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Measures on the Lattice of Closed Inner Ideals in a Spin Triple

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Two elements J and K of the complete lattice $\mathscr{I}(A)$ of weak*-closed inner ideals in a JBW*-triple A are said to be centrally orthogonal if there exists a weak*-closed ideal I in A such that $A_2(J) \subseteq A_2(I)$ and $A_2(K) \subseteq A_0(I)$, and are said to be rigidly collinear when $A_2(J) \subseteq A_1(K)$ and $A_2(K) \subseteq A_1(J)$, where, for j equal to 0, 1, or 2, $A_j(I)$, $A_j(J)$, and $A_j(K)$, are the components in the generalized Peirce decomposition of A relative to the weak*-closed inner ideals I, J, and K, respectively. A measure m on $\mathscr{I}(A)$ is a mapping from $\mathscr{I}(A)$ to $\mathbb{C}$ such that, if J and K are either centrally orthogonal or rigidly collinear, then

$$m(J \vee K) = m(J) + m(K).$$

A complex Hilbert space A endowed with a conjugation possesses a triple product and norm with respect to which it forms a JBW*-triple, known as a spin triple. In this paper the structure of the complete lattice $\mathscr{I}(A)$ of closed inner ideals in a spin triple A and the measures on it are investigated. It is shown that, if the dimension of A is greater than 5, then there are no non-zero measures on $\mathscr{I}(A)$. When the dimension of A is 5, non-zero measures exist and, up to multiplication by a constant, a unique measure exists that is invariant under automorphisms of A. When the dimension of A is 4, then A is triple isomorphic to the W^*-algebra of 2×2 complex matrices. In this case results of Bunce and Wright are used to show that there is an uncountable number of measures on $\mathscr{I}(A)$. The situation when the dimension of A is less than 4 is also described. © 2000 Academic Press

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1. INTRODUCTION

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a\ b\ c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable, and, for

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elements a, b, c, and d in A, satisfies the identity

$$[D(a,b),D(c,d)] = D(\{a \quad b \quad c\},d) - D(c,\{d \quad a \quad b\}),$$

where D is the mapping from $A \times A$ to the algebra of linear mappings from A to itself, defined by

$$D(a,b)c = \{a \quad b \quad c\},$$

is said to be a *Jordan*-triple*. When A is a Banach space and D is continuous from $A \times A$ to the Banach algebra of bounded linear operators on A and, for each element a in A, $D(a,a)$ is hermitian with non-negative spectrum and satisfies

$$\|D(a,a)\| = \|a\|^2,$$

the Jordan*-triple A is said to be a *JB*-triple*. If A is the dual of a Banach space A_* then A is said to be a *JBW*-triple*. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For the general theory of JBW*-triples the reader is referred to [4, 5, 12, 13, 17, 24, 28, 34–37, 41].

A subspace J of the JBW*-triple A is said to be a *subtriple* of A if $\{J \ J \ J\}$ is contained in J, is said to be an *inner ideal* in A if $\{J \ A \ J\}$ is contained in J, and is said to be an *ideal* in A if $\{A \ J \ A\}$ and $\{J \ A \ A\}$ are contained in J. Observe that a weak*-closed subtriple of a JBW*-triple is itself a JBW*-triple. Since the intersection of a family of weak*-closed inner ideals in A is a weak*-closed inner ideal in A, the set $\mathcal{I}(A)$ of weak*-closed inner ideals in A, when ordered by set inclusion, forms a complete lattice. A linear projection P on the JBW*-triple A is said to be a *structural projection* if, for all elements a, b, and c in A,

$$P\{a \quad Pb \quad c\} = \{Pa \quad b \quad Pc\}.$$

It was shown in [16, 19, 20] that the mapping $P \mapsto PA$ is a bijection from the set of structural projections on A to $\mathcal{I}(A)$ and, hence, that every structural projection is contractive and weak*-continuous, and the set of structural projections forms a complete lattice $\mathcal{S}(A)$ order isomorphic to $\mathcal{I}(A)$. The complete lattices $\mathcal{S}(A)$ and $\mathcal{I}(A)$ have quite complicated structures which have been investigated in [15, 16, 20, 21]. In particular, it is the case that a structural projection is an M-projection if and only if its range is an ideal in A [1, 2, 4, 10, 11, 28]. A JBW*-triple A, the only weak*-closed ideals in which are $\{0\}$ and A, is said to be a *JBW*-triple factor*. For each element J in $\mathcal{I}(A)$, the set $A_0(J)$ of elements a in A such that $D(a,J)$ is equal to zero is an element of $\mathcal{I}(A)$. Denoting by $P_2(J)$ and $P_0(J)$ the structural projections onto J [also written $A_2(J)$] and $A_0(J)$, respectively, the projection $P_1(J)$ which is equal to $\mathrm{id}_A - P_0(J) -$

$P_2(J)$ has range $A_1(J)$, a weak*-closed subspace of A, and the decomposition

$$A = A_0(J) \oplus A_1(J) \oplus A_2(J)$$

is said to be the *generalized Peirce decomposition* of A relative to J. Two elements J and K of the complete lattice $\mathscr{I}(A)$ of weak*-closed inner ideals in a JBW*-triple A are said to be *centrally orthogonal* (written $J \perp_c K$) if there exists a weak*-closed ideal I in A such that $A_2(J) \subseteq A_2(I)$ and $A_2(K) \subseteq A_0(I)$. Two elements J and K of $\mathscr{I}(A)$ are said to be *rigidly collinear* (written $J \top_r K$) when $J \subseteq A_1(K)$ and $K \subseteq A_1(J)$.

Observe that a W*-algebra A, for the properties of which the reader is referred to [39] and [40], endowed with the triple product defined, for elements a, b, and c in A, by

$$\{a \quad b \quad c\} = \tfrac{1}{2}(ab^*c + cb^*a),$$

is a JBW*-triple. Hence, weak*-closed subtriples of W*-algebras form JBW*-triples. In particular, weak*-closed subtriples of the W*-algebra $B(H)$ of bounded linear operators on a complex Hilbert space H form JBW*-triples, and any JBW*-triple A that is Jordan*-triple isomorphic, and, hence, using [35], isometric and weak*-isomorphic to such a JBW*-triple, is said to be a *special JBW*-triple*.

A JBW*-triple A that is isomorphic to a weak*-closed inner ideal in a W*-algebra B is said to be *rectangular*. The structure of the complete lattice $\mathscr{I}(A)$ in this case has been extensively studied in [22] and [23]. Some of the motivation for this study stems from the fact that W*-algebras are often taken to represent statistical quantum systems, in which case their weak*-closed inner ideals can be regarded as representing certain structural subsystems. This approach is closely related to the "quantum histories" approach to quantum systems discussed by Isham, Linden, Schreckenberg, and Wright [29–31, 44–46]. The statistics of such subsystems are represented by certain measures on $\mathscr{I}(A)$. To be precise, for an arbitrary JBW*-triple A, a complex-valued *measure* m on $\mathscr{I}(A)$ is a mapping from $\mathscr{I}(A)$ to $\mathbb{C}$ such that if J and K are either centrally orthogonal or rigidly collinear, then

$$m(J \vee K) = m(J) + m(K).$$

When A is rectangular and is contained in the W*-algebra B, using the Mackey–Gleason theorem for W*-algebras, proved by Bunce and Wright [6–8], it can be shown that such measures extend uniquely to sesquilinear functionals on the product of two hereditary sub-W*-algebras of B, always providing that these do not contain a weak*-closed ideal that is a W*-algebra of type I_2. It follows that, in this case, which may be termed the boson case, the measure-theoretic approach and the "quantum-histories" approach coincide.

This paper is concerned with the complementary situation, or the fermion case. A complex Hilbert space A endowed with a conjugation $a \mapsto a^-$ possesses a triple product and norm with respect to which it forms a JBW*-triple known as a *spin triple*. As far as special JBW*-triples are concerned, the class of spin triples is, in some sense, complementary to the class of rectangular JBW*-triples. In particular, it is the case that, provided that the dimension of the spin triple A is greater than 2, A is a JBW*-triple factor. Such JBW*-triples are closely related to the mathematical model used to describe the properties of fermions [3, 25]. For example, the type I_2 W*-algebra $M_2(\mathbb{C})$ of 2×2 complex matrices is a spin triple.

