $\mathscr{F}$ defines a $U(1)$ bundle $\mathcal{L}$ over Z_6 by taking $c_1(\mathcal{L}) = [\mathscr{F}/2\pi] \in H^2(Z_6, \mathbb{R})$. This is different from [160], where the $U(1)$ bundle was taken to be the tangent bundle to the fibres $T_{\text{fibres}}Z_6$, which gives the Euler class (4.1.17), rather than (4.1.16). We note that at the poles we have $c_1(\mathcal{L})\,|_{y=y_1,y_2} = -\frac{1}{2n_\pm}c_1(J)$, where we have defined $c_1(J) = [F_J/2\pi] \in H^2(Z_4, \mathbb{Z})$. In the anomaly polynomial we then write

$$c_1(R_{4d}) = c_1(R_{2d}) + c_1(\mathcal{L})\,, \tag{4.1.38}$$

where R_{2d} is the pull-back of a $U(1)$ bundle over Z_4. Notice that the twisting (4.1.38) will make sense globally only if the $d = 4$ R-charges of fields satisfy appropriate quantization conditions, and for gauge-invariant operators this is equivalent to the global regularity and flux quantization conditions imposed on the supergravity solutions, *cf.* the discussion below equation (4.1.28).

The $d = 2$ anomaly polynomial is obtained by integrating $\mathcal{A}_{4d}$ in (4.1.34) over Σ. Using

(4.1.37) and (4.1.38) we compute

$$\mathcal{A}_{2d} = \frac{2a_{4d}}{27}\Big[12\Big(\frac{1}{n_+}-\frac{1}{n_-}\Big)c_1(R_{2d})^2 - 6\Big(\frac{1}{n_+^2}-\frac{1}{n_-^2}\Big)c_1(R_{2d})c_1(J) + \Big(\frac{1}{n_+^3}-\frac{1}{n_-^3}\Big)c_1(J)^2\Big]\,. \tag{4.1.39}$$

The coefficient of $\frac{1}{2}c_1(L_i)c_1(L_j)$ in $\mathcal{A}_{2d}$ is $\operatorname{tr}\gamma^3 Q_iQ_j$, where the global symmetry Q_i is associated to the $U(1)$ bundle L_i over Z_4, and γ^3 is the $d=2$ chirality operator. On the other hand, c-extremization [43] implies that the $d=2$ superconformal $U(1)_R$ extremizes

$$c_{\text{trial}} = 3\operatorname{tr}\gamma^3 R_{\text{trial}}^2\,, \tag{4.1.40}$$

over the space of possible R-symmetries. We set

$$R_{\text{trial}} = R_{2d} + \varepsilon\, J\,, \tag{4.1.41}$$

and extremize the quadratic function of ε one obtains from (4.1.39) and (4.1.40). The extremal value is

$$\varepsilon_* = \frac{3n_-n_+(n_-+n_+)}{n_-^2+n_-n_++n_+^2}\,. \tag{4.1.42}$$

The right-moving central charge is given by (4.1.40) evaluated on the superconformal R-symmetry [43]. Substituting (4.1.42) into (4.1.40), (4.1.41) we find

$$c_r = \frac{(n_--n_+)^3}{n_-n_+(n_-^2+n_-n_++n_+^2)}\frac{4a_{4d}}{3}\,. \tag{4.1.43}$$

At leading order in N, $c_r = c_l \equiv c$ is the central charge of the SCFT, and we see that the field theory result (4.1.43) precisely matches the gravity result (4.1.29). Moreover, the R-symmetry (4.1.41), with $\varepsilon = \varepsilon_*$, precisely matches the supergravity R-symmetry (4.1.31).

4.2 Multi-charge spindles

In this section we construct a family of supersymmetric AdS_3 solutions to $D=5$, $U(1)^3$ gauged supergravity. These uplift on an S^5 internal space to corresponding $\mathrm{AdS}_3\times Y_7$ solu-

tions of type IIB supergravity. The local form of the solutions was first presented in [126], where they were obtained from multi-charge superstar solutions in [161]. Shortly afterwards the same solutions were rediscovered in [154], where they arise as the uplifts of near-horizon limits of a family $D = 5$ unbalanced black ring solutions. A novelty of this section is the global analysis that we perform for the allowed values of the parameters in the solution, which is very different to the approach taken in [154] and [126]. Instead we follow the approach from section 4.1. The resulting description will be crucial to match the dual field theory computation in section 4.2.6. In section 4.2.5 we also study a special subset of these solutions, which can then be uplifted to find $\mathrm{AdS}_3 \times Y_8$ solutions to $D = 11$ supergravity, for which we can then perform a similar analysis.

4.2.1 Local form of the solutions

The action of $D = 5$, $U(1)^3$ gauged supergravity is

$$S = \frac{1}{16\pi G_{(5)}} \int \Big\{ \mathrm{d}^5x \sqrt{-\det g} \Big[R + 4 \sum_{I=1}^{3} (X^{(I)})^{-1} - \frac{1}{2} \sum_{I=1}^{3} (X^{(I)})^{-2} (\partial X^{(I)})^2 - \frac{1}{4} \sum_{I=1}^{3} (X^{(I)})^{-2} (F^{(I)})^2 \Big] - F^{(1)} \wedge F^{(2)} \wedge A^{(3)} \Big\}. \tag{4.2.1}$$

Here $A^{(I)}$, $I = 1, 2, 3$, are the three $U(1)$ gauge fields, with field strengths $F^{(I)} = \mathrm{d}A^{(I)}$, and the three scalar fields $X^{(I)}$ satisfy the constraint $X^{(1)}X^{(2)}X^{(3)} = 1$. We have set the gauge coupling constant to 1. A consistent truncation to minimal gauged supergravity may be obtained by setting the scalars $X^{(I)} = 1$ and the three gauge fields equal $A^{(I)} = A$, $I = 1, 2, 3$, where to compare to the conventions in section 4.1 one should also rescale the gauge field A by factor of $\frac{3}{2}$.

Our starting point is the following supersymmetric AdS_3 solution to this theory [154]:

$$\begin{aligned} \mathrm{d}s_5^2 &= H(x)^{1/3} \left[\mathrm{d}s^2_{\mathrm{AdS}_3} + \frac{1}{4P(x)} \mathrm{d}x^2 + \frac{P(x)}{H(x)} \mathrm{d}\phi^2 \right], \\ A^{(I)} &= \frac{x - \alpha}{x + 3K_I} \mathrm{d}\phi, \\ X^{(I)} &= \frac{H(x)^{1/3}}{x + 3K_I}. \end{aligned} \tag{4.2.2}$$

Here $\mathrm{d}s^2_{\mathrm{AdS}_3}$ a unit radius metric on AdS_3, the metric functions are the polynomials

$$\begin{aligned} H(x) &\equiv (x+3K_1)(x+3K_2)(x+3K_3) = x^3+3c_1x+c_2\,, \\ P(x) &\equiv H(x)-(x-\alpha)^2\,, \end{aligned} \tag{4.2.3}$$

and α, K_I are constants, with the latter satisfying the constraint $K_1+K_2+K_3=0$. In the first line of (4.2.3) we have introduced the elementary symmetric polynomials for the parameters K_I, namely $c_1 \equiv 3(K_1K_2+K_2K_3+K_1K_3)$, $c_2 \equiv 27K_1K_2K_3$. We note that the solution to minimal gauged supergravity studied in section 4.1 is obtained by setting $K_I=0$ for all $I=1,2,3$, which is then parametrized by the single parameter α.

Any solution of $D=5$, $U(1)^3$ gauged supergravity may be uplifted (locally) on an S^5 internal space to a solution of type IIB supergravity [161]. The ten-dimensional metric is

$$L^{-2}\mathrm{d}s^2_{10} = W^{1/2}\mathrm{d}s^2_5 + W^{-1/2}\sum_{I=1}^{3}(X^I)^{-1}\left[\mathrm{d}\mu_I^2+\mu_I^2(\mathrm{d}\phi_I+A^{(I)})^2\right]\,, \tag{4.2.4}$$

while the self-dual five-form flux is

$$\begin{aligned} L^{-4}F_5 = (1+\star_{10})\Big\{&2\sum_{I=1}^{3}[(X^{(I)})^2\mu_I^2 - WX^{(I)}]\mathrm{vol}_5 + \frac{1}{2}\sum_{I=1}^{3}(X^{(I)})^{-1}\star_5\mathrm{d}X^{(I)}\wedge\mathrm{d}(\mu_I^2) \\ &+\frac{1}{2}\sum_{I=1}^{3}(X^{(I)})^{-2}\mathrm{d}(\mu_I^2)\wedge(\mathrm{d}\phi_I+A^{(I)})\wedge\star_5F^{(I)}\Big\}\,. \end{aligned} \tag{4.2.5}$$

Here we have introduced an arbitrary length scale parameter $L>0$ into the form of the solution, which will be fixed via flux quantization later. We have denoted the volume form on $\mathrm{d}s^2_5$ by vol_5, and $\star$ denotes the Hodge duality operator in the appropriate dimension. As in [126] we have corrected the sign of the second factor in (4.2.5), compared with the expression given in [161]. We have also introduced the warp factor function

$$W = \sum_{I=1}^{3}\mu_I^2X^{(I)} > 0\,. \tag{4.2.6}$$

Finally, $\{\mu_I,\phi_I\}$ form a system of polar coordinates on $S^5\subset\mathbb{R}^2\oplus\mathbb{R}^2\oplus\mathbb{R}^2$, where correspondingly $\sum_{I=1}^3\mu_I^2=1$ and the angular coordinates ϕ_I each have period 2π. Notice that the warp factor function (4.2.6) depends on both the internal S^5 coordinates μ_I and *a priori*

on the $D = 5$ coordinates, via the scalar fields $X^{(I)}$.

Applying the uplifting formula (4.2.4) to the AdS_3 solution (4.2.2) gives rise to the following warped $\mathrm{AdS}_3 \times Y_7$ metric

$$\mathrm{d}s_{10}^2 = L^2 W^{1/2} H(x)^{1/3} \left(\mathrm{d}s_{\mathrm{AdS}_3}^2 + \mathrm{d}s_{Y_7}^2\right) , \tag{4.2.7}$$

where the seven-dimensional metric on Y_7 is

$$\begin{aligned} \mathrm{d}s_{Y_7}^2 = & \frac{\mathrm{d}x^2}{4P(x)} + \frac{P(x)}{H(x)}\mathrm{d}\phi^2 \\ & + \frac{1}{WH(x)^{2/3}} \sum_{I=1}^{3} (x + 3K_I) \left[\mathrm{d}\mu_I^2 + \mu_I^2 \left(\mathrm{d}\phi_I + \frac{(x-\alpha)}{x+3K_I}\mathrm{d}\phi\right)^2\right] . \end{aligned} \tag{4.2.8}$$

One can verify that this is the same metric as that given in section 5.2 of [126], where we identify their coordinates and parameters, in terms of those presented here, as $w = x - \alpha$, $z = -2\phi$, $q_I = \alpha + 3K_I$. In particular, reference [126] also shows that the solution is supersymmetric, with the dual $d = 2$ CFT having $(0, 2)$ supersymmetry.

4.2.2 Global analysis and general solution

Given a local $\mathrm{AdS}_3 \times Y_7$ solution of the form (4.2.8), one would like to choose the parameters α, K_I and ranges of coordinates so as to obtain a smooth metric on a compact internal space Y_7. An analysis of this was carried out in [126] (see also [154]), where one looks at the degeneration loci of Killing vector fields on Y_7, which are constant linear combinations of the four vector fields ∂_ϕ and ∂_{ϕ_I}, $I = 1, 2, 3$. We instead follow the approach of section 4.1, first obtaining a suitably regular $D = 5$ $\mathrm{AdS}_3 \times \Sigma$ solution, where $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ is a two-dimensional orbifold known as a spindle. We then appropriately quantize the $U(1)^3$ gauge field fluxes so as to obtain a regular fibration of S^5 over this five-dimensional solution.

We begin then with the $D = 5$ metric in (4.2.2), which we rewrite as

$$\mathrm{d}s_5^2 = H(x)^{1/3} \left(\mathrm{d}s_{\mathrm{AdS}_3}^2 + \mathrm{d}s_{\Sigma}^2\right) , \tag{4.2.9}$$

with the two-dimensional surface Σ having coordinates x, ϕ. In order for the metric to have the correct signature and be non-singular we would like to choose the parameters so that

$H(x) > 0$ is strictly positive, and similarly $P(x) \geq 0$. In order that Σ forms a compact surface without boundary, we furthermore assume that $x \in [x_1, x_2]$ lies between two roots of the polynomial $P(x)$ in (4.2.3). In a certain regime of parameters, that we make explicit below, we find that the three roots x_i, $i = 1, 2, 3$, of $P(x)$ are real and positive, and we then take $0 < x_1 < x_2 < x_3$ so that $P(x) \geq 0$ for $x \in [x_1, x_2]$. Defining $\varrho_i = 2(x - x_i)^{1/2}$ for $i = 1, 2$ we find that as $x \to x_i$ the metric on the two-dimensional surface Σ approaches

$$\mathrm{d}s^2_\Sigma \simeq \frac{1}{4P'(x_i)} \left(\mathrm{d}\varrho_i^2 + \kappa_i^2 \varrho_i^2 \, \mathrm{d}\phi^2\right) , \qquad \text{where} \quad \kappa_i \equiv \left| \frac{P'(x_i)}{x_i - \alpha} \right| , \quad i = 1, 2 . \tag{4.2.10}$$

As for the solution in section 4.1, it is not possible to remove both conical singularities at the roots $x = x_i$ by making a single choice $\Delta\phi$ of the period for ϕ. Instead we impose

$$\Delta\phi = \frac{2\pi}{\kappa_1 n_+} = \frac{2\pi}{\kappa_2 n_-} , \tag{4.2.11}$$

where $n_\pm \in \mathbb{N}$. The resulting surface $\Sigma = \mathbb{WCP}^1_{[n_-, n_+]}$ is then again the orbifold known as a spindle. We remember that it is topologically a two-sphere, but with conical deficit angles $2\pi(1 - \frac{1}{n_\pm})$ at the poles $x = x_1, \, x_2$.

