Twisted coadmissible equivariant $\mathcal{D}$-modules on rigid analytic spaces

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Abstract

We aim to extend the results of Ardakov and Wadsley on representations of p-adic Lie groups and sheaves of equivariant $\mathcal{D}$-modules on rigid analytic spaces. Our main results are a canonical dimension estimate for coadmissible representations of a semisimple p-adic Lie group in a p-adic Banach space, and a Beilinson-Bernstein-type localisation for coadmissible equivariant $\mathcal{D}^{\lambda}$-modules, where λ is a ρ-dominant ρ-regular infinitesimal central character.

This thesis is split into two principal sections. The first section contains results on the structure of the dual nilpotent cone of a semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field K of positive characteristic p. When p is small, the structure of the adjoint and coadjoint orbits of G on $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively changes. We relax the hypotheses on p and prove that, under certain technical conditions, the dual nilpotent cone is a normal variety. Using this, we show that the canonical dimension of a coadmissible representation of a semisimple p-adic Lie group in a p-adic Banach space is either zero or at least half the dimension of a non-zero coadjoint orbit.

The second section extends the work of Ardakov on coadmissible equiv-

ariant $\mathcal{D}$-modules on rigid analytic spaces, by defining the category of coadmissible equivariant twisted $\mathcal{D}$-modules. We classify sheaves of differential operators on a rigid analytic variety $\mathbf{X}$ and use this to define the category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ in the case where λ is a regular ρ-dominant ρ-integral central character. In this particular case, we show that $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ is equivalent to Ardakov's category $\mathcal{C}_{\mathbf{X}/G}$ and use this to prove a twisted analogue of locally analytic equivariant Beilinson-Bernstein localisation on rigid analytic spaces.

The final part of this thesis is concerned with removing the integrality condition. We construct the enhanced completed skew-group algebra $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$ and realise the twisted completed skew-group algebra $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$ as a quotient, providing a natural framework to view coadmissible G-equivariant $\mathcal{D}^{\lambda}_{\mathbf{X}}$-modules as coadmissible modules over this algebra. We give a general definition of $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ for arbitrary ρ-dominant ρ-regular central character and discuss how to check this agrees with the definition given for the integral case. Following this, we show this category is equivalent to the category of $\widehat{U}(\mathfrak{g}, G)$-modules with fixed infinitesimal central character λ.

Contents

Chapter 1

Introduction

Let K be a non-archimedean field of mixed characteristic $(0, p)$ and let L be a finite extension of $\mathbb{Q}_p$ contained in K. The papers of Schneider and Teitelbaum, [54], [55], developed the theory of *locally analytic representations* of an L-analytic group G in locally convex topological vector spaces over K. These kinds of representations of G arise naturally in number theory, such as in the study of p-adic automorphic forms and the local Langlands program. It is hoped that the study of these representations will construct deep connections between representation theory and number theory via the p-adic local Langlands correspondence. The category of locally analytic representations that satisfy the important technical condition of *admissibility* turns out to be anti-equivalent to the category of *coadmissible* modules over the locally L-analytic distribution algebra $D(G, K)$ of G.

The theory of $\mathcal{D}$-modules can be used to better understand these representations. The Lie algebra $\mathfrak{g}$ of G has a natural action on any $D(G, K)$-module V. In the case where $\mathfrak{g}$ is semisimple, viewing V as a $\mathfrak{g}$-module allows us to localise V to the flag variety of G in the sense of Beilinson and Bernstein. The celebrated theory of Beilinson-Bernstein localisation, [7], provides an equivalence of categories:

$$\{\mathfrak{g} - \text{modules with trivial infinitesimal central character}\} \to$$

$$\{\mathcal{D} - \text{modules on the flag variety which are quasi-coherent as } \mathcal{O} - \text{modules}\}.$$

Ardakov and Wadsley studied the coadmissible representations of G, which are finitely generated modules over the completed group ring KG with coefficients in K, in [4]. These completed group rings may be realised as Iwasawa algebras, which are important objects in noncommutative Iwasawa theory.

