Canonical Extensions of Bounded Lattices and Natural Duality for Default Bilattices

Andrew Craig
St John's College
University of Oxford

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Abstract

This thesis presents results concerning canonical extensions of bounded lattices and natural dualities for quasivarieties of default bilattices. Part I is dedicated to canonical extensions, while Part II focuses on natural duality for default bilattices.

A canonical extension of a lattice-based algebra consists of a completion of the underlying lattice and extensions of the additional operations to the completion. Canonical extensions find rich application in providing an algebraic method for obtaining relational semantics for non-classical logics.

Part I gives a new construction of the canonical extension of a bounded lattice. The construction is done via successive applications of functors and thus provides an elegant exposition of the fact that the canonical extension is functorial.

Many existing constructions are described via representation and duality theorems. We demonstrate precisely how our new formulation relates to existing constructions as well as proving new results about complete lattices constructed from graphs. Part I ends with an analysis of the untopologised structures used in two methods of construction of canonical extensions of bounded lattices: the untopologised graphs used in our new construction, and the so-called 'intermediate structure'. We provide sufficient conditions for the intermediate structure to be a lattice and, for the case of finite lattices, identify when the dual graph is not a minimal representation of the lattice.

Part II applies techniques from natural duality theory to obtain dualities for quasivarieties of bilattices used in default logic. Bilattices are doubly-ordered algebraic structures which find application in reasoning about inconsistent and incomplete information. This account is the first attempt to provide dualities or representations when there is little interaction required between the two orders. Our investigations begin by using computer programs to calculate dualities for specific examples, before using purely theoretical techniques to obtain dualities for more general cases. The results obtained are extremely revealing, demonstrating how one of the lattice orders from the original algebra is encoded in the dual structure.

We conclude Part II by describing a new class of default bilattices. These provide an alternative way of interpreting contradictory information. We obtain dualities for two newly-described quasivarieties and provide insights into how these dual structures relate to previously described classes of dual structures for bilattices.

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Statement of Originality

I hereby declare all of the work presented in this thesis to be my own, except where otherwise referenced or acknowledged. The material in this thesis has not been submitted for an academic qualification at any other institution.

The work presented in Chapter 2 (Part I) is joint work with Miroslav Haviar and Hilary Priestley. A journal version of these results has been published as [19].

With the exception of Section 3.4, the results presented in Chapter 3 (Part I) are joint work with Miroslav Haviar. This work has been submitted for publication [18]. The work on the translation of graphs of the form $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underset{\sim}{\mathbf{2}}), E)$ to RS-frames in Chapter 4 (Part I) is also joint work with Miroslav Haviar.

Contents

Chapter 1

Introduction

This thesis consists of two parts, linked by the common theme of duality theory. Part I focuses on canonical extensions of bounded lattices, exploiting the connection between topological representation theorems and canonical extensions. Part II applies techniques from the theory of natural dualities to find and to understand dualities for particular quasivarieties of bilattices.

Duality theory provides methods for taking objects from one category, and translating them into objects in a dual category. The power of these methods lies in the fact that certain problems can be more easily visualised and solved in the dual category than in the category in which they are originally stated. A simple example is Stone's duality between finite Boolean algebras and finite sets [84]. This duality demonstrates that the entire structure of a finite Boolean algebra can be captured simply by considering the finite set of its atoms. One can then, for example, represent the product of two finite Boolean algebras by considering the disjoint union of their sets of atoms.

We are interested in dualities between classes of lattice-based algebras and classes of topological spaces (with additional structure). We will not, in general, be restricting ourselves to the case of finite algebras. The topology on the dual side plays an indispensable role in the representation of arbitrary infinite algebras. Indeed, it is subsets of the dual space satisfying topological (and possibly additional relational) conditions which will form the concrete representation of the lattice-based algebra. The dualities and representations that we make use of will represent the underlying lattice concretely either as a collection of sets, or as a collection of maps. In some cases a representation which does not encompass the morphisms between objects can be sufficient to help solve particular problems.

In the case of a lattice (or indeed a poset) with no additional operations, the canonical extension is a completion satisfying the characteristic properties of *density* and *compactness*. The first construction of the canonical extension was done by Jónsson and Tarski [71] for Boolean algebras with operators, and topological duality theory played a pivotal role. Their construction used Stone duality to represent the completion of the underlying Boolean algebra as an algebra of sets. A survey of the link between canonical extensions and duality theory is given in Chapter 1 of Part I.

Our work in Part I exploits the connection between duality theory and canonical extensions. We provide a new construction of the canonical extension, where the complete lattice is represented as a set of partial maps. We are able to use our approach via partial maps to demonstrate the functoriality of the canonical extension. Furthermore, we show how our new construction is linked to previous concrete constructions of the canonical extension for bounded lattices. Our results reinforce the idea that canonical extensions correspond to untopologised representations.

In Part II, we turn our attention to algebras equipped with two sets of lattice operations and a unary negation-type operation. First described by Ginsberg [61] in the 1980's, bilattices have found extensive application in artificial intelligence and logic programming. When lattices are used as algebraic models of logics, the meet and join operations are used to represent the conjunction and disjunction of logical statements. Bilattices are equipped with a second set of lattice operations and hence a second partial order. This second order represents "knowledge", and the meet and join operations model "consensus" and "gullibility". Representation and duality theorems for certain subclasses of bilattices are already known. We use tools from natural duality theory to investigate dual structures for classes of bilattices that cannot be accessed by existing representation methods. Our results demonstrate that the knowledge order is a critical feature of the dual structures.

This thesis does not attempt to answer questions directly related to non-classical logics or the logic of bilattices. Rather, it investigates, from a more abstract perspective, the algebras related to these logics using the tools of canonical extension and duality theory. In particular, in our work on canonical extensions we do not consider lattices with additional operations. We also do not attempt to explore deductive systems based on logical matrices involving bilattices.

The next section of this introductory chapter contains the necessary background on natural duality theory. These techniques play a role in both Part I and Part II. Detailed background and an introduction will be provided at the beginning of each part. Certain theoretical background and some useful definitions are contained in Appendix A, while relevant computational output used in Part II is presented in Appendix B. General background on order theory and lattices can be found in the book by Davey and Priestley [28]. As a standard reference on universal algebra, we recommend the monograph by Burris and Sankappanavar [14]. A complete treatment of Stone and Priestley duality is given in [28, Chapter 11]. Our only remark is that we will consider the Priestley dual space to be the set of prime filters, ordered by inclusion. We choose this setting as it parallels the way that the dual spaces are structured for natural duality.

1.1 Natural duality theory

Here we present the background that is required for a good understanding of the aims and methods of Chapter 2 of Part I. In addition, the concepts presented in this section form the foundation for the techniques used in Part II. The book by Clark and Davey [17] serves as a standard reference for natural duality. The theory of optimal dualities will be deferred until the introduction to Part II.

The theory of (basic) natural dualities concerns quasivarieties of algebras which are generated by a finite algebra $\mathbf{M} = \langle M; F \rangle$, where M is the underlying finite set and F is a finite set of finitary operations on M. That is, it is concerned with quasivarieties $\boldsymbol{\mathcal{A}}$, where $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M})$ for a finite algebra $\mathbf{M}$. Under suitable conditions, one can find a topological relational structure $\underset{\sim}{\mathbf{M}}$ with the same underlying set M such that a dual adjunction exists between the quasivariety $\boldsymbol{\mathcal{A}}$ and the category of topological relational structures $\boldsymbol{\mathfrak{X}}$ generated by $\underset{\sim}{\mathbf{M}}$. Specifically, the category $\boldsymbol{\mathfrak{X}}$ will be defined as $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$. That is, the category $\boldsymbol{\mathfrak{X}}$ consists of all isomorphic copies of closed substructures of non-empty products of the generating structure $\underset{\sim}{\mathbf{M}}$.

The primary examples of such natural dualities are those of Stone duality for Boolean algebras [84] and Priestley duality for distributive lattices [79]. In the case of $\boldsymbol{\mathfrak{B}}$ and $\boldsymbol{\mathfrak{D}}$, the algebra $\mathbf{M}$ is a two-element algebra, and $\underset{\sim}{\mathbf{M}}$ is the set $\{0, 1\}$, equipped with just the discrete topology and no relations in the Boolean case, while in the distributive case it has the discrete topology and the usual order relation $\leqslant$ (with $0 < 1$).

We use the notation $\mathbf{M}$ when we want to emphasise that the structure is being viewed as an algebra, and by $\underset{\sim}{\mathbf{M}}$ when we want it to be viewed as a set with the discrete topology and collections of finitary operations, finitary partial operations, and finitary relations. That is, $\underset{\sim}{\mathbf{M}} := \langle M; G, H, R, \mathcal{T} \rangle$. We are interested in the cases when the elements of the sets G,H, and R are such that all of the relations in R and the graphs of the (partial) operations in $G \cup H$ form subalgebras of some finite power of $\mathbf{M}$. If this is the case, we say that $\underset{\sim}{\mathbf{M}}$ is *algebraic over* $\mathbf{M}$.

Given a collection of algebraic relations and (partial) operations on $\underset{\sim}{\mathbf{M}}$ such that $\underset{\sim}{\mathbf{M}}$ is algebraic over $\mathbf{M}$, there is a natural method for getting a dual adjunction between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. We denote by $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ the $\boldsymbol{\mathcal{A}}$-homomorphisms from $\mathbf{A}$ into $\mathbf{M}$, and by $\boldsymbol{\mathcal{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ the continuous $(G \cup H \cup R)$-preserving maps from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$. The dually-adjoint hom-functors $\mathrm{D}\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{X}}$ and $\mathrm{E}\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}}$ are defined below.

$$\begin{aligned}
&\mathrm{D}\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{X}}, \quad \mathrm{D}(\mathbf{A}) = \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \quad \text{(a closed substructure of the product } \underset{\sim}{\mathbf{M}}^{A}\text{)}\\
&\mathrm{D}\colon \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{B}) \to \boldsymbol{\mathcal{X}}(\mathrm{D}(\mathbf{B}), \mathrm{D}(\mathbf{A})), \quad (\mathrm{D}(u))(f) = f \circ u \;\text{ for } f \in \mathrm{D}(\mathbf{B}) = \boldsymbol{\mathcal{A}}(\mathbf{B}, \mathbf{M})
\end{aligned}$$

The structure of $\underset{\sim}{\mathbf{M}}$ is defined pointwise on $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$. As an example, suppose that $r \in R$ is an n-ary relation on $\underset{\sim}{\mathbf{M}}$. Then for $\{\, f_i \mid 1 \leqslant i \leqslant n \,\} \subseteq \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ we have that $(f_1, f_2, \dots f_n) \in r_{\mathrm{D}(\mathbf{A})}$ if and only if for all $a \in A$, $(f_1(a), f_2(a), \dots f_n(a)) \in r$.

$$\begin{aligned}
&\mathrm{E}\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}}, \quad \mathrm{E}(\mathbf{A}) = \boldsymbol{\mathcal{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}}) \quad \text{(a subalgebra of the product } \mathbf{M}^{X}\text{)}\\
&\mathrm{E}\colon \boldsymbol{\mathcal{X}}(\mathbf{X}, \mathbf{Y}) \to \boldsymbol{\mathcal{A}}(\mathrm{E}(\mathbf{Y}), \mathrm{E}(\mathbf{X})), \quad (\mathrm{E}(v))(g) = g \circ v \;\text{ for } g \in \mathrm{E}(\mathbf{Y}) = \boldsymbol{\mathcal{X}}(\mathbf{Y}, \underset{\sim}{\mathbf{M}})
\end{aligned}$$

The Preduality Theorem [17, Theorem 1.5.2] confirms that these functors are well defined. What is absolutely critical here is the fact that the relations, operations, and partial operations of $G \cup H \cup R$ are all algebraic over $\mathbf{M}$. Given the above setup we can define embeddings $e_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ and $\epsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ by

$$\begin{aligned}
e_{\mathbf{A}}(a)(f) &= f(a) \quad \text{for } a \in A \text{ and } f \in \mathrm{D}(\mathbf{A}),\\
\epsilon_{\mathbf{X}}(x)(g) &= g(x) \quad \text{for } x \in X \text{ and } g \in \mathrm{E}(\mathbf{X}).
\end{aligned}$$

The embedding $e_{\mathbf{A}}$ is onto if every continuous, relation-preserving map from $\mathrm{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$ is of the form $e_{\mathbf{A}}(a)$ for some $a \in A$. We say that $\underset{\sim}{\mathbf{M}}$ yields a *duality* on $\boldsymbol{\mathcal{A}}$ if for every $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, the embedding $e_{\mathbf{A}}$ is an isomorphism. We will say that $\underset{\sim}{\mathbf{M}}$ yields a *full duality* on $\boldsymbol{\mathcal{A}}$ if $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ and for every $\mathbf{X}$ in the dual category $\boldsymbol{\mathcal{X}}$, the embedding $\epsilon_{\mathbf{X}}$ is an isomorphism. A subset X of M^S is *term-closed* (in M^S) if for every $y \in M^S \setminus X$ there exist S-ary term functions $\sigma_1, \sigma_2\colon M^S \to M$ on $\mathbf{M}$ that agree on X but not at y. If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ and every closed substructure of a power of $\underset{\sim}{\mathbf{M}}$ is term-closed then $\underset{\sim}{\mathbf{M}}$ is said to yield a *strong duality* on $\boldsymbol{\mathcal{A}}$.

The structure $\underset{\sim}{\mathbf{M}}$ is known as the *alter ego* of $\mathbf{M}$ and the set M is referred to as the *dualising object*. (The dualising object is sometimes called the ‘schizophrenic’ object in the older literature.)

This is all of the background on natural duality that is required for the work in Part I. Further theory will be provided in the introduction to Part II.

Part I

Canonical extensions of bounded lattices

Chapter 1

Introduction

The first part of this thesis focuses on canonical extensions of bounded lattices. Research into canonical extensions has largely been driven by its applications to logic: canonical extensions can provide an algebraic method to obtain relational semantics for a broad class of non-classical logics. We will give a brief outline of the role played by canonical extensions in a logical setting.

Semantic models, both algebraic and relational, are useful tools for studying a logic. The link between algebra and logic dates back to the 19th century, while relational (or Kripke) semantics have been in use since the 1950's. We refer the reader to the survey article by Font, Jansana and Pigozzi [42] for details of the development of algebraic logic, while the text by Blackburn, de Rijke and Venema [12] provides a treatment of relational semantics for modal logic. In our brief discussion here, we let Λ denote a propositional logic consisting of a formula algebra $\mathbf{Fm}_{\mathcal{L}}$ in a language $\boldsymbol{\mathcal{L}}$ and a consequence relation $\vdash_\Lambda$.

Algebraic semantics are usually obtained from the syntactic description of the logic Λ. The Lindenbaum–Tarski method, or a generalisation thereof, can be used to obtain a class of algebras which yields a complete algebraic semantics for Λ. (By complete we mean that the semantic notion of consequence exactly mirrors the syntactic consequence relation.) It is then possible to study a logic by studying its corresponding class of algebras.

Relational semantics have been successfully applied to modal and intuitionistic logics, and their description as "possible world" semantics provide a highly intuitive interpretation of the validity of modal formulas. The underlying objects are frames of the form $\langle W, R\rangle$ where W is a set of possible worlds, and R is a binary relation on W known as the accessibility relation. A model is a structure $\mathfrak{M} = \langle W, R, \models\rangle$ where $\langle W, R\rangle$ is a frame and $\models$ is a map from $\mathbf{Fm}_{\mathcal{L}}$ to $\wp(W)$, associating to each formula ϕ a set of possible worlds in which it is satisfied. In the absence of canonical extensions, the canonical frame is constructed by taking W to be the set of all maximal consistent subsets of the formula algebra. The use of canonical models is a powerful technique for obtaining complete relational semantics, although there do exist modal logics which are Kripke-incomplete.

Given a classs of algebras $\boldsymbol{\mathcal{A}}$ and an algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, we denote by $\mathbf{A}^\delta$ its canonical extension and by $\boldsymbol{\mathcal{A}}^+$ the class of concrete algebras such that $\mathbf{A}^\delta \in \boldsymbol{\mathcal{A}}^+$. A class of algebras $\boldsymbol{\mathcal{A}}$ is said to be *canonical* if for every $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, the canonical extension $\mathbf{A}^\delta \in \boldsymbol{\mathcal{A}}$. Given a logic Λ, we denote by $\boldsymbol{\mathcal{A}}_\Lambda$ the class of its corresponding algebraic models. If for each $\mathbf{A} \in \boldsymbol{\mathcal{A}}_\Lambda$ we have $\mathbf{A}^\delta \in \boldsymbol{\mathcal{A}}_\Lambda$ (that is, $\boldsymbol{\mathcal{A}}_\Lambda$ is canonical) then the class of relational frames dual to $\boldsymbol{\mathcal{A}}_\Lambda^+$ will yield complete relational semantics for the associated logic Λ. This is possible because the lifted algebraic operations can be completely described by their actions on the subset of A^δ which makes up the underlying set of the frame. A detailed description of the construction for the case of modal logics is given in Chapter 5 of the well-known text by Blackburn, de Rijke and Venema [12]. The two-sorted relational frames used in the case of non-distributive logics are briefly described in Section 4.4.

We are interested in the link between canonical extensions and duality theory. This connection has its origins in the use of Stone duality in the first construction of a canonical extension for Boolean algebras by Jónsson and Tarski in 1951 [71]. In their construction, the underlying lattice of the canonical extension occurs concretely as the power set of the Stone space dual to the Boolean algebra. Their work in fact focused on Boolean algebras *with operators* (additional operations which are join-preserving in every coordinate). However, in his 1994 survey article, Jónsson [70, Introduction] noted that although Boolean algebras with operators serve as the algebraic counterparts of modal logics, for many years little notice was taken of the algebraic methods from [71] available to logicians.

Gehrke and Jónsson [49] significantly widened the scope of the theory by constructing canonical extensions for distributive lattices. The construction made use of the topological duality of Priestley [79], just as the construction for Boolean algebras had used Stone duality. The canonical extension of a distributive lattice is the collection of up-sets of the underlying ordered set of its Priestley space.

Figure 1.1 provides an elegant description of the relationship between canonical extensions and duality theory in the Boolean and distributive cases. To move from a lattice to its canonical extension one can either make direct use of the canonical extension construction δ, or one can move first to the dual space of the lattice, then

$$\begin{tikzcd}
\boldsymbol{\mathcal{D}}^+ \arrow[r, bend left, "\mathrm{F}"] \arrow[d, bend left, "\flat"] & \boldsymbol{\mathcal{X}} \arrow[l, bend left, "\mathrm{G}"] \arrow[d, bend left, "\beta"] \\
\boldsymbol{\mathcal{D}} \arrow[r, bend left, "\mathrm{D}"] \arrow[u, bend left, "\delta"] & \boldsymbol{\mathcal{X}}_{\mathcal{T}} \arrow[l, bend left, "\mathrm{E}"] \arrow[u, bend left, "\flat"]
\end{tikzcd}$$

$$\begin{tikzcd}
\boldsymbol{\mathcal{B}}^+ \arrow[r, bend left, "\mathrm{F}"] \arrow[d, bend left, "\flat"] & \boldsymbol{\mathcal{Y}} \arrow[l, bend left, "\mathrm{G}"] \arrow[d, bend left, "\beta"] \\
\boldsymbol{\mathcal{B}} \arrow[r, bend left, "\mathrm{D}"] \arrow[u, bend left, "\delta"] & \boldsymbol{\mathcal{Y}}_{\mathcal{T}} \arrow[l, bend left, "\mathrm{E}"] \arrow[u, bend left, "\flat"]
\end{tikzcd}$$

Figure 1.1: The functors involved in the (natural) duality, canonical extension, and discrete duality for Boolean algebras and distributive lattices

forget the topology and see the canonical extension concretely as an algebra of sets coming from the untopologised dual space.

The diagram on the left has Priestley duality represented by the pair of functors D and E, while δ represents the canonical extension functor, and $\flat$ the functor which forgets the topological structure on the right and the additional lattice structure on the left. Here $\boldsymbol{\mathcal{X}}_{\mathcal{T}}$ is the category of Priestley spaces, $\boldsymbol{\mathcal{X}}$ is the category of posets, and $\boldsymbol{\mathcal{D}}^+$ is the category of doubly-algebraic distributive lattices with complete lattice homomorphisms. For the case of Boolean algebras, the diagram on the right uses the functors D and E from Stone duality, while F and G give the discrete duality between complete atomic Boolean algebras and sets. Here $\boldsymbol{\mathcal{Y}}_{\mathcal{T}}$ is the category of Stone spaces, and $\boldsymbol{\mathcal{Y}}$ is the category of sets.

A major achievement of this thesis is to replicate the square diagrams of Fig. 1.1 in the case of arbitrary bounded lattices. Although there is no natural duality available, we succeed in constructing functors defined in a similar spirit. This gives a categorical context in which to view canonical extensions of bounded lattices (see Chapter 2).

It should be noted that an alternative description of the category $\boldsymbol{\mathcal{D}}^+$ is given by Davey, Haviar and Priestley [23], where they reveal a duality attributed to Banaschewski [6]. This is a dual equivalence between the category of posets and the category of Boolean topological bounded distributive lattices. (The compact 0-dimensional topology on the topological lattices is the subspace topology from the product, as the lattices lie in the variety generated by the two-element distributive lattice with the discrete topology.)

Bounded lattices lie outside of the scope of natural duality as the variety is not finitely generated. Despite not having a suitable duality theory available, the theory of canonical extensions for bounded lattices has progressed rapidly in recent years.

The topics covered here show how the existing topological representation and duality theorems are inextricably linked to the theory of canonical extensions.

When looking at applications of canonical extensions to non-classical logics, the extensions of the additional operations are of interest, as these will determine whether or not a variety is canonical. We will use the terms *lattice expansion* or *lattice-based algebra* to refer to an algebraic structure that is a lattice equipped with additional operations.

In the original work on distributive lattices by Gehrke and Jónsson [49] it was shown that when the additional operations of a lattice expansion are operators (join-preserving in every argument), these extend to complete operators on the canonical extension. Gehrke and Jónsson later generalised this work to the case of monotone operations [50] and then later arbitrary additional operations [51]. The second of these papers introduced topology as a key tool in studying the extension of additional operations. It was shown that if operations satisfy certain topological continuity conditions on the topologised canonical extensions, then results regarding the composition and smoothness (see Definition 1.1.6) of the extensions will follow. The concepts of composition and smoothness of additional operations under canonical extensions are important in determining canonicity of a variety.

The rapid development of the theory of canonical extensions for distributive lattice expansions resulted in applications to the case of distributive modal algebras by Gehrke, Nagahashi and Venema [52], and to double quasi-operator algebras by Gehrke and Priestley [54], [55].

While the theory of additional operations in the case of distributive lattice expansions was shown to work particularly well, it has only recently been shown that in many cases this has little to do with the fact that the underlying lattice is distributive, but rather is a result of the lattice expansions being in a finitely generated lattice-based variety. We provide a brief summary of these developments at the end of this section.

We now turn our attention away from the general development of canonical extensions, and look closely at the main focus of Part I: canonical extensions of bounded lattices. The work of Gehrke and Harding [48] in 2001 described for the first time canonical extensions for non-distributive lattices. Their construction did not use duality, but instead used a polarity between the filters and ideals of the lattice, and the resulting completion was shown to possess the abstract properties (*density* and *compactness*) which characterise the canonical extension. Suitable duality theorems for Boolean algebras, distributive lattices and Heyting algebras have played a vital role in the development of canonical extension theory for these classes of algebras. We will show how bounded lattice representations and duality theorems are relevant to canonical extensions.

Perhaps the best known of the bounded lattice representation theorems is due to Urquhart [85]. The representation theorem of Ploščica [78] is essentially a recasting of Urquhart's construction, but in the spirit of natural duality theory (see Section 2.1). We in turn follow Ploščica's example and recast the duality of Allwein and Hartonas [2] as a duality based on partial maps into a two-element set. We will also use the representation theorem of Hartung [66], and make mention of the duality due to Hartonas and Dunn [65]. Lastly, we note the duality of Moshier and Jipsen [75]. Their concrete representation of the canonical extension falls outside of the general pattern that we observe as their duality is purely topological.

The above-mentioned approaches can be split into two groups. The first of these groups (Urquhart, Hartung, Ploščica) use dual spaces built from maximal elements (disjoint filter-ideal pairs, partial homomorphisms). This approach restricts to Priestley duality when the lattice is distributive. The dualities (not merely representations) in the second group (Allwein and Hartonas, Hartonas and Dunn, Moshier and Jipsen) all use a much larger dual space consisting of, respectively: the set of all disjoint filter-ideal pairs, the sets of all filters and all ideals, and the set of all filters. Even in a finite lattice, such dual spaces can have substantially more elements than the lattice itself. However, these approaches have the advantage that a duality, rather than just a representation, can be constructed. In the first group, only surjective homomorphisms can be represented on the dual side. In addition to giving rise to dualities, the bigger dual spaces are constructive—no choice principles are required to obtain the elements of the dual space.

Chapter 2 is inspired by the successful application of natural duality theory to canonical extensions, initiated by Haviar and Priestley [68]. This chapter combines (as far as is possible) natural duality and the existing lattice representation theory to describe canonical extensions for bounded lattices. The work contained in this chapter uses the representation theorem of Ploščica [78] to construct the canonical extension in the same manner as the Boolean and distributive cases can use the natural duality renditions of Stone and Priestley duality. Theorem 2.2.11 demonstrates that the method of using maximal partial maps into an object viewed first as an algebra and then as a topological relational structure can be successfully applied to construct the canonical extension of a bounded lattice.

The main reason for adopting the approach of Chapter 2 is to provide a categorical explanation for the fact that homomorphisms between bounded lattices extend to complete lattice homomorphisms between their canonical extensions. However, the difficulties in extending lattice homomorphisms were still evident. These difficulties were overcome by using the lesser-known duality theorem of Allwein and Hartonas [2]. This solution to the problem of the duals of morphisms comes at the cost of having a much larger dual space, but is effective in providing insight into the functorial way that lattice homomorphisms are lifted to complete lattice homomorphisms on the canonical extension.

In proving the results of Chapter 2, it became clear that there was a direct link between maximal partial maps and polarities. In addition to their construction via polarities, Gehrke and Harding assert without proof [48, Remark 2.10] that the canonical extension can be seen concretely as the stable sets of Urquhart's representation theorem [85]. In Chapter 3 we provide a proof of this assertion and detail exactly how various different constructions of canonical extensions of bounded lattices are related to one another. Furthermore, we identify amongst the maximal partial maps which form the canonical extension those which are the completely join-irreducible and completely meet-irreducible elements. The chapter ends with a description of a class of graph morphisms which lift to complete lattice homomorphisms between complete lattices.

Canonical extensions for posets (and poset-based algebras) were constructed by Dunn, Gehrke and Palmigiano [33]. Their construction uses the so-called *intermediate structure*, a poset formed by taking a quotient of the union of the up-directed down-sets and down-directed up-sets. The canonical extension is then realised as the MacNeille completion of this poset. This approach is a further application of polarities, and is used by Gehrke and Priestley [56] to provide an explanation of why lattice homomorphisms are lifted to complete lattice homomorphisms by the canonical extension construction. We give more details of this result in Section 4.1.

A further feature of the work on canonical extensions of posets [33] is the description of a duality between so-called *perfect lattices* and *perfect posets*. This duality was later described in the two-sorted setting of *RS-frames* by Gehrke [47]. A perfect lattice is a complete lattice in which the completely join-irreducible elements are join dense, and the completely meet-irreducible elements are meet dense. In both the poset and two-sorted setting, it is the completely join-irreducible and completely meet-irreducible elements that form the underlying set of the dual structure. The work in Chapter 4 uses RS-frames to provide a partial answer to a question raised by Ploščica regarding the dual graphs of lattices in his setting of maximal partial maps.

Before summarising the developments in the theory of canonical extensions in finitely generated lattice-based varieties, we note that a new construction of canonical extensions for bounded lattices has recently been put forward by Gouveia and Priestley [63]. The construction exploits the fact that a bounded lattice has both a meet semilattice and join semilattice reduct. Both of these varieties are finitely generated, and this property is exploited to provide an alternative light in which to view the extension of lattice homomorphisms.

The link between profinite completions and canonical extensions was developed by Bezhanishvili, Gehrke, Mines and Morandi [9] and Harding [64], before Gouveia [62] provided the result that for a lattice-based variety of finite type, the profinite completion and canonical extension coincide if and only if the variety is finitely generated.

The *natural extension* is a completion in finitely-generated lattice-based varieties developed by Davey, Gouveia, Haviar and Priestley [21]. It is shown to be an alternative concrete description of the profinite completion for algebras in a finitely generated lattice-based variety. Concretely, the natural extension of an algebra is the set of *all* relation-preserving functions from the dual space into the dualising object, with the algebra embedded as the *continuous* relation-preserving functions.

The works of Davey and Priestley [29, 30] and Gehrke and Vosmaer [58] have revealed why the lifting of additional operations to the canonical extension works particularly well in the case of a finitely generated lattice-based algebra. In [29] the operations are lifted with the underlying lattice to the canonical extension (viewed here as the natural extension). This differs from the usual method that uses the σ- and π-extensions (defined in Section 1.1). Some of the results of [29] overlap with those achieved by Gehrke and Vosmaer [58], although the methods used are different.

Before entering into the main chapters of the thesis, we provide some necessary background on canonical extensions. Each chapter has a brief introduction which contains any further definitions and concepts that may be required.

1.1 Canonical extensions

Throughout this thesis, except where explicitly stated otherwise, we will be looking at lattices that are bounded. Thus we let $\boldsymbol{\mathcal{L}}$ denote the variety of bounded lattices and for $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ we have $\mathbf{L} := \langle L; \wedge, \vee, 0, 1\rangle$. A *completion* of $\mathbf{L}$ is a pair $(e, \mathbf{C})$ where $e\colon L \hookrightarrow C$ is a lattice embedding and $\mathbf{C}$ is a complete lattice. We note that when the embedding is clear from the context, we will not make any reference to e, and will just refer to the completion $\mathbf{C}$.

Definition 1.1.1 ([48, Definition 2.5])**.** A completion $(e, \mathbf{C})$ of $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ is said to be *dense* if every element $x \in C$ can be expressed as a meet of joins and a join of meets of elements of L. That is,

$$x = \bigwedge \bigvee \{\, e(I) \mid x \leqslant \bigvee e(I) \,\} = \bigvee \bigwedge \{\, e(F) \mid \bigwedge e(F) \leqslant x \,\},$$

where $I, F \subseteq L$. A completion is said to be *compact* if whenever $A, B \subseteq L$ and $\bigwedge e(A) \leqslant \bigvee e(B)$, then there exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge e(A') \leqslant \bigvee e(B')$. Given $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, a *canonical extension* of $\mathbf{L}$ is a completion $(e, \mathbf{C})$ of $\mathbf{L}$ which is both dense and compact.

Definition 1.1.2. Let $\mathbf{L}$ be a bounded lattice and let $(e, \mathbf{C})$ be a completion of $\mathbf{L}$. The following statements are equivalent to the definition of compactness given in Definition 1.1.1:

(i) if $A, B \subseteq L$ and $\bigwedge e(A) \leqslant \bigvee e(B)$ then there exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge A' \leqslant \bigvee B'$;

(ii) given a filter F and an ideal I of L, then $\bigwedge e(F) \leqslant \bigvee e(I)$ if and only if $F \cap I \neq \emptyset$;

(iii) given any down-directed subset S of L and up-directed subset T of L with $\bigwedge e(S) \leqslant \bigvee e(T)$ then there exist $s \in S$ and $t \in T$ such that $s \leqslant t$.

The existence and uniqueness of canonical extensions for arbitrary bounded lattices was first shown by Gehrke and Harding [48].

Theorem 1.1.3 ([48, Propositions 2.6 and 2.7])**.** *Every lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ has a canonical extension $\mathbf{L}^\delta$, and $\mathbf{L}^\delta$ is unique up to an isomorphism which fixes $\mathbf{L}$.*

The elements of $\mathbf{L}^\delta$ which are meets of elements of $\mathbf{L}$ are called *filter* elements while those elements that are joins of elements of $\mathbf{L}$ are called *ideal* elements. We denote these two sets by $\mathbb{F}(\mathbf{L}^\delta)$ and $\mathbb{I}(\mathbf{L}^\delta)$. The motivation for the names of these sets is made clear by the following result.

Proposition 1.1.4 ([48, Lemma 3.3])**.** *Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and $\mathbf{L}^\delta$ its canonical extension. Then $\mathbb{F}(\mathbf{L}^\delta)$ and $\mathbb{I}(\mathbf{L}^\delta)$ are sublattices of $\mathbf{L}^\delta$ and are order-isomorphic to* $(\mathrm{Filt}(\mathbf{L}), \supseteq)$ *and* $(\mathrm{Idl}(\mathbf{L}), \subseteq)$ *respectively.*

In the older literature the elements of $\mathbb{F}(\mathbf{L}^\delta)$ and $\mathbb{I}(\mathbf{L}^\delta)$ are referred to as the *closed* and *open* elements of the canonical extension, because when $\mathbf{L}$ is a Boolean algebra they are concretely represented as the closed and open subsets of the Stone space dual to $\mathbf{L}$.

Before we look at extensions of maps, we state the following important result. It is a consequence of the uniqueness (up to isomorphism) of the canonical extension, and of [48, Definition 3.7].

Proposition 1.1.5. *Let* $\{\, \mathbf{L}_i \mid 1 \leqslant i \leqslant n \,\} \subseteq \boldsymbol{\mathcal{L}}$ *for some finite* n*. For each* i*, let* $\mathbf{L}_i^\partial$ *be dual lattice of* $\mathbf{L}_i$*, and* $\mathbf{L}_i^\delta$ *its canonical extension. Then* $(\mathbf{L}_i^\partial)^\delta$ *is isomorphic to* $(\mathbf{L}_i^\delta)^\partial$ *and* $(\mathbf{L}_1 \times \ldots \times \mathbf{L}_n)^\delta$ *is isomorphic to* $(\mathbf{L}_1^\delta \times \ldots \times \mathbf{L}_n^\delta)$*.*

When considering an n-ary operation on a lattice $\mathbf{L}$, the above result allows us to use the theory of maps $f\colon \mathbf{K} \to \mathbf{L}$ and apply it to the case where $\mathbf{K} = \mathbf{L}^n$.

Given a map between lattices, $f\colon \mathbf{L} \to \mathbf{K}$, the density condition suggest two natural possible extensions of f to the canonical extension. These extensions can be thought of as an approximation from above and an approximation from below.

Definition 1.1.6 ([48, Definition 4.1])**.** *Let* $\mathbf{L}, \mathbf{K} \in \boldsymbol{\mathcal{L}}$ *and let* $f\colon \mathbf{L} \to \mathbf{K}$*. For* $x \in \mathbf{L}^\delta$ *define*

$$f^\sigma(x) := \bigvee \{\, \bigwedge \{\, f(a) \mid a \in F \,\} \mid \bigwedge F \leqslant x \,\}$$

and

$$f^\pi(x) := \bigwedge \{\, \bigvee \{\, f(a) \mid a \in I \,\} \mid x \leqslant \bigvee I \,\}.$$

When $f^\sigma = f^\pi$*, the map* f *is said to be* smooth*.*

Both f^σ and f^π agree with f on L and, when compared pointwise, we always have the inequality $f^\sigma \leqslant f^\pi$. If an operation is not smooth, then a choice has to be made when looking at its canonical extension. An example of this choice occurs in the work of Almeida [3] where lattice expansions with various generalised negation operations are shown to be canonical. In that work, the π-extension of maps was used in order to prove canonicity results.

Non-canonicity of a variety can occur when the two extensions of the additional operations do not agree (i.e. the original operation is not smooth) but both extensions are required in order to satisfy one or more of the original algebra's equations. It was shown by Gehrke and Priestley [53] that the variety of MV-algebras is not canonical. In this instance, the non-canonicity was caused by an equation that required both the σ- and π-extension of an operation, but the operation was not smooth and so the terms of the equation cannot agree in general.

Chapter 2

Canonical extensions via partial maps

In this chapter we present a new approach to the canonical extension of a bounded lattice, based on the dual representation due to Ploščica [78]. The motivation behind this approach is to mimic as closely as possible the theory of natural dualities, and thereby obtain a categorical understanding not just of the canonical extension of the underlying lattices, but also of the extension of lattice homomorphisms.

As we have already mentioned, the original constructions of canonical extensions for Boolean algebras and distributive lattices made use of Stone and Priestley duality. Each of Stone duality for Boolean algebras, $\boldsymbol{\mathcal{B}}$, and Priestley duality for $\boldsymbol{\mathcal{D}}$ is an instance of a natural duality as outlined in Section 1.1. The functors D and E act on morphisms by composition, and this is a key feature of the way in which the duality operates. By modifying the functors D and E, one can obtain a functorial construction of the canonical extension. This highlights its relationship to the topological duality. The canonical extension construction for distributive lattices can then be viewed in the way shown in Fig. 2.1. In the figure, $\flat$ denotes the functor forgetting the topology and the functor G acts in the same manner as E does, but on untopologised rather than topologised structures. The codomain $\boldsymbol{\mathcal{D}}^+$ of G can be taken to be the category of doubly-algebraic distributive lattices with complete lattice homomorphisms. In fact, more is true: there is a hom-functor F adjoint to G such that F and G set up a dual equivalence between $\boldsymbol{\mathcal{D}}^+$ and $\boldsymbol{\mathcal{X}}$ (see the papers by Davey, Haviar and Priestley [23, 24]). However, the functor F is not required in order to factorise δ. On the other hand, we do make use of E:

$$\mathbf{L} \cong \mathrm{ED}(\mathbf{L}) \subseteq \mathbf{L}^{\delta} \quad \text{for all } \mathbf{L} \in \boldsymbol{\mathcal{D}}.$$

$$\begin{tikzcd}
\mathfrak{D}^{+} & \mathfrak{X} \arrow[l, "G"] \\
\mathfrak{D} \arrow[u, "\delta"] \arrow[r, shift left, "D"] & \mathfrak{X}_{\mathfrak{T}} \arrow[u, "\flat"'] \arrow[l, shift left, "E"]
\end{tikzcd}$$

Figure 2.1: Factorising the canonical extension functor on $\mathfrak{D}$

As mentioned in the introduction, the variety of bounded lattices falls outside the scope of the theory of natural dualities. Urquhart's representation [85] provides a concrete representation but does not provide a dual equivalence of categories due to unavoidable problems with the representation of morphisms (we encounter these in Section 2.3). However, the recasting of Urquhart's work by Ploščica [78] provides a useful setting for our aims.

Ploščica's extension of the Priestley representation is achieved by replacing total maps into $\{0, 1\}$, treated first as a lattice and then as a relational structure by appropriate maximally-defined partial maps of the same sort. Our conjecture was that the canonical extension of an arbitrary bounded lattice may be obtained in the same manner as in Fig. 2.1 by 'forgetting the topology' at the level of the Ploščica first dual. The principal result of Section 2.2, Theorem 2.2.11, confirms that this is indeed the case. A detailed discussion of the relationship between this new construction and existing approaches is contained in Chapter 3.

We encounter problems when trying to dualise and extend morphisms in the Ploščica setting. Our solution, which does give a functorial construction, requires the use of an enlarged dual space of **L**. Specifically, our factorisation after recasting the duality due to Allwein and Hartonas [1, 2] in the style of Ploščica.

Through our recasting we are able to retain one of the important features of a natural duality type framework. Our functors, although they are not true hom-functors, still operate in the same manner on morphisms. That is, they act on morphisms by composition. The main results for our categorical framework are obtained only after verifying that the functors we propose are indeed well-defined.

2.1 The framework for the construction

In this section we set up the framework within which we shall construct the canonical extension of a bounded lattice $\mathbf{L}$. In outline, we look at the structures and associated maps obtained by deleting the topology from the dual spaces employed in Ploščica's representation. We thereby arrive at a complete lattice which we shall later show serves as $\mathbf{L}^\delta$.

The central idea in Ploščica's representation of bounded lattices [78] is the replacement of total maps by partial maps. Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. A partial map $f\colon \mathbf{L}_1 \to \mathbf{L}_2$ between bounded lattices is called a *partial homomorphism* if its domain is a 0,1-sublattice of $\mathbf{L}_1$ and $f\colon\ \mathrm{dom}(f) \to \mathbf{L}_2$ is an $\boldsymbol{\mathcal{L}}$-homomorphism. A partial homomorphism is said to be *maximal* if there is no partial homomorphism properly extending it; such a map is referred to as an MPH for short. By Zorn's Lemma, every partial homomorphism can be extended to an MPH. For bounded lattices $\mathbf{L}$ and $\mathbf{K}$, we denote by $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \mathbf{K})$ the set of all MPH's from $\mathbf{L}$ to $\mathbf{K}$.

Let

$$\underline{\mathbf{2}} := \langle \{0,1\}; \vee, \wedge, 0, 1\rangle \quad \text{and} \quad \underset{\sim}{\mathbf{2}} := \langle \{0,1\}; \leqslant\rangle$$

denote, respectively, the two-element bounded lattice and the two-element ordered set with $0 < 1$. The topological structures $\underline{\mathbf{2}}_{\mathcal{T}}$ and $\underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ are obtained by adding the discrete topology $\mathcal{T}$ to $\underline{\mathbf{2}}$ and $\underset{\sim}{\mathbf{2}}$, respectively.

Following Ploščica [78], for any bounded lattice $\mathbf{L}$, the topological dual space of $\mathbf{L}$ is defined by equipping the set $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ with the binary relation E defined by the rule

$$(f, g) \in E \quad \text{iff } f(x) \leqslant g(x) \text{ for every } x \in \mathrm{dom}(f) \cap \mathrm{dom}(g);$$

and then endowing it with the topology $\mathcal{T}$, which has, as its subbasis of closed sets, all sets of the form

$$V_a = \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0 \,\} \quad \text{and} \quad W_a = \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1 \,\},$$

where $a \in L$. We let $\mathrm{D}(\mathbf{L}) = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathcal{T})$. The topology of $\mathrm{D}(\mathbf{L})$ is T_1 and moreover, it is compact (see [85, Lemma 6]). If the lattice $\mathbf{L}$ is distributive, then $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) = \boldsymbol{\mathcal{L}}(\mathbf{L}, \underline{\mathbf{2}})$, the relation E coincides with the pointwise partial order of maps and $\mathrm{D}(\mathbf{L})$ is the usual dual space of $\mathbf{L}$ in the Priestley duality [79].

Ploščica's representation of $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ is then obtained in the following way. One takes the family of continuous E-preserving partial maps from $\mathrm{D}(\mathbf{L})$ into $\underset{\sim}{\mathbf{2}}_{\mathcal{T}}$, where E on $\underset{\sim}{\mathbf{2}}$ is taken to be the usual order. The maximally-defined members of this set are then shown to form a lattice isomorphic to $\mathbf{L}$, the isomorphism being given by the natural evaluation map $e_{\mathbf{L}} \colon \mathbf{L} \to \mathrm{ED}(\mathbf{L})$. We recall further details of the construction in Section 2.2.

Our strategy for obtaining the canonical extension $\mathbf{L}^\delta$ of $\mathbf{L}$ will be to replace $\mathrm{D}(\mathbf{L})$ above by $\mathrm{D}^\flat(\mathbf{L})$. Here $\mathrm{D}^\flat$ is the composition ${}^\flat \circ \mathrm{D}$, where $\flat$ is the map forgetting the topology. Thus the category we shall use to build the canonical extension will be the category $\mathcal{G}$ of graphs $\mathbf{X} = (X, E)$ and partial maps which preserve E. (We note that the structures we call *graphs* would usually be referred to as *digraphs*. We also warn that structures of the form $\mathrm{D}^\flat(\mathbf{L})$ will have special properties which we shall need to exploit in due course.) Much of our work leading up to Theorem 2.2.11 involves checking that the ancillary results on the graphs with topology which underlie the Ploščica representation [78, Section 1] have appropriate analogues in the topology-free setting.

Initially we let $\mathbf{X} = (X, E)$ be any graph. Then, given two graphs $\mathbf{X} = (X, E_X)$ and $\mathbf{Y} = (Y, E_Y)$ we use the notation $\mathcal{G}(\mathbf{X}, \mathbf{Y})$ to denote the collection of total E-preserving maps from $\mathbf{X}$ to $\mathbf{Y}$. In Section 2.3 we will make use of the set of partial morphisms, $\mathcal{G}^{\mathrm{p}}(\mathbf{X}, \mathbf{Y})$. We note that, by Zorn's Lemma, every partial E-preserving map into $\underset{\sim}{\mathbf{2}}$ can be extended to a maximal partial E-preserving map; by 'maximal' we mean here that there is no partial E-preserving map properly extending it (in general such an extension will not be unique). We will denote by $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ the set of maximal partial E-preserving maps from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$. Let us use the shorthand MPE to refer to an element of such a set.

In the distributive case we can restrict to the situation in which E is a partial order and MPE's are simply total maps which are order-preserving. The order-preserving maps from a poset $\mathbf{X}$ into $\underset{\sim}{\mathbf{2}}$ always form a complete lattice under pointwise join and meet. In the case of MPE's on a graph the situation is more complicated.

We now work towards showing, for a general graph $\mathbf{X} = (X, E)$, that $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ does indeed form a complete lattice. As we shall see in Proposition 2.2.2 and the discussion preceding it, the similarity between Lemma 2.1.1 below and the corresponding result, [78, Lemma 1.3], concerning graphs with topology, is critical to the success of our approach to completions built from dual spaces. The key point is that the results involving maximal partial maps which involve topology (such as [78, Lemma 1.3]) do not change when the topology is removed, as it is the E-preserving property of these maps which limits their domains.

Most of the results in this chapter have two halves where the second half will follow by order duality. Thus we will show only one of the statements from each result.

Lemma 2.1.1. *Let* $\mathbf{X} = (X, E)$ *be a graph and* φ *a partial map from* $\mathbf{X}$ *to* $\underset{\sim}{\mathbf{2}}$*. Then* $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *if and only if*

(i) $\varphi^{-1}(0) = \{\, x \in X \mid \textit{there is no } y \in \varphi^{-1}(1) \textit{ with } (y, x) \in E \,\}$ *and*

(ii) $\varphi^{-1}(1) = \{\, x \in X \mid \textit{there is no } y \in \varphi^{-1}(0) \textit{ with } (x, y) \in E \,\}$.

Proof. Suppose that $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. It suffices to consider (i). Let $x \in \varphi^{-1}(0)$. Then for every $y \in \varphi^{-1}(1)$ we clearly have $1 = \varphi(y) \nleqslant \varphi(x) = 0$. As φ preserves E, we obtain $(y, x) \notin E$. Let $x \in X$ be such that there is no $y \in \varphi^{-1}(1)$ with $(y, x) \in E$. Then $\varphi \cup \{(x, 0)\}$ is a partial E-preserving map. The maximality of φ now yields $\varphi(x) = 0$.

Now assume that φ is a partial map from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$ satisfying (i) and (ii). If $x, y \in \operatorname{dom}(\varphi)$ such that $(x, y) \in E$ and $(\varphi, \varphi)(x, y) = (1, 0)$, this would contradict the assumption. Thus φ is E-preserving on its domain. Let $z \notin \operatorname{dom}(\varphi)$. Since $z \notin \varphi^{-1}(0)$ there exists $y \in \varphi^{-1}(1)$ such that $(y, z) \in E$. Thus there cannot exist an E-preserving extension ψ of φ with $\psi(z) = 0$. Similarly, by property (ii), we cannot have $\psi(z) = 1$ and so the domain of φ is maximal. □

Consider a family $\{\, \varphi_i \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \mid i \in I \,\}$ for an index set I. Then we can define the pointwise meet $p^{\bigwedge}_{\varphi_i}$ and pointwise join $p^{\bigvee}_{\varphi_i}$ as follows:

$$p^{\bigwedge}_{\varphi_i}(x) = \begin{cases} 1 & \text{if } x \in \bigcap_{i \in I} \varphi_i^{-1}(1), \\ 0 & \text{if } x \in \bigcup_{i \in I} \varphi_i^{-1}(0) \end{cases} \qquad \text{and} \qquad p^{\bigvee}_{\varphi_i}(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{i \in I} \varphi_i^{-1}(1), \\ 0 & \text{if } x \in \bigcap_{i \in I} \varphi_i^{-1}(0). \end{cases}$$

(The motivation for these definitions comes from Ploščica [78, Section 3].) To see that these partial maps are E-preserving, let $(p^{\bigwedge}_{\varphi_i}(x), p^{\bigwedge}_{\varphi_i}(y)) = (1, 0)$. Then there exists $j \in I$ with $\varphi_j(y) = 0$, and thus $(\varphi_j(x), \varphi_j(y)) = (1, 0)$ and so $(x, y) \notin E$.

We shall work with specific extensions of the maps $p^{\bigwedge}_{\varphi_i}$ and $p^{\bigvee}_{\varphi_i}$ to elements of $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. The motivation for our choice of maximal maps comes from Lemma 2.1.1. Define $e^{\bigwedge}_{\varphi_i} \colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ and $e^{\bigvee}_{\varphi_i} \colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ as follows:

$$e^{\bigwedge}_{\varphi_i}(x) = \begin{cases} 1 & \text{if } x \in \bigcap_{i \in I} \varphi_i^{-1}(1), \\ 0 & \text{if there is no } y \in \bigcap_{i \in I} \varphi_i^{-1}(1) \text{ with } (y, x) \in E; \end{cases}$$

$$e^{\bigvee}_{\varphi_i}(x) = \begin{cases} 1 & \text{if there is no } y \in \bigcap_{i \in I} \varphi_i^{-1}(0) \text{ with } (x, y) \in E, \\ 0 & \text{if } x \in \bigcap_{i \in I} \varphi_i^{-1}(0). \end{cases}$$

We claim that $e^{\bigwedge}_{\varphi_i}$ extends $p^{\bigwedge}_{\varphi_i}$. Let $x \in \bigcup_{i \in I} \varphi_i^{-1}(0)$, so that $x \in \varphi_j^{-1}(0)$ for some $j \in I$. Then there is no $y \in \bigcap_{i \in I} \varphi_i^{-1}(1)$ with $(y, x) \in E$, for otherwise $(\varphi_j(y), \varphi_j(x)) = (1, 0)$, which contradicts the preservation of E by φ_j. Analogously, $e^{\bigvee}_{\varphi_i}$ extends $p^{\bigvee}_{\varphi_i}$. The next lemma shows that $e^{\bigwedge}_{\varphi_i}$ and $e^{\bigvee}_{\varphi_i}$ are maximal partial E-preserving extensions of $p^{\bigwedge}_{\varphi_i}$ and $p^{\bigvee}_{\varphi_i}$, respectively.

Lemma 2.1.2. *Let* $\mathbf{X} = (X, E)$ *be a graph and let* $\{\, \varphi_i \mid i \in I \,\} \subseteq \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. *Then the maps* $e^{\bigwedge}_{\varphi_i}$ *and* $e^{\bigvee}_{\varphi_i}$ *are elements of* $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *extending* $p^{\bigwedge}_{\varphi_i}$ *and* $p^{\bigvee}_{\varphi_i}$*, respectively.*

Proof. We first show that $e^{\bigwedge}_{\varphi_i}$ preserves E. Suppose that $(e^{\bigwedge}_{\varphi_i}(x), e^{\bigwedge}_{\varphi_i}(y)) = (1, 0)$ where $x, y \in X$. Then $x \in \bigcap_{i \in I} \varphi_i^{-1}(1)$ and there is no $z \in \bigcap_{i \in I} \varphi_i^{-1}(1)$ with $(z, y) \in E$. Thus $(x, y) \notin E$ as required.

We now show the maximality of $e^{\bigwedge}_{\varphi_i}$. Let $\psi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ be a map extending $e^{\bigwedge}_{\varphi_i}$. Let $x \in X$ be such that $x \notin \mathrm{dom}(e^{\bigwedge}_{\varphi_i})$; thus $x \notin \bigcap_{i \in I} \varphi_i^{-1}(1)$ and so there exists $y \in \bigcap_{i \in I} \varphi_i^{-1}(1)$ with $(y, x) \in E$. We want to show that $x \notin \mathrm{dom}(\psi)$. Let $j \in I$ be such that $x \notin \varphi_j^{-1}(1)$. Since $\varphi_j \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$, by Lemma 2.1.1 there exists $z \in \varphi_j^{-1}(0)$ with $(x, z) \in E$. Hence $\varphi_j(z) = 0$ and so, by Lemma 2.1.1 again, there is no $w \in \varphi_j^{-1}(1)$ with $(w, z) \in E$. Hence there is no $v \in \bigcap_{i \in I} \varphi_i^{-1}(1)$ with $(v, z) \in E$, which gives us $e^{\bigwedge}_{\varphi_i}(z) = 0$. As ψ extends $e^{\bigwedge}_{\varphi_i}$, we obtain $\psi(z) = 0$. Moreover, as $e^{\bigwedge}_{\varphi_i}(y) = 1$, we have $\psi(y) = 1$.

Now suppose for a contradiction that $x \in \mathrm{dom}(\psi)$. Then $(y, x) \in E$ and $(x, z) \in E$ give us $1 = \psi(y) \leqslant \psi(x) \leqslant \psi(z) = 0$, which is false. $\square$

Having shown that our proposed definitions of arbitrary meet and join are indeed valid, we can conclude that the set $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ is a complete lattice.

Theorem 2.1.3. *Let* $\mathbf{X} = (X, E)$ *be a graph. Then the set* $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *ordered by the rule*

$$\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

is a complete lattice.

Proof. Obviously, the relation $\leqslant$ is reflexive and transitive. The antisymmetry of $\leqslant$ follows from the fact that for any $\varphi, \psi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$, if $\varphi^{-1}(1) = \psi^{-1}(1)$, then, by Lemma 2.1.1, $\varphi^{-1}(0) = \psi^{-1}(0)$, and hence $\varphi = \psi$.

To show that $\leqslant$ is a (complete) lattice order, let $\{\, \varphi_i \mid i \in I \,\} \subseteq \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. We claim that $\bigwedge \varphi_i = e^{\bigwedge}_{\varphi_i}$ which is an element of $\mathrm{C}(\mathbf{X})$ by Lemma 2.1.2.

First note that $(e^{\bigwedge}_{\varphi_i})^{-1}(1) = \bigcap_{i \in I} \varphi_i^{-1}(1) \subseteq \varphi_i^{-1}(1)$ for all $i \in I$. Thus in $(\mathrm{C}(\mathbf{X}), \leqslant)$, we have $e^{\bigwedge}_{\varphi_i} \leqslant \varphi_i$ for all $i \in I$. Now let $\psi \leqslant \varphi_i$ for all $i \in I$. This gives us that $\psi^{-1}(1) \subseteq (\bigcap_{i \in I} \varphi_i)^{-1}(1) = (e^{\bigwedge}_{\varphi_i})^{-1}(1)$, whence $\psi \leqslant e^{\bigwedge}_{\varphi_i}$ as required. $\square$

Before we move on to incorporate topology we insert a proposition concerning maps between graphs. We shall need this in Section 2.3, when we consider morphisms acting by composition.

Lemma 2.1.4. *Let* $\mathbf{X} = (X, E_X)$ *and* $\mathbf{Y} = (Y, E_Y)$ *be graphs and let* $\alpha \in \mathcal{G}(\mathbf{X}, \mathbf{Y})$. *Then for* $\varphi \in \mathcal{G}^{\mathrm{p}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$*, the map* $\varphi \circ \alpha$*, with the domain being determined by the set of elements for which the composition is defined, is a partial map from* $\mathbf{X}$ *to* $\underset{\sim}{\mathbf{2}}$ *which preserves* E*.*

Proof. Suppose that $(\varphi \circ \alpha)(x_1) = 1$ and $(\varphi \circ \alpha)(x_2) = 0$. That is, $\varphi(\alpha(x_1)) = 1$ and $\varphi(\alpha(x_2)) = 0$. Since φ is E_Y-preserving we have that $(\alpha(x_1), \alpha(x_2)) \notin E_Y$. Now since α is an E_X-preserving map, we must have that $(x_1, x_2) \notin E_X$. $\square$

In [78], a set equipped with a reflexive binary relation and a topology is called a topological graph. As the graph relation is not necessarily closed in the topology, here we prefer to use the term *graph with topology*. We consider the family $\mathcal{G}_{\mathcal{T}}$ of graphs with topology and make it into a category in the following way. A map $\varphi\colon (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ between graphs with topology is called a $\mathcal{G}_{\mathcal{T}}$*-morphism* if it preserves the binary relation and is continuous as a map from (X_1, τ_1) to (X_2, τ_2). A partial map $\varphi\colon (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ is called a *partial* $\mathcal{G}_{\mathcal{T}}$*-morphism* if its domain is a τ_1-closed subset of X_1 and the restriction of φ to its domain is a $\mathcal{G}_{\mathcal{T}}$-morphism. (We assume that $\mathrm{dom}(\varphi)$ inherits the binary relation and the topology from X_1.) A partial $\mathcal{G}_{\mathcal{T}}$-morphism is called maximal, or an MPM for short, if there is no partial $\mathcal{G}_{\mathcal{T}}$-morphism properly extending it. For a graph with topology $\mathbf{X}_{\mathcal{T}} = (X, E, \mathcal{T})$ we denote by $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ the set of MPM's from $\mathbf{X}_{\mathcal{T}}$ to $\underset{\sim}{\mathbf{2}}_{\mathcal{T}}$.

Our candidate for the canonical extension of a bounded lattice $\mathbf{L}$ is the complete lattice $\mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}}) = \mathrm{C}(\mathrm{D}^{\flat}(\mathbf{L}))$. The embedding will be given, as in the Ploščica representation (see Proposition 2.2.1 below) by the map $e_{\mathbf{L}}$, given by evaluation, onto the set of maps which are maximal among continuous partial morphisms into $\underset{\sim}{\mathbf{2}}_{\mathcal{T}}$. It is therefore necessary to reconcile the two versions of maximality—one with topology and the other without. This can be done for graphs in general, rather than just those arising from dual spaces.

Proposition 2.1.5. *Let* $\mathbf{X}_{\mathcal{T}} = (X, E, \mathcal{T})$ *be a graph with topology and* $\mathbf{X} = (X, E)$ *be its untopologised counterpart. Then* $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}}) \subseteq \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$*.*

Proof. We have already observed that Lemma 2.1.1 has a topological counterpart [78, Lemma 1.3]. Specifically, for $\varphi \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ we have

$$\begin{aligned}\varphi^{-1}(0) &= \{\, x \in X \mid \text{ there is no } y \in \varphi^{-1}(1) \text{ with } (y, x) \in E \,\};\\ \varphi^{-1}(1) &= \{\, x \in X \mid \text{ there is no } y \in \varphi^{-1}(0) \text{ with } (x, y) \in E \,\}.\end{aligned}$$

It now follows from Lemma 2.1.1 that $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. $\square$

The set $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ can be considered as a subposet of the poset $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ (recall Proposition 2.1.5), with the partial order on $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ given, as in [78], by

$$\varphi \leqslant \psi \quad \text{iff} \quad \varphi^{-1}(1) \subseteq \psi^{-1}(1).$$

If in particular $\mathbf{X}_{\mathcal{T}} = \mathrm{D}(\mathbf{L})$ then it was shown in [78] (see Proposition 2.2.1(iii) below) that the partial order $\leqslant$ on $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ is a lattice order, and the lattice $(\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}}), \leqslant)$ is clearly a sublattice of the lattice $\mathrm{C}(\mathbf{X}) = (\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}}), \leqslant)$ (see Theorem 2.1.3).

2.2 The canonical extension construction

Our next task is to establish that, when we take a bounded lattice $\mathbf{L}$ and the graph $\mathbf{X} = \mathrm{D}^{\flat}(\mathbf{L})$, then $\mathrm{C}(\mathbf{X})$ acts as a completion of $\mathbf{L}$. For this we call on the Ploščica representation, and so now need to recall further facts from [78]. The relationship between a lattice $\mathbf{L}$ and the set $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ is summed up in the following result.

Proposition 2.2.1 ([78, Lemmas 1.2 and 1.5 and Theorem 1.7]). *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $\mathrm{D}(\mathbf{L})$ *be the graph with topology defining the dual space of* $\mathbf{L}$. *For* $a \in L$, *let the evaluation map* $e_a \colon \mathrm{D}(\mathbf{L}) \to \underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ *be defined by*

$$e_a(f) = \begin{cases} f(a) & a \in \mathrm{dom}(f),\\ - & \textit{undefined otherwise.}\end{cases}$$

Then the following hold.

