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A new parametrisation of tilting modules for $GL(3, K)$ over characteristic 2

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Abstract

We determine the restrictions of tilting modules labelled by the largest weight of the groups $GL(3, K)$ over characteristic 2 to the finite subgroups $SL(3, 2)$, up to projective summands.

Keywords: General linear groups; Tilting modules; Auslander–Reiten quiver; Symmetric groups

Let K be an infinite field of prime characteristic $p > 0$, and let E be an n-dimensional vector space over K. Then E is a module for the general linear group $GL(n, K)$, with the natural action, and this gives a $GL(n, K)$-module action on the tensor space $E^{\otimes r}$ for each $r \geqslant 1$. To understand the decomposition of $E^{\otimes r}$ into indecomposable direct summands, as a $GL(n, K)$-module, is a very basic problem. One reason (amongst many others) why this is important, comes from the connection with symmetric groups, see [10,14]. The tensor space $E^{\otimes r}$ is also a module for the symmetric group $\mathcal{S}_r$, acting by place permutation; and this action commutes with that of $GL(n, K)$. A consequence of this is that the $GL(n, K)$-module structure encapsulates much of the representation theory of symmetric groups; and in particular of decomposition numbers.

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A parametrisation of the indecomposable summands of tensor space as $GL(n, K)$-modules is known. Let $\Lambda^+(n,r)$ be the set of all partitions of r into at most n parts, for each λ there is a distinguished indecomposable $GL(n, K)$-module $T(\lambda)$ with highest weight λ, known as 'tilting module.' Let $\Lambda_p^+(n,r)$ be the set of p-regular partitions in $\Lambda^+(n,r)$; a partition is p-regular if it does not have p (or more) equal parts. Then

$$E^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda^+(n,r)} T(\lambda)^{d_\lambda}, \tag{1}$$

where d_λ is the dimension of the simple $K\mathcal{S}_r$-module D^λ; here D^λ is the unique simple quotient of the Specht module S^λ of $\mathcal{S}_r$, see [15].

The tilting modules $T(\lambda)$ are characterised by the property that they have a filtration whose quotients are isomorphic to Weyl modules, and also a filtration with quotients isomorphic to dual Weyl modules (see [5]). Denote the dual Weyl module labelled by μ with $\nabla(\mu)$, and write $(T(\lambda) : \nabla(\mu))$ for the number of times $\nabla(\mu)$ occurs as a quotient in a filtration of $T(\lambda)$ by dual Weyl modules. It is known that for λ p-regular, the filtration multiplicity $(T(\lambda) : \nabla(\mu))$ is equal to the decomposition number $d_{\mu\lambda}$, that is the multiplicity of D^λ as a composition factor of the Specht module S^μ. Therefore, if one would understand the filtration multiplicities of the tilting modules, then one would understand the decomposition numbers for symmetric groups. The decomposition numbers for two-part partitions (that is, when $n = 2$) have been determined by James [11–13], long before tilting modules were known. More recently, the use of tilting modules has led to a new proof (see [7]); based on Donkin's 'twisted tensor product theorem' (see [5]). When $n = 3$ there are partial results in [13], and more in [16], but in general the problem of finding decomposition numbers for 3-part partitions is unsolved.

Here we consider $n = 3$ and $p = 2$. In this case, Donkin's twisted tensor product formula produces a substantial reduction. It shows that if one would understand the tilting modules labelled by 1-part partitions (r), for all r, then one could inductively find all $T(\lambda)$. There remains however a hard problem, and new information is needed.

In this paper, we study $T(r)$ as a module for the finite group $G := SL(3,2)$ of invertible matrices with entries in the field with two elements. We show that the restriction of $T(r)$ to $SL(3,2)$ has unique indecomposable non-projective summand, and we determine completely its submodule structure. Furthermore we show that this gives a parametrisation of the modules $T(r)$. That is, for $r \neq s$ the restrictions of $T(r)$ and of $T(s)$ to G are not isomorphic. We can do this because of the very exceptional representation theory of the group $SL(3,2)$ which allows us to make use of Auslander–Reiten theory of special biserial algebras [4]; furthermore we exploit the fact that tensoring with E induces an equivalence of the stable module category of KG. We also get a small amount of information on the restriction of $T(r)$ to the finite groups $SL(3,2^m)$.

The paper is organised as follows. The first section studies the effect of tensoring with the natural module E on the stable category of KG-modules. In the second section we summarise material on indecomposable KG-modules, in particular a presentation of the basic algebra of the principal block by quiver and relations, and description of string modules. In the next section we describe string modules in a given component of the stable Auslander–Reiten quiver, and in the fourth section we derive the submodule structure of the non-projective indecomposable summand of $T(r)$ as an $SL(3,2)$-module. In the final section we discuss the restriction of $T(r)$ to finite groups $SL(3,2^m)$ for arbitrary m.

All modules are finite-dimensional. For background we refer to [2,3,6].