One of the central results in [4] is an estimate for the canonical dimension of a coadmissible representation of a semisimple p-adic Lie group in a p-adic Banach space. With mild restrictions on p, Ardakov and Wadsley showed that this canonical dimension is either zero or at least half the dimension of a non-zero coadjoint orbit. The goal of the first part of this thesis is to weaken the restrictions on p. To do this, it is necessary to investigate the structure of split reductive algebraic groups when the prime p fails to be *very good* for G. We demonstrate that the dual nilpotent cone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a normal variety, and it admits an analogue of the Springer resolution, in the following cases:

Theorem A. Let $G = PGL_n$ and suppose $p|n$. Then the dual nilpotent cone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a normal variety.

Theorem B. Suppose G is of type G_2 and $p = 2$. Then the dual nilpotent cone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a normal variety.

Theorem A, along with the theory built up in [4, Section 9], allow us to prove the following result:

Theorem C. Let G be a compact p-adic analytic group whose Lie algebra is semisimple. Suppose that $G = PGL_n$, $p|n$, and $n > 2$. Let $G_{\mathbb{C}}$ be a complex semisimple algebraic group with the same root system as G, and let r be half the smallest possible dimension of a non-zero coadjoint $G_{\mathbb{C}}$-orbit. Then any coadmissible KG-module M that is infinite-dimensional over K satisfies $d(M) \geq r$.

More recently, Ardakov and Wadsley introduced the notion of a sheaf of infinite-order differential operators $\widehat{\mathcal{D}}$ on a smooth rigid analytic space $\mathbf{X}$. Heuristically, this is a rigid analytic quantisation of the cotangent bundle $T^*\mathbf{X}$ in the sense that the ordinary sheaf of finite-order differential operators $\mathcal{D}$ on a smooth algebraic variety X is an algebraic quantisation of T^*X. Furthermore, they defined the category of coadmissible $\widehat{\mathcal{D}}$-modules $\mathcal{C}_{\mathbf{X}}$, generalising the notion of coadmissible modules in the work of Schneider and Teitelbaum, and showed that this category behaves analogously to the category of coherent $\mathcal{D}$-modules on an algebraic variety.

Given a p-adic Lie group G acting continuously on a smooth rigid analytic space $\mathbf{X}$, Ardakov constructed the *completed skew-group algebra* $\widehat{\mathcal{D}}(\mathbf{X}, G)$. Under certain conditions on $\mathbf{X}$ and G, $\widehat{\mathcal{D}}(\mathbf{X}, G)$ turns out to be a Fréchet-Stein algebra in the sense of [55]. The constructions in [1], Section 3, allow us to form the abelian category $\mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ of coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. Ardakov defined a *localisation functor*:

$$\operatorname{Loc} : \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)} \to \operatorname{Frech}(G - \mathcal{D}_{\mathbf{X}}).$$

The category of coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules $\mathcal{C}_{\mathbf{X}/G}$ is a full subcategory of $\mathrm{Frech}(G-\mathcal{D}_{\mathbf{X}})$ whose objects are locally isomorphic to a sheaf of the form $\mathrm{Loc}(M)$, for $M \in \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$. Ardakov then proved an analogue of Beilinson-Bernstein localisation for admissible locally analytic representations with trivial infinitesimal central character.

The second goal of this thesis is to relax the hypothesis on the central character. More precisely, we aim to prove an analogue of Beilinson-Bernstein localisation for admissible locally analytic representations with ρ-dominant ρ-regular infinitesimal central character. This requires us to generalise the category $\mathcal{C}_{\mathbf{X}/G}$ to deal with sheaves of twisted differential operators. We parametrise sheaves of twisted differential operators $\mathcal{D}^{\lambda}_{\mathbf{X}}$ on a smooth rigid analytic variety $\mathbf{X}$ and use this to define a category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ of coadmissible G-equivariant $\mathcal{D}^{\lambda}_{\mathbf{X}}$-modules. We further show that this category is abelian. We prove the following important result regarding the ring-theoretic structure of $\widehat{\mathcal{D}}(\mathbf{X}, G)$ in this section:

Theorem D. Suppose the G-action on $\mathbf{X}$ is faithful, in the sense that the induced group homomorphism $\rho : G \to \mathrm{Aut}_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is injective. Then $\widehat{\mathcal{D}}(\mathbf{X}, G)$ may be realised as the inverse limit of a system of simple rings.