In what follows, the structure of the complete lattice $\mathcal{I}(A)$ of closed inner ideals in a spin triple A and the measures on it are investigated. The first main result of the paper is that, if the dimension of A is greater than 5, then there are no non-zero measures on $\mathcal{I}(A)$. When the dimension of A is equal to 5, for each complex number α, there exists a measure m on $\mathcal{I}(A)$ with $m(A)$ equal to α, and this is the unique such measure that is invariant under Jordan triple automorphisms of A. If the dimension of A is 4, then A is triple isomorphic to the W*-algebra $M_2(\mathbb{C})$ of 2×2 complex matrices, and, in this case, results of Bunce and Wright [9] can be used to show that there is an uncountable number of bounded measures on $\mathcal{I}(A)$. When the dimension of A is equal to 3, then A is isomorphic to the JBW*-algebra of 2×2 symmetric complex matrices, and once again, there is a multitude of measures on $\mathcal{I}(A)$. When the dimension of A is equal to 2, then A is no longer a JBW*-triple factor. Then, $\mathcal{I}(A)$ consists of two centrally orthogonal elements along with $\{0\}$ and A, and a measure on $\mathcal{I}(A)$ is determined by its values on the two non-trivial elements.

The paper is organized as follows. In Section 2 the definition and properties of spin triples are introduced. Although many of these have been discussed elsewhere [26, 27], this paper requires some results about spin triples of a rather different kind. Some of these can be found in [21]. In Section 3 the main results are proved.

2. SPIN TRIPLES

Let A be a complex Hilbert space. A *conjugation* $a \mapsto a^-$ on A is a conjugate linear mapping such that, for all elements a and b in A,

$$(a^-)^- = a, \qquad \langle a^-, b^- \rangle = \langle b, a \rangle.$$

The proof of the following result can be found in [27].

LEMMA 2.1. *Let A be a complex Hilbert space and let $a \mapsto a^-$ be a conjugation on A. For elements a, b, and c in A, let*

$$2\{a \quad b \quad c\} = \langle a, b\rangle c + \langle c, b\rangle a - \langle a, c^-\rangle b^-. \tag{2.1}$$

Then, with respect to the norm defined, for a in A, by

$$2\|a\|^2 = \langle a, a\rangle + \left(\langle a, a\rangle^2 - |\langle a, a^-\rangle|^2\right)^{1/2}, \tag{2.2}$$

A is a JBW-triple. Furthermore, the JBW*-triple norm is equivalent to the Hilbert space norm.*

The JBW*-triple A described in this lemma is said to be a *spin triple*. Since A is reflexive, subspaces of A are norm-closed if and only if they are weak*-closed. So as not to confuse the two notions of orthogonality present in a spin triple, the Hilbert space orthogonal complement of a subset M of A will be denoted by the symbol M^{perp}. The proofs of the following results are given in [21].

LEMMA 2.2. *Let A be a spin triple and let J be a closed subspace of A. Then the following hold.*

(i) *J is a subtriple of A if and only if either J and J^- coincide or $\langle J, J^-\rangle$ is equal to zero.*

(ii) *J is an inner ideal in A if and only if either J and A coincide or $\langle J, J^-\rangle$ is equal to zero.*

(iii) *If the dimension of A is greater than 2, then J is an ideal in A if and only if J is equal to A or to $\{0\}$.*

Notice that, since a spin triple A, of dimension greater than 2, has no non-trivial closed ideals it is a JBW*-factor.

Let A be a spin triple and let $G(A)$ be the complex Hilbert space that is the completion of the exterior algebra of A. For a in A, let d_a be the unique anti-derivation of $G(A)$ which decreases the order of a tensor by 1 and, for b in A, satisfies

$$d_a(b) = \langle a, b^-\rangle 1.$$

Let l_a be the bounded linear mapping on $G(A)$ defined for each element ξ in $G(A)$ by

$$l_\alpha(\xi) = a \wedge \xi.$$

Observe that, for each element a in A,

$$l_a^* = d_{a^-}.$$

For a in A, let $\phi(a)$ be the bounded linear operator on $G(A)$ defined by

$$\phi(a) = \frac{1}{\sqrt{2}}(l_a + d_a).$$

Then, for a in A,

$$\phi(a^-) = \phi(a)^*, \qquad \phi(a)^2 = \tfrac{1}{2}\langle a, a^- \rangle 1.$$

A simple calculation shows that, for elements a, b, and c in A,

$$\phi(\{a \quad b \quad c\}) = \tfrac{1}{2}(\phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a)).$$

Therefore, ϕ is a Jordan triple isomorphism from the JBW*-triple A into the W*-algebra $B(G(A))$ of bounded linear operators on $G(A)$. Therefore, A is a special JBW*-triple and ϕ is an isometry from A onto the sub-JBW*-triple $\phi(A)$ of $B(G(A))$ [35].

Recall that an element u in an arbitrary JBW*-triple A is said to be a *tripotent* if $\{u \ u \ u\}$ is equal to u. Let $\mathcal{U}(A)$ denote the set of tripotents in A. Observe that the zero element lies in $\mathcal{U}(A)$ and that all other elements of $\mathcal{U}(A)$ are of norm one. For each tripotent u in the JBW*-triple A the weak*-continuous conjugate linear operator $Q(u)$ and, for j equal to 0, 1, or 2, the weak*-continuous linear operators $P_j(u)$ are defined by

$$Q(u)a = \{u \quad a \quad u\}, \qquad P_2(u) = Q(u)^2,$$

$$P_1(u) = 2\big(D(u,u) - Q(u)^2\big), \qquad P_0(u) = \mathrm{id}_A - 2D(u,u) + Q(u)^2.$$

By the results of [4, 12, 13, 36], the linear operators $P_j(u)$ are weak*-continuous projections onto the eigenspaces $A_j(u)$ of $D(u,u)$ corresponding to eigenvalues $j/2$. The corresponding decomposition

$$A = A_0(u) \oplus A_2(u) \oplus A_2(u)$$

is said to be the *Peirce decomposition* of A relative to u. For j, k, and l equal to 0, 1, or 2, $A_j(u)$ is a sub-JBW*-triple such that $\{A_j(u) \ A_k(u) \ A_l(u)\}$ is contained in $A_{j-k+l}(u)$ when $j-k+l$ is equal to 0, 1, or 2, and $\{0\}$ otherwise. Moreover,

$$\{A_2(u) \quad A_0(u) \quad A\} = \{A_0(u) \quad A_2(u) \quad A\} = \{0\},$$

and $A_0(u)$ and $A_2(u)$ are inner ideals in A.

A pair u, v of elements of $\mathcal{U}(A)$ is said to be *orthogonal* if v is contained in $A_0(u)$. For two elements u and v of $\mathcal{U}(A)$, write $u \le v$ if $v - u$ is a tripotent orthogonal to u. A pair u, v is said to be *collinear* (written $u \top v$) if u lies in $A_1(v)$ and v lies in $A_1(u)$ and is said to be *rigidly collinear* (written $u \top_r v$) if $A_2(u)$ is contained in $A_1(v)$ and $A_2(v)$ is contained in $A_1(u)$. A subset Λ of $\mathscr{U}(A)$ is said to be *collinear* if, for all pairs u and v of distinct elements of Λ, $u \top v$. Notice that the empty subset and singleton subsets of $\mathscr{U}(A)$ are collinear subsets. Let $\mathscr{U}(A)^{\sim}$ be the partially ordered set $\mathscr{U}(A)$ with a largest element ω adjoined. For each element u in $\mathscr{U}(A)$, the set

$$\{u\}_{,} = \{x \in A_* : x(u) = \|x\| = 1\}$$

is a norm-closed fact of the unit ball $A_{*,1}$ in A_*. Define $\{\omega\}_{,}$ to be the set $A_{*,1}$. For a subset G of $A_{*,1}$, let

$$G' = \{a \in A : x(a) = \|a\| = 1\}.$$

Observe that G' is a weak*-closed face of the unit ball A_1 in A. The following result was proved in [17].