1. Tensoring with the natural module

1.1. Preliminaries

Let $A = KG$, the group algebra of $SL(3,2)$, over the field K of characteristic 2. Let Ω denote the Heller operator, that is $\Omega(M)$ is the kernel of a projective cover of the module M; and $\Omega^{-1}(M)$ is the cokernel of its injective hull. If M is indecomposable and not projective then so are $\Omega^{\pm}(M)$; recall that projective A-modules coincide with injective A-modules. For a general module M, we define $\Omega^0(M)$ by

$$M = \Omega^0(M) \oplus P,$$

where P is projective, and $\Omega^0(M)$ does not have a non-zero projective summand. For example, if L is simple then $\Omega(L) = \operatorname{rad} P(L)$, and $\Omega^{-1}(L) = P(L)/L$ where $P(L)$ is the projective module with simple quotient L.

We also note that $\Omega^k(\Omega^0 M) \cong \Omega^k(M)$ for any k.

For an $GL(3,K)$-module M we write $M{\downarrow_G}$ for the restriction to G. By general theory, the simple modules for G are precisely the restrictions of the simple modules for $GL(3,K)$ with 2-restricted highest weights, and these are K, E, E^* and W where W is the restriction of the Steinberg module; which is known to be simple projective, of dimension 8.

Lemma 1.1.

(a) *$E \otimes E^*{\downarrow_G} = K \oplus W$ and W is projective.*
(b) *The restriction $E^{\otimes r}{\downarrow_G}$ has a unique non-projective summand. That is, $\Omega^0(E^{\otimes r})$ is indecomposable.*
(c) *We have $\Omega^0(E^{\otimes r}) = \Omega^0(T(r))$.*

Proof. Part (a) is well known as already mentioned. This implies that the functor $E \otimes (-)$ induces an equivalence of the stable category KG-$\underline{\text{mod}}$ with inverse induced by $E^* \otimes (-)$. Consequently also tensoring with $E^{\otimes(r-1)}$ induces an equivalence of KG-$\underline{\text{mod}}$. Now E is indecomposable and not projective, consequently $E^{\otimes(r-1)} \otimes E$ also has a unique non-projective indecomposable summand, hence part (b) holds.

(c) Recall the direct sum decomposition of $E^{\otimes r}$ described in (1). By the Krull–Schmidt theorem there is a unique $\lambda \in \Lambda_2^+(3,r)$ such that $T(\lambda){\downarrow_G}$ is not projective, and moreover $d_\lambda = 1$. Now $d_\lambda = \dim D^\lambda = 1$ precisely when $D^\lambda = K$, the trivial module since K has characteristic 2. Hence $\lambda = (r)$. $\square$

So our task is now to understand the action of the functor $E \otimes (-)$ on the stable category of KG-modules. This is most easily done by exploiting the Auslander–Reiten quiver.

1.2. The Auslander–Reiten quiver

1.2.1. Definition and general properties

Suppose A is a finite-dimensional K-algebra. The Auslander–Reiten quiver $\Gamma(A)$ can be thought of as giving part of a presentation of the module category of A, by giving generators and some relations. It is the directed graph, with vertices the isomorphism classes of indecomposable

$$\begin{tikzcd}
 & {[X_1]} \arrow[dr] & \\
{[\Omega^2 N]} \arrow[ur] \arrow[dr] & & {[N]} \\
 & {[X_2]} \arrow[ur] &
\end{tikzcd}$$

Fig. 1. Mesh.

A-modules, we write $[M]$ for the vertex of the graph corresponding to M. The arrows $[M] \to [N]$ correspond to irreducible maps $f : M \to N$.

Recall that the homomorphism $f : M \to N$ is irreducible provided it is not split, and whenever f has a factorisation $f = h \circ g$ then either h is a split epimorphism, or g is a split monomorphism. If f is irreducible then f is 1–1 or onto (but not both). Suppose now $A = KG$, the group algebra of a finite group. Then the stable Auslander–Reiten quiver $\Gamma_s(A)$ of A is obtained from $\Gamma(A)$ by removing the projective modules and adjacent arrows. In our situation it is the union of infinitely many connected components.

Irreducible maps are collected in almost split sequences. The almost split sequence ending in N is a non-split short exact sequence of A-modules

$$0 \to \Omega^2(N) \xrightarrow{g} X \xrightarrow{h} N \to 0$$

such that any map $\Omega^2(N) \to Y$ which is not a split monomorphism factors through g. Equivalently, any map $Y \to N$ which is not a split epimorphism factors through h.

It was proved by Auslander and Reiten that given any indecomposable non-projective module N, there is a unique such almost split sequence. Furthermore they proved that there is an irreducible map $M \to N$ if and only if M is isomorphic to a direct summand of X, and if so then the irreducible map occurs as a component of h. Dually, there is an irreducible map $\Omega^2(N) \to M$ if and only if M is a direct summand of X, and if so then the irreducible map occurs as a component of g; see, for example, [3] or [6].

In our case, X has always precisely two indecomposable non-projective summands and they are non-isomorphic. Suppose they are X_1 and X_2, then the almost split sequence is represented in $\Gamma_s(A)$ by Fig. 1.

Usually we omit $[-]$ and identify the module with the corresponding vertex in the graph.