The main result of this section is as follows:

Theorem E. Let λ be a ρ-dominant ρ-regular integral weight of G. There is an equivalence of categories:

$$\theta_{\lambda}^{-1} \cdot \Gamma \cdot \gamma : \mathcal{C}^{\lambda}_{\mathbf{X}/G} \to \mathcal{C}^{\lambda}_{\widehat{U}(\mathfrak{g},G)}.$$

Further work allows us to remove the condition that the central character λ is integral. In the final chapter, we construct the *enhanced completed skew-group algebra* $\widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, G)$ by applying a similar construction as in [1, Section 3] to the enhanced tangent sheaf $\widetilde{\mathcal{T}}$, and then define the *twisted completed skew-group algebra* $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$ as a quotient of $\widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, G)$ by elements of $\mathfrak{h}_K$ that act by a certain fixed central character. We then construct the category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ of coadmissible equivariant twisted $\mathcal{D}$-modules, prove that this category is abelian, and discuss how one might check that this definition agrees with the construction given in Chapter 5 in the case where λ is integral. We then work towards a Beilinson-Bernstein-style localisation theorem for admissible locally L-analytic K-representations of G with ρ-regular ρ-dominant infinitesimal central character. Our final result is presented below:

Theorem F. Let $\mathbb{G}$ be a connected, simply connected, split semisimple affine algebraic group scheme over K, and let $\mathbf{X}$ be the rigid analytification of the flag variety of $\mathbb{G}$. Let G be a p-adic Lie group and let $\sigma : G \to \mathbb{G}(K)$ be a continuous group homomorphism. Let $\mathbb{H}$ be a flat affine algebraic group over $\mathcal{R}$, $\mathfrak{h}_K := \mathrm{Lie}(\mathbb{H}) \otimes_{\mathcal{R}} K$ and $\xi : \widetilde{\mathbf{X}} \to \mathbf{X}$ a locally trivial $\mathbf{H}$-torsor. Let λ be a ρ-dominant ρ-regular weight of $\mathfrak{h}_K$. Then the localisation functor:

$$(\mathrm{Loc}^{\lambda})^{\widehat{U}(\mathfrak{g},G)}_{\mathbf{X}} : \mathcal{C}^{\lambda}_{\widehat{U}(\mathfrak{g},G)} \to \mathcal{C}^{\lambda}_{\mathbf{X}/G}$$

is an equivalence of categories.

We conclude this introduction with a summary of the material contained in each chapter.

Chapter 2 summarises the background material required for this thesis. Following [47] and [38], we introduce integral group schemes, and outline the classification of split reductive algebraic groups over an algebraically closed field. We outline some of the basic constructions of rigid analytic geometry and Grothendieck topologies, and we define Fréchet-Stein algebras. Furthermore, we discuss the main constructions of $\mathcal{D}$-modules on rigid analytic spaces, in both the equivariant and non-equivariant settings, following [5] and [1].

Chapter 3 establishes our fundamental results on the structure of the nilpotent cone and its dual. We discuss the Springer resolution in positive characteristic in some detail, following [39], and show that the dual nilpotent cone is a normal variety. To do this, it is necessary to study both the structure of the Weyl group invariants $S(\mathfrak{h})^W$ and the structure of the coadjoint orbits of the action of G on $\mathfrak{g}^*$.

