Estimates for Small Eigenvalues of the Laplacian and Conformal Laplacian on Closed Manifolds



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Statement of Originality

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at another university. The work in Chapter 2 is the product of a collaboration with my supervisor Melanie Rupflin.

Abstract

A central and well-established theme in geometry is eigenvalue estimates for geometric operators on manifolds. In this thesis we obtain new estimates for small eigenvalues of the Laplacian and conformal Laplacian respectively, in two distinct geometric contexts.

In the first part of this thesis we consider eigenvalues of the Laplacian on closed hyperbolic surfaces. It is known that for such surfaces degenerating by the collapse of a single simple closed geodesic, the first eigenvalue to highest order depends only on the length of that geodesic and the topology of the limiting surface. We consider the behaviour of the eigenvalues in the more general case of surfaces degenerating by the collapse of multiple simple closed geodesics. We exploit the fact that the geometry of a closed hyperbolic surface of fixed genus depends on a finite number of parameters (up to pullback by diffeomorphisms), the Fenchel-Nielsen coordinates. We prove new local energy estimates for the eigenfunctions of small eigenvalues and use these to obtain estimates for the derivatives of the eigenvalues with respect to the Fenchel-Nielsen coordinates. These are then used to prove that in the case of multiple collapsing geodesics, the small eigenvalues of the Laplacian again to first order depend only on the lengths of the collapsing disconnecting geodesics and the topology.

In the second part of the thesis, we consider the behaviour of the first eigenvalue of the conformal Laplacian for particular families of metrics on closed manifolds of dimension at least three. It is known that the volume-normalised first positive eigenvalue of the conformal Laplacian is not bounded on any conformal class of metrics, contrasting with the behaviour of the Laplacian eigenvalues. We initiate an exploration of this difference in behaviour by proving sharp estimates for the first eigenvalue of the conformal Laplacian for particular families of conformal metrics, which we call asymptotically conical. We also obtain sharp estimates for the behaviour of the volume- and diameter-normalised eigenvalues for these metrics, giving new examples of conformal families of metrics for which the volume-normalised eigenvalues are unbounded.

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Chapter 1

Introduction

Spectral geometry is the study of the relationship between the spectra of differential operators on Riemannian manifolds and geometric properties of the manifold. This field, and in particular the study of geometric bounds for eigenvalues of the Laplace-Beltrami operator $-\Delta_g$, has a long history, and many techniques have been developed to explore the behaviour of eigenvalues in different geometric contexts.

In this thesis, we will study the small eigenvalues of the Laplacian $-\Delta_g$ on hyperbolic surfaces, as well as the first eigenvalue of the conformal Laplacian $-L_g := -\Delta_g + \frac{n-2}{4(n-1)}R_g$ for particular families of conformal metrics on closed manifolds of dimension $n \geq 3$. We make use of the variational characterisation of the eigenvalue problem to obtain new results on the behaviour of eigenvalues in each of these settings.

This chapter will provide an introduction to the research, first discussing the background of the problems and placing them into context, then describing our main results.

1.1 Basic Properties of the Laplacian Spectrum

In this section, we give an overview of some elementary results regarding the spectra of the operators we will study, and fix some notation. References for these basic results on the spectrum of the Laplacian on manifolds are the books of Chavel [13], Lablée [44], and Craioveanu, Puta, and Rassias [20].

Let (M, g) be a compact, connected, oriented Riemannian manifold without boundary. We denote by $-\Delta_g$ the Laplace-Beltrami operator acting on functions (hereafter referred to simply as the Laplacian). An important geometric invariant of the manifold is the spectrum of the Laplacian. Since M is compact, the spectrum consists of an unbounded, increasing sequence of non-negative real numbers

$$0 = \lambda_0(-\Delta_g) < \lambda_1(-\Delta_g) \leq \lambda_2(-\Delta_g) \leq \cdots \leq \lambda_k(-\Delta_g) \to \infty.$$

It is clear that any nonzero constant function on M is an eigenfunction of $-\Delta_g$ with eigenvalue 0. The first eigenvalue is simple and can be characterised by the variational property

$$0 = \lambda_0(-\Delta_g) = \inf \left\{ \frac{\int_M |\nabla_g u|_g^2 \, dV_g}{\int_M |u|^2 \, dV_g} : u \in C^\infty(M) \setminus \{0\} \right\}.$$

That is to say, $\lambda_0(-\Delta_g)$ is the infimum of the Rayleigh quotient over the space of non-zero smooth functions. Given the first k eigenvalues (counted with multiplicity) $\lambda_0(-\Delta_g) \leq \cdots \leq \lambda_{k-1}(-\Delta_g)$ and corresponding eigenfunctions $\varphi_0, \ldots, \varphi_{k-1}$, we can characterise $\lambda_k(-\Delta_g)$ as the infimum of the Rayleigh quotient over all non-zero smooth functions that are orthogonal in $L^2(M, g)$ to the span of $\{\varphi_i\}_{i=0}^{k-1}$, with the infimum being achieved if and only if the test function is an eigenfunction φ_k with

eigenvalue $\lambda_k(-\Delta_g)$.

In this thesis we will also consider eigenvalues of the conformal Laplacian, defined as

$$-L_g \coloneqq -\Delta_g + \frac{n-2}{4(n-1)} R_g,$$

where R_g denotes the scalar curvature of the metric g. In particular we observe that when $n = 2$ the operator $-L_g$ is just the Laplace-Beltrami operator. We note that the conformal Laplacian has been studied in many geometric contexts, as we briefly touch on in section 3.1; an in-depth discussion of these applications is beyond the scope of this thesis. The operator $-L_g$ enjoys a conformal covariance property; if a pair of Riemannian metrics are conformally related by $\tilde{g} = f^2 g$ for $f > 0$ a smooth function, then the corresponding conformal Laplacians are related by the transformation law:

$$-L_{\tilde{g}}(\cdot) = f^{-\frac{n+2}{2}}(-L_g(f^{\frac{n-2}{2}}\cdot)). \tag{1.1.1}$$

The spectrum of $-L_g$ on a compact manifold (M, g) again consists of an unbounded, increasing sequence of real numbers

$$\nu_1(-L_g) < \nu_2(-L_g) \leq \nu_3(-L_g) \leq \cdots \leq \nu_k(-L_g) \to \infty.$$

We remark that we follow convention in labelling the eigenvalues of $-L_g$ starting with index 1, while we label the eigenvalues of $-\Delta_g$ starting with index 0. This is due to the fact that $\lambda_0(-\Delta_g) = 0$ is trivial and carries no information about (M, g), so in both cases the first interesting eigenvalue is the one with index 1. Unlike the spectrum of $-\Delta_g$, it is not true in general that the spectrum of $-L_g$ is non-negative. However, we note that $\nu_1(-L_g)$ will be positive, negative, or zero

if and only if g admits a conformal metric $\tilde{g}$ such that the scalar curvature $R_{\tilde{g}}$ is pointwise positive, negative, or zero respectively. In fact, one can easily check using (1.1.1) that if $u > 0$ is a first eigenfunction for $-L_g$ (we can always take $u > 0$ since the first eigenfunction has constant sign), then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ will have scalar curvature $R_{\tilde{g}} = \frac{4(n-1)}{n-2} \nu_1(-L_g) u^{-\frac{4}{n-2}}$, which has the same sign as $\nu_1(-L_g)$ everywhere. The kth eigenvalue $\nu_k(-L_g)$ can be obtained by an iterative process as with the eigenvalues of $-\Delta_g$. The first eigenvalue is simple and is characterised as the infimum of the Rayleigh quotient

$$\frac{\int_M |\nabla_g u|_g^2 + \frac{n-2}{4(n-1)} R_g |u|^2 \, dV_g}{\int_M |u|^2 \, dV_g}$$

over all non-zero smooth functions u on M. Then given $\nu_1(-L_g), \ldots, \nu_{k-1}(-L_g)$ and corresponding eigenfunctions, we can obtain $\nu_k(-L_g)$ as the infimum of the Rayleigh quotient over all non-zero smooth functions on M that are orthogonal in $L^2(M, g)$ to the span of the previously obtained eigenfunctions.

1.2 Background and Description of Results

For an arbitrary closed manifold (M, g), obtaining a complete understanding of the eigenvalues and eigenfunctions of $-\Delta_g$ or of $-L_g$ is a difficult problem, since they can only be computed explicity in a small number of cases (e.g. for very simple domains such as rectangles, spheres, etc.). Research in spectral geometry has therefore been primarily concerned with the following two broad questions: Given a manifold (M, g), what can be said about its (conformal) Laplacian eigenvalues? And conversely, given all or some of the eigenvalues, what can be deduced about

the geometry of the manifold (or the conformal class of g)?

One of the earliest results along these lines is Weyl's asymptotic formula, the earliest version of which (for Euclidean domains of dimension $n = 2$ or 3) dates back to 1911 [66]:

$$\lim_{k\to\infty} \frac{\lambda_k(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}}\tilde{\omega}^{\frac{2}{n}}}{4\pi^2 k^{\frac{2}{n}}} = 1,$$

where $\tilde{\omega}$ is the volume of the Euclidean unit ball of dimension $n = \dim(M)$. Already from this result we see that the asymptotic behaviour of the Laplacian spectrum yields geometric information about (M, g), in particular its volume.

Weyl's formula concerns the asymptotic behaviour of the sequence $\lambda_k(-\Delta_g)$ as $k \to \infty$. This is just one of many aspects of the spectrum which have been studied by various authors in the course of the subject's long history. A summary of works done broadly in this area goes beyond the scope of this thesis. In this thesis, we restrict our focus to geometric bounds on small eigenvalues of $-\Delta_g$ and $-L_g$.

One foundational example of such a result is due to Cheeger [15], who in 1970 proved the following lower bound for $\lambda_1(-\Delta_g)$. Given $S \subset M$ a submanifold such that $M \setminus S$ has two connected components M_1 and M_2, write $A(S)$ for the $(n-1)$-dimensional area of S. Now define a constant $h = h(M, g)$ by

$$h = \inf \frac{A(S)}{\min_i(\mathrm{Vol}(M_i, g))},$$

where the infimum is taken over all submanifolds S as above. Then Cheeger's lower bound is

$$\lambda_1(-\Delta_g) \geq \frac{1}{4}h^2.$$

It was subsequently shown by Buser [11] in 1982 (see also Ledoux [45] for a later,

alternative proof of this estimate) that $\lambda_1(-\Delta_g)$ can also be bounded from above in terms of Cheeger's isoperimetric constant h:

$$\lambda_1(-\Delta_g) \leq 2\sqrt{k}(n-1)^2 h + 10h^2,$$

where $-(n-1)k$ is a lower bound for the Ricci curvature and $k \geq 0$.

We note that both of the bounds above apply to arbitrary closed manifolds (M, g). The addition of further geometric assumptions, such as constraints on the curvature, allows for more precise statements about the behaviour of eigenvalues to be made in specific contexts. For example, Cheng [16] showed that if the Ricci curvature is bounded below by $-(n-1)k$ (where we now make no assumption on the sign of k), and if $V(k, r)$ denotes the geodesic ball of radius r in the space form of curvature k, then

$$\lambda_1(M) \leq \lambda_1\left(V\left(-k, \frac{d}{2}\right)\right),$$

where $d = \operatorname{diam}(M, g)$. By estimating the first eigenvalue of geodesic balls in space forms, Cheng was thus able to show:

$$\text{If } \operatorname{Ric}_g(M) \geq 0, \text{ then } \lambda_1(-\Delta_g) \leq \frac{2n(n+4)}{d^2},$$
$$\text{If } \operatorname{Ric}_g(M) \geq n-1, \text{ then } \lambda_1(-\Delta_g) \leq \frac{n\pi^2}{d^2},$$
$$\text{If } \operatorname{Ric}_g(M) \geq -(n-1)k \text{ for } k > 0, \text{ then } \lambda_1(-\Delta_g) \leq \frac{(n-1)^2}{4}k + \frac{C_n}{d^2}.$$

Lower bounds for $\lambda_1(-\Delta_g)$ with constraints on the Ricci curvature are also known,

for example the following estimates of Li and Yau [47]:

$$\text{If } \mathrm{Ric}_g(M) \geq 0, \text{ then } \lambda_1(-\Delta_g) \geq \frac{\pi^2}{2d^2},$$

$$\text{If } \mathrm{Ric}_g(M) \geq -(n-1)k \text{ for } k \geq 0, \text{ then } \lambda_1(-\Delta_g) \geq \frac{\exp\left\{1-(1+4(n-1)^2d^2k)^{\frac{1}{2}}\right\}}{(n-1)d^2}.$$

In the above results, the geometric assumptions take the form of bounds on the Ricci curvature, and given such hypotheses we see that the eigenvalue $\lambda_1(-\Delta_g)$ can be controlled in terms of the curvature as well as the dimension and diameter of (M, g). It is natural to consider whether other geometric quantities such as dimension, volume, diameter, and/or curvature can be used to control the small eigenvalues of $-\Delta_g$ and $-L_g$. One motivation for this choice of geometric quantities with which to seek to control eigenvalues is Cheeger's Finiteness Theorem [14, Theorem 2.1], which states that given a positive integer n, an upper bound on the diameter, a lower bound on the volume, and upper and lower bounds on the sectional curvature, there exist only finitely many diffeomorphism classes of compact n-dimensional Riemannian manifolds satisfying those bounds. Thus, geometric control in terms of these quantities leads to topological restrictions. With this in mind, a problem of interest is the following: Given a closed Riemannian manifold (M, g), what can be said about the relationship between small eigenvalues of the (conformal) Laplacian and the dimension, volume, diameter, and/or curvature of g (or of metrics in the conformal class $[g]$)? The results in this thesis contribute towards the exploration of this problem.

Our first result in this thesis concerns the behaviour of small eigenvalues of $-\Delta_g$

on closed hyperbolic surfaces, that is two-dimensional manifolds with metrics of constant negative Gauss curvature $K_g \equiv -1$. Given such a surface of genus $\gamma \geq 2$, the metric g essentially depends (up to pullback by diffeomorphisms) on a finite number of real parameters, the so-called Fenchel-Nielsen coordinates $\{\ell_j, \psi_j\}_{j=1}^{3(\gamma-1)}$, which we describe in detail in subsection 2.2.3. The length coordinates $\{\ell_j\}_{j=1}^{3(\gamma-1)}$ are the lengths of certain simple closed geodesics in (M, g), and we remark that by results of Schoen, Wolpert and Yau [61] it is known that if these lengths are constrained to lie in a compact subset of $(0, \infty)$, then the first $2\gamma - 3$ eigenvalues of $-\Delta_g$ will also lie in a compact set. It is therefore of interest to study instead the case of surfaces degenerating by the collapse of simple closed geodesics, that is the behaviour of eigenvalues as $\ell_j \to 0$ for some indices j. This forms the first main part of this thesis, and the corresponding results will all be stated and proved in Chapter 2, which consists of joint work with my supervisor Melanie Rupflin. We are able to exploit the fact that the geometry depends on the finitely many parameters given by the Fenchel-Nielsen coordinates to apply estimates for the energy of eigenfunctions in order to prove the following bounds on the derivatives of small eigenvalues of $-\Delta_g$ with respect to these coordinates. (See Chapter 2 for preliminary definitions and Theorem 2.1.1 for a precise statement.)

Theorem. *Let (M, g) be a closed oriented hyperbolic surface of genus $\gamma \geq 2$ and let $\mathcal{E} = \{\sigma^1, \dots, \sigma^{3(\gamma-1)}\}$ be a collection of simple closed geodesics giving a pair of pants decomposition. Then there exists $\bar{\lambda} > 0$ and $C > 0$ such that any eigenvalue $\lambda \leq \bar{\lambda}$ of $-\Delta_g$ has the following dependence on the Fenchel-Nielsen coordinates ℓ_j and ψ_j:*

The derivatives with respect to the length coordinates ℓ_j satisfy

$$\left|\frac{\partial \lambda}{\partial \ell_j}\right| \leq C\lambda^2 \ell_j^{-2}$$

while the derivatives with respect to the twist coordinates satisfy

$$\left|\frac{\partial \lambda}{\partial \psi_j}\right| \leq C\lambda^2 \ell_j^2.$$

Furthermore, there exists a positive constant $\bar{c} = \bar{c}(\gamma)$ depending only on the genus such that for any $\alpha \in (0,1)$, any $0 < \bar{\delta} < \min\{\frac{\bar{c}}{2}, \operatorname{arsinh}(1)\}$, and any geodesic σ^j for which $\ell_j \leq 2\bar{\delta}$ but for which both ends of the corresponding collar $\mathcal{C}(\sigma^j)$ lie in the same connected component of $\bar{\delta}$-thick(M,g), we have the improved estimate

$$\left|\frac{\partial \lambda}{\partial \ell_j}\right| \leq C_\alpha \lambda^2 \bar{\delta}^{\alpha-2} \ell_j^{-\alpha}. \tag{1.2.1}$$

where C_α is now allowed to depend on α (but not on $\bar{\delta}$).

It is known by a result of Schoen, Wolpert and Yau [61] that if (M,g) is a closed hyperbolic surface for which a collection of simple closed geodesics decomposing M into $n+1$ connected components are all short compared to all other simple closed geodesics, then the first n eigenvalues of $-\Delta_g$ will be comparable to the sum of the lengths of these geodesics. We will use the C^1 estimates above to prove the following corollary, which says that in this situation the small eigenvalues essentially depend only on the lengths of the collapsing geodesics (and the choice of Fenchel-Nielsen coordinates), up to an error which is at least quadratic in the sum of the lengths of the geodesics. (See Corollary 2.1.3 for the precise statement).

Corollary. *In the situation described above, the first n eigenvalues of $-\Delta_g$ are essentially determined by the lengths $\ell_1, \dots, \ell_{j_0}$ of the collapsing decomposing geodesics in the sense that there exist functions $f_k : (0, 2\bar{\delta})^{j_0} \to \mathbb{R}^+$, $k = 1, \dots, n$ and a constant $C > 0$ such that the following holds true.*

$$|\lambda_k(M, g) - f_k(\ell_1, \dots, \ell_{j_0})| \leq C \left(\sum_{j=1}^{j_0} \ell_j \right)^2, \quad k = 1, \dots, n.$$

In particular, for hyperbolic metrics $g, \tilde{g}$ for which the lengths of the corresponding geodesics σ^j and $\tilde{\sigma}^j$ agree for $1 \leq j \leq j_0$, we have that

$$|\lambda_k(M, g) - \lambda_k(M, \tilde{g})| \leq C \left(\sum_{j=1}^{j_0} \ell_j \right)^2.$$

In proving this result, one has to overcome the difficulty that there are multiple collapsing geodesics, and the eigenvalue in principle depends on all of their lengths. However, not all of the collapsing geodesics play a significant role in the behaviour of the eigenvalue. In particular there may be geodesics which are short but which do not decompose the manifold into multiple connected components, and the lengths of these geodesics do not have a significant effect on the eigenvalue. We are able to control the dependence of the eigenvalues on the lengths of such geodesics using the estimate (1.2.1). This issue only arises when considering multiple collapsing geodesics, and so for example is not present in the earlier work of Große and Rupflin [28], who consider the behaviour of $\lambda_1(-\Delta_g)$ in the case of a single collapsing geodesic.

For our second main result in this thesis, we study the first eigenvalue of $-L_g$ on manifolds (M, g) of dimension $n \geq 3$ with positive scalar curvature. We remark

that fewer estimates for the eigenvalues of $-L_g$ are available than for $-\Delta_g$, and we discuss this difference in the known results at length in section 3.1. The analogues for $-L_g$ of certain well-established statements for $-\Delta_g$ are known to be false. For example, a celebrated result of Korevaar [43] for eigenvalues of $-\Delta_g$ gives the estimate

$$\sup_{g\in[g_0]} \lambda_k(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} \leq C([g_0])k^{\frac{2}{n}} \tag{1.2.2}$$

for a constant $C > 0$ depending only on $[g_0]$. If one simply replaces $\lambda_k(-\Delta_g)$ by $\nu_k(-L_g)$ in the above estimate, the resulting inequality is false, as shown by Amman and Jammes in [2], who studied the behaviour of the first positive eigenvalue of $-L_g$ for a family of asymptotically cylindrical metrics. One can ask whether it is possible to modify the inequality (1.2.2) to obtain a true statement for $\nu_k(-L_g)$. We hope to shed light on this problem by exploring the behaviour of a concrete example, studying a family of conformal metrics which generalises that considered by Amman and Jammes. The result presented in Chapter 3 of this thesis provides a sharp estimate for the first eigenvalue of $-L_g$ for families $\{g_L\}_{L\geq 1}$ of asymptotically conical metrics on M, parametrised by a parameter L, and defined in detail in section 3.2. We study the behaviour of $\nu_1(-L_{g_L})$ as $L \to \infty$, obtaining the following eigenvalue bounds by analysing the energy identity satisfied by the corresponding eigenfunction. (See Chapter 3 for further discussion and Theorem 3.2.1 for the precise statement).

Theorem. *Fix $\alpha \in (0,2)$ and a closed, connected submanifold $\Sigma \subset M$ of dimension $0 \leq k < \frac{(2-\alpha)(n-2)}{2}$, and let g_∞ be an asymptotically conical metric on $M \setminus \Sigma$ as defined in section 3.2, and with positive scalar curvature R_∞. Let $\{g_L\}_{L\geq 1}$ be a family of metrics on M given by truncations of g_∞ as described in section 3.2.*

Then there exist $L_0 \geq 1$ *and a constant* $C = C(g_\infty) > 0$ *such that for all* $L \geq L_0$ *we have*

$$\frac{1}{C} \leq \nu_{1,L} \leq C, \quad 0 < \alpha < 1$$
$$\frac{1}{C} \leq \nu_{1,L} L^{2(\alpha-1)} \leq C, \quad 1 \leq \alpha < 2.$$

Given this sharp estimate for $\nu_1(-L_g)$ for the family $\{g_L\}_{L \geq L_0}$ of conformal metrics, we then compare the behaviour of the eigenvalue as $L \to \infty$ with that of the diameter and volume of the manifold (M, g_L), obtaining the following estimates for $\alpha \in (1, 2)$, $k = 0$ and $L \geq L_0$ in Corollary 3.2.2 and Corollary 3.2.3:

$$\frac{1}{C} \leq \nu_1(-L_{g_L}) \mathrm{Vol}(M, g_L)^{\frac{2}{n}} \leq C$$
$$\frac{1}{C} \leq \nu_1(-L_{g_L}) \mathrm{diam}(M, g_L)^2 \leq C.$$

We note in particular that the volume estimate above differs from the $\alpha = 1$ (i.e. asymptotically cylindrical) case studied by Amman and Jammes in [2], since when $\alpha = 1$ the quantity $\nu_1(-L_{g_L}) \mathrm{Vol}(M, g_L)^{\frac{2}{n}}$ is unbounded as $L \to \infty$. Thus the example we study exhibits behaviour not seen in the $\alpha = 1$ case. We hope that the asymptotically conical metrics will serve as a guiding example for future work regarding the behaviour of $\nu_1(-L_{g_L})$.

1.3 Outline of the Thesis

In Chapter 1, we have introduced the context of the research contained in this thesis. Some elementary results on the spectra of the operators $-\Delta_g$ and $-L_g$ have been stated, and our main results have been described (though precise statements have been deferred to later chapters, where the relevant preliminary definitions

appear).

In Chapter 2, we will consider small eigenvalues of $-\Delta_g$ on hyperbolic surfaces. We will review some existing results regarding the dependence of these eigenvalues on the lengths of certain simple closed geodesics, and then state and prove the main result of the chapter, Theorem 2.1.1.

In Chapter 3, we will turn our attention to the behaviour of the first eigenvalue of $-L_g$ for asymptotically conical families of metrics on closed manifolds M of dimension $n \geq 3$. We will compare some known results for the eigenvalues of $-\Delta_g$ with those of $-L_g$ and identify some unexplored questions regarding the behaviour of $\nu_1(-L_g)$. We then state and prove Theorem 3.2.1, a first step towards the investigation of these questions.

The thesis concludes with Chapter 4, in which we will discuss the significance and limitations of our results.

Chapter 2

Laplacian Eigenvalue Estimates on Degenerating Hyperbolic Surfaces

In this chapter, we consider the behaviour of small eigenvalues of the Laplacian $-\Delta_g$ on closed oriented surfaces M of genus $\gamma \geq 2$ endowed with a hyperbolic metric g. We will prove new energy estimates for the Laplacian eigenfunctions corresponding to small eigenvalues on these surfaces, and apply them to obtain C^1 bounds for the dependence of the eigenvalues on the Fenchel-Nielsen coordinates in situations where the surface contains closed geodesics of short lengths that decompose the surface into several connected components. We will then use these bounds to prove that in this setting, the small eigenvalues of the Laplacian depend principally on the lengths of the collapsing disconnecting geodesics, up to an error which is at least quadratic in the sum of the lengths of the geodesics. This builds on and generalises the work of Große and Rupflin [28], who considered the behaviour of the first eigenvalue in the case of a single collapsing geodesic. The results in this chapter were obtained jointly with M. Rupflin.

2.1 Motivation and Statement of Results

Let (M, g) be a closed oriented surface of genus $\gamma \geq 2$, endowed with a hyperbolic metric g (that is, a metric of constant Gauss curvature $K_g \equiv -1$). We consider the behaviour of small eigenvalues of the Laplacian $-\Delta_g$ on (M, g) as the lengths of certain simple closed geodesics tend to zero. The study of the asymptotic behaviour of eigenvalues in this situation was initiated by Schoen, Wolpert and Yau in [61]. They proved that on surfaces with bounded negative curvature $-c \leq K_g \leq c' < 0$, as the surface degenerates by the collapse of simple closed geodesics decomposing M into $k+1$ components, the first k eigenvalues tend to zero at a rate which is linear with respect to the sum of the lengths of the corresponding geodesics. More precisely, the kth positive eigenvalue λ_k of $-\Delta_g$ satisfies the following bound for constants $c > 0$ and $C < \infty$ depending only on the genus

$$cL_k \leq \lambda_k \leq CL_k, \tag{2.1.1}$$

where $L_k = \inf \sum_{i \in I} L_g(\sigma^i)$, and where the infimum is taken over all collections $\{\sigma^i\}_{i \in I}$ of simple closed geodesics in (M, g) such that $M \setminus \cup_{i \in I} \sigma^i$ has $k+1$ connected components.

The asymptotics of small eigenvalues on hyperbolic surfaces were further studied by Burger in [8] and [9], who compared the eigenvalues on the surface M with the eigenvalues $\hat{\lambda}_k$ of the Laplacian on an associated weighted graph. Burger showed the following result for any hyperbolic surface (M, g) of genus $\gamma \geq 2$. Suppose there is a collection of simple closed geodesics decomposing M into $k+1$ connected components such that the length ℓ of the longest such geodesic satisfies $\frac{\ell}{\delta} \leq \varepsilon$

with $\hat{\delta}$ a lower bound for the injectivity radius of the surface away from the collars around the decomposing geodesics and $0 < \varepsilon \leq c$ for a constant c depending only on γ. Then

$$\frac{1}{2\pi^2}(1 - C\sqrt{\varepsilon}) \leq \frac{\lambda_j}{\hat{\lambda}_j} \leq \frac{1}{2\pi^2}(1 + C\varepsilon \log \varepsilon). \tag{2.1.2}$$

Since the weighted graph associated to the surface depends only on the surface topology and the lengths of the decomposing geodesics, one observes in particular that the error rates in the estimate (2.1.1) of Schoen-Wolpert-Yau should essentially depend only on these quantities.

A particular consequence of Burger's results in [9] is that in the case where there is a single collapsing disconnecting geodesic σ^1 of length ℓ_1 which is small compared to the lengths of all other simple closed geodesics, the first eigenvalue satisfies

$$C_{\text{top}} - C\sqrt{\ell_1} \leq \frac{\lambda_1}{\ell_1} \leq C_{\text{top}} + C\ell_1|\log(\ell_1)|, \tag{2.1.3}$$

where C_{top} is a topological constant, which depends only on the genera of the connected components of $M \setminus \sigma^1$.

Hence, if we have two metrics g and $\tilde{g}$ on a given surface M such that both surfaces contain a simple closed geodesic σ^1 (respectively $\tilde{\sigma}^1$ of the same length ℓ_1), which disconnects M into two components $M^{\pm}$ (respectively $\tilde{M}^{\pm}$ of the same genera), then

$$\ell_1^{-1}|\lambda_1(M, g) - \lambda_1(M, \tilde{g})| \leq C\sqrt{\ell_1}. \tag{2.1.4}$$

In [28], Große and Rupflin consider this setting, and improve the estimate (2.1.3)

of Burger, which leads in particular to an improvement of estimate (2.1.4),

$$|\lambda_1(M,g) - \lambda_1(M,\tilde{g})| \leq C\ell_1^2. \tag{2.1.5}$$

They prove these results as a consequence of sharp C^1 bounds they obtain for the dependence of the first eigenvalue λ_1 on the so-called Fenchel-Nielsen coordinates and which they show are optimal, see [28, Theorem 1.2] (we describe the Fenchel-Nielsen coordinates in section 2.2.3). The main result we present in this chapter is a generalisation of these C^1 bounds to surfaces with multiple collapsing geodesics. In the statement below and henceforth, $\mathcal{C}(\sigma)$ denotes the collar region around a short simple closed geodesic σ, as described by the well-known collar lemma which we recall in Lemma 2.2.2.

Theorem 2.1.1. *Let (M,g) be a closed oriented hyperbolic surface of genus $\gamma \geq 2$ and let $\mathcal{E} = \{\sigma^1, \ldots, \sigma^{3(\gamma-1)}\}$ be a collection of simple closed geodesics giving a pair of pants decomposition. Let $\eta \in (0, \operatorname{arsinh}(1))$ and $\bar{L} < \infty$ be such that $\mathcal{E}$ contains all simple closed geodesics in (M,g) of length at most 2η, and no geodesic in $\mathcal{E}$ has length greater than $\bar{L}$. Then there exists $\bar{\lambda} > 0$ and $C > 0$ depending only on $\eta, \bar{L}$, and γ such that any eigenvalue $\lambda = \lambda_\iota \leq \bar{\lambda}$ of $-\Delta_g$ has the following dependence on the Fenchel-Nielsen coordinates ℓ_j and ψ_j: For all $1 \leq j \leq 3(\gamma-1)$, the derivatives with respect to the length coordinates ℓ_j satisfy*

$$\left|\frac{\partial \lambda}{\partial \ell_j}\right| \leq C\lambda^2 \ell_j^{-2} \tag{2.1.6}$$

while the derivatives with respect to the twist coordinates satisfy

$$\left|\frac{\partial\lambda}{\partial\psi_j}\right| \leq C\lambda^2\ell_j^2. \tag{2.1.7}$$

Furthermore, there exists a positive constant $\bar{c} = \bar{c}(\gamma)$ *depending only on the genus such that for any* $\alpha \in (0,1)$*, any* $0 < \bar{\delta} < \min\{\frac{\bar{c}}{2}, \operatorname{arsinh}(1)\}$*, and any geodesic* σ^j *for which* $\ell_j \leq 2\bar{\delta}$ *but for which both ends of the corresponding collar* $\mathcal{C}(\sigma^j)$ *lie in the same connected component of* $\bar{\delta}$*-thick*(M,g)*, we have the improved estimate*

$$\left|\frac{\partial\lambda}{\partial\ell_j}\right| \leq C_\alpha\lambda^2\bar{\delta}^{\alpha-2}\ell_j^{-\alpha}. \tag{2.1.8}$$

where C_α *is now allowed to depend on* α *(in addition to* γ, η*, and* $\bar{L}$*), but not on* $\bar{\delta}$*.*

Remark 2.1.2. If $\lambda = \lambda_\iota$ is not simple then in general λ_ι is not a differentiable function of the Fenchel-Nielsen coordinates as different eigenvalue branches may cross, and in this case the estimates (2.1.6)–(2.1.8) need to be understood as bounds on the behaviour of these smooth branches of eigenvalues rather than as estimates on the function $g \mapsto \lambda_\iota(g)$. Indeed, we shall prove this result by showing that the above estimates hold true for any smooth family of metrics g_ε with $g_0 = g$ and associated smooth families $(\lambda_\varepsilon, u_\varepsilon)$ of corresponding eigenvalues and eigenfunctions. To be more precise, suppose g_ε is so that

$$\left.\frac{d\ell_i(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \delta_{ij}, \text{ and } \left.\frac{d\psi_i(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

for some $1 \leq j \leq 3(\gamma - 1)$ and for all i. Then we get

$$\left| \frac{d\lambda_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} \right| \leq C\lambda^2 \ell_j^{-2}.$$

Similarly we have the analogous estimates for the derivatives with respect to the twist coordinates and for the case where both ends of a collar lie in the same component of $\bar{\delta}$-thick(M, g). Thus by this slight abuse of notation we can make sense of the estimates (2.1.6)–(2.1.8) not just for simple eigenvalues but for smooth curves of eigenvalues and so with the above convention for all small eigenvalues of $-\Delta_g$ on hyperbolic surfaces.

