On Multiplicative Chaos in Number Theory



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This thesis presents research conducted under the supervision of Professor Jon Keating during the 2022–2026 academic years. This period has so far given rise to the following works.

[89] *Partial sums of random multiplicative functions with supercritical divisor twists.*

[15] *On the Fourier coefficients of critical Gaussian multiplicative chaos* (with L.-P. Arguin).

[14] *The Fyodorov–Hiary–Keating conjecture in mesoscopic intervals* (with L.-P. Arguin), accepted to the Annals of Probability.

[47] *Genus expansion for non-linear random matrix ensembles with applications to neural networks* (with N. M. Cirone and C. Salvi) accepted to Random Matrices: Theory and Applications subject to minor revisions.

[88] *A fixed-point approach to non-commutative central limit theorems,* Archiv der Mathematik.

The latter two works being orthogonal in topic, we restrict attention to the first three, which concern interactions between multiplicative chaos and number theory. The introduction provides the necessary background, though each chapter remains self-contained; we ask for the reader's patience with any ensuing repetition.

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Abstract

This thesis explores the interplay between number theory and multiplicative chaos.

The first chapter provides an overview of this connection and of our results. After introducing the notions of a logarithmically correlated field and Gaussian multiplicative chaos (GMC), we explain how they arise in the study of random multiplicative functions and of the value distribution of the Riemann zeta function $\zeta(s)$ on the critical line.

Let f be a Steinhaus random multiplicative function. It was conjectured by Helson that $\mathbb{E}\{|\sum_{n\leq x} f(n)|\} = o(\sqrt{x})$, and this was proved by Harper through a connection with *critical* GMC. In the second chapter, we prove conjecturally sharp bounds for small fractional moments of $\sum_{n\leq x} f(n)d_\alpha(n)$, where d_α is the α-divisor function and $\alpha \in (1,2)$, matching predictions from *supercritical* GMC. A similar argument is then used to give a short proof of Helson's conjecture, and derive improved bounds for the pseudomoments of the zeta function in a certain parameter range.

The third chapter (based on joint work with L.-P. Arguin) presents progress towards conjectures of Fyodorov, Hiary, and Keating on the distribution of $\max_{|h|\leq(\log T)^\theta} |\zeta(1/2 + it + ih)|$ for $t \in [T, 2T]$ and $\theta \in (-1, 0]$. Focusing on the *mesoscopic* setting $\theta < 0$, we derive a sharp (up to a constant) upper bound for the right tail of this quantity (meaning the proportion of t for which it exceeds a given threshold, after appropriate rescaling), and prove a bound for the "partition function" $\int_{|h|\leq(\log T)^\theta} |\zeta(1/2 + it + ih)|^2 \mathrm{d}h$ that holds for all $t \in [T, 2T]$ outside a set of size $o(T)$. These extend results of Arguin, Bourgade, and Radziwiłł and of Harper respectively, obtained at $\theta = 0$; we also give a new proof of the latter.

The asymptotic behaviour of Fourier coefficients of GMC in the large-frequency limit has been of great interest, in part because it is expected to mimic that of partial sums of random multiplicative functions. Recent work of Garban and Vargas proved that the Fourier coefficients of subcritical GMC converge to zero almost surely, and conjectured that this should hold at criticality. The fourth chapter (also joint with L.-P. Arguin) provides a first step towards this conjecture by showing that they converge to zero in probability at a logarithmic rate.

In a final, forward-looking chapter, we discuss a Poisson analogue to Gaussian multiplicative chaos, which we motivate by a connection with the multiplication table problem of Erdős.

Statement of originality

I declare that the work presented in this thesis is my own, except where indicated otherwise.

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Chapter 1

Introduction

"Maybe so, but something is going on with the primes."

— Pomerance (1997), imagining how Erdős and Kac might have replied to Einstein's *"God does not play dice"*.[1]

Random models occupy a central role in our modern understanding of prime numbers. Perhaps the simplest such heuristic is due to Cramér, and models their characteristic function $n \mapsto \mathbf{1}_{n \text{ is prime}}$ by the sequence of independent random variables

$$\xi_1 = 0, \xi_2 = 1, \quad \xi_n \sim \text{Bernoulli}\Big(\frac{1}{\log n}\Big) \text{ for } n \geq 3,$$

given that the primes have density $\approx 1/\log x$ by the prime number theorem (1.1.5). Despite its glaring deficiencies (e.g., for any $n \geq 3$, ξ_n and ξ_{n+1} could both equal 1 with positive probability), this model remarkably reproduces results on the distribution of prime gaps known to follow from the (widely-believed) Hardy-Littlewood conjectures [83, 165].

Any analytic object which encodes the distribution of the primes should then inherit their apparent randomness. The most famous example is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1,$$

whose zeros are known to be a Fourier-analytic dual to the primes via the so-called "explicit formula" (1.1.4). The study of their statistical properties was initiated by Montgomery [138], who, upon computing their pair correlation, indeed observed a striking agreement

[1] *"Die Theorie liefert viel, aber dem Geheimnis des Alten bringt sie uns kaum näher. Jedenfalls bin ich überzeugt, daß der Alte nicht würfelt."*; excerpt from a letter to Max Born, dated 4 December 1926.

with the statistics of eigenvalues of random matrices (cf. (1.1.7) and (1.1.10)). This gave rise to a now-widely-adopted paradigm of modelling the zeros of ζ by such eigenvalues, pushed further by the breakthrough work of Keating and Snaith [116], who modelled ζ itself by the characteristic polynomial of a random unitary matrix (see Conjecture 1.1.2).

This thesis focuses on another random object, whose connection to number theory was uncovered by influential conjectures of Fyodorov, Hiary, and Keating [71, 72] (themselves building on the philosophy of Montgomery, Keating and Snaith): a one-parameter family of random measures known as *Gaussian multiplicative chaos*. Constructed by Kahane [115] in the 1980s as a model for energy dissipation in turbulence, these measures have since been shown to govern the fluctuations of certain sums of multiplicative functions, as well as the behaviour of the zeta function in short intervals of the critical line.

The present introduction traces this emergent circle of ideas linking multiplicative number theory, Gaussian multiplicative chaos, and random matrices, with an emphasis on their interconnectedness; the main results of the thesis are introduced along the way. For a more thorough overview of the individual topics covered, the reader is referred to the excellent surveys [95, 4, 21, 156, 174, 150] consulted extensively in the preparation of this manuscript.

1.1 Random matrix theory and the Riemann zeta function

1.1.1 Basic theory of the Riemann zeta function

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

on $\Re(s) > 1$, where this sum is absolutely convergent. For such s, it admits a representation as a product over primes

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.1.1}$$

which, on multiplying formally, can be seen as an analytic restatement of the fundamental theorem of arithmetic. Noting that this product diverges like the harmonic series as $s \to 1^+$ on the real line, we also see that it encodes the infinitude of the primes.

The zeta function admits a meromorphic continuation to the entirety of $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1. It satisfies the functional equation

$$\xi(s) = \xi(1 - s), \quad \xi(s) := \zeta(s)\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right). \tag{1.1.2}$$

By studying this equation, one can show that $\zeta(s) = 0$ when s is a negative even integer (the *trivial* zeros, corresponding to the poles of $\Gamma(s/2)$), and that any other (*non-trivial*) zero must be confined to the *critical strip* $\Re(s) \in [0, 1]$. Furthermore, (1.1.2) is equivalent to the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \int f(x) e^{-2\pi i n x} \mathrm{d}x \tag{1.1.3}$$

(here, $f : \mathbb{R} \to \mathbb{R}$ is any function with sufficient decay at infinity), which is itself equivalent to the fact that the integers $\mathbb{Z}$ form a lattice in $\mathbb{R}$. From its definition alone, one sees that the zeta function combines both additive and multiplicative information about the integers, which begins to explain its standing in modern number theory.

Of particular interest are the zeros of ζ in the critical strip, for they govern the distribution of prime numbers. To see this, consider the *von Mangoldt* function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

which is, morally, a logarithmically weighted indicator function of the primes. For $\Re(s) > 1$, differentiating both sides of

$$\log \zeta(s) = -\sum_{p \text{ prime}} \log(1 - p^{-s}) = \sum_{p \text{ prime}} \sum_{k \geq 1} \frac{1}{k p^{ks}}$$

shows that $-\zeta'/\zeta$ is the Dirichlet series associated to Λ, that is,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}.$$

In other words, $-\zeta'/\zeta$ can be seen as a generating function for Λ, and this relationship can be exploited via Fourier analysis on the multiplicative group of positive real numbers $(\mathbb{R}_{>0}, \times)$, namely through Mellin inversion and a residue computation. This yields the remarkable *explicit formula*

$$\sum_{n\leq x} \Lambda(n) = x - \sum_{|t|\leq T} \frac{x^\rho}{\rho} + O\Big(\frac{x\log^2(xT)}{T} + \log x\Big), \quad \text{for all } x, T \geq 2, \tag{1.1.4}$$

where the sum is taken over non-trivial zeros $\rho = \sigma + it$ of ζ (see Theorem 5.1 in [122]).

Knowledge about the distribution of these zeros can therefore be used to count primes. The prime number theorem, which asserts that

$$\#\{p \leq x\} = \big(1 + o(1)\big)\mathrm{Li}(x), \quad \Big(\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}y}{\log y}\Big) \tag{1.1.5}$$

is equivalent to the fact that $\zeta(s) \neq 0$ on $\Re(s) = 1$, while the Riemann hypothesis, which asserts that all non-trivial zeros lie on the line $\Re(s) = 1/2$, is equivalent to an error term of $O_\varepsilon(x^{1/2+\varepsilon})$ for all $\varepsilon > 0$ in (1.1.5). As for the vertical distribution of these zeros, the classical Riemann-von Mangoldt formula asserts that if

$$\mathcal{N}(T) := \#\{\rho : \zeta(\rho) = 0,\ \Im(\rho) \in (0, T]\},$$

then

$$\mathcal{N}(T) = \frac{T}{2\pi}\log\Big(\frac{T}{2\pi e}\Big) + \frac{7}{8} + \frac{1}{\pi}\arg\zeta\Big(\frac{1}{2} + iT\Big) + O\big(T^{-1}\big), \tag{1.1.6}$$

implying that the average spacing between consecutive zeros near height T is proportional to $2\pi/\log T$.

1.1.2 A spectral interpretation for the zeros

Whereas the formula in (1.1.6) gives first-order information about the distribution of zeros in the critical strip, one can go further and ask about *correlations* between zeros. Let $\rho_n = 1/2 + i\gamma_n$ denote the non-trivial zeros (with multiplicity), ordered by increasing ordinate γ_n, and assume the Riemann hypothesis so that $\gamma_n \in \mathbb{R}$ for every n. One is then interested in the statistical properties of the sequence of $(\log T/2\pi)\gamma_n$ for $\gamma_n \in (T, 2T]$ (which we normalised to make consecutive gaps have unit mean, per (1.1.6)), whose pair correlation was first investigated by Montgomery [138].

Theorem 1.1.1 (Montgomery). *Let $\eta : \mathbb{R} \to \mathbb{R}$ be a Schwartz function with Fourier transform $\hat{\eta}$ satisfying* $\operatorname{supp}\hat{\eta} \subseteq (-1, 1)$. *Then*

$$\frac{2\pi}{T\log T}\sum_{\substack{T<\gamma_n,\gamma_m\leq 2T\\ \text{distinct}}}\eta\Big(\frac{\log T}{2\pi}(\gamma_n-\gamma_m)\Big)\sim\int\eta(x)\bigg(1-\Big(\frac{\sin(\pi x)}{\pi x}\Big)^2\bigg)\mathrm{d}x. \tag{1.1.7}$$

The zeros therefore exhibit repulsion on average, by contrast with, say, a Poisson point process with intensity 1, for which the analogue of (1.1.7) would yield $\sim \int \eta(x)\mathrm{d}x$.

This result is particularly striking in light of the fact that the (appropriately rescaled) eigenvalues of a random matrix in the Gaussian Unitary Ensemble[2], or GUE, have precisely the same pair correlation in the large-matrix limit (cf. Equation (1.1.10)). This came to Montgomery's attention following a now-famous encounter with Freeman Dyson in 1972, leading him to conjecture that (1.1.7) should hold without the support assumption on $\hat{\eta}$.

Now known as Montgomery's pair correlation conjecture, this claim is supported by extensive numerical computations due to Odlyzko [146], as well as extensions of Theorem 1.1.1 to three-point and n-point correlations, due to Hejhal [106], and Rudnick and Sarnak [160], respectively. All of these results align with a much older suggestion of Hilbert and Pólya, that the Riemann hypothesis should hold by virtue of the γ_n corresponding to the eigenvalues of a self-adjoint operator [36].

1.1.3 The value distribution of ζ and the Keating-Snaith model

The zeta function's zero distribution is intimately related to that of its values, which is also the subject of many conjectures. Perhaps the simplest of them to state is the Lindelöf hypothesis, which claims that

$$|\zeta(1/2+it)| \ll_\varepsilon (1+|t|)^\varepsilon \quad \text{for each } \varepsilon>0.$$

This is known to follow from the Riemann hypothesis, and for instance implies the existence of primes in all intervals $[x, x+x^\theta]$ for x large enough and $\theta > 1/2$. It can also be stated in terms of the *moments* of ζ

$$\mathcal{I}_k(T) := \frac{1}{T}\int_T^{2T}|\zeta(1/2+it)|^{2k}\mathrm{d}t, \quad k\geq 0, \tag{1.1.8}$$

being equivalent to the estimate $\mathcal{I}_k(T) \ll T^\varepsilon$ for every $\varepsilon > 0$ and $k \in \mathbb{N}$. The true order of magnitude of $\mathcal{I}_k(T)$ is, however, expected to be much smaller. It has been known since the work of Hardy–Littlewood [90] in 1918 and Ingham [111] in 1926 that $\mathcal{I}_1(T) \sim \log T$

[2]Meaning a complex Hermitian matrix sampled according to the probability density proportional to $\exp(-\frac{1}{2}\mathrm{Tr}(H^2))$ with respect to Lebesgue measure.

and $\mathcal{I}_2(T) \sim \frac{1}{2\pi^2}(\log T)^4$, respectively. Conditionally on the Riemann hypothesis, Harper has shown that $\mathcal{I}_k(T) \ll_k (\log T)^{k^2}$ for all $k \geq 0$ (building on work of Soundararajan [166]), and subsequent works have shown that this holds unconditionally for $0 \leq k \leq 2$ [105, 25, 103]. A matching lower bound is known unconditionally for all $k \geq 0$, following the work of many authors [153, 154, 151, 104, 6].

It is therefore natural to conjecture that for each $k \geq 0$,

$$\mathcal{I}_k(T) \sim C_k \cdot (\log T)^{k^2}$$

for some constant C_k, as is known for $k = 1, 2$ (with $C_1 = 1$ and $C_2 = \frac{1}{2\pi^2}$). While there had been some attempts to predict the values of C_3 and C_4 [51, 52], even conjectural formulae for larger and non-integer k remained elusive until the breakthrough work of Keating and Snaith [116], who, following Montgomery's philosophy, used a random-matrix-theoretic heuristic to predict the value of C_k for all $k \geq 0$.

Conjecture 1.1.2 (Keating-Snaith). *For any $k \geq 0$,*

$$\mathcal{I}_k(T) \sim a_k g_k \cdot (\log T)^{k^2}, \tag{1.1.9}$$

where

$$g_k = \frac{G(1+k)^2}{G(1+2k)}, \quad a_k = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)} \right)^2 p^{-m} \right\},$$

and G is the Barnes G-function.

The heuristic in question models ζ by the characteristic polynomial of a random (Haar-distributed) unitary matrix, the eigenvalue phases of such matrices being known to exhibit the same pair correlation as GUE eigenvalues in the large-matrix limit[3]: letting $M_N \in U(N)$ be Haar-distributed and $(e^{i\vartheta_1}, ..., e^{i\vartheta_N})$ denote its (random) eigenvalues, we have

$$\frac{1}{N}\mathbb{E}\Big\{ \sum_{i \neq j} f\big(\tfrac{N}{2\pi}(\vartheta_i - \vartheta_j)\big) \Big\} \underset{N\to\infty}{\sim} \int f(x)\left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)\mathrm{d}x, \tag{1.1.10}$$

for any continuous function $f : \mathbb{R} \to \mathbb{R}$ with compact support [136]. (Note that we rescaled the eigenangles ϑ_i to have unit mean spacing.) It is therefore reasonable to conjecture that the locally determined statistical properties of $\zeta(1/2 + it)$ (obtained by averaging over $t \in [T, 2T]$) might be shared by the characteristic polynomial $P_N(e^{i\vartheta}) := \det(I - M_N e^{-i\vartheta})$ (averaged over $M_N \in U(N)$) upon setting

$$N = \log \frac{T}{2\pi}, \tag{1.1.11}$$

[3] Provided one renormalises the eigenvalues to have unit mean spacing, using the semi-circle law.

so as to identify the mean density of the eigenangles ϑ_n with that of the non-trivial zeros (1.1.6). The relevant comparison is then between $\mathcal{I}_k(T)$ and $\mathbb{E}\{|P_N(e^{i\vartheta})|^{2k}\}$, which was evaluated explicitly in [116] using the Selberg integral formula.

Theorem 1.1.3 (Keating-Snaith). *For any $s \in \mathbb{C}$ satisfying $\Re(s) > -1$,*

$$\mathbb{E}\left\{|P_N(e^{i\vartheta})|^s\right\} = \prod_{j=1}^{N} \frac{\Gamma(j+s)\Gamma(j)}{\Gamma(j+s/2)^2} \tag{1.1.12}$$

where $\mathbb{E}$ denotes the average over $U(N)$ with respect to Haar measure.

In the leading order,

$$\prod_{j=1}^{N} \frac{\Gamma(j+2k)\Gamma(j)}{\Gamma(j+k)^2} \sim \frac{G(1+k)^2}{G(1+2k)} N^{k^2}, \quad \Re(k) > -1/2,$$

and one recognises the factor g_k in Conjecture 1.1.2, which encodes short-range correlations between the zeros; these are predicted to be *universal*, the same factor g_k for instance appearing in conjectural formulae for moments of Dirichlet L-functions [49]. The additional "arithmetic factor" a_k in (1.1.9) accounts for long-range zero correlations, which are specific to the zeta function, and are absent from the random matrix model.

Remark 1.1.4. *A less ad-hoc justification for this dichotomy was later provided by Gonek, Hughes and Keating [78], through a representation of ζ that combines a product over primes with a product over zeros. It is most easily understood under the assumption of the Riemann hypothesis, in which case it roughly reduces to*

$$\zeta(1/2+it) \approx \prod_{p \le X} \left(1 - \frac{1}{p^{1/2+it}}\right)^{-1} \cdot \prod_{|t-\gamma_n|<1/\log X} \left(i(t-\gamma_n)e^{\gamma_E} \log X\right), \tag{1.1.13}$$

where the γ_n denote ordinates of non-trivial zeros, γ_E is the Euler-Mascheroni constant and $X > 0$ is a parameter. It was conjectured in [78] that when X is not too large relative to T, averages of $\zeta(1/2+it)$ should split into averages of the respective products in (1.1.13)*, yielding the arithmetic and random-matrix-theoretic factors in Conjecture 1.1.2.*

Theorem 1.1.3 also sheds light on the value distribution of $\log|P_N(e^{i\vartheta})|$ when interpreted as a formula for its moment generating function. By repeatedly differentiating the expression therein and applying the method of moments, one can show [116, Equation (56)] that

$$\frac{\log|P_N(e^{i\vartheta})|}{\sqrt{(1/2)\log N}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } N \to \infty,$$

(where $\mathcal{N}(0,1)$ denotes a standard Gaussian random variable), which, using the identification $N = \log(T/2\pi)$, directly mirrors the following striking result of Selberg [162].

Theorem 1.1.5 (Selberg). *Let $T > 1$ and τ be uniformly distributed in $[T, 2T]$. Then*

$$\frac{\log|\zeta(1/2+i\tau)|}{\sqrt{(1/2)\log\log T}} \xrightarrow{d} \mathcal{N}(0,1).$$

Both Selberg and Keating–Snaith in fact went further and showed that, upon normalising by $\sqrt{(1/2)\log\log T}$ and $\sqrt{(1/2)\log N}$ respectively, $(\Re \log\zeta(1/2+i\tau), \Im\log\zeta(1/2+i\tau))$ and $(\Re\log P_N(e^{i\vartheta}), \Im\log P_N(e^{i\vartheta}))$ each converge jointly to a pair of independent, standard Gaussian random variables.

1.2 The conjectures of Fyodorov, Hiary and Keating

Selberg's theorem suggests that $\log|\zeta(1/2+it)|$ is typically of the order of $\sqrt{\log\log t}$ for large t, falling well short of the growth rate in the Lindelöf hypothesis. This invites the question of determining the true order of magnitude of $\max_{t\in[0,T]}|\zeta(1/2+it)|$, on which rather less is known. A 1928 result of Littlewood [128] asserts that on the Riemann hypothesis,

$$\max_{t\in[0,T]}|\zeta(1/2+it)| \ll \exp\Big(C\frac{\log T}{\log\log T}\Big), \quad \text{for some } C>0,$$

and remains, perhaps surprisingly, the strongest such bound (up to the value of the constant; see [41] for a proof that any $C > (\log 2)/2$ is permissible). An unconditional lower bound of the form

$$\max_{t\in[0,T]}|\zeta(1/2+it)| \geq \exp\left(C\Big(\frac{\log T\cdot\log\log\log T}{\log\log T}\Big)^{1/2}\right)$$

was recently proved by Bondarenko and Seip [34]. The gap between these bounds is considerable, and opinion as to which is closer to the truth remains divided (see, e.g., the conjecture of Farmer, Gonek, and Hughes [66], and the opposing views in Section 5.3 of the same work).

Faced with this difficulty, one is instead led to study the maximum of $|\zeta(1/2+it)|$ over a *typical*, *short* interval. Letting τ be uniformly distributed in $[T, 2T]$ as in Theorem 1.1.5, this amounts to studying the random variable

$$\mathscr{M}_T(\theta) := \max_{|h|\leq(\log T)^\theta} \log|\zeta(1/2+i\tau+ih)|, \quad \theta\in(-1,0],$$

for large T (the reason for not considering smaller θ will soon become apparent).

We restrict for now to the case $\theta = 0$, that is, to intervals of length of order one. To estimate the size of $\mathscr{M}_T(0)$, a natural first step is to discretise the underlying maximum by determining the scale of continuity of $h \mapsto |\zeta(1/2+i\tau+ih)|$. At height T, the zeta

function is well-approximated by a Dirichlet polynomial of length T, for instance through the Hardy-Littlewood approximation

$$\zeta(1/2+it+ih) = \sum_{n\le t} \frac{1}{n^{1/2+it+ih}} + O\big(T^{-1/2}\big), \quad T \le t \le 2T. \tag{1.2.1}$$

Noting that $n^{ih} = e^{ih\log n}$ oscillates at a scale of $1/\log T$ for $n \le T$, this suggests that it suffices to consider the maximum over a mesh of width $1/\log T$. Combining this observation with Theorem 1.1.5 yields the heuristic that $\mathscr{M}_T(0)$ should resemble the maximum of $K = 2\log T$ independent Gaussian random variables, each having variance $(1/2)\log\log T$.

Lemma 1.2.1. *Let $\{X_i\}_{i\le K}$ be a collection of independent Gaussian random variables with zero mean and variance $(1/2)\log\log T$, and $K = 2\log T$. Then for any fixed $y \in \mathbb{R}$,*

$$\mathbb{P}\Big(\max_{i\le K} X_i - \log\log T + \frac{1}{4}\log\log\log T \le y\Big) \underset{T\to\infty}{\longrightarrow} \exp\big(-\tfrac{1}{\sqrt{\pi}}e^{-2y}\big)$$

Proof. By independence,

$$\mathbb{P}\Big(\max_{i\le K} X_i \le V\Big) = \Big(1 - \frac{K\cdot\mathbb{P}(X_1 > V)}{K}\Big)^K$$

for any $V > 0$. Using the standard Gaussian estimate

$$\mathbb{P}\big(\mathcal{N}(0,\sigma^2) > V\big) = \frac{(1+o(1))}{\sqrt{2\pi}}\frac{\sigma}{V}\cdot e^{-V^2/(2\sigma^2)}, \quad \text{for } V \lll \sigma,$$

with $\sigma^2 = (1/2)\log\log T$, a direct computation shows that

$$K\cdot\mathbb{P}(X_1 > V) \to \tfrac{1}{\sqrt{\pi}}e^{-2y}, \quad \text{when } V = \log\log T - \frac{1}{4}\log\log\log T + y,$$

and the claim follows. □

One is then led to conjecture that[4]

$$\mathscr{M}_T(0) \overset{d}{=} \log\log T - \frac{1}{4}\log\log\log T + X + o_{\mathbb{P}}(1), \quad \text{as } T\to\infty, \tag{1.2.2}$$

where X is a Gumbel random variable with right tail $\mathbb{P}(X > y) \asymp e^{-2y}$, and in doing so, implicitly assumes that correlations between the values $\zeta(1/2+i\tau+ih)$ have little effect on the maximum. Were this the case, making such a heuristic rigorous would require a variant of Theorem 1.1.5 valid in the large deviations regime $V \sim \log\log T$, which has recently been proven by Arguin and Bailey [5] (see also [166, 6, 12]).

[4] We use $o_{\mathbb{P}}(1)$ to denote a random variable converging to zero in probability.

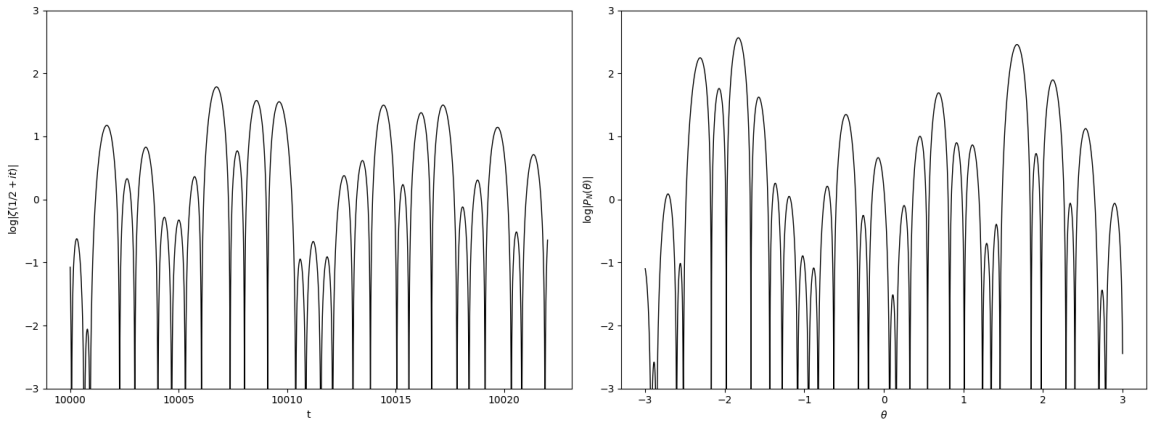


Figure 1.1: $\log|\zeta(1/2+it)|$ near $t = 10^5$ (left), compared to an instance of $\log|P_N(e^{i\vartheta})|$ for $\vartheta \in [-3,3]$, $N = 4000$ (right).

A more refined model should, however, take correlations into account. Having turned to *local* maxima, one can appeal to Keating and Snaith's characteristic polynomial model for $\zeta(1/2+i\tau)$ and liken $\mathscr{M}_T(\theta)$ to

$$\max_{|\vartheta|\le \pi N^{\theta}} \log|P_N(e^{i\vartheta})|, \qquad \text{for } N = \log(T/2\pi).$$

By comparison with a third model, originating in the theory of disordered systems, Fyodorov, Hiary and Keating [71, 72] conjectured a limiting distribution for $\max_{\vartheta}|P_N(e^{i\vartheta})|$, and in turn $\mathscr{M}_T(\theta)$, supported by extensive numerical evidence.

Conjecture 1.2.2 (Fyodorov-Hiary-Keating)**.** *Fix $\theta \in (-1,0]$. Then as $T \to \infty$,*

$$\mathscr{M}_T(\theta) \overset{d}{=} (1+\theta)\log\log T - \frac{3}{4}\log\log\log T + \mathscr{N}\left(0, \tfrac{|\theta|}{2}\log\log T\right) + X(\theta) + o_{\mathbb{P}}(1)$$

where $X(\theta)$ is a random variable with right tail $\mathbb{P}(X(\theta) > y) \asymp ye^{-2y}$, which is independent of the Gaussian.

At $\theta = 0$, this stands in stark contrast with (1.2.2), both in the second-order coefficient 3/4 and the tail of the limiting fluctuations. The latter places one squarely outside of classical extreme value theory altogether: the Fisher-Tippett-Gnedenko theorem asserts that the (appropriately recentred and scaled) maximum of independent, identically distributed random variables must converge to one of three distribution types (Gumbel, Fréchet and Weibull), none of which allow such a tail [77]. The discrepancy is accentuated in the *mesoscopic* setting, meaning $\theta < 0$, for which the heuristic behind (1.2.2), now with $K = 2(\log T)^{1+\theta}$, yields

$$\mathscr{M}_T(\theta) \overset{d}{=} \sqrt{1+\theta}\log\log T - \frac{1}{4\sqrt{1+\theta}}\log\log\log T + X(\theta) + o_{\mathbb{P}}(1)$$

for $X(\theta)$ of Gumbel type.

Conjecture 1.2.2 rather suggests that $h \mapsto \log|\zeta(1/2 + i\tau + ih)|$ belongs to the universality class of *log-correlated* processes, which are introduced in detail in the next section.

1.2.1 Logarithmically correlated processes and fields

Given a metric space (V, d) and $N \geq 0$, a centred stochastic process $X_N(\cdot)$ indexed by $v \in V$ is said to be *logarithmically correlated* (or *log-correlated*) if its covariances satisfy

$$\mathbb{E}\{X_N(u)X_N(v)\} = \log\left(\frac{1}{\mathrm{d}(u,v) \vee N^{-1}}\right) + g(u,v), \quad \forall u, v \in V, \tag{1.2.3}$$

for a bounded and sufficiently regular function $g : V \times V \mapsto \mathbb{R}$. One is then interested in the asymptotic distribution of $\max_{v\in V} X_N(v)$ as $N \to \infty$.

The study of such processes has a rich history in statistical mechanics, where they arise as energy landscapes of disordered systems; taking V to be the (finite) state space of a system with energy $X_N(v)$ at every state, the extrema of $(X_N(v))_{v\in V}$ can then be studied via the *partition function* $Z_N(\beta) = \sum_v e^{\beta X_N(v)}$ and its associated *free energy* $f_N(\beta) = \frac{1}{\beta N}\log Z_N(\beta)$. This approach was taken in an influential work of Derrida and Spohn [57], which studied a model in which V_N is the set of leaves of a K-ary tree, with correlations determined by the generation of the least common ancestor[5]. Their work showed that $Z_N(\beta)$ behaves analogously to the partition function associated to a collection of $|V| = K^N$ i.i.d. Gaussian variables (known as the *random energy model* [56]), exhibiting the same *freezing* phenomenon whereby $\lim_{N\to\infty} f_N(\beta)$ transitions from quadratic to linear growth in β.

Another motivation stems from the study of Gaussian *log-correlated fields* on a domain $D \subseteq \mathbb{R}^d$ which are, formally, Gaussian processes with a covariance kernel of the form

$$\mathbb{E}\{X(x)X(y)\} = \log\frac{1}{\|x-y\|} + g(x,y).$$

Such processes are not well-defined pointwise due to the singularity on the diagonal $x = y$, but can be realised as a random distribution in the sense of Schwartz. They play a prominent role in many areas of probability and mathematical physics including conformal field theory and the study of random surfaces, with the case $d = 2$ coinciding with the two-dimensional Gaussian free field [62]. Log-correlated processes (in the sense of (1.2.3)) often arise as regularisations of these fields, for instance upon convolving with an approximation of the identity φ_N satisfying $\lim_{N\to\infty}\varphi_N(x) = \delta(x)$ (see, e.g., §1.3.1).

[5] Their model is essentially a branching random walk, which we introduce in detail in §1.2.2

Returning to the setting of §1.2, we now show how the characteristic polynomial $P_N(e^{i\vartheta}) = \det(I - M_N e^{-i\vartheta})$ of a Haar-unitary matrix M_N gives rise to a log-correlated process. Denoting its eigenvalues by $(e^{i\vartheta_1}, ..., e^{i\vartheta_N})$, we begin with the expansion

$$\log|P_N(e^{i\vartheta})| = \sum_{k=1}^{N} \log\left|1 - e^{i(\vartheta_k - \vartheta)}\right| = -\sum_{j=1}^{\infty} \frac{\Re\left(e^{-ij\vartheta}\mathrm{Tr}(M_N^j)\right)}{j}, \tag{1.2.4}$$

which holds for $\vartheta \notin \{\vartheta_k\}_{k\leq N}$ (interpreting both sides as $-\infty$ otherwise), being the (conditionally convergent) Fourier expansion of the 2π-periodic and integrable function $\vartheta \mapsto \log|P_N(e^{i\vartheta})|$. Using the identity

$$\mathbb{E}\left\{\mathrm{Tr}(M_N^k)\overline{\mathrm{Tr}(M_N^n)}\right\} = (k \wedge N)\mathbf{1}_{n=k},$$

due to Diaconis and Shahshahani [58], a direct computation shows that

$$\mathbb{E}\left\{\log|P_N(e^{i\vartheta_1})|\log|P_N(e^{i\vartheta_2})|\right\} = \frac{1}{2}\log\frac{1}{|e^{i\vartheta_1} - e^{i\vartheta_2}| \vee N^{-1}} + O(1),$$

making $\vartheta \mapsto \sqrt{2}\log|P_N(e^{i\vartheta})|$ log-correlated. Furthermore, a genuine Gaussian log-correlated *field* can be recovered in the large-N limit by appealing to the following classical result [58].

Theorem 1.2.3 (Diaconis-Shahshahani). *Let M_N be an $N \times N$ Haar-unitary matrix and $k > 0$ a fixed integer. Then*

$$\left(\mathrm{Tr}(M_N), \mathrm{Tr}(M_N^2), \dots, \mathrm{Tr}(M_N^k)\right) \xrightarrow[N\to\infty]{d} \left(N_1, \sqrt{2}N_2, \dots, \sqrt{k}N_k\right)$$

where $(N_1, \dots, N_k)$ is a vector of independent standard complex Gaussian random variables[6].

While the above only holds for k fixed relative to N, together with (1.2.4) it nevertheless suggests that $\log|P_N(e^{i\vartheta})|$ should converge to the Gaussian field formally given by

$$X(\vartheta) = \sum_{k\geq 1} \frac{\Re(e^{-ik\vartheta}N_k)}{\sqrt{k}} \quad \left(\text{with } \mathbb{E}\left\{X(\vartheta_1)X(\vartheta_2)\right\} = \frac{1}{2}\log\frac{1}{|e^{i\vartheta_1} - e^{i\vartheta_2}|}\right) \tag{1.2.5}$$

in a suitable sense, namely one in which higher Fourier modes (for which the i.i.d. Gaussian approximation fails) are suppressed. One way of making this rigorous is by considering the following Sobolev spaces of generalised real-valued functions on the torus

$$H^s = \left\{ f(e^{i\vartheta}) = \sum_{k\in\mathbb{Z}} e^{ik\vartheta}\hat{f}_k, \quad \sum_{k\in\mathbb{Z}} (1 + |k|^2)^s |\hat{f}_k|^2 < \infty \right\}, \quad s \in \mathbb{R}, \tag{1.2.6}$$

[6] A complex standard Gaussian random variable is given by $Z = X + iY$, where X, Y are independent, mean zero, variance 1/2 (real) normal random variables.

equipped with the inner products

$$\langle f, g \rangle_{H^s} = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s \hat{f}_k \overline{\hat{g}_k}.$$

Letting H_0^s denote the subspace of $f \in H^s$ with $\hat{f}_0 = 0$, one directly checks that X (defined in (1.2.5)) is almost surely an element of H_0^{-s} for every $s > 0$, and it was indeed shown by Hughes, Keating and O'Connell [110] that $\log |P_N|$ converges to X in distribution in H_0^{-s}.

In an influential work, Carpentier and Le Doussal [40] observed that log-correlated processes which are translation-invariant in law, such as $\log |P_N(e^{i\vartheta})|$, are in some sense a continuous analogue of the tree-based model considered by Derrida and Spohn. Building on these works, a partition-function-based approach to studying their extrema was developed by Fyodorov and Bouchaud [70], and Fyodorov, Le Doussal, and Rosso [73], which was then used to conjecture the limiting distribution of the maximum of the centred Gaussian process

$$\mathbb{E}\{X_N(\omega_k) X_N(\omega_j)\} = \begin{cases} \log |\omega_j - \omega_k|^{-1} & \text{if } j \neq k \\ \log N & \text{if } j = k, \end{cases}$$

where $\{\omega_k\}_{k \leq N}$ are the N-th roots of unity.

Conjecture 1.2.4 (Fyodorov-Bouchaud). *As $N \to \infty$,*

$$\max_{k \leq N} \frac{1}{\sqrt{2}} X_N(\omega_k) = \log N - \frac{3}{4} \log \log N + Y + o_{\mathbb{P}}(1)$$

where Y is a random variable with density $4e^{2x} K_0(2e^x)$, and K_0 denotes the modified Bessel function of the second kind. In particular, $\mathbb{P}(Y > y) \asymp y e^{-2y}$.

Their approach was subsequently adapted by Fyodorov and Keating to formulate Conjecture 1.2.2; indeed, the maximum of $X_N(\omega_k)/\sqrt{2}$ can be seen as a toy model for that of $\log |P_N(e^{i\vartheta})|$, and one recognises the $\theta = 0$ version of Conjecture 1.2.2 directly by setting $N = \log(T/2\pi)$.

While powerful, significant obstacles remain to making the methods in [70, 72] rigorous, particularly in non-Gaussian settings such as that of Conjecture 1.2.2. The following section instead explains the shape of this conjecture through a probabilistic approach originating in the work of Bramson [37] on branching Brownian motion, which naturally lends itself to rigour, and serves as a principal tool throughout this thesis.

1.2.2 Extrema of log-correlated processes and Bramson's barrier method

Branching Brownian motion is the stochastic process obtained by first letting a particle evolve according to a standard Brownian motion, up to an independent and exponentially distributed random time T. At time T, the particle splits into two independent copies of itself, each proceeding to evolve as the initial particle did (starting from the point of fission). The process at time $t \geq 0$ is then determined by the locations $x_1(t), \dots, x_{N(t)}(t)$ of the $N(t)$ particles $u_1, \dots, u_{N(t)}$ existing at that time. Noting that $\mathbb{E}\{x_i(t)x_j(t)\} = \mathbb{E}\{Q(i,j)\}$, where

$$Q(i,j) := \sup\big\{s \leq t : u_i, u_j \text{ share an ancestor at time } s\big\},$$

$\{x_i(t)\}_i$ is log-correlated with respect to the ultrametric $\mathrm{d}(x_i(t), x_j(t)) = e^{-\mathbb{E}Q(i,j)}\mathbf{1}_{i\neq j}$ (cf. (1.3.5)).

It was observed by McKean [135] that the distribution of the maximal displacement $M_t = \max_{1\leq i\leq N(t)} x_i(t)$ is directly related to the F-KPP equation (named for Fisher [67], and Kolmogorov, Petrovsky, and Piscounov [121])

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u. \tag{1.2.7}$$

Letting $u(t,x) := \mathbb{P}(M(t) \leq x)$, he showed that $u(t,x)$ is the unique solution to (1.2.7) with Heaviside initial data $u(0,x) = \mathbf{1}_{x\geq 0}$. It had been known since [121] that (1.2.7) admits a travelling wave solution: that is,

$$u(t, m(t)+x) \to \omega(x), \qquad \text{uniformly in } x, \text{ as } t \to \infty,$$

where $\omega(x)$ is the unique solution to

$$\frac{1}{2}\omega_{xx} + \sqrt{2}\omega_x + \omega^2 - \omega = 0$$

(up to translation), and $m(t)$ is a centering term (the "front" of the wave) satisfying the asymptotic $m(t) \sim \sqrt{2}t$. However, it was using McKean's observation that Bramson [37] derived a sub-leading order correction for $m(t)$, showing that

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t, \tag{1.2.8}$$

by means of a probabilistic argument. A subsequent work of Lalley and Sellke [124] showed that $M_t - m(t)$ converges in distribution to a sum of two independent random variables: the first is Gumbel-distributed (cf. (1.2.2)), and the second is the almost sure limit of

$$\sum_{i=1}^{N(t)} \big(\sqrt{2}t - x_i(t)\big)e^{-\sqrt{2}(\sqrt{2}t - x_i(t))}$$

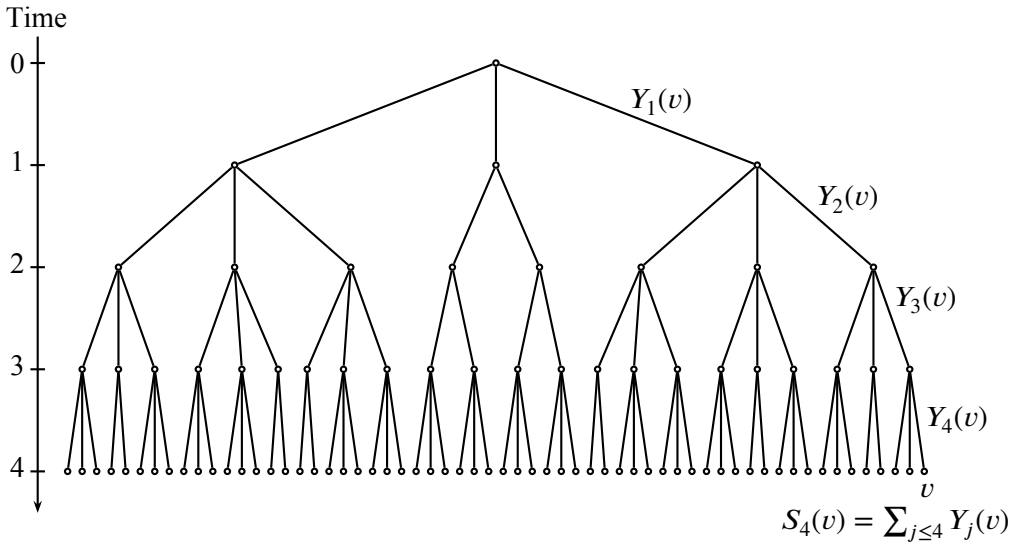


Figure 1.2: A branching random walk on a tree of depth $t = 4$.

as $t \to \infty$, known as the *derivative martingale*.

Building on the approach in [37], analogous limit theorems have been derived for the extrema of many other log-correlated processes, such as branching random walks (see [1, 20, 2, 38], as well as [27, 28]), the discrete two-dimensional Gaussian free field (see [39] and [29], respectively), and other examples (see, e.g., [59, 131]). All of these results share the same, manifestly universal structure: if $\{X_N(v)\}_{v\in V}$ is an approximately Gaussian process with covariance as in (1.2.3), whose *effective*[7] number of points is $\asymp N$, then

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{v\in V} \tfrac{1}{\sqrt{2}} X_N(v) - \log N + \frac{3}{4}\log\log N \leq y \right) = \mathbb{E}\left\{ \exp\left(-CZ_\infty e^{-cy} \right) \right\}, \quad (1.2.9)$$

for constants $C, c > 0$, where Z_∞ is the derivative martingale associated with X_N.

We illustrate how this structure arises through the example of a (Gaussian) *branching random walk*; a discrete analogue of branching Brownian motion, and a prototypical example of a log-correlated process. Let $\mathcal{T} = (V, E)$ be a rooted tree with $t \geq 0$ generations, with the k-th generation having $\approx e^k$ vertices; one could for instance take $\mathcal{T}$ to be a realisation of a Bienaymé-Galton-Watson tree with offspring distribution having mean $e = 2.718...$, but we deliberately keep this discussion informal. To each edge $e \in E$, attach an independent Gaussian random variable with mean zero and variance 1/2, and for each leaf[8] $v \in V$, we

[7] Either $\#V \asymp N$, or V is a one-dimensional manifold that can be covered by $\asymp N$ balls of radius N^{-1} (with respect to the metric d). In the latter case, it usually suffices to study the maximum on a set of $\asymp N$ points, since any two points within distance N^{-1} are almost perfectly correlated.

[8] A leaf is a vertex with out-degree equal to zero.

associate the random walk

$$S_k(v) := \sum_{j \le k} Y_j(v), \quad k \le t,$$

where $Y_j(v)$ is the random variable associated to the depth-j edge on the geodesic from the root to v (cf. Figure 1.2). A branching random walk is the name given to the process $\{S_t(v)\}_{v \in \mathscr{L}(t)}$, indexed by the set of leaves $\mathscr{L}(t)$ of $\mathscr{T}$; when multiplied by $\sqrt{2}$, it is log-correlated[9] in the sense of (1.2.3), using $N = e^t$ and $\mathrm{d}(u, v) = e^{-(u \wedge v)} 1_{u \neq v}$, where $(u \wedge v)$ is the depth of the least common ancestor between u and v.

To upper bound $\max_v S_t(v)$, one could use the *first moment method* to write

$$\mathbb{P}\Big(\max_{v \in \mathscr{L}(t)} S_t(v) > V \Big) = \mathbb{P}\Big(\#\big\{ v \in \mathscr{L}(t) \,:\, S_t(v) > V \big\} > 0 \Big) \le \mathbb{E}\Big\{ \sum_{v \in \mathscr{L}(t)} 1_{S_t(v) > V} \Big\}.$$

However, the right-hand side is $\approx e^t \cdot \mathbb{P}(\mathcal{N}(0, t/2) > V)$ since $\#\mathscr{L}(t) \approx e^t$, and a standard Gaussian estimate shows that this diverges as $t \to \infty$ for $V = t - \frac{3}{4}\log t + O(1)$. This should not come as a surprise in view of Lemma 1.2.1, and the fact that the first moment method is indifferent to correlations between the $X_t(v)$. To circumvent this issue, a common strategy is to restrict to a subset of "good" points $\mathscr{G} \subseteq \mathscr{L}(t)$ for which

$$\mathbb{E}\Big\{ \sum_{v \in \mathscr{G}} 1_{S_t(v) > t - (3/4)\log t + y} \Big\} \underset{y \to \infty}{\longrightarrow} 0, \tag{1.2.10}$$

while ensuring that $\mathbb{E}\{\#(\mathscr{L}(t) \setminus \mathscr{G})\}$ is sufficiently small. Identifying such a $\mathscr{G}$ can, however, be a challenging task in light of the intricate correlation structure of the process.

A key insight in [37] is that it suffices to exploit a simple consequence of this covariance structure, the *entropy gain* at intermediate generations: while $\{S_t(v)\}_{v \in \mathscr{L}(t)}$ is a collection of $\approx e^t$ random walks, there are only $\approx e^k$ distinct walks $S_k(v)$ at each time $k \le t$. This allows one to restrict to

$$\mathscr{G} = \Big\{ v \in \mathscr{L}(t) \,:\, S_k(v) \le y + m(k) + B(k), \forall k < t \Big\}, \quad m(k) := k\Big(1 - \frac{3}{4}\frac{\log t}{t}\Big), \tag{1.2.11}$$

for an appropriately chosen *barrier function* $B(t)$. Crucially, one can take this barrier to be smaller than the typical fluctuations of the *random walk bridge* $S_j(v) - (j/t)S_t(v)$ conditioned on $S_t(v) \approx y + m(k)$ (which are of the order of $\sqrt{k \wedge (t-k)}$). A valid choice is

$$B(k) = \kappa \log\big(1 + (k \wedge t - k)\big)$$

[9] We study $S_t(v)$ instead of $\sqrt{2}S_t(v)$ to facilitate later comparisons with the logarithm of the zeta function. Note that this will change the deterministic centering in (1.2.8) to $t - \frac{3}{4}\log t$ (cf. (1.2.11))

where $\kappa > 0$ is a large constant. The expectation in (1.2.10) then becomes

$$\approx e^t \cdot \mathbb{P}\Big(S_t(v_0) > m(t) + y, S_k(v_0) \le m(k) + y + B(k) \, \forall k < t \Big), \tag{1.2.12}$$

where v_0 is any fixed element of $\mathscr{L}(t)$, reducing the problem to that of estimating the probability that a Gaussian random walk achieves a large value at time t whilst having remained under a curve up to that time. This is a variant of the classical *ballot problem* in probability theory (see Proposition 3.6.9), which in this setting yields

$$\mathbb{P}\Big(S_k(v_0) \le m(k) + y + B(k), \forall k < t \,\Big|\, S_t(v_0) \Big) \asymp \frac{y\big(m(t) + y - S_t(v_0) + 1\big)}{t}. \tag{1.2.13}$$

By conditioning on the value of $S_t(v_0)$ in (1.2.12) and using this estimate, a Gaussian computation shows that the quantity therein is

$$\ll e^t \cdot \frac{y}{t} \frac{e^{-(m(t)+y)^2/t}}{\sqrt{t}} \ll y e^{-2y+o(1)} \cdot \frac{e^{t-t+(3/2)\log t}}{t^{3/2}} \ll y e^{-2y}$$

for any fixed $y > 1$. The additional factor of y/t, generated by the ballot theorem (1.2.13), is therefore responsible for the $-1/2 \log t$ shift in the order of the maximum and the polynomial fattening of the tail (cf. Lemma 1.2.1).

Up to the implicit constant, this tail bound is in fact sharp: a corresponding lower bound can be obtained using the Paley-Zygmund inequality

$$\mathbb{P}\Big(\max_{v \in \mathscr{L}(t)} S_t(v) > V \Big) \ge \mathbb{P}\Big(\max_{v \in \mathscr{G}} S_t(v) > V \Big) \ge \frac{\mathbb{E}\Big\{ \sum_{v \in \mathscr{G}} 1_{S_t(v) > V} \Big\}^2}{\mathbb{E}\Big\{ \big(\sum_{v \in \mathscr{G}} 1_{S_t(v) > V} \big)^2 \Big\}}$$

(or *second moment method*) after once again restricting to a good set $\mathscr{G}$. With additional care, both bounds can be honed into an asymptotic of the form $\sim C y e^{-2y}$, for an appropriate constant $C > 0$ (see Lemma 3 in [174]).

To go further and identify the full limiting distribution, we assume that such an asymptotic is at our disposal. For fixed $1 \le k \le t$, let $\mathscr{L}(k)$ denote the vertices in the k-th generation of $\mathscr{T}$, and consider the following decomposition

$$\max_{v \in \mathscr{L}(t)} S_t(v) = \max_{v \in \mathscr{L}(k)} \big(S_k(v) + M_{t-k}(v) \big), \quad M_{t-k}(v) := \max_{u \in \mathscr{L}_v(t)} \big(S_t(u) - S_k(u) \big),$$

where $\mathscr{L}_v(t)$ denotes the subset of leaves of $\mathscr{T}$ which are descendants of v. For each $v \in \mathscr{L}(k)$, $\{S_t(u) - S_k(u)\}_{u \in \mathscr{L}_v(t)}$ is a branching random walk in its own right (over the subtree rooted at v), and is independent of $M_{t-k}(v')$ for every $v' \neq v$ in $\mathscr{L}(k)$. It follows that

$$\lim_{t \to \infty} \mathbb{P}\Big(M_{t-k}(v) > (t-k) - \tfrac{3}{4} \log(t-k) + z \Big) \sim C z e^{-2z}, \quad \text{as } z \to \infty,$$

for every $v \in \mathscr{L}(k)$, and, letting $\mathscr{F}_k$ denote the σ-algebra generated by $\{S_k(v)\}_{v\in\mathscr{L}(k)}$, that

$$\mathbb{P}\Big(\max_{v\in\mathscr{L}(t)} S_t(v) \le m(t) + y \Big) = \mathbb{E}\Big\{ \prod_{v\in\mathscr{L}(k)} \mathbb{P}\big(M_{t-k}(v) \le m(t) + y - S_k(v) | \mathscr{F}_k \big) \Big\}, \tag{1.2.14}$$

for any fixed $y > 1$. Using the estimate $m(t) = (t-k) - (3/4)\log(t-k) + k + O(k/t)$, and the fact that

$$k - S_k(v) \ge k - \max_{u\in\mathscr{L}(k)} S_k(u) \gg \log k \quad \text{ for all } v \in \mathscr{L}(k)$$

with probability $1 - o(1)$ as $k \to \infty$, we conclude that (1.2.14) is

$$\underset{t\to\infty}{\sim} o(1) + \mathbb{E}\Big\{ \prod_{v\in\mathscr{L}(k)} \big(1 - C(y + k - S_k(v))e^{-2(y+k-S_k(v))}\big) \Big\},$$

where the $o(1)$ denotes a quantity which tends to zero as $k \to \infty$, independently of t. It follows that

$$\mathbb{P}\Big(\max_{v\in\mathscr{L}(t)} S_t(v) \le m(t) + y \Big) \underset{t\to\infty}{\sim} o(1) + \mathbb{E}\Big\{ \exp\big(- CZ_k e^{-2y} \big) \Big\} \tag{1.2.15}$$

for each fixed y and large enough k, where

$$Z_k := \sum_{v\in\mathscr{L}(k)} (k - S_k(v)) e^{2(S_k(v)-k)}. \tag{1.2.16}$$

With the right-hand side in (1.2.15) no longer depending on t, taking $t \to \infty$ followed by $k \to \infty$ shows that Z_k converges in distribution to some Z_∞ whose Laplace transform is given by

$$\lim_{k\to\infty} \lim_{t\to\infty} \mathbb{P}\Big(\max_{v\in\mathscr{L}(t)} S_t(v) \le m(t) + y \Big) = \mathbb{E}\Big\{ \exp\big(- CZ_\infty e^{-2y} \big) \Big\}.$$

The convergence in distribution of $\max_v S_t(v) - m(t)$ follows as a result, and the structure of this limit is now clear; the Gumbel random variable arises from a maximum of i.i.d. random variables (the $M_{t-k}(v)$ for $v \in \mathscr{L}(k)$), whereas the derivative martingale captures correlations acquired by the branching random walk in its early generations.

It remains to explain the shape of Conjecture 1.2.2 in the mesoscopic setting. In a branching random walk, this is analogous to fixing a leaf $v \in \mathscr{L}(t)$ and considering the maximum of $S_t(u)$ over $u \in B(v, e^{\theta t}) \cap \mathscr{L}(t)$, where $B(v, e^{\theta t})$ is the ball of radius $e^{\theta t}$ around v with respect to the metric $d(u, v) = e^{-(u\wedge v)} 1_{u\neq v}$. Equivalently, the maximum is being taken over leaves $u \in \mathscr{L}(t)$ for which $(v \wedge u) \ge |\theta| t$, and it follows that

$$\max_{(v\wedge u)\ge|\theta|t} S_t(u) = S_{t|\theta|}(v) + \max_{(v\wedge u)\ge|\theta|t} \big(S_t(u) - S_{t|\theta|}(u) \big) := S_{t|\theta|}(v) + M_{t(1+\theta)},$$

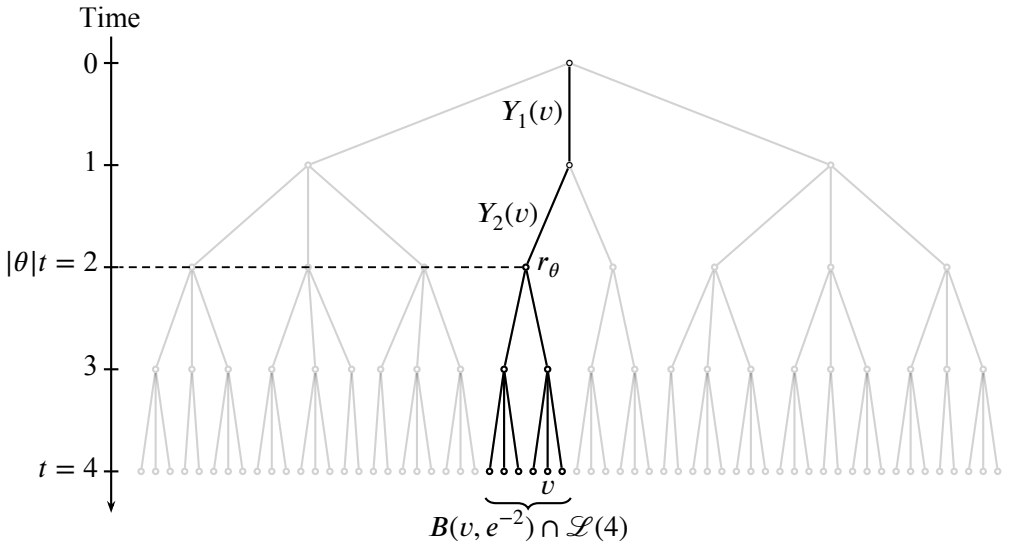


Figure 1.3: A mesoscopic subset of a branching random walk, for $t = 4$ and $|\theta| = 1/2$.

where $M_{t(1+\theta)}$ is once again the maximum of a branching random walk, this time over the subtree of depth $t(1+\theta)$ rooted at the least common ancestor of the vertices in $B(v, e^{\theta t}) \cap \mathscr{L}(t)$ (denoted by r_θ in Figure 1.3). By the discussion above and the fact that $S_{t|\theta|}(v) \overset{d}{=} \mathcal{N}(0, |\theta|t/2)$, we conclude that

$$\max_{(v\wedge u)\geq|\theta|t} S_t(v) \overset{d}{=} t(1+\theta) - \frac{3}{4}\log t(1+\theta) + \mathcal{N}\left(0, \tfrac{|\theta|}{2}t\right) + X(\theta) + o_{\mathbb{P}}(1),$$

as $t \to \infty$, where $X(\theta)$ is a Gumbel random variable shifted by the derivative martingale associated to the process $\{S_t(u) - S_{t|\theta|}(u)\}_u$. In particular, $\mathbb{P}(X(\theta) > y) \asymp ye^{-2y}$, and we recover the same form as Conjecture 1.2.2 for $\theta \in (-1, 0)$.

Despite continuous log-correlated processes such as $\log|P_N(e^{i\vartheta})|$ and $\log|\zeta(1/2 + i\tau + ih)|$ not enjoying the global hierarchical structure of $\{S_t(v)\}_{v\in\mathscr{L}(t)}$, a robust strategy to study their extrema consists of identifying an approximate branching structure within them, rendering the ideas above applicable. We now make this precise.

1.2.3 A branching random walk model for $\log|\zeta|$

Although the Euler product in (1.1.1) does not converge for $\Re(s) < 1$, one can still approximate $\zeta(1/2+it)$ in a statistical sense (a pointwise approximation by an Euler product being precluded by the existence of zeros on the critical line). By adapting a result of Soundararajan [166], it was for instance shown by Harper [93, Proposition 1] that for large T, there exists a subset $H \subseteq [T, T + 2\pi)$ of measure $\geq 1.99\pi$ on which

$$\log|\zeta(1/2+it)| = \Re\Big(\sum_{p\leq T} \frac{1}{p^{1/2+it}} \frac{\log(T/p)}{\log T}\Big) + O(1) \quad \forall t \in H \tag{1.2.17}$$

assuming the Riemann hypothesis. Ignoring the $\log(T/p)/\log T$ factor, this essentially corresponds to a first-order Taylor expansion of the logarithm of a truncated Euler product

$$\log\Big|\prod_{p\le T}\Big(1-\frac{1}{p^{1/2+it}}\Big)^{-1}\Big| = -\sum_{p\le T}\Re\log\big(1-p^{-1/2-it}\big) \approx \Re\Big(\sum_{p\le T}\frac{1}{p^{1/2+it}}\Big).$$

To understand the process $h\mapsto\log|\zeta(1/2+i\tau+ih)|$, one must therefore understand the distribution of $(p^{i\tau})_{p\le T}$ for large T. Recalling that τ is uniformly distributed on $[T,2T]$, it is reasonable to expect each $p^{i\tau}=e^{i\tau\log p}$ to equidistribute on the complex unit circle, as well as for the different $p^{i\tau}$ to be independent by linear independence of $\{\log p\}_{p\text{ prime}}$ over $\mathbb{Q}$. This is made rigorous by the following lemma, which is a consequence of the classical *mean-value theorem* for Dirichlet polynomials (see Lemma 3.6.4).

Lemma 1.2.5 (Lemma 3.4 in [8]). *Let $(Z_p)_{p\text{ prime}}$ be a sequence of independent random variables which are uniformly distributed on the complex unit circle. Let $x\ge 2$ be real, and for each prime p, let $a(p)$ be a complex number satisfying $|a(p)|\le 1$. Then*

$$\mathbb{E}\Big\{\Big(\frac{1}{2}\sum_{p\le x}\big(a(p)p^{-i\tau}+\overline{a(p)}p^{i\tau}\big)\Big)^2\Big\} = \frac{1}{2}\mathbb{E}\Big\{\Big|\sum_{p\le x}a(p)Z_p\Big|^2\Big\}+O\Big(\frac{x^2}{T}\Big)$$

Using the identity $\Re(z)=(z+\bar z)/2$, the above implies that

$$\mathbb{E}\Big\{\Re\Big(\sum_{p\le x}\frac{1}{p^{1/2+i\tau+ih_1}}\Big)\Re\Big(\sum_{p\le x}\frac{1}{p^{1/2+i\tau+ih_2}}\Big)\Big\}\underset{T\to\infty}{\sim}\frac{1}{2}\sum_{p\le x}\frac{\cos(|h_1-h_2|\log p)}{p}$$

whenever $x^2=o(T)$, which by the prime number theorem is approximately

$$\approx\frac{1}{2}\int_2^x\frac{\cos(|h_2-h_1|\log y)}{y\cdot\log y}\mathrm{d}y=\frac{1}{2}\log\frac{1}{|h_2-h_1|\vee(\log x)^{-1}}+O(1).$$

Despite this asymptotic not holding for $x=T$, it strongly suggests that $h\mapsto\log|\zeta(1/2+i\tau+ih)|$ is a log-correlated process, and would yield this claim rigorously were one able to approximate $\log|\zeta|$ by a shorter Dirichlet polynomial (of length equalling a small power of T, say). The former was indeed established by Bourgade [35] using an approximation of Selberg's, which also elucidates why Theorem 1.1.5 ought to be true, for it decomposes $\log|\zeta(1/2+i\tau)|$ into a sum of random variables that are approximately orthogonal (by Lemma 1.2.5), and whose variance accumulates to $(1/2)\log\log T$.

Lemma 1.2.5 also echoes the behaviour of the low Fourier modes in the expansion of $\log|P_N(e^{i\vartheta})|$ in §1.2 (Equation (1.2.4)), which we recall are independent as $N\to\infty$ provided only finitely many are being considered. In the same vein, the linear independence of $\{\log p\}_p$ over $\mathbb{Q}$ can be used to deduce that for distinct primes $p_1,\dots,p_K$,

$$(p_1^{i\tau},p_2^{i\tau},\dots,p_K^{i\tau})\xrightarrow[T\to\infty]{d}(Z_{p_1},Z_{p_2},\dots,Z_{p_K})$$

when K is fixed, with $(Z_p)_p$ being a collection of i.i.d. random variables on the complex unit circle (henceforth referred to as *Steinhaus* random variables).

We now identify a branching random walk in the process $(\log|\zeta(1/2+i\tau+ih)|, |h| \le 1)$, using a decomposition originating in a work of Arguin, Belius, and Harper [9] studying a random model for $\log|\zeta|$. We begin by exhibiting, for each h, an approximate random walk in $\log|\zeta(1/2+i\tau+ih)|$. Motivated by the asymptotic in (1.2.17), we introduce the partial sums

$$S_k(h) := \Re\Big(\sum_{p\le\exp(e^k)} \frac{1}{p^{1/2+i\tau+ih}}\Big), \quad k \le t := \log\log T. \tag{1.2.18}$$

The double-exponential indexing and definition of t reflect our intention of viewing S_t (and by extension, $\log|\zeta(1/2+i\tau+ih)|$) as a random walk with t steps. Indeed, letting

$$Y_k(h) := S_k(h) - S_{k-1}(h), \quad k \le t,$$

one can use Lemma 1.2.5 and the prime number theorem to show that

$$\mathbb{E}\{Y_k(h)^2\} \underset{T\to\infty}{\sim} \frac{1}{2} \sum_{\exp(e^{k-1})<p\le\exp(e^k)} \frac{1}{p} \approx \frac{1}{2}\int_{\exp(e^{k-1})}^{\exp(e^k)} \frac{\mathrm{d}x}{x\log x} = \frac{1}{2}\log\Big(\frac{e^k}{e^{k-1}}\Big) = \frac{1}{2},$$

for each large enough k (but still small relative to t). The central limit theorem then suggests that $Y_k \approx \mathcal{N}(0, 1/2)$ for each such k. Similarly,

$$\mathbb{E}\{Y_k(h_1)Y_j(h_2)\} = \begin{cases} 1/2 + o(1) & \text{if } j = k \text{ and } k \le (\log|h_1-h_2|^{-1}) \wedge t \\ o(1) & \text{otherwise,} \end{cases}$$

so that $S_k(h_1)$ and $S_k(h_2)$ can be seen as a pair of random walks with k, variance-1/2 increments, the first $\lfloor \log|h_1-h_2|^{-1}\rfloor$ of which are shared (mimicking two leaves in a branching random walk in §1.2.2). One also finds that $(S_k(h), |h| \le 1)$ should only contain around $\approx e^k$ distinct values, since $S_k(h) = \sum_{\ell\le k} Y_\ell(h)$ and $Y_\ell(h_1) \approx Y_\ell(h_2)$ whenever $|h_2-h_1| \le e^{-k}$.