LEMMA 2.3. *Let A be a JBW*-triple with predual A_*.*

(i) *The mapping $u \mapsto \{u\}_{,}$ is an order isomorphism from the partially ordered set $\mathscr{U}(A)^{\sim}$ of tripotents in A with a largest element adjointed onto the complete lattice $\mathscr{F}_n(A_{*,1})$ of all norm-closed faces of the closed unit ball $A_{*,1}$ in A_*, and, hence, $\mathscr{U}(A)^{\sim}$ is a complete lattice.*

(ii) *The mapping $u \mapsto (\{u\}_{,})'$ is an anti-order-isomorphism from $\mathscr{U}(A)^{\sim}$ onto the complete lattice $\mathscr{F}_{w*}(A_1)$ of weak*-closed faces of the closed unit ball A_1 in A and, for each element u in $\mathscr{U}(A)$,*

$$\left(\{u\}_{,}\right)' = u + A_0(u)_1.$$

By using the Krein−Milman Theorem, it is an immediate corollary of this result that the linear span of set $\mathscr{U}(A)$ is weak*-dense in A. The next two results, the straightforward proofs of which will be omitted, describe the set of tripotents in a spin triple and the corresponding Peirce decompositions.

LEMMA 2.4. *Let A be a spin triple with triple product and norm defined by* (2.1) *and* (2.2). *Let*

$$\mathscr{U}_1(A) = \{u \in A : \langle u, u\rangle = 1, \langle u, u^-\rangle = 0\},$$

$$\mathscr{U}_2(A) = \{u \in A : \langle u, u\rangle = 2, u^- = \lambda u, \lambda \in \mathbb{C}, |\lambda| = 1\}.$$

Then, the set $\mathscr{U}(A)$ of tripotents in A is equal to the union of $\mathscr{U}_1(A)$, $\mathscr{U}_2(A)$ and $\{0\}$.

LEMMA 2.5. *Let A be a spin triple and let $\mathscr{U}_1(A)$ and $\mathscr{U}_2(A)$ be as defined in Lemma 2.4. Then the following hold.*

(i) *For u in $\mathscr{U}_1(A)$, the Peirce decomposition of A is given by*

$$A_2(u) = \mathbb{C}u, \qquad A_0(u) = \mathbb{C}u^-, \qquad A_1(u) = \{u, u^-\}^{\text{perp}}.$$

(ii) *For u in $\mathscr{U}_2(A)$, the Peirce decomposition of A is given by*

$$A_2(u) = A, \qquad A_0(u) = A_1(u) = \{0\}.$$

Observe that a tripotent u in the JBW*-triple A is said to be *unitary* if $A_2(u)$ is equal to A. It follows that, in a spin triple A, every element u in $\mathscr{U}_2(A)$ is unitary.

LEMMA 2.6. *Let A be a non-zero spin triple and let $\mathscr{U}_1(A)$ and $\mathscr{U}_2(A)$ be the sets of tripotents in A defined as in Lemma 2.4. Then, the set $\mathscr{U}_2(A)$ is non-empty.*

Proof. Suppose that, on the contrary, $\mathscr{U}_2(A)$ is empty. Then, since, by Lemma 2.3, $\mathscr{U}(A)$ contains elements other than zero, there exists an element u in $\mathscr{U}_1(A)$. Let v be the element of A defined by

$$v = u + u^-.$$

Then, v^- is equal to v, and

$$\langle v, v\rangle = \langle u, u\rangle + \langle u, u^-\rangle + \langle u^-, u\rangle + \langle u^-, u^-\rangle = 2\langle u, u\rangle = 2.$$

It follows that v lies in $\mathscr{U}_2(A)$, giving a contradiction. This completes the proof of the lemma. ∎

LEMMA 2.7. *Let A be a spin triple and let $\mathscr{U}_1(A)$ and $\mathscr{U}_2(A)$ be the sets of tripotents in A defined as in Lemma 2.4. Then $\mathscr{U}_1(A)$ is empty if and only if A is at most one-dimensional.*

Proof. If A is equal to $\mathbb{C}$ then the only conjugation is complex conjugation, and there does not exist a complex number u of norm one of the square of which is equal to zero. Conversely, suppose that $\mathscr{U}_1(A)$ is empty. Then, since $\mathscr{U}(A)$ contains non-zero elements, there exists an element v in $\mathscr{U}_2(A)$. Hence, there exists a complex number λ of unit modulus such that v^- is equal to λv. Let B be the closed subspace $(\mathbb{C}v)^{\text{perp}}$ of A and let a lie in B. Then,

$$\langle a^-, v\rangle = \langle v^-, a\rangle = \langle \lambda v, a\rangle = \lambda\overline{\langle a, v\rangle} = 0,$$

and a^- also lies in B. By Lemma 2.2, B is a closed subtriple of A and is therefore also a spin triple. Suppose that u is an element of $\mathscr{U}_1(B)$. Then u

also lies in $\mathscr{U}_1(A)$, giving a contradiction. Let u be an element of $\mathscr{U}_2(B)$. Then, either u lies in $\mathscr{U}_2(A)$ or u is equal to zero. If the former is true, then v and u are orthogonal unitary tripotents in A. In this case there exist real numbers θ and ϕ in the interval $[0, \pi)$ such that

$$v^- = e^{2i\theta} v, \qquad u^- = e^{2i\phi} u.$$

Choosing

$$v_0 = e^{i\theta} v, \qquad u_0 = e^{i\phi} u,$$

v_0 and u_0 are orthogonal unitary tripotents in A such that

$$v_0^- = v_0, \qquad u_0^- = u_0.$$

Let

$$w = \tfrac{1}{2}(v_0 + iu_0).$$

Then,

$$w^- = \tfrac{1}{2}(v_0 - iu_0),$$

and

$$\langle w, w\rangle = \tfrac{1}{4}\langle v_0 + iu_0, v_0 + iu_0\rangle = \tfrac{1}{4}\langle v_0, v_0\rangle + \tfrac{1}{4}\langle u_0, u_0\rangle = 1,$$
$$\langle w, w^-\rangle = \tfrac{1}{4}\langle v_0 + iu_0, v_0 - iu_0\rangle = \tfrac{1}{4}\langle v_0, v_0\rangle - \tfrac{1}{4}\langle u_0, u_0\rangle = 0.$$

It follows that w lies in $\mathscr{U}_1(A)$, yielding a contradiction. Therefore, B is equal to $\{0\}$ and A is equal to $\mathbb{C}v$ as required. ∎

LEMMA 2.8. *Let A be a spin triple and let $\mathscr{U}_1(A)$ be the set of tripotents in A defined as in Lemma 2.4. Let u_1 and u_2 be elements of $\mathscr{U}_1(A)$. Then, the following conditions are equivalent.*

(i) $u_1 \in A_1(u_2)$.
(ii) $u_1^- \in A_2(u_2)$.
(iii) $u_1 \in A_1(u_2^-)$.
(iv) $u_1^- \in A_2(u_2^-)$.
(v) $u_2 \in A_1(u_1)$.
(vi) $u_2^- \in A_2(u_1)$.
(vii) $u_2 \in A_1(u_1^-)$.
(viii) $u_2^- \in A_1(u_1^-)$.
(ix) $\langle u_1, u_2\rangle = \langle u_1, u_2^-\rangle = 0$.

Proof. (i) ⇔ (ix) Observe that

$$\begin{aligned} 2D(u_2, u_2)u_1 &= \langle u_2, u_2\rangle u_1 + \langle u_1, u_2\rangle u_2 - \langle u_2, u_1^-\rangle u_2^- \\ &= u_1 + \langle u_1, u_2\rangle u_2 - \langle u_2, u_1^-\rangle u_2^-. \end{aligned}$$

If (ix) holds then

$$D(u_2, u_2)u_1 = \tfrac{1}{2}u_1,$$

and u_1 lies in $A_1(u_2)$ as required. Conversely, if u_1 lies in $A_1(u_2)$, then, by Lemma 2.5(i), u_1 lies in $\{u_2, u_2^-\}^{\text{perp}}$ and (ix) holds. The remaining assertions follow in a similar manner. ∎

This result has the following immediate corollary.