(i) *The map* $e_a \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ *for each* $a \in L$.

(ii) *Every* $\varphi \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ *is of the form* e_a *for some* $a \in L$.

(iii) *The map* $e_{\mathbf{L}} : \mathbf{L} \to \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ *given by evaluation,* $a \mapsto e_a$ $(a \in L)$, *is an isomorphism of* $\mathbf{L}$ *onto the lattice* $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, *ordered by* $\varphi \leqslant \psi$ *if and only if* $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$.

By combining preceding results we obtain the following proposition.

Proposition 2.2.2. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $\mathrm{D}(\mathbf{L})$ *be the graph with topology dual to* $\mathbf{L}$ *and* $\mathbf{X} = \mathrm{D}^\flat(\mathbf{L})$*. Then* $(e, \mathrm{C}(\mathbf{X}))$ *is a completion of* $\mathbf{L}$*, where* $e \colon a \mapsto e_a$ $(a \in L)$*.*

Proof. Theorem 2.1.3 tells us that $\mathrm{C}(\mathbf{X})$ is a complete lattice. The result now follows directly from Proposition 2.2.1 combined with the fact that every $\varphi \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, and in particular any evaluation map e_a, belongs to $\mathcal{G}^{\mathrm{mp}}(\mathrm{D}^\flat(\mathbf{L}), \underset{\sim}{\mathbf{2}})$, by Proposition 2.1.5. ☐

Proposition 2.2.2 identifies a completion for any bounded lattice constructed from the dual space of the lattice. When the lattice is distributive this certainly does give the canonical extension as introduced by Gehrke and Jónsson [49]. We would now like to prove that this completion supplies a canonical extension for an arbitrary bounded lattice. To achieve this, we need to examine more closely the structure of the dual space $\mathrm{D}(\mathbf{L})$ of a bounded lattice $\mathbf{L}$.

In [78, Section 2], Ploščica demonstrates how his dual representation for lattices relates to Urquhart's topological representation [85]. At the level of the dual spaces, the passage back and forth between Urquhart's dual representation and Ploščica's is set up by a bijection between maximal disjoint filter-ideal pairs in $\mathbf{L}$ (as employed by Urquhart) and $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$. Instead of carrying a single binary relation E, Urquhart's dual spaces are equipped with a pair of quasi-orders, $\leqslant_1$ and $\leqslant_2$. Interpreted in terms of MPH's, these two relations are defined on the set $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ as follows:

$$f \leqslant_1 g \iff f^{-1}(1) \subseteq g^{-1}(1) \quad \text{and} \quad f \leqslant_2 g \iff f^{-1}(0) \subseteq g^{-1}(0).$$

These quasi-orders $\leqslant_1$ and $\leqslant_2$ prove to be a valuable ancillary tool for working with graphs of the form $\mathrm{D}^\flat(\mathbf{L}) = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, and we shall use them, in an analogous manner but on bigger domains, in Section 2.3.

Lemma 2.2.3 ([78, Theorem 2.1])**.** *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $f, g \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$*. Then*

(i) $(f, g) \in E$ *if and only if there exists* $h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ *with* $f \leqslant_1 h$ *and* $g \leqslant_2 h$*;*

(ii) $f \leqslant_2 g$ *if and only if there is no* $h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ *with* $(h, g) \in E$ *and* $(h, f) \notin E$*;*

(iii) $f \leqslant_1 g$ *if and only if there is no* $h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ *with* $(g, h) \in E$ *and* $(f, h) \notin E$*.*

It is a consequence of (i) above that $\leqslant_1 \subseteq E$ and $\geqslant_2 \subseteq E$. This will be used repeatedly below. The following two lemmas describe properties of the set of MPH's which will prove crucial in enabling us to prove that $\mathrm{C}(\mathbf{X})$ is a dense completion of $\mathbf{L}$.

Lemma 2.2.4. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. *Let* $\mathbf{X} = \mathrm{D}^{\flat}(\mathbf{L})$ *and let* $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. *Then*

(i) *if* $f \notin \varphi^{-1}(0)$ *there exists* $g \in \varphi^{-1}(1)$ *such that* $f \leqslant_2 g$*;*

(ii) *if* $f \notin \varphi^{-1}(1)$ *there exists* $g \in \varphi^{-1}(0)$ *such that* $f \leqslant_1 g$*.*

Proof. (i) Given $f \notin \varphi^{-1}(0)$, we have by Lemma 2.1.1 that there exists $h \in \varphi^{-1}(1)$ such that $(h, f) \in E$. By Lemma 2.2.3(i) there exists $g \in X$ such that $h \leqslant_1 g$ and $f \leqslant_2 g$, and we now claim that $g \in \varphi^{-1}(1)$. If we suppose that $g \notin \varphi^{-1}(1)$, by Lemma 2.1.1 there must exist $u \in \varphi^{-1}(0)$ such that $(g, u) \in E$. Now $(g, u) \in E$ if and only if there exists $v \in X$ such that $g \leqslant_1 v$ and $u \leqslant_2 v$, again by Lemma 2.2.3(i). By the transitivity of $\leqslant_1$ we have that $h \leqslant_1 v$. Applying this with $u \leqslant_2 v$ gives $(h, u) \in E$. Since $u \in \varphi^{-1}(0)$ and $h \in \varphi^{-1}(1)$, this contradicts that φ is E-preserving. Thus $g \in \varphi^{-1}(1)$. The proof of (ii) is similar. ☐

Lemma 2.2.5. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, $\mathbf{X} = \mathrm{D}^{\flat}(\mathbf{L})$ *and* $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. *For* $f, g \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$

(i) *if* $f \leqslant_2 g$ *and* $\varphi(f) = 0$, *then* $\varphi(g) = 0$*;*

(ii) *if* $f \leqslant_1 g$ *and* $\varphi(f) = 1$, *then* $\varphi(g) = 1$.

Proof. Let $f \leqslant_2 g$ and $\varphi(f) = 0$ and suppose that $g \notin \varphi^{-1}(0)$. Then by Lemma 2.2.4(i) there exists $h \in \varphi^{-1}(1)$ such that $g \leqslant_2 h$. By the transitivity of $\leqslant_2$, we then see that $f \leqslant_2 h$ and so $(h, f) \in E$. Now $\varphi(h) = 1 \nleqslant 0 = \varphi(f)$, contradicting the fact that φ is E-preserving. Thus $\varphi(g) = 0$. ☐

We can now show that $\mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}})$, as an extension of $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, satisfies the density condition. That is, every element of $\mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}})$ can be written as both a join of meets and a meet of joins of elements of $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$.

Proposition 2.2.6. (**Density**) *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $\mathbf{X} = \mathrm{D}^{\flat}(\mathbf{L})$. *Then every element* $\varphi \in \mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}})$ *can be expressed as*

$$\varphi = \bigvee \Big\{ \bigwedge F \mid \bigwedge F \leqslant \varphi \Big\} = \bigwedge \Big\{ \bigvee I \mid \varphi \leqslant \bigvee I \Big\}$$

where F *ranges over the filters of* $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ *and* I *over the ideals.*

Proof. If $\psi = \bigvee \{ \bigwedge F \mid \bigwedge F \leqslant \varphi \}$, it is clear that $\psi \leqslant \varphi$. In order to show that $\varphi \leqslant \psi$, we must check that $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$. Therefore, using the definitions of joins and meets in $\mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}})$ in Lemma 2.1.2, and given $x \in \varphi^{-1}(1)$, we must show that if $y \in \bigcap \{ (\bigwedge F)^{-1}(0) \mid \bigwedge F \leqslant \varphi \}$, then $(x, y) \notin E$.

To do this, we claim that $\bigcap\{\,(\bigwedge F)^{-1}(0) \mid \bigwedge F \leqslant \varphi\,\} \subseteq \varphi^{-1}(0)$. Suppose that $f \notin \varphi^{-1}(0)$. By Lemma 2.2.4(i) there exists $g \in \varphi^{-1}(1)$ such that $f \leqslant_2 g$. Now let $\mathcal{F}_g := \{\, a \in L \mid g(a) = 1\,\}$ be a filter of $\mathbf{L}$. From Proposition 2.2.1(iii), we know that $F \subseteq \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ is a filter if and only if $F = \{\, e_b \mid b \in \mathcal{F}\,\}$ where $\mathcal{F}$ is a filter of $\mathbf{L}$. This gives us a filter G on $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, with $G = \{\, e_a \mid a \in \mathcal{F}_g\,\}$, and we will show that $\bigwedge G \leqslant \varphi$. If $h \in (\bigwedge G)^{-1}(1)$, then $h \in \bigcap\{\, e_a^{-1}(1) \mid a \in \mathcal{F}_g\,\}$. That is, for every $a \in L$ such that $g(a) = 1$, $h(a) = 1$. So $g^{-1}(1) \subseteq h^{-1}(1)$, giving us that $g \leqslant_1 h$. Now we use Lemma 2.2.5(ii) to get $\varphi(h) = 1$ and hence $(\bigwedge G)^{-1}(1) \subseteq \varphi^{-1}(1)$.

We have $\bigwedge G \leqslant \varphi$ and now show that $f \notin (\bigwedge G)^{-1}(0)$. By definition

$$\big(\bigwedge G\big)^{-1}(0) = \Big\{\, b \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \;\Big|\; \text{for all } c \in \bigcap\{\, e_a^{-1}(1) \mid a \in \mathcal{F}_g\,\}, (c, b) \notin E \,\Big\}$$

and clearly $g \in \bigcap\{\, e_a^{-1}(1) \mid a \in \mathcal{F}_g\,\}$. Now $f \leqslant_2 g$ gives us that $(g, f) \in E$ and so $f \notin (\bigwedge G)^{-1}(0)$. Clearly now $f \notin \bigcap\{\,(\bigwedge F)^{-1}(0) \mid \bigwedge F \leqslant \varphi\,\}$ and thus we have $\bigcap\{\,(\bigwedge F)^{-1}(0) \mid \bigwedge F \leqslant \varphi\,\} \subseteq \varphi^{-1}(0)$.

Now recall that $x \in \varphi^{-1}(1)$ and let $y \in \bigcap\{\,(\bigwedge F)^{-1}(0) \mid \bigwedge F \leqslant \varphi\,\}$. Then $(x, y) \in E$ would imply $\varphi(x) = 1 \nleqslant 0 = \varphi(y)$, contradicting φ being E-preserving. Thus $(x, y) \notin E$ and we have $x \in \psi^{-1}(1)$ as required. □

In Proposition 2.2.6 we proved very directly that $\mathrm{C}(\mathrm{D}^\flat(\mathbf{L}))$ supplies a dense completion of $\mathbf{L}$. We now want to prove that this completion is compact. En route we obtain characterisations of filter and ideal elements in $\mathrm{C}(\mathrm{D}^\flat(\mathbf{L}))$ which are of independent interest.

In order to describe the filter and ideal elements of the extension $\mathrm{C}(\mathrm{D}^\flat(\mathbf{L}))$ we need to look more closely at the topology on $\mathrm{D}(\mathbf{L}) = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathcal{T})$. We recall that $\mathcal{T}$ has subbasic closed sets of the form $V_a = \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0\,\}$ and $W_a = \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1\,\}$ for $a \in L$. We observe that from Lemma 2.2.5, for any $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathrm{D}^\flat(\mathbf{L}), \underset{\sim}{\mathbf{2}})$, we have that $\varphi^{-1}(1)$ is a $\leqslant_1$-increasing set, and $\varphi^{-1}(0)$ is a $\leqslant_2$-increasing set. We also note that the intersection of a collection of closed sets of the form W_b for $b \in L$ will be $\leqslant_1$-increasing, while the intersection of sets of the form V_a with $a \in L$ will be $\leqslant_2$-increasing.

The following result parallels [78, Lemma 1.4] though our approach in proving it is different.

Lemma 2.2.7. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $a \in L$. *Let* $\mathbf{X} = \mathrm{D}^\flat(\mathbf{L})$ *and let* $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathrm{D}^\flat(\mathbf{L}), \underset{\sim}{\mathbf{2}})$. *Then*

(i) $\varphi^{-1}(0) \cap W_a = \emptyset$ *implies* $\varphi^{-1}(0) \subseteq V_a$;

(ii) *$\varphi^{-1}(1) \cap V_a = \emptyset$ implies $\varphi^{-1}(1) \subseteq W_a$.*

Proof. We consider (i). Let $\varphi^{-1}(0) \cap W_a = \emptyset$. Suppose that $f \in \varphi^{-1}(0)$ and $f \notin V_a$. As $V_a = e_a^{-1}(0)$, by Lemma 2.2.4(i) there exists $g \in \varphi^{-1}(1)$ such that $f \leqslant_2 g$. Since $\varphi(f) = 0$, by Lemma 2.2.5(i) we obtain $\varphi(g) = 0$, a contradiction. $\square$

Lemma 2.2.8. *Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, let $\mathbf{X}_{\mathcal{T}} = \mathrm{D}(\mathbf{L})$ and let $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}})$. Then*

(i) *$\varphi^{-1}(1)$ is a $\mathcal{T}$-closed subset of $\mathbf{X}_{\mathcal{T}}$ if and only if $\varphi^{-1}(1) = \bigcap \{ W_b \mid b \in K \}$, for some $K \subseteq L$;*

(ii) *$\varphi^{-1}(0)$ is a $\mathcal{T}$-closed subset of $\mathbf{X}_{\mathcal{T}}$ if and only if $\varphi^{-1}(0) = \bigcap \{ V_a \mid a \in M \}$ for some $M \subseteq L$.*

Proof. We consider (i). The sufficiency is clear as each W_b is $\mathcal{T}$-closed. For the necessity, let $\varphi^{-1}(1)$ be a $\mathcal{T}$-closed subset of $\mathbf{X}_{\mathcal{T}}$. We let X denote the underlying set of $\mathbf{X}_{\mathcal{T}}$, that is, $X = \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$. Let

$$\boldsymbol{\mathcal{S}} := \{ W_b \mid b \in L,\ \varphi^{-1}(1) \subseteq W_b \}$$

be the family of all sets W_b containing $\varphi^{-1}(1)$ and let $K := \{ b \in L \mid W_b \in \boldsymbol{\mathcal{S}} \}$. Since $\varphi^{-1}(1) \subseteq W_b$ for each $b \in K$, it is obvious that $\varphi^{-1}(1) \subseteq \bigcap \{ W_b \mid b \in K \}$. The reverse inclusion will hold too provided we can show that $f \notin \varphi^{-1}(1)$ implies the existence of $W_b \in \boldsymbol{\mathcal{S}}$ such that $f \notin W_b$.

Let $f \notin \varphi^{-1}(1)$. By Lemma 2.1.1(ii), there exists $g \in \varphi^{-1}(0)$ such that $(f, g) \in E$. As $g \in \varphi^{-1}(0)$, for every $h \in \varphi^{-1}(1)$ we necessarily have $(h, g) \notin E$. By definition of the relation E this means that there exists $b_h \in L$ such that $h(b_h) = 1$ and $g(b_h) = 0$. We note that each set $X \setminus V_{b_h}$ is $\mathcal{T}$-open and $h \in X \setminus V_{b_h}$. Hence we have that $\varphi^{-1}(1) \subseteq \bigcup \{ X \setminus V_{b_h} \mid h \in \varphi^{-1}(1) \}$. Since the space $\mathbf{X}_{\mathcal{T}}$ is compact and $\varphi^{-1}(1)$ is a $\mathcal{T}$-closed subset of $\mathbf{X}_{\mathcal{T}}$ by hypothesis, there are $b_1, \dots, b_n \in L$ such that $g(b_1) = \dots = g(b_n) = 0$ and $\varphi^{-1}(1) \subseteq (X \setminus V_{b_1}) \cup \dots \cup (X \setminus V_{b_n})$. Let $b := b_1 \vee \dots \vee b_n \in L$. We have $g(b) = 0$ and $\varphi^{-1}(1) \subseteq X \setminus V_b$. The latter by Lemma 2.2.7 yields $\varphi^{-1}(1) \subseteq W_b$, and thus $W_b \in \boldsymbol{\mathcal{S}}$. Since $(f, g) \in E$ and $g(b) = 0$, we cannot have $f(b) = 1$. Hence $f \notin W_b$, as required. $\square$

Using the above lemma, we can now provide a characterisation of the filter and ideal elements of the canonical extension.

Proposition 2.2.9. **(Filter and ideal elements)** *Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, let $\mathbf{X}_{\mathcal{T}} = \mathrm{D}(\mathbf{L})$ and let $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}}) = \mathrm{C}(\mathbf{X})$. Then the three conditions in* (i) *are equivalent and the three conditions in* (ii) *are equivalent:*

(i) (1) *φ is a filter element of* $\mathrm{C}(\mathbf{X})$*;*

(2) $\varphi^{-1}(1) = \bigcap\{\, W_b \mid b \in K \,\}$*, for some* $K \subseteq L$*;*

(3) $\varphi^{-1}(1)$ *is a* $\mathcal{T}$*-closed subset of* $\mathbf{X}_{\mathcal{T}}$*.*

(ii) (1) *φ is an ideal element of* $\mathrm{C}(\mathbf{X})$*;*

(2) $\varphi^{-1}(0) = \bigcap\{\, V_a \mid a \in M \,\}$*, for some* $M \subseteq L$*;*

(3) $\varphi^{-1}(0)$ *is a* $\mathcal{T}$*-closed subset of* $\mathbf{X}_{\mathcal{T}}$*.*

Proof. The equivalences of (2) and (3) in (i) and (ii) come from Lemma 2.2.8. Now let (1) in (i) hold, that is, let φ be a filter element of $\mathrm{C}(\mathbf{X})$. Then $\varphi = \bigwedge \psi_i$ where $\{\, \psi_i \mid i \in I \,\} \subseteq \mathfrak{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ for some index set I. From the representation theorem (see Proposition 2.2.1), we know that each ψ_i is in fact e_{b_i} for some $b_i \in L$ and from the definition of a meet of elements of $\mathfrak{G}^{\mathrm{mp}}(\mathrm{D}^\flat(\mathbf{L}), \underset{\sim}{\mathbf{2}})$, we have that

$$\varphi^{-1}(1) = \bigcap_{i\in I} \psi_i^{-1}(1) = \bigcap_{i\in I} e_{b_i}^{-1}(1) = \bigcap_{i\in I} W_{b_i}.$$

Hence (1) implies (2). Now let (2) hold. Then $\varphi^{-1}(1) = \bigcap_{i\in I} e_{b_i}^{-1}(1)$, and thus $\varphi^{-1}(1)$ can be expressed as the meet of images of lattice elements in $\mathrm{C}(\mathbf{X})$, yielding (1). The equivalence of (1) and (2) in (ii) can be shown analogously. □

We now prove that our completion is indeed a compact completion. We note that we will make use of the equivalent definition of compactness stated in Definition 1.1.2(i) as this allows for the simplest possible proof.

Proposition 2.2.10. (**Compactness**) *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $\mathbf{X}_{\mathcal{T}} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underset{\sim}{\mathbf{2}}), E, \mathcal{T})$*. The lattice* $\mathrm{C}(\mathbf{X}) = \mathfrak{G}^{\mathrm{mp}}(\mathrm{D}^\flat(\mathbf{L}), \underset{\sim}{\mathbf{2}})$ *is a compact completion of* $\mathbf{L}$*.*

Proof. Let $A, B \subseteq L$ such that $\bigwedge\{\, e_a \mid a \in A \,\} \leqslant \bigvee\{\, e_b \mid b \in B \,\}$ in the completion $\mathrm{C}(\mathbf{X})$. Let $\varphi = \bigwedge\{\, e_a \mid a \in A \,\}$ and $\psi = \bigvee\{\, e_b \mid b \in B \,\}$. We have

$$\varphi^{-1}(1) = \bigcap_{a\in A} e_a^{-1}(1) = \bigcap_{a\in A} W_a \quad \text{and} \quad \psi^{-1}(0) = \bigcap_{b\in B} e_b^{-1}(0) = \bigcap_{b\in B} V_b.$$

Now $\varphi \leqslant \psi$ in $\mathrm{C}(\mathbf{X})$ means $\varphi^{-1}(1) \subseteq \psi^{-1}(1) \subseteq X \setminus \psi^{-1}(0)$, and hence

$$\bigcap_{a\in A} W_a \subseteq X \setminus \bigcap_{b\in B} V_b = \bigcup_{b\in B} (X \setminus V_b).$$

Since each W_a is $\mathcal{T}$-closed and each $X \setminus V_b$ is $\mathcal{T}$-open, the topological compactness of $\mathbf{X}_{\mathcal{T}}$ yields the existence of finite subsets $A' := \{a_1, \dots, a_r\} \subseteq A$ and $B' := \{b_1, \dots, b_s\} \subseteq B$ such that

$$W_{a_1} \cap \dots \cap W_{a_r} \subseteq (X \setminus V_{b_1}) \cup \dots \cup (X \setminus V_{b_s}).$$

We then let $a' := a_1 \wedge \cdots \wedge a_r$ and $b' := b_1 \vee \cdots \vee b_s$ and observe that

$$e_{a'}^{-1}(1) \;=\; W_{a'} \;=\; \bigcap_{a \in A'} W_a \;\subseteq\; \bigcup_{b \in B'} (X \setminus V_b) \;=\; X \setminus V_{b'} \;=\; X \setminus e_{b'}^{-1}(0).$$

By Lemma 2.2.7(ii) we get $e_{a'}^{-1}(1) \subseteq e_{b'}^{-1}(1)$, and then using the fact that $a \mapsto e_a$ is an order isomorphism we get $\bigwedge A' = a' \leqslant b' = \bigvee B'$. $\square$

The principal result of this section is now an immediate consequence of Propositions 2.2.6 and 2.2.10.

Theorem 2.2.11. (**Canonical extension**) *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *The lattice* $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *ordered by*

$$\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

is the canonical extension of $\mathbf{L}$.

We summarise in Fig. 2.2 what we have achieved. Here

$$\begin{aligned}
\mathrm{D}\colon \mathbf{L} &\longmapsto \mathbf{X}_{\mathcal{T}} := (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathcal{T}),\\
{}^{\flat} : \mathbf{X}_{\mathcal{T}} &\longmapsto \mathbf{X} := (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E),\\
\mathrm{G}\colon \mathbf{X} &\longmapsto \mathrm{C}(\mathbf{X}) := \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}}),
\end{aligned}$$

and $\mathbf{L} \hookrightarrow \mathbf{L}^{\delta}$ via $a \mapsto e_a$. The fact that the diagram commutes is the content of Theorem 2.2.11.

$$\begin{tikzcd}
\boldsymbol{\mathcal{L}}^{+} & \boldsymbol{\mathcal{G}} \arrow[l, "\mathrm{G}"'] \\
\boldsymbol{\mathcal{L}} \arrow[u, "\delta"] \arrow[r, "\mathrm{D}"] & \boldsymbol{\mathcal{G}}_{\mathcal{T}} \arrow[u, "\flat"']
\end{tikzcd}$$

Figure 2.2: Factorisation of ${}^{\delta}$ on objects in $\boldsymbol{\mathcal{L}}$

2.3 A categorical framework for the canonical extension

The canonical extension construction on $\boldsymbol{\mathcal{L}}$ is functorial; see [48, 56]. That is, homomorphisms between lattices are lifted to complete lattice homomorphisms between their canonical extensions. As we observed in the introduction to this chapter, in the distributive case there is a factorisation of the canonical extension functor in terms of functors obtained from the hom-functors setting up Priestley duality. In this section we seek an analogous result for $\boldsymbol{\mathcal{L}}$, insofar as this is possible. Specifically we seek to set up a commutative diagram as shown in Fig. 2.3, paralleling that shown for $\boldsymbol{\mathcal{D}}$ in Fig. 2.1.

$$\begin{tikzcd}
\boldsymbol{\mathcal{L}}^+ & \boldsymbol{\mathcal{Y}} \arrow[l, "\overline{\mathrm{G}}"] \\
\boldsymbol{\mathcal{L}} \arrow[u, "\delta"] \arrow[r, shift left, "\overline{\mathrm{D}}"] & \boldsymbol{\mathcal{Y}}_{\mathcal{T}} \arrow[u, "\flat"'] \arrow[l, shift left, "\overline{\mathrm{E}}"]
\end{tikzcd}$$

Figure 2.3: Seeking to factorise the functor ${}^{\delta}$ on $\boldsymbol{\mathcal{L}}$

In Fig. 2.3 we require:

- $\boldsymbol{\mathcal{L}}^+$ to be the category of complete lattices with complete lattice homomorphisms,
- $\boldsymbol{\mathcal{Y}}_{\mathcal{T}}$ to be a category of graphs with topology and $\boldsymbol{\mathcal{Y}}$ the corresponding category of graphs, obtained by forgetting the topology;
- $\overline{\mathrm{D}}$ and $\overline{\mathrm{E}}$ to set up a dual adjunction such that, for all $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, we have $\mathbf{L} \cong \overline{\mathrm{E}}\,\overline{\mathrm{D}}(\mathbf{L})$ with the isomorphism given by a natural evaluation map,
- $\overline{\mathrm{D}}$, $\overline{\mathrm{E}}$ and $\overline{\mathrm{G}}$ to be contravariant functors, sending an object C into some appropriately specified set of partial maps into, respectively, $\underline{\mathbf{2}}$, $\underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ and $\underset{\sim}{\mathbf{2}}$, and
- each of $\overline{\mathrm{D}}$, $\overline{\mathrm{E}}$ and $\overline{\mathrm{G}}$ to act on morphisms by composition.

Thinking in terms of maps between objects, rather than functors, we have seen (see Fig. 2.2) that we can obtain a diagram of this kind, based on the Ploščica representation. However, we encounter problems when trying to add morphisms to Ploščica's setup. For $u \in \boldsymbol{\mathcal{L}}(\mathbf{L}, \mathbf{K})$, the morphism $\mathrm{D}^{\flat}(u)\colon \mathrm{D}^{\flat}(\mathbf{K}) \to \mathrm{D}^{\flat}(\mathbf{L})$ would need to be given by $\mathrm{D}^{\flat}(u)(f) := f \circ u$, for all $f \in \mathrm{D}^{\flat}(\mathbf{K})$, where the domain is determined by the set of elements for which the composition is defined. Certainly, for f a maximal partial homomorphism, $f \circ u$ is a partial homomorphism. But, as the example in Fig. 2.4 demonstrates, $f \circ u$ need not be maximal.

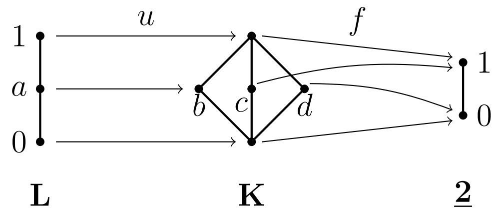


Figure 2.4: Failure of morphisms to act on MPH's by composition

We have already noted that Ploščica's representation is a recasting of Urquhart's topological representation for bounded lattices. Urquhart was able to set up a dual representation for surjective $\boldsymbol{\mathcal{L}}$-morphisms but not for $\boldsymbol{\mathcal{L}}$-morphisms in general. Therefore our example should come as no surprise. Moreover, it is very easy to see that $f \circ u$ is maximal if u is a surjective morphism and f an MPH.

A solution to the problem of making duality for lattices functorial was given by Hartung [67]. At the object level, topological contexts are considered, while the dual of a lattice homomorphism is a pair of so-called *multivalued functions*. This, however, will not suit what we are trying to achieve. Our approach is quite different in that instead of altering the morphisms on the dual side, we will use an enlarged object representation which will allow the duals of lattice homomorphisms to be considered.

We shall call on the topological representation of bounded lattices due to Allwein and Hartonas [1, 2]. This is in the same style as that of Urquhart, in that it makes use of disjoint filter-ideal pairs of $\mathbf{L}$ to construct a dual space for $\mathbf{L}$. However such pairs are not required to be maximal, as they are in Urquhart's representation. Allwein and Hartonas thereby overcome the problem encountered by Urquhart and are able to set up a dual equivalence between $\boldsymbol{\mathcal{L}}$ and a specified category of topological structures. This is achieved at the cost of working with a greatly enlarged dual space.

As an example, consider the five-element modular lattice $\underline{\mathbf{M}}_3$. Under Urquhart's representation, the dual space has six elements, whereas in the duality of Allwein and Hartonas the dual space has 13 elements. For a finite lattice, moving to a dual space which is bigger than the lattice itself constitutes a major sacrifice, and much of the appeal and power of duality for distributive lattices is lost. This could well explain why the work of Allwein and Hartonas is not particularly well-known. However, once the Allwein–Hartonas duality has been translated into the language of partial maps, we shall see that it offers exactly the setup that will help us to link the construction of the canonical extension to duality using functors, as in Fig 2.3.

We shall now outline what we have to do to carry out our programme.

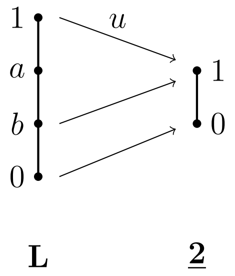


Figure 2.5: A partial homomorphism which is not special

First of all, we require a Ploščica-style presentation of the Allwein–Hartonas dual spaces. The first task is to recognise which partial homomorphisms from $\mathbf{L}$ into $\underline{\mathbf{2}}$ correspond to disjoint filter-ideal pairs and appropriately to equip the resulting set of partial homomorphisms with the structure of a graph with topology. We may consider the full subcategory $\mathcal{Y}_{\mathcal{T}}$ of $\mathcal{G}_{\mathcal{T}}$ whose objects are the enlarged dual spaces (in our graph formulation, given below) of members of $\boldsymbol{\mathcal{L}}$. We then let $\mathcal{Y}$ be the image of $\mathcal{Y}_{\mathcal{T}}$ under $^\flat$. As these graphs occur as the dual objects of lattices, we shall refer to the objects of $\mathcal{Y}$ as $\boldsymbol{\mathcal{L}}$*-graphs*.

The results of Allwein and Hartonas [1, 2] imply that there are contravariant functors $\overline{\mathrm{D}}\colon \boldsymbol{\mathcal{L}} \to \mathcal{Y}_{\mathcal{T}}$ and $\overline{\mathrm{E}}\colon \mathcal{Y}_{\mathcal{T}} \to \boldsymbol{\mathcal{L}}$ which are such that $\mathbf{L} \cong \overline{\mathrm{E}}\,\overline{\mathrm{D}}(\mathbf{L})$ for every $\mathbf{L} \in \boldsymbol{\mathcal{L}}$; we show that the isomorphism is set up by evaluation maps in the same manner as in Proposition 2.2.1 (see Propositions 2.3.3 and 2.3.4). There is now an obvious candidate for the canonical extension of $\mathbf{L}$: we forget the topology on $\overline{\mathrm{D}}(\mathbf{L})$ and take the set of all maximal partial morphisms into $\underset{\sim}{\mathbf{2}}$. Rather than verifying directly that we thereby obtain a dense and compact completion we instead set up an order-isomorphism between our new candidate for the canonical extension and the one based on the Ploščica representation. Finally we check out that we really do have a well-defined functor $\overline{\mathrm{G}}$ making the diagram in Fig. 2.3 commute, and at the same time use our diagram to confirm, in a transparent way, that $^\delta$ does lift each $\boldsymbol{\mathcal{L}}$-morphism to a complete lattice homomorphism. We now implement the strategy we have set out.

Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. Then the disjoint filter-ideal pairs of $\mathbf{L}$ are obviously in bijective correspondence with those partial homomorphisms f into $\underline{\mathbf{2}}$, for which $(f^{-1}(1), f^{-1}(0))$ is a (disjoint) filter-ideal pair of $\mathbf{L}$; let us call them *special partial homomorphisms*, SPH's for short, and denote the set of all such maps by $\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. We note that every maximal partial homomorphism into $\underline{\mathbf{2}}$ is special but that it is easy to find examples of partial homomorphisms into $\underline{\mathbf{2}}$ which fail to be special.

The presentation by Allwein and Hartonas now permits a translation into an equivalent formulation in terms of special partial homomorphisms (for the first dual) and continuous partial morphisms (for the second dual). The topology in [2] is defined in the same way as for the representation theorem of Ploščica. That is, let sets of the form

$$V_a = \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0 \,\} \quad \text{and} \quad W_a = \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1 \,\},$$

form a subbasis for the closed sets of the topology $\mathcal{T}$. We now define the relation E on $\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ by the rule

$$(f, g) \in E \text{ iff } f(x) \leqslant g(x) \text{ for every } x \in \mathrm{dom}(f) \cap \mathrm{dom}(g),$$

that is, we take the obvious extension of E, as defined earlier on the subset $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$. We denote the dual space of $\mathbf{L}$ by $\overline{\mathrm{D}}(\mathbf{L}) = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathcal{T})$. The dual space specified in [2], called an *enhanced L-space*, takes the form of a topological space equipped with two quasi-orders ($\leqslant_1$ and $\leqslant_2$) and a specified subbasis of closed sets.

We shall need the following two lemmas concerning SPH's. The first of these relates the relation E to quasi-orders $\leqslant_1$ and $\leqslant_2$ (defined on $\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ exactly as on $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$). For the case of $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ it appears as Lemma 2.2.3, which we carried over from [78]. Here we show that the result extends to all SPH's.

Lemma 2.3.1. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $f, g \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. *Then*

(i) $(f, g) \in E$ *if and only if there exists* $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ *with* $f \leqslant_1 h$ *and* $g \leqslant_2 h$*;*

(ii) $f \leqslant_2 g$ *if and only if there is no* $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ *with* $(h, g) \in E$ *and* $(h, f) \notin E$*;*

(iii) $f \leqslant_1 g$ *if and only if there is no* $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ *with* $(g, h) \in E$ *and* $(f, h) \notin E$.

Proof. (i) If $(f, g) \in E$ then $f^{-1}(1) \cap g^{-1}(0) = \emptyset$. Consider $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ defined by $h^{-1}(1) = f^{-1}(1)$ and $h^{-1}(0) = g^{-1}(0)$. Then $f \leqslant_1 h$ and $g \leqslant_2 h$. Conversely, if there exists $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ with $f^{-1}(1) \subseteq h^{-1}(1)$ and $g^{-1}(0) \subseteq h^{-1}(0)$ then clearly $f^{-1}(1) \cap g^{-1}(0) = \emptyset$ and so $(f, g) \in E$.

(ii) Suppose that $f \leqslant_2 g$ and let $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. Now suppose that $(h, f) \notin E$. This implies that there exists $a \in L$ such that $h(a) = 1$ and $f(a) = 0$. Clearly $g(a) = 0$ and so $(h, g) \notin E$.

Next, assume that $f \nleqslant_2 g$. This implies that there exists $a \in L$ such that $f(a) = 0$ and $g(a) \neq 0$. Consider $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ defined by $h^{-1}(1) = {\uparrow}a$ and $h^{-1}(0) = g^{-1}(0)$. Clearly $(h, g) \in E$ and $(h, f) \notin E$, completing the proof of (ii). Part (iii) is proved in the same way. □

We now derive an extension of Lemmas 2.2.5 and 2.2.7 from $\mathrm{D}^\flat(\mathbf{L})$ to $\overline{\mathrm{D}}^\flat(\mathbf{L})$:

Lemma 2.3.2. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. *Let* $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and let* $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ *and* $f, g \in Y$. *Then*

(i) *if* $f \leqslant_2 g$ *and* $\varphi(f) = 0$, *then* $\varphi(g) = 0$;

(ii) *if* $f \leqslant_1 g$ *and* $\varphi(f) = 1$, *then* $\varphi(g) = 1$.

Further, for $a \in L$, *we have*

(iii) $\varphi^{-1}(0) \cap W_a = \emptyset$ *implies* $\varphi^{-1}(0) \subseteq V_a$;

(iv) $\varphi^{-1}(1) \cap V_a = \emptyset$ *implies* $\varphi^{-1}(1) \subseteq W_a$.

Proof. Let $f \leqslant_2 g$ and $\varphi(f) = 0$. Then by Lemma 2.1.1, for all $h \in \varphi^{-1}(1)$ we have $(h, f) \notin E_Y$, that is, $h^{-1}(1) \cap f^{-1}(0) \neq \emptyset$. From $f \leqslant_2 g$ we have $f^{-1}(0) \subseteq g^{-1}(0)$, and then clearly for all $h \in \varphi^{-1}(1)$ we have $h^{-1}(1) \cap g^{-1}(0) \neq \emptyset$, whence $(h, g) \notin E_Y$. Thus $g \in \varphi^{-1}(0)$, again by Lemma 2.1.1.

Now consider part (iii). Let $\varphi^{-1}(0) \cap W_a = \emptyset$ and suppose that $f \notin V_a$ but $f \in \varphi^{-1}(0)$. We will show that this implies $\varphi^{-1}(0) \cap W_a \neq \emptyset$. Since $f(a) \neq 0$, consider the special partial homomorphism $g_f \colon \mathbf{L} \to \underline{\mathbf{2}}$ defined by $g_f^{-1}(1) = {\uparrow} a$ and $g_f^{-1}(0) = f^{-1}(0)$. Now $g_f \in W_a$, but since $f \leqslant_2 g_f$ we have by (i) that $g_f \in \varphi^{-1}(0)$. □

We are now ready to confirm that a lattice $\mathbf{L}$ can be recaptured from the graph with topology $\overline{\mathrm{D}}(\mathbf{L})$ in the same manner as, by Ploščica's representation, it is recaptured from $\mathrm{D}(\mathbf{L})$. We first reformulate the representation due to Allwein and Hartonas [1, 2] in terms of graphs with topology in the same way that Ploščica reformulated the L-spaces of Urquhart. We want to represent the lattice $\mathbf{L}$ by the maximal partial continuous E-preserving maps (MPM's) from $\overline{\mathrm{D}}(\mathbf{L})$ into $\underset{\sim}{\mathbf{2}}_{\mathcal{T}}$.

Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and let $\mathbf{Y}_{\mathcal{T}} = \overline{\mathrm{D}}(\mathbf{L})$ and $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. Let $a \in L$ and define $\bar{e}_a \colon \mathbf{Y}_{\mathcal{T}} \to \underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ for $f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ by

$$\bar{e}_a(f) = \begin{cases} f(a) & \text{if } a \in \mathrm{dom}(f), \\ - & \text{otherwise.} \end{cases}$$

Proposition 2.3.3. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and define* $\mathbf{Y}_{\mathcal{T}} = \overline{\mathrm{D}}(\mathbf{L})$, $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, *and* $\bar{e}_a$ *as above. Then* $\bar{e}_a \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{Y}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$. *Furthermore, if* $\bar{e}_a$ *is regarded as a map from* $\mathbf{Y}$ *to* $\underset{\sim}{\mathbf{2}}$, *then* $\bar{e}_a \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$.

Proof. We note that $\bar{e}_a^{-1}(1) = W_a$ and $\bar{e}_a^{-1}(0) = V_a$ are closed sets and hence, since the set $\mathrm{dom}(\bar{e}_a) = W_a \cup V_a$ is also closed, we have that $\bar{e}_a \colon \mathrm{dom}(\bar{e}_a) \to \underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ is continuous. Suppose $\bar{e}_a(f) = 1$ and $\bar{e}_a(g) = 0$. Then $(f, g) \notin E$ and so $\bar{e}_a$ is E-preserving. Now assume that $\phi \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{Y}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ and the domain of $\phi \colon \mathbf{Y}_{\mathcal{T}} \to \underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ properly extends the domain of $\bar{e}_a$. The first case to consider is $f \in \mathrm{dom}(\phi) \setminus \mathrm{dom}(\bar{e}_a)$ such that $\phi(f) = 1$. This implies that for all $g \in \bar{e}_a^{-1}(0)$, $(f, g) \notin E$. In particular it means that for $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underset{\sim}{\mathbf{2}})$ with $h^{-1}(1) = \{1\}$ and $h^{-1}(0) = {\downarrow}a$ we have $(f, h) \notin E$. Since $f^{-1}(1)$ is a filter, this can only happen if $f(a) = 1$ and so $f \in \mathrm{dom}(\bar{e}_a)$, a contradiction. Similarly one can show that if $\phi(f) = 0$ then $f(a) = 0$, a contradiction. Thus the domain of $\bar{e}_a$ is maximal.

The fact that $\bar{e}_a \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ follows from Proposition 2.1.5. □

The proof of part (i) of the following result mimics that of [78, Lemma 1.5]. We shall exploit the compactness of $\overline{\mathrm{D}}(\mathbf{L})$ [2, Lemma 3.17].

Proposition 2.3.4. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $\mathbf{Y}_{\mathcal{T}} = \overline{\mathrm{D}}(\mathbf{L})$.

(i) *Every* $\varphi \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{Y}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ *is of the form* $\bar{e}_a$ *for some* $a \in L$.

(ii) $\mathbf{L}$ *is order-isomorphic to* $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{Y}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ *via the map* $a \mapsto \bar{e}_a$.

Proof. Since φ is E-preserving, for any $f \in \varphi^{-1}(1)$ and $g \in \varphi^{-1}(0)$ we get that $(f, g) \notin E$. Thus there must exist an element $a_{fg} \in L$ such that $f(a_{fg}) = 1$ and $g(a_{fg}) = 0$. We form the set $U_{fg} = \{\, h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid h(a_{fg}) \neq 0 \,\}$, which is open since $U_{fg} = \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \backslash V_{a_{fg}}$. The collection $\{\, U_{fg} \mid f \in \varphi^{-1}(1) \,\}$ is a cover of $\varphi^{-1}(1)$ since $f \in U_{fg}$ for all $f \in \varphi^{-1}(1)$. Now $\varphi^{-1}(1)$ is compact as it is a closed subset of the compact space $\overline{\mathrm{D}}(\mathbf{L})$. Thus there is a finite set $\{\, a_i \mid 1 \leqslant i \leqslant n \,\} \subseteq g^{-1}(0)$ such that $\varphi^{-1}(1) \subseteq \{\, \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \backslash V_{a_i} \mid 1 \leqslant i \leqslant n \,\}$. We can now set $a_g = \bigvee \{\, a_i \mid 1 \leqslant i \leqslant n \,\}$. Since $g \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$, $g^{-1}(0)$ is an ideal and so $g(a_g) = 0$. Furthermore, we have that $\varphi^{-1}(1) \subseteq \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \backslash V_{a_g}$. From Proposition 2.1.5 we have that $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ and so using Lemma 2.3.2(iv) we get $\varphi^{-1}(1) \subseteq W_{a_g}$.

Now $\varphi^{-1}(0)$ is covered by open sets of the form $\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \backslash W_{a_g}$ for $g \in \varphi^{-1}(0)$. Since $\varphi^{-1}(0)$ is also compact, we can take a finite set

$$\{\, a^j \mid 1 \leqslant j \leqslant m \,\} \subseteq \{\, a_g \mid g \in \varphi^{-1}(0) \,\}$$

so that $\varphi^{-1}(0) \subseteq \bigcup \{\, \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \backslash W_{a^j} \mid 1 \leqslant j \leqslant m \,\}$. From above we can also see that $\varphi^{-1}(1) \subseteq \bigcap \{\, W_{a^j} \mid 1 \leqslant j \leqslant m \,\}$. If we set $a = \bigwedge \{\, a^j \mid 1 \leqslant j \leqslant m \,\}$ then $\varphi^{-1}(1) \subseteq W_a$ and $\varphi^{-1}(0) \subseteq \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \backslash W_a$. Now Lemma 2.3.2(iii) gives us that $\varphi^{-1}(0) \subseteq V_a$ and by the maximality of φ we now get $\varphi^{-1}(1) = W_a$ and $\varphi^{-1}(0) = V_a$ and so $\varphi = \bar{e}_a$.

From Proposition 2.3.3 and part (i) we have $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{Y}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}}) = \{\, \bar{e}_a \mid a \in L \,\}$. Now the map $a \mapsto \bar{e}_a$ is an order-isomorphism of $\mathbf{L}$ to $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{Y}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ since $a \leqslant b$ if and only if $\bar{e}_a^{-1}(1) \subseteq \bar{e}_b^{-1}(1)$. To see this, let $a \leqslant b$ and $f \in \bar{e}_a^{-1}(1)$ for $f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. Thus $f(a) = 1$ and since $f^{-1}(1)$ is a filter, we obtain $f(b) = 1$, whence $f \in \bar{e}_b^{-1}(1)$ as required. For the converse, let $a \nleqslant b$ in $\mathbf{L}$. Then $({\uparrow}a, {\downarrow}b)$ is a disjoint filter-ideal pair with an associated special partial homomorphism $f : \mathbf{L} \to \underline{\mathbf{2}}$ with $f^{-1}(1) = {\uparrow}a$ and $f^{-1}(0) = {\downarrow}b$. Then $f \in \bar{e}_a^{-1}(1)$ but $f \notin \bar{e}_b^{-1}(1)$. $\square$

We are finally set up to build a canonical extension of a bounded lattice $\mathbf{L}$, analogous to the one in Theorem 2.2.11 but now based on the larger dual space $\overline{\mathrm{D}}(\mathbf{L})$. We let $Y = \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ and shall show that the complete lattice of maximal partial E-preserving maps from the larger graph $\mathbf{Y} = (Y, E_Y)$ into $\underset{\sim}{\mathbf{2}}$ is order-isomorphic to the canonical extension from Theorem 2.2.11. To accomplish this we recall that $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$ and we define a map $\Psi \colon \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \to \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ such that for $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ and $f \in Y$,

$$(*) \qquad (\Psi(\varphi))(f) = \begin{cases} 1 & \text{if } \forall\, g \in \varphi^{-1}(0)\big((f,g) \notin E_Y\big), \\ 0 & \text{if } \forall\, h\big(\forall\, g \in \varphi^{-1}(0)\big((h,g) \notin E_Y \Rightarrow (h,f) \notin E_Y\big)\big), \\ - & \text{otherwise.} \end{cases}$$

The next proposition establishes some properties of Ψ and, most importantly, that it is well defined.

Proposition 2.3.5. *Let $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$ and let the map Ψ be defined by $(*)$ above. For every $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$,*

- (i) *$\varphi^{-1}(1) \subseteq (\Psi(\varphi))^{-1}(1)$ and $\varphi^{-1}(0) \subseteq (\Psi(\varphi))^{-1}(0)$;*
- (ii) *the map $\Psi(\varphi)$ is a maximal E-preserving map from $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ into $\underset{\sim}{\mathbf{2}}$, and consequently, $\Psi(\varphi) \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$.*

Proof. Consider (i). Let $f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ with $f \in \varphi^{-1}(1)$. By Lemma 2.1.1(ii), for all $g \in \varphi^{-1}(0)$, $(f,g) \notin E_X$ and so $(f,g) \notin E_Y$. Hence by the definition of $\Psi(\varphi)$, we get $f \in (\Psi(\varphi))^{-1}(1)$. If $f \in \varphi^{-1}(0)$, let us consider $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ such that if $g \in \varphi^{-1}(0)$, then $(h,g) \notin E_Y$. Clearly $(h,f) \notin E_Y$, which by the definition of $\Psi(\varphi)$ shows that $f \in (\Psi(\varphi))^{-1}(0)$.

Now consider (ii). Assume that $(\Psi(\varphi))(h) = 1$ and $(\Psi(\varphi))(f) = 0$ for some $h, f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. From the definition of $(\Psi(\varphi))^{-1}(1)$ we get that $(h,g) \notin E_Y$ for every $g \in \varphi^{-1}(0)$. Using this, we obtain from the definition of $(\Psi(\varphi))^{-1}(0)$ that $(h,f) \notin E_Y$. This shows that $\Psi(\varphi)$ is E_Y-preserving.

Suppose there exists an E-preserving map $\phi\colon \mathbf{Y} \to \underset{\sim}{\mathbf{2}}$ with $\mathrm{dom}(\Psi(\varphi)) \subsetneq \mathrm{dom}(\phi)$. Let $f \in \mathrm{dom}(\phi) \setminus \mathrm{dom}(\Psi(\varphi))$ for $f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. First, let us consider the case $\phi(f) = 1$. Since $f \notin (\Psi(\varphi))^{-1}(1)$, there exists $g \in \varphi^{-1}(0)$ such that $(f, g) \in E_Y$. Now $\varphi^{-1}(0) \subseteq (\Psi(\varphi))^{-1}(0)$ by (i), and since $(\Psi(\varphi))^{-1}(0) \subseteq \phi^{-1}(0)$, we get $g \in \phi^{-1}(0)$. This yields that ϕ is not E_Y-preserving, a contradiction. Now let $\phi(f) = 0$. Since $f \notin (\Psi(\varphi))^{-1}(0)$, there exists $g \in (\Psi(\varphi))^{-1}(1)$ with $(g, f) \in E_Y$. But then $g \in \phi^{-1}(1)$ and so ϕ is not E_Y-preserving, a contradiction. We have shown the maximality of $\Psi(\varphi)$. □

The following lemma will be deployed in the proof of Theorem 2.3.7.

Lemma 2.3.6. *Let* Ψ *be defined as in* $(*)$. *For* $\eta \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$, *let* $\varphi_\eta\colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ *be the restriction of* η *to* $\mathbf{X}$. *Then* $\varphi_\eta \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *and* $\Psi(\varphi_\eta) = \eta$.

Proof. Since φ_η is a restriction of an E_Y-preserving map onto $\mathbf{X}$, a subgraph of $\mathbf{Y}$, it is clear that φ_η will be E_X-preserving. Suppose there exists $\phi\colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ such that $\mathrm{dom}(\varphi_\eta) \subsetneq \mathrm{dom}(\phi)$. Consider first the case of $f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ with $f \in \phi^{-1}(1) \backslash \varphi_\eta^{-1}(1)$. This implies that there exists $g \in \eta^{-1}(0)$ such that $(f, g) \in E_Y$. Now consider $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ defined by $h^{-1}(1) = f^{-1}(1)$ and $h^{-1}(0) = g^{-1}(0)$. We can extend h to $h_* \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ and by Lemma 2.3.2 (i) we get $h_* \in \varphi_\eta^{-1}(0)$. But $\varphi_\eta^{-1}(0) \subseteq \phi^{-1}(0)$ and $(f, h) \in E_X$, contradicting ϕ being E_X-preserving. Using the same method we can show that the existence of $f \in \phi^{-1}(0) \backslash \varphi_\eta^{-1}(0)$ implies that ϕ cannot be E_X-preserving. Hence $\varphi_\eta \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$.

By the definition of the order of the lattice $\mathrm{C}(\mathbf{Y}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ (see Theorem 2.1.3), it suffices to prove $(\Psi(\varphi_\eta))^{-1}(1) = \eta^{-1}(1)$. If $f \in \eta^{-1}(1)$ then for all $g \in \eta^{-1}(0)$, we have $(f, g) \notin E_Y$ where $f, g \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. In particular this applies to all $g \in \eta^{-1}(0) \cap \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underset{\sim}{\mathbf{2}}) = \varphi_\eta^{-1}(0)$. Hence $f \in (\Psi(\varphi_\eta))^{-1}(1)$ and so $\eta^{-1}(1) \subseteq (\Psi(\varphi_\eta))^{-1}(1)$.

If $f \notin \eta^{-1}(1)$, there exists $g \in \eta^{-1}(0)$ such that $(f, g) \in E_Y$. Let $h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underset{\sim}{\mathbf{2}})$ be defined by $h^{-1}(1) = f^{-1}(1)$ and $h^{-1}(0) = g^{-1}(0)$. We know that we can extend h to an $h_* \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underset{\sim}{\mathbf{2}})$. Now $g^{-1}(0) \subseteq h_*^{-1}(0)$ and so by Lemma 2.3.2(i) we have $h_* \in \eta^{-1}(0) \cap \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) = \varphi_\eta^{-1}(0)$. Since $f^{-1}(1) \subseteq h_*^{-1}(1)$, we have $(f, h_*) \in E_Y$ and so clearly $f \notin (\Psi(\varphi_\eta))^{-1}(1)$. Thus $(\Psi(\varphi_\eta))^{-1}(1) \subseteq \eta^{-1}(1)$. □

The next result shows that the complete lattices $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ and $\mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ are order-isomorphic under Ψ, and that this map fixes the underlying lattice $\mathbf{L}$.

Theorem 2.3.7. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$*. Let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$*,* $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ *and let* Ψ *be the map defined by* $(*)$*. Then* $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ *is order-isomorphic to* $\mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{2})$ *under the map* Ψ*.*

Moreover, $\Psi(e_a) = \bar{e}_a$ *for any* $a \in L$*, where* $e_a \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ *and* $\bar{e}_a \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{2})$ *are the evaluation maps which represent the elements of* $\mathbf{L}$ *inside its completions* $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ *and* $\mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{2})$*, respectively.*

Proof. The surjectivity of Ψ has been proved in Lemma 2.3.6. That Ψ is an order-isomorphism can be seen from the fact that $\varphi \leqslant \psi$ if and only if $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$ (see Theorem 2.1.3), and this occurs if and only if $\psi^{-1}(0) \subseteq \varphi^{-1}(0)$ (by Lemma 2.1.1), which occurs if and only if $(\Psi(\varphi))^{-1}(1) \subseteq (\Psi(\psi))^{-1}(1)$ (by the definition of Ψ).

The equality $\Psi(e_a) = \bar{e}_a$ follows from Lemma 2.3.6 using the fact that the restriction of $\bar{e}_a$ to $\mathbf{X}$ is e_a. $\square$

Combining the preceding theorem with Theorem 2.2.11 we obtain the following.

Corollary 2.3.8. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$*. Then, with respect to the embedding set up by* $a \mapsto \bar{e}_a$*, the complete lattice* $\mathrm{C}(\mathbf{Y}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{2})$ *is the canonical extension of* $\mathbf{L}$*.*

We are almost ready to define the functors needed in our categorical approach. First, we make some observations about the relationship between the categories we defined earlier: $\mathcal{G}$, of graphs and E-preserving maps, and its topological analogue $\mathcal{G}_{\mathcal{T}}$, of topological graphs and continuous E-preserving maps.

From Lemma 2.3.1 we note that any $\boldsymbol{\mathcal{L}}$-graph $\mathbf{X} = (X, E)$ is automatically equipped with the quasi-orders $\leqslant_1$ and $\leqslant_2$, as previously defined. Thus, when considering $\mathbf{X}$ it is not necessary explicitly to state that $\leqslant_1$ and $\leqslant_2$ are part of the structure of $\mathbf{X}$. We note too that on $\underset{\sim}{2}$ (which is not an $\boldsymbol{\mathcal{L}}$-graph) we have quasi-orders $\leqslant_1$ and $\leqslant_2$ defined as follows:

$$\leqslant_1 = \ E \ = \leqslant \quad \text{and} \quad \leqslant_2 = \ E^{-1} \ = \geqslant .$$

We shall use $\underset{\sim}{2}$ to denote both the graph $(\{0,1\}, E)$ and the doubly-ordered set $(\{0,1\}, \leqslant_1, \leqslant_2)$.

The quasi-order relations $\leqslant_1$ and $\leqslant_2$ were used by Allwein and Hartonas [2] as well as by Urquhart [85]. Furthermore, Urquhart [85] defines maps $\ell\colon \wp(X) \to \wp(X)$ and $r\colon \wp(X) \to \wp(X)$, given by

$$\ell(A) = \{\, f \in X \mid \forall\, g \in A (f \nleqslant_1 g) \,\} \quad \text{and} \quad r(A) = \{\, f \in X \mid \forall\, g \in A (f \nleqslant_2 g) \,\},$$

for $A \subseteq X$. The maps ℓ and r are order-reversing on $(\wp(X), \subseteq)$ and form a Galois connection between the set of $\leqslant_2$-increasing subsets of X and the set of $\leqslant_1$-increasing subsets of X.

In his work, Urquhart uses the term *stable*, but we prefer to use the more descriptive terminology of Allwein and Hartonas (and others) and say that a subset $A \subseteq X$ is defined to be *ℓ-stable* if $\ell r(A) = A$ and *r-stable* if $r\ell(A) = A$. The following result is a simple consequence of the theory of Galois connections. It will be applied to sets of the form $\varphi^{-1}(1)$ and $\varphi^{-1}(0)$ where $\varphi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$, as we have from parts (i) and (ii) of Lemma 2.3.2 that $\varphi^{-1}(1)$ is $\leqslant_1$-increasing, and $\varphi^{-1}(0)$ is $\leqslant_2$-increasing.

Lemma 2.3.9. *Let $\mathbf{X} \in \mathfrak{Y}$ and let $A \subseteq X$. Then*

(i) *A is ℓ-stable if and only if $A = \ell(B)$ for some $\leqslant_2$-increasing $B \subseteq X$;*

(ii) *A is r-stable if and only if $A = r(B)$ for some $\leqslant_1$-increasing $B \subseteq X$.*

Now we define morphisms between $\boldsymbol{\mathcal{L}}$-graphs. It is obvious that we would want any such morphism to preserve the two quasi-orders $\leqslant_1$ and $\leqslant_2$. We also want morphisms to preserve the way in which these two quasi-orders interact with one another. This we achieve by requiring that inverse images of stable subsets of the codomain be stable sets in the domain of the morphism. We state this more precisely below.

Definition 2.3.10. *Let $\mathbf{X}$ and $\mathbf{Y}$ be $\boldsymbol{\mathcal{L}}$-graphs and $\alpha\colon \mathbf{X} \to \mathbf{Y}$. Then α is an $\boldsymbol{\mathcal{L}}$*-graph morphism *if*

(i) *α preserves $\leqslant_1$ and $\leqslant_2$;*

(ii) *if $A \subseteq Y$ is an ℓ-stable set, then $\alpha^{-1}(r_Y(A)) = r_X(\alpha^{-1}(A))$;*

(iii) *if $A \subseteq Y$ is an r-stable set, then $\alpha^{-1}(l_Y(A)) = l_X(\alpha^{-1}(A))$.*

We note that it is not difficult to show that the class of maps between graphs which satisfy the above conditions does indeed meet the criteria for a class of morphisms in a category. With this in hand we can give formal definitions of our categories $\mathfrak{Y}_{\mathcal{T}}$ and $\mathfrak{Y}$. The first of these consists of $\boldsymbol{\mathcal{L}}$-graphs with continuous $\boldsymbol{\mathcal{L}}$-graph morphisms while $\mathfrak{Y}$ is obtained by applying ${}^\flat$ to $\mathfrak{Y}_{\mathcal{T}}$. Lemma 2.3.11 serves to tell us that the category $\mathfrak{Y}$ (respectively $\mathfrak{Y}_{\mathcal{T}}$) is a full subcategory of $\mathfrak{G}$ (respectively $\mathfrak{G}_{\mathcal{T}}$).

Lemma 2.3.11. *Let $\mathbf{X} = (X, E_X)$ and $\mathbf{Y} = (Y, E_Y)$ be objects in $\mathfrak{Y}$ and consider the map $\alpha\colon \mathbf{X} \to \mathbf{Y}$. If α is an $\boldsymbol{\mathcal{L}}$-graph morphism, then α is E-preserving.*

Proof. Let $f, g \in X$ and suppose that $(f, g) \in E_X$. Then by Lemma 2.3.1(i) there exists $h \in X$ such that $f \leqslant_1 h$ and $g \leqslant_2 h$. Since α is $\leqslant_1$-preserving, we have that $\alpha(f) \leqslant_1 \alpha(h)$, and since α is $\leqslant_2$-preserving, we have $\alpha(g) \leqslant_2 \alpha(h)$. Thus there exists $y = \alpha(h) \in Y$ with the properties required by part (i) of Lemma 2.3.1, and so $(\alpha(f), \alpha(g)) \in E_Y$. $\square$

Not for the first time we are faced with the need to reconcile two notions which might possibly not coincide. We shall now prove that MPE's from $\boldsymbol{\mathcal{L}}$-graphs into $\underset{\sim}{\mathbf{2}}$ are exactly the maximal partial $\boldsymbol{\mathcal{L}}$-graph morphisms into $\underset{\sim}{\mathbf{2}}$.