After a little computation we find that the second equality in (4.2.11) may be written as

$$\frac{n_-}{n_+} = \left| \frac{x_2 - \alpha}{x_1 - \alpha} \right| \left| \frac{-1 + 2x_1 + x_2}{-1 + x_1 + 2x_2} \right| , \tag{4.2.12}$$

where the roots x_1, x_2 satisfy the equations

$$\begin{aligned} 2\alpha + 3c_1 &= x_1 + x_2 - x_1^2 - x_2^2 - x_1 x_2 , \\ c_2 &= \alpha^2 + x_1^2 x_2 + x_1 x_2^2 - x_1 x_2 , \end{aligned} \tag{4.2.13}$$

and we have eliminated x_3 via the relation $x_1 + x_2 + x_3 = 1$.

We will also need the fluxes of the gauge field strengths $F^{(I)} = \mathrm{d}A^{(I)}$ through the surface Σ. Using the expression for $A^{(I)}$ in (4.2.2) we compute

$$Q_I \equiv \frac{1}{2\pi} \int_\Sigma F^{(I)} = \frac{(x_2 - x_1)(\alpha + 3K_I)}{(x_1 + 3K_I)(x_2 + 3K_I)} \frac{\Delta\phi}{2\pi} . \tag{4.2.14}$$

As explained both in the previous section and in the previous chapter, for $\Sigma = \mathbb{WCP}^1_{[n_-, n_+]}$

the appropriate quantization condition on these fluxes is that

$$Q_I \equiv \frac{p_I}{n_- n_+}, \qquad \text{where} \quad p_I \in \mathbb{Z}. \tag{4.2.15}$$

Specifically, the circles parametrized by ϕ_I inside the internal space $S^5 \subset \mathbb{C}^3$ then give a well-defined orbifold circle fibration over Σ. Moreover, provided the integers p_I are coprime to both of $n_\pm$, the total space of this fibration is smooth.[3] This leads to a fibration

$$S^5 \hookrightarrow Y_7 \to \Sigma = \mathbb{WCP}_{[n_-,n_+]}, \tag{4.2.16}$$

where the total space Y_7 is a compact seven-manifold, and the twisting is determined by the integers p_I.

At this stage we have a three-parameter family of solutions, determined by the constants α, K_1, K_2, where the roots x_1, x_2 obey (4.2.13). We would like to express the solution in terms of the more physical flux parameters p_I and $n_\pm$, the ratio of the latter determined via (4.2.12). It turns out to be convenient to express quantities in terms of two of the three fluxes, which we take to be

$$p_1 \equiv q, \qquad p_2 \equiv p. \tag{4.2.17}$$

After some work, it is possible to solve the three equations (4.2.12), and (4.2.13) for α, K_1,

[3] See appendix A.1 for a detailed discussion of this.

K_2, x_1, and x_2 in terms of $n_\pm$, together with q and p. We find the rather unwieldy expressions

$$\begin{aligned}
x_1 &= \frac{1}{\mathcal{D}}\Big\{(p^2+pq+q^2)^2+3n_+pq(p+q+n_+)+2(n_+-n_-)(p+q)(p^2+q^2)\\
&\quad +(n_-^2+n_+^2)(p^2+q^2)-n_-n_+(3p^2+4pq+3q^2)+n_-n_+(n_--n_+)(p+q)\Big\}\,,\\
x_2 &= \frac{1}{\mathcal{D}}\Big\{(p^2+pq+q^2)^2-3n_-pq(p+q-n_-)+2(n_+-n_-)(p+q)(p^2+q^2)\\
&\quad +(n_-^2+n_+^2)(p^2+q^2)-n_-n_+(3p^2+4pq+3q^2)+n_-n_+(n_--n_+)(p+q)\Big\}\,,\\
K_1 &= \frac{1}{3\mathcal{D}}\Big\{3p^2(p+q)^2-(p^2+pq+q^2)^2+2(n_--n_+)(q^2-2p^2)(p+q)-2n_-n_+pq\\
&\quad -(n_-^2+n_+^2)(q^2-2p^2)+3n_-n_+(q^2-2p^2)-n_-n_+(n_--n_+)(q-2p)\Big\}\,,\\
K_2 &= \frac{1}{3\mathcal{D}}\Big\{3q^2(p+q)^2-(p^2+pq+q^2)^2+2(n_--n_+)(p^2-2q^2)(p+q)-2n_-n_+pq\\
&\quad -(n_-^2+n_+^2)(p^2-2q^2)+3n_-n_+(p^2-2q^2)-n_-n_+(n_--n_+)(p-2q)\Big\}\,,
\end{aligned} \tag{4.2.18}$$

where we have defined the denominator term

$$\mathcal{D} \equiv 3\left[(n_--n_+)(p+q)+n_-n_+-p^2-pq-q^2\right]^2. \tag{4.2.19}$$

The parameter α may then be determined from the first equation in (4.2.13).

The third flux is computed to be

$$Q_3 = \frac{n_--n_+-q-p}{n_-n_+}\,, \tag{4.2.20}$$

which notice also satisfies the quantization condition (4.2.15), with

$$p_1 = q\,, \qquad p_2 = p\,, \qquad p_3 = n_--n_+-q-p\,. \tag{4.2.21}$$

We then also have

$$Q_1+Q_2+Q_3 = \frac{n_--n_+}{n_-n_+}\,. \tag{4.2.22}$$

We find that the roots obey the assumed inequalities $0 < x_1 < x_2 < x_3$ provided

$$q, p > 0\,, \qquad q + p < n_- - n_+\,. \tag{4.2.23}$$

These inequalities also ensure that the denominator (4.2.19) is non-vanishing and hence $\mathcal{D} > 0$.

We now have a family of regular supersymmetric $\mathrm{AdS}_3 \times Y_7$ solutions, with Y_7 having the fibration structure (4.2.16), parametrized by the integers n_-, n_+, p and q. We address quantization of the five-form flux F_5 in the next subsection.

4.2.3 Central charge and R-symmetry

The AdS_3 solutions are, via the AdS/CFT correspondence, expected to be dual to $d = 2$, $(0,2)$ SCFTs. The central charge of these CFTs is computed in gravity again using the standard formula [45]

$$c = \frac{3L}{2G_{(3)}}\,, \tag{4.2.24}$$

where $G_{(3)}$ is the effective Newton constant in three dimensions. In turn the latter is computed via dimensional reduction on Y_7, and is given by [53]

$$\frac{1}{G_{(3)}} = \frac{L^7}{G_{(10)}} \int_{Y_7} W^2 H(x)^{4/3}\, \mathrm{vol}_7 = \frac{\pi^3 L^7 \Delta\phi}{2G_{(10)}}\,(x_2 - x_1)\,, \tag{4.2.25}$$

where we have explicitly evaluated the integrals over S^5 and the spindle Σ. The ten-dimensional Newton constant is

$$G_{(10)} = \frac{(2\pi)^7 g_s^2\, \ell_s^8}{16\pi}\,, \tag{4.2.26}$$

where ℓ_s and g_s are the constant string length and string coupling constant, respectively. The central charge (4.2.24) is then

$$c = \frac{3L^8 \Delta\phi}{32\pi^3 g_s^2\, \ell_s^8}\,(x_2 - x_1)\,. \tag{4.2.27}$$

In order to obtain a good string theory solution we must also quantize the flux of the closed five-form (4.2.5) through five-cycles in the ten-dimensional spacetime. In particular,

integrating F_5 through a copy of the S^5 fibre, at any point in the $D = 5$ spacetime, we find the total flux is

$$N \equiv \frac{1}{(2\pi\ell_s)^4 g_s}\int_{S^5} F_5 = \frac{L^4}{4\pi\ell_s^4\, g_s}\,. \tag{4.2.28}$$

Eliminating L in the central charge (4.2.27) using this last equation then gives

$$c = \frac{3}{2\pi}\Delta\phi(x_2 - x_1)N^2\,. \tag{4.2.29}$$

Finally, substituting the solution (4.2.18) into (4.2.29), we find the remarkably simple expression

$$c = \frac{3pq(n_- - n_+ - q - p)}{n_- n_+[(n_- - n_+)(p+q) + n_- n_+ - p^2 - pq - q^2]}N^2\,. \tag{4.2.30}$$

This is our final formula for the central charge, parametrized in terms of the integers n_-, n_+, p, q and N. It will be useful to note that (4.2.11) gives the period $\Delta\phi$ of ϕ to be

$$\frac{\Delta\phi}{2\pi} = \frac{(n_- - n_+)(p+q) + n_- n_+ - p^2 - pq - q^2}{n_- n_+(n_- + n_+)}\,, \tag{4.2.31}$$

an expression we will need momentarily.

The $U(1)$ R-symmetry of the dual $d = 2$, $(0,2)$ field theory is realized in the gravity solution as a Killing vector field $R_{2\mathrm{d}}$ of the $\mathrm{AdS}_3 \times Y_7$ gravity solution. This is defined as a certain bilinear in the Killing spinors on Y_7 [125], and combining the general results of the latter reference with the form of the solution given in [126] we can read off that this vector field is simply

$$R_{2\mathrm{d}} = \partial_\phi\,. \tag{4.2.32}$$

Here we have normalized the R-symmetry so that the Killing spinor on Y_7 has unit charge under the Lie derivative along this vector field, giving an explicit phase dependence of $\mathrm{e}^{\mathrm{i}\phi}$. On the other hand, this Killing spinor arises as a tensor product of the $D = 5$ Killing spinor with a spinor on the internal S^5. This ansatz preserves an $\mathcal{N} = 1$ spinor, out of the full $\mathcal{N} = 4$ supersymmetry of S^5, with the spinor on S^5 having charge $\frac{1}{2}$ under each of the three vector fields ∂_{ϕ_I} generating the $U(1)^3$ isometry of $S^5 \subset \mathbb{C}^3$. As discussed in section 4.1, we would like to choose a gauge for the $D = 5$ gauge fields such that the Killing spinor is uncharged

under the $U(1)$ isometry that rotates the spindle. Since a gauge transformation of the $A^{(I)}$ precisely leads to a phase rotation in the corresponding Killing spinor, which has unit charge under each $A^{(I)}$, this may be achieved via the gauge/coordinate transformation

$$\tilde{\phi}_I \;=\; \phi_I + \tfrac{2}{3}\,\phi\,, \qquad \tilde{A}^{(I)} \;=\; A^{(I)} - \tfrac{2}{3}\,\mathrm{d}\phi\,, \tag{4.2.33}$$

which ensures that $\mathrm{d}\phi_I + A^{(I)} = \mathrm{d}\tilde{\phi}_I + \tilde{A}^{(I)}$. Defining also

$$\varphi \;\equiv\; \frac{2\pi}{\Delta\phi}\,\phi\,, \tag{4.2.34}$$

so that $\Delta\varphi = 2\pi$, and completing the coordinate transformation (4.2.33) by defining

$$\tilde{\varphi} \;=\; \varphi\,, \tag{4.2.35}$$

we see that the R-symmetry vector (4.2.32) is

$$R_{2\mathrm{d}} \;=\; \partial_\phi \;=\; \frac{2\pi}{\Delta\phi}\,\partial_\varphi \;=\; \frac{2\pi}{\Delta\phi}\left(\sum_{I=1}^{3}\frac{\partial\tilde{\phi}_I}{\partial\varphi}\,\partial_{\tilde{\phi}_I} + \frac{\partial\tilde{\varphi}}{\partial\varphi}\,\partial_{\tilde{\varphi}}\right) \;=\; \frac{2}{3}\sum_{I=1}^{3}\partial_{\tilde{\phi}_I} + \frac{2\pi}{\Delta\phi}\,\partial_{\tilde{\varphi}}\,. \tag{4.2.36}$$

Note that $\partial_{\tilde{\varphi}}$ generates the $U(1)$ isometry of the spindle, that we refer to as $U(1)_{\mathcal{J}}$ in the subsequent discussion. The gauge transformation (4.2.33) leads to a phase $\mathrm{e}^{\mathrm{i}(\tilde{\phi}_1+\tilde{\phi}_2+\tilde{\phi}_3)/2}$ in the Killing spinor on Y_7, but this is now invariant under $U(1)_{\mathcal{J}}$. We may then identify the Killing vector field on S^5

$$R_{4\mathrm{d}} \;=\; \frac{2}{3}\sum_{I=1}^{3}\partial_{\tilde{\phi}_I}\,, \tag{4.2.37}$$

with the superconformal $\mathcal{N}=1$ R-symmetry before compactification of the theory on Σ, and from (4.2.36) hence write

$$R_{2\mathrm{d}} \;=\; R_{4\mathrm{d}} + \frac{n_-n_+(n_-+n_+)}{(n_--n_+)(p+q)+n_-n_+-p^2-pq-q^2}\,\partial_{\tilde{\varphi}}\,. \tag{4.2.38}$$

Here in the second term we have substituted for $\Delta\phi$ using (4.2.31). Equation (4.2.38) states that the $d=4$ $U(1)$ R-symmetry mixes with $U(1)_{\mathcal{J}}$ in flowing to the $d=2$ $U(1)$ R-symmetry in the IR. We shall recover this formula from a dual field theory calculation in section 4.2.6, along with the central charge (4.2.30).

4.2.4 Special cases

In this subsection we briefly analyse some interesting special cases of the general solutions of section 4.2.2, in particular making contact with the particular case of section 4.1.

Setting $p = q$ implies that the fluxes $Q_1 = Q_2$ are equal, and hence also $A^{(1)} = A^{(2)}$, and $X^{(1)} = X^{(2)}$. In this case (4.2.18) simplifies to

$$\begin{aligned} x_1 &= \frac{q\,(n_+ + q)[2n_-^2 - 2n_-(n_+ + 4q) + q\,(5n_+ + 9q)]}{3\,[n_-(n_+ + 2q) - q\,(2n_+ + 3q)]^2}\,, \\ x_2 &= -\frac{q\,(n_- - q)[2n_+^2 - 2n_+(n_- - 4q) - q\,(5n_- - 9q)]}{3\,[n_-(n_+ + 2q) - q\,(2n_+ + 3q)]^2}\,, \\ K_1 &= K_2 = \frac{q\,(n_- - n_+ - 3q)(n_+ + q)(n_- - q)}{9\,[n_-(n_+ + 2q) - q\,(2n_+ + 3q)]^2}\,, \end{aligned} \tag{4.2.39}$$

with corresponding central charge

$$c = \frac{3q^2(n_- - n_+ - 2q)}{n_-n_+[n_-(n_+ + 2q) - q\,(2n_+ + 3q)]}N^2\,. \tag{4.2.40}$$

This sub-family of solutions also uplift to solutions of $D = 11$ supergravity, as discussed in section 4.2.5.