In the situation considered by Große and Rupflin in [28], there is only a single collapsing disconnecting geodesic σ^1 of length less than $2\overline{\delta}$, and in particular there are no thin collars $\mathcal{C}(\sigma^j)$ with the property that both of their ends lie in the same component of $\bar{\delta}$-thick(M, g). Our estimates for the derivatives of the eigenvalue with respect to the twist coordinates (2.1.7) and for the derivatives with respect to the non-short geodesics (the second estimate in (2.1.6)) are of the same order as those obtained by Große and Rupflin, while our third estimate (2.1.8) for the derivatives with respect to the lengths of the short, non-disconnecting geodesics has no analogue in the previously considered setting.

We remark that work of Schoen, Wolpert and Yau [61] and of Burger [9] cited above is valid in the setting of degenerating hyperbolic surfaces with multiple short geodesics that we consider. Batchelor [5] has already obtained some estimates for the derivatives of small eigenvalues of $-\Delta_g$ with respect to the lengths of the collapsing geodesics (in our notation, for $\left|\frac{d\lambda}{d\ell}\right|$) in this setting. However, his estimates are only of order $O(\frac{1}{|\log \ell|})$, and he does not consider the derivatives

with respect to twist coordinates, nor with respect to the non-collapsing length coordinates.

As consequence of the above result we obtain C^0 estimates along the lines of (2.1.5) for small eigenvalues in the more general setting, building on the error estimates (2.1.2) obtained by Burger for the case of multiple collapsing geodesics. To state this result we let M be a closed oriented surface of genus $\gamma \geq 2$, and fix a collection $\{\bar{\sigma}^j\}_{j=1}^{j_0}$ of simple closed curves in M, none of which are homotopic to a point and no two of which are homotopic to each other. Suppose further that $M \setminus \cup_{j=1}^{j_0} \bar{\sigma}^j$ has $n+1 \geq 2$ connected components and that for any $1 \leq i \leq j_0$, $M \setminus \cup_{\substack{j=1 \\ j \neq i}}^{j_0} \bar{\sigma}^j$ has n connected components. The result below tells us that for any hyperbolic metric g for which the curves in the collection $\{\bar{\sigma}^j\}_{j=1}^{j_0}$ are homotopic to a collection of simple closed geodesics which are "short" (compared to other disconnecting simple closed geodesics), the first n eigenvalues of the Laplacian essentially depend only on the lengths of these geodesics. More precisely, we have

Corollary 2.1.3. *Let $\{\bar{\sigma}^j\}_{j=1}^{j_0}$ be a collection of simple closed curves in M as described above and let $\bar{\delta} \in (0, \min\{\frac{\bar{c}}{2}, \operatorname{arsinh}(1)\})$ be fixed, where $\bar{c} = \bar{c}(\gamma) > 0$ is a constant depending only on the genus of M. Then there exist functions $f_k : (0, 2\bar{\delta})^{j_0} \to \mathbb{R}^+$, $k = 1, \ldots, n$ and a constant $C > 0$ such that the following holds true. Let g be any hyperbolic metric on M which has the following two properties:*

1. *The lengths $\ell_j = L_g(\sigma^j)$ of the geodesics homotopic to $\bar{\sigma}^j$ are so that $\ell_j \in (0, 2\bar{\delta})$ for $j = 1, \ldots, j_0$.*

2. *Any other simple closed goedesic σ in (M, g) which has length $L_g(\sigma) < 2\bar{\delta}$ is so that both ends of the collar $\mathcal{C}(\sigma)$ lie in the same connected component*

of $\bar{\delta}$-thick(M, g).

Then the first n eigenvalues of $-\Delta_g$ are essentially determined by the lengths $\ell_1, \ldots, \ell_{j_0}$ in the sense that

$$|\lambda_k(M, g) - f_k(\ell_1, \ldots, \ell_{j_0})| \leq C \left(\sum_{j=1}^{j_0} \ell_j \right)^2, \quad k = 1, \ldots, n.$$

In particular, for metrics $g, \tilde{g}$ satisfying the assumptions above and for which the lengths of the corresponding geodesics σ^j and $\tilde{\sigma}^j$ agree for $1 \leq j \leq j_0$, we have that

$$|\lambda_k(M, g) - \lambda_k(M, \tilde{g})| \leq C \left(\sum_{j=1}^{j_0} \ell_j \right)^2.$$

Remark 2.1.4. As $M \setminus \cup_{j=1}^{j_0}$ has $n + 1$ connected components, we can equivalently replace condition 2 by the assumption that the set of all geodesics of length less than $2\bar{\delta}$ decomposes M into $n + 1$ connected components.

The remainder of this chapter is structured as follows: We begin in section 2.2 by stating a number of well-known results on the geometry of hyperbolic surfaces and on quadratic differentials which will be of use to us in proving our main results. In section 2.3 we state and prove some key energy estimates for eigenfunctions of the Laplacian to small eigenvalues. Lemma 2.3.1 and Proposition 2.3.2 are the main original results in this section — the former is a generalisation of a result in [28] whose proof follows similar lines, while the latter is entirely new, there being no analogous result in the setting considered in [28]. These results are then used in section 2.4 to prove our main result Theorem 2.1.1, and we conclude this chapter by proving Corollary 2.1.3 in section 2.4.

2.2 Preliminaries

In this section we recall some well-known results on the geometry of hyperbolic surfaces and on holomorphic quadratic differentials. References for these results are Buser [12] and Hummel [33].

The study of small eigenvalues of $-\Delta_g$ on hyperbolic surfaces has a rich history, see for example the books of Buser [12] and Bergeron [6] and the references therein for an overview. You may refer to the survey paper by Ballmann, Matthiesen, and Mondal [4] for an overview of some recent progress. There is in particular a well-studied relationship between the spectra of hyperbolic surfaces and aspects of number theory, in particular Selberg's eigenvalue conjecture, which is beyond the scope of this thesis; we direct the reader interested in this connection to the expository article of Sarnak [60].

2.2.1 Small Eigenvalues of the Laplacian on Hyperbolic Surfaces

In this subsection we collect some basic facts about the eigenvalues of the Laplacian on a hyperbolic surface, with a particular focus on the small eigenvalues which are the subject of our main results.

Recall that for any compact, connected, oriented Riemannian manifold (M, g) without boundary, the spectrum of the Laplacian consists of an unbounded, increasing sequence of nonnegative real numbers

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \lambda_k \to \infty \text{ as } k \to \infty.$$

It is clear that any nonzero constant function on M is an eigenfunction with eigenvalue 0. The first positive eigenvalue is characterised by the variational property

$$\lambda_1 = \inf\left\{ \frac{\int_M \|\nabla f\|^2}{\int_M f^2} : f \in H^1(M,g) \setminus \{0\}, \int_M f = 0 \right\}.$$

That is to say, λ_1 is the infimum of the Rayleigh quotient over the space of non-zero H^1 functions which are L^2-orthogonal to the constant functions. More generally, λ_k is the infimum of the Rayleigh quotient over the space of non-zero H^1 functions on M which are L^2-orthogonal to the span of the eigenfunctions corresponding to $\lambda_0, \ldots, \lambda_{k-1}$.

When (M,g) is a closed hyperbolic surface, the small eigenvalues of the Laplacian are of special interest, and in particular the distribution of eigenvalues in the interval $(0, \frac{1}{4}]$ is of importance in various geometric and number theoretic results. We note the following facts regarding the small eigenvalues, due to Buser [10], [12].

Theorem 2.2.1 (Buser)**.** *Let $\mathcal{R}_\gamma$ denote the set of all closed hyperbolic surfaces of genus $\gamma \geq 2$. For any $\delta > 0$ there exists $(M', g') \in \mathcal{R}_\gamma$ with $\lambda_{2\gamma-3}(M', g') \leq \delta$. On the other hand, there exists a universal constant $c > 0$ which is independent of γ and such that $\lambda_{2\gamma-2}(M,g) \geq c$ for any $(M,g) \in \mathcal{R}_\gamma$ and any $\gamma \geq 2$.*

We remark that the constant c is known to satisfy $c > 10^{-12}$ but it is conjectured that $c = \frac{1}{4}$.

2.2.2 The Collar Lemma

A fact we will use repeatedly throughout this work is that every small simple closed geodesic in a hyperbolic surface has a so-called "collar neighbourhood" which is topologically a cylinder and has a coordinate system for which we have an explicit

expression for the hyperbolic metric. This result is the well known Collar Lemma, due to Keen [40] and Randol [59].

Lemma 2.2.2. *(Keen-Randol, [59]) Let (M, g) be a closed oriented hyperbolic surface and let σ be a simple closed geodesic of length $\ell \leq 2\text{arsinh}(1)$. Then there is a neighbourhood $\mathcal{C}(\sigma)$ around σ, which we call a* collar*, and which is isometric to the cylinder $(-X(\ell), X(\ell)) \times S^1$ equipped with the metric $\rho^2(s)(ds^2 + d\theta^2)$ where*

$$\rho(s) = \rho_\ell(s) = \frac{\ell}{2\pi \cos(\frac{\ell s}{2\pi})} \quad \textit{and} \quad X(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\left(\frac{\ell}{2}\right)\right)\right).$$

We will always use these coordinates (s, θ) when working on collars. We also recall that on such a collar

$$\rho(s) \leq \text{inj}_g(s, \theta) \leq \pi\rho(s) \text{ and } \rho(s + \Lambda) \leq \rho(s)e^\Lambda \text{ for all } (s, \theta) \in \mathcal{C}(\sigma), \Lambda > 0.$$

and that

$$\sinh(\text{inj}_g(s, \theta)) \cdot \cos(\frac{\ell s}{2\pi}) = \sinh(\frac{\ell}{2}) \tag{2.2.1}$$

which yields in particular that $|\partial_s \text{inj}(s, \theta)| \leq \text{inj}(s, \theta)^2$.

We also recall here the following well-known fact: suppose $\mathcal{C}(\sigma)$ is a collar around a simple closed geodesic σ of length $\ell \in (0, 2\text{arsinh}(1)]$. Then $\delta-\text{thin}(\mathcal{C}(\sigma)) \coloneqq \{p \in \mathcal{C}(\sigma) : \text{inj}_g(p) < \delta\}$ is given in collar coordinates by

$$(-X_\delta(\ell), X_\delta(\ell)) \times S^1 \subset \mathcal{C}(\sigma), \text{ where } X_\delta(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arcsin\left(\frac{\sinh(\frac{\ell}{2})}{\sinh \delta}\right)\right) \tag{2.2.2}$$

for $\delta \geq \frac{\ell}{2}$, while $X_\delta(\ell) = 0$ for $\delta < \frac{\ell}{2}$.

We note that as a consequence of the collar lemma, hyperbolic surfaces admit a

natural *thick-thin* decomposition, which we make use of extensively. For any subset $F \subset M$ and $\delta \in (0, \text{arsinh}(1)]$ we define

$$\delta\text{−thin}(F, g) \coloneqq \{p \in F : \text{inj}_g(p) < \delta\} \text{ and } \delta\text{−thick}(F, g) \coloneqq \{p \in F : \text{inj}_g(p) \geq \delta\}.$$

We note that δ−thin(M, g) is contained in collar regions around simple closed geodesics as described above, and that the δ−thin part of any collar $\mathcal{C}(\sigma)$ furthermore satisfies

$$\text{Area}_g(\delta\text{−thin}(\mathcal{C}(\sigma))) \leq C\delta \tag{2.2.3}$$

for a universal constant C.

We also include here a useful form of the Poincaré inequality, which appears as Lemma 4.1 in [28].

Lemma 2.2.3. *Let (M, g) be a closed oriented hyperbolic surface and let M_1^δ denote a connected component of δ-thick(M, g) for some $\delta \in (0, \text{arsinh}(1)]$. Let γ^1 denote a boundary component of M_1^δ and suppose $v \in H^1(M, g)$ is so that $v \equiv 0$ on γ^1. Then we have*

$$\|v\|^2_{L^2(M_1^\delta, g)} + \delta^{-1}\|v\|^2_{L^2(\partial M_1^\delta, g)} \leq C\delta^{-1}\|\nabla v\|^2_{L^2(M_1^\delta, g)}$$

for a constant $C > 0$ depending only on the genus of M.

In this thesis we will need the following version of the trace estimate which is valid for functions whose average value on a boundary curve is zero.

Lemma 2.2.4. *Let (M, g) be a closed oriented hyperbolic surface and let M_1^δ denote a connected component of δ-thick(M, g) for some $\delta \in (0, \frac{1}{4}\text{arsinh}(1)]$. Let γ^1 denote*

a boundary component of M_1^δ. *Suppose* $v \in H^1(M, g)$ *is so that* $⨍_{\gamma^1} v\, dS_g = 0$. *Then the estimate*

$$\|v\|^2_{L^2(\partial M_1^\delta, g)} \leq C\|\nabla v\|^2_{L^2(M_1^\delta, g)}$$

holds for a constant $C > 0$ *depending only on the genus of* M.

The version of the lemma stated above in fact follows by applying Lemma 2.2.3 to a modification $\tilde{v}$ of v. We include a proof here for completeness.

Proof. We first note that the boundary curve γ^1 lies in a collar $\mathcal{C}(\sigma)$ as described by the Collar Lemma 2.2.2, for σ a simple closed geodesic of length $\ell \leq \frac{1}{2}\text{arsinh}(1)$. We can write γ^1 in collar coordinates as $\{X_\delta(\ell)\} \times S^1$, and we note that $\delta \leq \frac{1}{4}\text{arsinh}(1)$ implies $X(\ell) - X_\delta(\ell) \geq 1$.

We now modify v to obtain a function $\tilde{v}$ which satisfies the hypotheses of Lemma 2.2.3. To be more precise, first let $K_\delta(\sigma) := \{(s, \theta) \in \mathcal{C}(\sigma) : s \in [X_\delta(\ell), X_\delta(\ell) + 1]\}$. Now define $\tilde{v}$ by setting $\tilde{v} \equiv v$ on $M_1^\delta \setminus K_\delta(\sigma)$, while for $(s, \theta) \in K_\delta(\sigma)$ we set

$$\tilde{v}(s, \theta) = \varphi(s) ⨍_{\{s\}\times S^1} v(s, \theta)\, d\theta + (1 - \varphi(s))v(s, \theta),$$

where φ is the linear function of s on $[X_\delta(\ell), X_\delta(\ell) + 1]$ satisfying $\varphi(X_\delta(\ell)) = 1$ and $\varphi(X_\delta(\ell) + 1) = 0$. We note that $|\varphi'(s)| \leq 1$.

Note that $\tilde{v}(X_\delta(\ell), \theta) = ⨍_{\gamma^1} v\, dS_g = 0$, so we may apply Lemma 2.2.3 to $\tilde{v}$ to obtain

$$\frac{1}{\delta}\|\tilde{v}\|^2_{L^2(\partial M_1^\delta \setminus \gamma^1, g)} \leq \frac{C}{\delta}\|\nabla \tilde{v}\|^2_{L^2(M_1^\delta, g)}.$$

As $\tilde{v}$ agrees with v on all the boundary components of M_1^δ other than γ^1, we obtain the claimed L^2 bound on v on these boundary components if we check that

$\|\nabla\tilde{v}\|^2_{L^2(M_1^\delta,g)} \leq C\|\nabla v\|^2_{L^2(M_1^\delta,g)}$. As $\tilde{v} = v$ on $M_1^\delta \setminus K_\delta(\sigma)$, this will follow if we can show

$$\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\nabla\tilde{v}|^2\,d\theta\,ds \leq C\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\nabla v|^2\,d\theta\,ds.$$

We have

$$|\partial_s\tilde{v}(s,\theta)| \leq |\varphi'(s)|\left|v(s,\theta) - \fint_{\{s\}\times S^1} v\,dS_g\right| + \varphi(s)|\partial_s v(s,\theta)|.$$

So

$$\begin{aligned}\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\partial_s\tilde{v}|^2\,d\theta\,ds &\leq C\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}\left|v - \fint_{\{s\}\times S^1} v\,dS_g\right|^2 d\theta\,ds \\ &\qquad + C\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\partial_s v|^2\,d\theta\,ds \\ &\leq C\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\partial_\theta v|^2\,d\theta\,ds + C\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\partial_s v|^2\,d\theta\,ds,\end{aligned}$$

where in the second inequality we have applied the usual Poincaré inequality on the circles $\{s\}\times S^1$. For the angular derivative of $\tilde{v}$, we have the estimate

$$\int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\partial_\theta\tilde{v}|^2\,d\theta\,ds = \int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|1-\varphi|^2|\partial_\theta v|^2\,d\theta\,ds \leq \int_{X_\delta(\ell)}^{X_\delta(\ell)+1}\int_{S^1}|\partial_\theta v|^2\,d\theta\,ds.$$

On the other hand, the required bound on γ^1 follows from the usual Poincaré inequality on the unit cylinder, which tells us that if $\int_{\{0\}\times S^1} w\,d\theta = 0$, then

$$\int_{\{0\}\times S^1}|w|^2\,d\theta \leq C\int_{[0,1]\times S^1}|\nabla w|^2\,d\theta\,ds.$$

Applying this to v on the cylinder $[X_\delta(\ell), X_\delta(\ell)+1] \times S^1$ yields the claimed bound, as we have

$$\|v\|^2_{L^2(\gamma^1,g)} \leq C\delta \int_{\{X_\delta(\ell)\}\times S^1} |v|^2 \, d\theta \leq C\delta \int_{X_\delta(\ell)}^{X_\delta(\ell)+1} \int_{S^1} |\nabla v|^2 \, d\theta \, ds \leq C\delta \|\nabla v\|^2_{L^2(M^1_\delta,g)}.$$

□

2.2.3 The Fenchel-Nielsen Coordinates

Recall that closed oriented surfaces are topologically classified by genus, and that surfaces of genus $\gamma \geq 2$ admit hyperbolic metrics, that is Riemannian metrics g of constant Gauss curvature $K_g \equiv -1$. The aim of this subsection is to introduce the so-called Fenchel-Nielsen coordinates, which parametrise the moduli space of hyperbolic surfaces of a given genus. We begin by recalling the pair of pants decomposition of a hyperbolic surface.

Definition 2.2.5 (Definition 3.1.1 in [12])**.** A compact hyperbolic surface of genus 0 with three boundary components which are closed geodesics with respect to the hyperbolic metric is called a *Y-piece* or a *pair of pants*.

We have the following result which tells us that the geometry of a pair of pants is completely determined by the lengths of its boundary curves.

Theorem 2.2.6 (Theorem 3.1.7 in [12])**.** *For any triple of positive real numbers ℓ_1, ℓ_2, ℓ_3 there exists a unique (up to pullback by diffeomorphisms) pair of pants Y with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of length $L_g(\gamma_i) = \ell_i, i = 1, 2, 3$.*

Suppose now that we have two pairs of pants Y and $\tilde{Y}$ with boundary lengths ℓ_1, ℓ_2, ℓ_3 and $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$ respectively. If $\ell_1 = \tilde{\ell}_1$ then we can glue Y and $\tilde{Y}$ together

by identifying the corresponding boundary geodesics to obtain a new hyperbolic surface. To be more precise, let σ_1 and $\tilde{\sigma}_1$ be the boundary geodesics of Y and $\tilde{Y}$ respectively with length $\ell_1 = \tilde{\ell}_1$, and fix parametrisations of these geodesics on the circle S^1 (with the same orientation). For any real number ψ we can obtain a hyperbolic surface by making the identification

$$\sigma_1(t) = \tilde{\sigma}_1(\psi - t), \quad t \in S^1.$$

We note that in this construction we have introduced a new parameter ψ, which we call the *twist parameter*. It is important to note that the definition of ψ given here depends on our choice of parametrisation for the curves σ_1 and $\tilde{\sigma}_1$, but the results we prove in this thesis are independent of these choices as we will analyse the derivatives of the eigenvalues with respect to ψ.

Every closed oriented hyperbolic surface can be obtained by gluing together pairs of pants along their boundary components in this way. Indeed we have the following theorem.

Theorem 2.2.7 (Proposition 3.6 in [33])**.** *Let (M, g) be a closed oriented hyperbolic surface of genus $\gamma \geq 2$. Then we can find simple closed geodesics $\sigma_1, \ldots, \sigma_{3(\gamma-1)}$ in M such that the collection of surfaces obtained by cutting open M along these geodesics is isometric to a collection of hyperbolic pairs of pants. Moreover these geodesics may be chosen in such a way that their lengths are bounded above by a constant $\overline{L}(\gamma)$ which depends only on the genus.*

In the situation described above, the collection $\{\sigma_i\}_{i=1}^{3(\gamma-1)}$ of simple closed geodesics is said to give a *pair of pants decomposition* of the surface. We note that (up to pullback by diffeomorphisms, which in particular do not affect the Laplacian

eigenvalues) the hyperbolic metric g is then determined by the lengths of these geodesics together with the twist parameters $\{\psi_i\}_{i=1}^{3(\gamma-1)}$ which tell us how the pairs of pants are glued together to form (M, g).

Definition 2.2.8. Let (M, g) be a closed oriented hyperbolic surface of genus $\gamma \geq 2$ with a collection of simple closed geodesics $\{\sigma_i\}_{i=1}^{3(\gamma-1)}$ giving a pair of pants decomposition. The corresponding length and twist parameters $\{\ell_i, \psi_i\}_{i=1}^{3(\gamma-1)}$ are called *Fenchel-Nielsen coordinates* for (M, g).

2.2.4 Properties of Holomorphic Quadratic Differentials

In this subsection we recall a number of well-known properties (see for example the works of Wolpert [67, 68, 69]) of holomorphic quadratic differentials on hyperbolic surfaces, which we will use when proving the derivative estimates in section 2.4.

Given a closed oriented surface M of genus $\gamma \geq 2$, let $\mathcal{M}_{-1}$ denote the set of all Riemannian metrics g of Gauss curvature $K_g \equiv -1$ on M. It is well-known [65] that the tangent space to $\mathcal{M}_{-1}$ at g splits L^2-orthogonally like so

$$T_g\mathcal{M}_{-1} = \{L_X g : X \in \Gamma(TM)\} \oplus H(g), \tag{2.2.4}$$

where the first summand is tangential to pull-backs of the metric g by diffeomorphisms, while the second term is the so-called *horizontal space*, defined as the real part of the $3(\gamma - 1)$ (complex) dimensional vector space of holomorphic quadratic differentials $\mathcal{H}(M, g)$. We remark that $H(g)$ can be equivalently characterised as the space of all divergence- and trace-free $(0, 2)$ tensors on M.

The real parts of holomorphic quadratic differentials are sections of $T^*M \otimes T^*M$, and so we can and do compute inner products of such tensors using the standard

inner product induced by the hyperbolic metric g. For quadratic differentials we use the normalisation $|dz^2|_g = 2\rho^{-2}$ for $g = \rho^2(dx^2 + dy^2)$ so that in particular we have

$$\langle \mathrm{Re}(\Psi), \mathrm{Re}(\Phi)\rangle_{L^2(M,g)} = \frac{1}{2}\mathrm{Re}\langle \Psi, \Phi\rangle_{L^2(M,g)} \tag{2.2.5}$$

for $\Psi, \Phi \in \mathcal{H}(M, g)$.

We recall furthermore that on collars $\mathcal{C}(\sigma)$ as described by the Collar Lemma above, any holomorphic quadratic differential $\Upsilon \in \mathcal{H}(M, g)$ can be represented by its Fourier series

$$\Upsilon = \sum_{n=-\infty}^{\infty} b_n(\Upsilon) e^{n(s+i\theta)} dz^2, \quad b_n(\Upsilon) = b_n(\Upsilon, \mathcal{C}(\sigma)) \in \mathbb{C}, z = s + i\theta.$$

In particular we have that on each such collar Υ can be split orthogonally into a principal part $b_0(\Upsilon)dz^2$ and a collar decay part $\Upsilon - b_0(\Upsilon)dz^2$. When dealing with collars around some fixed collection $\{\sigma^j\}$ of geodesics we will use the abbreviation $b_0^j(\Upsilon) \coloneqq b_0(\Upsilon, \mathcal{C}(\sigma^j))$.

We note also that by a short calculation, on collars $\mathcal{C}(\sigma)$ around geodesics σ of length $\ell \leq 2\mathrm{arsinh}(1)$ we have using $|dz^2|_g = 2\rho^{-2}$

$$\|dz^2\|_{L^1(\mathcal{C}(\sigma),g)} = 8\pi X(\ell) \text{ and } \|dz^2\|^2_{L^2(\mathcal{C}(\sigma),g)} = 32\pi^5\ell^{-3} + O(1),$$

while we also have the pointwise estimates

$$\|dz^2\|_{L^\infty(\delta-\mathrm{thick}(\mathcal{C}(\sigma)),g)} \leq C\delta^{-2} \text{ and } \|dz^2\|_{L^\infty(\mathcal{C}(\sigma),g)} \leq C\ell^{-2}. \tag{2.2.6}$$

2.2.5 A Dual Basis for the Space of Holomorphic Quadratic Differentials

In order to analyse the dependence of eigenvalues on the length and twist parameters, we will use the observation of Große and Rupflin that the derivatives of the eigenvalue λ_k with respect to the Fenchel-Nielsen coordinates $\{\ell_i, \psi_i\}_{i=1}^{3(\gamma-1)}$ can be expressed in terms of L^2-inner products of the Hopf differential of the corresponding (normalised) eigenfunction with elements of a suitable basis for the space of (real parts of) holomorphic quadratic differentials on (M, g). We describe this characterisation of $\nabla\lambda_k$ and state a number of useful estimates for the basis elements due to Große and Rupflin [27] in this subsection.

Suppose we are given a collection of simple closed geodesics $\mathcal{E} = \{\sigma^i\}_{i=1}^{3(\gamma-1)}$ in (M, g) which decomposes the surface into pairs of pants. As described in subsection 2.2.3, such a decomposition allows us to define the Fenchel-Nielsen coordinates $\{\ell_i, \psi_i\}_{i=1}^{3(\gamma-1)}$. In [27], Große and Rupflin consider the basis $\{\Lambda^i, \Psi^i\}_{i=1}^{3(\gamma-1)}$ for $\mathcal{H}(M, g)$ which is dual to the real differentials of the Fenchel-Nielsen coordinates. To be more precise, for each $1 \leq i \leq 3(\gamma-1)$, consider $d\ell_i : T_g\mathcal{M}_{-1} \to \mathbb{R}$ given by

$$d\ell_i(k) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L_{g_\varepsilon}(\sigma^i(\varepsilon))$$

where g_ε is a smooth curve of metrics in $\mathcal{M}_{-1}$ with $g_0 = g$, $\partial_\varepsilon|_{\varepsilon=0} g_\varepsilon = k$, and $\sigma^i(\varepsilon)$ is the unique simple closed geodesic in (M, g_ε) homotopic to σ^i. Similarly we define $d\psi_i$ for each $1 \leq i \leq 3(\gamma-1)$. Then $\Lambda^i, \Psi^i \in \mathcal{H}(M, g)$ are defined by the duality relations

$$d\ell_j(\mathrm{Re}(\Lambda^i)) = \delta^i_j = d\psi_j(\mathrm{Re}(\Psi^i)) \text{ and } d\ell_j(\mathrm{Re}(\Psi^i)) = 0 = d\psi_j(\mathrm{Re}(\Lambda^i)) \text{ for all } i, j.$$

Große and Rupflin observe in [27] that any function $f : \mathcal{M}_{-1} \to \mathbb{R}$ which is defined in terms of geometric properties of (M, g) will be invariant under pull-back by diffeomorphisms, and hence as a result of the orthogonal splitting (2.2.4) of $T_g\mathcal{M}_{-1}$, the L^2-gradient of f lies in the horizontal space $H(g)$. This is true in particular when $f = \lambda_k$, the kth positive eigenvalue of the Laplacian, and we can characterise the derivatives of λ_k with respect to the Fenchel-Nielsen coordinates as

$$\frac{\partial \lambda_k}{\partial \ell_i} = \langle \nabla \lambda_k, \Lambda^i \rangle \text{ and } \frac{\partial \lambda_k}{\partial \psi_i} = \langle \nabla \lambda_k, \Psi^i \rangle. \tag{2.2.7}$$

We recall that the eigenvalue λ_k is differentiable at any g for which it is simple, and when this is the case we can express $\nabla \lambda_k$ in terms of the Hopf differential $\Phi(u_k, g)$ of the (normalised) eigenfunction u_k, which is given in local isothermal coordinates (x, y) by

$$\Phi(u_k, g) = (|\partial_x u_k|^2 - |\partial_y u_k|^2 - 2i\langle \partial_x u_k, \partial_y u_k \rangle) dz^2, \quad z = x + iy. \tag{2.2.8}$$

Indeed we have (see Lemma 3.5 in [28]):

Lemma 2.2.9. *Let (M, g) be a hyperbolic surface for which the kth eigenvalue λ_k is simple. Let u_k be the corresponding eigenfunction, normalised by $\|u_k\|_{L^2(M,g)} = 1$. Then the L^2-gradient of $\lambda_k : \mathcal{M}_{-1} \to \mathbb{R}$ is given by*

$$\nabla \lambda_k(g) = -\frac{1}{2} \mathrm{Re}(P_g^{\mathcal{H}}(\Phi(u_k, g))),$$

for $\Phi(u_k, g)$ the Hopf differential given by (2.2.8) and $P_g^{\mathcal{H}}$ the L^2-orthogonal projection from the space of L^2-quadratic differentials onto $\mathcal{H}(M, g)$.

Combining this result with (2.2.7) we obtain

$$\frac{\partial \lambda_k}{\partial \ell_i} = -\langle \mathrm{Re}(\Phi(u_k, g)), \mathrm{Re}(\Lambda^i)\rangle \text{ and } \frac{\partial \lambda_k}{\partial \psi_i} = -\langle \mathrm{Re}(\Phi(u_k, g)), \mathrm{Re}(\Psi^i)\rangle. \tag{2.2.9}$$

We will use the above expression, together with properties of the basis $\{\Lambda^i, \Psi^i\}_{i=1}^{3(\gamma-1)}$ obtained by Große and Rupflin in [27] and energy estimates for the corresponding eigenfunction u_k obtained in section 2.3, to prove estimates for the derivatives of the eigenvalue λ_k with respect to the Fenchel-Nielsen coordinates.

For completeness, we provide a proof of Lemma 2.2.9. In fact, a more general statement which in particular implies (2.2.9) is valid for smooth curves of eigenvalues, which will enable us to prove our main result for small eigenvalues of $-\Delta_g$ without having to assume they are simple. To be more precise, we claim that given any smooth family of metrics g_ε with $g_0 = g$ and u_ε a corresponding smooth curve of eigenfunctions to a smooth curve of eigenvalues λ_ε, we have

$$\begin{aligned} \left.\frac{d\lambda_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} &= -\frac{1}{2}\left\langle \mathrm{Re}\left(P_g^{\mathcal{H}}\left(\Phi(u,g)\right)\right), \partial_\varepsilon g_\varepsilon|_{\varepsilon=0}\right\rangle \\ &= -\frac{1}{2}\sum_{i=1}^{3(\gamma-1)}\left(\left\langle \mathrm{Re}\left(\Phi(u,g)\right), \mathrm{Re}(\Lambda^i)\right\rangle \left.\frac{d\ell_i(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}\right. \\ &\qquad \left. + \left\langle \mathrm{Re}\left(\Phi(u,g)\right), \mathrm{Re}(\Psi^i)\right\rangle \left.\frac{d\psi_i(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}\right) \end{aligned} \tag{2.2.10}$$

Indeed, given such a family of metrics we can exploit the L^2-orthogonal splitting (2.2.4) of $T_g\mathcal{M}_{-1}$ to write

$$\partial_\varepsilon g = \sum_{i=1}^{3(\gamma-1)} \mathrm{Re}(\Lambda^i)\frac{d\ell_i(g_\varepsilon)}{d\varepsilon} + \mathrm{Re}(\Psi^i)\frac{d\psi_i(g_\varepsilon)}{d\varepsilon} + L_X g$$

for a suitable vector field X. We note that as the eigenvalue λ is invariant under pullback by diffeomorphisms and $\left\langle \mathrm{Re}(P_g^{\mathcal{H}}(\Upsilon)), L_X g \right\rangle = 0$ for all $\Upsilon \in \mathcal{H}(M,g)$, it is enough to prove the claim for $\partial_\varepsilon g \in H(g)$, in which case the claim reduces to showing

$$\left.\frac{d\lambda_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{1}{2}\langle \mathrm{Re}(\Phi),\, \partial_\varepsilon g_\varepsilon|_{\varepsilon=0}\rangle.$$

Using the variational characterisation of the eigenvalue, we have

$$\lambda_\varepsilon = \frac{\int_M |\nabla_{g_\varepsilon} u_\varepsilon|^2_{g_\varepsilon}\, dV_{g_\varepsilon}}{\|u_\varepsilon\|^2_{L^2(M,g_\varepsilon)}}. \tag{2.2.11}$$

Differentiating (2.2.11) with respect to ε then gives

$$\begin{aligned}\left.\frac{d\lambda_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \frac{1}{\|u_{\varepsilon=0}\|^4_{L^2(M,g)}}\Bigg(&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\int_M |\nabla_{g_\varepsilon} u_\varepsilon|^2_{g_\varepsilon}\, dV_{g_\varepsilon}\right)\|u_{\varepsilon=0}\|^2_{L^2(M,g)} \\ &- \int_M |\nabla_g u_{\varepsilon=0}|^2_g\, dV_g \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int_M |u_\varepsilon|^2\, dV_{g_\varepsilon}\Bigg)\end{aligned}$$

We may assume without loss of generality that $\|u_{\varepsilon=0}\|^2_{L^2(M,g)} = 1$, so writing u for $u_{\varepsilon=0}$ the equation above simplifies to

$$\begin{aligned}\left.\frac{d\lambda_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int_M |\nabla_{g_\varepsilon} u|^2_{g_\varepsilon}\, dV_{g_\varepsilon} + \int_M \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} |\nabla_g u_\varepsilon|^2_g\, dV_g \\ - \lambda\int_M \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} |u_\varepsilon|^2\, dV_g - \lambda\int_M |u|^2 \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} dV_{g_\varepsilon}.\end{aligned}$$

We note that as $\partial_\varepsilon g \in H(g)$ and hence $\partial_\varepsilon g$ is trace-free, we have $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} dV_{g_\varepsilon} = 0$, so the last term above vanishes. Writing $E(u,g) = \|\nabla u\|^2_{L^2(M,g)}$ for (twice) the

Dirichlet energy of u on (M, g), we are left with

$$\begin{aligned}\left.\frac{d\lambda_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} &= \langle \nabla_g E(u,g),\, \partial_\varepsilon g|_{\varepsilon=0}\rangle + 2\int_M \nabla_g u \left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \nabla_g u_\varepsilon\right) dV_g \\ &\qquad - 2\lambda \int_M u \left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} u_\varepsilon\right) dV_g \\ &= -\frac{1}{2}\left\langle \mathrm{Re}\left(P_g^{\mathcal{H}}\left(\Phi(u,g)\right)\right),\, \partial_\varepsilon g_\varepsilon|_{\varepsilon=0}\right\rangle - 2\int_M \left(\Delta_g u + \lambda u\right) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} u_\varepsilon \, dV_g \\ &= -\frac{1}{2}\left\langle \mathrm{Re}\left(P_g^{\mathcal{H}}\left(\Phi(u,g)\right)\right),\, \partial_\varepsilon g_\varepsilon|_{\varepsilon=0}\right\rangle,\end{aligned}$$

where in the second line we use the well-known fact that the gradient of the Dirichlet energy is given by the Hopf differential and use integration by parts for the second term on the RHS. This proves the claim.