Proposition 2.3.12. *Let $\mathbf{X} = (X, E)$ be an $\boldsymbol{\mathcal{L}}$-graph, and let $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ be a partial map. Then the following are equivalent:*

(1) *φ is a maximal partial E-preserving map;*

(2) *$\varphi^{-1}(1) = \ell(\varphi^{-1}(0))$ and $\varphi^{-1}(0) = r(\varphi^{-1}(1))$;*

(3) *φ is a maximal partial $\boldsymbol{\mathcal{L}}$-graph morphism.*

Proof. We start by showing the equivalence of (1) and (2). If $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ and $g \in \varphi^{-1}(0)$ then, by Lemma 2.1.1, for all $f \in \varphi^{-1}(1)$ we have that $(f, g) \notin E$. Since $g \leqslant_2 f$ implies that $(f, g) \in E$, we have for all $f \in \varphi^{-1}(1)$ that $g \nleqslant_2 f$ and hence $g \in r(\varphi^{-1}(1))$. Thus $\varphi^{-1}(0) \subseteq r(\varphi^{-1}(1))$. If we suppose that $g \notin \varphi^{-1}(0)$, then by Lemma 2.1.1 we have that there exists $f \in \varphi^{-1}(1)$ such that $(f, g) \in E$. Now by Lemma 2.3.1(i) there exists h such that $f \leqslant_1 h$ and $g \leqslant_2 h$ and Lemma 2.3.2(ii) gives us that $h \in \varphi^{-1}(1)$. Thus $g \notin r(\varphi^{-1}(1))$ and hence $r(\varphi^{-1}(1)) = \varphi^{-1}(0)$. We can similarly show that $\ell(\varphi^{-1}(0)) = \varphi^{-1}(1)$.

Now assume (2) and let $A := \varphi^{-1}(1)$. To show that φ preserves E, let f, g be in the domain of φ, and suppose $(f, g) \in E$ but $\varphi(f) = 1$ and $\varphi(g) = 0$. From $\varphi(f) = 1$ we get $f \in \ell r(A) = \ell(\varphi^{-1}(0))$ and $\varphi(g) = 0$ gives $g \in r(\varphi^{-1}(1)) = r(A)$. Lemma 2.3.1(i) gives us that there exists h such that $f \leqslant_1 h$ and $g \leqslant_2 h$. Now $f \in \ell r(A)$ means that for all $q \in r(A)$ we have $f \nleqslant_1 q$ and hence $h \notin r(A)$. This implies that there exists $k \in A$ such that $h \leqslant_2 k$. The transitivity of $\leqslant_2$ then implies that $g \notin r(A)$, a contradiction. In order to see that φ is maximal, suppose that $\mathrm{dom}(\varphi) \subseteq \mathrm{dom}(\psi)$ for some $\psi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. It is clear that $A \subseteq \psi^{-1}(1)$ and $r(A) \subseteq \psi^{-1}(0)$. If $f \in \psi^{-1}(1)$ and $g \in \psi^{-1}(0)$, then $(f, g) \notin E$ and hence $f \nleqslant_1 g$ and $g \nleqslant_2 f$. Thus any $f \in \psi^{-1}(1)$ must be in $\ell(\psi^{-1}(0))$ and similarly $\psi^{-1}(0) \subseteq r(\psi^{-1}(1))$. We then get $\psi^{-1}(1) \subseteq \ell(\psi^{-1}(0)) \subseteq \ell r(A) = A$ and hence $A = \psi^{-1}(1)$. We also have $r(A) = \psi^{-1}(0)$ and so $\psi = \varphi$, showing that φ is maximal. Hence (1) holds.

Now assume (3). Note that $\{0\}$ is r-stable and $\{1\}$ is ℓ-stable in $\underset{\sim}{\mathbf{2}}$ as $r(\{1\}) = \{0\}$ and $\ell(\{0\}) = \{1\}$. From (ii) and (iii) in the definition of morphisms of $\boldsymbol{\mathcal{L}}$-graphs we get $\varphi^{-1}(0) = \varphi^{-1}(r(\{1\})) = r(\varphi^{-1}(1))$ and $\varphi^{-1}(1) = \varphi^{-1}(\ell(\{0\})) = \ell(\varphi^{-1}(0))$.

Finally, assuming (2), we get that for $f, g \in X$, if $\varphi(f) = 1$ and $\varphi(g) = 0$, then $f \in \ell(\varphi^{-1}(0))$ and $g \in r(\varphi^{-1}(1))$. By the definition of ℓ and r we see that $f \nleqslant_1 g$ and $g \nleqslant_2 f$. Thus φ is $\leqslant_1$- and $\leqslant_2$-preserving on its domain. Since $\{0\}$ and $\{1\}$ are, respectively, the only r- and ℓ-stable subsets of $\underset{\sim}{\mathbf{2}}$, we have that φ obeys properties (ii) and (iii) required of an $\boldsymbol{\mathcal{L}}$-graph morphism. We now suppose that there exists a partial $\boldsymbol{\mathcal{L}}$-graph morphism ψ from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$ such that $\mathrm{dom}(\varphi) \subseteq \mathrm{dom}(\psi)$. Suppose there exists $f \in \mathrm{dom}(\psi)$ but $f \notin \mathrm{dom}(\varphi)$, and suppose $\psi(f) = 1$. Since $f \notin \varphi^{-1}(1) = \ell(\varphi^{-1}(0))$, there exists $g \in \varphi^{-1}(0) \subseteq \psi^{-1}(0)$ such that $f \leqslant_1 g$. This implies that ψ is not $\leqslant_1$-preserving. Similarly, if we suppose that $\psi(f) = 0$ we see that ψ will not be $\leqslant_2$-preserving. Hence $\mathrm{dom}(\varphi)$ is maximal and we have shown (3). ☐

We are, at last, ready to set up the functors we require. We claim that we can define $\overline{\mathrm{D}}\colon \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{Y}}_{\mathcal{T}}$ and $\overline{\mathrm{G}}\colon \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{L}}^{+}$ as follows:

on objects: $\overline{\mathrm{D}}\colon \mathbf{L} \mapsto (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathcal{T})$
on morphisms: $\overline{\mathrm{D}}\colon u \mapsto - \circ u$

and

on objects: $\overline{\mathrm{G}}\colon \mathbf{Y} \mapsto \boldsymbol{\mathfrak{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$,
on morphisms: $\overline{\mathrm{G}}\colon \alpha \mapsto - \circ \alpha$.

In the action on morphisms of $\overline{\mathrm{D}}$ and $\overline{\mathrm{G}}$ the domain of the image map is the set on which the composite map is defined. In legitimising these definitions what is at stake is well-definedness: we must ensure that the images of morphisms under $\overline{\mathrm{D}}$ and $\overline{\mathrm{G}}$ are again morphisms for the categories concerned.

Proposition 2.3.13. *Let* $\mathbf{L}, \mathbf{K} \in \boldsymbol{\mathcal{L}}$ *and let* $u\colon \mathbf{L} \to \mathbf{K}$ *be a lattice homomorphism. Then* $\overline{\mathrm{D}}^{\flat}(u)\colon \overline{\mathrm{D}}^{\flat}(\mathbf{K}) \to \overline{\mathrm{D}}^{\flat}(\mathbf{L})$ *is an* $\boldsymbol{\mathcal{L}}$*-graph morphism.*

Proof. Let $f, g \in \overline{\mathrm{D}}^{\flat}(\mathbf{K})$ with $f \leqslant_1 g$ and $a \in ((\overline{\mathrm{D}}^{\flat}(u))(f))^{-1}(1) = (f \circ u)^{-1}(1)$. This gives us that $u(a) \in f^{-1}(1)$ and hence $u(a) \in g^{-1}(1)$. Thus $a \in (\overline{\mathrm{D}}^{\flat}(u))(g)^{-1}(1)$ and so $\overline{\mathrm{D}}^{\flat}(u)$ is $\leqslant_1$-preserving. Likewise, $\overline{\mathrm{D}}^{\flat}(u)$ is $\leqslant_2$-preserving.

Now consider $A \subseteq \overline{\mathrm{D}}^{\flat}(\mathbf{L})$ such that A is ℓ-stable. We want to show that $\overline{\mathrm{D}}^{\flat}(u))^{-1}$ preserves stable sets. That is, we want to show $(\overline{\mathrm{D}}^{\flat}(u))^{-1}(r(A)) = r((\overline{\mathrm{D}}^{\flat}(u))^{-1}(A))$.

Let $f \in \overline{\mathrm{D}}^\flat(\mathbf{K})$ such that $f \notin r((\overline{\mathrm{D}}^\flat(u))^{-1}(A))$. This implies that there exists an SPH $g \in (\overline{\mathrm{D}}^\flat(u))^{-1}(A)$ such that $f \leqslant_2 g$. We note that $g \in (\overline{\mathrm{D}}^\flat(u))^{-1}(A)$ if and only if $g \circ u \in A$. Now since $\overline{\mathrm{D}}^\flat(u)$ is $\leqslant_2$-preserving, we have that $(\overline{\mathrm{D}}^\flat(u))(f) = f \circ u \leqslant_2 g \circ u$ and hence $f \notin (\overline{\mathrm{D}}^\flat(u))^{-1}(r(A))$. Thus we have $(\overline{\mathrm{D}}^\flat(u))^{-1}(r(A)) \subseteq r((\overline{\mathrm{D}}^\flat(u))^{-1}(A))$.

Now if $f \notin (\overline{\mathrm{D}}^\flat(u))^{-1}(r(A))$ then $f \circ u \notin r(A)$ and so there exists $h \in A$ such that $f \circ u \leqslant_2 h$. Consider the map $g_h \colon \mathbf{K} \to \underline{\mathbf{2}}$ defined for $b \in \mathbf{K}$ by

$$g_h(b) = \begin{cases} 0 & \text{if } b \in f^{-1}(0), \\ 1 & \text{if } b \in {\uparrow}(u[h^{-1}(1)]), \\ - & \text{otherwise.} \end{cases}$$

We show that $g_h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. If $b \in {\uparrow}(u[h^{-1}(1)])$ then there exists $a \in h^{-1}(1)$ such that $u(a) \leqslant b$. Since $f \circ u \leqslant_2 h$ we have that $h^{-1}(1) \cap (f \circ u)^{-1}(0) = \emptyset$ and so $a \notin (f \circ u)^{-1}(0)$. If $b \in f^{-1}(0)$ then $u(a) \in f^{-1}(0)$, a contradiction. Hence $g_h^{-1}(1) \cap g_h^{-1}(0) = \emptyset$. We need to show that $g_h^{-1}(1)$ is a filter of $\mathbf{K}$, so we consider $b_1, b_2 \in g_h^{-1}(1)$. By the definition of g_h, there must exist $a_1, a_2 \in h^{-1}(1)$ such that $u(a_1) \leqslant b_1$ and $u(a_2) \leqslant b_2$. Now since $h^{-1}(1)$ is a filter of $\mathbf{L}$, we have $a_1 \wedge a_2 \in h^{-1}(1)$ and since $u(a_1 \wedge a_2)$ is a lower bound for $\{b_1, b_2\}$ we have that $u(a_1 \wedge a_2) \leqslant b_1 \wedge b_2$ and so $b_1 \wedge b_2 \in g_h^{-1}(1)$. Hence $g_h \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{K}, \underline{\mathbf{2}})$.

If $a \in h^{-1}(1)$ then $u(a) \in u[h^{-1}(1)]$ and so $u(a) \in g_h^{-1}(1)$. Hence $g_h(u(a)) = 1$ and we have $h \leqslant_1 g_h \circ u$. Now as $h \in A = \ell r(A)$ and using the definition of ℓ and the transitivity of $\leqslant_1$, we have that $g_h \circ u \in \ell r(A) = A$. Now clearly $f \leqslant_2 g_h$ and since $g_h \in (\overline{\mathrm{D}}^\flat(u))^{-1}(A)$ we have $f \notin r((\overline{\mathrm{D}}^\flat(u))^{-1}(A))$. Since we originally assumed that $f \notin (\overline{\mathrm{D}}^\flat(u))^{-1}(r(A))$, we now conclude that $r((\overline{\mathrm{D}}^\flat(u))^{-1}(A)) \subseteq (\overline{\mathrm{D}}^\flat(u))^{-1}(r(A))$. $\square$

In order to show the well-definedness of $\overline{\mathrm{G}}$, we must verify that the image of a morphism under $\overline{\mathrm{G}}$ takes MPE's to MPE's. To address this question we first prove a technical lemma.

Lemma 2.3.14. *Let* $\mathbf{X}, \mathbf{Y} \in \boldsymbol{\mathcal{Y}}$ *and let* $\alpha \colon \mathbf{X} \to \mathbf{Y}$ *be such that* $\alpha \in \boldsymbol{\mathcal{Y}}(\mathbf{X}, \mathbf{Y})$. *Further, let* $\varphi \in \boldsymbol{\mathfrak{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ *and* $f \in X$. *Then*

- (i) *if there exists* $m \in \varphi^{-1}(0)$ *such that* $(\alpha(f), m) \in E_Y$ *then there exists* $g_m \in X$ *such that* $\varphi(\alpha(g_m)) = 0$ *and* $(f, g_m) \in E_X$*;*
- (ii) *if there exists* $n \in \varphi^{-1}(1)$ *such that* $(n, \alpha(f)) \in E_Y$ *then there exists* $g_n \in X$ *such that* $\varphi(\alpha(g_n)) = 1$ *and* $(g_n, f) \in E_X$.

Proof. Considering (i), we have from Lemma 2.1.1 that $\alpha(f) \notin \varphi^{-1}(1)$. This implies that $f \notin \alpha^{-1}(\varphi^{-1}(1)) = \alpha^{-1}(\ell(\varphi^{-1}(0)))$. Now since α^{-1} preserves ℓ-stable sets, we have that $f \notin \ell(\alpha^{-1}(\varphi^{-1}(0)))$. This implies that there exists $g_m \in \alpha^{-1}(\varphi^{-1}(0))$ such that $f \leqslant_1 g_m$. Now clearly $\alpha(g_m) \in \varphi^{-1}(0)$ and $(f, g_m) \in E_X$. $\square$

Proposition 2.3.15. *Let* $\mathbf{X}, \mathbf{Y} \in \boldsymbol{\mathcal{Y}}$ *and let* $\alpha\colon \mathbf{X} \to \mathbf{Y}$ *be an* $\boldsymbol{\mathcal{L}}$*-graph morphism. Then for any* $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ *it is the case that* $(\overline{\mathrm{G}}(\alpha))(\varphi) \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$.

Proof. We have from Lemma 2.3.11 that α is E-preserving, and hence we can apply Lemma 2.1.4 to conclude that $(\overline{\mathrm{G}}(\alpha))(\varphi)$ is a partial E-preserving map from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$. We show that its domain is maximal.

Suppose that there exists an E-preserving map ψ extending $(\overline{\mathrm{G}}(\alpha))(\varphi)$ and that $f \notin \operatorname{dom}((\overline{\mathrm{G}}(\alpha))(\varphi))$ for some $f \in X$. We want to show that $f \notin \operatorname{dom}(\psi)$. Since $f \notin \operatorname{dom}(\varphi \circ \alpha)$ we know that $(\varphi \circ \alpha)(f) \neq 1$ and hence by Lemma 2.1.1 there must exist $m_0 \in \varphi^{-1}(0)$ such that $(\alpha(f), m_0) \in E_Y$. Similarly, since $f \notin (\varphi \circ \alpha)^{-1}(0)$ we know that $\varphi(\alpha(f)) \neq 0$ and hence by Lemma 2.1.1 there must exist $m_1 \in \varphi^{-1}(1)$ such that $(m_1, \alpha(f)) \in E_Y$.

Now from Lemma 2.3.14 we have $g_{m_0}, g_{m_1} \in \mathbf{X}$ such that $\varphi(\alpha(g_{m_1})) = 1$ with $(g_{m_1}, f) \in E_X$ and $\varphi(\alpha(g_{m_0})) = 0$ with $(f, g_{m_0}) \in E_X$. From this we can easily see that $g_{m_1} \in ((\overline{\mathrm{G}}(\alpha))(\varphi))^{-1}(1)$ whence $\psi(g_{m_1}) = 1$ and $g_{m_0} \in ((\overline{\mathrm{G}}(\alpha))(\varphi))^{-1}(0)$ whence $\psi(g_{m_0}) = 0$. If we now suppose, for contradiction, that $f \in \operatorname{dom}(\psi)$, then from $(g_{m_1}, f) \in E_X$ we obtain $1 = \psi(g_{m_1}) \leqslant \psi(f)$ and from $(f, g_{m_0}) \in E_X$ we obtain $\psi(f) \leqslant \psi(g_{m_0}) = 0$, which is impossible. $\square$

The next result confirms that the codomain of $\overline{\mathrm{G}}$ really is $\boldsymbol{\mathcal{L}}^{+}$, as our commuting diagram demands. The theorem which follows shows that under ${}^{\delta}$ an $\boldsymbol{\mathcal{L}}$-morphism lifts to an $\boldsymbol{\mathcal{L}}^{+}$-morphism. We emphasise that the proof of the first result is a completely routine definition chase. By contrast, lattice homomorphisms were regarded by Gehrke and Harding [48] as instances of additional operations which are join- and meet-preserving, and their liftings treated using the machinery developed to handle extensions of maps in general.

Proposition 2.3.16. *Let* $\mathbf{X}, \mathbf{Y} \in \boldsymbol{\mathcal{Y}}$ *and* $\alpha \in \boldsymbol{\mathcal{Y}}(\mathbf{X}, \mathbf{Y})$*. Then the map* $\overline{\mathrm{G}}(\alpha)$ *between the complete lattices* $\mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ *and* $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *is a complete lattice homomorphism. That is,* $\overline{\mathrm{G}}(\alpha) \in \boldsymbol{\mathcal{L}}^{+}(\overline{\mathrm{G}}(\mathbf{Y}), \overline{\mathrm{G}}(\mathbf{X}))$.

Proof. We prove that $\overline{\mathrm{G}}(\alpha)$ is meet-preserving. The fact that $\overline{\mathrm{G}}(\alpha)$ is join-preserving will then follow by a similar argument. Let $f \in X$ and consider the collection of maps

$\{\, \varphi_i \mid i \in I \,\} \subseteq \boldsymbol{\mathscr{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$. Then

$$\begin{aligned} f \in \big(\textstyle\bigwedge_{i\in I}(\overline{\mathrm{G}}(\alpha)(\varphi_i))\big)^{-1}(1) &\iff f \in \bigcap_{i\in I}(\overline{\mathrm{G}}(\alpha)(\varphi_i))^{-1}(1) \\ &\iff (\forall\, i \in I)\ (\varphi_i \circ \alpha)(f) = 1 \\ &\iff (\forall\, i \in I)\ \varphi_i(\alpha(f)) = 1 \\ &\iff \alpha(f) \in \bigcap_{i\in I} \varphi_i^{-1}(1) \\ &\iff \big(\textstyle\bigwedge_{i\in I}\varphi_i\big)(\alpha(f)) = 1 \\ &\iff \big((\textstyle\bigwedge_{i\in I}\varphi_i) \circ \alpha\big)(f) = 1 \\ &\iff f \in \big(\overline{\mathrm{G}}(\alpha)(\textstyle\bigwedge_{i\in I}\varphi_i)\big)^{-1}(1). \end{aligned}$$

□

We can now present the main result of this section.

Theorem 2.3.17. *Let $u \in \boldsymbol{\mathscr{L}}(\mathbf{L}, \mathbf{K})$ be a homomorphism of bounded lattices. Then*

$$\overline{\mathrm{G}}\,\overline{\mathrm{D}}^{\flat}(u) \colon \overline{\mathrm{G}}\,\overline{\mathrm{D}}^{\flat}(\mathbf{L}) \to \overline{\mathrm{G}}\,\overline{\mathrm{D}}^{\flat}(\mathbf{K})$$

given by the composition of functors $\overline{\mathrm{G}} \circ^{\flat} \circ \overline{\mathrm{D}}$ is a complete homomorphism of the corresponding canonical extensions.

Proof. This follows from Propositions 2.3.13 and 2.3.16. □

In conclusion, we sum up what we have achieved.

Theorem 2.3.18. *Let the categories $\boldsymbol{\mathscr{L}}$, $\boldsymbol{\mathscr{L}}^{+}$, $\boldsymbol{\mathscr{Y}}_{\mathfrak{J}}$ and $\boldsymbol{\mathscr{Y}}$ be as above and construct $\overline{\mathrm{D}}$ and $\overline{\mathrm{G}}$ as indicated. Then*

(i) $\overline{\mathrm{D}} \colon \boldsymbol{\mathscr{L}} \to \boldsymbol{\mathscr{Y}}_{\mathfrak{J}}$ *and* $\overline{\mathrm{G}} \colon \boldsymbol{\mathscr{Y}} \to \boldsymbol{\mathscr{L}}^{+}$ *are well-defined functors;*

(ii) *the functor $\overline{\mathrm{D}}$ has a right adjoint $\overline{\mathrm{E}}$ such that the unit of the adjunction is given by evaluation maps which are isomorphisms;*

(iii) *with the categories and functors as defined above, the diagram in Fig.* 2.3 *commutes, so that the canonical extension functor on $\boldsymbol{\mathscr{L}}$ factorises as the composition $\overline{\mathrm{G}} \circ^{\flat} \circ \overline{\mathrm{D}}$.*

Chapter 3

Reconciliation of constructions of canonical extensions

This chapter provides new insights into the relationship between different constructions of the canonical extension of a bounded lattice. The work in this chapter was prompted by results obtained while working on the canonical extension construction via MPE's contained in Chapter 2.

Different constructions of the canonical extension of a bounded lattice have proved useful for particular applications. The doubly-ordered set and resulting complex algebra (canonical extension) from Urquhart's representation [85] was used by Dzik, Orlowska and van Alten [34] to provide complete Kripke-style semantics for logics with negation. (We note, however, that they do not at any stage refer to it as the canonical extension.) By contrast, the categorical setting of Chapter 2 was used to provide a functorial explanation for the fact that lattice homomorphisms are lifted by canonical extensions to complete lattice homomorphisms.

Gehrke and Harding in [48, Remark 2.10] assert that the canonical extension of a bounded lattice $\mathbf{L}$ is isomorphic to the complete lattice of so-called ℓ-stable subsets of Urquhart's dual space of $\mathbf{L}$. Our first new result in this chapter shows directly that the ℓ-stable sets correspond to the maximal partial E-preserving maps (Theorem 3.1.1). We show that the complete lattice $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets of Ploščica's dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ of the lattice $\mathbf{L}$ (or, equivalently, the ℓ-stable subsets of Urquhart's dual space) ordered by inclusion, is order-isomorphic to the canonical extension $\mathrm{C}(\mathbf{X}) = \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ of $\mathbf{L}$. For this, we define an order-isomorphism $\Phi^{LS}_{CX}\colon \mathrm{LS}(\mathbf{X}) \to \mathrm{C}(\mathbf{X})$ and we prove that this order-isomorphism fixes the elements of $\mathbf{L}$, thus $\mathrm{LS}(\mathbf{X})$ is the canonical extension of $\mathbf{L}$ (Corollary 3.1.2). Next, we describe the sets $J^{\infty}(\mathrm{C}(\mathbf{X}))$ and $M^{\infty}(\mathrm{C}(\mathbf{X}))$ of the completely join- and completely meet-irreducible elements of the canonical extension $\mathrm{C}(\mathbf{X})$ of $\mathbf{L}$, and show that these sets are join- and meet-dense, respectively (Corollary 3.1.6 and Proposition 3.1.5). This is an an analogue of the result of Gehrke and Harding in [48, Lemma 3.4].

In Section 3.2 we turn to the framework of Formal Concept Analysis, a field developed by Ganter and Wille [45]. Here we show that the canonical extension $\mathbf{L}^{\delta} = \mathrm{C}(\mathbf{X})$ of $\mathbf{L}$ is isomorphic to the concept lattice of a specific context associated to the complement $E^{\complement}$ of the graph relation E. More precisely, for a graph $\mathbf{X} = (X, E)$ we consider the context $\mathbb{K}(\mathbf{X}) := (X, X, E^{\complement})$ and the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of this context. We show that for an arbitrary graph $\mathbf{X} = (X, E)$ (not necessarily coming from a lattice $\mathbf{L}$) there is an order-isomorphism $\Phi^{CL}_{CX} \colon \mathrm{CL}(\mathbb{K}(\mathbf{X})) \to \mathrm{C}(\mathbf{X})$ between the concept lattice and the lattice of MPE's (Proposition 3.2.1). When the graph is a dual of a lattice $\mathbf{L}$, that is $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, the order-isomorphism Φ^{CL}_{CX} fixes $\mathbf{L}$ and so the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the context $\mathbb{K}(\mathbf{X})$ is the canonical extension of the lattice $\mathbf{L}$ (Corollary 3.2.2). Theorem 3.2.3 shows a useful correspondence between the elements of the lattices $\mathrm{C}(\mathbf{X})$, $\mathrm{LS}(\mathbf{X})$ and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ and we apply it to translate the descriptions of the sets $\{\, J_x \mid x \in X \,\}$ and $\{\, M_y \mid y \in X \,\}$ of completely join- and completely meet-irreducible elements of $\mathrm{C}(\mathbf{X})$ into the language of the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ for the context $\mathbb{K}(\mathbf{X})$ (Corollary 3.2.5). We finally show that in case of an arbitrary graph $\mathbf{X}$, the sets $\{\, J_x \mid x \in X \,\}$ and $\{\, M_y \mid y \in X \,\}$, as defined in Corollary 3.2.5, are join- and meet-dense in the complete lattice $\mathrm{C}(\mathbf{X})$ (Proposition 3.2.6). On the one hand this nicely extends Proposition 3.1.5, and on the other hand, by combining it with a basic result of Formal Concept Analysis (Theorem 3.2.7), it gives an alternative argument to the reconciliation of the lattices $\mathrm{C}(\mathbf{X})$ and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ as proven in Proposition 3.2.1.

In Section 3.3 we first reconcile the concept lattice studied in Section 3.2 with the canonical extension $\mathrm{GH}(\mathbf{L})$ of $\mathbf{L}$ introduced by Gehrke and Harding [48]. We further recall our construction of the canonical extension $\mathrm{C}(\mathbf{Y})$ of $\mathbf{L}$ via its bigger dual graph $\mathbf{Y} = (\mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of 'special' partial homomorphisms from $\mathbf{L}$ into $\underline{\mathbf{2}}$. By employing the bigger dual $\mathbf{Y} = (\mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of a bounded lattice $\mathbf{L}$ due to Allwein and Hartonas [1, 2], we identify the order-isomorphism $\Phi^{GH}_{CL} : \mathrm{GH}(\mathbf{L}) \to \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ which fixes $\mathbf{L}$ (Proposition 3.3.1 and Proposition 3.3.2). Then we reconcile the canonical extensions $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ based on Ploščica's representation and $\mathrm{GH}(\mathbf{L})$ based on the Gehrke–Harding approach without using the Allwein–Hartonas dual space as a stepping stone (Theorem 3.3.5); we identify the order-isomorphism $\Phi^{CX}_{GH} \colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ which fixes $\mathbf{L}$.

In Figure 3.1 we provide a 'Reconciliation Diagram' which depicts the six different lattices studied in this chapter which all serve as the canonical extension of an

$$\begin{tikzcd}
 & \mathrm{CL}(\mathbb{K}(\mathbf{X})) \arrow[d, "\Phi^{CL}_{CX}"'] & \\
\mathrm{LS}(\mathbf{X}) \arrow[r, "\Phi^{LS}_{CX}"] & \mathrm{C}(\mathbf{X}) \arrow[r, "\Phi^{CX}_{GH}"] \arrow[d, "\Phi^{CX}_{CY}"'] & \mathrm{GH}(\mathbf{L}) \arrow[d, "\Phi^{GH}_{CL}"] \\
 & \mathrm{C}(\mathbf{Y}) & \mathrm{CL}(\mathbb{K}(\mathbf{Y})) \arrow[l, "\Phi^{CL}_{CY}"]
\end{tikzcd}$$

Figure 3.1: Reconciliation of the canonical extension constructions

arbitrary bounded lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. Three of these lattices are defined via the Ploščica dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$ of $\mathbf{L}$: the lattices $\mathrm{C}(\mathbf{X}) = \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ of MPE's, $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the concepts of the context $\mathbb{K}(\mathbf{X})$. Two of the lattices are defined via the Allwein–Hartonas dual graph $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of $\mathbf{L}$: the lattices $\mathrm{C}(\mathbf{Y}) = \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ of MPE's and $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ of the concepts of the context $\mathbb{K}(\mathbf{Y})$. The lattice $\mathrm{GH}(\mathbf{L})$ comes from a Galois connection between $\wp(\mathrm{Filt}(\mathbf{L}))$ and $\wp(\mathrm{Idl}(\mathbf{L}))$. The diagram also depicts all the transition order-isomorphisms between the canonical extensions fixing the original lattice $\mathbf{L}$ that we mentioned above. Each arrow is of course a bijective map, with the direction indicated in the diagram showing the direction of the proof.

The establishment of these order-isomorphisms and the understanding behind them forms the core of this chapter. In Section 3.4 we are able to use our reconciliation results to generalise our definition of $\boldsymbol{\mathcal{L}}$-graph morphisms to a class of morphisms that can be defined on an arbitrary graph. The point is that such a morphism between graphs will always lift to a complete lattice homomorphism between the complete lattices of MPE's coming from the graphs.

3.1 The complete lattice of MPE's

Much of the notation and terminology in this chapter follows directly from Chapter 2. We draw particular attention to the results of Lemmas 2.1.1, 2.2.3, 2.2.4, and 2.2.5 as these will be used repeatedly in this chapter. We also remind the reader that $\leqslant_1 \subseteq E$ and $\geqslant_2 \subseteq E$.

Important for the rest of this section will be the link between Ploščica's representation and that of Urquhart [85]. This is outlined by Ploščica [78, Section 2]. We refer the reader to page 39 for the definition of ℓ- and r-stable sets.

Ploščica [78, Theorem 2.2] shows that, for a lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and $Y \subseteq \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, the subset Y is a doubly closed ℓ-stable set if and only if $Y = \varphi^{-1}(1)$ for some $\varphi \in \mathfrak{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{2}_{\mathfrak{T}})$. Gehrke and Harding assert without proof that the canonical extension of a bounded lattice is isomorphic to the complete lattice of ℓ-stable subsets of Urquhart's dual space [48, Remark 2.10].

The first new result in this chapter shows directly that the ℓ-stable sets above do correspond to the maximal partial E-preserving maps. (We remark that the proof is similar to that used to prove [78, Theorem 2.2].)

Theorem 3.1.1. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *For any* $Y \subseteq X$ *the following are equivalent:*

(1) Y *is an* ℓ*-stable set;*

(2) $Y = \varphi^{-1}(1)$ *for some* $\varphi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$.

Proof. Consider an ℓ-stable subset $Y \subseteq X$, and define φ_Y as follows:

$$\varphi_Y(f) = \begin{cases} 1 & \text{if } f \in Y, \\ 0 & \text{if } f \in r(Y) \\ - & \text{otherwise.} \end{cases}$$

We first show that φ_Y preserves E. Let $f, g \in \mathrm{dom}(\varphi_Y)$ and suppose $(f, g) \in E$ but $\varphi_Y(f) = 1$ and $\varphi_Y(g) = 0$. From $\varphi_Y(f) = 1$ we get $f \in Y = \ell r(Y)$ and $\varphi_Y(g) = 0$ gives $g \in r(Y)$. Lemma 2.2.3(i) gives us that there exists an MPH h such that $f \leqslant_1 h$ and $g \leqslant_2 h$. Now $f \in \ell r(Y)$ means that for all $j \in r(Y)$ we have $f \nleqslant_1 j$ and hence $h \notin r(Y)$. This implies that there exists $k \in Y$ such that $h \leqslant_2 k$. The transitivity of $\leqslant_2$ then implies that $g \leqslant_2 k$ and hence $g \notin r(Y)$, a contradiction. In order to see that φ_Y is maximal, suppose that $\mathrm{dom}\, \varphi_Y \subseteq \mathrm{dom}\, \psi$ for some $\psi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. It is clear that $Y \subseteq \psi^{-1}(1)$ and $r(Y) \subseteq \psi^{-1}(0)$. If $f \in \psi^{-1}(1)$ and $g \in \psi^{-1}(0)$, then $(f, g) \notin E$ and hence $f \nleqslant_1 g$ and $g \nleqslant_2 f$. Thus any $f \in \psi^{-1}(1)$ must be in $\ell(\psi^{-1}(0))$ and similarly $\psi^{-1}(0) \subseteq r(\psi^{-1}(1))$. We then get $\psi^{-1}(1) \subseteq \ell(\psi^{-1}(0)) \subseteq \ell r(Y) = Y$ and hence $Y = \psi^{-1}(1)$. We also have $r(Y) = \psi^{-1}(0)$ and so $\psi = \varphi_Y$, showing that φ_Y is maximal.

Given $\varphi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$, define $Y = \varphi^{-1}(1)$. We will show that $r(Y) = \varphi^{-1}(0)$ and hence that Y is an ℓ-stable set. If $f \in \varphi^{-1}(0)$ then, by Lemma 2.1.1, for all

$g \in \varphi^{-1}(1)$ we have that $(g, f) \notin E$. Since $f \leqslant_2 g$ implies that $(g, f) \in E$, we have that $f \nleqslant_2 g$ and hence $f \in r(Y) = \{\, h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid (\forall k \in \varphi^{-1}(1))\ h \nleqslant_2 k \,\}$. Thus $\varphi^{-1}(0) \subseteq r(Y)$. If we suppose that $f \notin \varphi^{-1}(0)$, then by Lemma 2.2.4(i) we have that there exists $g \in \varphi^{-1}(1)$ such that $f \leqslant_2 g$. Thus $f \notin r(\varphi^{-1}(1)) = r(Y)$ and hence $r(Y) = \varphi^{-1}(0)$. We can similarly show that $\ell(\varphi^{-1}(0)) = \varphi^{-1}(1)$ and so conclude that $Y = \ell r(Y)$. ☐

Let $\mathrm{LS}(\mathbf{X})$ denote the complete lattice of ℓ-stable subsets of $\mathbf{X}$, ordered by inclusion. We define a map $\Phi^{LS}_{CX} \colon \mathrm{LS}(\mathbf{X}) \to \mathrm{C}(\mathbf{X})$ by $Y \mapsto \varphi_Y$. The previous theorem shows that it is a well-defined bijection. The next result, which is an easy consequence of the previous theorem, shows that Φ^{LS}_{CX} is an order-isomorphism fixing $\mathbf{L}$, thus the lattice $\mathrm{LS}(\mathbf{X})$ is the canonical extension of $\mathbf{L}$.

Corollary 3.1.2. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$.

(i) *The lattice* $\mathrm{LS}(\mathbf{X})$ *of ℓ-stable subsets of* $\mathbf{X}$ *is order-isomorphic to the canonical extension* $\mathrm{C}(\mathbf{X}) = \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *of* $\mathbf{L}$ *via the map* Φ^{LS}_{CX}.

(ii) *The lattice* $\mathrm{LS}(\mathbf{X})$ *of ℓ-stable subsets of* $\mathbf{X}$ *is the canonical extension of* $\mathbf{L}$ *via the embedding* $a \mapsto e_a^{-1}(1)$ $(a \in L)$.

Proof. Part (i) follows from Theorem 3.1.1 and the fact that for any ℓ-stable subsets Y, Z of $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, clearly $Y \subseteq Z$ if and only if $\varphi_Y^{-1}(1) \subseteq \varphi_Z^{-1}(1)$.

For part (ii) it remains to show that Φ^{LS}_{CX} is an order-isomorphism between the embedded copies of $\mathbf{L}$ in each of the complete lattices $\mathrm{LS}(\mathbf{X})$ and $\mathrm{C}(\mathbf{X})$.

From Proposition 2.2.2 we know that $\mathbf{L}$ is embedded in $\mathrm{C}(\mathbf{X})$ via the embedding $e \colon a \mapsto e_a$ $(a \in L)$, while from Ploščica's result [78, Theorem 2.2] mentioned above it, follows that $a \mapsto e_a^{-1}(1)$ $(a \in L)$ is an embedding of $\mathbf{L}$ into $\mathrm{LS}(\mathbf{X})$. Now we see that $\Phi^{LS}_{CX}(e_a^{-1}(1)) = e_a$ and for $a, b \in L$, we have $e_a^{-1}(1) \subseteq e_b^{-1}(1)$ if and only if $a \leqslant b$, if and only if $e_a \leqslant e_b$. ☐

The second new result in this section is a description of the completely join-irreducible and completely meet-irreducible elements of the canonical extension. We will denote these by $J^{\infty}(\mathbf{L}^{\delta})$ and $M^{\infty}(\mathbf{L}^{\delta})$ respectively. In order to identify the elements of $J^{\infty}(\mathbf{L}^{\delta})$ and $M^{\infty}(\mathbf{L}^{\delta})$, we shall need the following lemma.

Lemma 3.1.3. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *Then for every* $x \in X$, *the set* ${\uparrow_1} x := \{\, z \in X \mid x \leqslant_1 z \,\}$ *is ℓ-stable. Dually, for every* $y \in X$, *the set* ${\uparrow_2} y := \{\, z \in X \mid y \leqslant_2 z \,\}$ *is r-stable.*

Proof. It is clear that for any $Y \subseteq X$, we have $Y \subseteq \ell r(Y)$. For the reverse inclusion, if $w \notin {\uparrow_1} x$, we have by Lemma 2.2.3(iii) that there is y such that $(w, y) \in E$ and $(x, y) \notin E$. Now $(x, y) \notin E$ implies that for all u, either $x \nleqslant_1 u$ or $y \nleqslant_2 u$, and this in turn gives that $y \in r({\uparrow_1} x)$. From $(w, y) \in E$ we know there exists z such that $w \leqslant_1 z$ and $y \leqslant_2 z$. It is easy to see that $z \in r({\uparrow_1} x)$ and since $w \leqslant_1 z$ we get that $w \notin \ell r({\uparrow_1} x)$. Thus $\ell r({\uparrow_1} x) \subseteq {\uparrow_1} x$. □

For a bounded lattice $\mathbf{L}$ and the graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, we are now ready to reveal which elements of the canonical extension $\mathrm{C}(\mathbf{X})$ of $\mathbf{L}$ are the completely join-irreducibles and the completely meet-irreducibles. The proof of the first result in this direction is similar to the proof of the equivalent result due to Gehrke and Harding [48, Lemma 3.4].

Proposition 3.1.4. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$*. Then*

$$\{\, J_x \mid x \in X \,\} \subseteq J^{\infty}(\mathrm{C}(\mathbf{X})) \quad \textit{and} \quad \{\, M_y \mid y \in X \,\} \subseteq M^{\infty}(\mathrm{C}(\mathbf{X}))$$

where

$$J_x(z) = \begin{cases} 1 & \text{if } x \leqslant_1 z, \\ 0 & \text{if } z \in r({\uparrow_1} x) \end{cases} \qquad \textit{and} \qquad M_y(z) = \begin{cases} 1 & \text{if } z \in l({\uparrow_2} y), \\ 0 & \text{if } y \leqslant_2 z \end{cases}$$

for $z \in X$*.*

Proof. We first point out that as a consequence of Theorem 3.1.1 and Lemma 3.1.3, for any $x \in X$, we have $J_x \in \mathrm{C}(\mathbf{X})$. Next, we show that J_x is such that $J_x = \bigwedge F$ where F is a member of a maximal-disjoint filter-ideal pair on $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$. Since $x \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, we have that $x^{-1}(1)$ is such a filter on L, and so we consider the set of maps

$$F = \{\, e_a \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}}) \mid a \in x^{-1}(1) \,\}.$$

Since $\mathbf{L}$ is isomorphic to $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ via the map $a \mapsto e_a$ as indicated by Proposition 2.2.1, we have that F is a member of a maximally-disjoint filter-ideal pair on $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$. Similarly we can consider the corresponding ideal which comes from x, defined by $I = \{\, e_a \mid a \in x^{-1}(0) \,\}$.

Since J_x and $\bigwedge F$ are elements of $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$, in order to show that they are equal we need only to prove that $J_x^{-1}(1) = (\bigwedge F)^{-1}(1)$. By definition of the meet in $\mathrm{C}(\mathbf{X})$,

$$\big(\bigwedge F\big)^{-1}(1) = \bigcap \{\, e_a^{-1}(1) \mid a \in x^{-1}(1) \,\}.$$

Thus $z \in (\bigwedge F)^{-1}(1)$ if and only if for all $a \in x^{-1}(1)$, $e_a(z) = z(a) = 1$. That is, $z \in (\bigwedge F)^{-1}(1)$ if and only if $x^{-1}(1) \subseteq z^{-1}(1)$. This gives us the final required equivalence, that $z \in (\bigwedge F)^{-1}$ if and only if $z \in {\uparrow_1} x = J_x^{-1}(1)$.

Now assume that $J_x = \bigvee\{\, \varphi_k \mid k \in K \,\}$. Since the filter elements are join dense in $\mathrm{C}(\mathbf{X})$, we can consider the case where φ_k is a filter element for each $k \in K$. That is, for each $k \in K$, $\varphi_k = \bigwedge\{\, e_a \mid e_a \in F_k \,\}$ where F_k is a filter of $\boldsymbol{\mathcal{G}}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$. If J_x is not completely join-irreducible, then for every $k \in I$, $\varphi_k^{-1}(1) \subsetneq J_x^{-1}(1)$. That is, $\bigwedge F_k < \bigwedge F$ and hence $F \subsetneq F_k$ for each $k \in I$. Since F is maximally disjoint from I, there exists $b_k \in F_k \cap I$ for each $k \in K$. In $\mathrm{C}(\mathbf{X})$ we have

$$\bigwedge F \;=\; \bigvee_{k \in K}\bigl(\bigwedge\{\, e_a \mid e_a \in F_k \,\}\bigr) \;\leqslant\; \bigvee\{\, e_{b_k} \mid k \in K \,\} \;\leqslant\; \bigvee I.$$

By compactness, there must exist $e_a \in F \cap I$, a contradiction as they are disjoint. Hence there must exist $j \in K$ such that $J_x = \varphi_j = \bigwedge F_j$. □

Gehrke and Harding [48, Lemma 3.4] show additionally that $J^\infty(\mathbf{L}^\delta)$ is join-dense in $\mathbf{L}^\delta$ and that $M^\infty(\mathbf{L}^\delta)$ is meet-dense in $\mathbf{L}^\delta$. Our next result in this section shows this by repeatedly applying Lemma 2.2.4.

Proposition 3.1.5. *Let* $\mathbf{L}$ *be a bounded lattice and let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *Then for any* $\varphi \in \mathrm{C}(\mathbf{X})$,

$$\varphi = \bigvee\{\, J_x \mid x \in \varphi^{-1}(1) \,\} \;\text{ and }\; \varphi = \bigwedge\{\, M_y \mid y \in \varphi^{-1}(0) \,\}.$$

Proof. If $x \in \varphi^{-1}(1)$ and $y \in J_x^{-1}(1)$, we have $x \leqslant_1 y$. By Lemma 2.2.4(iv), $\varphi(y) = 1$ and so $J_x^{-1}(1) \subseteq \varphi^{-1}(1)$. Hence $\bigvee\{\, J_x \mid x \in \varphi^{-1}(1) \,\} \leqslant \varphi$.

We now show that $\varphi^{-1}(1) \subseteq (\bigvee J_x)^{-1}(1)$. Consider $z \notin \bigl(\bigvee J_x\bigr)^{-1}(1)$. By Lemma 2.2.4(ii) there exists $y \in \bigcap J_x^{-1}(0)$ such that $z \leqslant_1 y$. Now $y \in \bigcap J_x^{-1}(0)$ implies that for every $x \in \varphi^{-1}(1)$, $y \nleqslant_2 x$, for otherwise, by Lemma 2.2.4(iii), $J_x(x) = 0$, a contradiction. Now suppose that $z \in \varphi^{-1}(1)$. By Lemma 2.2.4(iv), $z \leqslant_1 y$ implies that $y \in \varphi^{-1}(1)$. Now $y \in \bigcap J_x^{-1}(0)$ implies that $y \nleqslant_2 y$, a contradiction. Thus $z \notin \varphi^{-1}(1)$ and so $\varphi = \bigvee\{\, J_x \mid x \in \varphi^{-1}(1) \,\}$.

The proof that $\varphi = \bigwedge\{\, M_y \mid y \in \varphi^{-1}(0) \,\}$ is similar. □

Our final result identifies the completely join-irreducible and completely-meet irreducible elements of the canonical extension $\mathrm{C}(\mathbf{X})$.

Corollary 3.1.6. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *Then*

$$J^\infty(\mathrm{C}(\mathbf{X})) = \{\, J_x \mid x \in X \,\} \quad \text{and} \quad M^\infty(\mathrm{C}(\mathbf{X})) = \{\, M_y \mid y \in X \,\}$$

where J_x *and* M_y *are defined as in Proposition 3.1.4.*

Proof. One direction of the inclusions has been shown in Proposition 3.1.4. Suppose that $\varphi \in J^\infty(\mathrm{C}(\mathbf{X}))$. By Proposition 3.1.5 we have $\varphi = \bigvee\{\, J_x \mid x \in \varphi^{-1}(1)\,\}$. Since φ is completely join-irreducible we have that $\varphi = J_x$ for some $x \in \varphi^{-1}(1)$. □

3.2 The canonical extension viewed as a concept lattice

In this section we first show that the complete lattice $\mathrm{C}(\mathbf{X})$, as constructed for the graph $\mathbf{X} = (X, E)$ in Theorem 2.1.3, is isomorphic to the concept lattice, in the framework of Formal Concept Analysis (FCA, for short), of a specific context associated to the complement of the relation E. A similar representation of lattices, though only in the finite case, was used by Wille in [87]. The addition of topology to a generalisation of Wille's contexts was used later to obtain a representation theorem for arbitrary lattices by Hartung [66].

We begin with some basics from Formal Concept Analysis (the main source is [45], but for our purposes, [28, Chapter 3] is sufficient). A *context* $\mathbf{K}$ is a triple (O, P, I) such that O and P are sets and $I \subseteq O \times P$. The elements of O are called *objects* and the elements of P are called *attributes* meaning properties of the objects. Then so-called *polar maps* (*polars*, for short) of the relation I set up a Galois connection

$$I_\triangleright : (\wp(O), \subseteq) \to (\wp(P), \supseteq) \quad \text{and} \quad I_\triangleleft : (\wp(P), \supseteq) \to (\wp(O), \subseteq)$$

defined by

$$I_\triangleright(U) = \{\, a \in P \mid (\forall\, o \in U)(o, a) \in I \,\} \;\text{ and }\; I_\triangleleft(V) = \{\, o \in O \mid (\forall\, a \in V)(o, a) \in I \,\}.$$

Hence one can associate to the context $\mathbf{K}$ a complete lattice

$$\mathrm{CL}(\mathbf{K}) = \{\, A \subseteq O \mid (I_\triangleleft \circ I_\triangleright)(A) = A \,\}$$

of Galois-closed sets. In the FCA literature the complete lattice $\mathrm{CL}(\mathbf{K})$ is called the *concept lattice of the context* $\mathbf{K}$. Its elements are usually referred to as pairs (A, B) where $I_\triangleright(A) = B$ and $I_\triangleleft(B) = A$. Such a pair is called a *concept* where A is referred to as the *extent* and B as the *intent* of the concept.

For our graphs $\mathbf{X} = (X, E)$ we shall consider the context

$$\mathbb{K}(\mathbf{X}) := (X, X, E^{\complement})$$

where the base set X of the graph $\mathbf{X}$ stands for both objects O and attributes P and the relation I is the complement of the graph relation: $E^{\complement} = (X \times X)\backslash E$. We define a Galois connection via polars

$$E^{\complement}_{\rhd} : (\wp(X), \subseteq) \to (\wp(X), \supseteq) \quad \text{and} \quad E^{\complement}_{\lhd} : (\wp(X), \supseteq) \to (\wp(X), \subseteq)$$

as above by

$$E^{\complement}_{\rhd}(Y) = \{\, x \in X \mid (\forall\, y \in Y)(y, x) \notin E \,\}$$

and

$$E^{\complement}_{\lhd}(Y) = \{\, z \in X \mid (\forall\, y \in Y)(z, y) \notin E \,\}.$$

The concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the context $\mathbb{K}(\mathbf{X}) = (X, X, E^{\complement})$, given by

$$\mathrm{CL}(\mathbb{K}(\mathbf{X})) = \{\, Y \subseteq X \mid (E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd})(Y) = Y \,\},$$

ordered by inclusion, will now be shown to be order-isomorphic to the lattice $\mathrm{C}(\mathbf{X})$ coming from the graph $\mathbf{X}$. Let $Y \subseteq X$ be an element of $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$. We define a map $\Phi^{CL}_{CX} \colon \mathrm{CL}(\mathbb{K}(\mathbf{X})) \to \mathrm{C}(\mathbf{X}), Y \mapsto \varphi_Y$ where $\varphi_Y \colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ is defined by

$$\varphi_Y(x) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \in E^{\complement}_{\rhd}(Y). \end{cases}$$

We emphasise that the first result below does not rely on the graph coming from a lattice. Rather, it establishes in the general case the relationship between MPE's and Galois-closed sets under the $E^{\complement}$ relation. The subsequent corollary highlights the fact that when the graph $\mathbf{X}$ does come from some lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, then the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ described above is the canonical extension of the lattice $\mathbf{L}$.

Proposition 3.2.1. *Let $\mathbf{X} = (X, E)$ be a graph. The concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the context $\mathbb{K}(\mathbf{X})$ is order-isomorphic to the lattice $\mathrm{C}(\mathbf{X})$ via the map Φ^{CL}_{CX}.*

Proof. We start by showing that the map Φ^{CL}_{CX} is well-defined. To this end, let $Y \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$ and consider $y \in \varphi_Y^{-1}(1)$ and $x \in \varphi_Y^{-1}(0)$. Since $y \in Y$, and $x \in E^{\complement}_{\rhd}(Y)$, we have $(y, x) \notin E$ and hence φ_Y is E-preserving. If $\operatorname{dom}\varphi_Y \subsetneq \operatorname{dom}\psi$ for some $\psi \in \mathrm{C}(\mathbf{X})$ which extends φ_Y, then there exists $x \in \operatorname{dom}(\psi) \setminus \operatorname{dom}(\varphi_Y)$. Since $x \notin E^{\complement}_{\rhd}(Y)$, there exists $y \in Y = \varphi_Y^{-1}(1)$ such that $(y, x) \in E$. Thus $x \notin \psi^{-1}(0)$ and so $\psi(x) = 1$. Now $x \notin \varphi_Y^{-1}(1)$ means that $x \notin Y = (E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd})(Y)$. This implies that there exists $z \in E^{\complement}_{\rhd}(Y)$ such that $(x, z) \in E$. But, $z \in \psi^{-1}(0)$, which contradicts ψ being E-preserving and hence φ_Y is maximal.

For $\psi \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$, let $Y = \psi^{-1}(1)$ and by part (i) of Lemma 2.1.1 we have $\psi^{-1}(0) = E^{\mathsf{C}}_{\rhd}(\psi^{-1}(1)) = E^{\mathsf{C}}_{\rhd}(Y)$. Applying part (ii) of Lemma 2.1.1, and using the definition of $E^{\mathsf{C}}_{\rhd}$ gives us $Y = \psi^{-1}(1) = (E^{\mathsf{C}}_{\lhd} \circ E^{\mathsf{C}}_{\rhd})(Y)$ and hence $Y \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$.

Thus we have a bijective correspondence between elements of $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ and MPE's from $\mathbf{X}$ into $\underset{\sim}{\mathbf{2}}$. We can further conclude that these two complete lattices are order-isomorphic. For $Y, Z \subseteq X$, clearly $Y \subseteq Z$ in $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ if and only if $\varphi_Y \leqslant \varphi_Z$ in $\mathrm{C}(\mathbf{X})$. $\square$

Corollary 3.2.2. *Let* $\mathbf{L}$ *be a bounded lattice. Let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and let* $\mathbb{K}(\mathbf{X}) = (X, X, E^{\mathsf{C}})$ *be the context associated to* $\mathbf{X}$*. Then the concept lattice* $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ *of the context* $\mathbb{K}(\mathbf{X})$ *is the canonical extension of the lattice* $\mathbf{L}$ *via the embedding* $a \mapsto e_a^{-1}(1)$ $(a \in L)$*.*

Proof. We shall show that the order-isomorphism $\Phi^{CL}_{CX} \colon \mathrm{CL}(\mathbb{K}(\mathbf{X})) \to \mathrm{C}(\mathbf{X})$ is an order-isomorphism between the embedded copies of $\mathbf{L}$ in each of the complete lattices $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ and $\mathrm{C}(\mathbf{X})$.

From Proposition 2.2.2 we know that $\mathbf{L}$ is embedded in $\mathrm{C}(\mathbf{X})$ via the embedding $e \colon a \mapsto e_a$ $(a \in L)$, while from Hartung's work [66, Theorem 2.1.8] it follows that $a \mapsto e_a^{-1}(1)$ $(a \in L)$ is an embedding of $\mathbf{L}$ into $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$. Now it is clear that the order-isomorphism Φ^{CL}_{CX} maps $\{\, e_a^{-1} \mid a \in L \,\}$ into $\{\, e_a \mid a \in L \,\}$, and that for $a, b \in L$, $a \leqslant b$ if and only if $e_a \leqslant e_b$, if and only if $e_a^{-1}(1) \subseteq e_b^{-1}(1)$. $\square$

So far we have presented three different approaches to the construction of the canonical extension of a bounded lattice $\mathbf{L}$ via its dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$: the lattices $\mathrm{C}(\mathbf{X}) = \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ of MPE's, $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets, and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the concepts of the context $\mathbb{K}(\mathbf{X})$. The next result summarises the equivalence of these approaches.

Theorem 3.2.3. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$*. Let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and let* $\mathbb{K}(\mathbf{X}) = (X, X, E^{\mathsf{C}})$ *be the context associated to* $\mathbf{X}$*. Then for* $Y \subseteq X$*, the statements in* (i) *are equivalent.*

- (i) (1) $Y = \varphi^{-1}(1)$ *for some* $\varphi \in \mathrm{C}(\mathbf{X})$*;*
  - (2) $Y \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$*;*
  - (3) $Y = E^{\mathsf{C}}_{\lhd}(B)$ *for some* $B \subseteq X$*;*
  - (4) $Y \in \mathrm{LS}(\mathbf{X})$*, that is,* Y *is an* ℓ*-stable set;*
  - (5) $Y = \ell(A)$ *for some* $\leqslant_2$*-increasing* $A \subseteq X$*.*

Dually, for $Z \subseteq X$*, the statements in* (ii) *are equivalent.*

(ii) (1) $Z = \varphi^{-1}(0)$ *for some* $\varphi \in \mathrm{C}(\mathbf{X})$*;*

(2) $E^{\complement}_{\triangleright}(Z) \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$*;*

(3) $Z = E^{\complement}_{\triangleright}(B)$ *for some* $B \subseteq X$*;*

(4) Z *is an* r*-stable set;*

(5) $Z = r(A)$ *for some* $\leqslant_1$*-increasing* $A \subseteq X$*.*

Proof. For part (i), the equivalence of (1) and (2) is the result of Proposition 3.2.1 above. The equivalence of (1) and (4) is the result of Theorem 3.1.1. The equivalences of (2) and (3), and of (4) and (5), are well known from the theory of Galois connections (see Lemma 2.3.9). $\square$

We can now use Theorem 3.2.3 to translate the characterisation of the completely join- and completely meet-irreducible elements of $\mathrm{C}(\mathbf{X})$ from the language of $\leqslant_1$ and $\leqslant_2$ (Corollary 3.1.6) into the language of the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$. We can then apply these translated definitions to extend Proposition 3.1.5 to the case of arbitrary graphs.

Before proving the following lemma, we note that in the case of an arbitrary graph $\mathbf{X} = (X, E)$ and $x \in X$, we have that $E^{\complement}_{\triangleright}(\{x\}) = \{\, y \in X \mid (x, y) \notin E \,\}$. In case $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ we define $\uparrow_1 x := \{z \in X \mid x \leqslant_1 z\}$, and similarly, $\uparrow_2 y := \{z \in X \mid y \leqslant_2 z\}$.

Lemma 3.2.4. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$*. For* $x \in X$ *we have* $\uparrow_1 x = (E^{\complement}_{\triangleleft} \circ E^{\complement}_{\triangleright})(\{x\})$*. Dually, for* $y \in X$ *we have* $\uparrow_2 y = (E^{\complement}_{\triangleright} \circ E^{\complement}_{\triangleleft})(\{y\})$*.*

Proof. It is clear from the definition that $x \in (E^{\complement}_{\triangleleft} \circ E^{\complement}_{\triangleright})(\{x\})$. By the equivalence of (1) and (3) in part (i) of Theoreom 3.2.3 we have that $(E^{\complement}_{\triangleleft} \circ E^{\complement}_{\triangleright})(\{x\}) = \varphi^{-1}(1)$ for some $\varphi \in \boldsymbol{\mathfrak{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. Now by applying Lemma 2.2.5 we see that $\uparrow_1 x \subseteq (E^{\complement}_{\triangleleft} \circ E^{\complement}_{\triangleright})(\{x\})$.

Now consider $z \in X$ such that $x \nleqslant_1 z$. By Lemma 2.2.3(iii) we have that there must exist y such that $(z, y) \in E$ and $(x, y) \notin E$. Now clearly $y \in E^{\complement}_{\triangleright}(\{x\})$ and so $z \notin (E^{\complement}_{\triangleleft} \circ E^{\complement}_{\triangleright})(\{x\})$. $\square$

Corollary 3.2.5. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$*. Then*

$$J^{\infty}(\mathrm{C}(\mathbf{X})) = \{\, J_x \mid x \in X \,\} \quad \textit{and} \quad M^{\infty}(\mathrm{C}(\mathbf{X})) = \{\, M_y \mid y \in X \,\}$$

where for $z \in X$

$$(**) \qquad J_x(z) = \begin{cases} 1 & \textit{if } z \in (E^{\complement}_{\triangleleft} \circ E^{\complement}_{\triangleright})(\{x\}), \\ 0 & \textit{if } z \in (E^{\complement}_{\triangleright})(\{x\}) \end{cases} \quad M_y(z) = \begin{cases} 1 & \textit{if } z \in (E^{\complement}_{\triangleleft})(\{y\}), \\ 0 & \textit{if } z \in (E^{\complement}_{\triangleright} \circ E^{\complement}_{\triangleleft})(\{y\}). \end{cases}$$

We now extend the result of Proposition 3.1.5 to the case of arbitrary graphs.

Proposition 3.2.6. *Let* $\mathbf{X} = (X, E)$ *be any graph and define maps* $J_x \colon \mathbf{X} \to \underset{\sim}{2}$ *and* $M_y \colon \mathbf{X} \to \underset{\sim}{2}$ *by* (**) *above. Then the set* $\{\, J_x \mid x \in X \,\}$ *is join-dense in* $\mathrm{C}(\mathbf{X})$ *and the set* $\{\, M_y \mid y \in X \,\}$ *is meet-dense in* $\mathrm{C}(\mathbf{X})$.

Proof. We will show that for any $\varphi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ we have $\varphi = \bigvee\{\, J_x \mid J_x \leqslant \varphi \,\}$. We note that by definition $\bigvee\{\, J_x \mid J_x \leqslant \varphi \,\} \leqslant \varphi$. For $x \in X$ we have

$$\begin{aligned}
J_x \leqslant \varphi \quad &\iff \quad J_x^{-1}(1) \subseteq \varphi^{-1}(1)\\
&\iff \quad \varphi^{-1}(0) \subseteq J_x^{-1}(0) = E_{\triangleright}^{\complement}(\{x\})\\
&\iff \quad (\forall z \in \varphi^{-1}(0))\ (x, z) \notin E\\
&\iff \quad x \in \varphi^{-1}(1).
\end{aligned}$$

Further, we see that

$$\begin{aligned}
\varphi \leqslant \bigvee\{\, J_x \mid J_x \leqslant \varphi \,\} \quad &\iff \quad \varphi^{-1}(1) \subseteq \big(\bigvee\{\, J_x \mid J_x \leqslant \varphi \,\}\big)^{-1}(1)\\
&\iff \quad \big(\bigvee\{\, J_x \mid J_x \leqslant \varphi \,\}\big)^{-1}(0) \subseteq \varphi^{-1}(0)\\
&\iff \quad \bigcap\{\, J_x^{-1}(0) \mid J_x \leqslant \varphi \,\} \subseteq \varphi^{-1}(0)\\
&\iff \quad \bigcap\{\, J_x^{-1}(0) \mid x \in \varphi^{-1}(1) \,\} \subseteq \varphi^{-1}(0).
\end{aligned}$$

Now, if $y \in \bigcap\{\, J_x^{-1}(0) \mid x \in \varphi^{-1}(1) \,\}$ we have that $(x, y) \notin E$ for all $x \in \varphi^{-1}(1)$ and hence, by Lemma 2.1.1, $y \in \varphi^{-1}(0)$. This shows $\varphi = \bigvee\{\, J_x \mid J_x \leqslant \varphi \,\}$. The proof that $\bigwedge\{\, M_y \mid \varphi \leqslant M_y \,\}$ is similar. □

Finally, one can set up maps $\mathbb{J} \colon \mathbf{X} \to \mathrm{C}(\mathbf{X})$ and $\mathbb{M} \colon \mathbf{X} \to \mathrm{C}(\mathbf{X})$ defined by

$$\mathbb{J} \colon x \mapsto J_x \quad \text{and} \quad \mathbb{M} \colon y \mapsto M_y.$$

These satisfy the properties of the maps γ and μ in the following basic result of FCA (noting that $\mathbb{J}(x) \leqslant \mathbb{M}(y)$ if and only if $(x, y) \notin E$).