The arguments in §1.2.2 therefore reduce the task of determining the maximum of $\log|\zeta(1/2+i\tau+ih)|$ to that of making the link between $\log|\zeta|$ and S_t rigorous, and ensuring that the many approximations made above hold with sufficiently small error. As the error term in Lemma 1.2.5 suggests, this can be a considerable challenge.

1.2.4 Progress on the conjectures

Conjecture 1.2.2 has seen rapid progress since its formulation, largely driven by the branching random walk picture developed in the previous section. The leading order of $\mathscr{M}_T(\theta)$

was determined concurrently by Najnudel [141] (conditionally on the Riemann hypothesis) and Arguin et al. [8] (unconditionally) for $\theta = 0$, and later by Arguin, Ouimet, and Radziwiłł [16] for $\theta \in (-1, 0)$. An upper bound for $\mathscr{M}_T(0)$ including the subleading correction of $-3/4 \log\log\log T$ was then proved by Harper [97], before the works of Arguin, Bourgade, and Radziwiłł [10, 11] settled the $\theta = 0$ case of Conjecture 1.2.2 in full.

Theorem 1.2.6 (Arguin-Bourgade-Radziwiłł). *For large enough T and $2 < y \leq \sqrt{\log\log T}$,*

$$\mathbb{P}\bigg(\max_{|h|\leq 1} \big|\zeta\big(\tfrac{1}{2} + i(\tau + h)\big)\big| > e^y \cdot \frac{\log T}{(\log\log T)^{3/4}} \bigg) \asymp y e^{-2y}.$$

Furthermore, the following sequence of random variables is tight

$$\frac{(\log\log T)^{3/4}}{\log T} \cdot \max_{|h|\leq 1} \big|\zeta\big(1/2 + i(\tau + h)\big)\big|, \quad T \text{ large}.$$

To successfully apply the strategy from the previous section, the proof of this theorem introduces a "recursive scheme" which roughly proceeds as follows. Letting

$$t_0 < t_1 < \cdots < t_{\mathscr{L}} \quad (\approx t := \log\log T),$$

denote a (carefully chosen) sequence of intermediate times, one wants to argue that for those h at which $|\zeta(1/2 + i\tau + ih)|$ achieves a large value, the partial sums $(S_k(h))_{k\leq t}$ must have remained in a corridor centered around $m(k) + y$

$$L(k) \leq S_k(h) - (y + m(k)) \leq U(k), \quad \text{where } m(k) := k\Big(1 - \frac{3}{4}\frac{\log t}{t}\Big),$$

for all times $k \leq t$, and a suitable choice of *upper* and *lower* barriers $L(k)$ and $U(k)$. This is done iteratively, in the sense that one first shows that this is true up to $k \leq t_0$, then up to $t_{\ell+1}$ under the assumption that it holds for t_ℓ, etc.; doing so restricts the possible values of the increments $Y_k(h) = S_k(h) - S_{k-1}(h)$ in a way that makes the error terms arising from applications of Lemma 1.2.5 manageable. The barriers thus play a dual role here: one is probabilistic, with $U(k)$ eventually delivering savings through the application of a ballot theorem (cf. §1.2.2), and the other number-theoretic, with both barriers enabling the approximation of S_k by a random model (via a mean-value theorem).

Crucially, this approach enables one not to work with the zeta function directly. The link between S_t and $\log|\zeta|$ is only made when constructing the lower barrier from time t_ℓ to $t_{\ell+1}$, which amounts to showing that the event

$$\left\{ \begin{array}{l} \log\big|\zeta(1/2 + i(\tau + ih))\big| > V, \\ S_k(h) - (m(k) + y) \in [L(k), U(k)]\ \forall k \leq t_\ell, \text{ and} \\ \exists j \in (t_\ell, t_{\ell+1}],\ S_j(h) < L(j) \end{array} \right\}$$

occurs with sufficiently small probability. After approximating the indicator of this event by (the square of) a Dirichlet polynomial, one can use Markov's inequality to bound its probability by

$$\mathbb{E}\Big\{\big|\zeta(1/2+i\tau+ih)\big|^4\big|\mathcal{Q}(1/2+i\tau+h)\big|^2\Big\},$$

for $\mathcal{Q}$ a Dirichlet polynomial of reasonable length. Bounding this expectation is then an instance of the well-studied problem of estimating a *twisted fourth moment* of the Riemann zeta function.

In Chapter 3, we adapt this recursive scheme to study the maximum over a mesoscopic interval and establish the following upper bound.

Theorem 1.2.7. *Let $\theta \in (-1,0)$. Then for large enough T and $2 < y \le \sqrt{\log\log T}$,*

$$\mathbb{P}\bigg(\max_{|h|\le(\log T)^\theta}\Big|\zeta\big(\tfrac{1}{2}+i(\tau+h)\big)\cdot P_\tau(\theta)\Big| > e^y\cdot\frac{(\log T)^{(1+\theta)}}{(\log\log T)^{3/4}}\bigg) \ll ye^{-2y}, \tag{1.2.19}$$

where $P_\tau(\theta)=\prod_{p\le\exp((\log T)^{|\theta|})}(1-p^{-1/2-i\tau})$.

This settles the upper bound of Conjecture 1.2.2 for $\theta<0$ in a strong form, as one can show that

$$\frac{\log|P_\tau(\theta)|}{\sqrt{(|\theta|/2)\log\log T}} \xrightarrow[T\to\infty]{d} \mathcal{N}(0,1)$$

using Lemma 1.2.5 and the method of moments.

We close with a brief account of progress on the random-matrix-theoretic counterpart of Conjecture 1.2.2, which goes further and predicts the limiting distribution of the recentred maximum of $(\log|P_N(e^{i\vartheta})|, \vartheta\in[0,2\pi))$. It has so far only been studied for $\theta=0$. The leading order of $\max_{\vartheta\in[0,2\pi)}\log|P_N(e^{i\vartheta})|$ was determined by Arguin, Belius and Bourgade [7], the subleading correction by Paquette and Zeitouni [147], and the tightness of the recentred maximum as $N\to\infty$ was proved by Chhaibi, Madaule and Najnudel [44]. Convergence in distribution was subsequently established by Paquette and Zeitouni [148], along with a description of the landscape near the extrema of the process. These works all rely on the identification of a branching structure in $\log|P_N(e^{i\vartheta})|$, with the latter two hinging on an orthogonal polynomial representation due to Killip and Nenciu [118].

Interestingly, the limiting distribution conjectured in [72, 71] for the recentred maximum of $(\log|P_N(e^{i\vartheta})|, \vartheta\in[0,2\pi))$ is the same as in Conjecture 1.2.4, which was later identified by Subag and Zeitouni [169] to be the distribution of a sum of two independent Gumbel random variables. One therefore expects the derivative martingale appearing in [148] to itself be Gumbel-distributed, but this remains an open problem.

1.3 The zeta function in short intervals and Gaussian multiplicative chaos

In light of the results in the previous section, one has come to expect the "partition function"

$$\mathscr{Z}_T^{(\beta)}(\theta) = \frac{1}{(\log T)^\theta} \int_{|h|\le(\log T)^\theta} |\zeta(1/2 + i\tau + ih)|^\beta \mathrm{d}h \tag{1.3.1}$$

to share some universal characteristics with the partition functions appearing in the works of Derrida-Spohn [57] and Fyodorov-Bouchaud [70] (cf. §1.2.1). It was indeed conjectured by Fyodorov-Hiary-Keating [71, 72] that $\mathscr{Z}_T^{(\beta)}(\theta)$ should undergo a similar *freezing transition*

$$\lim_{T\to\infty} \frac{\mathbb{E}\{\log \mathscr{Z}_T^{(\beta)}(\theta)\}}{\log\log T} = f(\theta,\beta) := \begin{cases} (\beta^2/4)(1+\theta) & \text{if } 0 \le \beta \le 2 \\ (\beta-1)(1+\theta) & \text{if } \beta > 2, \end{cases}$$

with the (sharp) threshold $\beta = 2$ representing the value of β beyond which $\mathscr{Z}_T^{(\beta)}(\theta)$ becomes dominated by a few large values near the level of the maximum.

Using the approximate branching structure identified in §1.2.3, Arguin, Ouimet and Radziwiłł [16] provided strong evidence for this claim by showing that

$$\mathscr{Z}_T^{(\beta)}(\theta) = (\log T)^{f(\theta,\beta)+o(1)} \quad \text{with probability } 1 - o(1) \text{ as } T \to \infty, \tag{1.3.2}$$

with a similar transition persisting for $\theta > 0$ (for different $f(\theta,\beta)$) but not for longer intervals. One can then ask whether the theory of log-correlated fields can be used to study the asymptotic distribution of $\mathscr{Z}_T^{(\beta)}(\theta)$ beyond the leading order, giving rise to the more general question of whether the exponential of a log-correlated field can be rigorously defined. This is precisely the motivation behind the theory of *Gaussian multiplicative chaos*, which we now introduce.

1.3.1 A brief introduction to Gaussian multiplicative chaos

The theory of Gaussian multiplicative chaos (GMC) was initiated by Kahane [115] in the 1980s, and rigorously constructs measures formally defined as

$$\mu_\gamma(\mathrm{d}x) \text{"}=\text{"} e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}X(x)^2} \sigma(\mathrm{d}x), \tag{1.3.3}$$

where X is a log-correlated Gaussian field on a locally compact metric space (D,ρ), σ is a reference (Radon) measure on D, and γ is a real-valued parameter. Such measures had first appeared in a work of Mandelbrot [133] as a model for energy dissipation and intermittency in the Kolmogorov-Obukhov theory of developed turbulence [133, 120, 145].

Following their construction in [115], they have found applications in numerous other areas of mathematics, for instance playing a central role in the probabilistic formulation of Liouville conformal field theory [157] (for an in-depth review, see [156] and the references therein).

For simplicity, we restrict to the case where $\sigma(\mathrm{d}x) = \mathrm{d}x$ is the Lebesgue measure on a domain $D \subseteq \mathbb{R}^d$, and X is log-correlated with respect to the standard (Euclidean) metric. The difficulty in defining (1.3.3) lies in the fact that X can only be defined as a random distribution (in the sense of Schwartz). In [115], the author constructs (1.3.3) for fields whose covariance K admits the decomposition

$$K(x, y) = \sum_{\ell=0}^{\infty} K_\ell(x, y),$$

for $(K_\ell)_{\ell \geq 1}$ a sequence of continuous, non-negative, positive-definite covariance kernels. For such "σ-positive" kernels, one can consider a sequence of Gaussian processes $(Y_\ell)_{\ell \geq 1}$ with respective covariances $(K_\ell)_{\ell \geq 1}$, and define the corresponding sequence of measures given by:

$$\mu_{\gamma,n}(A) := \int_A e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}X_n(x)^2}\mathrm{d}x, \text{ for all Borel } A \subseteq D, \tag{1.3.4}$$

where $X_n := \sum_{\ell \leq n} Y_\ell$. For each compact $A \subseteq D$, $(\mu_{\gamma,n}(A))_n$ is then a positive martingale, and standard arguments imply that $(\mu_{\gamma,n})_n$ converges weakly almost surely in the space of Radon measures on D as $n \to \infty$. The limiting measure, which we denote by μ_γ, was also shown not to depend on the specific sequence of $(K_\ell)_\ell$ in (1.3.4).

Kahane's construction in fact holds regardless of whether X is log-correlated, but such fields emerged as the only case for which the theory is genuinely non-trivial. Assuming that

$$\mathbb{E}\{X(x)X(y)\} = K(x, y) = \log_+ \frac{1}{\|x - y\|} + g(x, y)$$

for $g : D \times D \to \mathbb{R}$ a bounded continuous function, it was shown in [115] that μ_γ is non-degenerate if and only if $\gamma^2 < 2d$, which is known as the *subcritical* regime of GMC. For $\gamma^2 < d$, this follows from the bound

$$\mathbb{E}\{\mu_{\gamma,n}(A)^2\} \leq \iint_{A\times A} e^{\gamma^2 K(x,y)}\mathrm{d}x\mathrm{d}y \ll \iint_{A\times A} \|x - y\|^{-\gamma^2}\mathrm{d}x\mathrm{d}y,$$

for bounded $A \subseteq D$, the right-hand side being finite if and only if $\gamma^2 < d$. It follows that $(\mu_{\gamma,n}(A))_n$ is an L^2-martingale for $\gamma^2 < d$, converging to a strictly positive limit $\mu_\gamma(A)$

almost surely. This range of γ is sometimes referred to as the "L^2 phase" of GMC: it is more generally known that [115, 158]

$$\mathbb{E}\{\mu_\gamma(A)^p\} < \infty \iff p < \frac{2d}{\gamma^2} \quad \text{(for } A \subseteq D \text{ bounded)}$$

and the remainder of the subcritical range ($d < \gamma^2 < 2d$) is known as the "L^1 phase" (owing to $\mu_\gamma(A)$ having a finite expectation).

The theory has since been extended to more general classes of log-correlated fields [158, 164, 22, 113]. In the absence of σ-positivity, one can instead construct (1.3.3) by regularising X, convolving it with an approximation of the identity to eliminate the singularity on the diagonal. Letting $(\theta_\varepsilon)_\varepsilon$ be a family of mollifiers converging to Dirac mass at zero as $\varepsilon \to 0$, this gives rise to a family of log-correlated processes $X_\varepsilon = X * \theta_\varepsilon$ with covariances satisfying

$$\mathbb{E}\{X_\varepsilon(x)X_\varepsilon(y)\} = \log\frac{1}{|x-y| \vee \varepsilon} + O(1), \quad \text{as } \varepsilon \to 0^+, \tag{1.3.5}$$

(cf. [22, Lemma 3.5]), such that each X_ε is (almost surely) a Borel-measurable function of $x \in D$. The measures

$$\mu_{\gamma,\varepsilon}(\mathrm{d}x) = e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}X_\varepsilon(x)^2}\mathrm{d}x, \quad \varepsilon > 0 \tag{1.3.6}$$

are then well-defined, and the strategy is to define μ_γ as their limit as ε tends to zero. The main theorem in [22] indeed shows that for $\gamma^2 < 2d$, $\mu_{\gamma,\varepsilon}$ converges in probability as $\varepsilon \to 0^+$ for a wide class of mollifiers; the limit is moreover independent of the regularisation procedure, and coincides with the one obtained through the martingale approach when X has a σ-positive covariance kernel.

1.3.2 Critical and supercritical Gaussian multiplicative chaos

The parameter γ in (1.3.3) can be interpreted as modulating the sensitivity of μ_γ to "large values" of the underlying field X. As X is not defined pointwise, this is made precise by the notion of γ-thick points of the field, which are those belonging to the set

$$D_\gamma := \left\{ x \in D : \lim_{\varepsilon \to 0^+} \frac{X_\varepsilon(x)}{\log(1/\varepsilon)} = \gamma \right\}. \tag{1.3.7}$$

It was shown in [158] that for $\gamma^2 < 2d$, μ_γ almost surely assigns all of its mass to D_γ (meaning $\mu_\gamma(D \cap D_\gamma^c) = 0$), and this fact plays a central role in the construction of μ_γ in [22].

Recalling that $(X_\varepsilon(x), x \in D)$ is a log-correlated process with variance $\mathbb{E}\{X_\varepsilon(x)^2\} = \log(1/\varepsilon) + O(1)$ (cf. (1.3.5)), the concentration of μ_γ on γ-thick points also illuminates the

transition at $\gamma^2 = 2d$ through the ideas in §1.2.2: after identifying N with ε^{-1} in (1.2.9), one expects that

$$\max_{x\in D} X_\varepsilon(x) = \sqrt{2}\log(1/\varepsilon) - \frac{3}{2\sqrt{2}}\log\log(1/\varepsilon) + O(1) \tag{1.3.8}$$

with high probability when $d = 1$, and this asymptotic is in fact known to hold in higher dimensions upon substituting $\sqrt{2}$ by $\sqrt{2d}$ (see, e.g., [131]). This suggests that for $\gamma > \sqrt{2d}$, there are simply not enough points to support a non-trivial limiting measure μ_γ, aligning with the known fact that D_γ has Hausdorff dimension $d - \gamma^2/2$ in the subcritical phase [156].

However, one can still ask whether a non-trivial measure can be recovered for $\gamma \geq \sqrt{2d}$ through a different renormalisation procedure, in which (1.3.6) is scaled by a divergent quantity. For $\gamma = \gamma_c = \sqrt{2d}$, this has been answered in the affirmative by a series of works [61, 108, 113, 114] culminating in the following general result. Let $D \subseteq \mathbb{R}^d$ be compact for simplicity, and X be a log-correlated field on D with kernel

$$K(x, y) = \log\frac{1}{\|x - y\|} + g(x, y),$$

for some g satisfying a mild regularity condition. Lastly, let $(X_\varepsilon)_\varepsilon$ be a sequence of convolution approximations of X. Then there exists a measure μ'_{γ_c} on D such that the *derivative martingale*

$$\mu'_{\gamma_c,\varepsilon}(\mathrm{d}x) := \big(\gamma_c \mathbb{E}\{X_\varepsilon(x)^2\} - X_\varepsilon(x)\big) e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}X_\varepsilon(x)^2}\mathrm{d}x \tag{1.3.9}$$

converges weakly in probability to μ'_{γ_c}. The resulting measure, called *critical* GMC, is non-atomic and supported on a set of Hausdorff measure zero. It can also be obtained through the following limit

$$\lim_{\varepsilon\to 0^+} \sqrt{\log(1/\varepsilon)} \cdot \mu_{\gamma_c,\varepsilon}(\mathrm{d}x) = \sqrt{\frac{\pi}{2}} \cdot \mu'_{\gamma_c}(\mathrm{d}x),$$

which is known as the *Seneta-Heyde* normalisation of critical GMC. Both of these constructions trace their origins to the branching random walk literature [2, 27, 28], and reflect the fact that μ'_{γ_c} is supported on points x for which $(\gamma_c \log(1/\varepsilon) - X_\varepsilon(x))$ is of the order of $\sqrt{\log(1/\varepsilon)}$.

Critical GMC measures are closely connected to the maxima of log-correlated processes. A quick inspection of the derivative martingale Z_∞ in (1.2.16) reveals it to be a discrete analogue of the total mass of the measure μ'_γ, obtained through the normalisation procedure of the same name. This connection is anticipated to be exact in the case of the discrete two-dimensional Gaussian free field (see the discussion surrounding [155, Equation 1.16]). More generally, one expects the "universal" aspect of the fluctuations of Z_∞ to

be governed by a critical GMC measure, with Z_∞ itself typically retaining process-specific traits (particularly when this underlying process is not Gaussian). For a detailed treatment of critical GMC, see [150].

The connection with the extrema of log-correlated processes is all the more explicit for $\gamma > \gamma_c$, where a limit can still be recovered by considering

$$\nu_{\gamma,\varepsilon}(\mathrm{d}x) = \log(1/\varepsilon)^{\frac{3\gamma}{2\sqrt{2d}}} \varepsilon^{-(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}X_\varepsilon(x)^2} \mathrm{d}x. \tag{1.3.10}$$

As $\varepsilon \to 0^+$, this converges[10] to a fully atomic measure dubbed *supercritical* GMC [132, 24, 23], whose intensity is itself random and expressed in terms of a critical GMC measure. It is supported on points for which $X_\varepsilon(x)$ is within $O(1)$ of $\max_x X_\varepsilon(x)$, a fact directly visible both in the power of $\log(1/\varepsilon)$ and in the leading order in its normalisation (cf. (1.3.8)). At $d = 1$, the latter in particular implies that

$$\frac{\log \int e^{\gamma X_\varepsilon}}{\log(1/\varepsilon)} \sim -\Big(\frac{\gamma}{\sqrt{2}} - 1\Big)^2 + \frac{\gamma^2}{2} = \sqrt{2}\gamma - 1 \quad \text{with high probability as } \varepsilon \to 0^+,$$

thus providing an exact match with the freezing transition for $\mathscr{Z}_T^{(\beta)}(0)$ in (1.3.2) upon identifying $\log\log T$ with $\log(1/\varepsilon)$ and $\beta = \sqrt{2}\gamma$.

1.3.3 Moments of the zeta function in short intervals

Since [72, 71], evidence has accumulated in favour of the analogy between $\mathscr{Z}_T^{(\beta)}(\theta)$ and multiplicative chaos. We highlight two such results in particular. The first states that

$$\frac{|P_N(e^{i\vartheta})|^\beta}{\mathbb{E}\{|P_N(e^{i\vartheta})|^\beta\}} \frac{\mathrm{d}\vartheta}{2\pi} \xrightarrow[N\to\infty]{d} \mu_{\beta/\sqrt{2}}(\mathrm{d}\vartheta)$$

weakly in the space of Radon measures on $[0, 2\pi)$, where $\mu_{\beta/\sqrt{2}}$ is a GMC measure (with underlying field (1.2.5)). This was first established by Webb in the L^2 phase $\beta \in (-\sqrt{2}, \sqrt{2})$ [171], and later extended to $\beta \in [\sqrt{2}, 2)$ by Nikula, Saksman and Webb [144]. The second result, due to the latter two authors [161], proves the analogous claim for a measure built from the random Euler product

$$F_x(s) := \prod_{p\le x} \Big(1 - \frac{f(p)}{p^s}\Big)^{-1}, \tag{1.3.11}$$

[10] As shown in [24], convergence in probability does not hold in this context. The main theorem in [24] instead appeals to the notion of *stable* convergence.

where $(f(p))_{p \text{ prime}}$ is a sequence of i.i.d. Steinhaus random variables. They show that in the space of Radon measures on $[0,1]$,

$$\Big(\mathbf{1}_{\beta=2}\sqrt{\log\log x}+\mathbf{1}_{\beta\neq 2}\Big)\frac{|F_x(1/2+ih)|^\beta}{\mathbb{E}\big\{|F_x(1/2+ih)|^\beta\big\}} \xrightarrow[x\to\infty]{d} g_\beta(h)\lambda_\beta(\mathrm{d}h) \quad \text{for any } \beta\in(0,2], \tag{1.3.12}$$

with λ_β a GMC measure, and g_β a positive random continuous function such that for any $\ell \geq 1$, the norms $\|g_\beta\|_{C^\ell([0,1])}$ and $\|1/g_\beta\|_{C^\ell([0,1])}$ possess moments of all orders[11]. This result notably covers the critical case $\beta = 2$ as well, with the factor $\sqrt{\log\log x}$ being the Seneta-Heyde normalisation of λ_β.

Viewing $(F_T(1/2+ih), |h|\leq 1)$ as a random model for $(|\zeta(1/2+i\tau+ih)|, |h|\leq 1)$ (per §1.2.3) and recalling the classical second moment estimate $\mathbb{E}\{|\zeta(1/2+i\tau+ih)|^2\} \asymp \log T$ [90], this strongly suggests that

$$\mathscr{Z}_T^{(2)}(0) \asymp \frac{\log T}{\sqrt{\log\log T}}$$

should hold with high probability. This was indeed established by Harper in [97].

Theorem 1.3.1 (Harper). *Uniformly in large T and $A \geq 2$,*

$$\mathbb{P}\Big(\mathscr{Z}_T^{(2)}(0) > A\frac{\log T}{\sqrt{\log\log T}}\Big) \ll \frac{\log A \wedge \sqrt{\log\log T}}{A}. \tag{1.3.13}$$

For $\theta < 0$, it was anticipated in [72, Section 4(b)] that $\mathscr{Z}_T^{(2)}(\theta)$ should asymptotically split into the product of two uncorrelated random variables: the first being roughly log-normal, and the second behaving similarly to $\mathscr{Z}^{(2)}_{\exp((\log T)^{(1+\theta)})}(0)$ (referred to as the "reduced" partition function by the authors). In Chapter 3, we prove a mesoscopic extension to (1.3.13) which supports this prediction.

Theorem 1.3.2. *Fix $\theta \in (-1,0]$ and let $P_\tau(\theta)$ be as in the statement of Theorem 1.2.7. Then for any $\varepsilon > 0$,*

$$\mathbb{P}\Big(\mathscr{Z}_T^{(2)}(\theta)|P_\tau(\theta)|^2 \geq A\frac{(\log T)^{(1+\theta)}}{\sqrt{(1+\theta)\log\log T}}\Big) \leq CA^{-1+\varepsilon}$$

uniformly in large enough T and $A \leq \sqrt{\log\log T}$ when $\theta \in (-1,0)$. For $\theta = 0$,

$$\mathbb{P}\Big(\mathscr{Z}_T^{(2)}(0) \geq A\frac{\log T}{\sqrt{\log\log T}}\Big) \leq C\frac{\min\{\log A, \sqrt{\log\log T}\}}{A}$$

uniformly in large T and $A \geq 2$.

[11] The convergence here in fact holds weakly almost surely when $\beta \neq 2$

Our proof departs from that of Theorem 1.3.1, opting instead to express $\mathscr{Z}_T^{(2)}(\theta)$ in terms of the measure of level sets of $\log|\zeta(1/2+i\tau+ih)P_\tau(\theta)|$, taking inspiration from the works of Soundararajan [166] and Arguin-Bailey [5]. Combined with the barrier method of §1.2.2, this essentially reduces the problem to that of proving a "ballot theorem for $\log|\zeta(1/2+i\tau+ih)P_\tau(\theta)|$" (see Theorem 3.0.4) using the recursive scheme of [10].

This approach also offers a clearer path towards proving the corresponding *supercritical* bound for $\beta > 2$

$$\mathscr{Z}_T^{(\beta)}(\theta)|P_\tau(\theta)|^\beta \ll \frac{(\log T)^{(\beta-1)(1+\theta)}}{(\log\log T)^{3\beta/4}}, \quad \text{with high probability as } T\to\infty, \tag{1.3.14}$$

as evidenced by the fact that a simplification thereof suffices to establish the analogous bound for $\int|F_x(1/2+ih)|^\beta \mathrm{d}h$ in Chapter 2 (where F_x is defined in (1.3.11)). Beyond their role as a model for $\mathscr{Z}_T^{(\beta)}$, such integrals play a central role in another flourishing area of number theory, for which the Fyodorov-Hiary-Keating picture has proved equally fruitful: the study of *random multiplicative functions*, to which we now turn.

1.4 Random multiplicative functions

Many central problems in number theory can be recast as proving cancellation in partial sums of multiplicative functions, meaning those $f:\mathbb{N}\to\mathbb{C}$ satisfying $f(mn)=f(m)f(n)$ whenever $(m,n)=1$. For instance, the prime number theorem (1.1.5) is equivalent to the estimate $\sum_{n\le x}\mu(n)=o(x)$, while the Riemann hypothesis is equivalent to

$$\sum_{n\le x}\mu(n)\ll x^{1/2+\varepsilon},\quad \forall\varepsilon>0,$$

(see [170, Chapter 14]). This prompted Wintner [173] to ask whether this statement holds for a generic multiplicative function taking values ±1 on the primes: letting $(g(p))_{p\text{ prime}}$ be a sequence of i.i.d. random variables equal to $+1$ or -1 with equal probability, he introduced the function

$$g(n)=\begin{cases}\prod_{p|n}g(p) & \text{if } n \text{ is squarefree}\\ 0 & \text{otherwise,}\end{cases} \tag{1.4.1}$$

and proved that $\sum_{n\le x}g(n)\ll x^{1/2+\varepsilon}$ indeed holds almost surely.

Another model is obtained by taking $(f(p))_p$ to be a sequence of i.i.d. Steinhaus random variables and defining

$$f(n):=\prod_{p^a\|n}f(p)^a.$$

The resulting function is then completely multiplicative (meaning that $f(mn)=f(m)f(n)$ for all n,m), and is known as a *Steinhaus random multiplicative function*. It serves as a

natural model for statistics of Dirichlet characters $n \mapsto \chi(n)$ (averaged over χ) and of the "Archimedean" character $n \mapsto n^{it}$ (averaged over t). The latter is visible in Lemma 1.2.5, and more generally in the classical *Bohr correspondence*

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \Big|\sum_{n\le x} \frac{a_n}{n^{it}}\Big|^{2q} \mathrm{d}t = \mathbb{E}\Big\{\Big|\sum_{n\le x} a_n f(n)\Big|^{2q}\Big\}, \tag{1.4.2}$$

which holds for any sequence $(a_n)_n$ of complex numbers and $q > 0$ (see, e.g., Section 3.1 of [75]). The statistical behaviour of partial sums of random multiplicative functions has since been the subject of extensive investigation [42, 125, 92, 34, 86, 102, 101, 98, 96, 99, 134, 91, 82, 167, 119, 46], encompassing almost sure bounds, moment estimates, and limit theorems. For the purposes of this thesis, we focus on what is perhaps the simplest such problem to state: determining the asymptotic distribution of $\sum_{n\le x} f(n)$ as x tends to infinity.

1.4.1 Helson's conjecture and critical multiplicative chaos

Motivated by a problem in harmonic analysis, Helson [107] conjectured that $|\sum_{n\le x} n^{it}|$ should exhibit significant cancellation, being a multiplicative analogue of the Dirichlet kernel $D_x(\theta) = \sum_{|n|\le x} e^{2\pi i n\theta}$ for which

$$\int_0^1 |D_x(\theta)|\mathrm{d}\theta = \int_0^1 \Big|\frac{\sin((2x+1)\pi\theta)}{\sin(\pi\theta)}\Big|\mathrm{d}\theta \asymp \log x. \tag{1.4.3}$$

Viewed through the Bohr correspondence (1.4.2), his prediction reads as follows.

Conjecture 1.4.1 (Helson). *Let f be a Steinhaus random multiplicative function. Then*

$$\mathbb{E}\Big\{\Big|\sum_{n\le x} f(n)\Big|\Big\} = o(\sqrt{x}).$$

This contrasts with the $\asymp \sqrt{x}$ order of magnitude one would expect from, say, a sum of $\lfloor x\rfloor$ independent random variables with unit variance, providing a natural, albeit naïve, counter-conjecture to Helson's. While a central limit theorem for $\frac{1}{\sqrt{x}}\sum_{n\le x} f(n)$ was later precluded in [94], initial investigations on Conjecture 1.4.1 [33, 101] nevertheless pointed towards $\asymp \sqrt{x}$ being the true order of magnitude of $\sum_{n\le x} f(n)$ until a breakthrough work of Harper [98] revealed a more intricate picture.

Theorem 1.4.2 (Harper). *Uniformly in large x and $q \in [0,1]$,*

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} f(n)\Big|^{2q}\Big\} \asymp \big((1-q)\sqrt{\log\log x}+1\big)^{-q}. \tag{1.4.4}$$

The same bound also holds upon replacing f by g, defined in (1.4.1).

At $q = 1/2$, this yields a precise decay rate of $(\log\log x)^{-1/4}$, which interpolates to $\asymp 1$ near $q = 1$—as it must, since

$$\mathbb{E}\Big\{\Big|\sum_{n\le x} f(n)\Big|^2\Big\} = \sum_{n,m\le x} \mathbb{E}\big\{f(n)\overline{f(m)}\big\} = \sum_{n\le x} 1 = \lfloor x\rfloor.$$

Helson's conjecture therefore holds, but with a margin far slimmer than the analogy with the Dirichlet kernel would indicate[12]. The mechanism underlying Theorem 1.4.2 is instead rooted in the theory of GMC: Harper's argument roughly proceeds by showing that

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} f(n)\Big|^{2q}\Big\} \approx \mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_{-1}^{1}\big|F_x(1/2+ih)\big|^2\mathrm{d}h\Big)^q\Big\}, \tag{1.4.5}$$

with the integral on the right-hand side, as discussed in §1.3, being closely related to the total mass of a critical GMC measure. In particular, the *Seneta-Heyde* normalisation of $\sqrt{\log\log x}$ required to make the latter converge to a non-trivial limit (cf. Theorem 1.3.12) is absent, and is precisely the factor appearing on the right-hand side of (1.4.4).

We explain (1.4.5) by temporarily restricting the sum therein to integers whose largest prime factor $P(n)$ strictly exceeds $\sqrt{x}$. Using the fact that f is completely multiplicative, we can decompose the sum in question as

$$\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} f(n) = \sum_{\sqrt{x}<p\le x} f(p)\sum_{m\le x/p} f(m), \tag{1.4.6}$$

noting that the inner sums are now measurable with respect to the σ-algebra generated by $(f(p))_{p\le\sqrt{x}}$. In particular, upon conditioning on said σ-algebra, the right-hand side becomes a sum of independent random variables X_p with (conditional) variances

$$\mathbb{E}\Big\{|X_p|^2\,\Big|\,\big(f(p)\big)_{p\le\sqrt{x}}\Big\} = \Big|\sum_{m\le x/p} f(m)\Big|^2.$$

One might therefore expect, by the central limit theorem, that

$$\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} f(n) \approx \mathcal{N}(0,V_x), \quad \text{where } V_x := \sum_{\sqrt{x}<p\le x}\Big|\sum_{m\le x/p} f(m)\Big|^2, \tag{1.4.7}$$

and in particular that

$$\mathbb{E}\Big\{\Big|\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} f(n)\Big|^{2q}\Big\} \asymp \mathbb{E}\big\{V_x^q\big\}.$$

[12]Compare (1.4.3) with the estimate $(\int |D_x(\theta)|^2\mathrm{d}\theta)^{1/2} \asymp \sqrt{x}$.

The latter can indeed be established rigorously without Gaussian approximation: for an upper bound, the law of total expectation and Jensen's inequality directly yield

$$\mathbb{E}\Big\{\Big|\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} f(n)\Big|^{2q}\Big\} \le \mathbb{E}\Big\{\mathbb{E}\Big\{\Big|\sum_{\sqrt{x}<p\le x} f(p)\sum_{m\le x/p} f(m)\Big|^2 \Big| \big(f(p)\big)_{p\le\sqrt{x}}\Big\}^q\Big\},$$

which equals $\mathbb{E}\{V_x^q\}$ after expanding the square on the right-hand side. A corresponding lower bound can be obtained using Khintchine's inequality (see, e.g., [85, Lemma 3.8.1]), and one is left with the task of comparing $\mathbb{E}\{V_x^q\}$ to the right-hand side in (1.4.5).

This is achieved by approximating the outer sum in V_x by an integral, writing

$$\sum_{\sqrt{x}<p\le x}\Big|\sum_{m\le n/p} f(m)\Big|^2 \approx \frac{1}{\log x}\int_{\sqrt{x}}^{x}\Big|\sum_{m\le n/t} f(m)\Big|^2 \mathrm{d}t = \frac{x}{\log x}\int_1^{\sqrt{x}}\Big|\sum_{m\le z} f(m)\Big|^2\frac{\mathrm{d}z}{z^2}, \tag{1.4.8}$$

with the factor $1/\log x$ stemming from the density of the primes in $[\sqrt{x}, x]$ (see Equation (1.1.5)). As x tends to infinity, the range of integration in (1.4.8) tends to $[1,\infty)$, and a variant of Parseval's theorem gives

$$V_x \approx \frac{x}{\log x}\int_{\mathbb{R}}\frac{|F_{\sqrt{x}}(1/2+ih)|^2}{|1/2+ih|^2}\mathrm{d}h, \quad F_{\sqrt{x}}(1/2+ih) = \prod_{p\le\sqrt{x}}\Big(1-\frac{f(p)}{p^{1/2+ih}}\Big)^{-1}, \tag{1.4.9}$$

from which one can straightforwardly arrive to

$$\mathbb{E}\big\{V_x^q\big\} \approx x^q\cdot\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_{-1}^{1}\big|F_{\sqrt{x}}(1/2+ih)\big|^2\mathrm{d}h\Big)^q\Big\}. \tag{1.4.10}$$

To establish (1.4.5) for the full sum, the argument in [98] instead splits each n into its y-smooth (meaning $p|n \implies p\le y$) and y-rough ($p|n \implies p>y$) parts for a suitably chosen y. The integers n whose y-rough part m is small relative to x (say, $\le x^{3/4}$) are discarded using a standard sieve-theoretic estimate, and for those remaining, a variant of the argument above holds with $(f(m))_{m \text{ is } y-\text{rough}}$ playing the role of $(f(p))_{p>\sqrt{x}}$ (see §2.1 for more detail).

The bulk of the argument in [98] then lies in proving that

$$\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_{-1}^{1}\big|F_x(1/2+ih)\big|^2\mathrm{d}h\Big)^q\Big\} \asymp \Big((1-q)\sqrt{\log\log x}+1\Big)^{-q}, \tag{1.4.11}$$

which is achieved using a barrier argument (in the sense of §1.2.2), and requires proving a Girsanov-type theorem tailored to this setting. It was also anticipated in [98, Section 1.1]

that for q bounded away from 1, (1.4.11) could be proved by appealing to the aforementioned work of Saksman and Webb [161]; a recent work of Gorodetsky and Wong [82] confirms this by establishing the upper bound via this approach.

In Chapter 2, we develop a different approach to estimating low moments of $\int |F_x(1/2+ih)|^2 \mathrm{d}h$, proceeding along the lines of our proof of Theorem 1.3.2: expressing the integral in terms of the measure of level sets of $\log|F_x(1/2+ih)|$ eliminates the need for a Girsanov analogue, and allows for the use of a barrier method truer to Bramson's (cf. §1.2.2). We use this to give a short proof of the upper bound in (1.4.11) (including the uniformity in $q \in [0,1]$) and thus of Helson's conjecture (see §2.1). The approach moreover extends to moments of $\int |F_x(1/2+ih)|^\gamma \mathrm{d}h$ for larger γ, which is the subject of the next section.

With the order of magnitude of $\sum_{n\le x} f(n)$ identified, the question of whether this quantity converges to a distributional limit after appropriate rescaling was raised in [98], and recently resolved by Gorodetsky and Wong [81]. The form of the limit was first made precise by the same authors in [79, Conjecture 1.6], and refines the guess in (1.4.7) into

$$\frac{(\log\log x)^{1/4}}{\sqrt{x}} \sum_{n\le x} f(n) \xrightarrow[x\to\infty]{d} \mathcal{N}\left(0, V_\infty\right), \quad V_\infty = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{m_\infty(\mathrm{d}h)}{|1/2+ih|^2}, \tag{1.4.12}$$

where m_∞ is a random measure defined as the weak in-probability limit of

$$m_x(\mathrm{d}h) := \sqrt{\log\log x}\frac{|F_x(1/2+ih)|^2}{\mathbb{E}\left\{|F_x(1/2+ih)|^2\right\}}\mathrm{d}h$$

as $x \to \infty$ (in the space of Radon measures on $\mathbb{R}$). The existence of such a limit on a compact interval of $\mathbb{R}$ follows from the result of Saksman and Webb in (1.3.12), which also implies that $m_\infty(\mathrm{d}h) = g(h)\lambda(\mathrm{d}h)$, for g a random continuous function, and λ a critical GMC measure. Initial progress on (1.4.12) was then made by Hardy [91], who showed the convergence of the sum restricted to $P(n) > \sqrt{x}$ to $\mathcal{N}(0, \tilde{V}_\infty)$, for $\tilde{V}_\infty$ defined as the limit in distribution of

$$\tilde{V}_n = \frac{e^{-\gamma_E}\log 2}{2\pi}\int_{-n}^{n} \frac{g(h)}{|1/2+ih|^2}\lambda(\mathrm{d}h)$$

as $n \to \infty$ (and $\gamma_E = 0.5772...$ the Euler-Mascheroni constant). Finally, using martingale techniques, the conjecture was proven in full in [81], including a construction of m_∞ on the entire real line.

1.4.2 A supercritical analogue of Helson's conjecture

The results of the previous section establish a precise link between partial sums of random multiplicative functions and GMC. As the latter is a one-parameter theory, one is tempted

to ask which operation on $\sum_{n\le x} f(n)$ amounts to varying said parameter: this is essentially achieved by weighting (or "twisting") the sum by a (deterministic) multiplicative function g satisfying

$$\sum_{p\le x} |g(p)|^2 \sim \frac{\beta^2}{4}\frac{x}{\log x}. \tag{1.4.13}$$

The simplest example of such a twist is undoubtedly the divisor function d_α, defined through $\zeta(s)^\alpha = \sum_n d_\alpha(n)n^{-s}$ where this sum converges, which satisfies (1.4.13) for $\beta = 2\alpha$. In this instance, a generalisation of the argument from (1.4.6) to (1.4.10) can be used to show that the typical order of magnitude of sums of $d_\alpha(n)f(n)$ is governed by averages of $|F_x|^\beta$ (see, e.g., [75]). In [79, 80], the authors refine this idea by introducing a family of twists g for which a full limit theorem, analogous to (1.4.12), is established for $\beta \in (0,1)$:

$$\frac{1}{\sqrt{x}}\sum_{n\le x} f(n)g(n) \xrightarrow[x\to\infty]{d} \mathcal{N}\big(0, V_\infty(g)\big),$$

where $V_\infty(g)$ is formally given by

$$V_\infty(g) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{m_\infty^{(g)}(\mathrm{d}h)}{|1/2+ih|^2}, \quad m_\infty^{(g)}(\mathrm{d}h) := \prod_p \frac{|1+\sum_k f(p)^k g(p^k)p^{-k(1/2+ih)}|^2}{\mathbb{E}\{|1+\sum_k f(p)^k g(p^k)p^{-k(1/2+ih)}|^2\}}\mathrm{d}h.$$

Together with (1.4.12), this provides an essentially complete description of the limiting distribution of partial sums corresponding to *subcritical* and *critical* multiplicative chaos.

In §2.2, we prove an analogue of Theorem 1.4.2 in the *supercritical* phase.

Theorem 1.4.3. *Fix $\alpha \in (1,2)$. Then uniformly in all large x and $q \in [0, 1/\alpha]$,*

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} d_\alpha(n)f(n)\Big|^{2q}\Big\} \ll \frac{(\log x)^{2q(\alpha-1)}}{(\log\log x)^{(3\alpha q/2)}(1-\alpha q)^2+1}.$$

Using a generalisation of Harper's argument due to Gerspach [75, Proposition 3.6], the left-hand side is compared to the q-th moment of an Euler product integral

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} d_\alpha(n)f(n)\Big|^{2q}\Big\} \approx \mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_{-1}^{1}\big|F_x(\tfrac{1}{2}+ih)\big|^{2\alpha}\mathrm{d}h\Big)^q\Big\},$$

which we then analyse using our level-sets approach mentioned in §1.4.1.

Theorem 1.4.4. *Fix $\beta \in (2,4)$. Then uniformly in $q \in [0, 2/\beta]$ and all large x,*

$$\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_{-1}^{1}\big|F_x(\tfrac{1}{2}+ih)\big|^{\beta}\mathrm{d}h\Big)^q\Big\} \ll \frac{(\log x)^{(\beta-2)q}}{(\log\log x)^{(3\beta q/4)}(2-\beta q)^2+1}. \tag{1.4.14}$$

Upon identifying $\log(1/\varepsilon) = \log\log x$ and $\beta = \gamma/\sqrt{2}$, the right-hand side matches the normalisation of (one-dimensional) *supercritical* GMC (1.3.10) for q bounded away from $2/\beta$; this measure being known to have moments up to $1/\gamma$, this is consistent with the range of q in this theorem.

Continuing with this analogy, the integral in (1.4.14) is also expected to be supported on points h for which $|F_x(1/2+ih)|$ is within $O(1)$ of $\max_{h\in[0,1]}|F_x(1/2+ih)|$: the proof of (1.4.14) supports this view, with the contribution of such points matching the right-hand side therein (see Corollary 2.0.5, and the discussion that follows). As a byproduct of this proof, we also establish a conjecturally sharp upper bound for this maximum, which improves on the best known bound in [9] and can be seen as a random analogue of the Fyodorov-Hiary-Keating conjecture (cf. §1.2).

Corollary 1.4.5 (Maximum bound). *Uniformly in x large and $y > 2$,*

$$\mathbb{P}\Big(\max_{h\in[0,1]}|F_x(1/2+ih)| > \frac{\log x}{(\log\log x)^{3/4}}e^y\Big) \ll ye^{-2y}$$

Theorems 1.4.3 and 1.4.4 more generally provide evidence towards a limit theorem, in the spirit of (1.4.12), holding for $\frac{1}{\sqrt{x}}\sum_{n\le x} d_\alpha(n)f(n)$ when $\alpha \in (1,2)$ (and perhaps for more general twists g satisfying (1.4.13) with $\beta > 2$). Should a multiplicative chaos limit indeed persist in this regime, it would likely exhibit remarkably rich probabilistic structure.

1.4.3 An application to the pseudomoments of ζ^α

We close with another application of Theorem 1.4.4, pertaining to the so-called *pseudomoments* of ζ^α

$$\Psi_{2q,\alpha}(x) := \lim_{T\to\infty}\frac{1}{T}\int_T^{2T}\Big|\sum_{n\le x}\frac{d_\alpha(n)}{n^{1/2+it}}\Big|^{2q}\mathrm{d}t, \quad \alpha, q > 0. \tag{1.4.15}$$

For $\alpha = 1$, these were first introduced by Conrey and Gamburd [50] as a model for the moments $\mathcal{I}_q(T)$ (cf. (1.1.8)), which would arise from (1.4.15) by the Hardy-Littlewood approximation (1.2.1) were x allowed to grow with T.

By instead assuming that x is fixed, one simplifies the problem significantly by making the sum in (1.4.15) only depend on a fixed set of primes $(p_j)_{j\le K}$. Upon averaging over $t \in [T, 2T]$, this allows one to replace $(p_j^{it})_{j\le K}$ by a sequence of K, independent Steinhaus random variables using the mean-value theorem in (1.2.5). Doing so yields

$$\Psi_{2q,\alpha}(x) = \mathbb{E}\Big\{\Big|\sum_{n\le x}\frac{d_\alpha(n)f(n)}{\sqrt{n}}\Big|^{2q}\Big\},$$

which is then used in [50] to derive the asymptotic

$$\Psi_{2q,1}(x) \underset{x\to\infty}{\sim} a_q\gamma_q(\log x)^{q^2}, \quad q \in \mathbb{N}, \tag{1.4.16}$$

for a_q is the arithmetic constant in the Keating-Snaith conjecture (1.1.12), and γ_q the volume of a certain convex polytope.

The same work also argues that $\Psi_{2q,\alpha}$ admits a direct random-matrix-theoretic analogue in the following. Letting P_N be the characteristic polynomial of a Haar-unitary matrix M_N (as in §1.1.3), define its *secular coefficients* $c_n^{(N)}$ by

$$P_N(e^{i\vartheta}) = \sum_{j=0}^{N}(-1)^j c_j^{(N)} e^{-\vartheta ij}.$$

Then similarly to $\Psi_{2q,1}$, averages of the truncated sum

$$P_{N,K}(\vartheta) := \sum_{j=0}^{K}(-1)^j c_j^{(N)} e^{-\vartheta ij}, \quad K \text{ fixed},$$

isolate the contribution to the moments in Theorem 1.1.3 from "low Fourier modes" (in the sense of §1.2.3): each $c_n^{(N)}$ is a polynomial in finitely many traces $\mathrm{Tr}(M_N^j)$, which can be substituted by independent Gaussian random variables in the large N limit by Theorem 1.2.3. Letting γ_q be the constant from (1.4.16), Conrey and Gamburd make the connection to $\Psi_{2q,1}$ explicit by showing that

$$\gamma_q = \lim_{K\to\infty} \frac{\mathbb{E}\{|P_{N,K}(0)|^{2q}\}}{K^{q^2}},$$

where the expectation is taken over $U(N)$ for $N = qK$.

While the estimate $\Psi_{2q,1} \asymp (\log x)^{q^2}$ is known to persist for non-integer $q > 0$ following [32, 76], the order of $\Psi_{2q,\alpha}$ was shown by Gerspach [76] to undergo various transitions depending on the range of α and q. In his thesis [75, Conjecture 5.4], the same author conjectured the true order of magnitude of $\Psi_{2q,\alpha}$ for all $\alpha > 0, q > 0$, and in §2.3, we make progress on this conjecture in a certain parameter range.

Theorem 1.4.6. *Let $\alpha \in (1,2)$, $0 < q < 2(\alpha-1)/\alpha^2$ be fixed. Then uniformly for large x,*

$$\Psi_{2q,\alpha}(x) \ll \frac{(\log x)^{2q(\alpha-1)}}{(\log\log x)^{q(3\alpha/2)}}.$$

1.5 Fourier coefficients of multiplicative chaos

In an insightful work [168], Soundararajan and Zaman shed light on the connection between random multiplicative functions and GMC by analysing it in the *function field setting*, where the ring of integers $\mathbb{Z}$ is modeled by that of polynomials $\mathbb{F}_q[t]$ (for $\mathbb{F}_q$ a finite field with q elements and q a prime power). Positive integers of size $\approx x$ correspond to monic polynomials of degree n, and primes to those which are irreducible. One then recovers analogues of many classical results, including the "prime number theorem"

$$\#\big\{P \in \mathbb{F}_q[t] \,:\, P \text{ monic and irreducible, } \deg P = n\big\} = \frac{q^n}{n} + O\Big(\frac{q^{n/2}}{n}\Big)$$

and the law of quadratic reciprocity, to name a few (see [159] for a general introduction to number theory over function fields).

To study random multiplicative functions in this setting, we let $(f(P))_P$ be a sequence of independent Steinhaus random variables indexed by monic irreducible polynomials, and set $f(F) = f(P_1)^{a_1} \cdots f(P_k)^{a_k}$ whenever $F = P_1^{a_1} \cdots P_k^{a_k}$. The goal then becomes to understand the distribution of

$$\frac{1}{q^{n/2}} \sum_{\substack{F \in \mathbb{F}_q[t] \text{ monic,} \\ \deg(F)=n}} f(F) \tag{1.5.1}$$

as n tends to infinity. It was observed in [168] that as $q \to \infty$, this amounts to studying the coefficients $A(n)$ defined through the formal identity

$$\exp\Big(\sum_{k\geq 1} \frac{N_k}{\sqrt{k}} z^k\Big) = \sum_{n\geq 0} A(n) z^n, \tag{1.5.2}$$

where $\{N_k\}_{k\geq 1}$ is a sequence of i.i.d. standard complex Gaussian random variables. The behaviour of (1.5.1) (and by extension, $\frac{1}{\sqrt{x}}\sum_{n\leq x} f(n)$) should then resemble that of $A(n)$, and the main result in [168] for instance shows that $\mathbb{E}[|A(n)|] \asymp (\log n)^{1/4}$.

The analogy between $A(n)$ and $\frac{1}{\sqrt{x}}\sum_{n\leq x} f(n)$ has since been pushed much further [142, 143, 19], and a crucial first step in doing so is viewing the left-hand side in (1.5.2) as a genuine analytic object. Begin by introducing the *Gaussian analytic function*

$$\varphi(z) = \sum_{k\geq 1} \frac{N_k}{\sqrt{k}} z^k, \quad z \in \mathbb{D} = \{z \,:\, |z| < 1\}, \tag{1.5.3}$$

which we recognise as a complex analogue of the Gaussian field in §1.2.2 (Equation (1.2.5)). Here, too, one can view e^{φ} as arising from the the large-N limit of the characteristic poly-

nomial $P_N(z) = \det(I - zM_N^*)$ of a random unitary matrix[13]. Recalling the expansion

$$\log P_N(z) = -\sum_{k\geq 1} \frac{\mathrm{Tr}(M_N^{-k})z^k}{k},$$

Theorem 1.2.3 suggests that P_N should converge to e^{φ} in a suitable sense: it was indeed shown by Chhaibi and Najnudel [45, Proposition 3.1] that one can define these objects on the same probability space, on which

$$\sup_{|z|\leq r} |e^{\varphi(z)} - P_N(z)| \to 0 \quad \text{almost surely } \forall r < 1. \tag{1.5.4}$$

Despite not being defined on the unit circle, e^{φ} can still be made sense of as a random Schwartz distribution. For any trigonometric polynomial ϕ (and $z \mapsto \phi(z)$ its continuous harmonic extension to $\mathbb{D}$), one can define a one-parameter family of distributions $(\mathrm{HMC}_\alpha)_\alpha$ by setting

$$(\mathrm{HMC}_\alpha, \phi) := \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{\alpha}\varphi(re^{i\vartheta})}\overline{\phi(re^{i\vartheta})}\mathrm{d}\vartheta, \quad \alpha > 0. \tag{1.5.5}$$

(The existence of this limit is immediate by the almost sure analyticity of e^{φ} in $|z| < 1$.) First introduced in [142], this *holomorphic multiplicative chaos* HMC_α is then directly related to a GMC measure on the unit circle: formally, we write

$$\left|e^{\sqrt{\alpha}\varphi(e^{i\vartheta})}\right|^2 \mathrm{d}\vartheta = e^{2\sqrt{\alpha}\Re\varphi(e^{i\vartheta})}\mathrm{d}\vartheta,$$

and since $\Re\varphi(e^{i\vartheta})$ is the logarithmically correlated field in (1.2.5), we expect the "squared modulus" of HMC_α to give rise to a GMC measure $\mu_{\sqrt{2\alpha}}$ on $[0, 2\pi)$. The convergence

$$\lim_{r\to 1}(1-r^2)^\alpha \left|e^{\sqrt{\alpha}\varphi(re^{i\vartheta})}\right|^2 \mathrm{d}\vartheta = \mu_{\sqrt{2\alpha}}(\mathrm{d}\vartheta), \quad (\alpha < 1), \tag{1.5.6}$$

indeed follows rigorously by the convolution construction of GMC in [22], the left-hand side being obtained from $\Re\varphi(e^{i\vartheta})$ via convolution with the Poisson kernel.

Returning to (1.5.2), we can now identify $(A(n))_n$ with the sequence of Fourier coefficients of HMC_1: by Cauchy's theorem and the analyticity of e^{φ} inside the disc, we have that

$$A(n) = c_n(1) := (\mathrm{HMC}_1, \vartheta \mapsto e^{in\vartheta}),$$

and we more generally define $c_n(\alpha)$ to be the Fourier coefficients of HMC_α for any $\alpha > 0$.

[13] Note the use of a different convention in this context, namely $\det(I - zM_N^*)$ rather than $\det(I - z^{-1}M_N)$

By analogy with (1.4.12), one expects $A(n)$ to be related to the total mass of a GMC measure as $n \to \infty$. The identification with $c_n(1)$ reveals such a connection with little effort, with Parseval's theorem and (1.5.6) for instance implying that

$$\lim_{r\to 1}(1-r^2)^\alpha \sum_{n=0}^{\infty} |c_n(\alpha)|^2 r^{2n} = \lim_{r\to 1}(1-r^2)^\alpha \frac{1}{2\pi}\int_0^{2\pi} \left|e^{\sqrt{\alpha}\varphi(re^{i\vartheta})}\right|^2 \mathrm{d}\vartheta = \int_0^{2\pi} \mu_{\sqrt{2\alpha}}(\mathrm{d}\vartheta)$$

when $\alpha < 1$. Going further, a complete parallel to the limit theorems of Gorodetsky and Wong [79, 80, 81] was established in a sequence of works [142, 143, 19], culminating in the following analogue to (1.4.12) due to Atherfold and Najnudel

$$(\log n)^{1/4} c_n(1) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, M_{\sqrt{2}}\right),$$

where $M_{\sqrt{2}}$ is the total mass of a critical GMC measure on the circle. The same convergence was also shown to hold for the secular coefficients $c_n^{(N)}$ (the coefficients in the expansion $P_N(z) = \sum_{n\le N} c_n^{(N)} z^n$), provided $N = N_n$ tends to infinity fast enough relative to n.

Motivated in part by these connections to number theory and random matrices, Garban and Vargas [74] initiated the study of the Fourier coefficients of GMC itself. Letting X be the Gaussian field with covariances

$$\mathbb{E}\{X(\vartheta_1)X(\vartheta_2)\} = -\log|e^{i\vartheta_1} - e^{i\vartheta_2}| \tag{1.5.7}$$

and $(X_\varepsilon)_\varepsilon$ be a sequence of convolution approximations to X (see §1.3.1), they study the asymptotic behaviour of the sequence

$$\mathfrak{C}_n(\gamma) = \int_0^{2\pi} e^{in\vartheta}\mu_\gamma(\mathrm{d}\vartheta), \qquad \text{where } \mu_\gamma(\mathrm{d}\vartheta) := \lim_{\varepsilon\to 0^+} e^{\gamma X_\varepsilon(\vartheta) - (\gamma^2/2)\mathbb{E}X_\varepsilon(\vartheta)^2}\mathrm{d}\vartheta.$$

This task is considerably more delicate, as $\mathfrak{C}_n(\gamma)$ now depends on *all* of the Fourier modes of the underlying field X, as opposed to $c_n(\alpha)$, which is a polynomial of finite degree in the Gaussian variables N_k in (1.5.3). Nevertheless, one expects integration against $e^{in\vartheta}$ to mitigate the contribution from higher modes, and it is argued in [74, Section 1.2.3] that the integral

$$\int_0^{2\pi} e^{in\vartheta} e^{\gamma X_{1/n}(\vartheta)}\mathrm{d}\vartheta \quad \left(\approx n^{\gamma^2/2}\mathfrak{C}_n(\gamma), \text{ since } \mathbb{E}\{X_\varepsilon(\vartheta)^2\} \approx \log(1/\varepsilon)\right)$$

should display the same behaviour as $c_n(2\gamma^2)$. This claim is substantiated by the limit theorem

$$n^{(1-\gamma^2)/2}\mathfrak{C}_n(\gamma) \xrightarrow[n\to\infty]{d} \sqrt{\frac{\kappa}{2}}\cdot\mathcal{N}(0, M_{2\gamma}), \quad \gamma < \frac{1}{\sqrt{2}}$$

where

$$M_{2\gamma} = \int_0^{2\pi} \mu_{2\gamma}(\mathrm{d}\vartheta), \quad \text{and} \quad \kappa = \int_{\mathbb{R}} \frac{e^{2\pi i v}}{|v|^{\gamma^2}} \mathrm{d}v.$$

The main result in [74] shows that $\mathfrak{C}_n(\gamma)$ converges to zero almost surely as $n \to \infty$ when $\gamma < \sqrt{2}$, making μ_γ a so-called Rajchman measure. These form a class lying strictly between that of absolutely continuous and continuous measures (in this case on the circle), and have a long history of being studied in harmonic analysis [130] (see §4.0.1 for a discussion). A subsequent work of Lin, Qiu and Tan [126] refined this result by obtaining a polynomial decay rate, showing in particular that $\mathfrak{C}_n(\gamma) \asymp n^{-(\sqrt{2}-\gamma)^2/2}$ with probability one when $\gamma \in (1/\sqrt{2}, \sqrt{2})$.

At criticality ($\gamma = \sqrt{2}$), recall from §1.3.1 that the (appropriately normalised) GMC measure is non-atomic and supported on a set of Hausdorff dimension zero. Denoting its Fourier coefficients by $\mathfrak{C}_n(\sqrt{2})$ (with slight abuse of notation), this implies that $\mathfrak{C}_n(\sqrt{2})$ cannot decay at a polynomial rate, and by Wiener's theorem, that $\frac{1}{(2N+1)} \sum_{|n|\le N} |\mathfrak{C}_n(\sqrt{2})|^2$ converges to zero almost surely as $N \to \infty$. Beyond this, little is known, and the question of whether the Rajchman property persists was posed as an open problem in [74].

In Chapter 4, we make progress on this problem by showing that $\mathfrak{C}_n(\sqrt{2})$ converges to zero in probability[14], and decays like a negative power of $\log n$.

Theorem 1.5.1. *For any $a < 1/4$, $(\log n)^a \mathfrak{C}_n(\sqrt{2}) \to 0$ in probability as $n \to \infty$.*

The heuristic behind our proof also predicts the typical order of magnitude of $\mathfrak{C}_n(\gamma)$ for $\gamma < \sqrt{2}$ (see §4.0.4), suggesting a transition at $\gamma = \sqrt{2}$ seemingly echoed by the behaviour the partial sums $\sum_{n\le x} d_\alpha(n) f(n)$ as α approaches 2 (see Remark 4.0.6).

1.5.1 Poisson multiplicative chaos and the multiplication table problem

Let $p(k)$ denote the limit as $n \to \infty$ of the probability that a random permutation of n elements fixes a set of size k. Using a well-known analogy between integers and permutations [83], this models a famous question of Erdős asking to estimate the number of distinct entries in an $n \times n$ multiplication table. It was shown by Eberhard, Ford, and Green [64] that $p(k) \asymp k^{-\delta}(1 + \log k)^{-3/2}$, where $\delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.086071$ is the Erdős-Tenenbaum-Ford constant, in direct analogy with an older breakthrough of Ford [68] showing that

$$H(x, y) = \#\big\{n \le x : \text{has a divisor in } (y, 2y]\big\} \asymp \frac{x}{(\log y)^{\delta}(\log\log y)^{3/2}}, \quad 3 \le y \le \sqrt{x}.$$

[14] For simplicity, we prove this for a $\star$-scale invariant field on $[0, 1]$ as opposed to the field X in (1.5.7). This simplification was also used in [74], and the argument in §4 is expected to adapt straightforwardly to the measure constructed from (1.5.7)

Both of these results remained the state-of-the-art until a recent work of Green and Sawhney [84] established the asymptotic $p(k) \sim f(\{\log_2 k\})k^{-\delta}(\log k)^{-3/2}$ for an explicit function f.

It was noted in [168] that the problems of estimating $p(k)$ and $H(x, y)$ exhibit some features in common with the study of partial sums of random multiplicative functions and with the Fyodorov-Hiary-Keating conjectures, "although the link [...] is perhaps less clearly defined" (than with the study of $A(n)$). In a final, forward-looking chapter, §5, we apply the philosophy of the previous section to make the connection between $p(k)$ and multiplicative chaos precise: as $n \to \infty$, the number $N_k^{(n)}$ of k-sets fixed by a random permutation of n elements tends to a limiting random variable N_k, which we exhibit as the k-th Fourier coefficient of a Poisson analogue to the holomorphic multiplicative chaos. We also make a connection to a ratio of characteristic polynomials of random permutation matrices.

This provides a path towards potentially establishing a limit theorem for N_k as $k \to \infty$ (upon adequate renormalisation)—perhaps to a mixture of an infinitely divisible law with the mass of a Poisson counterpart to GMC on the circle, as a naïve analogy with c_n would suggest.

Chapter 2

Partial sums of random multiplicative functions through multiplicative chaos

This chapter is based on the work "Partial sums of random multiplicative functions with supercritical divisor twists" [89].

Let $(Z_p)_p$ denote a sequence of independent and identically distributed random variables indexed by the primes, which are uniformly distributed on the complex unit circle. A *Steinhaus random multiplicative function* $f : \mathbb{N} \to \mathbb{C}$ is defined as $f(p) = Z_p$ on the primes, and extended to $\mathbb{N}$ by making f completely multiplicative. Originally introduced to model Archimedean characters $n \mapsto n^{it}$, the study of partial sums of f and other, similarly defined random functions has grown into an active area of research in its own right [79, 80, 81, 167, 98, 96].

A landmark result in this area is Harper's [98] resolution of Helson's conjecture [107], which states that partial sums of $f(n)$ display a surprising amount of cancellation when compared to, say, sums of independent random variables. To be precise, Helson conjectured that $\mathbb{E}|\sum_{n\le x} f(n)| = o(\sqrt{x})$, and Harper showed that the following holds uniformly in $q \in [0, 1]$:

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} f(n)\Big|^{2q}\Big\} \asymp \big(\sqrt{\log\log x}(1-q)+1\big)^{-q}. \tag{2.0.1}$$

The key insight in [98] is that this phenomenon can be explained using a connection to the theory of Gaussian multiplicative chaos (GMC), whose relevance to number theory first emerged with the conjectures of Fyodorov, Hiary and Keating on the magnitude of the Riemann zeta function on typical short intervals of the critical line [71, 72].

Harper's argument is comprised of two main steps. The first consists of comparing moments of $x^{-1/2}\sum_{n\le x} f(n)$ to those of the integral of a random Euler product, by a sieve-theoretic argument and Parseval's theorem. The second is to realise that this integral approximates the total mass of a random measure—a smooth perturbation of *critical* GMC—

only when scaled by $\sqrt{\log\log x}$, which is known as the Seneta-Heyde normalisation of critical GMC [150]. The absence of this factor delivers the sought-after cancellation, and explains the right-hand side in (2.0.1). The bulk of the work lies in making this connection rigorous, which the author achieves by proving a non–Gaussian analogue of Girsanov's theorem, setting the stage for an application of the classical ballot theorem. A recent work of Gorodetsky and Wong [82] showed that by appealing to pre-existing results in the GMC literature instead, namely Kahane's convexity inequality and a coupling due to Saksman and Webb [161], one can significantly shorten this comparison step and recover (2.0.1), albeit without the uniformity in q near 1.