1.2.2. The standard sequence

Suppose L is a simple A-module L with projective cover $P(L)$. Then the almost split sequence ending at $P(L)/L$ is of the form

$$0 \to \operatorname{rad} P(L) \to P(L) \oplus \operatorname{rad} P(L)/L \to P(L)/L \to 0$$

and we call this the ‘standard sequence’ associated to L. This is the only almost split sequence in which $P(L)$ occurs. Hence a mesh in $\Gamma_s(A)$ corresponds almost always to an exact sequence, the only exceptions occur for the standard sequences.

When $L = K$, the trivial module, this standard sequence is of central importance for the study of almost split sequences because of the following, which is due to Auslander and Carlson [1].

Proposition 1.2. *Assume that M is an indecomposable KG-module whose dimension is not divisible by the characteristic of K. Then the almost split sequence starting in $\Omega(M)$ is of the form*

$$0 \to \Omega(M) \to P \oplus \Omega^0\big(M \otimes P(K)/K\big) \to \Omega^{-1}(M) \to 0,$$

where $P = P(M)$ if M is simple, and $P = 0$ otherwise.

1.3. The case $G = SL(3,2)$ over characteristic 2

We now return to the group $G = SL(3,2)$ and to our field K of characteristic 2. Then the middle term of the standard sequence associated to $L = K$ is of the form $P(K) \oplus E \oplus E^*$ (this is well known, and we will discuss modules for KG in detail below). Then Proposition 1.2 shows that, given any M which is indecomposable of odd dimension, then there are irreducible maps

$$\Omega(M) \to \Omega^0(M \otimes E), \qquad \Omega(M) \to \Omega^0\big(M \otimes E^*\big).$$

Furthermore, the proposition describes the almost split sequence starting with $\Omega(M)$. We apply this to $M = \Omega^0 T(r)$ and we get:

Corollary 1.3. *For $r \geqslant 0$, there are irreducible maps*

$$\Omega\big(T(r)\big) \to \Omega^0 T(r+1), \qquad \Omega\big(T(r)\big) \to \Omega^0 T(r-1).$$

When $r \geqslant 2$ then there are almost split sequences

$$0 \to \Omega\big(T(r)\big) \to \Omega^0 T(r+1) \oplus \Omega^0 T(r-1) \to \Omega^{-1}\big(T(r)\big) \to 0.$$

When $r = 0$ or $r = 1$, then the almost split sequence from Proposition 1.2 is the standard sequence associated to $T(0) = K$, or $T(1) = E$. The almost split sequences for KG-modules are well understood. We will now exploit this.

2. *KG*-modules

Let $A = KG$ where $G = SL(3,2)$ and K has characteristic 2. Indecomposable A-modules are classified and have a good combinatorial description.

2.1. Simple modules and projective modules

The group algebra A splits into two blocks; one which is simple and which has simple module W, and the other simple modules, that is K, E and E^* all belong to the same block (the principal block). The indecomposable projective modules in the principal block are easy to describe. If P is one, then $P/\operatorname{rad} P \cong \operatorname{soc} P$ and $\operatorname{rad} P/\operatorname{soc} P$ is the direct sum of two uniserial modules. Explicitly:

$$\begin{array}{ccc} & & \\ & K & \\ E & \oplus & E^*, \\ & K & \\ & & \end{array} \qquad \begin{array}{ccc} & E & \\ & & E^* \\ K & \oplus & E, \\ & & E^* \\ & E & \end{array} \qquad \begin{array}{ccc} & E^* & \\ & & E \\ K & \oplus & E^*. \\ & & E \\ & E^* & \end{array}$$

2.2. Indecomposable modules

The classification of indecomposable A-modules is now quite well known, several accounts appear in the literature, see, for example, [4] or [6]. The indecomposable modules come in two types, 'string modules' and 'band modules.' Simple modules are string modules. Furthermore, the class of string modules is closed under Ω, and is also closed under taking irreducible maps. Since $T(0)$ and $T(1)$ are simple, it follows from 1.3 that all modules $\Omega^k(T(r))$ are string modules; so we will not need to deal with band modules.

Roughly speaking, a string module can be described as 'glueing of uniserial modules.' We give two examples (these are actually $\Omega^0 T(3)$ and $\Omega^0 T(4)$) (see Fig. 2).
It is convenient to visualise string modules as representations of linear quivers, and parametrise them by words. These are based on a given presentation of the basic algebra by quiver and relations.