Theorem 3.2.7 ([45, Theorem 1.3])**.** *Let* $\mathbf{K} = (O, P, I)$ *be a context and consider the complete lattice* $\mathrm{CL}(\mathbf{K})$. *A lattice* $\mathbf{L}$ *is isomorphic to* $\mathrm{CL}(\mathbf{K})$ *if and only if there are mappings* $\gamma \colon O \to \mathbf{L}$ *and* $\mu \colon P \to \mathbf{L}$ *such that* $\gamma(O)$ *is join dense in* $\mathbf{L}$, $\mu(P)$ *is meet dense in* $\mathbf{L}$, *for all* $o \in O, a \in P$, $(o, a) \in I$ *if and only if* $\gamma(o) \leqslant \mu(a)$.

Now by combining Proposition 3.2.6 and Theorem 3.2.7 we are able to conclude an alternative argument to our first result in this section (proved in Proposition 3.2.1) stating that the lattices $\mathrm{C}(\mathbf{X})$ and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ are isomorphic.

3.3 The Gehrke–Harding construction

The original construction of a canonical extension $\mathbf{L}^\delta$ of a bounded lattice $\mathbf{L}$ by Gehrke and Harding [48, Proposition 2.6] uses a Galois connection between $\wp(\mathrm{Filt}(\mathbf{L}))$ and $\wp(\mathrm{Idl}(\mathbf{L}))$. The relation $(F, I) \in R$ iff $F \cap I \neq \emptyset$ gives rise to the Galois connection $R_\triangleright \colon \wp(\mathrm{Filt}(\mathbf{L})) \to \wp(\mathrm{Idl}(\mathbf{L}))$ and $R_\triangleleft \colon \wp(\mathrm{Idl}(\mathbf{L})) \to \wp(\mathrm{Filt}(\mathbf{L}))$ where the polars are given for $A \subseteq \mathrm{Filt}(\mathbf{L})$ and $B \subseteq \mathrm{Idl}(\mathbf{L})$ by

$$R_\triangleright(A) = \{\, I \in \mathrm{Idl}(\mathbf{L}) \mid (\forall\, F \in A)\ (F, I) \in R \,\}$$

and

$$R_\triangleleft(B) = \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid (\forall\, I \in B)\ (F, I) \in R \,\}.$$

The Galois-closed subsets of $\mathrm{Filt}(\mathbf{L})$, $\{\, A \subseteq \mathrm{Filt}(\mathbf{L}) \mid A = (R_\triangleleft \circ R_\triangleright)(A) \,\}$, ordered by inclusion, form the canonical extension of $\mathbf{L}$ [48]. To distinguish it from the previous constructions, we will denote it using the first letters of the surnames of its authors by $\mathrm{GH}(\mathbf{L})$. The lattice $\mathbf{L}$ is embedded in $\mathrm{GH}(\mathbf{L})$ via the embedding $a \mapsto A_a := \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid a \in F \,\}$. We observe that this embedding is exactly the embedding used by Hartonas and Dunn [65, Theorem 2.4] in their duality for bounded lattices. This provides yet another example of a duality theorem from which canonical extensions can be constructed.

We recall that Proposition 2.3.7 from Chapter 2 shows that the canonical extension constructed from the graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$ of maximal partial homomorphisms is order-isomorphic to the complete lattice of MPE's from the bigger graph $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of special partial homomorphisms into $\underset{\sim}{\mathbf{2}}$.

We now use this bigger dual $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of a bounded lattice $\mathbf{L}$. For the complement of this graph relation, $E_Y^{\complement} = (Y \times Y) \backslash E_Y$, on the bigger dual graph $\mathbf{Y}$, we shall consider the context $\mathbb{K}(\mathbf{Y}) := (Y, Y, E_Y^{\complement})$ and the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ of this context as defined in the previous section. Our first result in this section reconciles this concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ with the canonical extension $\mathrm{GH}(\mathbf{L})$ of $\mathbf{L}$ by Gehrke and Harding. The intuition behind this isomorphism is to map a Galois-closed set of filters A to the set of all SPH's whose filter part is a member of A.

Proposition 3.3.1. *Let* $\mathbf{L}$ *be a bounded lattice and let* $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$*. Then the complete lattice* $\mathrm{GH}(\mathbf{L}) = \{\, A \subseteq \mathrm{Filt}(\mathbf{L}) \mid A = (R_\triangleleft \circ R_\triangleright)(A) \,\}$ *is order-isomorphic to the concept lattice* $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ *via the isomorphism* $\Phi_{CL}^{GH} : \mathrm{GH}(\mathbf{L}) \to \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$*, defined by* $A \mapsto \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f^{-1}(1) \in A \,\}$.

Proof. In the proof we denote Φ^{GH}_{CL} simply by Φ and use $E^{\complement}_{\rhd}$ and $E^{\complement}_{\lhd}$ for the polars coming from $E^{\complement}_{Y}$. First we observe that for any $g \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$,

$$\begin{aligned} g \in E^{\complement}_{\rhd}(\Phi(A)) &\iff (\forall h \in \Phi(A))\ (h, g) \notin E_Y \\ &\iff (\forall h \in \Phi(A))\ h^{-1}(1) \cap g^{-1}(0) \neq \emptyset \\ &\iff (\forall F \in A)\ F \cap g^{-1}(0) \neq \emptyset \\ &\iff g^{-1}(0) \in R_{\rhd}(A). \end{aligned}$$

We have $\Phi(A) \in \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ since

$$\begin{aligned} f \in (E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd})(\Phi(A)) &\iff (\forall g \in E^{\complement}_{\rhd}(\Phi(A)))\ (f, g) \notin E_Y \\ &\iff (\forall g \in E^{\complement}_{\rhd}(\Phi(A)))\ f^{-1}(1) \cap g^{-1}(0) \neq \emptyset \\ &\iff (\forall I \in R_{\rhd}(A))\ f^{-1}(1) \cap I \neq \emptyset \\ &\iff f^{-1}(1) \in (R_{\lhd} \circ R_{\rhd})(A) = A \\ &\iff f \in \Phi(A). \end{aligned}$$

It is clear that Φ is 1-1. To see that Φ is onto, for any $Z \in \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$, we consider the set $A = \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid F = f^{-1}(1) \text{ for some } f \in Z \,\}$. It is easy to see that $\Phi(A) = Z$. Finally we show that $A \in \mathrm{GH}(\mathbf{L})$ by using similar arguments to those above. For any $f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$,

$$\begin{aligned} f^{-1}(1) \in (R_{\lhd} \circ R_{\rhd})(A) &\iff (\forall I \in R_{\rhd}(A))\ f^{-1}(1) \cap I \neq \emptyset \\ &\iff (\forall g \in E^{\complement}_{\rhd}(\Phi(A)))\ f^{-1}(1) \cap g^{-1}(0) \neq \emptyset \\ &\iff (\forall g \in E^{\complement}_{\rhd}(Z))\ f^{-1}(1) \cap g^{-1}(0) \neq \emptyset \\ &\iff (\forall g \in E^{\complement}_{\rhd}(Z))\ (f, g) \notin E_Y \\ &\iff f \in (E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd})(Z) = Z \\ &\iff f^{-1}(1) \in A. \end{aligned}$$

For $A, A' \subseteq \mathrm{Filt}(\mathbf{L})$ we clearly have $A \subseteq A'$ if and only if $\Phi(A) \subseteq \Phi(A')$, so Φ is an order-isomorphism. □

As before, we need to confirm that this isomorphism restricts to the representation of the lattice.

Proposition 3.3.2. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. *Let* $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ *and* $\mathbb{K}(\mathbf{Y}) := (Y, Y, E^{\complement}_{Y})$. *Then the map* $\Phi^{GH}_{CL} \colon \mathrm{GH}(\mathbf{L}) \to \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ *is an order-isomorphism between the embedded copies of* $\mathbf{L}$ *in each of the complete lattices.*

Proof. Consider the embeddings $a \mapsto A_a := \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid a \in F \,\}$ of $\mathbf{L}$ into $\mathrm{GH}(\mathbf{L})$ and $a \mapsto \bar{e}_a^{-1}(1)$ of $\mathbf{L}$ into $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$. Now for $\Phi := \Phi_{CL}^{GH}$ we have

$$\begin{aligned}\Phi(A_a) &= \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f^{-1}(1) \in A_a \,\}\\ &= \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid a \in f^{-1}(1) \,\}\\ &= \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1 \,\}\\ &= \bar{e}_a^{-1}(1).\end{aligned}$$

For $a, b \in L$ we obviously have $A_a \subseteq A_b \iff a \leqslant b \iff \bar{e}_a^{-1}(1) \subseteq \bar{e}_b^{-1}(1)$ and so we conclude that the embedded copies of $\mathbf{L}$ in $\mathrm{GH}(\mathbf{L})$ and in $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ are order-isomorphic via the isomorphism Φ. □

Our second result in this section will reconcile the approaches to the canonical extension of a bounded lattice based on Ploščica's representation and on the Gehrke–Harding approach without using the bigger Allwein–Hartonas dual space as a stepping stone.

We recall that for $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and its Ploščica dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, we have $a \in L$ embedded as e_a in $\mathrm{C}(\mathbf{X}) = \boldsymbol{\mathfrak{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ while by the Gehrke–Harding approach $a \in L$ is embedded as A_a in $\mathrm{GH}(\mathbf{L})$. We have

$$\begin{aligned}e_a^{-1}(1) &= \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1 \,\},\\ e_a^{-1}(0) &= \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0 \,\},\\ A_a &= \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid a \in F \,\}.\end{aligned}$$

We remember that elements of $\mathrm{C}(\mathbf{X})$ are MPE's from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$. Now, we want to assign to each MPE a set of filters of $\mathbf{L}$. In the search for a mapping $\Phi_{GH}^{CX} \colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ such that $\Phi_{GH}^{CX}(e_a) = A_a$, we require that every filter F in A_a must have the property that $a \in F$. We now extend this requirement to arbitrary elements of $\boldsymbol{\mathfrak{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. For $\varphi \in \boldsymbol{\mathfrak{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ we define a subset $\Phi(\varphi) \subseteq \mathrm{Filt}(\mathbf{L})$ by

$$\Phi(\varphi) := \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid (\forall\, f \in \varphi^{-1}(0))\ F \cap f^{-1}(0) \neq \emptyset \,\}.$$

The proof that the map $\Phi_{GH}^{CX} \colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ is well-defined follows easily from the definition. To show that Φ_{GH}^{CX} is an order isomorphism from $\mathrm{C}(\mathbf{X})$ to $\mathrm{GH}(\mathbf{L})$, we first show that it is an order embedding, and then that it is onto.

Lemma 3.3.3. *The map* $\Phi_{GH}^{CX} \colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L}), \varphi \mapsto \Phi(\varphi)$ *is an order embedding.*

Proof. Let $\varphi, \psi \in \mathrm{C}(\mathbf{X})$. It is clear that $\psi^{-1}(0) \subseteq \varphi^{-1}(0)$ implies that the set $\{\, F \in \mathrm{Filt}(\mathbf{L}) \mid (\forall f \in \varphi^{-1}(0))\ F \cap f^{-1}(0) \neq \emptyset \,\}$ is a subset of the larger set of filters $\{\, G \in \mathrm{Filt}(\mathbf{L}) \mid (\forall g \in \psi^{-1}(0))\ G \cap g^{-1}(0) \neq \emptyset \,\}$. Thus $\varphi \leqslant \psi$ yields that $\Phi(\varphi) \subseteq \Phi(\psi)$.

Now assume that $\varphi \nleqslant \psi$. This implies that $\psi^{-1}(0) \nsubseteq \varphi^{-1}(0)$ and so there exists $f \in \psi^{-1}(0)$ such that $\varphi(f) \neq 0$. Since φ is an MPE, there exists $g \in \varphi^{-1}(1)$ such that $(g, f) \in E$. Now $g \in \varphi^{-1}(1)$ implies that for all $h \in \varphi^{-1}(0)$, we have $g^{-1}(1) \cap h^{-1}(0) \neq \emptyset$. Thus $g^{-1}(1) \in \Phi(\varphi)$. However, $g^{-1}(1) \cap f^{-1}(0) = \emptyset$ means that $g^{-1}(1) \notin \Phi(\psi)$, and hence $\Phi(\varphi) \nsubseteq \Phi(\psi)$ as required. □

Lemma 3.3.4. *The map* $\Phi^{CX}_{GH}\colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L}), \varphi \mapsto \Phi(\varphi)$ *is onto.*

Proof. Let $A \in \mathrm{GH}(\mathbf{L})$ and consider the map $\varphi_A\colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ defined for $x \in X$ by

$$\varphi_A^{-1}(0) = \{\, f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid (\forall F \in A)\ F \cap f^{-1}(0) \neq \emptyset \,\}$$

and

$$\varphi_A^{-1}(1) = \{\, g \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid (\forall f \in \varphi_A^{-1}(0))\ (g, f) \notin E \,\}.$$

It is clear from the definition that φ_A is E-preserving. To show the maximality of φ_A, suppose that $\mathrm{dom}(\varphi_A) \subseteq \mathrm{dom}(\psi)$ for some E-preserving map $\psi\colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$. If $g \in \psi^{-1}(1)$ then since $\varphi_A^{-1}(0) \subseteq \psi^{-1}(0)$ we must have for all $f \in \varphi_A^{-1}(0)$ that $(g, f) \notin E$. Thus $g \in \varphi_A^{-1}(1)$. Now suppose that $f \in \psi^{-1}(0)$, but $f \notin \varphi_A^{-1}(0)$. This implies that there exists $F \in A$ such that $F \cap f^{-1}(0) = \emptyset$. Consider the partial homomorphism $h\colon \mathbf{L} \to \underline{\mathbf{2}}$ defined by $h^{-1}(1) = F$ and $h^{-1}(0) = f^{-1}(0)$. This can be extended to an MPH h_*. Then $(h_*, f) \in E$ as $h_*^{-1}(1) \cap h_*^{-1}(0) = \emptyset$ and $f^{-1}(0) \subseteq h_*^{-1}(0)$ imply $h_*^{-1}(1) \cap f^{-1}(0) = \emptyset$. Since ψ is E-preserving this implies that $h_* \notin \psi^{-1}(1)$ which in turn implies that $h_* \notin \varphi_A^{-1}(1)$. Now this means that there exists $g \in \varphi_A^{-1}(0)$ such that $(h_*, g) \in E$. But now $F \subseteq h_*^{-1}(1)$ implies that $F \cap g^{-1}(0) = \emptyset$, but this cannot be true since $g \in \varphi_A^{-1}(0)$. Thus f cannot exist and so the domain of φ_A is maximal.

We now want to show that $\Phi(\varphi_A) = A$. If $F \in A$, then by the definitions of $\varphi_A^{-1}(0)$ and $\Phi(\varphi_A)$ it follows that $F \in \Phi(\varphi_A)$. Assume now that $F \notin A$. Then $A \in \mathrm{GH}(\mathbf{L})$ implies that $F \notin (R_{\lhd} \circ R_{\rhd})(A)$ and so there exists $I \in R_{\rhd}(A)$ such that $F \cap I = \emptyset$. Now define a partial homomorphism f by $f^{-1}(1) = F$ and $f^{-1}(0) = I$. This can be extended to an MPH f_*. Now since $f_*^{-1}(0) \supseteq I$ and $I \in R_{\rhd}(A)$ we have that $f_* \in \varphi_A^{-1}(0)$. This shows that $F \notin \Phi(\varphi_A)$ since $f_*^{-1}(1) \cap f_*^{-1}(0) = \emptyset$ and $F \subseteq f_*^{-1}(1)$ imply $F \cap f_*^{-1}(0) = \emptyset$. Thus $\Phi(\varphi_A) = A$ as required. □

The lemmas above now yield the final result of this section.

Theorem 3.3.5. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. *Let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and* $\mathrm{C}(\mathbf{X}) = \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. *Then the map* $\Phi_{GH}^{CX} \colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ *given for* $\varphi \in \mathrm{C}(\mathbf{X})$ *by*

$$\Phi(\varphi) = \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid (\forall\, f \in \varphi^{-1}(0))\ F \cap f^{-1}(0) \neq \emptyset \,\}$$

is an order-isomorphism. Further, it preserves the embedded copies of $\mathbf{L}$ *in the lattices* $\mathrm{C}(\mathbf{X})$ *and* $\mathrm{GH}(\mathbf{L})$.

3.4 An application of the concept analysis approach

Here we show how our reconciliation in Section 3.2 can be used to simplify the results concerning morphisms from Section 2.3. When defining the category of $\boldsymbol{\mathcal{L}}$-graphs and $\boldsymbol{\mathcal{L}}$-graph morphisms in Definition 2.3.10, our aim was to replicate the untopologised conditions of the definition of morphisms in the category of enhanced L-spaces from Allwein and Hartonas [2, Definition 2.9]. We needed the morphisms between untopologised graphs to satisfy certain conditions in order to ensure that they would be extended by the functor $\overline{\mathrm{G}}$ (defined on page 42) to complete lattice homomorphisms between the complete lattices serving as the canonical extensions of the original bounded lattices.

We will now define a subcategory of the category $\boldsymbol{\mathcal{G}}$ of graphs and E-preserving maps. We will denote this category by $\boldsymbol{\mathcal{G}}^{+}$ and our goal is to define a functor G^{+} whose domain is the category $\boldsymbol{\mathcal{G}}^{+}$ and whose codomain is the category $\boldsymbol{\mathcal{L}}^{+}$ of complete lattices and complete lattice homomorphisms.

Definition 3.4.1. Let $\mathbf{X} = (X, E_X)$ and $\mathbf{Y} = (Y, E_Y)$ be graphs and let $\alpha \colon \mathbf{X} \to \mathbf{Y}$. We say that α is a *graph* $+$*-morphism* if the following hold for all $a, b \in X$ and $A \subseteq Y$:

(i) $\alpha^{-1}({}_{Y}E^{\complement}_{\lhd}(A)) = {}_{X}E^{\complement}_{\lhd}(\alpha^{-1}(A))$;

(ii) $\alpha^{-1}({}_{Y}E^{\complement}_{\rhd}(A)) = {}_{X}E^{\complement}_{\rhd}(\alpha^{-1}(A))$.

It is not difficult to check that graph $+$-morphisms are closed under composition and so they are indeed a class of morphisms. We denote by $\boldsymbol{\mathcal{G}}^{+}$ the category of graphs with graph $+$-morphisms. We write the subscript X and Y to the left of E so that we can keep track of the different E relations without interfering with our polar maps.

We note that the fact that a graph $+$-morphism is E-preserving follows from property (i) above. Furthermore, in the case that $\mathbf{X}$ and $\mathbf{Y}$ are $\boldsymbol{\mathcal{L}}$-graphs, a map

$\alpha\colon \mathbf{X} \to \mathbf{Y}$ is a graph +-morphism if and only if it is an $\boldsymbol{\mathcal{L}}$-graph morphism. This follows from Theorem 3.2.3.

Now we wish to define a functor G^{+} such that for any $\mathbf{X}, \mathbf{Y} \in \boldsymbol{\mathcal{G}}^{+}$, and for any $\alpha \in \boldsymbol{\mathcal{G}}^{+}(\mathbf{X}, \mathbf{Y})$, we have that $\mathrm{G}^{+}(\alpha)$ is a complete lattice homomorphism between the complete lattices $\boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ and $\boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. Our proposed functor $\mathrm{G}^{+}\colon \boldsymbol{\mathcal{G}}^{+} \to \boldsymbol{\mathcal{L}}^{+}$ will be defined as follows:

on objects: $\mathrm{G}^{+}\colon \mathbf{X} \mapsto \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$,

on morphisms: $\mathrm{G}^{+}\colon \alpha \mapsto - \circ \alpha$.

Our earlier results show that $\boldsymbol{\mathcal{G}}^{+}$ is well-defined on objects. The challenge will be to confirm that it is well-defined on morphisms. For $\mathbf{X}, \mathbf{Y} \in \boldsymbol{\mathcal{G}}^{+}$ and $\alpha \in \boldsymbol{\mathcal{G}}^{+}(\mathbf{X}, \mathbf{Y})$, we must show that for any $\varphi \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ we have that $\varphi \circ \alpha \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. It is clear that $\varphi \circ \alpha$ is a partial map from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$, and Lemma 2.1.4 confirms that it will be E_X-preserving, but we must show that $\varphi \circ \alpha$ is in fact an MPE.

Proposition 3.4.2. *Let* $\mathbf{X}, \mathbf{Y} \in \boldsymbol{\mathcal{G}}^{+}(\mathbf{X}, \mathbf{Y})$ *and let* $\alpha \in \boldsymbol{\mathcal{G}}^{+}(\mathbf{X}, \mathbf{Y})$. *Then for any* $\varphi \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ *we have that* $(\mathrm{G}^{+}(\alpha))(\varphi) \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$.

Proof. Consider the set $(\varphi \circ \alpha)^{-1}(1) = \alpha^{-1}(\varphi^{-1}(1))$. Proposition 3.2.1 tells us that since $\varphi \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$, we have that $\varphi^{-1}(1) = {}_{Y}E^{\complement}_{\lhd}(A)$ for some $A \subseteq Y$. Now we use property (ii) from the definition of a graph +-morphism to see that

$$\begin{aligned}(\varphi \circ \alpha)^{-1}(1) &= \alpha^{-1}(\varphi^{-1}(1))\\ &= \alpha^{-1}({}_{Y}E^{\complement}_{\lhd}(A))\\ &= {}_{X}E^{\complement}_{\lhd}(\alpha^{-1}(A)).\end{aligned}$$

In other words, we have shown that $(\varphi \circ \alpha)^{-1}(1) = {}_{X}E^{\complement}_{\lhd}(B)$ where $B = \alpha^{-1}(A) \subseteq X$. Now

$$\begin{aligned}(\varphi \circ \alpha)^{-1}(0) &= \alpha^{-1}(\varphi^{-1}(0))\\ &= \alpha^{-1}\big({}_{Y}E^{\complement}_{\rhd}({}_{Y}E^{\complement}_{\lhd}(A))\big)\\ &= {}_{X}E^{\complement}_{\rhd}\big(\alpha^{-1}({}_{Y}E^{\complement}_{\lhd}(A))\big)\\ &= {}_{X}E^{\complement}_{\rhd}\big({}_{X}E^{\complement}_{\lhd}(\alpha^{-1}(A))\big)\\ &= {}_{X}E^{\complement}_{\rhd}\big((\varphi \circ \alpha)^{-1}(1)\big).\end{aligned}$$

From the fact that $(\varphi \circ \alpha)^{-1}(0) = {}_{X}E^{\complement}_{\rhd}\big((\varphi \circ \alpha)^{-1}(0)\big)$ and $(\varphi \circ \alpha)^{-1}(1) = {}_{X}E^{\complement}_{\lhd}(B)$ for $B \subseteq X$ we have by Proposition 3.2.1 that $\varphi \circ \alpha \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. $\square$

We have shown that for $\mathbf{X}, \mathbf{Y} \in \mathcal{G}^+$ and $\alpha \in \mathcal{G}^+(\mathbf{X}, \mathbf{Y})$ we have that $\mathrm{G}^+(\alpha)$ is a map from $\mathrm{G}^+(\mathbf{Y}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{2})$ to $\mathrm{G}^+(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. The final task is to show that $\mathrm{G}^+(\alpha)$ is a complete lattice homomorphism.

Proposition 3.4.3. *Let* $\mathbf{X}, \mathbf{Y} \in \mathcal{G}^+$ *and let* $\alpha \in \mathcal{G}^+(\mathbf{X}, \mathbf{Y})$. *Then*

$$\mathrm{G}^+(\alpha) \in \boldsymbol{\mathcal{L}}^+(\mathcal{G}^+(\mathbf{Y}), \mathcal{G}^+(\mathbf{X})).$$

Proof. Let $\{\, \varphi_i \mid i \in I \,\} \subseteq \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{2})$. We need to show

$$\big(\mathrm{G}^+(\alpha)\big)\big(\bigwedge_{\mathrm{i\in I}} \varphi_{\mathrm{i}}\big) = \bigwedge_{\mathrm{i\in I}} \big(\mathrm{G}^+(\alpha)\big)(\varphi_{\mathrm{i}}).$$

We observe that for $x \in X$:

$$\begin{aligned}
x \in \big(\bigwedge_{i\in I}(\varphi_i \circ \alpha)\big)^{-1}(1) &\iff x \in \bigcap_{i\in I}(\varphi_i \circ \alpha)^{-1}(1) \\
&\iff x \in \bigcap_{i\in I} \alpha^{-1}(\varphi_i^{-1}(1)) \\
&\iff \alpha(x) \in \bigcap_{i\in I} \varphi_i^{-1}(1) \\
&\iff \alpha(x) \in \big(\bigwedge_{i\in I} \varphi_i\big)^{-1}(1) \\
&\iff x \in \alpha^{-1}\Big(\big(\bigwedge_{i\in I} \varphi_i\big)^{-1}(1)\Big) \\
&\iff x \in \Big(\big(\bigwedge_{i\in I} \varphi_i\big) \circ \alpha\Big)^{-1}(1). \qquad \square
\end{aligned}$$

In proving the above result for the functor G^+ we have changed the domain of the functor $\overline{\mathrm{G}}$ from $\boldsymbol{\mathcal{L}}$-graphs with $\boldsymbol{\mathcal{L}}$-graph morphisms to the larger class of arbitrary graphs with graph +-morphisms.

We remark that one of the crucial results here has been Proposition 3.2.1. We referred to this as the concept analysis approach, but in reality it is quite different from the intention of concept analysis where the two sets O and P are different sets representing objects and attributes. The benefit of using a single set X as the underlying set is that the definition of morphisms is more natural. The morphisms between contexts (equivalently polarities or RS-frames) are usually relations, and these can be intuitively difficult to handle.

Chapter 4

Untopologised structures used in the construction of $\mathbf{L}^{\delta}$

This chapter collects together results about the so-called *intermediate structure* as well as a further analysis of the particular graphs (of MPH's) used in Chapter 2 and Chapter 3. The first two sections examine the structure and properties of the poset that is the intermediate structure. The third section looks at the well-known example of Funayama [44], and shows how the canonical extension differs from the MacNeille completion (defined in Appendix A.1). The final section looks at the relationship between graphs of the form $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ and the *RS-frames* used by Gehrke [47]. We provide an answer in the finite case to a question posed by Ploščica [78, Section 3] regarding the minimality of dual graphs of lattices.

4.1 The intermediate structure

The first description of the intermediate structure was given in 1997 by Ghilardi and Meloni [59, Section 3]. They were interested in constructing canonical models for Boolean algebras with operators without the use of any choice principle. They defined a quasi-ordered set consisting of the filters and ideals of a Boolean algebra, and referred to it as the *intermediate level*. It was further shown how operations from the underlying algebra could be lifted to this intermediate level and then to the canonical model.

This idea of a structure which combined the filters and ideals of a lattice was used in the construction of the canonical extension of a poset by Dunn, Gehrke and Palmigiano [33]. The motivation for trying to expand the theory of canonical extensions to poset-based algebras was to obtain complete relational semantics for substructural logics. The specific posets that were dealt with in [33] were monotone poset expansions—posets equipped with additional operations that are either order-preserving or order-reversing in each argument. Complete relational semantics were obtained for the fusion-implication fragment of substructural logics [33, Section 6].

As they were operating in the context of a poset $\mathbf{P} = (P, \leqslant)$, the 'filters' were defined to be the down-directed up-sets and the 'ideals' were defined to be the up-directed down-sets. We note that in the case that the poset $\mathbf{P}$ is a lattice, the down-directed up-sets correspond to the usual lattice filters, and the up-directed down-sets correspond to the usual lattice ideals. Thus we will use the notation $\mathrm{Filt}(\mathbf{P})$ and $\mathrm{Idl}(\mathbf{P})$ for the case of both posets and lattices.

A *completion* of a poset $\mathbf{P}$ is a pair $(e, \mathbf{C})$ where $e \colon P \hookrightarrow C$ is an order embedding and $\mathbf{C}$ is a complete lattice. We will suppress the embedding when it is clear from the context. The formal definition of the canonical extension of a poset from [33] is given below.

Definition 4.1.1 ([33, Definitions 2.1 and 2.2])**.** Let $\mathbf{P} = (P, \leqslant)$ be a poset and $(e, \mathbf{C})$ a completion of $\mathbf{P}$. An element $x \in C$ is a *filter element* if $x = \bigwedge e(F)$ for some $F \in \mathrm{Filt}(\mathbf{P})$. Dually, $y \in C$ is an *ideal element* if $y = \bigvee e(I)$ for $I \in \mathrm{Idl}(\mathbf{P})$. The completion $\mathbf{C}$ is *dense* if every element of C is the join of the filter elements below it and the meet of the ideal elements above it. Further, $\mathbf{C}$ is *compact* if whenever $D \subseteq P$ is non-empty and down-directed, and $U \subseteq P$ is non-empty and up-directed such that $\bigwedge e(D) \leqslant \bigvee e(U)$, then there exist $x \in D$ and $y \in U$ such that $x \leqslant y$. A *canonical extension* of $\mathbf{P}$ is a dense and compact completion of $\mathbf{P}$.

We now define the intermediate structure, the major focus of this section.

Definition 4.1.2. Let $\mathbf{P}$ be a poset. Consider the structure $(\mathrm{Filt}(\mathbf{P}) \cup \mathrm{Idl}(\mathbf{P}), \preceq)$ where $\preceq$ is a binary relation on $\mathrm{Filt}(\mathbf{P}) \cup \mathrm{Idl}(\mathbf{P})$ defined for $F, G \in \mathrm{Filt}(\mathbf{P})$ and $I, J \in \mathrm{Idl}(\mathbf{P})$ by:

- $F \preceq G$ if $G \subseteq F$,
- $F \preceq I$ if $F \cap I \neq \emptyset$,
- $I \preceq J$ if $I \subseteq J$,
- $I \preceq F$ if for all $a \in I$, for all $b \in F$, $a \leqslant b$.

The relation $\preceq$ is a quasi-order. For $H_1, H_2 \in \mathrm{Filt}(\mathbf{P}) \cup \mathrm{Idl}(\mathbf{P})$, the equivalence relation $\sim$ is defined by $H_1 \sim H_2$ if and only if $H_1 \preceq H_2$ and $H_2 \preceq H_1$. Letting $\sqsubseteq \, = \, \preceq / \sim$, the *intermediate structure* is the quotient

$$\mathtt{IM}(\mathbf{P}) := \big((\mathrm{Filt}(\mathbf{P}) \cup \mathrm{Idl}(\mathbf{P}))/{\sim}\,,\, \sqsubseteq\,\big).$$

It is not difficult to see that for $H_1 \neq H_2 \in \mathrm{Filt}(\mathbf{P}) \cup \mathrm{Idl}(\mathbf{P})$, one has $H_1 \sim H_2$ if and only if $H_1 = {\uparrow}x$ and $H_2 = {\downarrow}x$ (or vice versa) for some $x \in P$.

Theorem 4.1.3 ([33, Theorem 2.5])**.** *Let* $\mathbf{P}$ *be a poset and* $\mathtt{IM}(\mathbf{P})$ *its intermediate structure. Then the MacNeille completion of the intermediate structure,* $\overline{\mathtt{IM}(\mathbf{P})}$*, is a canonical extension of* $\mathbf{P}$*.*

As pointed out in [33, Remark 2.7], the description of the canonical extension in Theorem 4.1.3, like the descriptions used by Ghilardi and Meloni [59] and by Gehrke and Harding [48], is clearly *constructive*. Other constructions that go via duality theory usually rely on some sort of choice principle.

Beyond its application to monotone poset expansions, Gehrke and Priestley [56] used the construction via the intermediate structure to provide a proof of the fact that a lattice homomorphism $f\colon \mathbf{L} \to \mathbf{K}$ extends to a complete lattice homomorphism $f^\delta\colon \mathbf{L}^\delta \to \mathbf{K}^\delta$. This was the first proof which did not use the f^σ and f^π liftings of additional operations to show that a lattice homomorphism between two lattices extends to a complete lattice homomorphism between their canonical extensions.

The work of Gehrke and Priestley made use of the so-called *cut-stable* maps between quasi-ordered sets defined by Erné [36]. We remember that for a quasi-ordered set $\mathbf{P} = (P, \preceq)$ and $A \subseteq P$, we have

$$A^u = \{\, b \in P \mid a \in A \Rightarrow a \preceq b \,\} \quad \text{and} \quad A^\ell = \{\, b \in P \mid a \in A \Rightarrow b \preceq a \,\}.$$

Definition 4.1.4 ([36, Section 2])**.** Let $\mathbf{P} = (P, \preceq)$ and $\mathbf{Q} = (Q, \preceq)$ be quasi-ordered sets. A map $f\colon P \to Q$ is *lower cut-stable* if for all $A \subseteq P$,

$$(f(A^u))^\ell = (f(A))^{u\ell},$$

and *upper cut-stable* if for all $B \subseteq P$,

$$(f(B^\ell))^u = (f(B))^{\ell u}.$$

A map is *cut-stable* if it is both lower cut-stable and upper cut-stable.

The motivation behind defining cut-stable maps is that every cut-stable map f between quasi-ordered sets $\mathbf{P} = (P, \preceq)$ and $\mathbf{Q} = (Q, \preceq)$ is lifted to a complete lattice homomorphism $\overline{f}\colon \overline{\mathbf{P}} \to \overline{\mathbf{Q}}$ [36, Theorem 3.1]. It is further shown that the category of complete lattices with complete lattice homomorphisms is a reflective subcategory of the category of quasi-ordered sets and cut-stable maps [36, Corollary 3.3].

Gehrke and Priestley provide a necessary and sufficient condition for an order-preserving map between posets to be lifted to a cut-stable map between the intermediate structures [56, Definition 4.3]. Furthermore, they show that for a lattice homomorphism between two lattices (which is clearly order-preserving), the homomorphism satisfies this necessary condition and thus extends to a cut-stable map between the intermediate structures [56, Theorem 4.6].

Our motivation for examining the intermediate structure was to use it to learn more about the order-theoretic structure of complete lattices which occur as the canonical extensions of bounded lattices. Our conjecture was that a better understanding of $\mathtt{IM}(\mathbf{L})$ could be combined with knowledge of the MacNeille completion to expose properties of $\mathbf{L}^{\delta}$. This has not been the final outcome, as the structure of $\mathtt{IM}(\mathbf{L})$ proved harder to understand than anticipated.

In our diagrams in this chapter, we will always use unfilled points to represent elements of the original lattice, while filled points will represent limit points that are added either at the level of the intermediate structure or in the final MacNeille completion of the intermediate structure.

4.2 The intermediate structure as a lattice

The role of the intermediate structure in the completion process is to add meets of non-principal filters and joins of non-principal ideals. We will refer interchangeably to the limit point $a = \bigwedge F$ and the filter F as an element of $\mathtt{IM}(\mathbf{L})$ (ignoring the quotient map to its equivalence class). In general, proofs will use the filter F, while diagrams will use $a = \bigwedge F$.

The example shown in Fig. 4.1 is a simple demonstration of how limit points are added by the formation of the intermediate structure. In this case, the intermediate structure is already a complete lattice and so no additional limit points are added by the subsequent MacNeille completion. This example is also a simple demonstration of how the intermediate structure and resulting canonical extension of a lattice can differ from the MacNeille completion of the lattice. The Chang algebra is $\mathbf{C} = \langle C; \oplus, \neg, 0 \rangle$ with the underlying set defined as

$$C := \{\, (0, a) \mid a \in \mathbb{Z}^{+} \,\} \cup \{\, (1, b) \mid b \in \mathbb{Z}^{-} \,\}.$$

We note that any MV-algebra is a distributive lattice as the lattice operations $\wedge, \vee$ are term-definable by $a \vee b := \neg(\neg a \oplus b) \oplus b$ and $a \wedge b := \neg(\neg a \vee \neg b)$. In this instance the lattice operations are clear as the underlying set has the lexicographic as its natural

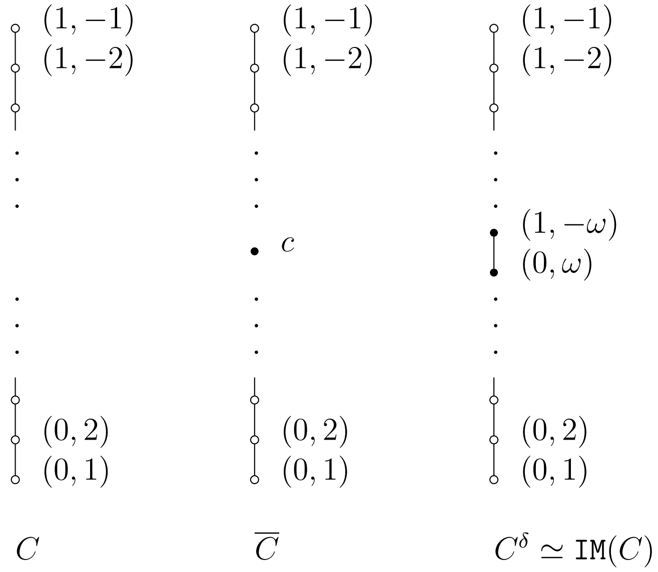


Figure 4.1: The Chang algebra showing how the MacNeille completion and the canonical extension of a bounded lattice can differ.

(linear) order. The full Chang algebra was used by Gehrke and Priestley [53] to show that the variety of MV-algebras is not canonical.

In Fig. 4.1, the element $(1, -\omega)$ represents the non-principal filter $\{\, (1, b) \mid b \in \mathbb{Z}^- \,\}$ in $\mathtt{IM}(\mathbf{C})$. Similarly $(0, \omega)$ represents the non-principal ideal $\{\, (0, a) \mid a \in \mathbb{Z}^+ \,\}$.

The example shown in Fig. 4.2 demonstrates that the formation of the intermediate structure by the addition of filter and ideal limit points will not always result in $\mathtt{IM}(\mathbf{L})$ being a complete lattice. The point $(-\omega, \omega)$ is the meet of the set $\{\, (-n, \omega) \mid n \in \omega \,\}$ and the join of the set $\{\, (-\omega, n) \mid n \in \omega \,\}$, and $(-\omega, \omega) \notin \mathtt{IM}(\mathbf{L})$.

In our introduction to canonical extensions in Section 1.1, Proposition 1.1.4 stated that both the set of filter elements and the set of ideal elements form sublattices of $\mathbf{L}^\delta$. The ordering of these elements is inherited from the canonical extension and coincides with the ordering on the intermediate structure. Since the MacNeille completion of a poset does not destroy existing joins, we have that the subposets $\mathrm{Filt}(\mathbf{L}) \subseteq \mathtt{IM}(\mathbf{L})$ and $\mathrm{Idl}(\mathbf{L}) \subseteq \mathtt{IM}(\mathbf{L})$ have joins and meets defined. In fact, some infinite meets and infinite joins will exist as well.

We have adapted the notation and statement of the following proposition, only quoting the relevant results.

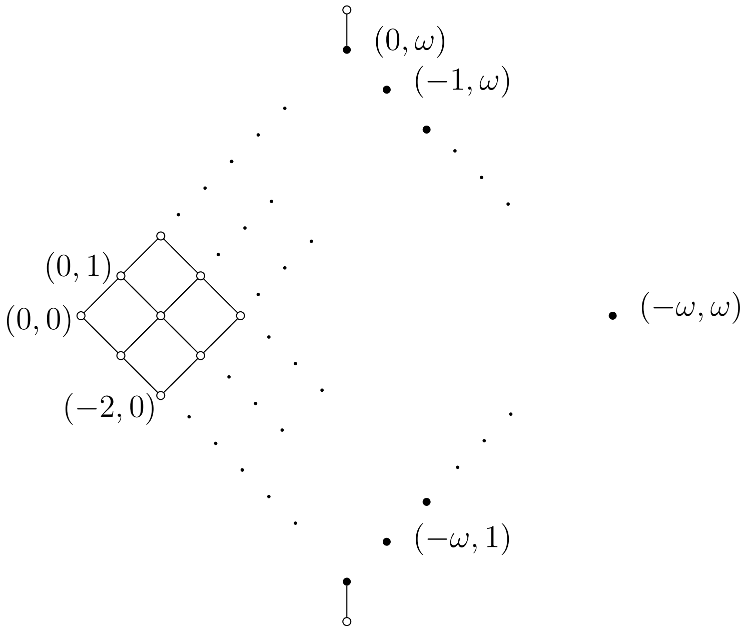


Figure 4.2: An example where $\mathtt{IM}(\mathbf{L}) \neq L^{\delta}$, noting that $(-\omega,\omega) \notin \mathtt{IM}(\mathbf{L})$.

Proposition 4.2.1 ([56, Proposition 3.3])**.** *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and consider the embedding* $e\colon \mathrm{Filt}(\mathbf{L}) \cup \mathrm{Idl}(\mathbf{L}) \mapsto \mathtt{IM}(\mathbf{L})$, $H \mapsto [H]$. *Let*

$$e(\mathrm{Filt}(\mathbf{L})) = \{\, e(F) \mid F \in \mathrm{Filt}(\mathbf{L}) \,\} \quad \text{and} \quad e(\mathrm{Idl}(\mathbf{L})) = \{\, e(I) \mid I \in \mathrm{Idl}(\mathbf{L}) \,\}.$$

Then

(i) $e(\mathrm{Filt}(\mathbf{L}))$ *is join-dense in* $\mathtt{IM}(\mathbf{L})$ *and* $e(\mathrm{Idl}(\mathbf{L}))$ *is meet-dense in* $\mathtt{IM}(\mathbf{L})$*;*

(ii) *in* $\mathtt{IM}(\mathbf{L})$*, arbitrary meets and finite joins exist for elements drawn from* $e(\mathrm{Filt}(\mathbf{L}))$ *and arbitrary joins and finite meets exist for elements drawn from* $e(\mathrm{Filt}(\mathbf{L}))$.

This greatly simplifies our search for an answer to the question of when the intermediate structure will be a lattice. The only meets and joins that might not exist are the meet and join of a filter and an ideal that are not comparable in the $\sqsubseteq$ order. We will now take a closer look at the structure of such incomparable filter-ideal pairs.

Given a lattice $\mathbf{L}$ and $a, b \in L$, we use the notation $a \perp b$ to indicate that a and b are *not comparable* in the lattice order. Similarly, for $F \in \mathrm{Filt}(\mathbf{L})$ and $I \in \mathrm{Idl}(\mathbf{L})$ we write $F \perp I$ if $\neg(F \sqsubseteq I)$ and $\neg(I \sqsubseteq F)$. It is clear that $F \perp I \iff I \perp F$. We denote by $\sqcap$ and $\sqcup$ the meet and join in $\mathtt{IM}(\mathbf{L})$.

Proposition 4.2.2. *Let* $F \in \mathrm{Filt}(\mathbf{L})$ *and* $I \in \mathrm{Idl}(\mathbf{L})$ *such that* $F \perp I$*. If either* $I = {\downarrow}x$ *or* $F = {\uparrow}y$ *for some* $x, y \in L$*, then* $F \sqcup I$ *and* $F \sqcap I$ *exist in* $\mathtt{IM}(\mathbf{L})$.

Proof. Suppose that $I = {\downarrow}x$ for some $x \in L$. Then $e(I) = e({\uparrow}x)$ and since finite meets and joins of elements from $e(\mathrm{Filt}(\mathbf{L}))$ exist in $\mathtt{IM}(\mathbf{L})$, we have that $I \sqcup F$ and $I \sqcap F$ exist. Dually, for $F = {\uparrow}y$ we consider the ideal ${\downarrow}y$ and obtain ${\downarrow}y \sqcup I$ and ${\downarrow}y \sqcap I$. ☐

Given $F \in \mathrm{Filt}(\mathbf{L})$ and $I \in \mathrm{Idl}(\mathbf{L})$ we define the following subsets:

$$F_{\perp}I := \{\, a \in F \mid \exists\, b \in I, a \perp b \,\} \quad \text{and} \quad I_{\perp}F := \{\, b \in I \mid \exists\, a \in F, b \perp a \,\}.$$

The set $I_{\perp}F$ can be thought of as the top cone of the ideal I, while $F_{\perp}I$ can be thought of as the bottom cone of F. This idea will be made clearer by the results that follow. From the definitions of $F_{\perp}I$ and $I_{\perp}F$ one can see the following equivalences:

$$F \perp I \quad\iff\quad F_{\perp}I \neq \emptyset \quad\iff\quad I_{\perp}F \neq \emptyset.$$

In certain cases when $F \perp I$ we can use $I_{\perp}F$ (respectively $F_{\perp}I$) to draw conclusions about the structure of I (respectively F). The next result shows that there are many cases in which $F \perp I$ but $F \sqcup I$ and $F \sqcap I$ do exist in $\mathtt{IM}(\mathbf{L})$.

Lemma 4.2.3. *Let* $F \in \mathrm{Filt}(\mathbf{L})$ *and* $I \in \mathrm{Idl}(\mathbf{L})$ *such that* $F \perp I$. *If* $I_{\perp}F \neq \emptyset$ *and* $|I_{\perp}F| < \omega$ *then* $I = {\downarrow}x$ *for some* $x \in L$. *Dually, if* $F_{\perp}I \neq \emptyset$ *and* $|F_{\perp}I| < \omega$ *then* $F = {\uparrow}y$ *for some* $y \in L$.

Proof. Suppose that $I_{\perp}F \neq \emptyset$. We first show that $I_{\perp}F$ is an upward closed subset of I. Let $b \in I_{\perp}F$ and suppose that $c \in {\uparrow}b \cap I$. Now if $c \leqslant a$ for all $a \in F$ then by transitivity $b \notin I_{\perp}F$. Thus there must exist $a_0 \in F$ such that $c \nleqslant a_0$. If $a_0 \leqslant c$ then $F \cap I \neq \emptyset$ which would imply $F \sqsubseteq I$. Thus $c \in I_{\perp}F$. Since $I_{\perp}F$ is finite we can consider $\bigvee(I_{\perp}F) \in L$. Now we will prove that $I = {\downarrow}(\bigvee I_{\perp}F)$. We must first show that every $c \in I \backslash (I_{\perp}F)$ is such that $c \leqslant b$ for some $b \in I_{\perp}F$. Given $c \in I \backslash (I_{\perp}F)$, clearly $c \ngeqslant b$ for any $b \in I_{\perp}F$. For any $b \in I_{\perp}F$ such that $c \perp b$ we can see that $c \leqslant c \vee b \in I_{\perp}F$. We now conclude that $I = {\downarrow}(\bigvee I_{\perp}F)$. ☐

The previous two results combine to give us a condition under which meets and joins will exist when $F \perp I$.

Proposition 4.2.4. *Let* $F \in \mathrm{Filt}(\mathbf{L})$ *and* $I \in \mathrm{Idl}(\mathbf{L})$ *such that* $F \perp I$. *If either* $0 < |I_{\perp}F| < \omega$ *or* $0 < |F_{\perp}I| < \omega$ *then* $F \sqcup I$ *and* $F \sqcap I$ *exist in* $\mathtt{IM}(\mathbf{L})$.

Proof. This is a consequence of Proposition 4.2.2 and Lemma 4.2.3. ☐

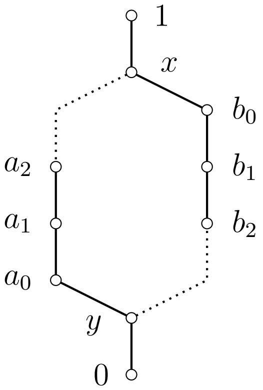


Figure 4.3: An example where $|F_{\perp}I|$ is not finite, but the join (and meet) still exist.

Unfortunately, Proposition 4.2.4 does not provide a complete characterisation of filter-ideal pairs without meets and joins, as the converse does not hold. This can be seen by the example in Fig. 4.3 with $F = \{\, b_i \mid i \in \omega \,\} \cup \{x, 1\}$ and $I = \{\, a_j \mid j \in \omega \,\} \cup \{y, 0\}$. Here we have $|F_{\perp}I| = \omega$ but still $F \sqcup I = {\uparrow}x = {\downarrow}x$ exists. This is in fact an example of a much more general occurrence of upper and lower bounds.

Proposition 4.2.5. *Let F and I be any filter and ideal in* $\mathtt{IM}(\mathbf{L})$ *such that $F \perp I$. Then there exists a least filter F_I such that $F \sqsubseteq F_I$ and $I \sqsubseteq F_I$. Dually, there exists a greatest ideal I_F such that $I_F \sqsubseteq F$ and $I_F \sqsubseteq I$.*

Proof. Consider the set $U = \{\, G \in \mathrm{Filt}(\mathbf{L}) \mid F \sqsubseteq G \text{ and } I \sqsubseteq G \,\}$. Now by part (ii) of Proposition 4.2.1 we have that the meet $\sqcap U$ exists. Clearly $F_I = \sqcap U$ is the least filter greater than both F and I in the $\sqsubseteq$ order. For the dual result, we consider $\bigsqcup V$ where $V = \{\, J \in \mathrm{Idl}(\mathbf{L}) \mid J \sqsubseteq F \text{ and } J \sqsubseteq I \,\}$. □

Given $F \perp I$, we will always have a filter and ideal candidate that is an upper bound for F and I. This is simply the principal filter (ideal) ${\uparrow}1$ (${\downarrow}1$). Dually, we always have that ${\downarrow}0 \sim {\uparrow}0$ is a lower bound. The above proposition shows us that we will always have a filter candidate, F_I, which can differ from ${\uparrow}1$, to act as the join of F and I (and dually). However, the construction of this filter F_I which acts as an upper bound does not tell us whether or not it is comparable to any possible ideal upper bounds that may occur. This situation (and the dual situation for I_F) is illustrated in Fig. 4.4. The dotted lines in Fig. 4.4 are not a precise representation of the set of ideals which are upper bounds for F and I. This set need not be a chain.

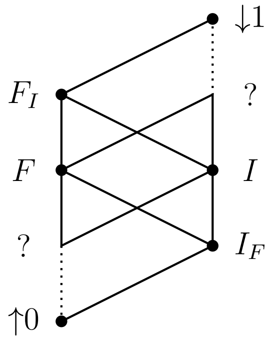


Figure 4.4: Possible meets and joins for $F \in \mathrm{Filt}(\mathbf{L})$ and $I \in \mathrm{Idl}(\mathbf{L})$ where $F \perp I$.

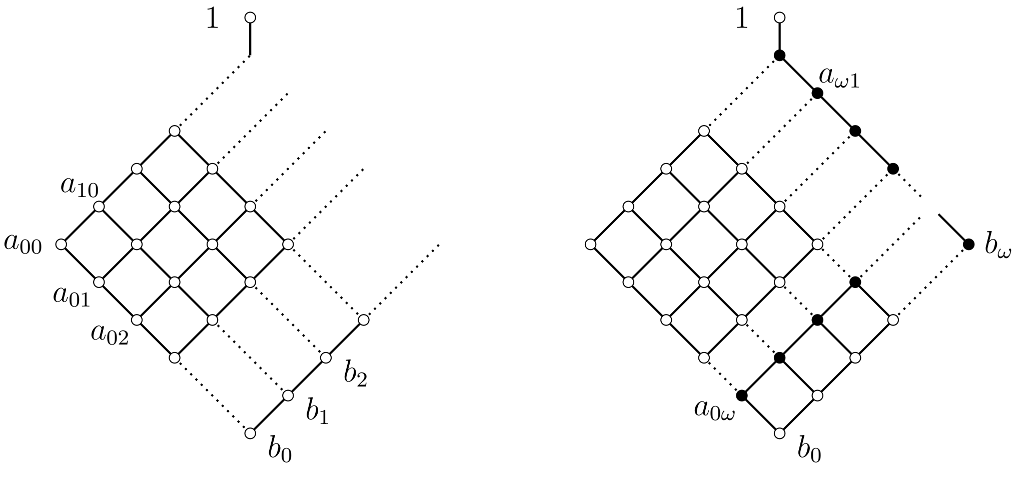
An answer to the question of whether the intermediate structure of a bounded lattice is a lattice was answered in the negative by Litak, Moshier and Suzuki [73]. Their counterexample is shown in Fig. 4.5.

Consider the ideal $I = \{\, b_i \mid i \in \omega \,\}$ and the filter $F_0 = \bigcup\{\, \uparrow a_{0j} \mid j \in \omega \,\}$. The limit point corresponding to I is $b_\omega = \bigvee I$, and the limit point corresponding to F is $a_{0\omega} = \bigwedge F_0$. One can see that $F_0 \perp I$. This is in fact true of $F_n = \{\, \uparrow a_{nj} \mid j \in \omega \,\}$ for any $n \in \omega$ (the corresponding limit point being $a_{n\omega}$). Fig. 4.5 shows us that for any $i \in \omega$, the join of b_ω and $a_{i,\omega}$ does not exist. The 'missing' point, $a_{\omega\omega}$ will be added once the poset $\mathtt{IM}(\mathbf{L})$ is completed by the MacNeille completion.

This is a specific example of having $I \perp F$ in $\mathtt{IM}(\mathbf{L})$ and every ideal which is greater than both I and F in the order of $\mathtt{IM}(\mathbf{L})$ is less than the only filter, $F' = \uparrow 1$, which is greater than I and F. Thus we get a descending chain of ideals all greater than I and F, all less than F_I, but with no least element. Thus $\mathtt{IM}(\mathbf{L})$ is not a lattice.

In [58, Example 3.11] it is noted that if a lattice $\mathbf{L}$ satisfies the ascending chain condition (ACC), then the intermediate structure is isomorphic to the filter completion of $\mathbf{L}$ ($\mathrm{Filt}(\mathbf{L})$ ordered by $\supseteq$). Dually, a lattice $\mathbf{L}$ satisfying the descending chain condition (DCC) will have its intermediate structure isomorphic to the ideal completion of $\mathbf{L}$ ($\mathrm{Idl}(\mathbf{L})$ ordered by $\subseteq$).

Our goal of describing the order structure of the intermediate structure has not been fully accomplished. It is not known whether the example from [73] is the only type of example of a non-lattice intermediate structure that can exist. A situation for which we have not been able to construct an example is the *bow-tie configuration*. This is where $F \perp I$ and there exists an ideal J that is the least upper bound of $\{F, I\}$ in $e(\mathrm{Idl}(\mathbf{L}))$, but $F_I \perp J$. It is not known to us whether or not this can occur. We close this section with a proposition that summarises our results.



$\mathbf{L}$ $\qquad$ $\mathtt{IM}(\mathbf{L})$

Figure 4.5: A lattice whose intermediate structure is not a lattice.

Proposition 4.2.6. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. *The order,* $\sqsubseteq$, *on* $\mathtt{IM}(\mathbf{L})$ *is a lattice order if one of the following conditions holds:*

(i) $\mathbf{L}$ *satisfies (ACC) or (DCC);*

(ii) *for all* $F \in \mathrm{Filt}(\mathbf{L}), I \in \mathrm{Idl}(\mathbf{L})$ *such that* $F \perp I$, *one of the following holds:*

  (1) $|F_\perp I| < \omega$,

  (2) $|I_\perp F| < \omega$,

  (3) $F_I \sqsubseteq J$ *for any* $J \in \mathrm{Idl}(\mathbf{L})$ *such that* $F \sqsubseteq J$ *and* $I \sqsubseteq J$, *and* $G \sqsubseteq I_F$ *for any* $G \in \mathrm{Filt}(\mathbf{L})$ *such that* $G \sqsubseteq F$ *and* $G \sqsubseteq I$.

4.3 Funayama's counterexample

In this brief section we present an example where the intermediate structure acts to preserve distributivity in the canonical extension construction. We do not present a formal proof, but rather indicate how the elements added by the intermediate structure help to ensure distributivity of the canonical extension. The example is due to Funayama [44] and was constructed to demonstrate that the MacNeille completion of a distributive lattice is not always distributive. There is also an example of the MacNeille completion not preserving distributivity due to Crawley [20], but his example is much harder to explain by way of illustration.

Funayama's example begins by considering a lattice $\mathbf{L}$ which is the product of three lattices. Two of these lattices are copies of the underlying lattice of the Chang algebra described on page 70. That is, $\mathbf{L}_1 := \langle C; \wedge, \vee\rangle$ and $\mathbf{L}_3 := \langle C; \wedge, \vee\rangle$. The remaining lattice is defined as $\mathbf{L}_2 = \langle\{q, p\}; \wedge, \vee\rangle$, ordered by $q < p$.

The lattice $\mathbf{L}_1 \times \mathbf{L}_2 \times \mathbf{L}_3$ can best be described as two layers of the grid formed by $\mathbf{L}_1 \times \mathbf{L}_3$. Each layer can be thought of as having four quadrants, each in bijective correspondence with (but not order-isomorphic to) $\omega \times \omega$. Since $\mathbf{L}$ is a product of chains, it is distributive. The lattice $\mathbf{K}$ which forms the counterexample is a distributive sublattice of $\mathbf{L}_1 \times \mathbf{L}_2 \times \mathbf{L}_3$. It consists of the 'left' and 'bottom' quadrants of the lower layer ($\mathbf{L}_1 \times \{q\} \times \mathbf{L}_3$), and the 'top' and 'left' quadrants of the upper layer ($\mathbf{L}_1 \times \{p\} \times \mathbf{L}_3$).

The MacNeille completion of $\mathbf{K}$ is shown in Fig. 4.6. Birkhoff [10] defined an element x of a lattice $\mathbf{L}$ to be *neutral* if for all $y, z \in L$, the lattice generated by $\{x, y, z\}$ is distributive. The non-distributivity of $\overline{\mathbf{K}}$ is caused by the non-neutral element (b_ω, q, d_ω).

Consider the set $X = \{(b_\omega, q, d_\omega), (b_0, p, c_0), (b_0, q, c_0)\}$. The sublattice of $\mathbf{K}$ generated by X has $X \cup \{(b_\omega, p, c_0), (b_0, q, d_\omega)\}$ as its underlying set. This sublattice is isomorphic to $\mathbf{N}_5$ and hence $\overline{\mathbf{K}}$ is not even modular. We did not have to choose the elements (b_0, p, c_0) and (b_0, q, c_0) to be the ones to join (b_ω, q, d_ω) in X. Any choice that takes one element from the bottom layer of the left quadrant, and the other from the top layer of the left quadrant will generate a lattice that is either isomorphic to $\mathbf{N}_5$, or has a sublattice isomorphic to $\mathbf{N}_5$.

In the canonical extension in Fig. 4.7, the limit points between quadrants added by the intermediate structure are all either limits of the top layer, or limits of the bottom layer. This ensures that all of the points in $K^\delta \backslash K$ are either limits of the top or bottom layer. This ensures that a limit point cannot be non-neutral. Take a point y from the bottom layer of the left quadrant and a point z from the top layer of the left quadrant. Then take a limit point x. If x is from the top layer, there will be a single join generated by $\{x, y, z\}$, and it will be on the top layer. However, there will be two meets generated, one on the top layer and one on the bottom. These two points act to preserve distributivity of the sublattice generated by $\{x, y, z\}$. Dually, if x was a limit point from the bottom layer, two joins (one top layer, one bottom layer) and one meet (bottom layer) would be generated.

We note that in Fig. 4.7 we have drawn in the lines which correspond to order relations which are implied by the transitivity of the partial order. This is simply to make the order relations between the different layers of $\mathbf{K}^\delta$ clearer to the reader.