The purpose of this chapter is to present a new and self-contained way to establish this comparison, through the study of *large deviations* of random Euler products. Compared to [98], this approach moves the Gaussian approximation step to an earlier stage of the argument, resulting in a short proof of the upper bound in (2.0.1). More importantly, it can also be used to prove a conjecturally sharp upper bound for partial sums of $f \cdot d_\alpha$ (Theorem 2.0.1), where d_α is the α-divisor function and $\alpha \in (1,2)$. In this regime, which corresponds to the *supercritical* phase of GMC, recovering double-logarithmic corrections similar to (2.0.1) requires a precise understanding of the maximum of random Euler products in short intervals. This also has applications to the study of *pseudomoments* of the Riemann zeta function, discussed further below.

2.0.1 Main results

Let f denote a Steinhaus random multiplicative function, and d_α denote the α-divisor function, defined through $\zeta(s)^\alpha = \sum_{n\geq 1} d_\alpha(n)n^{-s}$ where this series converges. The main result of this chapter is the following supercritical analogue of (2.0.1).

Theorem 2.0.1. *Fix $\alpha \in (1,2)$. Then uniformly in all large x and $q \in [0, 1/\alpha]$,*

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\leq x} d_\alpha(n)f(n)\Big|^{2q}\Big\} \ll \frac{(\log x)^{2q(\alpha-1)}}{(\log\log x)^{(3\alpha q/2)}(1-\alpha q)^2+1}.$$

The proof proceeds by studying averages of $|F_x|^{2\alpha}$, where $F_x(s) := \prod_{p\leq x}(1-f(p)p^{-s})^{-1}$ is the Euler product associated to f, truncated at x. Harper's argument in [98] carries this out when $\alpha = 1$, comparing the left-hand side to the q-th moment of $\int_0^1 |F_x(\frac{1}{2}+ih)|^2 \mathrm{d}h$. For larger α, the same comparison naturally gives rise to moments of $\int_0^1 |F_x(1/2+ih)|^{2\alpha}\mathrm{d}h$, and more generally suggests that the order of magnitude of partial sums twisted by a multiplicative function g is governed by averages of $|F_x|^\gamma$ whenever

$$\sum_{p\leq t} |g(p)|^2 \sim \frac{\gamma^2}{4}\frac{t}{\log t} \tag{2.0.2}$$

with a sufficiently strong error term (see [75], and the introduction of [79]). The divisor function d_α provides the simplest example of such a twist (for which $\gamma = 2\alpha$).

To illustrate the method, we begin by proving the following Euler product bound at $\gamma = 2$.

Proposition 2.0.2. *Uniformly in x sufficiently large and $q \in [0,1]$,*

$$\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_0^1 \big|F_x(1/2+ih)\big|^2 \mathrm{d}h\Big)^q\Big\} \ll \big(\sqrt{\log\log x}(1-q)+1\big)^{-q}. \tag{2.0.3}$$

Our approach will consist of studying this integral through the measure of level sets of $\log|F_x(s)|$. This will allow us to bypass the need for an approximate Girsanov's theorem as in [98], and reveals that the integral is dominated by those h for which

$$\log|F_x(1/2+ih)| = \log\log x - O(\sqrt{\log\log x}).$$

Combining this with the first step of Harper's argument in [98] yields a short proof of the upper bound therein (and thus of Helson's conjecture), which is given in full in Section 2.1.

We then adapt this approach to integrals of $|F_x|^\gamma$ for $\gamma > 2$, where the analysis becomes more delicate. We also handle averages off the critical line.

Theorem 2.0.3. *Fix $\gamma \in (2,4)$. Then uniformly in $q \in [0, 2/\gamma]$, all large x, $0 \le k \le \lfloor \log\log\log x\rfloor + 1$, $\sigma_k = 1/2 - k/\log x$, and $3 < y \le x^{e^{-k}}$,*

$$\mathbb{E}\Big\{\Big(\frac{1}{\log y}\int_0^1 \big|F_y(\sigma_k+ih)\big|^\gamma \mathrm{d}h\Big)^q\Big\} \ll \frac{(\log y)^{(\gamma-2)q}}{(\log\log y)^{(3\gamma q/4)}(2-\gamma q)^2+1}. \tag{2.0.4}$$

Upon making the necessary identifications, the exponents on the right-hand side match those found in the normalisation of *supercritical* GMC [132], which is also known to only have moments up to $q < 2/\gamma$. By contrast with the critical ($\gamma = 2$) case, the dominant contribution to the integral will now come from points surrounding the local maxima of $|F_y(\sigma_k+ih)|$ on $h \in [0,1]$. This leads us to study $\max_{h\in[0,1]}|F_y(\sigma_k+ih)|$ using ideas from the study of extrema of log-correlated fields [37, 38], and the following bound arises as an immediate corollary of the analysis.

Corollary 2.0.4 (Maximum bound)**.** *For x large and $1 < y \le \log\log x/\log\log\log x$,*

$$\mathbb{P}\Big(\max_{h\in[0,1]}|F_x(1/2+ih)| > \frac{\log x}{(\log\log x)^{3/4}}e^y\Big) \ll y\exp\Big(-2y - \frac{y^2}{\log\log x}\Big).$$

This can be seen as a random analogue of the Fyodorov-Hiary-Keating conjecture [71, 72], matching the best known bound in that setting [10], and improving the bound in [9] for $\max_{h\in[0,1]} |F_x(1/2+ih)|$ to $O(1)$ precision. By understanding the structure of these maxima, we can also bound the typical measure of level sets of $\log|F_x|$ near the height of the maxima.

Corollary 2.0.5 (Typical measure of level sets). *Let x be large and $1 < A \le (\log x)^{\frac{1}{\log\log\log x}}$. Then uniformly in $|y| < (\log\log x)/2$,*

$$\begin{aligned}\operatorname{meas}\Big\{ h \in [0,1] : |F_x(1/2+ih)| > \frac{\log x}{(\log\log x)^{3/4}} e^y \Big\} \\ \le (\log x)^{-1} A |\log A - y| \exp\Big(-2y - \frac{y^2}{\log\log x} \Big)\end{aligned}$$

with probability $1 - O((\log A)/A)$.

Theorem 2.0.3 and both of these corollaries display what is typical of Gaussian log-correlated processes (see, e.g., Lemma 4.2 in [55]), and are expected to be sharp. In particular, by taking A sufficiently large and $y = O(1)$ in Corollary 2.0.5, we find that the measure of points $h \in [0,1]$ for which $\log|F_x(1/2+ih)| = \log\log x - \frac{3}{4}\log\log\log x + O(1)$ is $\ll_A (\log x)^{-1}$ with high probability. This suggests that their contribution to the left-hand side in (2.0.4) should match the upper bound.

Lastly, the $(\log\log x)^{3\gamma q/4}$ saving in Theorem 2.0.3 leads to improved bounds for the *pseudomoments*

$$\Psi_{2q,\alpha}(x) := \lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \Big| \sum_{n\le x} \frac{d_\alpha(n)}{n^{1/2+it}} \Big|^{2q} \mathrm{d}t$$

of the Riemann zeta function, which were first introduced by Conrey and Gamburd [50]. Motivated by the classical problem of computing moments of the zeta function, they showed that $\Psi_{2q,1}(x) \sim a_q \gamma_q (\log x)^{q^2}$ when $q \in \mathbb{N}$ and $\alpha = 1$, where a_q is the "arithmetic" constant in the Keating-Snaith conjecture [116] and γ_q is the volume of a certain convex polytope. The order of magnitude $(\log x)^{q^2}$ was shown to persist to non-integer $q > 0$ in [32, 76].

A more nuanced picture has emerged when $\alpha > 1$: while $\Psi_{2q,\alpha} \asymp (\log x)^{(q\alpha)^2}$ for $q > 1/2$ [32], the order of magnitude for small $q > 0$ was determined up to $(\log\log x)$ factors by Gerspach [76] to be

$$\Psi_{2q,\alpha}(x) \asymp \begin{cases} (\log x)^{2(\alpha-1)q} (\log\log x)^{O(1)} & \text{if } 1 \le \alpha < 2 \text{ and } 0 < q \le 2(\alpha-1)/\alpha^2 \\ (\log x)^{(q\alpha)^2} & \text{if } 1 \le \alpha < 2 \text{ and } 2(\alpha-1)/\alpha^2 < q \le 1/2 \\ (\log x)^{q\alpha^2/2} (\log\log x)^{O(1)} & \text{if } \alpha \ge 2 \text{ and } 0 < q < 1/2, \end{cases} \tag{2.0.5}$$

following initial progress in [31]. In his thesis, he later conjectured the correct exponent of $(\log\log x)$ in the above, using heuristics based on work of Arguin-Ouimet-Radziwiłł [16] and the Fyodorov-Hiary-Keating conjectures (see Conjecture 5.4 in [75]). Our last result establishes the upper bound in this conjecture, in the first regime in (2.0.5).

Theorem 2.0.6. *Let $\alpha \in (1,2)$, $0 < q < 2(\alpha-1)/\alpha^2$ be fixed. Then uniformly for large x,*

$$\Psi_{2q,\alpha}(x) \ll \frac{(\log x)^{2q(\alpha-1)}}{(\log\log x)^{q(3\alpha/2)}}.$$

The proof combines Gerspach's original argument [75] with the bound from Theorem 2.0.3. It also requires us to extend the conclusion of Theorem 2.0.3 to integrals over *mesoscopic* intervals, meaning intervals of length $\asymp (\log y)^{\theta}$ for $\theta \in (-1,0]$. We achieve this using a tilted measure argument.

Organisation

Section 2.1 proves Helson's conjecture, by first reducing the claim to that of Proposition 2.0.2 (in Section 2.1.1), then proving said proposition in Sections 2.1.2 and 2.1.3. In Section 2.2, we show how to adapt our approach to prove Theorem 2.0.3; this relies on a result on the structure of the maxima of $|F_y(\sigma+ih)|$ on $[0,1]$, proved later in Section 2.2.2, where we also establish Corollaries 2.0.4 and 2.0.5. Finally, Section 2.3 proves Theorem 2.0.6, and the appendix compiles various Gaussian estimates used throughout the chapter.

2.1 A short proof of Helson's conjecture

This section proves the upper bound in (2.0.1). As observed by Harper [98], it suffices to do so for $q \geq 2/3$, since

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\leq x} f(n)\Big|^{2q}\Big\} \leq \mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\leq x} f(n)\Big|^{4/3}\Big\}^{3q/2} \ll (\log\log x)^{-q/2} \tag{2.1.1}$$

by Hölder's inequality and the claim for $q = 4/3$. We begin by showing that for $q \in [2/3, 1]$,

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\leq x} f(n)\Big|^{2q}\Big\} \ll \mathbb{E}\Big\{\Big(\frac{1}{\log y}\int_0^1 \big|F_y(1/2+ih)\big|^2 \mathrm{d}h\Big)^q\Big\} + o\big((\log\log y)^{-q/2}\big), \tag{2.1.2}$$

where $\log y = \log x/(\log\log x)^2$. This is the content of Section 2.1.1, which we emphasise is not new and is only included to make our proof of (2.0.1) self-contained. The main difficulty then lies in bounding the right-hand side in (2.1.2) (cf. Proposition 2.0.2), which is achieved in Sections 2.1.2 and 2.1.3.

2.1.1 Reduction to moments of integrals of Euler products

To prove (2.1.2), we follow the presentation of Gorodetsky and Wong [82], which streamlines Harper's argument from [98] by incorporating a simplification later introduced in his work on character sums [100, p. 13].

By the law of total expectation and Jensen's inequality, we begin by writing

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} f(n)\Big|^{2q}\Big\} \le \mathbb{E}\Big\{\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} f(n)\Big|^{2}\,\Big|\,\mathscr{F}_y\Big\}^{q}\Big\}, \tag{2.1.3}$$

where $\mathscr{F}_y$ is the $\sigma-$algebra generated by $(Z_p)_{p\le y}$. Using the multiplicativity of f, we can decompose the partial sum of $f(n)$ up to $n \le x$ as

$$\sum_{n\le x} f(n) = \sum_{\substack{n_1 n_2\le x\\ p|n_1 \implies p>y\\ p|n_2 \implies p\le y}} f(n_1 n_2) = \sum_{\substack{1\le n_1\le x\\ p|n_1 \implies p>y}} f(n_1) \sum_{\substack{n_2\le x/n_1\\ p|n_2 \implies p\le y}} f(n_2),$$

and use the orthogonality relation $\mathbb{E}\{f(n)\overline{f(m)}\} = \mathbf{1}(n = m)$. Note that this still holds upon conditioning on $\mathscr{F}_y$, provided m and n only have prime factors strictly greater than y. It follows that (2.1.3) is

$$\le \mathbb{E}\Big\{\Big(\sum_{\substack{1\le n_1\le x\\ p|n_1 \implies p>y}} \Big|\frac{1}{\sqrt{x}}\sum_{\substack{n_2\le x/n_1\\ p|n_2 \implies p\le y}} f(n_2)\Big|^{2}\Big)^{q}\Big\}. \tag{2.1.4}$$

The strategy then consists of smoothing the outer summation into an integral, in order to pick up the density of integers n_1 which are y–rough (meaning $p|n_1 \implies p > y$). This however requires n_1 to be large enough ($> x^{3/4}$, say), and we must therefore handle the sum over smaller n_1 separately.

We can separate the q-th moment of the sum over $1 \le n_1 \le x^{3/4}$ from the quantity in (2.1.4) by subadditivity of $x \mapsto x^q$ and Jensen's inequality, since $q < 1$. This gives

$$\Big(\sum_{\substack{1\le n_1\le x^{3/4}\\ p|n_1 \implies p>y}} \frac{1}{x}\mathbb{E}\Big\{\Big|\sum_{\substack{n_2\le x/n_1\\ p|n_2 \implies p\le y}} f(n_2)\Big|^{2}\Big\}\Big)^{q} = \Big(\sum_{\substack{1\le n_1\le x^{3/4}\\ p|n_1 \implies p>y}} \frac{\Psi(x/n_1, y)}{x}\Big)^{q},$$

where $\Psi(x, y)$ counts the number of y-smooth numbers $n \le x$ (meaning $p|n \implies p \le y$). Using a well-known estimate for Ψ (Theorem 5.3.1 in [48]), this is

$$\ll \Big(\sum_{\substack{1\le n_1\le x^{3/4}\\ p|n_1 \implies p>y}} \frac{(\log y)^{A}}{n_1} e^{-c\frac{\log(x/n_1)}{\log y}}\Big)^{q} \le (\log y)^{Aq} e^{-q\frac{c}{4}\frac{\log x}{\log y}}\Big(\sum_{1\le n_1\le x^{3/4}} \frac{1}{n_1}\Big)^{q}, \tag{2.1.5}$$

for a pair of absolute constants $A, c > 0$. Recalling that $\log y = \log x/(\log\log x)^2$, it follows that (2.1.5) is $\ll (\log y)^{(A+2)q} e^{-q(c/4)(\log x/\log y)} = o\big((\log\log y)^{-q/2}\big)$ uniformly over q.

We now turn to the sum over $n_1 \in (x^{3/4}, x]$ in Equation (2.1.4). By grouping terms according to the value r of $\lfloor x/n_1 \rfloor$, we can rewrite this sum as

$$\sum_{1\le r<x^{1/4}} \Big|\frac{1}{\sqrt{r}} \sum_{\substack{n_2\le r\\ p|n_2 \Longrightarrow p\le y}} f(n_2)\Big|^2 \sum_{\substack{x^{3/4}<n_1\le x\\ p|n_1 \Longrightarrow p>y\\ \lfloor x/n_1\rfloor = r}} \frac{1}{n_1}.$$

The inner sum over n_1 can now be estimated using the approximate density of y-rough numbers. Indeed, if we let $\Phi(x, y)$ count the number of such integers smaller than x, a standard sieve estimate yields

$$\sum_{\substack{x^{3/4}\le n_1\le x\\ p|n_1 \Longrightarrow p>y\\ \lfloor x/n_1\rfloor = r}} \frac{1}{n_1} \le \frac{\Phi(x/r, y) - \Phi(x/(r+1), y)}{x/r} \ll \frac{1}{r\cdot\log y}, \tag{2.1.6}$$

uniformly over $r \in [1, x^{1/4})$ (see [48], Theorem 6.2.5). It follows that

$$\begin{aligned}
&\mathbb{E}\Big\{\Big(\sum_{\substack{1\le n_1\le x\\ p|n_1 \Longrightarrow p>y}} \Big|\frac{1}{\sqrt{x}} \sum_{\substack{n_2\le x/n_1\\ p|n_2 \Longrightarrow p\le y}} f(n_2)\Big|^2\Big)^q\Big\}\\
&\quad\ll \frac{1}{(\log y)^q}\mathbb{E}\Big\{\Big(\sum_{1\le r<x^{1/4}} \Big|\frac{1}{r} \sum_{\substack{n_2\le r\\ p|n_2 \Longrightarrow p\le y}} f(n_2)\Big|^2\Big)^q\Big\}\\
&\quad\ll \frac{1}{(\log y)^q}\mathbb{E}\Big\{\Big(\int_0^\infty \Big|\sum_{\substack{n_2\le h\\ p|n_2 \Longrightarrow p\le y}} f(n_2)\Big|^2 \frac{\mathrm{d}h}{h^2}\Big)^q\Big\},
\end{aligned}$$

which by Parseval's theorem (in the form of Equation (5.26) in [140]) equals

$$\frac{1}{(\log y)^q}\mathbb{E}\Big\{\Big(\int_{\mathbb{R}} \frac{|F_y(1/2+ih)|^2}{|1/2+ih|^2}\mathrm{d}h\Big)^q\Big\}.$$

Noting that $(f(p)p^{-ih}, h \in [0,1])$ is equal in distribution to $(f(p)p^{-i(h+n)}, h \in [0,1])$ for any fixed $n \in \mathbb{Z}$, this is

$$\ll \frac{1}{(\log y)^q}\mathbb{E}\Big\{\Big(\int_0^1 |F_y(1/2+ih)|^2\mathrm{d}h\Big)^q\Big\}\Big(\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^q}\Big) \tag{2.1.7}$$

for $q \ge 2/3$. The claim follows since the sum over n is bounded.

2.1.2 Bounding moments of integrals of Euler products

We now turn to the proof of Proposition 2.0.2. Without loss of generality, assume that $x > C_0$ for a fixed, large constant $C_0 > 0$; we take $C_0 = 1000$ for concreteness. Set

$$t = t(x) = \lfloor \log\log x \rfloor.$$

For any $C_0 < j \le t$, we define

$$S_j(s) = \sum_{\exp(e^{C_0}) < p \le \exp(e^j)} X_p(s), \text{ where } X_p(s) := \Re\Big(\frac{Z_p}{p^s} + \frac{Z_p^2}{2p^{2s}}\Big), \tag{2.1.8}$$

so that $S_t(s)$ is a second-order approximation to $\log|F_x(s)|$. This choice of notation reflects our intention to view $S_t(1/2+ih)$ (and hence $\log|F_x(1/2+ih)|$) as a random walk with t increments $S_j(1/2+ih) - S_{j-1}(1/2+ih)$ of variance

$$\mathbb{E}\Big\{\big(S_j(1/2+ih) - S_{j-1}(1/2+ih)\big)^2\Big\} = \frac{1}{2}\sum_{\exp(e^{j-1}) < p \le \exp(e^j)} \Big(\frac{1}{p} + \frac{1}{4p^2}\Big),$$

which is roughly 1/2 by the prime number theorem (cf. Section 1.2.3).

Noting that

$$|F_x(1/2+ih)|^2 \ll e^{2S_t(1/2+ih) + \sum_p \sum_{k\ge 3} p^{-k/2}} \le e^{2S_t(1/2+ih) + 4\sum_p p^{-3/2}} \ll e^{2S_t(1/2+ih)}, \tag{2.1.9}$$

for any $h \in [0,1]$, it suffices to show that for any $q \in [0,1]$,

$$\mathbb{E}\big\{\mathscr{Z}_2^q\big\} \ll \big((1-q)\sqrt{t}+1\big)^{-q}, \quad \text{where } \mathscr{Z}_2 := \frac{1}{e^t}\int_0^1 e^{2S_t(1/2+ih)}\mathrm{d}h. \tag{2.1.10}$$

Furthermore, we may assume that $q < 1 - C/\sqrt{t}$ for any fixed constant $C > 0$; the desired bound is simply $\ll 1$ otherwise, in which case it follows directly from Hölder's inequality and the Laplace transform estimate in Lemma 2.4.2:

$$\mathbb{E}\big\{\mathscr{Z}_2^q\big\} \ll \mathbb{E}\big\{e^{2S_t(1/2+ih)}\big\}^q e^{-tq} \ll 1. \tag{2.1.11}$$

Assuming that $(1-q) \gg 1/\sqrt{t}$, we now proceed in two steps. The first will be to estimate the expectation of

$$\mathscr{Z}_2(A) := \frac{1}{e^t}\int_0^1 e^{2S_t(1/2+ih)}\mathbf{1}(h \in G_A)\mathrm{d}h,$$

where $G_A \subseteq [0,1]$ is a suitably chosen set of "good points" that we define shortly (cf. Proposition 2.1.1). We then leverage the fact that most points h are in G_A with high probability (cf. Lemma 2.1.3) to upgrade this estimate to (2.1.10). To define G_A, we make the

observation that $(S_t(1/2+ih))_{h\in[0,1]}$ is approximately a *logarithmically-correlated field*; that is, $S_t(1/2+ih)$ is approximately Gaussian for each h (cf. Lemma 2.4.4), and

$$\mathbb{E}\big\{S_t(1/2+ih)S_t(1/2+ih')\big\} \approx \frac{1}{2}\log\frac{1}{|h-h'|\vee e^{-t}}.$$

(This can be made precise by a straightforward application of a quantitative prime number theorem.) The extremal statistics of such stochastic processes have been extensively studied [4], beginning with the pioneering work of Bramson on branching Brownian motion [37]. Using a similar approach, we will prove in Lemma 2.1.3 that for large t,

$$\mathbb{P}\Big(\max_{h\in[0,1]} S_t(1/2+ih) \le m(t)+A\Big) \to 1 \text{ as } A\to\infty, \quad \text{where } m(t) := t-\frac{3}{4}\log t. \tag{2.1.12}$$

To be precise, we will argue that the paths $j\mapsto S_j(1/2+ih)$ for which $S_t(1/2+ih)\approx A+m(t)$ typically display linear growth, while remaining under a *barrier* $m(j)+B(j)+A$ at each time j. Crucially, we can pick $B(j)$ to be *smaller* than the typical fluctuations of a Brownian bridge from A to $m(t)+A$, which ultimately yields a logarithmic correction in the order of the maximum (an additional $-\frac{1}{2}\log t$).

For Proposition 2.0.2, we only require a weak version of this fact where $B(j)$ is relatively large. Let

$$G_{A,\sigma} = \{h\in[0,1] : S_j(\sigma+ih)\in[L_A(j),U_A(j)],\ \lfloor A/4\rfloor \le j\le t\}$$

for any $\sigma\in\mathbb{R}$ and $A>4C_0$, where

$$U_A(j) = A+j+2\log\big(1+j\wedge(t-j)\big), \quad L_A(j) = A-20j, \quad \forall \lfloor A/4\rfloor \le j\le t, \tag{2.1.13}$$

and $U_A(j)=-L_A(j)=\infty$ for smaller j. The lower barrier L_A is needed later in the proof, when approximating S_t by a bona fide Gaussian random walk. In this section, we only consider $\sigma=1/2$ and therefore omit the dependence on σ, writing $G_A=G_{A,1/2}$.

Proposition 2.1.1. *Uniformly in all large t and $4C_0<A\le 3\sqrt{t}$, $\mathbb{E}\big\{\mathscr{Z}_2(A)\big\}\ll A/\sqrt{t}$.*

Proof. We express the integral on the right-hand side in terms of the large deviation frequencies of S_t. Letting

$$\mathscr{S}(V,G_A) := \operatorname{meas}\{h\in[0,1] : S_t(1/2+ih)>V, h\in G_A\}, \tag{2.1.14}$$

for any $V>0$, we can rewrite $\mathbb{E}\{\mathscr{Z}_2(A)\}$ using Fubini's theorem as

$$\mathbb{E}\Big\{\frac{1}{e^t}\int_0^1\int_{-\infty}^{\infty} 2e^{2V}\mathbf{1}(h\in G_A, S_t(1/2+ih)>V)\mathrm{d}V\,\mathrm{d}h\Big\} = \frac{1}{e^t}\int_{-\infty}^{U_A(t)} 2e^{2V}\mathbb{E}\big\{\mathscr{S}(V,G_A)\big\}\mathrm{d}V. \tag{2.1.15}$$

To bound $\mathbb{E}\mathcal{S}(V, G_A)$, we derive the following bound on the large deviation probability of $S_t(1/2 + ih)$ when $h \in G_A$. Informally, it states that this probability is comparable to that of a Brownian bridge of length t, from 0 to V, remaining under $U_A(s)$ at all times $s \in [0, t]$. For later use, we state a more general result which applies to $S_t(\sigma + ih)$ for σ near 1/2, and another choice of envelope $U_A(s)$. The proof is technical and will be given in Section 2.1.3.

Lemma 2.1.2. *Let $h \in [0, 1]$ and x be large. Then there exists an absolute constant $C > 0$ such that the following holds. Fix $\sigma_k = 1/2 - k/\log x$ and $0 \le k \le \log\log\log x$. For any $A/4 < t \le \log\log x - k$, $4C_0 < A \le t$ and $t/4 \le V \le U_A(t)$,*

$$\mathbb{P}\big(S_t(\sigma_k + ih) > V, h \in G_{A,\sigma_k}\big) \le C \cdot \frac{A\big(U_A(t) - V + C\big)}{t} \frac{e^{-V^2/t}}{\sqrt{t}}.$$

Furthermore, the same bound holds upon replacing G_{A,σ_k} and U_A by $G^{\max}_{A,\sigma_k}$ and $U^{\max}_A$, defined as

$$U^{\max}_A(j) := \begin{cases} A + j\Big(1 - \frac{3}{4}\frac{\log t}{t}\Big) + 10^3 \cdot \log\big(1 + j \wedge (t - j)\big) & \text{if } j \ge \lfloor A/4 \rfloor \\ \infty & \text{if } j < \lfloor A/4 \rfloor, \end{cases} \tag{2.1.16}$$

and

$$G^{\max}_{A,\sigma_k} = \big\{h \in [0, 1] : S_j(\sigma_k + ih) \in [L_A(j), U^{\max}_A(j)],\ \forall j \le t\big\}, \tag{2.1.17}$$

respectively (cf. Equations (2.2.3) *and* (2.2.4)*).*

Proof. See Section 2.1.3. □

Noting that $(S_k(1/2 + ih))_{k \le t}$ is equal in distribution to $(S_k(1/2))_{k \le t}$ for any $h \in [0, 1]$, Fubini's theorem yields

$$\mathbb{E}\big\{\mathcal{S}(V, G_A)\big\} = \int_0^1 \mathbb{P}\big(S_t(1/2 + ih) > V, h \in G_A\big)\mathrm{d}h = \mathbb{P}\big(S_t(1/2) > V, 0 \in G_A\big),$$

and it follows by Lemma 2.1.2 that

$$\begin{aligned} \mathbb{E}\big\{\mathcal{Z}_2(A)\big\} &\ll \frac{1}{e^t}\int_{-\infty}^{t/4} e^{2V}\mathrm{d}V + \frac{1}{e^t}\int_{t/4}^{U_A(t)} \frac{A\big(U_A(t) - V + C\big)}{t}\frac{e^{2V - V^2/t}}{\sqrt{t}}\mathrm{d}V, \\ &\ll e^{-t/2} + \frac{1}{e^t}\int_0^{A+t} \frac{A\big(A + t - V + C\big)}{t}\frac{e^{2V - V^2/t}}{\sqrt{t}}\mathrm{d}V. \end{aligned} \tag{2.1.18}$$

Using the change of variables $u = (t - V)/\sqrt{t}$ and the assumption that $A \le 3\sqrt{t}$, the remaining integral is bounded by

$$\ll \frac{A}{\sqrt{t}}\int_{-A/\sqrt{t}}^{\sqrt{t}} (3 + o(1) + |u|)e^{-u^2}\mathrm{d}u \ll \frac{A}{\sqrt{t}}\int_0^{\infty} (1 + u)e^{-u^2}\mathrm{d}u \ll \frac{A}{\sqrt{t}}.$$

□

Lemma 2.1.3. *Uniformly in all large t and $4C_0 < A \le 3\sqrt{t}$, $\mathbb{P}\big(\exists h \notin G_A\big) \ll e^{-2A}$.*

Proof. A union bound over $j \le t$ yields

$$\mathbb{P}\big(\exists h \notin G_A\big) \le \sum_{A/4\le j\le t} \mathbb{P}\big(\exists h : S_j(1/2+ih) > U_A(j)\big) \tag{2.1.19}$$

$$+ \sum_{A/4\le j\le t} \mathbb{P}\big(\exists h : S_j(1/2+ih) < L_A(j)\big) \tag{2.1.20}$$

For each $j \le t$, let $(I_j)_j$ be a partition of $[0,1]$ into disjoint intervals I_j of width e^{-j}. Using the fact that $(S_j(1/2+ih), h \in I_j) \overset{d}{=} (S_j(1/2+ih), h \in [0, e^{-j}))$ for each I_j, another union bound yields

$$\begin{aligned}\mathbb{P}\big(\exists h : S_j(1/2+ih) > U_A(j)\big) &\le \sum_{I_j} \mathbb{P}\big(\max_{h\in I_j} S_j(1/2+ih) > U_A(j)\big)\\ &= e^j \mathbb{P}\big(\max_{h\in[0,e^{-j})} S_j(1/2+ih) > U_A(j)\big).\end{aligned}$$

We then claim that $\max_{h\in[0,e^{-j})} S_j(1/2+ih)$ is comparable in law to $S_j(1/2)$. This is made rigorous by Lemma 2.4.6, which we prove using a standard chaining argument in the appendix, and by which

$$e^j \cdot \mathbb{P}\big(\max_{h\in[0,e^{-j})} S_j(1/2+ih) > U_A(j)\big) \ll \exp\big(j - U_A(j)^2/j\big) \ll e^{-2A}\big(1 + j \wedge (t-j)\big)^{-2}, \tag{2.1.21}$$

uniformly in j. For the second sum in (2.1.19), we simply note that

$$\mathbb{P}\big(\exists h : S_j(1/2+ih) < L_A(j)\big) = \mathbb{P}\big(\exists h : -S_j(1/2+ih) > 20j - A\big), \tag{2.1.22}$$

and that the bound in Lemma 2.4.6 applies to $\max_{h\in[0,e^{-j})} -S_j(1/2+ih)$ (cf. Remark 2.4.7). We can therefore use the same argument used to bound $\mathbb{P}(\exists h : S_j(1/2+ih) > U_A(j))$, which in this case yields $\ll e^{j-400j+40A} \ll e^{-2A} j^{-2}$ provided $j \ge A/4$. Using this in (2.1.19) along with the estimate in (2.1.21) and summing over j proves the lemma. □

Proof of Proposition 2.0.2. We use an interpolation argument due to Soundararajan and Zaman [168]. Set $j_0 := 4C_0 + 1$ and consider the sequence $(n_j)_{j\ge j_0}$ defined by $n_j := \frac{j}{(1-q)}$. Recall that by (2.1.11), we may assume that $(1-q) \ge j_0/\sqrt{t}$. The range $j_0 \le j \le \lceil (1-q)\sqrt{t}\rceil + 1$ is then nonempty, and for each such j, $4C_0 < n_j \le 3\sqrt{t}$. In particular, Lemma 2.1.3 and Proposition 2.1.1 apply with $A = n_j$.

Using the fact that $\mathbf{1}([0,1] \subseteq G_{n_j}) \le \mathbf{1}(h_0 \in G_{n_j})$ for any fixed h_0, we decompose $\mathscr{Z}_2$ iteratively as:

$$\mathscr{Z}_2 \le \mathscr{Z}_2(n_{j_0}) + \sum_{j_0\le j\le K} \mathbf{1}(\exists h \notin G_{n_j})\mathscr{Z}_2(n_{j+1}) + \mathbf{1}(\exists h \notin G_{n_{K+1}})\mathscr{Z}_2 \tag{2.1.23}$$

where $K = \lceil \sqrt{t}(1-q) \rceil$. Taking the q-th moment of both sides and using the subadditivity of $x \mapsto x^q$,

$$\mathbb{E}\{\mathscr{Z}_2^q\} \leq \mathbb{E}\{\mathscr{Z}_2(n_{j_0})\}^q + \sum_{j_0 \leq j \leq K} \mathbb{E}\{\mathbf{1}(\exists h \notin G_{n_j}) \mathscr{Z}_2(n_{j+1})^q\} + \mathbb{E}\{\mathbf{1}(\exists h \notin G_{n_{K+1}}) \mathscr{Z}_2^q\},$$

which by Hölder's inequality and Lemma 2.1.3 is

$$\begin{aligned} &\ll \mathbb{E}\{\mathscr{Z}_2(n_{j_0})\}^q + \sum_{j_0 \leq j \leq K} \mathbb{P}(\exists h \notin G_{n_j})^{1-q} \mathbb{E}\{\mathscr{Z}_2(n_{j+1})\}^q + \mathbb{P}(\exists h \notin G_{n_{K+1}})^{1-q} \mathbb{E}\{\mathscr{Z}_2\}^q, \\ &\ll \mathbb{E}\{\mathscr{Z}_2(n_{j_0})\}^q + \sum_{j_0 \leq j \leq K} e^{-2j} \mathbb{E}\{\mathscr{Z}_2(n_{j+1})\}^q + e^{-2(K+1)} \mathbb{E}\{\mathscr{Z}_2\}^q. \end{aligned}$$

Using Proposition 2.1.1 and the estimate in (2.1.11), we conclude that

$$\mathbb{E}\{\mathscr{Z}_2^q\} \ll e^{-2(K+1)} + \left(\sqrt{t}(1-q)\right)^{-q} \sum_{j \leq K} (j+1)^q e^{-2j} \ll \left(\sqrt{t}(1-q)\right)^{-q}. \qquad \square$$

2.1.3 Proof of Lemma 2.1.2

We use a discretisation argument inspired by that of [10] (Section 7), albeit in a simpler setting. A similar idea was used by Harper in [98]. To alleviate the notation, we will assume without loss of generality that $h = 0$ and write $S_j = S_j(\sigma_k)$ for the remainder of this section.

Fix $r = \lfloor A/4 \rfloor$, and let $Y_j := S_j - S_{j-1}$ denote the j-th increment of S_t when $j > r$. It will be helpful to abuse notation by defining $Y_r := S_r$. To discretise the range of the $(Y_j)_{r \leq j \leq t}$, let $\mathscr{T} \subseteq \mathbb{R}^{t-r+1}$ denote the set of all (disjoint) tuples $(u_r, ..., u_t)$ with $u_j \in \Delta_j^{-1}\mathbb{Z}$, where $\Delta_j = j^4$ for $j > r$ and $\Delta_r = 1$. This mesh size ensures that $\sum_j \Delta_j^{-1} < 2$.

We begin with the following trivial inclusion

$$\{S_t \in [w, w+1), 0 \in G_A\} \subseteq \bigcup_{\mathscr{T}} \{Y_j \in [u_j, u_j + \Delta_j^{-1})\} \cap \{S_t \in [w, w+1), 0 \in G_A\}. \tag{2.1.24}$$

Furthermore, by definition of G_A, any $\mathbf{u} = (u_j)_j \in \mathscr{T}$ for which the intersection on the right-hand side is non-empty must necessarily satisfy the following constraints:

$$w - 2 \leq w - \sum_j \Delta_j^{-1} \leq \sum_{j=r}^{t} u_j \leq w + 1 \tag{2.1.25}$$

$$L_A(\ell) - 2 \leq L_A(\ell) - \sum_j \Delta_j^{-1} \leq \sum_{j=r}^{\ell} u_j \leq U_A(\ell), \quad \forall r \leq \ell \leq t. \tag{2.1.26}$$

This also forces $|u_j| \le 100j$ for each $j \in [r,t]$. Letting $\mathscr{T}'(w)$ be the set of all such $\mathbf{u}$, it follows that one can replace $\mathscr{T}$ by $\mathscr{T}'(w)$ in (2.1.24). By a union bound and the fact that the Y_j are independent, we therefore conclude that

$$\mathbb{P}\big(S_t \in [w,w+1), 0 \in G_A\big) \le \sum_{(u_j)_j \in \mathscr{T}'(w)} \prod_{j=r}^{t} \mathbb{P}\big(Y_j \in [u_j, u_j + \Delta_j^{-1})\big). \tag{2.1.27}$$

We now compare the probabilities on the right-hand side to a Gaussian counterpart. Namely, let $(\mathscr{N}_j)_{j\ge1}$ denote a sequence of real, independent centred Gaussian random variables of variance 1/2 when $j \le r$, and $V_j(\sigma_k) = \sum_{\exp(e^{j-1})<p\le\exp(e^j)} \frac{1}{2p^{2\sigma_k}} + \frac{1}{8p^{4\sigma_k}}$ otherwise. Define the random walk $\mathscr{G}_\ell = \sum_{1000\le j\le\ell} \mathscr{N}_j$ for $\ell \ge 1000$. Beginning with the $j = r$ term in (2.1.27), we can use the estimate in Lemma 2.4.2 to get

$$\mathbb{P}\big(Y_r \in [u_r, u_r+1)\big) \ll \frac{e^{-u_r^2/r}}{\sqrt{r}} \asymp \mathbb{P}\big(\mathscr{G}_r \in [u_r, u_r+1)\big). \tag{2.1.28}$$

For $j > r$, the more precise estimate in Lemma 2.4.4 yields

$$\begin{aligned}\mathbb{P}\big(Y_j \in [u_j, u_j+\Delta_j^{-1})\big) &= \mathbb{P}\big(\mathscr{N}_j \in [u_j, u_j+\Delta_j^{-1})\big) + O\big(e^{-ce^{j/2}}\big)\\ &= \mathbb{P}\big(\mathscr{N}_j \in [u_j, u_j+\Delta_j^{-1})\big)\cdot\big(1+O(j^{-3})\big),\end{aligned}$$

where we have used the fact that $|u_j| \le 100j$ to make the error multiplicative. Since $\prod_j(1+j^{-3}) < \infty$,

$$\mathbb{P}\big(S_t \in [w,w+1), 0 \in G_A\big) \ll \sum_{(u_j)_j \in \mathscr{T}'(w)} \mathbb{P}\big(\mathscr{G}_r \in [u_r, u_r+\Delta_r^{-1}),\, \mathscr{N}_j \in [u_j, u_j+\Delta_j^{-1})\,\forall r < j \le t\big).$$

We now undo the discretisation. For any $(u_j)_j \in \mathscr{T}'(w)$, the event in the right-hand side implies that

$$\forall r \le \ell \le t, \quad \mathscr{G}_\ell \le U_A(\ell) + \sum_{j\le\ell} \Delta_j^{-1} \le U_A(\ell) + 2, \text{ and } |\mathscr{G}_t - w| \le 1 + \sum_{j=r}^{t} \Delta_j^{-1}.$$

By summing over $(u_j)_j \in \mathscr{T}'(w)$, we conclude that

$$\begin{aligned}\mathbb{P}\big(S_t \in [w,w+1), 0 \in G_A\big) &\ll \mathbb{P}\big(|\mathscr{G}_t - w| < 3, \mathscr{G}_\ell \le U_A(\ell)+2, \forall \ell \le t\big)\\ &\le \sum_{i=-3}^{3} \mathbb{P}\big(\mathscr{G}_t \in [w+i, w+i+1), \mathscr{G}_\ell \le U_A(\ell)+2, \forall \ell \le t\big).\end{aligned} \tag{2.1.29}$$

By Lemma 2.4.1, $(r - C_0 - 1)/2 + \sum_{j=r+1}^{\ell} V_j(\sigma_k) = (\ell - C_0 - 1)/2 + O(1)$ uniformly in $\ell \le t$ and $0 \le k \le \log\log\log x$. When $k = 0$, $\sigma_k = 1/2$ and $V_j(\sigma_k) = 1/2 + O(e^{-c\sqrt{e^{r-1}}})$ by the same lemma. Otherwise,

$$V_j(\sigma_k) = \frac{1}{2}\Big(\mathrm{Ei}(ve) - \mathrm{Ei}(v)\Big) + O\Big(e^{-c\sqrt{e^{r-1}}} + \sum_{n > \exp(e^{r-1})} \frac{1}{n^{7/4}}\Big), \text{ where } v := \frac{2k}{\log x} e^{j-1} \le \frac{2}{e},$$

uniformly in $r < j \le t$. The error term is contained in $[-1/4, 1/4]$ by taking a larger C_0 if need be (recall that $r \ge C_0$), while the main term equals $\frac{1}{2}\int_v^{ev}(e^s/s)\mathrm{d}s \in [1/2, e^2/2]$ since $v \in (0, 2/e)$. In both cases, we can use Proposition 2.4.8 with $\kappa = 1/4$ to estimate each summand in (2.1.29), and conclude that

$$\mathbb{P}\big(S_t \in [w, w+1), 0 \in G_A\big) \ll \frac{A\big(U_A(t) - w + C\big)}{t} \frac{e^{-w^2/t}}{\sqrt{t}} \tag{2.1.30}$$

for some $C > 0$. By partitioning the event $\{S_t > V\}$ according to $S_t - V \in [v, v+1)$ for $v \in \mathbb{Z} \cap [0, U_A(t) - V]$, we conclude that

$$\begin{aligned}
\mathbb{P}\big(S_t > V, 0 \in G_A\big) &\ll \frac{e^{-V^2/t}}{t^{3/2}} \sum_v A\big(U_A(t) - V - v + C\big) e^{-2(V/t)v} \\
&\ll \frac{e^{-V^2/t}}{t^{3/2}} \sum_v A\big(U_A(t) - V - v + C\big) e^{-v/2} \\
&\ll \frac{e^{-V^2/t}}{t^{3/2}} A\big(U_A(t) - V + C'\big)
\end{aligned}$$

for some constant $C' > 0$.

2.2 Sums of random multiplicative functions twisted by d_α for $\alpha > 1$

We now prove Theorem 2.0.1. As before, the first step is a reduction to moments of an Euler product integral, which proceeds similarly to the argument in Section 2.1.1. In this case, however, we do not give the details, as the presence of the divisor function d_α introduces some complications: in place of (2.1.4), the same approach would yield

$$\mathbb{E}\Bigg\{\Bigg(\sum_{\substack{1 \le n_1 \le x \\ p|n_1 \Longrightarrow p > y}} |d_\alpha(n_1)|^2 \Bigg| \frac{1}{\sqrt{x}} \sum_{\substack{n_2 \le x/n_1 \\ p|n_2 \Longrightarrow p \le y}} f(n_2)\Bigg|^2\Bigg)^q\Bigg\}, \tag{2.2.1}$$

so that the sum of $1/n_1$ in (2.1.6) is replaced by one of $|d_\alpha(n_1)|^2/n_1$. Such a sum is then expected to have order $(r \cdot \log y)^{-1}(\log x/\log y)^{\alpha^2-1} = (r \cdot \log y)^{-1}(\log\log x)^{2(\alpha^2-1)}$ rather

than $(r\cdot\log y)^{-1}$, which is fatal, since the saving we are after in Theorem 2.0.1 itself a power of $\log\log x$.

As shown by Gerspach [75], this issue can be circumvented by appealing to Harper's original argument in [98], which decomposes $\sum_{n\le x} f(n)$ according to the size of $P(n)$, over ranges $P_i < P(n) \le P_{i+1}$ for a suitable sequence of $x = P_0 > P_1 > \cdots > P_K$. We refer the reader to [75, Proposition 3.6] for details of how to adapt this argument to $\sum_{n\le x} d_\alpha(n)f(n)$. Using this result, Theorem 2.0.1 follows readily from Theorem 2.0.3, as we now show.

Proof of Theorem 2.0.1 assuming Theorem 2.0.3. By Proposition 3.6 in [75],

$$\mathbb{E}\Big\{\Big|\frac{1}{\sqrt{x}}\sum_{n\le x} d_\alpha(n)f(n)\Big|^{2q}\Big\} \ll \sum_{0\le k\le K}\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_{\mathbb{R}}\frac{|F_{x^{e^{-k}}}(\frac12-\frac{k}{\log x}+ih)|^{2\alpha}}{|\frac12-\frac{k}{\log x}+ih|^2}\mathrm{d}h\Big)^q\Big\}+1,$$

for $q\in[0,1/\alpha]$, where $K=\lfloor\log\log\log x\rfloor$ for simplicity. Using subadditivity of $x\mapsto x^q$ and the rotational invariance in law of the $f(p)$ (as in Equation (2.1.7)), the right-hand side is

$$\ll \sum_{0\le k\le K}\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_0^1 \big|F_{x^{e^{-k}}}(\tfrac12-\tfrac{k}{\log x}+ih)\big|^{2\alpha}\mathrm{d}h\Big)^q\Big\}+1, \tag{2.2.2}$$

provided $q>1/2$. It suffices to prove the claim for such q, since the bound for $q\in[0,1/2]$ follows from the one at $q=\frac12(\frac12+\frac{1}{\alpha})$ by Hölder's inequality.

By Theorem 2.0.3 (with $y=x^{e^{-k}}$, $\gamma=2\alpha$ and $\sigma=1/2-k/\log x$), the quantity in (2.2.2) is bounded by

$$\ll \Big(\sum_{k\ge0} e^{-kq(2(\alpha-1)+1)}\Big)\frac{(\log x)^{2q(\alpha-1)}}{(\log\log x)^{q(3\alpha/2)}(1-\alpha q)^2+1}. \qquad \square$$

2.2.1 Proof of Theorem 2.0.3

Fix $0\le k\le\lfloor\log\log\log x\rfloor$, and let $\sigma_k=\frac12-\frac{k}{\log x}$. Letting y be as in the statement of the theorem, we once again take

$$t=t(y)=\lfloor\log\log y\rfloor,$$

and note that $t\le\log\log x-k$ by the assumption on y. We also note that we can make y (and t) large if need be without loss of generality.

The strategy follows that of Proposition 2.0.2, with two main differences. The first is that a much more precise upper barrier is required. For a given t, we pick

$$U_A^{\max}(j) := A + j\Big(1-\frac34\frac{\log t}{t}\Big)+\mathscr{C}\cdot\log\big(1+j\wedge(t-j)\big),\quad (j\ge\lfloor A/4\rfloor) \tag{2.2.3}$$

where $\mathscr{C}$ is any large enough constant (say, $\mathscr{C} = 10^3$), and $U_A^{\max}(j) = \infty$ for smaller j. Such a barrier is required because unlike the integral in (2.1.18), which is dominated by the range $V = t - O(\sqrt{t})$, the corresponding Gaussian integral when $\gamma > 2$ (cf. (2.2.6)) is dominated by the very largest values of V, namely $V = U_A^{\max}(t) + O(1)$. To obtain a sharp bound, this barrier must then be chosen so as to accurately reflect the true size of $\max_{h\in[0,1]} S_t(\sigma_k + ih)$ at time t. Indeed, $\max_{h\in[0,1]} S_j(\sigma_k + ih)$ is expected to fluctuate around $j - \frac{3}{4}\log j$ at a given time $j \le t$, and the barrier in (2.2.3) allows it to do so within a logarithmic bump at intermediate times, while ensuring that $\max_{h\in[0,1]} S_t(\sigma_k + ih)$ does not exceed $t - \frac{3}{4}\log t$ by more than A.

Having refined the upper barrier, we will need a version of Lemma 2.1.3 for the good set

$$G_{A,\sigma_k}^{\max} = \big\{h \in [0,1] : S_j(\sigma_k + ih) \in [L_A(j), U_A^{\max}(j)],\ \forall j \le t\big\}, \tag{2.2.4}$$

the proof of which is rather involved and thus postponed to Section 2.2.2.

Proposition 2.2.1. *Uniformly in all large t and $4C_0 < A \le t$,*

$$\mathbb{P}\big(\exists h \notin G_{A,\sigma_k}^{\max}\big) \ll A\exp\Big(-2A - \frac{A^2}{t} + \frac{3A}{2}\frac{\log t}{t}\Big).$$

Proof. See Section 2.2.2. □

The second way in which the proof differs from that of Proposition 2.0.2 is that if we let

$$\mathscr{Z}_{\gamma,\sigma_k} := \frac{1}{e^t}\int_0^1 e^{\gamma S_t(\sigma_k+ih)}\mathrm{d}h,$$

then the trivial bound for $\mathbb{E}\{\mathscr{Z}_{\gamma,\sigma_k}^q\}$ is not sharp in the leading order when $\gamma > 2$. Indeed, while Fubini's theorem and a Laplace transform estimate (Lemma 2.4.2) yield

$$\mathbb{E}\big\{\mathscr{Z}_{\gamma,\sigma_k}^q\big\} \le \mathbb{E}\big\{\mathscr{Z}_{\gamma,\sigma_k}\big\}^q \ll e^{qt(\gamma^2/4-1)} \tag{2.2.5}$$

for $q < 2/\gamma$, we will show that the exponent in the right-hand side can be brought down to $qt(\gamma - 2)$. Compared to the argument in Section 2.1, the interpolation argument below runs up to a much larger final value n_{K+1} to compensate for this discrepancy in the leading order.

Proof of Theorem 2.0.3. For any $4C_0 < A \le t(\gamma-2)/4$, define

$$\mathscr{Z}_{\gamma,\sigma_k}(A) := \frac{1}{e^t}\int_0^1 e^{\gamma S_t(\sigma_k+ih)}\mathbf{1}\big(h \in G_{A,\sigma_k}^{\max}\big)\mathrm{d}h.$$

Proceeding as in Equation (2.1.15), Fubini's theorem yields

$$\mathbb{E}\mathscr{Z}_{\gamma,\sigma_k}(A) \ll O(1) + \mathbb{E}\Big\{\frac{1}{e^t}\int_0^1 \int_{t/4}^{U_A^{\max}(t)} e^{\gamma V}\mathbf{1}\big(S_t(\sigma_k+ih)>V, h\in G_{A,\sigma_k}^{\max}\big)\mathrm{d}V\,\mathrm{d}h\Big\}$$
$$\ll O(1) + \frac{1}{e^t}\int_{t/4}^{U_A^{\max}(t)} e^{\gamma V}\mathbb{P}\big(S_t(\sigma_k)>V, h\in G_{A,\sigma_k}^{\max}\big)\mathrm{d}V.$$

By Lemma 2.1.2, this integral is bounded by a constant times

$$\frac{1}{e^t}\int_0^{U_A^{\max}(t)} \frac{e^{\gamma V - V^2/t}}{\sqrt{t}}\frac{A\big(U_A^{\max}(t)-V+C\big)}{t}\mathrm{d}V, \tag{2.2.6}$$

which, by the substitution $u = U_A^{\max}(t) - V$, is bounded by

$$\ll A\frac{e^{\gamma U_A^{\max}(t) - U_A^{\max}(t)^2/t}}{t^{3/2}}\frac{1}{e^t}\int_0^\infty (u+C)e^{(2U_A^{\max}(t)/t-\gamma)u-u^2/t}\mathrm{d}u$$
$$\ll \big(Ae^{(\gamma-2)A-\frac{A^2}{t}+\frac{3A}{2}\frac{\log t}{t}}\big)\frac{e^{(\gamma-2)t}}{t^{3\gamma/4}}\cdot\int_0^\infty (u+C)e^{(2-\gamma+2A/t)u}\mathrm{d}u$$
$$\ll \big(Ae^{(\gamma-2)A-\frac{A^2}{t}+\frac{3A}{2}\frac{\log t}{t}}\big)\frac{e^{(\gamma-2)t}}{t^{3\gamma/4}}, \tag{2.2.7}$$

uniformly in $4C_0 < A \le t(\gamma-2)/4$. We now use this estimate to bound $\mathbb{E}\big\{\mathscr{Z}_{\gamma,\sigma_k}^q\big\}$. Using the decomposition in (2.1.23) with $n_j = j$ and $j_0 = 4C_0+1$, we can bound $\mathbb{E}\big\{\mathscr{Z}_{\gamma,\sigma_k}^q\big\}$ by

$$\le \Big(\mathbb{E}\big\{\mathscr{Z}_{\gamma,\sigma_k}(n_{j_0})\big\}^q + \sum_{j_0\le j\le K}\mathbb{P}\big(\exists h\notin G_{n_j,\sigma_k}^{\max}\big)^{1-q}\mathbb{E}\big\{\mathscr{Z}_{\gamma,\sigma_k}(n_{j+1})\big\}^q\Big)$$
$$+\mathbb{E}\big\{\mathbf{1}\big(\exists h\notin G_{n_{K+1},\sigma_k}^{\max}\big)\mathscr{Z}_{\gamma,\sigma_k}^q\big\} \tag{2.2.8}$$

for any $K \ge j_0$. We pick $K = \lfloor t(\gamma-2)/4\rfloor - 1$, so that $4C_0 < n_j \le t(\gamma-2)/4$ for each $j_0 \le j \le K+1$. Proposition 2.2.1 and the bound in Equation (2.2.7) imply that the terms in brackets are

$$\ll \frac{e^{tq(\gamma-2)}}{t^{3\gamma q/4}}\sum_{j\ge1}(j+1)e^{-j^2/t - j(2-\gamma q-\frac{3}{2}\frac{\log t}{t})}. \tag{2.2.9}$$

We will show that this sum is $\ll \min((2-\gamma q)^{-2}, t)$, which implies the claim since

$$\frac{e^{tq(\gamma-2)}}{t^{3\gamma q/4}}\min\big((2-\gamma q)^{-2}, t\big) \ll \frac{e^{tq(\gamma-2)}}{t^{3\gamma q/4}(2-\gamma q)^2+1}.$$

We split according to whether $(2-\gamma q) \ge t^{-1/2}$ or $< t^{-1/2}$. In the former case, the minimum equals $(2-\gamma q)^{-2}$, and $(2-\gamma q - \frac{3}{2}\frac{\log t}{t}) \ge \frac{1}{2\sqrt{t}} > 0$ for large enough t. The sum in (2.2.9) is thus bounded by $(1-e^{-(2-\gamma q-\frac{3}{2}\frac{\log t}{t})})^{-2}$, which is $\ll (1-e^{-1})^{-2} \ll 1$ if $(2-\gamma q-\frac{3}{2}\frac{\log t}{t}) > 1$,

and $\ll (2-\gamma q-\frac{3}{2}\frac{\log t}{t})^{-2}$ otherwise; either way, this is $\ll (2-\gamma q)^{-2}$. For $(2-\gamma q) < t^{-1/2}$, the sum is $\ll t$ by completing the square:

$$\begin{aligned}\sum_{j\geq 1}(j+1)e^{-j^2/t-j(2-\gamma q-\frac{3}{2}\frac{\log t}{t})} &\leq \sum_{j\geq 1}(j+1)e^{-\frac{1}{t}(j-\frac{3}{4}\log t)^2+\frac{9(\log t)^2}{16t}}\\ &\ll \sum_{k}(|k|+\log t+1)e^{-k^2/t}\\ &\ll t.\end{aligned}$$

For the remaining term in (2.2.8), we use Hölder's inequality, the bound in Equation (2.2.5), and Proposition 2.2.1 to write

$$\ll \mathbb{P}\big(\exists h \notin G^{\max}_{n_{K+1},\sigma_k}\big)^{1-q}\mathbb{E}\big\{\mathscr{Z}_{\gamma,\sigma_k}\big\}^q \ll e^{qt(\gamma^2/4-1)}\cdot\Big((K+1)e^{(\frac{3}{2}\frac{\log t}{t}-2)(K+1)-(K+1)^2/t}\Big)^{1-q}.$$

By definition of K, this is

$$\ll t^{1+\frac{3}{2}\frac{(\gamma-2)}{4}(1-q)}e^{t\left((\gamma^2/4-1)q-(1-q)\left(\frac{(\gamma-2)}{2}+\frac{(\gamma-2)^2}{16}\right)\right)} \ll t^2 e^{qt(\gamma-2)}e^{\frac{f_\gamma(q)}{16}\cdot t},$$

for $\gamma\in(2,4)$, $q\in[0,2/\gamma]$, where $f_\gamma(q)=(\gamma-2)(q(5\gamma-2)-(\gamma+6))$. The claim then follows from the fact that f_γ is an increasing function of q satisfying $f_\gamma(2/\gamma)=-\gamma^{-1}(\gamma-2)^3<0$.

□

2.2.2 Proof of Proposition 2.2.1

To begin with, note that by the proof of Lemma 2.1.3 and the fact that $U_A(j) < U_A^{\max}(j)$ for $j \leq t/2$,

$$\begin{aligned}\sum_{j\leq t}\mathbb{P}\Big(\exists h : S_j(\sigma_k+ih) < L_A(j)\Big) &\ll \sum_{A/4\leq j\leq t} e^{j-A^2/j-400j+40A} \ll e^{-2A-A^2/t},\\ \sum_{j\leq t/2}\mathbb{P}\Big(\exists h : S_j(\sigma_k+ih) > U_A^{\max}(j)\Big) &\ll e^{-2A-A^2/t},\end{aligned}$$

uniformly in k and t. It therefore suffices to show that with the same uniformity,

$$\begin{aligned}\sum_{t/2<j\leq t}\mathbb{P}\Big(S_\ell(\sigma_k)\in[L_A(\ell),U_A^{\max}(\ell)]\,\forall \ell\leq j-1,\ \max_{h\in[0,1]}S_j(\sigma_k+ih) > U_A^{\max}(j)\Big)&\\ \ll Ae^{-2A-\frac{A^2}{t}+\frac{3A}{2}\frac{\log t}{t}}.&\end{aligned} \tag{2.2.10}$$

By translation invariance in law and a union bound, the left-hand side is bounded by

$$\sum_{t/2<j\leq t} e^j\cdot\mathbb{P}\Big(S_\ell(\sigma_k)\in[L_A(\ell),U_A^{\max}(\ell)]\,\forall\ell\leq j-1,\ \max_{h\in[0,e^{-j})}S_j(\sigma_k+ih) > U_A^{\max}(j)\Big).$$

We then split the event in each summand into two: for $\max_{h\in[0,e^{-j})} S_j(\sigma_k+ih)$ to cross $U_A^{\max}(j)$, either $S_j(\sigma_k) > U_A^{\max}(j)$, or $S_j(\sigma_k) \le U_A^{\max}(j)$ while $\max_{h\in[0,e^{-j})} S_j(\sigma_k+ih) - S_j(\sigma_k) > U_A^{\max}(j) - S_j(\sigma_k)$. This dichotomy was used by Arguin, Dubach and Hartung in [13] to prove Proposition 2.2.1 for a Gaussian analogue of S_j. In the first case, we have

$$\sum_{t/2<j\le t} e^j \cdot \mathbb{P}\Big(S_\ell(\sigma_k)\in[L_A(\ell),U_A^{\max}(\ell)]\,\forall \ell\le j-1, S_j(\sigma_k) > U_A^{\max}(j)\Big) \tag{2.2.11}$$

$$\le \sum_{\substack{L_A(j-1)\le u\le U_A^{\max}(j-1)\\ t/2<j\le t}} e^j \cdot \mathbb{P}\Big(S_\ell(\sigma_k)\in[L_A(\ell),U_A^{\max}(\ell)]\,\forall \ell, S_{j-1}(\sigma_k)\in[u,u+1)\Big) \tag{2.2.12}$$

$$\cdot\, \mathbb{P}\big(Y_j(\sigma_k) > U_A^{\max}(j)-u-1\big)$$

where $Y_j = S_j - S_{j-1}$. On the one hand, a Chernoff bound and the estimate in Equation (2.4.4) (summing only over $p\in(\exp(e^{j-1}),\exp(e^j)]$) yield

$$\mathbb{P}\big(Y_j(\sigma_k) > U_A^{\max}(j)-u-1\big) \ll e^{-10(U_A^{\max}(j)-u-1)} \ll e^{-10(U_A^{\max}(j-1)-u)}.$$

On the other hand, the proof of Lemma 2.1.2 (cf. Equation (2.1.30)) gives

$$\mathbb{P}\Big(S_\ell(\sigma_k)\in[L_A(\ell),U_A^{\max}(\ell)]\,\forall\ell\le j-1, S_{j-1}(\sigma_k)\in[u,u+1)\Big) \ll \frac{A\big(U_A^{\max}(j-1)-u+C\big)}{j-1}\frac{e^{-u^2/(j-1)}}{\sqrt{j-1}}.$$

(Note that this estimate holds for $u<0$ as well: in that case, we simply discard the first event and use Lemma 2.4.2 to get the bound $\ll e^{-u^2/(j-1)}/\sqrt{j-1}$, which is smaller than the right-hand side.) Using the change of variables $v = U_A^{\max}(j-1)-u$, it follows that the sum in (2.2.11) is

$$\ll A\sum_{t/2<j\le t}\frac{e^j}{j^{3/2}}\sum_{v\ge0}(v+C)e^{-10v-(U_A^{\max}(j-1)-v)^2/(j-1)} \tag{2.2.13}$$

$$\ll A\sum_{t/2<j\le t}\frac{e^j}{j^{3/2}}e^{-U_A^{\max}(j-1)^2/(j-1)}\sum_{v\ge0}(v+C)e^{-5v}. \tag{2.2.14}$$

Inserting the estimates $\sum_{v\ge0}(v+C)e^{-5v}\le 2C$ and

$$-\frac{U_A^{\max}(j-1)^2}{j-1} \le -\frac{A^2}{t} - 2A - (j-1) + \frac{3}{2}\log(j-1) + \frac{3A}{2}\frac{\log t}{t} - \mathscr{C}\cdot\log\big(1+(t-j+1)\big),$$

we conclude that

$$\begin{aligned} &A\sum_{t/2<j\le t}\frac{e^j}{j^{3/2}}e^{-U_A^{\max}(j-1)^2/(j-1)}\sum_{v\ge0}(v+C)e^{-5v}\\ &\ll Ae^{-2A-\frac{A^2}{t}+\frac{3A}{2}\frac{\log t}{t}}\sum_{j\le t}(t-j+1)^{-20}\\ &\ll Ae^{-2A-\frac{A^2}{t}+\frac{3A}{2}\frac{\log t}{t}} \end{aligned}$$

having picked $\mathscr{C} = 10^3$.

What remains is to show that

$$\sum_{t/2<j\le t} e^j \cdot \mathbb{P}\Big(S_\ell(\sigma_k) \in [L_A(\ell), U_A^{\max}(\ell)]\ \forall \ell \le j,\ \max_{h\in[0,e^{-j})} S_j(\sigma_k + ih) > U_A^{\max}(j)\Big) \tag{2.2.15}$$

satisfies the same bound. To this end, we partition according to $S_j(\sigma_k) \in [u, u+1)$ to get that the above is

$$\le \sum_{t/2<j\le t} \sum_{L_A(j)\le u\le U_A^{\max}(j)} e^j \cdot \mathbb{P}\Big(E_{j,u}^{(\sigma_k)},\ \max_{h\in[0,e^{-j})} S_j(\sigma_k + ih) - S_j(\sigma_k) > U_A^{\max}(j) - u - 1\Big), \tag{2.2.16}$$

where we used the shorthand

$$E_{j,u}^{(\sigma_k)} := \big\{ S_\ell(\sigma_k) \in [L_A(\ell), U_A^{\max}(\ell)]\ \forall \ell \le j,\ S_j(\sigma_k) \in [u, u+1)\big\}. \tag{2.2.17}$$

Our goal will be to show that each summand in (2.2.16) satisfies the bound

$$\ll e^{j-5(U_A^{\max}(j)-u)} \frac{A\big(U_A^{\max}(j) - u + C\big)}{j} \frac{e^{-u^2/j}}{\sqrt{j}}.$$

We will assume that $u < U_A^{\max}(j) - 1$ without loss of generality, since the desired bound otherwise follows directly by applying the bound in (2.1.30) to $\mathbb{P}(E_{j,u}^{(\sigma_k)})$.