So let Λ be the basic algebra of the principal block of KG. This is the algebra denoted by $D(3\mathcal{K})$ in [6], with slightly changed notation it is given by the quiver $\mathcal{Q}$ (see Fig. 3)
and Λ is the path algebra $K\mathcal{Q}/I$ where I is the ideal generated by

$$\begin{gathered} \beta\delta, \quad \delta\lambda, \quad \lambda\beta, \quad \bar\beta\bar\lambda, \quad \bar\lambda\bar\delta, \quad \bar\delta\bar\beta, \\ \beta\bar\beta - \bar\lambda\lambda, \quad \lambda\bar\lambda - (\bar\delta\delta)^2, \quad (\delta\bar\delta)^2 - \bar\beta\beta. \end{gathered}$$

Here the simple module at vertex 0 corresponds to K under a Morita equivalence between Λ and the principal block of KG; and the simple modules at vertex 1 and 2 correspond to E and E^*, respectively.

$$\begin{array}{ccccccccc} & & & & & & E^* & & \\ & & & & & / & & \backslash & \\ K & & & & E & & & & K, \\ & \backslash & & / & & & & & \\ & & E^* & & & & & & \end{array}$$

$$\begin{array}{ccccccccccccc} & & & & & & & & E^* & & & & E \\ & & & & & & & / & & \backslash & & / & \\ & & K & & & & E & & & & K & & \\ & / & & \backslash & & / & & & & & & & \\ E & & & & E^* & & & & & & & & \end{array} \quad , \quad \begin{array}{ccccccccccccc} E & & & & E^* & & & & & & & & \\ & \backslash & & / & & \backslash & & & & & & & \\ & & K & & & & E & & & & K & & \\ & & & & & & & \backslash & & / & & \backslash & \\ & & & & & & & & E^* & & & & E. \end{array}$$

Fig. 2. String modules.

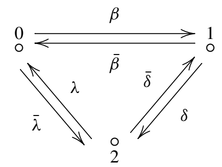

Fig. 3. Quiver.

Observe that these show that the algebra $\Lambda/\operatorname{soc}\Lambda$ is a 'string algebra,' and Λ itself is 'special biserial.' For details of this and the following we refer to [6]. String modules (which are not simple) are labelled by words in the alphabet $\{\beta, \bar{\beta}, \delta, \bar{\delta}, \lambda, \bar{\lambda}\}$, subject to an appropriate equivalence relation. Translating the second example above gives a representation of a linear quiver

$$\cdot \xleftarrow{\beta} \cdot \xrightarrow{\bar{\lambda}} \cdot \xleftarrow{\delta} \cdot \xleftarrow{\bar{\delta}} \cdot \xrightarrow{\lambda} \cdot \xleftarrow{\bar{\beta}} \cdot.$$

The word labelling the module is

$$\beta\bar{\lambda}^{-1}\delta\bar{\delta}\lambda^{-1}\bar{\beta}.$$

The same module can also be labelled by

$$\bar{\beta}^{-1}\lambda\bar{\delta}^{-1}\delta^{-1}\bar{\lambda}\beta^{-1}.$$

In general two words W_1 and W_2 label isomorphic string modules if and only if $W_1 = W_2$ or $W_1 = W_2^{-1}$.

Example 1.

(a) The module E is labelled by word of length zero at vertex 1, we write $E = M(e_1)$.
(b) Consider the module $E^{\otimes 2}$, it has only three composition factors and hence its restriction to G is indecomposable. It is isomorphic to the non-trivial direct summand of $\operatorname{rad} P(E)/E$, and it is labelled by $\bar{\delta}\delta$. We write $T(1) = E = M(e_1)$ and $T(2) = E^{\otimes 2} = M(\bar{\delta}\delta)$.

2.2.1. Irreducible maps

For special biserial algebras, an algorithm is known which describes the irreducible maps between string modules (see [4] or [6]). Roughly speaking, it says 'add or delete maximal uniserial strings' at the 'ends' of the string. Given a string module M, then it is easy to determine $\Omega^2(M)$ (and $\Omega^{-2}(M)$), and then as well to write down the indecomposable summands in the middle of the almost split sequences ending and starting at M.

The maximal uniserial strings are precisely the maximal uniserial quotients of the indecomposable projective modules, labelled by words as

$$\beta, \quad \bar{\beta}, \quad \lambda, \bar{\lambda}, \quad \delta\bar{\delta}\delta, \quad \bar{\delta}\delta\bar{\delta}. \tag{2}$$

Example 2. The kernel of a non-zero homomorphism $\operatorname{rad} P(K) \to E$ is maximal uniserial, so this map is irreducible. Similarly there is an irreducible map $\operatorname{rad} P(K) \to E^*$. Consider the irreducible maps ending at rad $P(K)$. We cannot have an inclusion map which corresponds to deleting a maximal uniserial string. Instead, there are precisely two ways to add a maximal uniserial string one for each end. This gives irreducible maps

$$M\big(\bar{\beta}^{-1}\lambda\bar{\delta}^{-1}\beta\big) \to M\big(\bar{\beta}^{-1}\lambda\big) = \operatorname{rad} P(K)$$

and

$$M\big(\bar{\lambda}^{-1}\delta\bar{\beta}^{-1}\lambda\big) \to M\big(\bar{\beta}^{-1}\lambda\big) = \operatorname{rad} P(K).$$

In general, given any string module M (other than a maximal uniserial string), there are precisely two irreducible maps ending at M and precisely two irreducible maps in $\Gamma_s(A)$ starting at M. By [4,9], any component consisting of string modules for A is, as a graph, isomorphic to $\mathbb{Z}A_\infty^\infty$. Furthermore, a simple module and its Ω-translate belong to different components.