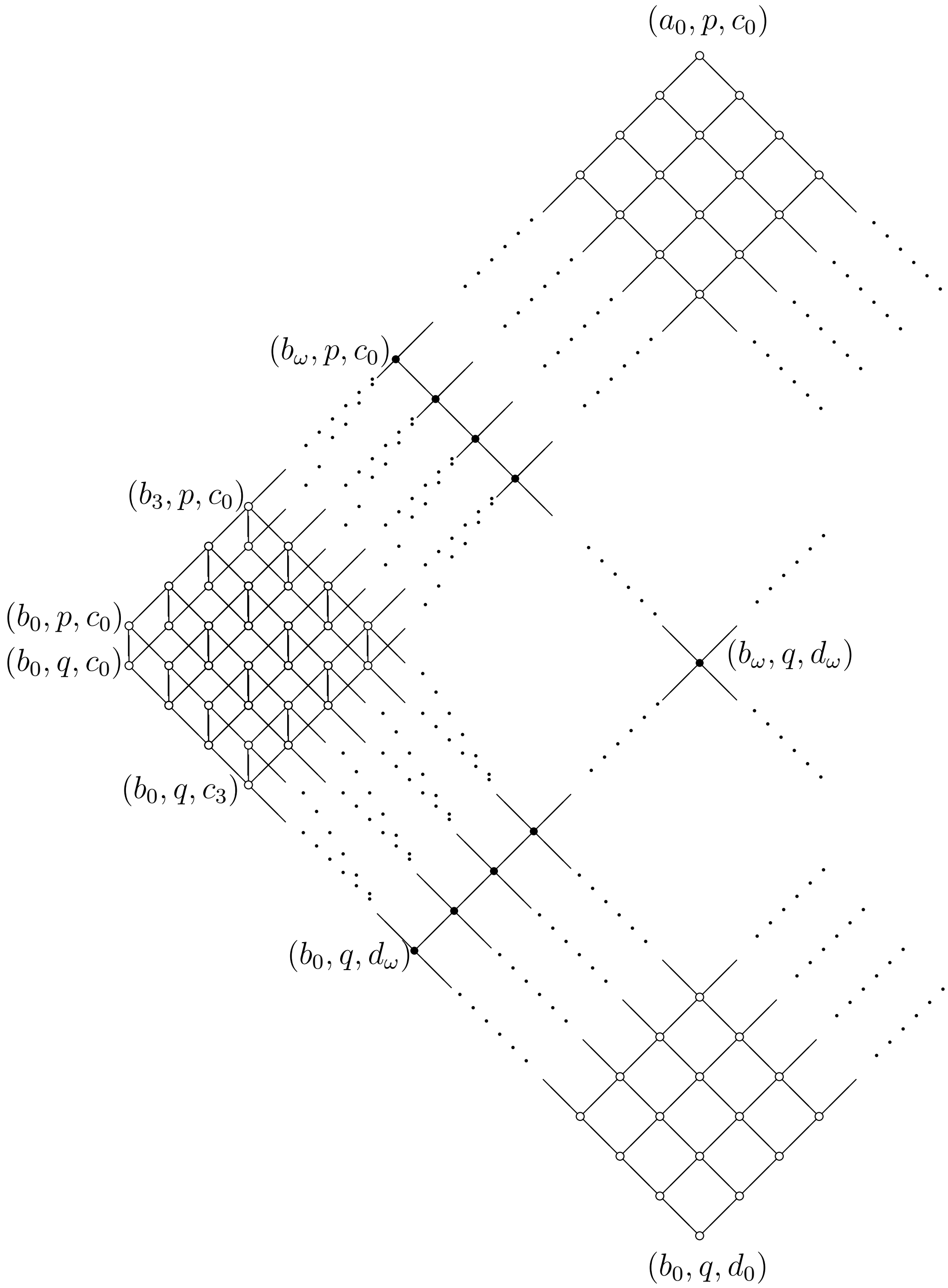


Figure 4.6: The MacNeille completion of the lattice **K**.

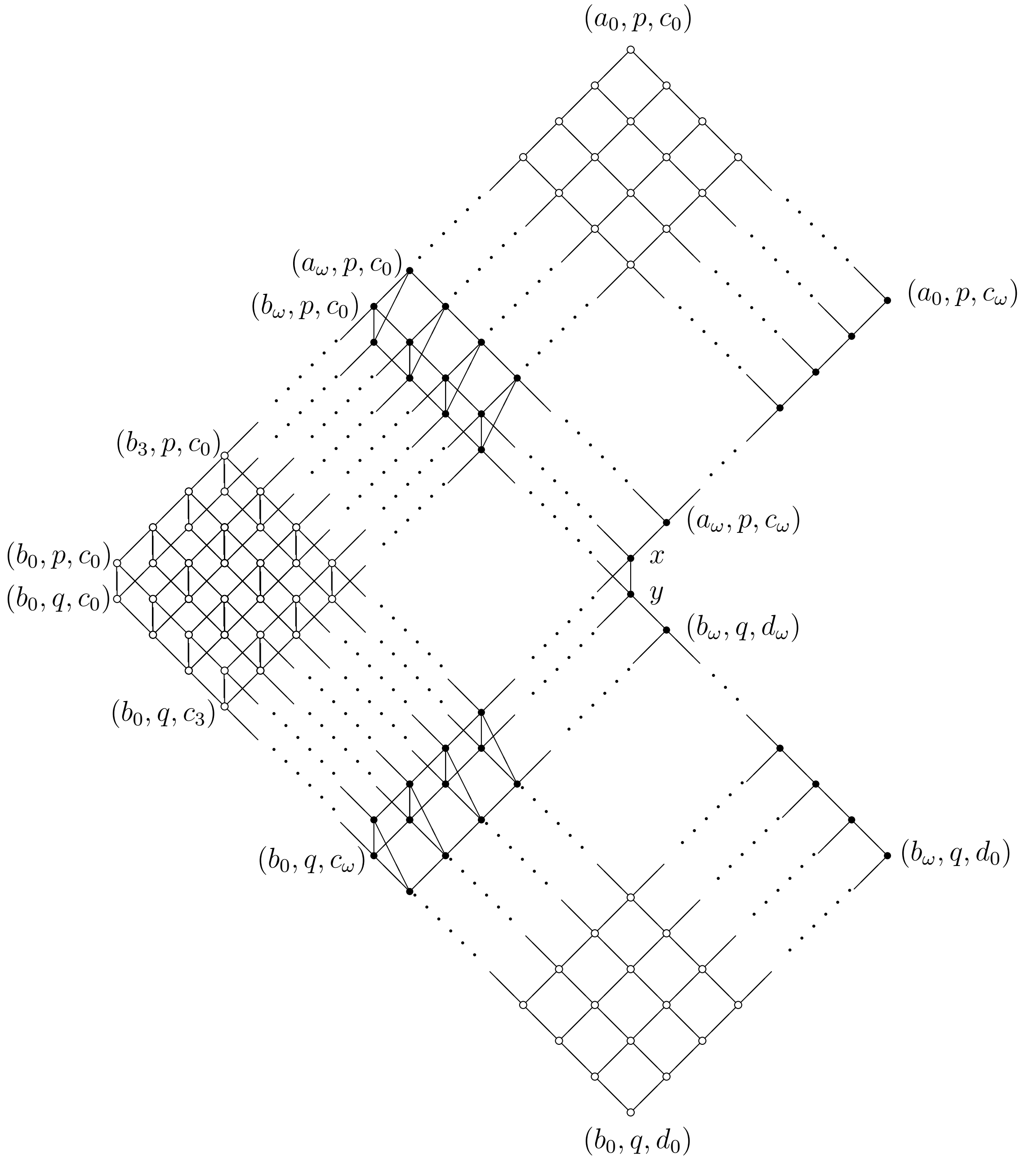


Figure 4.7: The canonical extension $\mathbf{K}^\delta$. The only elements which are not part of the intermediate structure are x and y. That is, $\{x, y\} = K^\delta \backslash \mathtt{IM}(K)$. The point x is the meet of the set $\{ (b_\omega, p, c_i) \mid i \in \omega \}$ and the join of the set $\{ (b_j, p, c_\omega) \mid j \in \omega \}$, and y is the meet of the set $\{(b_\omega, q, c_i) \mid i \in \omega \}$ and the join of $\{ (b_j, q, c_\omega) \mid j \in \omega \}$.

4.4 Dual graphs and RS-frames

In their work on canonical extensions of monotone poset expansions, Dunn, Gehrke and Palmigiano [33, Definition 2.9] define a complete lattice $\mathbf{C}$ to be *perfect* if the completely join-irreducible elements ($J^\infty(\mathbf{C})$) are join-dense and the completely meet-irreducible elements ($M^\infty(\mathbf{C})$) are meet-dense in the lattice. Further results from that paper show that the canonical extension of a poset (and hence also of a bounded lattice) is a perfect lattice [33, Corollary 2.10], and that there is a duality between perfect lattices and perfect posets [33, Section 4]. The definition of a perfect poset $\mathbf{P}$, in addition to the density requirements of perfect lattices, has the property that $P = J^\infty(\mathbf{P}) \cup M^\infty(\mathbf{P})$. The functor which takes the poset to the perfect lattice is the MacNeille completion. We note that the morphisms on the poset side of the duality have extra requirements in addition to being order-preserving.

Gehrke [47] gave an alternative presentation of this duality, whereby the perfect posets are presented as generalised Kripke-type frame structures called RS-frames. In Proposition 3.1.4 from Chapter 3 we identified each completely join- and completely meet-irreducible element of the canonical extension as coming from an element of the dual graph $(\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ of a lattice $\mathbf{L}$. Thus we would expect to be able to show some sort of correspondence between our dual graphs and RS-frames. Before recalling the basic definitions of RS-frames, we note that as objects, they are nothing more than a certain class of *polarities*. These structures originated with Birkhoff [11, Section V-7], and have since also been recast in the setting of Formal Concept Analysis.

A *frame* is a tuple (X, Y, R) where X and Y are non-empty sets and $R \subseteq X \times Y$ is a binary relation. Given a frame $\mathbf{F} = (X, Y, R)$, one can consider the usual Galois connection via the polar maps $R_\triangleright$ and $R_\triangleleft$ as follows:

$$\begin{aligned} R_\triangleright \quad : \quad \wp(X) &\to \wp(Y) \\ A &\mapsto \{\, y \in Y \mid (\forall\, x \in A)\ (x, y) \in R \,\} \end{aligned}$$

$$\begin{aligned} R_\triangleleft \quad : \quad \wp(Y) &\to \wp(X) \\ B &\mapsto \{\, x \in X \mid (\forall\, y \in B)\ (x, y) \in R \,\}. \end{aligned}$$

To the frame $\mathbf{F} = (X, Y, R)$ one can then associate a complete lattice

$$\mathrm{G}(\mathbf{F}) = \{\, A \subseteq X \mid (R_\triangleleft \circ R_\triangleright)(A) = A \,\}$$

of Galois-closed sets, ordered by inclusion.

For $x \in X$ and $y \in Y$ we define $xR := \{\, y \in Y \mid xRy \,\}$ and $Ry := \{\, x \in X \mid xRy \,\}$.

Definition 4.4.1 ([47, Section 2]). A frame $\mathbf{F} = (X, Y, R)$ is said to be a *separating frame*, or *S frame*, if for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$

(i) $x_1 \neq x_2$ implies $x_1 R \neq x_2 R$;

(ii) $y_1 \neq y_2$ implies $R y_1 \neq R y_2$.

Definition 4.4.2 ([47, Section 2]). An S frame $\mathbf{F} = (X, Y, R)$ is said to be *reduced*, or an *RS-frame*, if

(i) $\forall\, x \in X, \exists\, y \in Y$ such that $\neg(xRy)$ and $\forall\, w \in X$ if $w \neq x$ and $xR \subseteq wR$ then wRy;

(ii) $\forall\, y \in Y, \exists\, x \in X$ such that $\neg(xRy)$ and $\forall\, z \in Y$ if $z \neq y$ and $Ry \subseteq Rz$ then xRz.

We now wish to translate our dual graphs of MPH's into the language of RS-frames. We know from [48, Lemma 3.4] that the completely join-irreducible (respectively completely meet-irreducible) elements of the canonical extension correspond to the filters (respectively ideals) which are part of some maximal pair. This motivates us to define two equivalence relations on a graph $\mathrm{D}^\flat(\mathbf{L})$ by saying that two MPH's are considered $\sim_1$-equivalent (respectively $\sim_2$-equivalent) if their filter (respectively ideal) parts are equal as sets. This will happen if and only if their maximal filter (ideal) parts fail to intersect the same set of maximal ideals (filters). We can in fact mimic the definition of these equivalence relations on an arbitrary graph $\mathbf{X} = (X, E)$.

Definition 4.4.3. For an arbitrary graph $\mathbf{X} = (X, E)$ let the equivalence relations $\sim_1$ and $\sim_2$ on X be given such that for all $x, y, z \in X$

(i) $x \sim_1 y$ if $(\forall\, z \in X)\big((x, z) \in E \iff (y, z) \in E\big)$;

(ii) $x \sim_2 y$ if $(\forall\, z \in X)\big((z, x) \in E \iff (z, y) \in E\big)$.

Given a graph $\mathbf{X} = (X, E)$ we let $X_1 = X/{\sim_1}$ and $X_2 = X/{\sim_2}$. Where the same element $x \in X$ is being considered by representations in each of the quotient sets, we will denote its two different equivalence classes by $[x]_1 \in X_1$ and $[x]_2 \in X_2$. We define the relation $R_{\mathbf{X}}$ for $[x] \in X_1$ and $[y] \in X_2$ by

$$[x] R_{\mathbf{X}} [y] \iff (x, y) \notin E.$$

Now we define for any graph $\mathbf{X}$ the frame $\rho(\mathbf{X}) := (X_1, X_2, R_{\mathbf{X}})$.

The following lemma helps us to translate from the graph $\mathbf{X}$ to its associated frame $\rho(\mathbf{X})$. The proof follows easily from the definition of $R_{\mathbf{X}}$.

Lemma 4.4.4. *Let* $\mathbf{X} = (X, E)$ *be a reflexive graph and let* $\rho(\mathbf{X}) = (X_1, X_2, R_{\mathbf{X}})$. *Then for* $x, y \in X$ *we have*

(i) $y \in E_{\triangleright}^{\complement}(\{x\})$ *if and only if* $[y]_2 \in [x]_1 R_{\mathbf{X}}$;

(ii) $y \in E_{\triangleleft}^{\complement}(\{x\})$ *if and only if* $[y]_1 \in R_{\mathbf{X}}[x]_2$.

We require one further lemma which relates specifically to the case that $\mathbf{X}$ is the dual graph of MPH's of a lattice.

Lemma 4.4.5. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *For* $x, y \in X$. *Then*

(i) $E_{\triangleright}^{\complement}(\{x\}) \subseteq E_{\triangleright}^{\complement}(\{y\})$ *if and only if* $x^{-1}(1) \subseteq y^{-1}(1)$;

(ii) $E_{\triangleleft}^{\complement}(\{x\}) \subseteq E_{\triangleleft}^{\complement}(\{y\})$ *if and only if* $x^{-1}(0) \subseteq y^{-1}(0)$.

Proof. Let $E_{\triangleright}^{\complement}(\{x\}) \subseteq E_{\triangleright}^{\complement}(\{y\})$ and then suppose that $x^{-1}(1) \nsubseteq y^{-1}(1)$. That is, there exists $a \in L$ such that $x(a) = 1$ but $y(a) \neq 1$. Now consider the partial homomorphism z defined by $z^{-1}(1) = y^{-1}(1)$ and $z^{-1}(0) = {\downarrow}a$. We extend this to an MPH w, so now $w \in X$. Clearly $(y, w) \in E$ and so $w \notin E_{\triangleright}^{\complement}(\{y\})$. But by the assumption $w \notin E_{\triangleright}^{\complement}(\{x\})$ and this is a contradiction since $(x, w) \notin E$ and so $w \in E_{\triangleright}^{\complement}(\{x\})$. Thus $x^{-1}(1) \subseteq y^{-1}(1)$.

Now suppose that $x^{-1}(1) \subseteq y^{-1}(1)$ and let $z \in E_{\triangleright}^{\complement}(\{x\})$. This gives us that $x^{-1}(1) \cap z^{-1}(0) \neq \emptyset$ which implies that $y^{-1}(1) \cap z^{-1}(0) \neq \emptyset$. Thus $z \in E_{\triangleright}^{\complement}(\{y\})$. The proof of (ii) will follow by a dual argument. $\square$

Proposition 4.4.6. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and consider the graph* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *Then the frame* $\rho(\mathbf{X}) = (X_1, X_2, R_{\mathbf{X}})$ *is an RS-frame.*

Proof. Let $x, y \in X$ and suppose that $[x]_1 \neq [y]_1$. We apply Lemma 4.4.5. Without loss of generality, this implies that there exists $z \in X$ such that $(x, z) \in E$ but $(y, z) \notin E$. Now $[y]_1 R_{\mathbf{X}} [z]_2$ but $\neg([x]_1 R_{\mathbf{X}} [z]_2)$ and so $[x]_1 R_{\mathbf{X}} \neq [y]_1 R_{\mathbf{X}}$. The proof for $[x]_2 \neq [y]_2$ follows similarly. Thus the frame $\rho(\mathbf{X})$ is separated.

Now let $[x]_1 \in X_1$ and consider $[x]_2 \in X_2$. Since $(x, x) \in E$ we have that $\neg([x]_1 R_{\mathbf{X}} [x]_2)$. Now suppose $[w]_1 \in X_1$ such that $[w]_1 \neq [x]_1$ and $[x]_1 R_{\mathbf{X}} \subseteq [w]_1 R_{\mathbf{X}}$. Since $\rho(\mathbf{X})$ is separated we have that $[x]_1 R_{\mathbf{X}} \subsetneq [w]_1 R_{\mathbf{X}}$ and by Lemma 4.4.4(i) we have that $E_{\triangleright}^{\complement}(\{x\}) \subsetneq E_{\triangleright}^{\complement}(\{w\})$. Now by Lemma 4.4.5(i) we get that $x^{-1}(1) \subsetneq w^{-1}(1)$. Since $x^{-1}(1)$ is maximal with respect to being disjoint from $x^{-1}(0)$, we have that $w^{-1}(1) \cap x^{-1}(0) \neq \emptyset$. Thus $(w, x) \notin E$ and so $[w]_1 R_{\mathbf{X}} [x]_2$. $\square$

A bounded lattice can be represented by a topologised RS-frame. This fact is contained in the work of Hartung [66], and has recently been re-iterated in the language of RS-frames and canonical extensions by Gehrke and van Gool [57]. We use this knowledge to confront a question posed by Ploščica regarding the graphs that can be used to represent lattices.

Ploščica [78, Section 3] noted that there were cases where the graph of MPH's from $\mathbf{L}$ into $\underline{\mathbf{2}}$ was not minimal amongst those graphs with topology that could serve as a representation of the lattice $\mathbf{L}$. An example of this is the lattice and corresponding graphs shown in Fig. 4.8.

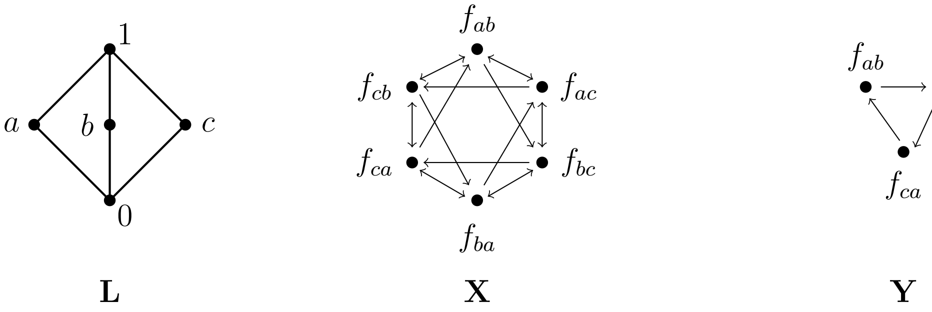


Figure 4.8: The modular lattice M_3 and two of its graph representations $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ and $\mathbf{Y}$ (defined below).

For $d, e \in \mathbf{L}$, let f_{de} be the MPH defined by $f_{de}^{-1}(1) = {\uparrow}d$ and $f_{de}^{-1}(0) = {\downarrow}e$. Now let $Y = \{f_{ab}, f_{bc}, f_{ca}\}$ and let $\mathbf{Y} = (Y, E_Y)$ where E_Y is our usual relation on $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, restricted to Y. We note that $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) = Y \cup \{f_{ba}, f_{cb}, f_{ac}\}$. It is straightforward to check that the lattice $\mathbf{L}$ is isomorphic to the lattice $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{Y}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, of MPM's, via the isomorphism $a \mapsto e_a$.

We will look at the case of finite graphs (and hence finite lattices). In the case of finite graphs with topology, we can simplify the question of which lattices they represent. Suppose that $\mathbf{X}_{\mathcal{T}} = (X, E, \mathcal{T})$ is a finite graph with topology for which $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ is a representation of a finite lattice $\mathbf{L}$. Now since $\mathbf{X}_{\mathcal{T}}$ is finite and is equipped with the discrete topology, every maximal partial E-preserving map is continuous. Thus $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$.

The following lemma shows that the results in Lemmas 2.2.5 and 2.3.2 for graphs of MPH's and SPH's coming from lattices can be generalised to graphs.

Lemma 4.4.7. *Let* $\mathbf{X} = (X, E)$ *be a graph and let* $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. *Then for any* $x, y \in X$,

(i) *if* $\varphi(x) = 1$ *and* $E_{\triangleright}(\{y\}) \subseteq E_{\triangleright}(\{x\})$ *then* $\varphi(y) = 1$*;*

(ii) *if* $\varphi(x) = 0$ *and* $E_{\triangleleft}(\{y\}) \subseteq E_{\triangleleft}(\{x\})$ *then* $\varphi(y) = 0$.

Proof. Suppose $\varphi(x) = 1$. If $\varphi(y) \neq 1$, then by Lemma 2.1.1(ii) we have that there exists $z \in \varphi^{-1}(0)$ such that $(y, z) \in E$. But since $E_{\triangleright}(\{y\}) \subseteq E_{\triangleright}(\{x\})$ we would then have $(x, z) \in E$, a contradiction. $\square$

In the next result, we need the reflexivity of the graph $\mathbf{X}$ to guarantee that if $E_{\triangleright}(\{a\}) = E_{\triangleright}(\{b\})$, then we get $(a, b) \in E$ and $(b, a) \in E$ (and similarly for $E_{\triangleleft}(\{a\}) = E_{\triangleleft}(\{b\})$).

Proposition 4.4.8. *Let* $\mathbf{X} = (X, E)$ *be a finite reflexive graph and suppose that there exists* $x \in X$ *with* $y, z \in X$ *such that* $E_{\triangleright}(\{x\}) = E_{\triangleright}(\{y\})$ *and* $E_{\triangleleft}(\{x\}) = E_{\triangleleft}(\{z\})$*. Then for* $\mathbf{X}' = (X \backslash \{x\}, E_{X'})$ *the lattices* $\mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *and* $\mathfrak{G}^{\mathrm{mp}}(\mathbf{X}', \underset{\sim}{\mathbf{2}})$ *are isomorphic.*

Proof. Define a map Γ on $\mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ by $\varphi \mapsto \varphi|_{X'}$. First we show that the image of an MPE φ is an MPE. It is clear that $\Gamma(\varphi)$ will be a partial E-preserving map from $\mathbf{X}'$ to $\underset{\sim}{\mathbf{2}}$. Suppose that there exists $\psi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}', \underset{\sim}{\mathbf{2}})$ such that $\mathrm{dom}(\Gamma(\varphi)) \subseteq \mathrm{dom}(\psi)$. Suppose there exists $w \in \mathrm{dom}(\psi)$ such that $w \notin \mathrm{dom}(\Gamma(\varphi))$. Now $w \neq x$ and $w \notin \mathrm{dom}(\Gamma(\varphi))$, hence $w \notin \mathrm{dom}(\varphi)$. Thus there must exist $v \in \varphi^{-1}(0)$ such that $(w, v) \in E$ and $u \in \varphi^{-1}(1)$ such that $(u, w) \in E$. If $v \neq x$ and $u \neq x$ then $\psi(w) \neq 1$ and $\psi(w) \neq 0$, a contradiction. If $v = x$ then since $E_{\triangleleft}(\{x\}) = E_{\triangleleft}(\{z\})$ we get that $(w, z) \in E$. By Lemma 4.4.7 we get $\varphi(z) = 0$ and thus $\psi(w) \neq 1$. If $u = x$ then by $E_{\triangleright}(\{x\}) = E_{\triangleright}(\{y\})$ we get $(y, w) \in E$. Then since $\varphi(y) = 1$ we see that $\psi(w) \neq 0$. This shows that there is no proper E-preserving extension of $\Gamma(\varphi)$ and hence $\Gamma(\varphi) \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}', \underset{\sim}{\mathbf{2}})$. The fact that Γ is injective follows from the fact that $\varphi^{-1}(1)$ and $\varphi^{-1}(0)$ completely determine one another.

Now we must show that Γ is surjective. Suppose that $\psi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}', \underset{\sim}{\mathbf{2}})$. The reflexivity of $\mathbf{X}$ gives $(x, x), (z, z) \in E$, and combined with $E_{\triangleleft}(\{x\}) = E_{\triangleleft}(\{z\})$ this implies $(z, x) \in E$ and $(x, z) \in E$. Now $E_{\triangleright}(\{x\}) = E_{\triangleright}(\{y\})$ implies that $(y, z) \in E$. This means that there are only five possibilities for the images of y and z under ψ:

(1) $\psi(y) = 0$, $\psi(z) = 0$,

(2) $\psi(y) = 1$, $\psi(z) = 1$,

(3) $\psi(y) = 0$, $\psi(z) = 1$,

(4) $\psi(y)$ undefined, $\psi(z) = 0$,

(5) $\psi(y) = 1$, $\psi(z)$ undefined.

For cases (1) and (4) we apply Lemma 4.4.7 and let $\varphi = \psi \cup \{(x, 0)\}$. For cases (2) and (5) we again apply Lemma 4.4.7, this time letting $\varphi = \psi \cup \{(x, 1)\}$. For case (3) we leave $\varphi(x)$ undefined. It is not difficult to check that in each case we get $\Gamma(\varphi) = \psi$.

The fact that $\varphi_1 \leqslant \varphi_2$ implies that $\Gamma(\varphi_1) \leqslant \Gamma(\varphi_2)$ follows simply from the fact that Γ restricts the domain of elements of $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. If $\Gamma(\varphi_1), \Gamma(\varphi_2) \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}', \underset{\sim}{\mathbf{2}})$ with $\Gamma(\varphi_1) \leqslant \Gamma(\varphi_2)$ then it will follow from the case evaluation above that $\varphi_1 \leqslant \varphi_2$. Thus Γ is an order-isomorphism. $\square$

We can easily apply the above proposition to graphs which come from lattices.

Corollary 4.4.9. *Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and let $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. Suppose there exists $f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ with $g, h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ such that $f^{-1}(1) = g^{-1}(1)$ and $f^{-1}(0) = h^{-1}(0)$. Then for $\mathbf{X}' = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \backslash \{f\}, E)$, the lattices $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ and $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}', \underset{\sim}{\mathbf{2}})$ are isomorphic.*

Proof. This follows from Proposition 4.4.8 and Lemma 4.4.5. $\square$

Applying Corollary 4.4.9 repeatedly on a finite graph allows us to reduce the number of elements of the graph to a minimal number. This result is perhaps unsurprising given our knowledge of Wille's representation of finite lattices [87]. This representation uses a context of join-irreducible elements and meet-irreducible elements, with the relation being the order from the lattice being represented. So long as each graph element that we delete still has the join-irreducible and meet-irreducible element that it represents captured by some other graph element, then it can be deleted without altering the lattice that it represents.

Part II

Natural dualities for quasivarieties of default bilattices

Chapter 1

Introduction

Bilattices were introduced in the late 1980's by Ginsberg [60] as a method for inference with incomplete and contradictory information. His theory of bilattices was inspired by the simple example introduced by Belnap [8, 7] more than 10 years earlier. Belnap's idea was that in certain situations a computer should base its decision making on four possible truth values rather than just the traditional two values of true and false. The underlying mathematical structure for this approach is shown in Fig. 1.1.

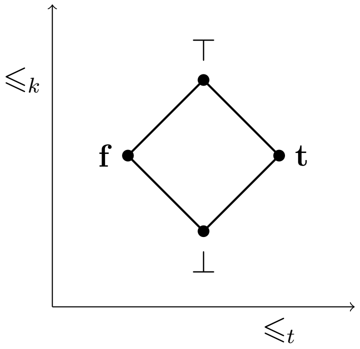


Figure 1.1: The four truth values proposed by Belnap.

The elements $\mathbf{t}$ and $\mathbf{f}$ represent "true" and "false", while the elements $\top$ and $\bot$ represent "contradiction" and "no information". The order represented on the vertical axis in Fig. 1.1 is the *knowledge* order ($\leqslant_k$), while the horizontal axis represents the *truth* order ($\leqslant_t$). (We note that the term 'information order' is used in some of the literature to refer to what we will call the knowledge order.)

Belnap proposed that a computer should have a truth value, $\top$, to assign to a statement which it had been told separately was both true and false. This is a very plausible idea in situations where the computer would be receiving information from different sources. Equally important is that a computer should also be able to make decisions based on incomplete information. The truth value $\bot$ is assigned to statements about which the computer has no information. A statement p which is assigned the truth value $\top$ is less true than a statement q which is assigned $\mathbf{t}$ as there is a source saying that p is false. On the other hand, more is known about p than is known about q as there are at least two different sources providing information.

First in [60], and then later in more detail in [61][1], Ginsberg introduced bilattices as a generalisation of Belnap's four-valued logic. Ginsberg's motivation was to use bilattices as a framework for inference with applications to artificial intelligence and logic programming. Ginsberg [61, p. 266] sums up this bilattice-based approach to inference:

> "...we will be describing the knowledge or beliefs held by an agent or inference system. Thus, we may say that one sentence p is "more true" than another sentence q; when we do so, we will mean only that the inference system has more reason to believe in the truth of p than it does to believe in the truth of q."

Soon after Ginsberg's first paper, much influential work was done by Fitting [37, 38]. The work of Fitting proved instrumental not only in developing the mathematical framework of bilattices, but also in furthering the applications to logic programming [39, 40].

Our contribution in Chapters 2 and 3 is to provide dual structures for quasi-varieties generated by bilattices used in default logic. What makes our work particularly interesting is the way in which the relations on the dual structures capture the knowledge order of the generating algebra. We first present the formal definitions underlying the theory of bilattices before detailing our contribution at the end of Section 1.2.

1.1 Bilattices

In this section we introduce the definitions and background theory of bilattices. In particular, we note the various representation theorems that exist for certain subclasses of bilattices. Most of the definitions are originally due to Ginsberg [61], although some of the terminology and notation has evolved over the years. There is quite a lot of variation in notation and terminology in the literature. The presentation here aligns most closely with the presentation in the survey article by Fitting [41].

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[1] We will use this second paper in our citation of definitions as this provides a more complete collection of Ginsberg's early definitions and results.

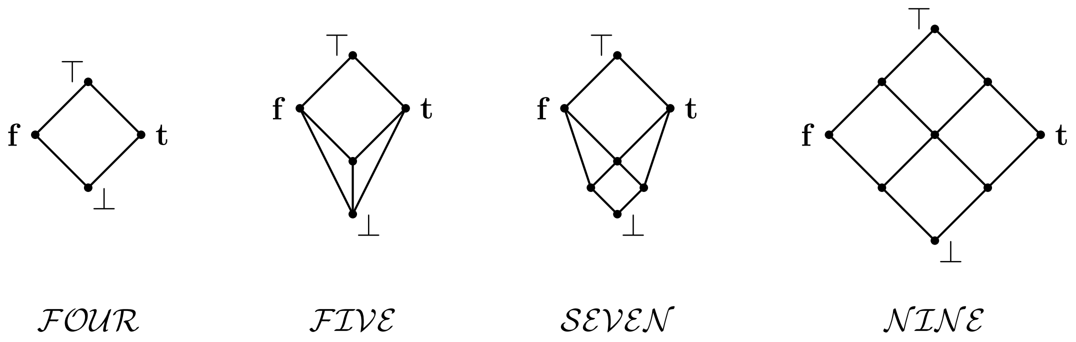


Figure 1.2: The bilattices $\mathcal{FOUR}$, $\mathcal{FIVE}$, $\mathcal{SEVEN}$ and $\mathcal{NINE}$. The vertical order is the knowledge order $\leqslant_k$, and the left-to-right order is the truth order $\leqslant_t$. The negation is defined as a reflection about the vertical axis.

Definition 1.1.1. A *pre-bilattice* is an algebra $\mathbf{B} = \langle B; \otimes, \oplus, \wedge, \vee\rangle$ such that $\langle B; \otimes, \oplus\rangle$ and $\langle B; \wedge, \vee\rangle$ are lattices. We denote by $\leqslant_k$ the order associated with $\langle B; \otimes, \oplus\rangle$ and $\leqslant_t$ the order associated with $\langle B; \wedge, \vee\rangle$.

The definition of a pre-bilattice has no requirements for any kind of relationship between the two lattice orders. Thus, with a change of signature, any lattice $\langle L; \sqcap, \sqcup\rangle$ can be considered as a pre-bilattice where each of $\leqslant_k$ and $\leqslant_t$ is either the original order $\sqsubseteq$ from L, or its dual $\sqsupseteq$. We note that Ginsberg's original definition [61, Definition 4.1] requires that both of the lattices $\langle B; \otimes, \oplus\rangle$ and $\langle B; \wedge, \vee\rangle$ are complete. In the development of bilattice theory since then, this requirement is not usually assumed and we will never make this assumption.

Definition 1.1.2. A *bilattice* is an algebra $\mathbf{B} = \langle B; \otimes, \oplus, \wedge, \vee, \neg\rangle$ such that the reduct $\langle B; \otimes, \oplus, \wedge, \vee\rangle$ is a pre-bilattice and $\neg$ is a unary operation which is $\leqslant_k$-preserving, $\leqslant_t$-reversing, and involutive.

Our first examples of bilattices are shown in Fig. 1.2. The knowledge order is simply the vertical order, while the truth order can be read off as follows: $a \leqslant_t b$ if there is a path from left to right connecting a and b. In the literature it is usual (as we have done) to include in the Hasse diagrams any redundant transitive $\leqslant_k$ relations that will assist in visually determining the $\leqslant_t$ order. When we take a closer look at specific bilattices we will find it easier to draw the two orders separately.

In some of the literature what we have called a *bilattice* above would be referred to as a 'bilattice with negation' while a pre-bilattice would be called a 'bilattice'.

In many cases one or both of the lattice orders will be considered to be bounded. In these cases, the bounds of the knowledge lattice are denoted by $\top$ and $\bot$, and the bounds of the truth lattice are denoted by $\mathbf{t}$ and $\mathbf{f}$. With or without bounds, the class of all bilattices (and pre-bilattices) can be equationally defined and hence the class forms a variety.

In addition to requiring some level of interaction between the negation operation and the two orders, it is natural to consider bilattices where there is a strong interaction between the two sets of lattice operations.

Definition 1.1.3 ([61, Definition 4.1])**.** Let $\mathbf{B} = \langle B; \otimes, \oplus, \wedge, \vee \rangle$ be a pre-bilattice. Then $\mathbf{B}$ is said to be *distributive* if for all $a, b, c \in B$ the identity

$$a \bullet (b * c) \approx (a \bullet b) * (a \bullet c)$$

holds whenever $\bullet, * \in \{\otimes, \oplus, \wedge, \vee\}$.

By adding a negation operation to the above definition one defines a *distributive bilattice*. Of the bilattices shown in Fig. 1.2, both $\mathcal{FOUR}$ and $\mathcal{NINE}$ are distributive bilattices, while $\mathcal{FIVE}$ and $\mathcal{SEVEN}$ are not. $\mathcal{FIVE}$ and $\mathcal{SEVEN}$ are of course distributive lattices in their k-orders. It is also possible to consider bilattices where each set of lattice operations is order-preserving with respect to the other order.

Definition 1.1.4 ([37, Definition 4.1])**.** A pre-bilattice is said to be *interlaced* if the k-lattice operations are monotone with respect to the truth order and if the t-lattice operations are monotone with respect to the knowledge order. That is,

$$\begin{aligned} a \leqslant_t b &\implies a \otimes c \leqslant_t b \otimes c, \\ a \leqslant_t b &\implies a \oplus c \leqslant_t b \oplus c, \\ a \leqslant_k b &\implies a \wedge c \leqslant_k b \wedge c, \\ a \leqslant_k b &\implies a \vee c \leqslant_k b \vee c. \end{aligned}$$

The conditions above could of course be expressed as equations, and hence the class of interlaced pre-bilattices forms a variety, as does the class of interlaced bilattices. Some authors have split the monotonicity conditions into separate groups. We will comment further on this after Theorem 1.2.3.

We note that if the truth operations preserve $\leqslant_k$, then $\leqslant_k \subseteq B^2$ will form a subuniverse of $\mathbf{B}^2$. In a bilattice, the negation operation will always erase the possibility of $\leqslant_t$ being a subuniverse of $\mathbf{B}^2$, although in an interlaced pre-bilattice $\leqslant_t$ will be a subuniverse. These observations will become important when we look for binary algebraic relations on $\mathbf{B}$ in our search for natural dualities for $\mathbb{ISP}(\mathbf{B})$.

1.2 Duality and representation for bilattices

The product construction of bilattices and pre-bilattices plays an important role in the representation theory of both distributive and interlaced bilattices.

Definition 1.2.1 ([38, Definition 4]). Let $\mathbf{L}_1 = \langle L_1; \sqcap_1, \sqcup_1 \rangle$ and $\mathbf{L}_2 = \langle L_2; \sqcap_2, \sqcup_2 \rangle$ be lattices. The operations of the product pre-bilattice

$$\mathbf{L}_1 \odot \mathbf{L}_2 = \langle L_1 \times L_2; \otimes, \oplus, \wedge, \vee \rangle$$

are defined for $(a_1, a_2), (b_1, b_2) \in L_1 \times L_2$ by

$$\begin{aligned}
(a_1, a_2) \otimes (b_1, b_2) &= (a_1 \sqcap_1 b_1, a_2 \sqcap_2 b_2)\\
(a_1, a_2) \oplus (b_1, b_2) &= (a_1 \sqcup_1 b_1, a_2 \sqcup_2 b_2)\\
(a_1, a_2) \wedge (b_1, b_2) &= (a_1 \sqcap_1 b_1, a_2 \sqcup_2 b_2)\\
(a_1, a_2) \vee (b_1, b_2) &= (a_1 \sqcup_1 b_1, a_2 \sqcap_2 b_2).
\end{aligned}$$

Note how the k-lattice operations are defined by the usual direct product. Another useful way of visualising the product pre-bilattice is via conditions on the order relations:

$$\begin{aligned}
(a_1, a_2) \leqslant_k (b_1, b_2) &\iff a_1 \sqsubseteq_1 b_1 \text{ and } a_2 \sqsubseteq_2 b_2,\\
(a_1, a_2) \leqslant_t (b_1, b_2) &\iff a_1 \sqsubseteq_1 b_1 \text{ and } b_2 \sqsubseteq_2 a_2.
\end{aligned}$$

The components a_1 and a_2, of the element $(a_1, a_2) \in L_1 \times L_2$, can be thought of as meaning "evidence for" and "evidence against". This way of thinking feeds into the way that a negation operation can be defined on the product structure. If $\mathbf{L}_1 = \mathbf{L}_2$, then for $(a, b) \in L_1^2$ the negation is defined as $\neg(a, b) = (b, a)$. It is easy to see that the negation will be $\leqslant_k$-preserving and $\leqslant_t$-reversing and an involution.

The following result demonstrates the importance of the product construction.

Theorem 1.2.2 ([38, Proposition 8]). *A bilattice* $\mathbf{B}$ *is a distributive bilattice if and only if* $\mathbf{B} \simeq \mathbf{L} \odot \mathbf{L}$ *for some distributive lattice* $\mathbf{L}$.

The following representation theorem for interlaced pre-bilattices was first proved by Avron [5], and has subsequently been generalised by Rivieccio [82] to the case of unbounded interlaced pre-bilattices. It was also proven via different methods by Movsisyan, Romanowska and Smith [77, Theorem 4.3].

Theorem 1.2.3 ([82, Theorem 2.1.7]). *Let* $\mathbf{B}$ *be a pre-bilattice. Then* $\mathbf{B}$ *is interlaced if and only if there exist two lattices* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *such that* $\mathbf{B} \simeq \mathbf{L}_1 \odot \mathbf{L}_2$.

The above theorem demonstrates the usefulness of the interlacing conditions. The bilattices that we examine in Chapter 2 and Chapter 3 are *not* interlaced, and hence no product representation is available. However, there are product representations available for weaker classes of pre-bilattices.

Pynko [80] called a pre-bilattice *regular* if the truth operations preserve the knowledge order. He proved that when the pre-bilattice is bounded in both orders, regularity is equivalent to being interlaced [80, Theorem 3.1]. Movsisyan [76] defines a *weakly interlaced* pre-bilattice to be a pre-bilattice where the truth operations preserve the knowledge order and one of the knowledge operations preserves the truth order. He further proves that every such pre-bilattice is isomorphic to the product pre-bilattice of two lattices.

Jipsen [69] recently showed (using the Waldmeister theorem prover) that for a bilattice (note the inclusion of negation), if $\otimes$ and $\oplus$ preserve $\leqslant_t$, then $\vee$ preserves $\leqslant_k$. This suggests that there is potential for further work (automated or otherwise) to determine the exact interdependence of the interlacing conditions. This could reveal exactly which weaker interlacing conditions admit a product representation, as well as determining when the negation and the bounds influence the interdependence of the various interlacing conditions. For example, Pynko shows that it is possible to have a bounded pre-bilattice such that $\wedge$ preserving $\leqslant_k$ and $\vee$ preserving $\leqslant_k$ are independent of one another [80, Example 5.1].

The following result uses the definition of the negation in the product construction to prove an analogue of Theorem 1.2.2 for the case of interlaced bilattices.

Theorem 1.2.4 ([82, Theorem 2.2.2]). *Let* $\mathbf{B}$ *be a bilattice. Then* $\mathbf{B}$ *is interlaced if and only if there exists a lattice* $\mathbf{L}$ *such that* $\mathbf{B} \simeq \mathbf{L} \odot \mathbf{L}$.

Duality and representation theorems for bilattices were initially focussed on product representations. Mobasher, Pigozzi, Slutzki and Voutsadakis [74] made use of the product representation of distributive bilattices to show that the category of distributive bilattices and the category of Priestley spaces are dually equivalent. Jung and Rivieccio [72] took a slightly different approach in their duality for distributive bilattices. They defined Priestley bispaces and show that this category is dually equivalent to the category of distributive bilattices.

More recently, Cabrer and Priestley [16] have used natural duality to show that the variety of bounded distributive bilattices (the variety generated by $\mathcal{FOUR}$) is dually equivalent to the category of Priestley spaces. This is the first approach to duality for bilattices that does not go via the product representation, and that takes full advantage of the rich algebraic structure available. The main result from [16] is stated below in the language of natural duality.

Theorem 1.2.5 ([16]). *Let* $B = \{\mathbf{t}, \mathbf{f}, \top, \bot\}$ *and consider the bilattice* $\mathcal{FOUR} = \langle B; \otimes, \oplus, \wedge, \vee, \neg, \mathbf{f}, \mathbf{t}\rangle$. *Then the dualising structure*

$$\underset{\sim}{\mathcal{FOUR}} := \langle B; \leqslant_k, \mathfrak{T}\rangle$$

yields a strong duality on $\mathbb{ISP}(\mathcal{FOUR}) = \mathbb{HSP}(\mathcal{FOUR})$.

This result is the point of departure for our work on duality for bilattices. In our class of examples the truth operations do not preserve $\leqslant_k$, and hence $\leqslant_k$ will not be an algebraic relation and cannot be used as a relation in the dualising structure. We shall use the techniques of natural duality to prove duality theorems for bilattices outside of the distributive and interlaced cases.

In addition to being strong, the duality given above for $\mathbb{ISP}(\mathcal{FOUR})$ is also *optimal* (see Definition 1.3.3). Our dualities for quasivarieties of non-interlaced bilattices will also turn out to be optimal.

$\mathcal{SEVEN}$

Figure 1.3: Ginsberg's bilattice for default logic drawn in its knowledge order (left) and truth order (right).

In Chapter 2 and Chapter 3 we examine, using natural duality theory, families of bilattices that have been developed for applications to default logic. The bilattice $\mathcal{SEVEN}$ (see Fig. 1.3) was first proposed by Ginsberg [61, Fig. 4] for use in inference with default logic. The bilattice $\mathcal{SEVEN}$ has two additional truth values, **dt** and **df**, which represent "true by default" and "false by default". The element $\mathbf{d}\top$ then represents the contradiction that arises if a statement is both true by default and false by default. The idea is further developed to include a hierarchy of default values, where the sequence of "true by default" truth values are decreasing in both the knowledge and truth order, while the sequence of "false by default" truth values are decreasing in the knowledge order while increasing in the truth order. Chapter 2 focuses on natural dualities for quasivarieties generated by such a hierarchy of default bilattices based on Ginsberg's $\mathcal{SEVEN}$.

Default logic was proposed by Reiter [81] in 1980. Such a system of reasoning is useful in situations where one does not have complete information about the truth or falsity of a set of sentences. However, if one knows certain other pieces of information, it may be possible to make inferences and hence consider a sentence to be "true by default".

One of the underlying assumptions is that inferences that are made based on default rules can, and should, be modified when further information becomes available. It is with this feature in mind that default logic is said to be *non-monotonic*. A logic is monotonic if it is not possible to add a formula to the set of theorems and then reduce the set of logical consequences. It is easy to see that in default logic, a sentence p could be inferred to be true by default, but when more information becomes available (for instance, if the sentence "not p" is added to the set of theorems), then p is no longer a consequence of the logic. This feature of being able to modify inferences based on additional information makes default logic a good candidate for bilattice-based reasoning.

A default theory $\mathbb{T}$ is a pair $\langle D, W\rangle$ where D is a set of rules of inference and W is a set of theorems, often called the *background theory*. A default rule $d \in D$ is denoted by

$$d := \frac{\alpha\colon \beta_1, \beta_2, \ldots, \beta_n}{\gamma}$$

where α, β_i, γ are all formulas in the language. The formula α is known as the *prerequisite*, the formulae β_i $(1 \leqslant i \leqslant n)$ are known as the *justifications* and γ is the *consequence* or *conclusion*. The rule d above says that if α is true, and γ being true does not contradict any of the β_i, then γ is true. Ginsberg defines valuations for default logic in terms of the bilattice $\mathcal{SEVEN}$ in [61, Section 7.3.2].

We begin Chapter 3 by introducing a six-element bilattice to serve as an alternative set of truth values for default logic. From this we are able to construct a hierarchy of prioritised default bilattices.

The investigations of dualities in both Chapter 2 and Chapter 3 were initially based on results obtained from computer output. In both cases it soon became apparent that there was potential for further investigation beyond the few examples that we had dealt with via computation.

We prove significant results which characterise the category of topological relational structures dual to the quasivariety of default bilattices. Our results show that a set of relations which capture the knowledge order on the dual structures can be used in order to obtain a duality. The method of proof makes use of globally minimal failsets (defined in the next section) allowing us to ensure that the dualities obtained are in fact optimal dualities. That is, the set of relations is a minimal such set that will yield a duality.

Before we present the important concepts from natural duality theory that we will use in our investigations of dualities, we remark on the two different tracks along which bilattice research has developed. The first of these tracks is the work of Ginsberg and Fitting where bilattices are used as what Fitting describes as *generalised truth-value spaces* [41, Section 2].

The second track uses bilattices to define deductive systems using logical matrices. This approach was initiated by Arieli and Avron [4] in 1996. They define a *logical bilattice* to be a pair $(\mathbf{B}, F)$ consisting of a bilattice $\mathbf{B}$ and a set of designated truth values $F \subseteq B$ known as a *prime bifilter*. A prime bifilter is a subset of B which is a prime filter in both the $\leqslant_k$ and $\leqslant_t$ lattice orders. In [4, Section 3], the prime bifilters are used to describe a semantics for the logical bilattice, and a Gentzen-style deductive system is developed and shown to be sound and complete with respect to the semantics. Building on the work by Arieli and Avron, a Hilbert-style presentation of bilattice based logics was developed by Rivieccio in his PhD thesis [82] (see also the paper by Bou and Rivieccio [13]).

A survey of these two areas of development is given by Gargov [46]: Part I is devoted to the generalised truth-value approach, while Part II examines bilattices as logical matrices.

In this work we do not engage with the logics of bilattices in the sense of Arieli and Avron or Bou and Rivieccio. We view bilattices purely as lattice-based algebras, and show how their rich algebraic and order-theoretic properties can be captured in their dual structures.

1.3 Dualising structures obtained via computation

This section follows on from the introduction to natural duality theory presented in Section 1.1 (page 3). There we defined what it meant for a structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ to yield a duality for a quasivariety $\mathbb{ISP}(\mathbf{M})$.

A further useful observation [17, Lemma 2.1.2] is that a topological structure with (partial) operations and relations $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ yields a duality on $\boldsymbol{\mathcal{A}}$ if and only if the topological relational structure $\underset{\sim}{\mathbf{M}}' = \langle M; R', \mathcal{T}\rangle$ yields a duality on $\boldsymbol{\mathcal{A}}$, where $R' = R \cup \{\, \mathrm{graph}(h) \mid h \in G \cup H \,\}$.

If a structure $\underset{\sim}{\mathbf{M}} = \langle M; R, \mathcal{T}\rangle$ yields a duality on $\boldsymbol{\mathcal{A}}$, then for every $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, and every $a \in A$, the evaluation maps $e_{\mathbf{A}}(a)\colon D(\mathbf{A}) \to M$ preserve every algebraic relation on $\mathbf{M}$ and hence the set of all finitary algebraic relations

$$\mathcal{B} := \bigcup \{\, \mathbb{S}(\mathbf{M}^n) \mid n \geqslant 1 \,\}$$

will yield a duality on $\boldsymbol{\mathcal{A}}$. This approach to showing that a duality exists is called the *Brute Force* method. These insights can be summed up in the following result.

Lemma 1.3.1 ([17, Lemma 1.3]). *Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ and let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Then the following are equivalent:*

(i) *there is some structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ which yields a duality on $\mathbf{A}$;*

(ii) *there is some purely (topological) relational structure $\underset{\sim}{\mathbf{M}} = \langle M; R, \mathcal{T}\rangle$ which yields a duality on $\mathbf{A}$;*

(iii) *the structure $\underset{\sim}{\mathbf{M}} = \langle M; \mathcal{B}, \mathcal{T}\rangle$ yields a duality on $\mathbf{A}$;*

(iv) *the evaluation maps $e_{\mathbf{A}}(a)\colon \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \to M$ $(a \in A)$ are the only continuous maps from $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ to M which preserve every finitary algebraic relation on $\mathbf{M}$.*

While it is comforting to know that the Brute Force method can be applied in order to obtain a duality, in practice one is more interested in a dualising structure with as few relations as possible.

A $(k+1)$-ary term $t(x_1, \dots, x_{k+1})$ is said to be a *$(k+1)$-ary near-unanimity term* on $\mathbf{A}$ if it satisfies the identities

$$t(a, b, \dots, b) \approx t(b, a, b, \dots, b) \approx \cdots \approx t(b, \dots, b, a) \approx b$$

for any $a, b \in A$. All lattice-based algebras $\mathbf{A}$ possess the *median term* which is defined for $a, b, c \in A$ by

$$t_m(a, b, c) := (a \vee b) \wedge (a \vee c) \wedge (b \vee c).$$

For any $a, b \in A$, we have

$$t_m(a, b, b) \approx t_m(b, a, b) \approx t_m(b, b, a) \approx b$$

and thus all lattice-based algebras have a ternary near-unanimity (NU) term. This fact is made extremely useful by the application of the following theorem.

Theorem 1.3.2 ([17, Theorem 2.3.4]). (**NU Duality Theorem**) *Let* $k \geqslant 2$ *and suppose that* $\mathbf{M}$ *has a* $(k+1)$*-ary near-unanimity term. Then* $\underset{\sim}{\mathrm{M}} := \langle M; \mathbb{S}(\underline{\mathbf{M}}^k), \mathcal{T}\rangle$ *yields a duality on* $\boldsymbol{\mathcal{A}}$*.*

This theorem tells us that any lattice-based quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ is dualisable, and that the relational structure on the dualising object $\underset{\sim}{\mathrm{M}}$ can be limited to just $\mathbb{S}(\mathbf{M}^2)$. This is a tremendous improvement on the size of the dualising set given by the Brute Force method. However, this set of subalgebras can still be extremely large, even when $\mathbf{M}$ is a small finite lattice. As an example, if $\mathbf{M} = \mathbf{M}_3$, the smallest modular non-distributive lattice, we have $|\mathbb{S}(\mathbf{M}^2)| = 3193$ [86]. Although we are guaranteed a duality by the entire set of binary algebraic relations (subalgebras of $\mathbf{M}^2$), we want to reduce the size of the set of required relations as much as possible, ideally to some minimal set of relations.

Definition 1.3.3. Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ and consider $\underset{\sim}{\mathrm{M}} = \langle M; G, H, R, \mathcal{T}\rangle$. We say that $\underset{\sim}{\mathrm{M}}$ yields an *optimal duality* on $\boldsymbol{\mathcal{A}}$ if $\underset{\sim}{\mathrm{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ and if any relation or (partial) operation is removed from the set $G \cup H \cup R$, then the new structure $\underset{\sim}{\mathrm{M}}'$ will no longer yield a duality on $\boldsymbol{\mathcal{A}}$.

In finding dualities for quasivarieties, we are often interested in finding optimal dualities as these are the simplest possible dual structures. The theory of optimal dualities was developed by Davey and Priestley [26, 27], beginning with their work on the varieties $\mathbf{B}_n$ of pseudocomplemented distributive lattices.

The NU Duality Theorem suggests a good starting point for finding an optimal duality for a quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$. It is in fact possible to carry out the entire optimisation procedure using a computer to perform the necessary calculations. First, one calculates the subalgebras of $\mathbf{M}^2$ and then one systematically works through each relation to see which combinations of relations are required in order to yield a duality. We were able to use a suite of computer programs written and used by Wegener [86] in order to perform the necessary calculations. Here we describe the theory that informs these calculations.

We remind the reader that the notation $\mathbf{S}$ will be used when we refer to a subalgebra (and want to treat it as such), and S will be used when we want to refer only to S viewed as an n-ary algebraic relation (a subset of M^n).

The concept of *entailment* is crucial in understanding how and why it is possible to reduce the number of algebraic relations required to yield a duality.

Definition 1.3.4. Let $\mathcal{R}$ be a set of finitary algebraic relations on $\mathbf{M}$, and let S be an n-ary algebraic relation (on $\mathbf{M}$). Then we say that $\mathcal{R}$ *entails* S if for every $\mathbf{A} \in \boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$, if $u\colon \mathrm{D}(\mathbf{A}) \to \mathrm{M}$ is a map which preserves $\mathcal{R}$, then u preserves S. This will be denoted $\mathcal{R} \vdash S$.

Entailment constructs were investigated by Davey, Haviar and Priestley [22]. They showed that whenever $\mathcal{R} \vdash S$, there are a finite number of *admissible constructs* that can be used to obtain S from $\mathcal{R}$. A full list is given in [17, Section 2.4.5] although we will only make use of the constructs of *trivial relations*, *intersection* and *converses*. The first of these implies that any set of relations $\mathcal{R}$ entails both $\Delta(\mathbf{M}^2)$ and $\mathbf{M}^2$. The intersection construct says that $\{S_1, S_2\} \vdash S_1 \cap S_2$ provided that $S_1 \cap S_2 \neq \emptyset$. For a binary relation S we denote its converse by $\breve{S}$. That is, $\breve{S} = \{\, (b, a) \mid (a, b) \in S \,\}$. The fact that $\{S\} \vdash \breve{S}$ is a consequence of the more complicated notion of subscript manipulation.

We draw attention to the fact that because binary relations entail their converse relations, we will seldom mention converse relations. Only in Chapter 2 where we completely describe the lattice $\mathbb{S}(\mathbf{M}^2)$ do we take an interest in converse relations. In our listing of subalgebras in Appendix B, for each subalgebra $\mathbf{S}$ we list either $\mathbf{S}$ or $\breve{\mathbf{S}}$, whichever is most meaningful in our interpretation. Likewise, our listing of the globally minimal failsets make no mention of the fact that for each relation S, one could also choose the converse relation $\breve{S}$.

In reducing the size of a set of relations $\mathcal{R}$ yielding a duality, we want to test each relation $S \in \mathcal{R}$ to see if we can remove it from the set $\mathcal{R}$ and have the smaller set $\mathcal{R}\backslash\{S\}$ still yield a duality. The term *test algebra* was first used by Davey and Priestley [26]. The phrase describes the fact that in order to test whether or not a relation S can be removed from a dualising set $\mathcal{R}$, we need only test if the set of relations $\mathcal{R}\backslash\{S\}$ will yield a duality on $\mathbf{S}$. The subalgebra $\mathbf{S}$ considered as an element of $\boldsymbol{\mathcal{A}}$, the "alternative persona" [27, p. 3674] of the algebraic relation S, is all that is needed to test whether or not the relation S is required to yield a duality on $\boldsymbol{\mathcal{A}}$. The following lemma is a formal statement of this result. It was first proved by Davey and Priestley [26, Propositions 2.2 and 2.3].

Lemma 1.3.5. *Let $\mathcal{R}$ be a collection of finitary algebraic relations on $\mathbf{M}$. If $\mathcal{R}$ yields a duality on the test algebra $\mathbf{S} \in \mathbb{S}(\mathbf{M}^n)$, then $\mathcal{R}$ entails S. In particular, if $\mathcal{R}$ yields a duality on $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ and if $\mathcal{R}\backslash\{S\}$ yields a duality on the test algebra $\mathbf{S} \in \mathbb{S}(\mathbf{M}^n)$, then $\mathcal{R}\backslash\{S\}$ yields a duality on $\boldsymbol{\mathcal{A}}$.*

We now proceed to explain how the test algebra method can be applied in practice. Let $\mathbf{M}$ be a finite algebra and let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$. Further, consider Ω to be any set of finitary algebraic relations on $\mathbf{M}$. For any n-ary relation $\mathbf{S} \in \Omega$ (considered as a subalgebra of $\mathbf{M}^n$) we consider the dual space $\mathrm{D}(\mathbf{S}) = (\boldsymbol{\mathcal{A}}(\mathbf{S}, \mathbf{M}), \Omega)$, where the relational structure of each $R \in \Omega$ on $\mathbf{S}$ is defined as on page 4.

Definition 1.3.6. Let $\mathbf{S} \in \Omega$ and let $\gamma\colon \mathrm{D}(\mathbf{S}, \mathbf{M}) \to \mathrm{M}$ be any map. Then

$$\mathrm{Fail}_{\mathbf{S}}(\gamma) := \{\, R \in \Omega \mid \gamma \text{ fails to preserve } R \,\}.$$

If $\mathbf{S} \in \mathrm{Fail}_{\mathbf{S}}(\gamma)$ then we say that $\mathrm{Fail}_{\mathbf{S}}(\gamma)$ is a *failset* for $\mathbf{S}$.

If γ is a map defined by evaluation, then for every $\mathbf{S} \in \Omega$ we will have $\mathrm{Fail}_{\mathbf{S}}(\gamma) = \emptyset$. Thus, the number of relations in a failset $\mathrm{Fail}_{\mathbf{S}}(\gamma)$ is a measure of how close γ is to being an evaluation map. For different maps $\gamma\colon \mathrm{D}(\mathbf{S}, \mathbf{M}) \to \mathrm{M}$ the set $\mathrm{Fail}_{\mathbf{S}}(\gamma)$ may be different. The important point is that as both $\boldsymbol{\mathcal{A}}(\mathbf{S}, \mathbf{M})$ and M are finite, there are finitely many possible maps γ and hence finitely many possibly failsets for $\mathbf{S}$.

Definition 1.3.7. Let $\mathbf{S} \in \Omega$. Then

$$\mathcal{F}_{\mathbf{S}} := \{\, \mathrm{Fail}_{\mathbf{S}}(\gamma) \mid \mathbf{S} \in \mathrm{Fail}_{\mathbf{S}}(\gamma) \,\}.$$

That is, $\mathcal{F}_{\mathbf{S}}$ is the family of all failsets of $\mathbf{S}$. We then define

$$\mathcal{F} := \bigcup \{\, \mathcal{F}_{\mathbf{S}} \mid \mathbf{S} \in \Omega \,\}.$$

Each failset is a collection of relations $\mathcal{R} \subseteq \Omega$. Thus we can order the collection of all failsets by containment. A *globally minimal failset* is a set $U \in \mathcal{F}$ such that U is minimal in $\mathcal{F}$.