COROLLARY 2.9. *Let A be a spin triple, and let Λ be a subset of $\mathcal{U}_1(A)$. Then Λ is collinear if and only if $\Lambda \cup \Lambda^-$ is an orthonormal set in the Hilbert space A.*

Recall that a *point space J* in a JBW*-triple A is a subspace of A such that, for each element a in J, the subspace $\{a\ A\ a\}$ is equal to $\mathbb{C}a$. Clearly, a point space is an inner ideal and every subspace of a point space is a point space.

LEMMA 2.10. *Let A be a spin triple and let J be a closed proper subspace of A. Then J is an inner ideal in A if and only if it is a point space.*

Proof. Every point space is an inner ideal. Conversely, using Lemma 2.2, for elements a in the inner ideal J and b in A,

$$\{a\ \ b\ \ a\} = \langle a, b \rangle a$$

and J is a point space. ∎

LEMMA 2.11. *Let J be a closed proper inner ideal in the spin triple A. Then, every orthonormal subset Λ of J is a collinear subset of elements in $\mathcal{U}_1(A)$.*

Proof. By Lemma 2.2, $\langle J, J^- \rangle$ is equal to $\{0\}$. Consequently, if Λ is an orthonormal subset of J then $\Lambda \cup \Lambda^-$ is an orthonormal subset in A. The assertion now follows from Corollary 2.9. ∎

The proof of the following result may be found in [21].

LEMMA 2.12. *Let A be a spin triple and let J be a closed proper inner ideal in A having an orthonormal basis Λ. Then, the structural projection $P_2(J)$ on A with range J is given by*

$$P_2(J) = \sum_{u \in \Lambda} P_2(u),$$

where convergence of this net is in the strong operator topology.

Let Λ be a collinear subset of $\mathcal{U}_1(A)$ and let $B(\Lambda)$ denote the closed subspace spanned by Λ. For any collinear subset Λ of $\mathcal{U}_1(A)$, Λ^- is also a collinear subset of $\mathscr{U}_1(A)$, and $B(\Lambda^-)$ coincides with $B(\Lambda)^-$. It follows, by Corollary 2.9, that $\langle B(\Lambda), B(\Lambda^-)\rangle$ is equal to $\{0\}$ and, therefore, by Lemma 2.2, that $B(\Lambda)$ is a closed inner ideal in A. It is now possible to turn to the first main result of this section.

THEOREM 2.13. *Let A be a spin triple. For every maximal collinear subset Λ of $\mathscr{U}_1(A)$, there exists a self-conjugate tripotent u_Λ in the set $\mathscr{U}_2(A) \cup \{0\}$ such that*

$$A = B(\Lambda) \oplus B(\Lambda^-) \oplus \mathbb{C}u_\Lambda, \qquad \langle u_\Lambda, B(\Lambda) \oplus B(\Lambda^-)\rangle = \{0\}. \quad (2.3)$$

Let Λ_1, Λ_2 be maximal collinear subsets of $\mathscr{U}_1(A)$. Then, there exists a unitary operator T on A such that

$$T\Lambda_1 = \Lambda_2, \qquad T\Lambda_1^- = \Lambda_2^-, \qquad Tu_{\Lambda_1} = u_{\Lambda_2}.$$

Moreover, for each element a in A,

$$(Ta)^- = Ta^-.$$

Proof. Let Λ be a maximal collinear subset of $\mathscr{U}_1(A)$. Suppose that u is an element of $\mathscr{U}_1(A)$ such that

$$\langle u, B(\Lambda)\rangle = \langle u, B(\Lambda^-)\rangle = 0.$$

Then, by Lemma 2.8, $\Lambda \cup \{u\}$ is a collinear subset in $\mathscr{U}_1(A)$, thereby contradicting the maximality of Λ. Let

$$N = (B(\Lambda) \oplus B(\Lambda^-))^{\text{perp}}.$$

Then, N is a closed subspace of A, and, conjugating,

$$A = B(\Lambda) \oplus B(\Lambda^-) \oplus N, \qquad A = B(\Lambda^-) \oplus B(\Lambda) \oplus N^-.$$

It follows that N coincides with N^- and, by Lemma 2.2, N is a subtriple of A. It is therefore itself a spin triple. From the preceding arguments, it can be seen that $\mathscr{U}_1(N)$ is empty. Therefore, either N is equal to $\{0\}$, or there exists an element u_Λ in $\mathscr{U}_2(N)$. In the latter case, by Lemma 2.7, N is equal to $\mathbb{C}u_\Lambda$. As in the proof of Lemma 2.7, u_Λ can be chosen to be self-conjugate.

Let Λ_1 and Λ_2 be maximal collinear subsets of $\mathscr{U}_1(A)$. Since unitary operators on A map tripotents into tripotents and preserve the norm, it follows, by (2.3), that u_{Λ_1} is equal to zero if and only if u_{Λ_2} is equal to zero. As a consequence, there exists a unitary operator T which satisfies the conditions. ∎

Let $U(A)$ denote the group of unitary operators on the Hilbert space A and let $U_0(A)$ denote the subgroup of $U(A)$ consisting of operators T such that, for each element a in A,

$$(Ta)^- = Ta^-.$$

COROLLARY 2.14. *Let A be a spin triple. A bounded linear operator T from A onto itself lies in the group $U_0(A)$ of unitary operators on A that commute with the conjugation on A if and only if, for each maximal collinear subset Λ of $\mathscr{U}_1(A)$ and self-conjugate tripotent u_Λ in $\mathscr{U}_2(A) \cup \{0\}$ such that*

$$A = B(\Lambda) \oplus B(\Lambda^-) \oplus \mathbb{C}u_\Lambda, \qquad \langle u_\Lambda, B(\Lambda) \oplus B(\Lambda^-)\rangle = \{0\},$$

the image $T(\Lambda)$ of Λ is a maximal collinear subset and Tu_Λ is a self-conjugate element of $\mathscr{U}_2(A) \cup \{0\}$ satisfying

$$A = B(T(\Lambda)) \oplus B(T(\Lambda^-)) \oplus \mathbb{C}Tu_\Lambda,$$

$$\langle Tu_\Lambda, B(T(\Lambda)) \oplus B(T(\Lambda^-))\rangle = \{0\}.$$

Proof. This follows immediately from the previous results. ∎

The result below, which describes the Jordan triple automorphism group of A, will be used in the following section. A proof can be found in [27].

LEMMA 2.15. *Let A be a spin triple, let* $\mathrm{Aut}(A)$ *be the group of Jordan triple automorphisms of A, and let $U_0(A)$ be the group of unitary operators on A that commute with the conjugation on A. Then, a bounded linear operator T lies in* $\mathrm{Aut}(A)$ *if and only if there exists a complex number μ of unit modulus and an element S of $U_0(A)$ such that*

$$T = \mu S.$$

The next result shows how Theorem 2.13 relates to an arbitrary closed inner ideal in A.

THEOREM 2.16. *Let A be a spin triple and let J be a closed proper inner ideal in A. Then, there exists a maximal collinear subset Λ in $\mathscr{U}_1(A)$ such that the closed inner ideal $B(\Lambda)$ spanned by Λ is given by*

$$B(\Lambda) = J \oplus (J^{\mathrm{perp}} \cap B(\Lambda)).$$

Proof. Let Γ be an orthonormal basis for J. By Lemma 2.11, Γ is a collinear subset in $\mathscr{U}_1(A)$. Let Λ be a maximal collinear subset in $\mathscr{U}_1(A)$ which contains Γ. Then, clearly, Λ^- is also a maximal collinear subset in $\mathscr{U}_1(A)$ and $B(\Lambda)$ has the required property. ∎

The following result describes the generalized Peirce decomposition of the spin triple A relative to a closed inner ideal J in A.