3. Auslander–Reiten components with string modules

We have seen how to relate $\Omega(T(r-1))$ to $\Omega^0 T(r-2)$ and $\Omega^0 T(r)$ via irreducible maps. We want to use this to determine the submodule structure of the $\Omega^0 T(r)$ generally, and to do so we use 'large scale' properties of the components of $\Gamma_s(KG)$ in which these modules lie.

3.1. *Labelling modules and maps*

For start we consider, more generally, a graph whose vertices are labelled by a subset of $\mathbb{Z} \times \mathbb{Z}$, as (a, b), and where we have arrows $(a, b) \xrightarrow{u_{a,b}} (a-1, b+1)$ and arrows $(a, b) \xrightarrow{d_{a,b}} (a-1, b-1)$. We draw the graph so that a is the column index and b is the row index. We agree that the column index increases from right to left, and the row index increases from bottom to top, and then all arrows will point to the right. We assume also that this graph is convex, as a subgraph of $\mathbb{Z} \times \mathbb{Z}$. In the application this will be a part of a component whose stable part is $\mathbb{Z}A_\infty^\infty$ and to which no projective module is attached.

The part of the graph around a vertex (a, b) is a subset of a graph of the form as shown in Fig. 4. Note that vertices on the 'diagonals' from SW to NE satisfy $a + b =$ constant; and the vertices on the 'diagonals' from NW to SE satisfy $b - a =$ constant.

Now suppose for each vertex (a, b) of this graph we have a module $M(a, b)$, and for each arrow in the graph we have a map, which we denote by the same name. If we are given that the 'meshes' in the graph correspond to short exact sequences then there are also short exact sequences involving modules further apart. The proof of the following is straightforward.

$$\begin{tikzcd}
 & (a,b+2) \arrow[dr] & & (a-2,b+2) \\
(a+1,b+1) \arrow[ur] \arrow[dr] & & (a-1,b+1) \arrow[ur] \arrow[dr] & \\
 & (a,b) \arrow[ur, "u_{a,b}"'] \arrow[dr, "d_{a,b}"] & & (a-2,b) \\
(a+1,b-1) \arrow[ur] & & (a-1,b-1) \arrow[ur] &
\end{tikzcd}$$

Fig. 4. Graph.

Proposition 3.1. *Suppose that for some $k \geqslant 1$, the set of vertices $\mathcal{V}_k = \{(a,b), (a,b-2k), (a-k, b \pm k)\}$ belong to the graph, and furthermore, assume that for each (e,f) such that (e,f), $(e-1, f \pm 1)$ and $(e-2,f)$ belong to the convex subset spanned by $\mathcal{V}_k$ we have an exact sequence*

$$0 \to M(e,f) \xrightarrow{[u,d]} M(e-1,f+1) \oplus M(e-1,f-1) \xrightarrow{\binom{d}{-u}} M(e-2,f) \to 0.$$

Assume also that all maps u occurring are one-to-one, and all maps d occurring are onto. Then there is also an exact sequence

$$0 \to M(a,b) \to M(a-k,b+k) \oplus M(a-k,b-k) \to M(a-2k,b) \to 0$$

in which the maps are compositions of maps d and compositions of maps u.

3.2. Sectional paths

In the above graph, we define a sectional path to be either a path along $a+b=$ constant, or along $b-a=$ constant.

Now assume that the graph above is a subset of a component whose stable part is isomorphic to $\mathbf{Z}A_\infty^\infty$. Suppose M is the module corresponding to a vertex (a,b). Then the module $\Omega^{\pm 2k}(M)$ corresponds to vertex $(a \pm 2k, b)$; and the modules corresponding to $(a-1, b \pm 1)$ correspond to the indecomposable non-projective middle terms of the almost split sequence starting with M.

In terms of representations, a sectional path in the component is a path

$$\cdots \to X_{r+1} \to X_r \to X_{r-1} \to X_{r-2} \cdots$$

such that for each r we have $X_r \ncong \Omega^2(X_{r-2})$. Taking the labelling as we did, this is consistent with the definition of a sectional path in the abstract graph.

The following is a direct consequence of the algorithm described in [4].

Lemma 3.2. *Suppose Λ is special biserial and symmetric, and assume that for each indecomposable projective module P, the subquotient* rad $P/$ soc P *is the direct sum of two non-zero uniserial modules. Let L be a string module which is simple, or uniserial* (*but not maximal uniserial*).

- (i) *Given a sectional path $\cdots X_r \to \cdots \to X_1 \to X_0 = L$, then for each r, the label of X_r is given by glueing a maximal uniserial string to an end of the label for X_{r-1}, and always glueing at the same end. The corresponding map is surjective.*
- (ii) *Given a sectional path $L = Y_0 \to Y_1 \to Y_2 \to \cdots$, then for each r, the label of Y_r is given by glueing a maximal uniserial string to an end of the label for Y_{r-1} and always glueing at the same end. The corresponding map is injective.*

Furthermore, knowing $X_1 \to L$, or $L \to Y_1$, already determines completely the sectional path in (i)*, or in* (ii).