Finally, setting all three charges equal gives, from (4.2.22),

$$p = q = \frac{1}{3}(n_- - n_+)\,. \tag{4.2.41}$$

This is precisely the solution to minimal gauged supergravity studied in 4.1, where we note that $p, q \in \mathbb{Z}$ then requires $n_- - n_+$ to be divisible by 3, as previously discussed.[4] Imposing (4.2.41), our central charge (4.2.30) and R-symmetry gauge field (4.2.38) reduce to the expressions given in (4.1.29) and (4.1.31).

4.2.5 M5 branes for two charge solution

In this subsection we construct a family of supersymmetric $\mathrm{AdS}_3 \times Y_8$ solutions to $D = 11$ supergravity. These arise by uplifting the $Q_1 = Q_2$ solutions of $D = 5$, $U(1)^3$ gauged

[4] In the notation of that section, we take the Kähler-Einstein four-manifold to be $\mathrm{KE}_4 = \mathbb{CP}^2$, and $k = 1$ then gives S^5 as the internal space. But this then requires $n_- - n_+$ to be divisible by the Fano index of $\mathbb{CP}^2$, which is $I = 3$. The parameters p and q in this section should not be confused with those in section 4.1, and also recall that one should rescale our current gauge field by a factor of $\frac{3}{2}$ to match to the previous conventions.

supergravity in section 4.2.4 on the Maldacena-Núñez solution [25], using the consistent truncation of [156]. We interpret these M-theory solutions as the near-horizon limit of N M5-branes wrapped on $\Sigma_g \times \Sigma$.

Romans supergravity and uplift

There are a number of different consistent truncations of type IIB supergravity on S^5, among them being both the $U(1)^3$ gauged supergravity of section 4.2.1, but also the Romans $\mathcal{N} = 4$, $SU(2) \times U(1)$ supergravity [155]. The latter preserves a different subgroup of the full $SO(6)$ R-symmetry of the internal S^5. The bosonic sector of the Romans theory contains a scalar field X, a triplet of $SU(2)$ gauge fields B^i_μ, $i = 1, 2, 3$, a $U(1)$ gauge field A_μ, and two two-forms. This theory can be further truncated by setting to zero the two-forms and truncating $SU(2)$ to its Abelian subgroup. It was shown in [162] that the resulting theory is the same as the $U(1)^3$ theory with two gauge fields $A^{(1)} = A^{(2)}$ and two scalar fields $X^{(1)} = X^{(2)}$ set equal, and in particular then

$$X = X^{(1)}\,, \qquad B^1_\mu = B^2_\mu = 0\,, \qquad B^3_\mu = A^{(1)}_\mu\,, \qquad A_\mu = A^{(3)}_\mu\,. \tag{4.2.42}$$

On the other hand, this Romans supergravity theory is also a consistent truncation of $D = 11$ supergravity, as shown in [156]. The vacuum AdS_5 solution uplifts to a warped $\mathrm{AdS}_5 \times N_6$ solution, which for a given internal space N_6 is dual to an $\mathcal{N} = 2$ SCFT in four dimensions. A particular example, studied in [156], is the Maldacena-Núñez solution, which is dual to the $d = 4$, $\mathcal{N} = 2$ SCFT living on M5-branes wrapping a Riemann surface, holomorphically embedded in a Calabi-Yau two-fold. More generally there are the $\mathrm{AdS}_5 \times N_6$ solutions of [27], corresponding to M5-branes wrapping a Riemann surface with punctures, and holographically dual to $d = 4$, $\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$. Our five-dimensional solutions with $Q_1 = Q_2$, given by (4.2.39), may thus also be uplifted to interesting classes of M5-brane solutions. Specifically, the M5-brane is wrapped on a Riemann surface (in general with punctures), which is then further wrapped on the spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ to obtain a two-dimensional theory.

In this section for simplicity we focus on the uplift on the Maldacena-Núñez solution. A solution to the $D = 5$ Romans theory with scalar field X, and Abelian gauge fields $B^3 = A^{(1)}$,

$A = A^{(3)}$, with $B^1 = B^2 = 0$ as in (4.2.42), uplifts to a $D = 11$ solution with metric [156][5]

$$\begin{aligned} L^{-2}\mathrm{d}s_{11}^2 = {} & 2^{-2/3}\Omega^{1/3}\mathrm{d}s_5^2 + 2^{1/3}X\Omega^{1/3}\left(\mathrm{d}\theta^2 + \mathrm{d}s_{\Sigma_g}^2\right) \\ & + 2^{1/3}X\Omega^{-2/3}\sin^2\theta\left(\mathrm{d}\psi + V + A^{(3)}\right)^2 + \frac{2^{-2/3}}{X^2}\,\Omega^{-2/3}\cos^2\theta D\mu_i D\mu_i\,. \end{aligned} \tag{4.2.43}$$

Here $\mathrm{d}s_5^2$ denotes the five-dimensional gauged supergravity metric, $\mathrm{d}s_{\Sigma_g}^2$ is the metric on a unit radius hyperbolic plane, quotiented by a discrete group of isometries to obtain a compact Riemann surface of genus $g > 1$, and we have introduced the warp factor function

$$\Omega \equiv \cos^2\theta + \frac{1}{2X^3}\sin^2\theta\,. \tag{4.2.44}$$

The coordinates θ, ψ, and μ_i, $i = 1, 2, 3$, are (constrained) coordinates on $S^4 \subset \mathbb{R}^2 \oplus \mathbb{R}^3$, where $\theta \in [0, \frac{\pi}{2}]$ describes the polar direction in the latter splitting. Smoothness of the metric (4.2.43) at $\theta = 0$ fixes the period $\Delta\psi = 2\pi$, and the local one-form V on Σ_g is such that $\mathrm{d}V = -\mathrm{vol}_{\Sigma_g}$. We note that

$$\mathrm{Vol}(\Sigma_g) = \int_{\Sigma_g}\mathrm{vol}_{\Sigma_g} = 4\pi(g-1) = -2\pi\,\chi(\Sigma_g)\,, \tag{4.2.45}$$

with $\chi(\Sigma_g)$ the Euler number of Σ_g. This identifies the $\mathbb{R}^2$ bundle over Σ_g, with unit circle in $\mathbb{R}^2 = \mathbb{C}$ having coordinate ψ, as $T^*\Sigma_g$. This is a local Calabi-Yau two-fold, with the $D = 11$ solution describing the near-horizon limit of a stack of M5-branes wrapped on the zero-section Σ_g. The M5-branes have corresponding normal space $\mathbb{R}^5 = \mathbb{C} \oplus \mathbb{R}^3$, and denoting the first factor by the complex line bundle $\mathcal{N}_1$, we see that

$$\mathcal{N}_1 = T^*\Sigma_g \otimes L_3\,, \tag{4.2.46}$$

where $A^{(3)}$ is a connection on a complex line bundle L_3 over the $D = 5$ spacetime. For our solution with $Q_1 = Q_2$ in (4.2.39), this is a line bundle over the spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ with charge Q_3.

Finally, μ_i, $i = 1, 2, 3$, are constrained coordinates describing a round $S^2 \subset \mathbb{R}^3$, with

[5]We note that in [156] the uplifting formula is stated for $D = 5$ gauge coupling set to $m_{\text{there}} = \frac{1}{2}$, while our $D = 5$ solutions have this quantity set to 1. We thus need to rescale our fields, and in the notation of [156] correspondingly $A^3_{\text{there}} = 2\sqrt{2}A^{(1)}$, $B_{\text{there}} = 2A^{(3)}$.

$\sum_{i=1}^3 \mu_i^2 = 1$, and the connection terms are given by

$$D\mu_1 \,=\, \mathrm{d}\mu_1 + 2A^{(1)}\mu_2\,, \qquad D\mu_2 \,=\, \mathrm{d}\mu_2 - 2A^{(1)}\mu_1\,, \qquad D\mu_3 \,=\, \mathrm{d}\mu_3\,. \tag{4.2.47}$$

Writing $\mu_1 = \sin\vartheta\cos\nu$, $\mu_2 = \sin\vartheta\sin\nu$, $\mu_3 = \cos\vartheta$ in spherical polar coordinates, the twisted metric on S^2 is

$$D\mu_i D\mu_i \,=\, \mathrm{d}\vartheta^2 + \sin^2\vartheta(\mathrm{d}\nu - 2A^{(1)})^2\,. \tag{4.2.48}$$

Since ν has $\Delta\nu = 2\pi$, we see that writing $\mathbb{R}^3 = \mathbb{C}\oplus\mathbb{R}$, the complex line $\mathbb{C}$ is twisted via L_1^2, with connection $2A^{(1)}$, so we have

$$\mathcal{N}_2 \,=\, L_1^2\,. \tag{4.2.49}$$

The total normal bundle of the M5-branes is then $\mathcal{N}_1\oplus\mathcal{N}_2\oplus\mathbb{R}$, with $\mathbb{R}$ the z-axis direction in $\mathbb{R}^3$.

The M-theory four-form G_4 may also be read off from the expression in [156], although we won't need its explicit form in what follows.

Central charge

We begin by writing the metric (4.2.43) as

$$\mathrm{d}s_{11}^2 \,=\, (16\,\Omega H(x))^{1/3}\left(\mathrm{d}s_{\mathrm{AdS}_3}^2 + \mathrm{d}s_{Y_8}^2\right)\,, \tag{4.2.50}$$

where we have uplifted the $D = 5$ solution of section 4.2.4 with $Q_1 = Q_2$. The eight-dimensional metric on Y_8 is then

$$\begin{aligned}\mathrm{d}s_{Y_8}^2 = {}& \frac{1}{4P(x)}\,\mathrm{d}x^2 + \frac{P(x)}{H(x)}\,\mathrm{d}\phi^2 + \frac{X}{2H(x)^{1/3}}\left(\mathrm{d}\theta^2 + \mathrm{d}s_{\Sigma_g}^2\right)\\ &+ \frac{X}{2\Omega H(x)^{1/3}}\sin^2\theta\left(\mathrm{d}\psi + V + A^{(3)}\right)^2 + \frac{1}{4X^2\Omega H(x)^{1/3}}\cos^2\theta D\mu_i D\mu_i\,.\end{aligned} \tag{4.2.51}$$

The central charge of the dual $d = 2$, $(0,\,2)$ field theories is again given by formula (4.2.24),

where $G_{(3)}$ is computed via dimensional reduction on Y_8. We find

$$\frac{1}{G_{(3)}} = \frac{L^8}{G_{(11)}}\int_{Y_8}(16\,\Omega H(x))^{3/2}\,\mathrm{vol}_8 = \frac{16\pi^2 L^8\Delta\phi}{3G_{(11)}}(x_2-x_1)\mathrm{Vol}(\Sigma_g)\,, \tag{4.2.52}$$

where in our conventions the eleven-dimensional Newton constant is

$$G_{(11)} = \frac{(2\pi)^8\,\ell_p^9}{16\pi}\,. \tag{4.2.53}$$

The central charge is then given by

$$c = \frac{L^9\Delta\phi}{2\pi^5\ell_p^9}(x_2-x_1)\mathrm{Vol}(\Sigma_g)\,. \tag{4.2.54}$$

We also need to impose flux quantization on the four-form in order to have a consistent M-theory solution. Integrating the expression for G_4 in [156] over a copy of the S^4 fibre we get the number of M5 branes

$$N \equiv \frac{1}{(2\pi\ell_p)^3}\int_{S^4} G_4 = \frac{L^3}{\pi\ell_p^3}\,. \tag{4.2.55}$$

Substituting for the length scale L in (4.2.54), $\mathrm{Vol}(\Sigma_g)$ given by (4.2.45), and also for x_1, x_2 in (4.2.39) and $\Delta\phi$ in (4.2.31) with $p=q$, we find the final central charge

$$c = \frac{4q^2(n_- - n_+ - 2q)}{n_-n_+[n_-(n_+ + 2q) - q(2n_+ + 3q)]}(g-1)N^3\,. \tag{4.2.56}$$

This is the central charge of a family of $d=2$, $(0,2)$ theories obtained by wrapping N M5-branes on $\Sigma_g\times\Sigma$, with $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ a spindle, and the integer q with $0<q<\frac{1}{2}(n_- - n_+)$ determines the twisting of the normal bundle of the M5-branes. We shall recover this from a dual theory calculation in section 4.2.6.

4.2.6 Field theory

Although our $\mathrm{AdS}_3\times Y_7$ solutions were first constructed in [126], the description in the latter reference meant that the dual $d=2$, $(0,2)$ SCFTs were left unidentified. However, as in section 4.1 our alternative $D=5$ construction of the solutions leads to a natural conjecture. Specifically, one begins with $\mathcal{N}=4$ SYM theory, which is holographically dual

to the $\mathrm{AdS}_5 \times S^5$ vacuum of the $D = 5$, $U(1)^3$ gauged supergravity theory. One then compactifies this theory on $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, together with background fluxes for the $U(1)^3$ Abelian symmetry given by (4.2.15). Recall here that $p_1 = q$, $p_2 = p$, $p_3 = n_- - n_+ - q - p$. The solution we have described suggests that this compactification of $\mathcal{N} = 4$ SYM flows to a $d = 2$, $(0,2)$ SCFT in the IR. We give evidence for this by computing the central charge and superconformal $U(1)$ R-symmetry using purely field theory methods, finding precise agreement with the supergravity results. There is a similar interpretation of the M5-brane solutions in section 4.2.5. At the end of this subsection we likewise compactify the theory on N M5-branes on $\Sigma_g \times \Sigma$, with appropriate background fluxes, and compute the central charge of the $d = 2$, $(0,2)$ SCFTs, again finding agreement with supergravity.

