

# Kato's Perturbation Theorem and Honesty Theory



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## Abstract

We study an additive perturbation theorem for substochastic semigroups which is known as Kato's Theorem. There are two previously-known generalisations of Kato's Theorem, namely for abstract state spaces and for  $KB$ -spaces. We prove a version of Kato's Theorem for a class of spaces which encompasses both, namely ordered Banach spaces with generating cone and monotone norm. We also study a property of the perturbed semigroup in Kato's Theorem known as honesty of the semigroup. We add a few results to the fairly extensive existing theory of honesty for Kato's Theorem for abstract state spaces. In light of our new generalisation of Kato's Theorem to ordered Banach spaces with monotone norm, we investigate generalising the theory of honesty to these spaces as well. The results for the general case are less complete as many of the results for the case of abstract state spaces depend on the additive norm structure of the space.

We also consider some new applications of honesty theory in abstract state spaces. We begin by applying honesty theory to the study of the heat equation on graphs. We prove that honesty of the heat semigroup coincides with a concept known as stochastic completeness of the graph which has been studied independently of honesty. We then look at the application of honesty theory to quantum dynamical semigroups. We show that honesty is the natural generalisation of the concept of conservativity of quantum dynamical semigroups. Conservative quantum dynamical semigroups are known to have certain "nice" properties. We show that similar properties hold for honest semigroups using honesty theory results. Finally, we consider a form of boundary perturbations in the context of transport semigroups. There exists an analogous theory of honesty for this set-up. We formulate a general result from which honesty results of both Kato's Theorem and transport semigroups can be derived.

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# Introduction

Substochastic semigroups are positive one parameter semigroups of operators which are contractions on the positive cone of the ordered Banach space they act in. These semigroups occur in a variety of situations. One of their major uses is modelling the time evolution of some quantity in a particular system, for example, the distribution of neutrons in a system of nuclear particles [4, 9, 46]. The semigroup acts on  $L^1$  and the norm of the semigroup then gives the total number of neutrons in the system at a given time. Other examples include modelling population changes in birth and death problems [9, Chapter 7] and modelling fragmentation problems which occur in physical and chemical phenomena such as rock fracture or depolymerisation [9, Chapter 8-9].

Substochastic semigroups also play an important role in probability theory where they occur in the guise of Markov or submarkov semigroups which are related to the study of Markov processes in  $L^1$ . In fact, the main generation theorem which we will study in this thesis originates from the study of the classical Kolmogorov differential equations. In particular, we are interested in a generation theorem for substochastic semigroups under a special class of positive additive perturbations. This theorem is known as Kato's Theorem as it was first proven by Kato in his search for solutions to the Kolmogorov differential equations in  $\ell^1$  [31].

Kato's Theorem is but one example of a result in the classical study of perturbations of semigroups of operators. The study of perturbations of semigroups is usually centered around two main questions:

- (i) Given the generator  $A$  of a strongly continuous ( $C_0$ -)semigroup, for which operators  $B$  can we find an extension  $G$  of  $A + B$  such that  $G$  also generates a  $C_0$ -semigroup?
- (ii) Suppose there exists such a perturbed generator  $G$  generating  $(V(t))_{t \geq 0}$ , what are the properties of the perturbed semigroup  $(V(t))_{t \geq 0}$ ?

Apart from Kato's Theorem, examples of results which give an answer to question (i) include the Bounded Perturbation Theorem [22, Theorem III.1.3] and the Miyadera Perturbation Theorem [22, Theorem III.3.14]. As for question (ii), there are several angles from which the problem can be tackled. We are often interested in ensuring that the perturbed semigroup retains the same properties as the original semigroup, for example if the original semigroup is analytic, we would like to know that the perturbed semigroup is analytic too. Occasionally, we study the properties of the perturbed semigroup independently of the original semigroup as we will see in this thesis. We will give more details after the next couple of paragraphs where we will first discuss Kato's Theorem briefly.

As mentioned above, we are mainly interested in (strongly continuous) substochastic semigroups. In particular, the solution to question (i) which we will focus on is Kato's Theorem. The main idea of the theorem states that if  $A$  is the generator of a substochastic semigroup in  $L^1(\Omega, \mu)$  and  $B$  is an operator which is positive on the domain of  $A$ ,  $D(A)$ , such that  $A$  and  $B$  satisfy the following condition:

$$\int (A + B)x \, d\mu \leq 0 \quad \text{for all } 0 \leq x \in D(A), \quad (1)$$

then there is an extension  $G$  of  $A + B$  that generates a perturbed substochastic semigroup  $(V(t))_{t \geq 0}$  [31, Theorem 1]. Note that in  $L^1$ , a substochastic semigroup is simply a positive contraction semigroup. The condition (1) ensures that the perturbed semigroup is contractive (or in more technical terms, the generator  $G$  is dissipative). The positivity of the operators  $R(\lambda, A) := (\lambda - A)^{-1}$ ,  $\lambda > 0$  and  $BR(\lambda, A)$ ,  $\lambda > 0$  and the series representation of the operator, namely

$$R(\lambda, G)x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x, \quad x \in X$$

allow us to deduce the positivity of the perturbed semigroup.

There exist generalisations of Kato's Theorem to spaces beyond  $L^1$ , namely to abstract state spaces and  $KB$ -spaces (See Chapters 2 and 3). Abstract state spaces are generalisations of  $L^1$ -spaces in that they are ordered Banach spaces with norm additive on the positive cone while  $KB$ -spaces are Banach lattices in which any non-negative, increasing and norm bounded sequence is norm convergent (see Section 1.1 for more information). Observe that  $L^1$  lies in both sets of spaces but neither set is a subset of the other. Abstract state spaces were introduced by Arlotti, Lods and Mokhtar-Kharroubi in [3] as a means of removing the lattice structure on  $L^1$ -spaces. The authors discovered that Kato's Theorem did not depend on this structure,

hence they were able to extend Kato's Theorem to this setting with minimal changes to his proof. The generalisation to  $KB$ -spaces however, does require an additional assumption, namely that the spectrum of  $BR(\lambda, A)$  lies within the closed unit disc for some  $\lambda > 0$ , in order to compensate for the loss of the additive norm structure. This generalisation was in fact, proven by Banasiak and Lachowicz in [10] a few years before the introduction of abstract state spaces.

Now let us consider the specific version of question (ii) which we will look at. The central theme of this thesis is a particular property of the perturbed semigroup in Kato's Theorem which is known as honesty of the semigroup. Historically, the study of this property originates from the study of the stochasticity of the semigroup, which is in turn motivated by the study of non-explosive systems. As we mentioned above, substochastic semigroups are often used to model real-world systems. Physical conservation laws often require that the modelled system is conservative or non-explosive. In semigroup terminology, this means that the semigroup should be stochastic. So for this conservative setting, we will say that the semigroup is honest if and only if it is stochastic. More generally, honest semigroups can be described as the semigroups which are faithful or consistent with the system they model. We will give more precise definitions in Section 2.2.2.

Note that Kato's Theorem only tells us of the existence of the perturbed semigroup  $(V(t))_{t \geq 0}$  but does not provide information about the honesty of the semigroup. Since honest semigroups are the "good" semigroups, a major part of the study of honesty theory involves finding characterisations of honesty which we can then use in applications to determine if the perturbed semigroup from Kato's Theorem is honest. One classical result in the characterisation of honesty tells us that honesty of the semigroup  $(V(t))_{t \geq 0}$  can be determined by identifying the exact relationship between the generator  $G$  and the operators  $A, B$ . Note that from Kato's Theorem, we know that  $G$  is an extension of the operator  $A + B$  but not the precise relationship. It turns out that  $(V(t))_{t \geq 0}$  is honest if and only if  $G = \overline{A + B}$  [3, Theorem 3.5]. So apart from the connection to the conservativity of the system, honesty of the semigroup is also interesting from a mathematical point of view because it allows us to determine when a core of  $A + B$  is also a core for  $G$ .

The honesty theory for Kato's Theorem has been quite thoroughly studied for the case of abstract state spaces (which includes  $L^1$ -spaces) using various approaches (see [4, 35, 3] for example). One notable approach is a functional approach using resolvent operators which was introduced by Voigt and Mokhtar-Kharroubi for the  $L^1$  case in [35] and extended to abstract state spaces by Arlotti, Lods and Mokhtar-Kharroubi in

[3]. Other notable approaches include a spectral approach [26] and a Dyson-Phillips series approach [3]. Unfortunately, the currently-known results in honesty theory do not extend to spaces beyond abstract state spaces despite the known existence of the version of Kato’s Theorem for  $KB$ -spaces. Thus, the extension of honesty theory to spaces beyond abstract state spaces will be one problem which we will investigate in this thesis.

Kato’s Theorem and the honesty theory associated with it have applications in many areas. Most of the known examples focus on the  $L^1$  case, for instance, birth and death problems and fragmentation problems. In this thesis, we will steer away from these classical examples. We will begin by looking at an application of Kato’s Theorem to an example based in  $\ell^2$ , namely the heat equation on graphs or networks. The theory of the heat equation on graphs is built on the discrete Laplacian, which has a self-adjoint realisation that generates a positive contractive heat semigroup on  $\ell^2$  [32, Section 1]. The link to  $\ell^1$  appears in the guise of Dirichlet forms and their associated set of compatible semigroups on  $\ell^p$ ,  $1 \leq p \leq \infty$ . This utilisation of Dirichlet forms as a means of studying the heat equation on graphs was introduced by Keller and Lenz in their seminal paper [32]. Using this link to  $\ell^1$  and the fact that the discrete Laplacian can be reformulated as the sum of two operators, we can show that Kato’s Theorem can be applied to this setting. The main consequence of this is that we can prove that honesty of the heat semigroup is equivalent to stochastic completeness of the graph, which is an important notion in the study of the heat equation on graphs that has previously been studied independently of honesty theory (see for example [21, 32, 48]). Stochastic completeness is related to the amount of heat in the graph and we say that a graph is stochastically complete if there is no loss of heat to “infinity” (heat is “conserved”) [32, p.195].

The second application we look at is an application to quantum dynamical semigroups. The generalisation of Kato’s Theorem to abstract state spaces allows us to apply our theory to quantum dynamical semigroups, which can be thought of as the noncommutative version of classical Markov semigroups on  $L^1$ . Quantum dynamical semigroups as defined in Fagnola’s comprehensive survey [24], act on  $\mathcal{L}(\mathfrak{H})$ , the space of bounded linear operators on a Hilbert space  $\mathfrak{H}$ , and they emerged from the study of quantum mechanics and quantum stochastic processes. The link between Kato’s Theorem (the original  $L^1$  version) and quantum dynamical semigroups has been known since the 1970s, when Davies in [19] showed that the techniques used in Kato’s paper [31] could be used to show the existence of a quantum dynamical semigroup. More precisely, the application of Kato’s Theorem is restricted to a special

class of quantum dynamical semigroups, namely those whose generators can be represented in Lindblad form (see Definition 5.1.9). This form is named after Lindblad, whose original generation theorem [24, Theorem 3.15] states that the generator of a uniformly continuous quantum dynamical semigroup is the sum of the generator of a substochastic quantum dynamical semigroup and a completely positive map. The more general class we study no longer restricts to uniformly continuous semigroups but still retains the decomposition of the generator into a sum of two operators which then allows us to apply Kato's Theorem. Note that although the relation with Kato's Theorem has long been identified, the link to honesty theory has yet to be established as honesty theory was developed in a systematic manner much more recently. However, just like the commutative case on  $L^1$ , there is also a notion of conservativity that has been studied independently of honesty theory (see [15, 16, 24] for example) and we will in fact, show that honesty is a generalisation of this notion.

Finally, we will look at the connections between Kato's Theorem and a different type of perturbation altogether, namely boundary perturbation. In particular, we will look at the transport equation with boundary conditions given by means of a boundary operator,  $H$ . This differs from additive perturbations because the transport operator acts on  $L^1(\Omega, \mu)$ , where  $\Omega$  is a sufficiently smooth, open subset of  $\mathbb{R}^n$  and  $\mu$  is the associated measure, while the boundary operator acts on functions defined on the boundary of  $\Omega$ . However, despite the differences between the two types of perturbations, we will see that there is a generation theorem for the transport semigroup which is structurally similar to Kato's Theorem. This theorem gives the existence of an extension  $\mathcal{G}$  of the transport operator associated with  $H$ , such that  $\mathcal{G}$  generates a substochastic semigroup on  $L^1(\Omega, \mu)$ . These similarities extend to honesty theory as well where honesty of the transport semigroup is usually couched in terms of mass loss in the system modelled by the transport semigroup. Indeed, there are many papers [6, 7, 8, 34] which detail the transfer of techniques from the study of honesty theory of Kato's Theorem to the case of transport semigroups, often produced by authors who studied both areas, for example, Arlotti, Lods, Mokhtar-Kharroubi and Banasiak. This success in the transfer of techniques and the analogous results obtained lead naturally to the question of whether one can obtain a general unifying theory from which results of both perturbation types can be derived without resorting to the actual transfer of techniques. This is the main problem which we will investigate for this application.

## 0.1 Summary of the Chapters

In the course of our study of Kato's Theorem and honesty theory, we will come across various concepts such as ordered spaces, positive operators and Dirichlet forms. We will introduce these concepts and the required auxiliary information in Chapter 1.

We will begin our study of Kato's Theorem in Chapter 2 where we will look at a generalisation of Kato's Theorem to the class of abstract state spaces. We will present the generation theorem by Arlotti, Lods and Mokhtar-Kharroubi [3, Theorem 2.1] in Section 2.1 before looking at the honesty theory related to it in the three subsequent sections. Our study of honesty theory begins in Section 2.2 with some background information, where we will highlight two particular approaches to honesty, namely the functional approach via resolvent operators and the spectral approach, as they will be utilised repeatedly in the rest of the thesis. We will then conclude the chapter by presenting some new characterisations and new approaches to honesty theory on abstract state spaces.

In Chapter 3, we will investigate generalisations of Kato's Theorem and honesty theory beyond abstract state spaces. The only previously known generalisation of Kato's Theorem beyond abstract state spaces is a result by Banasiak and Lachowicz in [10] who prove a version of Kato's Theorem for  $KB$ -spaces.  $KB$ -spaces are not strictly contained within the class of abstract state spaces and vice versa. In Section 3.1, we will prove a version of Kato's Theorem for a set of spaces which encompasses both  $KB$ -spaces and abstract state spaces, namely ordered Banach spaces with monotone norm. As a natural follow-up, we will look at honesty theory for this generalised version of Kato's Theorem. In Section 3.2.1, we will look at generalising the approaches used for the case of abstract state spaces. Unfortunately, we will see that the additivity of the norm in such spaces is essential for most of these approaches and the results cannot be easily generalised to spaces without additive norm. However, in Section 3.2.2, we will introduce an approach which does not depend on the additive structure and prove some new characterisations of honesty for the general case.

Our final three chapters cover applications of honesty theory. In Chapter 4, we will look at an application of Kato's Theorem in  $\ell^2$  in the set-up of the heat equation on graphs. Our main results will be in Section 4.2, where we will show that the discrete Laplacian can be recast in terms of additive perturbations and moreover, satisfies the conditions of Kato's Theorem. We will then show that honesty of the heat semigroup is equivalent to stochastic completeness of the graph. This equivalence then allows

us to apply honesty results in the study of stochastic completeness of graphs which we will do in Section 4.3.

In Chapter 5, we will study the application of Kato's Theorem and honesty theory to the class of quantum dynamical semigroups whose generators can be represented in Lindblad form. We will begin by describing precisely how Kato's Theorem can be applied to such semigroups in Section 5.1. In the following section, we will then see that honesty is equivalent to the conservativity of the quantum dynamical semigroup. However, conservativity is a concept that is restricted to the stochastic setting. Hence, we will show that honesty is the natural analogue of conservativity for the substochastic setting. Furthermore, previously known results for the conservative case can be extended to the substochastic case via honesty theory.

In our final chapter, we will look at boundary perturbations in the setting of transport theory. Section 6.1 will simply be an introduction to the transport operator and its associated boundary value problems. Only in Section 6.2 will we begin introducing the generation theorem for the transport semigroup. We will see that the generation theorem for the substochastic transport semigroup with positive, contractive boundary operator shares many similarities with Kato's Theorem. These similarities extend to the study of honesty theory of the transport semigroup which we will see in Section 6.3. We will then present a general unifying theory from which honesty results of both perturbation types can be derived. In the final section, we will look briefly at the strong stability of the perturbed semigroups in both Kato's Theorem and transport theory.

# Chapter 1

## Preliminaries

We introduce the preliminary information required in the subsequent chapters.

### 1.1 Ordered Banach Spaces and Positive Operators

Before beginning our discussion on ordered Banach spaces, let us clarify some notation. If  $X$  is a Banach space, we will use  $X^*$  to denote its dual space with duality pairing  $\langle \cdot, \cdot \rangle$ . If  $H$  is a Hilbert space, we will also use  $\langle \cdot, \cdot \rangle$  to denote the inner product. However, new notation will be introduced to differentiate the two whenever required. The following is a collection of results about ordered Banach spaces and positive operators from [1], [11] and [17, Appendix 2].

A real vector space  $X$  is called an ordered vector space if a partial ordering “ $\leq$ ” is defined in  $X$  with the additional property that  $x \leq y$  in  $X$  implies that  $x+z \leq y+z$  for all  $z \in X$  and  $\lambda x \leq \lambda y$  for all  $0 \leq \lambda \in \mathbb{R}$ . The positive cone of  $X$  is defined by  $X_+ = \{x \in X : x \geq 0\}$ . We say that a real Banach space  $(X, \|\cdot\|)$  is an ordered Banach space if  $X$  is an ordered vector space such that  $X_+$  is norm closed.

As we will see in our discussion on positive operators later in this section, it will generally be sufficient for us to work in real spaces. So henceforth, an ordered Banach space will always be a real space unless stated otherwise. However, it will sometimes be necessary to work in complex spaces, for example, when applying spectral theory. In order to move between real and complex spaces, we will apply the process of complexification, which we will describe next.

Let  $(X, \|\cdot\|)$  be a real Banach space. Define a linear structure on the complex

space  $X \times X$  by

$$\begin{aligned}(x, y) + (u, v) &= (x + u, y + v) \quad \text{for all } x, y, u, v \in X \\ (\alpha + i\beta)(x, y) &= (\alpha x - \beta y, \beta x + \alpha y) \quad \text{for all } x, y \in X, \alpha, \beta \in \mathbb{R}.\end{aligned}$$

It is easy to see that  $x \mapsto (x, 0)$  is a real, linear isomorphism from  $X$  onto a real, linear subspace of  $X \times X$  and so  $X_C := X \times X = X \oplus iX$ . We call  $X_C$  the *complexification* of  $X$ . If we equip  $X_C$  with the norm

$$\|x + iy\|_C = \sup_{0 \leq \theta \leq 2\pi} \|(\cos \theta)x + (\sin \theta)y\|, \quad x, y \in X,$$

then  $X_C$  is a Banach space. This follows since  $\|x\|, \|y\| \leq \|x + iy\|_C \leq \|x\| + \|y\|$ . In many cases, there is even a norm  $\|\cdot\|_{\tilde{C}}$  that makes  $X_C$  a Banach space with its natural norm and  $\|x\|_{\tilde{C}} = \|x\|$  for all  $x \in X$ . Such a norm is always equivalent to  $\|\cdot\|_C$  [17, p.268]. Finally, note that if  $X$  is a real ordered Banach space, we define the corresponding partial order on  $X_C$  by  $x \in (X_C)_+$  if and only if  $x \in X_+$ . In fact, we will say that a complex Banach space is an ordered Banach space if and only if it is the complexification of a real Banach space.

Now let us return to our discussion of (real) ordered Banach spaces. The norm on the ordered Banach space  $X$  is called  $M$ -monotone if there exists a constant  $M \geq 0$  such that  $0 \leq x \leq y$  implies that  $\|x\| \leq M\|y\|$ . If  $M = 1$ , then the norm is simply called monotone. The following result characterises ordered spaces with monotone norm.

**Proposition 1.1.1.** [11, Theorem 1.2.3] *Let  $X$  be an ordered Banach space. The following conditions are equivalent:*

- (i)  $\|\cdot\|_X$  is monotone.
- (ii) for each  $x \in X_+$  there is an  $x^* \in X_+^*$  such that  $\|x^*\|_{X^*} = 1$  and  $\langle x^*, x \rangle = \|x\|_X$ .

The positive cone of  $X$  is called generating if  $X = X_+ - X_+$ . By a Baire Category argument, if  $X_+$  is generating, there exists a constant  $M > 0$  such that each  $x \in X$  has a decomposition  $x = x_1 - x_2$  with  $x_i \in X_+$  and  $\|x_i\| \leq M\|x\|$ ,  $i = 1, 2$  [11, Proposition 1.1.2]. Given  $x, y \in X$ , we denote by  $[x, y]$  the order interval determined by  $x$  and  $y$ , i.e.  $[x, y] = \{z \in X : x \leq z \leq y\}$ . The cone  $X_+$  is normal if all order intervals are bounded. Examples of spaces with normal, generating cone are  $L^p$ -spaces and the self-adjoint part of  $C^*$ -algebras. The following proposition gives some alternative characterisations of ordered spaces with normal cones.

**Proposition 1.1.2.** [11, p.229, Proposition 1.2.1] *Let  $X$  be an ordered Banach space. The following are equivalent:*

- (i)  $X_+$  is normal.
- (ii) The norm is  $M$ -monotone for some  $M \geq 1$ .
- (iii)  $X$  has an equivalent monotone norm.

A class of ordered Banach spaces which we will be interested in is the class of Banach lattices. Let  $X$  be an ordered vector space. If for any two elements  $x, y \in X$ ,  $x \wedge y$  and  $x \vee y$  exist, then  $X$  is called a vector lattice (or Riesz space). Let  $x_+ = x \vee 0$ ,  $x_- = (-x) \vee 0$  and  $|x| = x \vee (-x)$ . Then  $x = x_+ - x_-$  and  $|x| = x_+ + x_-$ . A norm on the vector lattice  $X$  is called a Riesz norm if  $|x| \leq |y|$  implies  $\|x\| \leq \|y\|$ . A vector lattice space  $X$  is a Banach lattice if  $X$  is complete with respect to the Riesz norm.

A set of Banach lattices which will play an important role in this thesis are  $AL$ -spaces, i.e. Banach lattices  $X$  which satisfy

$$\|x + y\| = \|x\| + \|y\| \quad \text{for all } x, y \in X_+. \quad (1.1)$$

It can be shown that every  $AL$ -space is lattice isometric to an  $L^1(\Omega, \mu)$  space for some measure space  $(\Omega, \mu)$  [1, Theorem 4.27]. Another class of Banach lattices which will arise later are Kantorovich-Banach spaces ( $KB$ -spaces), which are defined as Banach lattices in which any non-negative, increasing and norm-bounded sequence is norm-convergent. These spaces include the  $L^p$ -spaces,  $1 \leq p < \infty$ , and in fact, all reflexive lattices. For more information on these spaces, see for example [1, pp. 232-238].

Finally, we introduce a class of ordered Banach spaces which we will call the class of abstract state spaces, following [3]. An abstract state space is a real ordered Banach space  $X$  with a generating positive cone  $X_+$  on which the norm is additive, i.e. (1.1) holds. The additivity of the norm implies that the norm is monotone, i.e.  $0 \leq x \leq y \Rightarrow \|x\| \leq \|y\|$ . In particular, the cone is normal (Proposition 1.1.2). It then follows that any norm-bounded monotone sequence of  $X_+$  is convergent.

The additivity of the norm on the positive cone allows us to extend the norm on the positive cone to a linear functional,  $\Psi$ , on  $X$ . More precisely,

$$\Psi : X \rightarrow \mathbb{R}, \quad \langle \Psi, x \rangle = \|x\|, \quad x \in X_+. \quad (1.2)$$

Note that  $\|\Psi\| = 1$ . This follows since given  $x \in X$  with  $x = x_1 - x_2$ ,  $x_1, x_2 \in X_+$ , we have  $|\langle \Psi, x \rangle| = |\|x_1\| - \|x_2\|| \leq \|x\|$ . This inequality combined with (1.2) gives us the result.

We now move on to operator theory. First, let us discuss some notation. If  $T$  is an operator on a Banach space  $X$ , we will use  $D(T)$  to denote its domain. We will also use  $\rho(T)$  to denote the resolvent set of  $T$ ,  $\sigma(T)$  to denote the spectrum of  $T$  and  $\sigma_p(T)$ ,  $\sigma_c(T)$ ,  $\sigma_r(T)$  to denote the point, continuous and residual spectrum of  $T$  respectively. We will use  $r_\sigma(T)$  to denote the spectral radius of  $T$  and  $s(T)$  to denote the spectral bound, i.e.  $\sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(T)\}$ . Finally, we use  $R(\lambda, T) := (\lambda - T)^{-1}$ ,  $\lambda \in \rho(T)$  to denote the resolvent operator of the operator  $T$ .

Now we present some information on positive operators. Suppose  $X$  and  $Y$  are ordered Banach spaces. A linear operator  $T$  from  $X$  into  $Y$  is called positive if  $T(X_+) \subseteq Y_+$ . We will often use  $T \geq 0$  to denote that  $T$  is a positive operator. Additionally, if  $S, T$  are positive operators, we will say that  $S \leq T$  if  $Sx \leq Tx$  for all  $x \in X_+$ . In many cases, positive linear operators are automatically bounded.

**Proposition 1.1.3.** *[17, Proposition A.2.11] If  $X_+$  is generating and  $Y_+$  is normal, then every positive linear operator  $T$  from  $X$  into  $Y$  is bounded.*

The next result is a perturbation result related to operators satisfying a different notion of positivity: We call an (unbounded) operator  $A$  in  $X$  resolvent positive if there exists  $w \in \mathbb{R}$  such that  $(w, \infty) \subset \rho(A)$  and  $R(\lambda, A) \geq 0$  for all  $\lambda > w$ .

**Theorem 1.1.4.** *[47, Theorem 1.1] Let  $X$  be an ordered Banach space with generating and normal cone. Let  $A$  be a resolvent positive operator in  $X$  and  $\lambda > s(A)$ . Let  $B : D(A) \rightarrow X$  be a positive operator. The following are equivalent:*

- (i)  $r_\sigma(BR(\lambda, A)) < 1$ .
- (ii)  $\lambda \in \rho(A + B)$  and  $R(\lambda, A + B) \geq 0$ .

*If one of these conditions is satisfied, then  $A + B$  is resolvent positive,  $s(A + B) < \lambda$  and for all  $x \in X$ ,*

$$R(\lambda, A + B)x = R(\lambda, A) \sum_{k=0}^{\infty} (BR(\lambda, A))^k x.$$

Finally, we justify why it is generally sufficient to work in real spaces when working with positive operators. Let us first introduce the process of complexification of operators.

Let  $X, Y$  be real Banach spaces and  $T : D(T) \subset X \rightarrow Y$ . Then  $T$  can be extended to  $X_C$  by

$$T_C(x + iy) = Tx + iTy, \quad x, y \in X, \quad D(T_C) = D(T) + iD(T).$$

In particular, if  $T \in \mathcal{L}(X, Y)$  with operator norm  $\|T\|$ , then  $\|T\| = \|T_C\|$  where  $\|T_C\|$  is the operator norm on  $\mathcal{L}(X_C, Y_C)$ . In fact, the mapping  $T \mapsto T_C$  is a real linear isomorphism from  $\mathcal{L}(X, Y)$  into  $\mathcal{L}(X_C, Y_C)$  and hence  $\mathcal{L}(X_C, Y_C)$  is the complexification of  $\mathcal{L}(X, Y)$  as a vector space. In general however, the operator norm of  $\mathcal{L}(X_C, Y_C)$  is not the complexification of the operator norm on  $\mathcal{L}(X, Y)$ .

Now recall that if  $X$  is a real ordered Banach space, we define the corresponding partial order on  $X_C$  by  $x \in (X_C)_+$  if and only if  $x \in X_+$ . Hence, if  $X$  has generating positive cone, any positive linear operator  $T$  on  $X_C$  is a real operator, i.e.  $T : X \rightarrow X$ . Therefore when dealing with positive operators, it suffices to consider real Banach spaces.

## 1.2 Semigroup Theory

As stated in the introduction, the main objects studied in this thesis are substochastic semigroups. We will mostly be interested in strongly continuous ( $C_0$ -)semigroups and will omit the  $C_0$  term whenever the assumption is obvious. The strong limit of operators will always be denoted by s-lim. Detailed information about  $C_0$ -semigroups can be found in [22], [37] or [2]. More specifically, for positive semigroups, see [9] or [17, Section III].

We begin with a basic definition of the generator of a  $C_0$ -semigroup:

**Definition 1.2.1.** [22, Definition II.1.2] *The generator  $A : D(A) \subset X \rightarrow X$  of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$  is the operator*

$$Ax := \lim_{h \downarrow 0} \frac{1}{h} (T(h)x - x)$$

*defined for every  $x$  in its domain*

$$D(A) := \{x \in X : \text{the map } t \mapsto T(t)x \text{ is differentiable}\}.$$

We are now ready to discuss substochastic semigroups. Let  $X$  be an ordered Banach space. A linear operator  $T$  in  $X$  is called substochastic (resp. stochastic) if  $T$  is positive and  $\|Tx\| \leq \|x\|$  (resp.  $\|Tx\| = \|x\|$ ) for all  $x \in X_+$ . Similarly, a semigroup  $(U(t))_{t \geq 0}$  in  $X$  is called substochastic (resp. stochastic) if  $U(t)$  is positive for all  $t \geq 0$  and  $\|U(t)x\| \leq \|x\|$  (resp.  $\|U(t)x\| = \|x\|$ ) for all  $x \in X_+$ ,  $t \geq 0$ . If  $X$  is a Banach lattice, for example an  $AL$ -space, this is equivalent to saying that  $(U(t))_{t \geq 0}$  is a semigroup of positive contractions. However, this generally does not hold in ordered Banach spaces with normal, generating cone but without lattice structure.

Instead, we can show that  $\|U(t)\| \leq 2M$  for all  $t \geq 0$  where  $M$  is the constant from the generating cone condition. Finally, we observe that a semigroup is positive if and only if its generator is resolvent positive [2, p.191]. In particular, if  $(U(t))_{t \geq 0}$  is a substochastic semigroup with generator  $A$ , then  $R(\lambda, A) \geq 0$  for all  $\lambda > 0$ .

One of the fundamental generation theorems for  $C_0$ -semigroups is the Hille-Yosida Theorem:

**Theorem 1.2.2** (Hille-Yosida). [22, Theorem II.3.8] *Let  $A$  be a linear operator on a Banach space and let  $w \in \mathbb{R}, M \geq 1$  be constants. The following are equivalent:*

(i)  *$A$  generates a strongly continuous semigroup  $(T(t))_{t \geq 0}$  satisfying*

$$\|T(t)\| \leq Me^{wt} \quad \text{for } t \geq 0.$$

(ii)  *$A$  is closed, densely defined and for every  $\lambda > w$ , we have  $\lambda \in \rho(A)$  and*

$$\|[(\lambda - w)R(\lambda, A)]^n\| \leq M \quad \text{for all } n \in \mathbb{N}.$$

(iii)  *$A$  is closed, densely defined and for every  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > w$ , we have  $\lambda \in \rho(A)$  and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}(\lambda) - w)^n} \quad \text{for all } n \in \mathbb{N}.$$

As we work with substochastic semigroups, we will also require an important generation theorem for  $C_0$ -semigroups of contractions. Recalling that we say that an operator  $A$  is dissipative if  $\|(\lambda - A)x\| \geq \|x\|$  for all  $\lambda > 0$ , all  $x \in D(A)$ , we have

**Theorem 1.2.3** (Lumer-Phillips). [22, Theorem II.3.15] *For a densely defined, dissipative operator  $A$  on a Banach space  $X$ , the following statements are equivalent:*

(i) *The closure  $\bar{A}$  of  $A$  generates a contraction semigroup.*

(ii)  *$\operatorname{Im}(\lambda - A)$  is dense in  $X$  for some (and hence all)  $\lambda > 0$ .*

Recall that in Section 1.1, we saw that it suffices to consider real Banach spaces when working with positive operators. Although the Hille-Yosida and Lumer-Phillips Theorems are stated for complex Banach spaces, we can see that they also hold in the real setting (see also [2, Section 3.3]).

We also have generation theorems specifically for substochastic semigroups such as Lemma 1.2.4 below whose proof is essentially like the proof of the Hille-Yosida Theorem but with additional positivity conditions.

**Lemma 1.2.4.** *Let  $X$  be an ordered Banach space. An operator  $A$  on  $X$  with dense domain generates a substochastic (resp. stochastic) semigroup if and only if for every  $\lambda > 0$ ,  $A$  has a resolvent  $R(\lambda, A)$  with domain  $X$  and  $\lambda R(\lambda, A)$  is a substochastic (resp. stochastic) operator.*

Using this lemma we can derive another generation result for substochastic semigroups on ordered spaces with monotone norm.

**Proposition 1.2.5.** *Let  $X$  be an ordered Banach space with monotone norm. A linear operator  $A$  with dense domain generates a substochastic semigroup on  $X$  if and only if*

- (i) *for any  $x \in D(A)_+$ , there is an  $x^* \in X_+^*$  such that  $\|x^*\|_{X^*} = 1$ ,  $\langle x^*, x \rangle = \|x\|_X$  and  $\langle x^*, Ax \rangle \leq 0$ , and*
- (ii) *for each  $\lambda > 0$  and  $x \in X$ , the equation*

$$\lambda y - Ay = x \tag{1.3}$$

*has a unique solution  $y = R(\lambda, A)x \in D(A)$  and  $R(\lambda, A)x \in X_+$  for all  $x \in X_+$ .*

*In particular, if  $X$  is an abstract state space, then  $x^*$  in condition (i) can be taken to be the functional  $\Psi$  for all  $x \in D(A)_+$ .*

*Proof.* The necessity follows directly from Lemma 1.2.4 and the fact that if  $A$  generates a substochastic semigroup  $(T(t))_{t \geq 0}$ , then for each  $x \in D(A)_+$  and  $x^*$  as in condition (ii) of Proposition 1.1.1,

$$\langle x^*, Ax \rangle = \lim_{t \rightarrow 0} \left\langle x^*, \frac{T(t)x - x}{t} \right\rangle \leq \lim_{t \rightarrow 0} \frac{1}{t} (\|T(t)x\| - \|x\|) \leq 0.$$

The sufficiency follows since for  $x \in X_+$  and  $\lambda > 0$ , we have

$$\begin{aligned} \|\lambda R(\lambda, A)x\| &= \langle (R(\lambda, A)x)^*, \lambda R(\lambda, A)x \rangle \\ &= \langle (R(\lambda, A)x)^*, x \rangle + \langle (R(\lambda, A)x)^*, AR(\lambda, A)x \rangle \\ &\leq \langle (R(\lambda, A)x)^*, x \rangle \leq \|x\| \end{aligned}$$

i.e.  $\|\lambda R(\lambda, A)x\| \leq \|x\|$ . Hence by Lemma 1.2.4,  $A$  generates a substochastic semigroup. The final assertion follows since the functional  $\Psi$  is positive and satisfies  $\langle \Psi, x \rangle = \|x\|$  for all  $x \in X_+$  by definition (see (1.2)) and  $\|\Psi\|_{X^*} = 1$ .  $\square$

Apart from existence of semigroups via explicit generation theorems, existence via approximation will also turn out to be important in this thesis. In particular, the Trotter-Kato Approximation Theorem given below will prove useful.

**Theorem 1.2.6** (Trotter-Kato). *[22, Theorem III.4.9] Let  $\mathfrak{G}(M, \omega)$ ,  $M \geq 1, \omega \in \mathbb{R}$  be the set of generators of  $C_0$ -semigroups  $(U(t))_{t \geq 0}$  satisfying  $\|U(t)\| \leq Me^{\omega t}$ . Suppose  $(A_n) \subset \mathfrak{G}(M, \omega)$ . For some  $\lambda_0 > \omega$  consider the following assertions:*

- (i) *There exists a densely defined operator  $A$  such that  $A_n x \rightarrow Ax$  for all  $x$  in a core  $D$  of  $A$  and such that  $\overline{\text{Im}(\lambda_0 - A)} = X$ .*
- (ii) *The operators  $R(\lambda_0, A_n)$  converge strongly to an operator  $R \in \mathcal{L}(X)$  which has dense range.*
- (iii) *The semigroups  $(U_n(t))_{t \geq 0}$ ,  $n \in \mathbb{N}$  converge strongly (and uniformly for  $t$  on bounded intervals) to a  $C_0$ -semigroup  $(U(t))_{t \geq 0}$  with generator  $B$  such that  $R = R(\lambda_0, B)$ .*

*Then the implications (i)  $\Rightarrow$  (ii)  $\Leftrightarrow$  (iii) hold. In particular, if (i) holds, then  $B = \overline{A}$ .*

The proof of the Trotter-Kato Theorem ([9, Theorem 3.43], see also [22, Proposition III.4.4]) yields an important fact which we will apply in Section 3.1:

**Proposition 1.2.7.** *Suppose  $(A_n) \subset \mathfrak{G}(M, \omega)$  for some  $M \geq 1, \omega \geq 0$ . Then the set  $\mathcal{S} := \{\mu : \mu > \omega, \text{s-lim}_{n \rightarrow \infty} R(\mu, A_n) \text{ exists}\}$  is either empty or  $\mathcal{S} = (\omega, \infty)$ .*

Another important formula for approximating semigroups is the Post-Widder Inversion Formula given in Proposition 1.2.8 below, which allows us to describe a semigroup in terms of its resolvents:

**Proposition 1.2.8.** *[22, Corollary III.5.5] Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on a Banach space  $X$  with generator  $A$ . Then*

$$T(t)x = \lim_{n \rightarrow \infty} \left[ \frac{n}{t} R\left(\frac{n}{t}, A\right) \right]^n x, \quad x \in X$$

*uniformly for  $t$  in compact intervals.*

Finally, as adjoint or dual semigroups will play an important role in Chapter 5, we give some preliminary information here, most of which comes from [43]. Let  $(U(t))_{t \geq 0}$  be a  $C_0$ -semigroup on a Banach space  $X$  with generator  $A$ . The adjoint semigroup  $(U^*(t))_{t \geq 0}$  is the semigroup on the dual space  $X^*$  defined by  $U^*(t) = (U(t))^*$  for all  $t \geq 0$ . Elementary calculations show that  $(U^*(t))_{t \geq 0}$  is a weak\*-continuous semigroup

but is not necessarily strongly continuous [43, p.3]. It can be shown however, that  $(U^*(t))_{t \geq 0}$  is strongly continuous if  $X$  is a reflexive space [22, Proposition I.5.14].

Let  $A^*$  denote the adjoint of  $A$ . Since  $A$  is closed and densely defined,  $A^*$  exists, is weak\*-densely defined and weak\*-closed. Moreover,  $A^*$  is the weak\* generator of  $(U^*(t))_{t \geq 0}$  [43, Theorem 1.2.3], i.e. for all  $x \in D(A^*)$ ,

$$A^*x = w^* \text{-}\lim_{t \rightarrow 0} \frac{U^*(t)x - x}{t} \text{ and } D(A^*) = \left\{ x \in X^* : w^* \text{-}\lim_{t \rightarrow 0} \frac{U^*(t)x - x}{t} \text{ exists} \right\}.$$

We will require one more property of  $A^*$ , but first, we need the definition of the weak\* integral.

Let  $(\Omega, \Sigma, \mu)$  be a finite measure space and  $X^*$  be a dual space. Let  $f : \Omega \rightarrow X^*$ . Then  $f$  is weak\*-measurable if the function  $\langle f(\cdot), x \rangle$  is measurable for each  $x \in X$ . Now suppose  $f : \Omega \rightarrow X^*$  is weak\*-measurable and suppose further that for each  $x \in X$ , the function  $\langle f(\cdot), x \rangle \in L^1(\Omega, \mu)$ . For  $E \in \Sigma$ , the map  $x_E^*$  defined by

$$\langle x_E^*, x \rangle = \int_E \langle f(\cdot), x \rangle d\mu$$

is called the weak\*-integral of  $f$  over  $E$  with respect to  $\mu$ . We will denote  $x_E^*$  by  $\text{weak}^* \int_E f d\mu$ . For more details, see [43, Appendix A.2].

**Proposition 1.2.9.** [43, Proposition 1.2.2] *Let  $(U(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$  with generator  $A$  and  $(U^*(t))_{t \geq 0}$  be its adjoint semigroup. Then  $\text{weak}^* \int_0^t U^*(s)x^* ds \in D(A^*)$  for all  $t > 0$  and  $x^* \in X^*$ , and*

$$A^* \left( \text{weak}^* \int_0^t U^*(s)x^* ds \right) = U^*(t)x^* - x^*.$$

### 1.3 Sesquilinear Forms

The theory of sesquilinear forms, which we will sometimes abbreviate as forms, is closely related to the theory of semigroups and will play an important role in Chapter 4. In this section, we will present a summary of important facts about forms, taken mostly from [20, Chapter 1], [36, Chapters 1 and 2] and [13, Chapter I]. Let us begin by clarifying some notation. We will use  $\mathfrak{H}$  to denote a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  throughout this section. Since forms are defined on inner product spaces, we need a different notion of positivity of an operator. We will say that an operator,  $T$  on  $D(T) \subseteq \mathfrak{H}$  is  $\mathfrak{H}$ -positive if  $\langle Tu, u \rangle \geq 0$  for all  $u \in D(T)$ .

Let  $\mathfrak{V}$  be a linear subspace of a Hilbert space  $\mathfrak{H}$  over  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$ . A sesquilinear form on  $\mathfrak{V}$  is a sesquilinear map  $Q : \mathfrak{V} \times \mathfrak{V} \rightarrow \mathbb{K}$ . We call  $\mathfrak{V}$  the domain of  $Q$  and denote it by  $D(Q)$ . We will only consider densely defined forms in this thesis.

First let us consider the case of bounded or continuous sesquilinear forms.

**Definition 1.3.1.** *A sesquilinear form  $Q : \mathfrak{H} \times \mathfrak{H} \rightarrow \mathbb{K}$  is bounded (or continuous) if there exists a constant  $M$  such that*

$$|Q(u, v)| \leq M \|u\| \|v\| \text{ for all } u, v \in \mathfrak{H}.$$

Bounded forms are associated with a unique bounded operator. More precisely,

**Proposition 1.3.2.** *[36, Proposition 1.2] A sesquilinear form  $Q : \mathfrak{H} \times \mathfrak{H} \rightarrow \mathbb{K}$  is bounded if and only if there exists a unique bounded linear operator  $T$  acting on  $\mathfrak{H}$  such that*

$$Q(u, v) = \langle Tu, v \rangle \text{ for all } u, v \in \mathfrak{H}.$$

*Remark 1.3.3.* This one-to-one correspondence holds in fact, for densely defined, bounded forms, i.e. forms satisfying  $|Q(u, v)| \leq M \|u\| \|v\|$  for all  $u, v \in D(Q)$  which is dense in  $\mathfrak{H}$ .

We will mostly be interested in positive symmetric forms in a real Hilbert space, i.e. forms which satisfy  $Q(u) := Q(u, u) \geq 0$  for all  $u \in D(Q)$  and  $Q(u, v) = Q(v, u)$  for all  $u, v \in D(Q)$ . So in the rest of this section, we will assume that  $\mathfrak{H}$  is a real Hilbert space and  $Q$  is a positive symmetric form unless stated otherwise.

We say  $Q$  is closed if for all sequences  $(u_n) \in D(Q)$  such that  $\lim_{n \rightarrow \infty} \|u_n - u\| = 0$  and  $\lim_{m, n \rightarrow \infty} Q(u_m - u_n) = 0$ , it follows that  $u \in D(Q)$  and  $\lim_{n \rightarrow \infty} Q(u_n - u) = 0$ . A form is said to be closable if it has a closed extension and the closure  $\overline{Q}$  is then the smallest closed extension. Now consider the form defined by

$$\langle u, v \rangle_Q := \langle u, v \rangle + Q(u, v), \quad u, v \in D(Q).$$

It can be shown that this is an inner product on  $D(Q)$ . We will define the corresponding form-norm by

$$\|u\|_Q = (\|u\|^2 + Q(u, u))^{\frac{1}{2}}, \quad u \in D(Q).$$

Then  $Q$  is closed if and only if  $D(Q)$  is complete under  $\|\cdot\|_Q$ .

This now allows us to define a weaker notion of continuity for densely defined forms. We will say that the form  $Q$  is continuous on  $D(Q)$  if there exists  $M > 0$  such that

$$|Q(u, v)| \leq M \|u\|_Q \|v\|_Q \text{ for all } u, v \in D(Q).$$

Note that if  $Q$  is a positive symmetric form, then it follows from the Cauchy-Schwarz inequality that  $|Q(u, v)| \leq \|u\|_Q \|v\|_Q$  for all  $u, v \in D(Q)$  so  $Q$  is continuous on  $D(Q)$  [36, Proposition 1.8].

Now recall from Proposition 1.3.2 that bounded forms are associated with a unique bounded operator. We can define an analogous notion for closed, densely defined forms. Let  $Q$  be a closed form on  $\mathfrak{H}$  with dense domain  $D(Q)$ . We associate an operator  $A$  with  $Q$ , given by

$$\begin{aligned} D(A) &= \{u \in D(Q) : \text{there exists } v \in \mathfrak{H} \text{ such that } Q(u, \phi) = \langle v, \phi \rangle \text{ for all } \phi \in D(Q)\}, \\ Au &= v. \end{aligned} \tag{1.4}$$

Note that  $v$  is well-defined since  $D(Q)$  is dense in  $\mathfrak{H}$ . We call  $A$  the operator associated with  $Q$ .

Conversely, let  $A$  be an operator on  $\mathfrak{H}$ . We say that  $A$  is induced by a form if there exists a densely defined, closed form  $Q$  on  $\mathfrak{H}$  such that  $A$  is associated with  $Q$ . Note that for the case when  $A$  is  $\mathfrak{H}$ -positive and self-adjoint,  $A$  is induced by the form

$$Q(u) = \langle A^{1/2}u, A^{1/2}u \rangle, \quad u \in D(A^{1/2}). \tag{1.5}$$

It turns out that this characterises closed, symmetric, positive forms:

**Proposition 1.3.4.** [20, Theorem 1.2.1] *If  $Q$  is a positive, symmetric form on  $\mathfrak{H}$  with domain  $D(Q)$ , the following are equivalent:*

- (i)  *$Q$  is the form of a  $\mathfrak{H}$ -positive, self-adjoint operator  $A$  in the sense of (1.5) and  $D(Q) = D(A^{1/2})$ .*
- (ii)  *$Q$  is closed.*

Next, we look at the relationship between symmetric forms and semigroups. By [22, Proposition II.3.27], a densely defined, self-adjoint operator  $A$  on  $\mathfrak{H}$  generates a  $C_0$ -semigroup if and only if there is  $C \in \mathbb{R}$  such that  $\langle Au, u \rangle \leq C \|u\|^2$  for all  $u \in D(A)$ . So if  $A$  is a  $\mathfrak{H}$ -positive, self-adjoint operator associated with a closed form  $Q_A$ , then  $-A$  generates a  $C_0$ -semigroup of contractions on  $\mathfrak{H}$  denoted by  $(e^{-tA})_{t \geq 0}$ . We are most interested in the special case when  $-A$  generates a substochastic semigroup in  $L^2$  that can be extended to all  $L^p$ -spaces. The classical results of Beurling and Deny given in Theorems 1.3.5 and 1.3.6 characterises when this occurs.

**Theorem 1.3.5.** [20, Theorem 1.3.2] *Let  $A$  be a  $\mathfrak{H}$ -positive self-adjoint operator on the real space  $\mathfrak{H} := L^2(\Omega, \mu)$ . The following are equivalent:*

- (i)  $0 \neq u \in D(Q_A) \Rightarrow |u| \in D(Q_A)$ . Also,  $u \in D(A), f \in D(Q_A)$  and  $f \geq 0$  imply  $\langle A^{1/2}f, A^{1/2}|u| \rangle \leq \left\langle f, \frac{u}{|u|}(Au) \right\rangle$ .
- (ii)  $u \in D(Q_A) \Rightarrow |u| \in D(Q_A)$  and  $\langle A^{1/2}|u|, A^{1/2}|u| \rangle \leq \langle A^{1/2}u, A^{1/2}u \rangle$ .
- (iii)  $R(\lambda, -A)$  is positive for all  $\lambda > 0$ .
- (iv)  $e^{-tA}$  is positive for all  $t \geq 0$ .

Although  $(e^{-tA})_{t \geq 0}$  is defined on  $L^2$ , it turns out that under certain conditions,  $(e^{-tA})_{t \geq 0}$  can be extended to all  $L^p$  for  $1 \leq p \leq \infty$  in the sense that for fixed  $t \geq 0$ ,  $e^{-tA}$  maps  $L^2 \cap L^p$  into  $L^2 \cap L^p$  and so can be extended to a contraction on  $L^p$ . In this case, we have a set of compatible semigroups on  $L^p$ , strongly continuous for  $1 \leq p < \infty$ , where a set of operators  $(A_p)$  acting on the spaces  $X_p$  respectively will be said to be compatible if the operators coincide on the intersection of the spaces.

**Theorem 1.3.6.** [20, Theorem 1.3.3] *Suppose  $A$  is an operator which satisfies the conditions of Theorem 1.3.5. The following are equivalent:*

- (i)  $e^{-tA}$  is a contraction on  $L^\infty$  for all  $t \geq 0$ .
- (ii)  $e^{-tA}$  is a contraction on  $L^p$  for all  $1 \leq p \leq \infty$  and  $t \geq 0$ .
- (iii) Let  $f \in D(Q_A)$  and let  $g \in L^2$  satisfy

$$|g(x)| \leq |f(x)|, \quad |g(x) - g(y)| \leq |f(x) - f(y)|$$

for all  $x, y \in \Omega$ . Then  $g \in D(Q_A)$  and  $Q(g) \leq Q(f)$ .

- (iv) If  $0 \leq f \in D(Q_A)$ , then  $f \wedge 1$  lies in  $D(Q_A)$  and  $Q(f \wedge 1) \leq Q(f)$ .

If  $A$  satisfies the conditions of Theorems 1.3.5 and 1.3.6, we will say that  $A$  is associated with a Dirichlet form  $Q_A$  and denote its associated semigroup extensions to the spaces  $L^p$  by  $(e^{-tA_p})_{t \geq 0}$  with generators  $-A_p$ .

Note that Theorems 1.3.5 and 1.3.6 describe the conditions for extending a substochastic semigroup on  $L^2$  to all  $L^p$ . Our next result (Proposition 1.3.8) gives a partial converse where we give conditions for extending a substochastic semigroup on  $L^1$  to a substochastic semigroup on  $L^2$  associated with a Dirichlet form. We will require the classical Riesz-Thorin Interpolation Theorem on interpolation of operators:

**Theorem 1.3.7** (Riesz-Thorin). [20, Theorem 1.1.5] Let  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$  and let  $S$  be a linear operator on  $L^{p_0} \cap L^{p_1}$  to  $L^{q_0} + L^{q_1}$  which satisfies  $\|Sf\|_{q_i} \leq M_i \|f\|_{p_i}$  for all  $f$  and  $i = 0, 1$ . Let  $0 < t < 1$  and define  $p, q$  by

$$\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_0}, \quad \frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_0}.$$

Then

$$\|Sf\|_q \leq M_1^t M_0^{1-t} \|f\|_p$$

for all  $f \in L^{p_0} \cap L^{p_1}$ . Hence  $S$  can be extended to a bounded operator from  $L^p$  to  $L^q$  with norm at most  $M_1^t M_0^{1-t}$ .

**Proposition 1.3.8.** Suppose  $T$  generates a positive  $C_0$ -semigroup of contractions  $(U(t))_{t \geq 0}$  on  $L^1$  and satisfies  $T = T^*$  on  $D(T) \cap D(T^*)$  which is dense in  $L^1 \cap L^\infty$ . Then  $(U(t))_{t \geq 0}$  can be extended to a set of compatible semigroups on  $L^p$ ,  $1 \leq p \leq \infty$ , denoted  $(U^{(p)}(t))_{t \geq 0}$ , which are strongly continuous for  $1 \leq p < \infty$ .

Moreover,  $(U^{(2)}(t))_{t \geq 0}$ , the semigroup on  $L^2$ , is associated with a Dirichlet form and the set of compatible semigroups associated with  $(U^{(2)}(t))_{t \geq 0}$  via the theory of Dirichlet forms coincides with the set  $(U^{(p)}(t))_{t \geq 0}$  extended from  $(U(t))_{t \geq 0}$ .

*Proof.* Note first that if you take  $p_0 = q_0 = 1$ ,  $p_1 = q_1 = \infty$  in the Riesz-Thorin Interpolation Theorem (Theorem 1.3.7), then any operator  $S$  which satisfies the conditions in Theorem 1.3.7, can be extended to  $L^p$  for all  $1 < p < \infty$ .

Fix  $t \geq 0$ . By assumption, we have  $U(t) : L^1 \cap L^\infty \subseteq L^1 \rightarrow L^1 \subseteq L^1 + L^\infty$  and  $\|U(t)f\|_1 \leq \|f\|_1$  for all  $f \in L^1 \cap L^\infty$ . By duality, we also have  $\|U^*(t)f\|_\infty \leq \|f\|_\infty$  for all  $f \in L^1 \cap L^\infty$ . Since  $T = T^*$  on  $D(T) \cap D(T^*)$  which is dense in  $L^1 \cap L^\infty$ , we have  $U^*(t) = U(t)$  on  $L^1 \cap L^\infty$  and so it follows that  $\|U(t)f\|_\infty \leq \|f\|_\infty$  for all  $f \in L^1 \cap L^\infty$ .