To see this, suppose for example, we know in (i) the structure of X_r and of X_{r-1}, the algorithm shows that there are two strings from which there are irreducible maps ending at X_r. One of them is $\Omega^2(X_{r-1})$ which can easily be computed from X_{r-1} but this is excluded and therefore X_{r+1} is uniquely determined.

4. The structure of $\Omega^0 T(r)$

The modules K and $\Omega(K)$ belong to different components, and these components are permuted by Ω. The simple module E lies in the component of $\Omega(K)$, by 1.2.2. Now it follows from 1.3 that all $\Omega^0 T(r)$ for r even belong to the same component as K, and the $\Omega^0 T(r)$ with r odd belong to the same component as E, and these are different components.

We label the modules in these two components, as in 3.1. In the component of K, we label K as $M(0,0)$. By 1.3, the modules $\Omega^0(T(b))$ for b even are then labelled as $M(0,b)$, and the modules $\Omega(T(b))$ with b odd are labelled as $M(1,b)$. It follows then that the general vertex with label $M(a,b)$ and $b \geqslant 0$ corresponds to $\Omega^a(T(b))$; here $a+b$ is always even.

Consider the component of E. We label E as $M(0,1)$, then similarly, by 1.3, the general module in this component labelled by $M(a,b)$ with $b \geqslant 0$ is $\Omega^a(T(b))$, but now $a+b$ is always odd. Now, different labels of vertices in these components correspond to non-isomorphic modules. We have therefore the following:

Corollary 4.1. *For $r \neq s$, the modules $\Omega^0 T(r)$ and $\Omega^0 T(s)$ are not isomorphic. Furthermore, for any k, ℓ, modules $\Omega^k(T(r))$ and $\Omega^\ell(T(s))$ are not isomorphic.*

This together with Lemma 3.2 allows us to describe sectional paths starting at E and at $T(2)$ $(= \Omega^0 T(2))$.

Lemma 4.2.

(a) *The sectional path ending with $\Omega(T(2)) \to E$ is of the form*

$$\cdots \to \Omega^k\bigl(T(k+1)\bigr) \to \Omega^{k-1}\bigl(T(k)\bigr) \to \cdots \to \Omega\bigl(T(2)\bigr) \to E.$$

The maps are onto.

(b) *The sectional path starting with $E \to \Omega^{-1}(T(2))$ is of the form*

$$E \to \Omega^{-1}\bigl(T(2)\bigr) \to \Omega^{-2}\bigl(T(3)\bigr) \to \cdots \to \Omega^{-k}\bigl(T(k+1)\bigr) \to \cdots.$$

The maps are one-to-one.

(c) *The sectional path ending with $\Omega(T(3)) \to T(2)$ is of the form*

$$\cdots \to \Omega^k\bigl(T(k+2)\bigr) \to \Omega^{k-1}\bigl(T(k+1)\bigr) \to \cdots \to \Omega\bigl(T(3)\bigr) \to T(2).$$

The maps are onto.

(d) *The sectional path starting with $T(2) \to \Omega^{-1}(T(3))$ is of the form*

$$T(2) \to \Omega^{-1}\bigl(T(3)\bigr) \to \Omega^{-2}\bigl(T(4)\bigr) \to \cdots \to \Omega^{-k}\bigl(T(k+2)\bigr) \to \cdots.$$

The maps are one-to-one.

To ensure that the hypotheses of 3.1 are satisfied, we have to avoid projective modules. The projective module $P(E)$ lies between $M(0,2)$ and $M(0,0) = K$, that is, in the 'mesh' of $\Omega(E)$ and $\Omega^{-1}(E)$, and the projective module $P(E^*)$ lies between $M(0,0)$ and $M(0,-2)$. The projective module $P(K)$ lies in the other component, between $M(0,1)$ and $M(0,-1)$. So if we work in the parts of the components with $b \geqslant 1$ then 3.1 can be applied. This gives several short exact sequences relating the modules we are interested in.

Theorem 4.3. *Suppose* $0 \leqslant s < r$ *and* $s \equiv r \bmod 2$.

(a) *Suppose first that* $r = 2k+1$ *and* $s = 2\ell+1$. *Then there is an exact sequence*

$$0 \to \Omega^{k-\ell}\big(T(k+\ell+1)\big) \to \Omega^0 T(s) \oplus \Omega^0 T(r) \to \Omega^{-(k-\ell)}\big(T(k+\ell+1)\big) \to 0.$$

(b) *If* $r = 2k$ *and* $s = 2\ell \geqslant 2$ *then there is an exact sequence*

$$0 \to \Omega^{k-\ell}\big(T(k+\ell)\big) \to \Omega^0 T(s) \oplus \Omega^0 T(r) \to \Omega^{-(k-\ell)}\big(T(k+\ell)\big) \to 0.$$

Proof. This will follow from the above observation. In the relevant part of the Auslander–Reiten component there is no projective module, so to apply Proposition 3.1 we only need to check that the maps $u_{a,b}$ are 1–1 and the maps $d_{a,b}$ are onto. This is the case, by 3.2 and 4.2. □