For the remaining u, we discretise the maximum over $h \in [0, e^{-j})$ using a chaining argument similar to the proof of Lemma 2.4.6. Following the argument therein from Equation (2.4.16) to (2.4.18) (ignoring the sum over q and the events B_q), we get the bound

$$\le \sum_{m\ge 0} \sum_{v\in H_m} e^j \cdot 2\mathbb{P}\Big(E_{j,u}^{(\sigma_k)}, S_j(\sigma_k + iv) - S_j(\sigma_k + iv_*) > \frac{U_A^{\max}(j) - u - 1}{e(m+1)^2}\Big), \tag{2.2.18}$$

where v_* denotes the closest point to v in H_{m+1} which is not equal to v. (Should there be two such points, we let v_* denote the smaller of the two.) By a Chernoff bound, this is

$$\le \sum_{m\ge 0} \sum_{v\in H_m} 2\exp\Big(j - \lambda_v \frac{U_A^{\max}(j) - u - 1}{(e(m+1)^2)}\Big) \mathbb{E}\Big\{ e^{\lambda_v(S_j(\sigma_k+iv) - S_j(\sigma_k+iv_*))}\Big\} \cdot \mathbb{Q}_v\big(E_{j,u}^{(\sigma_k)}\big), \tag{2.2.19}$$

where $\lambda_v = 100(m+1)^4$ for $v \in H_m$, and $\mathbb{Q}_v$ is the tilted measure defined through

$$\frac{\mathrm{d}\mathbb{Q}_v}{\mathrm{d}\mathbb{P}} = \frac{\exp\big(\lambda_v(S_j(\sigma_k + iv) - S_j(\sigma_k + iv_*))\big)}{\mathbb{E}\Big\{ \exp\big(\lambda_v(S_j(\sigma_k + iv) - S_j(\sigma_k + iv_*))\big)\Big\}}.$$

To bound the expectation in (2.2.19), note that $|v-v_*|^{-1}e^{-j} \asymp e^m$, and therefore that there exists a constant $c>0$ such that $\lambda_v \le c|v-v_*|^{-1}e^{-j}$ for all $v \in \cup_m H_m$. By the bound in (2.4.13), we thus get

$$\mathbb{E}\Big\{e^{\lambda_v(S_j(\sigma_k+iv)-S_j(\sigma_k+iv_*))}\Big\} \ll 1$$

uniformly in all parameters, and this can therefore be absorbed into the implied constant. Using the fact that $\#H_m \le e^m+1$, it follows that (2.2.19) is

$$\ll e^j \sup_{v\in\cup_m H_m} \mathbb{Q}_v\big(E_{j,u}^{(\sigma_k)}\big) \sum_{m\ge 0} e^{m-10(m+1)^2(U_A^{\max}(j)-u-1)} \ll e^{j-5(U_A^{\max}(j)-u)} \sup_{v\in\cup_m H_m} \mathbb{Q}_v\big(E_{j,u}^{(\sigma_k)}\big). \tag{2.2.20}$$

and we therefore need uniform estimates for the $\mathbb{Q}_v$-probability.

To do so, we make the crucial observation that for any such v, the independence of $(f(p))_p$ persists under $\mathbb{Q}_v$. Recalling the definition of $E_{j,u}^{(\sigma_k)}$ in (2.2.17), we can therefore compare $\mathbb{Q}_v(E_{j,u}^{(\sigma_k)})$ to a Gaussian counterpart by proceeding as in the proof of Lemma 2.1.2 (up to (2.1.29)), and leveraging the $\mathbb{Q}_v$-versions of Lemmas 2.4.2 and 2.4.4. This yields

$$\mathbb{Q}_v\big(E_{j,u}^{(\sigma_k)}\big) \ll \mathbb{P}\Big(|\mathscr{G}_j+\mu_j^{(v)}-u|<3,\ \mathscr{G}_\ell+\mu_\ell^{(v)} \le U_A^{\max}(\ell)+2, \forall \ell \le j\Big), \tag{2.2.21}$$

where $(\mathscr{G}_\ell)_\ell$ is the Gaussian random walk from Section 2.1.3, and

$$\mu_\ell^{(v)} := \sum_{s\le\ell} \nu_s^{(v)}, \quad \nu_s^{(v)} := \mathbb{E}_{\mathbb{Q}_v}\big\{S_s(\sigma_k)-S_{s-1}(\sigma_k)\big\}.$$

In other words, $\mathbb{Q}_v$ has the effect of adding a *drift* to the random walk $\mathscr{G}_\ell$. However, since

$$\sup_k \sup_{j\le t} \sup_{v\in\cup_m H_m} |\mu_j^{(v)}| < D$$

for some absolute constant $D>0$ by (2.4.8), this will essentially have no effect on the final bound. Indeed, the right-hand side in (2.2.21) is bounded by

$$\begin{aligned}&\ll \mathbb{P}\Big(\mathscr{G}_j \in [u-D', u+D'), \mathscr{G}_\ell \le U_A^{\max}(\ell)+D', \forall \ell \le j\Big)\\ &\le \sum_{n\in[u-D',u+D']\cap\mathbb{Z}} \mathbb{P}\Big(\mathscr{G}_j\in[n,n+1), \mathscr{G}_\ell \le U_A^{\max}(\ell)+D', \forall \ell\le j\Big),\end{aligned}$$

for some $D'>0$, which we can bound by

$$\ll \frac{A\big(U_A^{\max}(j)-u+C'\big)}{j}\frac{e^{-u^2/j}}{\sqrt{j}}$$

using Proposition 2.4.8, for some new constant C' depending on D'. By (2.2.20), we conclude that (2.2.16) is

$$\ll \sum_{t/2<j\le t} e^{j-5(U_A^{\max}(j)-u)} \frac{A\big(U_A^{\max}(j)-u+C'\big)}{j}\frac{e^{-u^2/j}}{\sqrt{j}}.$$

which can be estimated as in (2.2.13). □

2.2.3 Corollaries

We end this section by showing how Corollaries 2.0.4 and 2.0.5 follow from the argument in the previous section. To get Corollary 2.0.4, note that there exists a constant $C > 0$ such that

$$\mathbb{P}\Big(\max_{h\in[0,1]} |F_x(1/2+ih)| > \frac{\log x}{(\log\log x)^{3/4}} e^y\Big) \le \mathbb{P}\Big(\max_{h\in[0,1]} S_t(1/2+ih) > t - \frac{3}{4}\log t + C + y\Big) \tag{2.2.22}$$

by (2.1.9), where $t = t(x) = \lfloor \log\log x \rfloor$. By Proposition 2.2.1, this probability is

$$\ll ye^{-2y-y^2/t} + \mathbb{P}\Big(\max_{h\in[0,1]} S_t(1/2+ih) > t - \frac{3}{4}\log t + C + y \cap \big\{[0,1] \subseteq G^{\max}_{y+C,1/2}\big\}\Big),$$

and the second probability equals zero by definition of $G^{\max}_{y+C,1/2}$.

For Corollary 2.0.5, we let E denote the event therein and bound the probability of its complement by

$$\mathbb{P}\big(\exists h \notin G^{\max}_{(C+\log A),1/2}\big) + \mathbb{P}\Big((\neg E) \cap \big\{[0,1] \subseteq G^{\max}_{(C+\log A),1/2}\big\}\Big),$$

for the same constant C as in (2.2.22). The first term is $\ll (\log A)/A$ by Proposition 2.2.1. For the second, Markov's inequality and Fubini's theorem yield

$$\le \frac{\mathbb{P}\Big(S_t(1/2) > t - \frac{3}{4}\log t + C + y, 0 \in G^{\max}_{(C+\log A),1/2}\Big)}{(\log x)^{-1}A|\log A - y|e^{-2y-y^2/\log\log x}},$$

which is 0 when $y > \log A$, and $\ll (\log A)/A$ by Lemma 2.1.2 otherwise.

2.3 Low pseudomoments of the Riemann zeta function

Building on the results in the previous section, we now prove Theorem 2.0.6. In this section, we let $\sigma_k := 1/2 - 2(k+1)/\log x$, and begin by using Proposition 6.6 in [75], by which

$$\Psi_{2q,\alpha}(x) \ll (\log x)^{(q\alpha)^2} + \frac{1}{(\log x)^q}\sum_{k\le K}\mathbb{E}\Big\{\Big(\int_0^\infty \frac{|F_{x^{e^{-(k+1)}}}(\sigma_k+ih)|^{2\alpha}}{|2(k+1)/\log x + ih|^2}\mathrm{d}h\Big)^q\Big\}$$

for $K = \lfloor \log\log\log x \rfloor$. Noting that $\alpha^2 q < 2q(\alpha-1)$ when $\alpha \in (1,2)$ and $q \in (0, 2(\alpha-1)/\alpha^2)$, it suffices to show that the sum over k satisfies the claimed bound.

Let $F^{(k)} := F_{x^{e^{-(k+1)}}}$ for simplicity. By subadditivity of $x \mapsto x^q$, this sum is

$$\le \sum_{k\le K}\frac{1}{(\log x)^q}\mathbb{E}\Big\{\Big(\int_0^1 \frac{|F^{(k)}(\sigma_k+ih)|^{2\alpha}}{\big|\frac{2(k+1)}{\log x} + ih\big|^2}\mathrm{d}h\Big)^q\Big\} \tag{2.3.1}$$

$$+ \sum_{k\le K}\frac{1}{(\log x)^q}\mathbb{E}\Big\{\Big(\int_1^\infty \frac{|F^{(k)}(\sigma_k+ih)|^{2\alpha}}{h^2}\mathrm{d}h\Big)^q\Big\},$$

and Hölder's inequality and the translation invariance in law of $(F(\sigma_k+it), t\in[0,1])$ yield

$$\begin{aligned}&\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_1^\infty \frac{|F^{(k)}(\sigma_k+ih)|^{2\alpha}}{h^2}\mathrm{d}h\Big)^q\Big\}\\&\le \mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_1^\infty \frac{|F^{(k)}(\sigma_k+ih)|^{2\alpha}}{h^2}\mathrm{d}h\Big)^{q^*}\Big\}^{q/q^*}\\&\le \Big(\sum_{n\ge1}\frac{1}{n^{2q^*}}\Big)\mathbb{E}\Big\{\Big(\frac{1}{\log x}\int_0^1 |F^{(k)}(\sigma_k+ih)|^{2\alpha}\mathrm{d}h\Big)^{q^*}\Big\}^{q/q^*}\end{aligned}$$

for any $q^*\in(1/2,1/\alpha)$ and $k\le K$. The sum over n being finite, we conclude using Theorem 2.0.3 that[1]

$$\begin{aligned}\sum_{k\le K}\frac{1}{(\log x)^q}\mathbb{E}\Big\{\Big(\int_1^\infty \frac{|F^{(k)}(\sigma_k+ih)|^{2\alpha}}{h^2}\mathrm{d}h\Big)^q\Big\} &\ll \sum_{k\le K}\Big(\frac{e^{-kq^*}(\log x)^{2(\alpha-1)q^*}}{(\log\log x)^{3q^*\alpha/2}}\Big)^{q/q^*}\\&= \frac{(\log x)^{2(\alpha-1)q}}{(\log\log x)^{3q\alpha/2}}.\end{aligned}$$

To handle the first sum in (2.3.1), we decompose the range of integration into e-adic intervals and once again use subadditivity of $x\mapsto x^q$. This yields

$$\begin{aligned}&\sum_{k\le K}\frac{1}{(\log x)^q}\mathbb{E}\Big\{\Big(\int_0^1 \frac{|F^{(k)}(\sigma_k+ih)|^{2\alpha}}{|\frac{2(k+1)}{\log x}+ih|^2}\mathrm{d}h\Big)^q\Big\}\\&\ll \sum_{k\le K}\sum_{j\le J}\frac{T_{j-1}^{-2q}}{(\log x)^q}\mathbb{E}\Big\{\Big(\int_{T_{j-1}}^{T_j}|F^{(k)}(\sigma_k+ih)|^{2\alpha}\mathrm{d}h\Big)^q\Big\}\end{aligned}$$

where $T_j=(e^{2(k+1)}/\log x)e^j$ for each $j\ge1$, and $T_{-1}:=0$. The sum is taken up to $J_k=J(k,x)$, defined as the smallest integer for which $e^{J+2(k+1)}/\log x\ge1$.

To bound each summand, we make the observation that on $[T_{j-1},T_j]$, the contribution to $|F^{(k)}(\sigma_k+ih)|^{2\alpha}$ coming from primes up to $x_j=e^{1/T_j}$ is roughly constant over the range of integration; it should approximately equal $|F_{x_j}(\sigma_j)|^{2\alpha q}$, as suggested by Lemma 2.4.6. To leverage this fact, we introduce the a family of tilted measures $(\mathbb{P}_j)_{j\le J_k}$, defined through

$$\frac{\mathrm{d}\mathbb{P}_j}{\mathrm{d}\mathbb{P}}=\frac{|F_{x_j}(\sigma_k)|^{2\alpha q}}{\mathbb{E}\{|F_{x_j}(\sigma_k)|^{2\alpha q}\}}.$$

For each $j\le J_k$,

$$\begin{aligned}&\mathbb{E}\Big\{\Big(\int_{T_{j-1}}^{T_j}|F^{(k)}(\sigma_k+ih)|^{2\alpha}\mathrm{d}h\Big)^q\Big\}\\&=\mathbb{E}\Big\{\big|F_{x_j}(\sigma_k)\big|^{2\alpha q}\Big\}\cdot\mathbb{E}_{\mathbb{P}_j}\Big\{\Big(\int_{T_{j-1}}^{T_j}\frac{|F^{(k)}(\sigma_k+ih)|^{2\alpha}}{|F_{x_j}(\sigma_k+ih)|^{2\alpha}}\mathscr{E}_T(h)^{2\alpha}\mathrm{d}h\Big)^q\Big\}\end{aligned}$$

[1]Note that the shift from 1/2 here is by $2(k+1)/\log x$ rather than $(k+1)/\log x$, but one straightforwardly checks that the proof of Theorem 2.0.3 remains valid with this choice.

where $\mathscr{E}_T(h) := |F_{x_j}(\sigma_k + ih)|/|F_{x_j}(\sigma_k)|$ can be seen as an error term. We can then condition on the σ-algebra generated by $(f(p))_{x_j<p\le x^{e^{-(k+1)}}}$ and use Jensen's inequality to get the bound

$$\ll \mathbb{E}\Big\{ \big|F_{x_j}(\sigma_k)\big|^{2\alpha q} \Big\} \cdot \mathbb{E}\Big\{ \Big(\int_{T_{j-1}}^{T_j} \frac{|F^{(k)}(\sigma_k + ih)|^{2\alpha}}{|F_{x_j}(\sigma_k + ih)|^{2\alpha}} \mathbb{E}_{\mathbb{P}_j}\big\{ \mathscr{E}_T(h)^{2\alpha} \big\} \mathrm{d}h \Big)^q \Big\},$$

noting that $(|F^{(k)}(\sigma_k + ih)|/|F_{x_j}(\sigma_k + ih)|)_{h\in[T_{j-1},T_j]}$ is measurable with respect to said σ-algebra, and that its law under $\mathbb{P}_j$ is the same as under $\mathbb{P}$ by said independence. By the moment bound in (2.4.12),

$$\sup_{j\le J_k} \sup_{h<T_j} \mathbb{E}_{\mathbb{P}_j}\big\{ \mathscr{E}_T(h)^{2\alpha} \big\} \ll 1,$$

Since $\mathbb{E}\{|F_{x_j}(\sigma_k)|^{2\alpha q}\} \ll T_j^{-(\alpha q)^2}$ uniformly in j and k by Lemma 2.4.2, it follows that

$$\begin{aligned} &\sum_{\substack{k\le K \\ j\le J_k}} \frac{T_{j-1}^{-2q}}{(\log x)^q} \mathbb{E}\Big\{ \Big(\int_{T_{j-1}}^{T_j} |F^{(k)}(\sigma_k + ih)|^{2\alpha} \mathrm{d}h \Big)^q \Big\} \\ &\ll \sum_{\substack{k\le K \\ j\le J_k}} \frac{T_j^{-2q-(\alpha q)^2}}{(\log x)^q} \mathbb{E}\Big\{ \Big(\int_{T_{j-1}}^{T_j} \frac{|F^{(k)}(\sigma_k + ih)|^{2\alpha}}{|F_{x_j}(\sigma_k + ih)|^{2\alpha}} \mathrm{d}h \Big)^q \Big\} \end{aligned}$$

We're left with the task of bounding

$$\mathbb{E}\Big\{ \Big(\int_{T_{j-1}}^{T_j} \frac{|F^{(k)}(\sigma_k + ih)|^{2\alpha}}{|F_{x_j}(\sigma_k + ih)|^{2\alpha}} \mathrm{d}h \Big)^q \Big\} = \mathbb{E}\Big\{ \Big(\int_{0}^{(1-1/e)T_j} \frac{|F^{(k)}(\sigma_k + ih)|^{2\alpha}}{|F_{x_j}(\sigma_k + ih)|^{2\alpha}} \mathrm{d}h \Big)^q \Big\}. \quad (2.3.2)$$

To that end, we define the *shifted* field

$$S_k^{(j)}(\sigma_k + ih) = S_{k+t_j}\big(\sigma_k + ie^{t_j}h\big) - S_{t_j}\big(\sigma_k + ie^{t_j}h\big), \quad t_j := \log\log x_j,$$

where $S_k(s)$ is defined as in (2.1.8). By discarding higher order terms in the expansion of $\log|F^{(k)}/F_{x_j}|$ (cf. Equation (2.1.9) and using the change of variables $h \mapsto h/((1-1/e)T_j)$, the expectation in (2.3.2) is

$$\le T_j^q \cdot \mathbb{E}\Big\{ \Big(\int_0^1 e^{2\alpha S_{t-t_j}^{(j)}(\sigma_k+ih)} \mathrm{d}h \Big)^q \Big\}, \quad \text{where } t = \log\log x - (k+1).$$

This can then be studied exactly as in the proof of Theorem 2.0.3, replacing every occurence of S with $S^{(j)}$. Indeed, $S_{t-t_j}^{(j)}$ is now a sum of $t - t_j$ independent increments with variances

$$\sum_{\exp(e^{t_j+\ell-1})<p\le\exp(e^{t_j+\ell})} \frac{1}{2p^{2\sigma}} + \frac{1}{8p^{4\sigma}}, \quad \ell \le t - t_j,$$

and the t_j-shift in the range of $\log\log p$ in this sum only improves the error terms in the estimates in the appendix. We conclude that

$$T_j^q \cdot \mathbb{E}\Big\{ \Big(\int_0^1 e^{2\alpha S_{t-t_j}^{(j)}(\sigma_k+ih)} \mathrm{d}h \Big)^q \Big\} \ll T_j^q(e^{jq}) \cdot \frac{e^{j\cdot 2q(\alpha-1)}}{j^{q(3\alpha/2)}},$$

uniformly in j and k, and in turn that

$$\Psi_{2\alpha,q}(x) \ll (\log x)^{q\alpha^2} + \sum_{k\le K} \sum_{j\le J_k} e^{-kq} T_j^{-(\alpha q)^2} \frac{e^{j\cdot 2q(\alpha-1)}}{j^{q(3\alpha/2)}} \ll (\log x)^{q\alpha^2} + \sum_k e^{-Ak} \frac{e^{J_k\cdot 2q(\alpha-1)}}{J_k^{3q\alpha/2}}$$

where $A > 0$ is a constant. The theorem then follows by recalling that $J_k = \lceil \log\log x - 2(k+1) \rceil$.

2.4 Appendix

2.4.1 Gaussian comparison

Recall that $(Z_p)_p$ is a collection of i.i.d. Steinhaus random variables. It will be convenient to introduce

$$X_p^{(\sigma)}(h) := \Re\Big(\frac{Z_p}{p^{\sigma+ih}} + \frac{1}{2} \frac{Z_p^2}{p^{2(\sigma+ih)}} \Big), \quad I_j := \Big(\max\big\{ \exp(e^{C_0}), \exp(e^{j-1}) \big\}, \exp(e^j) \Big],$$

so that $S_j(\sigma+ih) = \sum_{\exp(e^{C_0})<p\le\exp(e^j)} X_p^{(\sigma)}(h)$ and $S_j(\sigma+ih) - S_{j-1}(\sigma+ih) = \sum_{p\in I_j} X_p^{(\sigma)}(h)$. We also let $V_j(\sigma) := \sum_{p\in I_j} \frac{1}{2p^{2\sigma}} + \frac{1}{8p^{4\sigma}}$.

Lemma 2.4.1 (Prime number theorem estimates). *Uniformly in $b > a > 2$, there exists a $c > 0$ for which*

$$\sum_{a<p\le b} \frac{1}{p} = \log\log b - \log\log a + O\big(e^{-c\sqrt{\log a}}\big), \tag{2.4.1}$$

and uniformly in $\sigma \in (1/2 - 1/\log b, 1/2)$,

$$\sum_{a<p\le b} \frac{1}{p^{2\sigma}} = \mathrm{Ei}\big((1-2\sigma)\log b\big) - \mathrm{Ei}\big((1-2\sigma)\log a\big) + O\big(e^{-c\sqrt{\log a}}\big) \tag{2.4.2}$$

where $\mathrm{Ei}(x) := \int_{-\infty}^x (e^s/s)\mathrm{d}s$. We also have that $\sum_{a\le p\le b} \frac{(\log p)^m}{p} = O\big((\log b)^m\big)$ for any $1 \le a \le b$.

Proof. This follows straightforwardly from Theorem 6.9 in [140] using integration by parts. □

Lemma 2.4.2 (Large deviation estimates). *Let $C > 0$ be an arbitrary constant. Then uniformly in all large x, $0 \le k \le \log\log\log x$, $\sigma = 1/2 - k/\log x$, $1 \le j \le \log\log x - k + 1$, $|\alpha| \le C$, and $|v| \le Cj$,*

$$\mathbb{E}\{e^{\alpha S_j(\sigma)}\} \ll e^{\frac{\alpha^2}{4}j}, \quad \text{and} \quad \mathbb{P}\big(S_j(\sigma) \in [v, v+1)\big) \ll \frac{e^{-v^2/j}}{\sqrt{j}}, \tag{2.4.3}$$

with implicit constants depending only on C.

Furthermore, the same bounds hold under the measures $\mathbb{Q} = \mathbb{Q}_{h_1,h_2,\sigma,\beta}$ defined through

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\exp\big(\beta(S_j(\sigma+ih_1) - S_j(\sigma+ih_2))\big)}{\mathbb{E}\{\exp\big(\beta(S_j(\sigma+ih_1) - S_j(\sigma+ih_2))\big)\}}$$

uniformly in $0 \le \beta \le C|h_1 - h_2|^{-1}e^{-j}$ and $h_1, h_2 \in [-e^{-j-1}, e^{-j-1}]$.

Proof. Since $|\alpha X_p^{(\sigma)}(0)|$ is bounded uniformly in $p \ge 2$ by our assumption on α, we can write

$$\log\mathbb{E}\{e^{\alpha S_j(\sigma)}\} = \sum_{p \le \exp(e^j)} \log\mathbb{E}\{e^{\alpha X_p^{(\sigma)}(0)}\} = \sum_{p \le \exp(e^j)} \log\Big(1 + \frac{\alpha^2}{4p^{2\sigma}} + O\big(\alpha^3 p^{-3\sigma}\big)\Big)$$

by Taylor expanding the exponential. Using Lemma 2.4.1 and the expansions $\log(1+x) = x + O(x^2)$ and $\mathrm{Ei}(x) = \gamma + \ln|x| + x + O(x^2)$,

$$\log\mathbb{E}\{e^{\alpha S_j(\sigma)}\} = \frac{\alpha^2}{4}\sum_{p \le \exp(e^j)} \frac{1}{p^{2\sigma}} + O\Big(\sum_p \alpha^6 p^{-3\sigma}\Big) \le \frac{\alpha^2}{4}j + C', \tag{2.4.4}$$

for some constant C' (depending on C). It follows that $\mathbb{E}\{e^{\alpha S_j(\sigma)}\} \ll e^{(\alpha^2/4)j}$ for any $|\alpha| \le C$.

To estimate the probability in (2.4.3), we rewrite it as

$$\mathbb{E}\{e^{\alpha S_j(\sigma)}\}\mathbb{E}_{\tilde{\mathbb{P}}}\{e^{-\alpha S_j(\sigma)}\mathbf{1}\big(S_j(\sigma) \in [v, v+1)\big)\} \tag{2.4.5}$$

$$\le \mathbb{E}\{e^{\alpha S_j(\sigma)}e^{-\alpha v - \alpha\mathbf{1}(v<0)}\}\tilde{\mathbb{P}}\big(S_j(\sigma) \in [v, v+1)\big)\} \tag{2.4.6}$$

for $\alpha = 2v/j$, where $\tilde{\mathbb{P}}$ is the measure given by $d\tilde{\mathbb{P}}/d\mathbb{P} = e^{\alpha S_j(\sigma)}/\mathbb{E}\{e^{\alpha S_j(\sigma)}\}$. The expectation is $\ll e^{-v^2/j}$ by the earlier estimate $\mathbb{E}\{e^{\alpha S_j(\sigma)}\} \ll e^{(\alpha^2/4)j}$. To estimate the remaining probability, note that under $\tilde{\mathbb{P}}$, $|X_p^{(\sigma)}(h)| \le \frac{3}{2}p^{-\sigma}$ for any h, from which it follows that

$$\sum_p \mathbb{E}_{\tilde{\mathbb{P}}}\Big\{\Big|X_p^{(\sigma)}(h) - \mathbb{E}_{\tilde{\mathbb{P}}}\{X_p^{(\sigma)}(h)\}\Big|^3\Big\} \le C'\sum_p p^{-3\sigma} < \infty$$

for some constant $C' > 0$. Furthermore, there exists a constant C_0 depending only on C such that

$$\frac{e^{\alpha X_p^{(\sigma)}(0)}}{\mathbb{E}\{e^{\alpha X_p^{(\sigma)}(0)}\}} = 1 + \alpha X_p^{(\sigma)}(0) + \frac{\alpha^2}{2}\Big(X_p^{(\sigma)}(0)^2 - \mathbb{E}\{X_p^{(\sigma)}(0)^2\}\Big) + O(\alpha^3 p^{-3\sigma})$$

uniformly for $p > C_0$, and we may assume that $C_0 < \exp(e^j)$ without loss of generality (the desired bound otherwise follows by bounding the probability in (2.4.5) by 1). It follows that

$$\begin{aligned}
\mathrm{Var}_{\tilde{\mathbb{P}}}\big(S_j(\sigma)\big) &= O(1) + \sum_{C_0 < p \le \exp(e^j)} \mathrm{Var}_{\tilde{\mathbb{P}}}\big(X_p^{(\sigma)}(0)\big) \\
&= O(1) + \sum_{C_0 < p \le \exp(e^j)} \mathbb{E}\Big\{ \frac{e^{\alpha X_p^{(\sigma)}(0)}}{\mathbb{E}\{e^{\alpha X_p^{(\sigma)}(0)}\}} \Big(X_p^{(\sigma)}(0) - \mathbb{E}\{X_p^{(\sigma)}(0)\}\Big)^2 \Big\} \\
&= O\big(1 + \sum_p p^{-4\sigma}\big) + \mathrm{Var}_{\mathbb{P}}\big(S_j(\sigma)\big) \asymp \mathrm{Var}_{\mathbb{P}}\big(S_j(\sigma)\big).
\end{aligned}$$

Using a standard Berry-Esseen bound and Lemma 2.4.1, we conclude that

$$\tilde{\mathbb{P}}\big(S_j(\sigma) \in [v, v+1)\big) \ll \mathrm{Var}_{\mathbb{P}}\big(S_j(\sigma)\big)^{-1/2} \ll j^{-1/2}.$$

The claim for $\mathbb{Q}$ follows from the same argument, provided one is armed with mean and variance estimates for $S_j(\sigma)$ under $\mathbb{Q}$, as well as a variance estimate under $\tilde{\mathbb{Q}}$ where $d\tilde{\mathbb{Q}}/d\mathbb{Q} = e^{\alpha S_j(\sigma)}/\mathbb{E}_{\mathbb{Q}}\{e^{\alpha S_j(\sigma)}\}$. We compute these directly, using the expansion

$$\begin{aligned}
\frac{e^{\beta(X_p^{(\sigma)}(h_1) - X_p^{(\sigma)}(h_2))}}{\mathbb{E}\{e^{\beta(X_p^{(\sigma)}(h_1) - X_p^{(\sigma)}(h_2))}\}} &= 1 + \beta\big(X_p^{(\sigma)}(h_1) - X_p^{(\sigma)}(h_2)\big) \\
&\quad + O\Big(\beta^2\big(X_p^{(\sigma)}(h_1) - X_p^{(\sigma)}(h_2)\big)^2 + \beta^2 \mathbb{E}\Big\{\big(X_p^{(\sigma)}(h_1) - X_p^{(\sigma)}(h_2)\big)^2\Big\}\Big), \qquad (2.4.7)
\end{aligned}$$

for $p \le \exp(e^j)$ large enough, and noting that the error term is

$$\ll \frac{\beta^2 |p^{-ih_1} - p^{-ih_2}|^2}{p^{2\sigma}} \le \frac{\beta^2 |h_1 - h_2|^2 (\log p)^2}{p^{2\sigma}} \ll \frac{1}{p^{2\sigma}}$$

uniformly for $p \le \exp(e^j)$ by our assumptions on β and $|h_1 - h_2|$. We therefore have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\{S_j(\sigma)\} &= O\Big(\sum_{p \le \exp(e^j)} p^{-3\sigma}\Big) + \beta \sum_{p \le \exp(e^j)} \mathbb{E}\{X_p^{(\sigma)}(0)\big(X_p^{(\sigma)}(h_1) - X_p^{(\sigma)}(h_2)\big)\} \\
&= O(1) + \frac{\beta}{2} \sum_{p \le \exp(e^j)} \frac{\cos(h_1 \log p) - \cos(h_2 \log p)}{p^{2\sigma}} \qquad (2.4.8) \\
&\ll \beta |h_1 - h_2| \sum_{p \le \exp(e^j)} \frac{\log p}{p} + 1, \qquad (2.4.9)
\end{aligned}$$

which is $O(1)$ by Lemma 2.4.1 since $\beta|h_1 - h_2| \le Ce^{-j}$. Similarly, we find that

$$\begin{aligned}\mathrm{Var}_{\mathbb{Q}}\big(S_j(\sigma)\big) &= \mathrm{Var}_{\mathbb{P}}\big(S_j(\sigma)\big) + O\Big(\beta\,|h_1 - h_2| \sum_p p^{-3\sigma}\log p\Big) \qquad (2.4.10)\\ &\qquad + O\Big(\beta^2|h_1 - h_2|^2 \sum_p p^{-4\sigma}(\log p)^2\Big)\\ &= \mathrm{Var}_{\mathbb{P}}\big(S_j(\sigma)\big) + O(1), \qquad (2.4.11)\end{aligned}$$

and $\mathrm{Var}_{\tilde{\mathbb{Q}}}\big(S_j(\sigma)\big) \asymp \mathrm{Var}_{\mathbb{Q}}\big(S_j(\sigma)\big)$. □

Remark 2.4.3. Letting $\sigma = 1/2 - k/\log x$ be as in Lemma 2.4.2, the same argument also yields the bound

$$\sup_{k\le\lfloor\log\log\log x\rfloor}\ \sup_{j\le\log\log x - k}\ \sup_{0<h<e^{-j}} \frac{\mathbb{E}\big\{e^{2\alpha S_j(\sigma+ih) - 2\alpha(1-q)S_j(\sigma)}\big\}}{\mathbb{E}\big\{e^{2\alpha q S_j(\sigma)}\big\}} < \infty \qquad (2.4.12)$$

for $q < 1$ and $\alpha < 2$ fixed. This estimate is needed in the proof of Theorem 2.0.6.

Lemma 2.4.4 (Berry-Esseen estimate)**.** *Let $x > 0$ be large enough and $h \in [0,1]$ be arbitrary. Let $\sigma = 1/2 - k/\log x$ for $0 \le k \le \log\log\log x$. Then there exists a constant $c > 0$ such that for $1 \le j \le \log\log x - k$ and any interval $A \subseteq \mathbb{R}$,*

$$\mathbb{P}\Big(\big(S_j(\sigma+ih) - S_{j-1}(\sigma+ih)\big) \in A\Big) = \mathbb{P}\big(\mathcal{N}_j \in A\big) + O\big(e^{-ce^{j/2}}\big),$$

where $\mathcal{N}_j$ is a real, centred Gaussian random variable with variance $V_j(\sigma)$.

Furthermore, for any $\mathbb{Q}$ defined as in the statement of Lemma 2.4.2, the same estimate holds under $\mathbb{Q}$ upon replacing $\mathcal{N}_j$ on the right-hand side by $\mathcal{N}_j + \nu_j$, where $\nu_j := \mathbb{E}_{\mathbb{Q}}\{S_j(\sigma+ih) - S_{j-1}(\sigma+ih)\}$

Proof. We begin with the claim for $\mathbb{P}$, which is essentially Lemma 20 in [10]. Note that the $X_p^{(\sigma)}(h)$ involved are centred, have variance in $[C^{-1}, C]$ for some $C > 0$, and satisfy $|X_p^{(\sigma)}(h)| < Cp^{-\sigma}$ deterministically. The Berry-Esseen theorem (Corollary 17.2 in [26]) thus yields

$$\Big|\mathbb{P}\Big(\big(S_j(\sigma+ih) - S_{j-1}(\sigma+ih)\big) \in A\Big) - \mathbb{P}\big(\mathcal{N}_j \in A\big)\Big| \ll \sum_{p\in I_j} p^{-3\sigma} = O\big(e^{-ce^{j/2}}\big).$$

Under $\mathbb{Q}$, we simply apply the Berry-Esseen theorem to $S_j(\sigma+ih) - S_{j-1}(\sigma+ih) - \nu_j = \sum_{p\in I_j} X_p^{(\sigma)}(h) - \mathbb{E}_{\mathbb{Q}}\big\{X_p^{(\sigma)}(h)\big\}$. □

2.4.2 Discretisation

Lemma 2.4.5 (Two-point estimates). *Let $C > 0$ be arbitrary and $x > 0$ be large enough. Let $0 \le k \le \log\log\log x$, $\sigma = 1/2 - k/\log x$, $j \le \log\log x - k$. Finally, let $-e^{-j-1} \le h_1, h_2 \le e^{-j-1}$, $0 \le V_1 \le Cj$, and $0 \le V_2 \le e^{2j}$. Then uniformly in all of these parameter ranges,*

$$\mathbb{P}\big(S_j(\sigma) \ge V_1, S_j(\sigma + ih_1) - S_j(\sigma + ih_2) \ge V_2\big) \ll_C \exp\Big(-\frac{V_1^2}{j} - \frac{cV_2^{3/2}}{e^j|h_2 - h_1|}\Big)$$

for a constant $c > 0$ which depends on C. Furthermore, if $\lambda \le C|h_1 - h_2|^{-1}e^{-j}$,

$$\mathbb{E}\big\{\exp\big(\lambda(S_j(\sigma + ih_1) - S_j(\sigma + ih_2)\big)\big\} \ll 1. \tag{2.4.13}$$

Proof. By a Chernoff bound, for any choice of $\lambda_1, \lambda_2 > 0$, this probability is bounded by

$$\mathbb{E}\Big\{\exp\Big(\lambda_1 S_k(\sigma) + \lambda_2\big(S_k(\sigma + ih_1) - S_k(\sigma + ih_2)\big)\Big)\Big\}\exp\big(-\lambda_1 V_1 - \lambda_2 V_2\big). \tag{2.4.14}$$

If $\lambda_1 \le C$ and $1 \le \lambda_2 \le |h_2 - h_1|^{-1}$, we can estimate this Laplace transform by Taylor expanding the exponential as in the proof of Lemma 2.1.2. This yields

$$\begin{aligned}&\mathbb{E}\Big\{\exp\Big(\lambda_1 S_j(\sigma) + \lambda_2\big(S_j(\sigma + ih_1) - S_j(\sigma + ih_2)\big)\Big)\Big\}\\ &\ll_C \exp\Big(\frac{\lambda_1^2}{4}j + c\lambda_2 e^j|h_1 - h_2| + c\big(\lambda_2 e^j|h_2 - h_1|\big)^2\Big)\end{aligned}$$

for some constant $c > 0$ depending on C. The first claim follows by using this in (2.4.14) and picking

$$\lambda_1 = V_1/j, \quad \lambda_2 = c\sqrt{V_2}\cdot e^{-j}|h_2 - h_1|^{-1},$$

assuming that V_2 is greater than a sufficiently large constant times $e^j|h_2 - h_1|$ (and in turn $e^{2j}|h_1 - h_2|^2$). We can do this without loss of generality, since the desired bound reduces to (2.4.3) for smaller V_2. The second claim follows by a similar argument. □

Lemma 2.4.6 (Maximum bound). *Let $C > 0$ be arbitrary. Then uniformly in all large x, $0 \le k \le \log\log\log x$, $\sigma = 1/2 - k/\log x$, $j \le \log\log x - k$ and $0 \le V \le Cj$,*

$$\mathbb{P}\Big(\max_{h\in[0,e^{-j})} S_j(\sigma + ih) > V\Big) \ll_C \exp(-V^2/j). \tag{2.4.15}$$

Proof. By the large deviations estimate in (2.4.3),

$$\mathbb{P}\Big(\max_{h\in[0,e^{-j})} S_j(\sigma+ih) > V\Big) \ll_C \exp(-V^2/j)+\mathbb{P}\Big(\max_{h\in[0,e^{-j})} S_j(\sigma+ih) > V, S_j(\sigma) \le V-2\Big).$$

To bound the probability on the right-hand side, we use a standard chaining argument which was adapted to this setting in [9] (Proposition 2.5). We include this argument here for completeness. Let

$$H_\ell = [0, e^{-j}] \cap e^{-j-\ell}\mathbb{Z}, \quad \ell \geq 0,$$

and assume without loss of generality that V is an integer. For every $q \in \{0, 1, ..., V-1\}$, let B_q be the event that $S_j(\sigma) \in [V-q-2, V-q-1]$, and B_{V-1} be the event that $S_j(\sigma) \leq 0$. Then

$$\mathbb{P}\Big(\max_{h \in [0, e^{-j})} S_j(\sigma + ih) > V, S_j(\sigma) \leq V-2 \Big) \tag{2.4.16}$$

$$\leq \sum_{q=0}^{V-2} \mathbb{P}\Big(B_q \cap \Big\{ \max_{h \in [0, e^{-j}]} S_j(\sigma + ih) - S_j(\sigma) \geq q \Big\} \Big). \tag{2.4.17}$$

We now decompose the summands on the right-hand side. Let $(h_\ell)_{\ell \geq 0}$ be an e-adic sequence tending to h with ℓ, satisfying $h_0 = 0$ and $h_\ell \in H_\ell$. Then by continuity of $h \mapsto S_j(\sigma + ih)$,

$$S_j(\sigma + ih) - S_j(\sigma) = \sum_{\ell \geq 0} S_j(\sigma + ih_{\ell+1}) - S_j(\sigma + ih_\ell)$$

and the series on the right-hand side converges almost surely. Furthermore, since

$$\sum_{\ell=0}^{\infty} \frac{1}{e(\ell+1)^2} \leq 1,$$

the following inclusion holds

$$\Big\{ S_j(\sigma + ih) - S_j(\sigma) \geq q \Big\} \subset \bigcup_{\ell \geq 0} \Big\{ S_j(\sigma + ih_{\ell+1}) - S_j(\sigma + ih_\ell) \geq \frac{q}{e(\ell+1)^2} \Big\}.$$

Since $(h_\ell)_{\ell \geq 0}$ was arbitrary, we conclude by a union bound that the right-hand side in (2.4.16) is

$$\leq \sum_{q=0}^{V-2} \sum_{\ell \geq 0} \sum_{h \in H_\ell} 2 \cdot \mathbb{P}\Big(B_q \cap \Big\{ S_j(\sigma + ih_*) - S_j(\sigma + ih) \geq \frac{q}{e(\ell+1)^2} \Big\} \Big), \tag{2.4.18}$$

where $h_* \neq h$ denotes the closest point to h in $H_{\ell+1}$. (Should there be two such points, we let h_* be the smaller of the two and note that the bound still holds due to the additional factor of 2, since the probability only depends on $|h - h_*|$.) Noting that $\#H_\ell = e^\ell + 1$, we can use the joint large deviations estimate of Lemma 2.4.5 to estimate each summand, and

conclude that (2.4.18) is

$$\begin{aligned}
&\ll_C \sum_{q=0}^{V-2} \sum_{\ell \geq 0} e^{\ell} \exp\Big(- \frac{(V-q-1)^2}{j} - ce^{\ell} \frac{q^{3/2}}{(\ell+1)^3} \Big) \\
&\ll_C \sum_{q=0}^{V-2} e^{-(V-q-1)^2/j - cq^{3/2}} \\
&\ll_C e^{-V^2/j}.
\end{aligned}$$

□

Remark 2.4.7. The conclusions of both Lemma 2.4.5 and Lemma 2.4.6 hold upon replacing S_j by $-S_j$, by the same proof.

2.4.3 Ballot theorem

Proposition 2.4.8. *Fix $0 < \kappa < 1$, $\delta > 0$ and $C_0 = 1000$. Then there exist constants $C, C' > 0$ depending only on κ and δ such that the following holds. Let $\{\mathcal{N}_j\}_{j \geq 1}$ be a collection of independent, centred Gaussian random variables with variances $\mathbb{E}\{\mathcal{N}_j^2\} \in (\kappa, \kappa^{-1})$ for each j, satisfying $|\sum_{C_0 \leq j \leq k} \mathbb{E}\{\mathcal{N}_j^2\} - (k - C_0 - 1)/2| < \delta$ for all $k \geq C_0$. Let $\mathcal{G}_k = \sum_{C_0 \leq j \leq k} \mathcal{N}_j$ for each such k.*

For any $A > 4C_0$, let $U_A(s) := \infty$ for $s < \lfloor A/4 \rfloor$, and for $s \geq \lfloor A/4 \rfloor$, let U_A be one of the following two functions:

$$U_A(s) = A + s + 2\log\big(1 + s \wedge (t-s)\big) \text{ or } U_A(s) = A + s\Big(1 - \frac{3}{4}\frac{\log t}{t}\Big) + 10^3 \log\big(1 + s \wedge (t-s)\big).$$

Then for either choice of U_A, any $4C_0 < A \leq 2t$, $t/2 \leq k \leq t$, and $w \in [0, U_A(k))$,

$$\mathbb{P}\Big(\mathcal{G}_k \in [w, w+1) \text{ and } \mathcal{G}_j \leq U_A(j) + 1 \text{ for all } j \leq k\Big) \leq C \cdot \frac{A\big(U_A(k) - w + C'\big)}{k} \frac{e^{-w^2/k}}{\sqrt{k}}.$$

Proof. For increments with equal variances $\mathbb{E}\{\mathcal{N}_j^2\} = 1/2$, this follows directly from [10, Proposition 5] by conditioning on the values of $\mathcal{G}_j$ at times $r := \lfloor A/4 \rfloor$ and k. To extend this proposition to the current setting, one follows the same proof as in [10] but in "variance time" $t_j := r + (k-r)\mathbb{E}\{(\mathcal{G}_j - \mathcal{G}_r)^2\}/\mathbb{E}\{(\mathcal{G}_k - \mathcal{G}_r)^2\}$, noting that $\{\mathcal{G}_j \leq U_A(j) + 1\} \subseteq \{\mathcal{G}_j \leq U_A(t_j) + C\}$ for some $C = C(\delta) > 0$, for either choice of U_A. The ballot theorem invoked in [10, Equation 101] is then replaced by [11, Lemma 20], which is the analogous bound for random walks with comparable-variance increments. □

Chapter 3

The Riemann zeta function on mesoscopic intervals of the critical line

This chapter is based on the work "The Fyodorov-Hiary-Keating conjecture on mesoscopic intervals", joint with Louis-Pierre Arguin [14].

The value distribution of the Riemann zeta function $\zeta(s)$ on the line $\Re(s) = 1/2$ is of fundamental importance in number theory. It is tightly linked to the distribution of its zeros, which encode the distribution of prime numbers via the explicit formula (1.1.4). A classical result in this vein is Selberg's central limit theorem [162, 163], which states that at a typical point $s = 1/2 + it$, $\log|\zeta(s)|$ behaves like a Gaussian random variable for large t. More precisely, if τ is uniformly distributed on $[T, 2T]$, then

$$\mathbb{P}\Big(\log|\zeta(1/2 + i\tau)| > \sqrt{\tfrac{1}{2}\log\log T}\cdot y\Big) \sim \int_y^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}}\mathrm{d}z, \quad y \in \mathbb{R} \tag{3.0.1}$$

as $T \to \infty$, suggesting a typical value of the order of $\sqrt{\log\log T}$ on such dyadic intervals. This Gaussian tail has since been shown to persist in the large deviations regime [166, 5, 6], as well as in higher dimensions [109], where one finds asymptotic independence between $\log|\zeta(1/2 + i\tau + ih)|$ and $\log|\zeta(1/2 + i\tau + ih')|$ at sufficiently distant shifts h and h'.

For smaller shifts, Bourgade [35] established a multidimensional central limit theorem whose limit has a non-trivial correlation structure, exhibiting *logarithmic* correlations (see [69, 54] for earlier work in this direction). These can play a significant role when studying the distribution of $\log|\zeta|$ on typical *short* intervals of the critical line, meaning either those of constant length, or *mesoscopic* intervals of length proportional to $(\log T)^\theta$ for $\theta \in (-1, 0)$. For instance, Fyodorov-Hiary-Keating [71] and Fyodorov-Keating [72] made the following striking conjecture about the distribution of local maxima of $\log|\zeta|$ on such intervals.

Conjecture 3.0.1 ([72], Section 2(c)(ii)). *Fix $\theta \in (-1, 0]$. Then*

$$\begin{aligned}&\max_{h \le \log^{\theta} T} \log\left|\zeta(\tfrac{1}{2} + i\tau + ih)\right| \\ &\stackrel{d}{=} (1+\theta)\log\log T - \frac{3}{4}\log\log\log T + \mathcal{N}\left(0, \tfrac{|\theta|}{2}\log\log T\right) + X(\theta) + o_{\mathbb{P}}(1),\end{aligned}$$

where $X(\theta)$ is a random variable with right tail $\mathbb{P}(X(\theta) > y) \asymp ye^{-2y}$, which is independent of the Gaussian.

This prediction is based on an analogous conjecture for the maximum of (the modulus of) the characteristic polynomial of a random unitary matrix, following the Keating-Snaith paradigm [116]. In that case, Fyodorov–Keating give the exact distribution for $\mathscr{M}$, which was recognised in [169] to be that of a sum of two independent Gumbel random variables.

Even at $\theta = 0$, the effect of the logarithmic correlations is already visible in the coefficient $-3/4$ multiplying $\log\log\log T$. Indeed, one can argue (e.g. using the Littlewood approximation (1.2.1) and Poisson summation) that the analysis of $\max_{|h|\le 1} \log|\zeta(1/2 + i\tau + ih)|$ can be reduced to a discrete set containing roughly $(\log T)$ points, each of which is approximately distributed like a Gaussian with variance $\frac{1}{2}\log\log T$ by Selberg's theorem. If one were to model the maximum by that of $(\log T)$ independent Gaussian random variables of the same variance, a straightforward calculation yields a sub-leading order of $-(1/4)\log\log\log T$ (see Lemma 1.2.1); a larger maximum, as one would expect by Slepian's lemma.

Conjecture 3.0.1 has seen significant progress since its formulation, particularly in the $\theta = 0$ case. Following initial progress by Najnudel [141], Arguin et al. [8], and Harper [97], the works of Arguin, Bourgade, and Radziwiłł [10, 11] showed that the sequence

$$\max_{|h|\le 1} \frac{(\log\log T)^{3/4}}{\log T}\left|\zeta(1/2 + i\tau + ih)\right|$$

is tight, using ideas from the study of branching random walks. They also show that the limiting right tail is proportional to ye^{-2y}. On the random-matrix-theoretic side, the leading order was established by Arguin, Belius and Bourgade [7], the subleading order by Paquette and Zeitouni [147] and the tightness of the recentred maximum by Chhaibi, Madaule and Najnudel [44]. Finally, a recent work of Paquette and Zeitouni [148] proved the distributional convergence of the maximum to a sum of a Gumbel random variable and a "derivative martingale"; it remains open whether the latter is itself Gumbel-distributed.

For $\theta < 0$, the maximum is only known in the leading order, following work by one of the authors, Ouimet, and Radziwiłł [16]. Our first result improves their upper bound to $O(1)$ precision.

Theorem 3.0.2 (Upper bound for the local maximum). *Let $C > 0$ be an arbitrary constant and $\theta \in (-1, 0]$ be fixed. Then uniformly in $2 \le y < C \log\log T / \log\log\log T$ and large enough T,*

$$\mathbb{P}\Big(\max_{|h| \le \log^\theta T} \big|\zeta\big(\tfrac{1}{2} + i(\tau + h)\big) P_\tau(\theta)\big| > e^y \frac{(\log T)^{(1+\theta)}}{(\log\log T)^{3/4}} \Big) \ll y e^{-2y - \frac{y^2}{(1+\theta)\log\log T}} \tag{3.0.2}$$

where $P_\tau(\theta) = \prod_{p \le \exp((\log T)^{|\theta|})} (1 - p^{-1/2 - i\tau})$, and the implicit constant depends only on C.

It follows by a standard moment computation (see, e.g., the proof of Selberg's theorem in [152]) that $\log|P_\tau(\theta)| / \sqrt{(|\theta|/2)\log\log T}$ converges in distribution to a standard Gaussian random variable as $T \to \infty$. Theorem 3.0.2 thus settles a strong form of the upper bound in Conjecture 3.0.1, and explicitly identifies the source of the Gaussian term therein.

The *moments* of ζ in short intervals

$$\mathscr{Z}_\beta(\theta) := \frac{1}{2\log^\theta T} \int_{|h| \le \log^\theta T} |\zeta(\tfrac{1}{2} + i(\tau + h))|^\beta \mathrm{d}h \tag{3.0.3}$$

are closely related to the maximum. It is useful to think of $\mathscr{Z}_\beta(\theta)$ as a (random) *partition function* at inverse temperature β, associated to a system whose states are indexed by the interval of radius $\log^\theta T$ around τ, and with energy $\log|\zeta(\frac{1}{2} + i(\tau + h))|$ at state h.

Upper and lower bounds for the leading order of $\mathscr{Z}_\beta(\theta)$ were established in [16] for all $\theta > -1$, with the latter holding unconditionally, and the former assuming the Riemann hypothesis when $\theta > 3$. In all cases, one finds that $\mathscr{Z}_\beta(\theta) \asymp (\log T)^{f(\theta,\beta)}$ for some $f(\theta, \beta)$ with high probability; the same work also shows that a *freezing transition* (conjectured in [72]) occurs at the critical parameter $\beta_c = 2\sqrt{1 + \min(\theta, 0)}$, where $f_\theta(\beta)$ transitions from being quadratic in β to linear. These bounds were later improved in [5] for $\theta \in [0, 3)$ and $\beta > 2\sqrt{1+\theta}$, by studying the measure of high points of $\log|\zeta|$.

In the case $\theta = 0, \beta = \beta_c = 2$, Harper [97] proved the precise estimate

$$\mathbb{P}\Big(\mathscr{Z}_2(0) \ge A \frac{\log T}{\sqrt{\log\log T}} \Big) \le C \cdot \frac{\min\{\log A, \sqrt{\log\log T}\}}{A}, \tag{3.0.4}$$

which implies that $\mathscr{Z}_2(0) = O(\log T / \sqrt{\log\log T})$ with high probability. This beats the trivial bound of $O(\log T)$, which follows from Markov's inequality and the classical second-moment estimate in Lemma 3.6.6. The additional $\sqrt{\log\log T}$ in the denominator was shown to stem from a connection to critical Gaussian multiplicative chaos [150], and is analogous to the so-called Seneta-Heyde normalisation in that setting.

For $\theta < 0$, it was anticipated in [72, Section 4(b)] that $\mathscr{Z}_2(\theta)$ should asymptotically split into the product of two uncorrelated random variables: the first being roughly log-normal,

and the second behaving similarly to $\mathscr{Z}_2(0)$ upon substituting T by $\exp((\log T)^{1+\theta})$. Towards this conjecture, our main result establishes an upper bound for this partition function in mesoscopic intervals. It features an analogue of the log-normal factor in [72, Section 4(b)] (recalling that $\log|P_\tau(\theta)|/\sqrt{(|\theta|/2)\log\log T}$ is asymptotically Gaussian), as well as a square root correction term as in Harper's result (which is recovered at $\theta = 0$).

Theorem 3.0.3 (Upper bound for the partition function at criticality). *Fix $\theta \in (-1, 0]$ and for any T, let $P_\tau(\theta)$ be as in Theorem 3.0.2. Define*

$$\mathscr{Z}_2(\theta) := \frac{1}{2\log^\theta T}\int_{|h|\le \log^\theta T} \left|\zeta(\tfrac{1}{2} + i(\tau + h))\right|^2 \mathrm{d}h.$$

Then for any $\varepsilon > 0$,

$$\mathbb{P}\Big(\mathscr{Z}_2(\theta)|P_\tau(\theta)|^2 \ge A\frac{(\log T)^{(1+\theta)}}{\sqrt{(1+\theta)\log\log T}}\Big) \ll_\varepsilon A^{-1+\varepsilon}$$

uniformly in large enough T and $A \le \sqrt{\log\log T}$ when $\theta \in (-1, 0)$. For $\theta = 0$,

$$\mathbb{P}\Big(\mathscr{Z}_2(0) \ge A\frac{\log T}{\sqrt{\log\log T}}\Big) \ll \frac{\min\{\log A, \sqrt{\log\log T}\}}{A}$$

uniformly in large T and $A \ge 2$.

Our proof departs from the approach in [97] and yields a new proof of Harper's bound at $\theta = 0$. We express $\mathscr{Z}_2(0)$ in terms of the typical measure of level sets of $\log|\zeta|$, similarly to the argument sketched in Section 4 of [5]. To implement it rigorously, we use a "partial barrier" strategy which we explain in Section 3.0.2.

3.0.1 Notation

Throughout our arguments, τ will always denote a random variable that is uniformly distributed in $[T, 2T]$. For simplicity, we will use the notation

$$\zeta_\tau(h) := \zeta(\tfrac{1}{2} + i(\tau + h)).$$

and let

$$t = \log\log T \quad \text{and} \quad t_\theta = t|\theta|,$$

noting that in this notation

$$t(1+\theta) = t - t_\theta.$$

For any $\ell \ge 1$, $\log_\ell t$ will denote the logarithm iterated ℓ times. We will use standard asymptotic notation, writing $f(T) = o(g(T))$ if $|f(T)/g(T)|$ tends to 0 as $T \to \infty$, and

$f(T) = O(g(T))$ or $f(T) \ll g(T)$ if $\limsup |f(T)/g(T)|$ is bounded. A subscripted parameter next to o, O or $\ll$ means that the implicit constant is allowed to depend on said parameter. Lastly, we will use standard number theoretic notation for the following arithmetical functions: $\omega(n)$ is the number of distinct primes dividing n, $\Omega(n)$ the number of such primes counted with multiplicity, and μ is the Möbius function, which equals -1 (resp. 1) on squarefree integers with an odd (resp. even) number of prime factors, and zero otherwise.

3.0.2 Branching random walk model and proof outlines

Our arguments are all centred around the branching random walk model described in Section 1.2.3, which we briefly recall. For $k \le t$, define the partial sums

$$S_k(h) = \sum_{C<p\le \exp(e^k)} \Re\left(\frac{1}{p^{1/2+i(\tau+h)}} + \frac{1}{2}\frac{1}{p^{1+2i(\tau+h)}}\right). \tag{3.0.5}$$

Each summand corresponds to a second-order Taylor approximation to $-\log|1-p^{-1/2-i\tau-ih}|$, and we view $S_k(h)$ as a proxy for $\log|\zeta(1/2+i\tau+ih)|$ when $k \approx t$. (The sum is started at a sufficiently large constant, in this case $C = \exp(e^{1000})$, for technical reasons.) We are then led to study the process $(S_k(h), |h| \le \log^\theta T)$, which by a standard estimate, has covariances

$$\mathbb{E}\left\{S_k(h)S_k(h')\right\} = \frac{1}{2}\sum_{p\le\exp(e^k)} \frac{\cos(|h-h'|\log p)}{p} + O\left(\frac{\exp(2e^k)}{T}\right). \tag{3.0.6}$$

For each h, we view $S_t(h)$ as an approximate random walk with t steps: ignoring the fact that the error term in (3.0.6) is large for $k \approx t$, the double-exponential indexing of the sum therein is chosen so as to make the increments $(S_k - S_{k-1})(h)$ have variance

$$\frac{1}{2}\sum_{\exp(e^{k-1})<p\le\exp(e^k)} \frac{1}{p} \approx \frac{1}{2}$$

by the prime number theorem. A similar calculation shows that for any h, h', the increments of $S_t(h)$ and $S_t(h')$ are highly correlated up to $k \approx \log|h-h'|^{-1}$, after which they decorrelate rapidly. At intermediate times $k \le t$, this implies that $S_k(h) \approx S_k(h')$ when $|h-h'| \le e^{-k}$, meaning that there are roughly e^k distinct "random walks" over $|h| \le 1$.

This leads one to compare $(S_t(h), |h| \le 1)$ to a branching random walk on a tree of depth t in which each node has $e = 2.718\ldots$ offspring on average: to each edge, one attaches a Gaussian random variable with variance 1/2, and to each leaf, the sum of the edge weights from that leaf to the root. The leaves then correspond to the values of $S_t(h)$ on a mesh of width $1/\log T$.

Upper bound for the maximum. On the interval $|h| \le \log^\theta T$, the processes $(S_k(h))_{k\le t}$ approximately share the increments up to $k \approx t_\theta = |\theta| \log\log T$ (as made precise by Proposition 3.0.5). The maximum of $\log|\zeta(\frac{1}{2}+i(\tau+h))| = \log|\zeta_\tau(h)|$ on this interval should thus roughly equal the sum of two random variables: the first is $S_{t_\theta}(0)$ (which we note equals $-\log|P_\tau(\theta)| + O(1)$), and the second is the maximum of $(S_k^{(t_\theta)}(h))_{|h|\le\log^\theta T}$, where

$$S_k^{(t_\theta)} := S_{t_\theta+k} - S_{t_\theta}. \tag{3.0.7}$$

The latter is itself modeled by the maximum of a branching random walk, this time over a tree of depth $t(1+\theta)$ (see, e.g. Figure 1.3).

To prove Theorem 3.0.2, we will therefore subtract $S_{t_\theta}(0)$ from $\log|\zeta_\tau(h)|$ and use the recursive scheme in [10] to bound the maximum of the difference (see Section 3.3). This scheme allows one to apply, in this number-theoretic setting, the *barrier method* developed by Bramson [37] to bound the maximal displacement in branching Brownian motion (explained in Section 1.2.2).Compared to the $\theta = 0$ case in [10], one must keep track of the "error" $S_{t_\theta}(h) - S_{t_\theta}(0)$ throughout the proof. We will also require uniform control in h, in the form of a Gaussian tail for $\max_{|h|\le e^{-t_\theta}} |S_{t_\theta}(h) - S_{t_\theta}(0)|$ (Proposition 3.0.5), derived in Section 3.2.

Upper bound for the partition function at criticality. We begin with the case $\theta = 0$, and express $\mathscr{Z}_2(0)$ in terms of the measure of level sets of $\log|\zeta_\tau(h)|$:

$$\mathscr{Z}_2(0) \asymp \int_{-\infty}^{\infty} e^{2V} \operatorname{meas}\big\{|h| \le 1 \,:\, \log|\zeta_\tau(h)| > V\big\} \mathrm{d}V.$$

By Theorem 3.0.2, we may truncate this integral at $M = m(t) + \log A$, since the maximum of $\log|\zeta_\tau(h)|$ only exceeds this threshold with probability $\ll 1/A$. Following the proof of Corollary 1.4 in [5], the claimed bound would follow if one could show that

$$\int_{-1}^{1}\int_{0}^{M} e^{2V}\,\mathbb{P}\big(\log|\zeta_\tau(h)| > V\big)\mathrm{d}V\,\mathrm{d}h \ll A\frac{e^t}{\sqrt{t}}. \tag{3.0.8}$$

However, the large deviations estimates for $\log|\zeta_\tau(h)|$ in [5, 12] show that the left-hand side is of the order of $\int_0^M e^{2V - V^2/t}/\sqrt{t} \asymp e^t$, the expectation of $\mathscr{Z}_2(0)$ being inflated by atypically large values of $\log|\zeta_\tau(h)|$.

To recover the additional factor of $1/\sqrt{t}$, it is helpful to once again liken the process $(\log|\zeta_\tau(h)|, |h| \le 1)$ to the approximate branching random walk $(S_t(h), |h| \le 1)$. Doing so suggests restricting to a good event on which $S_k(h)$ remains below a barrier $U_{\log A}(k)$ for all $k \le t$ and $|h| \le 1$. Were this a genuine branching random walk, one could use the

logarithmic correlation structure of $(S_t(h))_h$ to, roughly speaking, reduce the integrand in (3.0.8) to

$$e^{2V}\,\mathbb{P}\big(S_t(h) > V,\ S_k(h) \le m(k) + U_{\log A}(k)\,\forall k \le t\big),$$

for a $U_{\log A}$ lying *below* the typical fluctuations of a random walk conditioned to end at V. This would be computable via a ballot theorem (Proposition 3.6.9), and integrating the resulting estimate over $V \in [0, M]$ would recover the missing $1/\sqrt{t}$ (see, e.g., the proof of Proposition 2.1.1 in §2).

Adapting this approach to $\log|\zeta_\tau|$ requires one to control the deviations of the increments $S_k - S_{k-1}$ for large $k \approx t$, showing that their behaviour is close to Gaussian. This amounts to computing large moments of long Dirichlet polynomials, and the error in doing so (using Lemma 3.6.4) can quickly become unruly. The recursive scheme of [10] handles this by iteratively constructing both *upper* and *lower* barriers $U_{\log A}$ and $L_{\log A}$ which restrict the range of values that these increments can take. At every step of this iteration, one then only needs to consider Dirichlet polynomials short enough for Lemma 3.6.4 to be applicable.

However, this approach relies on the gap $U_{\log A}(k) - L_{\log A}(k)$ between these barriers being sufficiently small as k grows, shrinking to $O(1)$ at time t. This raises an obvious problem in our previous argument, which requires the same barrier $U_{\log A}$ to be taken uniformly over all V: any corresponding lower barrier would have to lie below $(k/t)V$, potentially resulting in a large gap for V away from $U_{\log A}(t) \approx M$.

To resolve this, we make the simple yet crucial observation that a barrier up to time $t_* = t/2$ suffices to get the sought-after bound. That this should be the case is not a priori clear; it follows from the fact that the resulting integral is localised at levels $V = t - O(\sqrt{t})$, for which a barrier up to t_* yields a bound of the same order as one up to t. This is not the case for larger V (e.g., for $V \approx M$, one is off by a factor of $\sqrt{t}$), but the Gaussian weight will suffice to offset the resulting error.

Concretely, we place ourselves on the event that $S_k(h) \in [L_{\log A}(k), U_{\log A}(k)]$ for all $k \le t_*$ for a suitable choice of barriers. This is straightforward, as t_* is small enough for the Mean Value Theorem for Dirichlet polynomials (Lemma 3.6.4) to apply directly, and one must then prove large deviation estimates for $\log|\zeta_\tau(h)|$ on this event. This still requires controlling the problematic increments $S_k - S_{k-1}$ for $k \in (t_*, t]$, but since no barrier is required at these times, this can be achieved by a variant of the recursive scheme developed in [5] in which one only restricts S_k at a sequence of sparse checkpoints $(t_\ell)_\ell$. This gives rise to the ($\theta = 0$ case of the) following result.

Theorem 3.0.4. *Fix $\theta \in (-1, 0]$ and $t_* = t_*(\theta) = (t - t_\theta)/2$. Let $C > 10$ be a large constant, and for any $A \geq 1$, define the barriers*

$$U_A(k) = A + k + C \log k, \quad L_A(k) = A - Ck, \quad \text{for } k > A/4,$$

and $U_A(k) = -L_A(k) = \infty$ otherwise. Then uniformly in large enough t, $|h| \leq 2$, $1 \leq A \leq \sqrt{t}$, and $t_* - t_*^{2/3} \leq V/2 < U_A(t_*)$,

$$\begin{aligned} \mathbb{P}\Big(\log |\zeta_\tau(h)| - S_{t_\theta}(h) > V, S_k^{(t_\theta)}(h) \in [L_A(k), U_A(k)], \forall k \leq t_* \Big) \\ \ll \frac{A\left(U_A(t_*) - V/2 + \sqrt{t_*}\right)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t - t_\theta}}. \end{aligned} \tag{3.0.9}$$

This theorem can be seen as an instance of a more general result which may be of independent interest: for a reasonable event E depending only on the increments $(S_k - S_{k-1})(h)$ for $k \leq t_*$, the same proof gives

$$\mathbb{P}\big\{\{\log |\zeta_\tau(h)| > V\} \cap E\big\} \ll \mathbb{P}\big\{\{\mathscr{W}_t > V\} \cap \tilde{E}\big\},$$

where $(\mathscr{W}_k)_{k \leq t}$ is a Gaussian random walk with t i.i.d. increments of variance 1/2, and $\tilde{E}$ is the event obtained from E by substituting $\mathscr{W}_k - \mathscr{W}_{k-1}$ for each $S_k - S_{k-1}$.