For a collection of sets $\mathcal{G}$, a *transversal* of $\mathcal{G}$ is a set T such that

(i) $T \cap G \neq \emptyset$ for each $G \in \mathcal{G}$,

(ii) for each $V \in T$, there exists $G \in \mathcal{G}$ such that $T \backslash \{V\} \cap G = \emptyset$.

The reason that we are interested in failsets is that they provide a method for calculating dualities. As we are always working with a lattice-based algebra $\mathbf{M}$, we can reduce Ω to $\mathbb{S}(\mathbf{M}^2)$. Although this may be extremely large, it is finite, and hence we can work through each $\mathbf{S} \in \mathbb{S}(\mathbf{M}^2)$ and calculate all of the (finitely many) possible failsets. As the total number of failsets is again finite, every failset contains a globally minimal failset. Thus we are able to make use of (iii) below.

Theorem 1.3.8 ([17, Theorem 8.3.1]). (**Optimal Duality by Failsets Theorem**) *Suppose that* Ω *yields a duality on* $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$, *and that* $\mathcal{R} \subseteq \Omega$. *Then*

(i) $\mathcal{R}$ *yields on duality on* $\boldsymbol{\mathcal{A}}$ *if and only if it intersects every failset;*

(ii) $\mathcal{R}$ *yields an optimal duality on* $\boldsymbol{\mathcal{A}}$ *if and only if it is a transversal of the family of failsets.*

Moreover, if every failset contains a globally minimal failset, then

(iii) $\mathcal{R}$ *yields an optimal duality if and only if it is a transversal of the globally minimal failsets.*

The starting point for our work in Chapter 2 and Chapter 3 is to take a bilattice $\mathbf{M}$, and the set of relations $\Omega = \mathbb{S}(\mathbf{M}^2)$, and to calculate the globally minimal failsets within Ω. A transversal of these globally minimal failsets then yields an optimal duality for the quasivariety $\mathbb{ISP}(\mathbf{M})$. An outline of the sequence of computations required to find an optimal duality for $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ starting from $\mathbb{S}(\mathbf{M}^2)$ is given in Appendix A.3.

During our analysis of the algebraic relations which will yield a duality on $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$, we will also be interested in the properties of the set $\mathbb{S}(\mathbf{M}^2)$, viewed as a lattice.

For an algebra $\mathbf{A}$, let $a \in A$. A subalgebra $\mathbf{B}$ of $\mathbf{A}$ is a *value* of $\mathbf{A}$ at a if $\mathbf{B}$ is maximal with respect to not containing a. The following lemma will help us to identify subalgebras which are values (particularly in the case that $\mathbf{A} = \mathbf{M}^2$).

Lemma 1.3.9 ([17, Lemma 8.5.1]). *A subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *is a value of* $\mathbf{A}$ *if and only if* $\mathbf{B}$ *is completely meet-irreducible in the lattice of subalgebras of* $\mathbf{A}$.

We remind ourselves that in an algebraic lattice (such as $\mathbb{S}(\mathbf{M}^2)$), every element is the meet of completely meet-irreducible elements. In a lattice of subalgebras, the meet operation is intersection, and thus by the entailment construct of intersection, we have that the completely meet-irreducible elements entail all of the other relations in $\mathbb{S}(\mathbf{M}^2)$.

Chapter 2

Natural dualities for quasivarieties of default bilattices

Distributive bilattices have a very rich algebraic structure. Not only do they afford a product representation, but as the variety of distributive bilattices is generated by $\mathcal{FOUR}$, they have a particularly simple natural duality. As with duality for bounded lattices, once one moves outside of the distributive case the situation becomes much more complicated. In the next two chapters we shall expose some the features of natural duality for quasivarieties of bilattices that are neither distributive nor interlaced.

Our initial investigations involved computations on the quasivariety generated by the default bilattice $\mathcal{SEVEN}$ proposed by Ginsberg [61]. However, it became apparent that using purely theoretical techniques we could generalise these results to a larger class of bilattices used in default logic.

$\mathcal{SEVEN}$

Figure 2.1: Ginsberg's bilattice for default logic drawn in its knowledge order (left) and truth order (right).

We first observe that $\mathcal{SEVEN}$ is not an interlaced bilattice. Neither $\wedge$ nor $\vee$

preserve $\leqslant_k$. This is witnessed by $\mathbf{d}\top \leqslant_k \mathbf{t}$ but $\mathbf{d}\top \wedge \bot = \mathbf{df} \nleqslant_k \bot = \mathbf{t} \wedge \bot$, and $\mathbf{dt} \leqslant_k \mathbf{f}$ but $\mathbf{dt} \vee \bot = \mathbf{dt} \nleqslant_k \bot = \mathbf{f} \vee \bot$.

The notion of bilattices which represent *prioritised defaults* was discussed by Ginsberg [61, Section 7.3.3]. In a prioritised default bilattice each additional pair of default truth values $\mathbf{f}_{n+1}$,$\mathbf{t}_{n+1}$ will be lower than their respective predecessors ($\mathbf{f}_n$ and $\mathbf{t}_n$) in the knowledge ordering. That is, the knowledge that the additional truth values represent is of a lower priority. In addition, each additional level of default truth values ($\mathbf{f}_{n+1}$ and $\mathbf{t}_{n+1}$) are thought of as $\mathbf{t}_{n+1}$ being 'less true' than its predecessor $\mathbf{t}_n$, while $\mathbf{f}_{n+1}$ is 'less false' than $\mathbf{f}_n$. That is, $\mathbf{t}_{n+1} \leqslant_t \mathbf{t}_n$ and $\mathbf{f}_{n+1} \geqslant_t \mathbf{f}_n$. We illustrate this idea in Fig. 2.2.

There is a wide range of applications of such prioritised default bilattices in artificial intelligence. One example is the design of a tutoring feedback system by Encheva and Tumin [35]. Their system uses a ten-element default bilattice to inform follow-up questions for an automated tutorial system when the initial responses are incomplete or inconsistent. Prioritised default bilattices have also been applied to visual surveillance by Shet, Harwood and Davis [83].

We remind the reader that we shall use boldface capital letters to refer to subalgebras, for example $\mathbf{S}$, and the non-bold letter, say S, to refer to the algebraic relation associated to the subalgebra. At points in this chapter we will be proving that certain subsets of $\mathbf{M}^2$ are indeed subuniverses of $\mathbf{M}^2$. Unless the letter say P comes straight after mention of a subalgebra $\mathbf{P}$, it should not be assumed that P is an algebraic relation, but merely some subset of M^2.

2.1 The bilattices $\mathbf{K}_n$

We define the set of elements of the n-level prioritised default bilattice as follows:

$$K_n = \{\bot, \mathbf{f}_0, \mathbf{t}_0, \top_0, \dots, \mathbf{f}_n, \mathbf{t}_n, \top_n\}.$$

Definition 2.1.1. Let $n \in \omega$ and consider the set K_n. Let $i, j \in \omega$ such that $0 \leqslant i < j \leqslant n$. For any $a \in K_n$ the binary relation $\leqslant_k$ is defined by

(i) $\bot \leqslant_k a \leqslant_k \top_0$,

(ii) $\mathbf{f}_j <_k \mathbf{f}_i$ and $\mathbf{t}_j <_k \mathbf{t}_i$,

(iii) $\mathbf{t}_j <_k \mathbf{f}_i$ and $\mathbf{f}_j <_k \mathbf{t}_i$,

(iv) $\mathbf{f}_j <_k \top_i$, $\mathbf{t}_j <_k \top_i$ and $\top_j <_k \top_i$,

(v) $\top_j <_k \mathbf{f}_i <_k \top_i$,

(vi) $\top_j <_k \mathbf{t}_i <_k \top_i$,

(vii) $a \leqslant_k a$.

Now suppose that $0 \leqslant i \leqslant j \leqslant n$ and $0 \leqslant m \leqslant n$. The binary relation $\leqslant_t$ is defined by

(i) $\mathbf{f}_i \leqslant_t \mathbf{f}_j$,

(ii) $\mathbf{t}_j \leqslant_t \mathbf{t}_i$,

(iii) $\mathbf{f}_m \leqslant_t \mathbf{t}_i$,

(iv) $\mathbf{f}_i \leqslant_t \top_j \leqslant_t \mathbf{t}_i$,

(v) $\mathbf{f}_m \leqslant_t \bot \leqslant_t \mathbf{t}_m$,

(vi) $\top_m \leqslant_t \top_m$ and $\bot \leqslant_t \bot$.

It is not difficult to see that both of the binary relations above are partial orders.

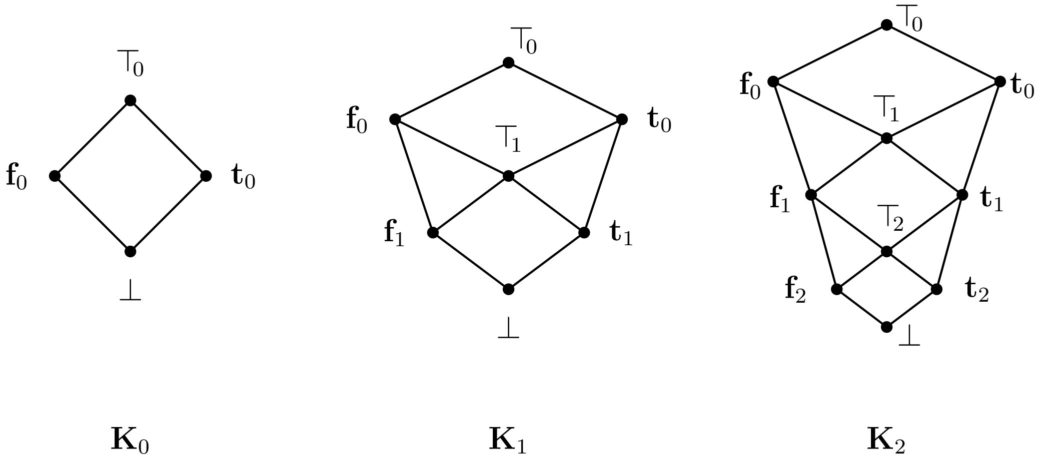


Figure 2.2: The bilattices $\mathbf{K}_0, \mathbf{K}_1$ and $\mathbf{K}_2$, drawn in their knowledge order.

Proposition 2.1.2. *Let $n \in \omega$ and consider the set K_n. The orders $\leqslant_k$ and $\leqslant_t$ are both lattice orders.*

The above proposition implies that there exist two sets of lattice operations on K_n, hence it can be considered as a pre-bilattice. We further define a negation operation for $a \in K_n$ and $i \in \omega$ such that $0 \leqslant i \leqslant n$ by

$$\neg a = \begin{cases} \mathbf{t}_i & \text{if } a = \mathbf{f}_i, \\ \mathbf{f}_i & \text{if } a = \mathbf{t}_i, \\ a & \text{otherwise.} \end{cases}$$

A subset of the truth values of K_n will be used as constants in the algebraic signature. These are values which only contain information about truth or falsity, but do not contain any information about contradictions or lack of information:

$$C_n = \{\mathbf{f}_0, \mathbf{t}_0, \ldots, \mathbf{f}_n, \mathbf{t}_n\}.$$

Finally, we define our bilattices to have an algebraic signature consisting of two sets of lattice operations, a negation operation and the relevant constants:

$$\mathbf{K}_n = \langle K_n; \otimes, \oplus, \wedge, \vee, \neg, C_n \rangle.$$

It is clear from the definition of C_n that the algebra $\mathbf{K}_n$ has the signature

$$(2, 2, 2, 2, 1, 0_{(1)}, 0_{(2)}, \ldots, 0_{(2n+1)}, 0_{(2n+2)}).$$

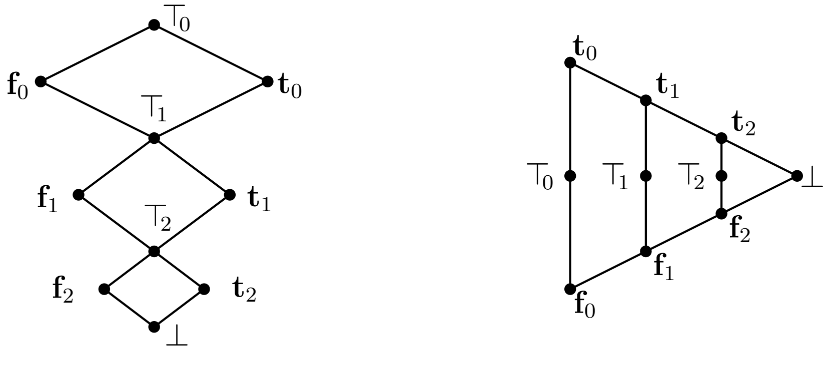


Figure 2.3: The bilattice $\mathbf{K}_2$ in its knowledge order (left) and truth order (right).

The following lemma contains important properties that will be used extensively in Section 2.3. The proof follows from the way in which the order is defined on the truth lattice. This is easily seen in Fig. 2.4.

Lemma 2.1.3. *For $n \in \omega$ consider the prioritised default bilattice $\mathbf{K}_n$ and let $i, j, m \in \{0, 1, \ldots, n\}$ with $i < j$. Then*

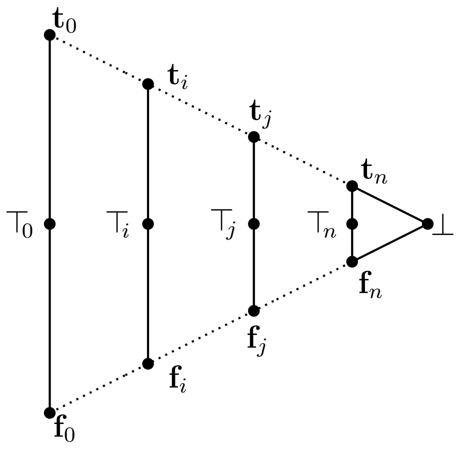


Figure 2.4: The bilattice $\mathbf{K}_n$ in its truth order.

(i) $\bot \wedge \top_m = \mathbf{f}_m$ *and* $\bot \vee \top_m = \mathbf{t}_m$,

(ii) $\top_i \vee \top_j = \mathbf{t}_i$ *and* $\top_i \wedge \top_j = \mathbf{f}_i$,

(iii) $\top_i \vee \mathbf{t}_j = \mathbf{t}_i$ *and* $\top_i \wedge \mathbf{t}_j = \mathbf{f}_i$,

(iv) $\top_i \vee \mathbf{f}_j = \mathbf{t}_i$ *and* $\top_i \wedge \mathbf{f}_j = \mathbf{f}_i$.

2.2 Homomorphic images

The homomorphic images of the bilattice $\mathbf{K}_n$ will play an important role in describing the duality for $\mathbb{ISP}(\mathbf{K}_n)$. This is because the homomorphisms from $\mathbf{K}_n$ to its homomorphic images will prove useful in describing subalgebras of $\mathbf{K}_n^2$.

Here we will define homomorphisms from the bilattice $\mathbf{K}_n$ to the bilattice $\mathbf{K}_m$ (for $m < n$). First, we must be quite clear about what we mean when writing $\mathbf{K}_m$ in this context. It is clear from the definition of the algebras $\mathbf{K}_n$ and $\mathbf{K}_m$, that the signature for $\mathbf{K}_n$ is different to the signature of $\mathbf{K}_m$ when $n \neq m$. The bilattice $\mathbf{K}_n$ has $2n+2$ constants in its signature, whereas the signature of the bilattice $\mathbf{K}_m$ contains $2m+2$ constants. However, it is still possible to define homomorphisms in this setting. Given $m < n$, we define a set of constants

$$E_{m,n} = \{\bot_1, \dots, \bot_{2(n-m)}\} \text{ where } |E_{m,n}| = 2(n-m).$$

Now we define the algebra $\mathbf{K}_{m,n}$ by

$$\mathbf{K}_{m,n} := \langle K_m; \otimes, \oplus, \wedge, \vee, \neg, C_m \cup E_{m,n} \rangle.$$

This is simply the algebra $\mathbf{K}_m$ with additional constants in the signature so that it is of the same type as $\mathbf{K}_n$.

We note that although $\bot$ is not an identified constant in $\mathbf{K}_m$, it is term-definable from $\mathbf{t}_m$ and $\mathbf{f}_m$ as $\bot = \mathbf{t}_m \otimes \mathbf{f}_m$. With the definition of $\mathbf{K}_{m,n}$ in mind, we can formally define a homomorphism $h\colon \mathbf{K}_n \to \mathbf{K}_{m,n}$. In the next proposition we make extensive use of the definitions and results from Appendix A.2.

Proposition 2.2.1. *Let $n \in \omega$ and consider the bilattice $\mathbf{K}_n$. For $m \in \omega$ such that $0 \leqslant m \leqslant n$, the map $h_{nm}\colon K_n \to K_m$ defined for $a \in K_n$ by*

$$h_{nm}(a) = \begin{cases} \bot_{(m)} & \text{if } a \leqslant_k \top_{m+1} \\ a_{(m)} & \text{otherwise} \end{cases}$$

is a homomorphism $h_{nm}\colon \mathbf{K}_n \to \mathbf{K}_{m,n}$.

Proof. The fact that h_{nm} preserves the constants in the signature of $\mathbf{K}_n$ is clear from the way that the constants in the signature of $\mathbf{K}_{m,n}$ are defined.

Consider the equivalence relation $\ker(h_{nm})$. The equivalence classes of $\ker(h_{nm})$ are $[a] = \{a\}$ for $a \notin {\downarrow_k}(\top_{m+1})$, and $[\top_{m+1}] = {\downarrow_k}(\top_{m+1})$. Each of these equivalence classes is a convex subset of K_n under the $\leqslant_k$ order. It is clear that each equivalence class is a sublattice of K_n. Now consider a quadrilateral: $\{a, b, c, d\} \subseteq K_n$ such that $a <_k b$, $c <_k d$ and $a \oplus d = b$ and $a \otimes d = c$. If $h_{nm}(a) = h_{nm}(b)$ we must have $\{a, b\} \subseteq {\downarrow}(\top_{m+1})$. Clearly then $c \leqslant_k b \leqslant_k \top_{m+1}$ and $d \leqslant_k b \leqslant_k \top_{m+1}$. If $b \oplus c = d$ and $b \otimes c = a$ and $h_{nm}(a) = h_{nm}(b)$ then $b = \mathbf{t}_i$ and $c = \mathbf{f}_i$ (or vice versa) for some $i \geqslant m + 1$ and so $d = \top_i \leqslant_k \top_{m+1}$ and so $\{a, b, c, d\} \subseteq {\downarrow}(\top_{m+1})$. These facts give us that $\ker(h_{nm})$ is compatible with $\otimes$ and $\oplus$ (see [28, Theorem 6.14]). As ${\downarrow_k}(\top_{m+1}) = {\downarrow_t}(\mathbf{t}_{m+1}) \cap {\uparrow_t}(\mathbf{f}_{m+1})$ (see Fig. 2.4 or Fig. 2.5), the equivalence classes of $\ker(h_{nm})$ are also convex in the $\leqslant_t$ order. The equivalence classes of h_{nm} are easily seen to be sublattices in the truth lattice. If $\{a, b, c, d\}$ is a quadrilateral in the $\leqslant_t$ order (with one of the two quadrilateral orderings described above) we must have either a not comparable to d, or b not comparable to c. In the first case, we must have $\{a, d\} \subseteq \{\, \top_i \mid 0 \leqslant i \leqslant n \,\} \cup \{\bot\}$. This gives us $a \vee d = b = \mathbf{t}_j$ where j is the least index such that $\top_j \in \{a, d\}$. Similarly, $a \wedge d = c = \mathbf{f}_j$. In the second case we must have $\{b, c\} \subseteq \{\, \top_i \mid 0 \leqslant i \leqslant n \,\} \cup \{\bot\}$, and so $b \wedge c = a = \mathbf{f}_j$ and $b \vee c = d = \mathbf{t}_j$. In either case, we can only have $h_{nm}(a) = h_{nm}(b)$ if $j \geqslant m + 1$. Then $\{a, b, c, d\} \subseteq {\downarrow_t}(\mathbf{t}_{m+1}) \cap {\uparrow_t}(\mathbf{f}_{m+1})$ and the quadrilateral condition is easily seen to be satisfied. Thus $\ker(h_{nm})$ is compatible with $\wedge$ and $\vee$ and hence preserves the lattice operations $\{\otimes, \oplus, \wedge, \vee\}$.

Lastly, we show that h_{nm} preserves the negation. Since $\neg$ is $\leqslant_k$-preserving, we have for $b \leqslant_k \top_{m+1}$ that $\neg b \leqslant_k \top_{m+1}$ and hence $\neg h_{nm}(b) = \neg(\bot_{(m)}) = \bot_{(m)} = h_{nm}(\neg b)$. $\square$

2.3 Subalgebras of $\mathbf{K}_n^2$

Here we prove some important results about the structure of the subalgebras of $\mathbf{K}_n^2$. We consider four-element sub-bilattices of $\mathbf{K}_n$ which we shall call *diamonds*. For m such that $0 \leqslant m < n$, we let $Q_m := \{\mathbf{f}_m, \mathbf{t}_m, \top_m, \top_{m+1}\}$ and for $m = n$ we have $Q_m := \{\mathbf{f}_m, \mathbf{t}_m, \top_m, \bot\}$.

The following result will be crucial when considering subalgebras of $\mathbf{K}_n^2$.

Lemma 2.3.1. *Consider the default bilattice* $\mathbf{K}_n$. *Then for any* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$ *it is the case that* $\Delta(\mathbf{K}_n^2) \subseteq S$.

Proof. For any m with $0 \leqslant m \leqslant n$ we have that $\mathbf{f}_m$ and $\mathbf{t}_m$ are constants in the signature of $\mathbf{K}_n$ and hence $(\mathbf{f}_m, \mathbf{f}_m)$ and $(\mathbf{t}_m, \mathbf{t}_m)$ are elements of any $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$. Since $\mathbf{S}$ is closed under $\oplus$, we have that $(\mathbf{f}_m, \mathbf{f}_m) \oplus (\mathbf{t}_m, \mathbf{t}_m) = (\top_m, \top_m) \in \mathbf{S}$ and that $(\mathbf{f}_n, \mathbf{f}_n) \otimes (\mathbf{t}_n, \mathbf{t}_n) = (\bot, \bot) \in \mathbf{S}$. $\square$

We will frequently be making use of the fact that for any $a \in K_n$, and any $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$, we have $(a, a) \in S$. Thus we will not refer each time to Lemma 2.3.1.

Although for all $n > 0$ we have that $\leqslant_k$ is not a subalgebra of $\mathbf{K}_n^2$, we will see throughout this section that the structure of the subalgebras closely resembles the $\leqslant_k$ order. The result below provides an early example of this pattern.

Proposition 2.3.2. (**Completing the diamond**) *Let* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$, *consider* Q_m *where* $0 \leqslant m \leqslant n$, *and let* $(a, b) \in Q_m^2 \cap <_k$. *If* $(a, b) \in S$ *then* $Q_m^2 \cap <_k \subseteq S$.

Proof. For $m < n$ we consider the set

$$Q_m^2 \cap <_k := \{(\mathbf{f}_m, \top_m), (\mathbf{t}_m, \top_m), (\top_{m+1}, \mathbf{f}_m), (\top_{m+1}, \mathbf{t}_m), (\top_{m+1}, \top_m)\}.$$

We will show if any one of these is an element of the subalgebra $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$, then $Q_m^2 \cap <_k \subseteq S$. The case of $m = n$ follows easily by replacing $\top_{m+1}$ with $\bot$.

Case 1: $(\mathbf{f}_m, \top_m) \in S$. Since $\mathbf{S}$ is closed under negation, we have that $(\mathbf{t}_m, \top_m) \in S$. Further, since $(\top_{m+1}, \top_{m+1}) \in S$ we have that $(\mathbf{f}_m, \top_m) \vee (\top_{m+1}, \top_{m+1}) = (\top_{m+1}, \mathbf{t}_m) \in S$ and $(\mathbf{t}_m, \top_m) \wedge (\top_{m+1}, \top_{m+1}) = (\top_{m+1}, \mathbf{f}_m) \in S$. Lastly we see that $(\top_{m+1}, \mathbf{t}_m) \wedge (\mathbf{t}_m, \top_m) = (\top_{m+1}, \top_m)$ and so $Q_m^2 \cap <_k \subseteq S$ whenever $(\mathbf{f}_m, \top_m) \in S$.

Case 2: $(\mathbf{t}_m, \top_m) \in S$. We apply the negation and follow Case 1.

Case 3: $(\top_{m+1}, \mathbf{f}_m) \in S$. Taking $(\top_m, \top_m)$ we see that $(\top_{m+1}, \mathbf{f}_m) \vee (\top_m, \top_m) = (\mathbf{t}_m, \top_m) \in S$. Now apply negation and the method from Case 1.

Case 4: $(\top_{m+1}, \mathbf{t}_m) \in S$. Apply negation and follow the previous case.

Case 5: $(\top_{m+1}, \top_m) \in S$. Using the fact that $(\top_{m+1}, \top_{m+1}) \in S$ we get $(\top_{m+1}, \top_m) \vee (\top_{m+1}, \top_{m+1}) = (\top_{m+1}, \mathbf{t}_m) \in S$ and then proceed as for Case 2. This also works for the case of $m = n$, simply by showing that from $(\bot, \top_m)$ we can get $(\bot, \top_m) \vee (\bot, \bot) = (\bot, \mathbf{t}_m) \in S$. □

The proof of the corresponding result for the diamond ordered by $\geqslant_k$ follows by replacing any applications of $\wedge$ with $\vee$ (and vice versa).

Proposition 2.3.3. *Let* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$*, consider* Q_m *where* $0 \leqslant m \leqslant n$*, and let* $(a, b) \in Q_m^2 \cap >_k$*. If* $(a, b) \in S$ *then* $Q_m^2 \cap >_k \subseteq S$*.*

We use the term 'collapse' when referring to a subset $A \subseteq K_n$ to mean that all of the elements of A are related to one another in a subalgebra of $\mathbf{K}_n^2$. The following sequence of results works towards showing, in Proposition 2.3.8, that if one of diamonds Q_m 'collapses', then all of the diamonds below it will also collapse. This collapsing is a crucial feature of the subalgebras of $\mathbf{K}_n^2$, and will greatly simplify working with the set of relations $\mathbb{S}(\mathbf{K}_n^2)$.

Lemma 2.3.4. *Let* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$ *and consider* Q_m *where* $0 \leqslant m \leqslant n$*. Then the following are equivalent:*

(1) $(\mathbf{f}_m, \mathbf{t}_m) \in S$;

(2) $(\mathbf{t}_m, \mathbf{f}_m) \in S$;

(3) $Q_m^2 \cap <_k \subseteq S$ *and* $Q_m^2 \cap >_k \subseteq S$;

(4) $Q_m^2 \subseteq S$.

Proof. We begin by noting that

$$Q_m^2 = (Q_m^2 \cap <_k) \cup (Q_m^2 \cap >_k) \cup \{(\mathbf{f}_m, \mathbf{t}_m), (\mathbf{t}_m, \mathbf{f}_m)\} \cup \Delta(Q_m^2).$$

The equivalence of (1) and (2) follows from the fact that $\mathbf{S}$ is closed under negation. Next, we show that (2) $\Rightarrow$ (4). By Lemma 2.3.1 we have that $(\top_m, \top_m) \in S$ and so we get that $(\mathbf{t}_m, \mathbf{f}_m) \vee (\top_m, \top_m) = (\mathbf{t}_m, \top_m) \in S$. Now we apply Proposition 2.3.2 and conclude that $Q^2_m \cap <_k \subseteq S$.

Next we note that $(\mathbf{t}_m, \mathbf{f}_m) \wedge (\top_m, \top_m) = (\top_m, \mathbf{f}_m) \in S$. Similarly, we apply Proposition 2.3.3 to see that $Q^2_m \cap >_k \subseteq S$, and hence $Q^2_m \subseteq S$.

Now we prove that (3) $\Rightarrow$ (1). From the assumption, we have that $(\mathbf{f}_m, \top_m)$ and $(\top_m, \mathbf{t}_m)$ are both in S. Thus we can see that $(\mathbf{f}_m, \top_m) \otimes (\top_m, \mathbf{t}_m) = (\mathbf{f}_m, \mathbf{t}_m) \in S$.

The fact that (4) $\Rightarrow$ (3) is obvious. □

Lemma 2.3.5. *Let $n, m \in \omega$ such that $0 \leqslant m \leqslant n$ and let $\mathbf{S} \in \mathbb{S}(\mathbf{K}^2_n)$. If $Q^2_m \cap <_k \subseteq S$ or $Q^2_m \cap >_k \subseteq S$ then $Q^2_{m+1} \cap <_k \subseteq S$ and $Q^2_{m+1} \cap >_k \subseteq S$.*

Proof. Since $Q^2_m \cap <_k \subseteq S$ we have $(\top_{m+1}, \top_m) \in S$. This implies that $(\top_{m+1}, \top_m) \wedge (\top_{m+2}, \top_{m+2}) = (\mathbf{f}_{m+1}, \mathbf{f}_m) \in S$. Using the fact that S is closed under negation, we get $(\mathbf{t}_{m+1}, \mathbf{t}_m) \in S$ and so $(\mathbf{f}_{m+1}, \mathbf{f}_m) \otimes (\mathbf{t}_{m+1}, \mathbf{t}_m) = (\top_{m+2}, \top_{m+1}) \in S$. Now we apply Proposition 2.3.2 to see that $Q^2_{m+1} \cap <_k \subseteq S$.

We also have $(\top_{m+1}, \mathbf{f}_m) \in S$ and so $(\top_{m+1}, \mathbf{f}_m) \vee (\top_{m+2}, \top_{m+2}) = (\mathbf{t}_{m+1}, \top_{m+2}) \in S$. By Proposition 2.3.3 we see that $Q^2_{m+1} \cap >_k \subseteq S$. The argument starting with $Q^2_m \cap >_k \subseteq S$ is similar. □

Proposition 2.3.6. *Let $n, m \in \omega$ such that $0 \leqslant m \leqslant n$ and consider the bilattice $\mathbf{K}_n$. Further, let $\mathbf{S} \in \mathbb{S}(\mathbf{K}^2_n)$ and suppose that there exists $(a, b) \in Q^2_m \cap S$ such that $a \neq b$. Then $Q^2_{m+1} \subseteq S$.*

Proof. Suppose $(a, b) \in Q^2_m \cap <_k$. By Proposition 2.3.2 we have that $Q^2_m \cap <_k \subseteq S$ and hence by Lemma 2.3.5 we have that $Q^2_{m+1} \cap <_k \subseteq S$ and $Q^2_{m+1} \cap >_k \subseteq S$. Now by Lemma 2.3.4 we have that $Q^2_{m+1} \subseteq S$. Similarly, if $(a, b) \in Q^2_m \cap >_k$ we can apply Proposition 2.3.3 to get $Q^2_m \cap >_k \subseteq S$ and hence both $Q^2_{m+1} \cap <_k \subseteq S$ and $Q^2_{m+1} \cap >_k \subseteq S$. Again, applying Lemma 2.3.4 gives the desired result. If $(a, b) = (\mathbf{t}_m, \mathbf{f}_m)$ or $(a, b) = (\mathbf{f}_m, \mathbf{t}_m)$, then we simply apply Lemma 2.3.4 and then Lemma 2.3.5 to get $Q^2_{m+1} \cap <_k \subseteq S$ and $Q^2_{m+1} \cap >_k \subseteq S$. We finish by applying Lemma 2.3.4 once more to conclude that $Q^2_{m+1} \subseteq S$. □

Lemma 2.3.7. *Let $n, m \in \omega$ such that $0 \leqslant m \leqslant n$ and consider the bilattice $\mathbf{K}_n$. Further, let $\mathbf{S} \in \mathbb{S}(\mathbf{K}^2_n)$. If $Q^2_m \subseteq S$, then $(\perp, \top_m) \in S$ and $(\top_m, \perp) \in S$.*

Proof. Since $Q^2_m \subseteq S$, we have $(\mathbf{t}_m, \mathbf{f}_m) \in S$. Now $(\mathbf{t}_m, \mathbf{f}_m) \wedge (\perp, \perp) = (\perp, \mathbf{f}_m) \in S$. Since $\mathbf{S}$ is closed under negation, we have that $(\perp, \mathbf{t}_m) \in S$. We then see that $(\perp, \mathbf{f}_m) \oplus (\perp, \mathbf{t}_m) = (\perp, \top_m) \in S$.

Next, we observe that $(\mathbf{t}_m, \mathbf{f}_m) \in S$ implies that $(\mathbf{t}_m, \mathbf{f}_m) \vee (\bot, \bot) = (\mathbf{t}_m, \bot) \in S$. By closure under negation we get $(\mathbf{f}_m, \bot) \in S$ and finally this gives us $(\mathbf{t}_m, \bot) \oplus (\mathbf{f}_m, \bot) = (\top_m, \bot) \in S$. $\square$

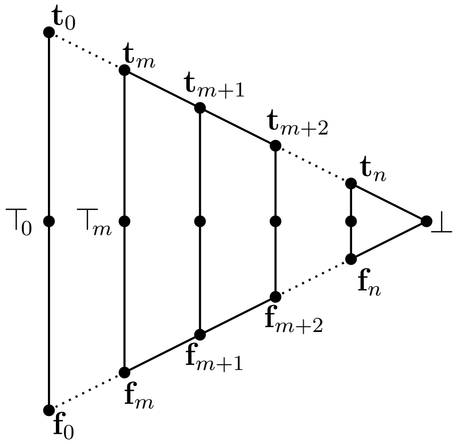


Figure 2.5: The bilattice $\mathbf{K}_n$ in its truth order.

We now have the necessary tools to show that, given certain conditions on elements of Q_m^2 being in S, all of the diamonds below Q_m will collapse in S.

Proposition 2.3.8. (**Collapsing the diamonds**) *Let* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$ *and let* $m \in \omega$ *such that* $0 \leqslant m \leqslant n$. *If* $\mathbf{S}$ *satisfies any of the equivalent conditions of Lemma 2.3.4, then* $(\downarrow_k \top_m)^2 \subseteq S$.

Proof. We assume that $Q_m^2 \subseteq S$ and apply Lemma 2.3.7 to see that $(\bot, \top_m) \in S$ and $(\top_m, \bot) \in S$. Now let $a, b \in K_n$ such that $a, b \leqslant_k \top_m$. Using the fact that $(a, a) \in S$ and $(b, b) \in S$, we can see that

$$\big((\bot, \top_m) \oplus (a, a)\big) \otimes \big((\top_m, \bot) \oplus (b, b)\big) \;=\; (a, \top_m) \otimes (\top_m, b) \;=\; (a, b) \in S,$$

and

$$\big((\bot, \top_m) \oplus (b, b)\big) \otimes \big((\top_m, \bot) \oplus (a, a)\big) \;=\; (b, \top_m) \otimes (\top_m, a) \;=\; (b, a) \in S. \qquad \square$$

We can now combine Proposition 2.3.8 with Proposition 2.3.6 to reveal another instance of the diamonds 'collapsing'. This corollary will enable us to draw conclusions about the elements of a subalegbra $\mathbf{S}$ when we only know about the existence of pairs taken from Q_m^2. The result below shows that in a subalgebra $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$, if we have $(a, b) \in S$ such that $(a, b) \in Q_m^2 \backslash (\Delta(Q_m^2))$, then all of the elements in all of the diamonds below Q_m are related to one another.

Corollary 2.3.9. *Let* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$ *and consider* Q_m *where* $0 \leqslant m < n$*. If there exists* $(a, b) \in Q_m^2$ *such that* $a \neq b$ *and* $(a, b) \in S$*, then* $(\downarrow_k \top_{m+1})^2 \subseteq S$*.*

Proof. Since $(a, b) \in S$ and $a \neq b$ we can apply Proposition 2.3.6 to get $Q_{m+1}^2 \subseteq S$. Now we use Proposition 2.3.8 to see that $(\downarrow_k \top_{m+1})^2 \subseteq S$. □

The final lemma in this series of results, shows that the elements of the collapsed diamonds below the element $\top_{m+1}$ are all related to the same elements as $\top_{m+1}$.

Lemma 2.3.10. *Let* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$*. If* $Q_m^2 \cap <_k \subseteq S$ *then for all* $a \leqslant_k \top_{m+1}$ *and for all* $b \in Q_m$*, we have* $(a, b) \in S$*. Dually, if* $Q_m^2 \cap >_k \subseteq S$*, then for all* $c \in Q_m$ *and for all* $d \leqslant_k \top_{m+1}$ *we have* $(c, d) \in S$*.*

Proof. Let $a \leqslant_k \top_{m+1}$ and take $b \in Q_m \backslash \{\top_{m+1}\}$. Clearly $\top_{m+1} <_k b$ and so $(\top_{m+1}, b) \in S$. By Corollary 2.3.9 we have that $(\downarrow_k \top_{m+1})^2 \subseteq S$ and hence $(a, \top_{m+1}) \in S$. Now $(a, \top_{m+1}) \otimes (\top_{m+1}, b) = (a, b) \in S$. □

Definition 2.3.11. Let $n \in \omega$ and consider the bilattice $\mathbf{K}_n$. For each $m \in \omega$ such that $0 \leqslant m \leqslant n$ we define $S_{nm} \subseteq K_n^2$ by

$$S_{nm} := \Delta(K_n^2) \cup (Q_m^2 \cap <_k) \cup \{\, (a, b) \mid a, b \leqslant_k \top_{m+1} \,\} \cup \{\, (c, d) \mid c \leqslant_k \top_{m+1}, d \in Q_m \,\}.$$

We note that if $m = n$ the expression for S_{nm} can be simplified. When $m = n$, there is no element $\top_{m+1}$ and so the last two sets in the union can be ignored. That is,

$$S_{nn} = \Delta(K_n^2) \cup (Q_n^2 \cap <_k).$$

The Hasse diagram style depictions of the sets S_{nm} are to be interpreted as a quasi-order, with the additional interpretation that $a, b \in \boxed{\text{a,b}}$ if and only if $(a, b) \in S_{nm}$ and $(b, a) \in S_{nm}$. For $n = 2$ we draw the sets S_{20}, S_{21}, S_{22} in Fig. 2.6. There are elements of K_n which are only related to themselves (such as $\top_0$ in S_{21}). Our diagrams are true to the sets S_{nm} but we do place them rather suggestively in the space available.

At this point, we only know that the S_{nm} are subsets of K_n^2. The next few results will reveal that the S_{nm} are in fact subuniverses of K_n^2 and hence algebraic relations.

Proposition 2.3.12. *Let* $n \in \omega$ *and consider the bilattice* $\mathbf{K}_n$*. Then* $S_{nn} \in \mathbb{S}(\mathbf{K}_n^2)$*.*

Proof. The fact that S_{nn} is closed under $\otimes$ and $\oplus$ is clear from the fact that for every $a \in K_n \setminus Q_n$ and every $(b, c) \in Q_n^2 \cap \leqslant_k$ we have that $a \geqslant_k b$ and $a \geqslant_k c$. Hence $(a, a) \otimes (b, c) = (b, c)$ and $(a, a) \oplus (b, c) = (a, a)$. To show S_{nn} is closed under $\wedge$ and $\vee$, we use the results in Lemma 2.1.3 and note how $\wedge$ and $\vee$ operate on combinations of elements of $\Delta(K_n^2)$ and Q_n. In the table below, we consider m such that $0 \leqslant m < n$.

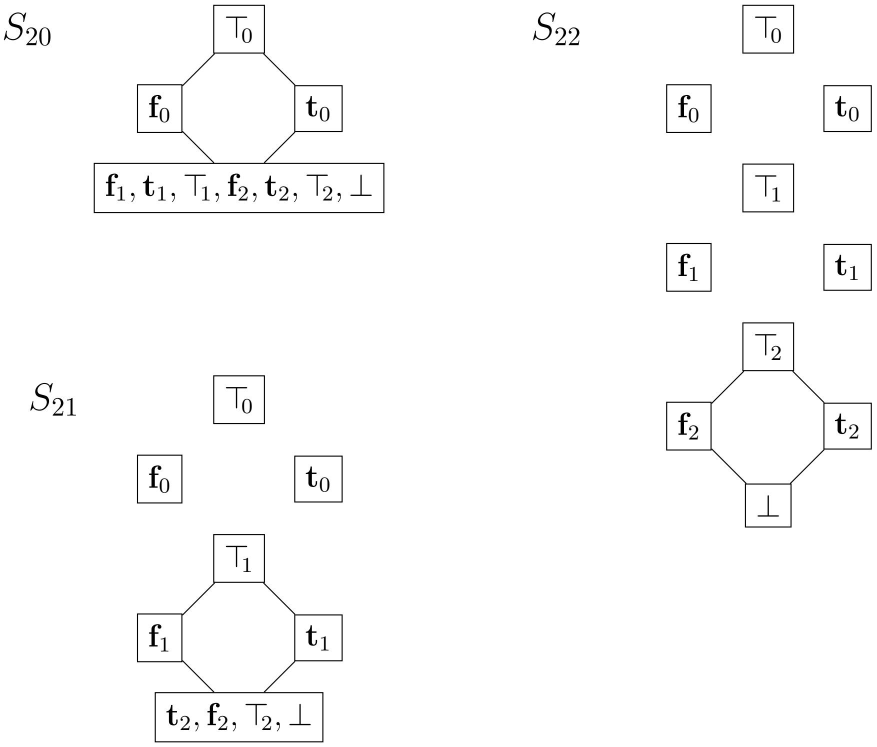


Figure 2.6: Subsets of K_2^2 drawn as quasi-orders.

	$\wedge(\mathbf{t}_m, \mathbf{t}_m)$	$\vee(\mathbf{t}_m, \mathbf{t}_m)$	$\wedge(\top_m, \top_m)$	$\vee(\top_m, \top_m)$	$\wedge(\mathbf{f}_m, \mathbf{f}_m)$	$\vee(\mathbf{f}_m, \mathbf{f}_m)$
$(\bot, \top_n)$	$(\bot, \top_n)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\bot, \top_n)$
$(\bot, \mathbf{t}_n)$	$(\bot, \mathbf{t}_n)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\bot, \mathbf{t}_n)$
$(\bot, \mathbf{f}_n)$	$(\bot, \mathbf{f}_n)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\bot, \mathbf{f}_n)$
$(\mathbf{f}_n, \top_n)$	$(\mathbf{f}_n, \top_n)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\mathbf{f}_n, \top_n)$
$(\mathbf{t}_n, \top_n)$	$(\mathbf{t}_n, \top_n)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\mathbf{t}_m, \mathbf{t}_m)$	$(\mathbf{f}_m, \mathbf{f}_m)$	$(\mathbf{t}_n, \top_n)$

Lastly, it is easy to see that S_{nn} is closed under negation. $\square$

We are working towards showing for any $n, m \in \omega$ with $0 \leqslant m \leqslant n$ that S_{nm} is a subalgebra of $\mathbf{K}_n^2$. To do this we will make use of the above result and the homomorphisms defined in Proposition 2.2.1. However, we first need to prove the necessary set correspondence.

Lemma 2.3.13. *Let $n, m \in \omega$ such that $0 \leqslant m \leqslant n$. Consider the subsets $S_{mm} \subseteq K_m^2$ and $S_{nm} \subseteq K_n^2$ and the homorphism $h_{nm} \colon \mathbf{K}_n \to \mathbf{K}_{m,n}$. Then*

$$(h_{nm}, h_{nm})^{-1}(S_{mm}) = S_{nm}.$$

Proof. If $m = n$, the claim is obviously true as $h_{nm} = h_{nn}$ is simply the identity homomorphism. Suppose that $m < n$ and recall from Proposition 2.2.1 that for

$a \in K_n$ we have $h_{nm}(a)$ defined by

$$h_{nm}(a) = \begin{cases} \bot_{(m)} & \text{if } a \leqslant_k \top_{m+1} \\ a_{(m)} & \text{otherwise.} \end{cases}$$

Now for $(a,b) \in S_{nm}$ we want to show that $(h_{nm}, h_{nm})(a,b) \in S_{mm}$. There are four cases for $(a,b) \in S_{nm}$.

<u>Case 1:</u> $(a,b) \in \Delta(K_n^2)$. Clearly $(h_{nm}, h_{nm})(a,b) = (h_{nm}(a), h_{nm}(a)) \in S_{mm}$.

<u>Case 2:</u> $(a,b) \in Q_m^2 \cap <_k$. This follows from the list of equations shown below:

$$\begin{aligned} (h_{nm}, h_{nm})(\top_{m+1}, \mathbf{f}_m) &= (\bot, \mathbf{f}_m), \\ (h_{nm}, h_{nm})(\top_{m+1}, \mathbf{t}_m) &= (\bot, \mathbf{t}_m), \\ (h_{nm}, h_{nm})(\top_{m+1}, \top_m) &= (\bot, \top_m), \\ (h_{nm}, h_{nm})(\mathbf{f}_m, \top_m) &= (\mathbf{f}_m, \top_m), \\ (h_{nm}, h_{nm})(\mathbf{t}_m, \top_m) &= (\mathbf{t}_m, \top_m). \end{aligned}$$

<u>Case 3:</u> $a, b \leqslant_k \top_{m+1}$. Here $(h_{nm}, h_{nm})(a,b) = (\bot, \bot) \in S_{mm}$.

<u>Case 4:</u> $a \leqslant_k \top_{m+1}$, $b \in Q_m$. If $b = \top_{m+1}$, then $(h_{nm}, h_{nm})(a,b) = (\bot, \bot)$, and otherwise we have $h_{nm}(b) = d$ where $d \in \{\mathbf{f}_m, \mathbf{t}_m, \top_m\}$ and hence $(h_{nm}, h_{nm})(a,b) = (\bot_{(m)}, d) \in Q_m^2 \cap <_k \subseteq S_{mm}$.

We have shown that $S_{nm} \subseteq (h_{nm}, h_{nm})^{-1}(S_{mm})$. Now suppose that $(a,b) \in (h_{nm}, h_{nm})^{-1}(S_{mm})$. That is, $(h_{nm}(a), h_{nm}(b)) \in S_{mm}$. All of the possibilities are covered by the implications below:

$$\begin{aligned} &(h_{nm}(a), h_{nm}(b)) \in \{(\bot, \mathbf{f}_m), (\bot, \mathbf{t}_m), (\bot, \top_m)\} \implies a \leqslant_k \top_{m+1}, b \in Q_m, \\ &(h_{nm}(a), h_{nm}(b)) = (\bot, \bot) \implies a, b \leqslant_k \top_{m+1}, \\ &(h_{nm}(a), h_{nm}(b)) = (a_{(m)}, a_{(m)}) \text{ for } a_{(m)} \neq \bot_{(m)} \implies a = b \in K_n \backslash ({\downarrow}\top_{m+1}), \\ &(h_{nm}(a), h_{nm}(b)) = (\mathbf{f}_m, \top_m) \implies (a,b) = (\mathbf{f}_m, \top_m) \in Q_m^2 \cap <_k, \\ &(h_{nm}(a), h_{nm}(b)) = (\mathbf{t}_m, \top_m) \implies (a,b) = (\mathbf{t}_m, \top_m) \in Q_m^2 \cap <_k \ . \end{aligned}$$

□

Now we can use the set equivalence above to show that all of the sets S_{nm} are indeed subuniverses.

Proposition 2.3.14. *Let $n \in \omega$ and consider the bilattice $\mathbf{K}_n$. If $m \in \omega$ such that $0 \leqslant m \leqslant n$, then $S_{nm} \in \mathbb{S}(\mathbf{K}_n^2)$.*

Proof. Proposition 2.2.1 says that there exists a homomorphism $h_{nm}\colon \mathbf{K}_n \to \mathbf{K}_{m,n}$. We also have from Proposition 2.3.12 that $S_{mm} \in \mathbb{S}(\mathbf{K}_m^2)$. Finally, using Lemma 2.3.13 we can conclude that $(h_{nm}, h_{nm})^{-1}(S_{mm}) = S_{nm}$ is a subuniverse of $\mathbf{K}_n^2$. □

Having shown that each of the $\mathbf{S}_{nm}$ are subalgebras of $\mathbf{K}_n^2$, we can go one step further and in fact show that all of subalgebras of $\mathbf{K}_n^2$ (except $\mathbf{K}_n^2$ itself) are formed by the $\mathbf{S}_{nm}$, their converses, and the intersections of these. We will need to use the following lemma to prove our main result.

Lemma 2.3.15. *Let* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$ *and suppose that* $(\top_i, b) \in S$ *for* $0 \leqslant i \leqslant n$ *and* $b <_k \top_i$*. Then* $(\top_i, \top_{i+1}) \in S$*. Dually, if* $(c, \top_i) \in S$ *for* $c <_k \top_i$*, then* $(\top_{i+1}, \top_i) \in S$*.*

Proof. First we consider the case that $b \leqslant_k \top_{i+1}$. Since $(\top_{i+1}, \top_{i+1}) \in S$ we have $(\top_i, b) \oplus (\top_{i+1}, \top_{i+1}) = (\top_i, \top_{i+1}) \in S$. If $b \nleqslant_k \top_{i+1}$ then either $b = \mathbf{t}_i$ or $b = \mathbf{f}_i$. In either case we can apply Proposition 2.3.3 to see that $Q_i^2 \cap >_k \subseteq S$ and hence $(\top_i, \top_{i+1}) \in S$. For $c \leqslant_k \top_{i+1}$ we see that $(c, \top_i) \oplus (\top_{i+1}, \top_{i+1}) = (\top_{i+1}, \top_i) \in S$. If $c = \mathbf{t}_i$ or $c = \mathbf{f}_i$ we use Proposition 2.3.2 to get $Q_i^2 \cap <_k \subseteq S$ and hence $(\top_{i+1}, \top_i) \in S$. □

We can now prove the main result which completely characterises the subalgebras of $\mathbf{K}_n^2$ for any $n \in \omega$. Given a subalgebra $\mathbf{S}$ which lies between $\Delta(\mathbf{K}_n^2)$ and $\mathbf{K}_n^2$, we will take elements $(a, b) \in S$ with $a \neq b$ that are "highest up" in the $\leqslant_k$ order, and use these elements to show that the subalgebra $\mathbf{S}$ falls into one of our easily described classes. We will make extensive use of the diamonds $\{\, Q_m \mid 0 \leqslant m \leqslant n \,\}$ and their properties.

Proposition 2.3.16. *Let* $n \in \omega$ *and consider the bilattice* $\mathbf{K}_n^2$*. If* $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$ *such that* $\mathbf{S} \neq \mathbf{K}_n^2$*, then for some* $m \in \{0, \dots, n\}$*,*

$$\mathbf{S} = \mathbf{S}_{nm} \quad \textit{or} \quad \mathbf{S} = \breve{\mathbf{S}}_{nm} \quad \textit{or} \quad \mathbf{S} = \mathbf{S}_{nm} \cap \breve{\mathbf{S}}_{nm}.$$

Proof. For $\mathbf{S} = \Delta(\mathbf{K}_n^2)$ it is easy to see that $\mathbf{S} = \mathbf{S}_{nn} \cap \breve{\mathbf{S}}_{nn}$. If $\mathbf{S} \neq \Delta(\mathbf{K}_n^2)$, the set of elements $T = \{\, (a, b) \in S \mid a \neq b \,\}$ is non-empty. Now let

$$A = \{\, a \mid \exists\, b \text{ such that } (a, b) \in T \,\} \quad \text{and} \quad D = \{\, d \mid \exists\, c \text{ such that } (c, d) \in T \,\}.$$

Let $i = \min\{\, k \mid A \cap Q_k \neq \emptyset \,\}$ and $j = \min\{\, \ell \mid D \cap Q_\ell \neq \emptyset \,\}$. There are four cases to consider.

Case 1: $i < j$. We know that there exists $(a, b) \in T$ with $a \in Q_i$. For any such (a, b), as $i < j$, we have that $b <_k \top_{i+1}$. Since $\Delta(\mathbf{K}_n^2) \subseteq \mathbf{S}$, we have $(\top_{i+1}, \top_{i+1}) \in S$ and thus we have that $(a, b) \oplus (\top_{i+1}, \top_{i+1}) = (a, \top_{i+1}) \in S$. If $a \neq \top_{i+1}$ we have

$(a, \top_{i+1}) \in T$, and so there exists $(c, d) \in T$ with $d \in Q_i$, contradicting the fact that $i < j$. Thus $a = \top_{i+1}$ and so either $b = \mathbf{f}_{i+1}$, $b = \mathbf{t}_{i+1}$ or $b \leqslant_k \top_{i+2}$. If $b \leqslant_k \top_{i+2}$, then $(a, b) \oplus (\top_{i+2}, \top_{i+2}) = (\top_{i+1}, \top_{i+2}) \in S$. Thus for any possible value of b we have that $Q_{i+1}^2 \cap >_k \neq \emptyset$ and so by completing the diamond (Proposition 2.3.3) we get $Q_{i+1}^2 \cap >_k \subseteq S$. This allows us to apply Corollary 2.3.9 and Lemma 2.3.10 and hence conclude that $\breve{S}_{n,i+1} \subseteq S$.

Now, because of the definitions of i and j, we cannot have any elements $(x, y) \in S$ (with $x \neq y$) where either x or y is an element of $K_n \backslash ({\downarrow_k} \top_i)$. Furthermore, if any element $(x, y) \in S$ such that $(x, y) \in Q_{i+1}^2 \backslash \geqslant_k$, then by Lemma 2.3.4 we will have $Q_{i+1}^2 \subseteq S$. This would mean that $(x, \top_{i+1}) \in S$ and hence $j = i$, a contradiction. Thus we have $\breve{S}_{n,i+1} = S$.

Case 2: $j < i$. We can pursue a similar argument to conclude that $i = j + 1$ and thus $S = S_{n,j+1}$.

Case 3: $i = j = 0$. Within this case we will look at four possible subcases. These subcases are determined by whether or not $\top_0$ is or isn't an element of A and/or D.
Case 3(a): $\top_0 \in A$ and $\top_0 \notin D$. Since $\top_0 \in A$ we have by Lemma 2.3.15 that $(\top_0, \top_1) \in S$ and hence by completing the diamond, $Q_0^2 \cap >_k \subseteq S$. Since $\top_0 \notin D$ we have that

$$\{(\top_1, \top_0), (\mathbf{f}_0, \top_0), (\mathbf{t}_0, \top_0), (\top_1, \mathbf{f}_0), (\top_1, \mathbf{t}_0)\} \cap S = \emptyset,$$

and so $Q_0^2 \cap S = Q_0^2 \cap \geqslant_k$. Thus $S = \breve{S}_{n,0}$.

Case 3(b): $\top_0 \notin A$ and $\top_0 \in D$. By Lemma 2.3.15 we have that $(\top_1, \top_0) \in S$ and thus $Q_0^2 \cap <_k \subseteq S$. Similar to the previous case, if $\top_0 \notin A$, we must have $Q_0^2 \cap S = Q_0^2 \cap \leqslant_k$ and so $S = S_{n,0}$.

Case 3(c): $\top_0 \in A \cap D$. This time Lemma 2.3.15 gives us that $(\top_0, \top_1) \in S$ and $(\top_1, \top_0) \in S$. By completing the diamond we get condition (3) of Lemma 2.3.4. We then apply Proposition 2.3.8 to see that $S = ({\downarrow} \top_0)^2 = K_n^2$, a contradiction to our original assumption.

Case 3(d): $\top_0 \notin A \cup D$. If we have $\{\mathbf{f}_0, \mathbf{t}_0\} \cap (A \cup D) \neq \emptyset$, then by completing the diamond we would always get either $\top_0 \in A$ or $\top_0 \in D$. Since $i = j = 0$, we must have $Q_0 \cap (A \cup D) = \{\top_1\}$. Now by Lemma 2.3.15 we get $(\top_1, \top_2) \in S$ and $(\top_2, \top_1) \in S$. This gives $Q_1^2 \subseteq S$ and hence by collapsing the diamonds we get $({\downarrow} \top_1)^2 \subseteq S$. Since

$Q_0 \cap (A \cup D) = \{\top_1\}$ we cannot have any other $(a, b) \in S$ with $a \neq b$, and thus $S = S_{n,0} \cap \breve{S}_{n,0}$.

Case 4: $i = j > 0$. We first note that $\top_i \notin A$ as then $A \cap Q_{i-1} \neq \emptyset$, contradicting the definition of i. Similarly, $\top_i = \top_j \notin D$, as this would contradict the definition of j. Thus $A \cap \{\mathbf{t}_i, \mathbf{f}_i, \top_{i+1}\} \neq \emptyset$ and $D \cap \{\mathbf{t}_i, \mathbf{f}_i, \top_{i+1}\} \neq \emptyset$.

Suppose that $\mathbf{t}_i \in A$. Since $j = i$, there must exist $b \in \{\mathbf{f}_i\} \cup {\downarrow_k} \top_{i+1}$ such that $(a, b) \in S$. If $b = \mathbf{f}_i$, then by Lemma 2.3.4 we have that $Q_i^2 \subseteq S$, again contradicting the definition of i and j. If $b \leqslant_k \top_{i+1}$ then we have

$$(a, b) \oplus (\top_{i+1}, \top_{i+1}) = (a, \top_{i+1}) = (\mathbf{t}_i, \top_{i+1}) \in S.$$

Applying Proposition 2.3.3 would give us $Q_{i+1}^2 \cap {>_k} \subseteq S$ and hence $\top_i \in A$, a contradiction. Thus $a \neq \mathbf{t}_i$. A similar proof shows that $a \neq \mathbf{f}_i$ and thus $Q_i \cap A = \{\top_{i+1}\}$.

Using what we have just shown, if $\mathbf{t}_i \in D$ there must exist $c \leqslant_k \top_{i+1}$ such that $(c, d) \in T$. But then $(c, d) \oplus (\top_{i+1}, \top_{i+1}) = (\top_{i+1}, \mathbf{t}_i) \in S$. Proposition 2.3.2 would give $\top_i \in D$, a contradiction. A similar argument will show that $\mathbf{f}_i \notin D$.

We have shown that if $i = j$, then $Q_i \cap A = \{\top_{i+1}\} = Q_i \cap D$. Now there must exist $b < \top_{i+1}$ and $c < \top_{i+1}$ such that $(\top_{i+1}, b) \in S$ and $(c, \top_{i+1}) \in S$. By Lemma 2.3.15 we get $(\top_{i+1}, \top_{i+2}) \in S$ and $(\top_{i+2}, \top_{i+1}) \in S$. We then apply Propositions 2.3.3 and 2.3.2 to complete the diamond in both directions. That is, S satisfies (3) from Lemma 2.3.4. Finally, we apply Proposition 2.3.8 to get that ${\downarrow_k} \top_{i+1} \subseteq S$. Since $A \cap \{\top_i, \mathbf{t}_i, \mathbf{f}_i\} = \emptyset$ and $D \cap \{\top_i, \mathbf{t}_i, \mathbf{f}_i\} = \emptyset$, we have that $S = S_{n,i} \cap \breve{S}_{n,i}$. ☐

We remind ourselves that, for any algebra $\mathbf{A}$, the collection of subalgebras of $\mathbf{A}$ is a complete lattice. Meets are defined by intersections, and, for subalgebras $\mathbf{S}_1$ and $\mathbf{S}_2$, the join $\mathbf{S}_1 \vee \mathbf{S}_2$ is defined to be the subalgebra generated by the set $S_1 \cup S_2$. That is,

$$\mathbf{S}_1 \vee \mathbf{S}_2 := \bigcap \{\, \mathbf{S} \in \mathbb{S}(\mathbf{A}) \mid S_1 \cup S_2 \subseteq S \,\}.$$

Using this definition of the lattice structure of the lattice $\mathbb{S}(\mathbf{K}_n^2)$, we are able to prove the following corollary. It is not clear whether the result is of any particular significance, or whether it is simply a mathematical curiosity.

Corollary 2.3.17. *Let $n \in \omega$ and consider the bilattice $\mathbf{K}_n$. The lattice of subalgebras of $\mathbf{K}_n^2$ is order-isomorphic to the k-lattice reduct of $\mathbf{K}_n$.*

Proof. We use the result of Proposition 2.3.16 and define an order isomorphism $\varphi\colon \mathbb{S}(\mathbf{K}_n^2) \to \mathbf{K}_n^\flat$ for $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$ as follows

$$\varphi(\mathbf{S}) = \begin{cases} \bot & \text{if } \mathbf{S} = \Delta(\mathbf{K}_n^2) \\ \mathbf{f}_m & \text{if } \mathbf{S} = \mathbf{S}_{nm} \\ \mathbf{t}_m & \text{if } \mathbf{S} = \breve{\mathbf{S}}_{nm} \\ \top_{m+1} & \text{if } \mathbf{S} = \mathbf{S}_{nm} \cap \breve{\mathbf{S}}_{nm} \\ \top_0 & \text{if } \mathbf{S} = \mathbf{K}_n^2. \end{cases}$$

□

2.4 Dualities and optimal dualities for $\mathbb{ISP}(\mathbf{K}_n)$

We know from the NU Duality Theorem that the set of subalgebras of $\mathbf{K}_n^2$ will always yield a duality on $\mathbb{ISP}(\mathbf{K}_n)$. Proposition 2.3.16 gave us a complete description of the subalgebras of $\mathbf{K}_n^2$. We can now combine this description of $\mathbb{S}(\mathbf{K}_n^2)$ with two basic entailment constructs to provide a more efficient duality than that provided by $\mathbb{S}(\mathbf{K}_n^2)$.