Thus, by Riesz-Thorin, the semigroup  $(U(t))_{t \geq 0}$  can be extended to a set of compatible semigroups on  $L^p$ ,  $1 \leq p \leq \infty$ , (strongly continuous for  $1 \leq p < \infty$ ), denoted  $(U^{(p)}(t))_{t \geq 0}$  with generators  $T_p$ . The semigroups are also positive since  $(U(t))_{t \geq 0}$  is positive. Moreover, since  $U(t) = U^*(t)$  on  $L^1 \cap L^\infty$ , we have  $U^{(2)}(t) = U^{(2)*}(t)$  i.e. the semigroup  $(U^{(2)}(t))_{t \geq 0}$  is self-adjoint. Hence  $(U^{(2)}(t))_{t \geq 0}$  satisfies the conditions of Theorems 1.3.5 and 1.3.6 and so  $Q_{T_2}$ , the form associated with  $(U^{(2)}(t))_{t \geq 0}$  is a Dirichlet form. By the theory of Dirichlet forms (Theorem 1.3.6), we know that  $(U^{(2)}(t))_{t \geq 0}$  extends to a set of compatible semigroups on  $L^p$ . These semigroups coincide with the set interpolated from  $(U(t))_{t \geq 0}$  as they coincide on  $L^1 \cap L^\infty$ , which is a dense subset of  $L^p$ , for all  $1 \leq p < \infty$ .  $\square$

As a large part of this thesis will focus on substochastic semigroups on  $L^1$ -spaces, we will be particularly interested in results relating the properties of a Dirichlet form to its associated substochastic semigroup on  $L^1$ . In Lemmas 1.3.9 and 1.3.10, we will let  $Q_A$  be a Dirichlet form with associated operator  $A$  and compatible semigroups  $(e^{-tA_p})_{t \geq 0}$  with generators  $-A_p$ .

**Lemma 1.3.9.** *[13, Lemma I.4.2.1.1] The space  $D(A_1) \cap L^\infty$  is a dense subspace of  $D(Q_A)$  and for all  $f \in D(A_1) \cap L^\infty$ ,  $g \in D(Q_A) \cap L^\infty$ ,*

$$Q_A(f, g) = \langle A_1 f, g \rangle.$$

**Lemma 1.3.10.** *[13, Lemma I.4.2.2.1] Let  $k \in D(Q_A) \cap L^1$ . If there exists  $h \in L^1$  such that for all  $g \in D(Q) \cap L^\infty$ ,*

$$Q_A(g, k) = \langle g, h \rangle,$$

*then  $k \in D(A_1)$  and  $A_1 k = h$ .*

Finally, as we work with positive semigroups, we will require a result relating the domination of semigroups to the domains of their associated forms. Let  $Q_A, Q_B$  be closed, densely defined positive forms, which are continuous on their respective domains in the space  $\mathfrak{H} := L^2(\Omega, \mu)$  with associated operators  $A, B$  and associated semigroups  $(e^{-tA})_{t \geq 0}$ ,  $(e^{-tB})_{t \geq 0}$ . We say that  $(e^{-tA})_{t \geq 0}$  is dominated by  $(e^{-tB})_{t \geq 0}$  if and only if

$$|e^{-tA} f| \leq e^{-tB} |f| \quad \text{for all } f \in \mathfrak{H}.$$

**Proposition 1.3.11.** *[36, Proposition 2.23] Suppose  $(e^{-tA})_{t \geq 0}$  and  $(e^{-tB})_{t \geq 0}$  are positive semigroups on  $L^2(\Omega, \mu)$ . If  $(e^{-tA})_{t \geq 0}$  is dominated by  $(e^{-tB})_{t \geq 0}$ , then  $D(Q_A) \subseteq D(Q_B)$ .*

## 1.4 Mean Ergodicity

In this section, we will discuss the notion of mean ergodicity of operators, which will occur repeatedly later, for example in Sections 2.3.1, 3.2.1, 6.3 and 6.4. For an in-depth survey on mean ergodicity of operators and semigroups, see [33].

A bounded linear operator  $T$  on a Banach space  $X$  is called mean ergodic if  $A_n^T := \frac{1}{n} \sum_{k=0}^{n-1} T^k$  converges strongly in  $X$  i.e.  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k x$  exists for all  $x \in X$  and uniformly ergodic if  $(A_n^T)$  converges in operator norm. We say that  $T$  is power bounded if  $\sup_n \|T^n\| < \infty$  and Cesàro bounded if  $\sup_n \|A_n^T\| < \infty$ . Finally,

$T$  is quasi-compact if there exists a compact operator  $K$  and  $m \in \mathbb{N}$  such that  $\|T^m - K\| < 1$ .

A fundamental result on the mean ergodicity of operators is the Mean Ergodic Theorem given below. We first define the following sets:

$$F := F(T) = \{x \in X : Tx = x\}, \quad N := \{x - Tx : x \in X\} = (I - T)X$$

$$F_* := \{x \in X^* : T^*x = x\}, \quad N_* := (I - T^*)X^*$$

**Theorem 1.4.1** (The Mean Ergodic Theorem). *[33, Theorem 2.1.3] Suppose  $T$  is a Cesáro bounded operator on  $X$  and assume*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|T^n x\| = 0 \tag{1.6}$$

for all  $x \in X$ . Then

$$X_{me} := \{x \in X : \lim_{n \rightarrow \infty} A_n^T x \text{ exists}\} = F \oplus \overline{N}.$$

The operator  $P$  defined by  $Px := \lim_{n \rightarrow \infty} A_n^T x$  is the projection of  $X_{me}$  onto  $F$ . We have  $P = P^2 = TP = PT$ . For any  $x \in X$ , the following are equivalent:

- (i)  $\lim_{n \rightarrow \infty} A_n^T x = 0$ .
- (ii)  $\langle f, x \rangle = 0$  for all  $f \in F_*$ .
- (iii)  $x \in \overline{N}$ .

As a corollary of the Mean Ergodic Theorem we have:

**Corollary 1.4.2.** *[41, Corollary 2.2] Let  $T$  be a bounded operator on a Banach space  $X$ . Then  $T$  is mean ergodic and  $\ker(I - T) = \{0\}$  if and only if  $T$  is Cesáro bounded,  $\lim_{n \rightarrow \infty} \frac{1}{n} \|T^n x\| = 0$  for all  $x \in X$  and  $X = \overline{\text{Im}(I - T)}$ .*

A similar result holds for uniformly ergodic operators and may be derived from the Uniform Ergodic Theorem and its proof [33, Theorem 2.2.1].

**Theorem 1.4.3.** *[41, Theorem 2.4] Let  $T$  be a bounded linear operator on a Banach space  $X$ . Then  $T$  is uniformly ergodic if and only if  $\lim_{n \rightarrow \infty} \frac{1}{n} \|T^n\| = 0$  and  $\text{Im}(I - T)$  is closed. Moreover, if  $T$  is uniformly ergodic and  $\ker(I - T) = \{0\}$ , then  $1 \in \rho(T)$ .*

Another result on uniform ergodicity which we require can be derived from the Yosida-Kakutani Uniform Ergodic Theorem (see [33, Remarks p.92, Theorem 2.2.8]).

**Proposition 1.4.4.** *Let  $T$  be a bounded linear operator on a Banach space  $X$ . If  $T$  is power bounded and quasi-compact, then  $T$  is uniformly ergodic.*

We will also apply the continuous version of the Mean Ergodic Theorem which is stated here for bounded semigroups.

**Theorem 1.4.5** (Mean Ergodic Theorem for Semigroups). *[2, Proposition 4.3.1] Suppose  $A$  generates a bounded semigroup  $(T(t))_{t \geq 0}$  and let  $x \in X$ . The following are equivalent:*

- (i) *There exists  $\lambda_n \downarrow 0$  such that  $\lambda_n R(\lambda_n, A)x$  converges weakly as  $n \rightarrow \infty$ .*
- (ii)  *$x_0 := \lim_{\lambda \rightarrow 0} \lambda R(\lambda, A)x$  exists.*
- (iii)  *$x \in \ker(A) + \overline{\text{Im}(A)}$ .*
- (iv)  *$x_1 := \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T(s)x ds$  exists.*

*Moreover,  $x_0 \in \ker(A)$ ,  $x - x_0 \in \overline{\text{Im}(A)}$  and  $x_0 = x_1$ .*

## Chapter 2

# Kato's Theorem and Honesty on Abstract State Spaces

In this chapter, we will introduce Kato's Perturbation Theorem for substochastic semigroups and the honesty theory related to it. In the first section, we will begin by briefly describing the background of Kato's Theorem. We will however, concentrate on a generalisation of Kato's Theorem to abstract state spaces as this is the case that will be the main focus in this chapter. As mentioned in the introduction, the study of Kato's Theorem would not be complete without the study of honesty theory. Thus in Section 2.2, we will present the honesty theory for Kato's Theorem in abstract state spaces, which is the most general form of honesty theory known currently. In Section 2.3, we will look at some new characterisations of honesty in abstract state spaces and finally, conclude the chapter with a section on the preservation of honesty under changes to the semigroup via perturbations or restrictions.

### 2.1 Generalisations of Kato's Perturbation Theorem

One of the main generation theorems which addresses the issue of positive perturbations of substochastic semigroups is Kato's classical theorem. Kato first presented this theorem in [31] for the special case of semigroups derived from Kolmogorov differential equations in  $\ell^1$ . However, he also noted in this paper that the same theorem (and proofs) held in any  $AL$ -space. It was 30 years later when the next major modification to this theorem was published, namely in [46]. In this paper, Voigt proved Kato's Theorem for general  $L^1(\Omega, \mu)$  spaces by applying Miyadera's Perturbation Theorem, in contrast to Kato's original paper, where the proof was via resolvent estimates and the Hille-Yosida Theorem.

**Theorem 2.1.1** (Kato). [31, Theorem 1], [46, Section 1] Let  $(\Omega, \mu)$  be a measure space and  $X = L^1(\Omega, \mu)$ . Suppose the operators  $A$  and  $B$  with  $D(A) \subseteq D(B) \subseteq X$  satisfy:

- (i)  $A$  generates a substochastic semigroup  $(U_A(t))_{t \geq 0}$ ,
- (ii)  $Bx \geq 0$  for  $x \in D(A)_+ := D(A) \cap X_+$ ,
- (iii)  $\int_{\Omega} (A+B)x \, d\mu \leq 0$  for all  $x \in D(A)_+$ .

Then there exists an extension  $G$  of  $A+B$  that generates a substochastic  $C_0$ -semigroup  $(V(t))_{t \geq 0}$  on  $X$ .

Recently, Arlotti, Lods and Mokhtar-Kharroubi [3] introduced the notion of abstract state spaces as generalisations of  $AL$ -spaces to more general ordered Banach spaces (see Section 1.1 for the full definition). They were inspired by Davies, who used a variant of such spaces in [18, pp.30-31], motivated by applications in probability theory and quantum statistical mechanics. They were then able to prove an analogue of Kato's Theorem in abstract state spaces (Theorem 2.1.2). Note however, that this theorem was first proven by Voigt and Thieme as an auxiliary result in [39] without defining the class of abstract state spaces explicitly and via a different proof.

**Theorem 2.1.2.** [3, Theorem 2.1],[39, Theorem 2.2] Suppose  $X$  is an abstract state space and suppose that the operators  $A$  and  $B$  with  $D(A) \subseteq D(B) \subseteq X$  satisfy:

- (i)  $A$  generates a substochastic semigroup  $(U_A(t))_{t \geq 0}$ ,
- (ii)  $Bx \geq 0$  for  $x \in D(A)_+$ ,
- (iii)  $\langle \Psi, (A+B)x \rangle \leq 0$  for all  $x \in D(A)_+$ .

Then there exists an extension  $G$  of  $A+B$  that generates a substochastic  $C_0$ -semigroup  $(V(t))_{t \geq 0}$  on  $X$ . The generator  $G$  satisfies for all  $\lambda > 0$  and  $x \in X$ ,

$$R(\lambda, G)x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x. \quad (2.1)$$

Finally,  $(V(t))_{t \geq 0}$  is the minimal substochastic  $C_0$ -semigroup whose generator is an extension of  $(A+B, D(A))$  in the following sense: if  $(\tilde{V}(t))_{t \geq 0}$  is another substochastic  $C_0$ -semigroup whose generator is an extension of  $(A+B, D(A))$ , then  $V(t) \leq \tilde{V}(t)$  for all  $t \geq 0$ .

The generalisation of Kato's Theorem to abstract state spaces in [3] is based on the fact that abstract state spaces are generalisations of  $AL$ -spaces. (If  $X = L^1(\Omega, \mu)$ , then  $\Psi$  is simply the constant function 1.) The similarities between the spaces mean that both Kato's and Voigt's proofs via the Hille-Yosida and Miyadera Theorem respectively, can be generalised to this case (see [3, Theorem 2.1] and [39, Theorem 2.2] respectively). However, in the proof via Miyadera's Theorem, one requires a generalisation of Miyadera's Theorem for positive semigroups (see [39, Theorem A.2]). The association of the Miyadera Theorem with Dyson-Phillips series also led naturally to a representation of the perturbed semigroup in the form of a Dyson-Phillips series given by Arlotti, Lods and Mokhtar-Kharroubi in [3, Theorem 2.3].

**Proposition 2.1.3.** *Suppose  $X$  is an abstract state space and the operators  $A, B, (U_A(t))_{t \geq 0}$ , and  $(V(t))_{t \geq 0}$  are as in Theorem 2.1.2. The semigroup  $(V(t))_{t \geq 0}$  has a Dyson-Phillips series representation, i.e.*

$$V(t) = \sum_{n=0}^{\infty} S_n(t) \quad \text{where} \quad S_0(t) = U_A(t), S_{n+1}(t) = \int_0^t S_n(t-s) B U_A(s) ds, n \in \mathbb{N}_0.$$

In the  $L^1$  setting, the application of the Miyadera Theorem also inspired Voigt to prove an approximation result for the perturbed semigroup in Kato's Theorem. Before presenting this result, we first consider the following lemma which gives a condition which ensures that an operator  $\tilde{B}$  satisfies Kato's Theorem.

**Lemma 2.1.4.** *Let  $X$  be an abstract state space and suppose  $A, B$  satisfy the hypotheses of Theorem 2.1.2. Suppose also that there exists an operator  $\tilde{B} : D(A) \rightarrow X$  such that  $0 \leq \tilde{B}x \leq Bx$  for all  $x \in D(A)_+$ . Then  $A, \tilde{B}$  also satisfy the hypotheses of Kato's Theorem.*

*Proof.* By assumption,  $\tilde{B}$  is positive on  $D(A)$  and  $\tilde{B}x \leq Bx$  for all  $x \in D(A)_+$ . Hence it follows that  $\langle \Psi, (A + \tilde{B})x \rangle \leq 0$  for all  $x \in D(A)_+$ . Therefore  $A, \tilde{B}$  also satisfy Kato's Theorem.  $\square$

In the next proposition, we will let  $(V_B(t))_{t \geq 0}$  denote the Kato semigroup derived from the operators  $A, B$ .

**Proposition 2.1.5.** *[46, Proposition 1.6] Suppose  $X = L^1(\Omega, \mu)$  and  $A, B$  satisfy the hypotheses of Theorem 2.1.1. Let  $(B_n)$  be a sequence of  $A$ -bounded operators in  $X$  such that for all  $x \in D(A)_+$ ,  $0 \leq B_n x \leq Bx$ ,  $n \in \mathbb{N}$ , and  $B_n x \rightarrow Bx$  as  $n \rightarrow \infty$ . Then  $V_B(t) = \text{s-lim}_{n \rightarrow \infty} V_{B_n}(t)$  for each  $t \geq 0$ .*

Kato's Theorem has applications in a variety of areas. Examples of applications in  $L^1(\Omega, \mu)$  include applications in biology such as birth and death problems and fragmentation problems [9, Chapters 7-9]. Other examples include piecewise-deterministic Markov processes [42] and kinetic theory or the transport equation [9, Chapters 10-11]. The following example is a classical application of Kato's Theorem to the transport equation.

**Example 2.1.6.** We will consider the linear transport equation (also known as the neutron transport equation or linear Boltzmann equation) with no incoming particles as boundary condition from [46, Section 3]. Let  $X$  be the space  $L^1(S \times V, \mu)$  where  $S \subset \mathbb{R}^n$ ,  $V \subset \mathbb{R}^n$  are locally compact in the induced topology and  $\mu = \lambda^n \times \rho$  where  $\lambda^n$  is the  $n$ -dimensional Lebesgue measure and  $\rho$  is a locally integrable Borel measure on  $V$ . The linear transport equation is given as

$$\frac{\partial f}{\partial t}(s, v, t) = -v \cdot \nabla_{\mathbf{x}} f(s, v, t) - h(s, v) f(s, v, t) + \int_V k(s, v, v') f(s, v', t) d\rho(v'),$$

$$\mathbf{x} = (s, v) \in S \times V, t \geq 0$$

where  $S$  is the region where neutron transport occurs while  $V$  is the set of velocities that the neutrons may assume. The term  $f(\cdot, \cdot, t) : S \times V \rightarrow [0, \infty)$  describes the neutron density at time  $t \geq 0$ . The region  $S$  is filled with material and the neutrons which move about in this space are scattered or absorbed by the material. The first term in the transport equation describes the free streaming of the neutrons, the second term is linked to the collision and absorption of neutrons while the final term describes the scattering of neutrons where particles at the point  $s$  with velocity  $v'$  become new particles with velocity  $v$  and the transition is given by  $k(s, v, v')$ . The full assumptions on the functions  $h$  and  $k$  will be given in the next paragraph.

It can be shown (see [44, Section 1] for example) that there exists a  $C_0$ -semigroup of free streaming,  $(U_0(t))_{t \geq 0}$ , whose generator  $T_0$  is a realisation of the formal operator  $T_0 f := -v \cdot \nabla_{\mathbf{x}} f$  on some suitable domain  $D(T_0)$  (see Section 6.1 for full details). We also suppose that  $h : S \times V \rightarrow [0, \infty]$  is measurable and will denote the maximal multiplication operator associated with  $h$  by the same symbol,  $h$ . Moreover, for simplicity, by some additive perturbation result (see for example [45]), we will assume that  $T := T_0 - h$  is the generator of a substochastic  $C_0$ -semigroup  $(U(t))_{t \geq 0}$  and  $D(T) = D(h)$ . It can be shown [46, Lemma 3.1] that for  $f \in D(T)_+$ , we have

$$\|hf\| \leq - \int T f d\mu. \tag{2.2}$$

We also assume that the background material described by  $h$  and  $k$  only scatters or absorbs the migrating particles i.e.  $k : S \times V \times V \rightarrow [0, \infty]$  is measurable and that

$$\int_V k(s, v, v') d\rho(v) \leq h(s, v') \quad \mu\text{-a.e.}$$

This condition allows us to define a  $h$ -bounded operator  $K$  on  $D(K) = D(h)$  by

$$Kf(s, v) := \int_V k(s, v, v') f(s, v') d\rho(v').$$

Then  $K$  is a positive operator with

$$\|Kf\| \leq \|hf\| \quad \text{for all } f \in D(h)_+. \quad (2.3)$$

Combining (2.2) and (2.3), we see that  $T$  and  $K$  satisfy condition (iii) in Theorem 2.1.1 and hence it follows that there is a substochastic  $C_0$ -semigroup whose generator is an extension of  $T + K$ .

We will consider the transport equation once more in Chapter 6 but this time in the context of boundary perturbations. In that chapter, we will look at how Kato's Theorem on additive perturbations relates to boundary perturbations of the transport semigroup. Another application which we will look at later is an application of Kato's Theorem in  $\ell^1$  to Laplacians on graphs in  $\ell^2$  in Chapter 4. Last but not least, the extension of Kato's Theorem to abstract state spaces allows us to apply Kato's Theorem to operators acting on the dual of a  $C^*$ -algebra, in particular to quantum dynamical semigroups, as we will see in Chapter 5.

## 2.2 Honesty Theory on Abstract State Spaces

Kato's Theorem provides sufficient conditions so that an extension  $G$  of  $A + B$  generates a substochastic semigroup but it does not describe the relationship between  $G$  and  $A + B$ . This relationship turns out to be vital in characterising the consistency between the semigroup and the system it models. Thus it is this relationship between  $G$  and  $A + B$  that we will study in the next three sections. In particular, we are interested in establishing when the generator  $G$  is precisely the closure of  $A + B$  and the study of this is called honesty theory.

This section will contain the necessary introduction to honesty theory. We will focus however, on the case of abstract state spaces as these are the most recent results to date in this area. In the rest of this chapter, Kato's Theorem will always refer to Theorem 2.1.2 and the semigroup  $(V(t))_{t \geq 0}$  will denote the perturbed semigroup in Kato's Theorem.

### 2.2.1 Background

The theory of substochastic semigroups in abstract state spaces is often used to model the time evolution of the states of a system. An example of this is given in Example 2.1.6 where the semigroup which represents the solution to the neutron transport equation models the time evolution of the density of neutrons in the system. Now consider the special case when the operators  $T, K$  in the example satisfy condition (iii) in Kato's Theorem with equality. Under this condition, the physical laws of conservation, which are encapsulated in terms of the differential equation, require that the described quantity should be preserved, i.e. the semigroup describing the evolution is conservative (stochastic). However, in many cases, the semigroup turns out to be not conservative even though the modelled system should have this property. This phenomenon is known as *dishonesty*. For a system modelled by a strictly substochastic semigroup, we have a loss term in the differential equation modelling the system. Dishonesty in this case would mean that the quantity described by the semigroup is lost from the system faster than predicted by the loss term. So the semigroup is not a faithful or honest representation of the system. In short, the semigroup is honest if and only if it is consistent with the system it models.

The use of the term “honesty” to describe this phenomenon appears to originate in a paper on Markov processes by Reuter [38], although his use of this term differs slightly from the definition we give here as he considered “honest processes” instead of honest semigroups. In fact, honesty theory has long been studied from a probabilistic point of view by considering Markov processes. The advantage of a functional analytic approach is that the loss term is treated separately in this case unlike in the case of Markov processes where the loss term is simply treated as an additional state. This is important as the loss term contains information about the process modelled, for example, mass loss by internal factors in the case of fragmentation problems [9, p.158].

The study of honesty of the perturbed semigroup in terms of Kato's Theorem in abstract state spaces dates to Kato's seminal paper [31] where he studied the stochasticity of the perturbed semigroup on  $\ell^1$ . This is in fact where the study of honesty of the semigroup originated, namely in the study of stochastic semigroups. More recently, Voigt and Mokhtar-Kharroubi in [35] introduced a more systematic approach to studying the problem on  $L^1$ , that is, via functionals and resolvent operators. Arlotti, Lods and Mokhtar-Kharroubi then extended their work to abstract state spaces in [3] and introduced some new functionals using the Dyson-Phillips series of the perturbed semigroup. We will describe the resolvent functional approach, which we will mostly refer to as the functional approach, in Section 2.2.2 as we will utilise these

functionals repeatedly later. We will also describe a second approach involving spectral theory in Section 2.2.3. However, there are other approaches to honesty theory which we will not delve into in this chapter. For example, Arlotti uses an approach involving extensions of functions in a vector space in order to derive conditions for honesty in [4]. We will also omit the Dyson-Phillips functional approach given in [3, Section 4].

## 2.2.2 Honesty via Functionals

We consider again the operators from Theorem 2.1.2. We are interested in the functional

$$a_0 : D(G) \rightarrow \mathbb{R}, \quad a_0(x) = -\langle \Psi, Gx \rangle.$$

Since  $(V(t))_{t \geq 0}$  is substochastic, it follows from Proposition 1.2.5 that  $\langle \Psi, Gx \rangle \leq 0$  for all  $x \in D(G)_+$ . Thus  $a_0$  is positive on  $D(G)$ . Moreover, from the definition, it is easy to see that  $a_0(x) \leq \|Gx\|$  for all  $x \in D(G)$ . Thus  $a_0$  is continuous on  $D(G)$  with respect to the graph norm. We denote the restriction of  $a_0$  to  $D(A)$  by  $a$ , i.e.

$$a_0|_{D(A)} = a : D(A) \rightarrow \mathbb{R}, \quad a(x) = -\langle \Psi, Ax + Bx \rangle. \quad (2.4)$$

Fix  $\lambda > 0$  and  $x \in X_+$ . By the positivity of  $R(\lambda, A)$  and  $BR(\lambda, A)$ , the sequence  $R^{(n)}x := \sum_{k=0}^n R(\lambda, A)(BR(\lambda, A))^k x$ ,  $n \in \mathbb{N}$  is non-decreasing and moreover, by (2.1), converges to  $R(\lambda, G)x$ . Therefore, we have  $a(R^{(n)}x) = a_0(R^{(n)}x) \leq a_0(R(\lambda, G)x)$  for all  $n \in \mathbb{N}$ , i.e. we have a bounded, monotone real sequence which must then be convergent. Taking  $x = x_+ - x_- \in X$ ,  $x_+, x_- \in X_+$ , we see that this convergence holds for any  $x \in X$ . Therefore, we can define a new functional on  $D(G)$  by

$$\bar{a}_\lambda(R(\lambda, G)x) = \sum_{k=0}^{\infty} a(R(\lambda, A)(BR(\lambda, A))^k x), \quad x \in X.$$

There is an alternative (but equivalent) definition of  $\bar{a}_\lambda$ . First, note that  $R(\lambda, G_r)x = \sum_{k=0}^{\infty} r^k R(\lambda, A)(BR(\lambda, A))^k x$  converges monotonically to  $R(\lambda, G)x$  as  $r \nearrow 1$  for all  $x \in X_+$ . The continuity of the embedding  $D(A) \hookrightarrow D(G)$  implies that the series  $R(\lambda, G_r)x$  converges in  $D(G)$  as well and thus

$$a(R(\lambda, G_r)x) = \sum_{k=0}^{\infty} r^k a(R(\lambda, A)(BR(\lambda, A))^k x)$$

converges in  $\mathbb{R}$  as  $r \nearrow 1$  for all  $x \in X_+$ . In fact, it turns out that

$$\bar{a}_\lambda(R(\lambda, G)x) = \lim_{r \rightarrow 1} a(R(\lambda, G_r)x), \quad x \in X.$$

It can be shown [3, Proposition 3.1] that

$$\bar{a}_\lambda|_{D(A)} = a, \quad \lambda > 0 \quad (2.5)$$

and that the definition of  $\bar{a}_\lambda$  is independent of  $\lambda$ . Thus we define  $\bar{a} := \bar{a}_\lambda$ . From the inequality  $a(R^{(n)}x) \leq a_0(R(\lambda, G)x)$  for  $x \in X_+$ , it follows that  $\bar{a}(R(\lambda, G)x) \leq a_0(R(\lambda, G)x)$ . Using this fact, we can also deduce that  $\bar{a}$  is continuous on  $D(G)$  with respect to the graph norm.

The two functionals now allow us to define a positive functional,  $\Delta_\lambda \in X^*$ ,

$$\langle \Delta_\lambda, x \rangle = a_0(R(\lambda, G)x) - \bar{a}(R(\lambda, G)x), \quad x \in X. \quad (2.6)$$

This functional will be key in characterising the honesty of the semigroup, but to see this, we need the technical definition of honesty as given in [3]. To motivate the definition, consider the following: For any  $x \in X_+$  and any  $t \geq 0$ , we have  $\int_0^t V(s)x ds \in D(G)$  with

$$V(t)x - x = G \int_0^t V(s)x ds.$$

Since the semigroup  $(V(t))_{t \geq 0}$  is positive, it follows that

$$\|V(t)x\| - \|x\| = -a_0 \left( \int_0^t V(s)x ds \right). \quad (2.7)$$

We define honesty by the following:

**Definition 2.2.1.** [3, Definition 3.8] *Let  $X$  be an abstract state space,  $x \in X_+$  and  $(V(t))_{t \geq 0}$  the perturbed semigroup in Theorem 2.1.2. The trajectory  $(V(t)x)_{t \geq 0}$  is said to be honest if and only if*

$$\|V(t)x\| - \|x\| = -\bar{a} \left( \int_0^t V(s)x ds \right) \quad \text{for all } t \geq 0. \quad (2.8)$$

*The semigroup  $(V(t))_{t \geq 0}$  is said to be honest if all trajectories are honest. Otherwise, the trajectory (resp. semigroup) is said to be dishonest.*

In fact, recalling that  $\bar{a} \leq a_0$ , we can say more precisely that the semigroup is dishonest if and only if there exists  $x \in X_+$  and  $t \geq 0$  such that

$$\|V(t)x\| + \bar{a} \left( \int_0^t V(s)x ds \right) < \|x\|. \quad (2.9)$$

Note that Definition 2.2.1 tells us that the semigroup is honest if and only if the difference  $\|x\| - \|V(t)x\|$ ,  $x \in X_+$  is given by  $\bar{a} \left( \int_0^t V(s)x ds \right)$ , which is bounded by  $a_0 \left( \int_0^t V(s)x ds \right)$  so  $\bar{a}$  is in some sense the ‘‘minimal’’ functional.

*Remark 2.2.2.* Observe that if we have equality in condition (iii) in Kato's Theorem, then  $\bar{a} = 0$ . Hence an honest semigroup in this case is simply a stochastic semigroup

Now let us see how Definition 2.2.1 relates to the loss functional  $\Delta_\lambda$ . By comparing (2.7) and (2.8), we see that  $(V(t)x)_{t \geq 0}$  is honest if and only if

$$a_0 \left( \int_0^t V(s)x ds \right) = \bar{a} \left( \int_0^t V(s)x ds \right) \quad \text{for all } t \geq 0,$$

or equivalently,

$$a_0 \left( \int_s^t V(s)x ds \right) = \bar{a} \left( \int_s^t V(s)x ds \right) \quad \text{for all } t \geq s \geq 0.$$

Additionally, recall that for any  $\lambda > 0$  and any  $x \in X$ ,

$$R(\lambda, G)x = \int_0^\infty e^{-\lambda t} V(t)x dt = \lambda \int_0^\infty e^{-\lambda t} \int_0^t V(s)x ds dt. \quad (2.10)$$

Now fix  $x \in X$ . Since  $t \mapsto \int_0^t V(s)x ds$  is continuous in  $D(G)$ , it follows that the outer integral in (2.10) converges in  $D(G)$  with respect to the graph norm. Since  $a_0$  and  $\bar{a}$  are continuous on  $D(G)$  with the graph norm, it follows that

$$a_0(R(\lambda, G)x) = \lambda \int_0^\infty e^{-\lambda t} a_0 \left( \int_0^t V(s)x ds \right) dt, \quad (2.11)$$

$$\bar{a}(R(\lambda, G)x) = \lambda \int_0^\infty e^{-\lambda t} \bar{a} \left( \int_0^t V(s)x ds \right) dt. \quad (2.12)$$

The uniqueness of the Laplace transform implies that

$$a_0 \left( \int_0^t V(s)x ds \right) = \bar{a} \left( \int_0^t V(s)x ds \right) \quad \text{if and only if } a_0(R(\lambda, G)x) = \bar{a}(R(\lambda, G)x).$$

Therefore  $(V(t))_{t \geq 0}$  is honest if and only if  $\Delta_\lambda = 0$ , i.e. no loss occurs. These calculations show that the functional  $\Delta_\lambda$  is a loss functional in the sense that it measures 'how far' a trajectory deviates from an honest one. In fact, this describes the equivalence of (i) and (iii) in Theorem 2.2.4 below.

However, before moving on to the theorem, let us first elaborate a little more on the functional  $\Delta_\lambda$ . In particular, an important property of  $\Delta_\lambda$  which will be required later is given by the following proposition.

**Proposition 2.2.3.** [3, Theorem 3.24] For any  $\lambda > 0$ , let  $(\psi_n(\lambda))_n \subset X^*$  be defined inductively by

$$\psi_{n+1}(\lambda) = (BR(\lambda, A))^* \psi_n(\lambda), \quad \psi_0(\lambda) = \Psi.$$

Then  $(\psi_n(\lambda))_n$  is non-increasing and converges in the weak\* topology of  $X^*$  to  $\psi(\lambda)$  with

$$(BR(\lambda, A))^*\psi(\lambda) = \psi(\lambda). \quad (2.13)$$

Moreover,  $\psi(\lambda) = \Delta_\lambda$  for all  $\lambda > 0$  and  $\Delta_\lambda$  is the maximal element of the set  $\{\varphi \in X^* : \varphi \leq \Psi\}$  which satisfies  $(BR(\lambda, A))^*\varphi = \varphi$ .

The next theorem is a restatement of [3, Theorem 3.5] and gives some well-known equivalent conditions for honesty (see [31, 35, 3] for example).

**Theorem 2.2.4.** *Suppose  $X$  is an abstract state space and  $A, B, (V(t))_{t \geq 0}$  are as in Theorem 2.1.2. Let  $\lambda > 0$ . The following are equivalent:*

- (i) *The semigroup  $(V(t))_{t \geq 0}$  is honest.*
- (ii)  *$\lim_{n \rightarrow \infty} \|[BR(\lambda, A)]^n x\| = 0$  for all  $x \in X_+$ .*
- (iii)  *$\Delta_\lambda = 0$ .*
- (iv)  *$G = \overline{A + B}$ .*
- (v) *The set  $\{(BR(\lambda, A))^n x\}_{n \in \mathbb{N}}$  is relatively weakly compact for all  $x \in X_+$ .*

Note that [3, Theorem 3.5] was proven by utilising the functionals  $a_0$  and  $\bar{a}$ . However, we omit the proof of the theorem here and simply note that (i)  $\Leftrightarrow$  (iii) follows from the discussion above, (v) follows easily from (ii) while (ii)  $\Leftrightarrow$  (iii) because  $\langle \Delta_\lambda, x \rangle = \lim_{n \rightarrow \infty} \langle \Psi, [BR(\lambda, A)]^n x \rangle$  for all  $x \in X$  (cf. Proposition 2.2.3) which can be shown by using some standard manipulations. The proof of the other implications require more work (see [3, Theorem 3.5] for full details).

Apart from characterising the honesty of the semigroup, there are other areas which have been studied in honesty theory in abstract state spaces. Although we do not consider them in this chapter, they are worth noting. Firstly, honesty is often studied in terms of trajectories and not the whole semigroup (see [3], [35] for example). Moreover, one can study the set of elements which generate honest trajectories. It can be shown that this set of elements is invariant under the semigroup and is an ideal of the space  $X$  (see [3, Section 3.3], [35, Section 2]). Finally, it is possible to work with a more general notion of honesty on an interval (see [35, Section 2]) and develop a more general version of the theory presented here.

### 2.2.3 Honesty via Spectral Theory

The second approach to honesty that we will highlight in this section is the spectral approach. The main advantage of this approach is that the proof does not depend on the additive norm structure of abstract state spaces. Hence it holds for any complex ordered Banach space. We will state the main result (Theorem 2.2.6) for complex Banach spaces as it will turn out more useful later. So we will assume that  $X$  is a complex Banach space in the first half of this section, namely from Lemma 2.2.5 to Lemma 2.2.10. Complex Banach spaces can be constructed from real spaces via the process of complexification (see Section 1.1).

Theorem 2.2.6 is actually a general result on spectral properties of operators  $A$  and  $B$  which are not necessarily positive. But first, we need a lemma which gathers a few elementary facts about the boundedness of the operator  $BR(\lambda, A)$  which will turn up repeatedly in the rest of this thesis.

**Lemma 2.2.5.** *[9, Remark 4.2],[41, Lemma 3.2] Suppose  $A$  and  $B$  are linear operators such that  $D(A) \subseteq D(B) \subseteq X$  and  $\rho(A) \neq \emptyset$ . Then*

- (i)  *$BR(\lambda, A)$  is bounded for all  $\lambda \in \rho(A)$  if and only if  $B$  is  $A$ -bounded.*
- (ii) *Suppose additionally that  $X$  is ordered and has normal, generating cone. Then  $BR(\lambda, A)$  is bounded for all  $\lambda \in \rho(A)$  if  $A$  generates a positive semigroup and  $B \geq 0$  on  $D(A)$ .*
- (iii) *Suppose  $A + B$  is closable. Then  $BR(\lambda, A)$  is bounded for all  $\lambda \in \rho(A)$ .*

In Theorem 2.2.6 we let  $A$  and  $B$  be linear operators on the complex Banach space  $X$  with  $D(B) \supseteq D(A)$  and  $(G, D(G))$  be an extension of  $(A + B, D(A))$  with  $\Lambda := \rho(A) \cap \rho(G) \neq \emptyset$ . Note that  $\rho(G) \neq \emptyset$  implies that  $G$  is closed and hence  $A + B$  is closable. Therefore  $BR(\lambda, A)$  is bounded for all  $\lambda \in \Lambda$ .

**Theorem 2.2.6.** *[9, Theorem 4.3] Suppose  $A + B$  has an extension  $G$  and  $\Lambda \neq \emptyset$ . Then*

- (i)  *$1 \notin \sigma_p(BR(\lambda, A))$  for any  $\lambda \in \Lambda$ .*
- (ii)  *$1 \in \rho(BR(\lambda, A))$  for some/all  $\lambda \in \Lambda$  if and only if  $D(G) = D(A)$  and  $G = A + B$ .*
- (iii)  *$1 \in \sigma_c(BR(\lambda, A))$  for some/all  $\lambda \in \Lambda$  if and only if  $D(G) \supsetneq D(A)$  and  $G = \overline{A + B}$ .*

(iv)  $1 \in \sigma_r(BR(\lambda, A))$  for some/all  $\lambda \in \Lambda$  if and only if  $G \supseteq \overline{A + B}$ .

The key to the proof of this theorem is the following two lemmas which describe the relation between the operator  $BR(\lambda, A)$  and the domains  $D(A) = D(A + B)$  and  $D(\overline{A + B})$ .

**Lemma 2.2.7.** [9, Lemma 4.4] *Let  $\lambda \in \Lambda$  and  $x \in X$ . Then  $R(\lambda, G)x \in D(A)$  if and only if  $x \in (I - BR(\lambda, A))X$ .*

**Lemma 2.2.8.** [9, Lemma 4.5] *For any  $\lambda \in \Lambda$*

$$D(\overline{A + B}) = R(\lambda, G)\overline{(I - BR(\lambda, A))X}.$$

We omit the proofs of the lemmas but merely note that both proofs apply the following useful observation, which follows from the equality  $(\lambda - A - B)R(\lambda, A) = I - BR(\lambda, A)$ .

**Lemma 2.2.9.** [41, Lemma 3.1] *Let  $\lambda \in \rho(A)$  and suppose  $BR(\lambda, A)$  is bounded in  $X$ . Then  $\text{Im}(\lambda - A - B) = \text{Im}(I - BR(\lambda, A))$ .*

Moreover, we note that  $\ker(\lambda - (A + B)^*)$  is the annihilator of  $\text{Im}(\lambda - A - B)$  and similarly for the operator  $I - BR(\lambda, A)$ . Hence, from Lemma 2.2.9, it follows that

**Lemma 2.2.10.** *Let  $\lambda \in \rho(A)$  and suppose  $BR(\lambda, A)$  is bounded in  $X$ . Then  $\ker(\lambda - (A + B)^*) = \ker(I - (BR(\lambda, A))^*)$ .*

We now want to use the spectral characterisation in Theorem 2.2.6 to obtain some characterisations of honest semigroups. For simplicity, we first consider the setting of Theorem 2.1.2 where  $X$  is an abstract state space and  $A, B, (V(t))_{t \geq 0}$  are as in Theorem 2.1.2. Then combining Theorem 2.2.4 and Theorem 2.2.6, we see that  $(V(t))_{t \geq 0}$  is honest if and only if  $1 \in \rho(BR(\lambda, A)) \cup \sigma_c(BR(\lambda, A))$  for some  $\lambda > 0$ . However  $1 \in \rho(BR(\lambda, A)) \cup \sigma_c(BR(\lambda, A))$  if and only if  $\text{Im}(I - BR(\lambda, A))$  is dense in  $X$ . Combining this with Lemma 2.2.9 and Lemma 2.2.10, we have the following characterisation of honesty in abstract state spaces:

**Proposition 2.2.11.** *Let  $X$  be an abstract state space and  $A, B, (V(t))_{t \geq 0}$  be as in Theorem 2.1.2. The semigroup  $(V(t))_{t \geq 0}$  is honest if and only if either of the following equivalent conditions hold:*

(i)  $\text{Im}(\lambda - (A + B)) = \text{Im}(I - BR(\lambda, A))$  is dense in  $X$  for some/all  $\lambda > 0$ .

(ii)  $\ker(\lambda - (A + B)^*) = \ker(I - (BR(\lambda, A))^*) = \{0\}$  for some/all  $\lambda > 0$ .

It is sometimes more useful to apply the complex version as we will see in Section 3.2.2, so let us look at the complexification of the operators in Kato's Theorem and how they relate to the real case. In the rest of this section, we let  $X$  be an abstract state space and  $A, B, G$  and  $(V(t))_{t \geq 0}$  be as in Theorem 2.1.2. As in Section 1.1, we will use  $A_C, B_C, G_C$  and  $(V_C(t))_{t \geq 0}$  to denote their respective complexified forms. Note that the definition of the complexification of operators indicates that  $G_C$  is the generator of  $(V_C(t))_{t \geq 0}$ .

First we show that we have a definition of complex honesty using functionals which agrees with that of honesty on real spaces. We apply the complexification of the functionals  $a_0$  and  $\bar{a}$ . Recall from Definition 2.2.1 that for abstract state spaces, the semigroup  $(V(t))_{t \geq 0}$  is honest if and only if

$$\|V(t)x\| - \|x\| = -\bar{a} \left( \int_0^t V(s)x \, ds \right) \quad \text{for all } x \in X_+, t \geq 0.$$

By observing that  $(X_C)_+ = X_+$ ,  $\|v\|_C = \|v\|$  for all  $v \in (X_C)_+$  and  $\bar{a}$  and  $(V(t))_{t \geq 0}$  are positive operators, it follows that this is equivalent to

$$\|V_C(t)x\|_C - \|x\|_C = -\bar{a}_C \left( \int_0^t V_C(s)x \, ds \right) \quad \text{for all } x \in (X_C)_+, t \geq 0.$$

Therefore, we have an analogous definition for honesty in the complexification of abstract state spaces in terms of complex functionals:

**Definition 2.2.12.** *Let  $x \in (X_C)_+$ . The trajectory  $(V_C(t)x)_{t \geq 0}$  is said to be honest if and only if*

$$\|V_C(t)x\|_C - \|x\|_C = -\bar{a}_C \left( \int_0^t V_C(s)x \, ds \right) \quad \text{for all } t \geq 0.$$

*The semigroup  $(V_C(t))_{t \geq 0}$  is said to be honest if all trajectories are honest. Otherwise, the trajectory (resp. semigroup) is said to be dishonest.*

Next we consider the relation  $G = \overline{A + B}$  under complexification. First, an elementary lemma:

**Lemma 2.2.13.** *If  $T$  is a closable operator on  $D(T) \subset X$  and  $T_C$  is its complex extension on  $D(T_C) \subset X_C$ , then  $\overline{T_C} = (\overline{T})_C$ .*

*Proof.* By definition,  $x + iy \in D(\overline{T_C})$  if and only if there exists  $x_n + iy_n \in D(T_C)$  and  $u + iv \in X_C$  such that  $x_n + iy_n \rightarrow x + iy$  and  $T_C(x_n + iy_n) = Tx_n + iTy_n \rightarrow u + iv$  in  $X_C$ . Now

$$\begin{aligned} x_n + iy_n \rightarrow x + iy \text{ in } X_C &\Leftrightarrow x_n \rightarrow x \text{ and } y_n \rightarrow y \in X \\ Tx_n + iTy_n \rightarrow u + iv \text{ in } X_C &\Leftrightarrow Tx_n \rightarrow u \text{ and } Ty_n \rightarrow v \in X. \end{aligned} \quad (2.14)$$

In other words,  $x + iy \in D(\overline{T_C})$  if and only if  $x, y \in D(\overline{T})$ , or equivalently,  $x + iy \in D((\overline{T})_C)$ . Moreover, from (2.14),  $\overline{T}x = u$  and  $\overline{T}y = v$  so  $\overline{T_C}(x + iy) = u + iv = \overline{T}x + i\overline{T}y = (\overline{T})_C(x + iy)$  for all  $x + iy \in D(\overline{T_C}) = D((\overline{T})_C)$ .  $\square$

**Corollary 2.2.14.**  $G_C = \overline{A_C + B_C}$  if and only if  $G = \overline{A + B}$ .

*Proof.* Let  $u = x + iy \in D(G_C)$ . Suppose  $G_C = \overline{A_C + B_C} = \overline{(A + B)_C}$ . Then,  $Gx + iGy = G_C u = \overline{(A + B)_C} u = \overline{(A + B)}_C u = \overline{A + B}x + i\overline{A + B}y$ . Hence  $G = \overline{A + B}$ . The converse follows by reversing the argument.  $\square$

From Definition 2.2.12, we have that  $(V(t))_{t \geq 0}$  is honest if and only if  $(V_C(t))_{t \geq 0}$  is honest. On the other hand, Corollary 2.2.14 tells us that  $G_C = \overline{A_C + B_C}$  if and only if  $G = \overline{A + B}$ , which is equivalent to  $(V(t))_{t \geq 0}$  being honest by Theorem 2.2.4. Therefore,  $(V_C(t))_{t \geq 0}$  is honest if and only if  $G_C = \overline{A_C + B_C}$ . Applying Theorem 2.2.6, we have the following analogue of Proposition 2.2.11 for the complex case.

**Proposition 2.2.15.** *The semigroup  $(V_C(t))_{t \geq 0}$  is honest if and only if either of the following equivalent conditions hold:*

- (i)  $\text{Im}(\lambda - (A_C + B_C)) = \text{Im}(I - B_C R(\lambda, A_C))$  is dense in  $X$  for some/all  $\lambda \in \mathbb{C}_+$  where  $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$ .
- (ii)  $\ker(\lambda - (A_C + B_C)^*) = \ker(I - (B_C R(\lambda, A_C))^*) = \{0\}$  for some/all  $\lambda \in \mathbb{C}_+$ .

We have just seen that real honesty is equivalent to complex honesty. Moreover, there exist characterisations for both real and complex honesty. So if we have operators satisfying Kato's Theorem on a real Banach space, we can also consider the complex extensions of our operators  $A, B, G$  and apply the complex characterisations of honesty (see Section 3.2.2 as well) to check if  $(V_C(t))_{t \geq 0}$  is honest which is then equivalent to honesty of the semigroup on the real space. Conversely, if we are working in a complex space (for instance in the case of quantum dynamical semigroups in Chapter 5), we can restrict to the real space and apply any of the real characterisations of honesty in order to study honesty in the complex space.

## 2.3 New Characterisations of Honesty

In this section, we will look at some new characterisations of honesty in abstract state spaces. Although we will prove each result directly in this section, we will see later in Section 6.3 that some of the results can be derived from a more general theory.

### 2.3.1 Honesty and Mean Ergodicity

We begin with a characterisation of honesty obtained by applying the Mean Ergodic Theorem. The advantage of this approach is that it allows us to characterise not only when the semigroup is honest but also characterise the exact form of the generator when it is honest. More precisely, we can find conditions which differentiate when the generator  $G = A + B$  and when  $G = \overline{A + B}$ . For the basics of ergodic theory, see Section 1.4.

The results in this section are in fact a variant of [41, Theorem 1.1, Theorem 3.3] where Tyran-Kamińska considers the abstract setting of a general Banach space where the operator  $A + B$  is dissipative and the generated semigroup is contractive. Our result for honesty of the Kato semigroup (Proposition 2.3.2) demonstrates that with the addition of positivity assumptions, dissipativity on the positive cone is sufficient and contractivity is not necessary.

As we saw in Theorem 2.2.4, the operator  $BR(\lambda, A)$  plays an important role in honesty. So we begin with an auxiliary lemma about the properties of  $BR(\lambda, A)$  which will turn up again repeatedly later (for example in Section 3.1, Proposition 4.3.3, Theorem 6.3.5).

**Lemma 2.3.1.** *Let  $X$  be an abstract state space and suppose  $A, B$  satisfy the conditions of Theorem 2.1.2. For all  $\lambda > 0$ , the operator  $BR(\lambda, A)$  is substochastic and hence, power bounded. Moreover,  $\ker(I - BR(\lambda, A)) = \{0\}$ .*

*Proof.* Fix  $\lambda > 0$ . Note first that since  $B : D(A) \rightarrow X$  is positive, we have that  $BR(\lambda, A)$  is positive and thus bounded (Lemma 2.2.5). Let  $y \in X_+$  and  $x = R(\lambda, A)y$  ( $\in D(A)_+$ ). Then

$$(A + B)x = (A + B)R(\lambda, A)y = -y + BR(\lambda, A)y + \lambda R(\lambda, A)y. \quad (2.15)$$

Applying condition (iii) of Theorem 2.1.2, we have

$$-\|y\| + \|BR(\lambda, A)y\| + \|\lambda R(\lambda, A)y\| \leq 0$$

and so  $\|BR(\lambda, A)y\| \leq \|y\|$ . Therefore  $BR(\lambda, A)$  is contractive on  $X_+$ . Moreover, by iterating this process, we have  $\|(BR(\lambda, A))^n y\| \leq \|y\|$  and so  $\|(BR(\lambda, A))^n\| \leq 2M$  where  $M$  is the constant from the generating cone condition described in Section 1.1. Therefore it follows that  $BR(\lambda, A)$  is in fact, a power-bounded operator.

To verify the second property, note first that since  $G \supset A + B$ , we have for all  $x \in X$  and  $\lambda > 0$ ,

$$(\lambda - G)R(\lambda, A)x = (\lambda - A - B)R(\lambda, A)x = (I - BR(\lambda, A))x. \quad (2.16)$$

Therefore,  $\ker(I - BR(\lambda, A)) \subset \ker(R(\lambda, A)) = \{0\}$ . Alternatively, this fact may be deduced directly from Theorem 2.2.6.  $\square$

Now we can prove the following characterisation of honesty:

**Proposition 2.3.2.** *Let  $X$  be an abstract state space and suppose  $A, B, (V(t))_{t \geq 0}$  are as in Theorem 2.1.2. Then the semigroup  $(V(t))_{t \geq 0}$  is honest if and only if  $BR(\lambda, A)$  is mean ergodic for some  $\lambda > 0$ . Moreover, the generator  $G = A + B$  if and only if  $BR(\lambda, A)$  is uniformly ergodic.*

*Proof.* Fix  $\lambda > 0$ . By Lemma 2.3.1, we have that  $\ker(I - BR(\lambda, A)) = \{0\}$  and  $BR(\lambda, A)$  is power-bounded. Therefore,  $BR(\lambda, A)$  is Cesàro bounded and moreover,  $\lim_{n \rightarrow \infty} \frac{1}{n} \|(BR(\lambda, A))^n x\| = 0$  for all  $x \in X$ . Hence by Corollary 1.4.2, the mean ergodicity of  $BR(\lambda, A)$  is equivalent to the condition  $X = \overline{\text{Im}(I - BR(\lambda, A))}$ . Applying Proposition 2.2.11, it follows that  $BR(\lambda, A)$  is mean ergodic if and only if  $(V(t))_{t \geq 0}$  is honest.

To prove the second assertion, we use the Uniform Ergodic Theorem. Since  $BR(\lambda, A)$  is power bounded, it satisfies the condition  $\lim_{n \rightarrow \infty} \frac{1}{n} \|(BR(\lambda, A))^n\| = 0$ . Hence by Theorem 1.4.3 and the fact that  $\ker(I - BR(\lambda, A)) = \{0\}$ , it follows that  $BR(\lambda, A)$  is uniformly ergodic if and only if  $\text{Im}(I - BR(\lambda, A)) = X$ . But  $\text{Im}(I - BR(\lambda, A)) = X$  if and only if  $I - BR(\lambda, A)$  is invertible (as  $\ker(I - BR(\lambda, A)) = \{0\}$ ). Hence by Theorem 2.2.6, it follows that  $G = A + B$  if and only if  $BR(\lambda, A)$  is uniformly ergodic.  $\square$

### 2.3.2 Honesty and Dual Operators

Next, we give a characterisation for honesty based on dual (or adjoint) operators. This characterisation in terms of duals of operators will have applications in Chapters 4 and 5. The result is given in terms of dishonesty as it will turn out to be more useful in applications.

The spectral characterisation in the previous section showed that honesty of the semigroup is related to the existence of eigenvectors of  $(BR(\lambda, A))^*$  and  $(A + B)^*$ . The following result tells us that it suffices to consider the existence of positive subeigenvectors of  $(BR(\lambda, A))^*$  and  $(A + B)^*$  instead of eigenvectors.

**Theorem 2.3.3.** *Let  $X$  be an abstract state space and suppose  $A, B, (V(t))_{t \geq 0}$  are as in Theorem 2.1.2. Fix  $\lambda > 0$ . The following are equivalent:*

- (i)  $(V(t))_{t \geq 0}$  is dishonest.