Lastly, to bound $\mathscr{Z}_2(\theta)$ when $\theta < 0$, we begin by factoring out the contribution coming from $S_{t_\theta}(h)$ in the integral, as it is (morally speaking) constant for all h. This is achieved using the following proposition.

Proposition 3.0.5. *Let $\theta \in (-1, 0]$ and $2 < x < e^{t(1+\theta)/2}$. Then there exist an absolute constant $c > 0$ such that*

$$\mathbb{P}\Big(\max_{|h| \leq 2e^{\theta t}} |S_{t_\theta}(h) - S_{t_\theta}(0)| > x \Big) \ll e^{-x^2/c}.$$

The rest of the argument to bound $\mathscr{Z}_2(\theta)$ is the same as the one used for $\mathscr{Z}_2(0)$, applied to $\log |\zeta_\tau(h)| - S_{t_\theta}(h)$ in place of $\log |\zeta_\tau(h)|$. The barrier event now restricts the evolution of the shifted walks $(S_k^{(t_\theta)}(h))_k$, and one uses the $\theta \in (-1, 0)$ case of Theorem 3.0.4.

3.0.3 Organisation

In Section 3.1, we prove Theorem 3.0.3 from Proposition 3.0.5 and Theorems 3.0.2 and 3.0.4. These theorems are then proved in Sections 3.2, 3.3 and 3.5, respectively. Section 3.4 contains proofs of auxiliary results which are used in the proof of Theorem 3.0.2. The appendix collates known results that are used throughout the chapter.

3.1 Upper bound for the partition function

We first prove Theorem 3.0.3 for $\theta = 0$. We may assume that $\log A \leq \sqrt{t}$, the desired bound otherwise following directly from Markov's inequality and Lemma 3.6.6.

Consider the following set of *good* h's, defined by

$$G_A = \{|h| \leq 1 \,:\, S_k(h) \in [L_{\log A}(k), U_{\log A}(k)] \text{ for all } k \leq t_*\},$$

where t_*, $L_{\log A}(k)$ and $U_{\log A}(k)$ are as in the statement of Theorem 3.0.4 (for $\theta = 0$). We are interested in the normalised measure of high points of the zeta function belonging to this good set, meaning

$$\mathcal{S}(V; G_A) := \frac{1}{2}\text{meas}\{|h| \leq 1 \,:\, \log|\zeta_\tau(h)| > V, h \in G_A\},$$

which we note is a random variable taking values in $[0, 1]$. We will also need

$$\mathcal{S}(V) := \frac{1}{2}\text{meas}\{|h| \leq 1 \,:\, \log|\zeta_\tau(h)| > V\}.$$

The proof follows the strategy outlined in [5] (proof of Corollary 1.4), with the main difference being that we use large deviation estimates for zeta *in the set* G_A (given by Theorem 3.0.4) to get an additional factor of $(t - t_\theta)^{-1/2}$. We begin by integrating by parts to get

$$\mathcal{I}_2(0) = -e^{2V}\mathcal{S}(V)\Big|_{-\infty}^{+\infty} + 2\int_{\mathbb{R}} e^{2V}\mathcal{S}(V)\mathrm{d}V.$$

The boundary term at $-\infty$ is equal to zero, since $\mathcal{S}(V) \leq 1$ by definition. To deal with the limit at $+\infty$, we place ourselves on the event

$$E_A := \{\forall h \in [-1, 1],\ \log|\zeta_\tau(h)| \leq m(t) + \log A \text{ and } h \in G_A\},$$

where $m(t) = t - \frac{3}{4}\log t$, on which said limit is zero (since $\mathcal{S}(V) = 0$ for $V > m(t) + \log A$). This can be done since $\mathbb{P}(E_A^c)$ is bounded by

$$\mathbb{P}\Big(\max_{|h|\leq 1} \log|\zeta_\tau(h)| > m(t) + \log A\Big) + \mathbb{P}\Big(\substack{\exists k \leq t/2,\\ \exists h \in [-1,1],} S_k(h) \notin [L_{\log A}(k), U_{\log A}(k)]\Big), \tag{3.1.1}$$

both of these terms being $\ll \log A \cdot e^{-2\log A} \ll 1/A$, by Theorem 3.0.2 and the proof of Proposition 3.3.4, respectively. Noting that $\mathcal{S}(V) = \mathcal{S}(V; G_A)$ on E_A, we are left with the task of bounding

$$\int_{\mathbb{R}} e^{2V}\mathcal{S}(V; G_A)\mathrm{d}V = \int_{-\infty}^{m(t)+\log A} e^{2V}\mathcal{S}(V; G_A)\mathrm{d}V.$$

Using the trivial bound $\mathcal{S}(V;G_A) \le \mathcal{S}(V) \le 1$, we can write

$$\int_{-\infty}^{m(t)+\log A} e^{2V}\,\mathcal{S}(V;G_A)\mathrm{d}V \le \int_{-\infty}^{t/4} e^{2V}\mathrm{d}V + \int_{t/4}^{m(t)+\log A} e^{2V}\,\mathcal{S}(V;G_A)\mathrm{d}V$$
$$\le e^{t/2} + \int_{t/4}^{m(t)+\log A} e^{2V}\,\mathcal{S}(V;G_A)\mathrm{d}V,$$

and since $e^{t/2} \le e^t/\sqrt{t}$ for $t \ge 1$, one must then show that the rightmost integral is of the desired order. This requires controlling $\mathcal{S}(V;G_A)$ on this range of V. To that end, we partition said range into sub-intervals of length of the order of $\sqrt{t}$, letting $(V_j, 1 \le j \le J)$ be an enumeration of the elements of $\left[\frac{t}{4}, m(t)+\log A\right] \cap (\sqrt{t}\mathbb{Z})$, and defining $V_0 = V_1 - \sqrt{t}$ and $V_{J+1} = V_J + \sqrt{t}$. The integral in question can then be partitioned into segments

$$I_j = \int_{V_j}^{V_{j+1}} e^{2V}\,\mathcal{S}(V;G_A)\mathrm{d}V, \quad 0 \le j \le J,$$

and we let E_j be the event that the integral I_j does not exceed a constant a_j times its expectation, namely

$$E_j := \left\{ I_j \le a_j \mathbb{E}\left\{I_j\right\} \right\}$$

for

$$a_j := \frac{A}{\log A} \cdot \begin{cases} \left(1 + \left(V_j - t\right)^2/t\right) & \text{if } V_j > t \\ \left(1 + \left(V_{j+1} - t\right)^2/t\right) & \text{if } V_j \le t \text{ and } V_{j+1} \le t \\ 1 & \text{if } V_j \le t \text{ and } V_{j+1} > t \end{cases}.$$

This choice of a_j ensures that on $[V_j, V_{j+1}]$,

$$a_j \le \frac{A}{\log A}\Big(1 + \frac{(V-t)^2}{t}\Big)$$

while also making the a_j^{-1} summable. By Markov's inequality and a union bound, it follows that the good event

$$G = E_A \cap \Big(\bigcap_j E_j\Big)$$

is such that

$$\mathbb{P}\big(G^c\big) \ll \sum_j a_j^{-1} + 1/A \ll \log(A)/A.$$

We can therefore place ourselves on the event G, where the following bound holds

$$\int_{t/4}^{m(t)+\log A} e^{2V}\,\mathcal{S}(V;G_A)\mathrm{d}V \le \sum_j a_j \int_{V_j}^{V_{j+1}} e^{2V}\,\mathbb{E}\left\{\mathcal{S}(V;G_A)\right\}\mathrm{d}V$$
$$\le \int_{t/4}^{m(t)+\log A} \frac{A}{\log A}\Big(1 + \frac{(V-t)^2}{t}\Big) e^{2V}\,\mathbb{E}\left\{\mathcal{S}(V;G_A)\right\}\mathrm{d}V. \tag{3.1.2}$$

By Fubini's theorem and Theorem 3.0.4,

$$\mathbb{E}\left\{\mathcal{S}(V;G_A)\right\} \ll \frac{\log(A)(U_{\log A}(t/2)-V/2+\sqrt{t})}{t}\frac{e^{-V^2/t}}{\sqrt{t}}, \tag{3.1.3}$$

for $V > 2(t_* - t_*^{2/3})$ in (3.1.2). For the remaining $V \in (t/4, 2(t_* - t_*^{2/3})]$ the main theorem in [5] gives the bound $\ll e^{-V^2/t}/\sqrt{t}$, and the integral in (3.1.2) is thus

$$\begin{aligned}
&\ll Ae^t \int_{t/4}^{2(t_*-t_*^{2/3})} \Big(1+\frac{(V-t)^2}{t}\Big)\frac{e^{-(V-t)^2/t}}{\sqrt{t}}\mathrm{d}V \\
&+ Ae^t \int_{2(t_*-t_*^{2/3})}^{m(t)+\log A} \Big(1+\frac{(V-t)^2}{t}\Big)\frac{(U_{\log A}(t/2)-V/2+\sqrt{t})}{t}\frac{e^{-(V-t)^2/t}}{\sqrt{t}}\mathrm{d}V.
\end{aligned}$$

Using the change of variables $u = (t-V)/\sqrt{t}$, the first of these terms is easily seen to satisfy the desired bound. For $V \in (2(t_* - t_*^{2/3}), m(t) + \log A]$ (and $\log A \le \sqrt{t}$), we can use the estimate

$$\frac{(U_{\log A}(t/2)-V/2+\sqrt{t})}{t} \ll \frac{1}{\sqrt{t}}(\sqrt{t}-V/\sqrt{t}+1) \ll \frac{1}{\sqrt{t}}(1+u),$$

to conclude that

$$\begin{aligned}
\mathscr{I}_2(0) &\ll A\frac{e^t}{\sqrt{t}} + A\frac{e^t}{\sqrt{t}}\int_{((3/4)\log t-\log A)/\sqrt{t}}^{2^{1/3}t^{1/6}}(1+u^2)(1+u)e^{-u^2}\mathrm{d}u \\
&\ll A\frac{e^t}{\sqrt{t}} + A\frac{e^t}{\sqrt{t}}\int_{u>0}(1+u^2)(1+u)e^{-u^2}\mathrm{d}u \\
&\ll A\frac{e^t}{\sqrt{t}}.
\end{aligned}$$

To prove the claim for $\theta \neq 0$, we proceed exactly as above while replacing every occurrence of ζ_τ with $\zeta_\tau e^{-S_{t_\theta}}$, S_k with $S_k^{(t_\theta)}$, and t with $t - t_\theta$. The application of Theorem 3.0.2 in Equation 3.1.1 is replaced with that of the following, similar bound

$$\mathbb{P}\Big(\max_{|h|\le e^{-t_\theta}} \log\left|\zeta_\tau(h)e^{-S_{t_\theta}(h)}\right| > m(t-t_\theta)+\log A\Big) \ll (\log A)e^{-2\log A} \ll \frac{1}{A},$$

which is a direct consequence of Theorem 3.0.2 and Lemma 3.6.8 (cf. Remark 3.3.3). We also use the $\theta < 0$ case of Theorem 3.0.4, and scale the factors a_j by $e^{-2\sqrt{c\log A}}$, where c is the constant from Proposition 3.0.5. Doing so shows that the probability that

$$\frac{1}{e^{t\theta}}\int_{|h|\le e^{\theta t}} |\zeta_\tau e^{-S_{t_\theta}}(h)|^2 dh \ll \frac{A}{e^{2\sqrt{c\log A}}}\frac{e^{t-t_\theta}}{\sqrt{t-t_\theta}} \tag{3.1.4}$$

holds is $1 - O(e^{2\sqrt{c\log A}}\log(A)/A)$.

Noting that $\sqrt{c\log A} < e^{t(1+\theta)/2}$ for large enough T since $A \le \sqrt{t}$, Proposition 3.0.5 also implies that

$$\max_{|h|\le \log^{\theta} T} |S_{t_\theta}(h) - S_{t_\theta}(0)| \le \sqrt{c\log A}$$

with probability $1 - O(1/A)$, and we can therefore place ourselves on the event where this inequality holds. This yields

$$e^{-2S_{t_\theta}(0)}\mathscr{Z}_2(\theta) \ll A\frac{e^{2\cdot\max_h |S_{t_\theta}(h)-S_{t_\theta}(0)|}}{e^{2\sqrt{c\log A}}}\frac{e^{t-t_\theta}}{\sqrt{t-t_\theta}} \le A\frac{e^{t-t_\theta}}{\sqrt{t-t_\theta}},$$

and the claim follows since $e^{-2S_{t_\theta}(0)} \asymp |P_\tau(\theta)|^2$.

3.2 Error process in the range of correlated primes

This section proves Proposition 3.0.5, which roughly states that over $|h| \le 2e^{t\theta}$, the contribution to $\log|\zeta_\tau(h)|$ coming from primes $p \le \exp(e^{t|\theta|})$ is typically close to its value at $h = 0$.

The claim is trivial for $\theta = 0$ as $S_{t_\theta} \equiv 0$. Otherwise, we can use Lemma 3.6.7 with $N = \exp(2e^{t_\theta})$ and $A = 100$ to write

$$\begin{aligned}&\mathbb{P}\Big(\max_{|h|\le 2e^{\theta t}} |S_{t_\theta}(0) - S_{t_\theta}(h)| > x\Big)\\ &\le \mathbb{P}\Big(\sum_{|j|\le 32} |S_{t_\theta}(0) - S_{t_\theta}(h_j)|^2 > x^2/C\Big) + \mathbb{P}\Big(\sum_{|j|>32} \frac{|S_{t_\theta}(0) - S_{t_\theta}(h_j)|^2}{(|j|^{100}+1)} > x^2/C\Big)\end{aligned} \tag{3.2.1}$$

where $h_j = h_j(t) = \pi j/(8e^{t_\theta})$, and $C > 0$ is a constant that is assumed to be large without loss of generality. We will bound these probabilities using the following moment estimate, whose proof is deferred to later in the section.

Lemma 3.2.1. *For any $h \in \mathbb{R}$ and any positive integer q satisfying $2q \le e^{t(1+\theta)}$,*

$$\mathbb{E}\Big\{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}\Big\} \ll \frac{(2q)!}{2^q q!}\big(e^{2t_\theta}|h|^2\big)^q.$$

To use this, we first note the bound

$$\mathbb{P}\Big(\sum_{|j|\le 32} |S_{t_\theta}(0) - S_{t_\theta}(h_j)|^2 > x^2/C\Big) \ll \sum_{|j|\le 32} \mathbb{P}\Big(|S_{t_\theta}(0) - S_{t_\theta}(h_j)|^2 > x^2/(64C)\Big).$$

By Markov's inequality and Lemma 3.2.1, this is

$$\ll \sum_{|j|\le 32} \frac{(2q)!}{2^q q!}\Big(\frac{64C\cdot e^{2t_\theta}|h_j|^2}{x^2}\Big)^q \ll \frac{(2q)!}{2^q q!}\Big(\frac{C'}{x^2}\Big)^q$$

for any positive integer q satisfying $2q \le e^{t(1+\theta)}$, where $C' = (32\pi)^2 C$. Picking $q = \lfloor x^2/2C' \rfloor$ (which is a valid choice by our assumption that $x^2 \le e^{t(1+\theta)}$, and the fact that $C > 1$), we can use Stirling's approximation to bound the right-hand side by

$$\ll \Big(\frac{2q}{e}\Big)^{2q}\Big(\frac{e}{2q}\Big)^q\Big(\frac{C'}{x^2}\Big)^q \ll e^{-q}\Big(\frac{2C'}{x^2}\cdot q\Big)^q \ll e^{-x^2/c},$$

for some constant $c > 0$, proving the claim.

To show that the remaining term in (3.2.1) is negligible, we first note that

$$\Big\{\sum_{|j|>32}\frac{|S_{t_\theta}(0)-S_{t_\theta}(h_j)|^2}{(|j|^{100}+1)} > \frac{x^2}{C}\Big\} \subseteq \bigcup_{|j|>32}\Big\{|S_{t_\theta}(0)-S_{t_\theta}(h_j)|^2 > \frac{x^2}{C}|j|^{50}\Big\}$$

since $\sum_{|j|>32}|j|^{50}(|j|^{100}+1)^{-1} < 1$. It follows that

$$\mathbb{P}\Big(\sum_{|j|>32}\frac{|S_{t_\theta}(0)-S_{t_\theta}(h_j)|^2}{(|j|^{100}+1)} > \frac{x^2}{C}\Big) \le \sum_{|j|>32}\mathbb{P}\Big(|S_{t_\theta}(0)-S_{t_\theta}(h_j)|^2 > \frac{x^2}{C}|j|^{50}\Big),$$

and we can bound each term in the resulting sum using Markov's inequality and Lemma 3.2.1. To be precise, picking $q = \lfloor x^2/(C\pi^2/32)\rfloor$ (which is $\le e^{t(1+\theta)}/2$ for $C > 64/\pi^2$) gives

$$\sum_{|j|>32}\mathbb{P}\Big(|S_{t_\theta}(0)-S_{t_\theta}(h_j)|^{2q} > \Big(\frac{x^2}{C}\Big)^q|j|^{50q}\Big) \ll \sum_{|j|>32}\frac{(2q)!}{2^q q!}\Big(\frac{C\pi^2}{64x^2}\Big)^q|j|^{-40q},$$

which is $\ll e^{-q}\sum_{|j|>1}|j|^{-2} \ll e^{-x^2/c}$ by Stirling's approximation, for a constant $c > 0$.

It remains to prove Lemma 3.2.1.

3.2.1 Proof of Lemma 3.2.1

We begin by introducing some notation. Let $(Z_p, p \text{ prime})$ be a sequence of independent and identically distributed random variables that are uniformly distributed on the unit circle. For any integer n with prime factorisation $\prod_i p_i^{\alpha_i}$ (where the p_i are distinct for different i), let

$$Z_n := \prod_{i=1} Z_{p_i}^{\alpha_i}.$$

Note that we have the orthogonality relation $\mathbb{E}Z_n\overline{Z_m} = 1_{n=m}$.

Consider the random variables

$$X_p(h) = \mathrm{Re}\left(Z_p p^{-1/2-ih} + \tfrac{1}{2}Z_p^2 p^{-1-2ih}\right),\ p \text{ prime}, \tag{3.2.2}$$

which replace $p^{-i\tau}$ by Z_p. The sums $\sum_{e^{1000}<\log p\le e^{t\theta}} X_p(0) - X_p(h)$ will serve as a random model for $S_{t_\theta}(0) - S_{t_\theta}(h)$. We will show that the former's moments approximate the latter's, and that they are roughly Gaussian.

We argue as in the proof of Lemma 16 in [10]. Let Φ be a smooth function on $\mathbb{R}$ satisfying $\Phi(x) \ge 0$ for all $x \in \mathbb{R}$, $\Phi(x) \gg 1$ for $x \in [-1,1]$, and $\hat{\Phi}$ is compactly supported in $[-1,1]$ (see [10], p. 37 for an example). Then for any collection $p_1, ..., p_k$ and $q_1, ..., q_\ell$ of (not necessarily distinct) primes satisfying $\prod_i p_i, \prod_j q_j \le T$,

$$\int_{\mathbb{R}} \left(\frac{p_1\cdots p_k}{q_1\cdots q_\ell}\right)^{it} \Phi\left(\frac{t}{2T}\right) dt = 2T\hat{\Phi}\left(2T\log\left(\frac{p_1\cdots p_k}{q_1\cdots q_\ell}\right)\right) = 2T\hat{\Phi}(0)\mathbf{1}_{p_1\cdots p_k = q_1\cdots q_\ell},$$

and the right-hand side equals $2T\hat{\Phi}(0)\mathbb{E}\left\{Z_{p_1\cdots p_k}\overline{Z}_{q_1\cdots q_\ell}\right\}$ by the aforementioned orthogonality relation.

It follows that for any h and any integer $q > 0$ satisfying $2q \le e^{t(1+\theta)}$, we can develop the product $(S_{t_\theta}(0) - S_{t_\theta}(h))^{2q}$ to get

$$\mathbb{E}\left\{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}\right\} \ll \mathbb{E}\left\{\left(\sum_{e^{1000}<\log p\le e^{t\theta}} X_p(0) - X_p(h)\right)^{2q}\right\}.$$

The right-hand side can then be compared to moments of a Gaussian analogue. Let

$$\mathcal{S}_1 = \sum_{e^{1000}<\log p\le e^{t\theta}} \mathrm{Re}\left(Z_p \cdot \frac{\Delta_1(p)}{p^{1/2}}\right), \quad \mathcal{S}_2 = \frac{1}{2}\sum_{e^{1000}<\log p\le e^{t\theta}} \mathrm{Re}\left(Z_p^2 \cdot \frac{\Delta_2(p)}{p}\right),$$

where $\Delta_j(p) := 1 - p^{-ijh}$, and similarly

$$\mathcal{G}_1 = \sum_{e^{1000}<\log p\le e^{t\theta}} \left(Y_p \cdot \frac{|\Delta_1(p)|}{p^{1/2}}\right), \quad \mathcal{G}_2 = \frac{1}{2}\sum_{e^{1000}<\log p\le e^{t\theta}} \left(Y_p \cdot \frac{|\Delta_2(p)|}{p}\right),$$

where the $(Y_p)_{p \text{ prime}}$ are centered, independent Gaussian random variables of variance 1/2. Noting that $\mathbb{E}\left\{(\mathrm{Re}Z_p)^{2m}\right\} \le \mathbb{E}\left\{Y_p^{2m}\right\}$ for any prime p and positive integer m, and that the odd moments of $\mathrm{Re}Z_p$ and Y_p are zero, the binomial theorem gives

$$\mathbb{E}\left\{\mathcal{S}_i^{2q}\right\} \le \mathbb{E}\left\{\mathcal{G}_i^{2q}\right\} \text{ for } i \in \{1,2\}.$$

We can therefore write

$$\begin{aligned}\mathbb{E}\left\{\left(\sum_{e^{1000}<\log p\le e^{t\theta}} X_p(0) - X_p(h)\right)^{2q}\right\} &= \mathbb{E}\left\{\left(\mathcal{S}_1 + \mathcal{S}_2\right)^{2q}\right\} \\ &\le \left(\mathbb{E}\left\{\mathcal{S}_1^{2q}\right\}^{1/2q} + \mathbb{E}\left\{\mathcal{S}_2^{2q}\right\}^{1/2q}\right)^{2q} \\ &\le \left(\mathbb{E}\left\{\mathcal{G}_1^{2q}\right\}^{1/2q} + \mathbb{E}\left\{\mathcal{G}_2^{2q}\right\}^{1/2q}\right)^{2q}.\end{aligned}$$

Since $\mathscr{G}_1$ and $\mathscr{G}_2$ are centred Gaussian random variables, we can then factor out the $2q$-th moment of a standard Gaussian from the right-hand side to get

$$\frac{(2q)!}{2^q q!}\big(\sigma_1+\sigma_2\big)^{2q}, \quad \text{where } \sigma_j^2 = \sum_{e^{1000}<\log p\le e^{t_\theta}} \frac{|\Delta_j(p)|^2}{2j^2p^j} \text{ for } j\in\{1,2\}.$$

Since $|\Delta_j(p)|^2 = 2-2\cos(j|h|\log p) \le j^2|h|^2(\log p)^2$, this is no greater than

$$\frac{(2q)!}{2^q q!}\big(2\sigma_1^2+2\sigma_2^2\big)^q \le \frac{(2q)!}{2^q q!}\left(|h|^2\sum_{e^{1000}<\log p\le e^{t_\theta}}\frac{(\log p)^2}{p}+|h|^2\sum_{e^{1000}<\log p\le e^{t_\theta}}\frac{(\log p)^2}{p^2}\right)^q.$$

Using Mertens' theorem and summation by parts, the first prime sum is $\frac{1}{2}e^{2t_\theta}+O(e^{t_\theta})$, and the second is $O(1)$. The total is then $\le e^{2t_\theta}$ for large t.

3.3 Upper bound for the maximum

In this section, we show how the recursive scheme in [10] can be adapted to prove Theorem 3.0.2, emphasising the ways in which the proof below differs from that of Theorem 2 in [10]. In the process, we will need three auxiliary lemmas (Lemma 3.3.7, Lemma 3.3.9 and Lemma 3.3.10), whose proofs are postponed to Section 3.4.

3.3.1 Initial reductions

Throughout this section, assume that $\theta\in(-1,0)$ (the $\theta=0$ case being the main result in [10]). To begin with, note that we may replace $|P_\tau(\theta)|$ in the statement of Theorem 3.0.2 by $e^{-S_{t_\theta}(0)}$, since $\log|P_\tau(\theta)| = -S_{t_\theta}(0)+O(1)$. By Lemma 3.6.8, it also suffices to study the maximum on the discrete set

$$\mathscr{T}_t^\theta = e^{-t-100}\mathbb{Z}\cap[-2e^{-t_\theta},2e^{-t_\theta}]. \tag{3.3.1}$$

Theorem 3.0.2 thus follows from the following.

Theorem 3.3.1. *Let $C>0$ be arbitrary and τ be uniformly distributed in $[T,2T]$. Then there exists an absolute constant c such that for large enough T and $c<y\le C(t-t_\theta)/\log(t-t_\theta)$,*

$$\mathbb{P}\Big(\max_{h\in\mathscr{T}_t^\theta}\log|\zeta_\tau(h)| > y+(t-t_\theta)-\frac{3}{4}\log(t-t_\theta)+S_{t_\theta}(0)\Big) \ll_C ye^{-2y-y^2/(t-t_\theta)}.$$

Consider the following subset of high points h of $\zeta_\tau e^{-S_{t_\theta}}$

$$H^\theta(y) = \left\{ h \in \mathscr{T}_t^\theta \ : \ |\zeta_\tau e^{-S_{t_\theta}}(h)| > e^y \frac{e^{t-t_\theta}}{(t-t_\theta)^{3/4}} \right\},$$

as well as the set

$$E^\theta(u) = \left\{ h \in \mathscr{T}_t^\theta \ : \ S_{t_\theta}(0) - S_{t_\theta}(h) \in [u, u+1) \right\}, \quad u \in \mathbb{R}, \tag{3.3.2}$$

which controls how close $S_{t_\theta}(h)$ is to $S_{t_\theta}(0)$ for $h \in \mathscr{T}_t^\theta$. Note that by Proposition 3.0.5, the probability that $h \in E^\theta(u)$ should obey a Gaussian tail provided u is not too large.

The following result bounds the probability that the set $H^\theta(y) \cap E^\theta(u)$ is non-empty. Heuristically, it gives an upper bound on the maximum of $\log|\zeta_\tau(h)| - S_{t_\theta}(h)$, and shows that this quantity behaves somewhat independently from the error $S_{t_\theta}(0) - S_{t_\theta}(h)$. This result implies Theorem 3.3.1 by essentially integrating over u, as we now show.

Theorem 3.3.2. *Let $C > 0$ be an arbitrary constant. Then for large enough t, $1 \le y < C(t-t_\theta)/\log(t-t_\theta)$ and $|u| \le C\sqrt{(t-t_\theta)}$, we have*

$$\mathbb{P}\big(\exists h \in H^\theta(y) \cap E^\theta(u)\big) \ll_C y e^{-2y - y^2/(t-t_\theta)} e^{-u^2/C_0}, \tag{3.3.3}$$

for an absolute constant $C_0 > 0$. Furthermore, $\mathbb{P}\big(\exists h \in H^\theta(y)\big) \ll y e^{-2y-y^2/(t-t_\theta)}$.

Proof of Theorem 3.3.1 from 3.3.2. Partitioning according to the value $u \in \mathbb{Z}$ of $S_{t_\theta}(0) - S_{t_\theta}(h)$, we have

$$\begin{aligned} &\mathbb{P}\Big(\max_{h \in \mathscr{T}_t^\theta} \log|\zeta_\tau(h)| > y + (t-t_\theta) - \frac{3}{4}\log(t-t_\theta) + S_{t_\theta}(0) \Big) \\ &\quad \ll \mathbb{P}\Big(\exists h \ : \ |S_{t_\theta}(0) - S_{t_\theta}(h)| > \sqrt{c\big(2y + y^2/(t-t_\theta)\big)} \Big) \\ &\qquad + \mathbb{P}\big(\exists h \in H^\theta(y)\big) + \sum_{-\sqrt{c(2y+y^2/(t-t_\theta))} < u < 0} \mathbb{P}\Big(\exists h \in H^\theta\big(y+u\big) \cap E^\theta(u)\Big), \end{aligned} \tag{3.3.4}$$

where c is the constant from Proposition 3.0.5. The cutoff $\sqrt{c(2y + y^2/(t-t_\theta))}$ was chosen to make the first term in (3.3.4) $\ll e^{-2y-y^2/(t-t_\theta)}$ by Proposition 3.0.5, and we note that $1 \le y + u \le y$ in this range provided $y > 3(c+1)$ and t is large enough. To bound what remains, we can therefore apply Theorem 3.3.2 to each term to get

$$\begin{aligned} &\ll y e^{-2y-y^2/(t-t_\theta)} + \sum_{-\sqrt{c(2y+y^2/(t-t_\theta))} < u < 0} (y+u) e^{-2(y+u) - (y+u)^2/(t-t_\theta)} e^{-u^2/C_0} \\ &\ll y e^{-2y-y^2/(t-t_\theta)} + y e^{-2y-y^2/(t-t_\theta)} \sum_u e^{-f(u)}. \end{aligned} \tag{3.3.5}$$

where $f(u) = 2u + (u^2 + 2yu)/(t - t_\theta) + u^2/C_0$. The claim then follows since

$$\sum_{u\in\mathbb{Z}} e^{-f(u)} \ll \int_{u\in\mathbb{R}} e^{-f(u)}\mathrm{d}u = \Big(\frac{\pi}{(t-t_\theta)^{-1} + C_0^{-1}}\Big)^{1/2} \exp\Big(\frac{C_0(t-t_\theta+y)^2}{(t-t_\theta)(C_0+t-t_\theta)}\Big) \ll 1.$$

□

Remark 3.3.3. The second claims in Lemma 3.6.8 and Theorem 3.3.2 together imply that

$$\mathbb{P}\Big(\max_{|h|\le e^{-t_\theta}} \log\big|\zeta_\tau(h)e^{-S_{t_\theta}(h)}\big| > y + (t-t_\theta) - \frac{3}{4}\log(t-t_\theta)\Big) \ll ye^{-2y-y^2/(t-t_\theta)}.$$

We record this bound here, as it is used in the proof of Theorem 3.0.3 in the case when $\theta \neq 0$ (see Section 3.1).

3.3.2 The recursive scheme

We now prove Theorem 3.3.2, focusing on the first claim (Equation (3.3.3)); the bound on $\mathbb{P}(\exists h \in H^\theta(y))$ follows from the same argument upon ignoring the event $E^\theta(u)$.

For each $h \in \mathscr{T}_t^\theta$, recall from Section 3.0.2 that $S_t(h)$ can be thought of as a random walk with t steps, and that the random variables $(S_k(h))_{|h|\le e^{-t_\theta}}$ are highly correlated for $k \le t_\theta$. We will treat this range of primes separately (via the event $E^\theta(u)$), and adapt the recursive scheme from [10] to prove large deviation estimates for the shifted walks

$$S_k^{(t_\theta)}(h) := S_{k+t_\theta}(h) - S_{t_\theta}(h), \quad 0 \le k \le t - t_\theta. \tag{3.3.6}$$

These serve as a proxy for $\log|\zeta_\tau e^{-S_{t_\theta}}(h)|$ when $k \approx t - t_\theta$.

To that end, we introduce the time scales

$$t_\ell := (t-t_\theta) - \mathfrak{s}\log_\ell(t-t_\theta) \tag{3.3.7}$$

where $\log_\ell$ is the $\ell-$th iterated logarithm (assuming $\ell \ge 1$) and $\mathfrak{s}$ is a large constant (for this section, $\mathfrak{s} = 10^6$ will do). The final time $\ell = \mathscr{L}$ is defined as the largest integer ℓ for which

$$\exp(10^6((t-t_\theta)-t_\ell)^{10^5} e^{t_{\ell+1}+t_\theta}) \le T^{1/100}. \tag{3.3.8}$$

While it is not immediately clear that such an $\mathscr{L}$ exists, a quick manipulation reduces this condition to $\log_\ell(t-t_\theta) > (10^8\mathfrak{s}^{10^5})^{1/(\mathfrak{s}-10^5)}$($\approx 4.64$ for our choice of $\mathfrak{s}$), from which this is more apparent. This definition also ensures that $(t-t_\theta) - t_{\mathscr{L}} = O(1)$ and $\log_{\mathscr{L}}(t-t_\theta) > 0$.

The recursive scheme consists of partitioning the set $H^\theta(y) \cap E^\theta(u)$ of high points h in terms of *decreasing good sets* $G(t_\ell) = G_\ell$, $0 \le \ell \le \mathscr{L}$, defined in (3.3.15) below. The probability in Theorem 3.3.2 can then be decomposed as

$$\begin{aligned}\mathbb{P}\big(\exists h \in H^\theta(y) \cap E^\theta(u)\big) \le{}& \mathbb{P}\big(\exists h \in H^\theta(y) \cap G_0^c \cap E^\theta(u)\big) \\ &+ \sum_{\ell=1}^{\mathscr{L}-1} \mathbb{P}\big(\exists h \in H^\theta(y) \cap (G_\ell \setminus G_{\ell+1}) \cap E^\theta(u)\big) \\ &+ \mathbb{P}\big(\exists h \in H^\theta(y) \cap G_{\mathscr{L}} \cap E^\theta(u)\big).\end{aligned} \tag{3.3.9}$$

The sets G_ℓ are built around the observation that the points $h \in H^\theta(y)$ at which $|\zeta_\tau e^{-S_{t_\theta}}|$ achieves a high value are those for which the (approximate) random walk $S_k^{(t_\theta)}$ remains within the corridor

$$(m(k) + L_y(k), m(k) + U_y(k)],$$

consisting of upper and lower *barriers*

$$L_y(k) := y - \begin{cases} \infty & \text{if } 1 \le k \le \lceil y/4 \rceil \\ 20k & \text{if } y/4 \le k \le t_0 := (t - t_\theta)/2 \\ 20\big((t - t_\theta) - k\big) & \text{if } t_0 < k < t - t_\theta \end{cases} \tag{3.3.10}$$

$$U_y(k) := y + \begin{cases} \infty & \text{if } 1 \le k \le \lceil y/4 \rceil \\ 10^3 \log k & \text{if } y/4 \le k \le t_0 = (t - t_\theta)/2 \\ 10^3 \log\big((t - t_\theta) - k\big) & \text{if } t_0 < k < t - t_\theta \end{cases} \tag{3.3.11}$$

centred at the linear interpolation from y to $y + (t - t_\theta) - \frac{3}{4}\log(t - t_\theta)$:

$$m(k) = \alpha k, \qquad \alpha = 1 - \frac{3}{4}\frac{\log(t - t_\theta)}{t - t_\theta}.$$

On these events G_ℓ, one can obtain large deviation estimates for $S_k^{(t_\theta)}$ by expressing the probabilities in question as moments of Dirichlet polynomials. The barriers restrict the length of these polynomials, so that their moments can be computed with small enough error via the mean-value theorem in Lemma 3.6.4 (see the proof of Lemmas 3.3.9 and 3.3.10). Were it not for these restrictions, estimating the probability that an increment $S_k^{(t_\theta)} - S_{k-1}^{(t_\theta)}$ is atypically large could force one to consider Dirichlet polynomials of length much larger than T in Lemma 3.6.4.

The form of these barriers is motivated as follows. If conditioned to end around $m(t - t_\theta)$, the approximate random walk $S_{t-t_\theta}^{(t_\theta)}$ (of length $t - t_\theta$) should typically fluctuate like a Brownian bridge, i.e. like $(k \wedge (t - t_\theta - k))^{1/2}$. We may therefore take any power of

$(k \wedge (t - t_\theta - k))$ greater than 1/2 in the lower barrier $L_y(k)$, and we choose the power 1 for simplicity. For the upper barrier, we keep with the branching random walk heuristic which suggests that one should (indeed must) pick a barrier lying below the typical fluctuations; we pick a large multiple of $\log(k \wedge (t - t_\theta - k))$ following Bramson [37], and this is ultimately responsible for the appearance of ballot-theorem-type savings in Lemmas 3.3.9 and 3.3.10 (cf. Proposition 3.6.9). For technical reasons, we do not impose any restrictions for small k in (3.3.11) and (3.3.10) (see, e.g., Equation (3.3.25) in the proof of Proposition 3.3.4).

From these barriers, we define the decreasing sets

$$B_\ell = B_{\ell-1} \cap \left\{ h \in \mathscr{T}_t^\theta \ : \ S_k^{(t_\theta)}(h) \le m(k) + U_y(k) \text{ for all } k \in (t_{\ell-1}, t_\ell] \right\}$$
$$C_\ell = C_{\ell-1} \cap \left\{ h \in \mathscr{T}_t^\theta \ : \ S_k^{(t_\theta)}(h) > m(k) + L_y(k) \text{ for all } k \in (t_{\ell-1}, t_\ell] \right\}$$

for $\ell \ge 1$, and

$$B_0 = \left\{ h \in \mathscr{T}_t^\theta \ : \ S_k^{(t_\theta)}(h) \le m(k) + U_y(k) \text{ for all } k \in [1, t_0] \right\}$$
$$C_0 = \left\{ h \in \mathscr{T}_t^\theta \ : \ S_k^{(t_\theta)}(h) > m(k) + L_y(k) \text{ for all } k \in [1, t_0] \right\}.$$

To relate ζ to the partial sums $S_k^{(t_\theta)}$, we follow [10] and define the following mollifiers:

$$\mathscr{M}_\ell(h) := \sum_{\substack{p|m \Rightarrow \log_2 p \in (t_{\ell-1}+t_\theta, t_\ell+t_\theta] \\ \Omega_\ell(m) \le (t_\ell - t_{\ell-1})^{10^5}}} \frac{\mu(m)}{m^{1/2+i\tau+ih}}, \tag{3.3.12}$$

for $\ell \ge 0$, adopting the convention $\mathscr{M}_{-1} \equiv 1$ and $t_{-1} = 1000 - t_\theta$ so that $\mathscr{M}_0$ is a sum over all primes $\exp(1000) \le p \le \exp(e^{t_\theta + t_0})$. Here, $\Omega_\ell(m)$ denotes the number of distinct prime factors of m in the interval $(\exp(e^{t_{\ell-1}+t_\theta}), \exp(e^{t_\ell+t_\theta})]$. The goal of the mollifier $\mathscr{M}_\ell(h)$ is to act as a "statistical" inverse for the contribution to ζ coming from the primes in the range $(\exp(e^{t_{\ell-1}+t_\theta}), \exp(e^{t_\ell+t_\theta})]$; heuristically, one would want to take

$$\prod_{\log_2 p \in (t_{\ell-1}+t_\theta, t_\ell+t_\theta]} \left(1 - \frac{1}{p^{1/2+i(\tau+h)}}\right) = \sum_{p|m \Rightarrow \log_2 p \in (t_{\ell-1}+t_\theta, t_\ell+t_\theta]} \frac{\mu(m)}{m^{1/2+i\tau+ih}},$$

but this Dirichlet polynomial has length far exceeding T, rendering Lemma 3.6.4 inapplicable. The restriction on Ω_ℓ in the definition of $\mathscr{M}_\ell$ truncates this polynomial, shortening it to an admissible length. (Note that for a typical integer n, $\Omega_\ell(n)$ should roughly have size $t_\ell - t_{\ell-1}$; the bound $(t_\ell - t_{\ell-1})^{10^5}$ therefore ensures that most integers are considered but none of them are too large.) We will also need the following refinement of $\mathscr{M}_\ell$ up to $k + t_\theta$ for $k \in (t_{\ell-1}, t_\ell]$:

$$\mathscr{M}_{\ell-1}^{(k)}(h) = \sum_{\substack{p|m \Rightarrow \log_2 p \in (t_{\ell-1}+t_\theta, k+t_\theta] \\ \Omega_\ell(m) \le (t_\ell - t_{\ell-1})^{10^5}}} \frac{\mu(m)}{m^{1/2+i\tau+ih}}, \quad \ell \ge -1.$$

In this notation, $\mathscr{M}_{\ell-1}^{(t_\ell)} = \mathscr{M}_\ell$ and the product $\mathscr{M}_{-1}\cdots\mathscr{M}_{\ell-1}\mathscr{M}_{\ell-1}^{(k)}$ will serve as an approximant for $e^{-S_{t_\theta+k}}$ (see Lemma 3.3.8).

To ensure that this approximation holds with small enough error, we introduce the complex version of the shifted partial sums $S_k^{(t_\theta)}$

$$\tilde{S}_k^{(t_\theta)}(h) = \sum_{e^{1000+t_\theta}<\log p\le e^{k+t_\theta}} \frac{1}{p^{1/2+i(\tau+h)}} + \frac{1}{2}\frac{1}{p^{1+2i(\tau+h)}}, \quad k \le t-t_\theta, \tag{3.3.13}$$

and restrict to points h belonging to the following sets:

$$A_\ell = A_{\ell-1} \cap \Big\{ h \in \mathscr{T}_t^\theta \ : \ |\tilde{S}_k^{(t_\theta)}(h) - \tilde{S}_{t_{\ell-1}}^{(t_\theta)}(h)| \le 10^3(t_\ell - t_{\ell-1}) \text{ for all } k \in (t_{\ell-1}, t_\ell] \Big\}$$
$$D_\ell = D_{\ell-1} \cap \big\{ h \in \mathscr{T}_t^\theta \ : \ |\zeta_\tau e^{-S_{k+t_\theta}}(h)| \le c_\ell |\zeta_\tau \mathscr{M}_{-1}\cdots\mathscr{M}_{\ell-1}^{(k)}(h)| + e^{-10^4((t-t_\theta)-t_{\ell-1})}$$
$$\text{for all } k \in (t_{\ell-1}, t_\ell]\big\},$$

where

$$c_\ell := \prod_{i=0}^{\ell}(1 + e^{-(t_{i-1}+t_\theta)}), \tag{3.3.14}$$

and $A_{-1} = D_{-1} = [-2\log^\theta T, 2\log^\theta T]$.

Putting everything together, we define the decreasing good sets to be

$$G_\ell = A_\ell \cap B_\ell \cap C_\ell \cap D_\ell, \quad \ell \ge -1. \tag{3.3.15}$$

While we only consider $\theta \in (-1,0)$, note that at $\theta = 0$, $S_k^{(t_\theta)} = S_k$ for every k and these definitions reduce to their counterparts in [10].

We evaluate the probabilities in (3.3.9) using the following propositions.

Proposition 3.3.4. *Fix $\theta \in (-1,0)$. Let $C > 0$ be arbitrary, $1 \le y < C(t-t_\theta)/\log(t-t_\theta)$ and $|u| \le C\sqrt{t-t_\theta}$. Then there exists a constant $K = K(C) > 0$ and an absolute constant $c > 0$ such that*

$$\mathbb{P}\Big(\exists h \in H^\theta(y) \cap G_0^c \cap E^\theta(u)\Big) \le K y e^{-2y-y^2/(t-t_\theta)} e^{-u^2/c}.$$

Proposition 3.3.5. *Fix $\theta \in (-1,0)$. Let $C > 0$ be arbitrary, $1 \le y < C(t-t_\theta)/\log(t-t_\theta)$, $|u| \le C\sqrt{t-t_\theta}$, and $\ell \ge 0$ satisfy (3.3.8). Then there exists a constant $K = K(C) > 0$ and an absolute constant $c > 0$ such that*

$$\mathbb{P}\Big(\exists h \in H^\theta(y) \cap (G_\ell \setminus G_{\ell+1}) \cap E^\theta(u)\Big) \le K \frac{y e^{-2y-y^2/(t-t_\theta)}}{\log_{\ell+1}(t-t_\theta)} \cdot e^{-u^2/c}.$$

Proposition 3.3.6. *Fix $\theta \in (-1,0)$. Let $C > 0$ be arbitrary, $1 \le y < C(t-t_\theta)/\log(t-t_\theta)$, and $|u| \le C\sqrt{t-t_\theta}$. Then there exists a constant $K = K(C) > 0$ and an absolute constant $c > 0$ such that*

$$\mathbb{P}\Big(\exists h \in H^\theta(y) \cap G_{\mathscr{L}} \cap E^\theta(u)\Big) \le K y e^{-2y-y^2/(t-t_\theta)} e^{-u^2/c}.$$

Proof of Theorem 3.3.2. Using Propositions 3.3.4, 3.3.5 and 3.3.6 in the decomposition of Equation (3.3.9), we get

$$\mathbb{P}\Big(\exists h \in H^\theta(y) \cap E^\theta(u)\Big) \ll_C y e^{-2y-y^2/(t-t_\theta)} e^{-u^2/c} + \sum_{1 \le \ell \le \mathscr{L}} \frac{y e^{-2y-y^2/(t-t_\theta)} e^{-u^2/c}}{\log_\ell(t-t_\theta)},$$

for some $c > 0$, and the claim then follows from the fact that $\sum_{1\le \ell \le \mathscr{L}} \frac{1}{\log_\ell(t-t_\theta)} = O(1)$. That this sum is bounded can be seen by the change of variables $n = \mathscr{L} - \ell$, which gives

$$\sum_{1 \le \ell \le \mathscr{L}} \frac{1}{\log_\ell(t-t_\theta)} = \frac{1}{\log_{\mathscr{L}}(t-t_\theta)} + \sum_{1 \le n \le \mathscr{L}-1} \frac{1}{\exp_n\big(\log_{\mathscr{L}}(t-t_\theta)\big)} \tag{3.3.16}$$

where $\exp_n$ denotes the exponential map iterated n times. Recalling that $\log_{\mathscr{L}}(t-t_\theta)$ is a constant by definition of $\mathscr{L}$ (cf. Equation (3.3.8)), it is clear that this sum is rapidly convergent in n. □

Propositions 3.3.4–3.3.6 are proved in the following three subsections. In their proofs, the implicit constants will be allowed to depend on the constant C in the statement of the corresponding proposition. For simplicity, we will also use the abbreviation

$$g(y,\theta) = y e^{-2y-y^2/(t-t_\theta)} \tag{3.3.17}$$

for the desired tail bound in what follows.

3.3.3 Proof of Proposition 3.3.4

The proof is based on the fact that the Dirichlet polynomials $S_k^{(t_\theta)}$ and S_{t_θ} behave to some extent like independent Gaussians, being supported on disjoint ranges of primes. This is for instance visible in the following lemma, required for the proof, whose proof is deferred to Section 3.4.

Lemma 3.3.7. *Let $C > 0$ be arbitrary, $y < C(t-t_\theta)/\log(t-t_\theta)$ and $|u| < C\sqrt{t-t_\theta}$. Suppose that $\mathcal{Q}$ is Dirichlet polynomial of length $N \le \exp(\frac{1}{100}e^t)$, supported on integers whose prime factors are all $> \exp(e^{t_\theta})$. Then there exists an absolute constant $c > 0$ such that*

$$\begin{aligned}&\mathbb{E}\left\{\max_{|h|\le 2\log^\theta T} |\mathcal{Q}(\tfrac12 + i\tau + ih)|^2 \cdot \mathbf{1}\big(h \in E^\theta(u)\big)\right\}\\ &\ll_C \big(e^{-t_\theta}\log N + 1\big)\,\mathbb{E}\left\{|\mathcal{Q}(\tfrac12 + i\tau)|^2\right\} e^{-u^2/c}.\end{aligned}$$

We now prove Proposition 3.3.4, beginning with a union bound to get

$$\begin{aligned}&\mathbb{P}\big(\exists h \in H^\theta(y) \cap E^\theta(u) \cap G_0^c\big)\\ &\le \mathbb{P}\big(\exists h \in A_0^c\big) + \mathbb{P}\big(\exists h \in B_0^c \cap E^\theta(u)\big)\\ &+ \mathbb{P}\big(\exists h \in D_0^c \cap A_0\big) + \mathbb{P}\big(\exists h \in C_0^c \cap E^\theta(u)\big).\end{aligned} \tag{3.3.18}$$

Recall that $t_0 = (t - t_\theta)/2$ and that $g(y,\theta) = ye^{-2y-y^2/(t-t_\theta)}$ (cf. Equation (3.3.17)). By a union bound on h and $k \le t_0 - t_{-1} = (t + t_\theta)/2$, and the Gaussian tail bound in Equation (3.6.3), we have

$$\mathbb{P}\big(\exists h \in A_0^c\big) \ll e^{t(1+\theta)}(t + t_\theta)\exp(-10^2(t_0 + t_\theta)) \ll g(y,\theta)e^{-u^2/c},$$

where the rightmost inequality holds in the assumed ranges of y and u.

For the second term in Equation (3.3.18), note that if there exists a $k \in [1, t_0]$ and h for which

$$S_k^{(t_\theta)}(h) > m(k) + U_y(k),$$

then k must be greater than $y/4$ (the right-hand side being infinite otherwise). It follows that if such a k exists, then

$$\sum_{y/4<k\le t_0} \max_{|h|\le 2\log^\theta T} \frac{|S_k^{(t_\theta)}(h)|^{2q_k}}{(y + k + 10\log k)^{2q_k}} \ge 1, \text{ for any sequence } (q_k)_k \subseteq \mathbb{Z}_{\ge 1}$$

and we thus have the bound

$$\mathbb{P}\big(\exists h \in B_0^c \cap E^\theta(u)\big) \le \sum_{y/4<k\le t_0} \mathbb{E}\left\{ \max_{|h|\le 2\log^\theta T} \frac{|S_k^{(t_\theta)}(h)|^{2q_k}}{(y + k + 10\log k)^{2q_k}} \mathbf{1}\big(h \in E^\theta(u)\big) \right\}. \tag{3.3.19}$$

Picking $q_k = \lceil (k + 10\log k + y)^2/k \rceil$, we find that $S_k^{(t_\theta)}(h)^{2q_k}$ is a Dirichlet polynomial of length $\exp(2q_k e^{k+t_\theta})(\le \exp(\frac{1}{100}e^t)$ for large enough t). We can thus apply Lemma 3.3.7 to each summand in (3.3.19) to get

$$\mathbb{P}\big(\exists h \in B_0^c \cap E^\theta(u)\big) \ll \sum_{y/4<k\le t_0} e^{-u^2/c}e^k q_k \mathbb{E}\left\{ \frac{|S_k^{(t_\theta)}(0)|^{2q_k}}{(y + k + 10\log k)^{2q_k}} \right\}.$$

Using the Gaussian moment estimate in Lemma 3.6.2, followed by Stirling's approximation as in the proof of Corollary 3.6.3 (cf. Equation (3.6.5)), we have that

$$\mathbb{E}\left\{ \frac{|S_k^{(t_\theta)}(0)|^{2q_k}}{(y + k + 10\log k)^{2q_k}} \right\} \ll e^{-q_k} \tag{3.3.20}$$

for each $y/4 < k \le t_0$. The sum is thus bounded by

$$\ll e^{-u^2/c} \sum_{y/4<k\le t_0} e^k \frac{(k+y)^2}{k} \exp\big(-(k+y+10\log k)^2/k\big) \tag{3.3.21}$$

$$\ll e^{-u^2/c} e^{-y^2/(t-t_\theta)-2y} \sum_{y/4<k\le t_0} e^k \frac{(k+y)^2}{k} e^{-k-20\log k} \tag{3.3.22}$$

$$\ll e^{-u^2/c} g(y,\theta) \sum_{y/4<k\le t_0} (k+y^2k^{-1})k^{-20} \tag{3.3.23}$$

$$\ll e^{-u^2/c} g(y,\theta), \tag{3.3.24}$$

where in the second line, we used the fact that $e^{-y^2/k} \le e^{-y^2/(t-t_\theta)}$ uniformly over $k \le t_0$.

We use a similar approach to deal with C_0^c, with the bound

$$\mathbb{P}\big(\exists h \in C_0^c \cap E^\theta(u)\big) \le \sum_{y/4<k\le t_0} \mathbb{E}\left\{ \max_{|h|\le \log^\theta T} \frac{|S_k^{(t_\theta)}(h)|^{2q_k}}{\big(20k-y\big)^{2q_k}} \mathbf{1}(h \in E^\theta(u)) \right\},$$

holding for any $q_k \ge 1$. By picking $q_k = \lceil (20k-y)^2/k \rceil$ (making $S_k^{(t_\theta)}(h)^{2q_k}$ a Dirichlet polynomial of length $\le \exp(2q_k e^{k+t_\theta}) \le \exp(\frac{1}{100}e^t)$) and applying Lemma 3.3.7, this is

$$\ll e^{-u^2/c} \sum_{y/4<k\le t_0} (q_k e^k) \mathbb{E}\left\{ \frac{|S_k^{(t_\theta)}(0)|^{2q_k}}{\big(20k-y\big)^{2q_k}} \right\}.$$

Using the Gaussian moment estimate in Lemma 3.6.2 for $S_k^{(t_\theta)}$ and Stirling's approximation as we did for Equation (3.3.20) gives

$$\ll e^{-u^2/c} \sum_{y/4<k\le t_0} q_k e^k \exp\Big(-\frac{(20k-y)^2}{k}\Big).$$

By the estimate $q_k \ll k$ in $y/4 < k \le t_0$ and elementary manipulations, this is

$$\ll e^{-u^2/c} \sum_{y/4<k\le t_0} k e^k \exp\Big(-400k - \frac{y^2}{k} + 40y\Big) \tag{3.3.25}$$

$$\ll e^{-u^2/c} \exp\big(-2y - y^2/(t-t_\theta)\big). \tag{3.3.26}$$

We conclude by proving that $\mathbb{P}(\exists h \in D_0^c \cap A_0)$ in Equation (3.3.18) is $\ll e^{-100t}$, from which the proposition follows since e^{-100t} is much smaller than $g(y,\theta)e^{-u^2/c}$ in our range of y and u. This requires following lemma, which justifies the use of mollifiers as a proxy for $e^{-(S_k^{(t_\theta)} - S_{t_{\ell-1}}^{(t_\theta)})}$.

Lemma 3.3.8 (Lemma 23 in [10]). *Let $\ell \geq 0$, $k \in (t_{\ell-1}, t_\ell]$ and suppose that $h \in A_\ell$. Then*

$$e^{-(S_k^{(t_\theta)}(h) - S_{t_{\ell-1}}^{(t_\theta)}(h))} \leq (1 + e^{-(t_{\ell-1}+t_\theta)})|\mathscr{M}_{\ell-1}^{(k)}(h)| + e^{-10^5(t_\ell - t_{\ell-1})}.$$

Proof. The proof is identical to that of Lemma 23 in [10], noting that their argument holds for our choice of times t_ℓ and mollifiers $\mathscr{M}_\ell^{(k)}$. □

Recall that $t_{-1} = 1000 - t_\theta$ by convention and $c_0 = 1 + e^{-1000}$ by definition (cf. (3.3.14)). This implies that $S_k^{(t_\theta)} - S_{t_{-1}}^{(t_\theta)} = S_{t_\theta + k}$. With this notation, if $h \in \mathscr{T}_t^\theta \cap A_0$ is such that $|\zeta_\tau(h)| \leq e^{100t}$, then the previous lemma implies that

$$\begin{aligned} |(\zeta_\tau e^{-S_{t_\theta+k}})(h)| &\leq c_0 |\zeta_\tau \mathscr{M}_{-1}^{(k)}(h)| + e^{100t - 10^5(t_0 - t_{-1})} \\ &\leq c_0 |\zeta_\tau \mathscr{M}_{-1}^{(k)}(h)| + e^{-10^4(t - t_\theta - t_{-1})}. \end{aligned}$$

It therefore suffices to show that $\mathbb{P}(\exists h : |\zeta_\tau(h)| > e^{100t}) \ll g(y,\theta)e^{-u^2/c}$. This follows immediately from a union bound over $h \in \mathscr{T}_t^\theta$, followed by Chebyshev's inequality, using a classical second moment bound for the zeta function (Lemma 3.6.6):

$$\mathbb{P}\big(\exists h \in \mathscr{T}_t^\theta : |\zeta_\tau(h)| > e^{100t}\big) \ll e^{t(1+\theta)} \cdot \frac{\mathbb{E}\left\{|\zeta_\tau(h)|^2\right\}}{e^{200t}} \ll e^{-199t+t}.$$

3.3.4 Proof of Proposition 3.3.5

We follow the strategy behind the proof of Proposition 2 in [10], making the necessary changes to consider the walk $S_k^{(t_\theta)}$ instead of S_k. We will need the following refinements of A_ℓ, B_ℓ, C_ℓ and D_ℓ to study increments of $S_k^{(t_\theta)}$ for $k \in (t_\ell, t_{\ell+1}]$:

$$\begin{aligned} A_\ell^{(k)} &= A_\ell \cap \{h \in \mathscr{T}_t^\theta \ : \ |\tilde{S}_j^{(t_\theta)}(h) - \tilde{S}_{t_{\ell-1}}^{(t_\theta)}(h)| \leq 10^3(t_\ell - t_{\ell-1}) \text{ for all } t_\ell < j \leq k\} \\ B_\ell^{(k)} &= B_\ell \cap \{h \in \mathscr{T}_t^\theta \ : \ S_j^{(t_\theta)}(h) \leq m(j) + U_y(j) \text{ for all } t_\ell < j \leq k\} \\ C_\ell^{(k)} &= C_\ell \cap \{h \in \mathscr{T}_t^\theta \ : \ S_j^{(t_\theta)}(h) > m(j) + L_y(j) \text{ for all } t_\ell < j \leq k\} \\ D_\ell^{(k)} &= D_\ell \cap \{h \in \mathscr{T}_t^\theta \ : \ |\zeta_\tau e^{-S_{j+t_\theta}}(h)| \leq c_{\ell+1}|\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell \mathscr{M}_\ell^{(k)}(h)| \\ &\qquad + e^{-10^4((t-t_\theta)-t_{\ell-1})} \text{ for all } t_\ell < j \leq k\} \end{aligned}$$

where $c_{\ell+1} = \prod_{i=0}^{\ell+1}(1 + e^{-t_{i-1}+t_\theta})$, and $A_\ell^{(t_{\ell+1})} = A_{\ell+1}$ (similarly for $B_{\ell+1}, C_{\ell+1}$ and $D_{\ell+1}$).

The proof relies on two lemmas, whose proofs are again deferred to Section 3.4. The intuition behind them is that the expectation of a product of random variables depending only on primes p in $(2, \exp(e^{t_\theta})]$, $(\exp(e^{t_\theta}), \exp(e^{t_\theta+k})]$ and $(\exp(e^{t_\theta+k}), \exp(e^t)]$, respectively, should decouple into the product of the corresponding expectations.

In the following lemma, the random variables in question are $\mathbf{1}(E^\theta(u))$, an indicator involving the sums $(S_j^{(t_\theta)})_j$ for $j \in (1,k]$, and the maximum over $|h| \le 2\log^\theta T$ of a (short enough) Dirichlet polynomial supported on primes $p > \exp(e^{t_\theta+k})$.

Lemma 3.3.9. *Let $\ell \ge 0$ satisfy (3.3.8). Suppose that $\mathcal{Q}$ is Dirichlet polynomial of length $N \le \exp(\frac{1}{100}e^t)$ supported on integers all of whose prime factors are $> \exp(e^{k+t_\theta})$. Let $C > 0$ be arbitrary, $1 \le y < C(t-t_\theta)/\log(t-t_\theta)$ and $|u| < C\sqrt{t-t_\theta}$, $L_y(k) < w - m(k) < U_y(k)$, we have*

$$\mathbb{E}\left\{\max_{|h|\le 2\log^\theta T} |\mathcal{Q}(\tfrac12+i\tau+ih)|^2 \cdot \mathbf{1}\big(h \in B_\ell^{(k)} \cap C_\ell^{(k)} \cap E^\theta(u),\, S_k^{(t_\theta)}(h) \in (w, w+1]\big)\right\}$$
$$\ll_C \mathbb{E}\left\{|\mathcal{Q}(\tfrac12+i\tau)|^2\right\}\left(e^{-t_\theta}\log N + e^k\big((t-t_\theta)-k\big)^{800}\right)$$
$$e^{-u^2/c} \times \frac{y(U_y(k)-w+m(k)+2)e^{-w^2/k}}{k^{3/2}}. \tag{3.3.27}$$

The last term in (3.3.27) arises from the application of a ballot theorem (Proposition 3.6.9), which is used to estimate the probability that the partial sum $S_j^{(t_\theta)}$ remains below $U_y(j)$ at every $j \le k$, and ends at $S_k^{(t_\theta)} \approx w$. The same term would arise were $S_k^{(t_\theta)}$ a genuine Gaussian random walk with variance-1/2 increments.

We will require a variant of Lemma 3.3.9 that also controls $\log|\zeta_\tau(h)| - S_{k+t_\theta}(h)$, which represents the contribution to $\log|\zeta_\tau(h)|$ from primes $p > \exp(e^{k+t_\theta})$. It hinges on a *twisted fourth moment* bound for the Riemann zeta function (Lemma 3.4.3), which bounds $\mathbb{E}\{|\zeta_\tau \mathscr{M}_{-1}\cdots\mathscr{M}_\ell\mathscr{M}_\ell^{(k)}(h)|^4 \cdot |D|^2\}$, for D a short enough Dirichlet polynomial supported on primes $p < \exp(e^{k+t_\theta})$. We once again obtain decoupling between random variables supported on disjoint ranges of primes (in this case: $p \in (2,\exp(e^{t_\theta})]$, $(\exp(e^{t_\theta}),\exp(e^{t_\theta+t_\ell})]$, $(\exp(e^{t_\theta+t_\ell}),\exp(e^{t_\theta+k})]$, and $p > \exp(e^{k+t_\theta})$), and we note that a twisted second moment would not be sufficient for our purposes.

Lemma 3.3.10. *Let $\ell \ge 0$ satisfy (3.3.8). Let $k \in [t_\ell, t_{\ell+1}]$, and let $\eta(m)$ be a sequence of complex coefficients with $|\eta(m)| \ll \exp(\frac{1}{1000}e^t)$ for all $m \ge 1$. Let*

$$\mathcal{Q}_\ell^{(k)}(h) := \sum_{\substack{p|m \implies p\in(\exp(e^{t_\ell+t_\theta}),\exp(e^{k+t_\theta})] \\ \Omega_{\ell+1}(m)\le(t_{\ell+1}-t_\ell)^{10^4}}} \frac{\eta(m)}{m^{1/2+i\tau+ih}}.$$

Lastly, let $C > 0$ be arbitrary, $1 \le y < C(t-t_\theta)/\log(t-t_\theta)$ and $|u| < C\sqrt{t-t_\theta}$. Then for

any $h \in [-2\log^\theta T, 2\log^\theta T]$*, and* $L_y(t_\ell) < w - m(k) < U_y(t_\ell)$*,*

$$\mathbb{E}\left\{|(\zeta_\tau \mathscr{M}_{-1}\cdots\mathscr{M}_\ell \mathscr{M}_\ell^{(k)}(h)|^4 \cdot |\mathscr{Q}_\ell^{(k)}(h)|^2 \cdot 1\left(h \in B_\ell \cap C_\ell \cap E^\theta(u),\ S_{t_\ell}^{(t_\theta)} \in (v, v+1]\right)\right\}$$
$$\ll_C e^{4((t-t_\theta)-k)}\mathbb{E}\left\{|\mathscr{Q}_\ell^{(k)}(h)|^2\right\} y(U_y(t_\ell) - v + m(t_\ell) + 2)e^{-v^2/t_\ell}(t-t_\theta)^{-3/2}e^{-u^2/c}.$$

We are now ready to prove Proposition 3.3.5. To begin with, a union bound gives

$$\mathbb{P}\big(\exists h \in H^\theta(y) \cap G_\ell \cap E^\theta(u)\big)$$
$$\le \mathbb{P}\big(\exists h \in H^\theta(y) \cap G_\ell \cap G_{\ell+1}^c \cap E^\theta(u)\big) + \mathbb{P}\big(\exists h \in H^\theta(y) \cap G_\ell \cap G_{\ell+1} \cap E^\theta(u)\big)$$

and the first term on the right-hand side is bounded above by

$$\mathbb{P}\big(\exists h \in A_{\ell+1}^c \cap H^\theta(y) \cap G_\ell \cap E^\theta(u)\big) \tag{3.3.28}$$
$$+ \mathbb{P}\big(\exists h \in D_{\ell+1}^c \cap A_{\ell+1} \cap H^\theta(y) \cap G_\ell \cap E^\theta(u)\big) \tag{3.3.29}$$
$$+ \mathbb{P}\big(\exists h \in C_{\ell+1}^c \cap D_{\ell+1} \cap A_{\ell+1} \cap H^\theta(y) \cap G_\ell \cap E^\theta(u)\big) \tag{3.3.30}$$
$$+ \mathbb{P}\big(\exists h \in B_{\ell+1}^c \cap C_{\ell+1} \cap A_{\ell+1} \cap H^\theta(y) \cap G_\ell \cap E^\theta(u)\big). \tag{3.3.31}$$

We will show that each of these probabilities is

$$\ll e^{-u^2/c}\frac{g(y,\theta)}{\big(\log_{\ell+1}(t-t_\theta)\big)^{100}}.$$

(The dominant term is (3.3.31).)