2.2.6 Properties of the Dual Basis

The basis elements $\{\Lambda^i, \Psi^i\}_{i=1}^{3(\gamma-1)}$ described above satisfy a number of estimates which will be of use to us in later proofs. We state these estimates without proof in Lemma 2.2.10 below, noting that these properties of the basis elements are not original results, but appear already in the work of Große and Rupflin [27].

Lemma 2.2.10 (c.f. Prop 1.1, Thm 1.2, and Thm 1.3 in [27])**.** *Let (M, g) be a closed hyperbolic surface of genus γ and let $\mathcal{E} = \{\sigma^1, \ldots, \sigma^{3(\gamma-1)}\}$ be a collection of simple closed geodesics giving a pair of pants decomposition. Let $\eta \in (0, \mathrm{arsinh}(1))$ and $\bar{L} < \infty$ be such that $\mathcal{E}$ contains all simple closed geodesics in (M, g) of length at most 2η, and no geodesic in $\mathcal{E}$ has length greater than $\bar{L}$. Then the associated dual basis $\{\Lambda^j, \Psi^j\}_{j=1}^{3(\gamma-1)}$ of $\mathcal{H}(M, g)$ satisfies the following. There exist constants $C > 0$*

depending only on γ, η, and $\bar{L}$ so that for any $\bar{\delta} \in (0, \operatorname{arsinh} 1)$,

$$\|\Lambda^j\|_{L^\infty(\delta-\text{thick}(M,g))} \leq C\delta^{-2}\ell_j \text{ and } \|\Psi^j\|_{L^\infty(\delta-\text{thick}(M,g))} \leq C\delta^{-2}\ell_j^3, \tag{2.2.12}$$

and the principal parts are so that for all $j, k \in \{1, \ldots, 3(\gamma-1)\}$,

$$\operatorname{Re}(b_0^k(\Lambda^j)) = -\frac{\ell_j}{2\pi^2}\delta_{kj} \qquad |\operatorname{Im}(b_0^k(\Lambda^j))| \leq C\ell_j\ell_k^3 \tag{2.2.13}$$

$$\operatorname{Re}(b_0^i(\Psi^j)) = 0 \qquad |\operatorname{Im}(b_0^k(\Psi^j))| \leq \begin{cases} C\ell_j^3 & k = j \\ C\ell_k^3\ell_j^3 & k \neq j \end{cases} \tag{2.2.14}$$

2.3 Energy Estimates

In our analysis of an eigenvalue we need two main estimates for the energy of the corresponding eigenfunction, which we state and prove in this section. The first, Lemma 2.3.1, gives a bound on the energy on the δ-thick part of the surface and is a generalisation of Lemma 3.1 in [28].

Lemma 2.3.1 (Compare [28] Lemma 3.1)**.** *For any $\gamma \geq 2$ there exists $C_0 \in \mathbb{R}$ so that for all closed oriented hyperbolic surfaces (M, g) of genus γ and every eigenfunction u of $-\Delta_g$ (normalised by $\|u\|_{L^2(M,g)} = 1$) to an eigenvalue λ we have*

$$\|\nabla u\|^2_{L^2(\delta\text{-thick}(M,g))} \leq C_0\delta^{-1}\lambda^2 \text{ for every } \delta > 0.$$

The second estimate gives a bound on the energy of the eigenfunction on the non-disconnecting δ-thin handles.

Proposition 2.3.2. *For any $\gamma \geq 2$ and any $\alpha \in (0, 1)$ there exists $\bar{\lambda} > 0$ and $C \in \mathbb{R}$*

so that the following holds true for any $\overline{\delta} \in (0, \frac{1}{4}\text{arsinh}(1)]$. *Let* (M, g) *be any closed oriented hyperbolic surface of genus* γ *and suppose that* $\sigma \subset M$ *is a simple closed geodesic with the property that both ends of the collar* $\mathcal{C}(\sigma)$ *lie in the same component of* $\overline{\delta}$*-thick*(M, g). *Then for any eigenfunction* u *of* $-\Delta_g$ *(normalised by* $\|u\|_{L^2(M,g)} = 1$*) to an eigenvalue* $\lambda \leq \bar{\lambda}$ *we have*

$$\|\nabla u\|^2_{L^2(\delta\text{-}thin(\mathcal{C}(\sigma)))} \leq \frac{C}{\bar{\delta}^{2-\alpha}} \lambda^2 \delta^{1-\alpha}$$

for any $0 < \delta \leq \overline{\delta}$.

To control the angular energy of eigenfunctions on such collars we will use the following standard energy estimate, which can for example be obtained by adjusting the proof of the analogue estimate for harmonic maps as found in [28]. Similar arguments have been used in many works, for example [58], [50], [64], [34].

Lemma 2.3.3 (compare e.g. Lemma 3.2 in [28], Lemma 2.4 in [34], Lemma 2.1 in [50], Lemma 2.13 in [64])**.** *Let* (M, g) *be any closed oriented hyperbolic surface, let* u *be any normalised eigenfunction of* $-\Delta_g$ *with* λ *the corresponding eigenvalue and let* $\mathcal{C}(\sigma) \subset M$ *be a collar around a simple closed geodesic of length* $\ell \leq \frac{3}{2}\text{arsinh}(1)$. *Then we can bound the angular energy* $\vartheta(s) \coloneqq \int_{\{s\}\times S^1} |u_\theta|^2 \, d\theta$ *in any point* (s, θ) *with* $\text{inj}_g(s, \theta) \leq \frac{3}{4}\text{arsinh}(1)$ *by*

$$\vartheta(s) \leq 2e^{-(X(\ell)-|s|)} \|\nabla u\|^2_{L^2(\frac{1}{2}\text{arsinh}(1)\text{-}thick(\mathcal{C}(\sigma)))} + \frac{1}{2}\lambda^2 \int_{\mathcal{C}(\sigma)} \rho^4 e^{-|s-q|} u^2 d\theta dq.$$

In particular there exists a universal constant C *so that for any such collar*

$$\int_{-X(\ell)}^{X(\ell)} \int_{S^1} |u_\theta|^2 \rho^{-4} ds d\theta \leq C \|\nabla u\|^2_{L^2(\frac{1}{2}\text{arsinh}(1)\text{-}thick(\mathcal{C}(\sigma)))} + C\lambda^2 \ell^{-1}. \tag{2.3.1}$$

As a short calculation shows that $\sup_{(q,\theta)\in\mathcal{C}(\sigma)}|\rho^2(q)e^{-|s-q|}| \leq C\rho^2(s)$, and $\rho(s) \leq C\,\mathrm{inj}_g(s,\theta)$ for a universal constant C, the above lemma implies in particular that

$$\vartheta(s) \leq C\,\mathrm{inj}_g(s,\theta)^2(\|\nabla u\|^2_{L^2(\frac{1}{2}\mathrm{arsinh}(1)\text{-thick}(\mathcal{C}(\sigma)))} + C\lambda^2)$$

for every s with $\mathrm{inj}_g(s,\theta) \leq \frac{3}{4}\mathrm{arsinh}(1)$. Together with Lemma 2.3.1, we have

$$\vartheta(s) \leq C\,\mathrm{inj}_g(s,\theta)^2\lambda^2 \tag{2.3.2}$$

for a constant $C > 0$ depending only on the genus.

2.3.1 Proofs of Energy Estimates

In this subsection we prove the properties of eigenfunctions stated earlier in this section, which are used in the proof of the main result about the eigenvalues. Throughout this section we sometimes denote the energy of a function v on a set $F \subset M$ by

$$E(v,F) \coloneqq \|\nabla v\|^2_{L^2(F,g)}.$$

The proofs of both Lemma 2.3.1 and Proposition 2.3.2 will be obtained by constructing suitable test functions $w \in H^1(M,g)$ with small energy on the relevant parts of the surface and then using the weak formulation of the eigenvalue equation $-\Delta_g u = \lambda u$ to note in particular that $\langle\nabla u, \nabla(w-u)\rangle_{L^2(M,g)} = \lambda\langle u, w-u\rangle_{L^2(M,g)}$ and thus that

$$\|\nabla u\|^2_{L^2(M,g)} \leq \|\nabla w\|^2_{L^2(M,g)} - 2\lambda\langle u, w-u\rangle_{L^2(M,g)}. \tag{2.3.3}$$

We present first an outline of the proof of Lemma 2.3.1. This is proved using very similar arguments as used to prove Lemma 3.1 in [28] except that we have to modify the construction on collars around geodesics σ that do not have the property that $M \setminus \sigma$ is disconnected and that we begin by proving the result for δ sufficiently small so that δ-thick$(M, g) = M$ and then inductively proving the claim for greater ranges of δ.

Proof of Lemma 2.3.1. Let $\{\sigma^i\}_{i=1}^m$ be the set of all simple closed geodesics in (M, g) of length at most 2arsinh(1) (note $m \leq 3(\gamma - 1)$), and suppose they are ordered by length so that, writing $\ell_i = L_g(\sigma^i)$, we have $\ell_1 \leq \cdots \leq \ell_m$. We first note that there exists C_1 depending only on the genus such that the result of the lemma holds with $C_0 = C_1$ for all $\delta \in (0, \frac{\ell_1}{2})$. To see this, let us define $L = \min\{\sum_{i\in I} L_g(\sigma^i)\}$, where the minimum is taken over all collections $\{\sigma^i\}_{i\in I}$ of simple closed geodesics in (M, g) such that $M \setminus \cup_{i\in I}\sigma^i$ has two connected components. Recall that by the results of Schoen-Wolpert-Yau [61], the first eigenvalue λ_1 satisfies $C^{-1}L \leq \lambda_1 \leq CL$ for some topological constant C. In particular for any eigenvalue $\lambda \neq 0$ we have $\lambda \geq \lambda_1 \geq C^{-1}L \geq C^{-1}\ell_1$, which implies the result for $\delta \in (0, \frac{\ell_1}{2})$ as $\|\nabla u\|^2_{L^2(\delta-\text{thick}(M,g))} = \|\nabla u\|^2_{L^2(M,g)} = \lambda$, while $\delta^{-1}\lambda \geq 2C^{-1} > 0$ for the same constant C. Hence the claim holds with $C_1 = \frac{1}{2}C$.

Suppose now that for some $j_0 \in \{1, \ldots, m\}$ there exists a constant C_{j_0} depending only on the genus such that

$$\|\nabla u\|^2_{L^2(\tilde{\delta}-\text{thick}(M,g))} \leq C_{j_0}\frac{\lambda^2}{\tilde{\delta}} \tag{2.3.4}$$

for all $\tilde{\delta} \in (0, \frac{\ell_{j_0}}{2})$. Given this assumption, we shall prove that there exists a constant C_{j_0+1} depending only on the genus (and hence allowed to depend on C_{j_0})

such that the claim of the lemma is true for $\delta \in (0, \frac{\ell_{j_0+1}}{2})$ if $j_0 < m$, respectively for all $\delta \in (0, \operatorname{arsinh}(1)]$ if $j_0 = m$. Repeating this procedure, the lemma is proved after at most $m+1$ steps, with $m \leq 3(\gamma-1)$ bounded in terms of the genus.

Note that if we choose $C_{j_0+1} \geq BC_{j_0}$ for some constant $B \geq 2$ depending only on the genus (to be chosen later), then the result holds trivially for $\delta \in (\frac{\ell_{j_0}}{2}, B\frac{\ell_{j_0}}{2}]$ as for such δ we have

$$\|\nabla u\|^2_{L^2(\delta-\text{thick}(M,g))} \leq \|\nabla u\|^2_{L^2(\frac{\ell_{j_0}}{2}-\text{thick}(M,g))} \leq 2C_{j_0}\frac{\lambda^2}{\ell_{j_0}} \leq C_{j_0}B\frac{\lambda^2}{\delta},$$

where the second inequality follows by applying (2.3.4) for a sequence $\tilde{\delta}_n$ in $(0, \frac{\ell_{j_0}}{2})$ with $\tilde{\delta}_n \nearrow \frac{\ell_{j_0}}{2}$, noting that $\{p \in M : \text{inj}_g(p) = \frac{\ell_{j_0}}{2}\}$ is a zero measure set.

Hence it suffices to prove the claim for $\delta \in (\frac{B\ell_{j_0}}{2}, \frac{\ell_{j_0+1}}{2})$ (if this interval is not empty, otherwise there is nothing to prove). So let $\delta \in (\frac{B\ell_{j_0}}{2}, \frac{\ell_{j_0+1}}{2})$. Note that this implies that the simple closed geodesics of length $\ell \leq 2\delta$ indeed have length $\ell \leq \frac{2\delta}{B}$. We now closely follow the proof of Lemma 3.1 in [28] to estimate the energy of u on $\delta-\text{thick}(M,g)$. To this end, consider the energy of the eigenfunction u on the connected components of $\{p : \text{inj}_g(p) > \delta\}$, and denote by M_1^δ the closure of the connected component on which this energy is maximal. Writing $\|\nabla u\|^2_{L^2(M_1^\delta,g)} = \mu\lambda^2\delta^{-1}$, we see that the claimed estimate is equivalent to proving an upper bound for $\mu = \mu(\delta)$ depending only on the genus.

Let $\{\tilde{M}_i\}$ denote the connected components of $M \setminus M_1^\delta$ (we note that the number of such components is bounded above by a constant depending only on γ), and for each i let $\gamma_i^1, \ldots, \gamma_i^{k_i}$ be the boundary curves of $\tilde{M}_i$ (again we have that for all i, k_i is bounded above in terms of the genus), noting $\partial M_1^\delta = \cup_i \cup_{j=1}^{k_i} \gamma_i^j$. We modify the construction of [28] as follows. Let $c_i^j = \fint_{\gamma_i^j} u\, dS_g$ be the mean value of u on each

boundary curve γ_i^j of M_1^δ. As in [28] we consider a comparison function w which is so that $w = c_1^1$ on M_1^δ except on small transition regions adjacent to each curve γ_i^j, on which we choose w as described in [28] so that $w|_{\gamma_i^j} = u + (c_1^1 - c_i^j)$. We denote the collar region containing γ_i^j by $\mathcal{C}_i^j$, and write $\mathcal{C}_{i,\delta}^j := \{p \in \mathcal{C}_i^j : \mathrm{inj}_g(p) \leq \delta\} \subset \tilde{M}_i$.

If $k_i = 1$ then we can define w on $\tilde{M}_i$ exactly as in [28], that is we set $w|_{\tilde{M}_i} = u + (c_1^1 - c_i^1)$. If $k_i \geq 2$ we either have that $\tilde{M}_i$ has two boundary curves γ_i^1, γ_i^2 which are both in the same collar, and $\tilde{M}_i$ is simply given by this collar $\mathcal{C}_{i,\delta}^1$, or else $\tilde{M}_i \setminus \cup_{j=2}^{k_i} \mathcal{C}_{i,\delta}^j \neq \emptyset$. In both cases defining the comparison function as in [28] would result in it being discontinuous, so we need to modify the definition of the comparison function from [28], and we will do this by adding a suitable function v_i^j on each collar $\mathcal{C}_{i,\delta}^j$ which is linear in s.

To be more precise, if $\tilde{M}_i = \mathcal{C}_{i,\delta}^1 = \mathcal{C}_{i,\delta}^2$ then we set $w = u + (c_1^1 - c_i^1) + v_i^2$ on $\tilde{M}_i$ for v_i^2 the linear function of s which takes the value 0 on γ_i^1 and the value $c_i^1 - c_i^2$ on γ_i^2. In the second case we choose $w = u + (c_1^1 - c_i^1)$ on $\tilde{M}_i \setminus \cup_{j=2}^{k_i} (\mathcal{C}_{i,\delta}^j)$ while on the collars $\mathcal{C}_{i,\delta}^j$ with $j \geq 2$ we set $w = u + (c_1^1 - c_i^1) + v_i^j$ for v_i^j the linear function of s which takes the value $v_i^j = c_i^1 - c_i^j$ on γ_i^j and the value $v_i^j = 0$ at the end of $\mathcal{C}_{i,\delta}^j$ contained in δ-thick$(\tilde{M}_i, g)$. We note that this gives $w \in H^1(M, g)$, so as $\|u\|_{L^2(M,g)} = 1$ we can apply (2.3.3) to get

$$\|\nabla u\|_{L^2(M,g)}^2 \leq \|\nabla w\|_{L^2(M,g)}^2 + 2\lambda \|w - u\|_{L^2(M,g)}.$$

We observe that w defined as above has the same structure as the test function considered in the proof of Lemma 3.1 in [28, (4.4), p. 28] except on the collars $\mathcal{C}_{i,\delta}^j$ with $j \geq 2$, where we have added only the linear functions v_i^j. As long as we account for the additional terms involving v_i^j we can thus still use the estimates

(4.7), (4.8) and (4.9) on the comparison function derived in [28, pp. 28–29]. When inserted into the inequality above, these estimates allow us to conclude that

$$\lambda = \|\nabla u\|^2_{L^2(M,g)} \leq \lambda \left(1 - \frac{\mu\lambda}{2\delta} + C\lambda\delta^4\right) + C\frac{\mu^{\frac{1}{2}}\lambda^2}{\delta} + R_1 + R_2 + R_3 \tag{2.3.5}$$

where we shall prove that the additional terms are bounded by

$$R_1 \coloneqq \sum_i \sum_{j=2}^{k_i} \|\nabla v_i^j\|^2_{L^2(\mathcal{C}^j_{i,\delta},g)} \leq \frac{C}{B}\frac{\mu\lambda^2}{\delta} \tag{2.3.6}$$

$$R_2 \coloneqq \left|2\sum_i \sum_{j=2}^{k_i} \int_{\mathcal{C}^j_{i,\delta}} \nabla u \cdot \nabla v_i^j \, ds d\theta\right| \leq C\mu^{-\frac{1}{2}}\frac{\mu\lambda^2}{\delta} \tag{2.3.7}$$

$$R_3 \coloneqq 2\lambda \sum_i \sum_{j=2}^{k_i} \|v_i^j\|_{L^2(\mathcal{C}^j_{i,\delta},g)} \leq C\mu^{-\frac{1}{2}}\delta^{\frac{1}{2}}\frac{\mu\lambda^2}{\delta}, \tag{2.3.8}$$

for constants C depending only on the genus. Choosing B sufficiently large such that $\frac{C}{B} \leq \frac{1}{4}$ and inserting these estimates into (2.3.5) we get

$$\lambda \leq \lambda - \frac{\mu\lambda^2}{\delta}\left(\frac{1}{4} - \delta^5\mu^{-1} - C\mu^{-\frac{1}{2}}\right)$$

which implies that the term in parentheses must be non-positive, and hence that μ must be bounded from above by a constant that depends only on the genus. It hence remains to prove the bounds (2.3.6), (2.3.7), (2.3.8).

We begin by considering the third term, R_3. A short calculation using the Poincaré inequality Lemma 2.2.4 gives that for each i,

$$|c_i^1 - c_i^j|^2 = \left(⨍_{\gamma_i^j} u - c_i^1 \, dS_g\right)^2 = \left(⨍_{\gamma_i^j} u - c_i^1 \, d\theta\right)^2 \leq C\int_{\gamma_i^j} |u - c_i^1|^2 \, d\theta \tag{2.3.9}$$

$$\leq \frac{C}{\delta}\int_{\gamma_i^j}|u-c_i^1|^2\,dS_g \leq \frac{C}{\delta}\|\nabla u\|^2_{L^2(M_1^\delta,g)} \leq \frac{C\mu}{\delta^2}\lambda^2.$$

As $|v_i^j| \leq |c_i^1 - c_i^j|$, we see that

$$\lambda\|v_i^j\|_{L^2(\mathcal{C}_{i,\delta}^j,g)} \leq C\frac{\mu^{\frac{1}{2}}\lambda^2}{\delta}(\mathrm{Area}_g(\mathcal{C}_{i,\delta}^j))^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}\frac{\mu^{\frac{1}{2}}\lambda^2}{\delta} = C\mu^{-\frac{1}{2}}\delta^{\frac{1}{2}}\frac{\mu\lambda^2}{\delta},$$

where we have used the fact that $\mathrm{Area}_g(\mathcal{C}_{i,\delta}^j) \leq C\delta \leq C$. Summing over the collars $\mathcal{C}_{i,\delta}^j$, $j \geq 2$, we obtain the desired estimate (2.3.8) for R_3.

We now turn to the estimates for the terms involving the gradient of v_i^j, namely R_1 and R_2. Note that as $2\delta \in (B\ell_{j_0}, \ell_{j_0+1})$, the length $\ell_i^j \leq 2\delta$ of the central geodesic for each collar $\mathcal{C}_{i,\delta}^j$ we consider satisfies $\ell_i^j \leq \frac{2\delta}{B}$. The δ-thin parts of such collars are described in collar coordinates by $(-X_\delta(\ell_i^j), X_\delta(\ell_i^j)) \times S^1$ where $X_\delta(\ell_i^j)$ is as in (2.2.2), so (as $B \geq 2$) it can be easily checked that $X_\delta(\ell_i^j) \geq \frac{2\pi^2}{3\ell_i^j}$. Putting this together with the estimate (2.3.9), we have

$$\|\nabla v_i^j\|^2_{L^2(\mathcal{C}_{i,\delta}^j,g)} = 2\pi\frac{|c_i^1 - c_i^j|^2}{2X_\delta(\ell_i^j)} \leq C\frac{\ell_i^j\mu\lambda^2}{\delta^2}. \tag{2.3.10}$$

Thus we obtain the bound

$$R_1 = \sum_i\sum_{j=1}^{k_i}\|\nabla v_i^j\|^2_{L^2(\mathcal{C}_{i,\delta}^j,g)} \leq C\frac{\mu\lambda^2}{\delta}\sum_i\sum_{j=1}^{k_i}\frac{\ell_i^j}{\delta} \leq \frac{C}{B}\frac{\mu\lambda^2}{\delta},$$

claimed in (2.3.6), where the last follows as $\ell_i^j \leq \ell_{j_0} \leq \frac{2\delta}{B}$.

Finally, we turn to R_2. Recall our assumption that there exists a constant C_{j_0} depending only on the genus such that the claim of the lemma holds for all $\tilde{\delta} \in (0, \frac{\ell_{j_0}}{2})$. We first note that this assumption in particular allows us to

apply (2.3.4) for a sequence $\tilde{\delta} \nearrow \frac{\ell_i^j}{2} \leq \frac{\ell_{j_0}}{2}$, and, noting that $\{p \in \mathcal{C}_{i,\delta}^j : \mathrm{inj}_g(p) = \frac{\ell_i^j}{2}\}$ is a zero set, yields in particular that on all collars $\mathcal{C}_{i,\delta}^j$ under consideration, we have the estimate

$$\|\nabla u\|^2_{L^2(\mathcal{C}_{i,\delta}^j,g)} \leq \|\nabla u\|^2_{L^2(\frac{\ell_i^j}{2}-\mathrm{thick}(M,g))} = \lim_{\tilde{\delta}\nearrow\frac{\ell_i^j}{2}} \|\nabla u\|^2_{L^2(\tilde{\delta}-\mathrm{thick}(M,g))} \leq C_{j_0}\frac{\lambda^2}{\ell_i^j}.$$

Together with (2.3.10), this allows us to bound R_2 as follows

$$\left|2\int_{\mathcal{C}_{i,\delta}^j} \nabla u \cdot \nabla v_i^j \, ds d\theta\right| \leq 2\|\nabla u\|_{L^2(\mathcal{C}_{i,\delta}^j,g)}\|\nabla v_i^j\|_{L^2(\mathcal{C}_{i,\delta}^j,g)} \leq C\left(C_{j_0}\frac{\lambda^2}{\ell_i^j}\right)^{\frac{1}{2}}\left(\frac{\ell_i^j\mu\lambda^2}{\delta^2}\right)^{\frac{1}{2}}$$

$$\leq CC_j^{\frac{1}{2}}\mu^{-\frac{1}{2}}\frac{\mu\lambda^2}{\delta},$$

which, when summed over the collars $\mathcal{C}_{i,\delta}^j$, gives the bound claimed in (2.3.7), completing the proof of the lemma. □

We now turn to the proof of Proposition 2.3.2. Let $\sigma \subset M$ be a simple closed geodesic of length less than $2\overline{\delta}$ such that both ends of the collar $\mathcal{C}(\sigma)$ lie in the same component of $\overline{\delta}$-thick(M,g). On the thin part of the collar $\mathcal{C}(\sigma)$ we know that u is well approximated by

$$\bar{u}(s) \coloneqq \fint_{\{s\}\times S^1} u(s,\theta)\, d\theta, \tag{2.3.11}$$

as the angular energy estimate (2.3.2) combined with Lemma 2.3.1 ensures that

$$|u(s,\theta) - \bar{u}(s)| \leq \int_{\{s\}\times S^1} |u_\theta|\, d\theta \leq C\vartheta(s)^{\frac{1}{2}} \leq C\lambda\rho(s).$$

We shall furthermore need bounds on $|\bar{u}(s)|$ and on $|\bar{u}(s) - \bar{u}(-s)|$. To formulate these bounds, suppose $\mathcal{C}(\sigma)$ is a collar around a simple closed geodesic of length ℓ, let $K \in \mathbb{N}$ be so that $\mathrm{inj}_g(X(\ell), \theta) \in [2^{K-1}\ell, 2^K\ell]$ and split the collar $\mathcal{C}(\sigma) = \bigcup_{j=1}^K F_j$, where

$$F_j \coloneqq \{p \in \mathcal{C}(\sigma) : \mathrm{inj}_g(p) \in [\delta_{j-1}, \delta_j]\} = \{(s,\theta) : s_{j-1} \leq |s| \leq s_j\}, \text{ for } \delta_j = 2^{j-1}\ell. \tag{2.3.12}$$

We note that (2.2.1), (2.2.3) and the resulting bound on $\partial_s \mathrm{inj}_g$ yield that

$$\frac{1}{2}\delta_j^{-1} \leq s_j - s_{j-1} \leq C\delta_j^{-1} \text{ and } \mathrm{Area}_g(F_i) \leq C\delta_i \tag{2.3.13}$$

for a universal constant C.

We first prove

Lemma 2.3.4. *There exist constants $\eta_1 > 0$ and C_1 so that for any hyperbolic surface (M, g), any simple closed geodesic σ of length $\ell \in (0, \mathrm{arsinh}(1))$ and any normalised eigenfunction u of $-\Delta_g$ to an eigenvalue $\lambda \leq \eta_1$ we have*

$$|\bar{u}(s_{j-1}) - \bar{u}(-s_{j-1})| \leq \left(1 - \frac{s_j - s_{j-1}}{s_j} + C_1\lambda\right)|\bar{u}(s_j) - \bar{u}(-s_j)|, \text{ for } j = 1, \dots, K. \tag{2.3.14}$$

Furthermore for any $\alpha \in (0,1)$ there exists a constant $C = C_\alpha$ that depends only on α so that for any $1 \leq i \leq j \leq K$ we have

$$\sup_{s \in [s_{j-1}, s_j]} \sup_{s' \in [s_{i-1}, s_i]} |\bar{u}(\pm s') - \bar{u}(\pm s)| \leq C\delta_i^{-\frac{\alpha}{2}} \max_{i \leq k \leq j} \delta_k^{-\frac{1}{2}(1-\alpha)} \|\nabla u\|_{L^2(F_k^\pm)},$$

in particular

$$\|\overline{u}\|_{L^\infty(F_i)} \leq C + C\delta_i^{-\frac{\alpha}{2}} \max_{i\leq k\leq K} \delta_k^{-\frac{1}{2}(1-\alpha)} \|\nabla u\|_{L^2(F_k)} \tag{2.3.15}$$

for a constant $C = C(\alpha)$.

Here $\bar{u}$ *is defined by* (2.3.11)*, while* s_j *is as in* (2.3.12).

We shall use this lemma to show that on collars whose ends lie in the same part of $\bar{\delta}$-thick(M, g) we have

Lemma 2.3.5. *For any* $\gamma \geq 2$ *there exists a constant* C *so that if* (M, g)*,* σ *and* $\bar{\delta}$ *are as in Proposition 2.3.2 and if* u *is a normalised eigenfunction to eigenvalue* $\lambda \leq \eta_1$ *then* $\bar{u}(s) \coloneqq \fint_{\{s\}\times S^1} u(s,\theta)\, d\theta$ *satisfies*

$$|\bar{u}(s_j) - \bar{u}(-s_j)|^2 \leq C\lambda^2 \bar{\delta}^{-2+\beta} \cdot \delta_j^{-\beta} \text{ for every } j \text{ with } \delta_j \leq \bar{\delta}$$

for $\beta \coloneqq 2\log_2(1 + C_1\lambda)$*, where* C_1 *and* η_1 *are as in Lemma 2.3.4 and* s_j *and* δ_j *are defined by* (2.3.12).

The proofs of Lemma 2.3.4 and Lemma 2.3.5 will be given below; we first see how they are used to prove Proposition 2.3.2.

Proof of Proposition 2.3.2. Given $\gamma \geq 2$ and $\alpha \in (0, 1)$ we choose $\bar{\lambda} \in (0, \eta_1)$ so that $\beta \coloneqq 2\log_2(1 + C_1\bar{\lambda}) \leq \alpha$, where η_1 and C_1 are as in Lemma 2.3.4. We note that this choice of $\bar{\lambda}$ ensures that we can apply both Lemma 2.3.4 and Lemma 2.3.5 to control the behaviour of any normalised eigenfunction u to eigenvalue $\lambda \leq \bar{\lambda}$.

Given any such u and we let $K_1 = K_1(\bar{\delta}, \ell)$ be so that $\bar{\delta} \in [\delta_{K_1-1}, \delta_{K_1})$ and hence so that $\bar{\delta}$-thin$(\mathcal{C}(\sigma)) \subset \bigcup_{j=1}^{K_1} F_j \subset 2\bar{\delta}$-thin$(\mathcal{C}(\sigma))$ where we keep using the

notation introduced in (2.3.12). We now let $\mu_k \geq 0$ be so that

$$\|\nabla u\|^2_{L^2(F_k)} = \mu_k \lambda^2 \delta_k^{1-\alpha}, \quad k = 1, \dots, K_1$$

and note that the claim of the proposition follows if we prove that

$$\overline{\mu} := \max_{1 \leq k \leq K_1} \mu_k \leq C \bar{\delta}^{-2+\alpha}$$

for a constant C that only depends on α and the genus.