Chapter 4 applies the results of Chapter 3 to representations of compact p-adic Lie groups. We prove our result concerning the canonical dimension of a coadmissible representation of a semisimple p-adic Lie group.

Chapter 5 classifies twisted differential operators on a smooth rigid variety $\mathbf{X}$ in terms of local automorphisms on a suitable cover for $\mathbf{X}$, consisting of affinoid subdomains admitting specific topological properties. We show that these local automorphisms may be extended to automorphisms of the completed skew-group algebra $\widehat{\mathcal{D}}(\mathbf{X}, G)$, which allows us to define the category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ of coadmissible G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules for some integral ρ-dominant ρ-regular central character

λ. We show that this category is equivalent as abelian categories to the category $\mathcal{C}_{\mathbf{X}/G}$ defined in [1, Definition 3.6.7], and we use this to prove our generalisation of [1, Theorem 6.4.9].

Chapter 6 generalises the results of Chapter 5 to drop the condition that λ is integral. We define a more general version of the category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ and prove that it is abelian. In general, $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ is not equivalent to $\mathcal{C}_{\mathbf{X}/G}$ and so the methods used in Chapter 5 break down. We construct the algebras $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$ and $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$, and prove our version of locally analytic Beilinson-Bernstein localisation directly, by defining a twisted localisation functor and showing that it is fully faithful and essentially surjective under certain conditions.

Chapter 2

Background

2.1 Algebraic groups

We begin with a basic introduction to the theory of algebraic group schemes, following [47, Section 1]. We assume the reader is familiar with basic algebraic geometry and category theory.

Fix k to be a base field, and let Alg_k denote the category of commutative k-algebras. An object A of Alg_k defines a functor $h^A : \mathrm{Alg}_k \to \mathrm{Set}$ via $R \mapsto \mathrm{Hom}_k(A, R)$.

Definition 2.1.1. A functor $F : \mathrm{Alg}_k \to \mathrm{Set}$ isomorphic to h^A for some A is *representable.*

This notion agrees with the notion of a representable functor from category theory, possibly from a more general source category than Alg_k.

Definition 2.1.2. Let (G, m) be a pair with G a representable functor $\mathrm{Alg}_k \to \mathrm{Set}$ and $m : G \times G \to G$ a k-morphism. An *affine group* over k is a group object (G, m) in the category of representable functors $\mathrm{Alg}_k \to \mathrm{Set}$. If G is represented by a finitely generated k-algebra, then G is an *affine algebraic group.*

Let $*$ denote the initial object in the category of k-schemes. The map m is a natural transformation $m : G \times G \to G$ such that there exist natural transformations $e : * \to G$ and $\mathrm{inv} : G \to G$ making the following diagrams commute:

$$\begin{CD} G \times G \times G @>{\mathrm{id} \times m}>> G \times G \\ @V{m \times \mathrm{id}}VV @VV{m}V \\ G \times G @>{m}>> G, \end{CD}$$

$$\begin{array}{ccccc} * \times G & \xrightarrow{e \times \mathrm{id}} & G \times G & \xleftarrow[\mathrm{id} \times e]{} & G \times * \\ & \searrow^{\cong} & \downarrow{\scriptstyle m} & \swarrow_{\cong} & \\ & & G, & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv},\mathrm{id})} & G \times G & \xleftarrow[(\mathrm{id},\mathrm{inv})]{} & G \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ * & \xrightarrow{e} & G & \xleftarrow[e]{} & *. \end{array}$$

This condition implies that, for any $R \in \mathrm{Alg}_k$, the map $m(R) : G(R) \times G(R) \to G(R)$ defines a group structure on $G(R)$. In practice, we will fix a k-algebra R: then the multiplication map $m(R)$ and the inversion map $\mathrm{inv}(R)$ are morphisms of algebraic varieties in the sense of [11, AG 5.1]. From now on, we will call an affine algebraic group an *algebraic group*, which is always assumed to be an affine scheme.