3.3.4.1 Bound on (3.3.28)

By definition of $A_{\ell+1}^c$, if there is a $k \in (t_\ell, t_{\ell+1}]$ and an h such that $|\tilde{S}_k^{(t_\theta)}(h) - \tilde{S}_{t_\ell}^{(t_\theta)}(h)| > 10^3((t-t_\theta)-t_\ell)$, then

$$\sum_{k\in(t_\ell,t_{\ell+1}]}\max_{|h|\le\log^\theta T}\frac{|\tilde{S}_k^{(t_\theta)}(h) - \tilde{S}_{t_\ell}^{(t_\theta)}(h)|^{2q}}{(10^3((t-t_\theta)-t_\ell))^{2q}} \ge 1, \quad \text{for all } q \ge 1.$$

This implies the bound

$$\mathbb{P}\big(\exists h \in A_{\ell+1}^c \cap G_\ell \cap E^\theta(u)\big)$$
$$\le \sum_{k\in(t_\ell,t_{\ell+1}]}\mathbb{E}\left\{\max_{|h|\le\log^\theta T}\frac{|\tilde{S}_k^{(t_\theta)}(h) - \tilde{S}_{t_\ell}^{(t_\theta)}(h)|^{2q}}{(10^3((t-t_\theta)-t_\ell))^{2q}}1\big(h \in G_\ell \cap E^\theta(u)\big)\right\}. \tag{3.3.32}$$

Picking $q = \lfloor 10^6((t-t_\theta)-t_\ell)^2/(k-t_\ell)\rfloor$ in Equation (3.3.32), we note that $|\tilde{S}_k^{(t_\theta)}(h) - \tilde{S}_{t_\ell}^{(t_\theta)}(h)|^{2q}$ is a Dirichlet polynomial of length $\le \exp(2qe^{k+t_\theta}) \le \exp(2\cdot 10^6((t-t_\theta)-$

$t_\ell)^2 e^{t_{\ell+1}+t_\theta}) \le \exp(e^t/100)$ (by the assumption that ℓ satisfies (3.3.8)). We can therefore bound each summand using Lemma 3.3.9 to get

$$\ll e^{-u^2/c} g(y,\theta) \sum_{k\in(t_\ell,t_{\ell+1}]} \big(q + ((t-t_\theta)-t_\ell)^{800}\big) e^{100((t-t_\theta)-t_\ell)} \mathbb{E}\left\{ \frac{|\tilde{S}_k^{(t_\theta)}(h) - \tilde{S}_{t_\ell}^{(t_\theta)}(h)|^{2q}}{\big(10^3((t-t_\theta)-t_\ell)\big)^{2q}} \right\}.$$

For our choice of q, we can apply a Gaussian moment estimate for the rightmost term (Lemma 3.6.1) along with Stirling's approximation as in the proof of Corollary 3.6.3 to get

$$\begin{aligned}\mathbb{E}\left\{ \frac{|\tilde{S}_k^{(t_\theta)}(h) - \tilde{S}_{t_\ell}^{(t_\theta)}(h)|^{2q}}{\big(10^3((t-t_\theta)-t_\ell)\big)^{2q}} \right\} &\ll \sqrt{q} e^{-q} \Big(\frac{q(k-t_\ell)}{10^3\big((t-t_\theta)-t_\ell\big)^2} \Big)^q \\ &\ll \sqrt{q} e^{-10^6((t-t_\theta)-t_\ell)^2/(k-t_\ell)} \\ &\ll \sqrt{q} e^{-10^6((t-t_\theta)-t_\ell)},\end{aligned}$$

uniformly in $k \in (t_\ell, t_{\ell+1}]$. Plugging this back into the sum, we have

$$\ll e^{-u^2/c} g(y,\theta) \sum_{k\in(t_\ell,t_{\ell+1}]} \sqrt{q} (q + ((t-t_\theta)-t_\ell))^{800} e^{(100-10^6)((t-t_\theta)-t_\ell)},$$

and summing over k then gives the desired bound:

$$\ll g(y,\theta) e^{-u^2/c} \big(\log_{\ell-1}(t-t_\theta)\big)^{-1}.$$

3.3.4.2 Bound on (3.3.29)

For any $h \in A_{\ell+1} \cap D_\ell$ we have

$$\begin{aligned}|\zeta_\tau e^{-S_{k+t_\theta}}(h)| &= |\zeta_\tau e^{-S_{t_\theta+t_\ell}}(h)| e^{-(S_k^{(t_\theta)} - S_{t_\ell}^{(t_\theta)})(h)} \\ &\le c_\ell |\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell(h)| e^{-(S_k^{(t_\theta)} - S_{t_\ell}^{(t_\theta)})(h)} + e^{-10^3((t-t_\theta)-t_{\ell-1})}, \qquad (3.3.33)\end{aligned}$$

using the definition of D_ℓ and $A_{\ell+1}$ to bound $|\zeta_\tau e^{-S_{t_\ell}^{(t_\theta)}}(h)|$ and $e^{-(S_k^{(t_\theta)} - S_{t_\ell}^{(t_\theta)})(h)}$ respectively. To bound (3.3.29), it then suffices to show that the following estimates hold with sufficiently high probability for each $h \in A_{\ell+1} \cap D_\ell$ and $k \in (t_\ell, t_{\ell+1}]$:

$$|(\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell)(h)| < e^{10^3((t-t_\theta)-t_\ell)}, \qquad (3.3.34)$$

$$|e^{-(S_k^{(t_\theta)} - S_{t_\ell}^{(t_\theta)})}(h)| \le (1 + e^{-t_\ell})|\mathscr{M}_\ell^{(k)}(h)| + e^{-10^5(t_{\ell+1}-t_\ell)}, \qquad (3.3.35)$$

since applying them in succession to the right-hand side of (3.3.33) implies that

$$|\zeta_\tau e^{-S_k+t_\theta}(h)| \le c_{\ell+1} |\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell \mathscr{M}_\ell^{(k)}(h)| + e^{-10^4((t-t_\theta)-t_\ell)}.$$

By Lemma 3.3.8, Equation (3.3.35) holds pointwise for each $h \in A_{\ell+1}$. As for (3.3.34), a union bound on h bounds the probability of its complement by

$$\begin{aligned}
&\ll \sum_{h \in \mathscr{T}_t^\theta} \mathbb{P}\Big(|(\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell)(h)| \geq e^{10^3((t-t_\theta)-t_\ell)}, h \in G_\ell \cap E^\theta(u) \Big) \\
&\ll e^{-4\cdot 10^3((t-t_\theta)-t_\ell)} e^{t-t_\theta} \mathbb{E}\left\{ |(\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell)(0)|^4 \cdot \mathbf{1}\big(0 \in G_\ell \cap E^\theta(u)\big) \right\}.
\end{aligned}$$

Using Lemma 3.3.10 with $\mathscr{Q} \equiv 1$ and $k = t_\ell$ to bound this expectation, we find that this is

$$e^{-u^2/c} g(y,\theta) e^{-4\cdot 10^3((t-t_\theta)-t_\ell)} e^{100((t-t_\theta)-t_\ell)} \ll e^{-u^2/c} \frac{g(y,\theta)}{\log_{\ell-1}(t-t_\theta)}. \tag{3.3.36}$$

3.3.4.3 Bound on (3.3.30)

By definition, for $h \in C_{\ell+1}^c$, there must be a k such that $S_k^{(t_\theta)}(h) \leq m(k) - 20((t-t_\theta)-k) + y$. We split $S_k^{(t_\theta)}(h)$ according to the value of $S_{t_\ell}^{(t_\theta)} \in [v, v+1]$ and $(S_k^{(t_\theta)} - S_{t_\ell}^{(t_\theta)})(h) \in [w, w+1]$, where $v, w \in \mathbb{Z}$, $|w| \leq 10^3((t-t_\theta)-t_\ell)$ and $w + v \leq m(k) - 20((t-t_\theta)-k) + y$. Since $h \in H^\theta(y)$, we also have that

$$|\zeta_\tau e^{-S_{k+t_\theta}}(h)| > V_\theta e^{-w-v},$$

where $V_\theta = e^y e^{(t-t_\theta)}(t-t_\theta)^{-3/4}$, and since $h \in D_{\ell+1}$, this implies that either

$$|\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell^{(k)}(h)| \gg V_\theta e^{-w-v}$$

or $\frac{1}{2} V_\theta e^{-w-u} \leq e^{-10^4(t-t_\theta-t_\ell)}$. However, the latter would force $S_{t_\ell}^{(t_\theta)}(h) > m(t_\ell) + U_y(t_\ell)$ for large enough t, contradicting the fact that $h \in G_\ell$. A union bound on $h \in \mathscr{T}_t^\theta$ and $k \in (t_\ell, t_{t_\ell+1}]$ therefore yields

$$\begin{aligned}
&\mathbb{P}\big(\exists h \in H^\theta(y) \cap C_{\ell+1}^c \cap D_{\ell+1} \cap A_{\ell+1} \cap G_\ell \cap E^\theta(u)\big) \\
&\ll \sum_{\substack{k \in (t_\ell, t_{\ell+1}] \\ h \in \mathscr{T}_t^\theta}} \sum_{\substack{w+v \leq m(k)+L_y(k) \\ |w| \leq 10^3(t_{\ell+1}-t_\ell) \\ L_y(t_\ell) \leq u - m(t_\ell) \leq U_y(t_\ell)}} \frac{e^{4w+4v}}{V_\theta^4} \mathbb{E}\Big\{ |(\zeta_\tau \mathscr{M}_{-1} \cdots \mathscr{M}_\ell^{(k)}(h)|^4 \cdot \frac{|(S_k^{(t_\theta)} - S_{t_\ell}^{(t_\theta)})(h)|^{2q}}{(1+w^{2q})} \\
&\qquad \cdot \mathbf{1}\Big(S_{t_\ell}^{(t_\theta)}(h) \in [u, u+1] \text{ and } h \in A_\ell \cap B_\ell \cap C_\ell \cap E^\theta(u) \Big) \Big\}.
\end{aligned} \tag{3.3.37}$$

Picking $q = \lfloor w^2/(k-t_\ell) \rfloor$ and applying Lemma 3.3.10 with the Dirichlet polynomial

$\mathcal{Q} = (S_k^{(t_\theta)} - S_{t_\ell}^{(t_\theta)})^{2q}$, we find that the above is

$$\ll e^{-u^2/c} \sum_{\substack{k\in(t_\ell,t_{\ell+1}]\\ w+v\le m(k)-20((t-t_\theta)-k)+y\\ |w|\le 10^3((t-t_\theta)-t_\ell)\\ L_y(t_\ell)\le v-m(t_\ell)\le U_y(t_\ell)}} \frac{e^{4w+4v}}{V_\theta^4} e^{5(t-t_\theta)-4k} e^{-\frac{w^2}{(k-t_\ell)}} \frac{y(U_y(t_\ell)-v+m(t_\ell)+2)e^{-v^2/t_\ell}}{t_\ell^{3/2}}. \quad (3.3.38)$$

It is left to show that this summation is $\ll g(y,\theta)(\log_\ell(t-t_\theta))^{-1}$. The restriction $L_y(t_\ell) \le v - m(t_\ell) \le U_y(t_\ell)$ implies that

$$0 \le U_y(t_\ell) - v + m(t_\ell) \le U_y(t_\ell) - L_y(t_\ell) \ll \log_\ell(t-t_\theta)$$

for all y. Using this in (3.3.38) and removing the restriction on v in that sum shows that it is

$$\ll ye^{-4y-u^2/c} \sum_{\substack{k\in(t_\ell,t_{\ell+1}]\\ w+v\le m(k)-20((t-t_\theta)-k)+y\\ |w|\le 10^3((t-t_\theta)-t_\ell)}} (t-t_\theta)^3 \log_\ell(t-t_\theta) \frac{e^{-v^2/t_\ell}}{t_\ell^{3/2}} e^{4w+4v} e^{(t-t_\theta)-4k} e^{-w^2/(k-t_\ell)},$$

recalling that $V_\theta = e^y e^{t-t_\theta}(t-t_\theta)^{-3/4}$. Making the change of variables $\bar{v} = v - m(t_\ell) - y$, $\bar{w} = w - m(k-t_\ell)$ and dropping the condition on $|w|$, this becomes

$$\ll ye^{-u^2/c} \sum_{\substack{k\in(t_\ell,t_{\ell+1}]\\ \bar{w}+\bar{v}\le -20((t-t_\theta)-k)\\ \bar{w}\in\mathbb{Z}}} (t-t_\theta)^3 \log_\ell(t-t_\theta) e^{4m(k)+(t-t_\theta)-4k} e^{4(\bar{w}+\bar{v})} \frac{e^{-\frac{(\bar{v}+m(t_\ell)+y)^2}{t_\ell}}}{t_\ell^{3/2}} e^{-\frac{(\bar{w}+m(k-t_\ell))^2}{(k-t_\ell)}}. \quad (3.3.39)$$

To estimate this, we begin by expanding the Gaussian term in $\bar{v}$ in the above to get

$$t_\ell^{-3/2} e^{-(\bar{v}+m(t_\ell)+y)^2/t_\ell} \ll t_\ell^{3/2} e^{-y^2/t_\ell - 2ym(t_\ell)/t_\ell - 2\bar{v}(y/t_\ell) - (m(t_\ell)+\bar{v})^2/t_\ell}.$$

Since $y \ll (t-t_\theta)/\log(t-t_\theta)$ by assumption, $e^{-2ym(t_\ell)/t_\ell} \ll e^{-2y}$, and $y/t_\ell = o(1)$. For our purposes, we will view $2y/t_\ell$ as a small constant which we henceforth denote by δ. Using the bound

$$\frac{e^{-(m(t_\ell)+\bar{v})^2/t_\ell}}{t_\ell^{3/2}} \le e^{-t_\ell} e^{-2\bar{v}},$$

we conclude that

$$t_\ell^{-3/2} e^{-(\bar{v}+m(t_\ell)+y)^2/t_\ell} \ll e^{-t_\ell} e^{-y^2/t_\ell - 2y - (2+\delta)\bar{v}}.$$

Plugging this back into (3.3.39) yields

$$\ll g(y,\theta)e^{-u^2/c}$$
$$\times \sum_{\substack{k\in(t_\ell,t_{\ell+1}]\\ \bar{w}+\bar{v}\leq -20((t-t_\theta)-k)\\ \bar{w}\in\mathbb{Z}}} (t-t_\theta)^3 \log_\ell(t-t_\theta) e^{4m(k)+(t-t_\theta)-4k-t_\ell} e^{4\bar{w}+(2-\delta)\bar{v}} e^{-\frac{(\bar{w}+m(k-t_\ell))^2}{(k-t_\ell)}}.$$

We perform the sum over $\bar{v}$ to get

$$\ll g(y,\theta)e^{-u^2/c}\log_\ell(t-t_\theta)$$
$$\times \sum_{\substack{k\in(t_\ell,t_{\ell+1}]\\ \bar{w}\in\mathbb{Z}}} (t-t_\theta)^3 e^{4m(k)+(t-t_\theta)-4k-t_\ell-20(2-\delta)((t-t_\theta)-k)} e^{(2+\delta)\bar{w}} e^{-\frac{(\bar{w}+m(k-t_\ell))^2}{(k-t_\ell)}},$$

followed by the sum over $\bar{w}$, yielding

$$\ll g(y,\theta)e^{-u^2/c} \sum_{k\in(t_\ell,t_{\ell+1}]} \log_\ell(t-t_\theta) e^{(t-t_\theta)-t_\ell}(k-t_\ell)^{1/2}$$
$$\times (t-t_\theta)^3 e^{4m(k)-(2+\delta)m(k-t_\ell)-4k-20(2-\delta)((t-t_\theta)-k)} e^{\frac{(2+\delta)^2}{4}(k-t_\ell)}.$$

Using the fact that $(k-t_\ell) \ll \log_\ell(t-t_\theta)$ along with the definition of $m(k)$ and $m(k-t_\ell)$, this is

$$\ll g(y,\theta)e^{-u^2/c} \sum_{k\in(t_\ell,t_{\ell+1}]} (\log_\ell(t-t_\theta))^{3/2}\cdot(t-t_\theta)^{3-3\left(\frac{k}{t-t_\theta}-\frac{2+\delta}{4}\left(1-\frac{k}{t-t_\theta}\right)\right)}$$
$$\times e^{(t-t_\theta)-t_\ell} e^{(k-t_\ell)((2+\delta)^2/4-(2+\delta))} e^{-20(2-\delta)((t-t_\theta)-k)}$$
$$\ll g(y,\theta)e^{-u^2/c} \sum_{k\in(t_\ell,t_{\ell+1}]} \log_\ell(t-t_\theta)^{3/2}\cdot(t-t_\theta)^{3-3\left(\frac{k}{t-t_\theta}-\frac{2+\delta}{4}\left(1-\frac{k}{t-t_\theta}\right)\right)}$$
$$\times e^{-19((t-t_\theta)-k)}.$$

If $\ell \geq 1$, then $k/(t-t_\theta) \sim 1$ and the powers of $(t-t_\theta)$ in the above cancel out to give a factor of the form $(t-t_\theta)^{c\log_\ell(t-t_\theta)/(t-t_\theta)}$ for a constant $c>0$, which is bounded in t. Otherwise, we can use a power of $e^{-((t-t_\theta)-k)}$ to offset the resulting positive power of $(t-t_\theta)$, recalling that $\log_0(t-t_\theta) := (t-t_\theta)$. It follows that the above is

$$\ll g(y,\theta)e^{-u^2/c} \sum_{k\in(t_\ell,t_{\ell+1}]} (\log_\ell(t-t_\theta))^{3/2} e^{-10((t-t_\theta)-k)} \ll g(y,\theta)e^{-u^2/c}(\log_\ell(t-t_\theta))^{-1}.$$

3.3.4.4 Bound on (3.3.31)

We show that

$$\mathbb{P}\big(\exists h \in (B_\ell \setminus B_{\ell+1}) \cap C_{\ell+1} \cap E^\theta(u)\big) \ll \frac{g(y,\theta)}{\big(\log_{\ell+1}(t-t_\theta)\big)^{100}} e^{-u^2/c}.$$

Using a union bound over $k \in [t_\ell, t_{\ell+1})$ and summing over all possible values $S_k^{(t_\theta)}(h) \in [w, w+1]$, $w \in \mathbb{Z}$, we bound the above by

$$\sum_{\substack{k\in[t_\ell,t_{\ell+1})\\ w\in[L_y(k),U_y(k)]}} \mathbb{P}\Big(\exists h : \overline{S_j^{(t_\theta)}}(h) < U_y(j)\,\forall j \le k,\ \overline{S_{k+1}^{(t_\theta)}}(h) > U_y(k+1),\ \overline{S_k^{(t_\theta)}}(h) \in (w, w+1],\ h \in E^\theta(u)\Big)$$

where $\overline{S_k^{(t_\theta)}} := S_k^{(t_\theta)} - m(k)$. Letting $V_{w,k} = U_y(k+1) - w$, using the fact that

$$S_{k+1}^{(t_\theta)} - S_k^{(t_\theta)} > U_y(k+1) + m(k+1) - m(k) - \overline{S_k^{(t_\theta)}} > U_y(k+1) - w + o\Big(\frac{\log(t-t_\theta)}{t-t_\theta}\Big)$$

and applying Markov's inequality shows that this sum is

$$\ll \sum_{\substack{k\in[t_\ell,t_{\ell+1})\\ w\in[L_y(k),U_y(k)]}} \mathbb{E}\Big\{ \max_{|h|\le \log^\theta T} \frac{|(S_{k+1}^{(t_\theta)} - S_k^{(t_\theta)} + 1)(h)|^{2q}}{(V_{w,k}+1)^{2q}} \tag{3.3.40}$$

$$* \mathbf{1}\Big(\overline{S_j^{(t_\theta)}}(h) < U_y(j), \forall j \le k, \overline{S_k^{(t_\theta)}}(h) \in (w, w+1],\ h \in E^\theta(u)\Big)\Big\}. \tag{3.3.41}$$

Picking $q = \lfloor (V_{w,k}+1)^2/10 \rfloor$, which is bounded by $400((t-t_\theta)-k)^2$ for $w \ge L_y(k)$, we note that the Dirichlet polynomial $(S_{k+1}^{(t_\theta)} - S_k^{(t_\theta)} + 1)^{2q}$ has length $\le \exp(1000((t-t_\theta)-k)^2 e^{k+1+t_\theta})$. We can therefore apply Lemma 3.3.9 to each summand, yielding

$$\ll e^{-u^2/c} \sum_{\substack{k\in[t_\ell,t_{\ell+1})\\ w\in[L_y(k),U_y(k)]}} e^k\big((t-t_\theta)-k\big)^{800} \tag{3.3.42}$$

$$\frac{\mathbb{E}\left\{|(S_{k+1}^{(t_\theta)} - S_k^{(t_\theta)} + 1)(0)|^{2q}\right\}}{(V_{w,k}+1)^{2q}} \frac{y(U_y(k) - w + 1)e^{-(w+m(k))^2/k}}{k^{3/2}}. \tag{3.3.43}$$

By Equation (3.6.2), the expectation is $\ll (2q)!/q! + 4^q$, which by Stirling's approximation is $\ll 100^q (q/e)^q$. For our choice of q, it then follows that

$$\frac{\mathbb{E}\left\{|(S_{k+1}^{(t_\theta)} - S_k^{(t_\theta)} + 1)(0)|^{2q}\right\}}{(V_{w,k}+1)^{2q}} \ll e^{-(V_{w,k}+1)^2/10},$$

and plugging this back into (3.3.42) yields

$$\ll y e^{-u^2/c} \sum_{\substack{k\in[t_\ell,t_{\ell+1})\\ w\in[L_y(k),U_y(k)]}} e^k\big((t-t_\theta)-k\big)^{800} e^{-(V_{w,k}+1)^2/10} \frac{(U_y(k)-w+1)e^{-(w+m(k))^2/k}}{k^{3/2}}.$$

To estimate this sum, we first make the change of variables $\bar{w} = w - y$ to get

$$\ll y e^{-u^2/c} \sum_{\substack{k\in[t_\ell,t_{\ell+1})\\ \bar{w}\in[L(k),U(k)]}} e^k\big((t-t_\theta)-k\big)^{800} e^{-(U(k+1)-\bar{w}+1)^2/10} \frac{(U(k)-\bar{w}+1)e^{-(\bar{w}+y+m(k))^2/k}}{k^{3/2}} \tag{3.3.44}$$

where $U(k) := U_y(k) - y$ and similarly for $L(k)$. Expanding the rightmost Gaussian term yields

$$\begin{aligned}\frac{e^{-(\bar{w}+y+m(k))^2/k}}{k^{3/2}} &= e^{-y^2/k-2y} e^{-2\bar{w}(y/k)} \frac{e^{-(\bar{w}+m(k))^2/k}}{k^{3/2}}\\ &\ll e^{-2y-y^2/(t-t_\theta)} e^{-\delta\bar{w}} e^{-k-2\bar{w}},\end{aligned}$$

where $\delta := (y/k) = o(1)$ can be seen as a small constant. Using this estimate in (3.3.44), and multiplying the resulting sum by $e^{(2+\delta)(U(k+1)-1)-(2+\delta)U(k)}$ (which is $O(1)$) yields

$$\begin{aligned}\ll g(y,\theta)e^{-u^2/c} \sum_{k\in[t_\ell,t_{\ell+1})} &\big((t-t_\theta)-k\big)^{800} e^{-(2+\delta)U(k)}\\ &\times \sum_{\bar{w}} (U(k)-\bar{w}+1) e^{-(U(k+1)-\bar{w}+1)^2/10-(2+\delta)(U(k+1)-\bar{w}+1)}.\end{aligned}$$

Notice that the sum over $\bar{w}$ is now finite, while the sum over k is

$$\ll \sum_{k\in[t_\ell,t_\ell+1)} \big((t-t_\theta)-k\big)^{800} e^{-10^3\log((t-t_\theta)-k)} \ll \big(\log_{\ell+1}(t-t_\theta)\big)^{-100},$$

which yields the desired bound.

3.3.5 Proof of Proposition 3.3.6

If $h \in H^\theta(y) \cap G_{\mathscr{L}}$ and $S^{(t_\theta)}_{t_{\mathscr{L}}} \in [v, v+1]$, then $|v - y - m(t_{\mathscr{L}})| \le 20((t-t_\theta)-t_{\mathscr{L}})$ and $|(\zeta_\tau e^{-S_{t_{\mathscr{L}}+t_\theta}})(h)| > V_\theta e^{-v}$ where $V_\theta = e^y e^{(t-t_\theta)}(t-t_\theta)^{-3/4}$. A union bound on $h \in \mathscr{T}_t^\theta$ thus gives

$$\begin{aligned}&\mathbb{P}\big(\exists h \in H^\theta(y)\cap G_{\mathscr{L}} \cap E^\theta(u)\big) \qquad (3.3.45)\\ &\ll e^{(t-t_\theta)} \sum_{\substack{|v-y-m(t_{\mathscr{L}})|\\ \le 20((t-t_\theta)-t_{\mathscr{L}})}} \mathbb{P}\Big(|\zeta_\tau e^{-S_{t_{\mathscr{L}}+t_\theta}}(h)| > V_\theta e^{-v},\ S^{(t_\theta)}_{t_{\mathscr{L}}} \in [v,v+1],\ h \in G_{\mathscr{L}} \cap E^\theta(u)\Big).\end{aligned}$$

Noting that $V_\theta e^{-v} > e^{-10^4((t-t_\theta)-t_{\mathscr{L}})}$ and that

$$|\zeta_\tau e^{-S_{k+t_\theta}}(h)| \le c_{\mathscr{L}}|\zeta_\tau \mathscr{M}_{-1}\cdots\mathscr{M}^{(k)}_{\mathscr{L}-1}(h)| + e^{-10^4((t-t_\theta)-t_{\mathscr{L}-1})}$$

since $h \in D_{\mathscr{L}}$, it follows that

$$V_\theta e^{-v} \ll |(\zeta_\tau \mathscr{M}_{-1}\cdots\mathscr{M}_{\mathscr{L}})(h)|.$$

We can therefore bound each summand in (3.3.45) using Markov's inequality and Lemma 3.3.10

$$\begin{aligned}&\mathbb{P}\{\exists h \in H^\theta(y)\cap G_{\mathscr{L}}\cap E^\theta(u)\}\\ &\ll e^{-4y-4(t-t_\theta)}(t-t_\theta)^3 e^{(t-t_\theta)}e^{-u^2/c}\sum_{\substack{|v-y-m(t_{\mathscr{L}})|\\ \le 20((t-t_\theta)-t_{\mathscr{L}})}} e^{4v}e^{4((t-t_\theta)-t_{\mathscr{L}})}y((t-t_\theta)-t_{\mathscr{L}})\frac{e^{-v^2/t_{\mathscr{L}}}}{t_{\mathscr{L}}^{3/2}}.\end{aligned}$$

Recall that $(t-t_\theta)-\mathscr{L} = O(1)$ by definition of $\mathscr{L}$ (cf. Equation (3.3.8)). It follows that the above is

$$\ll ye^{-4y-4(t-t_\theta)}(t-t_\theta)^3 e^{(t-t_\theta)}e^{-u^2/c}\sum_{\substack{|v-y-m(t_{\mathscr{L}})|\\ \le 20((t-t_\theta)-t_{\mathscr{L}})}}\frac{e^{4v-v^2/t_{\mathscr{L}}}}{t_{\mathscr{L}}^{3/2}},$$

and that this sum contains $O(1)$ many terms, each satisfying

$$\ll e^{4y+4m(t_{\mathscr{L}})-y^2/(t-t_\theta)-2y-t_{\mathscr{L}}}.$$

We conclude that

$$\mathbb{P}\{\exists h \in H^\theta(y)\cap G_{\mathscr{L}}\cap E^\theta(u)\} \ll g(y,\theta)e^{-u^2/c}\left((t-t_\theta)^3 e^{4m(t_{\mathscr{L}})-4t_{\mathscr{L}}}\right) \ll g(y,\theta)e^{-u^2/c}$$

since $\left((t-t_\theta)^3 e^{4m(t_{\mathscr{L}})-4t_{\mathscr{L}}}\right) = O(1)$.

3.4 Proofs of auxiliary results

3.4.1 Proof of Lemma 3.3.9

We begin with the proof of Lemma 3.3.9 and omit that of 3.3.7, which is similar and much simpler. We follow the strategy in [10, Section 7.2], where the main idea is to approximate the indicator appearing in (3.3.27) using Dirichlet polynomials. We first express it as a product of indicators involving the increments

$$Y_r^{(t_\theta)} := S_r^{(t_\theta)}, \quad Y_j^{(t_\theta)} := S_j^{(t_\theta)} - S_{j-1}^{(t_\theta)}, \quad j > r := \lceil y/4\rceil$$

(noting the slight abuse of notation). Approximating each of these by a Dirichlet polynomial then allows us to use Lemma 3.6.5 to decouple their second moments. This eventually leads to a Gaussian comparison argument and an application of the ballot theorem in Proposition 3.6.9.

To express the restrictions $S_j^{(t_\theta)} \in [m(j) + L_y(j), m(j) + U_y(j)]$, $r \le j \le k$, and $S_k^{(t_\theta)} \in (w, w+1]$ in terms of the increments, we partition the range of values of each $Y_j^{(t_\theta)}$ into intervals of the form $[u_j, u_j + \Delta_j^{-1}]$ where $u_j \in \Delta_j^{-1}\mathbb{Z}$ for a discretisation parameter

$$\Delta_j = \min(j, (t - t_\theta) - j)^4. \tag{3.4.1}$$

Note that $\sum_j \Delta_j^{-1} \le 1$. Define the set $\mathcal{I}_k(w) \subseteq \mathbb{R}^{k-r+1}$ of tuples $\mathbf{u} = (u_r, \cdots, u_k)$ with $u_j \in \Delta_j^{-1}\mathbb{Z}$, $r \le j \le k$, such that

$$\begin{aligned} &L_y(j) - 1 \le \sum_{i=r}^{j} u_i - m(j) \le U_y(j) + 1, \quad \forall r \le j \le k, \\ &\Big| \sum_{i=r}^{k} u_i - w \Big| \le 1. \end{aligned}$$

With this definition, one readily checks the inclusion

$$\begin{aligned} &\Big\{ S_j^{(t_\theta)}(h) - m(j) \in [L_y(j), U_y(j)], r \le j \le k, S_k^{(t_\theta)}(h) \in (w, w+1] \Big\} \\ &\subseteq \bigcup_{\mathbf{u} \in \mathcal{I}_k(w)} \Big\{ Y_j^{(t_\theta)}(h) \in [u_j, u_j + \Delta_j^{-1}), S_k^{(t_\theta)}(h) \in (w, w+1] \Big\}. \end{aligned} \tag{3.4.2}$$

We can then work on the right-hand side for an upper bound. Crucially, note that for $\mathbf{u} \in \mathcal{I}_k(w)$,

$$|u_j| \le U_y(j) - L_y(j) \le 40 \min(j, (t - t_\theta) - j) \le 100 \Delta_j^{1/4}, \quad \forall r \le j \le k. \tag{3.4.3}$$

We now approximate the indicator function of every interval $[u_j, u_j + \Delta_j^{-1}]$ by a polynomial (applying this approximation to $\mathbf{1}(Y_j^{(t_\theta)}(h) \in [u_j, u_j + \Delta_j^{-1}])$ then yields a Dirichlet polynomial as desired). Given $\Delta > 0$ and $A > 3$, let $G_{\Delta,A}$ be an entire function in $L^2(\mathbb{R})$ that approximates $\mathbf{1}(x \in [0, \Delta^{-1}])$ and satisfies, for some absolute constant $C > 0$, the following properties:

1. The Fourier transform $\hat{G}_{\Delta,A}$ is supported in $[-\Delta^{2A}, \Delta^{2A}]$.
2. For all $x \in \mathbb{R}$, $0 \le G_{\Delta,A} \le 1$.
3. $\mathbf{1}(x \in [0, \Delta^{-1}]) \le G_{\Delta,A}(x) \cdot (1 + Ce^{-\Delta^{A-1}})$.

4. We have $G_{\Delta,A}(x) \le \mathbf{1}(x \in [-\Delta^{-A/2}, \Delta^{-1} + \Delta^{-A/2}) + Ce^{-\Delta^{A-1}}$.

5. $\int_{\mathbb{R}} |\hat{G}_{\Delta,A}(x)| \mathrm{d}x \le 2\Delta^{2A}$.

The existence of such a function is given by Lemma 6 in [10]. We then define a truncated version of $G_{\Delta,A}$ given by

$$\mathscr{D}_{\Delta,A}(x) = \sum_{\ell \le \Delta^{10A}} \frac{(2\pi i x)^\ell}{\ell!} \int_{\mathbb{R}} \xi^\ell \hat{G}_{\Delta,A}(\xi) \mathrm{d}\xi, \tag{3.4.4}$$

and approximate $\mathbf{1}(Y_j^{(t_\theta)}(h) \in [u_j, u_j + \Delta_j^{-1}])$ by $\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)} - u_j)$. Note that $\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)} - u_j)$ is supported on integers n whose prime factors lie in $(\exp e^{j-1+t_\theta}, \exp e^{j+t_\theta}]$ and for which $\Omega(n) \le \Delta_j^{10A}$. This polynomial is of length $\le \exp(2\Delta_j^{10A} e^{j+t_\theta})$, and

$$\int_{\mathbb{R}} |\xi|^\ell |\hat{G}_{\Delta,A}(\xi)| \mathrm{d}\xi \le \Delta^{2A\ell} \int_{\mathbb{R}} |\hat{G}_{\Delta,A}| \mathrm{d}\xi \le 2\Delta^{2A\ell+2A}. \tag{3.4.5}$$

by the first and fifth properties of $G_{\Delta,A}$ in the list above. As a result, its coefficients are bounded by $\ll \Delta_j^{2A(\ell+1)}$.

This leads to the following.

Lemma 3.4.1 (Approximation of indicator functions by Dirichlet polynomials). *Let $\theta \in (-1, 0)$, $\ell \ge -1$, $A > 10$, and $y > 1$. Let $|u_0| > 1$, $k \ge 1$, and w be such that $L_y(k) \le w - m(k) \le U_y(k)$. Then for any fixed $\tau \in [T, 2T]$, we have*

$$\begin{aligned} &\mathbf{1}\Big(h \in B_\ell^{(k)} \cap C_\ell^{(k)} \cap E^\theta(u_0), S_k^{(t_\theta)}(h) \in [w, w+1]\Big) \\ &\le C \frac{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}}{(|u_0| - 1)^{2q}} \sum_{\mathbf{u} \in \mathscr{I}_k(w)} \prod_{j=r}^{k} |\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)}(h) - u_j)|^2 \end{aligned}$$

for some absolute constant $C > 0$ and any $q > 0$.

Proof. It suffices to show that for every $j \in [r, k]$ and $|u_j| \le 100 \min(j, (t - t_\theta) - j)$

$$\mathbf{1}(Y_j^{(t_\theta)}(h) \in [u_j, u_j + \Delta_j^{-1}]) \le |\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)}(h) - u_j)|^2 (1 + Ce^{-\Delta_j^{A-1}}), \quad \text{for } j > r, \tag{3.4.6}$$

and otherwise that

$$\mathbf{1}(Y_r^{(t_\theta)}(h) \in [u_r, u_r + \Delta_r^{-1}]) \le C |\mathscr{D}_{\Delta_r,A}(Y_r^{(t_\theta)}(h) - u_r)|^2 \tag{3.4.7}$$

for some constant $C > 0$. Noting that

$$\mathbf{1}(h \in E^\theta(u_0)) \le \frac{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}}{(|u_0| - 1)^{2q}},$$

the inclusion in (3.4.2) together with (3.4.6) and (3.4.7) would then imply that

$$1\Big(h \in B_\ell^{(k)} \cap C_\ell^{(k)} \cap E^\theta(u_0) : S_k^{(t_\theta)}(h) \in [w, w+1]\Big) \le \frac{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}}{(|u_0|-1)^{2q}} *$$

$$C \sum_{\mathbf{u} \in \mathcal{I}_k(w)} |\mathscr{D}_{\Delta_r,A}(Y_r^{(t_\theta)}(h) - u_r)|^2 \prod_{j=r+1}^{k} \Big(|\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)}(h) - u_j)|^2 (1 + Ce^{-\Delta_j^{A-1}})\Big),$$

and the lemma follows since $\prod_j (1 + Ce^{-\Delta_j^{A-1}})$ is bounded by an absolute constant.

The proof of (3.4.6) and (3.4.7) follows from the proof of Lemma 7 in [10]. We sketch the main steps. To prove (3.4.6), one first bounds the left-hand side from above by

$$|G_{\Delta_j,A}(Y_j^{(t_\theta)}(h) - u_j)|^2 (1 + Ce^{-\Delta_j^{A-1}}) = \Big| \int_{\mathbb{R}} e^{2\pi i \xi (Y_j^{(t_\theta)}(h) - u_j)} \hat{G}_{\Delta_j,A}(\xi) \mathrm{d}\xi \Big|^2 (1 + Ce^{-\Delta_j^{A-1}}).$$

By expanding the exponential up to order Δ_j^{10A}, this integral equals the sum of $\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)}(h) - u_j)$ and an error term $\mathscr{E}$, which is shown to be $\le e^{-\Delta_j^{4A}}$ in absolute value using the aforementioned properties of $G_{\Delta_j,A}$ and the fact that $|Y_j^{(t_\theta)}(h) - u_j| \le 10^4 \Delta_j^{1/4}$. Noting that $|\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)}(h) - u_j)| \ge 1/2$ whenever the left-hand side in (3.4.6) equals 1, this additive error $\mathscr{E}$ can then be made multiplicative, and the inequality follows.

The proof of (3.4.7) is similar. □

We now compare the increments $Y_j^{(t_\theta)}$ to the same random Steinhaus model used in Section 3.2. An application of the Berry-Esseen theorem is then used to compare the latter to Gaussian random variables, and by extension, $S_k^{(t_\theta)}$ to a bona fide random walk.

As in (3.2.2), let

$$X_p(h) = \mathrm{Re}\left(Z_p p^{-1/2-ih} + \tfrac{1}{2} Z_p^2 p^{-1-2ih}\right), \tag{3.4.8}$$

where the Z_p's are i.i.d. Steinhaus random variables. Let

$$\mathcal{S}_k(h) = \sum_{e^{1000} < \log p \le e^k} X_p(h), \text{ and } \mathcal{Y}_k(h) = \mathcal{S}_{k+t_\theta}(h) - \mathcal{S}_{k+t_\theta-1}(h), \quad 1 \le k \le t - t_\theta. \tag{3.4.9}$$

We also define

$$\mathscr{W}_k = \sum_{1000 \le \ell \le k} \mathscr{N}_\ell, \quad 1000 \le k \le t, \tag{3.4.10}$$

where the $\mathscr{N}_\ell' s$ are real Gaussian random variables with zero mean and variance 1/2.

Lemma 3.4.2 (Gaussian comparison). *Let $A = 20$ and $\ell \geq 0$ satisfy (3.3.8). Let $k \in (t_\ell, t_{\ell+1}]$, and assume that $1 < |u_0| < C\sqrt{t - t_\theta}$ for some constant C. Then one has for any $|h| \leq 2$,*

$$\mathbb{E}\left\{ \frac{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}}{(|u_0| - 1)^{2q}} \prod_{j=r}^{k} |\mathscr{D}_{\Delta_j, A}(Y_j^{(t_\theta)}(h) - u_j)|^2 \right\}$$
$$\leq (1 + C' e^{-ae^t}) \mathbb{E}\left\{ \frac{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}}{(|u_0| - 1)^{2q}} \right\} \prod_{j=r}^{k} \mathbb{E}\left\{ |\mathscr{D}_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j)|^2 \right\}$$

with $C', a > 0$ absolute constants, and $q = \lfloor (|u_0| - 1)^2/8 \rfloor$. Furthermore, for $w - m(k) \in [L_y(k), U_y(k)]$, we have

$$\sum_{\mathbf{u} \in \mathscr{I}_k(w)} \mathbb{E}\left\{ \frac{|S_{t_\theta}(0) - S_{t_\theta}(h)|^{2q}}{(|u_0| - 1)^{2q}} \right\} \prod_{j=r}^{k} \mathbb{E}\left\{ |\mathscr{D}_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j)|^2 \right\}$$
$$\leq C' e^{-u_0^2/16} \sum_{\mathbf{u} \in \mathscr{I}_k(w)} \mathbb{P}\Big(\mathscr{W}_r \in [u_r, u_r + \Delta_r^{-1}), \mathscr{N}_j \in [u_j, u_j + \Delta_j^{-1}],\ \forall r < j \leq k \Big). \quad (3.4.11)$$

Proof. This follows the proof of Lemma 8 in [10]. We give the broad lines.

The first claim follows from using Lemma 3.6.5 to factor out the $2q-$th moment of $S_{t_\theta}(0) - S_{t_\theta}(h)$, and the mean-value theorem for Dirichlet polynomials (Lemma 3.6.4) to compare $\mathbb{E}\big\{ \prod_j |\mathscr{D}_{\Delta_j, A}(Y_j^{(t_\theta)}(h) - u_j)|^2 \big\}$ to $\prod_j \mathbb{E}\big\{ |\mathscr{D}_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j)|^2 \big\}$ (noting the independence of the $\mathscr{Y}_j$). The error when using these lemmas is small since the length of the polynomial $(S_{t_\theta}(h) - S_{t_\theta}(0))^q \prod_{j=r}^{k} \mathscr{D}_{\Delta_j, A}(Y_j^{(t_\theta)}(h) - u_j)$ is at most

$$\exp\Big(2qe^{t_\theta} + 2 \sum_{r \leq j \leq k} e^{j + t_\theta} \Delta_j^{200} \Big) \leq \exp\Big(100 e^{t_{\ell+1} + t_\theta} ((t - t_\theta) - k)^{800} \Big) \leq T^{1/100},$$

for large enough T, by the condition (3.3.8). This proves the first inequality.

For the second inequality, we begin by using Lemma 3.2.1 and Stirling's approximation on the $2q$-th moment of $|S_{t_\theta}(0) - S_{t_\theta}(h)|$ to get the Gaussian tail $e^{-u_0^2/16}$.

To bound each $\mathbb{E}\{ |\mathscr{D}_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j)|^2 \}$ for $r < j \leq k$, one first argues that

$$\mathbb{E}\big\{ |\mathscr{D}_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j)|^2 \big\} \leq \mathbb{E}\big\{ |G_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j)|^2 \big\} + O\big(e^{-\frac{1}{8}\Delta_j^{6A}} \big) \quad (3.4.12)$$

by decomposing the left-hand side according to whether $|\mathscr{Y}_j(h) - u_j| \leq \Delta_j^{6A}$ or $> \Delta_j^{6A}$. In the former case, the inequality follows directly from the defining equation for $\mathscr{D}_{\Delta_j, A}$ (3.4.4) and the fact that $G_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j) \in (0, 1)$. For the latter, the Cauchy-Schwarz inequality yields

$$\mathbb{P}\big(|\mathscr{Y}_j(h) - u_j| > \Delta_j^{6A} \big)^{1/2} \mathbb{E}\big\{ |\mathscr{D}_{\Delta_j, A}(\mathscr{Y}_j(h) - u_j)|^4 \big\}^{1/2}.$$

The probability is then shown to be small by a Chernoff bound, while the fourth moment is straightforwardly bounded by (3.4.4) and an estimate similar to (3.4.5).

Using the fourth property of $G_{\Delta_j,A}$ (3.4.1) and the Berry-Esseen theorem [10, Lemma 20], it follows that

$$\mathbb{E}\big\{|G_{\Delta_j,A}(\mathscr{Y}_j(h)-u_j)|^2\big\} \le \mathbb{P}\big(\mathscr{Y}_j(h)\in[u_j-\Delta_j^{-A/2},u_j+\Delta_j^{-1}+\Delta_j^{-A/2}]\big)+O\big(e^{-\Delta_j^{A-1}}\big)$$

and this probability can be shown to equal $\mathbb{P}\big(\mathscr{N}_j\in[u_j,u_j+\Delta_j^{-1}]\big)\cdot(1+O(\Delta_j^{-A/4}))$ using the restrictions $|u_j|\le 100\min(j,t-t_\theta-j)$ (see (3.4.3)). The same argument also shows that $\mathbb{E}\{|\mathscr{D}_{\Delta_r,A}(\mathscr{Y}_r(h)-u_r)|^2\}\ll\mathbb{P}(\mathscr{W}_r\in[u_r,u_r+\Delta_r^{-1}])$, and the lemma follows since $\prod_j(1+O(\Delta_j^{-A/4}))$ is bounded by an absolute constant.

□

To complete the proof of Lemma 3.3.9, we need to deal with the maximum and the additional Dirichlet polynomial $\mathscr{Q}$. We note that Lemma 3.4.1 implies that

$$\begin{aligned}&\mathbf{1}\Big(h\in B_\ell^{(k)}\cap C_\ell^{(k)}\cap E^\theta(u_0)\text{ and }S_k^{(t_\theta)}(h)\in(w,w+1]\Big)\\&\ll\frac{|S_{t_\theta}(0)-S_{t_\theta}(h)|^{2q}}{(|u_0|-1)^{2q}}\sum_{\mathbf{u}\in\mathscr{I}_k(w)}\prod_{j=r}^{k}|\mathscr{D}_{\Delta_j,A}(Y_j^{(t_\theta)}(h)-u_j)|^2,\end{aligned}\tag{3.4.13}$$

where $q=\lfloor(|u_0|-1)^2/8\rfloor$ and $A=20$. The right-hand side can be written as a linear combination $\sum_{i\in\mathscr{J}}D_i$ of Dirichlet polynomials D_i, each of length

$$\le\exp\big(2qe^{t_\theta}+2\sum_{r\le j\le k}e^{j+t_\theta}\Delta_j^{200}\big)\le\exp\big(100e^{k+t_\theta}((t-t_\theta)-k)^{800}\big).$$

Therefore, multiplying (3.4.13) by an arbitrary Dirichlet polynomial $\mathscr{Q}$ of length $N\le\exp(\frac{e^t}{100})$ and discretising using Lemma 3.6.7 yields

$$\begin{aligned}&\mathbb{E}\left\{\max_{|h|\le 2\log^\theta T}|\mathscr{Q}(\tfrac12+i\tau+ih)|^2\mathbf{1}\Big(h\in B_\ell^{(k)}\cap C_\ell^{(k)}\cap E^\theta(u_0),\,S_k^{(t_\theta)}(h)\in(w,w+1]\Big)\right\}\\&\qquad\ll_C\big(e^{-t_\theta}\log N+e^k((t-t_\theta)-k)^{800}\big)\sum_{i\in\mathscr{J}}\mathbb{E}\left\{|\mathscr{Q}(\tfrac12+i\tau)|^2|D_i(\tfrac12+i\tau)|^2\right\},\end{aligned}\tag{3.4.14}$$

where $C>0$ denotes the constant in the lemma's statement. We used the fact that the expectations are roughly of the same order for all the relevant h's after discretising, except for the h's associated with very large j in Lemma 3.6.7 for which the contribution is negligible (see the paragraph after (8) in [10] for more detail).

Noting that the D_i are supported on primes up to $\exp(e^{k+t_\theta})$ while $\mathcal{Q}$ is supported on primes $> \exp(e^{k+t_\theta})$, we can use Lemma 3.6.5 to split each expectation in the sum in (3.4.14) and write

$$\mathbb{E}\left\{|\mathcal{Q}(\tfrac{1}{2}+i\tau)|^2|D_i(\tfrac{1}{2}+i\tau)|^2\right\} \le 2\mathbb{E}\left\{|\mathcal{Q}(\tfrac{1}{2}+i\tau)|^2\right\}\mathbb{E}\left\{|D_i(\tfrac{1}{2}+i\tau)|^2\right\}.$$

By definition of the D_i and Lemma 3.4.2, we have

$$\begin{aligned}&\sum_{i\in\mathcal{I}}\mathbb{E}\left\{|D_i(\tfrac{1}{2}+i\tau)^2|\right\}\\ &\le C'\sum_{\mathbf{u}\in\mathcal{I}_k(w)} e^{-u_0^2/16}\cdot\mathbb{P}\big(\mathscr{W}_r\in[u_r,u_r+\Delta_r^{-1}],\mathscr{N}_j\in[u_j,u_j+\Delta_j^{-1}],\ \forall r<j\le k\big),\end{aligned}$$

for $C'>0$ an absolute constant. By definition of $\mathcal{I}_k(w)$, this is

$$\le C'\ e^{-u_0^2/16}\cdot\mathbb{P}\big(\mathscr{W}_j\le m(j)+U_y(j)+2 \text{ for all } 1\le j\le k \text{ and } \mathscr{W}_k\in[w-2,w+2]\big).$$

It follows that for any $|h|\le 2\log^\theta T$,

$$\begin{aligned}&\mathbb{E}\left\{\max_{|h|\le\log^\theta T}|\mathcal{Q}(\tfrac{1}{2}+i\tau+ih)|^2\mathbf{1}\left(h\in B_\ell^{(k)}\cap C_\ell^{(k)}\cap E^\theta(u_0),\ S_k^{(t_\theta)}(h)\in(w,w+1]\right)\right\}\\ &\ll_C e^{-u_0^2/16}\cdot\mathbb{E}\left\{|\mathcal{Q}(\tfrac{1}{2}+i\tau)|^2\right\}(e^{-t_\theta}\log N+e^k((t-t_\theta)-k)^{800})\\ &\qquad\times\mathbb{P}\big(\mathscr{W}_j\le m(j)+U_y(j)+2 \text{ for all } 1\le j\le k \text{ and } \mathscr{W}_k\in[w-2,w+2]\big).\end{aligned}$$

After decomposing the rightmost factor into

$$\sum_{i\in\{-1,0,1,2\}}\mathbb{P}\big(\mathscr{W}_j\le m(j)+U_y(j)+2,\ \forall 1\le j\le k \text{ and } \mathscr{W}_k\in(w+i-1,w+i]\big),$$

we can bound each term in this sum using the ballot estimate (Proposition 3.6.9) (with y replaced by $y+2$) to get

$$\ll\frac{y(U_y(k)-w+m(k)+2)e^{-w^2/k}}{k^{3/2}}.$$

This proves Lemma 3.3.9.

3.4.2 Proof of Lemma 3.3.10

The argument follows that of the previous section, with the addition of a twisted fourth moment estimate for the zeta function.

For any $\ell \geq 0$, we call a Dirichlet polynomial $\mathcal{Q}$ *degree-k well-factorable* if it can be expressed as $\mathcal{Q}_\ell^{(k)}(s)\prod_{0\leq\lambda\leq\ell}\mathcal{Q}_\lambda(s)$, where

$$\mathcal{Q}_\lambda(s) = \sum_{\substack{p|m \Longrightarrow \log p\in(e^{t_{\lambda-1}+t_\theta},e^{t_\lambda+t_\theta}]\\ \Omega_\lambda(m)\leq 10(t_\lambda-t_{\lambda-1})^{10^4}}} \frac{\eta(m)}{m^s}, \text{ and } \mathcal{Q}_\ell^{(k)}(s) = \sum_{\substack{p|m \Longrightarrow \log p\in(e^{t_\ell+t_\theta},e^{k+t_\theta}]\\ \Omega_\ell(m)\leq 10(t_{\ell+1}-t_\ell)^{10^4}}} \frac{\eta(m)}{m^s}$$

for arbitrary coefficients η satisfying $|\eta(m)| \leq \exp(e^t/500)$ for every $m \geq 1$. With this definition, we have the following lemma.

Lemma 3.4.3 (Lemma 9 in [10]). *Let $\ell \geq 0$ satisfy (3.3.8), and $k \in [t_\ell, t_{\ell+1}]$. Let $\mathcal{Q}$ be a degree-k well-factorable Dirichlet polynomial. Then for any $|h| \leq 2$ and large enough t,*

$$\mathbb{E}\left\{|(\zeta_\tau\mathcal{M}_{-1}\cdots\mathcal{M}_\ell\mathcal{M}_\ell^{(k)})(h)|^4\cdot|\mathcal{Q}(\tfrac{1}{2}+i\tau+ih)|^2\right\} \ll e^{4(t-t_\theta-k)}\mathbb{E}\left\{|\mathcal{Q}(\tfrac{1}{2}+i\tau+ih)|^2\right\}.$$

Remark 3.4.4. The factor $e^{4(t-t_\theta-k)}$ differs from the one in [10] since the product of mollifiers here ranges over primes up to $\exp(e^{t_\theta+k})$ as opposed to $\exp(e^k)$. The lemma as stated in [10] is recovered at $\theta = 0$.

The intuition behind this result is that $\zeta_\tau\mathcal{M}_{-1}\cdots\mathcal{M}_\ell\mathcal{M}_\ell^{(k)}$ should be supported on primes larger than $\exp(e^{t_\theta+k})$, and is therefore expected to decouple from $\mathcal{Q}$. The exponential term $e^{4((t-t_\theta)-k)}$ then corresponds to $\mathbb{E}\{|(\zeta_\tau\mathcal{M}_{-1}\cdots\mathcal{M}_\ell\mathcal{M}_\ell^{(k)})(h)|^4\}$; it is also the moment generating function of a Gaussian random variable of mean 0 and variance $\frac{1}{2}(t-t_\theta-k)$, which agrees with the random walk heuristic for $\log|\zeta_\tau\mathcal{M}_{-1}\cdots\mathcal{M}_\ell^{(k)}| \approx S_{t-t_\theta}^{(t_\theta)} - S_k^{(t_\theta)}$.

By definition of the mollifiers $\mathcal{M}_\ell$ and the assumptions of Lemma 3.4.3, the length of the "twist" $(\mathcal{M}_{-1}\cdots\mathcal{M}_\ell\mathcal{M}_\ell^{(k)})^2\cdot\mathcal{Q}$ does not exceed $T^{1/100}$. This puts Lemma 3.4.3 within reach of current techniques, notably by those of [103] which were used in [10] to prove it.

We now return to the proof of Lemma 3.3.10. Arguing as in the previous section, we have

$$\begin{aligned}
&\mathbb{E}\left\{|(\zeta_\tau\mathcal{M}_{-1}\cdots\mathcal{M}_\ell^{(k)}(h)|^4\cdot|\mathcal{Q}_\ell^{(k)}(h)|^2\cdot\mathbf{1}\left(h\in B_\ell\cap C_\ell\cap E^\theta(u),\, S_{t_\ell}^{(t_\theta)}\in(v,v+1]\right)\right\}\\
&\ll \sum_{\mathbf{u}\in\mathcal{I}_{t_\ell}(v)}\mathbb{E}\Big\{|\zeta_\tau\mathcal{M}_{-1}\cdots\mathcal{M}_\ell^{(k)}(h)|^4\cdot|\mathcal{Q}_\ell^{(k)}(h)|^2\\
&\qquad\cdot\frac{|S_{t_\theta}(0)-S_{t_\theta}(h)|^{2q}}{(|u_0|-1)^{2q}}\prod_j|\mathcal{D}_{\Delta_j,A}(Y_k^{(t_\theta)}(h)-u_j)|^2\Big\},
\end{aligned}$$

where we picked $A = 20$ and $q := \lfloor(|u_0|-1)^2/8\rfloor$. Then for every $\mathbf{u}$ in the sum, the polynomial

$$\mathcal{Q}_\ell^{(k)}(h)(S_{t_\theta}(0)-S_{t_\theta}(h))^q\prod_j\mathcal{D}_{\Delta_j,A}(Y_k^{(t_\theta)}(h)-u_j)$$

is well-factorable and has length $< \exp(e^t/100)$ by the definitions of $\mathcal{Q}_\ell^{(k)}$ and $\mathscr{D}_{\Delta_j,A}$. It thus follows from Lemma 3.4.3 that

$$\begin{aligned}\sum_{\mathfrak{u}\in\mathscr{T}_{t_\ell}(v)} \mathbb{E}\Big\{ &|\zeta_\tau \mathscr{M}_{-1}\cdots \mathscr{M}_\ell^{(k)}(h)|^4 \cdot |\mathcal{Q}_\ell^{(k)}(h)|^2 \\ &\cdot \frac{|S_{t_\theta}(0)-S_{t_\theta}(h)|^{2q}}{(|u_0|-1)^{2q}} \prod_j |\mathscr{D}_{\Delta_j,A}(Y_k^{(t_\theta)}(h)-u_j)|^2 \Big\} \\ \ll e^{4((t-t_\theta)-t_{\ell+1})} \sum_{\mathfrak{u}\in\mathscr{T}_{t_\ell}(v)} \mathbb{E}\Big\{ &|\mathcal{Q}_\ell^{(k)}(h)|^2 \cdot \frac{|S_{t_\theta}(0)-S_{t_\theta}(h)|^{2q}}{(|u_0|-1)^{2q}} \\ &\times \prod_j |\mathscr{D}_{\Delta_j,A}(Y_k^{(t_\theta)}(h)-u_j)|^2 \Big\}.\end{aligned}$$

The lemma then follows by proceeding as in the previous section starting from Equation (3.4.14).

3.5 Constrained large deviation estimates

We now prove Theorem 3.0.4 by combining the arguments of the previous two sections with those in [5], which adapt the recursive scheme to prove large deviation estimates for $\log|\zeta(1/2+i\tau)|$.

Here, the situation is considerably simpler than in Section 3.3. First, since Theorem 3.0.4 is an estimate at a single point h, no (spatial) discretisation argument is needed. Second, unlike in Theorem 3.3.2, the primes $p < \exp(e^{t_\theta})$ and "error" $S_{t_\theta}(h) - S_{t_\theta}(0)$ play essentially no role. Throughout this section, we take $h = 0$, as the proof holds mutatis mutandis for general $|h| \le 2$.

Since we are working at a single point, the recursive scheme employed here replaces the good sets of Section 3.3 by good *events*, also denoted by G_ℓ. These restrict the behaviour of the sum $S_k^{(t_\theta)}(0)$ at a sequence of sparse checkpoints $k = t_\ell$, where

$$t_\ell = (t-t_\theta) - \mathfrak{s}\log_\ell(t-t_\theta)$$

for $\ell > 0$, and $\mathfrak{s} = 2\cdot 10^6$. (This is the same definition as in (3.3.7), but with a larger constant $\mathfrak{s}$.)

Our initial step, Proposition 3.5.1 below, goes up to time t_1. Accordingly, we want $\mathscr{M}_1$ to include primes up to $\exp(e^{t_1+t_\theta})$; we keep the same definition of $\mathscr{M}_\ell$ as in Equation

(3.3.12), but adopt the convention $t_0 = 1000 - t_\theta$ to get

$$\mathscr{M}_1(0) = \sum_{\substack{p|m \Rightarrow 1000 \le \log\log p \le t_1 + t_\theta \\ \Omega_1(m) \le (t_1+t_\theta)^{10^5}}} \frac{\mu(m)}{m^{1/2+i\tau}}.$$

Moreover, $S_{t_1}^{(t_\theta)} - S_{t_0}^{(t_\theta)} = S_{t_1+t_\theta}$. For simplicity, we will write $S_k, S_k^{(t_\theta)}, \zeta_\tau$ and $\mathscr{M}_\ell$ to denote each of these (random) functions evaluated at 0.

For $\ell \ge 1$ and $\kappa = V/(t - t_\theta)$, we introduce the events

$$\begin{aligned}
A_\ell &= A_{\ell-1} \cap \{|\tilde{S}_{t_\ell}^{(t_\theta)} - \tilde{S}_{t_{\ell-1}}^{(t_\theta)}| < 10^3(t_\ell - t_{\ell-1})\} \\
B_\ell &= B_{\ell-1} \cap \{S_{t_\ell}^{(t_\theta)} \le \kappa t_\ell + \mathfrak{s}_0 \log_\ell(t - t_\theta)\}, \\
C_\ell &= C_{\ell-1} \cap \{S_{t_\ell}^{(t_\theta)} > \kappa t_\ell - \mathfrak{s}_0 \log_\ell(t - t_\theta)\} \\
D_\ell &= D_{\ell-1} \cap \{|\zeta_\tau e^{-S_{t_\ell+t_\theta}}| \le c_\ell |\zeta_\tau \mathscr{M}_1 \cdots \mathscr{M}_\ell| + e^{-10^4(t_\ell - t_{\ell-1})}\}
\end{aligned}$$

where $c_\ell = \prod_{i=1}^{\ell}(1 + e^{-(t_{i-1}+t_\theta)})$, and $\mathfrak{s}_0 = (3/4)\mathfrak{s}$. At $\ell = 0$, we take

$$A_0 = B_0 = C_0 = D_0 = [T, 2T] \text{ (the sample space of } \tau).$$

Our good events are then defined as

$$G_\ell = A_\ell \cap B_\ell \cap C_\ell \cap D_\ell, \quad \ell \ge 0.$$

The choice of $\mathfrak{s}_0$ in the definition of B_ℓ and C_ℓ ensures that certain exponents are small enough in the proofs below, namely those in Equation (3.5.2), and (3.5.5)–(3.5.8). There is some flexibility in this choice; see [5, Equations (19)–(27)] for more detail.

Letting

$$G_A = \{S_k^{(t_\theta)} \in [L_A(k), U_A(k)] \, \forall k \le t_*\}, \quad t_* := (t - t_\theta)/2.$$

and

$$H^\theta = \{\log|\zeta_\tau e^{-S_{t_\theta}}| > V\},$$

we proceed similarly to how we did in Equation (3.3.9) and partition $H^\theta \cap G_A$ into

$$H^\theta \cap G_A = (H^\theta \cap G_A \cap G_1^c) \cup \bigcup_{\ell=1}^{\mathscr{L}-1} (H^\theta \cap G_A \cap G_\ell \setminus G_{\ell+1}) \cup (H^\theta \cap G_A \cap G_{\mathscr{L}}),$$

for $\mathscr{L}$ defined in Equation (3.3.8). The theorem then follows by a union bound and the following three propositions.

Proposition 3.5.1. *Let $\theta \in (-1,0]$. Then there exists $\delta > 0$ such that uniformly in large enough t and $t_* - t_*^{2/3} \le V/2 \le U_A(t_*)$,*

$$\mathbb{P}\big(H^\theta \cap G_A \cap G_1^c\big) \ll \frac{A\big(U_A(t_*) - V/2 + \sqrt{t_*}\big)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}} (t-t_\theta)^{-\delta}.$$

Proposition 3.5.2. *Let $\theta \in (-1,0]$ and $1 \le \ell \le \mathscr{L}-1$. Then there exists $\delta > 0$ such that uniformly in ℓ, large enough t and $t_* - t_*^{2/3} \le V/2 \le U_A(t_*)$,*

$$\mathbb{P}\big(H^\theta \cap G_A \cap G_\ell \setminus G_{\ell+1}\big) \ll \frac{A\big(U_A(t_*) - V/2 + \sqrt{t_*}\big)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}} \big(\log_\ell(t-t_\theta)\big)^{-\delta}.$$

Proposition 3.5.3. *Let $\theta \in (-1,0]$. Then uniformly in large enough t and $t_* - t_*^{2/3} \le V/2 \le U_A(t_*)$,*

$$\mathbb{P}\big(H^\theta \cap G_A \cap G_{\mathscr{L}}\big) \ll \frac{A\big(U_A(t_*) - V/2 + \sqrt{t_*}\big)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}}.$$

Proof of Theorem 3.0.4. Using the decomposition

$$\begin{aligned}&\mathbb{P}\big(H^\theta \cap G_A\big)\\ &= \mathbb{P}\big(H^\theta \cap G_A \cap G_1^c\big) + \sum_{\ell=1}^{\mathscr{L}-1} \mathbb{P}\big(H^\theta \cap G_A \cap G_\ell \setminus G_{\ell+1}\big) + \mathbb{P}\big(H^\theta \cap G_A \cap G_{\mathscr{L}}\big),\end{aligned}$$

the theorem follows from the propositions above, noting that the factors $(\log_\ell(t-t_\theta))^{-\delta}$ are summable (see (3.3.16)). □

A few heuristics are in order prior to proving these results. As in Section 3.3, we think of $\log|\zeta_\tau e^{-S_{t_\theta}}|$ as the partial sum $S^{(t_\theta)}_{t-t_\theta}$, which can itself be thought of as a Gaussian random walk of length $t-t_\theta$. The probability that such a walk's endpoint is in $[V, V+1]$ should roughly equal $e^{-V^2/(t-t_\theta)}/\sqrt{t-t_\theta}$.