We will prove this by applying (2.3.3) for a suitable test function $w \in H^1(M, g)$ which is chosen in particular so that $w - u$ is supported in $\mathcal{C}^{\delta_j} := \{p \in \mathcal{C}(\sigma) : \mathrm{inj}_g(p) \leq \delta_j\} = \bigcup_{1 \leq i \leq j} F_i$, where we choose $j \in \{1, \dots, K_1\}$ so that $\mu_j = \bar{\mu}$. For any such w we know from (2.3.3) that

$$\bar{\mu} \delta_j^{1-\alpha} \lambda^2 \leq \|\nabla u\|^2_{L^2(\mathcal{C}^{\delta_j})} \leq \|\nabla w\|^2_{L^2(\mathcal{C}^{\delta_j})} - 2\lambda \langle u, w - u \rangle_{L^2(\mathcal{C}^{\delta_j})}. \tag{2.3.16}$$

To bound $\bar{\mu}$ we hence want to construct such a comparison function w with small energy on $\mathcal{C}^{\delta_j}$. To this end we exploit the smallness of the angular energy to first interpolate between $\bar{u}(\pm s_j)$ and $u(\pm s_j, \theta)$ on small transition regions $K_j^{\pm} := \pm[s_j - 1, s_j] \times S^1$ and then linearly interpolate between these values $\bar{u}(\pm s_j)$ whose difference is controlled by Lemma 2.3.5. To be more precise, we define

$$w(s, \theta) = \begin{cases} \frac{1}{2}(\bar{u}(s_j) + \bar{u}(-s_j)) + \frac{1}{2}(\bar{u}(s_j) - \bar{u}(-s_j)) \frac{s}{(s_j - 1)} & \text{for } |s| \leq s_j - 1 \\ \bar{u}(\pm s_j) + \varphi^{\pm}(s)[u(\pm s_j, \theta) - \bar{u}(\pm s_j)] & \text{for } s_j - 1 < \pm s \leq s_j, \end{cases}$$

$\varphi^{\pm}$ linear with $\varphi^{\pm}(\pm s_j) = 1$ and $\varphi^{\pm}(\pm(s_j - 1)) = 0$.

The energy of w on the transition regions $K_j^{\pm}$ is controlled by $C\vartheta(\pm s_j) \leq C\lambda^2\delta_j^2$, compare (2.3.2), while the energy on the rest of $\mathcal{C}^{\delta_j}$ is simply given by $\frac{2\pi}{4(s_j-1)}|\bar{u}(s_j) - \bar{u}(-s_j)|^2$.

Lemma 2.3.5, combined with the fact that $s_j \geq c\ell^{-1}$, hence implies that

$$\|\nabla w\|^2_{L^2(\mathcal{C}^{\delta_j})} \leq C\lambda^2\ell\bar{\delta}^{-2+\beta} \cdot \delta_j^{-\beta} + C\lambda^2\delta_j^2 \leq C\bar{\delta}^{-2+\alpha}\delta_j^{1-\alpha}\lambda^2, \tag{2.3.17}$$

where we use in the last step that $\beta \leq \alpha$.

To estimate the second term in (2.3.16) we set $\bar{w} \coloneqq \fint_{\{s\}\times S^1} w\, d\theta$ and write $u = \bar{u} + \xi_u$, $w = \bar{w} + \xi_w$. We can then exploit that ξ_u and ξ_w are orthogonal to both $\bar{u}$ and $\bar{w}$ and that ξ_u and ξ_w are bounded pointwise by $C\sup_{|s|\leq s_j} \vartheta^{\frac{1}{2}}(s) \leq C\delta_j\lambda$ (using the angular energy estimate (2.3.2)) to bound

$$\begin{aligned}
-2\lambda\langle u, w-u\rangle_{L^2(\mathcal{C}^{\delta_j})} &= -2\lambda\langle \bar{u}, \bar{w}-\bar{u}\rangle_{L^2(\mathcal{C}^{\delta_j})} - 2\lambda\langle \xi_u, \xi_w - \xi_u\rangle_{L^2(\mathcal{C}^{\delta_j})} \\
&\leq \lambda\sum_{i=1}^{j}\|\bar{u}\|_{L^\infty(F_i)}\|\bar{w}-\bar{u}\|_{L^\infty(F_i)} \cdot \mathrm{Area}_g(F_i) + C\lambda^3\delta_j^2 \cdot \mathrm{Area}_g(\mathcal{C}^{\delta_j}) \\
&\leq C\lambda\sum_{i=1}^{j}(1 + \delta_i^{-\frac{\alpha}{2}}\lambda\bar{\mu}^{\frac{1}{2}})\|\bar{w}-\bar{u}\|_{L^\infty(F_i)}\delta_i + C\lambda^3\delta_j^3
\end{aligned} \tag{2.3.18}$$

where we have used (2.3.13) as well as the estimate (2.3.15) from Lemma 2.3.4 in the last step.

As $\bar{u} - w = 0$ at $\pm s_j$ we can apply Lemmas 2.3.4 and 2.3.5 to bound (writing

$F_i^{\pm} \coloneqq \{p = (s,\theta) \in F_i : \pm s \geq 0\})$

$$\begin{aligned}\|\bar{w} - \bar{u}\|_{L^\infty(F_i^{\pm})} &\leq \operatorname*{osc}_{\mathcal{C}^{\delta_j}} \bar{w} + \sup_{s\in[s_{i-1},s_i]} |\bar{u}(\pm s) - \bar{u}(\pm s_j)| \\ &\leq |\bar{u}(s_j) - \bar{u}(-s_j)| + C\delta_i^{-\frac{\alpha}{2}} \max_{i\leq k\leq j} \delta_k^{-\frac{1}{2}(1-\alpha)} \|\nabla u\|_{L^2(F_k^{\pm})} \\ &\leq C\lambda\bar{\delta}^{-1+\frac{\alpha}{2}}\delta_i^{-\frac{\alpha}{2}} + C\lambda\delta_i^{-\frac{\alpha}{2}}\bar{\mu}^{\frac{1}{2}} = C\lambda\delta_i^{-\frac{\alpha}{2}}(\bar{\delta}^{-1+\frac{\alpha}{2}} + \bar{\mu}^{\frac{1}{2}}).\end{aligned}$$

Inserting this into (2.3.18) and using (2.3.17) hence allows us to see that

$$\|\nabla w\|^2_{L^2(\mathcal{C}^{\delta_j})} - 2\lambda\langle u, w-u\rangle_{L^2(\mathcal{C}^{\delta_j})} \leq C\lambda^2\delta_j^{1-\alpha}(\bar{\delta}^{-2+\alpha} + \bar{\mu}^{\frac{1}{2}} + \lambda\bar{\mu}).$$

Combined with (2.3.16) this yields the claimed bound of $\bar{\mu} \leq C\bar{\delta}^{-2+\alpha}$ (after reducing $\bar{\lambda}$ if necessary) for a constant C that only depends on α and the genus and hence concludes the proof of the proposition.

□

To prove Lemma 2.3.4 we use the following simple technical lemma.

Lemma 2.3.6. *Let $\mathcal{C}(\sigma)$ be a hyperbolic collar around a simple closed geodesic of length $\ell \in (0, 2\text{arsinh}(1))$, let F_j and s_j be as in (2.3.12) and let $k_0 \in \{1,\dots,K\}$. Then for any smooth function $h : [-s_{k_0}, s_{k_0}] \to \mathbb{R}$ we can bound*

$$\|h - h(\pm s_{k_0})\|_{L^\infty(F_j^{\pm})} \leq C\delta_j^{-\frac{1}{2}} \sum_{k=j}^{k_0} 2^{-(k-j)/2} \|\nabla h\|_{L^2(F_k^{\pm})}, \quad j = 1,\dots k_0$$

where $F_j^{\pm} := F_j \cap \{\pm s \geq 0\}$. If furthermore $h(-s_{k_0}) = h(s_{k_0}) = 0$ then also we have

$$\|h\|_{L^2(\delta_{k_0}\text{-thin}(\mathcal{C}(\sigma)))} \leq C\|\nabla h\|_{L^2(\delta_{k_0}\text{-thin}(\mathcal{C}(\sigma)))}.$$

Here C is a universal constant and all norms of $h(s,\theta) = h(s)$ are computed with respect to the hyperbolic metric $g = \rho^2(ds^2 + d\theta^2)$.

Proof of Lemma 2.3.6. Using (2.3.13) we immediately obtain that for any $s \in [s_{j-1}, s_j]$

$$|h(s) - h(s_{k_0})| \leq \sum_{k=j}^{k_0} \int_{s_{k-1}}^{s_k} |\partial_s h| ds \leq C \sum_{k=j}^{k_0} \delta_k^{-\frac{1}{2}} \|\nabla h\|_{L^2(F_k^+)}$$
$$= C\delta_j^{-\frac{1}{2}} \sum_{k=j}^{k_0} 2^{-(k-j)/2} \|\nabla h\|_{L^2(F_k^+)}.$$

Combined with the analogue estimate for $s \in -[s_{j-1}, s_j]$ this gives the first claim of the lemma. As $\mathrm{Area}_g(F_j) \leq \mathrm{Area}_g(\delta_j\text{-thin}(\mathcal{C}(\sigma))) \leq C\delta_j$ we hence obtain that if additionally $h(s_{k_0}) = h(-s_{k_0}) = 0$ then we have

$$\|h\|^2_{L^2(F_j)} \leq C \Big(\sum_{k=j}^{k_0} 2^{-(k-j)/2} \|\nabla h\|_{L^2(F_k)} \Big)^2$$
$$\leq C \max_{k \in \{j,\dots,k_0\}} 2^{-(k-j)/2} \|\nabla h\|^2_{L^2(F_k)} \cdot \Big(\sum_{k=j}^{k_0} 2^{-(k-j)/4} \Big)^2$$
$$\leq C \max_{k \in \{j,\dots,k_0\}} 2^{-(k-j)/2} \|\nabla h\|^2_{L^2(F_k)}.$$

This results in the desired estimate on $\delta_{k_0}\text{-thin}(\mathcal{C}(\sigma)) \subset \bigcup_{j=1}^{k_0} F_j$ of

$$\|h\|^2_{L^2(\delta_{k_0}\text{-thin}(\mathcal{C}(\sigma)))} \leq C \sum_{1 \leq j \leq k \leq k_0} 2^{-(k-j)/2} \|\nabla h\|^2_{L^2(F_k)} \leq C \|\nabla h\|^2_{L^2(\delta_{k_0}\text{-thin}(\mathcal{C}(\sigma)))}.$$

□

Next we turn to the proof of Lemma 2.3.4, which bounds the growth of the odd part of the function $\bar{u}$ defined earlier.

Proof of Lemma 2.3.4. Let (M, g), σ and u be as in the lemma, let F_k and s_k be as in (2.3.12) and let $f(s) \coloneqq \frac{1}{2}(\bar{u}(s) - \bar{u}(-s))$ be the odd part of $\bar{u}$ on the collar $\mathcal{C}(\sigma)$. Given any fixed $j \in \{1, \dots, K\}$ we set $\tilde{f}(s) := f(s) - \frac{s}{s_j} f(s_j)$ and note that $\tilde{f}$ is odd and vanishes at $\pm s_j$. We can hence obtain a function $w \in H^1(M, g)$ by setting $w \equiv u$ on $M \setminus \delta_j\text{-thin}(\mathcal{C}(\sigma))$ while on $\delta_j\text{-thin}(\mathcal{C}(\sigma)) = (-s_j, s_j) \times S^1$ we choose

$$w(s, \theta) = u(s, \theta) + \tilde{f}(s).$$

We note that as $w - u = \tilde{f}$ is independent of θ, it is orthogonal to $u - \bar{u}$ in the sense of both L^2 and H^1. Using that $-\Delta_g u = \lambda u$ we hence know that

$$2\pi \int_{I_j} \bar{u}' \tilde{f}' \, ds = \langle \nabla u, \nabla(w - u) \rangle_{L^2(M,g)} = \lambda \langle u, w - u \rangle_{L^2(M,g)} = \lambda \langle \bar{u}, \tilde{f} \rangle_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))}.$$

where we write for short $I_j = [-s_j, s_j]$. Exploiting that $\tilde{f}$ is also orthogonal to the even part $\bar{u} - f$ of $\bar{u}$ we can reduce this equation further to get

$$\begin{aligned} \lambda \langle \tilde{f}, f \rangle_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))} &= 2\pi \int_{I_j} f' \tilde{f}' \, ds = 2\pi \int_{I_j} |\tilde{f}'|^2 \, ds + 2\pi \frac{f(s_j)}{s_j} \int_{I_j} \tilde{f}' ds \\ &= \|\nabla \tilde{f}\|^2_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))}, \end{aligned}$$

where in the last equality we used that $\tilde{f}(\pm s_j) = 0$. This implies that for every $\varepsilon > 0$,

$$\begin{aligned} \|\nabla \tilde{f}\|^2_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))} &= \lambda \|\tilde{f}\|^2_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))} + \lambda f(s_j) \langle \tilde{f}, \frac{s}{s_j} \rangle_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))} \\ &\leq (\varepsilon + \lambda) \|\tilde{f}\|^2_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))} + \varepsilon^{-1} \lambda^2 |f(s_j)|^2 \text{Area}(\delta_j\text{-thin}(\mathcal{C}(\sigma))) \\ &\leq C(\lambda + \varepsilon) \|\nabla \tilde{f}\|^2_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))} + C\varepsilon^{-1} \delta_j \lambda^2 |f(s_j)|^2 \end{aligned}$$

where C is a universal constant and where we used Lemma 2.3.6 as well as (2.2.3)

in the last step.

Choosing $\varepsilon = \eta_1 = \frac{1}{4C}$ allows us to absorb the first term into the left-hand side and thus yields that

$$\|\nabla \tilde{f}\|^2_{L^2(\delta_j\text{-thin}(\mathcal{C}(\sigma)))} \leq C\delta_j \lambda^2 |f(s_j)|^2$$

holds true for a universal constant C. Combined with (2.3.13) and the fact that $\tilde{f}(s_j) = 0$ we hence know that $|\tilde{f}(s_{j-1})| \leq \int_{s_{j-1}}^{s_j} |\tilde{f}'| ds \leq C\lambda |f(s_j)|$ and so obtain the first claim of the lemma that for some universal constant C_1,

$$|f(s_{j-1})| = |\tilde{f}(s_{j-1}) + \frac{s_{j-1}}{s_j} f(s_j)| \leq \left(1 - \frac{s_j - s_{j-1}}{s_j} + C_1\lambda\right) |f(s_j)|.$$

To obtain the second claim of the lemma we can apply Lemma 2.3.6 as well as $\delta_i^{-\frac{1}{2}} = \delta_i^{-\frac{\alpha}{2}} 2^{(k-i)\frac{1-\alpha}{2}} \delta_k^{-\frac{1}{2}(1-\alpha)}$ to get that for any $s \in [s_{j-1}, s_j]$ and any $s' \in [s_{i-1}, s_i]$, $i \leq j$, we have

$$\begin{aligned} |\bar{u}(\pm s') - \bar{u}(\pm s)| &\leq C\delta_i^{-\frac{\alpha}{2}} \sum_{k=i}^{j} 2^{-(k-i)\frac{\alpha}{2}} \delta_k^{-\frac{1}{2}(1-\alpha)} \|\nabla u\|_{L^2(F_k^{\pm})} \\ &\leq C\delta_i^{-\frac{\alpha}{2}} \max_{i \leq k \leq j} \delta_k^{-\frac{1}{2}(1-\alpha)} \|\nabla u\|_{L^2(F_k^{\pm})}, \end{aligned}$$

where we use that $\sum_{k \geq i} 2^{-\frac{\alpha}{2}(k-i)} \leq C_\alpha$ in the last step.

We then note that in the special case that $s' = \pm X_\ell$ we obtain

$$|\bar{u}(s) - \bar{u}(\pm X(\ell))| \leq C\delta_j^{-\frac{\alpha}{2}} \max_{k \geq j} \delta_k^{-\frac{1}{2}(1-\alpha)} \|\nabla u\|_{L^2(F_k)}, \tag{2.3.19}$$

which we'll now use to prove the third claim. As we have a uniform H^1 bound on u on

all of M, so in particular on the two connected components of $\frac{1}{2}\text{arsinh}(1)$-thick$(\mathcal{C}(\sigma))$, and as on these sets the trace theorem holds true with a universal constant, we know that $\|u\|_{L^2(\{\pm X(\ell)\}\times S^1)} \leq C$ for a universal constant C. Combining the resulting upper bound on $|\bar{u}(\pm X(\ell))|$ with (2.3.19) hence immediately gives the last claim of the lemma. □

Proof of Lemma 2.3.5. Fix $(s,\theta) \in \bar{\delta}\text{-thick}(\mathcal{C}(\sigma), g)$ and let $\delta = \text{inj}_g(s,\theta) \geq \bar{\delta}$. As the ends of the collar are contained in the same connected component of $\bar{\delta}$-thick(M,g), we can apply the Poincaré inequality Lemma (2.2.4) to $v \coloneqq u - \bar{u}(-s)$ on the corresponding component of δ-thick$(M,g) \subset \bar{\delta}$-thick$(M,g)$ to obtain

$$\begin{aligned}|\bar{u}(s) - \bar{u}(-s)| &\leq C\int_{\{s\}\times S^1} |v|\, d\theta \leq C\left(\int_{\{s\}\times S^1} |v|^2\, d\theta\right)^{\frac{1}{2}} \leq C\delta^{-\frac{1}{2}}\|v\|_{L^2(\{s\}\times S^1, g)} \\ &\leq C\delta^{-\frac{1}{2}}\|\nabla v\|_{L^2(\delta\text{-thick}(M,g))} \leq C\bar{\delta}^{-1/2}\|\nabla u\|_{L^2(\bar{\delta}\text{-thick}(M,g))},\end{aligned}$$

for a constant C depending only on the genus γ. We recall that the energy over $\bar{\delta}$-thick(M,g) is bounded by Lemma 2.3.1, so we deduce that for any $(s,\theta) \in \bar{\delta}\text{-thick}(\mathcal{C}(\sigma), g)$

$$|\bar{u}(s) - \bar{u}(-s)| \leq C\bar{\delta}^{-1/2}\|\nabla u\|_{L^2(\bar{\delta}\text{-thick}(M,g))} \leq C\bar{\delta}^{-1}\lambda.$$

Choosing k so that $\delta_{k-1} \leq \bar{\delta} \leq \delta_k$ and $\beta > 0$ as in the lemma so that $2^\beta = (1 + C_1\lambda)$ we can thus use (2.3.14) to bound

$$|\bar{u}(s_j) - \bar{u}(-s_j)| \leq (1 + C_1\lambda)^{k-j}|\bar{u}(s_k) - \bar{u}(-s_k)| \leq C\bar{\delta}^{-1}\lambda 2^{(k-j)\beta} = C\lambda\bar{\delta}^{-1+\beta}\delta_j^{-\beta}$$

for every j with $\delta_j \leq \bar{\delta}$. □

2.4 Proofs of Main Results

In this section we use the energy estimates for the eigenfunctions of the Laplacian obtained in section 2.3 together with properties of holomorphic quadratic differentials dual to Fenchel-Nielsen coordinates proved by Große and Rupflin in [27] (see subsection 2.2.4) to prove estimates for the derivatives of the eigenvalues.

Before proving Theorem 2.1.1, we recall a lemma of Große and Rupflin which allows us to control inner products of the Hopf differentials of eigenfunctions with holomorphic quadratic differentials.

Lemma 2.4.1. *[Lemma 3.7 in [28]]*

For any genus $\gamma \geq 2$ there exists a constant C so that the following holds: Let (M, g) be a closed oriented hyperbolic surface of genus γ and let u be any eigenfunction of $-\Delta_g$ normalised by $\|u\|_{L^2(M,g)} = 1$ to an eigenvalue λ. Let $a > 0$ be so that

$$\|\nabla u\|^2_{L^2(\delta-\mathrm{thick}(M,g))} \leq \frac{a}{\delta}\lambda^2 \text{ for every } 0 < \delta \leq \frac{1}{2}\mathrm{arsinh}(1). \tag{2.4.1}$$

Then the Hopf differential $\Phi = \Phi(u, g)$ satisfies the following estimates. For every $\Upsilon \in \mathcal{H}(M, g)$ and every simple closed geodesic σ of length $\ell \leq \mathrm{arsinh}(1)$,

$$|\langle \Upsilon - b_0(\Upsilon, \mathcal{C}(\sigma))dz^2, \Phi\rangle_{L^2(\mathcal{C}(\sigma),g)}| \leq Ca\lambda^2 \|\Upsilon\|_{L^2(\frac{1}{2}\mathrm{arsinh}(1)-\mathrm{thick}(M,g))} \tag{2.4.2}$$

$$|\mathrm{Re}(\langle idz^2, \Phi\rangle_{L^2(\mathcal{C}(\sigma),g)}| \leq C\sqrt{a}\lambda^2\ell^{-1}. \tag{2.4.3}$$

We note that by Lemma 2.3.1, we know that (2.4.1) holds with $a = C(\gamma)$, so that the hypotheses of the above lemma are satisfied in the setting we consider.

Recall from (2.2.10) that given any smooth family of metrics g_ε with $g_0 = g$

and u_ε a corresponding smooth curve of eigenfunctions to a smooth curve of eigenvalues λ_ε, we have

$$\left.\frac{d\lambda_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{1}{2}\sum_{i=1}^{3(\gamma-1)}\left(\left\langle \mathrm{Re}\left(\Phi(u,g)\right), \mathrm{Re}(\Lambda^i)\right\rangle \left.\frac{d\ell_i(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \left\langle \mathrm{Re}\left(\Phi(u,g)\right), \mathrm{Re}(\Psi^i)\right\rangle \left.\frac{d\psi_i(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}\right),$$

where $\Phi = \Phi(u,g)$ denotes the Hopf differential of the corresponding normalised eigenfunction u and $\{\Lambda^j, \Psi^j\}$ is the basis of $\mathcal{H}(M,g)$ described in subsection 2.2.5. So in order to prove Theorem 2.1.1, we need to show that for all $1 \leq j \leq 3(\gamma-1)$

$$\left|\left\langle \mathrm{Re}\left(\Phi(u,g)\right), \mathrm{Re}(\Lambda^j)\right\rangle\right| \leq C\lambda^2\ell_j^{-2} \tag{2.4.4}$$

$$\left|\left\langle \mathrm{Re}\left(\Phi(u,g)\right), \mathrm{Re}(\Psi^j)\right\rangle\right| \leq C\lambda^2\ell_j^{2}, \tag{2.4.5}$$

while if we are given $\bar{\delta} \in (0, \mathrm{arsinh}(1)]$ and a geodesic σ^j such that both ends of the corresponding collar $\mathcal{C}(\sigma^j)$ lie in the same connected component of $\bar{\delta}$-thick(M,g), we have

$$\left|\left\langle \mathrm{Re}\left(\Phi(u,g)\right), \mathrm{Re}(\Lambda^j)\right\rangle\right| \leq C_\alpha\lambda^2\ell_j^{-\alpha}\bar{\delta}^{\alpha-2} \tag{2.4.6}$$

for $C_\alpha = C(\alpha,\gamma)$ and $\alpha \in (0,1)$.

Proof of Theorem 2.1.1. Throughout this proof, C will denote a generic positive constant which may depend on η and on the genus γ, unless stated otherwise. We first observe that we have $\|\Phi(u,g)\|_{L^1(F,g)} = 2\|\nabla u\|^2_{L^2(F,g)}$ for any subset $F \subset M$. Thus in particular by Lemma 2.3.1 we have

$$\|\Phi(u,g)\|_{L^1(\eta-\mathrm{thick}(M,g))} \leq C_\gamma\eta^{-1}\lambda^2 \leq C\lambda^2 \tag{2.4.7}$$

for a constant $C > 0$ depending only on η and the genus of M.

Now let us consider the derivatives with respect to the length coordinates ℓ_j. As the collars $\mathcal{C}(\sigma^i)$, $i = 1, \dots, 3(\gamma-1)$ are disjoint and $M \setminus \cup_{i=1}^{3(\gamma-1)} \mathcal{C}(\sigma^i) \subset \eta\text{-thick}(M,g)$, we can use (2.2.5) and bound

$$\begin{aligned}\frac{1}{2}\left|\langle \mathrm{Re}(\Phi), \mathrm{Re}(\Lambda^j)\rangle_{L^2(M,g)}\right| &= \frac{1}{4}\left|\mathrm{Re}\langle \Phi, \Lambda^j\rangle_{L^2(M,g)}\right| \\ &\leq \frac{1}{4}\left|\mathrm{Re}\langle \Phi, \Lambda^j\rangle_{L^2(M\setminus\cup_{i=1}^{3(\gamma-1)}\mathcal{C}(\sigma^i),g)}\right| + \frac{1}{4}\sum_{i=1}^{3(\gamma-1)}\left|\mathrm{Re}\langle \Phi, \Lambda^j\rangle_{L^2(\mathcal{C}(\sigma^i),g)}\right|\end{aligned}$$

We estimate the first term using (2.4.7) and (2.2.12) to find

$$|\langle \Phi, \Lambda^j\rangle_{L^2(M\setminus\cup_{i=1}^{3(\gamma-1)}\mathcal{C}(\sigma^i),g)}| \leq \|\Phi\|_{L^1(\eta-\mathrm{thick}(M,g))}\|\Lambda^j\|_{L^\infty(\eta-\mathrm{thick}(M,g))} \leq C\lambda^2\ell_j.$$

On the thin collars, we split Λ^j into its principal and collar decay parts

$$\langle \Phi, \Lambda^j\rangle_{L^2(\mathcal{C}(\sigma^i),g)} = \langle \Phi, b_0^i(\Lambda^j)dz^2\rangle_{L^2(\mathcal{C}(\sigma^i),g)} + \langle \Phi, \Lambda^j - b_0^i(\Lambda^j)dz^2\rangle_{L^2(\mathcal{C}(\sigma^i),g)},$$

and estimate the collar decay term using (2.4.2) and (2.2.12) to obtain

$$|\langle \Phi, \Lambda^j - b_0^i(\Lambda^j)dz^2\rangle_{L^2(\mathcal{C}(\sigma^i),g)}| \leq C\lambda^2\|\Lambda^j\|_{L^2(\frac{1}{2}\mathrm{arsinh}(1)-\mathrm{thick}(M,g))} \leq C\lambda^2\ell_j.$$

Recall now from (2.2.13) that $\mathrm{Re}(b_0^i(\Lambda^j)) = -\frac{\ell_j}{2\pi^2}\delta_{ij}$, so when $i \neq j$ we have

$\mathrm{Re}(b_0^i(\Lambda^j)) = 0$. Hence, using (2.2.13) and (2.4.3) we find that for all $i \neq j$

$$\begin{aligned}|\mathrm{Re}\langle\Phi, b_0^i(\Lambda^j)dz^2\rangle_{L^2(\mathcal{C}(\sigma^i),g)}| &= |\mathrm{Re}(\langle\Phi, i\mathrm{Im}(b_0^i(\Lambda^j))dz^2\rangle_{L^2(\mathcal{C}(\sigma^i),g)})| \\ &\leq |\mathrm{Im}(b_0^i(\Lambda^j))||\mathrm{Re}\langle idz^2, \Phi\rangle_{L^2(\mathcal{C}(\sigma^i),g)}| \\ &\leq C\ell_j\ell_i^3 \cdot C\lambda^2\ell_i^{-1} \leq C\lambda^2\ell_j.\end{aligned}$$

For all $1 \leq j \leq 3(\gamma-1)$, we therefore find

$$|\langle\Phi, \Lambda^j\rangle| \leq C\ell_j\lambda^2 + |\langle\mathrm{Re}(b_0^j(\Lambda^j))dz^2, \Phi\rangle|.$$

We estimate the second term on the RHS using (2.2.13), (2.2.6), and (2.4.7), finding

$$\begin{aligned}|\langle\mathrm{Re}(b_0^j(\Lambda^j)dz^2), \Phi\rangle| &\leq |\mathrm{Re}(b_0^j(\Lambda^j))|\|dz^2\|_{L^\infty(\mathcal{C}(\sigma^j),g)}\|\Phi\|_{L^1(\frac{\ell_j}{2}\text{-thick}(M,g))} \\ &\leq C\ell_j \cdot C\ell_j^{-2} \cdot C\lambda^2\ell_j^{-1} \leq C\lambda^2\ell_j^{-2},\end{aligned}$$

where we have also used the fact that $\mathcal{C}(\sigma^j) \subset \frac{\ell_j}{2}\text{-thick}(M,g)$. This basic estimate holds for all $1 \leq j \leq 3(\gamma-1)$, giving the claimed bound (2.4.4) for the derivative of λ with respect to the length coordinate ℓ_j.

Now, given $\bar{\delta}$ and a geodesic σ^j such that both ends of the corresponding collar $\mathcal{C}(\sigma^j)$ lie in the same connected component of $\bar{\delta}$-thick(M,g), we can improve the bound on $|\langle\mathrm{Re}(b_0^j(\Lambda^j)dz^2), \Phi\rangle|$ and hence on $\left|\frac{1}{2}\mathrm{Re}\langle\Phi, \Lambda^j\rangle\right|$ as follows. On the collar $\mathcal{C}(\sigma^j)$, we let $\delta_i = 2^{i-1} \cdot \ell_j$ and decompose the collar region $\mathcal{C}(\sigma^j)$ around σ^j into hyperbolic cylinders F_i, such that $F_i = \{p \in \mathcal{C}(\sigma^j) : \mathrm{inj}_g(p) \in [\delta_{i-1}, \delta_i]\}$, as

in (2.3.12). Then, using Proposition 2.3.2, as well as (2.2.13) and (2.2.6) we have

$$\begin{aligned}
|\langle \Phi, b_0^j(\Lambda^j)dz^2\rangle_{L^2(\mathcal{C}(\sigma^j),g)}| &= \left|\sum_i \langle \Phi, b_0^j(\Lambda^j)dz^2\rangle_{L^2(F_i,g)}\right| \\
&\leq \sum_i \|\Phi\|_{L^1(F_i,g)} |b_0^j(\Lambda^j)| \|dz^2\|_{L^\infty(F_i,g)} \\
&\leq C\lambda^2\bar{\delta}^{\alpha-2}\sum_i \delta_i^{1-\alpha}\cdot \ell_j \cdot \delta_i^{-2} = C\lambda^2\bar{\delta}^{\alpha-2}\ell_j \sum_i \delta_i^{-(1+\alpha)} \\
&\leq C_\alpha \lambda^2 \bar{\delta}^{\alpha-2}\ell_j^{-\alpha},
\end{aligned}$$

where C_α depends also on $\alpha \in (0,1)$ as well as the usual dependence on η and γ. Hence for those collars $\mathcal{C}(\sigma^j)$ with both ends lying in the same component of $\bar{\delta}$-thick(M,g) we have

$$\left|\frac{1}{2}\mathrm{Re}\langle \Phi, \Lambda^j\rangle_{L^2(M,g)}\right| \leq C_\alpha \lambda^2 \bar{\delta}^{\alpha-2}\ell_j^{-\alpha}$$

as claimed in (2.4.6)

We now turn to the derivatives with respect to the twist coordinates ψ_j. As with the length coordinates, we begin by splitting the inner product into a term on the η-thick part of the surface, and additional terms on the collars $\mathcal{C}(\sigma^i)$ for $1 \leq i \leq 3(\gamma-1)$

$$\begin{aligned}
\frac{1}{2}\left|\langle \mathrm{Re}(\Phi), \mathrm{Re}(\Psi^j)\rangle_{L^2(M,g)}\right| &= \frac{1}{4}\left|\mathrm{Re}\langle \Phi, \Psi^j\rangle_{L^2(M,g)}\right| \\
&\leq \frac{1}{4}\left|\mathrm{Re}\langle \Phi, \Psi^j\rangle_{L^2(M\setminus \cup_{i=1}^{3(\gamma-1)}\mathcal{C}(\sigma^i),g)}\right| + \frac{1}{4}\sum_{i=1}^{3(\gamma-1)}\left|\mathrm{Re}\langle \Phi, \Psi^j\rangle_{L^2(\mathcal{C}(\sigma^i),g)}\right|
\end{aligned}$$

We estimate the first term using (2.4.7) and (2.2.12) to find

$$|\langle \Phi, \Psi^j \rangle_{L^2(\eta-\mathrm{thick}(M),g)}| \leq \|\Phi\|_{L^1(\eta-\mathrm{thick}(M,g))} \|\Psi^j\|_{L^\infty(\eta-\mathrm{thick}(M,g))} \leq C\lambda^2 \ell_j^3.$$

On the collars, we split Ψ^j into its principal and collar decay parts

$$\langle \Phi, \Psi^j \rangle_{L^2(\mathcal{C}(\sigma^i),g)} = \langle \Phi, b_0^i(\Psi^j) dz^2 \rangle_{L^2(\mathcal{C}(\sigma^i),g)} + \langle \Phi, \Psi^j - b_0^i(\Psi^j) dz^2 \rangle_{L^2(\mathcal{C}(\sigma^i),g)},$$

and estimate the collar decay term using (2.4.2) and (2.2.12) to obtain

$$|\langle \Phi, \Psi^j - b_0^i(\Psi^j) dz^2 \rangle_{L^2(\mathcal{C}(\sigma^i),g)}| \leq C\lambda^2 \|\Psi^j\|_{L^2(\frac{1}{2}\mathrm{arsinh}(1)-\mathrm{thick}(M,g))} \leq C\lambda^2 \ell_j^3.$$

It remains to bound the terms involving the principal parts of Ψ^j on the collars. We note from (2.2.14) that the real part of $b_0^i(\Psi^j)$ vanishes, so we have (using collar coordinates (s, θ))

$$\begin{aligned} \mathrm{Re}(b_0^i(\Psi^j) dz^2) &= \mathrm{Re}(b_0^i(\Psi^j))(ds^2 - d\theta^2) - \mathrm{Im}(b_0^i(\Psi^j))(ds \otimes d\theta + d\theta \otimes ds) \\ &= -\mathrm{Im}(b_0^i(\Psi^j))(ds \otimes d\theta + d\theta \otimes ds). \end{aligned}$$

Thus, recalling the definition of the Hopf differential (2.2.8) we see that the relevant inner product is bounded by

$$\left|\langle \mathrm{Re}(\Phi), \mathrm{Re}(b_0^i(\Psi^j) dz^2) \rangle_{L^2(\mathcal{C}(\sigma^i),g)}\right| \leq C \int_{\mathcal{C}(\sigma^i)} |u_s u_\theta \mathrm{Im}(b_0^i(\Psi^j)) \rho^{-2}| \, ds d\theta.$$

We may apply the angular energy estimate (2.3.1) to find that

$$\begin{aligned}\int_{\mathcal{C}(\sigma^i)} |u_s u_\theta \mathrm{Im}(b_0^i(\Psi^j))\rho^{-2}|\, ds\, d\theta &\leq \left(\int_{\mathcal{C}(\sigma^i)} |u_s|^2\, ds\, d\theta\right)^{\frac{1}{2}} \left(\int_{\mathcal{C}(\sigma^i)} \rho^{-4}|u_\theta|^2\, ds\, d\theta\right)^{\frac{1}{2}} |\mathrm{Im}(b_0^i(\Psi^j))| \\ &\leq \left(\int_{\mathcal{C}(\sigma^i)} |u_s|^2\, ds\, d\theta\right)^{\frac{1}{2}} \cdot \left(C\lambda^2 + C\ell_i^{-1}\lambda^2\right)^{\frac{1}{2}} |\mathrm{Im}(b_0^i(\Psi^j))|.\end{aligned}$$

we note that by Lemma 2.3.1 we have

$$\left(\int_{\mathcal{C}(\sigma^i)} |u_s|^2\, ds\, d\theta\right)^{\frac{1}{2}} \leq \|\nabla u\|_{L^2(\frac{\ell_i}{2}-\mathrm{thick}(M,g))} \leq C\lambda \ell_i^{-\frac{1}{2}},$$

so that altogether we find

$$\left|\langle \mathrm{Re}(\Phi), \mathrm{Re}(b_0^i(\Psi^j)dz^2)\rangle_{L^2(\mathcal{C}(\sigma^i),g)}\right| \leq C\lambda^2\ell_i^{-1}|\mathrm{Im}(b_0^i(\Psi^j))|.$$

To estimate $|\mathrm{Im}(b_0^i(\Psi^j))|$, we can use (2.2.14). While that result gives separate estimates for the cases $i = j$ and $i \neq j$, we see that in both cases, we have the estimate

$$|\mathrm{Im}(b_0^i(\Psi^j))| \leq C\ell_j^3.$$

So for all $1 \leq j \leq 3(\gamma - 1)$, we have

$$\left|\frac{1}{2}\mathrm{Re}\langle \Phi, \Psi^j\rangle_{L^2(M,g)}\right| \leq C\lambda^2\ell_j^2,$$

as claimed in (2.4.5).