Definition 2.1.3. A *homomorphism of algebraic groups* is a morphism $\phi : (G, m_G) \to (H, m_H)$ such that the following diagram commutes:

$$\begin{CD} G \times G @>{\phi \times \phi}>> H \times H \\ @V{m_G}VV @VV{m_H}V \\ G @>{\phi}>> H. \end{CD}$$

Definition 2.1.4. A *subgroup* of an algebraic group G is a subfunctor H of G such that, for each k-algebra A, $H(A)$ is a subgroup of $G(A)$. H is *normal* if, additionally, each $H(A)$ is a normal subgroup of $G(A)$.

In this way, a (normal) subgroup of an algebraic group is a representable subfunctor of the representable functor G such that each object in the essential image of G has a (normal) subgroup structure. In general, given a property P of an abstract group, G is said to have *property* P if $G(A)$ has that property for each k-algebra A.

Let A, R be k-algebras and $\Delta : A \to A \otimes A$ a homomorphism of k-algebras. Given a pair of k-algebra homomorphisms $f_1, f_2 : A \to R$, we may define a homomorphism $(f_1, f_2) : A \otimes A \to R$ by $(a_1, a_2) \mapsto f_1(a_1)f_2(a_2)$. Set $f_1 \cdot f_2 := (f_1, f_2) \circ \Delta$.

Definition 2.1.5. The pair (A, Δ) is a *Hopf algebra* over k if $(f_1, f_2) \mapsto f_1 \cdot f_2$ defines a group structure on $\mathrm{Hom}_k(A, R)$ for all objects $R \in \mathrm{Alg}_k$.

By the Yoneda lemma, (A, Δ) is a Hopf algebra if and only if there exist unique k-algebra homomorphisms:

$$\epsilon : A \to k,$$
$$S : A \to A.$$

such that:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta,$$

$$(\mathrm{id}, \epsilon) \circ \Delta = (\epsilon, \mathrm{id}) \circ \Delta,$$

$$(\mathrm{id}, S) \circ \Delta = \epsilon = (S, \mathrm{id}) \circ \Delta.$$

The morphisms Δ, ϵ and S are the *comultiplication map*, the *coidentity map*, and the *antipode* respectively.

Given an algebraic group G, its coordinate ring $\mathcal{O}(G)$ has the natural structure of a Hopf algebra, with the morphisms Δ, ϵ and S obtained by reversing the arrows in the commutative diagrams above Definition 2.1.3.

Definition 2.1.6. G is *connected* if the set of closed points $|G|$ is connected as a topological space.

We illustrate the definitions of an algebraic group with its associated Hopf algebra with the following basic examples.

Example 2.1.7. Define a functor $\mathbb{G}_a : \mathrm{Alg}_k \to \mathrm{Set}$ via $\mathbb{G}_a(A) := (A, +)$. This is the *additive group* over k with coordinate ring $\mathcal{O}(\mathbb{G}_a) = k[t]$, the polynomial ring in one variable. The Hopf algebra structure on $k[t]$ is given by the operations $\Delta(t) = 1 \otimes t + t \otimes 1, \epsilon(t) = 0, S(t) = -t$.

The *multiplicative group* over k is the functor $\mathbb{G}_m : \mathrm{Alg}_k \to \mathrm{Set}$ defined by $\mathbb{G}_m(A) := (A^\times, \times)$. The coordinate ring $\mathcal{O}(\mathbb{G}_m) = k[t, t^{-1}]$. The Hopf algebra structure on

$k[t, t^{-1}]$ is given by the operations $\Delta(t) = t \otimes t, \epsilon(t) = 1, S(t) = t^{-1}$.

Let $\overline{k}$ denote the algebraic closure of k.

Definition 2.1.8. An algebraic group over k is a *torus* if it becomes isomorphic to a product of copies of $\mathbb{G}_m$ over some finite separable extension of k. A torus over k is *split* if it is isomorphic to a product of copies of $\mathbb{G}_m$ over k.