D3-brane anomaly polynomial

Generalizing the approach of section 4.1, our starting point is the anomaly polynomial for $\mathcal{N} = 4$ SYM, with background gauge field fluxes $F^{(I)}$ for the $U(1)^3 \subset SO(6)$ Abelian global symmetry group of this theory. In the large N limit this anomaly polynomial reads (see, for example, [160])

$$\mathcal{A}_{4\mathrm{d}} = c_1(F^{(1)})c_1(F^{(2)})c_1(F^{(3)})\frac{N^2}{2}\,. \tag{4.2.57}$$

Here $c_1(F^{(I)})$ denote the first Chern classes of the $U(1)$ bundles with gauge field curvatures $F^{(I)}$, respectively.

We want to compactify this theory on $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, with fluxes given by (4.2.15), and compute the anomaly polynomial of the resulting $d = 2$ theory. For this we also need to take into account the $U(1)_{\mathcal{J}}$ global symmetry in $d = 2$, coming from the $U(1)_{\mathcal{J}}$ isometry of Σ. Geometrically, this means that we want to compute the anomaly polynomial (4.2.57), where the six-manifold Z_6 on which it is defined is the total space of a Σ fibration over Z_4. We may achieve this by introducing a corresponding connection form $\mathcal{A}_{\mathcal{J}}$ for $U(1)_{\mathcal{J}}$, and replacing $\mathrm{d}\varphi \to \mathrm{d}\varphi + \mathcal{A}_{\mathcal{J}}$. In doing so it is important that the $D = 5$ Killing spinor is independent of $\tilde{\varphi} = \varphi$, which is true in the tilded gauge defined by (4.2.33), (4.2.35). We thus introduce the connection one-forms on Z_6:

$$\mathscr{A}^{(I)} \equiv \left(\frac{x-\alpha}{x+3K_I} - \frac{2}{3}\right)\frac{\Delta\phi}{2\pi}(\mathrm{d}\varphi + \mathcal{A}_{\mathcal{J}}) \equiv \rho_I(x)(\mathrm{d}\varphi + \mathcal{A}_{\mathcal{J}})\,, \quad I = 1,2,3\,, \tag{4.2.58}$$

which by construction restricts to the supergravity gauge field $\tilde{A}^{(I)}$ on each Σ fibre. We then compute the curvature

$$\mathscr{F}^{(I)} \equiv \mathrm{d}\mathscr{A}^{(I)} = \rho_I'(x)\mathrm{d}x \wedge (\mathrm{d}\varphi + \mathcal{A}_{\mathcal{J}}) + \rho_I(x)\mathcal{F}_{\mathcal{J}}\,, \tag{4.2.59}$$

where $\mathcal{F}_{\mathcal{J}} = \mathrm{d}\mathcal{A}_{\mathcal{J}}$. By construction the integral of $\mathscr{F}^{(I)}$ over a fibre Σ gives the flux Q_I, as in (4.2.14). The curvature form $\mathscr{F}^{(I)}$ defines a $U(1)$ bundle $\mathcal{L}_I$ over Z_6 by taking $c_1(\mathcal{L}_I) = [\mathscr{F}^{(I)}/2\pi] \in H^2(Z_6, \mathbb{R})$. We also define $c_1(\mathcal{J}) = [\mathcal{F}_{\mathcal{J}}/2\pi] \in H^2(Z_4, \mathbb{Z})$. In the anomaly polynomial we then write

$$c_1(F^{(I)}) = \Delta_I\, c_1(R_{2\mathrm{d}}) + c_1(\mathcal{L}_I)\,, \tag{4.2.60}$$

where $R_{2\mathrm{d}}$ is the pull-back of a $U(1)$ bundle over Z_4. The trial R-charges Δ_I satisfy the constraint $\Delta_1 + \Delta_2 + \Delta_3 = 2$, which is equivalent to the superpotential (in $\mathcal{N} = 1$ language) having R-charge 2.

The $d = 2$ anomaly polynomial is then obtained by integrating $\mathcal{A}_{4\mathrm{d}}$ in (4.2.57) over Σ:

$$\mathcal{A}_{2\mathrm{d}} = \int_\Sigma \mathcal{A}_{4\mathrm{d}} = \frac{N^2}{2}\int_\Sigma c_1(F^{(1)})c_1(F^{(2)})c_1(F^{(3)})\,, \tag{4.2.61}$$

which reads

$$\begin{aligned}\mathcal{A}_{2\mathrm{d}} = \frac{N^2}{2}\int_\Sigma \Big\{&\Delta_1\Delta_2\Delta_3 c_1(R_{2\mathrm{d}})^3 + c_1(R_{2\mathrm{d}})^2\big[\Delta_2\Delta_3 c_1(\mathcal{L}_1) + \Delta_1\Delta_3 c_1(\mathcal{L}_2)\\ &+ \Delta_1\Delta_2 c_1(\mathcal{L}_3)\big] + c_1(R_{2\mathrm{d}})\big[\Delta_3 c_1(\mathcal{L}_1)c_1(\mathcal{L}_2) + \Delta_2 c_1(\mathcal{L}_1)c_1(\mathcal{L}_3)\\ &+ \Delta_1 c_1(\mathcal{L}_2)c_1(\mathcal{L}_3)\big] + c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)c_1(\mathcal{L}_3)\Big\}\,.\end{aligned} \tag{4.2.62}$$

After a computation using the explicit functions $\rho_I(x)$ defined in (4.2.58), we find the integral in (4.2.62) gives

$$\begin{aligned}\mathcal{A}_{2\mathrm{d}} = \frac{N^2}{2}\Bigg\{&\left[\frac{q\Delta_2\Delta_3}{n_-n_+} + \frac{p\Delta_1\Delta_3}{n_-n_+} + \frac{n_- - n_+ - p - q}{n_-n_+}\,\Delta_1\Delta_2\right] c_1(R_{2\mathrm{d}})^2\\ &+ \frac{\Delta_1 f_1(n_-, n_+, p, q) + \Delta_2 f_2(n_-, n_+, p, q) + \Delta_3 f_3(n_-, n_+, p, q)}{3n_-^2 n_+^2(n_- + n_+)}\, c_1(R_{2\mathrm{d}})c_1(\mathcal{J})\\ &+ \frac{g_1(n_-, n_+, p, q)\, g_2(n_-, n_+, p, q)}{9n_-^3 n_+^3(n_- + n_+)^2}\, c_1(\mathcal{J})^2\Bigg\}\,,\end{aligned} \tag{4.2.63}$$

where we have defined

$$\begin{aligned}
f_1(n_-,n_+,p,q) &\equiv -2(n_-^2+n_+^2)(p+q)+(n_--n_+)(2p^2+7pq+4q^2)\\
&\quad +2n_-n_+(-n_-+n_++2p+3q)-q(5p^2+5pq+2q^2)\,,\\
f_2(n_-,n_+,p,q) &\equiv -2(n_-^2+n_+^2)(p+q)+(n_--n_+)(2q^2+7pq+4p^2)\\
&\quad +2n_-n_+(-n_-+n_++2q+3p)-p(5q^2+5pq+2p^2)\,,\\
f_3(n_-,n_+,p,q) &\equiv -(n_--n_+)(2p^2+pq+2q^2)-2n_-n_+(p+q)\\
&\quad +(2p^2-pq+2q^2)(p+q)\,,\\
g_1(n_-,n_+,p,q) &\equiv -(n_--n_+)(4p^2+13pq+4q^2)+4(n_--n_+)^2(p+q)\\
&\quad +4n_-n_+(n_--n_+)+9pq(p+q)\,,\\
g_2(n_-,n_+,p,q) &\equiv (n_--n_+)(p+q)+n_-n_+-(p^2+pq+q^2)\,.
\end{aligned} \tag{4.2.64}$$

c-extremization and superconformal R-symmetry

The coefficient of $\frac{1}{2}c_1(L_i)c_1(L_j)$ in the anomaly polynomial $\mathcal{A}_{2\mathrm{d}}$ is precisely $\mathrm{Tr}\,\gamma^3\mathcal{Q}_i\mathcal{Q}_j$, where the global symmetry $\mathcal{Q}_i$ is associated to the $U(1)$ bundle L_i over Z_4, and γ^3 is the $d=2$ chirality operator. We know from c-extremization [43] that the $d=2$ superconformal $U(1)$ R-symmetry extremizes the trial function

$$c_{\mathrm{trial}} \;=\; 3\,\mathrm{Tr}\,\gamma^3 R_{\mathrm{trial}}^2\,, \tag{4.2.65}$$

over the space of possible R-symmetries. We accordingly set

$$R_{\mathrm{trial}} \;=\; R_{2\mathrm{d}}+\epsilon\,\mathcal{J}\,, \tag{4.2.66}$$

so that

$$c_{\mathrm{trial}} \;=\; 3\left(\mathrm{Tr}\,\gamma^3 R_{2\mathrm{d}}^2+2\epsilon\,\mathrm{Tr}\,\gamma^3 R_{2\mathrm{d}}\mathcal{J}+\epsilon^2\,\mathrm{Tr}\,\gamma^3\mathcal{J}^2\right)\,. \tag{4.2.67}$$

Next we can substitute for the traces using (4.2.63), and extremize over the trial R-charges ϵ, Δ_I, subject to the constraint $\Delta_1+\Delta_2+\Delta_3=2$. We find the extremal values of these

parameters to be

$$\epsilon_* = \frac{n_- n_+ (n_- + n_+)}{(n_- - n_+)(p+q) + n_- n_+ - p^2 - pq - q^2}\,, \qquad \Delta_1^* = \Delta_2^* = \Delta_3^* = \frac{2}{3}\,. \tag{4.2.68}$$

The right-moving central charge, which is equal to the central charge c in the large N limit, is then given by evaluating (4.2.65) on the superconformal R-symmetry with $\epsilon = \epsilon_*$ and $\Delta_I = \Delta_I^*$. We find

$$c = \frac{3pq(n_- - n_+ - p - q)}{n_- n_+ [(n_- - n_+)(p+q) + n_- n_+ - p^2 - pq - q^2]} N^2\,. \tag{4.2.69}$$

This precisely matches the gravity computation (4.2.30), and moreover the R-symmetry (4.2.66) with $\epsilon = \epsilon_*$ and $\Delta_I = \Delta_I^*$ precisely matches (4.2.38)!

M5-brane anomaly polynomial

The large N limit of anomaly polynomial for N M5-branes is given by (see *e.g.* [160])

$$\mathcal{A}_{6\mathrm{d}} = \frac{1}{24} p_2(R) N^3\,. \tag{4.2.70}$$

Here R denotes the $SO(5)_R$ symmetry of the M5-brane theory, which geometrically is identified with the normal bundle to the M5-brane in spacetime, and p_2 denotes the second Pontryagin class. The anomaly polynomial (4.2.70) is only valid to leading order in the large N limit, and more generally receives $O(N)$ corrections involving also Pontryagin classes of the tangent bundle of the M5-brane.

The supergravity solution in section 4.2.5 has normal bundle twisted via the Cartan $U(1) \times U(1) \subset SO(5)_R$. Denoting these as two complex line bundles $\mathcal{N}_1$, $\mathcal{N}_2$, as in section 4.2.5, we may write the Pontryagin class in terms of Chern classes so that

$$\mathcal{A}_{6\mathrm{d}} = \frac{1}{24} c_1(\mathcal{N}_1)^2 c_1(\mathcal{N}_2)^2 N^3\,. \tag{4.2.71}$$

As described in section 4.2.5, the M5-branes are wrapped on $\Sigma_g \times \Sigma$, with $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$

a spindle. From the metric we saw that the normal bundles are

$$\mathcal{N}_1 \,=\, T^*\Sigma_g \otimes L_3\,, \qquad \mathcal{N}_2 \,=\, L_1^2\,, \tag{4.2.72}$$

where $L_1 \cong L_2$ because we have set the first two charges equal, $Q_1 = Q_2$, and $T^*\Sigma_g$ is the cotangent bundle to the Riemann surface Σ_g, with total space being a Calabi-Yau two-fold. The corresponding first Chern class gives minus the Euler number of this Riemann surface:

$$\int_{\Sigma_g} c_1(T^*\Sigma_g) \,=\, 2(g-1)\,. \tag{4.2.73}$$

Since $c_1(\mathcal{N}_1) = c_1(T^*\Sigma_g) + c_1(F^{(3)})$, we may first integrate the anomaly polynomial (4.2.71) over Σ_g to obtain

$$\begin{aligned}\mathcal{A}_{4\mathrm{d}} \,&=\, \int_{\Sigma_g} \mathcal{A}_{6\mathrm{d}} \,=\, \frac{N^3}{24}\cdot 2\left(\int_{\Sigma_g} c_1(T^*\Sigma_g)\right) c_1(F^{(3)})\left[2c_1(F^{(1)})\right]^2 \\ &=\, c_1(F^{(3)})c_1(F^{(1)})^2\cdot\frac{2}{3}(g-1)N^3\,.\end{aligned} \tag{4.2.74}$$

We have thus effectively reduced to the four-dimensional anomaly polynomial (4.2.57), where we must take $F^{(1)} = F^{(2)}$ and the overall factor of $N^2/2$ should be replaced by $2(g-1)N^3/3$. The rest of the computation of the $d = 2$ anomaly polynomial, obtained by integrating $\mathcal{A}_{4\mathrm{d}}$ over the spindle, and extracting the central charge, then goes through *mutatis mutandis*. The final central charge is thus given by (4.2.69) with $p = q$, and replacing an overall factor of $N^2/2$ by $2(g-1)N^3/3$. We obtain

$$c \,=\, \frac{4(n_- - n_+ - 2q)q^2}{n_-n_+[n_-(n_+ + 2q) - q(2n_+ + 3q)]}(g-1)N^3\,. \tag{4.2.75}$$

This perfectly matches the supergravity result (4.2.56).