□

Next we turn to the proof of Corollary 2.1.3.

Proof of Corollary 2.1.3. Let $\{\sigma^j\}_{j=1}^{j_0}$ be a collection of disjoint simple closed curves as considered in the corollary, and let $0 < \bar{\delta} < \min\{\frac{\bar{c}}{2}, \operatorname{arsinh}(1)\}$ be fixed, where $\bar{c} = \bar{c}(\gamma) > 0$ is a constant to be chosen later. We recall the well-known fact that there exists a finite set of collections $\overline{\mathcal{E}}_i = \{\bar{\sigma}_i^j\}_{j=1}^{3(\gamma-1)}$, $i = 1, \dots, N(\gamma)$, of disjoint simple closed curves in M with $\bar{\sigma}_i^j = \sigma^j$ for $j = 1, \dots, j_0$, which decompose M into pairs of pants and which have the following property: Given any decomposing collection of simple closed curves $\tilde{\mathcal{E}} = \{\tilde{\sigma}^j\}_{j=1}^{3(\gamma-1)}$ with $\tilde{\sigma}^j = \sigma^j$ for $j = 1, \dots, j_0$, there exists an index $1 \leq i \leq N$ and a diffeomorphism $\tilde{f} : M \to M$ which maps $\tilde{\sigma}^j$ to $\bar{\sigma}_i^j$ for every j.

We want to define the function $f_k(\ell_1, \dots, \ell_{j_0})$ as the k^{th} eigenvalue of the Laplacian on a comparison surface $(M, g_{\underline{\ell}})$ for which the lengths of the geodesics corresponding to $\{\sigma^j\}_{j=1}^{j_0}$ are given by $\underline{\ell} \coloneqq (\ell_1, \dots, \ell_{j_0})$. To fix such a comparison surface, we choose constants $c_j > 2\operatorname{arsinh}(1)$ for $j = j_0 + 1, \dots, 3(\gamma - 1)$ and $\tilde{c}_j \in [0, 2\pi]$ for all $j = 1, \dots, 3(\gamma-1)$ and let $g_{\underline{\ell}}$ be a hyperbolic metric on M whose Fenchel-Nielsen coordinates $\{\ell_j^{(1)}, \psi_j^{(1)}\}_{j=1}^{3(\gamma-1)}$ with respect to $\overline{\mathcal{E}}_1$ are given by

$$\ell_j^{(1)}(g_{\underline{\ell}}) = \ell_j, \quad j = 1, \dots, j_0 \tag{2.4.8}$$

$$\ell_j^{(1)}(g_{\underline{\ell}}) = c_j, \quad j = j_0 + 1, \dots, 3(\gamma - 1) \tag{2.4.9}$$

$$\psi_j^{(1)}(g_{\underline{\ell}}) = \tilde{c}_j, \quad j = 1, \dots, 3(\gamma - 1). \tag{2.4.10}$$

Here and in the following we denote by $(\ell_j^{(i)}, \psi_j^{(i)})(g)$ the Fenchel-Nielsen coordinates of a metric g on M with respect to the decomposing collection of geodesics which are homotopic to $\overline{\mathcal{E}}_i$. Note that (2.4.8),(2.4.9), and (2.4.10) determine $g_{\underline{\ell}}$ up to pullback by diffeomorphisms and hence lead to a well-defined function $f_k(\ell_1, \dots, \ell_{j_0}) \coloneqq \lambda_k(M, g_{\underline{\ell}})$.

We note that we have a uniform lower bound $c_0 > 0$ on the length of any closed geodesic σ in $(M, g_{\underline{\ell}})$ which is not homotopic to one of $\sigma^1, \dots, \sigma^{j_0}$. To see this, note that either σ is homotopic to σ^j for some $j > j_0$, in which case $L_{g_{\underline{\ell}}}(\sigma) > 2\text{arsinh}(1)$, or else σ must intersect one of the curves σ^j for some j, and then $L_g(\sigma)$ is bounded below by the width of the collar $\mathcal{C}(\sigma^j)$ for some j, which is uniformly bounded from below by a constant c_0. In particular, the length coordinates $\ell_j^{(i)}(g_{\underline{\ell}})$ for $j \geq j_0 + 1$ with respect to any of the collections $\overline{\mathcal{E}}_i$ are bounded from below by c_0. As we have fixed the collections $\overline{\mathcal{E}}_i$ we also have a uniform upper bound $\bar{L} = \bar{L}(\gamma, \{\overline{\mathcal{E}}_i\}_{i=1}^N, \{c_j\}_{j=j_0+1}^{3(\gamma-1)})$ on the length coordinates $\ell_j^{(i)}(g_{\underline{\ell}})$ (independent of $\underline{\ell}$ but allowed to depend on the initial choice of the $\overline{\mathcal{E}}_i$ and the fixed constants c_j.)

Let now g be a metric as considered in the corollary. Let $\{\sigma^{j_0+1}, \dots, \sigma^{j_1}\}$ be the (possibly empty) collection of geodesics which are disjoint from $\{\sigma^1, \dots, \sigma^{j_0}\}$ and have length $L_g(\sigma^j) < 2\bar{\delta} \leq 2\text{arsinh}(1)$. Then we can extend $\{\sigma^1, \dots, \sigma^{j_1}\}$ to a decomposing collection of geodesics $\{\sigma^j\}_{j=1}^{3(\gamma-1)}$ in a way that ensures $\ell_j^{(i)} \leq \bar{L}_2(\gamma)$ for all $1 \leq j \leq 3(\gamma-1)$. Having chosen $\overline{\mathcal{E}}_i$ as above we know that there exists $i \in \{1, \dots, N\}$ and a diffeomorphism which maps σ^j to $\bar{\sigma}_i^j$ for all $j = 1, \dots, 3(\gamma-1)$. Hence, replacing g by the pullback metric, we can assume without loss of generality that σ^j is homotopic to $\bar{\sigma}_i^j$ for all j. Modifying this diffeomorphism if necessary, we can further assume that the corresponding twist coordinates are such that $\psi_j^{(i)}(g) \in [0, 2\pi]$ for all $1 \leq j \leq 3(\gamma-1)$. We recall that pulling back by diffeomorphisms in this way has no effect on the eigenvalues.

We now consider a curve of metrics $(g(t))_{t\in[0,1]}$ which interpolates between g and the comparison metric $g_{\underline{\ell}}$, $\underline{\ell} = (\ell_1, \dots, \ell_{j_0})$, constructed above. To be more precise, for $t \in [0, 1]$, we let $g(t)$ be a metric with Fenchel-Nielsen length coordinates (with

respect to $\overline{\mathcal{E}}_i$) of

$$\ell_j^{(i)}(g(t)) = (1-t)\ell_j^{(i)}(g) + t\ell_j^{(i)}(g_{\underline{\ell}})$$
$$\psi_j^{(i)}(g(t)) = (1-t)\psi_j^{(i)}(g) + t\psi_j^{(i)}(g_{\underline{\ell}})$$

Note that we have a uniform upper bound on the length coordinates $\ell_j^{(i)}(g(t)) \leq \bar{L}(\gamma)$ for all $1 \leq j \leq 3(\gamma-1)$, as well as a uniform lower bound of $\ell_j^{(i)}(g(t)) \geq 2\bar{\delta}$ for $j \geq j_1 + 1$, while at the same time for $j_0 + 1 \leq j \leq j_1$ we only know that $\ell_j^{(i)}(g(t)) \geq (1-t)\ell_j^{(i)}(g) + c_0 t \geq c_0 t$.

The corresponding geodesics $\sigma^j(g(t))$, $j_0 + 1 \leq j \leq j_1$, will be so that we can apply the improved estimate (2.1.8) for the dependence of the eigenvalue on those length coordinates. To see this, note that any closed geodesic in $M \setminus \cup_{j=1}^{j_1}\sigma^j(g(t))$ has length bounded below by a uniform constant $\bar{c} = \bar{c}(\bar{L}) > 0$. So we immediately get that both ends of each collar $\mathcal{C}(\sigma^j)(g(t))$ with $j_0 + 1 \leq j \leq j_1$ lie in the same connected component of $\frac{\bar{c}}{2}$-thick$(M, g(t)) \subset \bar{\delta}$-thick$(M, g(t))$, since we assumed that $\bar{\delta} \leq \frac{\bar{c}}{2}$ for $\bar{c}$ chosen in this way.

For $1 \leq k \leq n$ and $t \in [0,1]$, let $u_t^{(k)}$ denote a smooth curve of eigenfunctions of $-\Delta_{g(t)}$ with corresponding smooth curve of eigenvalues $\lambda^{(k)}(t)$ satisfying $\lambda^{(k)}(0) = \lambda_k(g(0))$[1]. Note that these smooth curves of eigenvalues may cross, so that $\lambda^{(k)}(1)$ does not necessarily equal $\lambda_k(g(1))$. By the results of Schoen, Wolpert and Yau [61] there exists a positive constant $c = c(\gamma, \bar{\delta})$ such that $\lambda_{n+1} \geq c$. At the same time, [61] also implies $\lambda_n(g(t)) \leq C\sum_{j=1}^{j_0}\ell_j \leq \min\{\frac{1}{2}c, \eta_1\}$ for all $t \in [0,1]$, where the last inequality holds provided we have $\sum_{j=1}^{j_0}\ell_j \leq \varepsilon = \varepsilon(\gamma, \bar{\delta})$. Note that if $\sum_{j=1}^{j_0}\ell_j \geq \varepsilon$ then the claim of the corollary holds trivially provided we take the

[1]We note that such curves can always be chosen, see for example the following result of Alekseevsky, Kriegl, Losik and Michor [1, Theorem 7.8].

constant C to be sufficiently large, so without loss of generality we henceforth assume $\sum_{j=1}^{j_0} \ell_j \leq \varepsilon$. This in particular allows us to apply our derivative estimates along the curves of eigenvalues $\lambda^{(k)}(t)$, while also guaranteeing that for all $t \in [0,1]$ the sets $\{\lambda_k(g(t))\}_{k=1}^n$ and $\{\lambda^{(k)}(t)\}_{k=1}^n$ coincide.

As the length coordinates $\ell_j^{(i)}(g(t))$ with $j \leq j_0$ are constant along the curve of metrics $g(t)$, we find that differentiating the eigenvalue $\lambda^{(k)}(t)$ along the curve $g(t)$ yields

$$\left|\frac{\partial \lambda^{(k)}(g(t))}{\partial t}\right| \leq 2\pi \sum_{j=1}^{3(\gamma-1)} \left|\frac{\partial \lambda^{(k)}}{\partial \psi_j^{(i)}}\right| + C \sum_{j=j_0+1}^{3(\gamma-1)} \left|\frac{\partial \lambda^{(k)}}{\partial \ell_j^{(i)}}\right|.$$

We see that the metrics $g(t)$ satisfy the hypotheses of our derivative estimates for every $t \in [0,1]$. We will use Theorem 2.1.1 with $\alpha = \frac{1}{2}$ to estimate the eigenvalue derivatives with respect to the Fenchel-Nielsen coordinates. Applying (2.1.7) to bound the twist derivatives, (2.1.8) with $\alpha = \frac{1}{2}$ to bound the length derivatives with $j = j_0 + 1, \ldots, j_1$, and (2.1.6) to bound the remaining length derivatives, we have

$$\left|\frac{\partial \lambda^{(k)}(g(t))}{\partial t}\right| \leq C(\lambda^{(k)}(t))^2 \sum_{j=1}^{3(\gamma-1)} (\ell_j^{(i)})^2 + C(\lambda^{(k)}(t))^2 \sum_{j=j_0+1}^{j_1} (\ell_j^{(i)})^{-\frac{1}{2}} + C(\lambda^{(k)}(t))^2 \sum_{j=j_1+1}^{3(\gamma-1)} (\ell_j^{(i)})^{-2},$$

for constants C depending only on $\gamma, \eta, \bar{L}$, and $\bar{\delta}$.

Using the fact that we have a uniform upper bound $\bar{L}$ for $\ell_j^{(i)}$ for all $j = 1, \ldots, 3(\gamma-1)$ as well as a uniform lower bound c_0 for $\ell_j^{(i)}$ for $j \geq j_1 + 1$, while for $j = j_0+1, \ldots, j_1$ we have the bound $\ell_j^{(i)}(g(t)) \geq c_0 t$, together with the result (2.1.1) of Schoen, Wolpert and Yau [61] which tells us that $\lambda^{(k)}(t) \leq C \sum_{j=1}^{j_0} \ell_j$, we obtain

the estimate

$$\left|\frac{\partial\lambda^{(k)}(g(t))}{\partial t}\right| \leq C\left(\sum_{j=1}^{j_0}\ell_j\right)^2\left(1+t^{-\frac{1}{2}}\right)$$

for every $t \in [0,1]$, where the constant C again depends only on $\gamma, \eta, \bar{L}$, and $\bar{\delta}$. Integrating this inequality for $t \in [0,1]$ hence yields the following bound

$$\begin{aligned}|\lambda^{(k)}(0) - \lambda^{(k)}(1)| &\leq C\left(\sum_{j=1}^{j_0}\ell_j\right)^2\int_0^1 1+t^{-\frac{1}{2}}\,dt\\ &\leq C\left(\sum_{j=1}^{j_0}\ell_j\right)^2.\end{aligned}$$

To conclude, we recall the sets $\{\lambda_k(g(t))\}_{k=1}^n$ and $\{\lambda^{(k)}(t)\}_{k=1}^n$ coincide for all $t \in [0,1]$ and apply the following elementary combinatorial fact: Given two ordered lists of real numbers $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$, and π any permutation of the set $\{1,\ldots,n\}$, the following is true:

$$\max_{1\leq k\leq n}|a_k - b_k| \leq \max_{1\leq k\leq n}|a_k - b_{\pi(k)}|.$$

Using this fact for $a_k = \lambda_k(g(0))$ and $b_k = \lambda_k(g(1))$, we see that the above bounds on $|\lambda^{(k)}(0) - \lambda^{(k)}(1)|$ for $k = 1,\ldots,n$ imply the claim of the corollary. □

Chapter 3

Conformal Laplacian Eigenvalue Estimates for Asymptotically Conical Families of Metrics

In this chapter we study the eigenvalues of the conformal Laplacian $-L_g = -\Delta_g + \frac{n-2}{4(n-1)}R_g$ for smooth metrics g on a closed manifold M. Unlike for the Laplacian, currently a Korevaar-type bound for eigenvalues of $-L_g$ is not available. We will explore possibilities of what estimates one might be able to prove by considering metrics g belonging to a fixed conformal class and which are truncated versions of certain asymptotically conical metrics on $M \setminus \Sigma$ for $\Sigma \subset M$ a closed submanifold. We will prove a sharp estimate for the first eigenvalue $\nu_1(-L_g)$ for these metrics when $n \geq 3$. This is inspired by work of Amman and Jammes [2], who considered truncations of asymptotically cylindrical metrics on compact manifolds. Although we do not succeed in proving a Korevaar-type bound, we hope that the metrics we study and the results we obtain will provide a guiding example for further

investigation of the eigenvalues of the conformal Laplacian.

3.1 Preliminaries

3.1.1 Summary of Known Results for the Laplacian

Let us begin by summarising some of the known results and important developments concerning the spectrum of the Laplacian $-\Delta_g$, where more results are available than for $-L_g$. The reader who is interested only in results for eigenvalues of $-L_g$ may go directly to subsection 3.1.2.

Let M^n be an n-dimensional closed manifold and let $\mathcal{M}(M)$ denote the set of all Riemannian metrics on M. An early example of an upper bound for Laplacian eigenvalues given by Yang and Yau [70] and subsequently improved by El Soufi and Ilias [21] asserts that if M is a closed orientable (two-dimensional) surface of genus γ, then the first eigenvalue $\lambda_1(-\Delta_g)$ satisfies

$$\sup_{g\in\mathcal{M}(M)} \lambda_1(-\Delta_g)\mathrm{Vol}(M,g) \leq 8\pi \left[\frac{\gamma+3}{2}\right], \tag{3.1.1}$$

where $\left[\frac{\gamma+3}{2}\right]$ denotes the integer part of $\frac{\gamma+3}{2}$. We note that similar estimates are known in the case of non-orientable surfaces, see for example Karpukhin [39].

The sharpness of these results has been studied by several authors. Equality in (3.1.1) can only be attained when $\gamma = 0$ (Hersch, [32]) or $\gamma = 2$ (Nayatani and Shoda, [56]). For all other values of γ, the inequality (3.1.1) is strict, see Nadirashvili [53] for $\gamma = 1$ and Karpukhin [38] for all other γ.

The result of Yang and Yau was extended to higher eigenvalues $\lambda_k(-\Delta_g), k \geq 1$

(still on orientable surfaces) by Korevaar [43, Theorem 0.5], who showed that

$$\sup_{g\in\mathcal{M}(M)} \lambda_k(-\Delta_g)\mathrm{Vol}(M,g) \leq C(\gamma+1)k,$$

for C a universal constant. Korevaar also proved an upper bound on higher dimensional manifolds if we restrict to a single conformal class [43, Theorem 0.3] (see also the work of Grigor'yan, Netrusov, and Yau [26]): If $n \geq 3$ and we fix a metric $g_0 \in \mathcal{M}(M)$, with $[g_0]$ denoting the corresponding conformal class, then we have[1]

$$\sup_{g\in[g_0]} \lambda_k(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} \leq C([g_0])k^{\frac{2}{n}} \tag{3.1.2}$$

for a constant $C > 0$ depending only on $[g_0]$. A lower bound for the normalised eigenvalue was obtained by Colbois and El Soufi [18], who showed that for $n \geq 2$

$$\sup_{g\in[g_0]} \lambda_k(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} \geq n\omega_n^{\frac{2}{n}}k^{\frac{2}{n}}, \tag{3.1.3}$$

where ω_n denotes the volume of the standard unit n-sphere. For the first eigenvalue $\lambda_1(-\Delta_g)$, equality in (3.1.3) is attained by the sphere $M = S^n$ with its standard conformal class. When $k = 2$ and $n = 2$, it is also known that equality is attained for the sphere S^2 with its standard conformal class due to a result of Nadirashvili [54].

We note that in contrast with the lower bound, the upper bound in (3.1.2) is not explicit; Korevaar's result provides us with no information about the constant C beyond the fact that it depends only on the conformal class $[g_0]$. Subsequent authors

[1]We note that the restriction to a single conformal class is necessary, since Colbois and Dodziuk [17] showed that for $n \geq 3$ we have $\sup_{g\in\mathcal{M}(M)} \lambda_1(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} = +\infty$.

have shed light on the nature of the constant by proving Korevaar-type estimates that depend explicitly on certain conformal invariants. One such approach uses the notion of m-dimensional *conformal volume* $V_c(m, M, [g_0])$ for an integer $m > 0$, which was originally introduced by Li and Yau in [48]. It is defined as

$$V_c(m, M, [g_0]) \coloneqq \inf_{\phi} \sup_{s} \{\mathrm{Vol}(M, (s \circ \phi)^* g_{\mathrm{can.}})\},$$

where the supremum is taken over all conformal diffeomorphisms s of the standard unit m-sphere $(S^m, g_{\mathrm{can.}})$ and the infimum is taken over all conformal immersions $\phi : (M, g_0) \to (S^m, g_{\mathrm{can.}})$. We note that $V_c(m, M, [g_0])$ may not be well-defined for all integers $m > 0$, since it may be the case for a given m that no conformal immersions $\phi : M \to S^m$ exist. However, $V_c(m, M, [g_0])$ will always be well-defined for m sufficiently large, since any compact manifold (M, g) may be embedded isometrically into $\mathbb{R}^N$ for some N by the Nash embedding theorem, and $\mathbb{R}^N$ admits an immersion into the torus $T^N \subset S^{2N-1}$ given by $(x_1, \ldots, , x_n) \mapsto (\cos x_1, \sin x_1, \ldots, \cos x_N, \sin x_N)$.

Li and Yau [48] used the notion of conformal volume to show the following bound on the first eigenvalue for surfaces, which was extended to all dimensions $n \geq 2$ by El Soufi and Ilias [22]:

$$\sup_{g \in [g_0]} \lambda_1(-\Delta_g)\mathrm{Vol}(M, g)^{\frac{2}{n}} \leq n V_c(m, M, [g_0])^{\frac{2}{n}}.$$

Kokarev [42] later proved a similar estimate for higher eigenvalues $\lambda_k(-\Delta_g), k \geq 1$, showing that for any integer $m > 0$ such that $V_c(m, M, [g_0])$ is well-defined, we

have

$$\sup_{g\in[g_0]} \lambda_k(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} \leq C(n,m)V_c(m,M,[g_0])^{\frac{2}{n}}k^{\frac{2}{n}},$$

for a positive constant $C(n,m)$ which depends only on the dimensions n and m.

A different approach was taken by Hassannezhad [29], who introduced the notion of *min-conformal volume* $V([g_0])$, defined as

$$V([g_0]) \coloneqq \inf\{\mathrm{Vol}(M,g) : g \in [g_0], \mathrm{Ricci}_g \geq -(n-1)\}.$$

She then proved the estimate

$$\sup_{g\in[g_0]} \lambda_k(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} \leq A_n V([g_0])^{\frac{2}{n}} + B_n k^{\frac{2}{n}},$$

for constants A_n and B_n depending only on the dimension n. In particular this tells us that given (M,g_0), there exists an integer $k_0 > 0$ depending only on the conformal class $[g_0]$ such that for all $k \geq k_0$, we have

$$\sup_{g\in[g_0]} \lambda_k(-\Delta_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} \leq B'_n k^{\frac{2}{n}}$$

for a constant B'_n depending only on n. That is, for sufficiently large k (where the meaning of "sufficiently large" depends on the conformal class), Korevaar's upper bound (3.1.2) holds with a constant depending only on the dimension.

The results selected above are only a small sample of the rich literature on the spectrum of the Laplacian. We direct the reader to [7, 19, 23, 24, 25, 31, 36, 37, 41, 52, 55] and the references therein.

3.1.2 Motivation

Let us turn to the discussion of the eigenvalues of the conformal Laplacian $-L_g$. The significance of the conformal Laplacian in many geometric contexts is well documented [3, 46, 62], a brief review of which would go beyond the scope of this thesis. Nevertheless, fewer results on eigenvalues of $-L_g$ are available compared to $-\Delta_g$.

As we saw in subsection 3.1.1, the volume-normalised eigenvalues of the Laplacian are a natural quantity to study. They have clear geometric meaning and, as the results mentioned above show, are bounded above and below on every conformal class of metrics, by geometric or (in the case of surfaces) topological constants. In contrast, we will see in this subsection that the analogue of Korevaar's bound (3.1.2) fails to hold for eigenvalues of the conformal Laplacian. This suggests that perhaps volume-normalised eigenvalues are not the most natural scale-invariant quantity to study in this setting. Our results in this chapter contribute to the investigation of which eigenvalue normalisations are natural or meaningful in the setting of the conformal Laplacian.

We assume from now on that $n \geq 3$. At first sight, one might be tempted to view $-L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$ as a perturbation of $-\Delta_g$ by the addition of a lower order term, and thus contemplate that many results analogous to those described above would also hold for the conformal Laplacian. One should however note that the additional term involving the scalar curvature R_g is second order in the metric. In consequence, the effect of the perturbation on the eigenvalues (considered as functions of g) is more subtle than it first appears. We note that upper bounds for eigenvalues of operators of the form $-\Delta_g + q$ for a potential q have been obtained

by Grigor'yan, Netrusov, and Yau [26, p. 151], and by Hassannezhad [30, Theorem 1.1].

An illustration of this difference in eigenvalue behaviour can be seen in the result of Amman and Jammes [2], who showed that the direct analogue of the Korevaar bound (3.1.2) does not hold for the eigenvalues ν_k of $-L_g$ for g belonging to a fixed conformal class $[g_0]$ on any closed manifold M. Indeed, among other results they showed that if $g_0 \in \mathcal{M}(M)$ and $\nu^+(-L_g)$ denotes the smallest positive eigenvalue of $-L_g$ for $g \in [g_0]$, then

$$\sup_{g\in[g_0]} \nu^+(-L_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} = +\infty. \tag{3.1.4}$$

Korevaar's bound (3.1.2) is a foundational result in the study of the spectrum of $-\Delta_g$. Given that (3.1.4) shows the direct analogue is false for $-L_g$, it is natural to ask how the inequality (3.1.2) could be modified for the conformal Laplacian operator in order to yield a true statement. Sire and Xu [63] showed that for $M = S^n$ with its standard conformal class, if one normalises the eigenvalue differently, then a Korevaar-type bound still holds. Before stating their result, we first observe that a different way of writing (3.1.4) is

$$\sup_{\substack{f\in C^2(M)\\ f>0}} \nu^+(-L_g)\left(\int_M f^n\, dV_{g_0}\right)^{\frac{2}{n}} = +\infty,$$

where $g = f^2 g_0 \in [g_0]$. Now if (S^n, g_0) denotes the standard sphere, the result

proved by Sire and Xu [63] is

$$\sup_{\substack{f\in C^2(M)\\ f>0}} \nu_k(-L_g)\int_{S^n} f^2\,dV_{g_0} \leq C(n)k^{\frac{2}{n}}.$$

Thus in this particular case of (S^n, g_0) the round sphere, unlike the volume-normalised eigenvalue considered by Amman and Jammes, the quantity $\nu_k \int_{S^n} f^2\,dV_{g_0}$ does satisfy a Korevaar-type bound. The behaviour of extremal eigenvalues with this normalisation has also recently been studied by Pérez-Ayala [57].

One may more generally consider the behaviour of the following scale-invariant quantity

$$\nu_1(-L_g)\cdot\left(\int_M f^p\,dV_{g_0}\right)^{\frac{2}{p}} \tag{3.1.5}$$

for different values of p. The result of Sire and Xu [63] says that when $p = 2$, this quantity is bounded in $[g_0]$ for g_0 the standard metric on the sphere, while the result of Amman and Jammes [2] implies that for $p = n$ it is unbounded in $[g_0]$ on any compact manifold for which $\nu_1(-L_{g_0}) = \nu^+(-L_{g_0})$ (which includes all compact manifolds with positive Yamabe constant).

The discussion above leads to the questions: What is the role of the exponent p? Is there a particular choice of $2 \leq p \leq n$ which is natural? A further interesting question is: What is the most general subclass of metrics in $[g_0]$ for which the eigenvalues of the conformal Laplacian satisfy a Korevaar-type bound? Among these problems, one which is of particular interest in this thesis is: Does there exist a closed manifold (M, g_0) for which

$$\sup_{\substack{f\in C^2(M)\\ f>0}} \nu_1(-L_g)\cdot\left(\int_M f^p\,dV_{g_0}\right)^{\frac{2}{p}} = +\infty$$

for some $2 < p < n$? In the present chapter, we begin our exploration of these questions in a specific context motivated by the work of Amman and Jammes [2]. While we are unable to answer them fully, we hope that our results together with that of Sire and Xu [63] provides a guiding example to steer towards satisfactory answers.

3.2 Definitions and Statement of Results

Before describing our set-up and due to its special relevance to the current work, let us briefly describe the construction of [2]. Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. In order to prove their result (3.1.4), Amman and Jammes begin by fixing a point $x_0 \in M$ and considering a so-called "asymptotically cylindrical" conformal metric $g_\infty = F_\infty^2 g_0$ on $M \setminus \{x_0\}$, where $F_\infty(x)$ blows up like $\mathrm{dist}_{g_0}(x, x_0)^{-1}$ in a neighbourhood of x_0. They then introduce a family of truncated metrics $g_L = F_L^2 g_0$ on M, for which $F_L \to F_\infty$ pointwise as $L \to \infty$, and show that along this family of metrics, the quantity $\nu^+(-L_g)\mathrm{Vol}(M, g)^{\frac{2}{n}}$ grows without bound.

In this chapter, we will consider a family of conformal metrics $\{g_L\}_{L \geq 1}$ which is similar to that considered by Amman and Jammes, but where F_∞ blows up like $\mathrm{dist}_{g_0}(x, \Sigma)^{-\alpha}$ for $0 < \alpha < 2$, with $\Sigma \subset M$ a closed submanifold of dimension $k < \frac{(2-\alpha)(n-2)}{2}$, and the conformal metric g_∞ has positive scalar curvature. In the case where $1 < \alpha < 2$, the metric $g_\infty = F_\infty^2 g_0$ opens up like a warped product of Euclidean space and a cone with cone vertex-angle $\theta = 2\arcsin(\alpha - 1) \in (0, \pi)$, and so we call these metrics *asymptotically conical* (see Proposition 3.3.2). We will obtain sharp estimates for the first eigenvalue of the conformal Laplacian on

(M, g_L) as $L \to \infty$.

Let us now describe our family of metrics $\{g_L\}_{L\geq 1}$ in more detail. Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Fix a closed submanifold $\Sigma \subset M$ of dimension $k < n$ and a constant $\alpha > 0$, where α and k satisfy

$$0 < \alpha < 2, \text{ and } k < \frac{(2-\alpha)(n-2)}{2}. \tag{3.2.1}$$

The reason for this condition will be clarified later. From here on, we write

$$r \coloneqq \mathrm{dist}_{g_0}(x, \Sigma).$$

It is well-known that there exists a tubular neighbourhood of Σ in which r is smooth. Let $F_\infty : M \setminus \Sigma \to \mathbb{R}$ be a positive smooth function such that

$$\begin{gathered} F_\infty(x) = Ar^{-\alpha} + e_\infty(x), \text{ where} \\ \left|\nabla^l_{g_0} e_\infty(x)\right| r^{\alpha-l} \leq \delta(r) \to 0 \text{ as } r \to 0 \text{ for } l = 0, 1, 2, \end{gathered} \tag{3.2.2}$$

for $\delta : [0, \infty) \to \mathbb{R}$ a nonnegative increasing function. Now we define

$$g_\infty \coloneqq F_\infty^2 g_0,$$

a conformal metric on $M \setminus \Sigma$. We will assume also that F_∞ is chosen so that this metric has positive scalar curvature:

$$R_\infty > 0. \tag{3.2.3}$$

We will see in Proposition 3.3.1 that under these assumptions the scalar curvature

R_∞ satisfies the global estimate

$$R_\infty(x) \geq \frac{1}{B} r^{2(\alpha-1)} \tag{3.2.4}$$

for a constant $B > 0$ depending only on F_∞. Some existence results for metrics of this type are provided in Example 3.3.3 and Example 3.3.4, while a necessary condition for the existence of such metrics is proved in Proposition 3.3.5.