4.1. The words labelling modules $\Omega^0 T(r)$

It remains to describe which of the maximal uniserial strings is attached at each step. We label the string modules by words, as suggested earlier. The words are given in terms of directed words, and every directed word is a subword of a maximal directed word. The maximal directed words are listed in (2). We write them as

$$B = \beta, \quad \bar{B} = \bar{\beta}, \quad D = \delta\bar{\delta}\delta, \quad \bar{D} = \bar{\delta}\delta\bar{\delta}, \quad L = \lambda, \quad \bar{L} = \bar{\lambda}.$$

A general string is written as $M(C_{2k}^{-1} C_{2k-1} C_{2k-2}^{-1} \ldots C_2^{-1} C_1)$ where the C_i are directed; here C_1 or C_{2k} (or both) are allowed to be of length zero. For example, if C is directed then $M(C)$ is uniserial. The end vertex of C_{2s+1} must be the start vertex of C_{2s+2}^{-1} and if the first ends in α then the second is not allowed to start in α, a similar property must hold for the end of C_{2s+2}^{-1} and the start of C_{2s+3}. We say that any word satisfying these conditions is admissible. If so then it labels an indecomposable module.

If both C_{2s+1} and C_{2s+2}^{-1} are maximal directed then the condition on the arrow α described just means that C_{2s+1} and C_{2s+2} are different, and since at any vertex we only have two maximal uniserial strings ending at that vertex this shows that given C_{2s+1}, then C_{2s+2} is already uniquely fixed. Suppose, for example, $C_{2s+1} = L$, this ends at vertex 0. So there are two maximal uniserial strings whose inverses start at vertex 0, and since L^{-1} is not admissible, it follows that $(C_{2s+1})^{-1} = \bar{B}^{-1}$. Similar properties hold for C_{2s+2}^{-1} and C_{2s+3}.

4.2. The modules $\Omega^{\pm r}(T(r+1))$ and $\Omega^0 T(2r+1)$

We will apply Theorem 4.3 with $s = 1$ and $s = 2$. The building blocks for the words describing the modules in question are strings made up of maximal uniserial strings, 'glued' by letters or inverse letters. Define a map on strings of the form $X = C_{2k}^{-1} C_{2k-1} \ldots C_2^{-1} C_1$ for which each C_i is maximal directed, by setting

$$\hat{X} := \bar{C}_1 \bar{C}_2^{-1} \ldots \bar{C}_{2k}^{-1}$$

that is we reverse the order of the C's and we replace each C by $\bar{C}$.

Let

$$S := \bar{B}^{-1}\lambda\bar{D}^{-1}\beta\bar{L}^{-1}\delta,$$

then S has the property that for each $a \geqslant 1$ also S^a is admissible. Moreover the string $\hat{S}$ has the same property, that is $\hat{S}^a$ is admissible for each $a \geqslant 1$.

Lemma 4.4. *Let $r = 3\bar{r} + r_0$ where $0 \leqslant r_0 \leqslant 2$.*

(a) *We have $\Omega^r(T(r+1)) = M(S_0 S^{\bar{r}})$ where*

$$S_0 = \begin{cases} \emptyset, & r_0 = 0, \\ \bar{L}^{-1}\delta, & r_0 = 1, \\ \bar{D}^{-1}\beta\bar{L}^{-1}\delta, & r_0 = 2. \end{cases}$$

(b) *We have $\Omega^{-r}(T(r+1)) = M(\hat{S}^{\bar{r}}\hat{S}_0)$.*

Proof. (a) Start with $T(1) = E$, then $\Omega(T(2)) = \bar{L}^{-1}\delta$. This already determines completely the sectional path, as explained in 3.2 and the statement follows by induction on r. Similarly one proves part (b). $\square$

The diagram, see Fig. 5, shows the first three of the modules described in part (a).

We set $D^* = \delta\bar{\delta}$, then $M(D^*) = T(2)$ as we have already seen. In order to determine $T(2r+1)$, we use now the short exact sequences

$$0 \to \Omega^r\big(T(r+1)\big) \to \Omega^0 T(2r+1) \oplus \Omega^0 T(1) \to \Omega^{-r}\big(T(r+1)\big) \to 0. \tag{3}$$

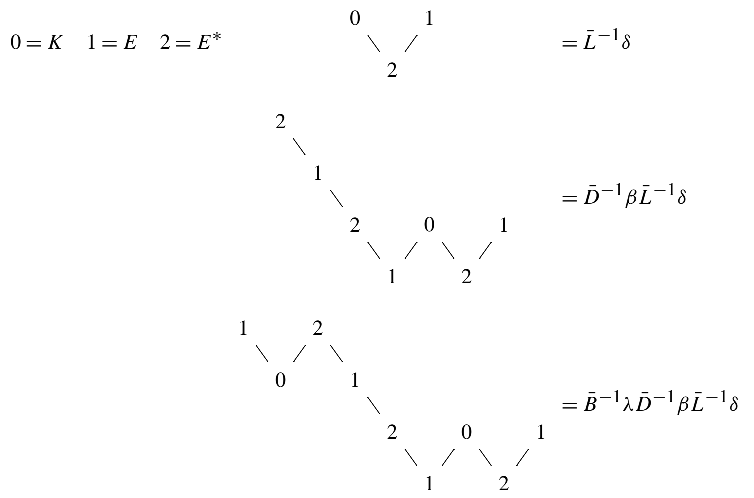

Fig. 5. $\Omega^r(T(r+1))$.