- (ii) There exists  $f_\lambda \in X^* \setminus \{0\}$  such that  $((BR(\lambda, A))^n f_\lambda)$  is weak\*-convergent and  $w^* \text{-}\lim_{n \rightarrow \infty} (BR(\lambda, A))^n f_\lambda \neq 0$ .
- (iii) There exists  $f_\lambda \in X_+^* \setminus \{0\}$  such that  $((BR(\lambda, A))^n f_\lambda)$  is weak\*-convergent and  $w^* \text{-}\lim_{n \rightarrow \infty} (BR(\lambda, A))^n f_\lambda \neq 0$ .
- (iv) There exists  $f_\lambda \in X_+^* \setminus \{0\}$  such that  $(BR(\lambda, A))^* f_\lambda \geq f_\lambda$ .
- (v) There exists  $f_\lambda \in X_+^* \setminus \{0\}$  such that  $(A + B)^* f_\lambda \geq \lambda f_\lambda$ .
- (vi) There exists  $f_\lambda \in X_+^* \setminus \{0\}$  such that  $(A + B)^* f_\lambda = \lambda f_\lambda$ .
- (vii) There exists  $f_\lambda \in X^* \setminus \{0\}$  such that  $(A + B)^* f_\lambda = \lambda f_\lambda$ .

*Proof.* (i)  $\Leftrightarrow$  (vii) follows directly from Proposition 2.2.11 (ii). (i)  $\Rightarrow$  (vi) because  $\Delta_\lambda \in \ker(I - (BR(\lambda, A))^*) = \ker(\lambda - (A + B)^*)$  by Proposition 2.2.3 and dishonesty of the semigroup implies  $\Delta_\lambda \neq 0$ . (vi)  $\Rightarrow$  (v) is obvious. To show (v)  $\Rightarrow$  (iv), note that  $BR(\lambda, A) - I = (A + B - \lambda)R(\lambda, A)$ . Hence  $(BR(\lambda, A))^* - I \supseteq R(\lambda, A^*)((A + B)^* - \lambda)$ . Since  $R(\lambda, A^*)$  is a positive operator, it follows that (v)  $\Rightarrow$  (iv).

To show (iv)  $\Rightarrow$  (iii), we first note that  $(BR(\lambda, A))^*$  is a power bounded operator as  $BR(\lambda, A)$  is. Moreover, (iv) and the positivity of  $(BR(\lambda, A))^*$  implies that for all  $x \in X_+$ ,  $\langle (BR(\lambda, A))^n f_\lambda, x \rangle \geq \langle (BR(\lambda, A))^{*(n-1)} f_\lambda, x \rangle$  for all  $n \in \mathbb{N}$ . Hence  $\{\langle (BR(\lambda, A))^n f_\lambda, x \rangle\}$  is a monotonically increasing sequence in  $\mathbb{R}$  which is bounded above, hence converges. Since the sequence is bounded below by  $\langle f_\lambda, x \rangle$  and  $f_\lambda \neq 0$ , it converges to a non-zero element. Since this holds for all  $x \in X_+$  and  $X_+$  is generating,  $w^* \text{-}\lim (BR(\lambda, A))^n f_\lambda$  exists and is non-zero.

(iii)  $\Rightarrow$  (ii) is clear. It remains to show (ii)  $\Rightarrow$  (i). Let  $F := w^* \text{-}\lim (BR(\lambda, A))^n f_\lambda$  and  $F_n := (BR(\lambda, A))^n f_\lambda$ ,  $n \in \mathbb{N}$ . Take  $x \in X$ . Then

$$\langle F_n, x \rangle = \langle (BR(\lambda, A))^* F_{n-1}, x \rangle = \langle F_{n-1}, BR(\lambda, A)x \rangle.$$

Letting  $n \rightarrow \infty$  on both sides, we have  $\langle F, x \rangle = \langle F, BR(\lambda, A)x \rangle = \langle (BR(\lambda, A))^* F, x \rangle$ , i.e.  $(I - (BR(\lambda, A))^*)F = 0$ . Thus by Proposition 2.2.11 (ii),  $(V(t))_{t \geq 0}$  is dishonest.  $\square$

### 2.3.3 Honesty and Uniqueness of the Kato Semigroup

Recall from Theorem 2.1.2 that the generator  $G$  of the perturbed semigroup in Kato's Theorem is an extension of  $A + B$ . It turns out that the Kato semigroup is the unique substochastic semigroup generated by an extension of  $A + B$  if and only if the Kato semigroup is honest. More precisely,

**Theorem 2.3.4.** *Let  $X$  be an abstract state space and suppose  $A, B$  satisfy the conditions of Theorem 2.1.2 with perturbed semigroup  $(V(t))_{t \geq 0}$  and generator  $G$ .*

- (i) *If  $(V(t))_{t \geq 0}$  is honest, then  $(V(t))_{t \geq 0}$  is the unique (substochastic) semigroup whose generator is an extension of  $A + B$ .*
- (ii) *If  $(V(t))_{t \geq 0}$  is dishonest, then there are infinitely many substochastic semigroups whose generators are extensions of  $A + B$ .*

We will need the following simple fact in our proof of Theorem 2.3.4:

**Lemma 2.3.5.** *Let  $T$  be the generator of a  $C_0$ -semigroup (or its adjoint) on a Banach space  $X$  and  $S$  be a closed extension of  $T$ . If  $S$  also generates a  $C_0$ -semigroup (resp. its adjoint), then  $S = T$ .*

This follows from the fact that if  $S$  is the generator of a  $C_0$ -semigroup, then  $\lambda - S$  is invertible for  $\lambda$  such that  $\operatorname{Re} \lambda$  is large enough. Hence, we have  $\lambda - S \supseteq \lambda - T$  and both are bijections, so they must be equal. The adjoint case follows since  $\rho(S) = \rho(S^*)$  where  $S^*$  is the generator of the adjoint semigroup.

*Proof of Theorem 2.3.4.* To prove (i), let  $(\tilde{V}(t))_{t \geq 0}$  be another (substochastic) semigroup with generator  $\tilde{G} \supset A + B$ . Then  $(A + B)^* \supset \tilde{G}^*$ . Since  $(V(t))_{t \geq 0}$  is honest,  $\overline{A + B} = G$ , i.e.  $(A + B)^* = G^*$  and so  $(A + B)^*$  generates an adjoint semigroup. Hence by Lemma 2.3.5, it follows that  $\tilde{G}^* = (A + B)^* = G^*$ . Thus for  $\lambda > 0$ ,  $R(\lambda, \tilde{G})^* = R(\lambda, \tilde{G}^*) = R(\lambda, G^*) = R(\lambda, G)^*$  and so  $R(\lambda, G) = R(\lambda, \tilde{G})$ . Therefore, by the Post-Widder Inversion Formula (Proposition 1.2.8),  $\tilde{V}(t) = V(t)$  for all  $t \geq 0$ , so  $(V(t))_{t \geq 0}$  is unique.

To prove (ii), we will construct an infinite set of substochastic semigroups whose generators are extensions of  $A + B$ . Fix  $x_0 \in X_+ \setminus \{0\}$  such that  $\|x_0\| \leq 1$ . Define  $\tilde{G}$  by

$$\tilde{G}x = Gx + (a_0 - \bar{a})(x)x_0, \quad x \in D(\tilde{G}). \quad (2.17)$$

Then  $\tilde{G}$  has domain  $D(G)$  and for  $x \in D(A + B) = D(A)$ , we have  $\tilde{G}x = Gx = (A + B)x$ , since by (2.4) and (2.5),  $a_0|_{D(A)} = a = \bar{a}|_{D(A)}$ . Hence  $\tilde{G}$  is an extension of  $A + B$ . It remains to show that  $\tilde{G}$  generates a substochastic semigroup. To do so, we check that  $\tilde{G}$  satisfies Proposition 1.2.5. Condition (i) is satisfied since for all  $x \in D(G)_+$ ,

$$\begin{aligned} \langle \Psi, \tilde{G}x \rangle &= \langle \Psi, Gx \rangle + (a_0 - \bar{a})(x) \langle \Psi, x_0 \rangle \\ &= -a_0(x)(1 - \|x_0\|) - \bar{a}(x) \|x_0\| \\ &\leq 0 \quad (\text{since } a_0, \bar{a} \text{ are positive functionals and } \|x_0\| \leq 1.) \end{aligned}$$

To show that condition (ii) is satisfied, substitute  $\tilde{G}$  into (1.3) to get

$$(\lambda - G)y + (\bar{a} - a_0)(y)x_0 = x.$$

Applying  $R(\lambda, G)$  to both sides, we have

$$y + (\bar{a} - a_0)(y)R(\lambda, G)x_0 = R(\lambda, G)x \quad (2.18)$$

$$\text{i.e. } y = R(\lambda, G)x + \alpha_x R(\lambda, G)x_0 \quad (2.19)$$

where  $\alpha_x := (a_0 - \bar{a})(y)$  is some constant depending on  $x$ .

To see that  $\alpha_x$  is unique for every  $x$ , apply  $a_0 - \bar{a}$  to (2.18) to get

$$\alpha_x(1 - (a_0 - \bar{a})(R(\lambda, G)x_0)) = (a_0 - \bar{a})(R(\lambda, G)x).$$

Now consider the coefficient of  $\alpha_x$ . Since  $\bar{a} \geq 0$  and  $x_0 \in X_+ \setminus \{0\}$ , it follows from the definition of  $a_0$  that

$$(a_0 - \bar{a})(R(\lambda, G)x_0) \leq -\langle \Psi, GR(\lambda, G)x_0 \rangle = \|x_0\| - \lambda \|R(\lambda, G)x_0\| < \|x_0\| \leq 1.$$

Hence  $1 - (a_0 - \bar{a})(R(\lambda, G)x_0) > 0$ . Therefore  $\alpha_x$  exists and is unique for all  $x \in X$ . Moreover,  $\alpha_x > 0$  if  $x \in X_+$  because  $a_0 \geq \bar{a}$ . Hence the solution  $y$  in (2.19) to (1.3) is unique and moreover positive if  $x \in X_+$ , and so condition (ii) of Proposition 1.2.5 is satisfied. Therefore  $\tilde{G}$  generates a substochastic semigroup. Since  $x_0$  was an arbitrary positive element with  $\|x_0\| \leq 1$ , it follows that we can construct infinitely many semigroups of this form by varying  $x_0$ .  $\square$

*Remark 2.3.6.* An alternative way of proving that  $\tilde{G}$  defined in (2.17) generates a substochastic semigroup is by applying Kato's Theorem with  $A_1x := Gx$  and  $B_1 := (a_0 - \bar{a})(x)x_0$  for  $x \in D(G)$ . The operator  $B_1$  is positive since  $a_0 \geq \bar{a}$  and  $x_0 \in X_+$  and we have shown in the proof above that  $A_1 + B_1$  satisfy condition (iii) in Kato's Theorem. Hence Kato's Theorem tells us that there is an operator  $G_1 \supseteq \tilde{G} \supseteq A + B$  that generates a substochastic semigroup.

The proof above shows that if  $(V(t))_{t \geq 0}$  is dishonest and  $X$  is not 1-dimensional, there are in fact infinitely many substochastic semigroups  $(\tilde{V}(t))_{t \geq 0}$  with generator  $\tilde{G} \supset A + B$  whose loss is "minimal" in the sense

$$\left\| \tilde{V}(t)x \right\| - \|x\| = -\bar{a} \left( \int_0^t \tilde{V}(s)x ds \right) \quad \text{for all } x \in X_+.$$

To see this, observe that (2.17) can be rewritten as

$$\tilde{G}x = Gx - \langle \Psi, Gx \rangle x_0 - \bar{a}(x)x_0 \quad \text{for all } x \in D(G)$$

where  $\|x_0\| \leq 1, x_0 \in X_+ \setminus \{0\}$ . Taking  $x_0$  satisfying  $\|x_0\| = 1$ , we have for all  $x \in D(G)$ ,

$$\langle \Psi, \tilde{G}x \rangle = (1 - \|x_0\|) \langle \Psi, Gx \rangle - \bar{a}(x) \|x_0\| = -\bar{a}(x).$$

Hence,

$$\|\tilde{V}(t)x\| - \|x\| = \left\langle \Psi, \tilde{G} \int_0^t \tilde{V}(s)x ds \right\rangle = -\bar{a} \left( \int_0^t \tilde{V}(s)x ds \right) \quad \text{for all } x \in X_+.$$

Theorem 2.3.4 is in fact a generalisation of [38, Theorem 6] where Reuter was interested in uniqueness of solutions to the backward Kolmogorov differential equations. As Kato's Theorem originated from studying solutions to Kolmogorov differential equations, it is quite natural that we have an analogous result regarding the uniqueness of solutions in Kato's setting too.

## 2.4 Preserving Honesty

In the study of semigroup properties, one is often interested in the preservation of the properties under modifications to the semigroup. In this section, we look at the preservation of honesty of the perturbed Kato semigroup. In particular, we study the preservation of honesty under a special class of perturbations as well as under restrictions to subspaces.

### 2.4.1 Honesty and Potentials

A common term which occurs in differential equations which model dynamical systems is an absorption or potential term. In semigroup language, this is often phrased in terms of adding or more precisely, taking away a (positive) potential term from the generator of the original semigroup. An example of this can be seen in the transport equation studied in Example 2.1.6. The operator  $T := T_0 - h$  where  $T_0$  is the free-streaming operator and  $h$  is the absorption term. In this case, the free-streaming operator generates the original semigroup  $(V(t))_{t \geq 0}$  while  $T_0 - h$  generates the new absorption semigroup,  $(V_h(t))_{t \geq 0}$ . Note that the relation  $V_h(t) \leq V(t)$  holds for all  $t \geq 0$  as  $h$  is positive. Other examples where absorption terms occur include piecewise deterministic Markov processes [42] and the heat equation on graphs (see Chapter 4). Thus in this section, we study conditions which ensure that the honesty or dishonesty of the original semigroup is retained by the absorption semigroup.

**Proposition 2.4.1.** *Let  $X$  be an abstract state space and suppose  $A, B$  satisfy the conditions of Kato's Theorem with honest perturbed semigroup  $(V(t))_{t \geq 0}$ . Let  $K$  be a positive operator such that there is an extension  $A_K$  of  $(A - K, D(A) \cap D(K))$  that generates a substochastic semigroup and  $A_K, B$  also satisfy Kato's Theorem with perturbed semigroup  $(V_K(t))_{t \geq 0}$ . If  $(D(A) \cap D(K))_+$  is dense in the graph norm in  $D(A)_+$ , then  $(V_K(t))_{t \geq 0}$  is also honest.*

*Proof.* Let  $S := (\lambda - A)(D(A) \cap D(K))_+$ . Elementary calculations show that if  $(D(A) \cap D(K))_+$  is dense in  $D(A)_+$ , then  $S$  is dense in  $X_+$ .

Let  $y \in S$ . Then

$$\begin{aligned} & (R(\lambda, A) - R(\lambda, A_K))y \\ &= R(\lambda, A_K)(\lambda - A_K - (\lambda - A))R(\lambda, A)y \\ &= R(\lambda, A_K)(\lambda - A + K - (\lambda - A))R(\lambda, A)y \quad (\text{as } R(\lambda, A)y \in (D(A) \cap D(K))_+) \\ &= R(\lambda, A_K)KR(\lambda, A)y \geq 0. \end{aligned}$$

Since  $S$  is dense in  $X_+$ ,  $R(\lambda, A), R(\lambda, A_K)$  are bounded and  $X_+$  is closed, it follows that  $R(\lambda, A_K) \leq R(\lambda, A)$ . Hence  $BR(\lambda, A_K) \leq BR(\lambda, A)$  and iterating,  $(BR(\lambda, A_K))^n \leq (BR(\lambda, A))^n$ . The result now follows by Theorem 2.2.4 (ii).  $\square$

If  $K$  is a bounded positive operator such that  $A - K$  generates a substochastic semigroup and  $A, B$  satisfy Kato's Theorem, then  $A - K, B$  also satisfy Kato's Theorem since  $\langle \Psi, (A - K + B)x \rangle = \langle \Psi, (A + B)x \rangle - \langle K, x \rangle \leq 0$  for all  $x \in D(A)_+$ . We will denote the perturbed semigroup by  $(V_K(t))_{t \geq 0}$ . Proposition 2.4.1 tells us that honesty is retained even after adding a bounded potential term to the generator. It turns out that in this case, dishonesty is retained as well:

**Proposition 2.4.2.** *Let  $X$  be an abstract state space and suppose  $A, B$  satisfy the conditions of Kato's Theorem with dishonest perturbed semigroup  $(V(t))_{t \geq 0}$ . Let  $K$  be a bounded positive operator such that  $A - K$  generates a substochastic semigroup. Then the perturbed semigroup  $(V_K(t))_{t \geq 0}$  is also dishonest.*

*Proof.* Theorem 2.3.4 (ii) implies that there are infinitely many extensions  $G_\alpha$  of  $A + B$  which generate substochastic semigroups. Since  $K$  is positive and bounded,  $G_\alpha - K$  also generates a semigroup for each  $\alpha$ . Moreover, since  $G_\alpha \supseteq A + B$  and  $K$  is bounded, it follows that  $G_\alpha - K \supseteq A - K + B$ . Hence there exist infinitely many semigroups whose generators are extensions of  $A - K + B$ . So by Theorem 2.3.4,  $(V_K(t))_{t \geq 0}$  is also dishonest.  $\square$

It turns out that for the case of preserving dishonesty, adding a bounded potential is in some sense sharp. We can find an example that shows that a dishonest semigroup can be converted into an honest semigroup by adding a potential that is “large enough”. We will not go into the full details now as it requires a lot more auxiliary information which will be covered in Chapter 4, but simply give a brief outline here. This example can be found in the study of Laplacians on graphs where the Laplacian is known to generate a substochastic heat semigroup on  $\ell^1$ . In Chapter 4, we will see that the heat semigroup can be seen as a perturbed semigroup derived from Kato’s Theorem and moreover honesty of the heat semigroup is equivalent to stochastic completeness of the graph. Then [32, Theorem 2] states that any graph can be modified to form a stochastically complete one by adding a (sufficiently large) potential term to the Laplacian. In other words, the heat semigroup generated by any graph Laplacian can be modified to become an honest one by increasing the potential term adequately. So even if the semigroup is dishonest, it can be converted into an honest one by adding a sufficiently large potential. So dishonesty is not always preserved under absorption.

Finally, let us apply these results to the case of the transport equation.

**Example 2.4.3.** Let us consider again the linear transport equation with no incoming particles as boundary condition from Example 2.1.6. So we let  $X = L^1(S \times V, \mu)$  where  $S \subset \mathbb{R}^n$ ,  $V \subset \mathbb{R}^n$  are locally compact in the induced topology and  $\mu = \lambda^n \times \rho$  where  $\lambda^n$  is the  $n$ -dimensional Lebesgue measure and  $\rho$  is a locally integrable Borel measure on  $V$ .

Recall that  $T_0$  is the generator of the  $C_0$ -semigroup of free streaming  $(U_0(t))_{t \geq 0}$  and  $h$  is the maximal multiplication operator associated with the function  $h(s, v)$ . In this case, we will assume that  $T = T_0 - h$ ,  $D(T) = D(h)$ , generates the substochastic  $C_0$ -semigroup  $(U(t))_{t \geq 0}$  and  $K$  is the positive operator defined by

$$Kf(s, v) := \int_V k(s, v, v') f(s, v') d\rho(v'), \quad f \in D(K) = D(T)$$

with

$$\|hf\| \leq - \int Tf d\mu, \quad \text{for all } f \in D(T)_+$$

and

$$\int_V k(s, v, v') d\rho(v) \leq h(s, v) \quad \mu\text{-a.e.} \quad (2.20)$$

In other words,  $T$  and  $K$  satisfy the conditions of Kato’s Theorem with perturbed semigroup  $(V(t))_{t \geq 0}$ .

Next we consider a second measurable function  $\tilde{h}(s, v) \geq 0$  and suppose that  $\tilde{h}$  (the maximal multiplication operator with  $\tilde{h}(s, v)$ ) is  $U(\cdot)$ -bounded (see [45, Definition 1.2]) i.e.  $\tilde{h}$  is  $T$ -bounded and there exist  $\alpha \in (0, \infty]$ ,  $\gamma \geq 0$  such that for all  $f \in D(T)$

$$\int_0^\alpha \left\| \tilde{h}U(t)f \right\| dt \leq \gamma \|f\|.$$

By [45, Corollary 2.10],  $T_{\tilde{h}} = T - \tilde{h}$  generates a substochastic  $C_0$ -semigroup. Moreover for  $f \in D(T)_+$ ,

$$\left\| (h + \tilde{h})f \right\| \leq - \int T f d\mu + \int \tilde{h}f d\mu = - \int T_{\tilde{h}}f d\mu.$$

and

$$\|Kf\| \leq \left\| (h + \tilde{h})f \right\|.$$

Therefore,  $T_{\tilde{h}}$  and  $K$  also satisfy the conditions of Kato's Theorem with perturbed semigroup denoted  $(V_{\tilde{h}}(t))_{t \geq 0}$ . Since by assumption  $\tilde{h}$  is  $T$ -bounded, we have  $(D(T) \cap D(\tilde{h}))_+$  dense in  $D(T)_+$  so by Proposition 2.4.1,  $(V_{\tilde{h}}(t))_{t \geq 0}$  is honest whenever  $(V(t))_{t \geq 0}$  is.

In the case when we have equality in (2.20), the background material is called a pure scatterer [45, p.463]. Proposition 2.4.1 and 2.4.2 tell us that adding a bounded  $\tilde{h}$  to  $h$  does not affect the honesty or dishonesty of the transport semigroup. In other words, a change from pure scatterer to impure does not affect honesty if the change is "small" enough.

## 2.4.2 Honesty and Restrictions to Subspaces

In this section, we are interested in the preservation of honesty under restrictions of the semigroup to subspaces. First, we will look at the relation between Kato's semigroup and its restriction to closed subspaces of  $X$ . We will see that the existence of a Dyson-Phillips series for the perturbed semigroup enables very simple proofs of the results. Recalling that in Kato's Theorem, we assume that  $A$  generates a substochastic semigroup denoted by  $(U_A(t))_{t \geq 0}$ , we have

**Lemma 2.4.4.** *Let  $X$  be an abstract state space and suppose  $A, B$  satisfy the hypotheses of Kato's Theorem. Suppose also that  $Y$  is an ordered, closed subspace of  $X$  such that  $U_A(t)Y \subseteq Y$  and  $B(D(A) \cap Y) \subseteq Y$ . Then  $B| := B|_{D(B) \cap Y}$  and  $A| := A|_{D(A) \cap Y}$  satisfy Kato's Theorem (in  $Y$ ).*

*Proof.* From [22, p.60], we know that  $A|$  generates a substochastic semigroup on  $Y$  which is  $U_A(t)| := U_A(t)|_Y$ . By assumption,  $B|$  is positive and  $D(B|) \supset D(A|)$ . Moreover  $\langle \Psi, (A| + B|)y \rangle = \langle \Psi, (A + B)y \rangle \leq 0$  for all  $y \in D(A|)_+$ . Therefore  $A|, B|$  satisfy Kato's Theorem on  $Y$ .  $\square$

Since  $A|, B|$  satisfy Kato's Theorem, there is an extension  $\tilde{G}$  of  $A| + B|$  such that  $\tilde{G}$  generates a substochastic semigroup  $(\tilde{V}(t))_{t \geq 0}$  on  $Y$ .

**Proposition 2.4.5.**  $\tilde{V}(t) = V(t)|_Y$  for all  $t \geq 0$ .

*Proof.* By Proposition 2.1.3,  $(V(t))_{t \geq 0}$  can be represented in the form of a Dyson-Phillips series, i.e.

$$V(t) = \sum_{n=0}^{\infty} S_n(t) \quad \text{where} \quad S_0(t) = U_A(t), S_{n+1}(t) = \int_0^t S_n(t-s) B U_A(s) ds, n \in \mathbb{N}.$$

Similarly,  $(\tilde{V}(t))_{t \geq 0}$  can be represented in the form of a Dyson-Phillips series, i.e.

$$\tilde{V}(t) = \sum_{n=0}^{\infty} \tilde{S}_n(t) \quad \text{where} \quad \tilde{S}_0(t) = U_A(t)|, \tilde{S}_{n+1}(t) = \int_0^t S_n(t-s) B| U_A(s) ds, n \in \mathbb{N}.$$

By comparing the series, we see that  $\tilde{V}(t) = V(t)|_Y$  for all  $t \geq 0$ .  $\square$

Our next proposition, which describes how honesty of the two semigroups is related, follows directly from the fact that  $\|(B|R(\lambda, A))^n y\|_Y = \|(BR(\lambda, A))^n y\|_X$  for all  $y \in Y$  and Theorem 2.2.4.

**Proposition 2.4.6.** *If  $(V(t))_{t \geq 0}$  is honest, then so is  $(\tilde{V}(t))_{t \geq 0}$ .*

# Chapter 3

## Kato's Theorem and Honesty beyond Abstract State Spaces

In this chapter, we will look at generalisations of Kato's Theorem beyond abstract state spaces. The existence of such generalisations leads naturally to the study of honesty theory on these spaces. Although there are previously known generalisations of Kato's Theorem beyond abstract state spaces, there are hardly any results on the honesty of the semigroup in these cases. Hence, we will introduce a notion of honesty on these spaces. As in abstract state spaces, we will be interested in characterisations of honesty of the semigroup. We will identify characterisations from abstract state spaces which extend beyond these spaces and present some new ones too. Unfortunately, it turns out that the results are less complete in the spaces beyond abstract state spaces.

### 3.1 Extension to Real Ordered Banach Spaces

We begin by presenting an extension of Kato's Theorem to  $KB$ -spaces which was proven by Banasiak and Lachowicz in [10]. We consider this to be the first generalisation of Kato's Theorem beyond abstract state spaces because in general,  $KB$ -spaces do not have additive norm on the positive cone. However,  $KB$ -spaces are Banach lattices, so the set of abstract state spaces is not contained within these spaces. Recalling from Proposition 1.1.1 that an ordered Banach space  $X$  has monotone norm if and only if for each  $x \in X_+$  there is an  $x^* \in X_+^*$  such that  $\|x^*\|_{X^*} = 1$  and  $\langle x^*, x \rangle = \|x\|_X$ , we have the following result:

**Theorem 3.1.1.** *[10, Theorem 3.2] Let  $X$  be a  $KB$ -space and suppose that the operators  $A$  and  $B$  with  $D(A) \subseteq D(B) \subseteq X$  satisfy:*

- (i)  $A$  generates a substochastic semigroup  $(U_A(t))_{t \geq 0}$ ,
- (ii)  $r_\sigma(BR(\lambda, A)) \leq 1$  for some  $\lambda > 0$ ,
- (iii)  $Bx \geq 0$  for  $x \in D(A)_+$ ,
- (iv) for any  $x \in D(A)_+$ , there is an  $x^* \in X_+^*$  such that  $\|x^*\| = 1$ ,  $\langle x^*, x \rangle = \|x\|$  and  $\langle x^*, (A + B)x \rangle \leq 0$ .

Then there is an extension  $G$  of  $(A + B, D(A))$  that generates a substochastic  $C_0$ -semigroup denoted by  $(V(t))_{t \geq 0}$ . The generator  $G$  satisfies for all  $\lambda > 0$  and  $x \in X$ ,

$$R(\lambda, G)x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x.$$

In order to prove Theorem 3.1.1, the authors in [10] noted that one key element required in the proof of Kato's Theorem in abstract state spaces was the convergence of any increasing, norm bounded sequence, which followed from the additive norm structure of the underlying space. Since  $KB$ -spaces are defined by precisely this convergence property, this enabled the extension of Kato's Theorem to such spaces. However, Theorem 3.1.1 does require an additional assumption, namely that  $r_\sigma(BR(\lambda, A)) \leq 1$  for some  $\lambda > 0$ , in order to compensate for the loss of the additive norm structure in the space. In spaces with norm additive on the positive cone, i.e. abstract state spaces, the condition  $r_\sigma(BR(\lambda, A)) \leq 1$  follows easily from the dissipativity condition on the positive cone, i.e. condition (iii) in Theorem 2.1.2, as this condition implies the power-boundedness of the operator  $BR(\lambda, A)$  (see the proof of Lemma 2.3.1).

As noted above, abstract state spaces are not a subset of  $KB$ -spaces as they do not necessarily have lattice structure while  $KB$ -spaces in turn do not all have norms which are additive on the positive cone, hence they are not necessarily abstract state spaces. We now prove a version of Kato's Theorem for a set of spaces which encompasses both sets of spaces, namely ordered Banach spaces with monotone norm, but with the additional assumption that for some  $\lambda > 0$ , the series  $R(\lambda) := \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k$  converges strongly. The proof of this theorem is based on that of [10, Theorem 3.2] and the additional condition is required to replace the loss of the convergence property which follows from the structure of  $KB$  and abstract state spaces. Although we can find no practical application for Theorem 3.1.2 currently, we give the full proof here for completeness and to demonstrate that the lattice structure is not essential to the proof of [10, Theorem 3.2].

**Theorem 3.1.2.** *Let  $X$  be a real, ordered Banach space with generating cone and monotone norm. Suppose the operators  $A$  and  $B$  with  $D(A) \subseteq D(B) \subseteq X$  satisfy:*

- (i)  $A$  generates a substochastic semigroup  $(U_A(t))_{t \geq 0}$ ,
- (ii)  $r_\sigma(BR(\lambda, A)) \leq 1$  for some  $\lambda > 0$ ,
- (iii)  $Bx \geq 0$  for  $x \in D(A)_+$ ,
- (iv) for any  $x \in D(A)_+$ , there is an  $x^* \in X_+^*$  such that  $\|x^*\|_{X^*} = 1$ ,  $\langle x^*, x \rangle = \|x\|_X$  and  $\langle x^*, (A + B)x \rangle \leq 0$ ,
- (v) the series

$$R(\lambda)x := \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x$$

converges for all  $x \in X$  and some  $\lambda > 0$  (possibly different from  $\lambda$  in (ii)).

Then there is an extension  $G$  of  $(A + B, D(A))$  that generates a substochastic  $C_0$ -semigroup denoted by  $(V(t))_{t \geq 0}$ . Moreover, the series  $R(\lambda)$  converges strongly for all  $\lambda > 0$  and the resolvent of  $G$  is given by this series, i.e. for all  $\lambda > 0$  and  $x \in X$ ,

$$R(\lambda, G)x = R(\lambda)x.$$

Finally,  $(V(t))_{t \geq 0}$  is the minimal substochastic  $C_0$ -semigroup whose generator is an extension of  $(A+B, D(A))$  in the following sense: if  $(\tilde{V}(t))_{t \geq 0}$  is another substochastic  $C_0$ -semigroup whose generator is an extension of  $(A + B, D(A))$ , then  $V(t) \leq \tilde{V}(t)$  for all  $t \geq 0$ .

The first step in proving the theorem, which we state as a lemma below, involves showing that  $A + rB$  generates a substochastic  $C_0$ -semigroup for  $0 < r < 1$ . We need the following observation:

*Remark 3.1.3.* Suppose for some  $\lambda_0 > 0$ ,  $A$  is a resolvent positive operator in  $X$ , with  $R(\lambda, A) \geq 0$  for  $\lambda \geq \lambda_0$  and  $B : D(A) \rightarrow X$  is a positive operator satisfying  $r_\sigma(BR(\lambda_0, A)) \leq c$ ,  $c \geq 0$ . Then  $r_\sigma(BR(\lambda, A)) \leq c$  for all  $\lambda > \lambda_0$ . Fix  $\lambda > \lambda_0$ . By applying the resolvent equation [22, Equation IV.1.2], the positivity of  $R(\lambda, A)$  and  $B$  on  $D(A)$ , we have

$$BR(\lambda, A) - BR(\lambda_0, A) = (\lambda_0 - \lambda)BR(\lambda_0, A)R(\lambda, A) \leq 0.$$

Therefore  $BR(\lambda_0, A)y \geq BR(\lambda, A)y$  for all  $y \in X_+$ . Iterating, it follows that  $\|(BR(\lambda_0, A))^n y\| \geq \|(BR(\lambda, A))^n y\|$  for all  $y \in X_+, n \in \mathbb{N}$ , and so

$$\|(BR(\lambda, A))^n\| \leq 2M \|(BR(\lambda_0, A))^n\|$$

where  $M$  is the constant from the generating cone condition (see Section 1.1). Therefore

$$r_\sigma(BR(\lambda, A)) \leq r_\sigma(BR(\lambda_0, A)) \leq c.$$

**Lemma 3.1.4.** *Under the conditions of Theorem 3.1.2,  $G_r := A + rB$ ,  $0 < r < 1$  generates a substochastic  $C_0$ -semigroup. Moreover, for all  $\lambda > 0$ , all  $x \in X$ , the resolvent of  $G_r$  satisfies*

$$R(\lambda, G_r)x = \sum_{k=0}^{\infty} R(\lambda, A)(rBR(\lambda, A))^k x. \quad (3.1)$$

*Proof.* We will begin by showing that  $A$  and  $rB$  satisfy the hypotheses of Theorem 1.1.4 for all  $\lambda > 0$ , hence (3.1) holds for all  $\lambda > 0, x \in X$ . Note first that by Proposition 1.1.2,  $X$  has normal cone, hence  $X$  satisfies the space conditions of Theorem 1.1.4. Now suppose  $BR(\lambda_0, A)$  satisfies condition (ii) in Theorem 3.1.2. By Remark 3.1.3, it follows that  $r_\sigma(BR(\lambda, A)) \leq 1$  for all  $\lambda \geq \lambda_0$  and hence  $r_\sigma(rBR(\lambda, A)) < 1$  for all  $\lambda > \lambda_0$ . Thus by Theorem 1.1.4, it follows that (3.1) holds for all  $\lambda \geq \lambda_0$ . It now remains to show that (3.1) holds for all  $\lambda \in (0, \lambda_0)$ .

Suppose  $x \in D(A)_+$  and  $x^*$  is as in condition (iv) of Theorem 3.1.2. Then for  $0 < r < 1$ , we have

$$\langle x^*, (A + rB)x \rangle = \langle x^*, (A + B)x \rangle + (r - 1) \langle x^*, Bx \rangle \leq \langle x^*, (A + B)x \rangle \leq 0.$$

Hence, for all  $\lambda \in \rho(G_r)$ , we have

$$\|(\lambda - G_r)x\| \|x^*\| \geq \langle x^*, (\lambda - G_r)x \rangle = \lambda \langle x^*, x \rangle - \langle x^*, G_r x \rangle \geq \lambda \|x\|.$$

Now let  $y \in X_+$ . Then  $x := R(\lambda, G_r)y \in D(A)_+$  and so

$$\lambda \|R(\lambda, G_r)y\| \leq \|y\|.$$

By iterating this inequality, we see that for all  $y \in X_+, \lambda \geq \lambda_0$  and  $n \in \mathbb{N}$ , we have

$$\|(R(\lambda, G_r))^n y\| \leq \frac{1}{\lambda^n} \|y\|. \quad (3.2)$$

Since  $X_+$  is generating, we can extend this estimate to the whole space to obtain

$$\|(R(\lambda, G_r))^n\| \leq \frac{2M}{\lambda^n} \quad \text{for all } n \in \mathbb{N}, \lambda \geq \lambda_0. \quad (3.3)$$

We will use (3.3) to show that  $(0, \lambda_0) \subset \rho(G_r)$ . Note first that (3.3) and the spectral radius formula tells us that  $r_\sigma(R(\lambda_0, G_r)) \leq \frac{1}{\lambda_0}$ . Now let  $\mu \in (0, \lambda_0)$  and suppose  $\mu \in \sigma(G_r)$ . Then  $\frac{1}{\lambda_0 - \mu} \in \sigma(R(\lambda_0, G_r))$  and  $\left| \frac{1}{\lambda_0 - \mu} \right| > \frac{1}{\lambda_0}$ . This contradicts the fact that  $r_\sigma(R(\lambda_0, G_r)) \leq \frac{1}{\lambda_0}$ , hence  $\mu \in \rho(G_r)$ . Therefore  $(0, \lambda_0) \subseteq \rho(G_r)$ . Finally, we note that from the series  $R(\lambda, G_r) = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k R(\lambda_0, G_r)^{k+1}$ , we also have that  $R(\lambda, G_r) \geq 0$  for all  $\lambda \in (0, \lambda_0)$ . Hence we may apply Theorem 1.1.4 to obtain that  $r_\sigma(rBR(\lambda, A)) < 1$  and (3.1) holds for all  $\lambda > 0$ .

To prove the first statement in the theorem, we note that  $G_r$  is closed, densely defined and satisfies (3.3). Hence, the Hille-Yosida Theorem (Theorem 1.2.2) tells us that  $(G_r, D(A))$  generates a  $C_0$ -semigroup  $(U_r(t))_{t \geq 0}$  for all  $0 < r < 1$ . The Post-Widder Inversion Formula (Proposition 1.2.8), and (3.2) implies that  $(U_r(t))_{t \geq 0}$  is a contraction on  $X_+$ . Moreover, the formula shows that  $(U_r(t))_{t \geq 0}$  is positive since  $R(\lambda, G_r)$  is positive for all  $\lambda > 0$ . Therefore,  $(U_r(t))_{t \geq 0}$  is substochastic.  $\square$

*Proof of Theorem 3.1.2.* We will show the existence of  $(V(t))_{t \geq 0}$  by checking that  $(R(\lambda, G_r))_r$  satisfies condition (ii) of the Trotter-Kato Theorem (Theorem 1.2.6). The first step is to show that  $R(\lambda_1)x = \lim_{r \rightarrow 1} R(\lambda_1, G_r)x$  for all  $x \in X$  where  $R(\lambda_1)$  satisfies condition (v) of Theorem 3.1.2. Let  $x \in X_+$  and  $\epsilon > 0$ . First note that for all  $0 < r < 1$  and all  $n \in \mathbb{N}$ , we have

$$0 \leq \sum_{k=n}^{\infty} (1 - r^k) R(\lambda_1, A) (BR(\lambda_1, A))^k x \leq \sum_{k=n}^{\infty} R(\lambda_1, A) (BR(\lambda_1, A))^k x.$$

So by condition (v) and the monotonicity of the norm, it follows that for  $n$  large enough,

$$\left\| \sum_{k=n}^{\infty} (1 - r^k) R(\lambda_1, A) (BR(\lambda_1, A))^k x \right\| < \epsilon.$$

On the other hand, for  $r$  close to 1, we have

$$\left\| \sum_{k=1}^{n-1} (1 - r^k) R(\lambda_1, A) (BR(\lambda_1, A))^k x \right\| < \epsilon.$$

Therefore  $\|R(\lambda_1)x - R(\lambda_1, G_r)x\| < 2\epsilon$ , so by the generating cone condition, we have the convergence we require. This convergence and the fact that  $G_r$  generates a substochastic semigroup for all  $0 \leq r < 1$  allow us to deduce from Proposition 1.2.7 that  $s\text{-}\lim_{r \rightarrow 1} R(\lambda, G_r)$  exists for all  $\lambda > 0$ . We will denote this limit by  $R_\lambda$ .

It now remains to show that  $R_\lambda$  has range dense in  $X$ . In particular, we want to show that for all  $x \in X$ ,

$$\lim_{\lambda \rightarrow \infty} \lambda R_\lambda x = \lim_{\lambda \rightarrow \infty} \lim_{r \rightarrow 1} \lambda R(\lambda, G_r)x = \lim_{r \rightarrow 1} \lim_{\lambda \rightarrow \infty} \lambda R(\lambda, G_r)x = x.$$

Hence, it suffices to show that for all  $x \in X$ ,

$$\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, G_r)x = x \quad (3.4)$$

uniformly for  $0 < r < 1$ .

Now let  $y \in D(A)$ ,  $\lambda > 0$  and  $0 < r < 1$ . Since  $R(\lambda, G_r)G_ry = \lambda R(\lambda, G_r)y - y$ , we have, by (3.3),

$$\|\lambda R(\lambda, G_r)y - y\| = \|R(\lambda, G_r)G_ry\| \leq \frac{2M}{\lambda} \|(A + rB)y\| \leq \frac{2M}{\lambda} (\|Ay\| + \|By\|).$$

Now fix  $\epsilon > 0$ . Since  $D(A)$  is dense in  $X$ , for  $x \in X$  and every  $\epsilon > 0$ , there exists  $y \in D(A)$  with  $\|y - x\| < \epsilon$ . Then

$$\begin{aligned} \|\lambda R(\lambda, G_r)x - x\| &\leq \lambda \|R(\lambda, G_r)(x - y)\| + \|y - x\| + \|\lambda R(\lambda, G_r)y - y\| \\ &\leq 2M\epsilon + \epsilon + \frac{2M}{\lambda} (\|Ay\| + \|By\|). \end{aligned}$$

Since  $\epsilon$  was arbitrary, this gives uniform convergence of (3.4) in  $r$  for all  $x \in X$ . So  $\text{Im}(R_\lambda)$  is dense in  $X$ .

Therefore, we may apply the Trotter-Kato Theorem to show that  $R_\lambda = R(\lambda, G)$  where  $G$  generates a  $C_0$ -semigroup  $(V(t))_{t \geq 0}$  satisfying

$$\lim_{r \rightarrow 1} U_r(t)x = V(t)x \quad \text{for all } x \in X$$

with the limit uniform for  $t$  in bounded intervals. Moreover,  $(V(t))_{t \geq 0}$  is also a substochastic  $C_0$ -semigroup since  $(V(t))_{t \geq 0}$  is the strong limit of substochastic  $C_0$ -semigroups.

To show that the series  $R(\lambda)$  converges strongly for all  $\lambda > 0$  and  $R(\lambda) = R(\lambda, G)$ , we will utilise the monotonicity of  $R(\lambda, G_r)x = \sum_{k=0}^{\infty} R(\lambda, A)(rBR(\lambda, A))^k x$ ,  $x \in X_+$  as  $r \nearrow 1$ . Let  $\lambda > 0$ . For every  $x \in X$  and  $n \in \mathbb{N}_0$ , we define

$$\begin{aligned} R_r^{(n)}(\lambda)x &:= \sum_{k=0}^n R(\lambda, A)(rBR(\lambda, A))^k x, \quad 0 \leq r < 1, \\ R^{(n)}(\lambda)x &:= \sum_{k=0}^n R(\lambda, A)(BR(\lambda, A))^k x. \end{aligned}$$

Now fix  $x \in X_+$  and  $\epsilon > 0$ . Choose  $r < 1$  such that  $\|R(\lambda, G)x - R(\lambda, G_r)x\| < \epsilon$  and for such  $r$ , choose an  $m$  such that  $\|R(\lambda, G_r)x - R_r^{(m)}(\lambda)x\| < \epsilon$ . Since  $R(\lambda, A)$  and  $BR(\lambda, A)$  are positive, for  $n > m$ , we have

$$R_r^{(m)}(\lambda)x \leq R^{(n)}(\lambda)x. \quad (3.5)$$

Moreover, since  $R^{(n)}(\lambda)x = \lim_{r \nearrow 1} R_r^{(n)}(\lambda)x$  and  $R_r^{(n)}(\lambda)x \leq R(\lambda, G_r)x$ , it follows that

$$R^{(n)}(\lambda)x = \lim_{r \nearrow 1} R_r^{(n)}(\lambda)x \leq \lim_{r \nearrow 1} R(\lambda, G_r)x = R(\lambda, G)x. \quad (3.6)$$

Combining (3.5) and (3.6), we have

$$\begin{aligned} 0 &\leq R(\lambda, G)x - R^{(n)}(\lambda)x \leq R(\lambda, G)x - R_r^{(m)}(\lambda)x \\ &= R(\lambda, G)x - R(\lambda, G_r)x + R(\lambda, G_r)x - R_r^{(m)}(\lambda)x \end{aligned}$$

By the monotonicity of the norm, it follows that  $\|R(\lambda, G)x - R^{(n)}(\lambda)x\| < 2\epsilon$  and so  $R(\lambda, G)x = \lim_{n \rightarrow \infty} R^{(n)}(\lambda)x$ . Since  $X_+$  is generating, we can conclude that  $R(\lambda, G)x = R(\lambda)x$  for all  $x \in X$ .

The proof that  $G$  is an extension of  $A + B$  and  $(V(t))_{t \geq 0}$  is the minimal semigroup generated by an extension of  $A + B$  can be done exactly as in [3, Theorem 2.1] as follows:

To show that  $G$  is an extension of  $A + B$ , fix  $\lambda > 0$  and  $x \in X$ . Let  $y \in D(A)$  and  $x = (\lambda - A)y$ . Then  $R^{(n)}(\lambda)x = y + R^{(n-1)}(\lambda)By$  for all  $n \in \mathbb{N}$ . Let  $n \rightarrow \infty$  to obtain  $R(\lambda, G)(\lambda - A)y = y + R(\lambda, G)By$ , i.e.  $(\lambda - A - B)y = (\lambda - G)y$  for all  $y \in D(A)$ . Therefore  $G$  is an extension of  $A + B$ .

Finally, we show that  $(V(t))_{t \geq 0}$  is minimal. Let  $(\tilde{V}(t))_{t \geq 0}$  be a substochastic  $C_0$ -semigroup in  $X$  whose generator  $G'$  is an extension of  $A + B$ . It is easy to show that for all  $\lambda > 0$ ,

$$R(\lambda, G') - R(\lambda, G_r) = R(\lambda, G')(G' - G_r)R(\lambda, G_r).$$

Since  $\text{Im}(R(\lambda, G_r)) = D(A)$ , it follows that

$$\begin{aligned} R(\lambda, G') - R(\lambda, G_r) &= R(\lambda, G')(A + B - A - rB)R(\lambda, G_r) \\ &= (1 - r)R(\lambda, G')BR(\lambda, G_r) \geq 0. \end{aligned}$$

Taking the strong limit as  $r \rightarrow 1$ , we have  $R(\lambda, G') \geq R(\lambda, G)$  and therefore by the Post-Widder Inversion Formula (Proposition 1.2.8),  $\tilde{V}(t) \geq V(t)$  for all  $t \geq 0$ .  $\square$

*Remark 3.1.5.* We can actually prove a slightly stronger result about the minimality of the semigroup  $(V(t))_{t \geq 0}$  (as was done for  $KB$ -spaces in [10, Proposition 3.10]). It turns out that  $(V(t))_{t \geq 0}$  is in fact the minimal substochastic  $C_0$ -semigroup whose generator is an extension of  $(A + B)|_D$  where  $D$  is a core of  $A$ . This follows since the generator  $G'$  of any such semigroup  $(V'(t))_{t \geq 0}$  is in fact an extension of  $A + B$ .

To see this, note that since  $D$  is a core of  $A$  and  $B$  is  $A$ -bounded, it follows that for all  $x \in D(A)$ , there exists  $(x_n) \subseteq D$  such that  $x_n \rightarrow x$  and  $(A + B)x_n \rightarrow (A + B)x$ .

Since  $G'$  is a closed operator and  $G' \supseteq (A + B)|_D$ , it follows that  $D(A) \subseteq D(G')$  and  $G' \supseteq A + B$ .

Theorem 3.1.2 gives sufficient conditions which ensure that positively perturbed generators of substochastic semigroups remain generators of substochastic semigroups. A natural question which follows is whether these conditions are also necessary. In particular, are conditions (ii), (iv) and (v) in Theorem 3.1.2 also necessary? We currently only have a positive answer for condition (iv) which follows from condition (i) in Proposition 1.2.5 and the fact that  $G \supseteq A + B$ .

**Proposition 3.1.6.** *Let  $X$  be a real, ordered Banach space with generating cone and monotone norm. Suppose  $(U(t))_{t \geq 0}$  is a substochastic semigroup on  $X$  with generator  $A$  and  $B : D(A) \rightarrow X$  a positive linear operator. If there exists an extension  $G$  of  $A + B$  that generates a substochastic  $C_0$ -semigroup  $(V(t))_{t \geq 0}$  on  $X$ , then  $A + B$  satisfies condition (iv) in Theorem 3.1.2, that is:*

*For any  $x \in D(A)_+$  there is an  $x^* \in X_+^*$  such that  $\|x^*\|_{X^*} = 1$ ,  $\langle x^*, x \rangle = \|x\|_X$  and  $\langle x^*, (A + B)x \rangle \leq 0$ .*

## 3.2 Honesty Theory beyond Abstract State Spaces

As we saw in Section 2.2.1, honesty theory via a functional approach was derived for Kato's Theorem in abstract state spaces. Currently, this theory does not extend to spaces beyond abstract state spaces even though versions of Kato's Theorem exist for spaces such as  $KB$ -spaces (Theorem 3.1.1). The functional approach cannot be easily generalised to general ordered Banach spaces because the boundedness of the functionals in abstract state spaces depend on the additivity of the norm on the positive cone. The lack of such functionals in general spaces means that we cannot use the definition of honesty as given in Definition 2.2.1 for general ordered Banach spaces. Hence, we will use a different definition here.

Our motivation for the definition we use in this section comes from the honesty theory for abstract state spaces. From Theorem 2.2.4, we can infer that the perturbed semigroup  $(V(t))_{t \geq 0}$  is honest if and only if  $G = \overline{A + B}$ . This indicates that the honesty of the perturbed semigroup in Kato's Theorem is closely related to the characterisation of its generator. Moreover, from a purely mathematical point of view, the study of when  $D(A + B)$  is a core of the extension  $G$  in Kato's Theorem is of interest on its own. Therefore we define honesty as the following:

**Definition 3.2.1.** *Suppose the conditions in Theorem 3.1.2 hold. The perturbed semigroup  $(V(t))_{t \geq 0}$  is honest if and only if  $G = \overline{A + B}$ .*

The corresponding definition for pointwise honesty holds, i.e. for  $x \in X_+$ , the trajectory  $(V(t)x)_{t \geq 0}$  is honest if and only if  $R(\lambda, G)x \in D(\overline{A + B})$ . However, in the rest of our results, we will concentrate on the honesty of the semigroup and not single trajectories.

We will first look at previously known characterisations of honesty for Kato's Theorem on abstract state spaces and attempt to generalise them to spaces without additive norm structure. We will then derive a new characterisation of honesty involving the uniqueness of solutions to an abstract Cauchy problem in general ordered Banach spaces in Section 3.2.2.

### 3.2.1 Modifying Conditions from Abstract State Spaces

As we saw in Section 2.2.1, honesty theory in abstract state spaces has been extensively studied (see [31], [35] and [3]). The results presented in Sections 2.3 and 2.4 also added to the theory of honesty for Kato's Theorem in abstract state spaces. In this section, we will study these results and see how they translate to spaces beyond abstract state spaces. We will see that the additivity of the norm on the positive cone of abstract state spaces plays an important role and the results become less complete in spaces which do not have this property. Finally, note that although the honesty results beyond abstract state spaces will be stated for operators satisfying Theorem 3.1.2, for all practical purposes, they are only useful for the case of  $KB$ -spaces since we noted in the previous section that Theorem 3.1.2 has no practical applications currently.

In particular, we will look at how the following characterisations of honesty on abstract state spaces given below (see Theorem 2.2.4, Proposition 2.3.2, Theorem 2.3.3) translate to general ordered Banach spaces.

**Theorem 3.2.2.** *Suppose  $X$  is an abstract state space and  $A, B, (V(t))_{t \geq 0}$  are as in Theorem 2.1.2. Let  $\lambda > 0$ . The following are equivalent:*

- (i) *The semigroup  $(V(t))_{t \geq 0}$  is honest.*
- (ii)  *$\lim_{n \rightarrow \infty} \|[BR(\lambda, A)]^n x\| = 0$  for all  $x \in X_+$ .*
- (iii)  *$\{(BR(\lambda, A))^n x\}_{n \in \mathbb{N}}$  converges weakly to 0 for all  $x \in X_+$ .*
- (iv) *The set  $\{[BR(\lambda, A)]^n x\}_{n \in \mathbb{N}}$  is relatively weakly compact for all  $x \in X_+$ .*

(v)  $w^* - \lim_{n \rightarrow \infty} (BR(\lambda, A))^n f = 0$  for all  $f \in X^*$ .

(vi) The operator  $BR(\lambda, A)$  is mean ergodic.

(vii)  $\ker(\lambda - (A + B)^*) = \{0\}$ .

First consider condition (vii). Recall that in Section 2.2.1, we noted that Theorem 2.2.6 holds for any complex Banach space  $X$ . Hence it holds for the operators  $A, B$  in Theorem 3.1.2 as well. Moreover for operators  $A, B$  which satisfy the hypotheses of Theorem 3.1.2, we have that  $BR(\lambda, A)$ ,  $\lambda > 0$  is a bounded operator too. Thus by Lemma 2.2.9, we have  $\text{Im}(\lambda - A - B) = \text{Im}(I - BR(\lambda, A))$  for  $\lambda > 0$ . Combining these, we see that condition (vii) above carries over to ordered Banach spaces with monotone norm as well, namely

**Proposition 3.2.3.** *Let  $X$  be a real, ordered Banach space with generating cone and monotone norm and  $A, B, (V(t))_{t \geq 0}$  be as in Theorem 3.1.2.  $(V(t))_{t \geq 0}$  is honest if and only if  $\ker(I - (BR(\lambda, A))^*) = \ker(\lambda - (A + B)^*) = \{0\}$  for all/some  $\lambda > 0$ .*

Next let us consider conditions (ii)–(v) of Theorem 3.2.2. It is clear that (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) and by the generating cone condition, (iii)  $\Leftrightarrow$  (v). It turns out that these conditions are sufficient to ensure the honesty of the semigroup.

**Proposition 3.2.4.** *Let  $X$  be a real, ordered Banach space with generating cone and monotone norm and  $A, B, (V(t))_{t \geq 0}$  be as in Theorem 3.1.2. Fix  $\lambda > 0$ . Any one of the following conditions is sufficient for the honesty of  $(V(t))_{t \geq 0}$ :*

(i)  $\lim_{n \rightarrow \infty} \|(BR(\lambda, A))^n x\| = 0$  for all  $x \in X$ .