LEMMA 2.17. *Let A be a spin triple and let J be a closed inner ideal in A.*

(i) *If J is of dimension 1 then there exists an element u in $\mathscr{U}_1(A)$ such that*

$$J = A_2(J) = A_2(u) = \mathbb{C}u, \qquad A_0(J) = A_0(u) = \mathbb{C}u^-,$$
$$A_1(J) = \{u, u^-\}^{\text{perp}}.$$

(ii) *If the dimension of J is greater than 1 then*

$$J = A_2(J), \qquad A_0(J) = \{0\}, \qquad A_1(J) = J^{\text{perp}}.$$

Proof. This follows from Lemma 2.5 and [21, Theorem 5.5]. ∎

Recall that a norm-closed inner ideal J in the JBW*-triple A is said to be a *Peirce inner ideal* if the weak*-continuous generalized Peirce projection $P_1(J)$ is contractive. Various other equivalent conditions are given in [21]. One such condition is that, for j, k, and l equal to 0, 1, or 2,

$$\{A_j(J) \quad A_k(J) \quad A_l(J)\} \subseteq A_{j+l-k}(J)$$

when $j + l - k$ is equal to 0, 1, or 2, and is equal to $\{0\}$ otherwise.

The following result describes the Peirce inner ideals in a spin triple.

THEOREM 2.18. *Let A be a spin triple and let J be a closed inner ideal in A. If the dimension of J is equal to 1 then J is Peirce. If the dimension of J is greater than 1 then J is Peirce if and only if J^- coincides with J^{perp}.*

Proof. This is immediate from Theorem 2.16 and Lemma 2.2. ∎

3. MEASURES ON THE LATTICE OF INNER IDEALS

Let A be a spin triple and let $\mathscr{I}(A)$ be the complete lattice of closed inner ideals in A. Recall that two elements J and K in $\mathscr{I}(A)$ are said to be *rigidly collinear* (written $J \top_r K$) if

$$J = A_2(J) \subseteq A_1(K), \qquad K = A_2(K) \subseteq A_1(J).$$

Observe that, if J is equal to $\{0\}$ then $J \top_r K$ if and only if K is equal to $\{0\}$. The following result describes rigid collinearity in the complete lattice $\mathscr{I}(A)$.

LEMMA 3.1. *Let A be a spin triple, let $\mathscr{I}(A)$ be the complete lattice of closed inner ideals in A, and let $\mathscr{U}_1(A)$ be the set of tripotents in A described in Lemma 2.4. Then the following hold.*

(i) *For elements u and v in* $\mathscr{U}_1(A)$, $A_2(u) \mathsf{T}_r A_2(v)$ *if and only if* $\{u, u^-\} \subseteq \{v, v^-\}^{\text{perp}}$.

(ii) *For an element u in* $\mathscr{U}_1(A)$ *and an element K of* $\mathscr{I}(A)$ *of dimension greater than* 1, $A_2(u) \mathsf{T}_r K$ *if and only if* $\{u, u^-\} \subseteq K^{\text{perp}}$.

(iii) *For elements J and K in* $\mathscr{I}(A)$, *both of dimension greater than* 1, $J \mathsf{T}_r K$ *if and only if* $J \subseteq K^{\text{perp}}$.

Proof. The result follows immediately from Lemmas 2.8 and 2.17. ∎

The next lemma describes how the supremum of two rigidly collinear elements of $\mathscr{I}(A)$ may be calculated.

LEMMA 3.2. *Let A be a spin triple, let* $\mathscr{I}(A)$ *be the complete lattice of closed inner ideals in A, and let* $\mathscr{U}_1(A)$ *be the set of tripotents in A described in Lemma* 2.4. *Then the following hold.*

(i) *For elements u and v in* $\mathscr{U}_1(A)$ *such that* $A_2(u) \mathsf{T}_r A_2(v)$,

$$A_2(u) \vee A_2(v) = A_2(u) \oplus A_2(v) = \text{lin}(\{u, v\}).$$

(ii) *For an element u in* $\mathscr{U}_1(A)$ *and an element K of* $\mathscr{I}(A)$, *of dimension greater than* 1, *such that* $A_2(u) \mathsf{T}_r K$,

$$A_2(u) \vee K = A_2(u) \oplus K = \text{lin}(\{u\} \cup K).$$

(iii) *For elements J and K in* $\mathscr{I}(A)$, *both of dimension greater than* 1, *such that* $J \mathsf{T}_r K$, *if* $J \oplus J^-$ *is orthogonal to* $K \oplus K^-$, *then*

$$J \vee K = J \oplus K,$$

and, if not, then

$$J \vee K = A.$$

Proof. (i) By Lemma 3.1(i), the set $\{u, v, u^-, v^-\}$ is orthonormal. It follows that the set

$$L = \text{lin}(\{u, v\})$$

is a closed subspace of A such that $\langle L, L^- \rangle$ is equal to zero. Therefore, by Lemma 2.2, L is an element of $\mathscr{I}(A)$ containing $A_2(u)$ and $A_2(v)$. It is clear that any element of $\mathscr{I}(A)$ containing $A_2(u)$ and $A_2(v)$ also contains L, and the proof of (i) is complete.

(ii) By Lemma 3.1(ii),

$$\langle u, K\rangle = \langle u^-, K\rangle = 0,$$

and, conjugating,

$$\langle u^-, K^-\rangle = \langle u, K^-\rangle = 0.$$

It follows that the subspace

$$L = \mathrm{lin}(\{u, K\})$$

is a closed subspace of A for which $\langle L, L^-\rangle$ is equal to zero. Hence, using Lemma 2.2, L is a closed inner ideal in A and the proof may be completed as in (i).

(iii) Let L be a closed inner ideal containing J and K. Since J and K are orthogonal, it follows that the closed subspace $J \oplus K$ is contained in L. If L is a point space then $J \oplus K$ is a point space. Observe that

$$\begin{aligned}\langle J, K^-\rangle + \langle J^-, K\rangle &= \langle J, J^-\rangle + \langle J, K^-\rangle + \langle J^-, K\rangle + \langle J^-, K^-\rangle \\ &= \langle J \oplus K, J^- \oplus K^-\rangle = \{0\}.\end{aligned}$$

It follows that $J \oplus K$ is a point space if and only if $\langle J, K^-\rangle$ is equal to zero and, hence, if and only if $J \oplus J^-$ is orthogonal to $K \oplus K^-$. In this case, as in the proof of (i), $J \vee K$ coincides with $J \oplus K$. Otherwise, L is not a point space and, therefore, must coincide with A, in which case, $J \vee K$ is equal to A. ∎

Recall that a spin triple A, of dimension greater than 2, has no non-trivial closed ideals, and, therefore, $\mathscr{I}(A)$ possesses no non-trivial pairs of centrally orthogonal elements. However, every non-zero element J of $\mathscr{I}(A)$ is centrally orthogonal to $\{0\}$. Consequently, a complex-valued mapping m on $\mathscr{I}(A)$ is a measure if, for each pair J and K of rigidly collinear elements of $\mathscr{I}(A)$,

$$m(J \vee K) = m(J) + m(K).$$

Observe that, since the pair J and K of elements of $\mathscr{I}(A)$ for which both J and K are equal to $\{0\}$ is rigidly collinear, it follows that, for every measure m on $\mathscr{I}(A)$, $m(\{0\})$ is equal to zero. The first main result of the paper follows.

THEOREM 3.3. *Let A be a spin triple of dimension greater than 5, and let m be a measure on the complete lattice $\mathscr{I}(A)$ of closed inner ideals in A. Then, for each element J of $\mathscr{I}(A)$,*

$$m(J) = 0.$$

Proof. Since the dimension of A is greater than 4, it follows from Theorem 2.13 that there exists a closed inner ideal J in A of dimension greater than 1, not equal to A. Let K be a closed subspace of J^{-}, of dimension greater than 1. Then, since J^{-} is also a closed inner ideal in A, not equal to A, it is a point space. It follows that K, being a subspace of a point space, is also a point space and hence an element of $\mathscr{I}(A)$ such that $J \top_{r} K$. However, $J \oplus J^{-}$ is not orthogonal to $K \oplus K^{-}$, and, by Lemma 3.2, $J \vee K$ coincides with A. Since m is a measure,

$$m(A) = m(J \vee K) = m(J) + m(K),$$

from which it follows that $m(K)$ is constant on closed subspaces K of J^{-} of dimension greater than 1.