The additional factor to the left of this Gaussian term in Propositions 3.5.1, 3.5.2 and 3.5.3 above arises from the event G_A, which restricts $S_k^{(t_\theta)}$ to lie under a barrier (starting at $A > 0$) up to time t_*. Were $S_k^{(t_\theta)}$ Gaussian, the probability of this happening and $S^{(t_\theta)}_{t_*} \approx u$ is

$$\ll \frac{A\big(U_A(t_*) - u + 1\big)}{{t_*}^{3/2}} e^{-u^2/t_*},$$

by the ballot theorem in Proposition 3.6.9. Being supported on different primes, $S^{(t_\theta)}_{t_*}$ and $S^{(t_\theta)}_{t-t_\theta} - S^{(t_\theta)}_{t_*}$ should behave independently, and if viewed as Gaussian increments, the probability of $H^\theta \cap G_A$ should therefore be

$$\ll \int_{u<U_A(t_*)} \frac{A\big(U_A(t_*) - u + 1\big)}{t_*} \frac{e^{-u^2/t_*}}{\sqrt{t_*}} \frac{e^{-(V-u)^2/(t_*)}}{\sqrt{t_*}} \mathrm{d}u.$$

This integral can then be estimated directly. By completing the square and using the substitution $v = u - V/2$, this is

$$\begin{aligned}
&= \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t_*}} \frac{A}{t_*} \int_{-\infty}^{U_A(t_*)-V/2} \frac{e^{-2v^2/t_*}}{\sqrt{t_*}} \big(U_A(t_*) - V/2 - v + 1\big) \mathrm{d}v \\
&\leq \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t_*}} \frac{A}{t_*} \Big(2\big(U_A(t_*) - V/2 + 1\big) + \int_0^\infty \frac{v e^{-2v^2/t_*}}{\sqrt{t_*}} \mathrm{d}v\Big) \qquad (3.5.1) \\
&\leq \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t_*}} \frac{A}{t_*} \Big(2\big(U_A(t_*) - V/2 + 1\big) + \mathbb{E}\big\{|X|\big\}\Big)
\end{aligned}$$

where X is a Gaussian random variable with variance $t_*/4$. Since $\mathbb{E}\big\{|X|\big\} \leq \sqrt{t_*} \asymp \sqrt{t - t_\theta}$, we recover the right-hand side in Theorem 3.0.4. The remainder of this section is devoted to making this argument rigorous.

3.5.1 Proof of Proposition 3.5.1

A union bound gives

$$\begin{aligned}
&\mathbb{P}\big(H^\theta \cap G_1^c \cap G_A\big) \\
&\quad \ll \mathbb{P}\big(A_1^c\big) + \mathbb{P}\big(A_1 \cap D_1^c\big) + \mathbb{P}\big(B_1^c \cap G_A\big) + \mathbb{P}\big(H^\theta \cap C_1^c \cap A_1 \cap D_1 \cap G_A\big),
\end{aligned}$$

We show that each of these terms satisfies the bound in 3.5.1. To begin with, the Gaussian tail bound in Corollary 3.6.3 gives $\mathbb{P}(A_1^c) \ll \sqrt{t_1 - t_0} \cdot \exp(-10^6(t_1 - t_0))$, which is clearly of the desired order for some $\delta > 0$.

To bound $\mathbb{P}(A_1 \cap D_1^c)$, we first note that $A_1 \cap \{|\zeta_\tau| \leq e^{2t}\} \subset A_1 \cap D_1$, since Lemma 3.3.8 (whose assumptions are satisfied on A_1) implies that

$$\begin{aligned}
\big|\zeta_\tau e^{-(S_{t_1}^{(t_\theta)} - S_{t_0}^{(t_\theta)})}\big| &\leq c_1 |\zeta_\tau \mathscr{M}_1| + e^{2t - 10^5(t_1 - t_0)} \\
&\leq c_1 |\zeta_\tau \mathscr{M}_1| + e^{-10^4(t_1 - t_0)}.
\end{aligned}$$

It follows that

$$\mathbb{P}\big(A_1 \cap D_1^c\big) \leq \mathbb{P}\big(|\zeta_\tau| > e^{2t}\big) \leq e^{-4t} \mathbb{E}\big\{|\zeta_\tau|^2\big\},$$

which $\leq e^{-3t}$ by Lemma 3.6.6, and thus small enough.

The bound on $\mathbb{P}(B_1^c \cap G_A)$ follows from Markov's inequality and Equation (3.6.1), as they together give

$$\mathbb{P}(B_1^c) \ll \frac{\mathbb{E}\big\{|S_{t_1}^{(t_\theta)}|^{2q}\big\}}{(\kappa t_1 + \mathfrak{s}_0 \log(t - t_\theta))^{2q}} \ll \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t - t_\theta}} (t - t_\theta)^{1 + \kappa^2 \mathfrak{s} - 2\kappa \mathfrak{s}_0}, \qquad (3.5.2)$$

and $1+\kappa^2\mathfrak{s}-2\kappa\mathfrak{s}_0 = 1-10^6+o(1) \le -10^5$.

Lastly, to estimate $\mathbb{P}(H^\theta \cap C_1^c \cap A_1 \cap D_1 \cap G_A)$, we partition according to the value of $S_{t_1}^{(t_\theta)}$ (in $A_1 \cap C_1^c$), to get the bound

$$\le \sum_{-10^3(t_1-t_0)<u<\kappa t_1 - \mathfrak{s}_0 \log t_1} \mathbb{P}\Big(\big\{ S_{t_1}^{(t_\theta)} \in (u,u+1], \big|\zeta_\tau e^{-S_{t_1+t_\theta}}\big| > e^{V-u-1} \big\} \cap D_1 \Big).$$

A short proof by contradiction (see Section 3.3.4.3) shows that if $|\zeta_\tau e^{-S_{t_1+t_\theta}}| > e^{V-u-1}$ and $h \in C_1^c \cap A_1 \cap D_1$, then $|\zeta_\tau \mathscr{M}_1| > \frac{1}{100} e^{V-u}$. Markov's inequality then implies that the previous sum is

$$\begin{aligned} &\le \sum_u \mathbb{P}\Big(\big\{ |\zeta_\tau \mathscr{M}_1| > \frac{1}{100} e^{V-u} \big\} \cap \big\{ S_{t_1}^{(t_\theta)} \in (u,u+1] \big\} \Big) \\ &\ll \sum_u e^{-4(V-u)} \mathbb{E}\Big\{ |\zeta_\tau \mathscr{M}_1|^4 \mathbf{1}\big\{ S_{t_1}^{(t_\theta)} \in (u,u+1] \big\} \Big\}. \end{aligned} \tag{3.5.3}$$

The sum over $u<0$ is $\ll te^{-4V} \ll e^{-3t_*}$ and can thus safely be ignored. For the remaining range, note that it follows directly from the proof of Lemma 3.3.10 that

$$\mathbb{E}\big\{ |\zeta_\tau \mathscr{M}_1|^4 \mathbf{1}(S_{t_1}^{(t_\theta)} \in \big(u,u+1]\big) \big\} \quad \ll e^{4(t-t_\theta-t_1)} \frac{e^{-u^2/t_1}}{\sqrt{t_1}}$$

for every $0 \le u \le 2t_1$. Applying this to the remaining terms in the summation in (3.5.3), we find that it is

$$\ll \sum_{0\le u<\kappa t_1 - \mathfrak{s}_0 \log t_1} e^{4((t-t_\theta)-t_1)} e^{-4(V-u)} \frac{e^{-u^2/t_1}}{\sqrt{t_1}}.$$

Using the change of variable $w = \kappa t_1 - u$, the summation above is bounded by

$$\begin{aligned} &\frac{e^{-\kappa^2 t_1}}{\sqrt{t_1}} e^{4(1-\kappa)((t-t_\theta)-t_1)} \sum_{w>\mathfrak{s}_0 \log(t-t_\theta)} e^{-(4-2\kappa)w} \\ &\ll \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}} (t-t_\theta)^{\mathfrak{s}\kappa^2+\mathfrak{s}(4-4\kappa)-2(2-\kappa)\mathfrak{s}_0}. \end{aligned} \tag{3.5.4}$$

This yields the claim since the exponent of $(t-t_\theta)$ is bounded by -10^5.

3.5.2 Proof of Propositions 3.5.2 and 3.5.3

These propositions follow from the same argument used to prove Propositions 2.2 and 2.3 in [5], suitably modified to include the event G_A. This modification simply amounts to replacing their Lemmas 2.6 and 2.7 with Lemmas 3.5.4 and 3.5.5 below, respectively, which play the role of Lemmas 3.3.9 and 3.3.10 in Section 3.3.

Lemma 3.5.4. *Let $\ell \geq 1$ satisfy (3.3.8). Let $\mathcal{Q}$ be a Dirichlet polynomial of length $N \leq \exp(\frac{1}{100}e^t)$ supported on integers all of whose prime factors are greater than $\exp(e^{t_\theta+t_\ell})$. Then for $|w-\kappa t_\ell| \leq \mathfrak{s}_0 \log_\ell(t-t_\theta)$ and large enough t, we have*

$$\mathbb{E}\big\{|\mathcal{Q}(\tfrac{1}{2}+i\tau)|^2 \mathbf{1}\big(B_\ell \cap C_\ell \cap G_A \cap \{S_{t_\ell}^{(t_\theta)} \in (w, w+1]\}\big)\big\} \\ \ll \mathbb{E}\Big\{|\mathcal{Q}(\tfrac{1}{2}+i\tau)|^2\Big\} \cdot \frac{A\big(U_A(t_*) - w(t_*/t_\ell) + \sqrt{t_*}\big)}{t_*} \frac{e^{-w^2/t_\ell}}{\sqrt{t_\ell}}.$$

Proof. Partitioning according to the value of $S_{t_*}^{(t_\theta)}$, the right-hand side in the lemma above is bounded by

$$\sum_{u \in [L_A(t_*), U_A(t_*)] \cap \mathbb{Z}} \mathbb{E}\big\{|\mathcal{Q}(\tfrac{1}{2}+i\tau)|^2 \mathbf{1}\big(B_\ell \cap C_\ell \cap \{S_{t_\ell}^{(t_\theta)} - S_{t_*}^{(t_\theta)} \in (w-u, w-u+1]\big) \\ \times \mathbf{1}\big(\{S_{t_*}^{(t_\theta)} \in [u, u+1]\} \cap G_A\big)\big\}.$$

Each term in the summation above can then be bounded as in the proof of Lemma 3.3.9 (see also Lemma 2.6 in [5]). Namely, one expresses the indicators in the expectation as a sum over tuples $(u_j)_j$ of products of the form $\prod_{Y_j^{(t_\theta)}} \mathbf{1}(Y_j^{(t_\theta)} \in [u_j, u_j + \Delta_j^{-1}])$, where $Y_j^{(t_\theta)}$ ranges over

$$Y_j^{(t_\theta)} \in \{S_r^{(t_\theta)}\} \cup \{S_j^{(t_\theta)} - S_{j-1}^{(t_\theta)}\}_{r<j\leq t_*} \cup \{S_{t_1}^{(t_\theta)} - S_{t_*}^{(t_\theta)}\} \cup \{S_{t_k}^{(t_\theta)} - S_{t_{k-1}}^{(t_\theta)}\}_{1<k\leq \ell},$$

for $r = \lceil A/4 \rceil$ and a suitable choice of mesh size Δ_j. After approximating each of these indicators by a Dirichlet polynomial $\mathcal{D}_{\Delta_j, A}$ (cf. Equation (3.4.4)), one uses Lemma 3.6.5 to decouple the second moment of their product from $\mathbb{E}\{|\mathcal{Q}(1/2+i\tau)|^2\}$, and Lemma 3.6.4 to bound said second moment by $\prod_j \mathbb{E}\{|\mathcal{D}_{\Delta_j,A}(\mathcal{Y}_j - u_j)|^2\}$, where $\mathcal{Y}_j$ is a Steinhaus model for $Y_j^{(t_\theta)}$ (cf. Equations (3.4.8) and (3.4.9)). Finally, undoing the Dirichlet polynomial approximation yields a product of probabilities, which are then bounded by a Gaussian counterpart using the Berry-Esseen theorem[1]. Altogether, this procedure yields the bound

$$\ll \sum_{u \in [L_A(t_*), U_A(t_*)] \cap \mathbb{Z}} \frac{A(U_A(t_*) - u + 1)}{t_*} \frac{e^{-u^2/t_*}}{\sqrt{t_*}} \frac{e^{-(w-u)^2/(t_\ell - t_*)}}{\sqrt{t_\ell - t_*}},$$

[1] In the $\theta = 0$ case, one uses the estimate in [10, Lemma 18] for the probabilities involving the first increment $S_r^{(t_\theta)} = S_r$ rather than the Berry-Esseen bound.

which by the change of variables $u = w(t_*/t_\ell) + v$ is

$$\ll \sum_{\substack{v+w(t_*/t_\ell) \\ \in \mathbb{Z}\cap[L_A(t_*),U_A(t_*)]}} \frac{A(U_A(t_*) - w(t_*/t_\ell) - v + 1)}{t_*} \frac{e^{-(w(t_*/t_\ell)+v)^2/t_*}}{\sqrt{t_*}} \frac{e^{-\frac{(w-w(t_*/t_\ell)-v)^2}{t_\ell - t_*}}}{\sqrt{t_\ell - t_*}}$$
$$= A\frac{e^{-w^2/t_\ell}}{t_*^{3/2}} \sum_{\substack{v+w(t_*/t_\ell) \\ \in \mathbb{Z}\cap[L_A(t_*),U_A(t_*)]}} (U_A(t_*) - w(t_*/t_\ell) - v + 1)\frac{e^{-v^2(t_\ell/t_*(t_\ell - t_*))}}{\sqrt{t_\ell - t_*}}.$$

This sum can then be estimated by comparison with the corresponding Gaussian integral. In this range of w, this is $\ll U_A(t_*) - w(t_*/t_\ell) + \sqrt{t_*}$ by completing the square as in (3.5.1). □

Combining the same argument with Lemma 3.4.3 also yields the following.

Lemma 3.5.5. *Let $\ell \geq 1$ satisfy (3.3.8). Then for $|w - \kappa t_\ell| \leq \mathfrak{s}_0 \log_\ell(t - t_\theta)$ and large enough t, we have*

$$\mathbb{E}\big\{|\zeta_\tau \mathscr{M}_1 \cdots \mathscr{M}_\ell|^4 \mathbf{1}\big(G_A \cap B_\ell \cap C_\ell, S_{t_\ell}^{(t_\theta)} \in [w, w+1]\big)\big\}$$
$$\ll e^{4((t-t_\theta)-t_\ell)} \cdot \frac{A\big(U_A(t_*) - w(t_*/t_\ell) + \sqrt{t_*}\big)}{t_*} \frac{e^{-w^2/t_\ell}}{\sqrt{t_\ell}}.$$

Proof of Proposition 3.5.2. By the proof of Proposition 2 in [5], replacing their Lemmas 2.6 and 2.7 with our Lemmas 3.5.4 and 3.5.5, we get the following estimates:

$$\mathbb{P}\big(G_A \cap A_{\ell+1}^c \cap G_\ell\big) \ll \frac{A\big(U_A(t_*) - V/2 + \sqrt{t_*}\big)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}} (\log_\ell(t-t_\theta))^{\kappa^2\mathfrak{s} - 10^6\mathfrak{s} + 2\kappa\mathfrak{s}_0}, \tag{3.5.5}$$

$$\mathbb{P}\big(G_A \cap B_{\ell+1}^c \cap G_\ell\big) \ll \frac{A\big(U_A(t_*) - V/2 + \sqrt{t_*}\big)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}} (\log_\ell(t-t_\theta))^{1/2 + \kappa^2\mathfrak{s} - 2\kappa\mathfrak{s}_0}, \tag{3.5.6}$$

$$\mathbb{P}\left(H^\theta \cap G_A \cap C_{\ell+1}^c \cap A_{\ell+1} \cap D_{\ell+1} \cap G_\ell\right) \tag{3.5.7}$$
$$\ll \frac{A\big(U_A(t_*) - V/2 + \sqrt{t_*}\big)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}} (\log_\ell(t-t_\theta))^{\mathfrak{s}(2-\kappa)^2 - 2(2-\kappa)\mathfrak{s}_0 + 1/2},$$

and finally

$$\mathbb{P}\big(G_A \cap D_{\ell+1}^c \cap A_{\ell+1} \cap G_\ell\big) \tag{3.5.8}$$
$$\ll \frac{A\big(U_A(t_*) - V/2 + \sqrt{t_*}\big)}{t_*} \frac{e^{-V^2/(t-t_\theta)}}{\sqrt{t-t_\theta}} (\log_{\ell-1}(t-t_\theta))^{\kappa^2\mathfrak{s} + 2\kappa\mathfrak{s}_0 - 4\mathfrak{s}(10^3 - 1)}$$

for any $\ell \geq 1$. In each of these, the exponent of the rightmost term is negative upon substituting $\mathfrak{s} = 2 \cdot 10^6$ and $\mathfrak{s}_0 = (3/4)10^6$. They therefore imply Proposition 3.5.2 by the union bound

$$\mathbb{P}\big(H^\theta \cap G_A \cap (G_\ell \smallsetminus G_{\ell+1})\big) \leq \mathbb{P}\big(G_A \cap A_{\ell+1}^c \cap G_\ell\big) + \mathbb{P}\big(G_A \cap B_{\ell+1}^c \cap G_\ell\big) + \mathbb{P}\big(H^\theta \cap G_A \cap C_{\ell+1}^c \cap A_{\ell+1} \cap D_{\ell+1} \cap G_\ell\big) + \mathbb{P}\big(G_A \cap D_{\ell+1}^c \cap A_{\ell+1} \cap G_\ell\big).$$

□

Proof of Proposition 3.5.3. Since $\mathbb{P}\big(H^\theta \cap G_A \cap G_{\mathscr{L}}\big) \leq \mathbb{P}\big(G_A \cap G_{\mathscr{L}}\big)$, it suffices to show that the latter satisfies the desired bound. Partitioning on the value of $u \in \mathbb{Z}$ of $S_{t_{\mathscr{L}}}^{(t_\theta)}$, we have that $\mathbb{P}\big(G_A \cap G_{\mathscr{L}}\big)$ is

$$\leq \sum_{-\mathfrak{s}_0 \log_{\mathscr{L}}(t-t_\theta) < u - \kappa t_{\mathscr{L}} < \mathfrak{s}_0 \log_{\mathscr{L}}(t-t_\theta)} \mathbb{E}\big\{1\big(B_{\mathscr{L}} \cap C_{\mathscr{L}} \cap G_A \cap \{S_{t_{\mathscr{L}}}^{(t_\theta)} \in (u, u+1]\}\big)\big\}.$$

By Lemma 3.5.4 (with $\mathcal{Q} \equiv 1$), this is

$$\ll \sum_{-\mathfrak{s}_0 \log_{\mathscr{L}}(t-t_\theta) < u - \kappa t_{\mathscr{L}} < \mathfrak{s}_0 \log_{\mathscr{L}}(t-t_\theta)} \frac{A\big(U_A(t_*) - u(t_*/t_{\mathscr{L}}) + \sqrt{t_*}\big)}{t_*} \frac{e^{-u^2/t_{\mathscr{L}}}}{\sqrt{t_{\mathscr{L}}}}.$$

Recalling that $\log_{\mathscr{L}}(t-t_\theta) = O(1)$ (cf. Equation (3.3.8)), a change of variables $w = u - \kappa t_{\mathscr{L}}$ in the previous sum yields

$$\ll \frac{e^{-\kappa^2(t-t_\theta)}}{\sqrt{t-t_\theta}} \sum_{-\mathfrak{s}_0 \log_{\mathscr{L}}(t-t_\theta) < w < \mathfrak{s}_0 \log_{\mathscr{L}}(t-t_\theta)} \frac{A\big(U_A(t_*) - w(t_*/t_{\mathscr{L}}) - V/2 + \sqrt{t_*}\big)}{t_*} e^{-2\kappa w},$$

and the sum is $\ll A(U_A(t_*) - V/2 + \sqrt{t_*})/t_*$. □

3.6 Appendix

3.6.1 Moments of the zeta function and of Dirichlet polynomials

Recall that

$$\tilde{S}_j(h) = \sum_{\exp(e^{1000}) < p \leq \exp(e^j)} \frac{1}{p^{1/2+i(\tau+h)}} + \frac{1}{2} \frac{1}{p^{1+2i(\tau+h)}}, \quad h \in \mathbb{R},$$

where $\exp(e^{1000})$ is a large constant, and $\Re \tilde{S}_j = S_j$. In particular, $S_0 \equiv \tilde{S}_0 \equiv 0$. The following (standard) moment estimate for Dirichlet polynomials follows from expanding $|\tilde{S}_k(h) - \tilde{S}_j(h)|^{2q}$ and using

$$\binom{q}{\alpha_1, \dots, \alpha_r}^2 \leq q! \binom{q}{\alpha_1, \dots, \alpha_r}$$

to bound the main contribution.

Lemma 3.6.1. *Fix $\theta \in (-1,0)$. Then for any $h \in [-2,2]$, and $0 \le j \le k \le t$ and $2q \le e^{t-k}$, we have*

$$\mathbb{E}\left\{|\tilde{S}_k(h) - \tilde{S}_j(h)|^{2q}\right\} \ll q!(k-j+C)^q.$$

for some constant $C > 0$, uniformly over h. If $k \ge j \ge t|\theta|$, this bound holds for $C = 1$.

Proof. This is a corollary of Lemma 3 in [166], with the constant C representing the error in estimating the resulting prime sums. □

We also require analogous estimates for the real part of $\tilde{S}_j$.

Lemma 3.6.2 (Lemma 16 in [10])**.** *Let $\theta \in (-1,0)$ be arbitrary. Then for any $h \in [-2,2]$, $t|\theta| \le j \le k$, and $2q \le e^{t-k}$ we have*

$$\mathbb{E}\left\{|S_k(h) - S_j(h)|^{2q}\right\} \ll \frac{(2q)!}{2^q q!}\left(\frac{k-j}{2}\right)^q, \tag{3.6.1}$$

as well as

$$\mathbb{E}\left\{|S_k(h) - S_j(h) + A|^{2q}\right\} \ll \frac{(2q)!}{q!}(k-j)^q + (2A)^{2q} \tag{3.6.2}$$

for any constant $A > 0$. Moreover, there exists a constant $C > 0$ such that for any $0 \le j < k$, $2q \le e^{t-k}$ we have

$$\mathbb{E}\left\{|S_k(h) - S_j(h)|^{2q}\right\} \ll \sqrt{q}\frac{(2q)!}{2^q q!}\left(\frac{k-j+C}{2}\right)^q.$$

Using Markov's inequality and Stirling's approximation, these bounds yield the following Gaussian tail estimates.

Corollary 3.6.3 (Gaussian tail for Dirichlet polynomials)**.** *Let $\theta \in (-1,0)$ be arbitrary. Then for any $h \in [-2,2]$, $t|\theta| \le j \le k \le t$, and $0 < V^2 \le (k-j)e^{t-k}/2$,*

$$\mathbb{P}\Big(|S_k(h) - S_j(h)| > V\Big) \ll \exp\Big(-\frac{V^2}{k-j}\Big), \tag{3.6.3}$$

$$\mathbb{P}\Big(|\tilde{S}_k(h) - \tilde{S}_j(h)| > V\Big) \ll \Big(1 + \frac{V}{(k-j+1)^{1/2}}\Big)\exp\left(-\frac{V^2}{k-j+1}\right). \tag{3.6.4}$$

For any $0 \le j \le k \le t$, the following bound also holds

$$\mathbb{P}\Big(|\tilde{S}_k(h) - \tilde{S}_j(h)| > V\Big) \ll \Big(1 + \frac{V}{(k-j+C)^{1/2}}\Big)\exp\left(-\frac{V^2}{k-j+C}\right)$$

for a constant $C > 0$.

Proof. Beginning with the first inequality, we may assume that $V^2 > k - j$ (the bound is trivial otherwise). Using Markov's inequality and the moment estimate in Lemma 3.6.2, we can write

$$\mathbb{P}\Big(|S_k(h) - S_j(h)| > V\Big) \ll \frac{\mathbb{E}\left\{|S_k(h) - S_j(h)|^{2q}\right\}}{V^{2q}} \ll \frac{1}{V^{2q}} \frac{(2q)!}{2^q q!} \Big(\frac{k-j}{2}\Big)^q$$

for any $1 \le q \le e^{t-k}/2$. Picking $q = \lfloor V^2/(k-j) \rfloor$ and using Stirling's approximation shows the above is

$$\ll \frac{1}{V^{2q}} \Big(\frac{2q}{e}\Big)^q \Big(\frac{k-j}{2}\Big)^q \le e^{-q} \ll e^{-V^2/(k-j)}, \tag{3.6.5}$$

which is the bound in Equation (3.6.3). The second and third bounds are proved in the same way, using the moment estimates in Lemma 3.6.1 and picking $q = \lfloor V^2/(k-j+1) \rfloor$ and $q = \lfloor V^2/(k-j+C) \rfloor$, respectively, where C is the constant in said lemma. □

We record the following mean-value theorem for Dirichlet polynomials (Lemma 3.6.4), due to Montgomery and Vaughan [139], and an immediate corollary of it which allows us to decouple averages of Dirichlet polynomials supported on disjoint ranges of primes (Lemma 3.6.5).

Lemma 3.6.4 (Mean–value theorem for Dirichlet polynomials)**.** *Let $T > 0$ and τ be uniformly distributed in $[T, 2T]$. Then*

$$\begin{aligned}\mathbb{E}\Big\{\Big|\sum_{n\le N} a(n) n^{-i\tau}\Big|^2\Big\} &= \Big(1 + O\Big(\frac{N}{T}\Big)\Big) \sum_{n\le N} |a(n)|^2 \\ &= \Big(1 + O\Big(\frac{N}{T}\Big)\Big) \mathbb{E}\Big\{\Big|\sum_{n\le N} a(n) Z_n\Big|^2\Big\}.\end{aligned}$$

Proof. This follows from Corollary 3 in [139]. □

Lemma 3.6.5 (Lemma 14 in [10])**.** *Let $h \in [-2, 2]$ and*

$$A(s) = \sum_{\substack{n\le N \\ p|n \implies p\le w}} \frac{a(n)}{n^s} \text{ and } B(s) = \sum_{\substack{n\le N \\ p|n \implies p> w}} \frac{b(n)}{n^s}$$

be Dirichlet polynomials with $N \le T^{1/4}$. Then we have

$$\begin{aligned}&\mathbb{E}\left\{|A(\tfrac{1}{2} + i\tau + ih)|^2 |B(\tfrac{1}{2} + i\tau + ih)|^2\right\} \\ &= \big(1 + O(T^{-1/2})\big)\, \mathbb{E}\left\{|A(\tfrac{1}{2} + i\tau + ih)|^2\right\} \mathbb{E}\left\{|B(\tfrac{1}{2} + i\tau + ih)|^2\right\}.\end{aligned}$$

Lastly, we use the following classical result for the second moment of ζ due to Hardy and Littlewood [90], see also Equation (1.79) in [112] for a quick proof of the upper bound.

Lemma 3.6.6. *For all $h \in [-2, 2]$, we have $\mathbb{E}\left\{|\zeta_\tau(h)|^2\right\} \sim \log T$.*

3.6.2 Discretisation

The following lemma allows us to discretise the maximum of a Dirichlet polynomial over a continuous interval to a countable mesh of points, and is essentially a consequence of the Poisson summation formula.

Lemma 3.6.7 (Lemma 27 in [10]). *Let $\mathcal{I}$ be a finite set of indices and D_l with $l \in \mathcal{I}$ be a sequence of Dirichlet polynomials of length $\leq N$. Then for any $A \geq 100$,*

$$\max_{|h|\leq 2\log^\theta T} \Big(\sum_{l\in\mathcal{I}} |D_l(\tfrac{1}{2} + i\tau + ih)|^2 \Big) \tag{3.6.6}$$

$$\ll_A \sum_{|j|\leq 16\log^\theta T \log N} \Big(\sum_{l\in\mathcal{I}} \Big| D_l\Big(\frac{1}{2} + i\tau + \frac{2\pi i j}{8\log N}\Big)\Big|^2 \Big) \tag{3.6.7}$$

$$+ \sum_{|j|>16\log^\theta T \log N} \frac{1}{1+|j|^A} \Big(\sum_{l\in\mathcal{I}} \Big| D_l\Big(\frac{1}{2} + i\tau + \frac{2\pi i j}{8\log N}\Big)\Big|^2 \Big). \tag{3.6.8}$$

Proof. The claim for $\theta = 0$ is Lemma 27 in [10], which is itself a consequence of Lemma 25 in the same work. The latter states that for any Dirichlet polynomial D of length N, $t, h_0 \in \mathbb{R}$, and $\varepsilon > 0$,

$$D\big(\tfrac{1}{2} + it + ih_0\big) = \frac{1}{2+\varepsilon} \sum_{h\in \frac{2\pi\mathbb{Z}}{(2+\varepsilon)\log N}} D\big(\tfrac{1}{2} + it + ih\big)\hat{V}\Big(\frac{(h-h_0)\log N}{2\pi}\Big),$$

where V is a smooth function which is compactly supported on $[-\varepsilon, 1+\varepsilon]$, and equals 1 on $[0,1]$. By applying this to each D_l^2, one straightforwardly deduces (3.6.6) using the fact that $\hat{V}(x) \ll_A (1+|x|)^{-A}$ for any $A > 0$.

We will show how the same argument handles the $\theta < 0$ case. If $h_j := 2\pi j/(8\log N)$ and $|j| > 16(\log^\theta T)(\log N)$, then for any $|h| \leq 2\log^\theta T$,

$$\Big|\frac{(h_j - h)\log N}{2\pi}\Big| \geq \frac{|j|}{8} - \frac{|h|\log N}{2\pi} \geq \frac{|j|}{8} - \frac{16(\log^\theta T)(\log N)}{16\pi} \geq \frac{|j|}{16}.$$

This implies that for any $A \geq 100$, and each $l \in \mathcal{I}$,

$$\max_{|h|\leq 2\log^\theta T} |D_l(\tfrac{1}{2} + i\tau + ih)|^2 \tag{3.6.9}$$

$$\ll_A \sum_{|j|\leq 16\log^\theta T \log N} \Big| D_l\Big(\frac{1}{2} + i\tau + \frac{2\pi i j}{8\log N}\Big)\Big|^2 \tag{3.6.10}$$

$$+ \sum_{|j|>16\log^\theta T \log N} \frac{1}{1+|j|^A} \Big| D_l\Big(\frac{1}{2} + i\tau + \frac{2\pi i j}{8\log N}\Big)\Big|^2. \tag{3.6.11}$$

The lemma follows by taking the maximum over $|h| \leq 2(\log T)^\theta$ on both sides and summing over l.

□

We also have the following discretisation result for the maximum of the zeta function. It can be seen as a consequence of the previous lemma, together with the classical fact that the zeta function is well-approximated by a Dirichlet polynomial of length T.

Lemma 3.6.8. *Fix $\theta \in (-1,0)$ and let $t_\theta = t|\theta|$, where $t = \log\log T$. Let $\mathscr{T}_t^\theta$ be a set of e^{-t-100}-well-spaced points in $[-2e^{-t_\theta}, 2e^{-t_\theta}]$. There exists an absolute constant $C > 1$ such that for any $V > 1$, $A > 100$*

$$\mathbb{P}\Big(\max_{|h|\le e^{-t_\theta}} |\zeta_\tau(h)e^{-S_{t_\theta}(0)}| > V\Big) \le \mathbb{P}\Big(\max_{h\in\mathscr{T}_t^\theta} |\zeta_\tau(h)e^{-S_{t_\theta}(0)}| > V/C\Big) + O_A(e^{-At}).$$

The same bound also holds upon replacing $S_{t_\theta}(0)$ by $S_{t_\theta}(h)$ on both sides of the inequality.

Proof. We begin with the bound for $\zeta_\tau(h)e^{-S_{t_\theta}(0)}$ in the case where $\theta < 0$, the $\theta = 0$ case being Lemma 26 in [10]. The claim for $\theta < 0$ will essentially follow from the same proof, as we now show.

By [30, Proposition 2], for any $\tau \in [T, 2T]$,

$$\zeta(1/2 + i\tau) = \sum_{n\le T} \frac{1}{n^{1/2+i\tau}}\Big(1 - \frac{\log n}{\log T}\Big)^{100} + O(T^{-100}) =: D(\tau) + O(T^{-100}). \tag{3.6.12}$$

Lemma 25 in [10] then implies that for any $|h_0| \le e^{-t_\theta}$,

$$|D(\tau + h_0)| \le \sum_{h\in e^{-t-100}\mathbb{Z}} |D(\tau + h)| \cdot \Big|\hat{V}\Big(\frac{(h - h_0)e^t}{2\pi}\Big)\Big| \tag{3.6.13}$$

where V is a smooth, compactly supported function which equals 1 on $[0, 1]$. It follows that there exists a constant $C > 0$ for which

$$|D(\tau + h_0)|e^{-S_{t_\theta}(0)} \le C \max_{h\in\mathscr{T}_t^\theta} |D(\tau + h)e^{-S_{t_\theta}(0)}| + \mathscr{E}(\tau)e^{-S_{t_\theta}(0)},$$

for any $|h_0| \le e^{-t_\theta}$, where

$$\mathscr{E}(\tau) = \sum_{\substack{h\in e^{-t-100}\mathbb{Z} \\ |h|>2e^{-t_\theta}}} |D(\tau + h)| \cdot \sup_{|h_0|\le e^{-t_\theta}} \Big|\hat{V}\Big(\frac{(h - h_0)e^t}{2\pi}\Big)\Big|.$$

We can therefore write

$$\mathbb{P}\Big(\max_{|h|\le e^{-t_\theta}} |\zeta_\tau(h)e^{-S_{t_\theta}(0)}| > V\Big) \le \mathbb{P}\Big(\max_{h\in\mathscr{T}_t^\theta} |\zeta_\tau(h)e^{-S_{t_\theta}(0)}| > \frac{V}{2C}\Big) + \mathbb{P}\Big(\mathscr{E}(\tau)e^{-S_{t_\theta}(0)} \ge \frac{1}{2}\Big)$$

and it remains to show that the second term is $O_A(e^{-At})$ for any $A > 100$. Using Lemma 3.6.2 and Stirling's approximation as in the proof of Lemma 3.6.3, we have that

$$\mathbb{P}\big(|S_{t_\theta}(0)| > At\big) \ll_A e^{-At}$$

for any $A > 100$, and in turn, by Markov's inequality, that

$$\begin{aligned}\mathbb{P}\Big(\mathscr{E}(\tau)e^{-S_{t_\theta}(0)} \geq \frac{1}{2}\Big) &\leq \mathbb{P}\Big(\mathscr{E}(\tau) \geq \frac{e^{-At}}{2}\Big) + O_A(e^{-At}) \\ &\ll_A e^{At}\mathbb{E}\big\{\mathscr{E}(\tau)\big\} + e^{-At}.\end{aligned}$$

Uniformly over $|h_0| \leq e^{-t_\theta}$, the $\hat{V}$ term in the definition of $\mathscr{E}(\tau)$ decays faster than any polynomial in $e^{t(1+\theta)}$; Lemma 3.6.4 and the Cauchy-Schwarz inequality then imply that $\mathbb{E}\{\mathscr{E}(\tau)\} \ll_A e^{-2At}$, and the desired claim follows.

To prove the analogous claim with $\zeta_\tau(h)e^{-S_{t_\theta}(h)}$ instead of $\zeta_\tau(h)e^{-S_{t_\theta}(0)}$, we first need a bound for $|\tilde{S}_{t_\theta}(h_0)|$ which is uniform over $|h_0| \leq 2e^{-t_\theta}$, holding with sufficiently high probability. In what follows, let $A > 100$ be arbitrary, and $q > 1$ be a positive integer that will be specified later. Applying Lemma 3.6.7 to $\tilde{S}_{t_\theta}^q$, we have

$$\max_{|h_0|\leq 2e^{-t_\theta}} |\tilde{S}_{t_\theta}(h_0)|^{2q} \ll_A \sum_{|j|\leq 16e^{-t_\theta}\log N} |\tilde{S}_{t_\theta}(h_j)|^{2q} + \sum_{|j|>16e^{-t_\theta}\log N} \frac{1}{1+|j|^A}|\tilde{S}_{t_\theta}(h_j)|^{2q}.$$

where $h_j = 2\pi j/(8\log N)$, and $N = \exp(2qe^{t_\theta})$.

For any q satisfying $2q \leq e^{t-t_\theta}$, Lemma 3.6.1 implies that $\mathbb{E}\{|\tilde{S}_{t_\theta}(h_j)|^{2q}\} \ll q!(t_\theta+C)^q$ uniformly over all $h_j \in \mathbb{R}$. Markov's inequality then yields

$$\mathbb{P}\Big(\max_{|h_0|\leq 2e^{-t_\theta}} |\tilde{S}_{t_\theta}(h_0)| > At\Big) \ll_A \mathbb{E}\Big\{\max_{|h_0|\leq 2e^{-t_\theta}} |\tilde{S}_{t_\theta}(h_0)|^{2q}\Big\}(At)^{-2q} \ll_A q\cdot q!\Big(\frac{t_\theta+C}{(At)^2}\Big)^q,$$

which, by picking $q = \lfloor At\rfloor \leq e^{t-t_\theta}/2$ and using Stirling's approximation, is $\ll_A e^{-At}$. We may therefore assume in what follows that $|\tilde{S}_{t_\theta}(h_0)| \leq At$ for $|h_0| \leq 2e^{-t_\theta}$.

Under this assumption, the proof of [10, Lemma 23] yields the bounds

$$e^{-S_{t_\theta}(h_0)} \leq 2|\mathscr{M}(h_0)| + e^{-t_\theta^{10^4}}, \quad |\mathscr{M}(h_0)| \leq 2e^{-S_{t_\theta}(h_0)} + e^{-t_\theta^{10^4}}, \tag{3.6.14}$$

where

$$\mathscr{M}(h) := \sum_{\substack{p|m \Longrightarrow \log_2 p\in(1000,t_\theta] \\ \Omega(m)\leq t_\theta^{10^5}}} \frac{\mu(m)}{m^{1/2+i\tau+ih}}.$$

For any $|h_0| \leq 2e^{-t_\theta}$, it follows that

$$|D(\tau+h_0)|e^{-S_{t_\theta}(h_0)} \leq 2|D(\tau+h_0)\mathscr{M}(h_0)| + e^{-t_\theta^{10^4}}|D(\tau+h_0)|,$$

and we may then apply [10, Lemma 25] to the polynomial $D\mathscr{M}$ (for D as defined in (3.6.12)), which has length $\leq T\exp(t_\theta^{10^5}e^{t_\theta}) \leq T^{1+o(1)}$. This yields

$$|D(\tau+h_0)\mathscr{M}(h_0)| \leq C\max_{h\in\mathscr{T}_t^\theta} |D(\tau+h)\mathscr{M}(h)| + \mathscr{E}_0(\tau), \tag{3.6.15}$$

for

$$\mathscr{E}_0(\tau) = \sum_{\substack{h \in e^{-t-100}\mathbb{Z} \\ |h| > 2e^{-t_\theta}}} |D(\tau+h)\mathscr{M}(h)| \cdot \sup_{|h_0| \le 2e^{-t_\theta}} \left| \hat{V}\Big(\frac{(h-h_0)e^t}{2\pi} \Big) \right|.$$

Noting once again that $\hat{V}$ decays faster than any polynomial in e^t (uniformly in $|h_0| \le 2e^{-t_\theta}$), another application of (3.6.14) allows us to conclude that

$$\begin{aligned} |D(\tau+h_0)|e^{-S_{t_\theta}(h_0)} \le & 4C \max_{h \in \mathscr{T}_t^\theta} |D(\tau+h)|e^{-S_{t_\theta}(h)} \\ & + 2\mathscr{E}_0(\tau) + (1+2C)e^{-t_\theta^{10^4}} \max_{|h| \le 2e^{-t_\theta}} |D(\tau+h)| \end{aligned}$$

for all $|h_0| \le 2e^{-t_\theta}$. Finally, combining this with (3.6.12) gives

$$\max_{|h| \le e^{-t_\theta}} |\zeta_\tau(h)|e^{-S_{t_\theta}(h)} \le C \max_{h \in \mathscr{T}_t^\theta} |\zeta_\tau(h)e^{-S_{t_\theta}(h)}| + \mathscr{E}(\tau),$$

where

$$\mathscr{E}(\tau) = 4C\Big(T^{-100}e^{At} + \mathscr{E}_0(\tau) + e^{-t_\theta^{10^4}} \max_{|h| \le 2e^{-t_\theta}} |D(\tau+h)| \Big).$$

The rest of the argument then proceeds analogously to the one for $\zeta_\tau(h)e^{-S_{t_\theta}(0)}$, using Lemmas 3.6.4 and 3.6.7, together with the Cauchy-Schwarz inequality, to conclude that $\mathbb{P}(\mathscr{E}(\tau) \ge 1/2) \ll_A e^{-At}$. □

3.6.3 Ballot theorem

As in Equation (3.4.10), let

$$\mathscr{W}_k = \sum_{1000 \le \ell \le k} \mathscr{N}_\ell$$

where each $\mathscr{N}_\ell$ is a centred, independent real Gaussian variable with variance 1/2. The following ballot theorem estimates the probability that this random walk ends around the value w after k steps, while having remained under a curved barrier at times $j \le k$.

Proposition 3.6.9 (Proposition 4 in [10]). *Uniformly in $t \ge 1, 1 \le y \le 2t, t/2 \le k \le t$ and $m(k) + L_y(k) - 4 \le w \le m(k) + U_y(k)$ (defined in Equations* (3.3.11) *and* (3.3.10)*),*

$$\mathbb{P}\big(\mathscr{W}_j \le m(j) + U_y(j) \forall j \le k, \mathscr{W}_k \in (w, w+1] \big) \ll (y+1) \frac{(U_y(k) + m(k) - w + 1)}{k} \frac{e^{-w^2/k}}{\sqrt{k}}.$$

Chapter 4

On the Fourier coefficients of critical Gaussian multiplicative chaos

This chapter is based on an eponymous joint work with Louis-Pierre Arguin [15].

The theory of Gaussian multiplicative chaos (GMC) was initiated by Kahane [115] in the 1980s and gives meaning to measures formally written as

$$\mu_\gamma(\mathrm{d}x) \text{“}= \text{”} e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}X(x)^2}\mathrm{d}x. \tag{4.0.1}$$

Here, X is a logarithmically correlated Gaussian field on $\mathbb{R}^d$, $\gamma > 0$ is an inverse temperature parameter, and $\mathrm{d}x$ denotes Lebesgue measure on some Borel set $D \subseteq \mathbb{R}^d$. Initially motivated by the need for a rigorous model for energy dissipation and intermittency in the Kolmogorov-Obukhov-Mandelbrot theory of developed turbulence [133, 120, 145], this theory has since found applications in many other fields, including finance [60], quantum gravity [108, 63], the extreme values of log-correlated fields [39, 59, 29, 38] (see also [156] and the references therein) and related problems in random matrix theory and number theory (e.g., [171, 144, 98, 100, 161]).

The difficulty in defining (4.0.1) lies in the fact that such a field X has a covariance kernel which is singular along the diagonal, being of the form

$$K(x,y) = \mathbb{E}[X(x)X(y)] = \log_+ \frac{1}{\|x-y\|} + f(x,y), \quad \log_+(x) = \max(\log x, 0)$$

for some smooth function $f : D\times D \longrightarrow \mathbb{R}$. It therefore cannot be defined pointwise and is instead a distribution (in the sense of Schwartz) with probability one. To rigorously define its exponential, one typically regularises X to obtain a family $(X_t)_t$ of (pointwise defined) Gaussian fields converging to X as $t \to \infty$, and defines μ_γ as the weak limit of

$$\mu_{\gamma,t}(\mathrm{d}x) = e^{\gamma X_t(x) - \frac{\gamma^2}{2}\mathbb{E}X_t(x)^2}\mathrm{d}x \tag{4.0.2}$$

as $t \to \infty$. This limit is known to be non-degenerate if $\gamma < \gamma_c = \sqrt{2d}$ [158, 22, 164]. At $\gamma = \gamma_c$, a non-trivial limit can still be obtained by considering the so-called *derivative martingale* [61]

$$\mu_{\gamma_c,t}(\mathrm{d}x) = \left(-X_t(x) + \gamma_c \mathbb{E}[X_t(x)^2]\right) e^{\gamma_c X_t(x) - \frac{\gamma_c^2}{2}\mathbb{E}X_t(x)^2} \mathrm{d}x, \tag{4.0.3}$$

or by multiplying (4.0.2) by a divergent factor known as the *Seneta-Heyde* normalization [149, 108], which depends on the regularization procedure. Both of these procedures yield the same limit up to a multiplicative constant, namely *critical* GMC, which is non-atomic and supported on a set of Hausdorff dimension zero [150]. In this chapter, our primary focus will be to study the asymptotic behavior of the Fourier coefficients of this measure.

4.0.1 Harmonic analysis of multifractal measures

Given a measure μ on the unit circle identified with $[0, 2\pi)$, one can ask whether or not its Fourier coefficients

$$\hat{\mu}(n) = \int_0^{2\pi} e^{-2\pi i n\theta} \mathrm{d}\mu(\theta)$$

vanish at infinity. Measures satisfying this condition are known as *Rajchman measures* and form a class lying strictly between those of absolutely continuous and continuous measures (on $[0, 2\pi)$). That these inclusions are proper is a consequence of Menšov's [137] famous construction of a singular Rajchman measure, and Wiener's characterization of continuous measures [172] as those for which the $\hat{\mu}(n)$ vanish *in mean*, respectively. Rajchman measures can alternatively be characterised by their common null sets (meaning sets to which they assign zero measure)—which are an intermediate between countable sets and sets of Lebesgue measure zero (see [129, 130])—and are more generally related to the notion of sets of multiplicity and uniqueness in harmonic analysis [117].

In a recent work [74], Garban and Vargas initiated the study of the Fourier coefficients of (one-dimensional) GMC, proving in particular that this measure is Rajchman if $\gamma < \sqrt{2}$ [1]. This resolved a longstanding open problem, and together with [65] was the first result of this kind for random multifractal measures. They also supplemented their result by a series of open problems that have since garnered some attention.

For instance, define the *Fourier dimension* of a measure $\dim_F(\mu)$ to be the largest s for which $\hat{\mu}(n) = O(n^{-s/2})$, and let μ_γ denote GMC on the unit circle with parameter γ. Using a fourth moment calculation, Garban and Vargas proved that $\dim_F(\mu) \geq (1 - \gamma^2/2)/2$ if

[1] To be precise, they prove this for the GMC arising from the Gaussian free field on the unit circle.

$\gamma < 1/\sqrt{2}$, and conjectured that

$$\dim_F(\mu_\gamma) = \begin{cases} 1-\gamma^2 & \text{if } 0 < \gamma < 1/\sqrt{2} \\ (\sqrt{2}-\gamma)^2 & \text{if } 1/\sqrt{2} \le \gamma < \sqrt{2}. \end{cases} \tag{4.0.4}$$

This would imply that $\dim_F(\mu_\gamma)$ coincides with the *correlation dimension* of μ_γ, defined as the maximal s for which

$$\iint_{[0,1]^2} \frac{\mu_\gamma(\mathrm{d}x)\mu_\gamma(\mathrm{d}y)}{|x-y|^s} < \infty \quad \text{a.s.},$$

thereby making μ_γ a *Salem measure*.

This conjecture (4.0.4) was recently settled by Lin, Qiu and Tan [126, 127] in a series of works that develop a general framework to study the Fourier dimension of multiplicative chaos measures (including but not limited to one-dimensional GMC). The latter builds on a previous work of Chen, Han, Qiu and Wang [43], which used the theory of *vector-valued martingales* to compute the Fourier dimension of the Mandelbrot canonical cascade measure, answering a question of Mandelbrot from 1976.

By contrast, the picture for critical GMC is far murkier. It is well-known that the Fourier dimension of a measure is bounded above by its Hausdorff dimension, which implies that $\dim_F(\mu_{\gamma_c}) = 0$ when μ_{γ_c} is a critical GMC measure (i.e., the in-probability limit of (4.0.3) as $t \to \infty$). To the best of our knowledge, there are no further results in this direction, and the main open problem in [74] was to determine whether or not critical GMC is Rajchman.

Our goal is to provide a first step towards the resolution of this conjecture by proving the following.

4.0.2 Main results

In what follows, let $\mu = \mu_{\sqrt{2}}$ be the critical Gaussian multiplicative chaos measure arising from a $\star$–scale invariant field X on $[0, 1]$ (defined in Section 4.1), meaning that if $(X_t)_{t\ge 0}$ is a sequence of $\star$-scale cutoff approximations to X,

$$\mu(\mathrm{d}\theta) := \lim_{t\to\infty} \mu_t(\mathrm{d}\theta), \qquad \text{where } \mu_t(\mathrm{d}\theta) := \sqrt{t}e^{\sqrt{2}X_t(\theta) - \mathbb{E}X_t(\theta)^2}\mathrm{d}\theta.$$

The limit is in probability in the space of Radon measures on $[0, 1]$. Let c_n denote the n-th Fourier coefficient of μ. The main result of this chapter is the following.

Theorem 4.0.1. *For any $\alpha < 1/4$, the sequence $((\log n)^\alpha c_n)_{n\ge 1}$ converges to zero in probability as n tends to infinity.*

The polylogarithmic decay stems from what can loosely be described as a freezing phenomenon, which takes effect as soon as $\gamma \geq 1/\sqrt{2}$. This is explained in Section 4.0.3, which also contains a heuristic explanation of why we believe the true order of magnitude of $|c_n|$ to be closer to $(\log n)^{-1}$.

Our proof is based on a second moment estimate for c_n on a good event. While the approach is similar in spirit to Berestycki [22] and Lacoin's [123] constructions of subcritical and critical GMC, respectively, the presence of an oscillating factor in the integral defining c_n here introduces significant complications. In particular, we require a more delicate notion of "good" points than in [22, 123], at which the underlying field $X_t(\theta)$ is controlled at scales $t \geq r_n$, where r_n is a function of the frequency n of the Fourier coefficient at hand. Doing so instead of controlling all scales makes the problem more tractable, at the cost of powers of $(\log n)$. An overview of the proof is given in Section 4.0.3.

By comparison, Garban and Vargas bypass the explicit study of c_n (when $\gamma > 1/\sqrt{2}$) by appealing to the Riemann-Lebesgue lemma. To show that $c_n \to 0$ almost surely, they note that it suffices by the latter to show that μ_γ^{*d} is in L^1 for sufficiently large d (where ^{*d} denotes d-fold convolution). Unfortunately, this approach breaks down at criticality as their argument would force one to take $d = \infty$.

4.0.3 Proof elements

Let X be a $\star$–scale invariant field as above, and consider the multiplicative chaos measure $\mu_\gamma := \lim_{t\to\infty} \mu_{\gamma,t}$ associated to X for $\gamma \in (1/\sqrt{2}, \sqrt{2}]$, where

$$\mu_{\gamma,t}(\mathrm{d}\theta) := \big(\sqrt{t} \cdot \mathbf{1}(\gamma = \sqrt{2}) + \mathbf{1}(\gamma \neq \sqrt{2})\big) e^{\gamma X_t(\theta) - (\gamma^2/2)t}. \tag{4.0.5}$$

Our goal is to estimate $\mathbb{E}[|c_{n,t}|^2]$, where $c_{n,t} := \int e^{in\theta} \mu_{\sqrt{2},t}(\mathrm{d}\theta)$.

Before this, it is helpful to understand the second moment of the total mass $\int \mu_{\gamma,t}(\mathrm{d}\theta)$ for $\gamma < \sqrt{2}$, without the oscillating factor $e^{in\theta}$. Since X_t is logarithmically correlated, Fubini's theorem implies that

$$\mathbb{E}\Big[\Big(\int \mu_{\gamma,t}(\mathrm{d}\theta)\Big)^2\Big] \approx \iint \frac{\mathrm{d}\theta_1 \mathrm{d}\theta_2}{|\theta_1 - \theta_2|^{\gamma^2}},$$

which is finite if and only if $\gamma^2 < 1$ (the so-called "L^2 phase" of GMC). For larger γ, the expectation is inflated by points θ where $X_t(\theta)$ is atypically large at many scales t. To circumvent this, one usually restricts the integral to "good" points $\theta \in G$ for which $(X_t(\theta), t \geq 0)$ is well-behaved[2]. At $\gamma = \sqrt{2}$, this is achieved by working on the event that

[2] This is the main idea in [22, 123], for instance.

$\{\forall \theta \in [0,1] : \theta \in G\}$, on which the bound

$$\iint \mu_{\sqrt{2},t}(\mathrm{d}\theta_1)\mu_{\sqrt{2},t}(\mathrm{d}\theta_2) \le \iint \mathbf{1}\{\theta_1, \theta_2 \in G\}\mu_{\sqrt{2},t}(\mathrm{d}\theta_1)\mu_{\sqrt{2},t}(\mathrm{d}\theta_2) \tag{4.0.6}$$

holds by monotonicity, and the right-hand side has finite expectation.

In our setting, one can proceed similarly and write

$$\mathbf{1}_{\{\forall\theta\in[0,1],\theta\in G\}}|c_{n,t}|^2 \le \iint e^{in(\theta_2-\theta_1)}\mathbf{1}\{\theta_1, \theta_2 \in G\}\mu_{\sqrt{2},t}(\mathrm{d}\theta_1)\mu_{\sqrt{2},t}(\mathrm{d}\theta_2) \tag{4.0.7}$$

where the inequality now uses the fact that the double integral is real-valued and positive, being equal to the modulus squared of the n-th Fourier coefficient of $\mathbf{1}\{\theta \in G\}\mu_t(\mathrm{d}\theta)$ (cf. (4.3.1)). If we hope to get savings from the oscillations in this integral, we then need some control over the regularity of

$$\mathbb{E}\left\{e^{\sqrt{2}\{X_t(\theta_1)+X_t(\theta_2)\}}\mathbf{1}\{\theta_1, \theta_2 \in G\}\right\}$$

as a function of the gap

$$\Delta = (\theta_2 - \theta_1).$$

This limits the choice of G to one for which the event $\{\theta_1, \theta_2 \in G\}$ is relatively simple, while being restrictive enough to make the expectation of $\mathbf{1}_{\{\forall\theta\in[0,1],\theta\in G\}}|c_{n,t}|^2$ of the correct order (at the very least, compensating for the additional $\sqrt{t}$ normalization present at criticality (4.0.5)).

We now explain why we believe that the order of $|c_n|$ is typically closer to $(\log n)^{-1}$. To begin with, a variant of the argument above reveals that the dominant contribution to (4.0.7) comes from $\log|\Delta|^{-1} \asymp \log n$. For concreteness, consider the contribution to $|c_{n,t}|^2$ from $\Delta \in [n^{-100}, n^{-1}]$, which is

$$\asymp te^{-2t}\iint_{|\theta_2-\theta_1|\in[n^{-100},n^{-1}]} e^{\sqrt{2}(X_t(\theta_1)+X_t(\theta_2))}\mathrm{d}\theta_1\mathrm{d}\theta_2. \tag{4.0.8}$$

(We ignore the $e^{in\Delta}$ factor since $n\Delta \le 1$ here.)

Since X_t is logarithmically correlated, one can show that for any θ_1, θ_2,

$$X_s(\theta_1) \approx X_s(\theta_2), \quad \forall s \le r(\Delta), \tag{4.0.9}$$

with high probability (see, e.g., Lemma 16 in [61]), where $r(\Delta) := \log|\Delta|^{-1}$ is the branching time. It follows that (4.0.8) is

$$\approx te^{-2t}\iint e^{2\sqrt{2}X_{r(\Delta)}(\theta_1)}e^{\sqrt{2}(X_t(\theta_1)-X_{r(\Delta)}(\theta_1))}e^{\sqrt{2}(X_t(\theta_2)-X_{r(\Delta)}(\theta_2))}\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Furthermore, $e^{\sqrt{2}(X_t(\theta_i)-X_{r(\Delta)}(\theta_i))}$ will typically be of the order of $e^{t-r(\Delta)}/\sqrt{t-r(\Delta)}$, which will allow us to bound the integral above by

$$\int_{n^{-100}}^{n^{-1}} e^{-2r(\Delta)}\Big(\int_0^1 e^{2\sqrt{2}X_{r(\Delta)}(\theta_1)}\mathrm{d}\theta_1\Big)\mathrm{d}\Delta \tag{4.0.10}$$

as $t \to \infty$ (noting the change of variables and that $r(\Delta)$ is bounded in t).

For large n, we recognise the integral over θ_1 as nothing but the (approximate, unnormalised) total mass of a *supercritical* chaos measure, this time of parameter $\tilde{\gamma} = 2\sqrt{2}$.[3] Recalling that $r(\Delta) = \log|\Delta|^{-1}$, this is known to have order

$$(\log|\Delta|^{-1})^{-(3\tilde{\gamma}/(2\sqrt{2}))}e^{r(\Delta)(\tilde{\gamma}^2/2-(\tilde{\gamma}/\sqrt{2}-1)^2)} = e^{3r(\Delta)}(\log|\Delta|^{-1})^{-3}$$

(see Theorem 2.2 in [132]). Inserting this into (4.0.10), we conclude that when $t \to \infty$,

$$|c_{n,t}|^2 \le C\int_{n^{-100}}^{n^{-1}} \frac{1}{(\log|\Delta|^{-1})^3}\frac{\mathrm{d}}{\Delta} = C\int_{\log n}^{100\log n}\frac{\mathrm{d}u}{u^3} = O\big((\log n)^{-2}\big)$$

with high probability, for some constant $C > 0$. In particular, this suggests a non-trivial contribution from scales $\Delta \approx n^{-a}$, even when $a > 0$ is large (see Remark 4.3.4).

4.0.4 Conjectures for subcritical chaos

While we do not attempt to make the heuristics from the previous section rigorous for $\gamma \in (1/\sqrt{2}, \sqrt{2})$, they can nonetheless be used to make predictions in that range. There, the analogue of the integral in (4.0.10) contains the total mass of a chaos with parameter $\tilde{\gamma} = 2\gamma$ and is therefore of the order of

$$\int_{n^{-100}}^{n^{-1}} e^{-\gamma^2 r(\Delta)}\Big(\frac{e^{r(\Delta)(2\gamma^2-(\sqrt{2}\gamma-1)^2)}}{(\log|\Delta|^{-1})^{3\gamma/\sqrt{2}}}\Big)\mathrm{d}\Delta.$$

By contrast with the critical case, the mass in this integral is concentrated around $\Delta \approx n^{-1}$, and we can factor out the correct normalization via the substitution $u = n\Delta$. What remains is approximately the integral of $1/|u|^\kappa$ over $[0,1]$ for some $\kappa < 1$, which is integrable at zero.

Conjecture 4.0.2. *Let $\gamma \in (1/\sqrt{2}, \sqrt{2})$, μ_γ denote the Gaussian multiplicative chaos measure on $[0,1]$ with parameter γ, and c_n its n-th Fourier coefficient. Then*

$$(\log n)^{3\gamma/(2\sqrt{2})}n^{(\sqrt{2}-\gamma)^2/2}|c_n|$$

converges in distribution to a (non-trivial) limiting random variable as $n \to \infty$.

[3] The emergence of such a chaos was also found by Garban and Vargas (see Theorem 1.3 in [74]), where it is *subcritical* since they only consider $2\gamma < \sqrt{2}$.

Remark 4.0.3. We recognise the Fourier dimension of μ_γ in the exponent of n here.

Lastly, when $\gamma = 1/\sqrt{2}$, the chaos of parameter $2\gamma = \sqrt{2}$ is critical and the heuristic above yields the following prediction.

Conjecture 4.0.4. *Let μ_γ denote the Gaussian multiplicative chaos measure on $[0,1]$ with parameter $\gamma = 1/\sqrt{2}$, and c_n its n-th Fourier coefficient. Then*

$$(\log n)^{1/4} n^{1/4} |c_n|$$

converges in distribution to a (non-trivial) limiting random variable as $n \to \infty$.

Remark 4.0.5. As explained in [74], Section 1.2.3, $n^{1/4}c_n$ is expected to behave like the n-th Fourier coefficient of the Holomorphic Multiplicative Chaos (HMC), which is known to be of order $(\log n)^{-1/4}$ [168].

Remark 4.0.6. This conjectured transition between an integral concentrated at $\Delta \approx n^{-1}$ for $\gamma < \sqrt{2}$, and one in which all scales $\Delta \approx n^{-a}$ contribute meaningfully at $\gamma = \sqrt{2}$, is reminiscent of the failure of Theorem 2.0.1 at $\alpha = 2$ (which corresponds to $\gamma = \sqrt{2}$ in this setting). There, one can no longer use Hölder's inequality and translation invariance in law of the corresponding random Euler product to reduce the analysis to the integral in (2.2.2); integrals over longer intervals must instead be considered.

Organisation

The chapter is organised as follows. Section 4.1 contains various preliminary results on logarithmically correlated fields needed for the proof of Theorem 4.0.1, which is then proved in Sections 4.2 and 4.3.

4.1 Preliminary estimates

4.1.1 Properties of ⋆-scale invariant fields.

We will work with chaos measures associated to a ⋆–scale invariant field X on $[0,1]$ (viewed as being 1–periodic) with ⋆–scale cut–off approximations $(X_t)_{t\geq 0}$ (see [150]). Our setup is essentially the same as the one in [61] for $d = 1$, and we refer to [3] for the construction of an approximating sequence X_t satisfying the properties below.

In what follows, $(X_t)_{t\geq 0}$ will denote a family of coupled, centered Gaussian fields on $[0,1]$ with covariances given by

$$\forall \theta_1 < \theta_2, \quad \mathbb{E}\{X_t(\theta_1)X_s(\theta_2)\} = K_{s\wedge t}(\theta_2 - \theta_1) := \int_1^{e^{s\wedge t}} \frac{k(u(\theta_2-\theta_1))}{u}\mathrm{d}u, \qquad (4.1.1)$$

for some fixed *seed function* $k \in C^2(\mathbb{R})$ associated to X, where $s \wedge t = \min(s,t)$. For convenience, we will assume that $k(0) = 1$, k is even and that it vanishes outside of the unit ball around the origin. The following properties then hold by construction:

1. $\mathbb{E}\{X_t(\theta)^2\} = t$ for every $\theta \in [0,1]$.
2. For every $\theta \in [0,1]$, $(X_t(\theta); t \geq 0)$ has independent increments, making $t \mapsto X_t(\theta)$ a standard Brownian motion B_t starting from zero for every θ.
3. $(X_t(\theta_1) - X_{t_0}(\theta_1); t \geq t_0)$ is independent from $(X_t(\theta_2) - X_{t_0}(\theta_2); t \geq t_0)$ for any θ_1, θ_2 satisfying $|\theta_1 - \theta_2| \geq e^{-t_0}$,

An exact description for the joint law of $(X_t(\theta_1), X_t(\theta_2))_t$ can be found in [61], Lemma 16, which roughly states that $X_k(\theta_1)$ and $X_k(\theta_2)$ are almost perfectly correlated up to time $k = \log 1/|\theta_1 - \theta_2|$ (after which they evolve independently, as per property (3) above). For our purposes, we will only need the following estimate reflecting this correlation structure.

Lemma 4.1.1. *Let* $\|k'\|_\infty = \sup_{x\in[-1,1]} |k'(x)|$ *and fix* $s > 0$. *Then uniformly in* $\Delta \in [-1,1] \setminus \{0\}$,

$$K_s(\Delta) = s \wedge (\log |\Delta|^{-1}) + O(\|k'\|_\infty). \tag{4.1.2}$$

Proof. Using the fact that k is supported on $[-1,1]$ and $k(0) = 1$,

$$K_s(\Delta) = \int_1^{e^s \wedge |\Delta|^{-1}} \frac{k(u|\Delta|)}{u} \mathrm{d}u = \int_1^{e^s \wedge |\Delta|^{-1}} \frac{\mathrm{d}u}{u} + \int_1^{e^s \wedge |\Delta|^{-1}} \frac{k(u|\Delta|) - k(0)}{u} \mathrm{d}u. \tag{4.1.3}$$

The first term is equal to the main term in (4.1.2). By the fundamental theorem of calculus,

$$|k(u|\Delta|) - k(0)| \leq \int_0^{u|\Delta|} |k'(x)| \mathrm{d}x \leq \|k'\|_\infty u|\Delta|$$

for any $u \in [1, e^s \wedge |\Delta|^{-1}]$. The second term in (4.1.3) is therefore $O(\|k'\|_\infty)$ by the triangle inequality. □

For any $t, r > 0$, we define the *shifted* field

$$\big(X_t^{(r)}(\theta)\big)_{\theta\in[0,1]} := \big((X_{t+r} - X_r)(\theta)\big)_{\theta\in[0,1]}, \tag{4.1.4}$$

noting that $(X_t^{(r)}(\theta))_{\theta\in[0,1]}$ is $\star-$scale invariant with kernel $k^{(r)}(x) := k(e^r x)$ since

$$\mathbb{E}\{X_s^{(r)}(\theta_1) X_s^{(r)}(\theta_2)\} = \Big(\int_1^{e^{s+r}} - \int_1^{e^s} \Big) \frac{k(u(x-y))}{u} \mathrm{d}u = \int_1^{e^s} \frac{k^{(r)}(u(x-y))}{u} \mathrm{d}u.$$

In particular, $t \mapsto X_t^{(r)}(\theta)$ is a standard Brownian motion for any $r > 0$ and $\theta \in [0, 1]$.