(ii)  $w - \lim_{n \rightarrow \infty} (BR(\lambda, A))^n x = 0$  for all  $x \in X$ .

(iii)  $\{(BR(\lambda, A))^n x\}_{n=0}^{\infty}$  is relatively weakly compact for all  $x \in X$ .

*Proof.* Note first that (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) follows easily from the definitions of weak limits and weak compactness so it suffices to show that (iii) implies honesty of  $(V(t))_{t \geq 0}$ .

Let  $y \in D(G)$ . Then there exists  $x \in X$  such that  $y := R(\lambda, G)x$ . Define  $y_n := \sum_{k=0}^n R(\lambda, A)[BR(\lambda, A)]^k x$ . Then  $y_n \in D(A + B)$  and converges to  $y = R(\lambda, G)x$  in  $X$  as  $n \rightarrow \infty$ . Moreover, we have

$$(\lambda - A - B)y_n = \sum_{k=0}^n (BR(\lambda, A))^k x - \sum_{k=0}^n (BR(\lambda, A))^{k+1} x = x - (BR(\lambda, A))^{n+1} x.$$

Since the sequence  $\{(BR(\lambda, A))^n x\}_{n=0}^\infty$  is relatively weakly compact, there is a subsequence  $\{(BR(\lambda, A))^{n_j} x\}_{j=0}^\infty$  that converges weakly in  $X$  to some element  $z \in X$ . This means that  $(\lambda - A - B)y_{n_j}$  converges weakly to  $x - BR(\lambda, A)z$  as  $j \rightarrow \infty$ . Since the graph of  $\overline{A + B}$  is closed (and equivalently weakly closed) and  $y_{n_j} \rightarrow y$ , it follows that  $y \in D(\overline{A + B})$  and moreover,  $(\lambda - \overline{A + B})y = x - BR(\lambda, A)z$ . Since  $y$  was an arbitrary element of  $D(G)$  and  $G \supseteq A + B$ , we can conclude that  $D(G) = D(\overline{A + B})$ . Therefore  $(V(t))_{t \geq 0}$  is honest.  $\square$

We can show that the conditions of Proposition 3.2.4 are also necessary for honesty if we know that  $G = A + B$ .

**Proposition 3.2.5.** *Let  $X$  be a real, ordered Banach space with generating cone and monotone norm and  $A, B, G$  be as in Theorem 3.1.2. Then  $G = A + B$  if and only if  $\lim_{k \rightarrow \infty} \|(BR(\lambda, A))^k\| = 0$ .*

*Proof.* Note that  $\lim_{k \rightarrow \infty} \|(BR(\lambda, A))^k\| = 0$  if and only if  $r_\sigma(BR(\lambda, A)) < 1$ . But  $r_\sigma(BR(\lambda, A)) < 1$  implies that  $1 \in \rho(BR(\lambda, A))$ , which by Theorem 2.2.6, is equivalent to  $G = A + B$ .  $\square$

The next result is a variant of Proposition 3.2.5 for a more general setting which was originally proven by Arlotti and Banasiak for Banach lattices with order continuous norm [9, Proposition 5.11].

**Proposition 3.2.6.** *Let  $X$  be an ordered Banach space with normal, generating cone. Suppose also that  $A$  is a linear operator on  $X$  which is resolvent positive for  $\lambda > \omega$  and  $B$  is a linear operator that is positive on  $D(A)$ . Suppose there is an extension  $G$  of  $(A + B, D(A))$  that generates a semigroup and the resolvent of  $G$  satisfies*

$$R(\lambda, G)x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x, \quad x \in X, \lambda \in \rho(G).$$

*Then  $G = A + B$  if and only if*

$$\sum_{k=0}^{\infty} (BR(\lambda, A))^k x$$

*converges for all  $x \in X$  and any  $\lambda > \omega$ .*

Proposition 3.2.6 can be proven in exactly the same way as [9, Proposition 5.11] as Arlotti and Banasiak restrict their result to Banach lattices with order continuous norm in order to apply [9, Proposition 5.10]. However [9, Proposition 5.10] is precisely

Theorem 1.1.4 restricted to the case of Banach lattices with order continuous norm and it holds more generally for all ordered Banach spaces with normal, generating cone.

Finally, we consider condition (vi) in Theorem 3.2.2, i.e. the mean ergodic characterisation of honesty. Propositions 3.2.7 and 3.2.8 below are also given for a more general set-up, namely we once again assume simply that  $X$  is a Banach space and suppose that  $A, B$  are linear operators with  $D(B) \supseteq D(A)$  and  $(G, D(G))$  is an extension of  $(A + B, D(A))$  such that  $\Lambda := \rho(A) \cap \rho(G) \neq \emptyset$ .

**Proposition 3.2.7.** *Suppose for some  $\lambda \in \Lambda$ ,  $BR(\lambda, A)$  is a mean ergodic operator. Then  $G = \overline{A + B}$ . In particular, if  $BR(\lambda, A)$  is uniformly ergodic, then  $G = A + B$ .*

*Proof.* From Theorem 2.2.6, we have that  $\ker(I - BR(\lambda, A)) = \{0\}$ . Combining this with the mean ergodicity of  $BR(\lambda, A)$  and applying Corollary 1.4.2, we have that  $X = \overline{\text{Im}(I - BR(\lambda, A))}$ . Thus Theorem 2.2.6 implies that  $G = \overline{A + B}$ . The second assertion follows by a similar argument using Theorem 1.4.3 and Theorem 2.2.6.  $\square$

The converse of Proposition 3.2.7 requires additional assumptions on  $BR(\lambda, A)$ .

**Proposition 3.2.8.** *Suppose  $A + B$  is closable and  $\lambda \in \rho(\overline{A + B}) \cap \rho(A) \neq \emptyset$ . Suppose also that  $BR(\lambda, A)$  is Cesàro bounded and satisfies (1.6). Then  $BR(\lambda, A)$  is mean ergodic. In particular, if  $A + B$  is closed and  $\lim_{n \rightarrow \infty} \frac{1}{n} \|(BR(\lambda, A))^n\| = 0$ , then  $BR(\lambda, A)$  is uniformly ergodic.*

*Proof.* Since  $A + B$  is closable, from Lemma 2.2.5 (iii), we know that  $BR(\lambda, A)$  is bounded. Moreover, from Theorem 2.2.6, we have that  $\ker(I - BR(\lambda, A)) = \{0\}$  and that  $1 \in \sigma_c(BR(\lambda, A))$  or  $1 \in \rho(BR(\lambda, A))$ . Hence  $X = \overline{\text{Im}(I - BR(\lambda, A))}$ . An application of Corollary 1.4.2 now tells us that  $BR(\lambda, A)$  is mean ergodic.

If  $A + B$  is closed, it follows from Theorem 2.2.6 that  $1 \in \rho(BR(\lambda, A))$  and so  $X = \text{Im}(I - BR(\lambda, A))$ . Then Theorem 1.4.3 implies that  $BR(\lambda, A)$  is uniformly ergodic.  $\square$

If we add positivity assumptions, it turns out that uniform ergodicity of  $BR(\lambda, A)$  is automatic if  $A + B$  generates a positive  $C_0$ -semigroup.

**Lemma 3.2.9.** *[41, Corollary 4.2] Let  $X$  be an ordered Banach space with normal, generating cone. Suppose  $A, B$  are linear operators such that  $A$  is resolvent positive and  $(B, D(A))$  is positive. If  $(A + B, D(A))$  is the generator of a positive  $C_0$ -semigroup, then  $BR(\lambda, A)$  is power-bounded and quasi-compact for all  $\lambda > s(A + B)$ .*

From Proposition 1.4.4, it follows that every power bounded and quasi-compact operator is uniformly ergodic, so Lemma 3.2.9 implies that  $BR(\lambda, A)$  is uniformly ergodic if  $A + B$  generates a positive semigroup. Noting that ordered Banach spaces with monotone norm have normal cone, we may use this to derive the following result for honesty in spaces with monotone norm.

**Proposition 3.2.10.** *Let  $X$  be a real, ordered Banach space with generating cone and monotone norm and  $A, B, (V(t))_{t \geq 0}$  be as in Theorem 3.1.2.*

(i) *Suppose  $BR(\lambda, A)$  is mean ergodic for some  $\lambda > 0$ . Then  $(V(t))_{t \geq 0}$  is honest. In particular, if  $BR(\lambda, A)$  is uniformly ergodic, then  $G = A + B$ .*

(ii) *Suppose  $(V(t))_{t \geq 0}$  is honest. If for some  $\lambda > 0$ ,  $BR(\lambda, A)$  is Cesáro bounded and satisfies (1.6), then  $BR(\lambda, A)$  is mean ergodic. In particular, if  $G = A + B$ , then  $BR(\lambda, A)$  is uniformly ergodic.*

*Proof.* First recall that by definition,  $(V(t))_{t \geq 0}$  is honest if and only if  $G = \overline{A + B}$ . Hence (i) follows directly from Proposition 3.2.7. (ii) follows from Proposition 3.2.8 and the discussion above where we saw that Lemma 3.2.9 and Proposition 1.4.4 imply that  $BR(\lambda, A)$  is uniformly ergodic if  $A + B$  generates a positive semigroup.  $\square$

Unlike in Proposition 3.2.10, we saw in Proposition 2.3.2 that in the special case of abstract state spaces, the mean ergodicity of  $BR(\lambda, A)$  for some  $\lambda > 0$  is both sufficient and necessary for the honesty of  $(V(t))_{t \geq 0}$ . Proposition 2.3.2 is fact a corollary of Proposition 3.2.10 since  $BR(\lambda, A)$  is power bounded for  $A, B$  satisfying Kato's Theorem in abstract state spaces (Lemma 2.3.1) and power bounded operators are Cesáro bounded and satisfy (1.6).

As mentioned in Section 2.3.1, the mean ergodic approach to honesty was inspired by [41] where Tyran-Kamińska considered dissipative operators and semigroups of contractions on general Banach spaces. Propositions 3.2.7 and 3.2.8 can be thought of as a minor generalisation of [41, Theorem 1.1, Theorem 3.3] with the semigroup-related properties of dissipativity and contractivity replaced by the existence of a closed extension of  $A + B$ . It is also worth noting that Tyran-Kamińska in [41] proves a variant of Kato's Theorem in real Banach lattices [41, Theorem 1.3] where she shows the existence of an honest semigroup under the additional condition that  $BR(\lambda, A)$  is mean ergodic. This result holds in fact, for all Banach spaces with generating cone and monotone norm which we can show by modifying the proof of Theorem 3.1.2.

**Theorem 3.2.11.** *Let  $X$  be an ordered Banach space with generating cone and monotone norm and suppose that the operators  $A$  and  $B$  with  $D(A) \subseteq D(B) \subseteq X$  satisfy:*

- (i)  $A$  generates a substochastic semigroup  $(U_A(t))_{t \geq 0}$ ,
- (ii)  $Bx \geq 0$  for  $x \in D(A)_+$ ,
- (iii) for any  $x \in D(A)_+$ , there is an  $x^* \in X_+^*$  such that  $\|x^*\| = 1$ ,  $\langle x^*, x \rangle = \|x\|$  and  $\langle x^*, (A + B)x \rangle \leq 0$ .

If for some  $\lambda > 0$  the operator  $BR(\lambda, A)$  is mean ergodic, then  $\overline{A + B}$  generates a substochastic  $C_0$ -semigroup.

*Sketch of Proof.* Note first that under the assumptions of Theorem 3.2.11, condition (ii) in Theorem 3.1.2 holds automatically because the mean ergodicity of  $BR(\lambda, A)$  implies that  $r_\sigma(BR(\lambda, A)) \leq 1$ . Moreover, the proof of Lemma 3.1.4 does not apply condition (v) of Theorem 3.1.2 so the existence of the semigroups  $(U_r(t))_{t \geq 0}$  generated by  $G_r, 0 < r < 1$  still holds without assuming condition (v). In order to obtain the perturbed semigroup as the strong limit of the semigroups  $(U_r(t))_{t \geq 0}$  in the proof of Theorem 3.1.2, we applied condition (ii) of the Trotter-Kato Theorem (Theorem 1.2.6). However, the assumption that  $BR(\lambda, A)$  is mean ergodic in Theorem 3.2.11 implies that  $\overline{\text{Im}(\lambda - A - B)} = X$ , so we can apply condition (i) of the Trotter-Kato Theorem instead in this case to conclude that  $\overline{A + B}$  generates a substochastic semigroup.  $\square$

Tyran-Kamińska proves Theorem 3.2.11 for the special case of Banach lattices by applying the Lumer-Phillips Theorem. The dissipativity of  $A + B$  follows from condition (iii) of Theorem 3.2.11 while the mean ergodicity of  $BR(\lambda, A)$  implies that  $\overline{\text{Im}(\lambda - A - B)} = X$ . Finally, note that in Theorem 3.2.11, we do not conclude that  $R(\lambda, \overline{A + B})$  has the series representation  $R(\lambda)$  because the assumptions in Theorem 3.2.11 do not include condition (v) of Theorem 3.1.2.

Finally, observe that Proposition 3.2.10 implies that any conditions ensuring the mean ergodicity of  $BR(\lambda, A)$  will be sufficient to ensure the honesty of the semigroup. This method allows us to derive some sufficient conditions for honesty on spaces beyond abstract state spaces. Recalling that an element  $x \in X_+$  is called a quasi-interior point if  $\langle f, x \rangle > 0$  for all  $f \in X_+^* \setminus \{0\}$  [11, pp.238-239], consider the following sufficient condition for honesty in abstract state spaces which is a restatement of [3, Theorem 3.14] and [3, Theorem 3.26].

**Proposition 3.2.12.** *Let  $X$  be an abstract state space and  $A, B, (V(t))_{t \geq 0}$  be as in Theorem 2.1.2. Suppose there exists a quasi-interior point  $x \in X_+$  such that  $BR(\lambda, A)x \leq x$  for some  $\lambda > 0$ . Then the semigroup  $(V(t))_{t \geq 0}$  is honest.*

To prove a version for general order continuous Banach lattices, we will apply stronger assumptions which follow from the following lemma:

**Lemma 3.2.13.** *[28, Corollary II.1] Let  $X$  be a Banach lattice with order continuous norm and  $T$  a positive contraction on  $X$ . If there is a quasi-interior point  $x \in X_+$  such that  $Tx \leq x$  and a strictly positive  $\phi \in X^*$  such that  $T^*\phi \leq \phi$ , then  $T$  is mean ergodic.*

**Proposition 3.2.14.** *Let  $X$  be a Banach lattice with order continuous norm and the operators  $A, B, (V(t))_{t \geq 0}$  be as in Theorem 3.1.2. Suppose for some  $\lambda > 0$ ,  $BR(\lambda, A)$  is contractive and there is a quasi-interior point  $x \in X_+$  such that  $BR(\lambda, A)x \leq x$  and a strictly positive  $\phi \in X^*$  such that  $(BR(\lambda, A))^*\phi \leq \phi$ . Then the semigroup  $(V(t))_{t \geq 0}$  is honest.*

*Remark 3.2.15.* The only result about honesty beyond abstract state spaces which we can currently find in the literature is Remark 3.9 in [10] where the authors observe that in  $L^p$ -spaces with  $p \in (1, \infty)$ , if  $\|BR(\lambda, A)\| \leq 1$ , then the semigroup  $(V(t))_{t \geq 0}$  is honest. Their argument uses the spectral approach (Theorem 2.2.6) and some results on the existence of primitive  $n$ th roots of unity in the spectrum of  $BR(\lambda, A)$ . The mean ergodic approach allows us to extend their result since in reflexive Banach spaces, all power-bounded operators are mean ergodic [33, Theorem 2.1.2]. This result is given in Proposition 3.2.16 below and is inspired by a similar result for dissipative operators and contraction semigroups in [41, Corollary 1.2].

**Proposition 3.2.16.** *Let  $X$  be a real, ordered Banach space with generating cone and monotone norm and  $A, B, (V(t))_{t \geq 0}$  be as in Theorem 3.1.2. If  $X$  is reflexive and  $BR(\lambda, A)$  is power-bounded for some  $\lambda > 0$ , then  $(V(t))_{t \geq 0}$  is honest.*

As we saw in this section, the conditions characterising honesty of Kato's semigroup in abstract state spaces is generally sufficient for honesty in spaces beyond abstract state spaces too. However, we are currently unable to show the necessity of these conditions as well in general spaces. We do have partial results, for example in Proposition 3.2.10, we showed that mean ergodicity is necessary for honesty if  $BR(\lambda, A)$  satisfies additional assumptions. Note however, that in the case where  $G = A + B$ , we can in fact show that the conditions are also necessary (Proposition 3.2.5 and Proposition 3.2.10). This indicates that the characterisations of  $G = \overline{A + B}$  obtained in abstract state spaces may depend on the additional structure from the additive norm of abstract state spaces. Unfortunately, we have so far been unable to construct any example to illustrate this fact. In the next section, we will derive a new characterisation of honesty which is independent of the additive norm structure.

### 3.2.2 A New Approach to Uniqueness and Honesty

We saw in Section 2.3.3 that Kato's semigroup on abstract state spaces is honest if and only if it is the unique semigroup whose generator is an extension of  $A + B$ . From the proof of Theorem 2.3.4, we see that statement (i) of the theorem, namely "honesty implies uniqueness" holds for any Banach space  $X$ . The proof of statement (ii) however, requires the application of the functionals  $a_0$  and  $\bar{a}$ . As we are currently unable to generalise the functional approach to spaces beyond abstract state spaces, we are unable to use this approach to prove statement (ii) for more general ordered Banach spaces. However, we will prove a variant of the result in this section, in which we characterise honesty by uniqueness of solutions to an abstract Cauchy problem in replacement of uniqueness of the Kato semigroup. To do so, we will apply results relating uniqueness of solutions of abstract Cauchy problems to eigenvectors of operators. In the rest of this section, we will assume that  $X$  is a complex Banach space and the operators  $A, B, G$  and  $(V(t))_{t \geq 0}$  denote the complexification of the operators in Kato's Theorem.

**Definition 3.2.17.** *Suppose  $X$  is a Banach space. Let  $\mathcal{A} : D(\mathcal{A}) \subseteq X \rightarrow X$  be a linear operator and  $x \in X$ . The initial value problem*

$$\begin{aligned} \dot{u}(t) &= \mathcal{A}u(t) & t \geq 0 \\ u(0) &= x \end{aligned} \tag{ACP}$$

*is called the abstract Cauchy problem associated to  $(\mathcal{A}, D(\mathcal{A}))$  and the initial value  $x$ .*

*A function  $u : \mathbb{R}_+ \rightarrow X$  is called a (classical) solution of (ACP) if  $u$  is continuously differentiable with respect to  $t$ ,  $u(t) \in D(\mathcal{A})$  for all  $t \geq 0$ , and (ACP) holds.*

A solution of (ACP) is said to be of *normal type  $w$*  if  $\limsup_{t \rightarrow \infty} t^{-1} \log \|u(t)\| = w < \infty$ . We shall say that a solution is of *normal type* if it is of normal type  $w$  for some  $w < \infty$ . A non-zero solution of (ACP) with initial condition 0 will be called a *nul-solution*. Note that if (ACP) has a nul-solution, then any solution to (ACP) for any non-zero  $x \in X$  is not unique. We need the following result on nul-solutions.

**Proposition 3.2.18.** *[9, Theorem 3.48] If  $\mathcal{A}$  is closed, a necessary and sufficient condition so that (ACP) has a nul-solution of normal type  $\leq w$  is that the characteristic equation*

$$\mathcal{A}x_\lambda = \lambda x_\lambda$$

*has a solution  $x_\lambda \neq 0$ , bounded and holomorphic in each half plane  $\operatorname{Re}(\lambda) \geq w + \epsilon$ ,  $\epsilon > 0$ .*

The theorem above will allow us to prove the following characterisation of honesty which we state in terms of dishonesty as it simplifies the proof.

**Theorem 3.2.19.** *Let  $X$  (resp.  $A, B$ ) denote the complexification of the space (resp. operators) satisfying Theorem 3.1.2. The following are equivalent:*

(i)  $(V(t))_{t \geq 0}$  is dishonest.

(ii) There exists a solution  $y_\lambda \in X^* \setminus \{0\}$  of

$$(A + B)^* y_\lambda = \lambda y_\lambda$$

for each  $\lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ .

(iii) There exists a solution  $y_\lambda \in X^* \setminus \{0\}$  of

$$(A + B)^* y_\lambda = \lambda y_\lambda$$

for each  $\lambda \in \mathbb{C}_+$ , which is bounded and holomorphic in  $\lambda$  on each half plane  $\operatorname{Re}(\lambda) \geq \epsilon$ ,  $\epsilon > 0$ .

(iv) The abstract Cauchy problem

$$\frac{d}{dt} u(t) = (A + B)^* u(t), \quad u(0) = 0$$

has a non-zero solution of normal type  $\leq 0$ .

In order to prove Theorem 3.2.19, we require two lemmas which are mild variants of [9, Lemma 3.50, Corollary 3.51] in that we also consider adjoint semigroups. The proofs follows almost exactly those of [9, Lemma 3.50, Corollary 3.51] but we give them here for the convenience of the reader.

**Lemma 3.2.20.** *Let  $X$  be a Banach space (resp. the dual of a Banach space). Suppose  $\mathcal{A}$  is a closed operator on  $X$  and  $T$  generates a bounded  $C_0$ -semigroup (resp. adjoint semigroup) on  $X$  with  $T \subset \mathcal{A}$ . Then for all  $\lambda \in \mathbb{C}_+$ ,*

$$D(\mathcal{A}) = \operatorname{Im}(P_\lambda) \oplus \ker(P_\lambda) = D(T) \oplus \ker(\lambda - \mathcal{A})$$

where  $P_\lambda = R(\lambda, T)(\lambda - \mathcal{A})$ .

*Proof.* To show that  $D(\mathcal{A})$  can be decomposed into the direct sum, it suffices to check that the mapping  $P_\lambda = R(\lambda, T)(\lambda - \mathcal{A})$  is a bounded linear projection of  $D(\mathcal{A})$  onto  $D(T)$ . Since  $T \subset \mathcal{A}$ , it follows easily that  $P_\lambda x = x$  for all  $x \in D(T)$  and  $P_\lambda^2 = P_\lambda$ . Moreover, we have  $\text{Im}(\lambda - \mathcal{A}) \supset \text{Im}(\lambda - T)$ , hence  $P_\lambda$  maps  $D(\mathcal{A})$  onto  $D(T)$ . By construction, it is a bounded mapping from  $D(\mathcal{A})$  onto  $D(T)$  equipped with their respective graph norms. Therefore, it follows that  $D(\mathcal{A}) = D(T) \oplus \ker(P_\lambda)$ . It remains to show that  $\ker(P_\lambda) = \ker(\lambda - \mathcal{A})$ . It is clear that  $\ker(\lambda - \mathcal{A}) \subset \ker(P_\lambda)$ . The reverse inclusion follows since  $R(\lambda, T)$  is injective.  $\square$

The next lemma tells us more about the properties of the eigenvectors of  $\mathcal{A}$ .

**Lemma 3.2.21.** *Let  $X$  be a Banach space (resp. the dual of a Banach space). Suppose  $\mathcal{A}$  is a closed operator on  $X$  and  $T$  generates a bounded  $C_0$ -semigroup (resp. adjoint semigroup) on  $X$  with  $T \subset \mathcal{A}$ . If  $D(\mathcal{A}) \setminus D(T) \neq \emptyset$ , then  $\sigma_p(\mathcal{A}) \supseteq \mathbb{C}_+$ . Moreover, there exists a non-zero, holomorphic (in the norm of  $X$ ) function  $e_\lambda : \mathbb{C}_+ \rightarrow X$ ,  $\lambda \mapsto e_\lambda$  such that  $e_\lambda \in \ker(\lambda - \mathcal{A})$  for all  $\lambda \in \mathbb{C}_+$  and  $e_\lambda$  is bounded on each half plane  $\text{Re}(\lambda) \geq \epsilon, \epsilon > 0$ .*

*Proof.* Let  $x \in D(\mathcal{A}) \setminus D(T)$  and  $\mathcal{A}x = f$ . Fix  $\lambda$  with  $\text{Re}(\lambda) > 0$  and define  $y := R(\lambda, T)(\lambda - \mathcal{A})x \in D(T)$ . From Lemma 3.2.20, we know that  $e'_\lambda := x - y \in \ker(\lambda - \mathcal{A})$ . Moreover,  $e'_\lambda = x - R(\lambda, T)(\lambda - \mathcal{A})x = x - \lambda R(\lambda, T)x + R(\lambda, T)f = -TR(\lambda, T)x + R(\lambda, T)f$ . Since  $\lambda \mapsto R(\lambda, T)$  is holomorphic on  $\mathbb{C}_+$ , it follows that  $\lambda \mapsto e'_\lambda$  is holomorphic on the same region.

Since  $T$  generates a bounded semigroup (or its adjoint) (we denote the bound by  $C$ ), by the Hille-Yosida Theorem (Theorem 1.2.2), we have  $\|R(\lambda, T)\| \leq \frac{C}{\text{Re}(\lambda)}$  for all  $\lambda \in \mathbb{C}_+$ . Now define  $e_\lambda := \frac{C}{\lambda} e'_\lambda$ . Then  $e_\lambda$  is holomorphic on  $\mathbb{C}_+$  as the multiplication of two holomorphic functions is holomorphic. Moreover,  $e_\lambda \in \ker(\lambda - \mathcal{A})$  and  $e_\lambda$  is bounded on each half plane  $\text{Re}(\lambda) \geq \epsilon, \epsilon > 0$ .  $\square$

*Proof of Theorem 3.2.19.* We begin by noting that (i)  $\Leftrightarrow$  (ii) follows from the complex version of Proposition 3.2.3 (cf. Proposition 2.2.15) while (iii)  $\Leftrightarrow$  (iv) follows from Proposition 3.2.18. Now (iii) implies (ii) is obvious. To complete the proof, we show that (i)  $\Rightarrow$  (iii). To show this, first note that (i) holds if and only if  $G \supsetneq \overline{A + B}$  i.e.  $G^* \subsetneq (A + B)^*$ . Therefore,  $D((A + B)^*) \setminus D(G^*) \neq \emptyset$ . (iii) now follows from Lemma 3.2.21.  $\square$

Note that condition (iv) in Theorem 3.2.19 can be rephrased as ‘‘The (ACP) with  $\mathcal{A} = (A + B)^*$  has a nul-solution’’. In other words, the (ACP) with  $\mathcal{A} = (A + B)^*$  and non-zero  $x \in X^*$  has non-unique solution. Additionally, recall that if  $(\mathcal{A}, D(\mathcal{A}))$  is the

generator of the  $C_0$ -semigroup  $(T(t))_{t \geq 0}$ , then for every  $x \in X$ , the map  $t \mapsto T(t)x$  is the unique mild solution of (ACP) [22, Proposition II.6.4]. Since  $A + B \subseteq G$  and  $G$  generates the  $C_0$ -semigroup  $(V(t))_{t \geq 0}$ , this indicates that condition (iv) is related to the uniqueness of the Kato semigroup as the semigroup whose generator is an extension of  $A + B$  (cf. Theorem 2.3.4). Unfortunately, we are currently unable to work out a precise characterisation of honesty based on the uniqueness of the semigroup from condition (iv) of Theorem 3.2.19.

# Chapter 4

## Stochastic Completeness of Graphs

In this chapter, we will explore an application of honesty theory in the study of the heat equation on graphs. We will see that although honesty theory of Kato's Theorem is based in  $\ell^1$  spaces, it has an  $\ell^2$  counterpart in the study of Laplacians on graphs. In particular, we will show that honesty is equivalent to a notion known as stochastic completeness which is related to the heat semigroup on graphs.

We begin by introducing the theory of the heat equation on graphs before proving the equivalence of the two concepts. We then demonstrate some further applications of honesty in the study of the heat equation on graphs. In this chapter, Kato's Theorem will refer to Theorem 2.1.1 unless stated otherwise.

### 4.1 Laplacians on Graphs

The main focus of this chapter is a concept known as stochastic completeness. The concept of stochastic completeness occurs in the study of the heat equation on a variety of geometric objects including manifolds [29] and graphs [21, 32]. This chapter however, will only focus on the case involving symmetric weighted graphs which define Laplacians on  $\ell^2$ . Stochastic completeness of graphs is related to the conservation or loss of heat in the graph. The loss of heat is attributed to two reasons; heat loss within the graph from internal factors, and heat loss by transport to "infinity". The notion of stochastic completeness occurs when we try to differentiate between the two methods of heat loss. More precisely, we say that a graph is stochastically complete if there is no loss of heat to "infinity". This study of heat loss is encapsulated mathematically in terms of a Laplacian,  $L$ , on the weighted sequence space  $\ell_m^2$  and the heat semigroup it generates,  $(e^{-tL})_{t \geq 0}$ .

Stochastic completeness for graphs has been studied in various settings. For example, Dodziuk and Mathai in [21] studied the bounded Laplacian by assuming a

uniform bound on the vertex degree of the graph. Wojciechowski in [48] on the other hand, studied locally finite graphs by mimicking the approach for the corresponding notion of stochastic completeness on Riemannian manifolds. Keller and Lenz then generalised these results by studying stochastic completeness of graphs with non-vanishing killing terms via non-local, regular Dirichlet forms on discrete sets in [32].

The two notions of stochastic completeness and honesty were introduced separately and as far as we know, have been studied independently to date. Although Keller and Lenz in [32], acknowledge the strong relation between the work on stochastic completeness of graphs and the work on Markov processes by Feller in [25] and Reuter in [38], which is in fact where the term honesty originated, there has been no formal attempt so far to link these two concepts together. The main result of this chapter shows the equivalence of the two notions (Theorem 4.2.7). It turns out that the heat semigroup is a substochastic semigroup and the Laplacian on graphs can be reformulated as the sum of two operators. This allows us to rephrase the theory of heat equations on graphs in terms of additive perturbations of substochastic semigroups and thus show the equivalence of stochastic completeness and honesty.

We will present the theory of stochastic completeness by considering regular Dirichlet forms on discrete sets as carried out by Keller and Lenz in [32]. We begin by introducing Laplacians on graphs.

Let  $V$  be a countable set. Let  $m$  be a measure on  $V$  with full support i.e.  $m$  is a map  $m : V \rightarrow (0, \infty)$ . We will consider functions on the measure space  $(V, m)$ . Of particular interest is the space  $\ell_m^p := \ell^p(V, m)$ ,  $1 \leq p < \infty$ , over  $\mathbb{R}$  defined by

$$\left\{ u : V \rightarrow \mathbb{R} : \sum_{x \in V} m(x) |u(x)|^p < \infty \right\}.$$

We will denote by  $\ell^\infty$  the space of bounded functions on  $V$  equipped with the supremum norm  $\|\cdot\|_\infty$ . This space does not depend on the choice of  $m$ . We will also let  $C_c := C_c(V)$  denote the space of finitely supported functions on  $V$ .

A symmetric weighted graph over  $V$  is a pair  $(b, c)$  consisting of a map  $b : V \times V \rightarrow [0, \infty)$  with  $b(x, x) = 0$  for all  $x \in V$  and a map  $c : V \rightarrow [0, \infty)$  satisfying:

- (i)  $b(x, y) = b(y, x)$  for all  $x, y \in V$ .
- (ii)  $\sum_{y \in V} b(x, y) < \infty$  for all  $x \in V$ .

$(V, b, c)$  then represents a weighted graph with vertex set  $V$  and  $b(x, y)$  the weight on the edge connecting the point  $x$  and  $y$ . If  $c(x) > 0$ , we think of  $x$  as connected to

the point “infinity” by an edge with weight  $c(x)$  and heat can flow out of the graph to “infinity” but not vice versa. The map  $c$  is also known as the killing term.

For each graph  $(V, b, c)$ , consider the closed form  $Q^M = Q_{b,c,m}^M$  defined on the Hilbert space  $\mathfrak{H} := \ell_m^2$  to  $[0, \infty]$  with diagonal given by

$$Q^M(u) := Q^M(u, u) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} b(x, y) (u(x) - u(y))^2 + \sum_{x \in V} c(x) u(x)^2.$$

We denote its restriction to  $C_c$  by  $Q^C := Q_{b,c}^C$  i.e.  $Q^C = Q^M|_{C_c}$ . Since  $Q^M$  is closed,  $Q^C$  is closable. We will denote its closure by  $Q = Q_{b,c,m}$  and its domain by  $D(Q)$  which is the closure of  $C_c$  under the  $\|\cdot\|_Q$ -norm. It is easy to see that  $Q$  is also positive. Hence, by Proposition 1.3.4, there exists a unique  $\mathfrak{H}$ -positive, self-adjoint operator  $L = L_{b,c,m}$  with domain  $D(Q) = D(L^{1/2})$  and  $Q(u) = \langle L^{1/2}u, L^{1/2}u \rangle$  for  $u \in D(Q)$ . It turns out that  $Q$  is in fact a regular Dirichlet form [32, Theorem 7] i.e.  $Q$  satisfies Theorem 1.3.5 and Theorem 1.3.6 and  $D(Q) \cap C_c$  is both dense in  $C_c$  with respect to the  $\|\cdot\|_\infty$ -norm and dense in  $D(Q)$  with respect to  $\|\cdot\|_Q$ . This means that  $-L$  generates a positive, contractive semigroup on  $\ell_m^2$  which can be extended to positive, contractive semigroups  $(e^{-tL_p})_{t \geq 0}$  on  $\ell_m^p$  for all  $p \in [1, \infty]$  with generators denoted  $-L_p$  and  $L = L_2$ . From the construction of  $Q$ , we can describe the action of the operator  $L$  explicitly.

Define the formal Laplacian  $\tilde{L}$  on the vector space

$$\tilde{F} := \left\{ u : V \rightarrow \mathbb{R} : \sum_{y \in V} |b(x, y)u(y)| < \infty \text{ for all } x \in V \right\}$$

by

$$\tilde{L}u(x) := \frac{1}{m(x)} \sum_{y \in V} b(x, y) (u(x) - u(y)) + \frac{c(x)}{m(x)} u(x)$$

for all  $x \in V$ . Note that  $\ell^\infty \subseteq \tilde{F}$ . Moreover, for any  $p \in [1, \infty]$ ,  $L_p f = \tilde{L}f$  for all  $f \in D(L_p) \subseteq \tilde{F}$  [32, Lemma 2.8, Theorem 9].

The notion of subgraphs and their relation to the original graphs will play an important role in this chapter. Let  $(V, b, c)$  be the weighted graph with measure  $m$  and  $W \subset V$  with measure  $m_W$  the restriction of  $m$  to  $W$ . A subgraph  $(W, b_W, c_W)$  of a weighted graph  $(V, b, c)$  is given by a subset  $W$  of  $V$  and the restriction  $b_W$  of  $b$  to  $W \times W$  and the restriction  $c_W$  of  $c$  to  $W$ . The subgraph  $(W, b_W, c_W)$  then gives rise to a regular Dirichlet form  $Q_{b_W, c_W, m_W} := \overline{Q_{b_W, c_W}^C}^{\|\cdot\|_{Q_{b_W, c_W, m_W}}}$  on  $\ell^2(W, m_W)$  with associated operator  $L_{b_W, c_W, m_W}$ .

Let  $i_W : \ell^2(W, m_W) \rightarrow \ell^2(V, m)$  be the canonical embedding and  $p_W : \ell^2(V, m) \rightarrow \ell^2(W, m_W)$  the canonical projection. It will also be useful to consider the form (defined on  $C_c(W)$ )

$$\begin{aligned} Q_W^C(u) &= Q(i_W u) \\ &= Q_{b_W^C, c_W^C}^C(u) + \sum_{x \in W} d_W(x) u^2(x) \end{aligned}$$

where  $d_W(x) := \sum_{y \in V \setminus W} b(x, y)$ , with corresponding formal operator

$$\begin{aligned} \tilde{L}_W u(x) &= \frac{1}{m(x)} \left( \sum_{y \in W} b(x, y)(u(x) - u(y)) + \left( \sum_{y \in V \setminus W} b(x, y) + c(x) \right) u(x) \right) \\ &= \frac{1}{m(x)} \left( \left( \sum_{y \in V} b(x, y) + c(x) \right) u(x) - \sum_{y \in W} b(x, y) u(y) \right), \quad x \in W. \end{aligned}$$

Alternatively, one can view  $Q_W := \overline{Q_W^C}^{\|\cdot\|_{Q_W}}$  as the form associated with the weighted graph  $(W, b_W^D, c_W^D)$  where  $b_W^D = b_W$  and  $c_W^D = c_W + d_W$ . Hence, a similar argument as above shows that  $Q_W$  is a regular Dirichlet form (see also [30, p.112]) and thus is associated with an operator  $L_W$  and the semigroup  $(e^{-tL_W})_{t \geq 0}$  on  $\ell^2(W, m_W)$ . For simplicity of notation, for  $f \in \ell^2(V, m)$ , we will write  $e^{-tL_W} f$  to mean  $i_W e^{-tL_W} (p_W f)$  and similarly for the resolvent operators.

The following proposition which tells us that the heat semigroup on a graph can be approximated by heat semigroups on its subgraphs will play an important role in the next section.

**Proposition 4.1.1.** *Let  $(V, b, c)$  be the weighted graph with measure  $m$  and  $(W_n)$  an increasing sequence of subsets of  $V$  satisfying  $V = \bigcup_{n=1}^{\infty} W_n$ . Then for any  $t \geq 0$  and  $f \in \ell^2(V, m)$ ,*

$$e^{-tL_{W_n}} f \xrightarrow[n \rightarrow \infty]{\ell^2(V, m)} e^{-tL} f.$$

*Proof.* Fix  $0 \leq f \in \ell^2(V, m)$ . Then [32, Theorem 11(a)] and the Dominated Convergence Theorem implies that  $R(\lambda, L_{W_n})f \xrightarrow[n \rightarrow \infty]{\ell^2(V, m)} R(\lambda, L)f$ . Since the resolvent operators are positive and every  $f \in \ell^2(V, m)$  has a decomposition,  $f = f^+ - f^-$  with  $f^+, f^- \geq 0$ , this convergence holds for all  $f \in \ell^2(V, m)$ . The strong convergence of the semigroups then follows from a Trotter Approximation Theorem for  $C_0$ -semigroups on approximating sequences of Banach spaces [40, Theorem 5.1].  $\square$

Now we can introduce the concept of stochastic completeness. We begin by introducing a function, formally defined as

$$M_t(x) := e^{-tL} \mathbb{1}(x) + \int_0^t \left( e^{-sL} \frac{c}{m} \right) (x) ds, \quad x \in V. \quad (4.1)$$

Note that the function is well-defined if  $\frac{c}{m} \in \ell_m^p$  for some  $p \in [1, \infty]$  (which may not necessarily hold) and it satisfies  $0 \leq M_t \leq 1$ . Although this bound can be shown by direct calculation, there is an alternative proof which we defer until Section 4.3 as it requires the results of Section 4.2 and Section 4.3. Moreover, from the differentiability of the semigroup  $(e^{-tL})_{t \geq 0}$  and the integral term, we have that the function  $t \mapsto M_t(x)$  is continuous and even differentiable for every  $x \in V$ . We are interested in when the function  $M_t$  is equal to  $\mathbb{1}$ .

**Theorem 4.1.2.** [32, Theorem 1] *Let  $(V, b, c)$  be a weighted graph and  $m$  a measure on  $V$ . Then for any  $\lambda > 0$ , the function*

$$w_\lambda := \int_0^\infty \lambda e^{-\lambda t} (\mathbb{1} - M_t) dt$$

*satisfies  $0 \leq w_\lambda \leq \mathbb{1}$ , solves  $(\tilde{L} + \lambda)w_\lambda = 0$  and is the largest non-negative  $u \leq \mathbb{1}$  with  $(\tilde{L} + \lambda)u \leq 0$ . In particular, the following are equivalent:*

- (i) *For any  $\lambda > 0$ , there exists  $u \in (\ell^\infty)_+ \setminus \{0\}$  with  $(\tilde{L} + \lambda)u \leq 0$ .*
- (ii) *For any  $\lambda > 0$ , there exists  $u \in (\ell^\infty) \setminus \{0\}$  with  $(\tilde{L} + \lambda)u = 0$ .*
- (iii) *For any  $\lambda > 0$ , there exists  $u \in (\ell^\infty)_+ \setminus \{0\}$  with  $(\tilde{L} + \lambda)u = 0$ .*
- (iv)  *$w_\lambda \neq 0$  for any  $\lambda > 0$ .*
- (v)  *$M_t(x) < 1$  for some  $x \in V$  and some  $t > 0$ .*
- (vi) *There exists a non-trivial, bounded, non-negative  $N : V \times [0, \infty) \rightarrow [0, \infty)$  satisfying  $\frac{d}{dt}N = -\tilde{L}N$ ,  $N(0) = 0$  in the sense that for every  $x \in V$ , the function  $t \mapsto N_x(t)$  is continuous on  $[0, \infty)$ , differentiable on  $(0, \infty)$  and for every  $t > 0$ ,  $N_t \in D(\tilde{L})$  and  $N$  satisfies  $\frac{d}{dt}N_x(t) = -\tilde{L}N_x(t)$ ,  $N_x(0) = 0$  for all  $t > 0$  and  $x \in V$ .*

We define stochastic completeness as precisely the situation when  $M_t = \mathbb{1}$  so the above result is in fact the main theorem characterising stochastic completeness. More precisely,

**Definition 4.1.3.** [32, Definition 1.1] *The weighted graph  $(V, b, c)$  with measure  $m$  is said to satisfy stochastic incompleteness at infinity ( $SI_\infty$ ) if it satisfies one (and thus all) of the equivalent assertions of Theorem 4.1.2. Otherwise,  $(V, b, c)$  is said to satisfy stochastic completeness at infinity ( $SC_\infty$ ).*

Note that if  $c = 0$ , i.e. the case of vanishing killing term, then  $M_t$  is simply  $e^{-tL}\mathbb{1}$ . Then stochastic completeness of the graph is equivalent to the semigroup being stochastic or conservative. More generally, the function  $M_t$  tells us in fact, that a graph is stochastically complete if there is no heat loss to “infinity”. This follows because the term  $e^{-tL}\mathbb{1}$  can be interpreted as the amount of heat contained in the graph at time  $t$  while the integral term can be interpreted as the amount of heat lost within the graph up to time  $t$  [32, p.195]. Therefore  $\mathbb{1} = M_t$  if and only if there is no heat lost to “infinity”.

## 4.2 The Equivalence of Stochastic Completeness and Honesty

The aim of this section is to show that honesty of the heat semigroup is in fact equivalent to stochastic completeness of the graph. The main difficulties in showing this equivalence stem from the fact that the Laplacian acts on  $\ell^2$  while Kato’s theory considers semigroups on  $\ell^1$ . Hence, one of the key steps in proving that honesty is equivalent to stochastic completeness is showing that the semigroup on  $\ell^2$  (graph case) and the semigroup on  $\ell^1$  (Kato case) are in fact compatible. The first step towards showing this is demonstrating that the theory of Laplacians on graphs fits into the framework of Kato’s Theorem.

We begin by reformulating the theory of Laplacians on graphs in terms of Kato’s Theorem. Since Kato’s Theorem is a result on perturbations, we consider  $-\tilde{L}$  as the sum of two operators  $A, B$  on  $\ell_m^1$  with

$$Au(x) = -\frac{1}{m(x)} \left( \sum_{y \in V} b(x, y) + c(x) \right) u(x), x \in V \text{ with } D(A) = \{u \in \ell_m^1 : Au \in \ell_m^1\} \quad (4.2)$$

and

$$Bu(x) = \frac{1}{m(x)} \sum_{y \in V} b(x, y)u(y), x \in V \text{ with } D(B) = D(A). \quad (4.3)$$

We can take the domain of  $B$  to be  $D(A)$  since for  $u \in D(A)$ ,

$$\|Bu\|_1 \leq \sum_{x \in V} \sum_{y \in V} |b(x, y)u(y)| = \sum_{y \in V} |u(y)| \sum_{x \in V} b(x, y) \leq \|Au\|_1.$$

First, we show that the decomposition  $A, B$  satisfies Kato's Theorem.

**Proposition 4.2.1.** *A and B satisfy the hypotheses of Kato's Theorem, hence there exists  $G \supseteq A + B$  that generates a  $C_0$ -semigroup of positive contractions on  $\ell_m^1$ ,  $(V(t))_{t \geq 0}$ .*

To prove Proposition 4.2.1, we simply check that  $A$  and  $B$  satisfy the conditions in Theorem 2.1.1. We begin by showing that  $A$  generates a substochastic semigroup on  $\ell_m^1$ .

**Lemma 4.2.2.** *A generates a  $C_0$ -semigroup of positive contractions on  $\ell_m^1$ . Moreover,  $A = \overline{A|_{C_c}}$ .*

*Proof.* Since  $A$  is a multiplication operator, it follows that it generates the semigroup of multiplication by the function  $(e^{at})_{t \geq 0}$  where  $a(x) = -\frac{1}{m(x)} \left( \sum_{y \in V} b(x, y) + c(x) \right)$ ,  $x \in V$  (see [22, Section II.2.9]). The fact that the semigroup is positive and contractive on  $\ell_m^1$  now follows directly from the construction of the semigroup and the fact that  $a \leq 0$ . The final assertion is elementary to prove, hence is omitted.  $\square$

In the rest of this chapter, we will denote the duality between  $\ell_m^1$  and  $\ell^\infty$  by  $\langle u, v \rangle_m = \sum_{x \in V} m(x)u(x)v(x)$  for all  $u \in \ell^\infty$ ,  $v \in \ell_m^1$ .

*Proof of Proposition 4.2.1.* We know from the above lemma that  $A$  generates a substochastic semigroup on  $\ell_m^1$  so it remains to consider the operator  $B$ . By definition, we have that  $D(B) = D(A)$ . Moreover,  $b(x, y) \geq 0$  for all  $x, y \in V$ , hence it follows immediately that  $B$  is a positive operator.

Finally, we have that

$$\langle \mathbb{1}, (A + B)u \rangle_m = \left\langle \mathbb{1}, -\tilde{L}u \right\rangle_m = - \sum_{x \in V} c(x)u(x) \leq 0$$

for all  $u \in D(A)_+$ . Therefore, we can conclude that  $A$  and  $B$  satisfy Kato's Theorem and the result follows.  $\square$

We now have (potentially) two semigroups on  $\ell_m^1$ , one of which comes from Kato's Theorem,  $(V(t))_{t \geq 0}$ , while the other originates from considering  $\tilde{L}$  on  $\ell_m^2$ ,  $(e^{-tL_1})_{t \geq 0}$ . The next step will be to show that the two semigroups coincide namely:

**Theorem 4.2.3.** *Let  $A, B$  be defined by (4.2) and (4.3) respectively. Then the heat semigroup on  $\ell_m^1$ ,  $(e^{-tL_1})_{t \geq 0}$  coincides with the perturbed semigroup  $(V(t))_{t \geq 0}$  derived from  $A$  and  $B$  in Kato's Theorem.*

We will give two proofs here; the first is via approximations of the semigroup while the second relies on properties of Dirichlet forms. We begin with some auxiliary information on subgraphs. In particular, we deduce new information about the subgraphs of  $(V, b, c)$  from the reformulation of the set-up of the Laplacian on graphs in terms of Kato's framework.

Let  $A, B$  denote the operators in Kato's Theorem associated with the weighted graph  $(V, b, c)$ ,  $W$  denote any subset of  $V$  and  $\tilde{L}_W$  the operator associated with the weighted graph  $(W, b_W^D, c_W^D)$  as described in Section 4.1. Note that since  $\tilde{L}_W$  is associated with the weighted graph  $(W, b_W^D, c_W^D)$ , it follows from Proposition 4.2.1 that the operators

$$\begin{aligned} A_W u(x) &= -\frac{1}{m(x)} \left( \sum_{y \in V} b(x, y) + c(x) \right) u(x), x \in W, \text{ with} \\ D(A_W) &= \{u \in \ell^1(W, m_W) : A_W u \in \ell^1(W, m_W)\} \end{aligned} \quad (4.4)$$

and

$$B_W u(x) = \frac{1}{m(x)} \sum_{y \in W} b(x, y) u(y), x \in W \text{ with domain } D(B_W) = D(A_W) \quad (4.5)$$

satisfy the hypotheses of Kato's Theorem on  $\ell^1(W, m_W)$  with associated Kato subgraph semigroup denoted  $(V_W(t))_{t \geq 0}$  and generator  $G_W$ .

The following extension of the Kato subgraph semigroup  $(V_W(t))_{t \geq 0}$  to  $\ell^1(V, m)$  will turn up repeatedly later. Let  $\tilde{B}_W$  be the extension of  $B_W$  to  $\ell^1(V, m)$  defined by

$$\tilde{B}_W u = i_W B_W (p_W u). \quad (4.6)$$

By definition,  $\tilde{B}_W$  is positive and  $\tilde{B}_W u \leq B u$  for all  $u \in D(A)_+$ . Hence it follows from Lemma 2.1.4 that  $A, \tilde{B}_W$  also satisfy Kato's Theorem on  $\ell^1(V, m)$  with perturbed semigroup denoted  $(\tilde{V}_W(t))_{t \geq 0}$  and generator  $\tilde{G}_W$ . We will refer to this semigroup as the extended Kato semigroup associated with the subgraph  $(W, b_W^D, c_W^D)$ .

$(\tilde{V}_W(t))_{t \geq 0}$  is an extension of  $(V_W(t))_{t \geq 0}$  in the following sense: From the definitions of  $A, A_W, B_W, \tilde{B}_W$ , it follows that for all  $u \in \ell^1(W, m_W)$ ,

$$R(\lambda, A) i_W u = i_W R(\lambda, A_W) u, \quad \tilde{B}_W R(\lambda, A) i_W u = i_W B_W R(\lambda, A_W) u. \quad (4.7)$$

Hence by Theorem 2.1.2 and the continuity of  $i_W$ , we have that

$$\begin{aligned} R(\lambda, \tilde{G}_W)i_W u &= \sum_{k=0}^{\infty} R(\lambda, A)(\tilde{B}_W R(\lambda, A))^k i_W u \\ &= i_W \left( \sum_{k=0}^{\infty} R(\lambda, A_W)(B_W R(\lambda, A_W))^k u \right) \\ &= i_W R(\lambda, G_W)u \end{aligned}$$

or equivalently,

$$\tilde{V}_W(t)(i_W u) = i_W V_W(t)u. \quad (4.8)$$

In addition to subgraphs, we will also need the following auxiliary lemmas. First, recall that if  $T$  is associated with a Dirichlet form, it generates  $C_0$ -semigroups of contractions on all  $\ell_m^p$  (see Theorem 1.3.5, Theorem 1.3.6), which we will denote  $(U_p(t))_{t \geq 0}$  with generators  $T_p$  or simply  $(U(t))_{t \geq 0}$  and  $T$  wherever they coincide.

**Lemma 4.2.4.** *Suppose  $T : D(T) \subset \ell_m^2 \rightarrow \ell_m^2$  is an operator associated with a Dirichlet form and generates the semigroup  $(U(t))_{t \geq 0}$ . Let  $H_1 \in \mathcal{L}(\ell_m^1)$ ,  $H_2 \in \mathcal{L}(\ell_m^2)$  such that  $H_1|_{\ell_m^1 \cap \ell_m^2} = H_2|_{\ell_m^1 \cap \ell_m^2}$ . Then the perturbed semigroups generated by  $T_1 + H_1$  on  $\ell_m^1$ ,  $(S_1(t))_{t \geq 0}$  and  $T_2 + H_2$  on  $\ell_m^2$ ,  $(S_2(t))_{t \geq 0}$  coincide on  $\ell_m^1 \cap \ell_m^2$ .*

*Proof.* Note first that we will use  $H$  to denote both  $H_1, H_2$  wherever they coincide.

To prove the result, simply consider the Dyson-Phillips series of the perturbed semigroups  $(S_1(t))_{t \geq 0}$  and  $(S_2(t))_{t \geq 0}$  which are respectively

$$S_1(t) = \sum_{n=0}^{\infty} U_{n,1}(t), \quad S_2(t) = \sum_{n=0}^{\infty} U_{n,2}(t)$$

where for  $i = 1, 2$

$$\begin{aligned} U_{0,i}(t)u &= U(t)u \quad \text{for all } u \in \ell_m^1 \cap \ell_m^2, \\ U_{n+1,i}(t)u &= \int_0^t U_{n,i}(t-s)H_i U_{0,i}(s)u \, ds \quad \text{for all } u \in \ell_m^i, n \geq 0 \\ &= \int_0^t U_{n,i}(t-s)H U(s)u \, ds \quad \text{for all } u \in \ell_m^1 \cap \ell_m^2, n \geq 0. \end{aligned}$$

Thus by induction on  $n$ , it follows that for all  $u \in \ell_m^1 \cap \ell_m^2$ , we have  $U_{n,1}(t)u = U_{n,2}(t)u$  for all  $n \in \mathbb{N}$  and all  $t \geq 0$ . Therefore  $(S_1(t))_{t \geq 0}$  and  $(S_2(t))_{t \geq 0}$  coincide on  $\ell_m^1 \cap \ell_m^2$ .  $\square$

The second lemma we require in order to prove Theorem 4.2.3 follows from the fact that if  $(f_n)$  is Cauchy in  $L^p(\mu)$ , then there exists  $f \in L^p(\mu)$  such that there is a subsequence  $(f_{n_j})$  satisfying  $f_{n_j} \rightarrow f$  a.e. and  $\|f_n - f\|_p \rightarrow 0$ .