Let u_1 be an element of $\mathscr{U}_1(A)$. Then, by Theorem 2.16, since the dimension of A is greater than 5, there exists a three-dimensional inner ideal J containing u_1. Let u_2 and u_3 be elements of $\mathscr{U}_1(A)$ such that u_1, u_2, and u_3 form a pairwise collinear set of elements in J. From Lemma 3.2 and the preceding remarks,

$$\begin{aligned} m(A_2(u_1)) &= m(J) - m(A_2(u_2) \oplus A_2(u_3)) \\ &= (m(A) - m(J^{-})) - (m(A) - m(J^{-})) = 0. \end{aligned}$$

By Theorem 2.16 and Lemmas 3.1 and 3.2, every finite-dimensional inner ideal J not equal to A is the supremum of a finite number of pairwise rigidly collinear one-dimensional inner ideals, and it follows from the preceding remarks that, for such J in $\mathscr{I}(A)$, $m(J)$ is equal to zero. Furthermore, since, for any norm closed inner ideal J in A, not equal to A, m is constant on closed subspaces of J, it can be seen that this constant must be zero. This completes the proof of the theorem. ∎

Observe that, by Lemma 2.15, for each element S in $U_0(A)$ and each complex number μ of unit modulus, the unitary mapping T, which is equal to μS, is an element of the group Aut(A) of Jordan triple automorphisms of A, and every element T of Aut(A) is of this form. It follows that, for each element J in $\mathscr{I}(A)$, the subspace TJ of A also lies in $\mathscr{I}(A)$, and it can easily be seen that the mapping $J \mapsto TJ$ is an order automorphism of the complete lattice $\mathscr{I}(A)$. A measure m on $\mathscr{I}(A)$ is said to be *invariant* if, for each element T in Aut(A) and each element J in $\mathscr{I}(A)$,

$$m(TJ) = m(J).$$

It is now possible to consider the case in which the dimension of A is equal to 5. It is shown not only that non-zero measures exist, but also that they can be chosen to be invariant.

THEOREM 3.4. *Let A be a spin triple of dimension 5. Then, for each complex number α, there exists a unique invariant measure m on the complete lattice $\mathscr{I}(A)$ of closed inner ideals in A such that $m(A)$ is equal to α.*

Proof. Let α be an arbitrary complex number. It follows from Theorem 2.16 that every element J of $\mathscr{I}(A)$ either is equal to $\{0\}$ or A or is of dimension 1 or 2. Define the mapping m on $\mathscr{I}(A)$ by

$$m(\{0\}) = 0, \qquad m(A) = \alpha, \qquad m(J) = \tfrac{1}{4}\alpha$$

if J is one-dimensional, and by

$$m(J) = \tfrac{1}{2}\alpha$$

if J is two-dimensional. From Lemma 3.2, it can be seen that two one-dimensional inner ideals $A_2(u_1)$ and $A_2(u_2)$ are rigidly collinear if and only if u_1 and u_2 form an orthonormal basis for the two-dimensional inner ideal $A_2(u_1) \oplus A_2(u_2)$ and, in this case,

$$A_2(u_1) \vee A_2(u_2) = A_2(u_1) \oplus A_2(u_2).$$

It follows that

$$m(A_2(u_1) \vee A_2(u_2)) = \tfrac{1}{2}\alpha = m(A_2(u_1)) + m(A_2(u_2)).$$

Lemma 3.1 shows that there are no rigidly collinear pairs of inner ideals, one of which is one-dimensional and one of which is two-dimensional. Moreover, a pair J and K of two-dimensional inner ideals is rigidly collinear if and only if K is equal to J^-, and in this case,

$$J \vee J^- = A.$$

It follows that

$$m(J \vee J^-) = m(A) = \alpha = m(J) + m(J^-).$$

Therefore, m is a measure on $\mathscr{I}(A)$. Let T be an element of $\mathrm{Aut}(A)$. Then, since T preserves the dimension of inner ideals, it can be seen that m is an invariant measure on $\mathscr{I}(A)$.

Suppose that m_0 is a further invariant measure on $\mathscr{I}(A)$ such that $m_0(A)$ is equal to α. Let B be a two-dimensional inner ideal in A with an orthonormal basis consisting of collinear elements u_1 and u_2 in $\mathscr{U}_1(A)$, and let u_0 be a self-conjugate element of $\mathscr{U}_2(A)$ such that

$$B \oplus B^- \oplus \mathbb{C}u_0 = A, \qquad \langle u_0, B \oplus B^- \rangle = \{0\}.$$

Suppose that

$$m_0(A_2(u_1)) = \alpha_1, \qquad m_0(A_2(u_2)) = \alpha_2, \qquad m_0(A_2(u_1^-)) = \beta_1,$$
$$m_0(A_2(u_2^-)) = \beta_2.$$

Using Lemma 3.2, it can be seen that

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = \alpha.$$

Define the linear operator T on A by

$$Tu_1 = u_2, \qquad Tu_2 = u_1, \qquad Tu_1^- = u_2^-, \qquad Tu_2^- = u_1^-, \qquad Tu_0 = u_0.$$

Then T lies in Aut(A) and, by the invariance of m_0,

$$\alpha_1 = \alpha_2, \qquad \beta_1 = \beta_2.$$

Similarly, defining the linear operator S on A by

$$Su_1 = u_1^-, \qquad Su_2 = u_2^-, \qquad Su_1^- = u_1, \qquad Su_2^- = u_2, \qquad Su_0 = u_0,$$

it can be seen that

$$\alpha_1 = \beta_1, \qquad \alpha_2 = \beta_2.$$

It follows that m_0 coincides with m and the proof is complete. ∎

Recall that a Jordan*-algebra A which is also a complex Banach space such that, for all elements a and b in A, $\|a^*\| = \|a\|$, $\|a \circ b\| \leq \|a\| \, \|b\|$, and $\|\{a \;\; a \;\; a\}\| = \|a\|^3$, where

$$\{a \quad b \quad c\} = a \circ (b^* \circ c) + (a \circ b^*) \circ c - b^* \circ (a \circ c)$$

is the *Jordan triple product* on A, is said to be a *Jordan C*-algebra* [43] or *JB*-algebra* [47]. A Jordan C*-algebra which is the dual of a Banach space is said to be a *Jordan W*-algebra* [14] or a *JBW*-algebra* [47]. With the triple product defined above, a Jordan C*-algebra is a JB*-triple and a Jordan W*-algebra is a JBW*-triple. Examples of JB*-algebras are C*-algebras and examples of JBW*-algebras are W*-algebras, in both cases equipped with the Jordan product

$$a \circ b = \tfrac{1}{2}(ab + ba).$$

The self-adjoint parts of JB*-algebras and JBW*-algebras are said to be *JB-algebras* and *JBW-algebras*, respectively. The set $\mathscr{P}(A)$ of self-adjoint idempotents, the *projections*, in a JBW*-algebra A forms a complete orthomodular lattice with respect to the partial ordering defined, for elements e and f in $\mathscr{P}(A)$ by $e \leq f$ if $e \circ f$ is equal to e, and the mapping $e \mapsto e^{\perp}$ defined by $e^{\perp}$ is equal to $1 - e$, where 1 is the unit in A. The set $\mathscr{S}(A)$ of self-adjoint elements s in A for which s^2 is equal to 1 is said to be the set of *symmetries* in A. Observe that the mapping $e \mapsto 2e - 1$ is a bijection from $\mathscr{P}(A)$ onto $\mathscr{S}(A)$. Let u be a tripotent in the JBW*-triple in A. Then, with respect to the product

$$(a, b) \mapsto a \circ b = \{a \quad u \quad b\},$$

unit u, and involution

$$a \mapsto a^{\dagger} = \{u \quad a \quad u\},$$

$A_2(u)$ is a JBW*-algebra. Furthermore, the Jordan triple product in the JBW*-algebra $A_2(u)$ coincides with the restriction to $A_2(u)$ of the triple product on A. For details of the results described above, the reader is referred to [26, 32, 36, 37, 41, 42].