Our argument will require some control on the large values of $(X_t(\theta))_{\theta\in[0,1]}$ at multiple scales t. For fixed, large t, the typical size of $\max_{\theta\in[0,1]} X_t(\theta)$ is known to fluctuate around

$$m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t \tag{4.1.5}$$

(see [61, 131]), which is what one expects from any log–correlated field (see, e.g. [4]). In fact, one can typically go further and show that the paths $s \mapsto X_s(\theta)$, $s \le t$, for which $X_t(\theta)$ is close to the maximum cannot fluctuate too wildly. Instead, they display approximately linear growth while remaining under a suitably chosen barrier function. This phenomenon was first leveraged by Bramson [37] to study the maximal displacement of branching Brownian motion, and underpins much of the subsequent work on extrema of log–correlated fields [38, 39, 2, 59].

For our purposes, we will only require the following upper bound of this type, due to Lacoin ([123], Proposition 2.6).

Proposition 4.1.2. *Let $(X_t)_{t\ge0}$ be defined as in Equation* (4.1.1) *and*

$$U(t) := \begin{cases} \sqrt{2}t - \frac{\log t}{2\sqrt{2}} + \frac{4 \log\log t}{\sqrt{2}} & \text{if } t \ge e, \\ \infty & \text{if } t < e. \end{cases} \tag{4.1.6}$$

Then as $A \to \infty$,

$$\mathbb{P}\Big(\exists t \ge 0, \exists \theta \in [0, 1], X_t(\theta) > U(t) + A\Big) \to 0.$$

Remark 4.1.3. While [123] assumes that the seed function k is in $C^\infty(\mathbb{R})$, the proof of Proposition 2.6 therein only requires that $k \in C^2(\mathbb{R})$.

4.1.2 Brownian estimates

We record the following estimates for the probability that a Brownian motion stays under a barrier starting at time e (in light of (4.1.6)). Such events will arise from restricting to samples for which the event in Proposition 4.1.2 holds.

Lemma 4.1.4. *Let B_t be a standard linear Brownian motion. Then there exists a constant $C > 0$ such that for any $t > e$, $a > 0$,*

$$\mathbb{P}\Big(\sup_{s\in[e,t]} B_s \le a\Big) \le C\frac{(a+1)}{\sqrt{t}+1}. \tag{4.1.7}$$

Furthermore, if $a, b > 0$ and $\mathbb{P}_{(0,a)}^{(t,b)}$ denotes the law of a Brownian bridge of length $t > e$ from a to b, then

$$\mathbb{P}_{(0,a)}^{(t,b)}\Big(B_s \geq 0, \forall s \in [e,t]\Big) \leq C\frac{(a+1)(b+1)}{t+1}. \tag{4.1.8}$$

Proof. Both inequalities are a direct consequence of the reflection principle after integrating over the value of B_e. □

4.2 Proof of the main theorem

Let X be a $\star-$scale invariant field with approximations $(X_t)_{t\geq 0}$, and define its associated (critical) GMC measure μ as

$$\mu(\mathrm{d}\theta) := \lim_{t\to\infty} \mu_t(\mathrm{d}\theta), \qquad \text{where } \mu_t(\mathrm{d}\theta) := \sqrt{t}e^{\sqrt{2}X_t(\theta)-\mathbb{E}X_t(\theta)^2}\mathrm{d}\theta.$$

This limit holds in probability, and it follows that

$$c_{n,t} = \int_0^1 e^{in\theta}\mu_t(\mathrm{d}\theta) \xrightarrow[t\to\infty]{\mathbb{P}} c_n = \int_0^1 e^{in\theta}\mu(\mathrm{d}\theta). \tag{4.2.1}$$

We want to show that $\{(\log n)^\alpha |c_n|\}_{n\geq 1}$ converges to zero in probability for any $\alpha < 1/4$, which we do by proving an upper bound of the correct order for the expectation of

$$|c_{n,t}|^2 = \iint_{[0,1]^2} e^{in(\theta_2-\theta_1)}\mu_t(\mathrm{d}\theta_1)\mu_t(\mathrm{d}\theta_2) \tag{4.2.2}$$

on a good event $E_{n,t}(A)$. This is the content of Proposition 4.2.1 below. As a byproduct of its proof, we identify the set of pairs $(\theta_1, \theta_2) \in [0,1]^2$ over which the integral in (4.2.2) yields a leading order contribution, and those for which the contribution is negligible in the limit (see Remark 4.3.4).

Proposition 4.2.1. *Let $A > 0$ and $0 < \delta < 1/4$ be fixed. Let $r_n := \delta \log n$. For any $t \geq r_n$, consider the event*

$$E_{n,t}(A) := \big\{\forall \theta \in [0,1],\, \theta \in G_{n,t}(A)\big\},$$

where

$$G_{n,t}(A) = \big\{\theta \in [0,1] :\ X_s(\theta) \leq U(s) + A,\ \forall s \in [r_n, t]\big\} \tag{4.2.3}$$

and $U(t)$ is defined in Equation (4.1.6)*. Then for any $\varepsilon > 0$, there exists an integer $N = N(A,\varepsilon,\delta) > 2$ such that for any $n \geq N$,*

$$\limsup_{t\to\infty} \mathbb{E}\Big\{|c_{n,t}|^2; E_{n,t}(A)\Big\} \ll_A (\log n)^{-1/2+\varepsilon},$$

where the implicit constant is uniform in n.

The choice of good event $E_{n,t}(A)$ will be explained in Section 4.3. For the time being, we show how Theorem 4.0.1 follows straightforwardly from this proposition.

Proof of Theorem 4.0.1 assuming Proposition 4.2.1. Fix $\varepsilon > 0$ and $\alpha < 1/4$. By (4.2.1) and the Portmanteau theorem,

$$\mathbb{P}\Big(|c_n|(\log n)^\alpha > \varepsilon\Big) \le \limsup_{t\to\infty} \mathbb{P}\Big(|c_{n,t}|(\log n)^\alpha > \varepsilon\Big)$$

for any $n > 0$. By definition of $E_{n,t}(A)$, the pre-limit quantity on the right-hand side is bounded above by

$$\mathbb{P}\Big(\mathbf{1}_{E_{n,t}(A)}|c_{n,t}|(\log n)^\alpha > \varepsilon\Big) + \mathbb{P}\Big(\exists s \in [0,\infty), \exists \theta \in [0,1] : X_s(\theta) > U(s) + A\Big), \tag{4.2.4}$$

for any $A > 0$. By Proposition 4.1.2, we can make the second term in (4.2.4) arbitrarily small in a way that is uniform in n. We have that

$$\sup_{n>2} \mathbb{P}\Big(\exists s \in [0,\infty), \exists \theta \in [0,1] : X_s(\theta) > U(s) + A\Big) < \varepsilon/2, \tag{4.2.5}$$

for sufficiently large A, and it follows that for such an A and any $n > 0$,

$$\mathbb{P}\Big(|c_n|(\log n)^\alpha > \varepsilon\Big) \le \limsup_{t\to\infty} \mathbb{P}\Big(\mathbf{1}_{E_{n,t}(A)}|c_{n,t}|(\log n)^\alpha > \varepsilon\Big) + \varepsilon/2.$$

Using Chebyshev's inequality, we can then write

$$\limsup_{t\to\infty} \mathbb{P}\Big(\mathbf{1}_{E_{n,t}(A)}|c_{n,t}|(\log n)^\alpha > \varepsilon\Big) \le \frac{(\log n)^{2\alpha}}{\varepsilon^2} \limsup_{t\to\infty} \mathbb{E}\big\{|c_{n,t}|^2; E_{n,t}(A)\big\},$$

and the claim follows by Proposition 4.2.1 upon taking $n \to \infty$. □

What's left is to prove Proposition 4.2.1.

4.3 Second moment estimate on a good event

4.3.1 Initial reductions

This section proves Proposition 4.2.1. To bound $|c_{n,t}|\mathbf{1}_{E_{n,t}(A)}$, we first want to extract a local condition (meaning a condition on each pair of points θ_1, θ_2) in (4.2.2) from the global event $E_{n,t}(A)$, noting that one cannot directly upper bound the integrand due to the presence of the oscillating factor $e^{in(\theta_2-\theta_1)}$. We instead note the following pointwise inequality on the sample space

$$\begin{aligned}\mathbf{1}_{E_{n,t}(A)}|c_{n,t}|^2 &= \mathbf{1}_{E_{n,t}(A)} \iint_{[0,1]^2} e^{in(\theta_2-\theta_1)} \mathbf{1}\Big\{\theta_1, \theta_2 \in G_{n,t}(A)\Big\} \mu_t(\mathrm{d}\theta_1)\mu_t(\mathrm{d}\theta_2) \qquad (4.3.1)\\ &\le \iint_{[0,1]^2} e^{in(\theta_2-\theta_1)} \mathbf{1}\Big\{\theta_1, \theta_2 \in G_{n,t}(A)\Big\} \mu_t(\mathrm{d}\theta_1)\mu_t(\mathrm{d}\theta_2).\end{aligned}$$

We used the fact that the indicator in the integrand is redundant in the first line and positivity of the double integral on the second, noting that it equals the modulus squared of the n–th Fourier coefficient of

$$\tilde{\mu}_t(\mathrm{d}\theta) := 1\Big\{\theta \in G_{n,t}(A)\Big\}\mu_t(\mathrm{d}\theta).$$

It therefore suffices to prove the analogous claim (to Proposition 4.2.1) for the Fourier coefficients of $\tilde{\mu}_t$.

To that end, fix $0 < \delta < 1/4$ and partition $[0,1]^2$ into the following two ranges:

$$(\mathrm{I}) = \{(\theta_1, \theta_2) : |\theta_2 - \theta_1| \in [\Delta_n, 1]\} \tag{4.3.2}$$

$$(\mathrm{II}) = \{(\theta_1, \theta_2) : |\theta_2 - \theta_1| \in [0, \Delta_n)\}, \tag{4.3.3}$$

where

$$\Delta_n =: en^{-\delta}, \text{ noting that } \log \Delta_n^{-1} = r_n - 1.$$

Denote the contribution to (4.2.2) coming from each of these regions by

$$\mathscr{C}_{n,t}(\#) := \iint_{(\#)} e^{in(\theta_2-\theta_1)} 1\Big\{\theta_1, \theta_2 \in G_{n,t}(A)\Big\}\mu_t(\mathrm{d}\theta_1)\mu_t(\mathrm{d}\theta_2) \tag{4.3.4}$$

for $\# \in \{\mathrm{I}, \mathrm{II}\}$.

The restriction on δ is technical, and is necessary for our bound for the contribution from (I) (see (4.3.17)) to be of the correct order. On (II), we will ignore the $e^{in(\theta_2-\theta_1)}$ by the triangle inequality. While this may seem excessively inefficient—since $\delta < 1$, $e^{in(\theta_2-\theta_1)}$ will still oscillate with n—it will become apparent that any $\delta > 0$ yields the same bound (due to the observation in Remark 4.3.4).

4.3.2 The oscillatory range

We first study the contribution from (I) where $e^{in(\theta_2-\theta_1)}$ oscillates very rapidly.

Proposition 4.3.1. *Let $A > 0$. Then*

$$\lim_{t\to\infty} \mathbb{E}\big\{\mathscr{C}_{n,t}(\mathrm{I})\big\} = o_A\big((\log n)^{-2}\big) \tag{4.3.5}$$

for large enough n.

Proof. Let $\Delta = (\theta_2 - \theta_1)$. By the change of variables $(\theta, \Delta) = (\theta_1, \theta_2 - \theta_1)$ and Fubini's theorem,

$$\begin{aligned}\mathbb{E}\{\mathscr{C}_{n,t}(\mathrm{I})\} &\ll te^{-2t} \iint_{(\mathrm{I})} e^{in(\theta_2-\theta_1)}\mathbb{E}\Big\{e^{\sqrt{2}(X_t(\theta_1)+X_t(\theta_2))}1\big\{\theta_1, \theta_2 \in G_{n,t}(A)\big\}\Big\}\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ &= te^{-2t}\int_{\Delta_n}^{1} e^{in\Delta}\mathbb{E}\Big\{e^{\sqrt{2}(X_t(0)+X_t(\Delta))}1\big\{0, \Delta \in G_{n,t}(A)\big\}\Big\}\mathrm{d}\Delta,\end{aligned}$$

where in the second line, we used the translation invariance of the law of X_t in the following sense

$$\big(X_s(\theta), X_s(\theta+\Delta)\big)_{s\le t} \overset{d}{=} \big(X_s(0), X_s(\Delta)\big)_{s\le t}.$$

The next step will be to exploit the cancellation coming from $e^{in\Delta}$ using integration by parts, for which we first need to transform the expectation in the integrand into a tractable function of Δ. We aim to leverage the fact that since $G_{n,t}(A)$ only restricts $X_s(0)$ and $X_s(\Delta)$ at times $s \ge r_n$, it should, in principle, only depend on $(X_s(0), X_s(\Delta))_{s\le r_n}$ through the value of $(X_{r_n}(0), X_{r_n}(\Delta))$. Furthermore, since $\log|\Delta|^{-1} \le r_n - 1$ when $|\Delta| \ge \Delta_n$, the integrand's dependence on Δ should only figure in the joint density of $(X_{r_n}(0), X_{r_n}(\Delta))$.

We make this precise under a tilted measure $\mathbb{Q}_\Delta$, defined through

$$\frac{\mathrm{d}\mathbb{Q}_\Delta}{\mathrm{d}\mathbb{P}} = e^{\sqrt{2}(X_{r_n}(0)+X_{r_n}(\Delta))-\mathbb{E}\{(X_{r_n}(0)+X_{r_n}(\Delta))^2\}}. \tag{4.3.6}$$

Noting that

$$\mathbb{E}\{(X_{r_n}(0)+X_{r_n}(\Delta))^2\} = 2r_n + 2K_{r_n}(|\Delta|),$$

(for K_{r_n} in (4.1.1)) we can write

$$\begin{aligned} &te^{-2t}\int_{\Delta_n}^1 e^{in\Delta}\mathbb{E}\big\{e^{\sqrt{2}(X_t(0)+X_t(\Delta))}\mathbf{1}\big\{0,\Delta\in G_{n,t}(A)\big\}\big\}\mathrm{d}\Delta \\ &\ll te^{-2t+2r_n}\int_{\Delta_n}^1 e^{in\Delta+2K_t(\Delta)}\mathbb{E}_{\mathbb{Q}_\Delta}\big\{e^{\sqrt{2}(X^{(r_n)}_{t-r_n}(0)+X^{(r_n)}_{t-r_n}(\Delta))}\mathbf{1}\big\{0,\Delta\in G_{n,t}(A)\big\}\big\}\mathrm{d}\Delta. \end{aligned}$$

By properties (2) and (3) in Section 4.1, we know that for $\Delta \ge \Delta_n$, the processes

$$X^{(r_n)}_s(0) \text{ and } X^{(r_n)}_s(\Delta) \text{ for } 0 \le s \le t - r_n \tag{4.3.7}$$

are independent, identically distributed standard Brownian motions. They are also independent from the σ–algebra generated by the field X_s up to time $s \le r_n$, and their distribution is thus unchanged under $\mathbb{Q}_\Delta$. By Girsanov's theorem, the mean of $X_{r_n}(0)$ under $\mathbb{Q}_\Delta$ is shifted by

$$\sqrt{2}\cdot\mathrm{Cov}\Big(X_{r_n}(0), X_{r_n}(0)+X_{r_n}(\Delta)\Big) = \sqrt{2}\big(r_n + K(\Delta)\big) =: \mu_\Delta \tag{4.3.8}$$

where $K = K_{r_n}$, and the same is true for $X_{r_n}(\Delta)$ since $K(\Delta) = K(|\Delta|)$. The covariance between $X_{r_n}(0)$ and $X_{r_n}(\Delta)$ remains unchanged under $\mathbb{Q}_\Delta$.

Using the decomposition

$$\begin{aligned} \mathbf{1}\big\{0,\Delta\in G_{n,t}(A)\big\} = \mathbf{1}\big\{&X_{r_n}(0), X_{r_n}(\Delta) \le U(r_n)+A\big\} \\ &\times \mathbf{1}\big\{X^{(r_n)}_s(0) \le U(s+r_n)+A-X_{r_n}(0), \forall s\in(0,t-r_n]\big\} \\ &\times \mathbf{1}\big\{X^{(r_n)}_s(\Delta) \le U(s+r_n)+A-X_{r_n}(\Delta), \forall s\in(0,t-r_n]\big\} \end{aligned}$$

and the law of total expectation, we therefore have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_\Delta}\Big\{ e^{\sqrt{2}(X^{(r_n)}_{t-r_n}(0)+X^{(r_n)}_{t-r_n}(\Delta))}\mathbf{1}\big\{0,\Delta\in G_{n,t}(A)\big\}\Big\}\\
&=\mathbb{E}_{\mathbb{Q}_\Delta}\Big\{\mathbf{1}\big\{X_{r_n}(0),X_{r_n}(\Delta)\le U(r_n)+A\big\}\times\\
&\mathbb{E}\Big\{e^{\sqrt{2}X^{(r_n)}_{t-r_n}(0)}\mathbf{1}\big\{X^{(r_n)}_s(0)\le U(s+r_n)+A-X_{r_n}(0),\forall s\in(0,t-r_n]\big\}\Big|\,X_{r_n}(0)\Big\}\\
&\mathbb{E}\Big\{e^{\sqrt{2}X^{(r_n)}_{t-r_n}(\Delta)}\mathbf{1}\big\{X^{(r_n)}_s(\Delta)\le U(s+r_n)+A-X_{r_n}(\Delta),\forall s\in(0,t-r_n]\big\}\Big|\,X_{r_n}(\Delta)\Big\}\Big\}.
\end{aligned}$$

Performing the outer integral on positions $(X_{r_n}(0), X_{r_n}(\Delta))$ then shows that the above equals

$$f(\Delta):=\iint_{(-\infty,U(r_n)+A]^2}p_\Delta(u_1,u_2)\beta(u_1)\beta(u_2)\mathrm{d}u_1\mathrm{d}u_2,$$

where

$$\begin{aligned}
p_\Delta(u_1,u_2)=&\frac{1}{2\pi\sqrt{r_n^2-K(\Delta)^2}}\\
&\times\exp\Big(-\frac{1}{2}\frac{r_n((u_1-\mu_\Delta)^2+(u_2-\mu_\Delta)^2)-2K(\Delta)(u_1-\mu_\Delta)(u_2-\mu_\Delta)}{\big(r_n^2-K(\Delta)^2\big)}\Big)
\end{aligned}$$

is the density of a pair (X_1, X_2) of (real) Gaussian random variables with mean (μ_Δ, μ_Δ) and covariance

$$\mathbb{E}\{(X_1-\mu_\Delta)(X_2-\mu_\Delta)\}=K(\Delta),\quad \mathbb{E}\{(X_1-\mu_\Delta)^2\}=\mathbb{E}\{(X_2-\mu_\Delta)^2\}=r_n,$$

and

$$\beta(u):=\mathbb{E}\Big\{e^{\sqrt{2}B_{t-r_n}}\mathbf{1}\big\{B_s\le U(s+r_n)+A-u,\forall s\in(0,t-r_n]\big\}\Big\},$$

for $(B_t)_{t\ge0}$ a standard Brownian motion. As intended, the dependence on Δ is now solely contained in p_Δ.

We now turn to our main task, and show that

$$\limsup_{t\to\infty}te^{-2t+2r_n}\int_{\Delta_n}^1 e^{in\Delta+2K(\Delta)}f(\Delta)\mathrm{d}\Delta=o_A\big((\log n)^{-2}\big).\tag{4.3.9}$$

Integration by parts shows that the left-hand side equals

$$\limsup_{t\to\infty}te^{-2t+2r_n}\Big(\frac{e^{in\Delta+2K(\Delta)}f(\Delta)}{in}\Big|_{\Delta_n}^1-\int_{\Delta_n}^1\frac{e^{in\Delta}\partial_\Delta\big(e^{2K(\Delta)}f(\Delta)\big)}{in}\mathrm{d}\Delta\Big).\tag{4.3.10}$$

We first handle the boundary term by deriving a uniform bound on $f(\Delta)$. Under the change of measure

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}=e^{\sqrt{2}B_{t-r_n}-(t-r_n)},$$

the mean of $(B_s)_{s\le t-r_n}$ is shifted by $\sqrt{2}s$ at time s by Girsanov's theorem, and it follows that

$$\begin{aligned}\beta(u) &= e^{t-r_n}\tilde{\mathbb{P}}\big(B_s \le U(s+r_n)+A-u, \forall s\in(0,t-r_n]\big)\\ &\le e^{t-r_n}\mathbb{P}\big(B_s \le A+\sqrt{2}r_n-u, \forall s\in(e,t-r_n]\big)\\ &\ll \big(A+r_n+|u|\big)\frac{e^{t-r_n}}{\sqrt{t-r_n}}\end{aligned} \tag{4.3.11}$$

for any $u\in\mathbb{R}$, by Equation (4.1.7) and the fact that $U(s)\le\sqrt{2}s$ for $s>e$. The implicit constant in (4.3.11) is uniform over u, t and n. Using this pointwise bound and the Cauchy-Schwarz inequality, we deduce that

$$f(\Delta)\le\frac{e^{2(t-r_n)}}{t-r_n}\mathbb{E}\Big\{\big(A+r_n+|X_1|\big)\big(A+r_n+|X_2|\big)\Big\}\ll_A\frac{r_n^2e^{2(t-r_n)}}{t-r_n} \tag{4.3.12}$$

for any $\Delta\in[\Delta_n,1]$, and in turn that

$$\limsup_{t\to\infty} te^{-2t+2r_n}\frac{e^{in\Delta+2K(\Delta)}f(\Delta)}{in}\Big|_{\Delta_n}^1\ll_A\frac{r_n^2e^{2K(\Delta_n)}}{n}\ll_A\frac{r_n^2e^{2r_n}}{n}. \tag{4.3.13}$$

Since $r_n=\delta\log n$ and $\delta<1/4$, this is clearly $o_A((\log n)^{-2})$.

To handle the second term inside the limit in (4.3.10), we first note that it is

$$\begin{aligned}&\ll_A te^{-2t+2r_n}\frac{1}{n}\Big(\sup_{\Delta\in[\Delta_n,1]}f(\Delta)|K'(\Delta)|e^{2K(\Delta)}+e^{2K(\Delta)}f'(\Delta)\Big)\\ &\ll_A\frac{e^{2K(\Delta_n)}}{n}\Big(r_n^2\cdot\sup_{\Delta\in[\Delta_n,1]}|K'(\Delta)|+te^{-2t+2r_n}\cdot\sup_{\Delta\in[\Delta_n,1]}f'(\Delta)\Big)\end{aligned} \tag{4.3.14}$$

for large enough t, by the triangle inequality and the uniform bound for $f(\Delta)$ in (4.3.12). Using the fact that $e^{2K(\Delta_n)}\le n^{2\delta}$ and that K' is uniformly bounded by

$$|K'(\Delta)|\le\int_1^{e^{r_n}}|k'(u\Delta)|\mathrm{d}u\le e^{r_n}\|k'\|_\infty\ll n^\delta, \tag{4.3.15}$$

on $[\Delta_n,1]$ (cf. Equation (4.1.1)), we conclude that (4.3.14) is

$$\begin{aligned}&\ll r_n^2n^{3\delta-1}+n^{2\delta-1}(te^{-2t+2r_n})\sup_{\Delta\in[\Delta_n,1]}f'(\Delta)\\ &\ll o\big((\log n)^{-2}\big)+n^{2\delta-1}(te^{-2t+2r_n})\sup_{\Delta\in[\Delta_n,1]}f'(\Delta).\end{aligned} \tag{4.3.16}$$

It remains to bound $f'(\Delta)$ uniformly over $[\Delta_n,1]$. Recalling that

$$f(\Delta)=\iint_{(-\infty,U(r_n)+A]^2}p_\Delta(u_1,u_2)\beta(u_1)\beta(u_2)\mathrm{d}u_1\mathrm{d}u_2,$$

and noting that $\beta(u_1)\beta(u_2)$ does not depend on Δ, differentiating under the integral yields

$$f'(\Delta) = \iint_{(-\infty, U(r_n)+A]^2} \partial_\Delta p_\Delta(u_1, u_2)\beta(u_1)\beta(u_2)\mathrm{d}u_1\mathrm{d}u_2.$$

That we can do so is justified by the dominated convergence theorem, noting the Gaussian decay of p_Δ and the estimate in (4.3.11). The latter also implies that

$$f'(\Delta) \ll \frac{e^{2(t-r_n)}}{t-r_n} \iint_{\mathbb{R}^2} |\partial_\Delta p_\Delta(u_1, u_2)|(A + r_n + |u_1|)(A + r_n + |u_2|)\mathrm{d}u_1\mathrm{d}u_2.$$

Differentiating the bivariate Gaussian density, we find that

$$\partial_\Delta p_\Delta(u_1, u_2) = p_\Delta(u_1, u_2) \cdot K'(\Delta)\bigg(\frac{K(\Delta)}{r_n^2 - K(\Delta)^2}\Big(1 - \frac{r_n(\bar{u}_1^2 + \bar{u}_2^2) - 2K(\Delta)\bar{u}_1\bar{u}_2}{r_n^2 - K(\Delta)^2}\Big) + \frac{\bar{u}_1\bar{u}_2}{r_n^2 - K(\Delta)^2} + \frac{\sqrt{2}(\bar{u}_1 + \bar{u}_2)}{r_n + K(\Delta)}\bigg)$$

where $\bar{u}_i = u_i - \mu_\Delta$ for $i = 1, 2$. For $r_n \geq 1$ and $|\Delta| \geq en^{-\delta} = e^{-r_n+1}$,

$$r_n \pm K(\Delta) \geq 1, \quad \text{and} \quad r_n^2 - K(\Delta)^2 \geq r_n^2 - (r_n - 1)^2 \geq r_n,$$

from which it follows that

$$\begin{aligned} |\partial_\Delta p_\Delta(u_1, u_2)| &\leq p_\Delta(u_1, u_2) \cdot |K'(\Delta)|\Big(1 + (\bar{u}_1^2 + \bar{u}_2^2) + 2|\bar{u}_1\bar{u}_2| + \frac{|\bar{u}_1\bar{u}_2|}{r_n} + \sqrt{2}|\bar{u}_1 + \bar{u}_2|\Big) \\ &\ll p_\Delta(u_1, u_2) \cdot |K'(\Delta)|\Big(1 + \bar{u}_1^2 + \bar{u}_2^2 + |\bar{u}_1| + |\bar{u}_2|\Big), \end{aligned}$$

and in turn that $f'(\Delta)$ is

$$\ll \frac{e^{2(t-r_n)}|K'(\Delta)|}{t-r_n}\mathbb{E}\Big\{\big(A + r_n + |X_1|\big)\big(A + r_n + |X_2|\big) \cdot \big(1 + |\bar{X}_1|^2 + |\bar{X}_2|^2 + |\bar{X}_1| + |\bar{X}_2|\big)\Big\},$$

where $\bar{X}_i = X_i - \mu_\Delta$. This expectation is $\ll_A r_n^3$ by the Cauchy-Schwarz inequality, and since $|K'(\Delta)| \leq n^\delta$ by Equation (4.3.15), we conclude that $f'(\Delta) \ll_A e^{2(t-r_n)}r_n^3 n^\delta/(t - r_n)$. Inserting this into (4.3.16) gives

$$\limsup_{t\to\infty} n^{2\delta-1}(te^{-2t+2r_n}) \sup_{\Delta\in[\Delta_n,1]} f'(\Delta) \ll_A n^{3\delta-1}r_n^3 = o_A\big((\log n)^{-2}\big), \tag{4.3.17}$$

and the bound in (4.3.9) then follows from Equations (4.3.13), (4.3.16), and (4.3.17). □

4.3.3 Dominant contribution

Having shown that $\mathbb{E}\mathscr{C}_{n,t}(\mathrm{I})$ can be discarded, it remains to bound the contribution coming from $\mathbb{E}\mathscr{C}_{n,t}(\mathrm{II})$. We prove the following bound.

Proposition 4.3.2. *Let $\mathscr{C}_{n,t}(\mathrm{II})$ be as in* (4.3.4)*. Then for any $\varepsilon > 0$,*

$$\limsup_{t\to\infty} \mathbb{E}\big\{\mathscr{C}_{n,t}(\mathrm{II})\big\} = O_A\big((\log n)^{-1/2+\varepsilon}\big)$$

for large enough $n > 0$.

We begin by introducing the auxiliary function

$$\mathscr{F}_{n,t}(\Delta) = te^{-2t} \cdot \mathbb{E}\Big\{ 1\big\{0, \Delta \in G_{n,t}(A)\big\} e^{\sqrt{2}(X_t(0)+X_t(\Delta))} \Big\} \tag{4.3.18}$$

noting that Fubini's theorem and the triangle inequality yield

$$\mathbb{E}\big\{\mathscr{C}_{n,t}(\mathrm{II})\big\} \ll \int_0^{\Delta_n} \mathscr{F}_{n,t}(\Delta)\mathrm{d}\Delta. \tag{4.3.19}$$

It therefore suffices to prove that the right-hand side is $\ll (\log n)^{-1/2+\varepsilon}$ as $t \to \infty$. The bulk of the work lies in proving the following uniform upper bound for $\mathscr{F}_{n,t}$ whose proof is deferred to Section 4.4.

To state it, we introduce the *branching time*

$$r(\Delta) = t \wedge \big(\log|\Delta|^{-1}\big).$$

Proposition 4.3.3. *Let $\mathscr{F}_{n,t}$ be defined as in Equation* (4.3.18)*. Then there exists a constant $C(A) = C > 0$ such that for large enough $t, n > 0$, $\mathscr{F}_{n,t}(\Delta)$ satisfies the bound*

$$\mathscr{F}_{n,t}(\Delta) \le C\sqrt{r_n} \cdot \frac{e^{r(\Delta)}}{r(\Delta)} \cdot \frac{t}{(t - r(\Delta)) + 1} \cdot \frac{\log\big(r(\Delta)\big)^4}{(r(\Delta) - r_n) \vee 1} \tag{4.3.20}$$

uniformly in $\Delta \in [0, \Delta_n]$.

Proof of Proposition 4.3.2 assuming Proposition 4.3.3. For large enough $n > 0$, we can use Proposition 4.3.3 to bound the integral in (4.3.19), beginning with the range $[0, e^{-t}]$ where

$$\limsup_{t\to\infty} \int_0^{e^{-t}} \mathscr{F}_{n,t}(\Delta)\mathrm{d}t \ll_A \sqrt{r_n} \lim_{t\to\infty} te^{t-t}\frac{1}{t}\Big(\frac{\log(t)^4}{t - r_n}\Big) = 0.$$

On $[e^{-t}, \Delta_n]$, Proposition 4.3.3 yields

$$\int_{e^{-t}}^{\Delta_n} \mathscr{F}_{n,t}(\Delta)\mathrm{d}\Delta \ll_A \sqrt{r_n} \int_{e^{-t}}^{\Delta_n} \frac{1}{r(\Delta)} \frac{t}{(t - r(\Delta)) + 1} \frac{\log(r(\Delta))^4}{(r(\Delta) - r_n) \vee 1} \frac{1}{\Delta}\mathrm{d}\Delta.$$

We first take care of the range where $r(\Delta)$ is close to r_n. On $[\Delta_n/e^3, \Delta_n]$, say, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{\Delta_n/e^3}^{\Delta_n} \mathscr{F}_{n,t}(\Delta)\mathrm{d}\Delta \\
&\ll \limsup_{t\to\infty} \frac{\sqrt{r_n}\log(r_n+2)^4}{r_n-1}\frac{t}{t-r_n+2}\int_{\Delta_n/e^3}^{\Delta_n}\frac{\mathrm{d}\Delta}{\Delta} \\
&\ll \frac{(\log r_n)^4}{\sqrt{r_n}}\int_{r_n-1}^{r_n+2}\mathrm{d}\log(\Delta^{-1}) \ll \frac{(\log r_n)^4}{\sqrt{r_n}}.
\end{aligned}$$

Using the substitution $u = r(\Delta) = \log\Delta^{-1}$, we split the remaining range into $u \in [r_n+2, t/2)$ and $u \in [t/2, t)$. The contribution from the latter is

$$\begin{aligned}
&\ll \sqrt{r_n}\int_{t/2}^{t}\frac{1}{u}\frac{t}{(t-u)+1}\frac{\log(u)^4}{(u-r_n)\vee 1}\mathrm{d}u \\
&\ll \sqrt{r_n}\frac{t}{t-1}\frac{\log(t)^4}{(t-1-r_n)} + \sqrt{r_n}t\log(t)^4\int_{t/2}^{t-1}\frac{1}{u(u-r_n)(t-u)}\mathrm{d}u \\
&\ll \sqrt{r_n}\frac{\log(t)^5}{t} \to 0 \text{ as } t\to\infty.
\end{aligned}$$

On the former, we have $t/(t-u) \le 2$ and we conclude that

$$\begin{aligned}
\limsup_{t\to\infty}\int_{e^{-t}}^{\Delta_n}\mathscr{F}_{n,t}(\Delta)\mathrm{d}\Delta &\ll_A \sqrt{r_n}\int_{r_n+2}^{\infty}\frac{1}{u}\frac{\log(u)^4}{(u-r_n)}\mathrm{d}u \\
&\ll_A \frac{\log(2r_n)^4}{\sqrt{r_n}}\int_2^{r_n}\frac{\mathrm{d}x}{x} + \sqrt{r_n}\int_{r_n}^{\infty}\frac{\log(2x)^4}{x^2}\mathrm{d}x \\
&\ll_A \frac{(\log\log n)^5}{\sqrt{\log n}},
\end{aligned}$$

which is $\ll_A (\log n)^{-1/2+\varepsilon}$ for any $\varepsilon > 0$ as claimed. □

Remark 4.3.4. For any $a > b > \delta$, the same argument shows that

$$\lim_{t\to\infty}\int_{n^{-a}}^{n^{-b}} e^{in\Delta}\mathscr{F}_{n,t}(\Delta)\mathrm{d}\Delta \ll_A \frac{1}{(\log n)^{1/2-\varepsilon}}$$

whereas for any $s_1(n) \le s_2(n)$ both tending to $+\infty$ in n,

$$\lim_{t\to\infty}\int_{n^{-s_2(n)}}^{n^{-s_1(n)}} e^{in\Delta}\mathscr{F}_{n,t}(\Delta)\mathrm{d}\Delta = o_A\Big(\frac{1}{(\log n)^{1/2-\varepsilon}}\Big).$$

This suggests that the main contribution to $\mathbb{E}\{\mathscr{C}_{n,t}(\mathrm{II})\}$ (and in turn $\mathbb{E}\{|c_{n,t}|^2; E_{n,t}(A)\}$) comes from pairs (θ_1, θ_2) for which $|\theta_2 - \theta_1| \approx n^{-a}$ for *all* scales $a > \delta$.

4.4 Upper bound for $\mathscr{F}_{n,t}(\Delta)$

This section proves Proposition 4.3.3. Recalling that

$$\mathscr{F}_{n,t}(\Delta) = te^{-2t}\mathbb{E}\Big\{e^{\sqrt{2}(X_t(0)+X_t(\Delta))}1\big\{X_s(*) \le U(s) + A, \forall s \in [r_n, t], * \in \{0, \Delta\}\big\}\Big\}$$

an application of Girsanov's theorem shows that $\mathscr{F}_{n,t}(\Delta)$ equals

$$te^{2K_t(\Delta)}\mathbb{P}\Big(X_s(*) \le A + U(s) - \sqrt{2}(s + K_s(\Delta)), \forall s \in [r_n, t], * \in \{0, \Delta\}\Big). \tag{4.4.1}$$

The goal will be to prove that this quantity satisfies the bound

$$\ll_A \sqrt{r_n}\frac{e^{r(\Delta)}}{r(\Delta)}\frac{t}{(t - r(\Delta)) + 1} \cdot \frac{\log(r(\Delta) - r_n)^4}{(r(\Delta) - r_n) \vee 1},$$

by arguing similarly to the proof of Proposition 2.6 in [123]. We require a slightly sharper estimate than the one proved there, which we obtain by utilizing the lower order terms in $U(s)$.

Assume for now that $r(\Delta) - r_n > 1$. Observe first that by symmetry, we can add the restriction $X_{r(\Delta)}(0) \le X_{r(\Delta)}(\Delta)$ to the event in (4.4.1) since doing so can only change its probability by a factor of 1/2. Secondly, we can use the estimate

$$K_s(\Delta) = s \wedge (\log|\Delta|^{-1}) + O(\|k'\|_\infty) = s \wedge r(\Delta) + O(1),$$

(cf. Equation (4.1.2)), which is uniform in $s \ge 0$ and $\Delta \in [0, \Delta_n]$. It follows that there exists a constant $C = C(A) > 0$ for which the intersection of $\{X_{r(\Delta)}(0) \le X_{r(\Delta)}(\Delta)\}$ with the event in (4.4.1) is contained in $E_1 \cap E_2 \cap E_3$, where

$$\begin{aligned}
E_1 &:= \Big\{X_s(0) \le C + U(s) - 2\sqrt{2}s, \forall s \in [r_n, r(\Delta)]\Big\} \\
E_2 &:= \Big\{X_s^{(r(\Delta))}(0) \le C - \sqrt{2}r(\Delta) - X_{r(\Delta)}(0), \forall s \in [0, t - r(\Delta)]\Big\} \\
E_3 &:= \Big\{X_s^{(r(\Delta))}(\Delta) \le C - \sqrt{2}r(\Delta) - X_{r(\Delta)}(0), \forall s \in [0, t - r(\Delta)]\Big\}
\end{aligned}$$

Note that to get the inequality in E_3, we used (and subsequently dropped) the restriction $X_{r(\Delta)}(0) \le X_{r(\Delta)}(\Delta)$.

We thus need to bound

$$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{E}\Big\{1_{E_1}\mathbb{P}\big(E_2 \cap E_3\big|\Sigma_\Delta\big)\Big\}, \tag{4.4.2}$$

where $\Sigma_\Delta = \sigma((X_s(\theta))_{\theta\in[0,1], s\le r(\Delta)})$. This will require a "linearised" version of $U(s)$. For $r(\Delta) \ge e$, define the *slope*

$$\alpha_\Delta = \alpha = \Big(\sqrt{2} - \frac{\log r(\Delta)}{2\sqrt{2}r(\Delta)}\Big),$$

so that

$$U(s) \le \alpha s + \frac{4}{\sqrt{2}} \log\log r(\Delta) \text{ for all } s \in [e, r(\Delta)], \tag{4.4.3}$$

with equality at $s = r(\Delta)$. For simplicity, we also define

$$V_{\Delta,A}(s) = V(s) = C + \frac{4}{\sqrt{2}} \log\log r(\Delta) + (\alpha - 2\sqrt{2})s$$

which is greater than $U(s) - 2\sqrt{2}s$ for all $s \in [e, r(\Delta)]$.

We can now estimate (4.4.2). Letting $(B_s)_{s \le r(\Delta)}$ denote a standard Brownian motion, we note on the one hand that

$$\begin{aligned}
&\mathbb{P}\Big(E_1 \,\Big|\, X_{r_n}(0) = y, X_{r(\Delta)}(0) = z \Big) \\
&\le \mathbb{P}\Big(X_s^{(r_n)}(0) \le V(s + r_n) - y, \forall s \in [0, r(\Delta) - r_n] \,\Big|\, X_{r_n}(0) = y, X_{r(\Delta)}(0) = z \Big) \\
&= \mathbb{P}\big(B_s \le V(s + r_n) - y, \forall s \in [0, r(\Delta) - r_n] \,\big|\, B_{r(\Delta)-r_n} = z - y \big),
\end{aligned}$$

and that the right-hand side can be estimated using Equation (4.1.8). Conditionally on $B_{r(\Delta)-r_n} = z - y$, the inequality in the event therein can be rewritten as

$$\begin{aligned}
\big(B_s - \frac{s}{r(\Delta) - r_n} B_{r(\Delta)-r_n} \big) + \frac{s}{r(\Delta) - r_n} \big((\alpha - 2\sqrt{2})(r(\Delta) - r_n) + (z - y) \big) \\
+ \big(y - V(r_n) \big) \le 0,
\end{aligned} \tag{4.4.4}$$

with $(B_s - \frac{s}{r(\Delta)-r_n} B_{r(\Delta)-r_n})_{s \le r(\Delta)-r_n}$ now distributed like a standard[4] Brownian bridge. Recall that in general, a Brownian motion $(W_s)_{s \le t}$ conditioned on $W_0 = a$ and $W_t = b$ is equal in distribution to

$$\Big(a + \frac{s}{t}(b - a) + \beta_s \Big)_{s \le t},$$

for $(\beta_s)_{s \le t}$ a standard bridge. The probability that (4.4.4) holds for every $s \in [0, r(\Delta) - r_n]$ therefore equals

$$\begin{aligned}
&\mathbb{P}^{(r(\Delta)-r_n, z-V(r(\Delta)))}_{(0, y-V(r_n))} \big(B_s \le 0, \forall s \in [0, r(\Delta) - r_n] \big) \\
&= \mathbb{P}^{(r(\Delta)-r_n, V(r(\Delta))-z)}_{(0, V(r_n)-y)} \big(B_s \ge 0, \forall s \in [0, r(\Delta) - r_n] \big),
\end{aligned}$$

which is

$$\ll \Big(\frac{(V(r_n) - y + 1)(V(r(\Delta)) - z + 1)}{(r(\Delta) - r_n) + 1} \wedge 1 \Big)$$

by Equation (4.1.8).

[4] Beginning and ending at zero.

On the other hand, Equation (4.1.7) yields

$$\mathbb{P}\big(E_2 \cap E_3 \,\big|\, \Sigma_\Delta\big) \ll_A \Big(\frac{(V(r(\Delta)) - X_{r(\Delta)}(0) + 1)^2}{((t - r(\Delta)) + 1)} \wedge 1\Big).$$

By integrating over the values of

$$(y, x) = (X_{r_n}(0), X_{r(\Delta)}(0) - X_{r_n}(0))$$

we can therefore conclude that $\mathscr{F}_{n,t}(\Delta)$ is

$$\begin{aligned} &\ll te^{2K_t(\Delta)}\mathbb{E}\Big\{\mathbf{1}_{E_1}\mathbb{P}\big(E_2 \cap E_3 \,\big|\, X_{r(\Delta)}(0)\big)\Big\} \\ &\ll_A \frac{te^{2r(\Delta)}}{(r(\Delta) - r_n)(t - r(\Delta) + 1)} \\ &\times \int_{-\infty}^{V(r_n)} \frac{e^{-\frac{y^2}{2r_n}}}{\sqrt{r_n}} \int_{-\infty}^{V(r(\Delta))-y} \frac{e^{-\frac{x^2}{2(r(\Delta)-r_n)}}}{\sqrt{r(\Delta) - r_n}} F(y, (x + y))\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{4.4.5}$$

where

$$F(a, b) := (V(r_n) - a + 1)(V(r(\Delta)) - b + 1)^3.$$

The change of variables $z = x + y$ yields

$$\begin{aligned} &\frac{te^{2r(\Delta)}}{(r(\Delta) - r_n)(t - r(\Delta) + 1)} \\ &\times \int_{-\infty}^{V(r_n)} \int_{-\infty}^{V(r(\Delta))} \frac{e^{-z^2/2r(\Delta)}}{\sqrt{r(\Delta)}} \frac{e^{-(y-\frac{r_n}{r(\Delta)}z)^2/2(\frac{r_n}{r(\Delta)}(r(\Delta)-r_n))}}{\sqrt{(r_n/r(\Delta))(r(\Delta) - r_n)}} F(y, z)\mathrm{d}z\mathrm{d}y. \end{aligned}$$

We first evaluate the integral over y, which equals

$$\mathbb{E}\Big\{\mathbf{1}\big(X < V(r_n)\big)\big(V(r_n) - X + 1\big)\Big\}, \text{ where } X \sim \mathcal{N}\Big(\mu = \frac{r_n}{r(\Delta)}z, \sigma^2 = \frac{r_n(r(\Delta) - r_n)}{r(\Delta)}\Big).$$

This can be decomposed into

$$\big(V(r_n) - \mu + 1\big)\mathbb{P}(X < V(r_n)) + \mathbb{E}\big\{\mathbf{1}(X < V(r_n))\big(X - \mu\big)\big\} \tag{4.4.6}$$

$$\leq \big(V(r_n) - \mu + 1\big) + \sigma\mathbb{E}\Big\{Z\mathbf{1}\big(Z < (V(r_n) - \mu)/\sigma\big)\Big\} \tag{4.4.7}$$

(for Z a standard Gaussian), which is equal to

$$(V(r_n) - \mu + 1) + \frac{\sigma}{\sqrt{2\pi}}e^{-(V(r_n)-\mu)^2/(2\sigma^2)}.$$

Inserting this into (4.4.6) yields the bound

$$\begin{aligned}
&\ll \frac{te^{2r(\Delta)}}{(r(\Delta)-r_n)(t-r(\Delta)+1)} \\
&\quad \times \Big(\int_{-\infty}^{V(r(\Delta))} \frac{e^{-z^2/2r(\Delta)}}{\sqrt{r(\Delta)}} \Big(V(r_n) - \frac{r_n}{r(\Delta)} z + 1 \Big) \big(V(r(\Delta)) - z + 1 \big)^3 \mathrm{d}z \\
&\qquad + \sigma \int_{-\infty}^{V(r(\Delta))} \frac{e^{-z^2/2r(\Delta) - (V(r_n) - \frac{r_n}{r(\Delta)} z)^2/2\sigma^2}}{\sqrt{r(\Delta)}} \big(V(r(\Delta)) - z + 1 \big)^3 \mathrm{d}z \Big)
\end{aligned}$$

which by the substitution $\bar{z} := V(r(\Delta)) - z$ equals

$$\begin{aligned}
&\frac{te^{2r(\Delta)}}{(r(\Delta)-r_n)(t-r(\Delta)+1)} \Big(\int_0^\infty \frac{e^{-(V(r(\Delta))-\bar{z})^2/2r(\Delta)}}{\sqrt{r(\Delta)}} \Big(\frac{r_n}{r(\Delta)} \bar{z} + 1 \Big) \big(\bar{z} + 1 \big)^3 \mathrm{d}\bar{z} \\
&\qquad + \sigma \int_0^\infty \frac{e^{-(V(r(\Delta))-\bar{z})^2/2r(\Delta) - \bar{z}^2/2(r_\Delta - r_n)}}{\sqrt{r(\Delta)}} \big(\bar{z} + 1 \big)^3 \mathrm{d}\bar{z} \Big) \\
&\leq \frac{(1+\sigma)te^{2r(\Delta)}}{(r(\Delta)-r_n)(t-r(\Delta)+1)} \int_0^\infty \frac{e^{-(V(r(\Delta))-\bar{z})^2/2r(\Delta)}}{\sqrt{r(\Delta)}} (\bar{z}+1)^4 \mathrm{d}\bar{z} \\
&\leq \frac{(1+\sigma)te^{2r(\Delta)}}{(r(\Delta)-r_n)(t-r(\Delta)+1)} \frac{e^{-V(r(\Delta))^2/2r(\Delta)}}{\sqrt{r(\Delta)}} \int_0^\infty e^{\frac{V(r(\Delta))}{r(\Delta)}\bar{z}} (\bar{z}+1)^4 \mathrm{d}\bar{z}. \qquad (4.4.8)
\end{aligned}$$

Recalling that

$$V(r(\Delta)) = C + \frac{4}{\sqrt{2}} \log\log r(\Delta) - \sqrt{2} r(\Delta) - \frac{1}{2\sqrt{2}} \log r(\Delta),$$

the remaining integral is $\ll \int_0^\infty e^{-\bar{z}} (\bar{z}+1)^4 \mathrm{d}z < \infty$ for large enough n. Expanding the Gaussian term in (4.4.8) and noting that $\sigma \leq \sqrt{r_n}$, we conclude that

$$\mathscr{F}_{n,t}(\Delta) \ll_A \frac{te^{r(\Delta)}}{(r(\Delta)-r_n)(t-r(\Delta)+1)} \frac{\sqrt{r_n}}{r(\Delta)} (\log r(\Delta))^4,$$

which is the claimed bound.

When $r(\Delta) - r_n < 1$, the claim follows by repeating the argument above while replacing E_1 by

$$\Big\{ X_{r_n}(0) \leq C + U(r_n) - 2\sqrt{2} r_n \Big\},$$

and $r(\Delta)$ by r_n in E_2 and E_3.

Chapter 5

On Poisson multiplicative chaos and the multiplication table problem

Let $p(k)$ be the limit as n tends to infinity of the probability that a random permutation in S_n fixes a set of size k. By a well-known analogy between integers and permutations, this models the question of whether a very large random integer has a divisor in the interval $[e^k, e^{k+1})$, and by extension, that of estimating the number of distinct entries in an $n \times n$ multiplication table as $n \to \infty$. The purpose of this short chapter is to establish a connection between the problem of estimating $p(k)$, and a generalisation thereof, and the theory of multiplicative chaos.

Let S_n denote the symmetric group on n elements, and $\sigma_n \in S_n$ be sampled uniformly at random. Consider the sequence of random variables

$$N_k^{(n)} = \#\big\{A \subseteq \{1, \dots, n\} \,:\, |A| = k,\ A \text{ is fixed by } \sigma_n\big\},$$

(so that $p(k) = \lim_{n\to\infty} \mathbb{P}(N_k^{(n)} > 0)$), and introduce the *cycle counts* of σ_n

$$\big(C_1(\sigma_n), \dots, C_n(\sigma_n)\big), \quad C_j(\sigma) := \#\text{ of cycles of length } j \text{ in } \sigma.$$

It is not difficult to see that $N_k^{(n)}$ only depends on σ_n through this sequence: each fixed k-set $A \subseteq \{1, \dots, n\}$ is obtained by first picking an element of

$$\Big\{(a_1, \dots, a_k) \,:\, a_j \le C_j(\sigma_n) \text{ for all } j,\ \sum_{j\le k} j a_j = k\Big\},$$

which counts the number of j-cycles of σ_n contained in A for each j, then selecting these cycles in one of

$$\prod_j \binom{C_j(\sigma_n)}{a_j}$$

many ways. This yields the following equality of formal power series

$$\sum_{k=0}^{\infty} N_k^{(n)} z^k = \prod_{j=1}^{\infty} (1+z^j)^{C_j(\sigma_n)}, \tag{5.0.1}$$

which (by the binomial theorem) implies that each $N_k^{(n)}$ is a polynomial in *finitely many* cycle counts $C_j(\sigma_n)$

$$N_1^{(n)} = C_1(\sigma_n),\ N_2^{(n)} = C_2(\sigma_n) + \frac{1}{2}C_1(\sigma_n)(C_1(\sigma_n)-1), \dots .$$

A classical result of Arratia, Barbour and Tavaré [17] asserts that for fixed k

$$\big(C_1(\sigma_n), \dots, C_k(\sigma_n)\big) \xrightarrow[n\to\infty]{d} \big(P_1, \dots, P_k\big) \tag{5.0.2}$$

where the right-hand side is a vector of independent Poisson random variables, each P_j having parameter $1/j$. Using (5.0.1) and the continuous mapping theorem, this implies that

$$N_k^{(n)} \xrightarrow[n\to\infty]{d} N_k, \quad N_k := [z^k] \prod_{j=1}^{\infty} (1+z^j)^{P_j}$$

where $[z^k]G(z)$ denotes the k-th coefficient in the (formal) Taylor expansion of $G(z)$ about the origin.

Our aim is then to promote this from a formal statement to one regarding the Fourier coefficients of a random distribution, which we view as a Poisson analogue to the Holomorphic Multiplicative Chaos of [142]. We begin with the following lemma.

Lemma 5.0.1. *Let* log *denote the principal branch of the logarithm. Then the series* $F(z) = \sum_{j\geq 1} \log(1+z^j)P_j$ *defines a holomorphic function almost surely in* $|z|<1$.

Proof. By expanding the logarithm, we can express $F(z)$ as

$$\sum_{j\geq 1} P_j \log(1+z^j) = \sum_{j\geq 1} P_j \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} z^{jk} = \sum_{n\geq 1} \Big(\sum_{j|n} \frac{(-1)^{n/j+1}}{n/j} P_j \Big) z^n := \sum_{n\geq 1} a_n z^n. \tag{5.0.3}$$

To prove the claim, it thus suffices to show that

$$\sum_{j,k} \frac{P_j}{k} r^{jk} < \infty, \quad \text{a.s.}$$

for any $0<r<1$, as this justifies the rearrangement in (5.0.3) and implies that $\sum_{n\geq 1} |a_n| r^n < \infty$. For a fixed r, one indeed has the bound

$$\mathbb{E}\Big\{ \sum_{n\geq 1} |a_n| r^n \Big\} \leq \sum_{n\geq 1} \frac{\tau(n)}{n} r^n, \tag{5.0.4}$$

where $\tau(n) = \sum_{j|n} 1$ is the divisor function. Using the standard bound $\tau(n) \ll_\varepsilon n^\varepsilon$ for fixed $\varepsilon > 0$, it follows that the right-hand side in (5.0.4) is finite, and that $F(z)$ converges absolutely almost surely for $|z| < r$.

To extend this to the entire disk $\{|z| < 1\}$, let $(r_m)_m \subset (0,1)$ be a countable sequence converging to 1, and E_m be the event that $F(z)$ converges absolutely for all $|z| < r_m$. Then since $\mathbb{P}(E_m) = 1$ for every m, it follows by countable subadditivity that $\mathbb{P}(\cup_m E_m^c) \leq \sum_m \mathbb{P}(E_m^c) = 0$, so that $\mathbb{P}(\cap_m E_m) = 1$. □

For any trigonometric polynomial ϕ, and $z \mapsto \phi(z)$ its continuous harmonic extension to the unit disk, define the *holomorphic Poisson multiplicative chaos* HPMC_γ as the random distribution acting as follows on ϕ

$$\big(\mathrm{HPMC}_\gamma, \phi\big) := \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} e^{\gamma F(re^{i\vartheta})} \overline{\phi(re^{i\vartheta})} \mathrm{d}\vartheta.$$

Having shown that $F(z)$ is analytic inside the unit disk, the existence of this limit is immediate, and it suffices to uniquely determine this distribution. It also follows by Cauchy's theorem that

$$\big(\mathrm{HPMC}_1, \vartheta \mapsto e^{ik\vartheta}\big) = N_k, \qquad \text{for each } k \geq 1,$$

meaning that $p(k)$ is the probability that the k-th Fourier coefficient of HPMC_1 is greater than or equal to 1.

The nomenclature is chosen in light of the following. On the circle, we can proceed formally and write

$$\begin{aligned}
\mathrm{Cov}\big\{\Re F(e^{i\vartheta_1}), \Re F(e^{i\vartheta_2})\big\} &= \sum_{j\geq 1} \frac{1}{j} \log|1 + e^{ij\vartheta_1}| \log|1 + e^{ij\vartheta_2}| \\
&= \sum_{j\geq 1} \frac{1}{j} \sum_{m,n} (-1)^{m+n} \frac{\cos(jn\vartheta_1)\cos(jm\vartheta_2)}{nm} \\
&= \frac{1}{2} \sum_{m,n} \frac{(-1)^{n+m}}{nm} \sum_{j\geq 1} \frac{\cos(j(n\vartheta_1 - m\vartheta_2)) + \cos(j(n\vartheta_1 + m\vartheta_2))}{j}.
\end{aligned}$$

revealing a covariance structure which is highly sensitive to the Diophantine properties of the pair $(\vartheta_1, \vartheta_2)$. That said, it is reasonable to expect that for nice enough pairs (for which the integer linear combinations $\pm n\vartheta_1 \pm m\vartheta_2$ are not often close to zero), the diagonal term in the above dominates and yields

$$\approx \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \log \frac{1}{|\vartheta_1 - \vartheta_2|} = \frac{\pi^2}{12} \log \frac{1}{|\vartheta_1 - \vartheta_2|} \qquad \text{as } |\vartheta_1 - \vartheta_2| \to 0, \tag{5.0.5}$$

due to $\{jn(\vartheta_1 + \vartheta_2)\}$ rapidly mixing. This makes $\Re F(e^{i\vartheta})$ in some sense log-correlated.

Such a heuristic is to be taken with a grain of salt, but it is worth noting that a field with a similar covariance structure (only without the alternating sign $(-1)^{n+m}$) was studied rigorously by Cook and Zeitouni [53], in their work on the maximum of the characteristic polynomial $\det(I - zM_n)$ of a random permutation matrix M_n. Noting that $\log|\det(I - zM_n)| = \sum_j C_j(M_n)\log|1 - z^j|$, where $C_j(M_n)$ denotes the number of j-cycles in the permutation represented by M_n, they proceed by comparison to the Poisson field $\sum_j P_j \log|1 - e^{i\vartheta j}|$, for which they indeed uncover a logarithmic correlation structure using an adaptation of the Hardy-Littlewood circle method. We also mention that an older work of Hambly, Keevash and O'Connell [87] showed that

$$\frac{\log|\det(I - e^{i\vartheta}M_n)|}{\sqrt{\frac{\pi^2}{12}\log n}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1),$$

when ϑ satisfies an appropriate Diophantine condition; one recognises in this result the constant in (5.0.5).

The field $F(z)$ studied here is also directly related to random permutation matrices. Noting the elementary identity

$$\frac{\det(I - z^2M_n)}{\det(I - zM_n)} = \prod_j \frac{(1 - z^{2j})^{C_j(M_n)}}{(1 - z^j)^{C_j(M_n)}} = \prod_j (1 + z^j)^{C_j(M_n)},$$

HPMC_1 can be seen as a sort of distributional limit for this ratio on the unit circle, and $\mathfrak{R}F$ as a limit for the real part of its logarithm. We for instance prove the following analogue of [110, Theorem 2.5].

Theorem 5.0.2. *For $a < 0$, let H_0^a be the space of generalised functions on the torus defined in* (1.2.6)*. Let $R(\vartheta)$ be given by the formal series $\mathfrak{R}F(e^{i\vartheta})$, and*

$$R_n(e^{i\vartheta}) := \log\left|\frac{\det(I - e^{2i\vartheta}M_n)}{\det(I - e^{i\vartheta}M_n)}\right|.$$

Then $R \in H_0^a$ for all $a < 0$, and the law of R_n converges weakly to that of R as $n \to \infty$.

Proof. We follow the argument in [110], making the necessary modifications. First, note that

$$(\widehat{R})_k = \mathbf{1}_{k\neq 0} \cdot \frac{1}{2|k|} \sum_{j\,||\,|k|} (-1)^{k/j+1} j P_j$$

by the expansion $\log|1 + e^{ij\vartheta}| = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k}\cos(kj\vartheta)$. A straightforward calculation then shows that

$$\mathbb{E}\{(\widehat{R})_k^2\} \leq \frac{\sigma_1(|k|) + \tau(|k|)^2}{k^2} \leq \frac{(|k| + \tau(|k|))\tau(|k|)}{k^2}, \quad \Big(\text{where } \sigma_1(k) = \sum_{d|k} d\Big),$$

and since $\tau(k) \ll_\varepsilon k^\varepsilon$ for any $\varepsilon > 0$, we conclude that $\mathbb{E}\{\|R\|^2_{H_0^a}\} < \infty$ for any $a < 0$. This proves that $R \in H_0^a$.

To prove convergence, we use the same expansion for $\log|1 + e^{i\vartheta}|$ in R_n to get

$$(\widehat{R_n})_k = \mathbf{1}_{k\neq 0} \cdot \frac{1}{2|k|} \sum_{j \mid |k|} (-1)^{k/j+1} j C_j(\sigma_n). \tag{5.0.6}$$

Convergence on cylinder sets then follows from (5.0.2) and the continuous mapping theorem. To prove tightness in H_0^a, one relies on the fact that the unit ball of H_0^b is compact in H_0^a when $a < b < 0$. This reduces the task to proving that $\sup_n \mathbb{E}\{\|R_n\|^2_{H_0^b}\} < \infty$, as Markov's inequality would then give

$$\sup_n \mathbb{P}\big(\|R_n\|_{H_0^b} > q\big) \to 0, \quad \text{as } q \to \infty.$$

Such a uniform bound indeed follows from the identities

$$\mathbb{E}\big\{C_j(M_n)^2\big\} = \frac{1}{j}\mathbf{1}_{j\leq n} + \frac{1}{j^2}\mathbf{1}_{2j\leq n}, \quad \mathbb{E}\big\{C_j(M_n)C_\ell(M_n)\big\} = \frac{1}{j\ell}\mathbf{1}_{j+\ell\leq n}, \quad (\text{for } j \neq \ell)$$

(see [18, Equation 5]); inserting this into (5.0.6) shows that for $k \neq 0$,

$$\begin{aligned}\mathbb{E}\big\{(\widehat{R_n})_k^2\big\} &\leq \frac{1}{4|k|^2}\Bigg(\sum_{j \mid |k|} j^2\Big(\frac{1}{j} + \frac{1}{j^2}\Big) + \sum_{j\neq\ell,\, j,\ell \mid |k|} j\ell \cdot \frac{1}{j\ell}\Bigg) \\ &\leq \frac{\tau(|k|)}{|k|} + \frac{\tau(|k|)^2}{k^2},\end{aligned}$$

and thus that $\sup_n \mathbb{E}\{\|R_n\|^2_{H_0^b}\} \ll \sum_k |k|^{2b-1}\tau(|k|) < \infty$. □

The natural next step in this theory is to ask whether one can construct a measure corresponding to $e^{\gamma \Re F(e^{i\vartheta})}\mathrm{d}\vartheta$, for $\gamma > 0$. Following Kahane [115], an obvious first attempt would be to consider the truncated field $G_n(e^{i\vartheta}) = \sum_{j\leq n} P_j \log|1 + e^{i\vartheta j}|$, and define

$$\mu_{n,\gamma}(\mathrm{d}\vartheta) := \frac{e^{\gamma G_n(e^{i\vartheta})}}{\mathbb{E}\big\{e^{\gamma G_n(e^{i\vartheta})}\big\}}\mathrm{d}\vartheta, \quad \gamma > 0.$$

It is then straightforward to check that for any Borel $A \subseteq [0, 2\pi]$, the masses $(\mu_{n,\gamma}(A))_{n\geq 1} := (\int_A \mu_{n,\gamma}(\mathrm{d}\vartheta))_{n\geq 1}$ form a positive martingale, and therefore converge almost surely to a limit. Should such a limit be non-trivial, the procedure to upgrade the convergence of $\mu_{n,\gamma}(A)$ to that of $\mu_{n,\gamma}$ to a limiting measure would be routine, but it is not clear whether this is the

case. Unlike for Gaussian multiplicative chaos,

$$\begin{aligned}\mathbb{E}\Big\{\mu_{n,\gamma}\big([0,2\pi]\big)^2\Big\} &= \iint_{[0,2\pi]^2} \frac{\mathbb{E}\big\{e^{\gamma(G_n(e^{i\vartheta_1})+G_n(e^{i\vartheta_2}))}\big\}}{\mathbb{E}\big\{e^{\gamma G_n(e^{i\vartheta_1})}\big\}\mathbb{E}\big\{e^{\gamma G_n(e^{i\vartheta_2})}\big\}}\mathrm{d}\vartheta_1\mathrm{d}\vartheta_2 \\ &= \iint_{[0,2\pi]^2} \exp\Big(\sum_{j=1}^{n}\frac{(|1+e^{i\vartheta_1 j}|^\gamma-1)(|1+e^{i\vartheta_2 j}|^\gamma-1)}{j}\Big)\mathrm{d}\vartheta_1\mathrm{d}\vartheta_2 \\ &\geq \exp\Big(\sum_{j=1}^{n}\frac{1}{j}\Big(\frac{1}{2\pi}\int_0^{2\pi}\big(|1+e^{i\vartheta j}|^\gamma-1\big)\mathrm{d}\vartheta\Big)^2\Big) \\ &= \exp\Big(\sum_{j=1}^{n}\frac{(m_\gamma-1)^2}{j}\Big),\end{aligned}$$

where

$$m_\gamma = \frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\vartheta}|^\gamma\mathrm{d}\vartheta > \exp\Big(\frac{\gamma}{2\pi}\int_0^{2\pi}\log|1+e^{i\vartheta}|\mathrm{d}\vartheta\Big) = 1.$$

It follows that for any $\gamma > 0$, $\mathbb{E}\{\mu_{n,\gamma}\big([0,2\pi]\big)^2\}$ diverges at least like a positive power of n; in the language of GMC theory, there is no "L^2 phase". We nevertheless conjecture that a non-trivial limiting measure should be obtainable from $e^{\gamma G_n(e^{i\vartheta})}\mathrm{d}\vartheta$, potentially using a different renormalisation procedure. This is the subject of ongoing work.

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