**Lemma 4.2.5.** *Suppose  $(f_n) \subset \ell_m^1 \cap \ell_m^2, g_1 \in \ell_m^1, g_2 \in \ell_m^2$  such that  $f_n \rightarrow g_1$  in  $\ell_m^1$  and  $f_n \rightarrow g_2$  in  $\ell_m^2$ . Then  $g_1 = g_2$ .*

We are now ready to prove Theorem 4.2.3.

*Proof of Theorem 4.2.3.* Choose an increasing sequence of finite subsets  $F_n \subseteq V$ ,  $n \in \mathbb{N}$ , such that  $\cup_n F_n = V$ . Define  $b_n(x, y) = ((\chi_{F_n \times F_n})b)(x, y)$  where  $\chi_W$  denotes the indicator function for the set  $W$ . Then the operators  $B_n, n \in \mathbb{N}$  defined by

$$B_n u(x) = \sum_{y \in V} b_n(x, y) u(y) \quad \text{for all } x \in V$$

are bounded operators in  $\ell_m^1$  and  $\ell_m^2$ . Taking  $A_2$  to be the multiplication operator with maximal domain, i.e.

$$A_2 u(x) = -\frac{1}{m(x)} \left( \sum_{y \in V} b(x, y) + c(x) \right) u(x), \quad x \in V, \quad \text{with}$$

$$D(A_2) = \{u \in \ell_m^2 : A_2 u \in \ell_m^2\},$$

similar arguments to those in Lemma 4.2.2 show that  $A_2$  generates a positive  $C_0$ -semigroup of contractions and is self-adjoint. Hence  $A_2$  is associated with a Dirichlet form with compatible set of generators denoted  $A_p$ . Now fix  $n \in \mathbb{N}$ . Taking  $T_2 = A_2$ ,  $T_1 = A_1$  and  $H_1 = H_2 = B_n$ , we see that  $A_1, A_2, B_n$  satisfy the conditions of Lemma 4.2.4. Hence, for each  $n \in \mathbb{N}$ ,  $t \geq 0$  and  $u \in \ell_m^1 \cap \ell_m^2$ ,  $U_n^{(1)}(t)u = U_n^{(2)}(t)u$  where  $(U_n^{(1)}(t))_{t \geq 0}$  is the semigroup generated by  $A_1 + B_n$  on  $\ell_m^1$  and  $(U_n^{(2)}(t))_{t \geq 0}$  is the semigroup generated by  $A_2 + B_n$  on  $\ell_m^2$ .

Let us consider first the semigroups  $(U_n^{(1)}(t))_{t \geq 0}, n \in \mathbb{N}$  on  $\ell_m^1$ . Recall that  $A$  and  $B$  denote the operators in (4.2) and (4.3). Since  $A$  is a multiplication operator, it is easy to see that the operator  $A_1$  is precisely the operator  $A$ . It also follows from the definitions that  $B_n u \leq B u$  for all  $u \in D(A)_+$ , all  $n \in \mathbb{N}$ . Hence by Lemma 2.1.4,  $(U_n^{(1)}(t))_{t \geq 0}$  is the Kato semigroup generated by  $A + B_n$ . Moreover, for  $u \in D(A)_+$ ,

$$\|B u - B_n u\|_{\ell_m^1} = \sum_{x \in F_n} \sum_{y \in V \setminus F_n} b(x, y) u(y) + \sum_{x \in V \setminus F_n} \sum_{y \in V} b(x, y) u(y) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore it follows from Proposition 2.1.5 that for all  $t \geq 0$ ,  $U_n^{(1)}(t)$  converges strongly to  $V(t)$  on  $\ell_m^1$  where  $(V(t))_{t \geq 0}$  is the semigroup from Kato's construction in Proposition 4.2.1.

Now consider the semigroups  $(U_n^{(2)}(t))_{t \geq 0}, n \in \mathbb{N}$  on  $\ell_m^2$ . We will show that  $U_n^{(2)}(t)$  converges strongly in  $\ell_m^2$  to  $e^{-tL}$  for all  $t \geq 0$ . Fix  $u \in \ell_m^2$  and  $t \geq 0$ . We have

$$\|U_n^{(2)}(t)u - e^{-tL}u\|_2 \leq \|U_n^{(2)}(t)(u - \chi_{F_n}u)\|_2 + \|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\|_2.$$

Note that  $\left\|U_n^{(2)}(t)(u - \chi_{F_n}u)\right\|_2 \leq \|u - \chi_{F_n}u\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ , so it remains to consider  $\left\|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\right\|_2$ . We begin by noting that since  $F_n$  is finite for all  $n$ ,  $\chi_{F_n}u \in \ell_m^1 \cap \ell_m^2$ . Hence  $U_n^{(2)}(t)(\chi_{F_n}u) = U_n^{(1)}(t)(\chi_{F_n}u)$ . Now for fixed  $n$ , consider once again the semigroup  $(U_n^{(1)}(t))_{t \geq 0}$ . By construction, it follows that  $(U_n^{(1)}(t))_{t \geq 0}$  is the extended Kato semigroup associated with the subgraph  $(F_n, b_{F_n}^D, c_{F_n}^D)$  described above. Recalling that  $(V_{F_n}(t))_{t \geq 0}$  denotes the Kato subgraph semigroup associated with  $(F_n, b_{F_n}^D, c_{F_n}^D)$ , it follows from (4.8) that

$$\left\|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\right\|_2 = \left\|i_{F_n}V_{F_n}(t)(p_{F_n}u) - e^{-tL}u\right\|_2.$$

Finally, since  $F_n$  is finite for every  $n$ , it follows that  $V_{F_n}(t) = e^{-tL_{F_n}}$  for all  $t \geq 0$ . Hence

$$\left\|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\right\|_2 = \left\|i_{F_n}e^{-tL_{F_n}}(p_{F_n}u) - e^{-tL}u\right\|_2$$

and this converges to 0 by Proposition 4.1.1. Therefore  $\left\|U_n^{(2)}(t)u - e^{-tL}u\right\|_2 \rightarrow 0$  as  $n \rightarrow \infty$  for all  $u \in \ell_m^2$ .

Since  $U_n^{(1)}(t)$  converges strongly to  $V(t)$  on  $\ell_m^1$  and  $U_n^{(2)}(t)$  converges strongly to  $e^{-tL}$  in  $\ell_m^2$  for all  $t \geq 0$ , we may conclude by Lemma 4.2.5 that  $V(t)f = e^{-tL}f$  for all  $f \in \ell_m^1 \cap \ell_m^2$  and all  $t \geq 0$ .  $\square$

We now turn to a second proof of Theorem 4.2.3 using quadratic forms instead of approximations by finite subgraphs. In this proof, we will require some information about the generator  $G$  in Kato's Theorem given in the following lemma whose proof follows exactly like that of Lemma 10 in [31] (see also [9, Theorem 6.20]).

**Proposition 4.2.6.** *The generator  $G$  is a restriction of the maximal operator  $-L_1^{\max}$  where  $L_1^{\max}u = \tilde{L}u$  for all  $u \in D(L_1^{\max}) = \{u \in \ell_m^1 : \tilde{L}u \in \ell_m^1\}$ .*

*Second proof of Theorem 4.2.3.* Let  $(e_x)$  denote the normalised standard basis elements. Then from Proposition 4.2.6, it follows  $\langle Gu, e_x \rangle_m = -\tilde{L}u(x)$  for all  $u \in D(G)$ . Moreover, from direct calculation, we have that

$$\tilde{L}e_x(z) = \begin{cases} \frac{1}{m(x)^2} \sum_y b(x, y) + \frac{c(x)}{m(x)^2} & \text{if } z = x \\ -\frac{1}{m(x)m(z)} b(x, z) & \text{if } z \neq x. \end{cases}$$

Thus it follows that

$$\left\langle u, -\tilde{L}e_x \right\rangle_m = -\frac{1}{m(x)} \sum_{y \in V} b(x, y)(u(x) - u(y)) - \frac{c(x)}{m(x)} u(x) = -\tilde{L}u(x).$$

Therefore

$$\langle Gu, e_x \rangle_m = -\tilde{L}u(x) = \left\langle u, -\tilde{L}e_x \right\rangle_m. \quad (4.9)$$

Since  $\tilde{L}e_x \in \ell^\infty$ , it follows that  $C_c \subseteq D(G^*)$ . From Lemma 4.2.2, we also know that  $C_c \in D(G)$ , hence  $C_c \subseteq D(G) \cap D(G^*)$  and so  $D(G) \cap D(G^*)$  is dense in  $\ell_m^1 \cap \ell^\infty$ . Moreover, from (4.9), it also follows that  $G = G^*$  on  $D(G) \cap D(G^*)$ . Thus, by Proposition 1.3.8, we have that the Kato semigroup  $(V(t))_{t \geq 0}$  can be extended to a set of compatible semigroups on  $\ell_m^p$ , denoted  $(V^{(p)}(t))_{t \geq 0}$ , with generators  $G_p$  and moreover,  $Q_{G_2} := Q_{(-G_2)}$ , the form associated with the generator  $G_2$  is a Dirichlet form. Hence, from Lemma 1.3.9, we have that  $D(G) \cap \ell^\infty$  is a dense subspace of  $D(Q_{G_2})$  and for all  $u \in D(G) \cap \ell^\infty$ ,  $v \in D(Q_{G_2}) \cap \ell^\infty$ , we have

$$Q_{G_2}(u, v) = \langle -Gu, v \rangle. \quad (4.10)$$

Since  $C_c \subseteq D(G) \cap \ell^\infty$ , we have that  $C_c \subseteq D(Q_{G_2})$ . Moreover, since  $G_2|_{C_c} = -\tilde{L}|_{C_c}$ , we can show by (4.10) and the definition of  $Q$  that  $Q_{G_2}|_{C_c} = Q^M|_{C_c}$ . As  $Q_{G_2}$  is closed, we have

$$Q_{G_2} \supseteq Q = \overline{Q^M|_{C_c}}. \quad (4.11)$$

To complete the proof, we will show that  $D(Q_{G_2}) \subseteq D(Q)$ .

To begin, we show that  $-L_1$  (the generator of the semigroup  $(e^{-tL_1})_{t \geq 0}$ ) is an extension of  $(A+B)|_{C_c}$ . Since  $-(A+B)|_{C_c} \subseteq \tilde{L}$  and  $L_1 \subseteq \tilde{L}$ , it suffices to show that  $C_c \subseteq D(L_1)$ . Let  $k \in C_c \subseteq D(Q) \cap \ell_m^1$  and  $g \in D(Q) \cap \ell^\infty \subseteq D(Q_{G_2}) \cap \ell^\infty$ . Then

$$\begin{aligned} Q(g, k) &= Q_{G_2}(g, k) && \text{(since } Q \subseteq Q_{G_2}\text{)} \\ &= Q_{G_2}(k, g) && \text{(since } Q_{G_2} \text{ symmetric)} \\ &= \langle -Gk, g \rangle && \text{(from (4.10))} \\ &= \langle g, -Gk \rangle. \end{aligned}$$

Since  $-Gk \in \ell_m^1$ , and  $g$  was an arbitrary element of  $D(Q) \cap \ell^\infty$ , this means that we have found an element  $h \in \ell_m^1$  such that for all  $g \in D(Q) \cap \ell^\infty$ ,  $Q(g, k) = \langle g, h \rangle$ . Hence by Lemma 1.3.10, we have that  $k \in D(L_1)$  and  $-L_1k = Gk$ . Therefore  $C_c \subseteq D(L_1)$  and so  $-L_1 \supseteq (A+B)|_{C_c}$ . Since  $C_c$  is a core for  $A$ , by Remark 3.1.5, it follows that  $V(t) \leq e^{-tL_1}$  i.e.  $V^{(2)}(t) \leq e^{-tL}$  for all  $t \geq 0$ . From this inequality and the positivity of the semigroup  $(V(t))_{t \geq 0}$ , we have

$$|V^{(2)}(t)f| \leq V^{(2)}(t)|f| \leq e^{-tL}|f| \quad \text{for all } f \in \ell_m^2, t \geq 0.$$

In other words,  $(V^{(2)}(t))_{t \geq 0}$  is dominated by  $(e^{-tL})_{t \geq 0}$ . Hence, Proposition 1.3.11 tells us that  $D(Q_{G_2}) \subseteq D(Q)$ . Combining this with (4.11), we conclude that  $Q = Q_{G_2}$  or equivalently,  $e^{-tL} = V^{(2)}(t)$ ,  $t \geq 0$ .  $\square$

If we impose an extra geometric condition on the graph, i.e.

$$\text{For any sequence } (x_n) \subseteq V \text{ with } b(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N}, \text{ we have} \quad (\text{GP})$$

$$\sum_{n \in \mathbb{N}} m(x_n) = \infty,$$

then the proof of Theorem 4.2.3 simplifies considerably. One of the results which enables this simplification is the theorem that gives the precise form of the generator in this case. In particular, Keller and Lenz show in [32, Theorem 5] that  $-L_1$  is in fact the maximal operator in this case, i.e.  $-L_1 = -L_1^{\max}$ . To complete the proof we will apply Proposition 4.2.6 and Lemma 2.3.5, which we recall, states that if  $T$  is the generator of a  $C_0$ -semigroup on a Banach space and  $S$  is a closed extension of  $T$  that also generates a  $C_0$ -semigroup, then  $S = T$ . Combining Proposition 4.2.6 with the fact that  $-L_1 = -L_1^{\max}$  and thus  $-L_1^{\max}$  generates a  $C_0$ -semigroup, it follows immediately from Lemma 2.3.5 that  $G = -L_1^{\max} = -L_1$ .

We now come to the main aim of this section.

**Theorem 4.2.7.** *Stochastic completeness ( $SC_\infty$ ) of the weighted graph  $(V, b, c)$  is equivalent to honesty of the semigroup on  $\ell^1(V, m)$ ,  $(e^{-tL_1})_{t \geq 0}$ .*

We will show the equivalence of the two concepts by showing that  $w_\lambda = \Delta_\lambda$  for some, or equivalently, all  $\lambda > 0$ , where  $\Delta_\lambda$  is the functional defined by (2.6) in Section 2.2.2 and  $w_\lambda$  is as defined in Theorem 4.1.2. To do so, we must first give a precise description of  $(A + B)^*$ . Recall from Lemma 4.2.2 that  $A = \overline{A|_{C_c}}$ . In particular, this means  $C_c \subset D(A)$  and so we can define  $G_{\min} := (A + B)|_{C_c} = -\tilde{L}|_{C_c}$ .

**Lemma 4.2.8.** *Let  $(V, b, c)$  be a weighted graph with measure  $m$ . Suppose  $A, B, \tilde{L}$ , and  $G_{\min}$  are as defined above. Then  $(A + B)^* = G_{\min}^* = -L_\infty^{\max}$  where*

$$L_\infty^{\max} u = \tilde{L}u \text{ for all } u \in D(L_\infty^{\max}) = \{u \in \ell^\infty : \tilde{L}u \in \ell^\infty\}.$$

*Proof.* The first step in the proof is to show that  $G_{\min}^* = -L_\infty^{\max}$ . By definition,  $v^* \in D(G_{\min}^*)$  if and only if there exists  $u^* \in \ell^\infty$  such that  $\langle G_{\min} u, v^* \rangle_m = \langle u, u^* \rangle_m$  for all  $u \in D(G_{\min})$  and then  $G_{\min}^* v^*$  is defined to be  $u^*$ . Since  $D(G_{\min}) = C_c$  and  $C_c$  is the span of the normalised standard basis elements  $(e_x)_{x \in V}$ , it suffices to consider the elements  $(e_x)_{x \in V}$ . Then calculations similar to those in the second proof of Theorem 4.2.3 show that for all  $x \in V$ ,

$$\langle G e_x, v^* \rangle_m = -\frac{1}{m(x)} \left( \sum_{y \in V} b(x, y) - c(x) \right) v^*(x) + \frac{1}{m(x)} \sum_{y \in V} b(x, y) v^*(y) = -\tilde{L} v^*(x).$$

So  $v^* \in D(G_{\min}^*)$  if and only if  $-\tilde{L}v^* \in \ell^\infty$  and in this case,  $G_{\min}^*v^* = -\tilde{L}v^*$ . Since we require that  $-\tilde{L}v^* \in \ell^\infty$ , this can be restated as  $v^* \in D(G_{\min}^*)$  if and only if  $v^* \in D(L_\infty^{\max})$  and in this case  $G_{\min}^*v^* = -L_\infty^{\max}v^*$ . Therefore  $G_{\min}^* = -L_\infty^{\max}$ .

The next step involves showing that  $\overline{G_{\min}} = \overline{A + B}$ . In other words, we show that  $C_c$  is a core for  $\overline{A + B}$ . However, this follows since  $C_c$  is a core for  $A$  and  $B$  is an  $A$ -bounded operator. The lemma now follows from  $G_{\min}^* = \overline{G_{\min}}^* = \overline{A + B}^* = (A + B)^*$ .  $\square$

*Proof of Theorem 4.2.7.* Fix  $\lambda > 0$ . We begin by showing that  $w_\lambda$  satisfies the eigenvalue problem  $(\lambda - (A + B)^*)w_\lambda = 0$  while  $\Delta_\lambda$  satisfies  $(\tilde{L} + \lambda)\Delta_\lambda = 0$ . From Theorem 4.1.2, we know that  $w_\lambda$  satisfies  $(\tilde{L} + \lambda)w_\lambda = 0$ . But  $w_\lambda$  is bounded and  $L_\infty^{\max} \subset \tilde{L}$ , so this can be equivalently rewritten as  $(L_\infty^{\max} + \lambda)w_\lambda = 0$ . By Lemma 4.2.8, this is equivalent to saying  $(\lambda - (A + B)^*)w_\lambda = 0$ . Noting that Proposition 2.2.3 and Lemma 2.2.10 implies that  $\Delta_\lambda$  satisfies  $(\lambda - (A + B)^*)\Delta_\lambda = 0$ , we can reverse this argument with  $\Delta_\lambda$  replacing  $w_\lambda$  to prove the second assertion.

Combining Lemma 2.2.10 and Proposition 2.2.3, we see that  $\Delta_\lambda$  is the maximal element in  $\{f \in \ell_+^\infty : f \leq \mathbb{1}\}$  that satisfies  $(\lambda - (A + B)^*)f = 0$  and so  $w_\lambda \leq \Delta_\lambda$ . Similarly, Theorem 4.1.2 states that  $w_\lambda$  is the largest non-negative  $f \leq \mathbb{1}$  such that  $(L_\infty^{\max} + \lambda)f \leq 0$ . Hence  $\Delta_\lambda \leq w_\lambda$  and so  $\Delta_\lambda = w_\lambda$ . Therefore, Theorem 2.2.4 and Theorem 4.1.2 imply that stochastic completeness at infinity is equivalent to honesty of the Kato semigroup  $(V(t))_{t \geq 0}$ , which is equal to the heat semigroup  $(e^{-tL_1})_{t \geq 0}$  on  $\ell_m^1$  by Theorem 4.2.3.  $\square$

*Remark 4.2.9.* The proof of Theorem 4.2.7 tells us in fact, that  $(SC_\infty)$  is equivalent to honesty of the Kato semigroup  $(V(t))_{t \geq 0}$  described in Proposition 4.2.1, independently of Theorem 4.2.3. The role of Theorem 4.2.3 is to connect  $(SC_\infty)$  of the given graph to its associated heat semigroup.

Note that in the proof of Theorem 4.2.7, we showed that

**Proposition 4.2.10.** *Let  $\lambda > 0$  and suppose  $w_\lambda$  is as defined in Theorem 4.1.2 and  $\Delta_\lambda$  is as defined in (2.6). Then  $w_\lambda = \Delta_\lambda$ .*

This fact will turn up again in Section 4.3.

### 4.3 Applications of Honesty in Weighted Graphs

The equivalence of honesty and stochastic completeness allows us to derive new characterisations of stochastic completeness from Theorem 2.2.4.

**Corollary 4.3.1.** *Let  $(V, b, c)$  be a weighted graph and  $A, B$  be as defined in (4.2) and (4.3). The following are equivalent:*

- (i)  $(V, b, c)$  satisfies  $(SC_\infty)$ .
- (ii)  $\lim_{n \rightarrow \infty} \|[BR(\lambda, A)]^n u\| = 0$  for all  $u \in (\ell_m^1)_+$ , some  $\lambda > 0$ .
- (iii)  $-L_1 = \overline{A + B}$ .

We demonstrate how condition (ii) of Corollary 4.3.1 may be applied to a graph. This example covers the case studied in [48] where they consider infinite but locally finite graphs, under the counting measure and with no killing term.

**Example 4.3.2.** Consider an infinite but locally finite connected graph  $(V, b, 0)$ , with  $m$  the counting measure and  $b(x, y) = 1$  if  $x$  is connected to  $y$  by an edge and 0 otherwise. We will use  $x \sim y$  to denote that  $x$  is connected to  $y$  by an edge and  $d_x$  to denote the degree of  $x$ , i.e. the number of edges emanating from  $x$ . In this case

$$\tilde{L}u(x) = \sum_{y \sim x} (u(x) - u(y)), \quad x \in V.$$

**Proposition 4.3.3.** *The graph  $(V, b, 0)$  is stochastically complete if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda + d_y} \sum_{x \in V} \sum_{\{i_1 \sim x\}} \sum_{\{i_2 \sim i_1\}} \cdots \sum_{\{(i_{n-1} \sim i_{n-2}) \cap (i_{n-1} \sim y)\}} \prod_{k=1}^{n-1} \frac{1}{\lambda + d_{i_k}} = 0$$

for all  $y \in V$  and some  $\lambda > 0$ .

*Proof.* From Corollary 4.3.1, we have that  $(V, b, 0)$  is stochastically complete if and only if  $\lim_{n \rightarrow \infty} \|[BR(\lambda, A)]^n u\| = 0$  for all  $u \in (\ell_m^1)_+$ . Since  $BR(\lambda, A)$  is power-bounded (Lemma 2.3.1), it suffices to check that  $\lim_{n \rightarrow \infty} \|[BR(\lambda, A)]^n u\| = 0$  for all  $u$  in a dense subset of  $(\ell_m^1)_+$ . In particular,  $\ell_m^1$  is the closed linear span of the standard basis,  $(e_x)_{x \in V}$ , hence  $(V, b, 0)$  is stochastically complete if and only if

$$\lim_{n \rightarrow \infty} \|[BR(\lambda, A)]^n e_x\| = 0 \text{ for all } x \in V. \quad (4.12)$$

Since  $BR(\lambda, A)$  is a positive operator, (4.12) is equivalent to

$$\lim_{n \rightarrow \infty} \sum_{x \in V} c_{xy}^{(n)} = 0 \text{ for all } y \in V$$

where  $c_{xy}^{(n)} = \langle e_x, (BR(\lambda, A))^n e_y \rangle$ ,  $x, y \in V$ .

It remains to calculate  $c_{xy}^{(n)}$ . To do so, note first that since  $b(x, y) = 1$  if and only if  $x \sim y$ , and  $R(\lambda, A)u(x) = \frac{u(x)}{\lambda + d_x}$ , we have

$$c_{xy} := c_{xy}^{(1)} = \begin{cases} \frac{1}{\lambda + d_y} & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that  $c_{xy}^{(n+1)} = \sum_{i \in V} c_{xi} c_{iy}^{(n)}$  and induction, we find that

$$c_{xy}^{(n)} = \frac{1}{\lambda + d_y} \sum_{\{i_1 \sim x\}} \sum_{\{i_2 \sim i_1\}} \cdots \sum_{\{(i_{n-1} \sim i_{n-2}) \cap (i_{n-1} \sim y)\}} \prod_{k=1}^{n-1} \frac{1}{\lambda + d_{i_k}}$$

and the result follows.  $\square$

The characterisation of  $(SC_\infty)$  in Proposition 4.3.3 is fairly complicated and not easy to apply. Consider for example the special case of model trees, namely trees whose vertex degree is constant on spheres of radius  $r$  from a fixed root vertex  $x_0$  (see [48, Section 3.2] for more details). Wojciechowski shows that a model tree satisfies  $(SC_\infty)$  if and only if  $\sum_{r=1}^{\infty} \frac{1}{d_{r-1}} + \frac{1}{d_0}$  diverges where  $d_r$  is the degree of vertices of distance  $r$  from  $x_0$  [48, Theorem 3.2.1]. Even in this simple case, it is not clear how the condition in Proposition 4.3.3 reduces to this form.

As a second example, consider the case when the Laplacian is bounded. Keller and Lenz [32, Remark (a) p.195] note that if  $\tilde{L}$  gives rise to a bounded operator on  $\ell^\infty(V)$ , then the graph is stochastically complete. Their justification for this is that condition (ii) in Theorem 4.1.2 must fail for  $\lambda$  large enough whenever  $\tilde{L}$  is bounded. Corollary 4.3.1 allows us to derive an alternative proof of this statement. To see this, note first that if  $\tilde{L}$  is bounded on  $\ell^\infty$ , then by duality,  $\tilde{L}$  is bounded on  $\ell_m^1$ . Since  $A + B \subseteq -L_1 \subseteq -\tilde{L}$  in  $\ell_m^1$ , the boundedness of  $\tilde{L}$  implies that we must have  $\overline{A + B} = -L_1$ .  $(SC_\infty)$  of the graph now follows from condition (iii) of Corollary 4.3.1.

The bounded Laplacian has been studied in a variety of contexts, for example in [21] and [48]. In [21], Dodziuk and Mathai studied graphs satisfying the set-up of Example 4.3.2 but with the additional condition that the degree of the graph is bounded, that is, there is  $M > 0$  such that  $d_x \leq M$  for all  $x \in V$ . They prove that the graphs in this case are stochastically complete by studying bounded solutions of the initial value problem:

$$\begin{aligned} \tilde{L}u + \frac{\partial u}{\partial t} &= 0 \\ u(x, 0) &= u_0(x). \end{aligned}$$

Wojciechowski on the other hand in [48], considers a normalised version of the Laplacian on connected, locally finite graphs which he calls the bounded Laplacian,  $\Delta_{bd}$  where

$$\Delta_{bd}f(x) = f(x) - \frac{1}{d_x} \sum_{y \sim x} f(y), \quad x \in V$$

which acts on functions  $f$  such that  $\sum_{x \in V} (f(x))^2 d_x < \infty$ . In the terminology we used in this chapter,  $\Delta_{bd}$  is simply the Laplacian associated with the weighted graph  $(V, b, 0)$  with  $b(x, y) = 1$  if  $x$  is connected to  $y$  by an edge and 0 otherwise and measure on the space given by  $m(x) = d_x, x \in V$ . Simple calculations [48, p.57] then show that  $\|\Delta_{bd}\| \leq 2$ . Hence any graph is stochastically complete with respect to this operator. Wojciechowski proves this by showing directly that positive solutions satisfying the eigenvalue problem  $\Delta_{bd}v(x) = \lambda v(x)$  for  $\lambda < 0$  are unbounded.

Next, we demonstrate an application of honesty theory on subgraphs. In particular, we show that the criteria for  $(SI_\infty)$  involving subgraphs in [32, Theorem 4] can be derived from the following condition for honesty which we state for abstract state spaces. Recalling from Lemma 2.1.4 that if  $A, B$  satisfy Kato's Theorem (Theorem 2.1.2) and the operator  $\tilde{B} : D(A) \rightarrow X$  satisfies  $0 \leq \tilde{B}u \leq Bu$  for all  $u \in D(A)_+$ , then  $A, \tilde{B}$  also satisfy Kato's Theorem, we have

**Proposition 4.3.4.** *Let  $X$  be an abstract state space and suppose  $A, B$  satisfy the conditions of Theorem 2.1.2 with perturbed semigroup  $(V(t))_{t \geq 0}$ . Suppose also that there exists an operator  $\tilde{B}$  such that  $0 \leq \tilde{B}u \leq Bu$  for all  $u \in D(A)_+$  with perturbed semigroup denoted  $(\tilde{V}(t))_{t \geq 0}$ . If the semigroup  $(V(t))_{t \geq 0}$  is honest, then so is  $(\tilde{V}(t))_{t \geq 0}$ .*

*Proof.* Fix  $\lambda > 0$ . Since both  $B$  and  $\tilde{B}$  are positive on  $D(A)$  and  $\tilde{B}u \leq Bu$  for all  $u \in D(A)_+$ , we have by positivity of  $R(\lambda, A)$  that  $0 \leq \tilde{B}R(\lambda, A) \leq BR(\lambda, A)$ . Iterating, we have for all  $n \in \mathbb{N}$ ,  $0 \leq (\tilde{B}R(\lambda, A))^n \leq (BR(\lambda, A))^n$  and so

$$\left\| (\tilde{B}R(\lambda, A))^n u \right\| \leq \|(BR(\lambda, A))^n u\|$$

for all  $u \in X_+$ . The result now follows since from Theorem 2.2.4, we have that  $(V(t))_{t \geq 0}$  (resp.  $(\tilde{V}(t))_{t \geq 0}$ ) is honest if and only if for some  $\lambda > 0$  and all  $u \in X_+$  we have that  $\|(BR(\lambda, A))^n u\| \rightarrow 0$  (resp.  $\|(\tilde{B}R(\lambda, A))^n u\| \rightarrow 0$ ).  $\square$

To derive [32, Theorem 4] as a corollary, we consider the extended Kato semigroup associated with the subgraph  $(W, b_W^D, c_W^D)$  with operators  $A, \tilde{B}_W$  and  $(\tilde{V}_W(t))_{t \geq 0}$  as defined in Section 4.2.

**Corollary 4.3.5.** [32, Theorem 4] Let  $(V, b, c)$  be a weighted graph with measure  $m$ . Then  $(SI_\infty)$  holds whenever there exists  $W \subseteq V$  such that the weighted graph  $(W, b_W^D, c_W^D)$  satisfies  $(SI_\infty)$ .

*Proof.* Let  $A, B, (V(t))_{t \geq 0}$  denote the operators in Kato's Theorem associated with the weighted graph  $(V, b, c)$  and  $A_W, B_W$  the operators associated with the weighted subgraph  $(W, b_W^D, c_W^D)$  as defined in (4.4) and (4.5). By Corollary 4.3.1, there exists  $u \in \ell^1(W, m_W)$  such that

$$\|(B_W R(\lambda, A_W))^n u\|_{\ell^1(W, m_W)} \rightarrow 0. \quad (4.13)$$

Now consider once again the extended Kato semigroup associated with the subgraph  $(W, b_W^D, c_W^D)$  with operators  $\tilde{B}_W$  and  $(\tilde{V}_W(t))_{t \geq 0}$  as defined by (4.6) and (4.8). From (4.7), for all  $u \in \ell^1(W, m_W)$ ,

$$\left\| (\tilde{B}_W R(\lambda, A))^n i_W u \right\|_{\ell^1(V, m)} = \|(B_W R(\lambda, A_W))^n u\|_{\ell^1(W, m_W)}.$$

Hence, by (4.13) and Theorem 2.2.4, the semigroup  $(\tilde{V}_W(t))_{t \geq 0}$  is dishonest. Since  $\tilde{B} \leq B$ , Proposition 4.3.4 implies that  $(V(t))_{t \geq 0}$  is also dishonest. The result now follows from Theorem 4.2.7.  $\square$

Finally, let us examine more closely how the theory of stochastic completeness fits into the theory of honesty. First, note that in order to prove Theorem 4.2.7, we showed that condition (iii) of Theorem 2.2.4 is equivalent to the negation of condition (iv) of Theorem 4.1.2. There are of course, alternative methods of proving this. For example, consider conditions (i)–(iii) in Theorem 4.1.2. From the proof of Theorem 4.2.7, we can deduce that the statement “ $u \in \ell^\infty$  satisfies  $(\tilde{L} + \lambda)u = 0$ ” is equivalent to saying that “ $u$  is bounded and satisfies  $(\lambda - (A + B)^*)u = 0$ ”. Hence conditions (i)–(iii) in Theorem 4.1.2 are equivalent to conditions (v)–(vii) in Theorem 2.3.3.

It turns out that condition (v) in Theorem 4.1.2 can also be derived from abstract results in honesty theory. Henceforth we will work with the honesty theory of Kato's Theorem in abstract state spaces. The functionals  $\Psi, a_0, \bar{a}$  and  $\Delta_\lambda$  which will occur here are as defined in Chapter 2 and  $\langle \cdot, \cdot \rangle$  will denote the duality between  $X$  and  $X^*$ . Let us begin by considering the function  $M_t$  defined by (4.1). It turns out that there is an analogous abstract form. Define  $\mathcal{M}_t, t \geq 0$  by

$$\langle \mathcal{M}_t, u \rangle = \langle \Psi, V(t)u \rangle + \bar{a} \left( \int_0^t V(s)u \, ds \right), \quad u \in X.$$

Then

$$\begin{aligned}
\langle \Psi - \mathcal{M}_t, u \rangle &= \langle \Psi, u - V(t)u \rangle - \bar{a} \left( \int_0^t V(s)u \, ds \right) \\
&= \left\langle \Psi, -G \int_0^t V(s)u \, ds \right\rangle - \bar{a} \left( \int_0^t V(s)u \, ds \right) \\
&= a_0 \left( \int_0^t V(s)u \, ds \right) - \bar{a} \left( \int_0^t V(s)u \, ds \right).
\end{aligned}$$

Moreover, from (2.11) and (2.12) we have that for all  $u \in X$ ,

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t} \langle \Psi - \mathcal{M}_t, u \rangle \, dt &= \lambda \int_0^\infty e^{-\lambda t} \left( a_0 \left( \int_0^t V(s)u \, ds \right) - \bar{a} \left( \int_0^t V(s)u \, ds \right) \right) dt \\
&= a_0(R(\lambda, G)u) - \bar{a}(R(\lambda, G)u) \\
&= \langle \Delta_\lambda, u \rangle.
\end{aligned}$$

Now let us investigate how the graph case fits into the abstract case. Consider the function  $w_\lambda$  from Theorem 4.1.2. Let  $u \in (\ell_m^1)_+$ . Then

$$\begin{aligned}
\langle w_\lambda, u \rangle_m &= \sum_{x \in V} m(x)w_\lambda(x)u(x) \\
&= \sum_{x \in V} \lambda \int_0^\infty e^{-\lambda t} (\mathbb{1} - M_t)(x)m(x)u(x) \, dt \\
&= \lambda \int_0^\infty e^{-\lambda t} \sum_{x \in V} (\mathbb{1} - M_t)(x)m(x)u(x) \, dt \quad (\text{Monotone Conv. Theorem}).
\end{aligned}$$

By linearity, we have  $\langle w_\lambda, u \rangle_m = \lambda \int_0^\infty e^{-\lambda t} \sum_{x \in V} (\mathbb{1} - M_t)(x)m(x)u(x) \, dt$  for all  $u \in X$ . Since by Proposition 4.2.10,  $\Delta_\lambda = w_\lambda$  or equivalently,  $\langle \Delta_\lambda, u \rangle = \langle w_\lambda, u \rangle_m$  for all  $u \in X$ , it follows that  $\lambda \int_0^\infty e^{-\lambda t} \langle \Psi - \mathcal{M}_t, u \rangle \, dt = \lambda \int_0^\infty e^{-\lambda t} \sum_{x \in V} (\mathbb{1} - M_t)(x)m(x)u(x) \, dt$ . Hence by uniqueness of the Laplace transform, we have  $\langle \Psi - \mathcal{M}_t, u \rangle = \langle \mathbb{1} - M_t, u \rangle_m$  for all  $u \in \ell_m^1$  and almost all  $t \geq 0$ . From this equality, it follows that condition (v) of Theorem 4.1.2 can be derived directly from the definition of honesty as given in Definition 2.2.1. This holds since the remarks after Definition 2.2.1 (see (2.9)) tell us that the semigroup is dishonest if and only if there exists  $u \in X_+$ ,  $t > 0$  such that

$$\|V(t)u\| + \bar{a} \left( \int_0^t V(s)u \, ds \right) < \|u\|. \quad (4.14)$$

By definition of  $\mathcal{M}_t$ , (4.14) is equivalent to

$$\langle \mathcal{M}_t, u \rangle < \langle \Psi, u \rangle.$$

Since we have just shown that  $\mathcal{M}_t = M_t$  in the  $\ell_m^1$  graph case and  $\ell_m^1$  has a (normalised) canonical Schauder basis  $(e_x)$ , it follows that dishonesty as defined by (4.14) is equivalent to

$$\langle M_t, e_x \rangle_m < \langle \mathbb{1}, e_x \rangle_m \text{ for some basis vector } e_x, x \in V \text{ and some } t > 0$$

which is precisely condition (v) of Theorem 4.1.2.

The argument above also allows us to show that  $0 \leq M_t \leq 1$  as noted in Section 4.1. Since Definition 2.2.1 and (4.14) tell us that  $0 \leq \langle \mathcal{M}_t, u \rangle \leq \langle \Psi, u \rangle$  for all  $u \in X_+$  and all  $t \geq 0$ , it follows that

$$0 \leq \langle M_t, e_x \rangle_m \leq \langle \mathbb{1}, e_x \rangle_m \text{ for all basis vectors } e_x, x \in V \text{ and all } t > 0.$$

In other words,  $0 \leq M_t \leq 1$ .

Deriving condition (vi) from the conditions for honesty is trickier. In fact, we currently do not have the precise analogue of condition (vi) in abstract form. In proving condition (vi) in Theorem 4.1.2, Keller and Lenz show that  $\mathbb{1} - M_t$  satisfy the conditions of the function  $N$  in the sense that  $(\mathbb{1} - M_t)(x)$  is differentiable for all  $x \in V$  and  $\mathbb{1} - M_t \in D(\tilde{L})$  for each  $t > 0$ . The natural analogue which we might expect for the abstract case is that  $\langle \Psi - \mathcal{M}_t, u \rangle$  is differentiable for every  $u \in X$ ,  $\Psi - \mathcal{M}_t \in D((A + B)^*)$  for each  $t > 0$  and satisfies for all  $u \in X$ ,

$$\frac{d}{dt} \langle (\Psi - \mathcal{M}_t), u \rangle = \langle (A + B)^*(\Psi - \mathcal{M}_t), u \rangle.$$

We first consider the differentiability of  $\mathcal{M}_t$ . In this case, standard calculations show that  $\langle \mathcal{M}_t, u \rangle : [0, \infty) \rightarrow (D(G), \|\cdot\|_{D(G)})$  is differentiable on  $(0, \infty)$  for all  $u \in D(G)$ . Since  $C_c \subset D(G)$ , this is consistent with the fact that  $M_t(x) = \langle M_t, e_x \rangle$  is differentiable for all  $x \in V$ . However, the fact that  $\Psi - \mathcal{M}_t \in D((A + B)^*)$  is not obvious. In fact, the case of Laplacians on graphs covered in this chapter already provides a counterexample to the hypothesis that  $\Psi \in D((A + B)^*)$  since  $\tilde{L}\mathbb{1} = \frac{c}{m}$  so  $\mathbb{1} \notin D(L_\infty^{\max}) = D((A + B)^*)$  unless  $\frac{c}{m} \in \ell^\infty$ . However, it should be noted that there is a similar result characterising honesty involving the existence of a solution to the ACP:  $\frac{d}{dt}u(t) = (A + B)^*u(t), u(0) = 0$  in Theorem 3.2.19 but this result requires the application of spectral theory and complex spaces.

# Chapter 5

## Quantum Dynamical Semigroups

In this chapter, we will look at an application of Kato's Theorem and honesty theory to a special set of quantum dynamical semigroups. We will begin by demonstrating how Kato's Theorem on abstract state spaces can be applied to quantum dynamical semigroups in the first section before looking at honesty in the next. As in the classical case where we are interested in determining when the semigroup is stochastic or honest, we are also interested in the corresponding notion for quantum dynamical semigroups. In particular, there is a notion known as conservativity of the quantum dynamical semigroup which has long been studied, and we will show that this is equivalent to honesty in the stochastic case. We will then show that in the more general substochastic case (which has not been studied before), honesty is the corresponding analogue of conservativity. The results in honesty theory then allow us to derive some results about the uniqueness of the semigroup and the domain of the generator (for the substochastic case) which have been previously proven for the stochastic case. We then finish up with some concrete examples of applications of honesty to quantum dynamical semigroups. In this chapter, Kato's Theorem will always refer to Theorem 2.1.2.

### 5.1 Quantum Dynamical Semigroups and Kato's Theorem

In Chapter 2 we saw that the development of Kato's Theorem was inspired by the study of classical Kolmogorov differential equations on  $\ell^1$ , which are in turn linked to the study of stochastic processes. The non-commutative analogue of stochastic processes was introduced by von Neumann in the 1930s but was systematically developed only in the second half of the 1970s [24]. This non-commutative counterpart is linked

to the study of quantum mechanics and hence is now known as the study of quantum stochastic processes or quantum flows. The counterpart to a Markov process in the classical setting is a quantum Markov process while the corresponding semigroups are known as quantum Markov semigroups or quantum dynamical semigroups.

In this chapter, we will not delve into the theory of general quantum Markov semigroups but instead, study a special set of quantum dynamical semigroups where Kato's Theorem on abstract state spaces can be applied. The generalisation of Kato's Theorem to abstract state spaces allows us to use this theorem to construct quantum dynamical semigroups since they are defined on a Banach space which has as its predual, an abstract state space.

We will begin by presenting the theory of quantum dynamical semigroups following that given by Fagnola in [24]. Fagnola considers quantum dynamical semigroups defined on the space of bounded operators on a complex Hilbert space  $\mathfrak{H}$ ,  $\mathcal{L}(\mathfrak{H})$ , which is a von Neumann algebra. He constructs a minimal quantum dynamical semigroup based on Chung's construction of the minimal solution of Feller-Kolmogorov equations for countable state Markov chains, the proof of which we will only sketch. However, he notes that an alternative method for constructing the semigroup is by considering the predual space and applying Kato's methods (but not the actual theorem itself) from [31], as was done by Davies in [19]. This application of Kato's methods to quantum dynamical semigroups were also noted by others including Arlotti, Lods and Mokhtar-Kharroubi in [3] but we have yet to find any literature which applies Kato's Theorem directly to quantum dynamical semigroups or which actually shows that the two methods are equivalent. For completeness, in this section we will demonstrate that the semigroup obtained by Fagnola via Chung's method coincides with the semigroup obtained by applying Kato's Theorem (Theorem 2.1.2).

Before describing Fagnola's construction of a minimal quantum dynamical semigroup, let us first discuss some auxiliary information about the space  $\mathcal{L}(\mathfrak{H})$ . Note that in the rest of this chapter,  $\mathfrak{H}$  will denote a complex Hilbert space and  $\langle \cdot, \cdot \rangle$  will denote the inner product on  $\mathfrak{H}$ . Also,  $\mathcal{L}(\mathfrak{H})_+$  is the cone consisting of  $\mathfrak{H}$ -positive operators.

First recall that  $\mathcal{L}(\mathfrak{H})$  has predual isometrically isomorphic to  $\mathfrak{T}(\mathfrak{H})$ , the space of trace class operators on  $\mathfrak{H}$ , equipped with the trace norm,  $\|\cdot\|_{tr}$ . The duality is given by

$$\langle \rho, x \rangle_{\mathfrak{T}(\mathfrak{H}), \mathcal{L}(\mathfrak{H})} = \text{Tr}(\rho x), \quad \rho \in \mathfrak{T}(\mathfrak{H}), x \in \mathcal{L}(\mathfrak{H}).$$

We will use this duality repeatedly in the rest of this chapter. In particular, we will use this duality to enable us to apply Kato's Theorem (on abstract state spaces) and its related results to the theory of quantum dynamical semigroups on  $\mathcal{L}(\mathfrak{H})$ . This

follows because the predual space  $\mathfrak{T}(\mathfrak{H})$  is the complexification of the space of self-adjoint trace-class operators  $\mathfrak{T}_s(\mathfrak{H})$ , which is a real ordered Banach space with trace norm additive on the positive cone, i.e. it is an abstract state space. Note that the positive cone of  $\mathfrak{T}(\mathfrak{H})$  is the set of  $\mathfrak{H}$ -positive operators of trace class. Moreover, the functional  $\Psi$  which we saw in Chapter 2 is simply the trace functional in this case.

The subspace of  $\mathfrak{T}(\mathfrak{H})$  consisting of the rank one operators  $|u\rangle\langle v|$ ,  $u, v \in \mathfrak{H}$ , defined by

$$|u\rangle\langle v|\varphi := \langle v, \varphi \rangle u, \quad \varphi \in \mathfrak{H}$$

will play an important role in this chapter. Note that for  $u, v \in \mathfrak{H}$ ,  $x \in \mathcal{L}(\mathfrak{H})$ , we have  $\text{Tr}(x|u\rangle\langle v|) = \langle v, xu \rangle$ . Moreover,  $\||u\rangle\langle v|\|_{tr} = \|u\| \|v\| = \||u\rangle\langle v|\|_{\infty}$  where  $\|\cdot\|_{\infty}$  refers to the usual operator (sup-)norm. We will often apply the following elementary lemma about the convergence of rank one operators.

**Lemma 5.1.1.** *Suppose  $u, v, (u_n), (v_n) \in \mathfrak{H}$  satisfy  $\|u_n - u\| \rightarrow 0$  and  $\|v_n - v\| \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\||u\rangle\langle v| - |u_n\rangle\langle v_n|\|_{tr} \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* By elementary calculations, we have

$$\begin{aligned} \||u\rangle\langle v| - |u_n\rangle\langle v_n|\|_{tr} &= \||u\rangle\langle v - v_n| - |u - u_n\rangle\langle v_n|\|_{tr} \\ &\leq \|u\| \|v - v_n\| + \|u - u_n\| \|v_n\| \rightarrow 0 \end{aligned}$$

since  $(v_n)$  is bounded and  $v_n \rightarrow v, u_n \rightarrow u$ . □

We also require a short note on topologies of operator algebras.

**Definition 5.1.2.** *Let  $(x_\alpha)$  be a net in  $\mathcal{L}(\mathfrak{H})$  and let  $x \in \mathcal{L}(\mathfrak{H})$ . We say that*

- (i)  $(x_\alpha)$  converges  $\sigma$ -weakly to  $x$  if the sum  $\sum_n \langle v_n, x_\alpha u_n \rangle \rightarrow \sum_n \langle v_n, x u_n \rangle$  for every pair of sequences  $(v_n), (u_n)$  of elements of  $\mathfrak{H}$  such that the series  $\sum_n \|v_n\|^2$  and  $\sum_n \|u_n\|^2$  converge.
- (ii)  $(x_\alpha)$  converges strongly to  $x$  if  $x_\alpha u \rightarrow x u$  for every  $u \in \mathfrak{H}$ .

Note that  $\sigma$ -weak convergence is simply weak\*-convergence, i.e.  $(x_\alpha)$  converges  $\sigma$ -weakly to  $x$  if and only if for every trace class operator  $\rho \in \mathfrak{T}(\mathfrak{H})$ ,  $\text{Tr}(x_\alpha \rho)$  converges to  $\text{Tr}(x \rho)$ .

Finally, we introduce the notion of complete positivity:

**Definition 5.1.3.** [24, Definition 2.5] Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $*$ -sub-algebras of  $\mathcal{L}(\mathfrak{H})$ . The linear map  $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{B}$  is called completely positive if for every  $n \in \mathbb{N}$ , and every family  $a_1, \dots, a_n$  of  $\mathcal{A}$  and every family  $b_1, \dots, b_n$  of  $\mathcal{B}$ , we have

$$\sum_{p,q=1}^n b_p^* \mathcal{T}(a_p^* a_q) b_q \geq 0.$$

Complete positivity is a stronger notion than positivity. Therefore, it is unsurprising that completely positive maps have special properties which positive maps do not necessarily have (see [24, Section 2.2] for more details).

Now we are ready to define the main objects of this chapter, namely quantum dynamical semigroups.

**Definition 5.1.4.** [24, Definition 3.1] Let  $\mathcal{A}$  denote a  $W^*$ -algebra of operators acting on a Hilbert space  $\mathfrak{H}$ . A quantum dynamical semigroup on  $\mathcal{A}$  is a family  $(\mathcal{T}(t))_{t \geq 0}$  of bounded operators on  $\mathcal{A}$  with the following properties:

- (i)  $\mathcal{T}(0)a = a$  for all  $a \in \mathcal{A}$ .
- (ii)  $\mathcal{T}(t+s)a = \mathcal{T}(t)\mathcal{T}(s)a$  for all  $s, t \geq 0$  and all  $a \in \mathcal{A}$ .
- (iii)  $\mathcal{T}(t)$  is completely positive for all  $t \geq 0$ .
- (iv)  $\mathcal{T}(t)$  is a  $\sigma$ -weakly continuous operator in  $\mathcal{A}$  for all  $t \geq 0$ .
- (v) For every  $a \in \mathcal{A}$ , the map  $t \mapsto \mathcal{T}(t)a$  is continuous with respect to the  $\sigma$ -weak topology on  $\mathcal{A}$ .

**Definition 5.1.5.** [24, Definition 3.2] The infinitesimal generator of the quantum dynamical semigroup  $(\mathcal{T}(t))_{t \geq 0}$  is the operator  $\mathcal{G}^*$  whose domain  $D(\mathcal{G}^*)$  is the space of elements  $a \in \mathcal{A}$  for which there exists an element  $b \in \mathcal{A}$  such that  $b = \lim_{t \rightarrow 0} \frac{\mathcal{T}(t)a - a}{t}$  in the  $\sigma$ -weak topology and  $\mathcal{G}^*a = b$ .

Since the quantum dynamical semigroup  $(\mathcal{T}(t))_{t \geq 0}$  on  $\mathcal{L}(\mathfrak{H})$  satisfies conditions (iv) and (v) of Definition 5.1.4, it follows that  $(\mathcal{T}(t))_{t \geq 0}$  induces a predual semigroup  $(\mathcal{S}(t))_{t \geq 0}$  on  $\mathfrak{T}(\mathfrak{H})$  defined by

$$\langle \mathcal{S}(t)\rho, x \rangle_{\mathfrak{T}(\mathfrak{H}), \mathcal{L}(\mathfrak{H})} = \langle \rho, \mathcal{T}(t)x \rangle_{\mathfrak{T}(\mathfrak{H}), \mathcal{L}(\mathfrak{H})} \quad \text{for all } \rho \in \mathfrak{T}(\mathfrak{H}), x \in \mathcal{L}(\mathfrak{H}), t \geq 0.$$

Equivalently, this may be stated via the generator of the semigroup, i.e.  $\mathcal{G}$  is the generator of  $(\mathcal{S}(t))_{t \geq 0}$  if and only if  $\mathcal{G}^*$  is the generator of  $(\mathcal{T}(t))_{t \geq 0}$  [43, Theorem 1.2.3], [11, p.252]. For a general semigroup, the predual semigroup may not necessarily

be strongly continuous. However, if  $(\mathcal{T}(t))_{t \geq 0}$  is a quantum dynamical semigroup, the predual semigroup is in fact strongly continuous. This follows from condition (v) in Definition 5.1.4 since the map  $t \mapsto \mathcal{T}(t)x, x \in \mathcal{L}(\mathfrak{H})$  being continuous with respect to the  $\sigma$ -weak topology implies that  $(\mathcal{S}(t))_{t \geq 0}$  is weakly continuous and hence by [22, Theorem I.5.8], is equivalently strongly continuous. To ensure that the notation in this chapter concurs with those in the previous chapters, we will always denote the generator of a quantum dynamical semigroup as an adjoint operator, for example  $\mathcal{G}^*$ ; more precisely, as the adjoint of the generator of the predual semigroup.

In the rest of this chapter, we will assume that the following premise holds unless stated otherwise:

**Premise 5.1.6.** *Suppose  $Y$  generates a  $C_0$ -semigroup of contractions  $(P(t))_{t \geq 0}$  in  $\mathfrak{H}$ . Suppose also the sequence of operators  $(L_l)_{l=1}^\infty$  are such that  $D(L_l) \supseteq D(Y)$  and for all  $u \in D(Y)$ , we have*

$$\langle u, Yu \rangle + \langle Yu, u \rangle + \sum_{l=1}^{\infty} \langle L_l u, L_l u \rangle \leq 0. \quad (5.1)$$

It will prove useful later to consider the sesquilinear form  $\Upsilon(x), x \in \mathcal{L}(\mathfrak{H})$  with domain  $D(Y) \times D(Y) \subseteq \mathfrak{H} \times \mathfrak{H}$  given by

$$\Upsilon(x)[v, u] = \langle v, xYu \rangle + \langle Yv, xu \rangle + \sum_{l=1}^{\infty} \langle L_l v, xL_l u \rangle. \quad (5.2)$$

Under the right conditions, the form  $\Upsilon(x)$  will be induced by a closed linear operator in the sense of (1.4). In these cases, we will denote the closed operator associated with it by  $W(x)$ . Assuming Premise 5.1.6 holds, Fagnola [24, Chapter 3], via Chung's iterative construction, shows that:

**Proposition 5.1.7.** [24, Theorem 3.22] *There exists a minimal quantum dynamical semigroup  $(T(t))_{t \geq 0}$  satisfying*

$$\langle v, (T(t)x)u \rangle = \langle v, xu \rangle + \int_0^t \Upsilon(T(s)x)[v, u] ds \quad \text{for all } u, v \in D(Y) \quad (5.3)$$

and  $T(t)\mathbb{1} \leq \mathbb{1}$  for all  $t \geq 0$ . The semigroup is minimal in the sense that for any quantum dynamical semigroup  $(U(t))_{t \geq 0}$  on  $\mathcal{L}(\mathfrak{H})$  which is a solution to (5.3) and for any  $x \in \mathcal{L}(\mathfrak{H})_+$ , we have  $T(t)x \leq U(t)x$  for all  $t \geq 0$ .