4.3 Discussion

In this chapter we have described different families of five-dimensional solutions with a number of new properties, that have raised several directions for future research. First, we described $\mathrm{AdS}_3\times\Sigma$ solutions to $D = 5$ minimal gauged supergravity, with $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, that despite

having orbifold singularities at the $D = 5$ level, could be uplifted on specific SE_5 manifolds to obtain completely regular solutions to type IIB supergravity. We interpret our solutions as the near horizon limit of black brane solutions associated with D3-branes wrapped on the orbifold Σ. Usually this kind of construction preserves supersymmetry through the topological twist, but we have seen that in this case that does not happen. The R-symmetry flux through the spindle is not proportional to the Euler number (as it happens in the topological twist) and, centrally, the Killing spinors on the spindle are not constant. Our results suggest that there should exist black string solutions which approach AdS_5 in the UV and $\mathrm{AdS}_3 \times \Sigma$ in the IR. Such solutions will reveal the precise deformations of the $d = 4$, $\mathcal{N} = 1$ SCFTs that can then flow to the $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs dual to the $\mathrm{AdS}_3 \times \mathcal{M}_7$ of [153], including the way in which the D3-brane wrapping the spindle is preserving supersymmetry. We made a precision test of this interpretation by computing the central charge and superconformal R-symmetry of the dual $d = 2$ field theories using anomaly polynomials and c-extremization and found agreement with the gravity result.

Then we considered a multi-charge extension of these solutions, that is, a five-parameter family of $\mathrm{AdS}_3 \times Y_7$ solutions of type IIB supergravity, in which Y_7 is the total space of an S^5 fibration over a spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$. In addition to the coprime positive integers $n_- > n_+$ specifying Σ, the solution is also characterized by positive integers q, p that determine the twisting of the S^5 over Σ, together with the quantized five-form flux N through the S^5. Setting $q = p = \frac{1}{3}(n_- - n_+)$ recovers the minimal gauged solutions, and we have thus generalized those solutions via the addition of the twisting parameters q, p. As in the minimal case we interpret these as the near-horizon limit of N D3-branes wrapping the spindle Σ, with q and p now determining the background $U(1)^3$ Abelian R-symmetry fluxes via (4.0.5), (4.0.6). As a consistency check on this interpretation, we have reproduced the gravity formulas for the central charge and superconformal R-symmetry from a dual field theory computation, starting with the anomaly polynomial of $\mathcal{N} = 4$ SYM and performing an appropriate twisted compactification on Σ. Similarly, we have constructed a four-parameter family of $\mathrm{AdS}_3 \times Y_8$ solutions of $D = 11$ supergravity, which we interpret as the near-horizon limit of N M5-branes wrapping $\Sigma_g \times \Sigma$, and we have reproduced the gravitational formula for the central charge from a dual field theory computation.

Again we find for these solutions that the Killing spinors on Σ are not simply given by

a topological twist, like we found for the minimal case and for the black hole solutions of chapter 3. The physical interpretation of this, in terms of wrapped branes, is still somewhat obscure, and it would be interesting to find an analogous class of black string solutions in $D = 5$, $U(1)^3$ gauged supergravity that have our solutions as a near-horizon limit. Alternatively, one could directly attempt to study $\mathcal{N} = 4$ SYM on a rigid supersymmetric background $\mathbb{R}^2 \times \Sigma$, with background fluxes (4.0.5), (4.0.6), making more explicit the twisting of the fields and their boundary conditions at the conical singularities of Σ. In particular, one might envisage that additional data needs to be specified at the conical singularities, in describing the behaviour of the fields. We note that the anomaly polynomial method gives the correct supergravity result, despite the presence of the conical deficit singularities on Σ, and it would be interesting to justify this more carefully in such an analysis.

In section 4.2.5 we uplifted the $D = 5$, $Q_1 = Q_2$ solutions on the Maldacena-Núñez solution, but we also commented that one can uplift on other internal six-manifolds N_6 to obtain solutions to $D = 11$ supergravity. Such solutions [27] describe M5-branes wrapped on a Riemann surface, in general with punctures, which are then further wrapped on the spindle Σ to obtain $d = 2$, $(0, 2)$ SCFTs. It would be interesting to investigate these solutions in more detail, in particular computing the central charge in gravity and via the anomaly polynomial for the theory on N M5-branes.

Since the appearance in the literature of the work described in the last two chapters, many new similar solutions have been found. In [163] the authors also consider multi-charge generalizations of the minimal $\mathrm{AdS}_3 \times \Sigma$ solutions while also including rotation (thus it contains some overlap with the work of section 4.2). Also linked to the multi-charge solutions of section 4.2 is the work of [164] where they show how a tuning of the parameters in (4.2.3) can lead to solutions associated to D3-branes wrapping a topological disk. These kind of topological disk solutions started with the work of [165, 166], where the authors note that their geometry can be thought of as a "half-spindle". The study of these geometries was continued in [144, 167–170].

Another class of solutions associated to M5-branes wrapping a spindle was described in [171]. The authors find that solutions of this form exist when adding matter couplings, that is, they find $\mathrm{AdS}_5 \times \Sigma$ solutions of $D = 7$ gauged supergravity coupled to two $U(1)$ gauge fields. Again they find that supersymmetry is not preserved through a topological twist, since

the spin connection on the spindle is not equal to the R-symmetry gauge fields and the Killing spinors are in fact not constant. However, they find that the R-symmetry flux is equal to the Euler character of the spindle, as in the topological twist, and opposed to the findings of chapter 3 and 4 in this thesis. At the time this was dubbed "topologically a topological twist". Moreover, the work of [172] analyzes $\mathrm{AdS}_4 \times \Sigma$ solutions to $D = 6$ supergravity, associated to a stack of D4-branes wrapped on a spindle; and the authors find that these constructions also preserve supersymmetry through the topologically topological twist. In [147] a systematic approach was taken to analyze all possible constructions of spin^c spinors on a spindle, and find that there are just two possibilities: the "twist" (the above topologically topological twist) and the "anti-twist" (as the solutions described in this chapter). They also find some new *twist* solutions in the landscape of M2 and D3-branes wrapping spindles that we have described.

At the time of writing these have been the most fruitful directions since the appearance of these ideas in the literature. Many interesting questions however remain. From the gravity side, firstly, what type of singularities are permitted in lower-dimensional supergravity theories that can then be removed by uplifting to a completely regular solution? Secondly, it is often claimed that supersymmetry requires a topological twist when branes wrap a compact manifold, while the near horizon solutions from the last two chapters are a counterexample, and it would be interesting to understand this more systematically. Our results also suggest many questions on the field theory side. When defining a SCFT on a spindle what additional data should be specified at the orbifold points? Our holographic analysis shows that SCFTs dual to regular SE manifolds can only be placed on certain spindles and there is an apparent obstruction for those dual to irregular SE manifolds; why is this? The anomaly polynomial technique we employed is based on smooth manifolds and yet it gives a consistent result in the context of the orbifolds we studied, in the large N limit. It would be interesting to justify this more systematically and determine sub-leading contributions. All of these intriguing questions are left for future research and will undoubtedly sharpen our understanding on both sides of the holographic duality.

Appendix A

Details on accelerating black holes

A.1 Circle fibrations over spindles

In section 3.3.1 we showed that the $D = 4$ PD black hole metrics uplift to regular $D = 11$ solutions that on a fixed t, r slice are topologically Lens space $S^3/\mathbb{Z}_{\mathfrak{q}}$ bundles over the KE_6. On the other hand, we explained that this same space may also be viewed as an SE_7 fibration over a spindle/weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$. Fixing a point on the KE_6 implies that the Lens space fibre $S^3/\mathbb{Z}_{\mathfrak{q}}$ is a circle bundle over the weighted projective space, where recall $\mathfrak{q} = (k/I)(n_- - n_+)$. In this appendix we spell this out in a little more detail, discussing circle bundles over spindles more generally.

We begin with S^3, embedded inside $\mathbb{C}^2$ as the unit sphere $S^3 = \{|z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$, with z_1, z_2 standard complex coordinates. We may then consider the weighted circle action

$$(z_1, z_2) \to (\lambda^{n_+} z_1, \lambda^{n_-} z_2)\,, \tag{A.1.1}$$

where $\lambda = \mathrm{e}^{\mathrm{i}\theta} \in U(1)$, $n_\pm \in \mathbb{N}$, and note that for the action to be effective we need $\mathrm{hcf}(n_+, n_-) = 1$. The quotient $S^3/U(1) = \mathbb{WCP}^1_{[n_-,n_+]}$ is by definition a weighted projective space. This is a complex orbifold which is topologically a two-sphere with conical angles $2\pi/n_\pm$ at the poles. This is also known as a spindle. In terms of the action (A.1.1), the poles arise from $z_2 = 0$ and $z_1 = 0$, respectively, where in the first case all powers of the primitive n_+-th root of unity $\lambda = \mathrm{e}^{2\pi\mathrm{i}/n_+}$ act trivially, while in the second case this is true for $\lambda = \mathrm{e}^{2\pi\mathrm{i}/n_-}$.

It is a straightforward exercise to compute the Chern number of this fibration. That is, introduce a (2π)-period coordinate ν along the weighted $U(1)$ action, and corresponding connection one-form $\mathscr{A}$. This can be done starting from the round metric on S^3, with corresponding term $(\mathrm{d}\nu + \mathscr{A})^2$ for the metric on the circle fibre. The result[1] is

$$\int_{\mathbb{WCP}^1_{[n_-,n_+]}} \frac{\mathrm{d}\mathscr{A}}{2\pi} = \frac{1}{n_+ n_-}\,, \tag{A.1.2}$$

where notice the overall sign is a matter of convention. By construction, the total space of this circle fibration over $\mathbb{WCP}^1_{[n_-,n_+]}$ is S^3.

We may now consider a more general fibration with

$$\int_{\mathbb{WCP}^1_{[n_-,n_+]}} \frac{\mathrm{d}\mathscr{A}}{2\pi} = \frac{\mathtt{r}}{n_+ n_-}\,, \tag{A.1.3}$$

and $\mathtt{r} \in \mathbb{Z}$, and we will take $\mathtt{r} > 0$ in what follows. We shall denote the corresponding complex line bundle, on which $\mathscr{A}$ is a connection, by $O(\mathtt{r})$. In terms of the original construction of $\mathbb{WCP}^1_{[n_-,n_+]}$ as a quotient, notice that ν has period $2\pi/\mathtt{r}$. The total space of this circle fibration is then $S^3/\mathbb{Z}_{\mathtt{r}}$, where the $\mathbb{Z}_{\mathtt{r}}$ action is generated by

$$(z_1, z_2) \to (\omega_{\mathtt{r}}^{n_+} z_1, \omega_{\mathtt{r}}^{n_-} z_2)\,, \tag{A.1.4}$$

where $\omega_{\mathtt{r}} \equiv \mathrm{e}^{2\pi \mathrm{i}/\mathtt{r}}$ is a primitive $\mathtt{r}$th root of unity. In general the action (A.1.4) is not free. Specifically, the circle $S^1 = \{z_2 = 0\}$ is fixed by $\mathbb{Z}_{\mathrm{hcf}(\mathtt{r},n_+)}$, while the circle $S^1 = \{z_1 = 0\}$ is fixed by $\mathbb{Z}_{\mathrm{hcf}(\mathtt{r},n_-)}$. Thus the total space is a smooth manifold (rather than an orbifold) if and only if $\mathtt{r}$ has no common factor with either n_+ or n_-.

The main case of interest in the main text is when

$$\mathtt{r} = \mathtt{q} = \frac{k}{I}(n_- - n_+)\,. \tag{A.1.5}$$

Recall here that I/k is an integer that divides $n_- - n_+$. The gauged supergravity connection

[1] Write the metric on S^3 as $ds^2 = d\theta^2 + \cos^2\vartheta d\phi_1^2 + \sin^2\vartheta d\phi_2^2$ with $\vartheta \in [0, \pi/2]$ and $\Delta\phi_i = 2\pi$. The weighted $U(1)$ action is $V = n_+\partial_{\phi_1} + n_-\partial_{\phi_2}$. Introduce new coordinates $\phi_1 = n_+\nu$ and $\phi_2 = n_-\nu + (n_+)^{-1}\mu$ with $\Delta\nu = \Delta\mu = 2\pi$. In the new coordinates $V = \partial_\nu$, and the metric can be written as a $U(1)$ fibration over $\mathbb{WCP}^1_{[n_-,n_+]}$ as follows: $ds^2 = \Lambda(d\nu + \mathscr{A}d\mu)^2 + d\vartheta^2 + \frac{\cos^2\vartheta \sin^2\vartheta}{\Lambda} d\mu^2$, where $\mathscr{A} = \frac{n_-}{n_+}\frac{\sin^2\vartheta}{\Lambda}$ and $\Lambda = n_+^2\cos^2\vartheta + n_-^2\sin^2\vartheta$.

$2A$ is a connection on $O(\mathtt{q})$, as one sees for example in equation (3.4.16). Since $\mathrm{hcf}(n_+, n_-) = 1$, it immediately follows that $\mathrm{hcf}(\mathtt{q}, n_\pm) = 1$, and the total space of the circle fibration is smooth. In fact the construction in section 3.3.1 indirectly implies that this is diffeomorphic to the Lens space $S^3/\mathbb{Z}_\mathtt{q} = L(\mathtt{q}, 1)$. We may see this directly as follows. Since $\mathrm{hcf}(\mathtt{q}, n_+) = 1$, we can find integers $a, b \in \mathbb{Z}$ such that $an_+ + b\mathtt{q} = 1$. It follows that $a\frac{n_+}{\mathtt{q}} = -b + \frac{1}{\mathtt{q}}$, and hence

$$(\omega_\mathtt{q}^{n_+})^a = \mathrm{e}^{2\pi \mathrm{i} a n_+/\mathtt{q}} = \omega_\mathtt{q}\,. \tag{A.1.6}$$

Thus the $\mathbb{Z}_\mathtt{q}$ action (A.1.4) is equivalent to

$$(z_1, z_2) \to (\omega_\mathtt{q} z_1, \omega_\mathtt{q}^{an_-} z_2)\,, \tag{A.1.7}$$

which is the definition of the Lens space $L(\mathtt{q}, an_-)$. On the other hand, we have $an_- = a\left(n_+ + \frac{I}{k}\mathtt{q}\right) = 1 + \mathtt{q}(\frac{I}{k}a - b) \cong 1 \bmod \mathtt{q}$. Thus $L(\mathtt{q}, an_-) \cong L(\mathtt{q}, 1) = S^3/\mathbb{Z}_\mathtt{q}$, as we wanted to show.