Proposition 2.4.1. *Let $n \in \omega$ and consider the quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{K}_n)$. Let $\mathcal{R}_n = \{\, S_{nm} \mid 0 \leqslant m \leqslant n \,\}$. Then the dualising structure*

$$\underset{\sim}{\mathbf{K_n}} = \langle K_n; \mathcal{R}_n, \mathfrak{T} \rangle$$

yields a duality on $\boldsymbol{\mathcal{A}}$.

Proof. The NU Duality Theorem tells us that $\mathbb{S}(\mathbf{K}_n^2)$ yields a duality on $\mathbb{ISP}(\mathbf{K}_n)$. We know that $\Delta(\mathbf{K}_n^2)$ and $\mathbf{K}_n^2$ are always entailed by any set of relations, and further we see that the set of relations $\{\, \breve{S}_{nm} \mid 0 \leqslant m \leqslant n \,\}$ is entailed by $\mathcal{R}$ via the construct of converses, and that the set of relations $\{\, S_{nm} \cap \breve{S}_{nm} \mid 0 \leqslant m \leqslant n \,\}$ is entailed via converses and intersections. □

The number of relations in $\mathbb{S}(\mathbf{K}_n^2)$ is $3n + 4$. The dualities given by the above proposition require only $n+1$ relations, a substantial improvement. As we will see in Theorem 2.4.7, these dualising structures are in fact optimal. In order to show that the dualities are optimal, we first set out to characterise the dual spaces of the $\mathbf{S}_{nm}$, viewed as algebras. To do this, we will first prove some seemingly unrelated results.

Proposition 2.4.2. *Let $n \in \omega$ and consider the bilattice $\mathbf{K}_n$. The congruence lattice $\mathrm{Con}(\mathbf{K}_n)$ is order-isomorphic to the $(n+2)$-element chain.*

Proof. In order for $\theta \subseteq K_n^2$ to be a congruence, it must be both an equivalence relation and a subalgebra of $\mathbf{K}_n^2$. It is clear that for $m \in \{0,\dots,n\}$, the set S_{nm} is not an equivalence relation as it is not symmetric. On the other hand, it is easy to check that each of the relations $S_{nm} \cap \breve{S_{nm}}$ is an equivalence relation. Hence the congruence lattice of $\mathbf{K}_n$ is simply the sublattice of $\mathbb{S}(\mathbf{K}_n^2)$ with the underlying set

$$L = \{K_n^2\} \cup \{\, S_{nm} \cap \breve{S_{nm}} \mid 0 \leqslant m \leqslant n \,\}. \qquad \square$$

Corollary 2.4.3. *For each $n \in \omega$, the bilattice $\mathbf{K}_n$ is subdirectly irreducible.*

Proof. Using Theorem A.2.5, we simply observe that $\mathbf{K}_n$ has a minimum congruence $\theta = S_{n,n-1} \cap \breve{S_{n,n-1}}$ in $\mathrm{Con}(\mathbf{K}_n)\backslash(\Delta(\mathbf{K}_n))$. $\square$

The projection maps π_1 and π_2 are defined by

$$\pi_1 \colon \mathbf{K}_n^2 \to \mathbf{K}_n, \quad (a,b) \mapsto a \qquad \text{and} \qquad \pi_2 \colon \mathbf{K}_n^2 \to \mathbf{K}_n, \quad (a,b) \mapsto b.$$

For any $\mathbf{S} \in \mathbb{S}(\mathbf{K}_n^2)$, the restrictions of π_1 and π_2 to $\mathbf{S}$ are homomorphisms from $\mathbf{S}$ to $\mathbf{K}_n$. These restrictions are denoted by ρ_1 and ρ_2. That is, for $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{K}_n)$, we have $\{\rho_1,\rho_2\} \subseteq \boldsymbol{\mathcal{A}}(\mathbf{S},\mathbf{K}_n)$.

The condition below and the lemma which follows it are due to Davey and Priestley [27]. In combination, they provide a very simple way of describing the elements of the dual space of subalgebras of $\mathbf{M}$ under certain conditions. Let $\mathbf{M}$ be a finite algebra and consider the following condition for $\mathbf{S} \in \mathbb{S}(\mathbf{M}^n)$:

(H) every homomorphism from $\mathbf{S}$ to $\mathbf{M}$ is of the form $e \circ \rho_i$ for some $i \in \{1,\dots,n\}$ where e is a partial endomorphism of $\mathbf{M}$ and the image of ρ_i is contained in the domain of e.

Lemma 2.4.4 ([27, Lemma 6.6])**.** *Assume that* $\mathbf{M}$ *generates a congruence-distributive variety and that every subalgebra of* $\mathbf{M}$ *is subdirectly irreducible. Then condition* (H) *holds for every subalgebra of* $\mathbf{M}^n$*.*

We use this result to characterise the dual spaces of the algebras $\mathbf{S}_{nm}$.

Lemma 2.4.5. *Let $n \in \omega$ and consider the quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{K}_n)$. Further, let ρ_1, ρ_2 be the projection maps defined above and let $m \in \omega$ such that $0 \leqslant m \leqslant n$. Then $\boldsymbol{\mathcal{A}}(\mathbf{S}_{nm},\mathbf{K}_n) = \{\rho_1,\rho_2\}$.*

Proof. The only subalgebra of $\mathbf{K}_n$ is $\mathbf{K}_n$ itself, and Corollary 2.4.3 tells us that $\mathbf{K}_n$ is subdirectly irreducible. Since $\mathbf{K}_n$ is a lattice, it generates a congruence distributive variety. Thus condition (H) holds for all $\mathbf{S}_{nm}$. To conclude, we observe that the only partial endomorphism on $\mathbf{K}_n$ is the identity map with domain K_n. $\square$

The result above gives us vital information about the dual spaces of each of the subalgebras S_{nm}. The fact that the dual spaces consist of only the two projections ρ_1 and ρ_2 will be used to construct the failset maps in our main theorem.

In our original proof of Lemma 2.4.5, we supposed that there exists $h \in \boldsymbol{\mathcal{A}}(\mathbf{S}_{nm}, \mathbf{K}_n)$ such that for some $(a, b) \in S_{nm}$ with $a \neq b$, we had $h(a, b) \notin \{a, b\}$. The proof then proceeded to show, for all possible cases of $(a, b) \in S_{nm}$ with $a \neq b$, that h could not be a homomorphism. Our new approach is more elegant, but we will have to revert to this bare-hands approach in Section 3.5.

We denote by $(S_{nm})_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$ the relation S_{nm} on $\mathrm{D}(\mathbf{S}_{\mathrm{nm}})$ and remind ourselves of how the relational structure on the dual space of homomorphisms is defined (see page 4). For any binary relation R we say that $(\rho_1, \rho_2) \in R_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$ if and only if for all $(a, b) \in S_{nm}$ we have $(\rho_1(a, b), \rho_2(a, b)) = (a, b) \in R$.

Proposition 2.4.6. *Let $n \in \omega$ and consider the quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{K}_n)$. For $m \in \{0, \dots, n\}$, the relation S_{nm} is absolutely unavoidable in a natural duality for $\boldsymbol{\mathcal{A}}$.*

Proof. We note from Lemma 2.4.5 that $\boldsymbol{\mathcal{A}}(\mathbf{S}_{nm}, \mathbf{K}_n) = \{\rho_1, \rho_2\}$. We now wish to construct a failset map $\gamma_m \colon \mathrm{D}(\mathbf{S}_{nm}) \to \underset{\sim}{\mathbf{K}_{\mathbf{n}}}$.

In order to prove that S_{nm} is an absolutely unavoidable relation, we must first show that γ_m fails to preserve S_{nm} and then that γ_m does preserve every other relation on $\mathrm{D}(\mathbf{S}_{\mathrm{nm}})$.

We define γ_m as follows:

$$\gamma_m(h) = \begin{cases} \top_m & \text{if } h = \rho_1 \\ \mathbf{t}_m & \text{if } h = \rho_2 \end{cases}$$

It is trivially true for S_{nm} that $(\rho_1, \rho_2) \in (S_{nm})_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$. From the way that we have defined γ_m it is also clear that $(\gamma_m(\rho_1), \gamma_m(\rho_2)) = (\top_m, \mathbf{t}_i) \notin S_{nm}$ and thus we see that γ_m fails to preserve the relation S_{nm}.

We note that for all S_{ni} where $i \in \omega$ such that $0 \leqslant i \leqslant n$, we have $(\rho_1, \rho_1) \in (S_{ni})_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$ and $(\rho_2, \rho_2) \in (S_{ni})_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$. These are trivially preserved by γ_m as $(\top_m, \top_m) \in S_{ni}$ and $(\mathbf{t}_i, \mathbf{t}_i) \in S_{ni}$.

Now consider S_{nj} where $j \in \omega$ such that $0 \leqslant j \leqslant n$ but $j \neq m$. If $m < j$ then $S_{nj} \subsetneq S_{nm}$ and so clearly $(\rho_1, \rho_2) \notin (S_{nj})_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$. Also, $(\rho_2, \rho_1) \notin S_{nj}$, as $(\top_{m+1}, \mathbf{f}_m) \in S_{nm}$ but $(\mathbf{f}_m, \top_{m+1}) \notin S_{nj}$. Thus γ_m preserves the relation S_{nj} on $\mathrm{D}(\mathbf{S}_{\mathrm{nm}})$.

Finally, we consider S_{nj} where $j < m$. In this case we have $Q^2_m \subseteq S_{nj}$ and $S_{nm} \subsetneq S_{nj}$. This gives us that $(\rho_1, \rho_2) \in (S_{nj})_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$. Since $Q^2_m \subseteq S_{nj}$ we have that $\breve{S}_{nm} \subseteq S_{nj}$ and hence $(\rho_2, \rho_1) \in (S_{nj})_{\mathrm{D}(\mathbf{S}_{\mathrm{nm}})}$. Now, $(\gamma_m(\rho_1), \gamma_m(\rho_2)) = (\top_m, \mathbf{t}_m) \in S_{nj}$

and $(\gamma_m(\rho_2), \gamma_m(\rho_1)) = (\mathbf{t}_m, \top_m) \in S_{nj}$ and so γ_m preserves S_{nj}. Thus the failset for $\gamma_m = \{S_{nm}\}$. $\square$

Theorem 2.4.7. *Let $n \in \omega$ and consider the quasivariety $\mathbb{ISP}(\mathbf{K}_n)$ generated by the bilattice $\mathbf{K}_n$. Let*

$$\mathcal{R}_n = \{\, S_{nm} \mid 0 \leqslant m \leqslant n\}.$$

Then the dualising structure $\underset{\sim}{\mathbf{K}}_n = \langle K_n; \mathcal{R}, \mathfrak{T}\rangle$ yields an optimal duality on $\mathbb{ISP}(\mathbf{K}_n)$.

Proof. This is an immediate consequence of Propositions 2.4.1 and 2.4.6. $\square$

We have succeeded in identifying an easily-described set of relations on $\mathbf{K}_n^2$ that will yield an optimal duality on $\mathbb{ISP}(\mathbf{K}_n)$. We now want to know whether this duality is *full*. That is, we want to know whether or not every topological-relational structure is dually represented as an algebra. The usual method for proving fullness is to prove the stronger condition that the duality is *strong*. However, it is noted that there are dualities that are full but not strong, and thus this method cannot always be applied.

Once again we will benefit from the fact that we are dealing with a lattice-based algebra, as this significantly simplifies what is required in order to prove that our dualities are strong. First, we need to introduce the concept of irreducibility [17, Chapter 3].

The congruence lattice of a finite algebra $\mathbf{A}$ is algebraic, and hence every element of $\mathrm{Con}(\mathbf{A})$ is the meet of the completely meet-irreducible elements of $\mathrm{Con}(\mathbf{A})$. For $\mathbf{A}$, $\mathrm{irr}(\mathbf{A})$ is the least number of completely meet-irreducible congruences such that their intersection is $\Delta(\mathbf{A}^2)$. The *irreducibility index* of a finite algebra $\mathbf{M}$ is

$$\mathrm{Irr}(\mathbf{M}) = \max\{\, \mathrm{irr}(\mathbf{A}) \mid \mathbf{A} \in \mathbb{S}(\mathbf{M})\,\}.$$

Although we need to know this theory for the general case, for $\mathbf{K}_n$, we get $\mathrm{Irr}(\mathbf{K}_n) = 1$ as $\mathbf{K}_n$ has no proper subalgebras, and it is subdirectly irreducible. In the theorem below, we have that J is the set of all one-element subalgebras of $\mathbf{M}$, $\mathcal{P}_n$ is the set of all (partial) n-ary operations on $\mathbf{M}$, and $\mathcal{B}_k$ is the set of all k-ary relations on $\mathbf{M}$. Lastly, we note that $G \cup H \cup R$ *strongly entails* $G' \cup H' \cup R'$ if it entails $G' \cup H' \cup R'$ and $G \cup H$ hom-entails $G' \cup H'$.

Theorem 2.4.8 ([17, Theorem 3.3.8])**.** (**NU Strong Duality Theorem**) *Let $k \geqslant 2$ and suppose that $\mathbf{M}$ has a $(k+1)$-ary near-unanimity term. If*

$$\underset{\sim}{\mathbf{M}} = \langle M; J, H, \mathcal{B}_k, \mathfrak{T}\rangle \;\; \textit{where} \;\; H = \bigcup\{\, \mathcal{P}_n \mid 1 \leqslant n \leqslant \mathrm{Irr}(\mathbf{M})\,\},$$

then any structure that strongly entails $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathbb{ISP}(\mathbf{M})$.

The fact that we have strong dualities for the quasivarieties $\mathbb{ISP}(\mathbf{K}_n)$ will now follow easily.

Theorem 2.4.9. *The duality yielded by* $\underset{\sim}{\mathbf{K_n}} = \langle K_n; \mathcal{R}_n, \mathcal{T}\rangle$ *on* $\mathbb{ISP}(\mathbf{K}_n)$ *is strong.*

Proof. We have that $\mathrm{Irr}(\mathbf{K}_n) = 1$ and note that $\mathbf{K}_n$ has no unary partial algebraic operations besides the identity, and it has no one-element subalgebras. Thus our structures $\underset{\sim}{\mathbf{K_n}}$ strongly entail the structures required by Theorem 2.4.8. $\square$

Having shown that the dualising structure given by the subalgebras $\mathbf{S}_{nm}$ (for $0 \leqslant m \leqslant n$) is both optimal and strong, we make a further observation as to how this set of dualising relations encodes the knowledge order $\leqslant_k$ on $\mathbf{K}_n$. We will not be able to do this using the subsets Q_m of K_n which have been so useful thus far. Instead, we consider the sets $W_m = \{\mathbf{f}_m, \mathbf{t}_m, \top_m\}$ for $m \in \{0, \dots, n\}$.

Proposition 2.4.10. *Let* $n \in \omega$ *and consider the bilattice* $\mathbf{K}_n$ *and the set of relations* $\mathcal{R}_n = \{\, S_{nm} \mid 0 \leqslant m \leqslant n \,\}$. *For* $a, b \in K_n$ *we have that* $a <_k b$ *if and only if there exists* $S \in \mathcal{R}_n$ *such that* $(a, b) \in S$ *and* $(b, a) \notin S$.

Proof. Suppose that $a <_k b$. If $a, b \in W_m$ for some $m \in \{0, \dots, n\}$, then by Definition 2.3.11 we have that $(a, b) \in S_{nm}$ and $(b, a) \notin S_{nm}$. If $a \in W_j$ and $b \in W_i$ for $0 \leqslant i < j \leqslant n$, then $(a, b) \in S_{ni}$ and $(b, a) \notin S_{ni}$. The final case that we must consider is if $a = \bot$ and $b \in W_m$ for $0 \leqslant m \leqslant n$. Again we have that $(a, b) \in S_{nm}$ but $(b, a) \notin S_{nm}$.

Now suppose that there exists $S \in \mathcal{R}_n$ such that $(a, b) \in S$ but $(b, a) \notin S$. We take $S = S_{nm}$ for some $m \in \{0, \dots, n\}$. Looking at the definition of S_{nm}, we have that either $(a, b) \in Q_m^2 \cap <_k$, or $(a, b) \in \{\, (c, d) \mid c \leqslant_k \top_{m+1}, d \in Q_m \,\}$. In either case it is clear that $a <_k b$. $\square$

This last result is very intriguing. The work of Cabrer and Priestley [16] has shown that the knowledge order $\leqslant_k$ is the only relation required on the dual structures in order to dually represent all of the algebraic structure of the bilattices. Although this is not exactly possible in our situation, all the dualities that we have proven above have the feature that the knowledge order of the original structure is captured in the dual structures. We make further comments on this topic in Section 3.6.

2.5 The varieties $\mathbb{HSP}(\mathbf{K}_n)$

Up to this point we have only considered the *quasivarieties* generated by the bilattices $\mathbf{K}_n$. To end this chapter, we make some remarks about dualities for $\boldsymbol{\mathcal{V}}(\mathbf{K}_n)$, the *varieties* generated by the bilattices $\mathbf{K}_n$. We recall the definition of the homomorphic images of the $\mathbf{K}_n$ in Section 2.2 and define $\mathfrak{K}_n = \{\, \mathbf{K}_{m,n} \mid 0 \leqslant m \leqslant n \,\}$.

Proposition 2.5.1. *Consider the bilattice* $\mathbf{K}_n$. *Then* $\boldsymbol{\mathcal{V}}(\mathbf{K}_n) = \mathbb{ISP}(\mathfrak{K}_n)$.

Proof. It is clear that $\mathbf{K}_n$ has no proper subalgebras. Proposition 2.2.1 tells us that there are at least $n+1$ homomorphic images of $\mathbf{K}_n$. The kernel of every homomorphism is a congruence, and we know from Proposition 2.4.2 that $\mathrm{Con}(\mathbf{K}_n)$ has $n+1$ elements. Hence $\mathfrak{K}_n = \mathbb{HS}(\mathbf{K}_n)$. Since $\boldsymbol{\mathcal{V}}(\mathbf{K}_n)$ is lattice-based, it is congruence distributive [14, Theorem II-12.3]. Thus we can apply Theorem A.2.6 to see that the set of subdirectly irreducible algebras in $\boldsymbol{\mathcal{V}}(\mathbf{K}_n)$ is exactly $\mathfrak{K}_n$. ☐

The theory of multisorted dualities applies to a quasivariety generated by a finite set of finite algebras. This was first explored by Davey and Priestley [25]; a useful summary can be found in [17, Section 7.1]. From the result above, we see that it would be possible to use these techniques to investigate dualities for the varieties $\boldsymbol{\mathcal{V}}(\mathbf{K}_n)$ $(n \in \omega)$. As a cautionary remark, we note that the question of whether or not such dualities will be *full* will be a difficult problem to solve. The usual method for showing that a duality is full is to show that it is strong. The problem of determining whether or not multisorted dualities are strong has been looked at by Davey and Talukder [31] for varieties of Heyting algebras. They are able to prove strongness, but only for varieties of Heyting algebras satisfying quite specific conditions.

Chapter 3

An alternative family of default bilattices

Here we introduce an alternative family of bilattices for use in prioritised default logic. A criticism that can be levelled at the seven-element default bilattice due to Ginsberg [61] is the fact that the element $\mathbf{d}\top$ is both the k-meet of $\mathbf{t}$ and $\mathbf{f}$, and the k-join of $\mathbf{dt}$ and $\mathbf{df}$. That is, $\mathbf{t} \otimes \mathbf{f} = \mathbf{d}\top = \mathbf{dt} \oplus \mathbf{df}$.

If an agent is told that a certain statement is both true and false, the level of *agreement* or *consensus* is modelled by the bilattice element $\mathbf{t} \otimes \mathbf{f}$. The k-join $\mathbf{dt} \oplus \mathbf{df}$ represents the total knowledge that an agent has if it is told that something is both true by default and false by default. However, it is not necessarily clear that $\mathbf{t} \otimes \mathbf{f}$ should represent the same degree of knowledge and truth as the k-join $\mathbf{dt} \oplus \mathbf{df}$. Thus we find it worthwhile to consider an alternative bilattice to $\mathcal{SEVEN}$, and also to consider a family of bilattices to act as an alternative to the family $\{\, \mathbf{K}_n \mid n \in \omega \,\}$ studied in Chapter 2.

The main difference between our new family of bilattices and the prioritised default bilattices in the style of $\mathcal{SEVEN}$ is that there is no distinction between the level at which the contradictions or agreements take place. That is, for $n, m \in \omega$ with $0 \leqslant m \leqslant n$ we propose that

$$\mathbf{t}_0 \oplus \mathbf{f}_0 \;=\; \top \;=\; \mathbf{t}_m \oplus \mathbf{f}_m \quad \text{and} \quad \mathbf{t}_0 \otimes \mathbf{f}_0 \;=\; \bot \;=\; \mathbf{t}_m \otimes \mathbf{f}_m.$$

In addition for any $i, j \in \{0, \dots, n\}$,

$$\mathbf{t}_i \otimes \mathbf{f}_j = \bot \quad \text{and} \quad \mathbf{t}_i \oplus \mathbf{f}_j = \top.$$

We give an illustration of such a proposed bilattice structure with default truth values $\mathbf{t}_1$ and $\mathbf{f}_1$ in Fig. 3.1.

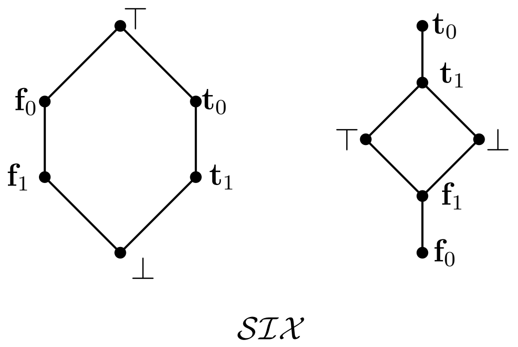


Figure 3.1: A six-element bilattice for modelling default logic drawn with both its knowledge (left) and truth (right) orders.

3.1 The bilattices $\mathbf{J}_n$

In a similar fashion to the way that we defined the bilattices $\mathbf{K}_n$ in Chapter 2, we now define prioritised default bilattices which extend the motivation behind the six-element bilattice of Fig 3.1 to n-levels of default truth values. Let

$$J_n = \{\bot, \mathbf{f}_0, \mathbf{t}_0, \ldots, \mathbf{f}_n, \mathbf{t}_n, \top\}.$$

Definition 3.1.1. Let $n \in \omega$ and consider the set J_n. Let $i, j, m \in \omega$ such that $0 \leqslant i < j \leqslant n$ and $0 \leqslant m \leqslant n$. For any $a \in K_n$ the binary relation $\leqslant_k$ is defined by

(i) $\bot <_k \mathbf{f}_m <_k \top$,

(ii) $\bot <_k \mathbf{t}_m <_k \top$,

(iii) $\mathbf{t}_j <_k \mathbf{t}_i$,

(iv) $\mathbf{f}_j <_k \mathbf{f}_i$,

(v) $a \leqslant_k a$.

Now suppose that $0 \leqslant i \leqslant j \leqslant n$ and $0 \leqslant m \leqslant n$. The relation $\leqslant_t$ is defined by

(i) $\mathbf{f}_i \leqslant_t \mathbf{f}_j$,

(ii) $\mathbf{t}_j \leqslant_t \mathbf{t}_i$,

(iii) $\mathbf{f}_m \leqslant_t \mathbf{t}_i$,

(iv) $\mathbf{f}_m \leqslant_t \bot$ and $\bot \leqslant_t \mathbf{t}_m$,

(v) $\mathbf{f}_m \leqslant_t \top$ and $\top \leqslant_t \mathbf{t}_m$,

(vi) $\bot \leqslant_t \bot$ and $\top \leqslant_t \top$.

It is clear that the definition of the order relations $\leqslant_k$ and $\leqslant_t$ will produce the lattice orders shown in Fig. 3.2.

Proposition 3.1.2. *Let $n \in \omega$ and consider the set J_n. The orders $\leqslant_t$ and $\leqslant_k$ define lattice orders on J_n.*

Let $n, m \in \omega$ with $0 \leqslant m \leqslant n$. We can define a unary involutive operation $\neg$ as follows for $a \in J_n$:

$$\neg a = \begin{cases} a & \text{if } a \in \{\top, \bot\}, \\ \mathbf{t}_m & \text{if } a = \mathbf{f}_m, \\ \mathbf{f}_m & \text{if } a = \mathbf{t}_m. \end{cases}$$

The operation $\neg$ as defined above preserves the $\leqslant_k$ order and reverses the $\leqslant_t$ order on J_n. We further define the set of constants $C_n := \{\mathbf{f}_0, \mathbf{t}_0, \ldots, \mathbf{f}_n, \mathbf{t}_n\}$. Finally, we define the bilattices $\mathbf{J}_n (n \in \omega)$ by

$$\mathbf{J}_n := \langle J_n; \otimes, \oplus, \wedge, \vee, \neg, C_n \rangle.$$

We note that for any $n \geqslant 1$, $\mathbf{J}_n$ is not interlaced. We have $\mathbf{f}_0 \leqslant_k \top$ but $\mathbf{f}_0 \wedge \bot = \mathbf{f}_0 \nleqslant_k \mathbf{f}_n = \top \wedge \bot$.

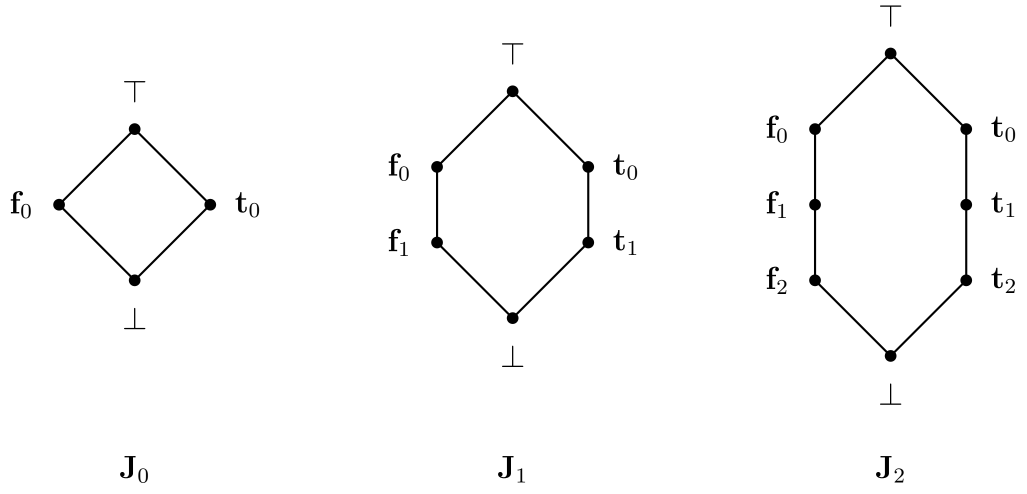


Figure 3.2: The bilattices $\mathbf{J}_0, \mathbf{J}_1$ and $\mathbf{J}_2$, drawn with their knowledge order.

3.2 Optimal dualities for $\mathbf{J}_1$ and $\mathbf{J}_2$

The bilattices $\mathbf{J}_n$ are connected to the $\mathbf{K}_n$ in the sense that they have both been designed to interpret default information. We will now see if there is any similarity in their dualising structures. We refer the reader to Appendix B.3 for a list of subalgebras and globally minimal failsets for $\mathbf{J}_1$. The dual structures for $\mathbb{ISP}(\mathbf{J}_1)$ are quite simple, consisting of just two absolutely unavoidable relations, $\mathbf{S}_2$ and $\mathbf{S}_4$. In Fig. 3.3 we draw the relations $\mathbf{S}_2$ and $\mathbf{S}_4$.

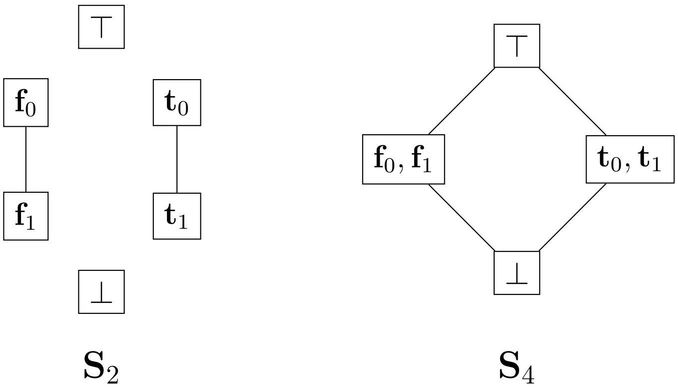


Figure 3.3: The binary relations $\mathbf{S}_2$ and $\mathbf{S}_4$ drawn as quasi-orders.

Theorem 3.2.1. *Consider the bilattice* $\mathbf{J}_1$ *and let* $\mathcal{R} = \{\mathbf{S}_2, \mathbf{S}_4\}$. *Then*

$$\underset{\sim}{\mathbf{J}}_1 := \langle J_1; \mathcal{R}, \mathcal{T} \rangle$$

yields an optimal duality on the quasivariety $\mathbb{ISP}(\mathbf{J}_1)$.

Proof. The set of relations $\mathcal{R}$ is a transversal of the globally minimal failsets of $\mathbf{J}_1$. □

The duality given in Theorem 3.2.1 has features that are similar to the dualities obtained for the bilattices in Chapter 2. We note that the k-lattice order of the bilattice $\mathbf{J}_1$ can be recovered from the subalgebras used in the duality.

Proposition 3.2.2. *Consider the bilattice* $\mathbf{J}_1$ *and* $\mathcal{R} = \{\mathbf{S}_2, \mathbf{S}_4\}$ *and let* $a, b \in J_1$. *Then* $a <_k b$ *if and only if there exists* $\mathbf{S} \in \mathcal{R}$ *such that* $(a, b) \in \mathbf{S}$ *and* $(b, a) \notin \mathbf{S}$.

The dualities obtained for $\mathbb{ISP}(\mathbf{K}_1)$ and for $\mathbb{ISP}(\mathbf{J}_1)$ are similar in that the two subalgebras of $\mathbf{M}^2$ required to yield a duality are, respectively, the maximal subalgebra contained in the binary relation $\leqslant_k$, and the subalgebra generated by $\leqslant_k$.

We refer the reader to Appendix B.4 for the complete computational output of the subalgebras of $\mathbf{J}_2^2$. The bilattices that we have dealt with thus far have produced globally minimal failsets all of which contain only one relation. Before even looking at the failset data for $\mathbf{J}_2$, it is clear that we are dealing with a much more complicated algebra. There are 28 subalgebras (17 up to converses) and thus the globally minimal failsets are unsurprisingly not as simply structured as those from the $\mathbf{K}_n$ and $\mathbf{J}_1$. In order to help reveal why certain relations are so prominent in the globally minimal failsets, we look at the set of all subalgebras. With the help of the Universal Algebra Calculator (UAC) [43], we draw the subalgebra lattice $\mathbb{S}(\mathbf{J}_2^2)$ in Fig. 3.4.

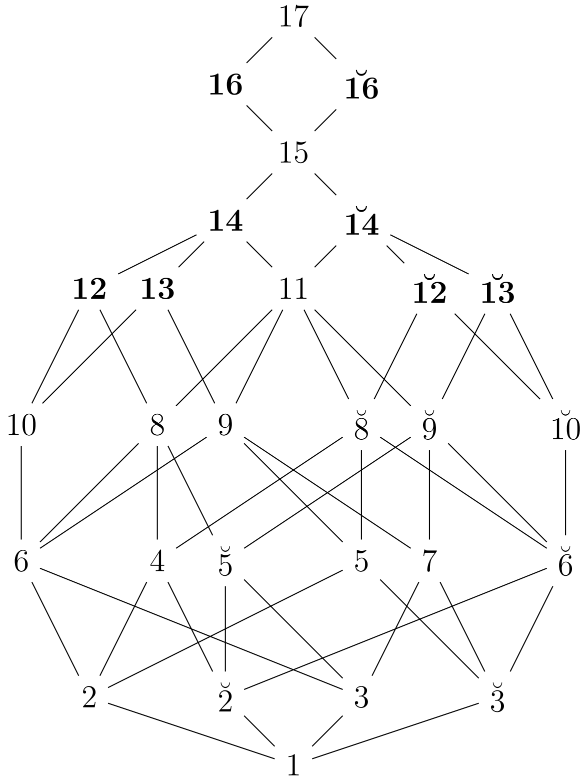


Figure 3.4: The algebraic lattice $\mathbb{S}(\mathbf{J}_2^2)$ with its completely meet-irreducible elements shown in bold.

From Fig. 3.4 we can identify that the subalgebras $\mathbf{S}_{12}$, $\mathbf{S}_{13}$, $\mathbf{S}_{14}$ and $\mathbf{S}_{16}$ (and their converses) are the only completely meet-irreducible elements of $\mathbf{J}_2^2$. From Lemma 1.3.9 we know that each of these subalgebras is a value of $\mathbf{J}_2^2$. That is, for each $\mathbf{S} \in \{\mathbf{S}_{12}, \mathbf{S}_{13}, \mathbf{S}_{14}, \mathbf{S}_{16}\}$ there exists $(a, b) \in J_2^2$ such that $\mathbf{S}$ is maximal in $\mathbb{S}(\mathbf{J}_2^2)$ such that $(a, b) \notin S$. We denote by $\mathrm{Val}(\mathbf{S})$ the set of elements of $\mathbf{J}_2^2$ at which $\mathbf{S}$ is a value. We have the following for the values of $\mathbf{J}_2^2$ (ignoring converses):

$$
\begin{aligned}
\mathrm{Val}(\mathbf{S}_{12}) &= \{(\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_0, \mathbf{t}_1)\},\\
\mathrm{Val}(\mathbf{S}_{13}) &= \{(\mathbf{f}_1, \mathbf{f}_2), (\mathbf{t}_1, \mathbf{t}_2)\},\\
\mathrm{Val}(\mathbf{S}_{14}) &= \{(\mathbf{f}_0, \mathbf{f}_2), (\mathbf{t}_0, \mathbf{t}_2)\},\\
\mathrm{Val}(\mathbf{S}_{16}) &= \big(\{\top\} \times J_2 \backslash \{\top\}\big) \cup \big(J_2 \backslash \{\bot\} \times \{\bot\}\big) \cup \big(\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\} \times \{\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2\}\big)\\
&\quad \big(\{\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2\} \times \{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\}\big).
\end{aligned}
$$

The elements of $\mathbf{J}_2^2$ at which $\mathbf{S}_{12}$, $\mathbf{S}_{13}$, $\mathbf{S}_{14}$ are values are all elements of the order $\geqslant_k$. Equally, the first two subsets of elements at which $\mathbf{S}_{16}$ is a value are elements of $\geqslant_k$. The remaining elements at which $\mathbf{S}_{16}$ is a value show that any relation between one of the $\mathbf{f}_i$ and one of the $\mathbf{t}_j$ in a subalgebra $\mathbf{S}$ results in $\mathbf{S}$ collapsing to $\mathbf{J}_2^2$.

In Fig. 3.5 we draw the subalgebras $\mathbf{S}_{12}, \mathbf{S}_{13}$ and $\mathbf{S}_{16}$.

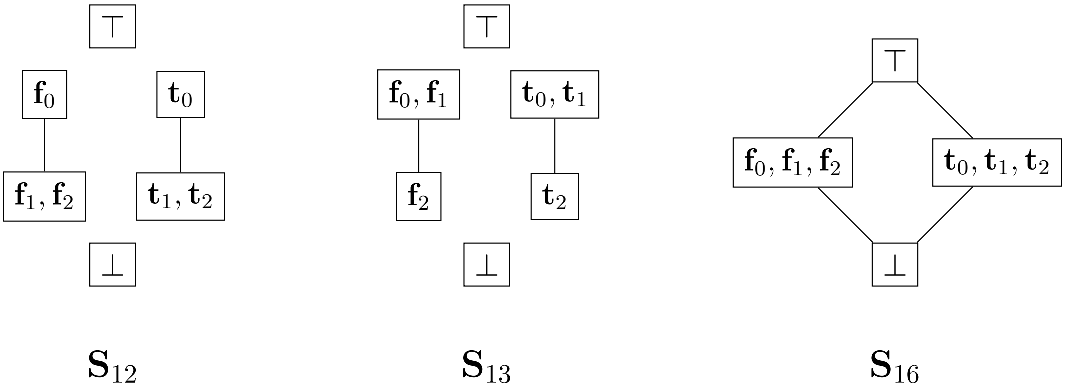


Figure 3.5: The binary relations $\mathbf{S}_{12}, \mathbf{S}_{13}$ and $\mathbf{S}_{16}$ drawn as quasi-orders.

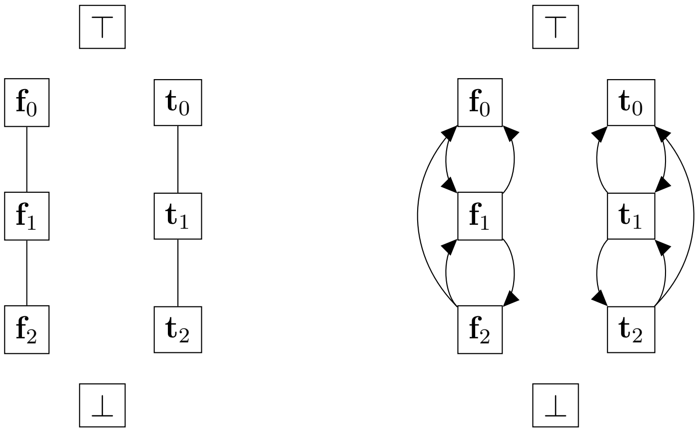
We observe that $\mathbf{S}_{10}$ is the maximal subalgebra contained in $\leqslant_k$. However, $\mathbf{S}_{10} = \mathbf{S}_{12} \cap \mathbf{S}_{13}$ and so is entailed by the set $\mathcal{R}$ below.

Theorem 3.2.3. *Consider the bilattice* $\mathbf{J}_2$ *and let* $\mathcal{R} = \{\mathbf{S}_{12}, \mathbf{S}_{13}, \mathbf{S}_{16}\}$. *Then the structure*

$$\underset{\sim}{\mathbf{J}_2} := \langle J_2; \mathcal{R}, \mathcal{T}\rangle$$

yields a duality on $\mathbb{ISP}(\mathbf{J}_2)$.

The duality presented in Theorem 3.2.3 again gives us the result that $a <_k b$ if and only if there exists $\mathbf{S} \in \mathcal{R}$ such that $(a, b) \in S$ and $(b, a) \notin S$. This is the 'magnification' that we observed on the $\mathbf{K}_n$ and on $\mathbf{J}_1$ whereby the $\leqslant_k$ order can be recovered by using all of the subalgebras in the dualising structure. However, we note that the relations $\mathbf{S}_{12}$ and $\mathbf{S}_{13}$ are not absolutely unavoidable. Furthermore, there



$\mathbf{S}_{10}$ $\mathbf{S}_{14}$

Figure 3.6: The subalgebras $\mathbf{S}_{10}$ and $\mathbf{S}_{14}$.

exists a choice of relations that will give you a duality which (even up to converses) will not allow you to recover the order $\leqslant_k$ by the procedure described above.

Given the complexity of the subalgebra lattice for $\mathbf{J}_2^2$, we are not able to find a strategy for generalising the dualities for $\mathbf{J}_n$. Homomorphic images played a role in Chapter 2 and so we investigate these to see if they might help us to interpret the subalgebra data.

3.3 Homomorphic images

As with $\mathbf{K}_n$, the homomorphic images require careful consideration of the signature of the algebra. At first, it may seem that there is only one possible homomorphic image of $\mathbf{J}_2$. This would be the map that takes $\mathbf{t}_1$ to $\mathbf{t}_1$ and $\mathbf{f}_1$ to $\mathbf{f}_1$. This would seem to be the only natural way for the map to indeed be a homomorphism preserving the necessary constants. However, it is possible, via a relabelling of the constants in the signature, to have two six-element homomorphic images of $\mathbf{J}_2$. The two homomorphisms are shown in Fig. 3.7, noting in each case that $h(\top) = \top$ and $h(\bot) = \bot$.

What is required in order to ensure that the maps in Fig. 3.7 are homomorphisms is a subtle change in the signature. We define

$$\mathbf{J}_{1,2,0} := \langle J_1; \otimes, \oplus, \wedge, \vee, \neg, \mathbf{f}_0, \mathbf{t}_0, \mathbf{f}_0, \mathbf{t}_0, \mathbf{f}_1, \mathbf{t}_1 \rangle$$

and

$$\mathbf{J}_{1,2,1} := \langle J_1; \otimes, \oplus, \wedge, \vee, \neg, \mathbf{f}_0, \mathbf{t}_0, \mathbf{f}_1, \mathbf{t}_1, \mathbf{f}_1, \mathbf{t}_1 \rangle.$$

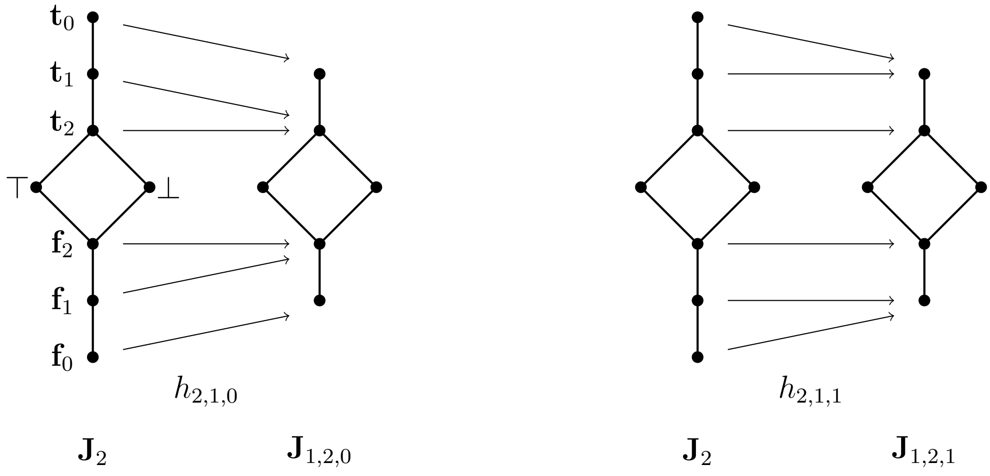


Figure 3.7: The two six-element homomorphic images of $\mathbf{J}_2$.

These homomorphisms help to give us some further insight into the relations that produce optimal dualities for $\mathbb{ISP}(\mathbf{J}_2)$. It is not difficult to calculate that

$$\mathbf{S}_{12} = (h_{2,1,0}, h_{2,1,0})^{-1}(\mathbf{S}_2) \quad \text{and} \quad \mathbf{S}_{13} = (h_{2,1,1}, h_{2,1,1})^{-1}(\mathbf{S}_2).$$

This is similar to the situation that we observed for the bilattices $\mathbf{K}_n$. Here we are taking the inverse homomorphic images of the maximal subalgebra contained in $\leqslant_k$ (on $\mathbf{J}_1$) and are obtaining two relations which, when combined with the inverse homomorphic image of the $\leqslant_k$ order on $\mathbf{J}_0$, produce an optimal duality for $\mathbb{ISP}(\mathbf{J}_2)$. It may be possible to extend these observations to the whole family of quasivarieties $\{\, \mathbb{ISP}(\mathbf{J}_n) \mid n \in \omega \,\}$, but it is clear that the problem will be become substantially more complex as we increase n. For $\mathbf{J}_3$, there will be three homomorphisms into $\mathbf{J}_2$ and three more homomorphisms into $\mathbf{J}_1$, as well as a homomorphism into $\mathbf{J}_0$.

It is quite straightforward to see that in general, the number of homomorphic images of $\mathbf{J}_n$ with underlying set $\mathbf{J}_m$ (for $0 \leqslant m \leqslant n$) is $\binom{n}{m}$. This can be seen by considering just the ‘true’ constants $\{\, \mathbf{t}_i \mid 0 \leqslant i \leqslant m \,\}$ for $\mathbf{J}_m$. Calculating the number of homomorphisms from $\mathbf{J}_n$ to $\mathbf{J}_m$ is equivalent to calculating the number of ways of ordering the $m + 1$ truth constants into $n + 1$ empty slots such that $\mathbf{t}_0$ always goes into slot 0 and $\mathbf{t}_m$ always goes into slot n. The remaining $\mathbf{t}_i$ must all be used at least once and must preserve their indexing order in the slots. The homomorphic images of an algebra correspond to congruences on the algebra. In Fig. 3.8 we illustrate the congruence lattices for the first four algebras in the family $\{\, \mathbf{J}_n \mid n \in \omega \,\}$, again drawn with the assistance of UAC [43].

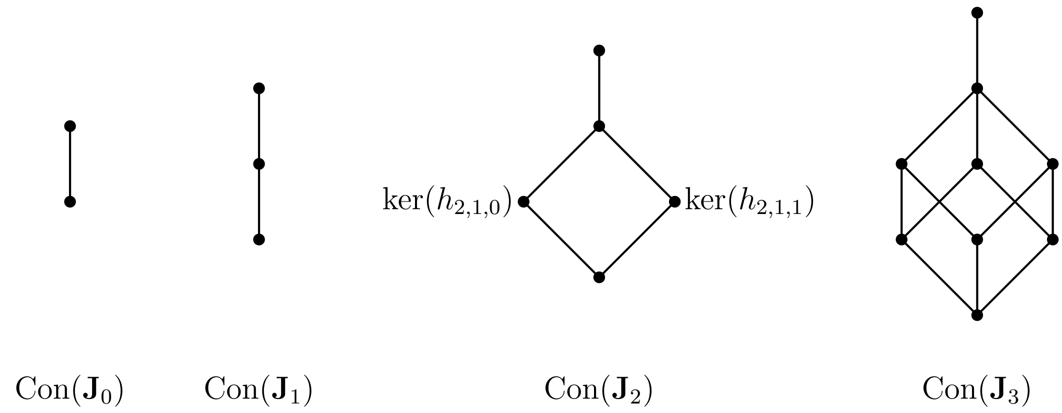


Figure 3.8: The congruence lattices of some algebras from the family $\{\,\mathbf{J}_n \mid n \in \omega\,\}$.

A study of all of the bilattices $\mathbf{J}_n$ using these homomorphic images would no doubt prove interesting, but it would quickly become difficult to keep track of the subalgebras, inverse images of subalgebras and how each of them is connected to the $\leqslant_k$ order. Some preliminary calculations yielded 200 subalgebras of $\mathbf{J}_3^2$ (107 up to converses). We do not undertake such an investigation here, but rather focus on the subalgebras which are the inverse images of the $\leqslant_k$ order on $\mathbf{J}_0$.

The following proposition shows that there exists a homomorphism from each of the $\mathbf{J}_n$ to $\mathbf{J}_0$. This will be used in the next section on subalgebras of $\mathbf{J}_n^2$. To define such a homomorphism, we must again alter the signature of $\mathbf{J}_0$. We define

$$\mathbf{J}_{0,n} := \langle J_0; \otimes, \oplus, \wedge, \vee, \neg, \mathbf{f}_{0(0)}, \mathbf{t}_{0(0)}, \mathbf{f}_{0(1)}, \mathbf{t}_{0(1)}, \dots, \mathbf{f}_{0(n)}, \mathbf{t}_{0(n)} \rangle.$$

In the proof below, we assume that an element a is an element of J_n, but will use the notation $a_{(0)}$ if we are referring to a as an element of J_0.

Proposition 3.3.1. *Let $n \in \omega$ and consider the bilattice $\mathbf{J}_n$, and the bilattice $\mathbf{J}_{0,n}$ defined above. Then the map $h_n \colon \mathbf{J}_n \to \mathbf{J}_{0,n}$ defined by*

$$h_n(a) = \begin{cases} a_{(0)} & \text{if } a \in \{\bot, \top\} \\ \mathbf{f}_{0(0)} & \text{if } a \in \{\mathbf{f}_0, \dots, \mathbf{f}_n\} \\ \mathbf{t}_{0(0)} & \text{if } a \in \{\mathbf{t}_0, \dots, \mathbf{t}_n\} \end{cases}$$

is a homomorphism.

Proof. The fact that h_n preserves all of the constants and the negation is clear from the definition. We must now show that for any $a, b \in J_n$, and for any $* \in \{\otimes, \oplus, \wedge, \vee\,\}$, we have $h_n(a * b) = h_n(a) * h_n(b)$. Since all of the lattice operations are idempotent, we need only consider the cases where $a \neq b$. First, we deal with $\otimes$ and $\oplus$. Suppose that either $a = \top$ or $b = \top$. Without loss of generality (as all the lattice operations are commutative), we assume that $a = \top$. Now we see that

$$h_n(a \otimes b) = h_n(b) = h_n(b) \otimes \top_{(0)} = h_n(a) \otimes h_n(b)$$

and

$$h_n(a \oplus b) = h_n(a) = \top_{(0)} = \top_{(0)} \oplus h_n(b) = h_n(a) \oplus h_n(b).$$

The proof is similar if $\bot \in \{a, b\}$. Now we suppose that $\{a, b\} \cap \{\top, \bot\} = \emptyset$. If $\{a, b\} \subseteq \{\mathbf{t}_0, \ldots, \mathbf{t}_n\}$ or $\{a, b\} \subseteq \{\mathbf{f}_0, \ldots, \mathbf{f}_n\}$ then since $h_n(a) = h_n(b)$ we easily get $h_n(a \otimes b) = h_n(a) \otimes h_n(b)$ and $h_n(a \oplus b) = h_n(a) \oplus h_n(b)$. Finally, suppose that $a \in \{\mathbf{t}_0, \ldots, \mathbf{t}_n\}$ and $b \in \{\mathbf{f}_0, \ldots, \mathbf{f}_n\}$. Then

$$h_n(a \otimes b) = h_n(\bot) = \bot_{(0)} = \mathbf{t}_{0(0)} \otimes \mathbf{f}_{0(0)} = h_n(a) \otimes h_n(b)$$

and

$$h_n(a \oplus b) = h_n(\top) = \top_{(0)} = \mathbf{t}_{0(0)} \oplus \mathbf{f}_{0(0)} = h_n(a) \oplus h_n(b).$$

We have shown that h_n preserves the operations $\otimes$ and $\oplus$, and now consider $\wedge$ and $\vee$. If $a \in \{\mathbf{t}_0, \ldots, \mathbf{t}_n\}$ and $b \in \{\mathbf{f}_0, \ldots, \mathbf{f}_n\}$ then

$$h_n(a \wedge b) = h_n(b) = \mathbf{f}_{0(0)} = \mathbf{f}_{0(0)} \wedge \mathbf{t}_{0(0)} = h_n(a) \wedge h_n(b)$$

and

$$h_n(a \vee b) = h_n(a) = \mathbf{t}_{0(0)} = \mathbf{t}_{0(0)} \vee \mathbf{f}_{0(0)} = h(a) \vee h(b).$$

If $\{a, b\} \subseteq \{\mathbf{f}_0, \ldots, \mathbf{f}_n\}$ or $\{a, b\} \subseteq \{\mathbf{t}_0, \ldots, \mathbf{t}_n\}$ then it is clear that both $h_n(a \vee b) = h_n(a) \vee h_n(b)$ and $h_n(a \wedge b) = h_n(a) \wedge h_n(b)$. If $a = \top$ and $b = \bot$, then $h_n(a \wedge b) = h_n(\mathbf{f}_n) = \mathbf{f}_{0(0)} = \bot_{(0)} \wedge \top_{(0)} = h_n(a) \wedge h_n(b)$, and similarly for $h_n(a \vee b)$. The last general case to consider is if $a \in \{\bot, \top\}$ and $b \in \{\mathbf{f}_0, \mathbf{t}_0, \ldots, \mathbf{f}_n, \mathbf{t}_n\}$. One example from these cases is if $a = \top$ and $b = \mathbf{t}_i$ for some i such that $0 \leqslant i \leqslant n$. Then

$$h_n(a \wedge b) = h_n(\top) = \top_{(0)} = \top_{(0)} \wedge \mathbf{t}_{0(0)} = h_n(a) \wedge h_n(b).$$

The remaining subcases will follow similarly by using (iv) and (v) from the definition of the $\leqslant_t$ order on $\mathbf{J}_n$. $\square$

3.4 Subalgebras of $\mathbf{J}_n^2$

We have already seen an important type of subalgebra emerge from the computations on $\mathbf{J}_1$ and $\mathbf{J}_2$. In the case of $\mathbf{J}_1$, the subalgebra $\mathbf{S}_4 \in \mathbb{S}(\mathbf{J}_1^2)$ is the subalgebra generated by $\leqslant_k$ on $\mathbf{J}_1$. Similarly, $\mathbf{S}_{16} \in \mathbb{S}(\mathbf{J}_2^2)$ is the subalgebra generated by $\leqslant_k$ on $\mathbf{J}_2$. The basic form of $\mathbf{S}_4 \in \mathbb{S}(\mathbf{J}_1^2)$ and $\mathbf{S}_{16} \in \mathbb{S}(\mathbf{J}_2^2)$ is that of the order $\leqslant_k$, but with the elements $\{\mathbf{t}_0, \dots, \mathbf{t}_n\}$ and $\{\mathbf{f}_0, \dots, \mathbf{f}_n\}$ all related to one another. We will proceed to show that the subalgebra generated by $\leqslant_k$ on $\mathbf{J}_n$ is always of this form.

We saw in the previous chapter how the presence of certain elements in a subalgebra of $\mathbf{K}_n^2$ forced other elements into the subalgebra. We referred to this as *collapsing*. We now show that a similar collapsing will happen in a subalgebra of $\mathbf{J}_n^2$ whenever either $\top$ or $\bot$ is related to anything other than itself.

Lemma 3.4.1. *Let $n \in \omega$ and consider the bilattice $\mathbf{J}_n$. Then for $\mathbf{S} \in \mathbb{S}(\mathbf{J}_n^2)$, the following are equivalent:*

(i) $(a, \top) \in S$ *for some* $a \in J_n$ *with* $a \neq \top$;

(ii) $(\bot, a) \in S$ *for some* $a \in J_n$ *with* $a \neq \bot$;

(iii) $(b, \top) \in S$ *for all* $b \in J_n$;

(iv) $(\bot, b) \in S$ *for all* $b \in J_n$.

Proof. The implications (iii) ⇒ (i) and (iv) ⇒ (ii) are of course trivial. Now suppose that (i) holds. If $a = \bot$, then we simply use the fact that $(b, b) \in S$ and see that $(\bot, \top) \oplus (b, b) = (b, \top) \in S$. If $a = \mathbf{f}_i$ or $a = \mathbf{t}_i$ for some $i \in \omega$ such that $0 \leqslant i \leqslant n$, we see that $(a, \top) \otimes \neg(a, \top) = (\bot, \top) \in S$. Then we can apply the first case to get $(b, \top) \in S$ for any $b \in J_n$. The dual method can be applied to show that (ii) ⇒ (iv).

If (i) holds, and if $a = \bot$, then we can conclude (ii). If (i) holds and $a \neq \bot$, then we see that $(a, \top) \otimes \neg(a, a) = (\bot, \neg a) \in S$ and hence (i) ⇒ (ii). Finally, suppose that (ii) holds, and that $a \neq \top$. We have $(\bot, a) \oplus \neg(a, a) = (\neg a, \top) \in S$. If $a = \top$, then (i) follows trivially and we have (ii) ⇒ (i). □

We now show that any of the conditions listed above results in the collapsing of the sets $\{\mathbf{t}_0, \dots, \mathbf{t}_n\}$ and $\{\mathbf{f}_0, \dots, \mathbf{f}_n\}$.

Lemma 3.4.2. *Let $\mathbf{S} \in \mathbb{S}(\mathbf{J}^2)$ and suppose that S satisfies one of the equivalent conditions of Lemma 3.4.1. Then $\{\mathbf{t}_0, \dots, \mathbf{t}_n\}^2 \subseteq S$ and $\{\mathbf{f}_0, \dots, \mathbf{f}_n\}^2 \subseteq S$.*

Proof. Let $i, j \in \{0, 1, \dots, n\}$. Taking $(\bot, \mathbf{t}_i) \in S$ and $(\mathbf{t}_j, \top) \in S$ we see that $(\bot, \mathbf{t}_i) \vee (\mathbf{t}_j, \top) = (\mathbf{t}_j, \mathbf{t}_i) \in S$. Similarly, $(\bot, \mathbf{f}_i) \wedge (\mathbf{f}_j, \top) = (\mathbf{f}_i, \mathbf{f}_j) \in S$. □

The converse versions of these results can easily be proven using similar methods. We state them here for later reference.

Lemma 3.4.3. *Let $n \in \omega$ and consider the bilattice $\mathbf{J}_n$. Then for $\mathbf{S} \in \mathbb{S}(\mathbf{J}_n^2)$, the following are equivalent:*

(i) *$(\top, a) \in S$ for some $a \in J_n$ with $a \neq \top$;*

(ii) *$(a, \bot) \in S$ for some $a \in J_n$ with $a \neq \bot$;*

(iii) *$(\top, b) \in S$ for all $b \in J_n$;*

(iv) *$(b, \bot) \in S$ for all $b \in J_n$.*

Lemma 3.4.4. *Let $\mathbf{S} \in \mathbb{S}(\mathbf{J}^2)$ and suppose that S satisfies one of the equivalent conditions of Lemma 3.4.3. Then $\{\mathbf{t}_0, \dots, \mathbf{t}_n\}^2 \subseteq S$ and $\{\mathbf{f}_0, \dots, \mathbf{f}_n\}^2 \subseteq S$.*

We want an easy description of the subalgebra of $\mathbf{J}_n^2$ generated by the relation $\leqslant_k$ on $\mathbf{J}_n$. Thus we define the set S_{nn} by

$$S_{nn} := \big(\{\bot\} \times J_n\big) \cup \big(J_n \times \{\top\}\big) \cup \{\mathbf{t}_0, \dots, \mathbf{t}_n\}^2 \cup \{\mathbf{f}_0, \dots, \mathbf{f}_n\}^2.$$

From Lemma 3.4.2 we can see that the set S_{nn} will be contained in the subalgebra generated by $\leqslant_k$ on $\mathbf{J}_n$. We now show that S_{nn}, equipped with all of the necessary operations and constants inherited from $\mathbf{J}_n$, is a subalgebra of $\mathbf{J}_n^2$.

Proposition 3.4.5. *The structure $\mathbf{S}_{nn}$ is a subalgebra of $\mathbf{J}_n^2$. Furthermore, it is the subalgebra generated by $\leqslant_k$ on $\mathbf{J}_n$.*

Proof. We will use the homomorphism defined in Proposition 3.3.1 to show that S_{nn} is indeed a subuniverse of $\mathbf{J}_n^2$. To this end we simplify the list of elements of S_{nn} by defining

$$A = \{\bot\} \times J_n, \quad B = J_n \times \{\top\}, \quad C = \{\mathbf{t}_0, \dots, \mathbf{t}_n\}^2, \quad D = \{\mathbf{f}_0, \dots, \mathbf{f}_n\}^2.$$

We denote by $\leqslant_{k(0)}$ the subalgebra of $\mathbf{J}_{0,n}^2$. We want to show $(h_n, h_n)^{-1}(\leqslant_{k(0)}) = S_{nn}$. Suppose that $(x, y) \in S_{nn}$. If $(x, y) \in A$, then $h_n(x) = \bot_{(0)} \leqslant_{k(0)} h_n(y)$, and if $(x, y) \in B$, then $h_n(x) \leqslant_{k(0)} \top_{(0)} = h_n(y)$. If $(x, y) \in C$, then $h_n(x) = \mathbf{t}_{0(0)} = h_n(y)$ and if $(x, y) \in D$ then $h_n(x) = \mathbf{f}_{0(0)} = h_n(y)$. So for every possible $(x, y) \in S_{nn}$ we have that $h_n(x) \leqslant_{k(0)} h_n(y)$ and so $S_{nn} \subseteq (h_n, h_n)^{-1}(\leqslant_{k(0)})$.