Fagnola constructs the semigroup by defining for every  $t \geq 0$ , a sequence of linear contractions  $\{T^{(n)}(t)\}_n$  iteratively, namely for all  $t \geq 0$ ,  $x \in \mathcal{L}(\mathfrak{H})$  and  $u, v \in D(Y)$ ,

$$\begin{aligned}\langle v, T^{(0)}(t)xu \rangle &= \langle P(t)v, xP(t)u \rangle \\ \langle v, T^{(n+1)}(t)xu \rangle &= \langle P(t)v, xP(t)u \rangle + \sum_{l=1}^{\infty} \int_0^t \langle L_l P(t-s)v, T^{(n)}(s)xL_l P(t-s)u \rangle ds.\end{aligned}$$

For all  $x \in \mathcal{L}(\mathfrak{H})_+$  and every  $t \geq 0$ , the sequence  $\{T^{(n)}(t)x\}_n$  is increasing and bounded from above by  $\|x\| \cdot \mathbb{1}$ . Therefore, the limit  $\lim_{n \rightarrow \infty} \langle u, T^{(n)}(t)xu \rangle$  exists for all  $u \in \mathfrak{H}$ . The semigroup is then given by this limit, i.e.

$$\langle v, T(t)xu \rangle = \lim_{n \rightarrow \infty} \langle v, T^{(n)}(t)xu \rangle \quad \text{for all } u, v \in \mathfrak{H}.$$

Note that equation (5.3) can be restated in terms of the generator of the quantum dynamical semigroup:

**Proposition 5.1.8.** *A contractive quantum dynamical semigroup  $(\mathcal{T}(t))_{t \geq 0}$  satisfies (5.3) if and only if its generator  $\mathcal{G}^*$  satisfies*

$$\langle v, (\mathcal{G}^*x)u \rangle = \Upsilon(x)[v, u] \quad \text{for all } u, v \in D(Y), x \in D(\mathcal{G}^*). \quad (5.4)$$

*Proof.* Suppose  $(\mathcal{T}(t))_{t \geq 0}$  satisfies (5.3). Fix  $u, v \in D(Y)$  and  $x \in D(\mathcal{G}^*)$ . Since  $\text{Tr}(x|u\rangle\langle v|) = \langle v, xu \rangle$ , (5.3) can be rewritten as  $\text{Tr}((\mathcal{T}(t)x|u\rangle\langle v|) - \text{Tr}(x|u\rangle\langle v|) = \int_0^t \Upsilon(\mathcal{T}(s)x)[v, u] ds$ . Hence,

$$\begin{aligned}\frac{1}{t} \text{Tr}((\mathcal{T}(t)x - x)|u\rangle\langle v|) \\ = \frac{1}{t} \int_0^t \left( \langle Yv, (\mathcal{T}(s)x)u \rangle + \langle v, (\mathcal{T}(s)x)Yu \rangle + \sum_{l=1}^{\infty} \langle L_l v, (\mathcal{T}(s)x)L_l u \rangle \right) ds.\end{aligned} \quad (5.5)$$

The continuity of  $t \mapsto \mathcal{T}(t)x$  with respect to the  $\sigma$ -weak topology implies that the maps  $s \mapsto \langle Yv, (\mathcal{T}(s)x)u \rangle$ ,  $s \mapsto \langle v, (\mathcal{T}(s)x)Yu \rangle$ ,  $s \mapsto \langle L_l v, (\mathcal{T}(s)x)L_l u \rangle$ ,  $l \in \mathbb{N}$  are continuous. Moreover, since  $(\mathcal{T}(t))_{t \geq 0}$  is contractive, we have for all  $l \in \mathbb{N}$ ,  $|\langle L_l v, (\mathcal{T}(s)x)L_l u \rangle| \leq \|x\| \|L_l u\| \|L_l v\|$ . But by (5.1), we have

$$\begin{aligned}\sum_{l=1}^{\infty} \|L_l v\| \|L_l u\| &\leq \left( \sum_{l=1}^{\infty} \|L_l v\|^2 \right)^{1/2} \left( \sum_{l=1}^{\infty} \|L_l u\|^2 \right)^{1/2} \\ &\leq (-2 \text{Re} \langle v, Yv \rangle)^{1/2} (-2 \text{Re} \langle u, Yu \rangle)^{1/2}.\end{aligned} \quad (5.6)$$

Thus, by the Weierstrass M-test, the map  $s \mapsto \sum_{l=1}^{\infty} \langle L_l v, (\mathcal{T}(s)x)L_l u \rangle$  is continuous. Therefore we can let  $t \rightarrow 0$  in (5.5) to obtain  $\langle v, (\mathcal{G}^*x)u \rangle = \Upsilon(x)[v, u]$ .

Conversely, suppose  $(\mathcal{T}(t))_{t \geq 0}$  satisfies (5.4). We begin by observing that the form  $\Upsilon(x)[v, u]$ ,  $x \in \mathcal{L}(\mathfrak{H})$ ,  $u, v \in D(Y)$  can be restated as

$$\Upsilon(x)[v, u] = \text{Tr} \left( x \left( |Yu\rangle\langle v| + |u\rangle\langle Yv| + \sum_{l=1}^{\infty} |L_l u\rangle\langle L_l v| \right) \right) \quad (5.7)$$

since  $\sum_{l=1}^{\infty} |L_l u\rangle\langle L_l v|$  converges in trace norm by (5.6). On the other hand, from Proposition 1.2.9, we have for  $x \in \mathcal{L}(\mathfrak{H})$ ,

$$\mathcal{T}(t)x - x = \mathcal{G}^* \text{weak}^* \int_0^t \mathcal{T}(s)x ds.$$

Hence

$$\begin{aligned} & \langle v, (\mathcal{T}(t)x)u \rangle - \langle v, xu \rangle \\ &= \left\langle v, \left( \mathcal{G}^* \text{weak}^* \int_0^t \mathcal{T}(s)x ds \right) u \right\rangle \\ &= \Upsilon \left( \text{weak}^* \int_0^t \mathcal{T}(s)x ds \right) [v, u] \quad (\text{by assumption}) \\ &= \text{Tr} \left( \left( \text{weak}^* \int_0^t \mathcal{T}(s)x ds \right) \left( |Yu\rangle\langle v| + |u\rangle\langle Yv| + \sum_{l=1}^{\infty} |L_l u\rangle\langle L_l v| \right) \right) \quad (\text{by (5.7)}) \\ &= \int_0^t \text{Tr} \left( (\mathcal{T}(s)x) \left( |Yu\rangle\langle v| + |u\rangle\langle Yv| + \sum_{l=1}^{\infty} |L_l u\rangle\langle L_l v| \right) \right) ds \\ & \hspace{15em} (\text{by definition of weak}^* \text{ integral}) \\ &= \int_0^t \Upsilon(\mathcal{T}(s)x)[v, u] ds \quad (\text{by (5.7)}). \end{aligned}$$

Therefore  $(\mathcal{T}(t))_{t \geq 0}$  satisfies (5.3).  $\square$

**Definition 5.1.9.** *We say that the generator  $\mathcal{G}^*$  of a quantum dynamical semigroup can be represented in Lindblad form if there exists operators  $Y$ ,  $(L_l)$  on  $\mathfrak{H}$  satisfying Premise 5.1.6 such that*

$$\langle v, (\mathcal{G}^*x)u \rangle = \Upsilon(x)[v, u]$$

for all  $x \in D(\mathcal{G}^*)$  and all  $u, v \in D(Y)$ .

Next, we look at the construction via Kato's Theorem of a quantum dynamical semigroup whose generator can be represented in Lindblad form. Since Kato's Theorem is stated for real spaces, we will restrict to the space of self adjoint operators during the discussion of this construction. In particular, the space we will be working in will be the space of self adjoint trace class operators,  $\mathfrak{T}_s(\mathfrak{H})$ .

First, consider the semigroup  $(U(t))_{t \geq 0}$  in  $\mathfrak{T}_s(\mathfrak{H})$  defined by  $U(t)\rho = P(t)\rho P(t)^*$ . Note that since  $(P(t))_{t \geq 0}$  is a  $C_0$ -semigroup of contractions, so is  $(P(t)^*)_{t \geq 0}$ . It turns out that  $(U(t))_{t \geq 0}$  is also a  $C_0$ -semigroup of contractions with generator we will denote by  $A$  (see for example [22, Section I.3.16]). In [19], Davies considers the case where we have equality in equation (5.1) and shows that the operator  $A$  and an appropriately defined  $B$  (see Lemma 5.1.11, Corollary 5.1.14) satisfy in our terminology, the conditions in Theorem 2.1.2. His methods also hold for the more general case (with inequality in (5.1)) with minor modifications. We describe this method in more detail below as it will be useful later in this chapter.

To determine the domain of the generator  $A$ , Davies introduces a positive, one-to-one map

$$\pi : \mathfrak{T}_s(\mathfrak{H}) \rightarrow \mathfrak{T}_s(\mathfrak{H}), \quad \pi(\rho) = R(1, Y)\rho R(1, Y)^*$$

and considers the subspace  $\mathcal{D}_s := \pi(\mathfrak{T}_s(\mathfrak{H}))$ . Then [19, Lemma 2.1] holds in this case as well since the inequality (5.1) has no role in the proof.

**Lemma 5.1.10.** [19, Lemma 2.1] *The domain  $\mathcal{D}_s$  is dense in  $\mathfrak{T}_s(\mathfrak{H})$ . Let  $\rho \in \mathcal{D}_s$  and  $\epsilon > 0$ . Then there exist  $\rho_1, \rho_2 \in (\mathcal{D}_s)_+ := \mathcal{D}_s \cap \mathfrak{T}_s(\mathfrak{H})_+$  such that*

$$\rho = \rho_1 - \rho_2, \quad \|\rho_1\|_{tr} + \|\rho_2\|_{tr} < \|\rho\|_{tr} + \epsilon. \quad (5.8)$$

Moreover,  $\mathcal{D}_s$  is a core for  $A$  and for all  $\rho \in \mathcal{D}_s$

$$A\rho = Y\rho + \rho Y^* \quad (5.9)$$

in the sense that  $Y\rho$  is a trace class operator while  $\rho Y^*$  is a restriction of the operator  $(Y\rho)^*$  which is also trace class.

Now let us consider the operator  $B$ . The next two lemmas are the analogues of [19, Lemma 2.2, Lemma 2.3] and can be proven almost exactly as in [19]. The only changes required are changes from equalities to inequalities at the appropriate points, hence the proofs are omitted.

**Lemma 5.1.11.** *The formula*

$$B\rho = \sum_{l=1}^{\infty} L_l R(1, Y) \pi^{-1}(\rho) (L_l R(1, Y))^* \quad (5.10)$$

with the series converging in the trace norm defines a positive linear map  $B : \mathcal{D}_s \rightarrow \mathfrak{T}_s(\mathfrak{H})$  such that

$$\mathrm{Tr}(A\rho + B\rho) \leq 0 \quad \text{for all } \rho \in (\mathcal{D}_s)_+. \quad (5.11)$$

**Lemma 5.1.12.** *For all  $\lambda > 0$ , the map  $BR(\lambda, A)$  from  $\mathcal{D}_s$  into  $\mathfrak{T}_s(\mathfrak{H})$  has a unique, positive, bounded linear extension  $J_\lambda : \mathfrak{T}_s(\mathfrak{H}) \rightarrow \mathfrak{T}_s(\mathfrak{H})$  such that  $\|J_\lambda\| \leq 1$ .*

*Remark 5.1.13.* Since  $A$  is resolvent positive,  $R(\lambda, A)\mathcal{D}_s \subset \mathcal{D}_s$  and  $B$  is positive on  $\mathcal{D}_s$ , it follows from (5.11) that for all  $\rho \in (\mathcal{D}_s)_+$ ,

$$\|BR(\lambda, A)\rho\|_{tr} \leq -\text{Tr}(AR(\lambda, A)\rho) = \text{Tr}(\rho) - \lambda \text{Tr}(R(\lambda, A)\rho) \leq \|\rho\|_{tr}.$$

Then (5.8) implies that  $\|BR(\lambda, A)\rho\|_{tr} \leq \|\rho\|_{tr}$  for all  $\rho \in \mathcal{D}_s$ , that is,  $B$  is  $A$ -bounded on  $\mathcal{D}_s$ . Davies uses this to prove the existence of  $J_\lambda$  in Lemma 5.1.12.

The results above allow us to derive an important corollary. We give the complete proof here as some details were omitted in [19].

**Corollary 5.1.14.** *The map  $B$  has a positive extension  $B' : D(A) \rightarrow \mathfrak{T}_s(\mathfrak{H})$  such that*

$$\text{Tr}(A\rho + B'\rho) \leq 0 \quad \text{for all } \rho \in D(A)_+. \quad (5.12)$$

*Proof.* We define

$$B'\rho = J_1(I - A)\rho.$$

We begin by showing that  $B'$  is an extension of  $B$ . Since  $B$  is  $A$ -bounded on  $\mathcal{D}_s$  (by Remark 5.1.13), it suffices to show that  $B'\rho = B\rho$  for all  $\rho$  in a core of  $A$  which lies in  $\mathcal{D}_s$ . In particular, we will show that  $B'\rho = B\rho$  for all  $\rho \in \pi^2(\mathfrak{T}_s(\mathfrak{H})) := \pi(\pi(\mathfrak{T}_s(\mathfrak{H}))) \subset \mathcal{D}_s$ . To see that  $\pi^2(\mathfrak{T}_s(\mathfrak{H}))$  is a core for  $A$ , simply note that density in  $\mathfrak{T}_s(\mathfrak{H})$  follows because  $\pi^2(\mathfrak{T}_s(\mathfrak{H}))$  contains the finite rank operators whose eigenvectors lie in  $D(Y^2)$  (see also Lemma 5.1.17) while invariance of  $\pi^2(\mathfrak{T}_s(\mathfrak{H}))$  under  $(U(t))_{t \geq 0}$  follows directly from the definition of  $\pi$ . So let  $\rho = \pi^2(\sigma)$  for some  $\sigma \in \mathfrak{T}_s(\mathfrak{H})$ , that is,  $\rho \in \pi^2(\mathfrak{T}_s(\mathfrak{H}))$ . Then by (5.9), we have that  $A\rho = \pi(Y\pi(\sigma) + \pi(\sigma)Y^*) \in \mathcal{D}_s$  and so  $(I - A)\rho \in \mathcal{D}_s$ . Therefore  $B'\rho = J_1(I - A)\rho = BR(1, A)(I - A)\rho = B\rho$  and so  $B'$  is an extension of  $B$ . Moreover,  $\|B'R(1, A)\rho\|_{tr} = \|J_1\rho\|_{tr} \leq \|\rho\|_{tr}$  for all  $\rho \in \mathfrak{T}_s(\mathfrak{H})$ . Therefore  $B'$  is  $A$ -bounded.

To show that the inequality (5.12) holds, let  $\rho \in D(A)_+$  and consider

$$\rho_\epsilon = R(1, \epsilon Y)\rho R(1, \epsilon Y)^*, \quad \epsilon > 0.$$

It is easy to see that  $\rho_\epsilon$  is self-adjoint if  $\rho$  is. Moreover, by the resolvent equation (see for example [22, Equation IV.1.2]), we have

$$(1 - \epsilon^{-1})R(1, Y)R(\epsilon^{-1}, Y) + R(1, Y) = R(\epsilon^{-1}, Y),$$

and so it follows that

$$\rho_\epsilon = R(1, Y)((1 - \epsilon^{-1})R(1, \epsilon Y) + \epsilon^{-1}I)\rho((1 - \epsilon^{-1})R(1, \epsilon Y)^* + \epsilon^{-1}I)R(1, Y)^*.$$

Thus,  $\rho_\epsilon \in (\mathcal{D}_s)_+$ . Moreover, the map  $\rho \mapsto \rho_\epsilon$  is bounded independently of  $\epsilon$  as

$$\|\rho_\epsilon\|_{tr} \leq \|R(1, \epsilon Y)\|_\infty \|\rho\|_{tr} \|R(1, \epsilon Y)^*\|_\infty \leq \|\rho\|_{tr}.$$

We will show that  $\rho_\epsilon \rightarrow \rho$  in trace norm for all  $\rho \in \mathfrak{T}_s(\mathfrak{H})$ . Consider the rank one operator  $\rho := |u\rangle\langle v|$ ,  $u, v \in \mathfrak{H}$ . Then elementary calculations show that  $\rho_\epsilon = |u_\epsilon\rangle\langle v_\epsilon|$  where  $u_\epsilon = R(1, \epsilon Y)u$  and  $v_\epsilon = R(1, \epsilon Y)v$ . By [22, Lemma II.3.4],  $u_\epsilon \rightarrow u$  and  $v_\epsilon \rightarrow v$  as  $\epsilon \rightarrow 0$ . Hence by Lemma 5.1.1, it follows that  $\rho_\epsilon \rightarrow \rho$  in trace norm as  $\epsilon \rightarrow 0$ . Since the (self-adjoint) finite rank operators are dense in  $\mathfrak{T}_s(\mathfrak{H})$  and the map  $\rho \mapsto \rho_\epsilon$  is uniformly bounded, it follows that  $\rho_\epsilon \rightarrow \rho$  for all  $\rho \in \mathfrak{T}_s(\mathfrak{H})$ .

Now for  $\rho \in D(A)$ , we have  $A\rho \in \mathfrak{T}_s(\mathfrak{H})$  and  $A\rho_\epsilon = (A\rho)_\epsilon$ . Therefore we can conclude that  $\rho_\epsilon \rightarrow \rho$  and  $A\rho_\epsilon \rightarrow A\rho$  as  $\epsilon \rightarrow 0$ . Since  $B'$  is  $A$ -bounded, it follows that  $B\rho_\epsilon \rightarrow B'\rho$ . Therefore by Lemma 5.1.11, for all  $\rho \in D(A)_+$ ,  $B'\rho \geq 0$  and

$$\mathrm{Tr}(A\rho + B'\rho) = \lim_{\epsilon \rightarrow 0} \mathrm{Tr}(A\rho_\epsilon + B\rho_\epsilon) \leq 0.$$

□

Henceforth we will identify  $B$  with  $B'$  and simply denote it by  $B$ . With this we have:

**Proposition 5.1.15.** *A and B satisfy Theorem 2.1.2 and so there exists a minimal perturbed semigroup  $(\tilde{S}(t))_{t \geq 0}$  with generator  $\tilde{G}$  an extension of  $A + B$ . Moreover,  $R(\lambda, \tilde{G})$  satisfies*

$$R(\lambda, \tilde{G})\rho = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k \rho \text{ for all } \rho \in \mathfrak{T}_s(\mathfrak{H}).$$

We have just described two methods of constructing a minimal quantum dynamical semigroup with  $Y, (L_t)$  satisfying Premise 5.1.6; one via Fagnola's method (Proposition 5.1.7) and the other via Kato's Theorem (Proposition 5.1.15). The remainder of this section will be devoted to showing that the two semigroups coincide.

Note that the semigroup from Kato's Theorem acts in the space  $\mathfrak{T}_s(\mathfrak{H})$  while Fagnola's semigroup acts in the space  $\mathcal{L}(\mathfrak{H})$ . In order to show that the semigroups coincide, we will first transfer the semigroups to the same space, namely  $\mathfrak{T}(\mathfrak{H})$ . Recalling that  $\mathfrak{T}(\mathfrak{H})$  is simply the complexification of  $\mathfrak{T}_s(\mathfrak{H})$ , we will henceforth work with the complexifications of the operators  $A, B, (U(t))_{t \geq 0}, (\tilde{S}(t))_{t \geq 0}, \tilde{G}$  but retain the

same notation. We can do so because we saw in Section 2.2.3 that honesty in the complexified space is equivalent to honesty in the real space. To transfer Fagnola's semigroup to  $\mathfrak{T}(\mathfrak{H})$  on the other hand, we will utilise the fact that every quantum dynamical semigroup on  $\mathcal{L}(\mathfrak{H})$  induces a predual semigroup on  $\mathfrak{T}(\mathfrak{H})$ . We will denote by  $(S(t))_{t \geq 0}$  the predual semigroup of Fagnola's minimal quantum dynamical semigroup  $(T(t))_{t \geq 0}$  identified in Proposition 5.1.7. We now show that  $(S(t))_{t \geq 0}$  coincides with  $(\tilde{S}(t))_{t \geq 0}$ .

**Theorem 5.1.16.** *Let  $(S(t))_{t \geq 0}$  be the predual semigroup of the minimal quantum dynamical semigroup  $(T(t))_{t \geq 0}$  in Proposition 5.1.7 and  $(\tilde{S}(t))_{t \geq 0}$  be the perturbed semigroup in Proposition 5.1.15. Then  $\tilde{S}(t)\rho = S(t)\rho$  for all  $\rho \in \mathfrak{T}(\mathfrak{H})$ ,  $t \geq 0$ .*

In order to prove the theorem, we require some auxiliary information. First let us consider a few important subspaces, beginning with

$$\mathcal{V} = \mathcal{V}_1 := \text{Span}\{|u\rangle\langle v| : u, v \in D(Y)\}.$$

We will also occasionally require the spaces

$$\mathcal{V}_n := \text{Span}\{|u\rangle\langle v| : u, v \in D(Y^n)\}, n \in \mathbb{N}, n \geq 2.$$

Moreover, the map  $\pi$  can be extended to  $\mathfrak{T}(\mathfrak{H})$  and we will be interested in the spaces

$$\mathcal{D} := \pi(\mathfrak{T}(\mathfrak{H})) = \mathcal{D}_s + i\mathcal{D}_s, \quad \pi^n(\mathfrak{T}(\mathfrak{H})), n \in \mathbb{N}, n \geq 2$$

**Lemma 5.1.17.** *For all  $n \in \mathbb{N}$ ,  $\mathcal{V}_n \subset \mathcal{D}$  and moreover,  $\mathcal{V}_n$  is a core for  $A$ .*

*Proof.* Since  $\mathcal{V}_{n+1} \subseteq \mathcal{V}_n$ , it suffices to show that  $\mathcal{V} \subseteq \mathcal{D}$  to prove the first statement. Fix  $u, v \in D(Y)$ . Then by inspection, we see that  $|u\rangle\langle v| = R(1, Y)\rho R(1, Y)^*$  if  $\rho$  is the rank-one operator defined by  $\rho := |(I - Y)u\rangle\langle (I - Y)v|$ . Therefore  $|u\rangle\langle v| \in \mathcal{D}$  and so  $\mathcal{V} \subset \mathcal{D}$ .

Next, we show that  $\mathcal{V}$  is dense in  $\mathfrak{T}(\mathfrak{H})$ . Since  $Y$  generates a  $C_0$ -semigroup, it follows that  $D(Y)$  is dense in  $\mathfrak{H}$  and so for fixed  $u, v \in \mathfrak{H}$ , there exist  $(u_n), (v_n) \in D(Y)$  such that  $u_n \rightarrow u$  and  $v_n \rightarrow v$ . By Lemma 5.1.1, it follows that  $\| |u\rangle\langle v| - |u_n\rangle\langle v_n| \|_{tr} \rightarrow 0$ . In other words,  $\mathcal{V}$  is dense in the space of finite rank operators. Since the finite rank operators are dense in  $\mathfrak{T}(\mathfrak{H})$ , it follows that  $\mathcal{V}$  is dense in  $\mathfrak{T}(\mathfrak{H})$ .

Finally, observe that  $U(t)(|u\rangle\langle v|) = P(t)|u\rangle\langle v|P(t)^* = |P(t)u\rangle\langle P(t)v|$  for all  $t \geq 0$ . Since  $P(t)u \in D(Y)$  for all  $u \in D(Y)$  and all  $t \geq 0$ , we have that  $U(t)(|u\rangle\langle v|) \in \mathcal{V}$ . Therefore  $\mathcal{V}$  is invariant under  $(U(t))_{t \geq 0}$  and so  $\mathcal{V}$  is a core for  $A$ .

A similar argument shows that  $\mathcal{V}_n$  is a core for  $A$  for all  $n \geq 2$  since  $D(Y^n)$  is a core for  $Y$  and  $D(Y^n)$  is invariant under the semigroup  $(P(t))_{t \geq 0}$ .  $\square$

*Remark 5.1.18.* Recall that in the proof of Corollary 5.1.14 we showed that  $\pi^2(\mathfrak{T}_s(\mathfrak{H}))$  is a core for  $A|_{\mathfrak{T}_s(\mathfrak{H})}$ . We can in fact show more generally that  $\pi^n(\mathfrak{T}(\mathfrak{H}))$ ,  $n \in \mathbb{N}$  are cores for  $A$ . A similar proof as that of Lemma 5.1.17 shows in fact that  $\mathcal{V}_n \subseteq \pi^n(\mathfrak{T}(\mathfrak{H}))$  for all  $n \geq 2$ . Furthermore, it is easy to see that  $\pi^n(U(t)\sigma) = U(t)(\pi^n\sigma)$  for all  $\sigma \in \mathfrak{T}(\mathfrak{H})$  and all  $t \geq 0$ . Therefore  $\pi^n(\mathfrak{T}(\mathfrak{H}))$ ,  $n \in \mathbb{N}$  are also cores for  $A$ .

It will also be useful to know how the operators  $A, B$  act on the operators in  $\mathcal{V}$ .

**Lemma 5.1.19.** *For all  $|u\rangle\langle v| \in \mathcal{V}$  and  $x \in \mathcal{L}(\mathfrak{H})$*

$$\mathrm{Tr}(x(A|u\rangle\langle v|)) = \langle Yv, xu \rangle + \langle v, xYu \rangle \quad \text{and} \quad \mathrm{Tr}(x(B|u\rangle\langle v|)) = \sum_{l=1}^{\infty} \langle L_l v, xL_l u \rangle.$$

*In particular,*

$$\mathrm{Tr}(x((A+B)|u\rangle\langle v|)) = \Upsilon(x)[v, u].$$

*Proof.* Fix  $|u\rangle\langle v| \in \mathcal{V}$  and  $x \in \mathcal{L}(\mathfrak{H})$ . Note first that elementary calculations show that  $Y|u\rangle\langle v| + |u\rangle\langle v|Y^* = |Yu\rangle\langle v| + |u\rangle\langle Yv|$ . Since  $\mathcal{V} \subset \mathcal{D}$  by Lemma 5.1.17, and hence (5.9) holds for  $\rho = |u\rangle\langle v|$  (Lemma 5.1.10), we have

$$\mathrm{Tr}(x(A|u\rangle\langle v|)) = \mathrm{Tr}(x(|Yu\rangle\langle v| + |u\rangle\langle Yv|)) = \langle v, xYu \rangle + \langle Yv, xu \rangle.$$

On the other hand, by Lemma 5.1.11, for  $\rho \in \mathcal{D}$ , (5.10) holds and from the proof of Lemma 5.1.17,  $\pi^{-1}(|u\rangle\langle v|) = |(I-Y)u\rangle\langle (I-Y)v|$ . Hence for  $\varphi \in \mathfrak{H}$ ,

$$\begin{aligned} (B|u\rangle\langle v|)\varphi &= \sum_{l=1}^{\infty} L_l R(1, Y) |(I-Y)u\rangle\langle (I-Y)v| (L_l R(1, Y))^* \varphi \\ &= \sum_{l=1}^{\infty} L_l R(1, Y) \langle (I-Y)v, (L_l R(1, Y))^* \varphi \rangle (I-Y)u \\ &= \sum_{l=1}^{\infty} \langle L_l v, \varphi \rangle L_l u \\ &= \sum_{l=1}^{\infty} |L_l u\rangle\langle L_l v| \varphi. \end{aligned}$$

Therefore,  $\mathrm{Tr}(x(B|u\rangle\langle v|)) = \sum_{l=1}^{\infty} \langle L_l v, xL_l u \rangle$ . The final assertion follows directly from the definition of  $\Upsilon(x)[v, u]$ .  $\square$

**Corollary 5.1.20.** *The generator  $\tilde{G}^*$  of the adjoint semigroup  $(\tilde{S}^*(t))_{t \geq 0}$  of  $(\tilde{S}(t))_{t \geq 0}$  can be represented in Lindblad form.*

*Proof.* Let  $u, v \in D(Y)$  and  $x \in D(\tilde{G}^*)$ . Then by Lemma 5.1.17,  $|u\rangle\langle v| \in \mathcal{V} \subseteq D(A)$ , so by Lemma 5.1.19 we have,  $\langle v, (\tilde{G}^*x)u \rangle = \text{Tr}((\tilde{G}^*x)|u\rangle\langle v|) = \text{Tr}(x(\tilde{G}|u\rangle\langle v|)) = \text{Tr}(x((A+B)|u\rangle\langle v|)) = \Upsilon(x)[v, u]$ .  $\square$

We also require some information about Fagnola's construction,  $(T(t))_{t \geq 0}$  with generator  $G^*$ . In particular, Fagnola shows in [24, Proposition 3.25] that the resolvent of  $G^*$  is given by

$$R(\lambda, G^*)x = \sum_{k=0}^{\infty} Q_{\lambda}^k(P_{\lambda}(x)), \quad x \in \mathcal{L}(\mathfrak{H}), \lambda > 0 \quad (5.13)$$

with the series convergent in the strong operator topology, where  $P_{\lambda}$  and  $Q_{\lambda}$ ,  $\lambda > 0$  are linear positive maps in  $\mathcal{L}(\mathfrak{H})$  defined by

$$\langle v, P_{\lambda}(x)u \rangle = \int_0^{\infty} e^{-\lambda s} \langle P(s)v, xP(s)u \rangle ds \quad (5.14)$$

$$\langle v, Q_{\lambda}(x)u \rangle = \sum_{l=1}^{\infty} \int_0^{\infty} e^{-\lambda s} \langle L_l P(s)v, xL_l P(s)u \rangle ds \quad (5.15)$$

for  $x \in \mathcal{L}(\mathfrak{H})$ ,  $u, v \in D(Y)$ . Moreover, the maps  $P_{\lambda}$  and  $Q_{\lambda}$  are normal, completely positive and satisfy the norm estimates  $\|P_{\lambda}\| \leq \lambda^{-1}$  and  $\|Q_{\lambda}\| \leq 1$  [24, Proposition 3.24]. We will rephrase the operators  $P_{\lambda}$  and  $Q_{\lambda}$  in terms of the operators  $A$  and  $B$ .

**Lemma 5.1.21.** *Suppose  $P_{\lambda}, Q_{\lambda}$  are as defined in (5.14) and (5.15) and  $A, B$  are as in Proposition 5.1.15. Then  $P_{\lambda} = R(\lambda, A)^* = R(\lambda, A^*)$  and  $Q_{\lambda} = (BR(\lambda, A))^*$  for all  $\lambda > 0$ .*

*Proof.* Fix  $\lambda > 0$  and observe that since  $R(\lambda, A)\rho = \int_0^{\infty} e^{-\lambda s} P(s)\rho P(s)^* ds$  for all  $\rho \in \mathfrak{T}(\mathfrak{H})$ , it follows by elementary calculations that for  $|u\rangle\langle v| \in \mathcal{V}$ ,

$$R(\lambda, A)|u\rangle\langle v| = \int_0^{\infty} e^{-\lambda s} |P(s)u\rangle\langle P(s)v| ds$$

where the integral is absolutely convergent in  $\mathfrak{T}(\mathfrak{H})$  and even in the graph norm of  $A$ . Hence, for  $u, v \in D(Y)$ ,  $x \in \mathcal{L}(\mathfrak{H})$ ,

$$\begin{aligned} \langle v, (R(\lambda, A)^*x)u \rangle &= \text{Tr}(x(R(\lambda, A)|u\rangle\langle v|)) \\ &= \text{Tr}\left(x \int_0^{\infty} e^{-\lambda s} |P(s)u\rangle\langle P(s)v| ds\right) \\ &= \int_0^{\infty} e^{-\lambda s} \text{Tr}(x|P(s)u\rangle\langle P(s)v|) ds \\ &= \int_0^{\infty} e^{-\lambda s} \langle P(s)v, xP(s)u \rangle ds \\ &= \langle v, P_{\lambda}(x)u \rangle. \end{aligned}$$

Since  $P_\lambda(x) \in \mathcal{L}(\mathfrak{H})$  for all  $x \in \mathcal{L}(\mathfrak{H})$ , it follows that  $P_\lambda = R(\lambda, A)^*$ . Similarly,

$$\begin{aligned}
& \langle v, ((BR(\lambda, A))^*x)u \rangle \\
&= \text{Tr}(x(BR(\lambda, A)|u\rangle\langle v|)) \\
&= \text{Tr}\left(x\left(B\int_0^\infty e^{-\lambda s}|P(s)u\rangle\langle P(s)v| ds\right)\right) \\
&= \int_0^\infty e^{-\lambda s} \text{Tr}(x(B|P(s)u\rangle\langle P(s)v|)) ds \quad (\text{as } |P(s)u\rangle\langle P(s)v| \in D(B)) \\
&= \sum_{l=1}^\infty \int_0^\infty e^{-\lambda s} \langle L_l P(s)v, xL_l P(s)u \rangle ds \quad (\text{by Lemma 5.1.19}) \\
&= \langle v, Q_\lambda(x)u \rangle.
\end{aligned}$$

Therefore,  $Q_\lambda = (BR(\lambda, A))^*$ . □

Now we can show that the two semigroups are equal.

*Proof of Theorem 5.1.16.* From (5.13) and Lemma 5.1.21, we have that

$$R(\lambda, G)^*x = R(\lambda, G^*)x = \sum_{k=0}^\infty Q_\lambda^k(P_\lambda(x)) = \sum_{k=0}^\infty (BR(\lambda, A))^k R(\lambda, A)^*x$$

for all  $x \in \mathcal{L}(\mathfrak{H})$  with the series convergent in the strong operator topology. Since by Proposition 5.1.15, we know that  $\sum_{k=0}^\infty R(\lambda, A)(BR(\lambda, A))^k \rho$  converges in trace norm for all  $\rho \in \mathfrak{T}(\mathfrak{H})$  and the trace functional is continuous on  $\mathfrak{T}(\mathfrak{H})$ , it follows that

$$R(\lambda, G)\rho = \sum_{k=0}^\infty R(\lambda, A)(BR(\lambda, A))^k \rho = R(\lambda, \tilde{G})\rho$$

for all  $\rho \in \mathfrak{T}(\mathfrak{H})$ . Therefore, by the Post-Widder Inversion Formula (Proposition 1.2.8),  $S(t)\rho = \tilde{S}(t)\rho$  for all  $\rho \in \mathfrak{T}(\mathfrak{H})$  and all  $t \geq 0$ . □

In the remainder of this chapter, unless stated otherwise,  $(T(t))_{t \geq 0}$  will denote the minimal quantum dynamical semigroup with generator  $G^*$  identified in Proposition 5.1.7 with associated form  $\Upsilon$  satisfying Premise 5.1.6 and  $(S(t))_{t \geq 0}$  will always denote its predual semigroup with generator  $G$ .

*Remark 5.1.22.* One can also prove Theorem 5.1.16 by using the minimality of the semigroups, that is, Fagnola's semigroup  $(T(t))_{t \geq 0}$  is the minimal semigroup whose generator can be represented in Lindblad form (Proposition 5.1.7) while the Kato semigroup  $(\tilde{S}(t))_{t \geq 0}$  is the minimal semigroup whose generator is an extension of  $A+B$  (Theorem 2.1.2). Corollary 5.1.20 tells us that the adjoint semigroup of  $(\tilde{S}(t))_{t \geq 0}$

can be represented in Lindblad form. To complete the proof, we only need to show that any quantum dynamical semigroup satisfying (5.3) has predual semigroup whose generator is an extension of  $A + B$ . We will in fact prove this in the next section (Lemma 5.2.14).

This duality between the quantum dynamical semigroups satisfying Proposition 5.1.7 and the semigroups satisfying Kato's Theorem allows us to derive some of the properties of the quantum dynamical semigroups from Kato's Theorem, for example, the minimality of the semigroup or the series representation of the resolvent of the generator. In the next section, we will see further examples of how results related to Kato's Theorem prove useful in the study of quantum dynamical semigroups.

## 5.2 Applying Honesty Theory to Quantum Dynamical Semigroups

In the previous section, we saw that the generator of a given quantum dynamical semigroup in Lindblad form can be viewed as the dual of a perturbed generator of a substochastic semigroup in the setting of Kato's Theorem. A natural question to look at next is the role of honesty in the study of quantum dynamical semigroups. It turns out that if we have equality in (5.1), then honesty of the predual semigroup is equivalent to a notion known as conservativity of the quantum dynamical semigroup (Proposition 5.2.2).

Conservativity of the quantum dynamical semigroup is a notion which has long been studied (see [15, 16, 24] for example). The main reason for the interest in conservativity is that it is related to the non-explosion of the system [16], [24, Section 3.6]. However, the study of conservativity is also of interest because conservative quantum dynamical semigroups turn out to be the semigroups with “nice” properties. For example, if the minimal semigroup is conservative, then it is the unique semigroup satisfying (5.3) [24, Corollary 3.23]. Moreover, one can give a precise description of the domain of the generator if the minimal semigroup is conservative [24, Proposition 3.33], [27, Theorem 4.1]. This is important as we saw in the previous section that the domain of the generator of the quantum dynamical semigroup is difficult to determine precisely and is often defined in terms of a form.

We begin by giving the definition of conservativity.

**Definition 5.2.1.** *A quantum dynamical semigroup  $(T(t))_{t \geq 0}$  is called Markov or conservative or identity preserving if  $T(t)\mathbb{1} = \mathbb{1}$  for every  $t \geq 0$ .*

A necessary condition for  $(T(t))_{t \geq 0}$  to be conservative is for

$$\Upsilon(\mathbb{1})[v, u] = \frac{d}{dt} \langle v, (T(t)\mathbb{1})u \rangle |_{t=0} = 0$$

for all  $u, v \in D(Y)$ . So when we speak of conservativity, we will only consider the case when we have equality in (5.1).

**Proposition 5.2.2.** *Suppose  $(T(t))_{t \geq 0}$  is the minimal quantum dynamical semigroup identified in Proposition 5.1.7 with  $\Upsilon$  satisfying Premise 5.1.6 with equality in (5.1). Then the semigroup  $(T(t))_{t \geq 0}$  is conservative if and only if its predual semigroup  $(S(t))_{t \geq 0}$  is honest.*

*Proof.* Since  $\Upsilon$  satisfies (5.1) with equality, the predual semigroup being honest is equivalent to it being stochastic (Remark 2.2.2). In this context, this means that  $S(t)$  is trace-preserving for all  $t \geq 0$ , i.e.  $\text{Tr}(S(t)\rho) = \text{Tr}(\rho)$  for all  $\rho \in \mathfrak{T}_s(\mathfrak{H})$  and this is equivalent to  $(T(t))_{t \geq 0}$  being conservative since  $\text{Tr}(S(t)\rho) = \text{Tr}((S(t)\rho)\mathbb{1}) = \text{Tr}(\rho(T(t)\mathbb{1}))$  for all  $\rho \in \mathfrak{T}(\mathfrak{H}), t \geq 0$ .  $\square$

The equivalence between honesty and conservativity allows us to derive some of the previously known conditions for conservativity from honesty theory. For example, conditions (ii) and (iii) in Proposition 5.2.3 can be found in [24, Theorem 3.28]. It should be noted however, that [24, Theorem 3.28] was obtained directly and not via an honesty theory approach. Moreover, we obtain some new characterisations of conservativity by applying Theorem 3.2.2, Proposition 3.2.3, Lemma 5.1.21 and Proposition 5.2.2.

**Proposition 5.2.3.** *Suppose  $(T(t))_{t \geq 0}$  is the minimal quantum dynamical semigroup identified in Proposition 5.1.7 with  $\Upsilon$  satisfying Premise 5.1.6 with equality in (5.1). Let  $\lambda > 0$  and  $Q_\lambda$  as defined in (5.15). The following are equivalent:*

- (i) *The semigroup  $(T(t))_{t \geq 0}$  is conservative.*
- (ii) *The sequence of operators  $\{Q_\lambda^n(\mathbb{1})\}_{n \geq 0}$  converges  $\sigma$ -weakly to 0.*
- (iii) *If for some  $x \in \mathcal{L}(\mathfrak{H})$ , we have  $Q_\lambda x = x$ , then  $x = 0$ .*
- (iv) *The operator  $Q_{\lambda*}$  is mean ergodic, where  $Q_{\lambda*}$  denotes the predual operator of  $Q_\lambda$ .*
- (v)  *$\lim_{n \rightarrow \infty} \|Q_{\lambda*}^n \rho\|_{tr} = 0$  for all  $\rho \in \mathfrak{T}_s(\mathfrak{H})_+$ .*

(vi) For each  $\rho \in \mathfrak{T}_s(\mathfrak{H})_+$ , the sequence  $\{Q_{\lambda^*}^n \rho\}_{n \geq 0}$  is weakly compact with 0 as a weak cluster point.

We now generalise the notion of conservativity to the class of minimal quantum dynamical semigroups constructed in Proposition 5.1.7 by transferring the concept of honesty from Kato's Theorem to these quantum dynamical semigroups:

**Definition 5.2.4.** *Let  $(T(t))_{t \geq 0}$  be the minimal quantum dynamical semigroup identified in Proposition 5.1.7 with  $\Upsilon$  satisfying Premise 5.1.6. The semigroup  $(T(t))_{t \geq 0}$  is said to be honest if and only if its predual semigroup  $(S(t))_{t \geq 0}$  is honest in the sense of Definition 2.2.1.*

We will show that honesty is the natural analogue of conservativity in the strictly substochastic case. As we mentioned above, conservativity is important because it allows us to characterise uniqueness of the semigroup and also the domain of its generator. It turns out that honesty also allows us to do the same for the substochastic case as we will show in Corollary 5.2.12 and Proposition 5.2.13.

As in the conservative case, we are also interested in characterising the honesty of the quantum dynamical semigroups. Since Proposition 5.2.3 was derived from characterisations of honesty in Theorem 3.2.2 and Proposition 3.2.3, it follows that conditions (ii) to (vi) in Proposition 5.2.3 also characterise honesty of the minimal semigroup  $(T(t))_{t \geq 0}$ . These conditions can also be used to show that other previously known characterisations of conservativity (which were proven without using honesty theory methods) also characterise honesty. For example, consider the following characterisation of conservativity based on the map  $\Upsilon : \mathcal{L}(\mathfrak{H}) \rightarrow \text{Sesquilinear Forms}$  from [24].

**Proposition 5.2.5.** *[24, Proposition 3.3.1] Let  $(T(t))_{t \geq 0}$  be the minimal quantum dynamical semigroup identified in Proposition 5.1.7 with  $\Upsilon$  satisfying Premise 5.1.6 with equality in (5.1). Then  $(T(t))_{t \geq 0}$  is conservative if and only if  $\ker(\lambda - \Upsilon) = \{0\}$  for some/all  $\lambda > 0$ .*

To prove a version of this for honesty, we need the following lemma which describes the relationship between the eigenvalues of the map  $\Upsilon : \mathcal{L}(\mathfrak{H}) \rightarrow \text{Sesquilinear Forms}$  and the operator  $Q_\lambda, \lambda > 0$  defined in (5.15).

**Lemma 5.2.6.** *Fix  $\lambda > 0$ . Then for all  $x \in \mathcal{L}(\mathfrak{H})$ , we have  $\Upsilon(x) = \lambda x$  if and only if  $Q_\lambda(x) = x$ .*

We omit the proof of the lemma as it can be proven almost exactly as in [24, Proposition 3.30]. This result combined with Proposition 5.2.3 (iii) then allows us to deduce the analogue of Proposition 5.2.5 for the case of honesty.

**Corollary 5.2.7.** *Let  $(T(t))_{t \geq 0}$  be the minimal quantum dynamical semigroup identified in Proposition 5.1.7 with  $\Upsilon$  satisfying Premise 5.1.6. The semigroup  $(T(t))_{t \geq 0}$  is honest if and only if  $\ker(\lambda - \Upsilon) = \{0\}$  for some/all  $\lambda > 0$ .*

Lemma 5.2.6 also allows us to derive an alternative description of  $\ker(\lambda - \Upsilon)$ . Since we have  $Q_\lambda = (BR(\lambda, A))^*$  (Lemma 5.1.21) and from Lemma 2.2.9 we know that  $\ker(I - (BR(\lambda, A))^*) = \ker(\lambda - (A + B)^*)$ , we have as a corollary:

**Corollary 5.2.8.**  $\ker(\lambda - \Upsilon) = \ker(\lambda - (A + B)^*)$  for all  $\lambda > 0$ .

Now let us look at the domains of generators of these minimal semigroups. We will see that honesty theory allows us to give two different descriptions of the domain of the generator of an honest semigroup, one in terms of cores (Proposition 5.2.9) and the other, a description of the actual domain (Proposition 5.2.11). The description in terms of cores given below is an extension of [24, Proposition 3.32] which states that  $\mathcal{V}$  is a core for  $G$  if and only if  $(T(t))_{t \geq 0}$  is conservative.

**Proposition 5.2.9.** *The minimal quantum dynamical semigroup  $(T(t))_{t \geq 0}$  is honest if and only if any of the spaces  $\mathcal{V}_n$  or  $\pi^n(\mathfrak{T}(\mathfrak{H}))$ ,  $n \in \mathbb{N}$  is a core for  $G$ , the generator of the predual semigroup  $(S(t))_{t \geq 0}$ .*

*Proof.* Theorem 2.2.4 tells us that  $(T(t))_{t \geq 0}$  is honest if and only if  $G = \overline{A + B}$ . Since  $B$  is  $A$ -bounded, a subspace  $\mathcal{S} \subset D(A)$  is a core for  $\overline{A + B}$  if and only if  $\mathcal{S}$  is a core for  $A$ . Lemma 5.1.17 and Remark 5.1.18 tell us that for each  $n \in \mathbb{N}$ ,  $\mathcal{V}_n$  and  $\pi^n(\mathfrak{T}(\mathfrak{H}))$  are cores for  $A$  and hence are cores for  $\overline{A + B}$ . Therefore  $(T(t))_{t \geq 0}$  is honest if and only if any of the spaces  $\mathcal{V}_n$  or  $\pi^n(\mathfrak{T}(\mathfrak{H}))$ ,  $n \in \mathbb{N}$  is a core for  $G$ .  $\square$

Next, we will give a precise description of the domain of the generator of an honest semigroup. But first, we need some auxiliary information. Recall from (5.2) that  $\Upsilon(x)$  is a sesquilinear form on  $D(Y) \times D(Y)$  for all  $x \in \mathcal{L}(\mathfrak{H})$ . So if  $\Upsilon(x)$  is closed, we can associate an operator  $W(x)$  with the form  $\Upsilon(x)$  in the sense of (1.4). Define

$$\mathcal{F} := \{x \in \mathcal{L}(\mathfrak{H}) : \text{there exists } W(x) \in \mathcal{L}(\mathfrak{H}) \text{ such that} \\ \Upsilon(x)[v, u] = \langle v, W(x)u \rangle \text{ for all } u, v \in D(Y)\}.$$

**Lemma 5.2.10.**  $\mathcal{F} = D((A + B)^*)$  and  $W(x) = (A + B)^*x$  for all  $x \in \mathcal{F}$ .

*Proof.* We begin by showing that  $W \supseteq (A + B)^*$ . Let  $x \in D((A + B)^*)$ ,  $u, v \in D(Y)$ . Then by Lemma 5.1.19,

$$\langle v, ((A + B)^*x)u \rangle = \text{Tr}(((A + B)^*x)|u\rangle\langle v|) = \text{Tr}(x((A + B)|u\rangle\langle v|)) = \Upsilon(x)[v, u].$$

Thus  $\Upsilon(x)$  is given by a bounded operator, namely  $(A + B)^*x$  so  $x \in \mathcal{F}$  and  $W \supseteq (A + B)^*$ .

Now let  $x \in \mathcal{F}$  and  $u, v \in D(Y)$ . Then by Lemma 5.1.19,

$$\text{Tr}(W(x)|u\rangle\langle v|) = \langle v, W(x)u \rangle = \Upsilon(x)[v, u] = \text{Tr}(x((A + B)|u\rangle\langle v|)).$$

So there exists  $y = W(x) \in \mathcal{L}(\mathfrak{H})$  such that  $\text{Tr}(x((A + B)|u\rangle\langle v|)) = \text{Tr}(y|u\rangle\langle v|)$  for all  $u, v \in D(Y)$ . Since by Lemma 5.1.17 we have that  $\mathcal{V}$  is a core for  $A + B$ , it follows that  $\text{Tr}(x((A + B)\rho)) = \text{Tr}(y\rho)$  for all  $\rho \in D(A + B)$ . Therefore  $x \in D((A + B)^*)$  and  $W(x) = y = (A + B)^*x$ . So  $W \subseteq (A + B)^*$ .  $\square$

Lemma 5.2.10 allows us to give the following precise description of the domain of the generator when the semigroup is honest because by Theorem 2.2.4, the semigroup is honest if and only if  $G = \overline{A + B}$ , i.e. if and only if  $G^* = (A + B)^* = W$ .

**Proposition 5.2.11.** *The minimal quantum dynamical semigroup  $(T(t))_{t \geq 0}$  is honest if and only if  $D(G^*) = \mathcal{F}$  and  $G^*x = W(x)$  for all  $x \in D(G^*)$ .*

As a corollary, we have a characterisation of an honest quantum dynamical semigroup in terms of the form  $\Upsilon(x)$ . A similar result was proven for the case of conservativity in [24, Proposition 3.33] and [27, Theorem 4.1] but with different proofs as they did not apply honesty theory results.

**Corollary 5.2.12.** *The minimal quantum dynamical semigroup  $(T(t))_{t \geq 0}$  is honest if and only if the domain of its generator  $G^*$  is the space of all elements  $x \in \mathcal{L}(\mathfrak{H})$  such that the form  $\Upsilon(x)$  on  $D(Y) \times D(Y)$ ,  $(v, u) \mapsto \Upsilon(x)[v, u]$  is norm continuous.*

*Proof.* From Proposition 1.3.2 and Remark 1.3.3, the form  $\Upsilon(x)$  on  $D(Y) \times D(Y)$  is norm continuous if and only if there exists an operator  $W(x) \in \mathcal{L}(\mathfrak{H})$  such that  $\Upsilon(x)[v, u] = \langle v, W(x)u \rangle$  for all  $u, v \in D(Y)$ . So the set of all elements  $x \in \mathcal{L}(\mathfrak{H})$  such that the form  $\Upsilon(x)$  is norm continuous is precisely  $\mathcal{F}$ . The result now follows from Proposition 5.2.11.  $\square$

Lemma 5.2.10 also allows us to show that honesty characterises uniqueness of semigroups satisfying (5.3). This result is an extension of [24, Corollary 3.23], which tells us that a minimal quantum dynamical semigroup which is conservative is unique.

**Proposition 5.2.13.** *The minimal quantum dynamical semigroup  $(T(t))_{t \geq 0}$  is honest if and only if it is the unique contractive quantum dynamical semigroup on  $\mathcal{L}(\mathfrak{H})$  satisfying equation (5.3).*

To prove Proposition 5.2.13 using honesty theory, we require a result describing the relationship between the generators of quantum dynamical semigroups satisfying (5.3) and the operators  $A, B$  in Kato's Theorem.

**Lemma 5.2.14.** *Suppose  $(\mathcal{T}(t))_{t \geq 0}$  is a contractive quantum dynamical semigroup on  $\mathcal{L}(\mathfrak{H})$  with generator  $\mathcal{G}^*$ . Then  $(\mathcal{T}(t))_{t \geq 0}$  satisfies (5.3) for all  $u, v \in D(Y)$  if and only if  $\mathcal{G}^* \subseteq (A + B)^*$ .*

*Proof.* By Proposition 5.1.8,  $(\mathcal{T}(t))_{t \geq 0}$  satisfies (5.3) if and only if its generator  $\mathcal{G}^*$  satisfies (5.4). Since  $\mathcal{G}^*x \in \mathcal{L}(\mathfrak{H})$  for all  $x \in D(\mathcal{G}^*)$ , it follows from Lemma 5.2.10 that (5.4) holds if and only if  $D(\mathcal{G}^*) \subseteq D((A + B)^*)$  and  $\mathcal{G}^*x = (A + B)^*x$  for all  $x \in D(\mathcal{G}^*)$ .  $\square$

*Proof of Proposition 5.2.13.* We begin by noting that if  $S, T$  are closed and densely defined operators, then it follows from [2, Proposition B.10] that  $S \subseteq T$  if and only if  $T^* \subseteq S^*$ . Since generators of  $C_0$ -semigroups are closed and densely defined, and  $(A + B)^* = \overline{A + B}^*$ , it follows that if  $\mathcal{G}$  is a generator of a  $C_0$ -semigroup, then  $\mathcal{G}^* \subseteq (A + B)^*$  if and only if  $A + B \subseteq \mathcal{G}$ .