A.2 Non-accelerating case: $\alpha = 0$

The principal focus of chapter 3 was to study accelerating black holes with $\alpha \neq 0$. However, for completeness we discuss here the case when $\alpha = 0$, also known as the Kerr-Newman-AdS spacetime [129].

We set $\alpha = 0$ in (3.2.1) and then rescale

$$\phi = \varphi\, \Xi^{-1}\,, \tag{A.2.1}$$

with $\Xi = 1 - a^2$, to ensure that the metric is well-defined at $\theta = 0$ and $\theta = \pi$, with $\varphi \in [0, 2\pi)$. The metric now reads[2]

$$ds^2 = -\frac{Q}{\Sigma}\big(dt - \frac{a\sin^2\theta}{\Xi}\,d\varphi\big)^2 + \frac{\Sigma}{Q}\,dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta\big(adt - \frac{r^2 + a^2}{\Xi}d\varphi\big)^2, \tag{A.2.2}$$

[2] We can compare with metric as given in [133], which has $g = 0$. We should make the identification $m^{\text{there}}(1 + 2\sinh^2\delta) = m$, $m^{\text{there}}\sinh 2\delta = e$ and $\tilde{r} = r$ as well as $A^{\text{there}} = -2A$.

with

$$\begin{aligned}\Sigma &= r^2 + a^2\cos^2\theta\,,\\ P &= 1 - a^2\cos^2\theta\,,\\ Q &= (r^2+a^2)(1+r^2) - 2mr + e^2 + g^2\,,\end{aligned} \tag{A.2.3}$$

and the gauge field given by

$$A = -e\frac{r}{\Sigma}\big(dt - \frac{a\sin^2\theta}{\Xi}d\varphi\big) + g\frac{\cos\theta}{\Sigma}\big(adt - \frac{r^2+a^2}{\Xi}d\varphi\big)\,. \tag{A.2.4}$$

When $\alpha = 0$, the conditions for preservation of supersymmetry need to be carried out again. We find that the integrability conditions now give the BPS constraints

$$0 = m\,g\,, \tag{A.2.5}$$

$$\begin{aligned}0 = {}& e^4 - 2a^2e^4 + a^4e^4 + 2e^2g^2 - 4a^2e^2g^2 + 2a^4e^2g^2 - 4e^4g^2 + g^4 - 2a^2g^4 + a^4g^4\\ & - 8e^2g^4 - 4g^6 - 2e^2m^2 - 2a^2e^2m^2 - 2g^2m^2 - 2a^2g^2m^2 + m^4\,.\end{aligned} \tag{A.2.6}$$

With $m > 0$ we therefore[3] take $g = 0$. The second equation can then be solved and with $a, e \geq 0$ we conclude that for BPS black holes we should take

$$g = 0\,, \qquad m = (1+a)e\,. \tag{A.2.7}$$

By studying the roots of Q we can determine that we have an extremal BPS black hole provided that

$$e = \sqrt{a}(1+a)\,, \tag{A.2.8}$$

and hence

$$m = \sqrt{a}(1+a)^2\,, \tag{A.2.9}$$

[3] As in [123] setting $m = 0$ is a solution to the BPS equations, but it gives no real positive roots to Q and hence no event horizon.

with the horizon at $r_+ = \sqrt{a}$. From (3.2.14) we can then write the total electric charge as

$$G_{(4)}Q_e = \frac{\sqrt{a}}{1-a}\,, \tag{A.2.10}$$

and from (A.5.6) the total angular momentum is

$$G_{(4)}J = \frac{a\sqrt{a}}{(1-a)^2} = \frac{G_{(4)}Q_e}{2}\left(\sqrt{1+4(G_{(4)}Q_e)^2}-1\right)\,. \tag{A.2.11}$$

Note that when $\alpha = 0$, this expression for the angular momentum, in the gauge we are using, can also be obtained from a Komar integral (A.5.1), as is often used in the literature. Finally, from (3.2.12) the entropy of these black holes is given by

$$S_{BH} = \frac{\pi}{G_{(4)}}\frac{a}{1-a} = \frac{\pi}{2G_{(4)}}(\sqrt{1+4(G_{(4)}Q_e)^2}-1) = \frac{\pi}{G_{(4)}}\frac{J}{Q_e}\,, \tag{A.2.12}$$

in agreement with the literature.

A.3 Near horizon limit

In this appendix we shall consider the near horizon limit of the PD solution (3.2.1)–(3.2.3), in the BPS and extremal case. To this end, we first write the function $Q(r)$ as

$$Q(r) = (r-r_+)^2\,\left(x_0 + x_1\,r + x_2\,r^2\right)\,, \tag{A.3.1}$$

where the constants x_i are defined by

$$\begin{aligned} x_0 &= -\alpha^2\left(a^2+e^2+g^2\right) + a^2 + 4\alpha\,g\,r_+ - 3\left(\alpha^2-1\right)r_+^2 + 1\,,\\ x_1 &= 2\alpha\,g - 2\left(\alpha^2-1\right)r_+\,,\\ x_2 &= 1-\alpha^2\,. \end{aligned} \tag{A.3.2}$$

The horizon radius, r_+, is subject to the following condition:

$$\begin{aligned}
&-4\alpha\left(a^2+e^2\right)+4\alpha^3\left(a^2+e^2+g^2\right)-5\alpha g^2\\
&+g\left(\left(a^2-5\right)\alpha^2+\alpha^4\left(-\left(a^2+e^2+g^2\right)\right)+6\right)r_+\\
&+\alpha\left(-2a^2\left(\alpha^2-1\right)^2-2\alpha^4\left(e^2+g^2\right)+\alpha^2\left(2e^2+5g^2+2\right)-2\right)r_+^2=0\,,
\end{aligned}\tag{A.3.3}$$

with the parameters obeying the BPS constraints.[4]

A convenient way to find the near horizon solution is to implement the following coordinate transformation,

$$r\to r_+ +\lambda\, s\,\rho\,,\quad t\to\lambda^{-1}\,s\,\tau\,,\quad \phi\to\phi'+\lambda^{-1}\,s\,W\frac{\Delta\phi}{2\pi}\,\tau\,,\tag{A.3.4}$$

where s is a constant, and then take the $\lambda\to 0$ limit. Here W is given by

$$W=\frac{a}{r_+^2+a^2}\frac{2\pi}{\Delta\phi}\,,\tag{A.3.5}$$

with $\partial_t+W\frac{\Delta\phi}{2\pi}\partial_\phi=\partial_t+W\partial_\varphi$ a null generator of the horizon. It is convenient to choose

$$s=\sqrt{\frac{r_+^2+a^2}{x_0+x_1\,r_++x_2\,r_+^2}}\,,\tag{A.3.6}$$

and we then find that the near horizon metric reads, after dropping the primes from the new coordinates,

$$ds_4^2=\lambda(\theta)\left(-r^2\,dt^2+\frac{dr^2}{r^2}\right)+\frac{d\theta^2}{\tilde{P}(\theta)}+\frac{(r_+^2+a^2)^2\,\sin^2\theta\,\tilde{P}(\theta)}{(1-\alpha\,r_+\,\cos\theta)^4}\left(d\phi+v r\,dt\right)^2\,,\tag{A.3.7}$$

where

$$\begin{aligned}
\tilde{\lambda}(\theta)&=\frac{a^2\cos^2\theta+r_+^2}{(1-\alpha r_+\cos\theta)^{\,2}\left(x_0+x_1\,r_++x_2\,r_+^2\right)}\,,\\
\tilde{P}(\theta)&=\frac{(1-\alpha r_+\cos\theta)^{\,2}}{a^2\cos^2\theta+r_+^2}\,P(\theta)\,,\\
v&=\frac{2ar_+}{\left(a^2+r_+^2\right)\left(x_0+x_1\,r_++x_2\,r_+^2\right)}\,.
\end{aligned}\tag{A.3.8}$$

[4] In the non-accelerating case, with $\alpha=g=0$, this condition degenerates and one instead has $r_+=\sqrt{a}$.

For the gauge field, we find that there is a piece that is singular in the $\lambda \to 0$ limit, but it can be removed by a gauge transformation. We thus implement the gauge transformation

$$A \to A - \frac{s\, r_+\, e}{\lambda\left(a^2 + r_+^2\right)}\, dt - \frac{e(a^2 - r_+^2)}{2ar_+} d\phi\,, \tag{A.3.9}$$

where the second term is included for convenience. Taking the $\lambda \to 0$ limit we then obtain the near horizon gauge field

$$A = -\frac{\left(a^2 + r_+^2\right)\left(a^2 e\cos^2\theta + 2agr_+\cos\theta - er_+^2\right)}{2ar_+\left(a^2\cos^2\theta + r_+^2\right)}\,\left(d\phi + v\, r\, dt\right)\,. \tag{A.3.10}$$

We can now show that the near horizon metric (A.3.7) and gauge field (A.3.10) are equivalent to the solution (3.4.10). The easiest way to see this is to start from (A.3.7), and perform the following coordinate transformations:

$$\tau \to -t\,, \quad \rho \to r\,, \quad y \to \frac{c_1 + c_2\,\cos\theta}{1 - \alpha\, r_+\,\cos\theta}\,, \quad z \to -\kappa\,\phi\,, \tag{A.3.11}$$

where

$$\begin{aligned} c_1 &= \frac{2\,\alpha\, r_+^3}{\sqrt{\left(x_0 + x_1\, r_+ + x_2\, r_+^2\right)\left(a^2 + \alpha^2\, r_+^4\right)}}\,, \\ c_2 &= \frac{2\, a^2}{\sqrt{\left(x_0 + x_1\, r_+ + x_2\, r_+^2\right)\left(a^2 + \alpha^2\, r_+^4\right)}}\,, \\ \kappa &= \sqrt{\frac{x_0 + x_1\, r_+ + x_2\, r_+^2}{a^2 + \alpha^2\, r_+^4}}\,\left(r_+^2 + a^2\right)\,. \end{aligned} \tag{A.3.12}$$

The parameters are then identified as follows:

$$\begin{aligned} \mathtt{j} &= \frac{2\, a\, r_+}{\sqrt{\left(x_0 + x_1\, r_+ + x_2\, r_+^2\right)\left(a^2 + \alpha^2\, r_+^4\right)}}\,, \\ \mathtt{a} &= 4\frac{-a^2\, g + 2\, a\, e\,\alpha\, r_+^2 + g\,\alpha^2\, r_+^4}{\left(x_0 + x_1\, r_+ + x_2\, r_+^2\right)\left(a^2 + \alpha^2\, r_+^4\right)}\,. \end{aligned} \tag{A.3.13}$$

Let us now turn to the non-rotating case, $a = 0$. As discussed in section 3.5.1, in the BPS

and extremal limit we can express all parameters and horizon radius in terms of α via

$$e = 0\,, \quad g = \frac{\sqrt{1-\alpha^2}}{\alpha^2}\,, \quad m = \frac{\sqrt{1-\alpha^2}}{\alpha^3}\,, \quad r_+ = \frac{\sqrt{5-4\alpha^2}-1}{2\alpha\sqrt{1-\alpha^2}}\,. \tag{A.3.14}$$

With $Q(r) = (r-r_+)^2\,(x_0+x_1\,r_++x_2\,r_+^2)$, we can now obtain simple expressions for x_0, x_1, x_2:

$$\begin{aligned} x_0 &= \frac{-2\alpha^2+\sqrt{5-4\alpha^2}+3}{2\alpha^2}\,, \quad x_1 = \frac{\sqrt{1-\alpha^2}\left(\sqrt{5-4\alpha^2}+1\right)}{\alpha}\,, \\ x_2 &= 1-\alpha^2\,. \end{aligned} \tag{A.3.15}$$

We then take the near horizon limit with the coordinate transformation (A.3.4), where we note that now since $a = 0$ we have $W = 0$, as expected since we have switched off rotation. Taking the limit $\lambda \to 0$, and again dropping the primes from the new coordinates, we get the near horizon metric

$$\begin{aligned} ds^2 = \frac{r_+^2}{(1-\alpha\,r_+\,\cos\theta)^2}\Bigg[&\frac{1}{x_0+x_1\,r_++x_2\,r_+^2}\left(-r^2\,dt^2+\frac{dr^2}{r^2}\right) \\ &+\frac{d\theta^2}{P(\theta)}+P(\theta)\,\sin^2\theta\,d\phi^2\Bigg]\,, \end{aligned} \tag{A.3.16}$$

and gauge field

$$A = -g\,\cos\theta\,d\phi\,, \tag{A.3.17}$$

where we recall that all parameters are constrained by the BPS and extremality conditions (A.3.14). It can be shown that this solution is equivalent to the non-rotating case ($\mathtt{j} = 0$) of the metric (3.4.10). To see this, starting from (3.4.10) with $\mathtt{j} = 0$, one needs to change coordinates and identify the parameters as follows:

$$\begin{aligned} &\tau \to -t\,, \quad \rho \to r\,, \quad y \to \frac{2\left(\sqrt{5-4\alpha^2}-1\right)}{\sqrt{5-4\alpha^2}\left(\left(1-\sqrt{5-4\alpha^2}\right)\cos\theta+2\sqrt{1-\alpha^2}\right)}\,, \\ &z \to -\frac{\sqrt{5-4\alpha^2}}{\alpha^2}\,\phi\,, \quad \mathtt{a} \to \frac{4\sqrt{1-\alpha^2}}{5-4\alpha^2}\,. \end{aligned} \tag{A.3.18}$$

Recall that $\sqrt{3}/2 < \alpha < 1$, which corresponds to the range $0 < \mathtt{a} < 1$.

A.4 The non-rotating $\mathrm{AdS}_2 \times Y_9$ solutions

We briefly outline how the $D = 11$ $\mathrm{AdS}_2 \times Y^9$ solutions found in [126] and further discussed in appendix D of [140] can also be viewed as solutions of $D = 4$ minimal gauged supergravity. The analysis of regularity and flux quantization which was carried out in [140] is different to what we have done in chapter 3, and we explain how they are related.