Given an algebraic group G over k, its toral subgroups have a partial ordering by inclusion. A *maximal torus* is a maximal element of this set. An algebraic group G is *split* if it contains a split maximal torus. Note that, if k is algebraically closed, every torus over k is automatically split.

Definition 2.1.9. The *radical* RG of G is the largest smooth connected solvable normal subgroup of G. G is *semisimple* if $RG(\overline{k}) = 1$.

G is *unipotent* if every non-zero representation of the group has a non-zero fixed vector. The *unipotent radical* R_uG of G is the largest smooth connected unipotent normal subgroup of G. G is *reductive* if $R_uG(\overline{k}) = 1$.

Important examples of semisimple and reductive groups are given below.

Example 2.1.10. Let V be a k-vector space of dimension n. There is a k-group functor $GL(V) : \mathrm{Alg}_k \to \mathrm{Set}$ defined by $GL(V)(A) := \mathrm{End}_A(V \otimes A))^{\times}$. By choosing a basis for V, we may identify $GL(V)$ with the group of all invertible $n \times n$ matrices over A, $GL_n(A)$. It is an algebraic k-group with coordinate ring:

$$\mathcal{O}(GL_n) = k[T_{ij} \mid 1 \leq i, j \leq n]_{\{(\det)^m | m \in \mathbb{N}\}}.$$

GL_n is a reductive algebraic group. Fix a k-algebra A: then the radical of $GL_n(A)$ is the largest connected solvable normal subgroup of $GL_n(A)$, which can be identified with the multiplicative group of non-zero scalar multiples of the identity matrix. Hence $R(GL_n) \cong \mathbb{G}_m$, so $R_u(GL_n) = 1$.

The determinant defines a homomorphism of algebraic k-groups $GL(V) \to \mathbb{G}_m$. Its kernel is denoted $SL(V)$. Similarly we define $SL_n(k)$; this is a semisimple algebraic group since its radical can be identified with the scalar multiples of the identity matrix which are contained in SL_n.

Definition 2.1.11. A *linear algebraic group* is a smooth closed subgroup scheme of GL_n over k for some $n \in \mathbb{N}$.

Definition 2.1.12. G is *simple* if it is semisimple, non-abelian, and every proper normal algebraic subgroup is trivial. It is *almost simple* if, instead, every proper normal algebraic group is finite.

Example 2.1.13. The algebraic group SL_n is almost simple. If $n > 1$, the algebraic group $PGL_n := GL_n/\mathbb{G}_m$ is simple.

2.2 Representations of algebraic groups

Fix an algebraically closed ground field K of arbitrary characteristic. Write $G := G(K)$, the K-points of an algebraic group scheme G over $k \subseteq K$. This variety, together with multiplication $m(K) : G(K) \times G(K) \to G(K)$, defines an algebraic group in the sense of [11, AG 1.1]. Let x be a point of G, from [11, AG 15.5, 16.1], it follows that $T(G)_x = \mathrm{Der}_K(\mathcal{O}(G), K(x))$, where $T(G)_x$ is the tangent space of G at x.

Let $A := \mathcal{O}(G)$ and let $e_x : A \to K(x)$ be the evaluation map at x, $e_x(f) = f(x)$. Let D be a K-linear derivation of A, $D \in \mathrm{Der}_K(A, A)$, and let $D_x := e_x \circ D$. Then D_x is an element of $T(G)_x$.

Fix $g \in G$ and let $\lambda_g : A \to A$ be the function on A defined by left translation: $(\lambda_g f)(x) = f(g^{-1}x)$ for $x \in G, f \in A$. Define:

$$\mathrm{Lie}(G) := \{D \in \mathrm{Der}_K(A, A) \mid \lambda_x \circ D = D \circ \lambda_x \forall x \in G\}.$$