Now recall that we denote the generator of  $(T(t))_{t \geq 0}$  by  $G^*$  and suppose that there is another quantum dynamical semigroup  $(\mathcal{T}(t))_{t \geq 0}$  satisfying (5.3) with generator denoted  $\mathcal{G}^*$ . Then by Lemma 5.2.14, both  $\mathcal{G}^*, G^* \subseteq (A + B)^*$  and thus  $A + B \subseteq \mathcal{G}$  and  $A + B \subseteq G$ . So there exist at least two substochastic predual semigroups which have generators that are extensions of  $A + B$ . By Theorem 2.3.4, this occurs if and only if the minimal semigroup is dishonest. Therefore  $(T(t))_{t \geq 0}$  is unique if and only if it is honest.  $\square$

We will conclude this section by giving two examples of applications of honesty theory to quantum dynamical semigroups. The first is an example for the stochastic (conservative) case while the second is an example for the strictly substochastic case. As the stochastic case has been quite thoroughly studied, we will simply present a previously known example from [24] which we will consider again in Section 6.4. Since quantum dynamical semigroups as defined by Fagnola in Definition 5.1.4 act on the dual space of (the complexification of) an abstract state space, it is unsurprising that quantum dynamical semigroups exhibit circumstances where it may be easier to apply

the conditions for honesty involving the dual operator,  $(BR(\lambda, A))^*$  as we will see in Example 5.2.15.

**Example 5.2.15.** [24, Example 3.4] Let  $\mathfrak{H}$  be the Hilbert space  $\ell^2(\mathbb{N})$  over  $\mathbb{C}$  with canonical orthonormal basis  $(\mathbf{e}_n)_{n \geq 0}$ . Let  $(\alpha(n))_{n \geq 0} \subset \mathbb{C}$  be a sequence such that  $\alpha(n) \neq 0$  for all  $n \geq 0$  and  $S$  be the left shift operator on  $\mathfrak{H}$  defined by  $S\mathbf{e}_n = \mathbf{e}_{n+1}$ . Consider the operator  $\Lambda$  in  $\mathfrak{H}$  defined by

$$D(\Lambda) = \left\{ u = (u_n) \in \mathfrak{H} : \sum_{n \geq 0} |\alpha(n)|^2 |u_n|^2 < \infty \right\}, \quad \Lambda u := \sum_{n \geq 0} \alpha(n) u_n \mathbf{e}_n.$$

We will define  $Y$  and  $L_1$  to be the operators with domain  $D(|\Lambda|^2)$  where  $|\Lambda|^2 u := \sum_{n \geq 0} |\alpha(n)|^2 u_n \mathbf{e}_n$  and

$$Y u := -\frac{1}{2} \sum_{n \geq 0} |\alpha(n)|^2 u_n \mathbf{e}_n \quad L_1 u := S \Lambda u \quad \text{for all } u \in D(|\Lambda|^2)$$

and  $L_l = 0$  for  $l \geq 2$ . Elementary calculations and the fact that  $S$  is an isometry show that the operators  $Y$  and  $(L_l)_{l \geq 1}$  satisfy Premise 5.1.6 with equality in (5.1). So we let  $(T(t))_{t \geq 0}$  denote the minimal quantum dynamical semigroup associated with the operators  $Y$  and  $(L_l)_{l \geq 1}$ .

Fagnola then proves the following necessary and sufficient condition for the conservativity of  $(T(t))_{t \geq 0}$ .

**Proposition 5.2.16.**  $(T(t))_{t \geq 0}$  is conservative if and only if  $\sum_{n \geq 0} \frac{1}{|\alpha(n)|^2}$  diverges.

To prove the proposition, he uses an inductive proof to show first that for all  $k \geq 1$ ,

$$\langle \mathbf{e}_m, Q_\lambda^k(\mathbb{1}) \mathbf{e}_n \rangle = \langle \mathbf{e}_m, \mathbf{e}_n \rangle \prod_{j=0}^{k-1} \frac{|\alpha(n+j)|^2}{\lambda + |\alpha(n+j)|^2}. \quad (5.16)$$

He then applies condition (ii) of Proposition 5.2.3 which by (5.16), is equivalent to  $\lim_{k \rightarrow \infty} \prod_{j=0}^{k-1} \frac{|\alpha(n+j)|^2}{\lambda + |\alpha(n+j)|^2} = 0$  for all  $n \in \mathbb{N}_0$ . This is then shown to be equivalent to the divergence of  $\sum_{n \geq 0} \frac{1}{|\alpha(n)|^2}$  by elementary calculations. See [24] for a full proof.

We now give an application of honesty to a strictly substochastic semigroup. As honesty theory for the strictly substochastic case has yet to be studied in the literature, we will modify an example from [23, Section 2] (which was used for the study of conservativity) to form a strictly substochastic semigroup. Our modification will be based on classical  $L^1$  examples of stochastic and substochastic semigroups. In the classical case, one modification which affects the stochasticity of semigroups is the

addition of a potential or absorption term to the generator (see for instance, Example 2.1.6 for the case of transport semigroups or the case of Laplacians on graphs in Chapter 4). As we will see in Example 5.2.17, we will use a similar modification to obtain a substochastic quantum dynamical semigroup.

**Example 5.2.17.** We begin by looking at the example from [23]. Let  $\mathfrak{H} := L^2(\mathbb{R}, \mathbb{C})$ , the space of complex-valued square-integrable functions on the real line and let the operators  $Y, (L_l)_{l \in \mathbb{N}}$  be defined as:

$$\begin{aligned} (Yu)(x) &= \frac{1}{2}\sigma(x)^2u''(x), & D(Y) &= \{u \in \mathfrak{H} : u', u'' \in \mathfrak{H}\}, \\ (Lu)(x) &= L_1u(x) = \sigma(x)u'(x), & D(L) &= \{u \in \mathfrak{H} : u' \in \mathfrak{H}\}, \\ & \text{where } \sigma(x) \text{ is a complex-valued function on } \mathbb{R} \text{ with } |\sigma| \leq 1 & (5.17) \\ (L_lu)(x) &= 0, & l &\geq 2. \end{aligned}$$

For simplicity, we will consider two cases, namely  $\sigma(x) = 1$  and  $\sigma(x) = -ie^{ix}$ . Note that if  $\sigma(x) = 1$ , then  $Y$  is simply the Laplacian on  $\mathbb{R}$ . Thus it follows that (some realisation of)  $Y$  is a self-adjoint operator which generates a substochastic semigroup on  $\mathfrak{H}$  and  $D(L) \supseteq D(Y)$ . In fact, it can be shown, see [23], that these operators satisfy Premise 5.1.6 with equality. More importantly,

**Proposition 5.2.18.** [23, Theorem 2.1, Remark 3.5] *The semigroup is honest if  $\sigma(x) = 1$  and dishonest if  $\sigma(x) = -ie^{ix}$ .*

More generally, if  $\sigma$  is a real-valued bounded smooth function in  $\mathbb{R}$  with bounded derivatives of all orders or if  $\sigma$  is multiplied by a complex phase independent of  $x$ , then the minimal quantum dynamical semigroup constructed from  $Y$  and  $L$  above is conservative (or honest) [23, Remark 3.5]. Such semigroups occur in the dilation and quantum extension of classical diffusion processes on  $\mathbb{R}$ .

We use a simple modification in order to obtain the strictly substochastic example. First consider the Schrödinger operator, defined formally by  $S_K := \frac{1}{2}\Delta - K$  for some measurable function  $K$ . This operator is related to diffusion processes with absorption and the Laplacian  $S = \frac{1}{2}\Delta$  is simply a special case of the Schrödinger operator with  $K = 0$ . Under some additional conditions, the Schrödinger operator is a self-adjoint operator generating a substochastic semigroup in  $\mathfrak{H}$  (see [45] for example). We will consider the case when  $K$  is a strictly positive, bounded, real-valued function and we will also let  $K$  denote the operator of multiplication with this function. We now define

$$(Y_Ku)(x) := \frac{1}{2}\sigma(x)^2u''(x) - K(x)u(x), \quad D(Y_K) = \{u \in \mathfrak{H} : u', u'' \in \mathfrak{H}\}$$

and leave  $(L_l)_{l \in \mathbb{N}}$  as defined in (5.17).

Once again, we consider the two cases above, namely  $\sigma(x) = 1$  and  $\sigma(x) = -ie^{ix}$ . Since  $K$  is multiplication with a strictly positive function,  $Y_K$  and  $(L_l)_{l \in \mathbb{N}}$  satisfy (5.1) with a strict inequality. Hence the minimal quantum dynamical semigroup associated with  $Y_K, (L_l)_{l \in \mathbb{N}}$  is strictly substochastic.

From Section 5.1, we know that the formal operators

$$\begin{aligned} A_K \rho &:= Y_K \rho + \rho Y_K^* = Y \rho + \rho Y^* - (K \rho + \rho K), \quad \rho \in D(A) \text{ and} \\ B \rho &:= L_1 \rho L_1, \quad \rho \in D(B) \end{aligned}$$

satisfy Kato's Theorem. Since the operator  $K$  is bounded, so is the operator  $\mathcal{K} : \mathfrak{T}(\mathfrak{H}) \rightarrow \mathfrak{T}(\mathfrak{H}), \mathcal{K} \rho := K \rho + \rho K$ . Hence we may apply our results relating potentials and honesty from Section 2.4.1 to derive some results on the honesty of this semigroup. In particular, combining Proposition 2.4.1 and Proposition 2.4.2 with Proposition 5.2.18, it follows that

**Proposition 5.2.19.** *The minimal quantum dynamical semigroup associated with  $Y_K$  and  $L$  is honest for the case  $\sigma(x) = 1$  and dishonest for the case  $\sigma(x) = -ie^{ix}$ .*

Therefore, if  $\sigma(x) = 1$  (or more generally,  $\sigma$  is a real-valued, bounded, smooth function in  $\mathbb{R}$  with bounded derivatives of all orders), then the minimal semigroup is the unique semigroup satisfying (5.3) (Proposition 5.2.13). Moreover, by Corollary 5.2.12, in this case we have a precise description of the domain of the generator of the minimal semigroup, namely  $D(G^*)$  is given by all elements  $x \in \mathcal{L}(\mathfrak{H})$  such that the form  $\Upsilon(x)$  on  $D(Y) \times D(Y), (v, u) \mapsto \Upsilon(x)[v, u]$  is norm continuous.

## Chapter 6

# Transport Semigroups - Boundary Perturbations

In this chapter, we will look at boundary perturbations in the context of the transport semigroup. Although the theory of boundary perturbations differs greatly from the theory of additive perturbations, we will see that there are results in the perturbation theory of transport semigroups which closely resemble those of Kato's Theorem. Moreover, there is also a theory of honesty with respect to these perturbations which mirrors that of Kato's Theorem. We will begin by introducing the set-up of the transport operator in Section 6.1 followed by the generation theorem for the transport semigroup in Section 6.2. We will then consider the similarities between the honesty theory of Kato's Theorem and the corresponding results for the transport semigroup in Section 6.3 and subsequently present a result which unifies both theories. Finally, we look briefly at strong stability of both semigroups, which can be thought of as the reverse of honesty.

### 6.1 The Transport Operator

We begin by considering once again the classical transport equation with no re-entry boundary conditions from Example 2.1.6. In that example, we applied Kato's Theorem to show that there was an extension of the operator  $T + K$  which generates a substochastic  $C_0$ -semigroup. Moreover, we noted that the property of  $T := T_0 - h$  generating a substochastic  $C_0$ -semigroup rested on yet another branch of additive perturbation theory for  $C_0$ -semigroups. These applications of additive perturbation theory indicate that the study of the transport equation begins by investigating generation theorems for the operator  $T_0$  and indeed, it is a more general version of this

operator that we will study in this chapter. We also note that historically, the operator  $T_0$  was the main interest in many studies in transport theory, see for example [44, 12, 5, 6].

In this section, we will introduce the transport operator and its related notions. We will consider the transport equation with general external fields and general boundary conditions following the set-up of [5, 6]. We let  $\mathbb{R}^n$  be endowed with a positive Radon measure  $\mu$  and  $\Omega$  be a sufficiently smooth open subset of  $\mathbb{R}^n$  with  $\mathcal{F}$  a restriction to  $\Omega$  of a time independent globally Lipschitz vector field  $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . The vector field induces the outgoing and incoming parts of the boundary  $\Gamma_+, \Gamma_-$ , and we define a boundary operator  $H$  which will usually be a linear bounded operator acting between the trace spaces  $L^1_{\pm} := L^1(\Gamma_{\pm}, \mu_{\pm})$ . More precise details will follow in this section and the next.

The transport equation is given as:

$$\frac{\partial f}{\partial t}(\mathbf{x}, t) = -\mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, t > 0$$

with boundary conditions

$$f|_{\Gamma_-}(\mathbf{y}, t) = H(f|_{\Gamma_+})(\mathbf{y}, t), \quad \mathbf{y} \in \Gamma_-, t > 0$$

and initial condition

$$f(\mathbf{x}, 0) = f_0(\mathbf{x}).$$

Define the flow  $\Phi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  such that for  $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}$ , the mapping  $\mathbb{R} \ni t \mapsto \Phi(\mathbf{x}, t)$  is the unique solution to the problem

$$\frac{d\mathbf{X}}{dt}(t) = \mathcal{F}(\mathbf{X}(t)) \quad \text{for all } t \in \mathbb{R}; \quad \mathbf{X}(0) = \mathbf{x} \in \Omega.$$

We assume that the measure  $\mu$  is invariant under the flow.

An example of this type of equation is the Vlasov equation, where  $\Omega$  is the cylindrical domain  $\Omega = D \times \mathbb{R}^3 \subset \mathbb{R}^6$  with  $D$  a sufficiently smooth open subset of  $\mathbb{R}^3$  and the field  $\mathcal{F}$  given by

$$\mathcal{F}(\mathbf{x}) = (v, \mathbf{F}(r, v)) \text{ for any } \mathbf{x} = (r, v) \in \Omega$$

where  $\mathbf{F}$  is a time independent force field over  $D \times \mathbb{R}^3$  [6, p.2]. Other examples include applications to flows on networks where the measure  $\mu$  is then supported on graphs [6, p.3].

As in Example 2.1.6, the transport equation allows us to define what we will call, transport operators (see for example the paragraph after Proposition 6.1.1 and

Definition 6.2.1). In this case, we are mainly interested in what is known as the free streaming operator in  $X = L^1(\Omega, \mu)$ . Before giving the definition of this operator however, we first need some preliminary information. As we are interested in boundary perturbations, the trace theory of the functions in  $X$  will be of particular importance. The theory we present below was developed by Arlotti, Banasiak and Lods in [5] as an extension of the trace theory developed by Cessenat in [14] for functions in  $L^p(\mathbb{R}^n, m)$ , where  $m$  is the Lebesgue measure [6, p.3].

We are mainly interested in the times when the solution lies within  $\Omega$  so we define the stay times of the characteristic curves in  $\Omega$ . For any  $\mathbf{x} \in \Omega$ , define

$$\tau_{\pm}(\mathbf{x}) = \inf\{s > 0 : \Phi(\mathbf{x}, \pm s) \notin \Omega\}$$

with the convention that  $\inf \emptyset = \infty$ . This then allows us to define the outgoing and incoming parts of the boundary  $\partial\Omega$ ,

$$\Gamma_{\pm} := \{\mathbf{y} \in \partial\Omega : \text{there exists } \mathbf{x} \in \Omega \text{ with } \tau_{\pm}(\mathbf{x}) < \infty \text{ such that } \mathbf{y} = \Phi(\mathbf{x}, \pm\tau_{\pm}(\mathbf{x}))\}.$$

We then extend the definition of  $\tau_{\pm}$  to  $\Gamma_{\pm}$  as follows: For  $\mathbf{y} \in \Gamma_{\pm}$ , let  $\tau_{\pm}(\mathbf{y}) = 0$  and  $\tau_{\mp}(\mathbf{y}) = \tau_{+}(\mathbf{x}) + \tau_{-}(\mathbf{x})$  where  $\mathbf{x}$  is such that  $\mathbf{y} = \Phi(\mathbf{x}, \pm\tau_{\pm}(\mathbf{x}))$ . In other words,  $\tau_{\mp}(\mathbf{y})$  is the length of the characteristic curve having  $\mathbf{y}$  as its outgoing (resp. incoming) end point.

The characteristic curves are not assumed to be of finite length, so we define

$$\Omega_{\pm} = \{\mathbf{x} \in \Omega : \tau_{\pm}(\mathbf{x}) < \infty\}, \quad \Omega_{\pm\infty} = \{\mathbf{x} \in \Omega : \tau_{\pm}(\mathbf{x}) = \infty\}$$

and

$$\Gamma_{\pm\infty} = \{\mathbf{y} \in \Gamma_{\pm} : \tau_{\mp}(\mathbf{y}) = \infty\}.$$

It can be shown [5, Section 2] that there exist unique Borel measures  $\mu_{\pm}$  on  $\Gamma_{\pm}$  such that the integral of  $f \in X$  over  $\Omega$  can be calculated as the integral over all characteristic curves, of the integrals of  $f$  over each characteristic curve.

**Proposition 6.1.1.** [5, Proposition 2.10] *There exist unique Borel measures  $\mu_{\pm}$  on  $\Gamma_{\pm}$  such that for any  $f \in X$ ,*

$$\int_{\Omega_{\pm}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_{\pm}} d\mu_{\pm}(\mathbf{y}) \int_0^{\tau_{\mp}(\mathbf{y})} f(\Phi(\mathbf{y}, \mp s)) ds$$

and

$$\int_{\Omega_{\pm} \cap \Omega_{\mp\infty}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_{\pm\infty}} d\mu_{\pm}(\mathbf{y}) \int_0^{\infty} f(\Phi(\mathbf{y}, \mp s)) ds.$$

We will often use the space of functions defined on  $\Gamma_{\pm}$ ,  $L^1_{\pm} := L^1(\Gamma_{\pm}, \mu_{\pm})$ .

We can now define the maximal transport operator,  $\mathcal{T}_M$  as the negative of the weak derivative along the characteristic curves. More precisely, let  $\mathfrak{S}$  be the set of all measurable functions  $\varphi$  with compact support in  $\Omega$  such that for any  $\mathbf{x} \in \Omega$ , the mapping

$$(-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \ni s \mapsto \varphi(\Phi(\mathbf{x}, s))$$

is continuously differentiable and the map  $\Omega \ni \mathbf{x} \mapsto \frac{d}{ds}\varphi(\Phi(\mathbf{x}, s))$  is bounded. The domain of  $\mathcal{T}_M$  is then defined as all  $f \in X$  for which there exists a function  $g \in X$  such that

$$\int_{\Omega} g(\mathbf{x})\varphi(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega} f(\mathbf{x}) \frac{d}{ds}\varphi(\Phi(\mathbf{x}, s))|_{s=0} d\mu(\mathbf{x}) \quad \text{for all } \varphi \in \mathfrak{S}.$$

Then

$$\mathcal{T}_M f := g.$$

Note that  $\mathcal{T}_M$  is closed as the test functions are compactly supported.

A fundamental property of  $\mathcal{T}_M$  is that every  $f \in D(\mathcal{T}_M)$  admits an absolutely continuous representation along characteristic curves [5, Theorem 3.1]. More precisely, for any  $f \in D(\mathcal{T}_M)$ , there exists a representative  $f^{\#}$  of  $f$  such that for  $\mu$ -a.e.  $\mathbf{x} \in \Omega$  and any  $-\tau_-(\mathbf{x}) \leq t_1 \leq t_2 \leq \tau_+(\mathbf{x})$ ,

$$f^{\#}(\Phi(\mathbf{x}, t_1)) - f^{\#}(\Phi(\mathbf{x}, t_2)) = \int_{t_1}^{t_2} [\mathcal{T}_M f](\Phi(\mathbf{x}, s)) ds.$$

An important consequence of this result is that it allows us to define the trace operators

$$B^+ f(\mathbf{y}) := \lim_{s \rightarrow 0^+} f^{\#}(\Phi(\mathbf{y}, -s)), \text{ where the limit exists for a.e. } \mathbf{y} \in \Gamma_+,$$

$$B^- f(\mathbf{y}) := \lim_{s \rightarrow 0^+} f^{\#}(\Phi(\mathbf{y}, s)), \text{ where the limit exists for a.e. } \mathbf{y} \in \Gamma_-.$$

Although the traces of functions in  $D(\mathcal{T}_M)$  do not necessarily belong to  $L^1_{\pm}$ , they do belong to an  $L^1$ -space with a different measure. Define the measures  $\xi_{\pm}$  as

$$d\xi_{\pm}(\mathbf{y}) := \min\{\tau_{\mp}(\mathbf{y}), 1\} d\mu_{\pm}(\mathbf{y}) \quad \mathbf{y} \in \Gamma_{\pm}.$$

**Proposition 6.1.2.** [6, Theorem 3.1] *For any  $f \in D(\mathcal{T}_M)$ , the trace  $B^{\pm} f$  belongs to  $Y_{\pm} := L^1(\Gamma_{\pm}, \xi_{\pm})$  with*

$$\|B^{\pm} f\|_{Y_{\pm}} \leq \|f\|_X + \|\mathcal{T}_M f\|_X.$$

Now that we have defined the transport operator and its related notions, let us look at how the transport operator is related to substochastic semigroups. In particular, we will look at solutions to simple boundary value problems involving the transport operator in this section before moving on to consider boundary perturbations in the form of a boundary operator in the next.

Let us begin by looking at the simplest unperturbed case. Like the operator  $A$  in the case of additive perturbations, we have a similar base operator in boundary perturbations, namely, the free streaming operator with no re-entry boundary condition,  $\mathcal{T}_0$  defined by:

$$\mathcal{T}_0 f = \mathcal{T}_M f, \text{ for all } f \in D(\mathcal{T}_0) \text{ where } D(\mathcal{T}_0) = \{f \in D(\mathcal{T}_M) : B^- f = 0\}.$$

It is shown in [3, Theorem 4.1] that  $\mathcal{T}_0$  is the generator of a substochastic  $C_0$ -semigroup  $(U_0(t))_{t \geq 0}$  in  $X$  where

$$U_0(t)f(\mathbf{x}) = f(\Phi(\mathbf{x}, -t))\chi_{\{t < \tau_-(\mathbf{x})\}}(\mathbf{x}), \quad \mathbf{x} \in \Omega, f \in X. \quad (6.1)$$

Alternatively, we may think of  $\mathcal{T}_0$  in the following manner: For any  $g \in X$  and  $\lambda > 0$ , the boundary value problem (BVP)

$$(\lambda - \mathcal{T}_M)f = g, \quad B^- f = 0 \quad (6.2)$$

has a unique solution  $f = R(\lambda, \mathcal{T}_0)g$ .

We can also find solutions to BVPs with more general boundary conditions. For example, consider the BVP

$$(\lambda - \mathcal{T}_M)f = g, \quad B^- f = u. \quad (6.3)$$

Then we have the following existence theorem:

**Theorem 6.1.3.** [5, Theorem 4.2] *Given  $\lambda > 0$ ,  $u \in L_-^1$  and  $g \in X$ , the function*

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} g(\Phi(\mathbf{x}, -s))e^{-\lambda s} ds + u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x})))e^{-\lambda \tau_-(\mathbf{x})}\chi_{\{\tau_-(\mathbf{x}) < \infty\}}(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

*is the unique solution to boundary value problem (6.3). Moreover,  $B^+ f \in L_+^1$  and*

$$\|B^+ f\|_{L_+^1} + \lambda \|f\|_X \leq \|u\|_{L_-^1} + \|g\|_X \quad (6.4)$$

*with equality if  $g \geq 0$  and  $u \geq 0$ .*

From this theorem, it follows that

$$\mathfrak{W} := \{f \in D(\mathcal{T}_M) : B^- f \in L_-^1\} \subseteq \{f \in D(\mathcal{T}_M) : B^+ f \in L_+^1\}.$$

It turns out that we actually have equality [6, Proposition 3.2], i.e.

$$\mathfrak{W} := \{f \in D(\mathcal{T}_M) : B^- f \in L_-^1\} = \{f \in D(\mathcal{T}_M) : B^+ f \in L_+^1\}. \quad (6.5)$$

Theorem 6.1.3 and (6.5) then induce a particularly important corollary, namely

**Corollary 6.1.4** (Green's Formula). [6, Corollary 4.2] *Let  $f \in \mathfrak{W}$ . Then*

$$\int_{\Omega} T_M f \, d\mu = \int_{\Gamma_-} B^- f \, d\mu - \int_{\Gamma_+} B^+ f \, d\mu.$$

It turns out that Theorem 6.1.3 may in fact be extended to  $u \in Y_-$ ,

**Theorem 6.1.5.** [6, Theorem 3.2] *Given  $\lambda > 0$ ,  $u \in Y_-$  and  $g \in X$ , the function*

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} g(\Phi(\mathbf{x}, -s)) e^{-\lambda s} \, ds + u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x}))) e^{-\lambda \tau_-(\mathbf{x})} \chi_{\{\tau_-(\mathbf{x}) < \infty\}}(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

*is the unique solution to the boundary value problem (6.3).*

In order to simplify notation and represent the solutions to BVP (6.3) in an abstract form, we now introduce a set of operators. For any  $\lambda > 0$  and  $u \in Y_-$ , define

$$\begin{aligned} \Xi_\lambda : Y_- &\rightarrow X, & [\Xi_\lambda u](\mathbf{x}) &= u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x}))) e^{-\lambda \tau_-(\mathbf{x})} \chi_{\{\tau_-(\mathbf{x}) < \infty\}}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ M_\lambda : Y_- &\rightarrow Y_+, & [M_\lambda u](\mathbf{y}) &= B^+ \Xi_\lambda u(\mathbf{y}) \\ & & &= u(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) e^{-\lambda \tau_-(\mathbf{y})} \chi_{\{\tau_-(\mathbf{y}) < \infty\}}(\mathbf{y}), & \mathbf{y} \in \Gamma_+, \\ C_\lambda : X &\rightarrow X, & [C_\lambda f](\mathbf{x}) &= R(\lambda, \mathcal{T}_0) f(\mathbf{x}) \\ & & &= \int_0^{\tau_-(\mathbf{x})} f(\Phi(\mathbf{x}, -s)) e^{-\lambda s} \, ds, & \mathbf{x} \in \Omega, \\ G_\lambda : X &\rightarrow L_+^1, & [G_\lambda f](\mathbf{y}) &= B^+ C_\lambda f(\mathbf{y}) \\ & & &= \int_0^{\tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) e^{-\lambda s} \, ds, & \mathbf{y} \in \Gamma_+. \end{aligned}$$

First, note that by Theorem 6.1.5 with  $g = 0$ , it follows that  $\Xi_\lambda$  is a lifting operator which, to a given  $u \in Y_-$ , associates a function  $f := \Xi_\lambda u \in D(\mathcal{T}_M)$  so that  $B^- f = u$ . More precisely, we have for  $u \in Y_-$ ,

$$(\lambda - \mathcal{T}_M) \Xi_\lambda u = 0, \quad B^- \Xi_\lambda u = u, \quad B^+ \Xi_\lambda u = M_\lambda u. \quad (6.6)$$

Useful relations involving  $C_\lambda$  and  $G_\lambda$  on the other hand include

$$(\lambda - \mathcal{T}_M)C_\lambda f = f, \quad B^- C_\lambda f = 0, \quad B^+ C_\lambda f = G_\lambda f \quad \text{where } f \in X. \quad (6.7)$$

It can be shown that the operators are bounded on their respective domains [6, pp.13-14]. Of particular interest to us is the bound on the operator  $M_\lambda$ . Since  $\Xi_\lambda$  maps  $Y_-$  into  $D(\mathcal{T}_M)$ , by Proposition 6.1.2, we see that  $M_\lambda$  maps  $Y_-$  into  $Y_+$ . We actually know more, namely given  $u \in Y_-$ ,  $M_\lambda u \in L_+^1$  if and only if  $u \in L_-^1$  [6, Lemma 3.1]. It turns out that both

$$\|M_\lambda\|_{\mathcal{L}(Y_-, Y_+)} \leq 1 \quad \text{and} \quad \|M_\lambda\|_{\mathcal{L}(L_-^1, L_+^1)} \leq 1. \quad (6.8)$$

To see this, consider  $M_\lambda : Y_- \rightarrow Y_+$ . Note that if  $\mathbf{y} \in \Gamma_+$ , then  $\mathbf{z} = \Phi(\mathbf{y}, -\tau_-(\mathbf{y})) \in \Gamma_-$  with  $\tau_+(\mathbf{z}) = \tau_-(\mathbf{y})$ . Therefore for  $u \in Y_+$ ,

$$\begin{aligned} \|M_\lambda u\|_{Y_+} &= \int_{\Gamma_+ \setminus \Gamma_{+\infty}} |u(\Phi(\mathbf{y}, -\tau_-(\mathbf{y})))| e^{-\lambda\tau_-(\mathbf{y})} \min\{\tau_-(\mathbf{y}), 1\} d\mu_+(\mathbf{y}) \\ &= \int_{\Gamma_- \setminus \Gamma_{-\infty}} |u(\mathbf{z})| e^{-\lambda\tau_+(\mathbf{z})} \min\{\tau_+(\mathbf{z}), 1\} d\mu_-(\mathbf{y}) \\ &\leq \int_{\Gamma_-} |u(\mathbf{z})| d\xi_-(\mathbf{z}) = \|u\|_{Y_-}. \end{aligned}$$

A similar argument shows that  $\|M_\lambda\|_{\mathcal{L}(L_-^1, L_+^1)} \leq 1$ .

Finally, observe that using these operators, we can restate the unique solution to (6.3) in Theorem 6.1.5 as

$$f = C_\lambda g + \Xi_\lambda u. \quad (6.9)$$

Theorem 6.1.3 shows that for any  $u \in L_-^1$ , we can find  $f \in D(\mathcal{T}_M)$  such that its trace on  $\Gamma_-$  is precisely  $u$ . A natural question to ask is whether a similar result holds for  $u \in L_+^1$  too. The answer to this question is in the affirmative as given in the proposition below but with the additional requirement that the constructed function,  $f$ , has trace 0 on  $\Gamma_-$ .

**Proposition 6.1.6.** [6, Proposition 2.3] *Given  $h \in L_+^1$ , we can find  $f \in D(\mathcal{T}_M)$  such that  $B^- f = 0$  and  $B^+ f = h$ . In particular, for  $\mathbf{x} \in \Omega$*

$$f(\mathbf{x}) = \begin{cases} h(\Phi(\mathbf{x}, \tau_+(\mathbf{x}))) \frac{\tau_-(\mathbf{x})e^{-\tau_+(\mathbf{x})}}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty \\ h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})))e^{-\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) = \infty \text{ and } \tau_+(\mathbf{x}) < \infty \\ 0 & \text{if } \tau_+(\mathbf{x}) = \infty. \end{cases}$$

The construction in Proposition 6.1.6 tells us that we can find a  $g \in X$  such that  $f$  is the solution to the BVP

$$(\lambda - \mathcal{T}_M)f = g, \quad B^+f = h, \quad \text{for a given } h \in L_+^1.$$

More precisely, given any  $u \in L_+^1$ , we can find  $f \in D(\mathcal{T}_M)$  such that  $B^-f = 0$  and  $B^+f = u$ . This is equivalent to saying that  $f \in D(\mathcal{T}_0)$  i.e. for any  $u \in L_+^1$ , there exists  $g \in X$  such that  $(\lambda - \mathcal{T}_0)f = g$  and  $u = B^+f = B^+C_\lambda g = G_\lambda g$ . Thus we have as a corollary:

**Corollary 6.1.7.** *The operator  $G_\lambda : X \rightarrow L_+^1$  is surjective for all  $\lambda > 0$ .*

Finally, we need a result which describes when a given pair of functions  $\psi_\pm \in Y_\pm$  are the traces of a function  $f \in D(\mathcal{T}_M)$ .

**Proposition 6.1.8.** *[6, Proposition 3.1] Let  $\psi_\pm \in Y_\pm$ . There exists  $f \in D(\mathcal{T}_M)$  such that  $B^\pm f = \psi_\pm$  if and only if  $\psi_+ - M_\lambda \psi_- \in L_+^1$  for some/all  $\lambda > 0$ . In particular, if  $\psi_+ - M_\lambda \psi_- = G_\lambda g$  for some  $g \in X$ , then there exists  $f \in D(\mathcal{T}_M)$  such that  $B^\pm f = \psi_\pm$  and  $(\lambda - \mathcal{T}_M)f = g$ .*

*Proof.* Suppose there exists  $f \in D(\mathcal{T}_M)$  such that  $B^\pm f = \psi_\pm$ . Set  $g = f - \Xi_\lambda \psi_-$ . Then by (6.6), we have  $g \in D(\mathcal{T}_M)$  with  $(\lambda - \mathcal{T}_M)g = (\lambda - \mathcal{T}_M)f$  and  $B^-g = B^-f - B^-\Xi_\lambda \psi_- = \psi_- - \psi_- = 0$ . Moreover, we have  $B^+g = \psi_+ - M_\lambda \psi_-$ . Since  $B^-g \in L_-^1$ , it follows from (6.5) that  $B^+g = \psi_+ - M_\lambda \psi_- \in L_+^1$ .

Conversely, let  $\psi_+ - M_\lambda \psi_- \in L_+^1$ . Since  $G_\lambda$  is surjective (Corollary 6.1.7), there exists  $g \in X$  such that  $\psi_+ - M_\lambda \psi_- = G_\lambda g$ . Define  $f = C_\lambda g + \Xi_\lambda \psi_-$ . Then  $f \in D(\mathcal{T}_M)$  with  $(\lambda - \mathcal{T}_M)f = (\lambda - \mathcal{T}_M)C_\lambda g + (\lambda - \mathcal{T}_M)\Xi_\lambda \psi_- = g$  by (6.6) and (6.7). Moreover,  $B^+f = B^+C_\lambda g + B^+\Xi_\lambda \psi_- = G_\lambda g + M_\lambda \psi_- = \psi_+ - M_\lambda \psi_- + M_\lambda \psi_- = \psi_+$  and  $B^-f = B^-C_\lambda g + B^-\Xi_\lambda \psi_- = \psi_-$  by (6.6) and (6.7).  $\square$

We now define  $\mathfrak{E}$  as the space of elements  $(\psi_+, \psi_-) \in Y_+ \times Y_-$  such that  $\psi_+ - M_\lambda \psi_- \in L_+^1$  for some/all  $\lambda > 0$ . Equivalently,  $\mathfrak{E}$  is just the space of elements  $(\psi_+, \psi_-) \in Y_+ \times Y_-$  such that there exists  $f \in D(\mathcal{T}_M)$  such that  $\psi_\pm$  are its traces. For fixed  $\lambda > 0$ , we equip  $\mathfrak{E}$  with the norm

$$\|(\psi_+, \psi_-)\|_{\mathfrak{E}} = \|\psi_+\|_{Y_+} + \|\psi_-\|_{Y_-} + \|\psi_+ - M_\lambda \psi_-\|_{L_+^1}.$$

This norm makes it a Banach space [6, p.15]. In the next corollary, which follows mostly from Proposition 6.1.8 and its proof (see also [6, Corollary 3.1]),  $D(\mathcal{T}_M)$  is endowed with the graph norm.

**Corollary 6.1.9.** *The trace mapping  $\mathcal{B} : D(\mathcal{T}_M) \rightarrow \mathfrak{E}$ ,  $f \mapsto (B^+f, B^-f)$  is continuous, surjective and has right inverse given by*

$$\mathcal{B}^{-1}(\psi_+, \psi_-) = C_\lambda g + \Xi_\lambda \psi_-$$

where  $g$  satisfies  $\psi_+ - M_\lambda \psi_- = G_\lambda g$ .

## 6.2 Boundary Perturbations and Transport Semigroups

In the previous section, we considered solutions to boundary value problems involving the transport operator with fixed, boundary conditions (incoming and outgoing) which were independent of each other. In this section, we will look at existence results for boundary value problems where the boundary conditions are interdependent. The dependence is given in terms of a boundary operator  $H$  and the solutions will be stated in terms of generation theorems for  $C_0$ -semigroups. In particular, we will consider  $H$  as a boundary perturbation and present some generation results with respect to this boundary perturbation, much as we did in the previous chapters for the additive perturbation operator  $B$ . As we mentioned at the beginning of this chapter, we will see that there is a generation theorem for transport semigroups which shares many similarities with Kato's Theorem. In this section, we will study this perturbation theorem and its similarities to Kato's Theorem. Although the results in this section are not new, the presentation of the results is novel as we will prove the theorem in a manner which mirrors the proof of Kato's Theorem.

We consider a (possibly unbounded) linear operator  $H$  from  $Y_+$  to  $Y_-$  with domain  $D(H)$  and assume that  $\text{Graph}(H) \subset \mathfrak{E}$ .

**Definition 6.2.1.** *The transport operator with boundary operator  $H$ ,  $\mathcal{T}_H$  is defined as  $\mathcal{T}_H f = \mathcal{T}_M f$  for any  $f \in D(\mathcal{T}_H)$  where*

$$\begin{aligned} D(\mathcal{T}_H) &= \{f \in D(\mathcal{T}_M) : \mathcal{B}f = (B^+f, B^-f) \in \text{Graph}(H)\} \\ &= \{f \in D(\mathcal{T}_M) : B^-f = HB^+f\}. \end{aligned}$$

Note that by Corollary 6.1.9, for any  $h \in D(H)$ , there is an  $f \in D(\mathcal{T}_H)$  such that  $B^+f = h$ , so the operator is well-defined. The perturbation theorem we are most interested in holds for the case when  $H : L_+^1 \rightarrow L_-^1$ .

**Theorem 6.2.2.** *Let  $H \in \mathcal{L}(L_+^1, L_-^1)$  be a positive operator with  $\|H\| \leq 1$ . Then there exists an extension  $\mathcal{G}$  of  $\mathcal{T}_H$  that generates a substochastic  $C_0$ -semigroup,  $(V_H(t))_{t \geq 0}$  in  $X$ . Moreover, its resolvent is given by*

$$R(\lambda, \mathcal{G})f = C_\lambda f + \sum_{n=0}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda f \quad \text{for all } f \in X, \lambda > 0. \quad (6.10)$$

$(V_H(t))_{t \geq 0}$  is the minimal substochastic  $C_0$ -semigroup whose generator is an extension of  $\mathcal{T}_H$  in the following sense: if  $(\tilde{V}(t))_{t \geq 0}$  is another substochastic  $C_0$ -semigroup whose generator is an extension of  $\mathcal{T}_H$ , then  $V_H(t) \leq \tilde{V}(t)$  for all  $t \geq 0$ .

We will prove this theorem by considering the case for  $\|H\| < 1$  first and then taking the limit of semigroups  $(U_{H_r}(t))_{t \geq 0}$  as  $r \rightarrow 1$ . We give the proof in this section as it will better demonstrate the complexities of considering boundary perturbations and the differences with respect to additive perturbations in Kato's Theorem.

This theorem has been presented in various forms in different contexts. One of the earliest forms of this theorem was for the case of the free streaming operator in the classical linear transport equation [44]. In this case,  $T_H f := -v \cdot \nabla_{\mathbf{x}} f(s, v, t)$  where  $\mathbf{x} = (s, v)$  (see Example 2.1.6). Voigt applies an iteration technique and the Banach Fixed-Point Theorem to prove that  $\mathcal{T}_H$  generates a substochastic  $C_0$ -semigroup for  $\|H\| < 1$  [44, Theorem 4.3]. He then gives sufficient conditions for the existence of an extension  $\mathcal{G}$  of  $\mathcal{T}_H$  that generates a substochastic  $C_0$ -semigroup in the case  $\|H\| = 1$  [44, Lemma 4.6]. However, in both cases, he does not give the series representation of the resolvent as in (6.10).

Another early form of the theorem which is worth noting was given by Beals and Protopopescu in [12, Theorems 9-13]. In their paper, they give sufficient conditions for the existence and uniqueness of solutions for a nonautonomous, inhomogeneous transport equation before presenting some generation theorems for the transport semigroup as the solutions to the autonomous, homogeneous case. Moreover, the results are presented in greater generality in that they do not assume the positivity or norm-boundedness of the operator  $H$ . Finally, it should be noted that Beals and Protopopescu consider functions in all  $L^p$ -spaces,  $1 \leq p < \infty$ , although they note that better results can be obtained in the  $L^1$  case.

Arlotti and Banasiak present this theorem for a variant of the Vlasov equation in [9, Chapter 10]. In particular, they consider the case when  $\mathcal{F}(\mathbf{x}) = (v, \mathbf{F}(r, v))$  for any  $\mathbf{x} = (r, v) \in \Omega$  and the streaming operator  $\mathcal{T}_H - v$ , where  $\mathcal{T}_H$  is the free streaming operator as described in Section 6.1 and  $v$  is a multiplication operator with some suitable function (cf.  $T_0 - h$  in Example 2.1.6).

The theorem and its proof which we present here will be a modification of that in [6]. The proof mirrors that of additive perturbations in that we prove the case for  $\|H\| < 1$  first (Corollary 6.2.5) before obtaining the semigroup  $(V(t))_{t \geq 0}$  generated by an extension of  $\mathcal{T}_H$  for the case  $\|H\| = 1$  by a limiting procedure. However, the proof for the case when  $\|H\| < 1$  requires a bit more work than in the analogous case for Kato's Theorem in abstract state spaces where we may apply a general perturbation result such as the Miyadera Perturbation Theorem. In this case, we will revert to the classical Lumer-Phillips Theorem in order to prove our result for  $\|H\| < 1$ . It should be noted that one of the differences between additive perturbations and boundary perturbations is that the domains of the operators  $\mathcal{T}_{rH}$  with different boundary operators  $rH$ ,  $0 < r < 1$ , cannot be compared as the order of the operators ( $rH \leq r'H$  for  $r \leq r'$ ,  $H \geq 0$ ) does not translate to any order on the domains of the related transport operators.

The first step in our proof is the following lemma which describes the relation between the image of  $I - M_\lambda H$  and the image of  $\lambda - \mathcal{T}_H$  (cf. Lemma 2.2.9 for the corresponding version for Kato's Theorem).

**Lemma 6.2.3.** [6, Lemma 4.2] *For all  $\lambda > 0$ ,  $(I - M_\lambda H)D(H) = L_+^1$  if and only if  $(\lambda - \mathcal{T}_H)D(\mathcal{T}_H) = X$ .*

*Proof.* Suppose  $(I - M_\lambda H)D(H) = L_+^1$  and let  $g \in X$ . We need to show that the boundary value problem

$$(\lambda - \mathcal{T}_M)f = g, \quad B^- f = HB^+ f \tag{6.11}$$

has a solution  $f \in D(\mathcal{T}_H)$ . Since  $G_\lambda g \in L_+^1$ , by assumption, there exists  $\psi \in D(H)$  such that  $\psi - M_\lambda H\psi = G_\lambda g$ . By Proposition 6.1.8, there exists  $f \in D(\mathcal{T}_M)$  such that  $B^+ f = \psi$ ,  $B^- f = H\psi = HB^+ f$  and  $(\lambda - \mathcal{T}_M)f = g$ . Therefore we have shown that  $f$  is the required solution to BVP (6.11). Since  $g \in X$  was arbitrary, it follows that  $(\lambda - \mathcal{T}_H)D(\mathcal{T}_H) = X$ .

Conversely, suppose  $(\lambda - \mathcal{T}_H)D(\mathcal{T}_H) = X$  and let  $\psi \in L_+^1$ . Since  $G_\lambda$  is surjective (Corollary 6.1.7), there exists  $g \in X$  such that  $\psi = G_\lambda g$ . By assumption, for such  $g$  there exists  $f \in D(\mathcal{T}_M)$  which is a solution to BVP (6.11). Moreover, by (6.9),  $f = C_\lambda g + \Xi_\lambda B^- f = C_\lambda g + \Xi_\lambda HB^+ f$ . Hence  $B^+ f = B^+ C_\lambda g + B^+ \Xi_\lambda HB^+ f = G_\lambda g + M_\lambda HB^+ f = \psi + M_\lambda HB^+ f$ . Therefore,  $\psi = (I - M_\lambda H)(B^+ f)$ . Since  $\psi \in L_+^1$  was arbitrary and  $B^+ f \in D(H)$ , we can conclude that  $(I - M_\lambda H)D(H) = L_+^1$ .  $\square$

Applying this lemma, we can prove a generation result for transport operators with contractive boundary operators under an extra assumption.

**Proposition 6.2.4.** *Let  $H \in \mathcal{L}(L_+^1, L_-^1)$  with  $\|H\| \leq 1$  and suppose that  $I - M_\lambda H$  is invertible on  $L_+^1$  with a continuous inverse. Then  $\mathcal{T}_H$  generates a  $C_0$ -semigroup of contractions in  $X$ . Moreover, its resolvent is given by*

$$R(\lambda, \mathcal{T}_H)f = C_\lambda f + \Xi_\lambda H(I - M_\lambda H)^{-1}G_\lambda f, \quad f \in X, \lambda > 0. \quad (6.12)$$

*Proof.* Let  $f \in D(\mathcal{T}_H)$  and  $g = (\lambda - \mathcal{T}_H)f$ . Then  $f$  is the solution of the boundary value problem (6.11). Hence from (6.4), we have

$$\begin{aligned} \|B^+ f\|_{L_+^1} + \lambda \|f\|_X &\leq \|HB^+ f\|_{L_-^1} + \|g\|_X \\ \text{i.e. } \lambda \|f\|_X - \|(\lambda - \mathcal{T}_H)f\|_X &\leq \|HB^+ f\|_{L_-^1} - \|B^+ f\|_{L_+^1} \leq 0. \end{aligned}$$

Therefore,  $\mathcal{T}_H$  is a dissipative operator.

Next, recall that  $M_\lambda u \in L_+^1$  if and only if  $u \in L_-^1$ . Hence, our assumption that  $I - M_\lambda H$  is invertible with continuous inverse and  $H \in \mathcal{L}(L_+^1, L_-^1)$  implies that  $(I - M_\lambda H)D(H) = L_+^1$ . Thus it follows from Lemma 6.2.3 that  $(\lambda - \mathcal{T}_H)D(\mathcal{T}_H) = X$ . Therefore by the Lumer-Phillips Theorem (Theorem 1.2.3),  $\mathcal{T}_H$  generates a  $C_0$ -semigroup of contractions on  $X$ .

To show that (6.12) holds, recall from (6.9) that the unique solution to BVP (6.11) is given by  $f = C_\lambda g + \Xi_\lambda B^- f = C_\lambda g + \Xi_\lambda HB^+ f$ . Then  $B^+ f = G_\lambda g + M_\lambda HB^+ f$ , i.e.  $B^+ f = (I - M_\lambda H)^{-1}G_\lambda g$ . Substituting this into  $f = C_\lambda g + \Xi_\lambda HB^+ f$ , we get equation (6.12).  $\square$

**Corollary 6.2.5.** *Let  $H \in \mathcal{L}(L_+^1, L_-^1)$  with  $\|H\| < 1$ . Then  $\mathcal{T}_H$  generates a  $C_0$ -semigroup of contractions in  $X$ . Moreover, its resolvent is given by*

$$R(\lambda, \mathcal{T}_H)f = C_\lambda f + \sum_{n=0}^{\infty} \Xi_\lambda H(M_\lambda H)^n G_\lambda f \quad f \in X, \lambda > 0. \quad (6.13)$$

*Proof.* Since  $\|H\| < 1$  and  $\|M_\lambda\|_{\mathcal{L}(L_-^1, L_+^1)} \leq 1$  (see (6.8)), we have  $\|M_\lambda H\| < 1$  for all  $\lambda > 0$ . From the Neumann series, it follows that  $I - M_\lambda H$  is invertible with inverse  $(I - M_\lambda H)^{-1} = \sum_{n=0}^{\infty} (M_\lambda H)^n$ . The result now follows from Proposition 6.2.4.  $\square$

Note that if additionally  $H$  is positive, it follows from the series representation (6.13) that  $R(\lambda, \mathcal{T}_H)$  is positive. Thus, the generated semigroup is substochastic in this case.

There is a generalisation of Proposition 6.2.4 to operators  $H : D(H) \subset Y_+ \rightarrow Y_-$  with  $\|H\psi\|_{L_-^1} \leq \|\psi\|_{L_+^1}$  for all  $\psi \in D(H) \cap L_+^1$  (Theorem 6.2.6). The norm bound on the restriction of  $H$  ensures the dissipativity of  $\mathcal{T}_H$  while the other conditions in

the theorem ensure that  $I - M_\lambda H$  is invertible with continuous inverse. Moreover, it turns out that these conditions are not only sufficient but also necessary. In this theorem, we equip  $D(H)$  with the norm

$$\|\psi\|_{D(H)} := \|\psi\|_{Y_+} + \|H\psi\|_{Y_-} + \|(I - M_\lambda H)\psi\|_{L_+^1}$$

under which it is a Banach space. Note that  $\|\cdot\|_{D(H)}$  is equivalent to  $\|\cdot\|_{L_+^1}$  under the conditions of Proposition 6.2.4. The authors in [6] prove Theorem 6.2.6 first before deducing Proposition 6.2.4 as a corollary.

**Theorem 6.2.6.** [6, Theorem 4.1] *Let  $H : D(H) \subset Y_+ \rightarrow Y_-$  satisfy*

- (i) *the graph of  $H$  is a closed subspace of  $\mathfrak{E}$ ,*
- (ii)  *$(I - M_\lambda H)D(H)$  is a dense subspace of  $L_+^1$ ,*
- (iii) *there is a constant  $C > 0$  such that*

$$\|(I - M_\lambda H)\psi\|_{L_+^1} \geq C(\|\psi\|_{Y_+} + \|H\psi\|_{Y_-}) \quad \text{for all } \psi \in D(H),$$

- (iv)  *$D(H) \cap L_+^1$  is dense in  $D(H)$  under the norm  $\|\cdot\|_{D(H)}$ ,*
- (v) *the restriction of  $H$  to  $L_+^1$  is a contraction i.e.*

$$\|H\psi\|_{L_-^1} \leq \|\psi\|_{L_+^1} \quad \text{for all } \psi \in D(H) \cap L_+^1.$$

*Then  $\mathcal{T}_H$  generates a  $C_0$ -semigroup of contractions in  $X$ . Conversely, if  $\mathcal{T}_H$  generates a  $C_0$ -semigroup of contractions and  $D(\mathcal{T}_H) \cap \mathfrak{W}$  is dense in  $D(\mathcal{T}_H)$  endowed with the graph norm, then  $H$  satisfies conditions (i)–(v) above.*

Now we are ready to prove Theorem 6.2.2.

*Proof of Theorem 6.2.2.* From Corollary 6.2.5, it follows that for  $H_r := rH$ ,  $r \in [0, 1)$ , the transport operator  $\mathcal{T}_{H_r}$  generates a positive contraction semigroup denoted by  $(U_{H_r}(t))_{t \geq 0}$ . It is easy to see from the series representation of the resolvents (6.13), that for  $0 \leq r \leq r' < 1$ , we have that  $R(\lambda, H_r) \leq R(\lambda, H_{r'})$  for  $\lambda > 0$ . By the Post-Widder Inversion Formula (Proposition 1.2.8), it follows that  $U_{H_r}(t) \leq U_{H_{r'}}(t)$  for all  $t \geq 0$ . Moreover,  $\|U_{H_r}(t)\| \leq 1$  for all  $r \in [0, 1)$ ,  $t \geq 0$ . Hence for  $f \in X_+$ ,  $(U_r(t)f)$  is a bounded, monotonically increasing sequence as  $r \nearrow 1$ , so it converges. From the fact that  $X$  has generating cone, we can conclude that the limit

$$V_H(t) := \text{s-lim}_{r \rightarrow 1} U_{H_r}(t)$$

exists for all  $t \geq 0$ . The proof that  $(V_H(t))_{t \geq 0}$  is a substochastic  $C_0$ -semigroup follows from the fact that  $(V_H(t))_{t \geq 0}$  is the strong limit of substochastic  $C_0$ -semigroups. The series representation (6.10) of the resolvent follows by considering the monotone limits of  $R(\lambda, \mathcal{T}_{H_r})$  (see the proof of [6, Lemma 6.1] for example). The proof that  $(V_H(t))_{t \geq 0}$  is a substochastic  $C_0$ -semigroup and the proof of the series representation of the resolvent are in fact very similar to the proofs for Kato's Theorem for abstract state spaces (see the proof of [3, Theorem 2.1]). The main difference between the proof of this theorem and Kato's Theorem is in showing that  $\mathcal{G}$  is an extension of  $\mathcal{T}_H$ .

Let  $f \in D(\mathcal{T}_H)$ . Then  $f$  is a solution to BVP (6.11). Since  $HB^+f \in L_-^1$ , by (6.9),  $f$  can be written in the form

$$f = C_\lambda g + \Xi_\lambda HB^+f. \quad (6.14)$$

Now define  $f_n = C_\lambda g + \sum_{k=0}^n \Xi_\lambda H(M_\lambda H)^k G_\lambda g$ . From the series representation (6.10) for the resolvent of  $\mathcal{G}$ , we know that  $f_n$  converges to  $R(\lambda, \mathcal{G})g$  in  $X$  as  $n \rightarrow \infty$ . To complete the proof, we will show that  $f_n$  converges to  $f$  in  $X$  as  $n \rightarrow \infty$ .