Consider the metric $ds^2(Y_9)$ given in (3.36) of [126]:

$$ds^2(Y_9) = w(y)Dz^2 + \frac{q(y)}{y^6 w(y)}D\psi^2 + \frac{4}{q(y)}dy^2 + \frac{16}{y^2}ds^2_{KE_6}\,, \tag{A.4.1}$$

where $Dz = dz - g(y)D\psi$, $D\psi = d\psi + 4B$, with $dB = 2J$ and

$$\begin{aligned} q(y) &= y^4 - 4y^2 + 4\mathtt{a}y - \mathtt{a}^2\,,\\ g(y) &= \frac{\mathtt{a} - y}{y^3 - 3y + 2\mathtt{a}}\,,\\ w(y) &= \frac{y^3 - 3y + 2\mathtt{a}}{y^3}\,. \end{aligned} \tag{A.4.2}$$

If we complete the square using the ψ coordinate we immediately find

$$ds^2(Y_9) = \frac{1}{y^2}\Big[D\psi + (1 - \frac{\mathtt{a}}{y})dz\Big]^2 + \frac{4}{q}dy^2 + \frac{q}{y^4}dz^2 + \frac{16}{y^2}ds^2_{KE_6}\,. \tag{A.4.3}$$

Assembling the $D = 11$ metric, as described in [126], we obtain

$$\begin{aligned} ds^2_{11} &= \frac{2}{3}\left[y^2 ds^2(\mathrm{AdS}_2) + \frac{4y^2}{q}dy^2 + \frac{q}{y^2}dz^2 + \Big[D\psi + (1 - \frac{\mathtt{a}}{y})dz\Big]^2 + 16ds^2_{KE_6}\right]\,,\\ &= \frac{32}{3}\left[\frac{1}{4}\Big\{\frac{y^2}{4}ds^2(\mathrm{AdS}_2) + \frac{y^2}{q}dy^2 + \frac{q}{4y^2}dz^2\Big\}\right.\\ &\qquad\left. + \Big\{\Big[\frac{1}{4}d\psi + B + \frac{1}{4}(1 - \frac{\mathtt{a}}{y})dz\Big]^2 + ds^2_{KE_6}\Big\}\right]\,. \end{aligned} \tag{A.4.4}$$

Comparing this with (3.1.2) we see this is precisely of the form to give a solution of minimal

$D = 4$ gauged supergravity with

$$\begin{aligned} ds_4^2 &= \frac{y^2}{4} ds^2(\mathrm{AdS}_2) + \frac{y^2}{q} dy^2 + \frac{q}{4y^2} dz^2, \\ A &= \frac{1}{2}(1 - \frac{\mathtt{a}}{y}) dz \,, \end{aligned} \tag{A.4.5}$$

after choosing $L = (32/3)^{1/2}$. Finally, as a check, the four-form flux in [126] can be written as

$$G = \left(\frac{32}{3}\right)^{3/2} \left[\tfrac{3}{8}\mathrm{vol}_4 - \tfrac{1}{2} *_4 F \wedge J\right] \,, \tag{A.4.6}$$

in agreement with (3.1.2).

We now return to the metric as written in (A.4.1) and recall the analysis demonstrating regularity, as discussed in [126]. The analysis of [126] begins by showing that after taking $\Delta\psi = 2\pi$, then ψ, y parametrize a smooth two-sphere. Then one shows that this two-sphere can be fibred over the KE_6 space to give an eight-dimensional manifold. Next, by choosing the period of z to be $2\pi l$, for suitably defined l, we obtain a good circle fibration over the eight-dimensional manifold. To implement the latter one shows that the periods over a basis of two-cycles on the eight-dimensional space are suitably quantized. A basis can be taken to be the S^2 fibre at a fixed point on the KE_6 together with a basis of two-cycles on KE_6 sitting at the one of two poles on the S^2, say $y = y_2$. The conditions are satisfied if and only if

$$g(y_3) - g(y_2) = l\mathtt{q} \,, \qquad g(y_2) = l\mathtt{p}/I \,, \tag{A.4.7}$$

with integers $\mathtt{p}, \mathtt{q}$. If $\mathtt{p}, \mathtt{q}$ have no common factor then the nine-dimensional space Y_9 is simply-connected. As shown in [126] these conditions imply

$$\begin{aligned} \mathtt{a} &= \frac{I\mathtt{q}(2\mathtt{p} + I\mathtt{q})}{2\mathtt{p}^2 + 2I\mathtt{pq} + I^2\mathtt{q}^2} \,, \\ l^2 &= \frac{I^2(2\mathtt{p}^2 + 2I\mathtt{pq} + I^2\mathtt{q}^2}{2\mathtt{p}^2(\mathtt{p} + I\mathtt{q})^2} \,. \end{aligned} \tag{A.4.8}$$

This construction can also be viewed as a fibration of a Lens space $S^3/\mathbb{Z}_\mathtt{q}$ over the KE_6. To

see this more explicitly we can use canonical coordinates on $S^3/\mathbb{Z}_{\mathtt{q}}$, with the $\mathbb{Z}_{\mathtt{q}}$ identification acting on a Hopf fibre coordinate. We consider[5] $\psi = \gamma$ and $z = \bar{\alpha} + \lambda\gamma$ and we fix the constant λ momentarily. Observe that this implies $\partial_\psi = \partial_\gamma - \lambda\partial_{\bar{\alpha}}$ and $\partial_z = \partial_{\bar{\alpha}}$. In the new coordinates the metric reads

$$ds^2_{Y_9} = w(y)(d\bar{\alpha} - \tilde{g}D\gamma - 4\lambda B)^2 + \frac{q}{y^6 w(y)}D\gamma^2 + \frac{4}{q}dy^2 + \frac{16}{y^2}ds^2_{KE_6}\,, \tag{A.4.9}$$

with $D\gamma = d\gamma + 4B$ and $\tilde{g}(y) \equiv g(y) - \lambda$. If we take $\Delta\gamma = 2\pi$ then γ, y again parametrize an S^2, which is fibred over the KE_6 manifold. We now consider the $\bar{\alpha}$ circle fibred over this eight-dimensional base. To see the Lens space structure, we fix a point on the KE_6 base and fix the constant λ by demanding $\tilde{g}(y_3) + \tilde{g}(y_2) = 0$, which ensures that $\bar{\alpha}$ is a Hopf fibre coordinate on the resulting $S^3/\mathbb{Z}_{\mathtt{q}}$. More specifically, we choose $\lambda = \frac{l}{2}(\mathtt{q} + 2\mathtt{p}/I)$ and then we have

$$\tilde{g}(y_3) = -\tilde{g}(y_2) = \frac{l\mathtt{q}}{2}\,. \tag{A.4.10}$$

Taking $\Delta\gamma = 2\pi$ and $\Delta\bar{\alpha} = 2\pi l$ then implies that we have an $S^3/\mathbb{Z}_{\mathtt{q}}$ fibre. That the $S^3/\mathbb{Z}_{\mathtt{q}}$ is suitably fibred over the KE_6 base is easily demonstrated: taking a basis of two-cycles, $\Sigma_a(y_2)$, to be again located at $y = y_2$, for example we calculate

$$\begin{aligned}\frac{1}{2\pi l}\int_{\Sigma_a(y_2)} d[d\bar{\alpha} - \tilde{g}D\gamma - 4\lambda B] &= -\frac{1}{2\pi l}(\tilde{g}(y_2) + \lambda)\int_{\Sigma_a(y_2)}\rho \\ &= -\mathtt{p}s_a\,,\end{aligned} \tag{A.4.11}$$

for integers s_a.

We can also introduce coordinates on $S^3/\mathbb{Z}_{\mathtt{q}}$ via $\bar{\alpha}_\pm = \gamma \mp \frac{1}{l\mathtt{q}}\bar{\alpha}$. In these coordinates we can write the vectors in the original ψ, z coordinates as $\partial_z = -\frac{1}{lq}(\partial_+ - \partial_-)$ and $\partial_\psi = \partial_+ + \frac{\mathtt{p}}{I\mathtt{q}}(\partial_+ - \partial_-)$, with the latter directly analogous to (3.3.18). This underscores that while in the regularity construction using the z, ψ coordinates, $\Delta\psi = 2\pi$ was required to ensure that y, ψ formed an S^2, it is not the case, in general, that ψ is a periodic coordinate on the three-dimensional Lens space, i.e. under the motion of ∂_ψ by 2π one does not return to the

[5] To compare with [126] we should identify $\bar{\alpha} = \alpha^{there}$.

same point.[6]

A final point is that the R-symmetry Killing vector can be obtained from [126], and we find that it is given by

$$R = 2\partial_\psi + 2\partial_z \,. \tag{A.4.12}$$

This agrees with our general formula (3.4.49) in the $\mathtt{j} = 0$ limit.

A.5 Angular momentum

In this section we discuss the angular momentum of the PD black holes (3.2.1), (3.2.3). We will also calculate the angular momentum of the near horizon AdS_2 solutions that we constructed in appendix A.3, and explain how these two quantities differ in general. Finally, we shall derive the formula (3.0.2).

A common way of defining the angular momentum is via a Komar integral associated to the spacelike Killing vector $k = \partial_\varphi$, where φ is the angular coordinate on the spindle with period 2π (see (3.2.10)). For the associated one-form k one takes

$$J_K(r) \equiv \frac{1}{16\pi G_{(4)}} \int_{\Sigma_r} *dk \,, \tag{A.5.1}$$

where the integral is over the surface Σ_r, parametrized by θ, ϕ at fixed r and t. While this integral is gauge invariant, it clearly depends on the radial coordinate r since, in general, $d * dk \neq 0$.

Instead we will adopt a different definition which does not depend on the radial position where the integral is done, and hence it can equally be evaluated at the black hole horizon. While this an attractive feature, and is the definition that is expected to appear in the First Law [128], the price we pay is that the integral changes under gauge transformations. We first define the two-form

$$Z = dk + 4(A \cdot k)F \,, \tag{A.5.2}$$

[6]This simple, but potentially confusing, point can be made even more explicit. If we forget about the KE_6 we note that for *any* value of λ we still obtain a Lens space parametrized by $\bar{\alpha}, \gamma$ with $\Delta\gamma = 2\pi$ and $\Delta\bar{\alpha} = 2\pi l$. However, since $\partial_\psi = \partial_\gamma - \lambda\partial_{\bar{\alpha}}$ it is clear that the orbits of ∂_ψ will not close in general.

for an arbitrary Killing vector k which satisfies, on-shell,

$$\nabla_\mu Z^{\mu\nu} = -k^\nu \mathcal{L}\,, \tag{A.5.3}$$

where $\mathcal{L}$ is the Lagrangian. For $k = \partial_\varphi$ we can then define the angular momentum via the horizon integral

$$J(A) = \frac{1}{16\pi G_{(4)}} \int_{\Sigma_{r_+}} *Z\,. \tag{A.5.4}$$

With the gauge field as in (3.2.3), we obtain

$$J \equiv J(A) = \frac{1}{G_{(4)}} m\, a \left(\frac{\Delta\phi}{2\pi}\right)^2 . \tag{A.5.5}$$

For the simpler case of the Kerr-Newman-AdS black holes with $\alpha = 0$ as in (A.2.2), we then have

$$G_{(4)} J = \frac{ma}{(1-a^2)^2}\,, \tag{A.5.6}$$

in agreement with the results given in [128, 133], for example. Note also that for the case of $\alpha = 0$, the expressions (A.5.5), (A.5.6) can also be obtained from the Komar integral (A.5.1); we have not checked whether this also happens to be true when $\alpha \neq 0$ (and one should be aware that the conformal boundary is not obtained simply by taking $r \to \infty$).

We would also like to calculate the angular momentum using the near horizon AdS_2 solution (3.4.10). However, (A.5.4) is not gauge invariant and since in the derivation of (3.4.10) we performed some gauge transformations (see in particular eq. (A.3.9)), some care is required. We first note that if we consider gauge transformations of the form

$$\tilde{A} = A + \alpha_1\, dt + \alpha_2\, d\phi\,, \tag{A.5.7}$$

for constant α_i, then

$$J(\tilde{A}) = J(A) + \alpha_2\, Q_e \frac{\Delta\phi}{2\pi}\,. \tag{A.5.8}$$

Notice, in particular, that this does not depend on α_1. Demanding that the gauge field of the full black hole solution is regular on the horizon effectively fixes α_1, as we explain in section A.5.1. However, the freedom in choosing α_2 is associated with how one defines the electric potential, as we discuss in section A.5.2. In section A.5.3 we then calculate the angular momentum for the near horizon AdS_2 solution (3.4.10) and explain how it is related to that of the black hole (A.5.5).

A.5.1 Regularity of the gauge field on the horizon

We now investigate the gauge transformation that is required in order to ensure that the full black hole solution (3.2.1), (3.2.3) is well-defined on the horizon of the black hole. To do so, we Wick rotate $t \to \mathrm{i}\, t_E$, as well as $a \to \mathrm{i}\, a_E$, $e \to \mathrm{i}\, e_E$, and define the Euclidean angular velocity

$$\Omega_E \equiv \mathrm{i}\, W \equiv -\frac{a_E}{r_+^2 - a_E^2}\frac{2\pi}{\Delta\phi}\,. \tag{A.5.9}$$

We next change coordinates $t_E \to \tilde{t}_E$ and $\phi \to \tilde{\phi} + \frac{\Delta\phi}{2\pi}\Omega_E\, t_E$, and consider the metric near the horizon $r \sim r_+$, where r_+ is the outer horizon, defined by the largest positive root of $Q(r)$. We find

$$ds^2 \simeq (r - r_+) f(\theta) d\tilde{t}_E^2 + \frac{f(\theta)}{4\kappa^2\,(r - r_+)} dr^2 + h(\theta) d\tilde{\phi}^2 + (r - r_+)\, g(\theta)\, d\tilde{\phi}\, d\tilde{t}_E\,, \tag{A.5.10}$$

with κ the surface gravity. Changing variables via

$$x = \sqrt{r - r_+}\, \cos(\kappa\, \tilde{t}_E)\,, \qquad y = \sqrt{r - r_+}\, \sin(\kappa\, \tilde{t}_E)\,, \tag{A.5.11}$$

with (x, y) parametrizing $\mathbb{R}^2$, we get

$$ds^2 \simeq \frac{f(\theta)}{\kappa^2}(dx^2 + dy^2) + \frac{g(\theta)}{\kappa}(x\, dy - y\, dx) d\tilde{\phi} + h(\theta) d\tilde{\phi}^2\,, \tag{A.5.12}$$

which is clearly regular near the origin $x = y = 0$.