Let A be a non-zero spin triple and let u be an element of $\mathscr{U}_2(A)$. Since, for some θ in $[0, \pi)$, u^- is equal to $e^{2i\theta}u$, by defining 1 to be equal to $ie^{i\theta}u$, it can be seen that 1 is an element of $\mathscr{U}_2(A)$ such that 1^- is equal to -1. Since 1 is unitary, A coincides with $A_2(1)$ and, from above, is a JBW*-algebra. Observe that each element a in A has a unique orthogonal decomposition

$$a = \tfrac{1}{2}\langle a, 1\rangle 1 + \left(a - \tfrac{1}{2}\langle a, 1\rangle 1\right) = \alpha_a 1 + a_0,$$

where $\langle 1, a_0\rangle$ is equal to zero. In the JBW*-algebra $A_2(1)$,

$$(\alpha_a 1 + a_0)^{\dagger} = \{1 \quad a \quad 1\} = \overline{\alpha}_a 1 + a_0^-, \tag{3.1}$$

and

$$\begin{aligned}(\alpha_a 1 + a_0) &\circ (\alpha_b 1 + b_0) \\ &= \{a \quad 1 \quad b\} = \left(\alpha_a \alpha_b + \tfrac{1}{2}\langle a_0, b_0^-\rangle\right)1 + (\alpha_a b_0 + \alpha_b a_0).\end{aligned} \tag{3.2}$$

Observe that, by (2.2),

$$\begin{aligned}2\|\alpha_a 1 + a_0\|^2 = {} & 2|\alpha_a|^2 + \langle a_0, a_0\rangle \\ & + \left(\left(2|\alpha_a|^2 + \langle a_0, a_0\rangle\right)^2 - \left|\langle a_0, a_0^-\rangle - 2\alpha_a^2\right|^2\right)^{1/2},\end{aligned} \tag{3.3}$$

and, by (3.1), $\alpha_a 1 + a_0$ is self-adjoint if and only if

$$\overline{\alpha_a} = \alpha_a, \qquad a_0^- = a_0.$$

In the JBW-algebra consisting of self-adjoint elements of $A_2(1)$, by (3.2) and (3.3),

$$(\alpha_a 1 + a_0) \circ (\alpha_b 1 + b_0) = \left(\alpha_a \alpha_b + \tfrac{1}{2}\langle a_0, b_0 \rangle\right)1 + (\alpha_a b_0 + \alpha_b a_0)$$

and

$$\|\alpha_a 1 + a_0\| = |\alpha_a| + \frac{1}{\sqrt{2}} \langle a_0, a_0 \rangle^{1/2}.$$

Such a JBW-algebra is said to be a *spin factor*. By slight abuse of notation the Jordan W*-algebra that is the complexification of a spin factor will also said to be a spin factor [14]. The details of the properties of spin factors may be found in [26]. In particular, for each cardinal n greater than 2, up to isomorphism, there exists a unique spin factor of dimension n.

It is now possible to consider spin triples of dimension less than 5. The first case to be considered is that when A is of dimension 4. In this case, A is Jordan*-triple isomorphic to the W*-algebra $M_2(\mathbb{C})$ of 2×2 complex matrices. Indeed, there exists such a unique isomorphism ϕ from $M_2(\mathbb{C})$ onto the spin factor $A_2(1)$ constructed, as above, by means of a tripotent 1 in $\mathscr{U}_2(A)$ mapping the identity in $M_2(\mathbb{C})$ to 1. The inner product of two elements a and b in $M_2(\mathbb{C})$ is given by

$$\langle a, b \rangle = \mathrm{Tr}(ab^*),$$

where, for each element a in $M_2(\mathbb{C})$, $\mathrm{Tr}(a)$ denotes the trace of a. Furthermore, the conjugation $a \mapsto a^-$ coincides with the adjoint operation on $M_2(\mathbb{C})$. It follows from Lemma 2.15 that every Jordan triple automorphism T of $M_2(\mathbb{C})$ is of the form μS, where μ is a complex number of unit modulus and S is a unitary operator on the Hilbert space $M_2(\mathbb{C})$ that commutes with the conjugation. Consequently, S is a linear isometry on the W*-algebra on $M_2(\mathbb{C})$ that commutes with the adjoint operation. It follows from [33] and Wigner's Theorem that, for each element T of $\mathrm{Aut}(M_2(\mathbb{C}))$ there exists a unique complex number μ, of unit modulus, an element s of the set $\mathscr{S}(M_2(\mathbb{C}))$ of symmetries in $M_2(\mathbb{C})$, and a unitary or anti-unitary operator u on $\mathbb{C}^2$ such that, for each element a in $M_2(\mathbb{C})$,

$$Ta = \mu s u^* a u. \tag{3.4}$$

Furthermore, for each complex number μ of unit modulus, each element s of $\mathscr{S}(M_2(\mathbb{C}))$, and each unitary or anti-unitary operator u on $\mathbb{C}^2$, the linear operator T on $M_2(\mathbb{C})$, defined by (3.4), is an element of $\mathrm{Aut}(M_2(\mathbb{C}))$.

As a consequence of the results of [38] and [18], it follows that the partially ordered set $\mathscr{T}(A)\backslash\{0\}$ is order isomorphic to the partially ordered set $(\mathscr{P}(M_2(\mathbb{C}))\backslash\{0\}) \times (\mathscr{P}(M_2(\mathbb{C}))\backslash\{0\})$, where $\mathscr{P}(M_2(\mathbb{C}))$ is the complete orthomodular lattice of projections in $M_2(\mathbb{C})$.

A bounded complex-valued mapping ν on $\mathscr{P}(M_2(\mathbb{C}))$ such that, for each element e in $\mathscr{P}(M_2(\mathbb{C}))$,

$$\nu(e) + \nu(1 - e) = \nu(1),$$

is said to be a *bounded complex measure* on $\mathscr{P}(M_2(\mathbb{C}))$. The results of [9] show that there exists an isometric linear isomorphism η from $\mathbb{R}^3$ onto the subspace of $M_2(\mathbb{C})$ consisting of self-adjoint elements of trace zero, defined, for each element (x_1, x_2, x_3) in $\mathbb{R}^3$, by

$$\eta(x_1, x_2, x_3) = x_1 s_1 + x_2 s_2 + x_3 s_3, \tag{3.5}$$

where s_1, s_2, and s_3 are the elements of $\mathscr{S}(M_2(\mathbb{C}))$ defined by

$$s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad s_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

the Pauli spin matrices. The mapping η, defined in (3.5), maps the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ onto the set $\mathscr{S}(M_2(\mathbb{C}))\backslash\{-1, 1\}$ of non-trivial symmetries in $M_2(\mathbb{C})$. The following lemma is proved in exactly the same way as the main result of [9].

LEMMA 3.5. *Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and let η be the isometric linear isomorphic from $\mathbb{R}^3$ into $M_2(\mathbb{C})$, defined by* (3.5), *that sends $\mathbb{S}^2$ onto the set $\mathscr{S}(M_2(\mathbb{C}))\backslash\{-1, 1\}$. Then, for each complex number α, there exists a bijection $\psi \mapsto \nu_{\psi, \alpha}$ from the set V of odd, bounded complex-valued functions ψ on $\mathbb{S}^2$, onto the set of bounded complex measures ν on $\mathscr{P}(M_2(\mathbb{C}))$ such that*

$$\nu(1) = \alpha,$$

defined, for ψ in V and e, a one-dimensional projection in $\mathscr{P}(M_2(\mathbb{C}))$, by

$$\nu_{\psi, \alpha}(0) = 0, \qquad \nu_{\psi, \alpha}(1) = \alpha, \qquad 2\nu_{\psi, \alpha}(e) = \psi\big(\eta^{-1}(2e - 1)\big) + \alpha. \tag{3.6}$$

Using the remarks above it is now possible to consider measures on the complete lattice $\mathscr{I}(A)$ of closed inner ideals in the four-dimensional spin triple A.