The family of truncated metrics $\{g_L\}_{L\geq 1}$ on M we consider will be given by

$$g_L \coloneqq F_L^2 g_0, \tag{3.2.5}$$

for smooth functions $0 < F_L \leq F_\infty$ which will be assumed to satisfy

$$\begin{gathered} F_L(x) = F_\infty(x) \text{ for } r > L^{-1} \\ \frac{1}{C} L^{-\alpha} \leq F_L(x) \leq C L^{-\alpha} \text{ for } r \leq L^{-1} \end{gathered} \tag{3.2.6}$$

for a fixed constant C independent of L.

We are interested in the behaviour as $L \to \infty$ of the first eigenvalue of the following eigenvalue problem

$$-L_{g_L} \varphi_L = \nu \varphi_L, \tag{3.2.7}$$

where $-L_{g_L} = -\Delta_{g_L} + \frac{n-2}{4(n-1)} R_{g_L}$ denotes the conformal Laplacian on (M, g_L). We will prove a sharp estimate for the rate at which the first eigenvalue of the conformal Laplacian decays as $L \to \infty$ in the family of metrics described above and consider

the growth of the following scaling-invariant quantities

$$\nu_1(-L_{g_L})\mathrm{Vol}(M,g_L)^{\frac{2}{n}}$$
$$\nu_1(-L_{g_L})\mathrm{diam}(M,g_L)^2$$
$$\nu_1(-L_{g_L})\left(\int_M F_L^p\, dV_{g_0}\right)^{\frac{2}{p}}.$$

Our first result is the following sharp estimate for the eigenvalue $\nu_1(-L_{g_L})$:

Theorem 3.2.1. *Let (M,g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Fix $0 < \alpha < 2$ and let $\Sigma \subset M$ be a closed submanifold of dimension k satisfying* (3.2.1)*. Let F_∞ be a smooth positive function on $M \setminus \Sigma$ satisfying* (3.2.2) *and such that the conformal metric $g_\infty = F_\infty^2 g_0$ on $M \setminus \Sigma$ satisfies* (3.2.3)*. Consider the asymptotically conical family of metrics $\{g_L\}_{L\geq 1}$ on M of the form* (3.2.5)*, where F_L satisfies* (3.2.6)*. Let $\nu_{1,L}$ denote the first eigenvalue of the conformal Laplacian $-L_{g_L}$ on (M,g_L). Then there exist $L_0 \geq 1$ and a constant $C > 0$ depending only on n, A, α, δ, B (as defined in* (3.2.2) *and* (3.2.4)*) such that for all $L \geq L_0$ we have*

$$\begin{aligned} \frac{1}{C} \leq \nu_{1,L} \leq C, \quad &\text{if } 0 < \alpha < 1 \\ \frac{1}{C} \leq \nu_{1,L} L^{2(\alpha-1)} \leq C, \quad &\text{if } 1 \leq \alpha < 2. \end{aligned} \tag{3.2.8}$$

In the setting where $\nu_1(-L_g) = \nu^+(-L_g)$ for all $g \in [g_0]$ (which for example is the case when (M,g_0) has positive Yamabe constant), recall Amman and Jammes [2] proved

$$\sup_{g\in[g_0]} \nu_1(-L_g)\mathrm{Vol}(M,g)^{\frac{2}{n}} = +\infty. \tag{3.2.9}$$

We observe that by taking $\alpha = 1$ in our estimate (3.2.8) and noting that the volume

of (M, g_L) grows without bound as $L \to \infty$, we obtain (3.2.9), so Theorem 3.2.1 recovers the result of Amman and Jammes. For other values of α, writing our eigenvalue bounds (3.2.8) in terms of the volume and diameter gives us the following corollaries.

Corollary 3.2.2. *Under the assumptions of Theorem 3.2.1, we have*

$$\begin{aligned}
\sup_{L\geq L_0} \nu_{1,L}\mathrm{Vol}(M, g_L)^{\frac{2}{n}} &< \infty \text{ for } 0 < \alpha < \frac{n-k}{n} \\
\sup_{L\geq L_0} \nu_{1,L}\mathrm{Vol}(M, g_L)^{\frac{2}{n}} &= \infty \text{ for } \alpha = \frac{n-k}{n} \\
\sup_{L\geq L_0} \nu_{1,L}\mathrm{Vol}(M, g_L)^{\frac{2}{n}} &< \infty \text{ for } \frac{n-k}{n} < \alpha < 2 \text{ and } k = 0 \\
\sup_{L\geq L_0} \nu_{1,L}\mathrm{Vol}(M, g_L)^{\frac{2}{n}} &= \infty \text{ for } \frac{n-k}{n} < \alpha < 2 \text{ and } k > 0.
\end{aligned} \tag{3.2.10}$$

We also have the following for the behaviour of the diameter-normalised eigenvalue:

Corollary 3.2.3. *Under the assumptions of Theorem 3.2.1, we have:*

$$\begin{aligned}
\sup_{L\geq L_0} \nu_{1,L}\mathrm{diam}(M, g_L)^2 &< \infty \text{ for } \alpha \neq 1 \\
\sup_{L\geq L_0} \nu_{1,L}\mathrm{diam}(M, g_L)^2 &= \infty \text{ for } \alpha = 1.
\end{aligned} \tag{3.2.11}$$

The estimates (3.2.10) and (3.2.11) show that when $k = 0$ and $\alpha > 1$ (so that the angle θ at which the cone at infinity opens satisfies $0 < \theta < \pi$), the volume- and diameter-normalised eigenvalues both remain bounded as $L \to \infty$, in contrast with the asymptotically cylindrical case. We note also that when $k > 0$ and $\frac{n-k}{k} < \alpha < 2$, the metrics $\{g_L\}_{L\geq 1}$ are new examples of families for which the direct analogue of

Korevaar's bound (3.1.2) fails to hold for $\nu_1(-L_g)$.

Remark 3.2.4. We remark that in the course of proving Corollary 3.2.2 and Corollary 3.2.3, we obtain the following estimates for the volume- and diameter-normalised eigenvalue, telling us the rate of growth as $L \to \infty$ in the cases where the normalised eigenvalue is unbounded. (Here we write $A_1 \sim A_2$ to mean $\frac{1}{C}A_1 \leq A_2 \leq CA_1$ for a constant C depending on at most $n, A, \alpha, \delta, B, g_0$):

$$\nu_{1,L}\mathrm{Vol}(M, g_L)^{\frac{2}{n}} \sim \begin{cases} 1 & 0 < \alpha < \frac{n-k}{n} \\ (\log L)^{\frac{2}{n}} & \alpha = \frac{n-k}{n} \\ L^{2\alpha - 2\frac{n-k}{n}} & \frac{n-k}{n} < \alpha < 1 \\ L^{2\alpha - 2\frac{n-k}{n}} & \alpha = 1 \text{ and } k > 0 \\ L^{2-2\frac{n-k}{n}} & 1 < \alpha < 2. \end{cases} \tag{3.2.12}$$

For the diameter-normalised eigenvalues, we obtain the following estimates:

$$\nu_{1,L}\mathrm{diam}(M, g_L)^2 \sim \begin{cases} 1 & 0 < \alpha < 1 \\ (\log L)^2 & \alpha = 1 \\ 1 & 1 < \alpha < 2. \end{cases}$$

Our main result (3.2.8) can also be used to study the scale-invariant quantity (3.1.5) alluded to earlier for the family of metrics $\{g_L\}_{L \geq 1}$.

Corollary 3.2.5. *Let $F_L, \nu_{1,L}$, and L_0 be as in Theorem 3.2.1, and suppose in*

addition that $\alpha = 1$. *Then*

$$\sup_{L \geq L_0} \nu_{1,L} \left(\int_M F_L^p \, dV_{g_0} \right)^{\frac{2}{p}} = \infty \text{ for } p \geq n - k. \tag{3.2.13}$$

We note that in this Corollary we have the additional assumption that $\alpha = 1$. Taking other values of α complicates the proof but does not yield an improved result. Recall that when $p = n$ the quantity appearing on the LHS in (3.2.13) is known to be unbounded on any conformal class on a compact manifold for which $\nu_1(-L_g) = \nu^+(-L_g)$ by a result of Amman and Jammes [2]. Corollary 3.2.5 shows that this quantity can in fact be unbounded for some smaller values of p. To see this, note that if we take k to be the largest integer which satisfies (3.2.1) for $\alpha = 1$, then $p = n - k$ is the smallest integer that is strictly larger than $\frac{n+2}{2}$. This exponent is strictly smaller than n when $n \geq 5$. On the other hand, when (M, g_0) is the standard sphere the quantity appearing on the LHS of (3.2.13) is finite when $p = 2$ by a result of Sire and Xu [63]. The following question remains: Do there exist closed manifolds (M, g_0) such that

$$\sup_{\substack{f \in C^2(M) \\ f > 0}} \nu_1(-L_{f^2 g_0}) \cdot \left(\int_M f^p \, dV_{g_0} \right)^{\frac{2}{p}} = +\infty$$

for some $2 < p \leq \frac{n+2}{2}$?

In the following section, we will discuss the existence and behaviour of the asymptotically conical metrics g_∞, and see some examples of such metrics.

3.3 Remarks on Asymptotically Conical Metrics

One of our standing assumptions in this chapter is (3.2.3), that is positivity of the scalar curvature of the asymptotically conical metric g_∞ on $M \setminus \Sigma$. The scalar curvature R_∞ is in fact bounded from below by a power of the distance function r, as the following proposition shows.

Proposition 3.3.1. *Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Fix a closed submanifold $\Sigma \subset M$ of dimension k and a constant α satisfying* (3.2.1)*. Let $F_\infty : M \setminus \Sigma \to \mathbb{R}$ be a positive smooth function satisfying* (3.2.2) *and* (3.2.3)*. Then we have the following global estimate on $M \setminus \Sigma$*

$$R_\infty(x) \geq \frac{1}{B} r^{2(\alpha-1)}, \tag{3.3.1}$$

for a constant $B > 0$ which depends only on F_∞.

We remark that the limiting case $k = \frac{(2-\alpha)(n-2)}{2}$ may be of interest (see page 123 for further discussion), but goes beyond the scope of this thesis and will not be discussed in detail here.

Proof. We first consider the behaviour of R_∞ close to the submanifold Σ. Recall R_∞ is related to R_0 by the formula

$$R_\infty = F_\infty^{-2} \left[R_0 + \frac{4(n-1)}{n-2} F_\infty^{-\frac{n-2}{2}} \left(-\Delta_{g_0} F_\infty^{\frac{n-2}{2}} \right) \right]. \tag{3.3.2}$$

We will apply the following well-known formulae for the gradient and Laplacian of

the distance function r:

$$|\nabla_{g_0} r|^2_{g_0} = 1, \text{ and } \Delta_{g_0} r = \frac{n-k-1}{r} + O(r)$$

Using these formulae as well as (3.2.2), we can compute the limit

$$\lim_{r\to 0} R_\infty r^{2(1-\alpha)} = \frac{\alpha(n-1)}{A^2}\left[(2-\alpha)(n-2) - 2k\right]. \tag{3.3.3}$$

Our constraints (3.2.1) on α and k imply that the RHS above is strictly positive. This suffices to show that the desired curvature estimate (3.3.1) holds in a small neighbourhood of Σ. The complement of such a neighbourhood is a compact set, where both the scalar curvature and the distance function are bounded above and below by constants depending only on F_∞, and hence (3.3.1) holds for a suitably large constant B. □

Suppose (M, g) is a Riemannian manifold, and we have a sequence of scale factors $\mu_i \to 0$ and points $p_i \in M$ with $p_i \to p$ such that that the rescaled sequence of pointed metric spaces $(M, \mu_i g, p_i)$ converges in the pointed Gromov-Hausdorff sense to a pointed metric space (X', p'). Then we say (X', p') is a tangent cone to (M, g) at p. The following Proposition explains why we call the metrics g_∞ asymptotically conical.

Proposition 3.3.2. *Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Fix $1 < \alpha < 2$ and let $\Sigma \subset M$ be a closed submanifold of dimension $k < n$. Let $F_\infty : M \setminus \Sigma \to \mathbb{R}$ be a smooth positive function satisfying* (3.2.2) *and define a conformal metric $g_\infty = F_\infty^2 g_0$ on $M \setminus \Sigma$. For any point $p \in \Sigma$, the tangent cones at p of $(M \setminus \Sigma, g_\infty)$ are warped products of the Euclidean space $\mathbb{R}^k$ with a Euclidean*

cone of dimension $n-k$ and cone-vertex angle θ satisfying $\sin(\frac{\theta}{2}) = \alpha - 1$.

Proof. Choose a sequence $R_i \to 0$ and for each i fix $p_i \in M \setminus \{p\}$ such that for all i, $\mathrm{dist}_{g_0}(p_i, p) = R_i$. For sufficiently large i, we have $R_i < \mathrm{inj}(M, g_0)$, so the tail of the sequence (p_i) is contained in the maximal domain in M on which the inverse of the exponential map at p (with respect to g_0) is a diffeomorphism. Discarding some points from the beginning of the sequence if necessary, we henceforth assume that this is the case for all points p_i. Choose coordinates $\{y^i\}$ on $T_pM \cong \mathbb{R}^n$ such that the tangent space to $\exp_p^{-1}(\Sigma)$ at 0 is $\mathbb{R}^k \coloneqq \{(y^1, \dots, y^k, 0, \dots, 0)\} \subset \mathbb{R}^n$. Let $x = \exp_p y$ for $|y| < \mathrm{inj}(M, g_0)$, so that $\{x^i\}$ is a geodesic normal coordinate system at $p \in M$ (for the metric g_0). Now we introduce dilations $\Phi_i : \mathbb{R}^n \to \mathbb{R}^n$ defined for each i by

$$\Phi_i(z) = R_i z.$$

In the dilated coordinates $y = R_i z$, the points p_i become $\tilde{p}_i = R_i^{-1} \exp_p^{-1}(p_i) \in \mathbb{R}^n$, a sequence lying on the Euclidean unit sphere in $\mathbb{R}^n$. Let us define the rescaled pullback metrics

$$g_\infty^{R_i} \coloneqq R_i^{2(\alpha-1)} \Phi_i^* \exp_p^* g_\infty.$$

For each i, the components of this metric with respect to the coordinates $\{z^i\}$ on $\mathbb{R}^n$ satisfy

$$\left(g_\infty^{R_i}\right)_{ij} = A^2 \mathrm{dist}_{R_i^{-2}\Phi_i^* \exp_p^* g_0}(z, \Phi_i^{-1} \exp_p^{-1}(\Sigma))^{-2\alpha} \delta_{ij} + O(R_i^\alpha),$$

As $i \to \infty$, the sequence of rescaled metrics $g_\infty^{R_i}$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{R}^k)$ to the

following metric on $\mathbb{R}^n \setminus \mathbb{R}^k$:

$$g_\infty^C := A^2|z'|^{-2\alpha} g_{\text{Eucl.}},$$

where here $z = \lim_{i\to\infty} R_i^{-1} y$ and we write $z = (z^1, \dots, z^k, z') \in \mathbb{R}^n$, so that $|z'|$ is the Euclidean distance of z from $\mathbb{R}^k \subset \mathbb{R}^n$. We may choose a subsequence along which $\tilde{p}_i$ converges to some point $\tilde{p}$ on the unit sphere in $\mathbb{R}^n \setminus \mathbb{R}^k$. The pointed space $(\mathbb{R}^n \setminus \mathbb{R}^k, g_\infty^C, \tilde{p})$ is the tangent cone at p of $(M \setminus \Sigma, g_\infty)$. We can write the metric g_∞^C as

$$g_\infty^C = A^2|z'|^{-2\alpha} \left(g_{\text{Eucl.}}^{\mathbb{R}^k}(z^1, \dots, z^k) + g_{\text{Eucl.}}^{\mathbb{R}^{n-k}}(z') \right),$$

and we see that $(\mathbb{R}^n \setminus \mathbb{R}^k, g_\infty^C)$ is a warped product of the Euclidean space $\mathbb{R}^k$ and the manifold $\mathbb{R}^{n-k} \setminus \{0\}$ equipped with metric

$$A^2|z'|^{-2\alpha}(dz_{k+1}^2 + \cdots + dz_n^2) = A^2\rho^{-2\alpha}(d\rho^2 + \rho^2\sigma_{n-k-1}), \tag{3.3.4}$$

where in the last equality we express the metric in polar coordinates with radial coordinate $\rho = |z'|$, and where σ_{n-k-1} denotes the metric on the standard unit sphere of dimension $n-k-1$. Making the change of coordinates $r = \frac{1}{\rho}$, we see that the metric (3.3.4) can also be expressed as

$$A^2 r^{2(\alpha-2)} \left(dr^2 + r^2 \sigma_{n-k-1} \right). \tag{3.3.5}$$

For comparison, consider now the $n-k$-dimensional Euclidean cone in $\mathbb{R}^{n-k+1}$ defined by the equation $x_1^2 + \cdots + x_{n-k}^2 - t^2 x_{n-k+1}^2 = 0$ for some fixed $t > 0$. Note

that the vertex angle of this cone is θ satisfying $t = \tan(\frac{\theta}{2})$. The metric on the cone as a hypersurface in $\mathbb{R}^{n-k+1}$ can be written in the form

$$g_{\text{cone}}^{n-k} = 4r^{2(\alpha-2)}\left(dr^2 + \frac{t^2}{1+t^2}(\alpha-1)^{-2}r^2\sigma_{n-k-1}\right),$$

for any $1 < \alpha < 2$. Comparing with (3.3.5), we see that when $1 < \alpha < 2$ the metric on the tangent cone of $(M \setminus \Sigma, g_\infty)$ at p is (up to a homothety) the same as that of a warped product of $\mathbb{R}^k$ with a Euclidean cone of dimension $n-k$ with vertex angle determined by the condition

$$\frac{t^2}{1+t^2}(\alpha-1)^{-2} = 1 \iff \sin(\frac{\theta}{2}) = \alpha - 1.$$

In particular, when $\alpha \to 1$ we have $\theta \to 0$ and the conical part of the metric approaches a cylinder, while when $\alpha = 2$ the cone angle is $\theta = \pi$ and the conical part of the metric opens up in a flat plane. □

Example 3.3.3. We observe that a choice of F_∞ satisfying the assumptions (3.2.2) and (3.2.3) always exists when (M, g_0) has positive Yamabe constant and $\Sigma = \{p\}$ is a single point.

Proof. First we note that when (M, g_0) has positive Yamabe constant, then for every $p \in M$ there exists a unique smooth function Γ_p on $M \setminus \{p\}$ such

$$-L_{g_0}\Gamma_p = \delta_p$$

in the sense of distributions, where δ_p denotes the Dirac delta distribution at p, and moreover Γ_p is strictly positive (see for example [46, Lemma 6.1]). The asymptotic

behaviour near p of the function

$$G = (n-2)\omega_n \Gamma_p,$$

where ω_n denotes the volume of the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$ was studied by Lee and Parker [46], and their results in particular imply that the conformal factor

$$F_\infty = AG^{-\frac{\alpha}{2-n}}$$

satisfies (3.2.2), as can be verified by a short calculation using the asymptotic expansion for G given in [46, Lemma 6.4]. We will show that with this choice of F_∞, the metric g_∞ also satisfies the positive curvature condition $R_\infty > 0$. The curvature can be expressed in terms of R_{g_0} by the formula

$$R_\infty = A^{-2}G^{-\frac{2\alpha}{n-2}}\left[R_{g_0} + \frac{4(n-1)}{n-2}G^{-\frac{\alpha}{2}}\left(-\Delta_{g_0}G^{\frac{\alpha}{2}}\right)\right]. \tag{3.3.6}$$

Now, using the fact that $-L_{g_0}G = 0$ away from p, we have

$$\begin{aligned}
-\Delta_{g_0}G^{\frac{\alpha}{2}} &= -\frac{\alpha}{2}G^{\frac{\alpha}{2}-1}\Delta_{g_0}G - \frac{\alpha(\alpha-2)}{4}G^{\frac{\alpha}{2}-2}|\nabla_{g_0}G|^2_{g_0} \\
&= -\frac{\alpha(n-2)}{8(n-1)}G^{\frac{\alpha}{2}}R_{g_0} - \frac{\alpha(\alpha-2)}{4}G^{\frac{\alpha}{2}-2}|\nabla_{g_0}G|^2_{g_0}.
\end{aligned}$$

Plugging this into (3.3.6), we have

$$R_\infty = A^{-2}G^{-\frac{2\alpha}{n-2}}\left[R_{g_0}\left(1-\frac{\alpha}{2}\right) + \frac{\alpha(2-\alpha)(n-1)}{n-2}G^{-2}|\nabla_{g_0}G|^2_{g_0}\right],$$

so we see that in order to have $R_\infty > 0$ we must have

$$R_{g_0}\left(1-\frac{\alpha}{2}\right)+\frac{\alpha(2-\alpha)(n-1)}{n-2}G^{-2}|\nabla_{g_0}G|^2_{g_0}>0. \tag{3.3.7}$$

We note that since we assume (M, g_0) has positive Yamabe constant, g_0 admits a conformal metric with positive scalar curvature, so without loss of generality we can choose such a metric from the start and assume $R_{g_0} > 0$. Now note that both terms in (3.3.7) are positive since $0 < \alpha < 2$. We have thus shown that if (M, g_0) has positive Yamabe constant and $\Sigma = \{p\}$, then (after replacing g_0 by a suitable conformal metric if necessary), the metric $g_\infty = G^{-\frac{2\alpha}{2-n}}g_0$ on $M \setminus \{p\}$ has the desired asymptotic behaviour near p and has positive scalar curvature. □

Example 3.3.4. Suppose (M, g_0) is the standard sphere S^n with $n \geq 3$ and $\Sigma \subset S^n$ is a closed submanifold of dimension $k \leq \frac{n-2}{2}$. By a result of Mazzeo and Pacard [51], $M \setminus \Sigma$ admits a complete conformal metric $g = f^2 g_0$ with constant positive scalar curvature. The function f is necessarily singular at Σ since g is a complete metric, and in fact is of order $r^{-\frac{n+2}{4}}$ near Σ. Writing $\alpha = \frac{n+2}{4}$, we see that $\alpha < 2$ provided $n \leq 5$. So for $3 \leq n \leq 5$, these metrics are examples of asymptotically conical metrics with constant positive scalar curvature.

A common feature of the examples above is that in both cases the background manifold (M, g_0) has positive Yamabe constant. Indeed, example 3.3.3 shows that when Σ is a single point, (M, g_0) having positive Yamabe constant is a sufficient condition for the existence of an asymptotically conical conformal metric g_∞ on $M \setminus \Sigma$ satisfying (3.2.2) and (3.2.3). We now show that for general $\Sigma \subset M$ of dimension k satisfying (3.2.1), (M, g_0) having positive Yamabe constant is in fact a necessary condition for such metrics to exist.

Proposition 3.3.5. *Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Fix $0 < \alpha < 2$ and a closed submanifold $\Sigma \subset M$ of dimension k satisfying (3.2.1). Suppose there exists a smooth positive function $F_\infty : M \setminus \Sigma \to \mathbb{R}$ and such that the conformal metric $g_\infty = F_\infty^2 g_0$ on $M \setminus \Sigma$ has strictly positive scalar curvature. Then (M, g_0) has positive Yamabe constant.*

Proof. Positivity of the Yamabe constant for (M, g_0) is equivalent to the statement

$$J(\varphi) \coloneqq \int_M |\nabla_{g_0}\varphi|^2_{g_0} + \frac{n-2}{4(n-1)} R_{g_0} |\varphi|^2 \, dV_{g_0} \geq 0 \tag{3.3.8}$$

for all $\varphi \in C^\infty(M)$, with equality if and only if $\varphi = 0$. We consider first test functions $\varphi \in C_c^\infty(M \setminus \Sigma)$. To prove the inequality (3.3.8), we use the "Hardy trick" (see for example [35, Lemma A.1]). Given such a φ, we write it as

$$\varphi = F_\infty^{\frac{n-2}{2}} \psi$$

for $\psi \in C_c^\infty(M \setminus \Sigma)$. We note that such a decomposition is always possible, since $F_\infty > 0$ on $M \setminus \Sigma$. Plugging this into (3.3.8) gives

$$\begin{aligned} J(\varphi) &= \int_M \left| \nabla_{g_0}(F_\infty^{\frac{n-2}{2}} \psi) \right|^2_{g_0} + \frac{n-2}{4(n-1)} \left| F_\infty^{\frac{n-2}{2}} \psi \right|^2 dv_{g_0} \\ &= \int_M \langle \nabla_{g_0}(F_\infty^{\frac{n-2}{2}}), \nabla_{g_0}(F_\infty^{\frac{n-2}{2}} \psi^2) \rangle_{g_0} + \frac{n-2}{4(n-1)} R_{g_0} F_\infty^{\frac{n-2}{2}} (F_\infty^{\frac{n-2}{2}} \psi^2) \, dV_{g_0} \\ &\qquad + \int_M F_\infty^{n-2} |\nabla_{g_0}\psi|^2_{g_0} \, dV_{g_0} \\ &= \int_M -L_{g_0}(F_\infty^{\frac{n-2}{2}}) F_\infty^{\frac{n-2}{2}} \psi^2 \, dV_{g_0} + \int_M F_\infty^{n-2} |\nabla_{g_0}\psi|^2_{g_0} \, dV_{g_0}, \end{aligned} \tag{3.3.9}$$

where in the last line we have integrated by parts and recognised the conformal Laplacian $-L_{g_0}$. Now, from the formula (3.3.2) relating the scalar curvatures of g_0

and g_∞, we can see that

$$-L_{g_0} F_\infty^{\frac{n-2}{2}} = \frac{n-2}{4(n-1)} R_\infty F_\infty^{\frac{n+2}{2}} > 0.$$

It follows that the first term in (3.3.9) is nonnegative, and discarding this term we find

$$J(\varphi) \geq \int_M F_\infty^{n-2} |\nabla_{g_0} \psi|_{g_0}^2 \, dV_{g_0} \geq 0,$$

with equality if and only if $\psi \equiv 0$ if and only if $\varphi \equiv 0$. We have thus proven the inequality (3.3.8) for $\varphi \in C_c^\infty(M \setminus \Sigma)$. Now recall k satisfies (3.2.1), and hence in particular $k \leq n-2$. Thus $C_c^\infty(M \setminus \Sigma)$ is dense in $H^1(M, g_0)$, and we consequently have the desired inequality for all $\varphi \in C^\infty(M)$. □

3.4 Proofs of Main Results

In this section we present a proof of Theorem 3.2.1. Let $-L_\infty$ denote the conformal Laplacian on $(M \setminus \Sigma, g_\infty)$. Equation (3.2.7) can be written in terms of $-L_\infty$. To see this, recall that if a pair of Riemannian metrics are conformally related by $\tilde{g} = f^2 g$ for $f > 0$, then the corresponding conformal Laplacians are related by the following transformation law:

$$-L_{\tilde{g}}(\cdot) = f^{-\frac{n+2}{2}}(-L_g(f^{\frac{n-2}{2}}\cdot)).$$

Thus, if we set (recalling that φ_L was defined as the first eigenfunction of $-L_{g_L}$ on M)

$$\psi_L \coloneqq \left(\frac{F_L}{F_\infty}\right)^{\frac{n-2}{2}} \varphi_L,$$

then for every $L \geq 1$, ψ_L is a smooth function on M which satisfies the equation

$$-L_\infty \psi_L = \nu_{1,L} \left(\frac{F_L}{F_\infty} \right)^2 \psi_L \text{ on } M \setminus \Sigma. \tag{3.4.1}$$

It is well known that φ_L can be chosen to be positive, and hence we also have $\psi_L > 0$ on $M \setminus \Sigma$.

We now state the lower and upper bounds on $\nu_{1,L}$ claimed in Theorem 3.2.1 as separate propositions, which we will prove in turn.

Proposition 3.4.1. *Under the assumptions of Theorem 3.2.1 there exist $L_0 \geq 1$ and a constant $C > 0$ depending only on n, A, α, δ, B such that for all $L \geq L_0$ we have*

$$\nu_{1,L}(-L_{g_L}) \geq \begin{cases} \frac{1}{C} & 0 < \alpha < 1 \\ \frac{1}{C} L^{2(1-\alpha)} & 1 \leq \alpha < 2. \end{cases} \tag{3.4.2}$$

Proposition 3.4.2. *Under the assumptions of Theorem 3.2.1, there exist $L_0 \geq 1$ and a constant $C > 0$ depending only on n, A, α, δ, B such that for all $L \geq L_0$ we have*

$$\nu_{1,L}(-L_{g_L}) \leq \begin{cases} C & 0 < \alpha < 1 \\ C L^{2(1-\alpha)} & 1 \leq \alpha < 2. \end{cases}$$

In the proofs that follow, C denotes a generic positive constant depending only on n, A, α, δ, B, which may change from line to line.

3.4.1 Proof of Eigenvalue Lower Bound

Proof of Proposition 3.4.1. We prove the lower bound (3.4.2) for the eigenvalue $\nu_{1,L}$ using energy estimates, exploiting the positivity of R_∞. The energy identity

satisfied by ψ_L is

$$\int_M |\nabla_{g_\infty}\psi_L|^2_{g_\infty} + \frac{n-2}{4(n-1)}R_\infty|\psi_L|^2\,dV_{g_\infty} = \nu_{1,L}\int_M \left(\frac{F_L}{F_\infty}\right)^2|\psi_L|^2\,dV_{g_\infty}.$$

Discarding the derivative terms, we get

$$\int_M \frac{n-2}{4(n-1)}R_\infty|\psi_L|^2\,dV_{g_\infty} \leq \nu_{1,L}\int_M \left(\frac{F_L}{F_\infty}\right)^2|\psi_L|^2\,dV_{g_\infty}.$$

Now we use the bounds (3.3.1), (3.2.6) to obtain

$$\begin{aligned}&\int_{\{r\leq L^{-1}\}} r^{2(\alpha-1)}|\psi_L|^2\,dV_{g_\infty} + \int_{\{L^{-1}\leq r\leq 1\}} r^{2(\alpha-1)}|\psi_L|^2\,dV_{g_\infty} + \int_{\{r\geq 1\}}|\psi_L|^2\,dV_{g_\infty}\\ &\quad\leq C_*\nu_{1,L}\left(\int_{\{r\leq L^{-1}\}} L^{2\alpha}r^{2\alpha}|\psi_L|^2\,dV_{g_\infty} + \int_{\{L^{-1}\leq r\leq 1\}}|\psi_L|^2\,dV_{g_\infty} + \int_{\{r\geq 1\}}|\psi_L|^2\,dV_{g_\infty}\right)\end{aligned}$$

for a constant C_* depending at most on n, A, α, δ, B. We now consider the cases $0<\alpha<1$ and $1\leq\alpha<2$ separately. When $0<\alpha<1$, moving everything to the LHS and using the fact that $r\leq L^{-1}$, which implies $L^{2\alpha}\leq r^{-2}$ in the first integral, while $r\leq 1$ which implies$1\leq r^{2(\alpha-1)}$ in the second integral, we obtain

$$\begin{aligned}(1-C_*\nu_{1,L})\Bigg[\int_{\{r\leq L^{-1}\}} r^{2(\alpha-1)}|\psi_L|^2\,dV_{g_\infty} + \int_{\{L^{-1}\leq r\leq 1\}} r^{2(\alpha-1)}|\psi_L|^2\,dV_{g_\infty}\\ + \int_{\{r\geq 1\}}|\psi_L|^2\,dV_{g_\infty}\Bigg]\leq 0.\end{aligned}$$

The integrals are nonnegative and cannot simultaneously vanish since ψ_L is nontrivial, so we must have

$$1-C_*\nu_{1,L}\leq 0 \implies \nu_{1,L}\geq\frac{1}{C_*}.$$

In the case $1 \leq \alpha < 2$, again moving everything to the LHS, this time using the fact that $r \leq L^{-1}$ which implies $L^{2\alpha} r^{2\alpha} \leq L^{2(\alpha-1)} r^{2(\alpha-1)}$ in the first integral, while $r \geq L^{-1}\{textwhichimplies 1 \leq L^{2(\alpha-1)} r^{2(\alpha-1)}$ in the second integral, we obtain

$$(1 - C_* \nu_{1,L} L^{2(\alpha-1)}) \left[\int_{\{r \leq L^{-1}\}} r^{2(\alpha-1)} |\psi_L|^2 \, dV_{g_\infty} + \int_{\{L^{-1} \leq r \leq 1\}} r^{2(\alpha-1)} |\psi_L|^2 \, dV_{g_\infty} \right] + (1 - C_* \nu_{1,L}) \int_{\{r \geq 1\}} |\psi_L|^2 \, dV_{g_\infty} \leq 0.$$

The integrals are all nonnegative and cannot simultaneously vanish since ψ_L is nontrivial, so we must have

$$1 - C_* \nu_{1,L} L^{2(\alpha-1)} \leq 0, \text{ or } 1 - C_* \nu_{1,L} \leq 0.$$

In either case, the desired lower bound follows. □

3.4.2 Proof of Eigenvalue Upper Bound — Spherical Case

Before proving Proposition 3.4.2, we first give a proof of an intermediate result where the main ideas are more transparent (this is particularly true if the reader makes the additional assumption that all functions are rotationally symmetric). The reader who is only interested in the proof of the main result Theorem 3.2.1 may skip this subsection.