For the gauge field given in (3.2.3), we find

$$\begin{aligned} A &= f_1(r,\theta)d\tilde{t}_E + f_2(r,\theta)d\tilde{\phi}\,, \\ &= \frac{f_1(r,\theta)}{\kappa}\frac{x\,dy - y\,dx}{x^2+y^2} + f_2(r,\theta)d\tilde{\phi}\,. \end{aligned} \tag{A.5.13}$$

Since $f_2(r_+,\theta)$ is finite, the last term is well behaved at the horizon. However, in the gauge (3.2.3) we find that $f_1(r,\theta)$ does not vanish fast enough as $r \to r_+$ to ensure regularity. Let us consider the gauge transformation given in (A.5.7) and choose

$$\alpha_1 = \frac{r_+\, e}{r_+^2 + a^2} - \frac{\Delta\phi}{2\pi} W \alpha_2\,, \tag{A.5.14}$$

with α_2 arbitrary. We then find

$$\tilde{A} = \frac{\tilde{f}_1(r,\theta)}{\kappa}\frac{x\,dy - y\,dx}{x^2+y^2} + \left(f_2(r,\theta) + \alpha_2\right)\, d\tilde{\phi}\,, \tag{A.5.15}$$

with $\tilde{f}_1(r,\theta) \equiv f_1(r,\theta) - \frac{r_+\, e_E}{r_+^2 - a_E^2}$ and, as $r \to r_+$,

$$\tilde{f}_1(r,\theta) \sim (r - r_+)F(\theta) = (x^2+y^2)\,F(\theta)\,. \tag{A.5.16}$$

Thus, this gauge field $\tilde{A}$ is regular on the horizon, for any value of α_2. As we discussed above, the choice of α_1 does not affect the value of the angular momentum of the black hole, $J(A)$ as defined in (A.5.4), but the choice of α_2 will.

Notice that the value of α_1 in (A.5.14) with α_2 given by

$$\alpha_2 = \frac{e(a^2 - r_+^2)}{2ar_+}\,, \tag{A.5.17}$$

exactly matches the gauge transformation that we performed in (A.3.9). That the value of α_1 agrees is exactly as expected, since this is the gauge transformation which was required in that analysis in order to have a regular gauge field in the near horizon limit. The last term in (A.3.9) corresponds to the specific choice of α_2 given in (A.5.17) and, as noted below (A.3.9), was added for convenience, a point we return to in section A.5.3.

A.5.2 The electric potential

To further illuminate the role of gauge transformations parametrized by α_2, we discuss the electric potential Φ. We define $\Phi = \Phi_\infty - \Phi_H$, where [122]

$$\Phi_\infty = \frac{1}{4\pi\, G_{(4)} Q_e\, \beta} \int_{\partial M} \sqrt{-h}\, n_a\, F^{ab}\, A_b\,, \tag{A.5.18}$$

is the potential as $r \to \infty$. Here h_{ab} is the induced metric and n_a the normal vector to the hypersurface $r \to \infty$, while $\beta = 1/T$ is the inverse temperature. On the other hand we define

$$\Phi_H = \ell \cdot A|_{r\to r_+}\,, \tag{A.5.19}$$

to be the potential on the horizon $r = r_+$, where $\ell = \partial_t + \frac{\Delta\phi}{2\pi} W\, \partial_\phi = \partial_t + W\, \partial_\varphi$ is the null generator of the horizon and W was defined in (A.5.9). In general, Φ depends on the choice of gauge for the gauge field.

For the black hole solutions in the gauge (3.2.3) we find $\Phi_\infty(A) = 0$, $\Phi_H(A) = -\frac{r_+\, e}{r_+^2 + a^2}$ and hence

$$\Phi(A) = \frac{r_+\, e}{r_+^2 + a^2}\,. \tag{A.5.20}$$

On the other hand after a general gauge transformation of the form (A.5.7), with general α_1 and α_2, we find $\Phi_\infty(\tilde{A}) = \alpha_1$, $\Phi_H(\tilde{A}) = \alpha_1 + \frac{\Delta\phi}{2\pi} W \alpha_2 - \frac{e r_+}{a^2 + r_+^2}$ with

$$\Phi(\tilde{A}) = \frac{r_+\, e}{r_+^2 + a^2} - \frac{a}{r_+^2 + a^2}\, \alpha_2\,. \tag{A.5.21}$$

Thus $\Phi(\tilde{A})$ depends on α_2 but not α_1. We also observe that the choice of α_1 that makes the gauge field regular on the horizon given in (A.5.14) has the feature that $\Phi_H(\tilde{A}) = 0$.

It is interesting to observe that while $\Phi(A)$ and $J(A)$ both depend on the choice of gauge transformations parametrized by α_2, the combination $WJ + \Phi Q_e$ does not. This will similarly be true for the first law, which necessarily should be gauge invariant.

A.5.3 The angular momentum from the near horizon solution

Let us now consider computing the angular momentum for the near horizon solution (3.4.10). Suppose that in taking the near horizon limit in appendix A.3, we had instead started with the regular gauge field given by (A.5.7), with α_1 given by (A.5.14) and $\alpha_2 = 0$. Then there would be no need to perform the singular gauge transformation in (A.3.9), and we would arrive at a gauge field for the near horizon solution which we will call A_1, which we give below. On the other hand in the expression for the gauge field given in (A.3.9), which we will call A_2, we did an additional gauge transformation associated with α_2 as given in (A.5.17). Specifically, we have

$$\begin{aligned} A_2 &= -\frac{\left(a^2+r_+^2\right)\left(a^2 e\cos^2\theta + 2agr_+\cos\theta - er_+^2\right)}{2ar_+\left(a^2\cos^2\theta + r_+^2\right)}\left(d\phi + v\, r\, dt\right), \\ &= h(y)\left(dz + \mathtt{j}\,\rho\, d\tau\right), \\ A_1 &= A_2 + \frac{e(a^2-r_+^2)}{2ar_+}d\phi\,. \end{aligned} \tag{A.5.22}$$

Notice that A_2 (as in (A.3.10)) has the appealing feature that it is clearly AdS_2 invariant[7]; indeed this was the motivation for carrying out the additional gauge transformation in (A.3.9).

Now from the discussion in this appendix we know that $J(A_1)$ will be the angular momentum that agrees with that of the black hole solution in (A.5.5). Indeed we find

$$J \equiv J(A_1) = \frac{1}{G_{(4)}} m\, a \left(\frac{\Delta\phi}{2\pi}\right)^2. \tag{A.5.23}$$

However, this *differs* from the angular momentum calculated in the AdS_2 invariant gauge A_2. Specifically, if we denote this by J_{AdS_2} we have

$$\begin{aligned} J_{AdS_2} \equiv J(A_2) &= \frac{1}{G_{(4)}}\frac{1}{4}\mathtt{j}\sqrt{1-\mathtt{j}^2}\left(\frac{\Delta z}{2\pi}\right)^2, \\ &= \frac{1}{G_{(4)}}\frac{\mathtt{j}\sqrt{1-\mathtt{j}^2}\left(n_-^2+n_+^2\right)}{8\left(1-2\mathtt{j}^2\right)n_-^2 n_+^2}, \end{aligned} \tag{A.5.24}$$

[7] Let $P = \partial_t$, $D = t\,\partial_t - r\,\partial_r$ and $K = -\frac{1}{2}(t^2 + r^{-2})\partial_t + t\,r\,\partial_r$ be the standard generators for the isometries of AdS_2. We can lift this to an action on $AdS_2 \times S^1$ using P, D and $\tilde{K} \equiv K + \frac{1}{v\,r}\partial_\phi$, which satisfy the same algebra. We then notice that the one form $D\phi = d\phi + v\,r\,dt$, which appears in (A.5.22), is invariant, satisfying $\mathcal{L}_{P,D,\tilde{K}} D\phi = 0$.

where Δz is given in (3.4.15). Using this, together with (3.4.18), we find that

$$J_{\mathrm{AdS}_2} = Q_e \frac{\sqrt{8n_-^2 n_+^2 (G_{(4)}Q_e)^2 + n_-^2 + n_+^2}}{2\sqrt{2}n_- n_+} \,. \tag{A.5.25}$$

Comparing to the similar expression (3.4.20) for J, we deduce that

$$J_{\mathrm{AdS}_2} - J = \frac{Q_e}{4} \frac{n_- + n_+}{n_- n_+} \,, \tag{A.5.26}$$

which is precisely (3.0.2).

A.6 Killing spinor equations

We first recall the Killing spinors on a SE_7. The metric is given by

$$ds^2 = ds^2(KE_6) + (\tfrac{1}{4}d\psi + \sigma)^2 \,. \tag{A.6.1}$$

We take purely imaginary $D = 7$ gamma matrices with $\rho_{1234567} = -\mathrm{i}$. The Killing spinor equation for the SE_7 is taken to be

$$\mathcal{D}_A \chi = \frac{\mathrm{i}}{2} \rho_A \chi \,. \tag{A.6.2}$$

After introducing the obvious orthonormal frame, we can solve this equation as in e.g. [173]. We impose the following projections on the Killing spinor:

$$\rho_{12}\chi = \rho_{34}\chi = \rho_{56}\chi = \mathrm{i}\chi, \qquad \Rightarrow \qquad \rho_7 \chi = \chi \,, \tag{A.6.3}$$

and find that

$$\chi = \mathrm{e}^{\mathrm{i}\psi/2}\chi_0 \,, \tag{A.6.4}$$

where χ_0 is a spinor on KE_6 satisfying

$$D_m\chi_0 = 2\mathrm{i}\sigma_m\chi_0\,, \tag{A.6.5}$$

where here D_m is the covariant derivative on the KE_6, which always has a solution. Notice in particular, that the spinor has the dependence $\mathrm{e}^{\mathrm{i}\psi/2}$ mentioned in section 3.1.

We turn now to the $D = 11$ KS equation as given in [136]:

$$\nabla_M\varepsilon + \frac{1}{12\times 4!}\Gamma_M{}^{N_1N_2N_3N_4}G_{N_1N_2N_3N_4}\varepsilon - \frac{1}{6\times 3!}\Gamma^{N_1N_2N_3}G_{MN_1N_2N_3}\varepsilon = 0\,. \tag{A.6.6}$$

We decompose the $D = 11$ Clifford algebra via

$$\begin{aligned}\Gamma_a &= -\mathrm{i}\gamma_a\gamma_5\otimes 1_8\,, \qquad & a &= 0,1,2,3\,,\\ \Gamma_{A+3} &= \gamma_5\otimes\rho_A\,, & A &= 1,\ldots,7\,.\end{aligned} \tag{A.6.7}$$

where $\gamma_5 \equiv -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$. We then substitute the $D = 11$ uplift of the $D = 4$ solution given (3.1.2) into (A.6.6). The directions of the equation tangent to the SE_7 are satisfied with χ solving (A.6.2), as above. For the remaining directions, using a frame adapted to ds_4^2, we find

$$\left(\nabla_a - \mathrm{i}A_a + \frac{1}{2}\gamma_a + \frac{\mathrm{i}}{4}F_{bc}\gamma^{bc}\gamma_a\right)\epsilon = 0\,, \tag{A.6.8}$$

in agreement with e.g. (2.1) of [123] (with a different sign choice in the definition of γ_5).

In order to construct spinor bilinears in section 3.4.3 it is helpful to identify various intertwiners. In $D = 7$ we take $A_7 = C_7 = 1_8$ and hence $\rho_i = \rho_i^\dagger = -\rho_i^T$. For $D = 4$, for the gamma matrices as in (3.4.28) we can take

$$\begin{aligned}A_4 &= \sigma^1\otimes\sigma^3\,, \qquad & A_4\gamma_a\,A_4^{-1} &= \gamma_a^\dagger\,, \qquad & A_4^\dagger &= A_4\,,\\ C_4 &= \sigma^2\otimes\sigma^1\,, & C_4^{-1}\,\gamma_a\,C_4 &= -\gamma_a^T\,, & C_4^T &= -C_4\,.\end{aligned} \tag{A.6.9}$$

We can also define $D_4 = C_4\,A_4^T = -\sigma^3\otimes\sigma^2$ and $\tilde{D}_4 = \gamma_5 D_4 = i1\otimes\sigma^1$. Notice that $D_4D_4^* = -1$, so cannot be used for a Majorana condition. However, $\tilde{D}_4\tilde{D}_4^* = +1$, so we can

define a four-dimensional Majorana spinor as one satisfying $\epsilon = \tilde{D}_4\,\epsilon^*$. We can define the barred $D = 4$ spinor as $\bar{\epsilon} = \epsilon^\dagger A_4$.

For the $D = 11$ intertwiners we then have

$$\begin{aligned} &A_{11} = A_4 \otimes C_7 \to A_{11}\,\Gamma_M\,A_{11}^{-1} = -\Gamma_M^\dagger\,, \qquad && A_{11}^\dagger = A_{11}\,, \\ &C_{11} = \gamma_5 C_4 \otimes C_7 \to C_{11}^{-1}\,\Gamma_M\,C_{11} = -\Gamma_M^T\,, \qquad && C_{11}^T = -C_{11}\,, \end{aligned} \tag{A.6.10}$$

where $\gamma_5 C_4 = \sigma^1 \otimes \sigma^2$. We also have $D_{11} = C_{11}\,A_{11}^T = \tilde{D}_4 \otimes 1_8$ and we note that $D_{11} D_{11}^* = +1$ so that the $D = 11$ Majorana condition is $\varepsilon = D_{11}\,\varepsilon^*$. We can also define the barred $D = 11$ spinor as $\bar{\varepsilon} = \varepsilon^\dagger A_{11}$. Thus if we have a $D = 11$ spinor $\varepsilon = \epsilon \otimes \chi$ which is possibly complex (i.e. constructed from two $D = 11$ Majorana spinors) we have

$$\varepsilon = \epsilon \otimes \chi\,, \qquad \bar{\varepsilon} = \bar{\epsilon} \otimes \chi^\dagger\,. \tag{A.6.11}$$

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