THEOREM 3.6. *Let A be a spin triple of dimension* 4 *and let ϕ be the Jordan*-triple isomorphism from A onto the JBW*-triple $M_2(\mathbb{C})$ of 2×2 complex matrices defined above. Let V be the set of odd bounded complex-valued functions ψ on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, and, for each such ψ and* *each complex number α, let $\nu_{\psi,\alpha}$ be the bounded complex measure on $\mathscr{P}(M_2(\mathbb{C}))$ defined by (3.6). Then, for each pair ψ_1 and ψ_2 of elements of V, and each pair α_1 and α_2 of complex numbers, there exists a bounded measure $m_{\psi_1\otimes\psi_2}$ on the complete lattice $\mathscr{I}(A)$ of inner ideals in A such that*

$$m_{\psi_1\otimes\psi_2}(A) = \alpha_1\alpha_2,$$

defined, for elements e and f of $\mathscr{P}(M_2(\mathbb{C}))\backslash\{0\}$, by

$$m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})f)\big) = \nu_{\psi_1,\alpha_1}(e)\,\nu_{\psi_2,\alpha_2}(f).$$

The measure $m_{\psi_1\otimes\psi_2}$ is invariant if and only if

$$\psi_1 = \psi_2 = 0,$$

in which case, for one-dimensional elements e and f in $\mathscr{P}(M_2(\mathbb{C}))$,

$$m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})f)\big) = \tfrac{1}{4}\alpha_1\alpha_2.$$

Proof. From the preceding remarks, the mapping $m_{\psi_1\otimes\psi_2}$ is a well defined. By [22, Theorem 5.3], two non-zero elements $\phi^{-1}(e_1M_2(\mathbb{C})f_1)$ and $\phi^{-1}(e_2M_2(\mathbb{C})f_2)$ are rigidly collinear if and only if, either e_1 is equal to e_2 and f_1 is orthogonal to f_2, or e_1 is orthogonal to e_2 and f_1 is equal to f_2. Let e and f be elements of $\mathscr{P}(M_2(\mathbb{C}))$ not equal to 0 or 1. Then,

$$\begin{aligned}
&m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})f) \vee \phi^{-1}((1-e)M_2(\mathbb{C})f)\big)\\
&\quad= m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})f \vee (1-e)M_2(\mathbb{C})f)\big)\\
&\quad= m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(M_2(\mathbb{C})f)\big)\\
&\quad= \nu_{\psi_1,\alpha_1}(1)\,\nu_{\psi_2,\alpha_2}(f)\\
&\quad= \nu_{\psi_1,\alpha_1}(e)\,\nu_{\psi_2,\alpha_2}(f) + \nu_{\psi_1,\alpha_1}(1-e)\,\nu_{\psi_2,\alpha_2}(f)\\
&\quad= m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})f)\big) + m_{\psi_1\otimes\psi_2}\big(\phi^{-1}((1-e)M_2(\mathbb{C})f)\big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})f) \vee \phi^{-1}(eM_2(\mathbb{C})(1-f))\big)\\
&\qquad= m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})f)\big) + m_{\psi_1\otimes\psi_2}\big(\phi^{-1}(eM_2(\mathbb{C})(1-f))\big),
\end{aligned}$$

and it follows that $m_{\psi_1 \otimes \psi_1}$ is a bounded complex measure on $\mathscr{I}(A)$ such that

$$m_{\psi_1 \otimes \psi_2}(A) = \alpha_1 \alpha_2.$$

Let T be the element of $\mathrm{Aut}(M_2(\mathbb{C}))$ defined by (3.4). Then, for elements e and f in $\mathscr{P}(M_2(\mathbb{C}))\backslash\{0\}$,

$$\begin{aligned} T(eM_2(\mathbb{C})f) &= \mu su^* eM_2(\mathbb{C}) fu \\ &= (us)^* e(us) su^* M_2(\mathbb{C}) uu^* fu \\ &= (us)^* e(us) M_2(\mathbb{C}) u^* fu, \end{aligned}$$

since the mapping $a \mapsto su^*au$ maps $M_2(\mathbb{C})$ onto itself. The linear operator us is unitary if u is unitary and anti-unitary if u is antiunitary, and the mapping $u \mapsto us$ is a bijection from the set of unitary operators on $\mathbb{C}^2$ to itself and a bijection from the set of anti-unitary operators on $\mathbb{C}^2$ to itself. It follows that the measure $m_{\psi_1 \otimes \psi_2}$ is invariant if and only if both ν_{ψ_1, α_1} and ν_{Ψ_2, α_2} are invariant under the mappings $e \mapsto v^*ev$, where v is unitary or anti-unitary. However, for any pair e_1 and e_2 of projections, there exists a unitary operator u such that u^*e_1u and e_2 coincide. Therefore, the measure $m_{\psi_1 \otimes \psi_2}$ is invariant if and only if both ψ_1 and ψ_2 are constant on $\mathbb{S}^2$. Since ψ_1 and ψ_2 are odd, this is possible if and only if both are zero. In this case, it can be seen from (3.6), that, if e is one-dimensional, then

$$\nu_{\psi_1, \alpha_1}(e) = \tfrac{1}{2}\alpha_1, \qquad \nu_{\psi_2, \alpha_2}(e) = \tfrac{1}{2}\alpha_2,$$

and the proof is complete. ∎

It follows from this result that, when the spin triple A is of dimension 4, the complete lattice $\mathscr{I}(A)$ of inner ideals in A has an abundance of measures upon it.

The next case to be considered is that when the spin triple A is of dimension 3.

THEOREM 3.7. *Let A be a spin factor of dimension 3 and let $\mathscr{I}(A)$ be the complete lattice of inner ideals in A. Then, every complex-valued function m on $\mathscr{I}(A)$ such that $m(\{0\})$ is equal to zero is a measure on $\mathscr{I}(A)$.*

Proof. It follows from Theorem 2.16 that all non-trivial elements of $\mathscr{I}(A)$ are of the form $A_2(u)$, where u is an element of $\mathscr{U}_1(A)$. From Lemma 3.1, it can be seen that there are no non-trivial pairs of rigidly collinear elements of $\mathscr{I}(A)$. It follows that any complex-valued function m on $\mathscr{I}(A)$ which vanishes on $\{0\}$ is a measure. ∎

A spin triple A of dimension 2 is no longer a JBW*-triple factor. Indeed, if u is an element of $\mathscr{U}_1(A)$, then $\{u, u^-\}$ forms an orthonormal basis for A. Moreover, an element $\alpha u + \beta u^-$ lies in $\mathscr{U}_1(A)$ if and only if one of α or β is zero and the other is of modulus one, and lies in $\mathscr{U}_2(A)$ if and only if both α and β are of modulus one. Furthermore,

$$\|\alpha u + \beta u^-\| = \max\{|\alpha|, |\beta|\}.$$

Therefore A can be regarded as the M-sum of two copies of the JBW*-triple $\mathbb{C}$. In this case there are no pairs of rigidly collinear elements of $\mathscr{I}(A)$, and a complex-valued function m on $\mathscr{I}(A)$ is a measure if and only if, for each pair J and K of centrally orthogonal elements in $\mathscr{I}(A)$,

$$m(J \vee K) = m(J) + m(K).$$

THEOREM 3.8. *Let A be a spin triple of dimension 2. Then the following hold.*

(i) *There exists a pair $\{u, u^-\}$ in $\mathscr{U}_1(A)$, unique up to multiplication by complex numbers of unit modulus, that forms an orthonormal basis for A.*

(ii) *Every element of $\mathscr{U}_1(A)$ is of the form αu or βu^-, and every element of $\mathscr{U}_2(A)$ is of the form $\alpha u + \beta u^-$, where α and β are complex numbers of unit modulus.*

(iii) *The elements of $\mathscr{I}(A)$ are $\{0\}$, $A_2(u)$, $A_2(u^-)$, and A.*

(iv) *For each pair γ and δ of elements of $\mathbb{C}$, there exists a measure $m_{\gamma,\delta}$ on $\mathscr{I}(A)$ defined by*

$$m_{\gamma,\delta}(\{0\}) = 0, \qquad m_{\gamma,\delta}(A_2(u)) = \gamma, \qquad m_{\gamma,\delta}(A_2(u^-)) = \delta,$$
$$m_{\gamma,\delta}(A) = \gamma + \delta.$$

Proof. (i), (ii), and (iii) are easy calculations from the results quoted above. To prove (iv), observe that $A_2(u)$ and $A_2(u^-)$ are centrally orthogonal, and, therefore, every measure m on $\mathscr{I}(A)$ is determined by its values on these two elements of $\mathscr{I}(A)$. Furthermore,

$$m(A) = m(A_2(u) \vee A_2(u^-)) = m(A_2(u)) + m(A_2(u^-)).$$

This completes the proof of the theorem. ∎

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