Let $f \in A, y \in G$. By definition:

$$(\lambda_x \circ D)f(y) = Df(x^{-1} \cdot y) = D_{x^{-1} \cdot y} f.$$

By [11, AG 16.1], $(D \circ \lambda_x)f(y)$ is the image of D_y under the differential at y of the translation $g \mapsto x^{-1} \cdot g$, denoted by $x^{-1} \cdot D_y$. Hence:

$$\mathrm{Lie}(G) = \{D \in \mathrm{Der}_K(A, A) \mid x \cdot D_y = D_{x \cdot y} \forall x, y \in G\}.$$

Theorem 2.2.1. *The map $v : D \mapsto e_1 \circ D$ is an isomorphism of vector spaces, sending Lie(G) onto $T(G)_1$.*

Proof. This is [11, Theorem 3.4]. □

Definition 2.2.2. The *Lie algebra* $\mathfrak{g}$ of G is $T(G)_1$, endowed with the restricted Lie algebra structure of $\mathrm{Lie}(G)$ induced by v.

$\mathfrak{g}$ may be identified with the Lie algebra of left invariant vector fields on G. The assignment $G \mapsto \mathfrak{g}$ is functorial: given a morphism $\alpha : G \to G'$ of algebraic groups, its differential $d\alpha_1 : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras.

Definition 2.2.3. The *adjoint action* of G on $\mathfrak{g}$ is the differential of the conjugation action. Explicity, given $g \in G$, $X \in \mathfrak{g}$:

$$g \cdot X := \mathrm{Ad}_g(X).$$

Whenever $G \leq GL_n(K)$ for some n, we may identify the adjoint action $g \cdot X = gXg^{-1}$.

There is also an adjoint action of $\mathfrak{g}$ on itself. If $X, Y \in \mathfrak{g}$, we define $\mathrm{ad}_Y(X) := [Y, X]$.

Definition 2.2.4. An element $X \in \mathfrak{g}$ is *nilpotent* if the adjoint operator is a nilpotent operator on $\mathfrak{g}$.

Definition 2.2.5. A *Cartan subalgebra* of $\mathfrak{g}$ is a nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is self-normalising: if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$.

Let $G := G(K)$ be a split reductive algebraic K-group, T a maximal torus in G.

Definition 2.2.6. The *character group* of T is $X(T) := \mathrm{Hom}_K(T, \mathbb{G}_m)$. Dually, the *cocharacter group* of T is $Y(T) := \mathrm{Hom}_K(\mathbb{G}_m, T)$.

Proposition 2.2.7. The associated Lie algebra $\mathfrak{t}$ to T is a *toral subalgebra*. If G is split semisimple, a toral subalgebra $\mathfrak{t}$ can be identified with a Cartan subalgebra $\mathfrak{h}$. Furthermore, all maximal tori are conjugate.

Proof. This is [11, Corollary 11.3]. □

By [38, I.2.5(1)], $X(T) \cong \mathbb{Z}^r$ for some r, and hence is a free abelian group. The *rank* of G equals r. Any T-module M has a direct sum decomposition into weight spaces by [38, 1.2.11(3)]:

$$M = \bigoplus_{\lambda \in X(T)} M_\lambda.$$

Definition 2.2.8. λ is a *weight* of M if $M_\lambda \neq 0$.

Applying this argument to the T-module $\mathfrak{g}$, let R be the set of non-zero weights of $\mathfrak{g}$. Then $\mathfrak{g}$ has a corresponding decomposition of the form:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \text{ and}$$
$$\mathfrak{g}_0 = \mathfrak{t}.$$

Definition 2.2.9. R is the *root system* of G with respect to T. Elements of R are the *roots* of T.

For each $\alpha \in R$ there is a *root homomorphism*:

$$x_\alpha : \mathbb{G}_a \to G,$$

with $tx_\alpha(a)t^{-1} := x_\alpha(\alpha(t)(a))$ for any K-algebra A and all $t \in T, a \in A$, such that the tangent map dx_α induces an isomorphism:

$$dx_\alpha : \text{Lie } (\mathbb{G}_a) \to (\text{Lie } G)_\alpha.$$

This root homomorphism is unique up to multiplication by a non-zero element in K. The functor $A \mapsto x_\alpha(\mathbb{G}_a(A))$ is a closed subfunctor of G, denoted by U_α. This is the *root subgroup* of G corresponding to α. Then x_α is an isomorphism $\mathbb{G}_a \to U_\alpha$, and Lie $(U_\alpha) =$ (Lie $G)_\alpha$.