Proposition 3.4.3. *Let (M, g_0) be the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$ and let $\Sigma = \{N\}$ be the north pole. Under the assumptions of Theorem 3.2.1, there exist $L_0 \geq 1$ and a constant $C > 0$ depending only on n, A, δ, B such that for all $L \geq L_0$*

we have

$$\nu_{1,L} \leq \begin{cases} C & 0 < \alpha < 1 \\ CL^{2(1-\alpha)} & 1 \leq \alpha < 2. \end{cases} \tag{3.4.3}$$

Proof. After stereographic projection $(S^n \setminus \{N\}, g_0)$ is isometric to the manifold

$$\left(\mathbb{R}^n, \left(\frac{2}{1+r^2}\right)^2 (dr^2 + r^2\sigma)\right),$$

where $r^2 \coloneqq \sum_{i=1}^n x_i^2$ and σ denotes the metric of the standard $(n-1)$-sphere. We now express our assumptions on F_∞ in this setting. Our asymptotically conical metric g_∞ on $\mathbb{R}^n$ (equivalently on $S^n \setminus \{N\}$) satisfies

$$g_\infty \coloneqq F_\infty^2 g_0,$$

where $F_\infty : \mathbb{R}^n \to \mathbb{R}$ is a smooth positive function such that

$$\begin{gathered} F_\infty(x) = Ar^\alpha + e_\infty(x), \text{ with} \\ \left|\nabla_{g_0}^l e_\infty(x)\right| r^{-\alpha+l} \leq \delta(r) \to 0 \text{ as } r \to \infty \text{ for } l = 0, 1, 2 \end{gathered} \tag{3.4.4}$$

for some constants A and $0 < \alpha < 2$, and δ a nonnegative decreasing function. We note that these conditions are equivalent to (3.2.2). In this setting, the formula (3.3.3) giving the limiting behaviour of R_∞ near the singular point becomes

$$\lim_{r\to\infty} R_\infty r^{2(\alpha-1)} = \frac{\alpha(2-\alpha)(n-1)(n-2)}{4A^2}. \tag{3.4.5}$$

Our assumptions (3.4.4) on the behaviour of F_∞ immediately imply the following

limit, which will be of use to us later:

$$\lim_{r\to\infty} \frac{2F_\infty}{1+r^2} r^{2-\alpha} = 2A. \tag{3.4.6}$$

The family of truncated metrics $\{g_L\}_{L\geq 1}$ on $\mathbb{R}^n$ we consider will be given by

$$g_L \coloneqq F_L^2 g_0,$$

for smooth functions $0 < F_L \leq F_\infty$ which will be assumed to satisfy

$$\begin{aligned} F_L &= F_\infty \text{ for } |x| < L \\ F_L(x) &\leq CL^\alpha \text{ for } |x| \geq L \\ \tilde{F}_L(y) \coloneqq F_L\left(\frac{y}{|y|^2}\right) &\text{ is smooth at } y = 0 \end{aligned} \tag{3.4.7}$$

for a fixed constant C independent of L. We note that the last of these conditions ensures that for each L, the function F_L on $\mathbb{R}^n$ pulls back to a smooth function on S^n via stereographic projection. It implies in particular that F_L has a finite limit as $|x| \to \infty$.

In polar coordinates on $\mathbb{R}^n$, equation (3.4.1) satisfied by ψ_L is

$$\begin{aligned} &-\left(\frac{2F_\infty}{1+r^2}\right)^{-n} r^{1-n}\frac{\partial}{\partial r}\left(\left(\frac{2F_\infty}{1+r^2}\right)^{n-2} r^{n-1}\frac{\partial \psi_L}{\partial r}\right) \\ &\quad -\left(\frac{2F_\infty}{1+r^2}\right)^{-n} r^{1-n}\mathrm{div}_\sigma\left(\left(\frac{2F_\infty}{1+r^2}\right)^{n-2} r^{n-3}\nabla_\sigma \psi_L\right) + \frac{n-2}{4(n-1)} R_\infty \psi_L \\ &\qquad\qquad = \nu_{1,L}\left(\frac{F_L}{F_\infty}\right)^2 \psi_L, \end{aligned} \tag{3.4.8}$$

where div_σ and ∇_σ represent the divergence and gradient on S^{n-1} respectively.

We now make a change of variables $\rho = \mu r$ on $\mathbb{R}^n$ for $\mu > 0$. We will use a subscript μ to denote functions expressed in the transformed coordinates, so that for a given function $f(r, \theta)$ on $\mathbb{R}^n$ we have

$$f_\mu(\rho, \theta) \coloneqq f(\frac{\rho}{\mu}, \theta). \tag{3.4.9}$$

Under this transformation, the PDE (3.4.8) becomes

$$\begin{aligned} &- \left(\frac{2\mu F_{\infty,\mu}}{\mu^2 + \rho^2}\right)^{-n} \rho^{1-n} \frac{\partial}{\partial \rho} \left(\left(\frac{2\mu F_{\infty,\mu}}{\mu^2 + \rho^2}\right)^{n-2} \rho^{n-1} \frac{\partial \psi_{L,\mu}}{\partial \rho} \right) \\ &- \left(\frac{2\mu F_{\infty,\mu}}{\mu^2 + \rho^2}\right)^{-n} \rho^{1-n} \mathrm{div}_\sigma \left(\left(\frac{2\mu F_{\infty,\mu}}{\mu^2 + \rho^2}\right)^{n-2} \rho^{n-3} \nabla_\sigma \psi_{L,\mu} \right) + \frac{n-2}{4(n-1)} R_{\infty,\mu} \psi_{L,\mu} \\ &\qquad = \nu_{1,L} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 \psi_{L,\mu}, \end{aligned} \tag{3.4.10}$$

for $(\rho, \theta) \in (0, \infty) \times S^{n-1}$.

To prove the upper bound for $\nu_{1,L} L^{2(\alpha-1)}$ in this special case, we will compare the eigenvalue $\nu_{1,L}$ with that of the same operator on a finite annulus $\frac{a}{\mu} < r < \frac{b}{\mu}$, where $\mu = \mu(L)$ and the fixed constants a, b will be chosen later. Making use of the variational characterisation of the eigenvalue and observing the limiting behaviour as $L \to \infty$, we will obtain upper bounds for $\nu_{1,L}$ of the form (3.4.3).

We return to equation (3.4.10) and observe that

$$\frac{1}{C} \mu^{2(\alpha-1)} \leq R_{\infty,\mu} \leq C \mu^{2(\alpha-1)}$$

at $\rho = 1$, for a constant $C > 0$ which depends on F_∞ but not on μ. Multiplying

both sides of (3.4.10) by $\mu^{2(1-\alpha)}$ gives

$$\begin{aligned}\mathcal{L}_\mu u \coloneqq &-\left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^{-n}\rho^{1-n}\frac{\partial}{\partial\rho}\left(\left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^{n-2}\rho^{n-1}\frac{\partial\psi_{L,\mu}}{\partial\rho}\right)\\ &-\left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^{-n}\rho^{1-n}\mathrm{div}_\sigma\left(\left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^{n-2}\rho^{n-3}\nabla_\sigma\psi_{L,\mu}\right)\\ &+\frac{n-2}{4(n-1)}R_{\infty,\mu}\mu^{2(1-\alpha)}\psi_{L,\mu}=\nu_{1,L}\mu^{2(1-\alpha)}\left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2\psi_{L,\mu}. \quad (3.4.11)\end{aligned}$$

Now fix a finite interval $[a,b]$ with $0<a<b<\infty$. It is well-known that there exists a unique $\bar{\nu}_L>0\in\mathbb{R}$ and smooth function $\bar{\psi}_L>0$ on $(a,b)\times S^{n-1}$ with $\bar{\psi}_L(a,\theta)=\bar{\psi}_L(b,\theta)=0$ for all $\theta\in S^{n-1}$ such that

$$\mathcal{L}_\mu\bar{\psi}_L=\bar{\nu}_L\mu^{2(1-\alpha)}\left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2\bar{\psi}_L \quad (3.4.12)$$

in $A_{a,b}\coloneqq[a,b]\times S^{n-1}$. It is also known that the constant $\nu_{1,L}\mu^{2(1-\alpha)}$ appearing on the RHS of (3.4.11) is less than or equal to the constant $\bar{\nu}_L\mu^{2(1-\alpha)}$ appearing on the RHS of (3.4.12), that is

$$\nu_{1,L}\mu^{2(1-\alpha)}\leq\bar{\nu}_L\mu^{2(1-\alpha)}. \quad (3.4.13)$$

For completeness, we give the proof here. First we define the energy

$$\begin{aligned}E_{L,\mu}(u)\coloneqq\int_{S^{n-1}}\int_a^b\Bigg\{&\left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^{n-2}\rho^{n-1}\left(|\partial_\rho u|^2+\rho^{-2}|\nabla_\sigma u|_\sigma^2\right)\\ &+\frac{n-2}{4(n-1)}R_{\infty,\mu}\mu^{2(1-\alpha)}\left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^n\rho^{n-1}|u|^2\Bigg\}\,d\rho\,dV_\sigma. \quad (3.4.14)\end{aligned}$$

The weighted eigenvalue $\bar{\nu}_L \mu^{2(1-\alpha)}$ satisfies the variational principle

$$\bar{\nu}_L \mu^{2(1-\alpha)} = \inf_{\substack{u \in C_c^\infty(A_{a,b}) \\ u \not\equiv 0}} \mathcal{R}_{L,\mu}(u), \tag{3.4.15}$$

where we define the Rayleigh quotient $\mathcal{R}_{L,\mu}$ by

$$\mathcal{R}_{L,\mu}(u) \coloneqq \frac{E_{L,\mu}(u)}{\int_{S^{n-1}} \int_a^b \left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^n \rho^{n-1} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 |u|^2 \, d\rho \, dV_\sigma}.$$

Thus, the claim that $\bar{\nu}_L \mu^{2(1-\alpha)} \geq \nu_{1,L} \mu^{2(1-\alpha)}$ is equivalent to the statement

$$E_{L,\mu}(u) - \nu_{1,L} \mu^{2(1-\alpha)} \int_{S^{n-1}} \int_a^b \left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^n \rho^{n-1} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 |u|^2 \, d\rho \, dV_\sigma \geq 0 \tag{3.4.16}$$

for all $u \in C_c^\infty(A_{a,b})$. We will use the Hardy trick to prove the inequality above. Define a function $v \in C_c^\infty([a,b] \times S^{n-1})$ by setting $u = \psi_{L,\mu} v$. Such a decomposition is always possible since $\psi_{L,\mu}$ is smooth and strictly positive on $(a,b) \times S^{n-1}$ and u has compact support in $A_{a,b}$. Inserting $u = \psi_{L,\mu} v$ into the energy (3.4.14) yields

$$\begin{aligned} E_{L,\mu}(u) = \int_{S^{n-1}} \int_a^b \Bigg\{ \left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^{n-2} \rho^{n-1} \big\{ |\psi_{L,\mu}|^2 (|\partial_\rho v|^2 + \rho^{-2} |\nabla_\sigma v|_\sigma^2) \\ + \partial_\rho \psi_{L,\mu} \partial_\rho (v^2 \psi_{L,\mu}) + \rho^{-2} \langle \nabla_\sigma \psi_{L,\mu}, \nabla_\sigma (v^2 \psi_{L,\mu}) \rangle_\sigma \big\} \\ + \frac{n-2}{4(n-1)} R_{\infty,\mu} \mu^{2(1-\alpha)} \left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2}\right)^n \rho^{n-1} |\psi_{L,\mu} v|^2 \Bigg\} \, d\rho \, dV_\sigma. \end{aligned}$$

Using the weak formulation of the PDE (3.4.12) satisfied by $\psi_{L,\mu}$ with test function $v^2 \psi_{L,\mu}$, we obtain

$$E_{L,\mu}(u) = \int_{S^{n-1}} \int_a^b \left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2} \right)^{n-2} \rho^{n-1} \left\{ |\psi_{L,\mu}|^2 (|\partial_\rho v|^2 + \rho^{-2} |\nabla_\sigma v|_\sigma^2) \, d\rho \, dV_\sigma \right.$$
$$\left. + \nu_{1,L} \mu^{2(1-\alpha)} \int_{S^{n-1}} \int_a^b \left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2} \right)^{n} \rho^{n-1} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}} \right)^2 |\psi_{L,\mu} v|^2 \, d\rho \, dV_\sigma, \quad (3.4.17) \right.$$

Noting that the first integral in (3.4.17) is nonnegative, we have thus shown that

$$E_{L,\mu}(u) \geq \nu_{1,L} \mu^{2(1-\alpha)} \int_{S^{n-1}} \int_a^b \left(\frac{2\mu^\alpha F_{\infty,\mu}}{\mu^2+\rho^2} \right)^{n} \rho^{n-1} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}} \right)^2 |u|^2 \, d\rho \, dV_\sigma,$$

which is equivalent to (3.4.16), as desired. We therefore have the inequality $\nu_{1,L} \mu^{2(1-\alpha)} \leq \bar{\nu}_L \mu^{2(1-\alpha)}$ as claimed.

Fix an arbitrary function $\tilde{u} \in C_c^\infty(A_{a,b})$. For every $L \geq 1$, we have $\bar{\nu}_L \mu^{2(1-\alpha)} \leq \mathcal{R}_{L,\mu}(\tilde{u})$ by (3.4.15). Combining this with (3.4.13) gives

$$\nu_{1,L} \mu^{2(1-\alpha)} \leq \mathcal{R}_{L,\mu}(\tilde{u}). \quad (3.4.18)$$

If we set $\mu = 1, a = \frac{1}{2}, b = 1$, then the numerator of the Rayleigh quotient $\mathcal{R}_{L,1}(\tilde{u})$ becomes

$$\int_{S^{n-1}} \int_{\frac{1}{2}}^1 \left\{ \left(\frac{2F_{\infty,1}}{1+\rho^2} \right)^{n-2} \rho^{n-1} (|\partial_\rho \tilde{u}|^2 + \rho^{-2} |\nabla_\sigma \tilde{u}|_\sigma^2) \right.$$
$$\left. + \frac{n-2}{4(n-1)} R_{\infty,1} \left(\frac{2F_{\infty,1}}{1+\rho^2} \right)^{n} \rho^{n-1} |\tilde{u}|^2 \right\} d\rho \, dV_\sigma. \quad (3.4.19)$$

We see from (3.4.7) that $F_{L,1} \equiv F_{\infty,1}$ when $\rho < L$. So with this choice of μ, a, b, the denominator of $\mathcal{R}_{L,1}(\tilde{u})$ becomes

$$\int_{S^{n-1}} \int_{\frac{1}{2}}^1 \left(\frac{2F_{\infty,1}}{1+\rho^2} \right)^{n} \rho^{n-1} |\tilde{u}|^2 \, d\rho \, dV_\sigma. \quad (3.4.20)$$

Putting together (3.4.19) and (3.4.20), we see that $\mathcal{R}_{L,1}(\tilde{u})$ is a constant independent of L. So by setting $\mu = 1, a = \frac{1}{2}, b = 1$ and recalling (3.4.18), we obtain the bound $\nu_{1,L} \leq C$, which is the desired upper bound when $0 < \alpha < 1$.

In the case $1 \leq \alpha < 2$, we can obtain an improved upper bound by setting $\mu = mL^{-1}$ for a constant $m > 0$ to be chosen later. We proceed to estimate $\mathcal{R}_{L,\mu}(\tilde{u})$ as $L \to \infty$ and hence $\mu = mL^{-1} \to 0$. Consider the integrand of $E_{L,\mu}(\tilde{u})$. We observe that (3.4.6) and (3.4.5) imply the following limits, which are uniform for $\rho \in [a, b]$:

$$\lim_{\mu \to 0} \frac{2\mu^{\alpha} F_{\infty,\mu}}{\mu^2 + \rho^2} = 2A\rho^{\alpha-2}. \tag{3.4.21}$$

$$\lim_{\mu \to 0} R_{\infty,\mu} \mu^{2(1-\alpha)} = \frac{\alpha(2-\alpha)(n-1)(n-2)}{4A^2} \rho^{2(1-\alpha)}.$$

We thus deduce that the integral $E_{L,\mu}(\tilde{u})$ converges to

$$\int_{S^{n-1}} \int_a^b \Bigg\{ \left(2A\rho^{\alpha-2}\right)^{n-2} \rho^{n-1} \left(|\partial_\rho \tilde{u}|^2 + \rho^{-2} |\nabla_\sigma \tilde{u}|_\sigma^2 \right) + \frac{\alpha(2-\alpha)(n-2)^2}{16A^2} \rho^{2(1-\alpha)} \left(2A\rho^{\alpha-2}\right)^n \rho^{n-1} |\tilde{u}|^2 \Bigg\} \, d\rho \, dV_\sigma.$$

We can treat the denominator in a similar way. The denominator of $\mathcal{R}_{L,\mu}(\tilde{u})$ is

$$\int_{S^{n-1}} \int_a^b \left(\frac{2\mu^{\alpha} F_{\infty,\mu}}{\mu^2 + \rho^2} \right)^n \rho^{n-1} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}} \right)^2 |\tilde{u}|^2 \, d\rho \, dV_\sigma.$$

We observe by (3.4.7) and (3.4.9) that $\frac{F_{L,\mu}}{F_{\infty,\mu}} \equiv 1$ for $\rho \leq m$. Having fixed the interval $[a, b]$, we are free to choose $m \geq b$, so that the denominator of $\mathcal{R}_{L,\mu}(\tilde{u})$ becomes

$$\int_{S^{n-1}} \int_a^b \left(\frac{2\mu^{\alpha} F_{\infty,\mu}}{\mu^2 + \rho^2} \right)^n \rho^{n-1} |\tilde{u}|^2 \, d\rho \, dV_\sigma.$$

Using (3.4.21) we easily see that this integral converges to

$$\int_{S^{n-1}} \int_a^b \left(2A\rho^{\alpha-2}\right)^n \rho^{n-1} |\tilde{u}|^2 \, d\rho \, dV_\sigma.$$

Combining the limits for the numerator and denominator, we therefore find that $\mathcal{R}_{L,\mu}(\tilde{u})|_{\mu=mL^{-1}}$ converges as $L \to \infty$, to a limit which depends on F_∞, a, and b but not on m (provided $m \geq b$). Recalling (3.4.18), we obtain the following estimate: For every $m \geq b$,

$$\limsup_{L\to\infty} \nu_{1,L} L^{2(\alpha-1)} \leq C(F_\infty, a, b) m^{2(\alpha-1)}. \tag{3.4.22}$$

Fixing any choice of a, b, m with $m \geq b$ (for example $a = 1, b = 2, m = b$) will give us the desired upper bound on $\nu_{1,L}$ when $1 \leq \alpha < 2$, finishing the proof. □

Remark 3.4.4. If the limit

$$\zeta := \lim_{\substack{L\to\infty \\ \mu=mL^{-1}}} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 \tag{3.4.23}$$

were well-defined and positive, with sufficiently strong convergence, then it would be possible to extend estimate (3.4.22) to any choice of $m > 0$. For example, if we have $F_L(x) \geq \frac{1}{C}|L|^\alpha$ when $|x| \geq L$ for some constant $C > 0$, then the limit in (3.4.23) exists as a weak limit in $L^p(A_{a,b})$ for any $p \in (1, \infty)$ and is uniformly positive.

Suppose that such additional assumptions hold, so that the denominator in the

Rayleigh quotient converges to

$$\int_{S^{n-1}} \int_a^b \left(2A\rho^{\alpha-2}\right)^n \rho^{n-1}\zeta|\tilde{u}|^2 \, d\rho \, dV_\sigma.$$

As $\nu_{1,L}L^{2(\alpha-1)} \leq m^{2(\alpha-1)}\mathcal{R}_L(\tilde{u})|_{\mu=mL^{-1}}$, we would obtain

$$\limsup_{L\to\infty} \nu_{1,L}L^{2(\alpha-1)} \leq m^{2(\alpha-1)}C(F_\infty, a, b, m)$$

for a positive constant $C(F_\infty, a, b, m)$ which is continuous in m. As we have seen in the proof, for $m \geq b$ the constant in fact has no dependence on m. On the other hand if we choose $m < a$, then it is possible to show $\zeta \sim m^{2\alpha}$ on $[a, b]$, and in this case $C(F_\infty, m) \leq m^{2\alpha}C(F_\infty)$, so we would obtain the estimate

$$\nu_{1,L}L^{2(\alpha-1)} \leq m^{-2}C(F_\infty, a, b)$$

for $m < a$. Combining this with the estimate (3.4.22) for $m \geq b$ and using continuity in m, one has

$$\limsup_{L\to\infty} \nu_{1,L}L^{2(\alpha-1)} \leq C(F_\infty, a, b)m^{-2}(1+m)^{2\alpha}.$$

In particular we observe that letting $m \to 0$ or $m \to \infty$ causes the bound to degenerate, giving no new estimate.

3.4.3 Proof of Eigenvalue Upper Bound — General Case

We now present a proof of Proposition 3.4.2, completing the proof of our main result Theorem 3.2.1. We note that unlike the case of the punctured sphere described

above, the stereographic projection and the dilation group are not available to us in the more general setting. We will instead transfer our problem to a domain in $\mathbb{R}^n$ by using the exponential map.

Proof of Proposition 3.4.2. As in the special case considered in subsection 3.4.2, we will proceed by comparing the eigenvalue $\nu_{1,L}$ with that of a related operator on a smaller domain. We note however that in the general case our choice of domain is different due to the lack of the dilatation group. Fix a point $p \in \Sigma$ and let $\exp_p : T_pM \to M$ be the exponential map at p on (M, g_0). Choose standard coordinates $\{y^i\}$ on $T_pM \cong \mathbb{R}^n$ and let $x = \exp_p y$ for $|y| < \mathrm{inj}(M, g_0)$, so that $\{x^i\}$ is a geodesic normal coordinate system at $p \in M$ (for the metric g_0). We note that the components (with respect to the coordinates $\{y^i\}$) of the pullback metric $\exp_p^* g_0$, which is defined on the Euclidean ball of radius $\mathrm{inj}(M, g_0)$ centred at $0 \in \mathbb{R}^n$, satisfy

$$(\exp_p^* g_0)_{ij} = \delta_{ij} + O(|y|^2).$$

Now we introduce the dilation $\Phi_\mu : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi_\mu(z) = \mu z.$$

We pullback the metric once more, letting $y = \mu z$. We introduce the rescaled metric $\bar{g}_{0,\mu} \coloneqq \mu^{-2}\Phi_\mu^* \exp_p^* g_0$, whose components with respect to the induced coordinates $\{z^i\}$ satisfy

$$(\bar{g}_{0,\mu})_{ij} = \delta_{ij} + \mu^2 O(|z|^2).$$

Without loss of generality we may assume that the coordinates $\{y^i\}$ on $\mathbb{R}^n$ were chosen so that the tangent space to $\exp_p^{-1}(\Sigma)$ at 0 is $\mathbb{R}^k \subset \mathbb{R}^n$. Now fix a

ball $\tilde{B} \subset (\mathbb{R}^n, \bar{g}_{0,\mu})$ such that the ray from 0 to the centre of $\tilde{B}$ is perpendicular to $\mathbb{R}^k$, and so that $\tilde{B} \cap \Phi_\mu^{-1} \exp_p^{-1}(\Sigma) = \emptyset$. Suppose the centre of the ball $\tilde{B}$ is at a distance $a > 0$ from 0 in $(\mathbb{R}^n, \bar{g}_{0,\mu})$ and it has radius $\frac{a}{2}$. We note that $\exp_p \circ \Phi_\mu(\tilde{B})$ is an open set in (M, g_0), which lies in the annular region between the geodesic spheres of radii $\mu\frac{a}{2}$ and $\mu\frac{3a}{2}$ centred at p, and that for sufficiently small μ this set does not intersect Σ.

How does pulling back by these diffeomorphisms affect the equation satisfied by ψ_L? Given any function $f : M \to \mathbb{R}$, we can pullback to a function $f_\mu : \mathbb{R}^n \to \mathbb{R}$ by setting

$$f_\mu(z) \coloneqq f\left(\exp_p(\mu z)\right)$$

We have the conformal metric

$$\bar{g}_{\infty,\mu} \coloneqq \mu^{2\alpha} F_{\infty,\mu}^2 \bar{g}_{0,\mu},$$

whose components with respect to the coordinates $\{z^i\}$ on $\mathbb{R}^n$ satisfy

$$(\bar{g}_{\infty,\mu})_{ij} = A^2 \mathrm{dist}_{\bar{g}_{0,\mu}}\left(z, \Phi_\mu^{-1} \exp_p^{-1}(\Sigma)\right)^{-2\alpha} \delta_{ij} + O(\mu^\alpha). \tag{3.4.24}$$

After pulling back by $\exp_p \circ \Phi_\mu$ the equation satisfied by ψ_L becomes

$$\mathcal{L}_\mu \psi_{L,\mu} \coloneqq -\Delta_{\bar{g}_{\infty,\mu}} \psi_{L,\mu} + \frac{n-2}{4(n-1)} R_{\bar{g}_{\infty,\mu}} \psi_{L,\mu} = \mu^{2(1-\alpha)} \nu_{1,L} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 \psi_{L,\mu}, \tag{3.4.25}$$

with this expression being valid on the Euclidean ball of radius $\mu^{-1}\mathrm{inj}(M, g_0) \subset \mathbb{R}^n$ centred at 0.

As in the special case considered in the previous section, there exists a unique

real number $\tilde{\nu}_{L,\mu} > 0$ and smooth function $\bar{\psi}_{L,\mu} > 0$ on $\tilde{B}$ with $\bar{\psi}_{L,\mu} = 0$ on $\partial\tilde{B}$ such that

$$\mathcal{L}_\mu \bar{\psi}_{L,\mu} = \bar{\nu}_{L,\mu}\mu^{2(1-\alpha)} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 \bar{\psi}_{L,\mu}.$$

It is also known that

$$\nu_{1,L}\mu^{2(1-\alpha)} \leq \bar{\nu}_{L,\mu}\mu^{2(1-\alpha)}. \tag{3.4.26}$$

The proof of these facts proceeds as in the case considered in subsection 3.4.2. Indeed, we have

$$\bar{\nu}_{L,\mu}\mu^{2(1-\alpha)} = \inf_{\substack{u\in C_c^\infty(\tilde{B}) \\ u\not\equiv 0}} \mathcal{R}_{L,\mu},$$

where $\mathcal{R}_{L,\mu}$ is the Rayleigh quotient

$$\mathcal{R}_{L,\mu} \coloneqq \frac{\int_{\tilde{B}} \left(\left|\nabla_{\bar{g}_{\infty,\mu}} u\right|^2_{\bar{g}_{\infty,\mu}} + \frac{n-2}{4(n-1)} R_{\bar{g}_{\infty,\mu}} |u|^2 \right) dV_{\bar{g}_{\infty,\mu}}}{\int_{\tilde{B}} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 |u|^2 \, dV_{\bar{g}_{\infty,\mu}}}.$$

Thus, the claim that $\bar{\nu}_{L,\mu}\mu^{2(1-\alpha)} \geq \nu_{1,L}\mu^{2(1-\alpha)}$ is equivalent to the statement

$$\int_{\tilde{B}} \left(\left|\nabla_{\bar{g}_{\infty,\mu}} u\right|^2_{\bar{g}_{\infty,\mu}} + \frac{n-2}{4(n-1)} R_{\bar{g}_{\infty,\mu}} |u|^2 \right) dV_{\bar{g}_{\infty,\mu}} - \nu_{1,L}\mu^{2(1-\alpha)} \int_{\tilde{B}} \left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2 |u|^2 \, dV_{\bar{g}_{\infty,\mu}} \geq 0 \tag{3.4.27}$$

for all $u \in C_c^\infty(\tilde{B})$. We once again use the Hardy trick to prove the inequality above. Define a function $v \in C_c^\infty(\tilde{B})$ by setting $u = \psi_{L,\mu} v$. Such a decomposition is always possible since $\psi_{L,\mu}$ is smooth and strictly positive on $\tilde{B}$ and u has compact support in $\tilde{B}$. Inserting $u = \psi_{L,\mu} v$ into the numerator of the Rayleigh quotient

yields

$$\int_{\tilde{B}}\left(|\psi_{L,\mu}|^2\left|\nabla_{\bar{g}_{\infty,\mu}}v\right|^2_{\bar{g}_{\infty,\mu}}+\langle\nabla_{\bar{g}_{\infty,\mu}}\psi_{L,\mu},\nabla_{\bar{g}_{\infty,\mu}}v^2\psi_{L,\mu}\rangle_{\bar{g}_{\infty,\mu}}+\frac{n-2}{4(n-1)}R_{\bar{g}_{\infty,\mu}}|\psi_{L,\mu}v|^2\right)dV_{\bar{g}_{\infty,\mu}}.$$

Using the weak formulation of the PDE (3.4.25) satisfied by $\psi_{L,\mu}$ with test function $v^2\psi_{L,\mu}$, we see that this equals

$$\begin{aligned}\int_{\tilde{B}}|\psi_{L,\mu}|^2\left|\nabla_{\bar{g}_{\infty,\mu}}v\right|^2_{\bar{g}_{\infty,\mu}}dV_{\bar{g}_{\infty,\mu}}+\nu_{1,L}\mu^{2(1-\alpha)}\int_{\tilde{B}}\left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2|\psi_{L,\mu}v|^2\,dV_{\bar{g}_{\infty,\mu}}\\ \geq\nu_{1,L}\mu^{2(1-\alpha)}\int_{\tilde{B}}\left(\frac{F_{L,\mu}}{F_{\infty,\mu}}\right)^2|\psi_{L,\mu}v|^2\,dV_{\bar{g}_{\infty,\mu}}.\end{aligned}$$

We have thus shown (3.4.27) as desired, and the inequality (3.4.26) follows.

Fix an arbitrary function $\tilde{u}\in C_c^\infty(\tilde{B})$, then for every $L\geq 1$ we have $\bar{\nu}_{L,\mu}\mu^{2(1-\alpha)}\leq\mathcal{R}_{L,\mu}(\tilde{u})$. Combining this with (3.4.26), we have

$$\nu_{1,L}\mu^{2(1-\alpha)}\leq\mathcal{R}_{L,\mu}(\tilde{u}) \tag{3.4.28}$$

We claim that if we make a judicious choice of $\mu=\mu(L)$, then $\mathcal{R}_{L,\mu}(\tilde{u})$ converges as $L\to\infty$. Our choice of μ will be different in the two cases $0<\alpha<1$ and $1\leq\alpha<2$ and is motivated by the proof of the special case Proposition 3.4.3 considered in the previous subsection. First recall that the centre of the ball $\tilde{B}$ is at a distance $a>0$ from 0 in $(\mathbb{R}^n,\bar{g}_{0,\mu})$, and it has radius $\frac{a}{2}$. Recall also that $\exp_p\circ\Phi_\mu(\tilde{B})$ is an open set in (M,g_0) which lies in the annular region between the geodesic spheres of radii $\mu\frac{a}{2}$ and $\mu\frac{3a}{2}$ centred at p, and that for all sufficiently small μ, this set does not intersect Σ.