Theorem 4.5. *We have $\Omega^0 T(3) = M(\bar{L}^{-1} D^* L^{-1})$. For $2r+1 \geqslant 5$ we have*

$$\Omega^0 T(2r+1) = M\big(S_0 S^{\bar{r}-1} U D^* \hat{U} \hat{S}^{\bar{r}-1} \hat{S}_0\big),$$

where $U = \bar{B}^{-1}\lambda\bar{D}^{-1}\beta\bar{L}^{-1}$, so that $U\delta = S$.

This follows from the previous lemma, and by using the exact sequence (3). The next diagram illustrates the construction of $\Omega^0 T(3)$.

$$0 \to \Omega\big(T(2)\big) \to \Omega^0 T(3) \oplus T(1) \to \Omega^{-1}\big(T(2)\big) \to 0$$

$$\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & & & 2 & & & & & & & & \\
0 & & & & 1 & & & & & & & / & & \backslash & & & & & 2 & & \\
 & \backslash & & / & & \to & 0 & & & & 1 & & & & 0 & \oplus 1 \to & & / & & \backslash & \\
 & & 2 & & & & & \backslash & & / & & & & & & & 1 & & & & 0 \\
 & & & & & & & & 2 & & & & & & & & & & & &
\end{array}$$

4.3. The modules $\Omega^{\pm}(T(r+2))$ and $\Omega^0 T(2r+2)$

Now let R be the word

$$R = \bar{D}^{-1}\beta\bar{L}^{-1}\delta\bar{B}^{-1}\lambda$$

Note that R is a 'rotation' of S, again every power of R is admissible.

Lemma 4.6. *Let $r = 3\bar{r} + r_0$ where $0 \leqslant r_0 \leqslant 2$.*

(a) *We have $\Omega^r(T(r+2)) = M(R_0 R^{\bar{r}} (D^*)^{-1})$ where*

$$R_0 = \begin{cases} \emptyset, & r_0 = 0, \\ \bar{B}^{-1}\lambda, & r_0 = 1, \\ \bar{L}^{-1}\delta\bar{B}^{-1}\lambda, & r_0 = 2. \end{cases}$$

(b) *We have $\Omega^{-r}(T(r+2)) = M((D^*)^{-1}\hat{R}^{\bar{r}}\hat{R}_0)$.*

Proof. (a) Start with $T(2) = M(D^*) \cong M((D^*)^{-1})$, then $\Omega(T(3)) = M(\bar{B}^{-1}L(D^*)^{-1})$. This already determines completely the sectional path, as explained in 3.2 and the statement follows by induction on r. Similarly one proves part (b). □

Now we exploit the exact sequence

$$0 \to \Omega^r\big(T(r+2)\big) \to \Omega^0 T(2r+2) \oplus \Omega^0 T(2) \to \Omega^{-r}\big(T(r+2)\big) \to 0. \tag{4}$$

Theorem 4.7. *For $r \geqslant 1$ we have*

$$\Omega^0 T(2r+2) = M\big(R_0 R^{\bar{r}} \big(D^*\big)^{-1} \big(\hat{R}_0^{\bar{r}} \hat{R}_0\big)\big).$$

This follows from the previous lemma, together with (4).

5. Possible generalisations

One would like to know whether one can understand the structure of $T(r)$ to arbitrary finite subgroups $SL(3, 2^m)$ of the general linear group $GL(3, K)$. Write G_m for this subgroup.

Lemma 5.1. *The module $E^{\otimes r}{\downarrow}_{G_m}$ has a unique indecomposable direct summand of odd dimension, and all other indecomposable direct summands have dimensions divisible by* 8. *If $T(r)_m$ is this summand then $T(r)_m$ is a direct summand of the restriction of $T(r)_{m+1}$; and $\Omega^0(T(r)) = T(r)_1$.*

Proof. We know that the restriction of $E^{\otimes r}$ to G_1 has a unique non-projective summand, and it clearly has odd dimension (since projectives have even dimension but the dimension of the tensor space is odd). Furthermore, all indecomposable projective modules for G_1 have dimensions divisible by 8. This implies the statements directly. $\square$

Since the group algebra of G_m is free as a module over the group algebra of G_k for $k < m$, it follows that $\Omega^s(T(r)_m)$ restricted to G_k is isomorphic to $\Omega^s(T(r)_k)$ modulo projective summands. So it would be entirely consistent if there were exact sequences of the form as in 4.3 but replacing $\Omega^0 T(r)$ by $T(r)_m$ for more general m (which may depend on r), and allowing projective summands in the middle of the sequence. If this were the case then there would be a realistic chance to understand more of the submodule structure of the tilting modules.

The graph structure of the stable Auslander–Reiten quiver is however different when $m \geqslant 2$, see [8] and it is not of use in the same way as for $m = 1$.

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