We now show the reverse containment. Suppose that $(x, y) \in (h_n, h_n)^{-1}(\leqslant_{k(0)})$. This implies that $h_n(x) \leqslant_{k(0)} h_n(y)$. If $h_n(x) = \bot_{(0)}$, then we must have $x = \bot$ and hence $(x, y) \in A$. If $h_n(y) = \top_{(0)}$ then we get $y = \top$ and $(x, y) \in B$. If we have $h_n(x) \neq \bot_{(0)}$ and $h_n(y) \neq \top_{(0)}$ then we must have either $h_n(x) = \mathbf{t}_{0(0)} = h_n(y)$ or $h_n(x) = \mathbf{f}_{0(0)} = h_n(y)$. If the first of these cases occurs, then $x \in \{\mathbf{t}_0, \dots, \mathbf{t}_n\}$ and $y \in \{\mathbf{t}_0, \dots, \mathbf{t}_n\}$ and so $(x, y) \in C$. Similarly if $h_n(x) = \mathbf{f}_{0(0)}$ we get $(x, y) \in D$. Thus $S_{nn} = (h_n, h_n)^{-1}(\leqslant_{k(0)})$ and so $S_{nn} \in \mathbb{S}(\mathbf{J}_n^2)$.

Next, we will show that any subalgebra of $\mathbf{J}_n^2$ that contains $\leqslant_k$ is either $\mathbf{J}_n^2$ itself, or $\mathbf{S}_{nn}$. Let $\mathbf{R} \in \mathbb{S}(\mathbf{J}_n^2)$ such that $S_{nn} \subsetneq R$. Suppose that $(\top, a) \in R$ for some $a \neq \top$. From this we can get that for all $a \in J_n$ we have $(a, \top), (\top, a), (a, \bot), (\bot, a) \in R$. Now let $a, b \in J_n$. Then we see that $(a, \bot) \oplus (\bot, b) = (a, b) \in R$, and hence $R = J_n^2$. Similarly, if $(b, \bot) \in R$ for some $b \neq \bot$, we get $R = J_n^2$. Suppose that $(\top, a) \notin R$ for all $a \neq \top$ and that $(b, \bot) \notin R$ for all $b \neq \bot$. The only other way that we can have $S_{nn} \subsetneq R$ is if there exists some $i, j \in \{0, 1, \dots, n\}$ such that $(\mathbf{f}_i, \mathbf{t}_j) \in R$. This is without loss of generality, since the negation gives us $(\mathbf{t}_i, \mathbf{f}_j) \in R$. Now $(\mathbf{f}_i, \mathbf{t}_j) \oplus (\mathbf{t}_j, \mathbf{t}_j) = (\top, \mathbf{t}_j) \in R$, and so again we get $R = J_n^2$. □

We will need converse a version of this result in our proof of Proposition 3.5.3.

Proposition 3.4.6. *The structure* $\breve{\mathbf{S}}_{nn}$ *is a subalgebra of* $\mathbf{J}_n^2$. *Furthermore, it is the subalgebra generated by* $\geqslant_k$ *on* $\mathbf{J}_n$.

3.5 Absolutely unavoidable relations within $\mathbb{S}(\mathbf{J}_n^2)$

As can be seen by the set of globally minimal failsets for $\mathbb{ISP}(\mathbf{J}_2)$, we are not able to match the results of Chapter 2 in providing a simple description of the relations which will always provide an optimal duality for $\mathbb{ISP}(\mathbf{J}_n)$. However, we are able to prove that the relation $\mathbf{S}_{nn}$ is always an absolutely unavoidable relation. As we commented earlier, this subalgebra is the subalgebra generated by the $\leqslant_k$ order on $\mathbf{J}_n$.

To assist us in proving that S_{nn} is an absolutely unavoidable relation, we first show that the restricted projections ρ_1 and ρ_2 are the only elements of the dual space of $\mathbf{S}_{nn}$ (when viewed as an algebra). To do this, we show that any map $h \colon \mathbf{S}_{nn} \to \mathbf{J}_n$ with $h \neq \rho_1$ and $h \neq \rho_2$ will not satisfy the properties required to be a homomorphism.

Lemma 3.5.1. *Let* $n \in \omega$ *and consider the bilattice* $\mathbf{J}_n$. *Let* $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{J}_n)$ *and consider the subalgebra* $\mathbf{S}_{nn} \in \mathbb{S}(\mathbf{J}_n^2)$. *If* $h \in \boldsymbol{\mathcal{A}}(\mathbf{S}_{nn}, \mathbf{J}_n)$ *then for all* $(a, b) \in S_{nn}$ *either* $h(a, b) = a$ *or* $h(a, b) = b$.

Proof. Suppose there exists a homomorphism $h\colon \mathbf{S}_{nn} \to \mathbf{J}_n$ such that there exists $(a,b) \in S_{nn}$ with $a \neq b$ such that $h(a,b) \neq a$ and $h(a,b) \neq b$.

There are five cases that we need to consider in order to show that such an h cannot be a homomorphism. Let $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$. Then

$$(a,b) \in \big\{(\bot,\top), (\mathbf{t}_i,\mathbf{t}_j), (\mathbf{f}_i,\mathbf{f}_j), (\bot,\mathbf{t}_i), (\bot,\mathbf{f}_i), (\mathbf{t}_i,\top), (\mathbf{f}_i,\top)\big\}.$$

We note that for $(a,b) = (\mathbf{t}_i,\mathbf{t}_j)$ we only need to cover cases where $|i-j| > 1$ as otherwise the fact that h preserves the $\leqslant_k$ order would force $h(\mathbf{t}_i,\mathbf{t}_j) = \mathbf{t}_i$ or $h(\mathbf{t}_i,\mathbf{t}_j) = \mathbf{t}_j$. We note that proofs of the cases for $(\bot,\mathbf{f}_i)$ and $(\bot,\mathbf{t}_i)$ will be similar, as will the cases of $(\mathbf{t}_i,\top)$ and $(\mathbf{f}_i,\top)$.

Case 1: Suppose that $(a,b) = (\bot,\top)$ and furthermore that $h(a,b) = \mathbf{f}_i$ for some $i \in \omega$ with $0 \leqslant i \leqslant n$. Now

$$\neg h(\bot,\top) = \neg\mathbf{f}_i = \mathbf{t}_i \neq \mathbf{f}_i = h(\bot,\top) = h(\neg\bot,\neg\top).$$

A similar proof shows that if $h(\bot,\top) = \mathbf{t}_j$ for any $0 \leqslant j \leqslant n$, then h cannot be a homomorphism.

Case 2: Suppose that $(a,b) = (\mathbf{t}_i,\mathbf{t}_j)$. We assume that $i < j$. Since h must preserve the $\leqslant_k$ order, the only way that we can have $h(\mathbf{t}_i,\mathbf{t}_j) \notin \{\mathbf{t}_i,\mathbf{t}_j\}$ is if $h(\mathbf{t}_i,\mathbf{t}_j) = \mathbf{t}_\ell$ where $i < \ell < j$. Now we have

$$h(\mathbf{t}_i,\mathbf{t}_j) \vee h(\mathbf{t}_j,\mathbf{t}_i) = h(\mathbf{t}_i,\mathbf{t}_i) = \mathbf{t}_i.$$

This implies that $\mathbf{t}_\ell \vee h(\mathbf{t}_j,\mathbf{t}_i) = \mathbf{t}_i$ and since $\mathbf{t}_j <_t \mathbf{t}_\ell <_t \mathbf{t}_i$ we see that $h(\mathbf{t}_j,\mathbf{t}_i) = \mathbf{t}_i$. Now

$$h(\mathbf{t}_i,\mathbf{t}_j) \wedge h(\mathbf{t}_j,\mathbf{t}_i) = \mathbf{t}_\ell \wedge \mathbf{t}_i = \mathbf{t}_\ell \neq \mathbf{t}_j = h(\mathbf{t}_i \wedge \mathbf{t}_j, \mathbf{t}_j \wedge \mathbf{t}_i).$$

This same argument can be applied to $(a,b) = (\mathbf{t}_j,\mathbf{t}_i)$, and to $(a,b) = (\mathbf{f}_i,\mathbf{f}_j)$ and $(a,b) = (\mathbf{f}_j,\mathbf{f}_i)$.

Case 3: Suppose $(a,b) = (\bot,\mathbf{f}_i)$. Since h must preserve $\leqslant_k$, we have that

$$\bot = h(\bot,\bot) \leqslant_k h(\bot,\mathbf{f}_i) \leqslant_k h(\mathbf{f}_i,\mathbf{f}_i) = \mathbf{f}_i.$$

So we suppose that $h(\bot,\mathbf{f}_i) = \mathbf{f}_j$ where $\mathbf{f}_j <_k \mathbf{f}_i$ (and hence $i < j$). Now consider

$$h(\mathbf{f}_i,\mathbf{f}_j) = h(\mathbf{f}_i,\top) \wedge h(\mathbf{f}_j,\mathbf{f}_j) = h(\mathbf{f}_i,\top) \wedge \mathbf{f}_j.$$

We have from Case 2 that $h(\mathbf{f}_i, \mathbf{f}_j) \in \{\mathbf{f}_i, \mathbf{f}_j\}$. If $h(\mathbf{f}_i, \mathbf{f}_j) = \mathbf{f}_j$ then we clearly have $h(\mathbf{f}_i, \top) \geqslant_t \mathbf{f}_j$. Since h must be $\leqslant_t$-preserving, this implies that $h(\mathbf{f}_i, \top) \in \{\top, \mathbf{f}_j, \mathbf{f}_{j+1}, \ldots, \mathbf{f}_n\}$. But now h must also be $\leqslant_k$-preserving and so we must have $h(\mathbf{f}_i, \top) \in \{\top, \mathbf{f}_i, \mathbf{f}_{i-1}, \ldots, \mathbf{f}_0\}$. Thus we must have $h(\mathbf{f}_i, \top) = \top$. We can then see that

$$h(\mathbf{f}_i, \top) \wedge h(\bot, \mathbf{f}_i) \;=\; \top \wedge \mathbf{f}_j \;=\; \mathbf{f}_j \neq \mathbf{f}_i \;=\; h(\mathbf{f}_i, \mathbf{f}_i) \;=\; h(\mathbf{f}_i \wedge \bot, \top \wedge \mathbf{f}_i)$$

and so h cannot be a homomorphism. Since $h(\mathbf{f}_i, \mathbf{f}_j) \neq \mathbf{f}_j$, we must have $h(\mathbf{f}_i, \mathbf{f}_j) = \mathbf{f}_i$. In this case we see that

$$h(\mathbf{f}_i, \top) \vee h(\bot, \mathbf{f}_i) \;=\; \mathbf{f}_i \vee \mathbf{f}_j \;=\; \mathbf{f}_j.$$

However, $h(\mathbf{f}_i, \top) \vee h(\bot, \mathbf{f}_i) = h(\bot, \top)$ and by Case 1 we have that $h(\bot, \top) \in \{\bot, \top\}$. Thus $(a, b) \neq (\bot, \mathbf{f}_i)$ for any $i \in \{0, 1, \ldots, n\}$.

For the case of $h(\bot, \mathbf{t}_i) = \mathbf{t}_j$ $(i < j)$, one considers $h(\mathbf{t}_i, \top) \vee h(\mathbf{t}_j, \mathbf{t}_j)$ in the first step and then proceeds to show that $h(\mathbf{t}_i, \mathbf{t}_j) \notin \{\mathbf{t}_i, \mathbf{t}_j\}$.
<u>Case 4:</u> Suppose that $(a, b) = (\mathbf{f}_i, \top)$. Since h must be $\leqslant_k$-preserving and $\mathbf{f}_i \leqslant_k \top$ we have $\mathbf{f}_i = h(\mathbf{f}_i, \mathbf{f}_i) \leqslant_k h(\mathbf{f}_i, \top) \leqslant_k h(\top, \top) = \top$. Thus if $h(\mathbf{f}_i, \top) \notin \{\mathbf{f}_i, \top\}$ we must have $h(\mathbf{f}_i, \top) = \mathbf{f}_j$ for $\mathbf{f}_i <_k \mathbf{f}_j$ $(j < i)$. But since h must also be $\leqslant_t$-preserving, we must have $\mathbf{f}_i = h(\mathbf{f}_i, \mathbf{f}_i) \leqslant_t h(\mathbf{f}_i, \top) \leqslant_t h(\top, \top) = \top$. Now $h(\mathbf{f}_i, \top) = \mathbf{f}_j <_t \mathbf{f}_i$, a contradiction. □

We can now use the above lemma to show that the dual space of $\mathbf{S}_{nn}$ will consist only of the restricted projections.

Proposition 3.5.2. *Let $n \in \omega$ and consider the bilattice $\mathbf{J}_n$. Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{J}_n)$ and consider the subalgebra $\mathbf{S}_{nn} \in \mathbb{S}(\mathbf{J}_n^2)$. Then $\boldsymbol{\mathcal{A}}(\mathbf{S}_{nn}, \mathbf{J}_n) = \{\rho_1, \rho_2\}$.*

Proof. From Lemma 3.5.1 we know that any $h \in \boldsymbol{\mathcal{A}}(\mathbf{S}_{nn}, \mathbf{J}_n)$ has to map $(a, b) \in S_{nn}$ to one of its coordinates. Now suppose that there exists $h \in \boldsymbol{\mathcal{A}}(\mathbf{S}_{nn}, \mathbf{J}_n)$ such that there exists $(a, b) \in S_{nn}$ and $(c, d) \in S_{nn}$ with $h(a, b) = a$ and $h(c, d) = d$. That is, h still maps every element of S_{nn} to one of its coordinates, but h is neither ρ_1 nor ρ_2. (Note that we are also assuming that $a \neq b$, $c \neq d$ and that $(a, b) \neq (c, d)$.) We will show that for all possible combinations of (a, b) and (c, d), the map h cannot be a homomorphism. As above, the cases that need to be considered (with $i, j \in \{0, 1, \ldots, n\}$ and $i \neq j$) are:

$$(a, b), (c, d) \;\in\; \big\{(\bot, \top), (\mathbf{t}_i, \mathbf{t}_j), (\mathbf{f}_i, \mathbf{f}_j), (\bot, \mathbf{t}_i), (\bot, \mathbf{f}_i), (\mathbf{t}_i, \top), (\mathbf{f}_i, \top)\big\}.$$

As before, the proof for the case of $(\mathbf{t}_i, \mathbf{t}_j)$ (similarly $(\bot, \mathbf{t}_i)$, $(\mathbf{t}_i, \top)$) can be converted to the case of $(\mathbf{f}_i, \mathbf{f}_j)$ (or $(\bot, \mathbf{f}_i)$, $(\mathbf{f}_i, \top)$) by applying the negation operation and, in some cases, using the dual lattice operation.

Case 1: $(a,b) = (\bot,\top)$. We now need to consider all possible cases for (c,d) with $h(c,d) = d$. If $(c,d) = (\mathbf{t}_i,\mathbf{t}_j)$, then $h((\bot,\top)\oplus(\mathbf{t}_i,\mathbf{t}_j)) = h(\mathbf{t}_i,\top) \in \{\mathbf{t}_i,\top\}$. However, $h(\bot,\top)\oplus h(\mathbf{t}_i,\mathbf{t}_j) = \bot\oplus\mathbf{t}_j = \mathbf{t}_j$. The case of $(c,d) = (\mathbf{f}_i,\mathbf{f}_j)$ follows from an application of the negation operator. Suppose now that $(c,d) = (\bot,\mathbf{t}_i)$. Then $h((\bot,\top)\oplus(\bot,\mathbf{t}_i)) = h(\bot,\top) = \bot$. However, $h(\bot,\top)\oplus h(\bot,\mathbf{t}_i) = \bot\oplus\mathbf{t}_i = \mathbf{t}_i$. Finally, suppose that $(c,d) = (\mathbf{t}_i,\top)$. We have $h((\bot,\top)\wedge(\mathbf{t}_i,\top)) = h(\bot,\top) = \bot$, but $h(\bot,\top)\wedge h(\mathbf{t}_i,\top) = \bot\wedge\top = \mathbf{f}_n$.

Case 2: $(a,b) = (\mathbf{t}_i,\mathbf{t}_j)$. If $(c,d) = (\bot,\top)$, then $h((\mathbf{t}_i,\mathbf{t}_j)\otimes(\bot,\top)) = h(\bot,\mathbf{t}_j) \in \{\bot,\mathbf{t}_j\}$. However, $h(\mathbf{t}_i,\mathbf{t}_j)\otimes h(\bot,\top) = \mathbf{t}_i\otimes\top = \mathbf{t}_i$. Now suppose that $(c,d) = (\mathbf{t}_k,\top)$ where $0 \leqslant k \leqslant n$. It is easy to see that $h(\mathbf{f}_k,\top) = h(\neg\mathbf{t}_k,\neg\top) = \neg h(\mathbf{t}_k,\top) = \top$. We then have $h((\mathbf{t}_i,\mathbf{t}_j)\otimes(\mathbf{f}_k,\top)) = h(\bot,\mathbf{t}_j) \in \{\bot,\mathbf{t}_j\}$ but $h(\mathbf{t}_i,\mathbf{t}_j)\otimes h(\mathbf{f}_k,\top) = \mathbf{t}_i\otimes\top = \mathbf{t}_i$. If $(c,d) = (\bot,\mathbf{t}_k)$ we can show that $h(\bot,\mathbf{f}_k) = \mathbf{f}_k$. Now $h((\mathbf{t}_i,\mathbf{t}_j)\oplus(\bot,\mathbf{f}_k)) = h(\mathbf{t}_i,\top) \in \{\mathbf{t}_i,\top\}$. However, from the previous subcase, we know that $h(\mathbf{t}_i,\top) \neq \top$. Hence $h(\mathbf{t}_i,\top) = \mathbf{t}_i$ but $h(\mathbf{t}_i,\mathbf{t}_j)\oplus h(\bot,\mathbf{f}_k) = \mathbf{t}_i\oplus\mathbf{f}_k = \top$. The final subcase is $(c,d) = (\mathbf{t}_k,\mathbf{t}_m)$ where $k,m \in \{0,\ldots,n\}$. From the second subcase, we know that $h(\bot,\mathbf{t}_k) = \bot$ and hence $h(\bot,\mathbf{f}_k) = \bot$. Now $h((\bot,\mathbf{f}_k)\oplus(\mathbf{t}_k,\mathbf{t}_m)) = h(\mathbf{t}_k,\top)$. From the third subcase, we see that we must have $h(\mathbf{t}_k,\top) = \mathbf{t}_k$. However, $h(\bot,\mathbf{f}_k)\oplus h(\mathbf{t}_k,\mathbf{t}_m) = \bot\oplus\mathbf{t}_m = \mathbf{t}_m$.

Case 3: $(a,b) = (\bot,\mathbf{t}_i)$. Let $(c,d) = (\bot,\top)$. Now $h((\bot,\mathbf{t}_i)\wedge(\bot,\top)) = h(\bot,\top) = \top$ but $h(\bot,\mathbf{t}_i)\wedge h(\bot,\top) = \bot\wedge\top = \mathbf{f}_n$. Suppose $(c,d) = (\mathbf{t}_k,\mathbf{t}_m)$ for $k,m \in \{0,\ldots,n\}$. Since $h(\bot,\mathbf{t}_i) = \bot$ we get that $h(\bot,\mathbf{f}_i) = h(\neg\bot,\neg\mathbf{t}_i) = \neg h(\bot,\mathbf{t}_i) = \bot$. Now $h((\bot,\mathbf{f}_i)\oplus(\mathbf{t}_k,\mathbf{t}_m)) = h(\mathbf{t}_k,\top) \in \{\mathbf{t}_k,\top\}$. However, $h(\bot,\mathbf{f}_i)\oplus h(\mathbf{t}_k,\mathbf{t}_m) = \bot\oplus\mathbf{t}_m = \mathbf{t}_m$. Lastly, suppose that $(c,d) = (\mathbf{t}_k,\top)$ where $0 \leqslant k \leqslant n$. Then $h((\bot,\mathbf{t}_i)\wedge(\mathbf{t}_k,\top)) = h(\bot,\top) \in \{\bot,\top\}$. However, $h(\bot,\mathbf{t}_i)\wedge h(\mathbf{t}_k,\top) = \bot\wedge\top = \mathbf{f}_n$.

Case 4: $(a,b) = (\mathbf{t}_i,\top)$. If $(c,d) = (\bot,\top)$, then $h((\mathbf{t}_i,\top)\otimes(\bot,\top)) = h(\bot,\top) = \top$, but $h(\mathbf{t}_i,\top)\otimes h(\bot,\top) = \mathbf{t}_i\otimes\top = \mathbf{t}_i$. Suppose that $(c,d) = (\mathbf{t}_k,\mathbf{t}_m)$ (where $k,m \in \{0,\ldots,n\}$). Since $h(\mathbf{t}_i,\top) = \mathbf{t}_i$, we have that $h(\mathbf{f}_i,\top) = h(\neg\mathbf{t}_i,\neg\top) = \neg h(\mathbf{t}_i,\top) = \mathbf{f}_i$. Now $h((\mathbf{f}_i,\top)\otimes(\mathbf{t}_k,\mathbf{t}_m)) = h(\bot,\mathbf{t}_m) \in \{\bot,\mathbf{t}_m\}$ and $h(\mathbf{f}_i,\top)\otimes h(\mathbf{t}_k,\mathbf{t}_m) = \mathbf{f}_i\otimes\mathbf{t}_m = \bot$. If $h(\bot,\mathbf{t}_m) = \mathbf{t}_m$ then h is not a homomorphism. So, suppose that $h(\bot,\mathbf{t}_m) = \bot$. This implies that $h(\bot,\mathbf{f}_m) = \bot$, which in turn implies that $h(\bot,\mathbf{f}_m)\oplus h(\mathbf{t}_k,\mathbf{t}_m) = \bot\oplus\mathbf{t}_m = \mathbf{t}_m$. However, $h((\bot,\mathbf{f}_m)\oplus(\mathbf{t}_k,\mathbf{t}_m)) = h(\mathbf{t}_k,\top) \in \{\mathbf{t}_k,\top\}$. Lastly, we let

$(c,d) = (\bot, \mathbf{t}_k)$ $(0 \leqslant k \leqslant n)$. We see that $h((\mathbf{t}_i, \top) \wedge (\bot, \mathbf{t}_k)) = h(\bot, \top) \in \{\bot, \top\}$, but $h(\mathbf{t}_i, \top) \wedge h(\bot, \mathbf{t}_k) = \mathbf{t}_i \wedge \mathbf{t}_k \in \{\mathbf{t}_i, \mathbf{t}_k\}$. $\square$

Proposition 3.5.3. *Let $n \in \omega$ and consider the bilattice $\mathbf{J}_n$. The relation S_{nn} is absolutely unavoidable on the dualising structure $\underset{\sim}{\mathbf{J}_{\mathbf{n}}}$.*

Proof. We will demonstrate that there exists a map $\gamma \colon \boldsymbol{\mathcal{A}}(\mathbf{S}_{nn}, \mathbf{J}_n) \to J_n$ such that γ fails to preserve S_{nn} on $\mathrm{D}(\mathbf{S}_{\mathrm{nn}})$ but preserves every other binary algebraic relation on $\mathrm{D}(\mathbf{S}_{\mathrm{nn}})$. For $h \in \boldsymbol{\mathcal{A}}(\mathbf{S}_{nn}, \mathbf{J}_n)$ we define a map γ as follows:

$$\gamma(h) = \begin{cases} \top & \text{if } h = \rho_1 \\ \bot & \text{if } h = \rho_2. \end{cases}$$

We have $(\rho_1, \rho_2) \in (S_{nn})_{\mathrm{D}(\mathbf{S}_{\mathrm{nn}})}$ as for any $(a,b) \in S_{nn}$ it is clear that $(\rho_1(a,b), \rho_2(a,b)) = (a,b) \in S_{nn}$. It is trivially true that for any relation $S \in \mathbb{S}(\mathbf{J}_n^2)$ we have $(\rho_1, \rho_1) \in S_{\mathrm{D}(\mathbf{S}_{\mathrm{nn}})}$ and $(\rho_2, \rho_2) \in S_{\mathrm{D}(\mathbf{S}_{\mathrm{nn}})}$.

Lemma 3.4.1 and Proposition 3.4.5 combine to tell us that every $\mathbf{S} \neq \mathbf{S}_{nn}$ (apart from $\mathbf{J}_n^2$) does not contain the element $(\bot, \top)$, and that $S \subseteq S_{nn}$. Thus we have, for every $\mathbf{S} \in \mathbb{S}(\mathbf{J}_n^2)$ such that $\mathbf{S} \notin \{\mathbf{J}_n^2, \mathbf{S}_{nn}\}$, that $(\rho_1(\bot, \top), \rho_2(\bot, \top)) = (\bot, \top) \notin S$ and hence $(\rho_1, \rho_2) \notin S_{\mathrm{D}(\mathbf{S}_{\mathrm{nn}})}$. Using the converse results, Lemma 3.4.3 and Proposition 3.4.6, we see that (ρ_2, ρ_1) is only an element of $(\overset{\smile}{\mathbf{S}}_{nn})_{\mathrm{D}(\mathbf{S}_{\mathrm{nn}})}$.

Thus for every $S \in \mathbb{S}(\mathbf{J}_n^2)$ such that $S \neq S_{nn}$, the map γ trivially preserves every relation $S_{\mathrm{D}(\mathbf{S}_{\mathrm{nn}})}$. However, γ does not preserve S_{nn}. We see this as $(\rho_1, \rho_2) \in (S_{nn})_{\mathrm{D}(\mathbf{S}_{\mathrm{nn}})}$ but $(\gamma, \gamma)(\rho_1, \rho_2) = (\top, \bot) \notin S_{nn}$. Thus the relation S_{nn} is an absolutely unavoidable relation in the duality for $\mathbb{ISP}(\mathbf{J}_n)$. $\square$

3.6 The role of $\leqslant_k$ in dualities for bilattices

Throughout the investigation of dualities for bilattices in Chapter 2 and the current chapter, we have become aware that the order $\leqslant_k$ was somehow, although absent from the set of dualising relations, playing a role in the dualising structures. In these examples, the relation $\leqslant_k$ is not itself available as an algebraic relation, but we can see it encoded in the dual structures.

Perhaps, it should not have been surprising to see the $\leqslant_k$ order appearing in the dual structures for bilattices. From failset calculations done on finitely generated quasivarieties of lattices by Wegener [86, Chapter 2], we know that the lattice order often plays an important role in the dualising structures of lattice-based quasivarieties. In the absence of non-lattice operations, it is of course an algebraic relation. Wegener's work revealed that for the quasivarieties $\mathbb{ISP}(\mathbf{M}_3)$, $\mathbb{ISP}(\mathbf{M}_4)$, and $\mathbb{ISP}(\mathbf{N}_5)$, the lattice order was always an absolutely unavoidable relation. When studying the quasivariety of modular ortholattices $\mathbb{ISP}(\mathbf{MO}_4)$, the order was no longer present in the duality. This is of course because the negation operation reverses the order. This is similar to the way that $\leqslant$ is an algebraic relation for $\mathbb{ISP}(\underline{\mathbf{2}})$ for $\underline{\mathbf{2}} \in \boldsymbol{\mathfrak{D}}$, but there are no non-trivial algebraic relations for $\mathbb{ISP}(\underline{\mathbf{2}})$ when $\underline{\mathbf{2}} \in \boldsymbol{\mathfrak{B}}$.

Although in our examples the $\leqslant_t$ operations do not preserve the $\leqslant_k$ order, the bilattice negation does, and so this leaves fragments and collapses of the $\leqslant_k$ order available as algebraic relations.

The smallest non-interlaced bilattice is a five-element bilattice first presented by Ginsberg [61, Fig. 6]. The bilattice $\mathcal{FIVE}$ is drawn in Fig. 3.9 and its algebraic signature is defined below:

$$\mathcal{FIVE} := \langle \{\top, \mathbf{t}, \mathbf{f}, a, \bot\}; \otimes, \oplus, \wedge, \vee, \neg, \mathbf{t}, \mathbf{f}\rangle.$$

The negation operation fixes the elements $\top$, $\bot$ and a, and maps $\mathbf{t}$ to $\mathbf{f}$ and vice versa.

Figure 3.9: The bilattice $\mathcal{FIVE}$ drawn with its $\leqslant_k$ (left) and $\leqslant_t$ (right) orderings.

At the beginning of our investigations into dualities for quasivarieties of non-interlaced bilattices, we performed calculations to obtain dualities for the quasivariety $\mathbb{ISP}(\mathcal{FIVE})$, with the aim of obtaining useful clues to direct us when searching for dualities for other quasivarieties of bilattices. However, the results did not prove to be particularly revealing. The failset calculations revealed that there was only one subalgebra, $\mathbf{S}$, of $\mathcal{FIVE}^2$ that was required for the dualising structure. The ten-element subalgebra $\mathbf{S}$ contains the elements listed below.

$$\begin{aligned} S = {} & \{(a,a),(a,\mathbf{f}),(a,\mathbf{t}),(a,\top),(\mathbf{f},\mathbf{f}),(\mathbf{f},\top),(\mathbf{t},\mathbf{t}),(\mathbf{f},\top),(\top,\top),(a,\bot)\} \\ = {} & \big(\leqslant_k \cap \{\top,\mathbf{f},\mathbf{t},a\}^2\big) \cup \{(a,\bot)\}. \end{aligned}$$

Proposition 3.6.1. *The dualising structure* $\underset{\sim}{\mathcal{FIVE}} = \langle \{\top, \mathbf{f}, \mathbf{t}, a, \bot\}; S, \mathcal{T}\rangle$ *yields an optimal duality on* $\mathbb{ISP}(\mathcal{FIVE})$.

The algebraic relation $\mathbf{S}$ does not fit at all with the pattern that we have observed in our other dualities for bilattices. There are two major differences between the bilattice $\mathcal{FIVE}$ and the families of bilattices $\mathbf{K}_n$ and $\mathbf{J}_n$.

The first difference is that not every element of $\mathcal{FIVE}$ is term-definable from the set of constants in the algebraic signature. If we were to change the signature of $\mathcal{FIVE}$ so that all of the elements were term-definable, it would have the result that $\{(\top,\top), (\mathbf{f},\mathbf{f}), (\mathbf{t},\mathbf{t}), (a,a)\}$ would no longer be a subalgebra of $\mathcal{FIVE}^2$. Thus $\Delta(\mathcal{FIVE}^2)$ would be contained in every subalgebra and this would have the additional consequence that $\mathbb{S}(\mathcal{FIVE}^2) = \{\Delta(\mathcal{FIVE}^2), \mathcal{FIVE}^2\}$ and hence the category of spaces dual to $\mathbb{ISP}(\mathcal{FIVE})$ would simply be the category of Boolean spaces.

Secondly, there does not exist a homomorphism from $\mathcal{FIVE}$ to $\mathcal{FOUR}$ $(\mathbf{K}_0, \mathbf{J}_0)$. The subalgebra generated by $\leqslant_k$ is simply the whole algebra $\mathcal{FIVE}^2$. In each of the previous cases, the duality for $\mathbb{ISP}(\mathbf{K}_n)$ and $\mathbb{ISP}(\mathbf{J}_n)$ always had the inverse homomorphic image of $\leqslant_k$ (on $\mathcal{FOUR}$) as an unavoidable relation. This suggests grounds for further research.

We know from our earlier observations for the case of $\mathbb{ISP}(\mathbf{J}_2)$ that the inverse homomorphic image of an absolutely unavoidable relation is not necessarily absolutely unavoidable (see $\mathbf{S}_{12} = (h_{2,1,0}, h_{2,1,0})(\mathbf{S}_2)$).

Our first question in this line of thinking would be the following:

- Let $\mathbf{A}$ be a finite *bilattice* such that there exists a homomorphism $h\colon \mathbf{A} \to \mathcal{FOUR}$. Are the relations in the natural duality for $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{A})$ always influenced by the order $\leqslant_k$?

We are being deliberately vague in our use of the phrase “influenced by”. As we noted earlier, in the case of the bilattices $\mathbf{K}_n$ and $\mathbf{J}_n$ the inverse image (under the homomorphism $h\colon \mathbf{B} \to \mathcal{FOUR}$) of the $\leqslant_k$ order is absolutely unavoidable. However, this is the strongest possible statement that we can make about the role of the relation $(h,h)^{-1}(\leqslant_k)$. It would not be surprising to find that there are some bilattices for which a failset for the relation $(h,h)^{-1}(\leqslant_k)$ exists, but that it is not absolutely unavoidable.

An even more general question would be:

- Given a finite algebra $\mathbf{A}$ and a homomorphic image $\mathbf{B}$ of $\mathbf{A}$, is the natural duality for $\mathbb{ISP}(\mathbf{A})$ influenced by the absolutely unavoidable relations on the dualising object $\underset{\sim}{\mathbf{B}}$?

We should note that in the case of $\mathcal{FOUR}$, the algebraic relation $\leqslant_k$ is (up to converses) the *only* non-trivial subalgebra of $\mathcal{FOUR}^2$. Thus it is no surprise that it should play a role in the duality for $\mathbb{ISP}(\mathcal{FOUR})$. However, the situation would be vastly more complicated if the duality on the algebra $\mathbf{B}$ had some combination of relations, some or none of which was absolutely unavoidable.

In the cases where we are able to prove that relations are absolutely unavoidable, we did so using the fact that the dual space of the relation (subalgebra) consisted only of the projection maps ρ_1 and ρ_2. It may be possible to attempt some of the questions above in that very restricted setting, or in the setting covered by Lemma 2.4.4.

3.7 Piggyback dualities for default bilattices

We conclude this chapter with some comments about other methods that could be used to obtain natural dualities for the quasivarieties $\mathbb{ISP}(\mathbf{K}_n)$ and $\mathbb{ISP}(\mathbf{J}_n)$. The theory of piggyback dualities was first described by Davey and Werner [32] and further developed to the case of multisorted dualities by Davey and Priestley [25]. The brief summary given in Appendix A follows the exposition in Chapter 7 of [17].

The basic idea is that given an algebra $\mathbf{M}$ which has a term-definable bounded distributive lattice reduct $\mathbf{M}^\flat$ (the $^\flat$ notation meaning that we have forgotten some of the structure) one can use the underlying distributive lattice structure to help determine a relational structure on the dualising object which will yield a duality. From the NU Duality Theorem it is already known that $\mathbb{ISP}(\mathbf{M})$ is dualisable, but the piggyback duality serves to reduce the number of relations (taken from $\mathbb{S}(\mathbf{M}^2)$) that are used in order to yield a duality. There is no guarantee that the duality will be optimal, but by taking maximal subalgebras contained in the $(w_i, w_j)^{-1}(\leqslant)$ it helps to reduce the number of relations as much as possible, although it does favour large relations.

Dualities for the quasivarieties generated by the bilattices $\mathbf{K}_n$ and $\mathbf{J}_n$ (for $n \in \omega$) can be obtained using the piggybacking method. This follows from the fact that the bilattices $\mathbf{K}_n$ are distributive lattices in their k-lattice reducts, while the bilattices $\mathbf{J}_n$ are distributive lattices in their t-lattice reducts. We note that the bilattices $\mathbf{K}_n$ do not have *bounded* distributive lattice reducts, but the bounds of the k-lattice are term definable. Thus we can piggyback over the duality for bounded distributive lattices.

Although the piggybacking technique is possible to do by hand for some small examples, finding the maximal subalgebras contained in the $(w_i, w_j)^{-1}(\leqslant)$ can be difficult. We have written a series of programs which can be used in conjunction with those of Wegener [86] to calculate the maximal subalgebras contained in $(w_i, w_j)^{-1}(\leqslant)$ for carrier maps w_i, w_j.

The piggyback method will yield a natural duality for $\mathbb{ISP}(\mathbf{M})$. It is also possible to obtain what is known as a *restricted Priestley duality*. This is obtained by adding structure to the Priestley spaces in order to capture the additional structure on the algebra side. This additional structure could be in the form of algebraic relations, or (partial) operations.

The restricted Priestley dualities have the disadvantage that they do not provide the clear description of free algebras that is available when using a natural duality. However, moving between the dual spaces encountered in restricted Priestley dualities and the underlying lattices of the distributive lattice-based algebras is a much more intuitive process. The concrete representation of the algebras as the up-sets of the Priestley dual spaces is an important feature of the restricted Priestley dualities.

A better understanding of this translation process between natural dualities and restricted Priestley dualities for algebras with a distributive lattice reduct is currently being pursued by Cabrer and Priestley [15]. Their work is motivated by the fact that coproducts of distributive lattices can be easily described using their Priestley dual spaces. Thus an easy translation between Priestley dual spaces and natural dual spaces can provide an easier way of describing coproducts of algebras which have a distributive lattice reduct.

Appendix A

Theory

Here we present some definitions and results that will be used in the rest of the thesis. The main sources are Davey and Priestley [28], Burris and Sankappanavar [14] and Clark and Davey [17].

A.1 Order theory

A *quasi-ordered set* is a set P with a binary relation $\preceq$ such that $\preceq$ is reflexive and transitive. A quasi-ordered set $\mathbf{P} = (P, \preceq)$ whose relation $\preceq$ is also anti-symmetric is called a *partially ordered set* or *poset*. A partial ordering will usually be denoted $\leqslant$.

Definition A.1.1. (MacNeille Completion) Let $\mathbf{P} = (P, \preceq)$ be a quasi-ordered set. Define for $A \subseteq P$ the sets

$$A^u = \{\, b \in P \mid a \in A \Rightarrow a \preceq b \,\} \quad \text{and} \quad A^\ell = \{\, b \in P \mid a \in A \Rightarrow b \preceq a \,\}.$$

Then the *MacNeille completion* of $\mathbf{P}$ is the complete lattice

$$\overline{\mathbf{P}} := \big(\{\, A \subseteq P \mid A^{u\ell} = A \,\}, \subseteq \big).$$

When $\mathbf{P}$ is a poset, P is embedded into $\overline{P}$ via the embedding $a \hookrightarrow {\downarrow}a$.

The MacNeille completion is also known as the *Dedekind–MacNeille completion*, the *completion by cuts*, and the *normal completion.*

Definition A.1.2. Let $\mathbf{L}$ be a complete lattice. Then $x \in L$ is *completely join-irreducible* if for any $A \subseteq L$ such that $x = \bigvee A$, then $x \in A$. Dually, $y \in L$ is *completely meet-irreducible* if for any $B \subseteq L$ with $y = \bigwedge B$, we have $y \in B$.

A.2 Universal algebra

An *algebra* is a structure $\mathbf{A} = \langle A; F\rangle$ where A is the underlying set and F is a finite set of finitary operations on A. The set F is known as the *type* of $\mathbf{A}$. A *subuniverse* of $\mathbf{A}$ is a subset $B \subseteq A$ such that B is closed under the operations in F. If $\mathbf{A}$ and $\mathbf{B}$ are of the same type F and $B \subseteq A$, then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if for each operation f, $f^{\mathbf{B}}$ is the restriction of $f^{\mathbf{A}}$ to B.

Suppose $\mathbf{A}$ and $\mathbf{B}$ are two algebras of the same type F. A *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $h\colon \mathbf{A} \to \mathbf{B}$ such that for all $f \in F$,

$$h(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(h(a_1), \dots, h(a_n)).$$

If h is onto, then $\mathbf{B}$ is known as a *homomorphic image* of $\mathbf{A}$. For a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ the *kernel* of h is the set

$$\ker(h) := \{\, (a, b) \in A^2 \mid h(a) = h(b) \,\}.$$

A *variety* $\boldsymbol{\mathcal{A}}$ is a class of algebras of type F such that $\boldsymbol{\mathcal{A}}$ is closed under subalgebras, homomorphic images and direct products. We denote by $\boldsymbol{\mathcal{V}}(\mathbf{A})$ the *variety generated by* $\mathbf{A}$. A variety $\boldsymbol{\mathcal{A}}$ is *finitely generated* if $\boldsymbol{\mathcal{A}} = \mathbb{HSP}(K)$ where K is some finite set of finite algebras.

A *quasi-identity* is an identity or a formula of the form

$$(p_1 \approx q_1 \ \& \ p_2 \approx q_2 \ \& \ \dots \ \& \ p_n \approx q_n) \to p \approx q.$$

A class of algebras $\boldsymbol{\mathcal{A}}$ is a *quasivariety* if $\boldsymbol{\mathcal{A}}$ is closed under subalgebras, direct products, ultraproducts, and isomorphic images. A quasivariety $\boldsymbol{\mathcal{A}}$ is *finitely generated* if $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(K)$ for some finite set K of finite algebras. A quasivariety can be axiomatised by quasi-identities.

Definition A.2.1. Consider an algebra $\mathbf{A}$. An equivalence relation θ (reflexive, symmetric, transitive) on the set A is *congruence* on $\mathbf{A}$ if it is a subalgebra of $\mathbf{A}^2$. The equality relation $\Delta(\mathbf{A}^2) = \{\, (a, a) \mid a \in A \,\}$ is the smallest congruence on $\mathbf{A}$. The largest congruence relation is $\theta = A^2$.

For a congruence θ on $\mathbf{A}$, the *quotient algebra of* $\mathbf{A}$ *by* θ, denoted $\mathbf{A}/\theta$ is the algebra with underlying set A/θ and with the operations in F defined by

$$f^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = f^{\mathbf{A}}(a_1, \dots, a_n)/\theta.$$

It is clear that the quotient algebras of $\mathbf{A}$ will be of the same type as $\mathbf{A}$. Congruences are closed under arbitrary intersection and hence the lattice of all congruences of an algebra $\mathbf{A}$ form a complete lattice, denoted $\mathrm{Con}\mathbf{A}$.

For an algebra $\mathbf{A}$ and $\theta \in \mathrm{Con}(\mathbf{A})$ the *natural map* $\nu_\theta \colon A \to A/\theta$ is given by $\nu_\theta(a) = a/\theta$. The relationship between congruences on an algebra $\mathbf{A}$ and the homomorphic images of $\mathbf{A}$ is demonstrated by the following results.

Theorem A.2.2 ([14, Theorem II-6.8])**.** *Let* $h \colon \mathbf{A} \to \mathbf{B}$ *be a homomorphism. Then* $\ker(h)$ *is a congruence on* $\mathbf{A}$*.*

Theorem A.2.3 ([14, Theorem II-6.10])**.** *Let* $\mathbf{A}$ *be an algebra and* $\theta \in \mathrm{Con}(\mathbf{A})$*. The natural map* ν_θ *is an onto homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}/\theta$*.*

Theorem A.2.4 ([14, Theorem II-6.12])**.** (**First Isomorphism Theorem**) *Let* $h \colon \mathbf{A} \to \mathbf{B}$*. Then there is an isomorphism* g *from* $\mathbf{A}/\ker(h)$ *to* $\mathbf{B}$ *defined by* $h = g \circ f$ *where* f *is the natural homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}/\ker(h)$*.*

A subalgebra $\mathbf{B}$ of the direct product $\prod\{\, A_i \mid i \in I \,\}$ is a *subdirect product* of the algebras $\{\, \mathbf{A}_i \mid i \in I \,\}$ if, for each $j \in I$, the image of $\mathbf{B}$ under π_j is the whole algebra $\mathbf{A}_j$.

An embedding $u \colon \mathbf{B} \to \prod\{\, \mathbf{A}_i \mid i \in I \,\}$ of an algebra $\mathbf{B}$ into a direct product is called a *subdirect representation* of $\mathbf{B}$ if the image $u(\mathbf{B})$ is a subdirect product of the algebras $\{\, \mathbf{A}_i \mid i \in I \,\}$.

A subdirect representation u of $\mathbf{B}$ is *trivial* if, for some $i \in I$, the composite map $\pi_i \circ u \colon \mathbf{B} \to \mathbf{A}_i$ is an isomorphism. The algebra $\mathbf{B}$ is said to be *subdirectly irreducible* if every subdirect representation of $\mathbf{B}$ is trivial.

The following theorem presents an equivalent definition of a subdirectly irreducible algebra. This definition is easier to implement in practice.

Theorem A.2.5 ([14, Theorem II-8.4])**.** *An algebra* $\mathbf{A}$ *is subdirectly irreducible if and only if* $\mathbf{A}$ *is trivial or there is a minimum congruence in* $\mathrm{Con}\backslash\{\Delta(A^2)\}$*. If such a minimum congruence* θ *exists then* $\theta = \bigcap \big(\mathrm{Con}\mathbf{A}\backslash\{\Delta(A^2)\}\big)$*, a principal congruence.*

The following result will be useful when looking at the algebras that generate a variety.

Theorem A.2.6 ([14, Corollary IV-6.10])**.** *Suppose that* $\mathcal{K}$ *is a finite set of finite algebras and the variety* $\boldsymbol{\mathcal{V}}(\mathcal{K})$ *is congruence distributive. Then every subdirectly irreducible algebra in* $\boldsymbol{\mathcal{V}}(\mathcal{K})$ *is in* $\mathbb{HS}(\mathcal{K})$ *and* $\boldsymbol{\mathcal{V}}(\mathcal{K}) = \mathbb{ISP}\,\mathbb{HS}(\mathcal{K})$*.*

A.3 Programs for computation of failsets

Given a finite algebra $\mathbf{M}$, the sequence of computer programs written by Wegener [86] follows a number of steps before producing the globally minimal failsets. We provide a basic summary of the sequence of calculations that must be carried out in order to obtain optimal dualities for the quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$.

(1) Calculate $\mathbb{S}(\mathbf{M}^2)$.

(2) Reduce $\mathbb{S}(\mathbf{M}^2)$ to a list of subalgebras unique up to isomorphism (test algebras).

(3) Calculate $\boldsymbol{\mathcal{A}}(\mathbf{S}, \mathbf{M})$ for each test algebra $\mathbf{S}$.

(4) For each $R \in \mathbb{S}(\mathbf{M}^2)$, calculate which pairs of elements of $\boldsymbol{\mathcal{A}}(\mathbf{S}, \mathbf{M})$ are in R.

(5) Attempt to build a (failset) map γ from $\boldsymbol{\mathcal{A}}(\mathbf{S}, \mathbf{M})$ to M which does not preserve $\mathbf{S}_{\mathrm{D}(\mathbf{S})}$.

(6) For each failset map $\gamma_i \colon \boldsymbol{\mathcal{A}}(\mathbf{S}, \mathbf{M}) \to \underset{\sim}{\mathbf{M}}$, calculate which relations $R \in \mathbb{S}(\mathbf{M}^2)$ are not preserved by γ_i.

(7) For each test algebra $\mathbf{S}$, collate the minimal failsets (ordered by containment).

(8) Sort the minimal failsets to find the globally minimal failsets (again ordered by containment).

A.4 Piggyback dualities

In short, the technique of *piggybacking* uses a known duality for a quasivariety $\boldsymbol{\mathcal{B}}$ and uses this to find a duality for a quasivariety $\boldsymbol{\mathcal{A}}$ when every $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ has a term-reduct in $\boldsymbol{\mathcal{B}}$. The source for the explanation and results presented below is Chapter 7 of [17].

Theorem A.4.1 ([17, Theorem 7.2.1])**.** *Let* $\mathbf{M}$ *be a finite algebra which has a term-reduct in* $\boldsymbol{\mathcal{D}}$*, and let* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M})$*. Let* Ω *be a subset of* $\boldsymbol{\mathcal{D}}(\mathbf{M}^\flat, \underline{\mathbf{2}})$*. Let*

$$\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T} \rangle$$

where

(i) *R is the set of all $\boldsymbol{\mathcal{A}}$-subalgebras of $\mathbf{M}^2$ which are maximal in*

$$(w_i, w_j)^{-1}(\leqslant) := \{\, (a, b) \in M^2 \mid w_i(a) \leqslant w_j(b) \,\},$$

where $w_i, w_j \in \Omega$,

(ii) $G \subseteq End(\mathbf{M})$ *satisfies the separation condition*

> (S) *for all* $a \neq b$ *in* M *we have* $w(a) \neq w(b)$, *for some* $w \in \Omega$, *or* $w(g(a)) \neq w(g(b))$ *where* g *is the composite of a finite number of maps from* G,

(iii) $\mathcal{T}$ *is the discrete topology.*

Then $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\boldsymbol{\mathcal{A}}$.

The maps $w_i \in \Omega$ where $w_i \colon \mathbf{M}^\flat \to \underline{\mathbf{2}}$ are known as the *carrier maps*. Condition (S) ensures that these maps (possibility in combination with some subset of the endomorphism monoid of $\mathbf{M}$) separate the points of M. The bilattices in Chapter 2 are distributive in their k-lattice reducts, while those in Chapter 3 are distributive in their t-lattice reducts.

In the examples that we have considered in this thesis, the endomorphism monoid $\mathrm{End}(\mathbf{M})$ consists of only the identity map, and so we require the full set of distributive lattice homomorphisms from $\mathbf{M}^\flat$ to $\underline{\mathbf{2}}$ in order to separate the points of $\mathbf{M}$.

Appendix B

Computational output

Here we list the output of our computations for the bilattices $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{J}_1$, $\mathbf{J}_2$. We note that for each of these bilattices the only automorphism on $\mathbf{A}^2$ is the converse operation $\mathbf{S} \mapsto \breve{\mathbf{S}}$. We only list the subalgebras up to automorphism, but do observe for which subalgebras we have $\mathbf{S} = \breve{\mathbf{S}}$. We also note that, strictly speaking, we are not listing the subalgebras, but rather the *subuniverses* which are the underlying sets of the subalgebras.

B.1 The bilattice $\mathbf{K}_1$

Subalgebras of $\mathbf{K}_1^2$

$$\mathbf{S}_1 = \Delta(K_1^2)$$

$$\mathbf{S}_2 = \Delta(K_1^2) \cup \{(\mathbf{f}_1, \top_1), (\mathbf{t}_1, \top_1), (\bot, \top_1), (\bot, \mathbf{f}_1), (\bot, \mathbf{t}_1)\}$$

$$\mathbf{S}_3 = \{(\top_0, \top_0), (\mathbf{f}_0, \mathbf{f}_0), (\mathbf{t}_0, \mathbf{t}_0)\} \cup \{\top_1, \mathbf{f}_1, \mathbf{t}_1, \bot\}^2$$

$$\begin{aligned}\mathbf{S}_4 = \{(\top_0, \top_0), (\mathbf{f}_0, \top_0), (\mathbf{f}_0, \mathbf{f}_0), (\mathbf{t}_0, \top_0), (\mathbf{t}_0, \mathbf{t}_0)\} \cup{}& \\ &(\{\top_1, \mathbf{f}_1, \mathbf{t}_1, \bot\} \times \{\top_0, \mathbf{f}_0, \mathbf{t}_0\}) \cup \{\top_1, \mathbf{f}_1, \mathbf{t}_1, \bot\}^2\end{aligned}$$

$$\mathbf{S}_5 = K_1^2$$

We note that $\breve{\mathbf{S}}_3 = \mathbf{S}_3$ and so $|\mathbb{S}(\mathbf{K}_1^2)| = 7$.

Globally minimal failsets

$$GMF_1(\mathbf{K}_1) = \{\mathbf{S}_2\}$$

$GMF_2(\mathbf{K}_1) = \{\mathbf{S}_4\}$

B.2 The bilattice $\mathbf{K}_2$

Subalgebras of $\mathbf{K}_2^2$

$\mathbf{S}_1 = \Delta(K_2^2)$

$\mathbf{S}_2 = \Delta(K_2^2) \cup \{(\bot, \mathbf{f}_2), (\bot, \mathbf{t}_2), (\bot, \top_2), (\mathbf{f}_2, \top_2), (\mathbf{t}_2, \top_2)\}$

$\mathbf{S}_3 = \{(\top, \top), (\mathbf{f}_0, \mathbf{f}_0), (\mathbf{t}_0, \mathbf{t}_0), (\top_1, \top_1), (\mathbf{f}_1, \mathbf{f}_1), (\mathbf{t}_1, \mathbf{t}_1)\} \cup \{\bot, \mathbf{f}_2, \mathbf{t}_2, \top_2\}^2$

$$\begin{aligned}\mathbf{S}_4 = \{(\top, \top), (\mathbf{f}_0, \mathbf{f}_0), (\mathbf{t}_0, \mathbf{t}_0)\} \cup \{(\mathbf{f}_1, \top_1), (\mathbf{t}_1, \top_1)\} \cup \\ (\{\bot, \mathbf{f}_2, \mathbf{t}_2, \top_2\} \times \{\mathbf{f}_1, \mathbf{t}_1, \top_1\}) \cup \{\bot, \mathbf{f}_2, \mathbf{t}_2, \top_2\}^2\end{aligned}$$

$\mathbf{S}_5 = \{(\top, \top), (\mathbf{f}_0, \mathbf{f}_0), (\mathbf{t}_0, \mathbf{t}_0)\} \cup \{\bot, \mathbf{f}_2, \mathbf{t}_2, \top_2, \mathbf{f}_1, \mathbf{t}_1, \top_1\}^2$

$$\begin{aligned}\mathbf{S}_6 = \{(\top, \top), (\mathbf{f}_0, \mathbf{f}_0), (\mathbf{t}_0, \mathbf{t}_0)\} \cup \{(\mathbf{f}_0, \top_0), (\mathbf{t}_0, \top_0)\} \cup \\ (\{\bot, \mathbf{f}_2, \mathbf{t}_2, \top_2, \mathbf{f}_1, \mathbf{t}_1, \top_1\} \times \{\mathbf{f}_0, \mathbf{t}_0, \top_0\}) \\ \cup \{\bot, \mathbf{f}_2, \mathbf{t}_2, \top_2, \mathbf{f}_1, \mathbf{t}_1, \top_1\}^2\end{aligned}$$

$\mathbf{S}_7 = K_2^2$

We note that $\mathbf{S}_3 = \breve{\mathbf{S}}_3$ and $\mathbf{S}_5 = \breve{\mathbf{S}}_5$ and so $|\mathbb{S}(\mathbf{K}_2)^2| = 10$.

Globally minimal failsets

$GMF_1(\mathbf{K}_2) = \{\mathbf{S}_2\}$

$GMF_2(\mathbf{K}_2) = \{\mathbf{S}_4\}$

$GMF_3(\mathbf{K}_2) = \{\mathbf{S}_6\}$

B.3 The bilattice $\mathbf{J}_1$

Subalgebras of $\mathbf{J}_1^2$

$\mathbf{S}_1 = \Delta(J_1^2)$

$\mathbf{S}_2 = \Delta(J_1^2) \cup \{(\mathbf{f}_1, \mathbf{f}_0), (\mathbf{t}_1, \mathbf{t}_0)\}$

$\mathbf{S}_3 = \Delta(J_1^2) \cup \{(\mathbf{f}_1, \mathbf{f}_0), (\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_1, \mathbf{t}_0), (\mathbf{t}_0, \mathbf{t}_1)\}$

$\mathbf{S}_4 = \Delta(J_1^2) \cup \{(\mathbf{f}_1, \mathbf{f}_0), (\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_1, \mathbf{t}_0), (\mathbf{t}_0, \mathbf{t}_1)\} \cup$
$\{(\bot, \mathbf{f}_1), (\bot, \mathbf{f}_0), (\bot, \mathbf{t}_1), (\bot, \mathbf{t}_0), (\bot, \top)\} \cup \{(\mathbf{f}_1, \top), (\mathbf{f}_0, \top), (\mathbf{t}_1, \top), (\mathbf{t}_0, \top)\}$

$\mathbf{S}_5 = J_1^2$

We note that $\mathbf{S}_3 = \breve{\mathbf{S}}_3$ and so $|\mathbb{S}(\mathbf{J}_1^2)| = 7$.

Globally minimal failsets

$GMF_1(\mathbf{J}_1) = \{\mathbf{S}_2\}$

$GMF_2(\mathbf{J}_1) = \{\mathbf{S}_4\}$

B.4 The bilattice $\mathbf{J}_2$

Subalgebras of $\mathbf{J}_2^2$

$\mathbf{S}_1 = \Delta(J_2^2)$

$\mathbf{S}_2 = \Delta(J_2^2) \cup \{(\mathbf{t}_2, \mathbf{t}_1), (\mathbf{f}_2, \mathbf{t}_1)\}$

$\mathbf{S}_3 = \Delta(J_2^2) \cup \{(\mathbf{t}_1, \mathbf{t}_0), (\mathbf{f}_1, \mathbf{f}_0)\}$

$\mathbf{S}_4 = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_1, \mathbf{f}_2), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_1, \mathbf{t}_2)\}$

$\mathbf{S}_5 = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_0, \mathbf{t}_1)\}$

$\mathbf{S}_6 = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_1, \mathbf{t}_0)\}$

$\mathbf{S}_7 = \Delta(J_2^2) \cup \{(\mathbf{f}_1, \mathbf{f}_0), (\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_1, \mathbf{t}_0), (\mathbf{t}_0, \mathbf{t}_1)\}$

$\mathbf{S}_8 = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_1, \mathbf{f}_2), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_1, \mathbf{t}_2), (\mathbf{t}_1, \mathbf{t}_0)\}$

$\mathbf{S}_9 = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_1, \mathbf{t}_0), (\mathbf{t}_0, \mathbf{t}_1)\}$

$\mathbf{S}_{10} = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_2, \mathbf{f}_0), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_2, \mathbf{t}_0), (\mathbf{t}_1, \mathbf{t}_0)\}$

$\mathbf{S}_{11} = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_1, \mathbf{f}_2), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_1, \mathbf{t}_2), (\mathbf{t}_1, \mathbf{t}_0), (\mathbf{t}_0, \mathbf{t}_1)\}$

$\mathbf{S}_{12} = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_1, \mathbf{f}_2), (\mathbf{f}_2, \mathbf{f}_0), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_1, \mathbf{t}_2), (\mathbf{t}_2, \mathbf{t}_0), (\mathbf{t}_1, \mathbf{t}_0)\}$

$\mathbf{S}_{13} = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_2, \mathbf{f}_0), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{f}_0, \mathbf{f}_1), (\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_2, \mathbf{t}_0), (\mathbf{t}_1, \mathbf{t}_0), (\mathbf{t}_0, \mathbf{t}_1)\}$

$$\mathbf{S}_{14} = \Delta(J_2^2) \cup \{(\mathbf{f}_2, \mathbf{f}_1), (\mathbf{f}_1, \mathbf{f}_2), (\mathbf{f}_2, \mathbf{f}_0), (\mathbf{f}_1, \mathbf{f}_0), (\mathbf{f}_0, \mathbf{f}_1)\} \cup \\ \{(\mathbf{t}_2, \mathbf{t}_1), (\mathbf{t}_1, \mathbf{t}_2), (\mathbf{t}_2, \mathbf{t}_0), (\mathbf{t}_1, \mathbf{t}_0), (\mathbf{t}_0, \mathbf{t}_1)\}$$

$\mathbf{S}_{15} = \{(\bot, \bot), (\top, \top)\} \cup \{\mathbf{f}_2, \mathbf{f}_1, \mathbf{f}_0\}^2 \cup \{\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_0\}^2$

$\mathbf{S}_{16} = \big(\{\bot\} \times J_2\big) \cup \{\mathbf{f}_2, \mathbf{f}_1, \mathbf{f}_0\}^2 \cup \{\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_0\}^2 \cup \big(J_2 \backslash \{\bot\} \times \{\top\}\big)$

$\mathbf{S}_{17} = J_2^2$

For each $\mathbf{S} \in \{\mathbf{S}_1, \mathbf{S}_4, \mathbf{S}_7, \mathbf{S}_{11}, \mathbf{S}_{15}, \mathbf{S}_{17}\}$ we have $\mathbf{S} = \breve{\mathbf{S}}$ and so $|\mathbb{S}(\mathbf{J}_2^2)| = 28$.

Globally minimal failsets

$GMF_1(\mathbf{J}_2) = \{\mathbf{S}_2, \mathbf{S}_5, \mathbf{S}_6, \mathbf{S}_9, \mathbf{S}_{10}, \mathbf{S}_{13}\}$

$GMF_2(\mathbf{J}_2) = \{\mathbf{S}_3, \mathbf{S}_5, \mathbf{S}_6, \mathbf{S}_8, \mathbf{S}_{10}, \mathbf{S}_{12}\}$

$GMF_3(\mathbf{J}_2) = \{\mathbf{S}_4, \mathbf{S}_8, \mathbf{S}_{12}\}$

$GMF_4(\mathbf{J}_2) = \{\mathbf{S}_7, \mathbf{S}_9, \mathbf{S}_{13}\}$

$GMF_5(\mathbf{J}_2) = \{\mathbf{S}_{16}\}$

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