Note first that  $f - f_n = \Xi_\lambda H(B^+f - \sum_{k=0}^n (M_\lambda H)^k G_\lambda g)$ . From (6.14), it follows that  $B^+f = G_\lambda g + M_\lambda HB^+f$ , i.e.  $(I - M_\lambda H)B^+f = G_\lambda g$ . Substituting this into the series  $\sum_{k=0}^n (M_\lambda H)^k G_\lambda g$ , we have

$$\sum_{k=0}^n (M_\lambda H)^k G_\lambda g = \sum_{k=0}^n (M_\lambda H)^k (I - M_\lambda H)B^+f = B^+f - (M_\lambda H)^{n+1}B^+f.$$

Hence  $f - f_n = \Xi_\lambda H(M_\lambda H)^{n+1}B^+f$ . Furthermore, note that by (6.10), the series  $\sum_{k=0}^\infty \Xi_\lambda H(M_\lambda H)^k G_\lambda g$  converges in  $X$  for all  $g \in X$ . Since  $G_\lambda$  is surjective (Corollary 6.1.7), this means that  $\sum_{k=0}^\infty (M_\lambda H)^k u$  is convergent in  $X$  for any  $u \in L_+^1$ . Since  $B^+f \in L_+^1$ , it follows that  $\sum_{k=0}^\infty \Xi_\lambda H(M_\lambda H)^k B^+f$  converges in  $X$  and so  $\Xi_\lambda H(M_\lambda H)^{n+1}B^+f \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,  $f_n \rightarrow f$  and so  $D(\mathcal{T}_H) \subseteq D(\mathcal{G})$ .

To complete the proof, we show that  $\mathcal{G}f = \mathcal{T}_H f = \mathcal{T}_M f$  for all  $f \in D(\mathcal{T}_H)$ . To do so, it suffices to show that  $\mathcal{T}_M$  is an extension of  $\mathcal{G}$ . Let  $f = R(\lambda, \mathcal{G})g$  and  $f_n$  be as constructed above. Note that  $f_n \in D(\mathcal{T}_M)$  for all  $n \in \mathbb{N}_0$  and  $\mathcal{T}_M f_n = \lambda f_n - g$  since  $(\lambda - \mathcal{T}_M)f_n = (\lambda - \mathcal{T}_M)C_\lambda g + \sum_{k=0}^n (\lambda - \mathcal{T}_M)\Xi_\lambda H(M_\lambda H)^k G_\lambda g = g$  by (6.6) and (6.7). Hence,  $\mathcal{T}_M f_n \rightarrow \lambda f - g$  as  $n \rightarrow \infty$ . Substituting  $g = (\lambda - \mathcal{G})f$ , we have  $\mathcal{T}_M f_n \rightarrow \mathcal{G}f$  as  $n \rightarrow \infty$ . Since  $\mathcal{T}_M$  is closed in  $X$ , this shows that  $\mathcal{G}f = \mathcal{T}_M f$ .

The proof of the final statement about the minimality of the semigroup  $(V_H(t))_{t \geq 0}$  uses similar ideas as the case of additive perturbations, hence we omit it here. However, the proof is more technical and requires a more precise description of the generator  $\mathcal{G}$  (Proposition 6.2.8). For full details of the proof, see [8, Theorem 3.1].  $\square$

As mentioned above, we can give a precise description of the perturbed generator  $\mathcal{G}$  in Theorem 6.2.2. It turns out that  $\mathcal{G}$  is also a transport operator  $\mathcal{T}_{\mathcal{H}}$  with the boundary operator  $\mathcal{H}$  an extension of  $H$ . We will not give the full proof here but merely outline the steps required. The full details can be found in [6, Section 6]. The first step in proving this result is showing that  $\mathcal{G}$  acts like  $\mathcal{T}_M$  on  $D(\mathcal{G})$  as we did in the proof of Theorem 6.2.2. We can in fact, show more:

**Lemma 6.2.7.** [6, Lemma 6.1] *For any  $\lambda > 0$  and  $f \in X$ , the series (6.10) converges in  $D(\mathcal{T}_M)$  endowed with the graph norm. Moreover,  $\mathcal{T}_M$  is an extension of  $\mathcal{G}$ .*

Next, observe that for  $f_n := C_\lambda g + \sum_{k=0}^n \Xi_\lambda H(M_\lambda H)^k G_\lambda g$ ,  $g \in X$ , we have by (6.6) that

$$B^+ f_n = B^+ C_\lambda g + \sum_{k=0}^n B^+ \Xi_\lambda H(M_\lambda H)^k G_\lambda g = \sum_{k=0}^{n+1} (M_\lambda H)^k G_\lambda g$$

and

$$B^- f_n = B^- C_\lambda g + \sum_{k=0}^n B^- \Xi_\lambda H(M_\lambda H)^k G_\lambda g = \sum_{k=0}^n H(M_\lambda H)^k G_\lambda g.$$

Moreover, from the proof of Theorem 6.2.2 and the above lemma, we have that  $f_n \rightarrow f := R(\lambda, \mathcal{G})g$  and  $\mathcal{T}_M f_n \rightarrow \mathcal{T}_M f$  in  $X$ . Hence a natural deduction would be that  $\mathcal{G} = \mathcal{T}_{\mathcal{H}}$  with  $\mathcal{H} : D(\mathcal{H}) \subseteq Y_+ \rightarrow Y_-$  defined as:

$$D(\mathcal{H}) = \left\{ \psi \in Y_+ : \text{there exists } u \in L_+^1 \text{ such that } \psi = \psi_u = \sum_{k=0}^{\infty} (M_1 H)^k u \right\}$$

and

$$\mathcal{H}\psi = \sum_{k=0}^{\infty} H(M_1 H)^k u \quad \text{for all } \psi \in D(\mathcal{H}).$$

It turns out that this definition is well-defined [6, Lemma 6.1]. Moreover,  $\mathcal{H}$  satisfies the following properties.

**Proposition 6.2.8.** [6, Proposition 6.1] *Let  $H$  be as in Theorem 6.2.2 and  $\mathcal{H}$  as defined above. Then  $\mathcal{H}$  is an extension of  $H$  i.e.  $L_+^1 \subset D(\mathcal{H})$  and  $\mathcal{H}\psi = H\psi$  for all  $\psi \in L_+^1$ . Moreover,  $\text{Graph}(\mathcal{H}) \subset \mathfrak{E}$  and  $\mathcal{H}\psi \in L_-^1$  if and only if  $\psi \in L_+^1$ .*

Now we can provide a description of  $\mathcal{G}$ .

**Theorem 6.2.9.** [6, Theorem 6.3] *Let  $H$  and  $\mathcal{G}$  be as in Theorem 6.2.2. Then,  $\mathcal{G} = \mathcal{T}_{\mathcal{H}}$  where  $\mathcal{T}_{\mathcal{H}}$  is the transport operator with boundary operator  $\mathcal{H}$  defined as above.*

That  $D(\mathcal{G}) \subset D(\mathcal{T}_{\mathcal{H}})$  follows from the remarks after Lemma 6.2.7. The reverse inclusion can be proven in a similar way as the proof that  $\mathcal{G}$  is an extension of  $\mathcal{T}_H$  in Theorem 6.2.2. See [6, Theorem 6.3] for full details.

*Remark 6.2.10.* Choosing  $\lambda = 1$  in the definition of  $\mathcal{H}$  is not essential. In fact, it can be shown [6, Corollary 6.2] that  $\psi \in D(\mathcal{H})$  if and only if for any  $\lambda > 0$ , there exists  $u \in L_+^1$  such that  $\psi = \sum_{k=0}^{\infty} (M_\lambda H)^k u$  where the series converges in  $Y_+$ . In such a case, we have  $\mathcal{H}\psi = \sum_{k=0}^{\infty} H(M_\lambda H)^k u$ .

Finally, we note that for the free streaming operator in the classical transport equation, there is an alternative characterisation of  $\mathcal{G}$  in [7, Theorem 3.6] which utilises extensions of the operators  $H$ ,  $B^\pm$ ,  $M_\lambda$ , and  $\Xi_\lambda$  in more general (non-Banach) vector spaces of functions.

### 6.3 Honest Theory of Kato's Theorem and Transport Theory

In the previous section, we saw that the generation theorem for the transport semigroup with positive contractive boundary operator (Theorem 6.2.2) shares many similarities with Kato's Theorem. In particular, let us note that the generator of the transport semigroup is an extension of the operator  $\mathcal{T}_H$  while in Kato's Theorem, the generator of the perturbed semigroup,  $G$ , is an extension of the operator  $A + B$ . In the study of Kato's Theorem, the characterisation of when  $G = \overline{A + B}$  turns out to be important and is in fact, the basis for the study of honesty theory of Kato's Theorem. In this section, we will see that there is a corresponding theory for Theorem 6.2.2. In this case, the phenomenon of dishonesty is usually used to describe transport systems with conservative boundary conditions, i.e.  $\|Hu\|_{L_-^1} = \|u\|_{L_+^1}$  for all  $u \in L_+^1$ , which experience mass loss. In fact, it is this characterisation of mass loss that has been most often studied for the transport semigroup while the term ‘‘honesty’’ as well as the links with the honesty theory of Kato's Theorem has rarely been emphasized. Thus in this section, we will present the honesty theory of the transport semigroup in a form which mirrors that of Kato's Theorem to better showcase the similarities between the two theories. We will only consider the case where  $\|H\| = 1$  since for the case  $\|H\| < 1$ , we know that the generator  $\mathcal{G} = \mathcal{T}_H$  (Corollary 6.2.5) and the system is thus always honest.

Recall from Section 2.2.1 that we introduced the functional approach to honesty theory in abstract state spaces where we defined the functionals  $a_0 : D(G) \rightarrow \mathbb{R}$ ,

$a_0(x) = -\langle \Psi, Gx \rangle$  and  $\bar{a} : D(G) \rightarrow \mathbb{R}$ ,  $\bar{a}(R(\lambda, G)x) = \lim_{r \rightarrow 1} a(R(\lambda, G_r)x)$ ,  $x \in X$ . Moreover, we saw that the functional  $\langle \Delta_\lambda, x \rangle = a_0(R(\lambda, G)x) - \bar{a}(R(\lambda, G)x)$ ,  $x \in X$  played an important role in characterising honesty. We will begin our study of honesty for the transport semigroup by defining the analogous functionals for the transport semigroup. Firstly, the analogue of  $a_0$  is given by

$$c_0 : D(\mathcal{G}) \rightarrow \mathbb{R}, \quad c_0(f) = - \int_{\Omega} \mathcal{G}f \, d\mu.$$

As the focus in this section is on boundary perturbations, it is sometimes more useful to state the functional in terms of the trace operators. To do so, note that from the proof of Theorem 6.2.2 we have that the resolvents  $R(\lambda, \mathcal{G}) := \text{s-lim}_{r \rightarrow 1} R(\lambda, \mathcal{T}_{H_r})$  for all  $\lambda > 0$ . Combining this with Green's Formula, it follows that for  $f \in X_+$ ,  $\lambda > 0$ ,

$$c_0(R(\lambda, \mathcal{G})f) = \lim_{r \rightarrow 1} \left( \|B^+ R(\lambda, \mathcal{T}_{H_r})f\|_{L_+^1} - \|rH(B^+ R(\lambda, \mathcal{T}_{H_r})f)\|_{L_-^1} \right).$$

To motivate the definition of the second functional, we consider the formal expression

$$\|g|_{\Gamma_+}\|_{L_+^1} - \|H(g|_{\Gamma_+})\|_{L_-^1} \quad \text{for } g \in D(\mathcal{G})_+. \quad (6.15)$$

Note that (6.15) does not make sense in general because  $g|_{\Gamma_+}$  may not be in  $L_+^1$  even for  $g \in D(\overline{\mathcal{T}_H})$ . However, if  $g \in D(\mathcal{T}_H)$  and  $H$  is isometric, then the expression (6.15) is equal to 0. So an honest semigroup would potentially minimize the difference in (6.15). Thus our second functional  $\bar{c}$  is essentially (6.15) in a form which makes sense. Now fix  $\lambda > 0$ . Since all  $g \in D(\mathcal{G})$  can be written as  $R(\lambda, \mathcal{G})f$  for some  $f \in X$ , and (6.10) holds, it follows that a reasonable form of (6.15) might be

$$\begin{aligned} & \lim_{m \rightarrow \infty} \left\| B^+ \left( C_\lambda f + \sum_{n=0}^m \Xi_\lambda H(M_\lambda H)^n G_\lambda f \right) \right\|_{L_+^1} \\ & - \lim_{m \rightarrow \infty} \left\| HB^+ \left( C_\lambda f + \sum_{n=0}^m \Xi_\lambda H(M_\lambda H)^n G_\lambda f \right) \right\|_{L_-^1}. \end{aligned}$$

To ensure that the limit exists, we take  $f \in X_+$ . Then by the positivity of the operators  $\Xi_\lambda$ ,  $M_\lambda$ ,  $G_\lambda$ , and  $H$  we have that the partial sums are monotonically increasing and bounded from above, hence convergent. Thus we define the second functional as

$$\bar{c}(R(\lambda, \mathcal{G})f) = \lim_{m \rightarrow \infty} \left( \left\| \sum_{n=0}^m (M_\lambda H)^n G_\lambda f \right\|_{L_+^1} - \left\| H \sum_{n=0}^m (M_\lambda H)^n G_\lambda f \right\|_{L_-^1} \right) \quad \text{for } f \in X_+.$$

The functional extends to all  $f \in X$  by linearity since  $X$  has generating cone. As in the additive case, it can be shown [6, Theorem 6.4], [34, Theorem 8] that there exists a non-negative functional  $\beta_\lambda$  in  $X$  such that

$$\langle \beta_\lambda, G_\lambda f \rangle = c_0(R(\lambda, \mathcal{G})f) - \bar{c}(R(\lambda, \mathcal{G})f) \text{ for all } f \in X.$$

Moreover, it can be shown that  $\langle \beta_\lambda, h \rangle = \lim_{m \rightarrow \infty} \|(M_\lambda H)^{m+1} h\|_{L^1_+}$  for all  $h \in L^1_+$  [34, Proof of Theorem 8] and  $\beta_\lambda$  is the maximal positive element  $\psi \in L^\infty(\Gamma_+, d\mu_+)$ , such that  $\psi \leq 1$  a.e. and  $(M_\lambda H)^* \psi = \psi$  [34, Lemma 15].

Using these functionals, one can define honesty mirroring Definition 2.2.1:

**Definition 6.3.1.** *Let  $H$  and  $(V_H(t))_{t \geq 0}$  be as in Theorem 6.2.2. The trajectory  $(V_H(t)u)_{t \geq 0}$ ,  $u \in X$  is said to be honest if and only if*

$$\|V_H(t)u\| - \|u\| = -\bar{c} \left( \int_0^t V_H(s)u \, ds \right) \quad \text{for all } t \geq 0. \quad (6.16)$$

The  $C_0$ -semigroup  $(V_H(t))_{t \geq 0}$  is said to be honest if all trajectories are honest. Otherwise, the trajectory (resp. semigroup) is said to be dishonest.

Moreover, we can obtain virtually the same theorem as Theorem 2.2.4 for honesty of the transport semigroup but with  $G$  replaced by  $\mathcal{G}$ ,  $A + B$  replaced by  $\mathcal{T}_H$  and  $BR(\lambda, A)$  replaced by  $M_\lambda H$ .

**Theorem 6.3.2.** *Suppose  $H$  and  $\mathcal{T}_H$  satisfy Theorem 6.2.2 with  $\|H\| = 1$  and let  $\lambda > 0$ . The following are equivalent:*

- (i) *The semigroup  $(V_H(t))_{t \geq 0}$  is honest.*
- (ii)  *$\lim_{n \rightarrow \infty} \|(M_\lambda H)^n \psi\| = 0$  for all  $0 \leq \psi \in L^1_+$ .*
- (iii)  *$\beta_\lambda = 0$ .*
- (iv)  *$\mathcal{G} = \overline{\mathcal{T}_H}$ .*
- (v) *The set  $\{(M_\lambda H)^n \psi\}_{n \in \mathbb{N}}$  is relatively weakly compact for all  $0 \leq \psi \in L^1_+$ .*

For a proof, see [34, Sections 3 and 4], [6, Proposition 6.2, Theorem 6.5]. Finally, we note that there are also analogous characterisations of honesty for the transport semigroup using series representations [8, Section 4] and also a spectral characterisation [6, Theorem 6.6] (cf. Section 2.2).

We have just seen that the results in honesty theory for boundary perturbations in transport theory can be obtained from those of Kato's Theorem in abstract state

spaces by simply replacing the operator  $BR(\lambda, A)$  with the operator  $M_\lambda H$ . Although the results can be proven using analogous methods, more often than not, the proofs cannot be transferred verbatim from additive perturbations to transport theory because of the role of the boundary operator in transport theory. Hence, a natural question to ask is whether one can find a general theory which would cover both cases. In what follows, we will give one general result from which results of both cases can be derived. This result is a minor generalisation of [42, Proposition 2.1] to abstract state spaces and substochastic operators.

**Proposition 6.3.3.** *Let  $X$  be an abstract state space and  $T$  be a substochastic operator on  $X$ . The following are equivalent:*

- (i)  $T$  is mean ergodic and  $\ker(I - T) = \{0\}$ .
- (ii)  $T$  is strongly stable, i.e.  $\lim_{n \rightarrow \infty} \|T^n x\| = 0$  for all  $x \in X$ .
- (iii)  $\ker(I - T^*) = \{0\}$ .
- (iv)  $\ker(I - T^*) \cap X_+^* = \{0\}$ .
- (v)  $w^* - \lim_{n \rightarrow \infty} T^{*n} \Psi = 0$ .
- (vi)  $w^* - \lim_{n \rightarrow \infty} T^{*n} f = 0$  for all  $f \in X_+^*$ .
- (vii) If  $f \in X_+^*$  and  $T^* f \geq f$ , then  $f = 0$ .
- (viii)  $w - \lim_{n \rightarrow \infty} T^n x = 0$  for all  $x \in X$ .
- (ix) For each  $x \in X$ ,  $\{T^n x\}_{n=0}^\infty$  is relatively weakly compact with 0 as a weak cluster point.

*Proof.* We begin by noting that (i)  $\Leftrightarrow$  (iii) follows directly from the Mean Ergodic Theorem (Theorem 1.4.1 (ii)) since  $T$  is a power bounded operator. Next, we note that (ii) is equivalent to  $\lim_{n \rightarrow \infty} \|T^n x\| = 0$  for all  $x \in X_+$  since  $X_+$  has generating positive cone. Hence (ii) follows from (i) since for  $x \in X_+$ ,  $n \in \mathbb{N}$ ,

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x \right\| = \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x\| \geq \|T^{n-1} x\|$$

by the additivity of the norm of  $X$  and the contractivity of  $T$  on the positive cone. To show (ii)  $\Rightarrow$  (iii), suppose  $0 \neq f \in \ker(I - T^*)$ . Then there exists  $x \in X$  such that  $0 < |\langle f, x \rangle|$ . Since  $T^* f = f$ , we have

$$0 < |\langle f, x \rangle| = |\langle T^{*n} f, x \rangle| = |\langle f, T^n x \rangle| \leq \|f\| \|T^n x\| \rightarrow 0,$$

a contradiction. Therefore,  $f = 0$ .

Now (ii)  $\Rightarrow$  (viii)  $\Rightarrow$  (ix) is clear. We prove (ix)  $\Rightarrow$  (ii). Fix  $x \in X_+$ . From (ix), we have a subsequence  $(T^{n_k}x)_k$  which converges weakly to 0. In particular, we have  $\langle \Psi, T^{n_k}x \rangle = \|T^{n_k}x\| \rightarrow 0$ . Since  $(\|T^n x\|)_n$  is always convergent (as  $T$  is a contraction on  $X_+$ ), we must have  $\lim_{n \rightarrow \infty} \|T^n x\| = 0$ . Therefore  $\lim_{n \rightarrow \infty} \|T^n x\| = 0$  for all  $x \in X_+$  and so (ii) holds.

Next we show that (iv)–(vii) are equivalent. To show (iv)  $\Rightarrow$  (v), note first that since  $T$  is substochastic, we have for all  $x \in X_+$ ,  $\langle \Psi, T^{n+1}x \rangle \leq \langle \Psi, T^n x \rangle \leq \langle \Psi, x \rangle$ . Hence we have  $0 \leq \langle T^{*(n+1)}\Psi, x \rangle \leq \langle T^{*n}\Psi, x \rangle \leq \langle \Psi, x \rangle$ . So  $(\langle T^{*n}\Psi, x \rangle)_n$  is a positive, bounded, monotonically decreasing sequence in  $\mathbb{R}$ , hence it converges. By linearity, this holds for all  $x \in X$ , so it follows that  $T^{*n}\Psi$  converges in the weak\* topology to some element  $h \in X_+^*$ . Since  $T^*h = h$ , it follows from (iv) that  $h = 0$ . For (v)  $\Rightarrow$  (vi), let  $f \in X_+^*$  and  $x \in X_+$ . Then  $\langle f, x \rangle \leq \|f\| \|x\| = \|f\| \langle \Psi, x \rangle$ . Hence  $f \leq \|f\| \Psi$  and so  $T^{*n}f \leq \|f\| T^{*n}\Psi$ . Therefore (vi) holds. To show (vi)  $\Rightarrow$  (vii), suppose there exists  $0 \neq f \in X_+^*$  such that  $T^*f \geq f$ . Then it follows that  $T^{*n}f \geq T^{*(n-1)}f \geq f$ . Since  $T^*$  is power bounded, it follows that for all  $x \in X_+$ ,  $(\langle T^{*n}f, x \rangle)$  is a bounded, monotonically increasing sequence in  $\mathbb{R}$  hence, converges. Since  $T^{*n}f \geq f \neq 0$  for all  $n$ , it must follow that  $w^* \text{-}\lim_{n \rightarrow \infty} T^{*n}f \neq 0$ . Finally, (vii)  $\Rightarrow$  (iv) is clear.

We also note that (iii)  $\Rightarrow$  (iv) is obvious. To complete the proof, we show (v)  $\Rightarrow$  (ii). From (v), we have that  $\lim_{n \rightarrow \infty} \langle T^{*n}\Psi, x \rangle = 0$  for all  $x \in X_+$ , i.e.  $\lim_{n \rightarrow \infty} \langle \Psi, T^n x \rangle = \lim_{n \rightarrow \infty} \|T^n x\| = 0$  for all  $x \in X_+$  and this is sufficient to conclude that (ii) holds.  $\square$

*Remark 6.3.4.* If the set of quasi-interior points is non-empty, then the equivalent conditions in Proposition 6.3.3 are also equivalent to the following:

$$(ii') \lim_{n \rightarrow \infty} \|T^n x\| = 0 \text{ holds for some quasi-interior } x \in X_+.$$

Clearly (ii)  $\Rightarrow$  (ii'). The proof of (ii')  $\Rightarrow$  (iii) is similar to that of (ii)  $\Rightarrow$  (iii) since by definition,  $x \in X_+$  is a quasi-interior point if and only if  $\langle g, x \rangle > 0$  for all  $g \in X_+^* \setminus \{0\}$  [11, pp.238-239].

We now demonstrate how Theorem 6.3.3 can be used to derive the results on honesty for Kato's Theorem in abstract state spaces. First, recall from Lemma 2.3.1 that  $BR(\lambda, A)$  is a positive contraction on the positive cone and  $\ker(I - BR(\lambda, A)) = \{0\}$ . Hence the operator  $BR(\lambda, A)$  satisfies the conditions for  $T$  in Proposition 6.3.3. The following theorem now follows from Theorem 2.2.4 and Proposition 6.3.3.

**Theorem 6.3.5.** *Suppose  $A, B$  satisfy the conditions of Theorem 2.1.2 with perturbed semigroup  $(V(t))_{t \geq 0}$  and let  $\lambda > 0$ . The following are equivalent:*

- (i) The semigroup  $(V(t))_{t \geq 0}$  is honest.
- (ii) The operator  $BR(\lambda, A)$  is mean ergodic.
- (iii)  $\lim_{n \rightarrow \infty} \|(BR(\lambda, A))^n x\| = 0$  for all  $x \in X$ .
- (iv)  $\ker(I - (BR(\lambda, A))^*) = \{0\}$ .
- (v)  $\ker(I - (BR(\lambda, A))^*) \cap X_+^* = \{0\}$ .
- (vi)  $w^* \text{-}\lim_{n \rightarrow \infty} (BR(\lambda, A))^{*n} \Psi = 0$ .
- (vii)  $w^* \text{-}\lim_{n \rightarrow \infty} (BR(\lambda, A))^{*n} f = 0$  for all  $f \in X_+^*$ .
- (viii) If  $f \in X_+^*$  and  $(BR(\lambda, A))^* f \geq f$ , then  $f = 0$ .
- (ix)  $w \text{-}\lim_{n \rightarrow \infty} (BR(\lambda, A))^n x = 0$  for all  $x \in X$ .
- (x) For each  $x \in X$ ,  $\{(BR(\lambda, A))^n x\}_{n=0}^\infty$  is relatively weakly compact with 0 as a weak cluster point.

Note that the conditions in Theorem 6.3.5 can be found in Chapter 2. For example, conditions (iii) and (x) can be found in Theorem 2.2.4 while condition (ii) can be found in Proposition 2.3.2. Now we do the same for the case of transport theory. This time, we check that the operator  $M_\lambda H$  satisfies Theorem 6.3.3.

**Lemma 6.3.6.** *Suppose  $H$  satisfies the conditions of Theorem 6.2.2 and let  $\lambda > 0$ . Then  $M_\lambda H : L_+^1 \rightarrow L_+^1$  is a positive contraction on  $L_+^1$  with  $\ker(I - M_\lambda H) = \{0\}$ .*

*Proof.* The contractivity and positivity of  $M_\lambda H$  follows from the contractivity and positivity of both  $M_\lambda$  (see (6.8)) and  $H$ . We prove the second assertion by contradiction. So suppose there is a  $\psi \neq 0$ ,  $\psi \in L_+^1$  such that  $\psi = M_\lambda H \psi = B^+ \Xi_\lambda H \psi$ . By considering the definition of the trace operator, we see that this means that  $\Xi_\lambda H \psi \neq 0$ . Moreover, since  $H \psi \in L_-^1$ , we have  $(\lambda - \mathcal{T}_M) \Xi_\lambda H \psi = 0$  and  $B^- \Xi_\lambda H \psi = H \psi = H B^+ \Xi_\lambda H \psi$ . In other words,  $\Xi_\lambda H \psi \in D(\mathcal{T}_H)$  with  $(\lambda - \mathcal{T}_H) \Xi_\lambda H \psi = 0$ . This contradicts the fact that  $\lambda \in \rho(\mathcal{G})$ . Therefore  $\ker(I - M_\lambda H) = \{0\}$ .  $\square$

Now we can combine Theorem 6.3.2, Proposition 6.3.3 and Lemma 6.3.6 to derive the analogue of Theorem 6.3.5 for transport semigroups (cf. [6, Section 6]). Moreover, in  $L_+^1$ , we know that quasi-interior points exist as they are the strictly positive points in  $L_+^1$ , thus we have an extra condition in this case.

**Theorem 6.3.7.** *Suppose  $H$  and  $\mathcal{T}_H$  satisfy Theorem 6.2.4 with  $\|H\| = 1$  and suppose  $\lambda > 0$ . The following are equivalent:*

- (i) *The semigroup  $(V_H(t))_{t \geq 0}$  is honest.*
- (ii) *The operator  $M_\lambda H$  is mean ergodic.*
- (iii)  *$\lim_{n \rightarrow \infty} \|(M_\lambda H)^n \psi\| = 0$  for all  $\psi \in L_+^1$ .*
- (iv) *There is  $\psi \in L_+^1$ ,  $\psi > 0$  a.e. such that  $\lim_{n \rightarrow \infty} \|(M_\lambda H)^n \psi\| = 0$ .*
- (v)  *$\ker(I - (M_\lambda H)^*) = \{0\}$ .*
- (vi) *If for any  $0 \leq f \in L_+^\infty := L^\infty(\Gamma_+)$  we have  $(M_\lambda H)^* f = f$ , then  $f = 0$  a.e.*
- (vii)  *$\mu_+ \{\mathbf{y} \in \Gamma_+ : g_\lambda(\mathbf{y}) > 0\} = 0$  where  $g_\lambda(\mathbf{y}) = \lim_{n \rightarrow \infty} (M_\lambda H)^{*n} \mathbb{1}(\mathbf{y})$ .*
- (viii)  *$w^* \text{-}\lim_{n \rightarrow \infty} (M_\lambda H)^{*n} f = 0$  a.e. for all  $0 \leq f \in L_+^\infty$ .*
- (ix) *If  $0 \leq f \in L_+^\infty$  and  $(M_\lambda H)^* f \geq f$ , then  $f = 0$  a.e.*
- (x)  *$w \text{-}\lim_{n \rightarrow \infty} (M_\lambda H)^n \psi = 0$  a.e. for all  $\psi \in L_+^1$ .*
- (xi) *For each  $\psi \in L_+^1$ ,  $\{(M_\lambda H)^n \psi\}_{n=0}^\infty$  is relatively weakly compact with 0 as a weak cluster point.*

Note that some of the conditions such as conditions (ii) and (ix) in Theorem 6.3.7 are new in the study of honesty of the transport semigroup while others can be found in the existing literature, see for example Theorem 6.3.2. We will conclude this section by applying one of the results in Theorem 6.3.7 to a one-dimensional example introduced in [44]. In particular, we will apply condition (viii) and our proof will be similar to that of [7, Proposition 5.7] as we are applying the dual version of the condition they utilised.

**Example 6.3.8.** Let  $\Omega = \bigcup_{n=0}^\infty I_n$  where  $I_n = (a_n, b_n)$  with  $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}$  two non-decreasing sequences that satisfy  $-\infty < a_n < b_n < a_{n+1}$  for all  $n \in \mathbb{N}_0$  and  $a_n \xrightarrow{n \rightarrow \infty} \infty$ . Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$  and the field  $\mathcal{F} : \mathbb{R} \rightarrow \mathbb{R}$  be given as  $\mathcal{F}(\mathbf{x}) = 1$  for all  $\mathbf{x} \in \mathbb{R}$ .

Under this setup we have

$$\begin{aligned} \Phi(\mathbf{x}, t) &= \mathbf{x} + t \text{ for all } \mathbf{x}, t \in \mathbb{R} \\ \Gamma_- &= \{a_n\}_{n \in \mathbb{N}_0}, \quad \Gamma_+ = \{b_n\}_{n \in \mathbb{N}_0} \text{ with } \mu_\pm = \text{counting measure} \\ \tau_-(\mathbf{x}) &= \mathbf{x} - a_n \text{ for } a_n < \mathbf{x} < b_n. \end{aligned}$$

Now let  $H : L_+^1 \rightarrow L_-^1$  be defined by

$$(H\psi)(a_n) = \begin{cases} 0 & \text{if } n = 0 \\ \psi(b_{n-1}) & \text{if } n \geq 1. \end{cases}$$

Physically, this means that  $H$  sends the function from the exit point of one interval onward into the entry point of the next interval. By construction,  $H$  is a positive boundary operator of norm 1, so it satisfies the conditions of Theorem 6.2.2. In fact,  $H$  is isometric.

By definition, we have  $M_\lambda H\psi(\mathbf{y}) := (H\psi)(\Phi(\mathbf{y}, -\tau_-(\mathbf{y})))e^{-\lambda\tau_-(\mathbf{y})}$ . Therefore

$$M_\lambda H\psi(b_0) = 0; \quad M_\lambda H\psi(b_k) := (H\psi)(a_k)e^{-\lambda(b_k - a_k)} = \psi(b_{k-1})e^{-\lambda(b_k - a_k)}, \quad k \geq 1.$$

We can now describe the action of  $(M_\lambda H)^*$ . Note that if  $\psi \in L_+^1$  and  $\phi \in L_+^\infty$ , we have

$$\begin{aligned} \int_{\Gamma_+} (M_\lambda H\psi)(\mathbf{y})\phi(\mathbf{y}) d\mu_+(\mathbf{y}) &= \sum_{k=0}^{\infty} (M_\lambda H\psi)(b_k)\phi(b_k) \\ &= \sum_{k=1}^{\infty} \psi(b_{k-1})e^{-\lambda(b_k - a_k)}\phi(b_k) \\ &= \sum_{k=0}^{\infty} \psi(b_k)e^{-\lambda(b_{k+1} - a_{k+1})}\phi(b_{k+1}). \end{aligned}$$

Thus,  $(M_\lambda H)^*\phi(b_k) = \phi(b_{k+1})e^{-\lambda(b_{k+1} - a_{k+1})}$ ,  $k \geq 0$  and by iterating, for  $n \in \mathbb{N}$ , we have

$$(M_\lambda H)^{*n}\phi(b_k) = \phi(b_{k+n})e^{-\lambda\sum_{j=1}^n (b_{k+j} - a_{k+j})}, \quad k \geq 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} (M_\lambda H)^{*n}\mathbb{1}(b_k) = e^{-\lambda\sum_{j=1}^{\infty} (b_{k+j} - a_{k+j})}.$$

Since  $\Gamma_+$  is a discrete set of points with counting measure, condition (viii) of Theorem 6.3.7 indicates that the semigroup  $(V_H(t))_{t \geq 0}$  is stochastic if and only if

$$\sum_{j=1}^{\infty} (b_{k+j} - a_{k+j}) \text{ diverges for all } k \geq 0.$$

We compare this result with the necessary and sufficient condition obtained in [7, Proposition 5.7] and [8, Proposition 5.1], namely

$$\sum_{j=0}^{\infty} (b_j - a_j) \text{ diverges.} \tag{6.17}$$

It is easy to see that the conditions are equivalent. However, the proof in [7] applied condition (iii) in Theorem 6.3.7, while the proof in [8] applied techniques involving the series representation of the semigroup  $(V_H(t))_{t \geq 0}$ . Moreover, from condition (ii) in Theorem 6.3.7, it follows that (6.17) is equivalent to  $M_\lambda H$  being mean ergodic, while from condition (vi) in Theorem 6.3.7, (6.17) holds if and only if the equation

$$\phi(b_k) = \phi(b_{k+1})e^{-\lambda(b_{k+1}-a_{k+1})}, \quad k \geq 0$$

has no bounded, positive, non-trivial solution  $\phi$ .

## 6.4 Strong Stability

In this section, we study the strong stability of the perturbed semigroup in Kato's Theorem in abstract state spaces as well as in transport theory. Strong stability can be thought of as the reverse of stochasticity in some sense since stochastic semigroups are never strongly stable. The results in this section are motivated by [42, Proposition 2.4, Theorem 3.5]. The results in [42] were derived for the semigroups which satisfy Kato's Theorem in  $L^1$ -spaces (Theorem 2.1.1) with equality in condition (iii). We will extend these results to all semigroups satisfying Kato's Theorem in abstract state spaces (Theorem 2.1.2). Furthermore, we will also show that similar results can be obtained for the semigroups in transport theory satisfying Theorem 6.2.2. As in the case of honesty, we will see that there is also a general result which enables us to derive results for the strong stability of the perturbed semigroup in both Kato's Theorem and transport theory.

We say that a semigroup is strongly stable if  $\lim_{t \rightarrow \infty} \|S(t)x\| = 0$  for all  $x \in X$ . The following theorem is the analogue of [42, Proposition 2.4] for strong stability of semigroups on abstract state spaces.

**Proposition 6.4.1.** *Let  $(S(t))_{t \geq 0}$  be a substochastic semigroup on an abstract state space  $X$  with generator  $A$ . The following are equivalent.*

- (i)  $(S(t))_{t \geq 0}$  is strongly stable.
- (ii) For every  $x \in X_+$ ,  $\lim_{\lambda \downarrow 0} \lambda R(\lambda, A)x = 0$ .
- (iii)  $\text{Im}(A)$  is dense in  $X$ .

*Proof.* The equivalence of (ii) and (iii) follows from Theorem 1.4.5 so it suffices to show (i)  $\Leftrightarrow$  (ii).

(i)  $\Rightarrow$  (ii). Suppose  $(S(t))_{t \geq 0}$  is strongly stable. Then  $\text{s-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(r) dr = 0$ . Hence (ii) follows from Theorem 1.4.5.

(ii)  $\Rightarrow$  (i). Now let  $x \in X_+$ . Then

$$\begin{aligned} 0 &= \lim_{\lambda \rightarrow 0} \langle \Psi, \lambda R(\lambda, A)x \rangle \\ &= \lim_{\lambda \rightarrow 0} \lambda \int_0^\infty e^{-\lambda t} \langle \Psi, S(t)x \rangle dt \\ &= \lim_{\lambda \rightarrow 0} \lambda \int_0^\infty e^{-\lambda t} \|S(t)x\| dt. \end{aligned} \quad (6.18)$$

Since  $(S(t))_{t \geq 0}$  is substochastic, this implies that  $\lim_{t \rightarrow \infty} \|S(t)x\|$  exists. By [2, Lemma 4.2.12], it follows that  $\lim_{t \rightarrow \infty} \|S(t)x\|$  converges to the same limit as (6.18). Hence  $\lim_{t \rightarrow \infty} \|S(t)x\| = 0$  for all  $x \in X_+$  and the result follows.  $\square$

We now use Proposition 6.4.1 to derive some conditions for strong stability of the perturbed semigroup in Kato's Theorem and in transport theory. We begin with Kato's Theorem on abstract state spaces and once again utilise the functionals  $a_0, \bar{a}, \Delta_\lambda$  introduced in [3] in the study of honesty (see Section 2.2.2). In the rest of this section,  $X$  will denote an abstract state space,  $A, B$  the operators satisfying the conditions in Theorem 2.1.2 and  $(V(t))_{t \geq 0}$  the perturbed semigroup with generator  $G$ .

First consider the functional  $a_0$ . Let  $x \in X_+$  and note that from the resolvent equation, we have  $R(\mu, G) \leq R(\lambda, G)$  for  $0 < \lambda \leq \mu$ . Since  $a_0$  is a positive operator on  $D(G)$ , we have  $a_0(R(\mu, G)x) \leq a_0(R(\lambda, G)x)$  for  $0 < \lambda \leq \mu$ . Moreover,  $a_0(R(\lambda, G)x) \leq \|GR(\lambda, G)x\| \leq 2\|x\|$  for all  $\lambda > 0$ . Hence  $\lim_{\lambda \downarrow 0} a_0(R(\lambda, G)x)$  exists. By linearity, the limit exists for all  $x \in X$ . A similar argument shows that the limit  $\lim_{\lambda \downarrow 0} \bar{a}(R(\lambda, G)x)$  exists as well and so

$$\lim_{\lambda \downarrow 0} \langle \Delta_\lambda, x \rangle = \lim_{\lambda \downarrow 0} a_0(R(\lambda, G)x) - \lim_{\lambda \downarrow 0} \bar{a}(R(\lambda, G)x)$$

exists for all  $x \in X$ .

**Proposition 6.4.2.** *The semigroup  $(V(t))_{t \geq 0}$  is strongly stable if and only if*

$$\lim_{\lambda \downarrow 0} \langle \Delta_\lambda, x \rangle = \langle \Psi, x \rangle - \lim_{\lambda \downarrow 0} \bar{a}(R(\lambda, G)x)$$

for all  $x \in X_+$ .

*Proof.* Let  $x \in X_+$  and note that  $\lambda \|R(\lambda, G)x\| = \|x + GR(\lambda, G)x\|$ . Proposition 6.4.1 implies that  $(V(t))_{t \geq 0}$  is strongly stable if and only if  $\lim_{\lambda \downarrow 0} \langle \Psi, x + GR(\lambda, G)x \rangle = 0$

for all  $x \in X_+$ . By definition of the functional  $a_0$ , this condition is equivalent to  $\lim_{\lambda \downarrow 0} a_0(R(\lambda, G)x) = \langle \Psi, x \rangle$  for all  $x \in X_+$ . Applying this to the definition of  $\Delta_\lambda$ , this is equivalent to  $\lim_{\lambda \downarrow 0} \langle \Delta_\lambda, x \rangle = \langle \Psi, x \rangle - \lim_{\lambda \downarrow 0} \bar{a}(R(\lambda, G)x)$  for all  $x \in X_+$ .  $\square$

*Remark 6.4.3.* Observe that for  $x \in X_+$ ,

$$a_0(R(\lambda, G)x) = \langle \Psi, -GR(\lambda, G)x \rangle = \langle \Psi, x \rangle - \lambda \langle \Psi, R(\lambda, G)x \rangle \leq \langle \Psi, x \rangle.$$

Thus  $(V(t))_{t \geq 0}$  is strongly stable if and only if we maximize the loss functional while  $(V(t))_{t \geq 0}$  is honest if and only if we minimize the loss.

Note that in the case where the semigroup in Kato's Theorem satisfies condition (iii) in Theorem 2.1.2 with equality (which was the  $L^1$  set-up studied in [42]), the functional  $\bar{a} = 0$ . Thus the semigroup is strongly stable if and only if  $\lim_{\lambda \downarrow 0} \Delta_\lambda = \Psi$ . Since  $\Delta_\lambda = \text{w}^* \text{-}\lim_{n \rightarrow \infty} (BR(\lambda, A))^{*n} \Psi$  (see Proposition 2.2.3), we have the following corollary:

**Corollary 6.4.4.** *If  $A, B$  satisfies the conditions of Theorem 2.1.2 with equality in condition (iii), the semigroup  $(V(t))_{t \geq 0}$  is strongly stable if and only if*

$$\text{w}^* \text{-}\lim_{\lambda \downarrow 0} \lim_{n \rightarrow \infty} (BR(\lambda, A))^{*n} \Psi = \Psi. \quad (6.19)$$

As an application, we consider once more the set of quantum dynamical semigroups identified in Theorem 5.1.7 with  $\Upsilon$  satisfying Premise 5.1.6. As in Section 5.1, we let  $(T(t))_{t \geq 0}$  denote the minimal quantum dynamical semigroup on  $\mathcal{L}(\mathfrak{H})$  satisfying Theorem 5.1.7 while  $(S(t))_{t \geq 0}$  will denote its predual semigroup. Since strong stability of the predual semigroup is defined as  $\lim_{t \rightarrow \infty} \|S(t)\rho\| = 0$  for all  $\rho \in \mathfrak{T}(\mathfrak{H})_+$ , we say that the quantum dynamical semigroup  $(T(t))_{t \geq 0}$  is strongly stable if the operator  $T(t)\mathbb{1}$  converges  $\sigma$ -weakly to 0 as  $t \rightarrow \infty$  i.e.  $\text{w}^* \text{-}\lim_{t \rightarrow \infty} T(t)\mathbb{1} = 0$ .

Returning to Example 5.2.15, we have the following:

**Proposition 6.4.5.** *The quantum dynamical semigroup  $(T(t))_{t \geq 0}$  in Example 5.2.15 is strongly stable if and only if  $\sum_{n \geq 0} \frac{1}{|\alpha(n)|^2}$  converges.*

*Proof.* From Theorem 6.4.4, we have that  $(T(t))_{t \geq 0}$  is strongly stable if and only if

$$\text{w}^* \text{-}\lim_{\lambda \downarrow 0} \lim_{k \rightarrow \infty} Q_\lambda^k(\mathbb{1}) = \mathbb{1}.$$

Combining this with (5.16), it follows that  $(T(t))_{t \geq 0}$  is strongly stable if and only if

$$\lim_{\lambda \downarrow 0} \lim_{k \rightarrow \infty} \prod_{j=0}^{k-1} \frac{|\alpha(n+j)|^2}{\lambda + |\alpha(n+j)|^2} = 1 \quad \text{for all } n \geq 0.$$

From the proof of Proposition 5.2.16, we have  $p_n(\lambda) := \lim_{k \rightarrow \infty} \prod_{j=0}^{k-1} \frac{|\alpha(n+j)|^2}{\lambda + |\alpha(n+j)|^2} \neq 0$  for all  $n \geq 0$  if and only if  $\sum_{n \geq 0} \frac{1}{|\alpha(n)|^2}$  converges. We will show that  $\lim_{\lambda \downarrow 0} p_n(\lambda) = 1$  for all  $n \geq 0$ .

Fix  $n \geq 0$ . Note first that  $\lim_{k \rightarrow \infty} \lim_{\lambda \downarrow 0} \prod_{j=0}^{k-1} \frac{|\alpha(n+j)|^2}{\lambda + |\alpha(n+j)|^2} = 1$  so  $\lim_{\lambda \downarrow 0} p_n(\lambda) = 1$  if and only if we can interchange the limit. This however, follows from some elementary calculations and an application of the Monotone Convergence Theorem.  $\square$

Now we consider the case of transport theory. In order to derive the corresponding results on strong stability for the transport semigroup  $(V_H(t))_{t \geq 0}$  satisfying Theorem 6.2.2 from Proposition 6.4.1, we will apply the functionals  $c_0$ ,  $\bar{c}$  and  $\beta_\lambda$  as defined in Section 6.3 instead. In the rest of this chapter  $(V_H(t))_{t \geq 0}$  will denote the transport semigroup satisfying Theorem 6.2.2 and all the the notation related to it will be as defined in Sections 6.1, 6.2 and 6.3.

We begin with the analogue of Proposition 6.4.2 for the transport semigroup which can be deduced from a similar argument as in Proposition 6.4.2.

**Proposition 6.4.6.** *The semigroup  $(V_H(t))_{t \geq 0}$  is strongly stable if and only if*

$$\lim_{\lambda \downarrow 0} \langle \beta_\lambda, G_\lambda f \rangle = \|f\| - \lim_{\lambda \downarrow 0} \bar{c}(R(\lambda, G)f)$$

for all  $f \in (L^1)_+$ .

For the special case when the boundary operator  $H$  is isometric, that is, when  $\|H\psi\|_{L^1_-} = \|\psi\|_{L^1_+}$  for all  $\psi \in L^1_+$ , we obtain the following analogue of Corollary 6.4.4 by applying the fact that  $\langle \beta_\lambda, u \rangle = \lim_{n \rightarrow \infty} \|(M_\lambda H)^{n+1} u\|_{L^1_+}$  for all  $u \in L^1_+$ .

**Corollary 6.4.7.** *Suppose  $\|H\psi\|_{L^1_-} = \|\psi\|_{L^1_+}$  for all  $\psi \in L^1_+$ . Then the semigroup  $(V_H(t))_{t \geq 0}$  is strongly stable if and only if*

$$\mu_+ \{ \mathbf{y} \in \Gamma_+ : \liminf_{\lambda \downarrow 0} g_\lambda(\mathbf{y}) < 1 \} = 0$$

where  $g_\lambda(\mathbf{y}) = \lim_{n \rightarrow \infty} (M_\lambda H)^{*(n+1)} \mathbb{1}(\mathbf{y})$ .

# Bibliography

- [1] Charalambos D. Aliprantis and Owen Burkinshaw. *Positive operators*. Springer, Dordrecht, 2006. Reprint of the 1985 original.
- [2] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.
- [3] L. Arlotti, B. Lods, and M. Mokhtar-Kharroubi. On perturbed substochastic semigroups in abstract state spaces. *Z. Anal. Anwend.*, 30(4):457–495, 2011.
- [4] Luisa Arlotti. A perturbation theorem for positive contraction semigroups on  $L^1$ -spaces with applications to transport equations and Kolmogorov's differential equations. *Acta Appl. Math.*, 23(2):129–144, 1991.
- [5] Luisa Arlotti, Jacek Banasiak, and Bertrand Lods. A new approach to transport equations associated to a regular field: trace results and well-posedness. *Mediterr. J. Math.*, 6(4):367–402, 2009.
- [6] Luisa Arlotti, Jacek Banasiak, and Bertrand Lods. On general transport equations with abstract boundary conditions. The case of divergence free force field. *Mediterr. J. Math.*, 8(1):1–35, 2011.
- [7] Luisa Arlotti and Bertrand Lods. Substochastic semigroups for transport equations with conservative boundary conditions. *J. Evol. Equ.*, 5(4):485–508, 2005.
- [8] Luisa Arlotti and Bertrand Lods. Transport semigroup associated to positive boundary conditions of unit norm: a Dyson-Phillips approach. *Discrete Contin. Dyn. Syst. Ser. B*, 19(9):2739–2766, 2014.
- [9] Jacek Banasiak and Luisa Arlotti. *Perturbations of positive semigroups with applications*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2006.

- [10] Jacek Banasiak and Mirosław Lachowicz. Around the Kato generation theorem for semigroups. *Studia Math.*, 179(3):217–238, 2007.
- [11] Charles J. K. Batty and Derek W. Robinson. Positive one-parameter semigroups on ordered Banach spaces. *Acta Appl. Math.*, 2(3-4):221–296, 1984.
- [12] R. Beals and V. Protopopescu. Abstract time-dependent transport equations. *J. Math. Anal. Appl.*, 121(2):370–405, 1987.
- [13] Nicolas Bouleau and Francis Hirsch. *Dirichlet forms and analysis on Wiener space*, volume 14 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1991.
- [14] Michel Cessenat. Théorèmes de trace  $L^p$  pour des espaces de fonctions de la neutronique. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(16):831–834, 1984.
- [15] A. M. Chebotarëv. The theory of dynamical semigroups and its applications. In *Probability theory and mathematical statistics, Vol. I (Vilnius, 1989)*, pages 217–227. “Mokslas”, Vilnius, 1990.
- [16] A. M. Chebotarev and F. Fagnola. Sufficient conditions for conservativity of minimal quantum dynamical semigroups. *J. Funct. Anal.*, 153(2):382–404, 1998.
- [17] Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter. *One-parameter semigroups*, volume 5 of *CWI Monographs*. North-Holland Publishing Co., Amsterdam, 1987.
- [18] E. B. Davies. *Quantum theory of open systems*. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1976.
- [19] E. B. Davies. Quantum dynamical semigroups and the neutron diffusion equation. *Rep. Mathematical Phys.*, 11(2):169–188, 1977.
- [20] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.
- [21] Józef Dodziuk and Varghese Mathai. Kato’s inequality and asymptotic spectral properties for discrete magnetic Laplacians. In *The ubiquitous heat kernel*, volume 398 of *Contemp. Math.*, pages 69–81. Amer. Math. Soc., Providence, RI, 2006.

- [22] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [23] F. Fagnola and L. Pantaleón Martínez. Are sufficient conditions for conservativity of minimal quantum semigroups necessary? *Math. Notes*, 91(5-6):851–856, 2012. Translation of *Mat. Zametki* **91** (2012), no. 6, 908–919.
- [24] Franco Fagnola. Quantum Markov semigroups and quantum flows. *Proyecciones*, 18(3):144, 1999.
- [25] William Feller. On boundaries and lateral conditions for the Kolmogorov differential equations. *Ann. of Math. (2)*, 65:527–570, 1957.
- [26] Giovanni Frosali, Cornelis V. M. van der Mee, and Francesco Mugelli. A characterization theorem for the evolution semigroup generated by the sum of two unbounded operators. *Math. Methods Appl. Sci.*, 27(6):669–685, 2004.
- [27] Julio C. García and Roberto Quezada. Hille-Yosida estimate and nonconservativity criteria for quantum dynamical semigroups. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 7(3):383–394, 2004.
- [28] Nassif Ghoussoub. Subadditive ergodic theory in general function spaces. In *Seminar on functional analysis, 1983/1984*, volume 20 of *Publ. Math. Univ. Paris VII*, pages 157–177. Univ. Paris VII, Paris, 1984.
- [29] Alexander Grigor'yan. Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 36(2):135–249, 1999.
- [30] Alexander Grigor'yan and Jiaxin Hu. Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces. *Invent. Math.*, 174(1):81–126, 2008.
- [31] Tosio Kato. On the semi-groups generated by Kolmogoroff's differential equations. *J. Math. Soc. Japan*, 6:1–15, 1954.
- [32] Matthias Keller and Daniel Lenz. Dirichlet forms and stochastic completeness of graphs and subgraphs. *J. Reine Angew. Math.*, 666:189–223, 2012.

- [33] Ulrich Krengel. *Ergodic theorems*, volume 6 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.
- [34] Mustapha Mokhtar-Kharroubi. On collisionless transport semigroups with boundary operators of norm one. *J. Evol. Equ.*, 8(2):327–352, 2008.
- [35] Mustapha Mokhtar-Kharroubi and Jürgen Voigt. On honesty of perturbed sub-stochastic  $C_0$ -semigroups in  $L^1$ -spaces. *J. Operator Theory*, 64(1):131–147, 2010.
- [36] El Maati Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.
- [37] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [38] G. E. H. Reuter. Denumerable Markov processes and the associated contraction semigroups on  $l$ . *Acta Math.*, 97:1–46, 1957.
- [39] H. R. Thieme and J. Voigt. Stochastic semigroups: their construction by perturbation and approximation. In *Positivity IV—theory and applications*, pages 135–146. Tech. Univ. Dresden, Dresden, 2006.
- [40] H. F. Trotter. Approximation of semi-groups of operators. *Pacific J. Math.*, 8:887–919, 1958.
- [41] Marta Tyran-Kamińska. Ergodic theorems and perturbations of contraction semigroups. *Studia Math.*, 195(2):147–155, 2009.
- [42] Marta Tyran-Kamińska. Substochastic semigroups and densities of piecewise deterministic Markov processes. *J. Math. Anal. Appl.*, 357(2):385–402, 2009.
- [43] Jan van Neerven. *The adjoint of a semigroup of linear operators*, volume 1529 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.
- [44] Jürgen Voigt. Functional analytic treatment of the initial boundary value problem for collisionless gases. Habilitationsschrift, 1980.
- [45] Jürgen Voigt. Absorption semigroups, their generators, and Schrödinger semigroups. *J. Funct. Anal.*, 67(2):167–205, 1986.

- [46] Jürgen Voigt. On substochastic  $C_0$ -semigroups and their generators. In *Proceedings of the conference on mathematical methods applied to kinetic equations (Paris, 1985)*, volume 16, pages 453–466, 1987.
- [47] Jürgen Voigt. On resolvent positive operators and positive  $C_0$ -semigroups on  $AL$ -spaces. *Semigroup Forum*, 38(2):263–266, 1989. Semigroups and differential operators (Oberwolfach, 1988).
- [48] Radoslaw Krzysztof Wojciechowski. *Stochastic completeness of graphs*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)—City University of New York.