Now, in the case $0<\alpha<1$, we set $\mu=\bar{\mu}$ a constant chosen small enough so that

$\exp_p \circ \Phi_{\bar{\mu}}(\tilde{B}) \cap \Sigma = \emptyset$. There exists a constant $\bar{L} > 1$ such that $\mathrm{dist}_{g_0}(\exp_p \circ \Phi_{\bar{\mu}}(\tilde{B}), \Sigma) > \bar{L}^{-1}$. For all $L \geq \bar{L}$, we have that $\exp_p \circ \Phi_{\bar{\mu}}(\tilde{B})$ lies in the part of M where $F_L \equiv F_\infty$, which implies $F_{L,\bar{\mu}} \equiv F_{\infty,\bar{\mu}}$ in $\tilde{B}$. We have

$$\begin{aligned}
\bar{\nu}_{L,\bar{\mu}} \bar{\mu}^{2(1-\alpha)} &\leq \mathcal{R}_{L,\bar{\mu}}(\tilde{u}) \\
&= \frac{\int_{\tilde{B}} \left|\nabla_{\bar{g}_{\infty,\bar{\mu}}} \tilde{u}\right|^2_{\bar{g}_{\infty,\bar{\mu}}} + \frac{n-2}{4(n-1)} R_{\bar{g}_{\infty,\bar{\mu}}} |\tilde{u}|^2 \, dV_{\bar{g}_{\infty,\bar{\mu}}}}{\int_{\tilde{B}} \left(\frac{F_{L,\bar{\mu}}}{F_{\infty,\bar{\mu}}}\right)^2 |\tilde{u}|^2 \, dV_{\bar{g}_{\infty,\bar{\mu}}}} \\
&= \frac{\int_{\tilde{B}} \left|\nabla_{\bar{g}_{\infty,\bar{\mu}}} \tilde{u}\right|^2_{\bar{g}_{\infty,\bar{\mu}}} + \frac{n-2}{4(n-1)} R_{\bar{g}_{\infty,\bar{\mu}}} |\tilde{u}|^2 \, dV_{\bar{g}_{\infty,\bar{\mu}}}}{\int_{\tilde{B}} |\tilde{u}|^2 \, dV_{\bar{g}_{\infty,\bar{\mu}}}}.
\end{aligned}$$

Note that the RHS does not depend on L. Recalling (3.4.28), we thus have

$$\nu_{1,L} \leq C$$

for a constant C which depends on F_∞ and n, but which does not depend on L. This is the desired upper bound when $0 < \alpha < 1$.

The argument above also holds for $1 \leq \alpha < 2$, but in this case we can obtain an improved estimate by making a different choice of $\mu = \mu(L)$. To see this, let us write $\Sigma_{L^{-1}} \coloneqq \{x \in M : r < L^{-1}\}$. Now note that given a constant $M > \frac{2}{a}$, there exists $\bar{L} > 1$ such that for all $L \geq \bar{L}$ we have $\tilde{B} \cap \frac{L}{M} \exp_p^{-1}(\Sigma_{L^{-1}}) = \emptyset$. Thus, if we fix such an M and set $\mu(L) = ML^{-1}$, we have $F_{L,\mu} \equiv F_{\infty,\mu}$ on $\tilde{B}$ for all $L \geq \bar{L}$.

We proceed to estimate $\mathcal{R}_{L,\mu}(\tilde{u})$ as $L \to \infty$ and hence $\mu = mL^{-1} \to 0$. Consider the integrand in the numerator of the Rayleigh quotient. We observe that (3.3.3) and (3.4.24) imply the following limits, which are uniform for $z \in \tilde{B}$:

$$\lim_{\mu \to 0} \left|\nabla_{\bar{g}_{\infty,\mu}} \tilde{u}\right|^2_{\bar{g}_{\infty,\mu}} = A^{-2} \mathrm{dist}_{g_0}(\exp_p(z), \Sigma)^{2\alpha} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial z^i}\right)^2$$

$$\lim_{\mu\to 0} R_{\bar{g}_{\infty,\mu}} = \mathrm{dist}_{g_0}(\exp_p(z),\Sigma)^{2(\alpha-1)} A^{-2}\alpha(n-1)[(2-\alpha)(n-2)-2k]$$

$$\lim_{\mu\to 0} dV_{\bar{g}_{\infty,\mu}} = A^n \mathrm{dist}_{g_0}(\exp_p(z),\Sigma)^{-n\alpha}\, dz$$

Writing $\rho(z) \coloneqq \mathrm{dist}_{g_0}(\exp_p(z),\Sigma)$, we thus deduce that the numerator of the Rayleigh quotient converges to

$$\int_{\tilde{B}} \Bigg(A^{-2}\rho(z)^{2\alpha} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial z^i}\right)^2 + \frac{1}{4}\rho(z)^{2(\alpha-1)} A^{-2}\alpha(n-2)[(2-\alpha)(n-2)-2k]|\tilde{u}|^2 \Bigg) A^n \rho(z)^{-n\alpha} dz.$$

We can treat the denominator in a similar way. For $L \geq \bar{L}$, we have that $F_{L,\mu} \equiv F_{\infty,\mu}$ on $\tilde{B}$, and so the denominator of the Rayleigh quotient is

$$\int_{\tilde{B}} |\tilde{u}|^2\, dV_{\bar{g}_{\infty,\mu}},$$

which converges as $\mu \to 0$ to

$$\int_{\tilde{B}} |\tilde{u}|^2 A^n \rho(z)^{-n\alpha}\, dz$$

Combining the limits for the numerator and denominator, we therefore find that $\mathcal{R}_{L,\mu}(\tilde{u})|_{\mu=mL^{-1}}$ converges as $L \to \infty$, to a limit which depends on A, α, n, k. Recalling (3.4.28), we find that

$$\limsup_{L\to\infty} \nu_{1,L} L^{2(\alpha-1)} \leq M^{2(\alpha-1)} C(A,\alpha,n,k).$$

Fixing any choice of $M > \frac{2}{a}$ gives the upper bound claimed in the theorem

for $1 \leq \alpha < 2$. □

3.4.4 Proofs of Corollaries

Next we turn to the proof of Corollary 3.2.2 and Corollary 3.2.3. The corollaries will follow from combining our eigenvalue estimates in Theorem 3.2.1 with the following estimates for the volume and diameter of (M, g_L). In both of these lemmas, we write $A_1 \sim A_2$ to mean $\frac{1}{C}A_1 \leq A_2 \leq CA_1$ for a constant $C > 0$ depending on at most $n, A, \alpha, \delta, B, g_0$.

Lemma 3.4.5. *Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Fix $0 < \alpha < 2$ and let $\Sigma \subset M$ be a closed submanifold of dimension $k < \frac{(2-\alpha)(n-2)}{2}$. Let $\{g_L\}_{L\geq 1}$ be a family of conformal metrics on M satisfying the hypotheses of Theorem 3.2.1. We have the following estimate for the volume of (M, g_L):*

$$\mathrm{Vol}(M, g_L) \sim \begin{cases} 1 & 0 < \alpha < \frac{n-k}{n} \\ \log L & \alpha = \frac{n-k}{k} \\ L^{n\alpha+(k-n)} & \frac{n-k}{n} < \alpha < 2. \end{cases}$$

Lemma 3.4.6. *Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Fix $0 < \alpha < 2$ and let $\Sigma \subset M$ be a closed submanifold of dimension $k < \frac{(2-\alpha)(n-2)}{2}$. Let $\{g_L\}_{L\geq 1}$ be a family of conformal metrics on M satisfying the hypotheses of*

Theorem 3.2.1. We have the following estimate for the diameter of (M, g_L)*:*

$$\operatorname{diam}(M, g_L) \sim \begin{cases} 1 & 0 < \alpha < 1 \\ \log L & \alpha = 1 \\ L^{\alpha - 1} & 1 < \alpha < 2. \end{cases} \tag{3.4.29}$$

We will prove these lemmas in turn, and then use them to prove Corollary 3.2.2 and Corollary 3.2.3. Throughout these proofs, for any $\varepsilon > 0$ we write $\Sigma_\varepsilon := \{x \in M : r < \varepsilon\}$. Let ε' be a constant chosen small enough so that the tubular neighbourhood Σ_ε has smooth boundary for all $\varepsilon \leq \varepsilon'$.

Proof of Lemma 3.4.5. For the lower bound on the volume, we have

$$\begin{aligned} \operatorname{Vol}(M, g_L) &= \int_M F_L^n \, dV_{g_0} \\ &\geq \frac{1}{C} + \int_{\Sigma_{\varepsilon'} \setminus \Sigma_{L^{-1}}} F_L^n \, dV_{g_0} \\ &\geq \frac{1}{C} \left(1 + \int_{\Sigma_{\varepsilon'} \setminus \Sigma_{L^{-1}}} r^{-n\alpha} \, dV_{g_0} \right) \\ &\geq \frac{1}{C} \left(1 + \int_{L^{-1}}^{\varepsilon'} r^{-n\alpha} r^{n-k-1} \, dr \right) \\ &\geq \begin{cases} \frac{1}{C} & \alpha < \frac{n-k}{n} \\ \frac{1}{C} \log L & \alpha = \frac{n-k}{n} \\ \frac{1}{C} L^{n\alpha + (k-n)} & \alpha > \frac{n-k}{n} \end{cases}, \end{aligned}$$

where we have used (3.2.2) and the fact that $F_L(x) = F_\infty(x)$ when $r > L^{-1}$. For

the upper bound on the volume, we have

$$\begin{aligned}
\mathrm{Vol}(M, g_L) &= \int_M F_L^n \, dV_{g_0} \\
&\leq C \int_{M \setminus \Sigma_{L^{-1}}} r^{-n\alpha} \, dV_{g_0} + C \int_{\Sigma_{L^{-1}}} L^{n\alpha} \, dV_{g_0} \\
&\leq C \int_{L^{-1}}^{C} r^{-n\alpha} r^{n-k-1} \, dr + C L^{n\alpha} \int_0^{L^{-1}} r^{n-k-1} \, dr \\
&\leq \begin{cases} C & \alpha < \frac{n-k}{n} \\ C \log L & \alpha = \frac{n-k}{n} \\ C L^{n\alpha + (k-n)} & \alpha > \frac{n-k}{n} \end{cases},
\end{aligned}$$

where we have used the upper bound in (3.2.6) to estimate $F_L(x)$ when $r < L^{-1}$. □

Proof of Lemma 3.4.6. We start with the lower bound and consider separately the cases $1 < \alpha < 2$, $\alpha = 1$, and $0 < \alpha < 1$.

Suppose first that $1 < \alpha < 2$. Let $x_1, x_2 \in M$ be such that $\mathrm{dist}_{g_0}(x_1, \Sigma) = 2L^{-1}$ and $\mathrm{dist}_{g_0}(x_2, \Sigma) = L^{-1}$. Let $\gamma : [0, 1] \to M$ be any curve from x_1 to x_2. Define $0 \leq t_0 < t_0' \leq 1$ by

$$\begin{aligned}
t_0 &\coloneqq \sup\{t \in [0, 1] : \mathrm{dist}_{g_0}(\gamma(t), \Sigma) = 2L^{-1}\} \\
t_0' &\coloneqq \inf\{t \in [t_0, 1] : \mathrm{dist}_{g_0}(\gamma(t), \Sigma) = L^{-1}\},
\end{aligned}$$

so that in particular we have

$$L^{-1} \leq \mathrm{dist}_{g_0}(\gamma(t), \Sigma) \leq 2L^{-1} \text{ for all } t \in [t_0, t_0'].$$

We note also that

$$\text{length}_{g_0}(\gamma|_{[t_0,t_0']}) \geq L^{-1}.$$

Putting this all together we then have

$$\begin{aligned}\text{length}_{g_L}(\gamma) &= \int_\gamma F_L|\dot{\gamma}|_{g_0} \\ &\geq \int_{t_0}^{t_0'} \frac{1}{C}L^\alpha|\dot{\gamma}(t)|_{g_0}\,dt \\ &\geq \frac{1}{C}L^\alpha \text{length}_{g_0}(\gamma|_{[t_0,t_0']}) \geq \frac{1}{C}L^{\alpha-1}.\end{aligned}$$

This gives the desired lower bound when $\alpha > 1$.

When $\alpha = 1$, we can obtain an improved lower bound as follows: Let $x_1, x_2 \in M$ be such that $\text{dist}_{g_0}(x_1, \Sigma) = \varepsilon'$ and $\text{dist}_{g_0}(x_2, \Sigma) = L^{-1}$. We will assume that L is sufficiently large that $L^{-1} < \varepsilon'$, noting that if the desired lower bound (3.4.29) on the diameter holds for such L, then by enlarging the constant C if necessary it will hold for all $L \geq 1$. Let $\gamma : [0,1] \to M$ be any curve from x_1 to x_2. Let n be the smallest integer such that $\frac{\varepsilon'}{2^{n+1}} < L^{-1}$. For $i = 0, \ldots, n-1$, we define

$$\begin{aligned}t_i &:= \sup\left\{t \in [0,1] : \text{dist}_{g_0}(\gamma(t), \Sigma) = \frac{\varepsilon'}{2^i}\right\} \\ t_i' &:= \inf\left\{t \in [t_0,1] : \text{dist}_{g_0}(\gamma(t), \Sigma) = \frac{\varepsilon'}{2^{i+1}}\right\},\end{aligned}$$

so that $0 \leq t_0 < t_0' \leq \cdots \leq t_i < t_i' \leq \cdots \leq t_{n-1} < t_{n-1}' \leq 1$ and for every i we have

$$\frac{\varepsilon'}{2^{i+1}} \leq \text{dist}_{g_0}(\gamma(t), \Sigma) \leq \frac{\varepsilon'}{2^i} \text{ for all } t \in [t_i, t_i'].$$

We note also that for every i we have

$$\text{length}_{g_0}(\gamma|_{[t_i,t_i']}) \geq \frac{\varepsilon'}{2^i}.$$

Now we have

$$\begin{aligned}\text{length}_{g_L}(\gamma) &= \int_\gamma F_L|\dot{\gamma}|_{g_0} \\ &\geq \sum_{i=0}^{n-1} \int_{t_i}^{t_i'} \frac{1}{C}\text{dist}_{g_0}(\gamma(t),\Sigma)^{-1}|\dot{\gamma}(t)|_{g_0}\,dt \\ &\geq \frac{1}{C}\sum_{i=0}^{n-1} \frac{2^{i+1}}{\varepsilon'}\text{length}_{g_0}(\gamma|_{[t_i,t_i']}) \geq \frac{1}{C}n.\end{aligned}$$

Recall n satisfies $\frac{\varepsilon'}{2^{n+1}} < L^{-1}$, which implies $n \geq \frac{1}{C}\log L$, and the desired lower bound on $\text{diam}(M, g_L)$ for $\alpha = 1$ follows.

When $0 < \alpha < 1$, we claim that the diameter of (M, g_L) is bounded below by a constant independent of L. This can be easily seen from the fact that far from Σ, the metric g_L does not depend on L. If we fix a pair of points $x_1, x_2 \in M \setminus \Sigma_{L^{-1}}$ (which we can certainly do for sufficiently large L), then we have $\text{diam}(M, g_L) \geq \text{dist}_{g_L}(x_1, x_2) \geq \frac{1}{C}$.

We now turn to the proof of the upper bounds on the diameter. To prove an upper bound on $\text{diam}(M, g_L)$, I need to show that for any pair of points $x_1, x_2 \in M$, the distance between them with respect to the metric g_L satisfies an appropriate upper bound. It suffices to prove the desired bounds for L large enough that $L^{-1} < \varepsilon'$, and we henceforth assume that this is so. We consider three cases: when $x_1, x_2 \in M \setminus \Sigma_{L^{-1}}$, when $x_1 \in M \setminus \Sigma_{L^{-1}}$ but $x_2 \in \Sigma_{L^{-1}}$, and when $x_1, x_2 \in \Sigma_{L^{-1}}$ respectively.

First, suppose $x_1, x_2 \in M \setminus \Sigma_{L^{-1}}$. Then there exists a constant C independent of L such that $\mathrm{dist}_{g_L}(x_1, x_2) \leq C$, since the metric g_L does not depend on L outside $\Sigma_{L^{-1}}$.

Now suppose $x_1 \in M \setminus \Sigma_{L^{-1}}$ and $x_2 \in \Sigma_{L^{-1}}$. Consider a geodesic ray with respect to the background metric g_0 which passes through x_2 and which is normal to Σ (with respect to g_0). Let x' be a point on this ray which lies on the boundary of $\Sigma_{L^{-1}}$, and x'' a point on the ray which lies outside $\Sigma_{\varepsilon'}$. We let γ denote the part of this curve between x_2 and x'', and estimate its length with respect to g_L. First, let γ_1 denote the part of γ which runs from x_2 to x', and γ_2 the part which runs from x' to x'', so that $\gamma = \gamma_1 \cup \gamma_2$. For the length of γ_1, we have the estimate

$$\begin{aligned}\mathrm{length}_{g_L}(\gamma_1) &= \int_{\gamma_1} F_L |\dot{\gamma}|_{g_0} \\ &\leq CL^{\alpha}\mathrm{length}_{g_0}(\gamma_1) \leq CL^{\alpha-1}.\end{aligned}$$

For the length of γ_2, we have the estimate

$$\begin{aligned}\mathrm{length}_{g_L}(\gamma_2) &= \int_{\gamma_2} F_L |\dot{\gamma}|_{g_0} \\ &\leq C \int_{L^{-1}}^{\varepsilon'} t^{-\alpha}\, dt \\ &\leq \begin{cases} C & 0 < \alpha < 1 \\ C \log L & \alpha = 1 \\ CL^{\alpha-1} & 1 < \alpha < 2. \end{cases}\end{aligned}$$

Finally, we note that $\mathrm{dist}_{g_L}(x'', x_1) \leq C$, since both x'' and x_1 lie outside $\Sigma_{L^{-1}}$.

Putting all these bounds together, we find that

$$\mathrm{dist}_{g_L}(x_1, x_2) \leq \begin{cases} C & 0 < \alpha < 1 \\ C \log L & \alpha = 1 \\ C L^{\alpha - 1} & 1 < \alpha < 2 \end{cases},$$

as desired.

Finally, suppose $x_1, x_2 \in \Sigma_{L^{-1}}$. This case can be treated in a similar way to the previous one. Consider a geodesic (with respect to g_0) that is perpendicular to Σ and which runs through x_1. Arguing as above, we see that the part of this curve from x_1 to some point x' lying outside $\Sigma_{\varepsilon'}$ has length (with respect to g_L) at most

$$\begin{cases} C & 0 < \alpha < 1 \\ C \log L & \alpha = 1 \\ C L^{\alpha - 1} & 1 < \alpha < 2. \end{cases}$$

Similarly, we can construct a curve from x_2 to a point x'' lying outside $\Sigma_{\varepsilon'}$, whose length with respect to g_L satisfies the same upper bound. Then x' and x'' can be connected by a curve which has length bounded above by a constant independent of L. We deduce that we must have the upper bound

$$\mathrm{dist}_{g_L}(x_1, x_2) \leq \begin{cases} C & 0 < \alpha < 1 \\ C \log L & \alpha = 1 \\ C L^{\alpha - 1} & 1 < \alpha < 2 \end{cases},$$

and the desired upper bound on $\mathrm{diam}(M, g_L)$ follows. □

With Lemmas 3.4.5 and 3.4.6 in hand, we are ready to prove Corollary 3.2.2 and Corollary 3.2.3.

Proof of Corollary 3.2.2. Combining the volume estimates from Lemma 3.4.5 with the eigenvalue estimates from Theorem 3.2.1 yields the following estimates for the volume-normalised eigenvalue, valid for $L \geq L_0 = L_0(n, A, \alpha, B, \delta)$:

$$\nu_{1,L}\mathrm{Vol}(M, g_L)^{\frac{2}{n}} \sim \begin{cases} 1 & 0 < \alpha < \frac{n-k}{n} \\ (\log L)^{\frac{2}{n}} & \alpha = \frac{n-k}{n} \\ L^{2\alpha - 2\frac{n-k}{n}} & \frac{n-k}{n} < \alpha < 1 \\ L^{2\alpha - 2\frac{n-k}{n}} & \alpha = 1 \text{ and } k > 0 \\ L^{2-2\frac{n-k}{n}} & 1 < \alpha < 2. \end{cases}$$

We see that as $L \to \infty$, the RHS above remains bounded in the case $0 < \alpha < \frac{n-k}{n}$ and in the case where $k = 0, \alpha > \frac{n-k}{n} = 1$, while in all other cases it grows without bound, as claimed in the statement of the Lemma. □

Proof of Corollary 3.2.3. Combining the diameter estimates from Lemma 3.4.6 with the eigenvalue estimates from Theorem 3.2.1 yields the following estimates for the diameter-normalised eigenvalue, again valid for $L \geq L_0$:

$$\nu_{1,L}\mathrm{diam}(M, g_L)^2 \sim \begin{cases} 1 & 0 < \alpha < 1 \\ (\log L)^2 & \alpha = 1 \\ 1 & 1 < \alpha < 2. \end{cases}$$

We see that as $L \to \infty$, the RHS is bounded if and only if $\alpha \neq 1$, as claimed in the statement of the Lemma. □

We end this section with a proof of Corollary 3.2.5.

Proof. Arguing as in the proof of the volume bounds in Corollary 3.2.2, we obtain the estimate

$$\int_M F_L^p \, dV_{g_0} \sim \begin{cases} 1 & \alpha < \frac{n-k}{p} \\ \log L & \alpha = \frac{n-k}{p} \\ L^{p\alpha+(k-n)} & \alpha > \frac{n-k}{p}. \end{cases} \tag{3.4.30}$$

Combining this with the eigenvalue estimates (3.2.8), we find that when $\alpha = 1$ we have

$$\nu_{1,L} \left(\int_M F_L^p \, dV_{g_0} \right)^{\frac{2}{p}} \sim \begin{cases} 1 & p < n-k \\ (\log L)^{\frac{2}{p}} & p = n-k \\ L^{2+2\frac{k-n}{p}} & p > n-k. \end{cases}$$

We see that the eigenvalue functional grows to infinity as $L \to \infty$ provided $p \geq n-k$, as claimed. □

Remark 3.4.7. We note that if we do not restrict our attention to the special case $\alpha = 1$, we can use (3.4.30) and the eigenvalue estimates (3.2.8) to obtain sharp estimates for the behaviour of the functional (3.1.5) for the full range of metrics considered in this chapter, i.e. for $0 < \alpha < 2$ and all integers $0 \leq k < \frac{(2-\alpha)(n-2)}{2}$.

We include these estimates here for completeness. When $0 < \alpha < 1$, we have

$$\nu_{1,L}\left(\int_M F_L^p \, dV_{g_0}\right)^{\frac{2}{p}} \sim \begin{cases} 1 & \alpha < \frac{n-k}{p} \\ (\log L)^{\frac{2}{p}} & \alpha = \frac{n-k}{p} \\ L^{2\alpha+2\frac{k-n}{p}} & \alpha > \frac{n-k}{p} \end{cases},$$

while for $1 \leq \alpha < 2$ we have

$$\nu_{1,L}\left(\int_M F_L^p \, dV_{g_0}\right)^{\frac{2}{p}} \sim \begin{cases} L^{2(1-\alpha)} & \alpha < \frac{n-k}{p} \\ (\log L)^{\frac{2}{p}} L^{2(1-\alpha)} & \alpha = \frac{n-k}{p} \\ L^{2+2\frac{k-n}{p}} & \alpha > \frac{n-k}{p}. \end{cases}$$

Chapter 4

Conclusion and Outlook

In this chapter we will discuss the significance and limitations of our results, and identify some further questions which may serve as possible directions of future research.

4.1 Eigenvalues of the Laplacian on Hyperbolic Surfaces

The main result we presented in Chapter 2 of this thesis, Theorem 2.1.1, is a generalisation of Theorem 1.2 in [28] to the case of surfaces which degenerate in a more complicated way, i.e. with multiple collapsing geodesics. In [28], Große and Rupflin use their result to obtain sharp error estimates for the first eigenvalue, improving on estimates obtained by Burger in [9]. Corollary 2.1.3 is an analogous result that holds in the more general setting we consider.

We note that moving from the case of a single collapsing geodesic to multiple collapsing geodesics allows us to prove results for multiple small eigenvalues, and not

just the first eigenvalue $\lambda_1(-\Delta_g)$. However, this generalisation creates additional subtleties to the problem which needed to be addressed in our work.

One example of such an issue is that when considering multiple eigenvalues there is the possibility that curves of eigenvalues cross as we vary the metric. In general the function $g \mapsto \lambda_k(-\Delta_g)$ is not differentiable with respect to the Fenchel-Nielsen coordinates unless $\lambda_k(-\Delta_g)$ is simple, so the interpretation of our derivative estimates for smooth curves of eigenvalues and corresponding eigenfunctions needed to be clarified. This was addressed in Remark 2.1.2 and in the proof of Lemma 2.2.9.

Another issue arising in the setting we consider is the possible existence of collapsing non-disconnecting geodesics, i.e. those geodesics labelled by indices $j_0 + 1 \leq j \leq j_1$ in the proof of Corollary 2.1.3. While the lengths of these geodesics are small, the dependence of the small eigenvalues of $-\Delta_g$ on their lengths is less significant than for the disconnecting geodesics $\{\sigma^j\}_{j=1}^{j_0}$. In particular, the derivative estimate (2.1.6) applied for these geodesics is not sufficient to prove the claim of Corollary 2.1.3. We overcome this difficulty by obtaining the improved estimate (2.1.8) for these geodesics. This estimate has no analogue in the setting considered by Große and Rupflin in [28].

One of the aspects of the work of Große and Rupflin in [28] that our results do not touch on is the sharpness (or otherwise) of the derivative estimates in Theorem 2.1.1. The analogous estimates in [28] are shown to be sharp by the construction of families of metrics for which the derivatives of the eigenvalue (with respect to the Fenchel-Nielsen coordinates) satisfy appropriate lower bounds. A problem of interest which is not addressed by the present work is the construction and analysis of such metrics in the generalised setting of multiple collapsing geodesics.

The result of Große and Rupflin [28] also gives the following estimate for the derivative of the first eigenvalue with respect to the length of the single collapsing geodesic

$$\left|\frac{\partial\lambda_1}{\partial\ell_1} - \frac{\lambda_1}{\ell_1}\right| \leq C\ell_1|\log(\ell_1)|.$$

In contrast, the derivative estimates (2.1.6) we prove in Theorem 2.1.1 hold for all $1 \leq j \leq 3(\gamma - 1)$ and do not provide any additional information for the derivatives with respect to the lengths of the collapsing disconnecting geodesics $\sigma^1, \ldots, \sigma^{j_0}$. Obtaining more precise knowledge of the dependence of the eigenvalues on the lengths of these geodesics is an interesting problem which requires further analysis. We remark that similar arguments to those used to prove Theorem 2.1.1 can be used to show that for $1 \leq j \leq j_0$,

$$\left|\frac{\partial\lambda}{\partial\ell_j} - \frac{1}{2\pi}\ell_j \int_{\mathcal{C}(\sigma^j)} (|u_s|^2 - |u_\theta|^2)\rho^{-2}\, ds\, d\theta\right| \leq C\lambda^2\ell_j$$

suggesting that the behaviour of the derivatives with respect to the lengths of the small disconnecting geodesics depends principally on the behaviour of the eigenfunction u on the corresponding collar $\mathcal{C}(\sigma^j)$. This dependence may in general be complicated and depend on the relative lengths of the small geodesics ℓ_i.

4.2 Eigenvalue Estimates for the Conformal Laplacian

The main result we presented in Chapter 3 of this thesis, Theorem 3.2.1, provides a sharp estimate for $\nu_1(-L_g)$ for a family $\{g_L\}_{L\geq 1}$ of conformal metrics on a closed

manifold M of dimension $n \geq 3$ as $L \to \infty$. This family of metrics was obtained as truncations of an asymptotically conical metric g_∞ with positive scalar curvature, a choice inspired by work of Amman and Jammes [2].

In our framework, the metrics g_L depend on parameters $0 \leq k < \frac{(2-\alpha)(n-2)}{2}$ and $0 < \alpha < 2$. When $\alpha \in (1, 2)$, the metric g_∞ is asymptotically conical in the sense of Proposition 3.3.2, and the value of α determines the angle at which the cone at infinity opens up. When $k = 0$ and $\alpha = 1$, g_∞ is asymptotically cylindrical; in this case the metrics we consider are the same as those studied by Amman and Jammes [2]. We recover a special case of a result of theirs which shows that the direct analogue for $\nu_k(-L_g)$ of Korevaar's estimate (3.1.2) for $\lambda_k(-\Delta_g)$ is false. By obtaining the estimate (3.2.12) we also find that the metrics $\{g_L\}_{L\geq 1}$ with $k > 0$ and $\frac{n-k}{k} \leq \alpha < 2$ are new examples of conformal families for which the volume-normalised eigenvalues are unbounded. We also use our eigenvalue estimate (3.2.8) to prove Corollary 3.2.2, which tells us that a Korevaar-type bound does in fact hold for our asymptotically conical family when $k = 0$ and $\alpha > 1$.

The eigenvalue normalisation considered by Sire and Xu in [63] and the volume-normalisation considered by Amman and Jammes in [2] can be considered as two possible choices from a continuum of normalisations of the form (3.1.5), where each choice of the exponent p yields a different way of normalising the eigenvalue. An interesting question is the following: what is the largest value of p for which a Korevaar-type bound holds for the first eigenvalue of $-L_g$? Given the results of Amman and Jammes in [2], we know that such a bound does not hold when $p = n$, while the result of Sire and Xu in [63] shows that in the conformal class of the standard sphere, a Korevaar-type bound does hold when $p = 2$. In Corollary 3.2.5, we consider the behaviour of these normalised eigenvalues for arbitrary p for the

metrics $\{g_L\}_{L\geq 1}$, and see that a Korevaar-type bound cannot hold when $p \geq n-k$, for $0 \leq k < \frac{n-2}{2}$ an integer. The smallest possible value of $p = n-k$ satisfying the constraint on k is the smallest integer strictly larger than $\frac{n+2}{2}$, which can also be written as $\left[\frac{n+2}{2}\right] + 1$, where $[x]$ denotes the integer part of x.

If k were not constrained to be an integer (i.e. if we allow the subset Σ to be of non-integer dimension), then it is possible that this could be improved to $p > \frac{n+2}{2}$. Having a non-integer dimension for Σ introduces issues regarding the regularity and properties of the distance function $\mathrm{dist}_{g_0}(x, \Sigma)$, which we used extensively in proving the results in this thesis. The question of whether and to what extent it would be possible to prove similar results in this setting is a potential direction of future research in this area.

In this thesis we do not consider the limiting case $k = \frac{(2-\alpha)(n-2)}{2}$. In this case we do not have the global lower bound (3.3.1) on the scalar curvature in terms of the distance function $r = \mathrm{dist}_{g_0}(x, \Sigma)$, since the constant B degenerates. If one could prove our result in this limiting case, then in particular the result of Corollary 3.2.5 might be extended to $k = \frac{n-2}{2}$. We note that when (M, g_0) is the standard sphere S^n, then for $k \leq \frac{n-2}{2}$, there exist complete conformal metrics on $M \setminus \Sigma$ with constant positive scalar curvature by a result of Mazzeo and Pacard [51]. On the other hand, it is known by a result of Schoen and Yau [62] that for $k > \frac{n-2}{2}$ no such metrics can exist. Thus the dimension $k = \frac{n-2}{2}$ plays a crucial role in the existence of solutions to the singular Yamabe problem, and a question which remains unexplored by this thesis is whether the appearance of this dimension in the present work is somehow connected to this phenomenon.

In the context of Laplacian eigenvalue estimates (as in the results of Hassannezhad [29]) and in the study of the Yamabe problem and its fully nonlinear

generalisations (by Li and Nguyen [49]), the existence of a lower Ricci bound can be helpful. It would be interesting to see if a connection exists between a lower Ricci bound and eigenvalues of the conformal Laplacian. We note that for the asymptotically conical metrics we consider, in the special case of the tangent cone at infinity the eigenvalues of the Ricci curvature are

$$\frac{\alpha(n-k-2)}{|z'|^2} - (n-2)\alpha^2, \text{ with multiplicity } k, \tag{4.2.1}$$

$$\frac{\alpha(2n-k-4)}{|z'|^2} - (n-2)\alpha^2, \text{ with multiplicity } n-k-1, \tag{4.2.2}$$

$$\frac{\alpha(\alpha(n-2)-k)}{|z'|^2} - (n-2)\alpha^2, \text{ with multiplicity } 1. \tag{4.2.3}$$

The eigenvalues (4.2.1) and (4.2.2) are both positive for sufficiently small $|z'|$ (i.e. close to the singular set $\mathbb{R}^k \subset \mathbb{R}^n$). The eigenvalue (4.2.3) is positive for small $|z'|$ if $\alpha > \frac{k}{n-2}$. When $\alpha = \frac{k}{n-2}$ the eigenvalue is a negative constant, while for $\alpha < \frac{k}{n-2}$ it is negative and unbounded below as $|z'| \to 0$. So the Ricci curvature is bounded from below unless $\alpha < \frac{k}{n-2}$, and in this case it has a single negative eigenvalue near the singular set.

We hope that this concrete example serves as a useful example in beginning to understand the difference in behaviour between the eigenvalues of $-\Delta_g$ and of $-L_g$. At this stage in our study, a question which remains unexplored is the determination of which properties of the metrics $\{g_L\}_{L\geq 1}$ are responsible for the change in behaviour in the cases where the volume-normalised eigenvalue remains bounded (i.e. $k = 0, \alpha > 1$) compared to the cases where the volume-normalised eigenvalue blows up as $L \to \infty$ (i.e. $k = 0, \alpha = 1$ and $k > 0, \alpha \geq \frac{n-k}{k}$). A greater understanding of this question could shed light on the problem of determining the

most general subclass of metrics in $[g_0]$ for which $\nu_1(-L_g)$ satisfies a Korevaar-type bound. For example, suppose one could find some geometric property satisfied by the metrics g_L in the case $k = 0, \alpha > 1$ which behaves differently in the other cases mentioned above. It would be interesting then to see if other families of conformal metrics constructed to have this property would also give eigenvalues which satisfy a Korevaar-type bound.

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