Let $Y(T)$ be the cocharacter group of the maximal torus T. Given $\lambda \in X(T)$, $\phi \in Y(T)$, $\lambda \circ \phi \in \text{End}(\mathbb{G}_m) \cong \mathbb{Z}$, so there is a unique integer $\langle \lambda, \phi \rangle$ such that $\lambda \circ \phi$ is the map $a \mapsto a^{\langle \lambda, \phi \rangle}$ for each K-algebra A. The pairing $\langle , \rangle$ on $X(T) \times Y(T)$ is bilinear and induces an isomorphism:

$$Y(T) \cong \text{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z}).$$

For any $\alpha \in R$, there is a homomorphism $\phi_\alpha : SL_2 \to G$ such that for a suitable normalisation of x_α and $x_{-\alpha}$:

$$\phi_\alpha\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right) = x_\alpha(a),$$
$$\phi_\alpha\left(\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\right) = x_{-\alpha}(a),$$

for any A and $a \in A$. In this case, define:

$$n_\alpha(a) := x_\alpha(a)x_{-\alpha}(a^{-1})x_\alpha(a) = \phi_\alpha\left(\begin{bmatrix} 0 & a \\ a^{-1} & 0 \end{bmatrix}\right) \in N_G(T)(A),$$

and there is an element $\alpha^\vee \in Y(T)$ such that:

$$\alpha^\vee(a) = \phi_\alpha\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) \in T(A)$$

for $a \in A^\times$ and any K-algebra A. $\alpha^\vee$ is the *coroot* corresponding to α.

The set R together with the map $\alpha \to \alpha^\vee$ forms a root system in the sense of [15, Chapter 6, 1.1], and the set $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$ is the *dual root system* of R. Define $s_\alpha(\lambda) := \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. This defines, for each $\alpha \in R$, a reflection on $X(T)$, which can be extended to a reflection on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ by extending $\alpha^\vee \in Y(T) \cong X(T)^*$ to $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.2.10. The *Weyl group* of G is $W := \langle s_\alpha \mid \alpha \in R \rangle$.

Proposition 2.2.11. Let A be an integral K-algebra. There is a group isomorphism:

$$W \cong \left(\frac{N_G(T)}{T}\right)(A) \cong \frac{N_G(T)(A)}{T(A)}.$$

Moreover, W is finite.

Proof. Let $g \in N_G(T)(A)$. Then g acts through conjugation on T_A and so acts linearly on the $\mathbb{Z}$-modules $X(T(A))$ and $Y(T(A))$. If A is integral, then $X(T(A)) \cong$

$X(T)$ and $Y(T(A)) \cong Y(T)$ and so there are actions on $X(T)$ and $Y(T)$ which preserves the pairing. The action of any $n_\alpha(a)$ for $\alpha \in R, a \in A$ on $X(T)$ agrees with the action of s_α, and so, for integral A, there are isomorphisms:

$$W \cong \left(\frac{N_G(T)}{T}\right)(A) \cong \frac{N_G(T)(A)}{T(A)},$$

where the last equality follows from the fact that each generator s_α of W has a representative $n_\alpha(1)$ in $N_G(T)(A)$, and hence so has any $w \in W$. Furthermore, W is finite by [38, I.8.5(a)]. □

Definition 2.2.12. A *positive root system* $R^+ \subseteq R$ is a subset of R satisfying the conditions:

(a) for any $\alpha \in R$, exactly one of the roots $\alpha, -\alpha$ are contained in R^+,

(b) for any two distinct $\alpha, \beta \in R^+$ such that $\alpha + \beta \in R$, $\alpha + \beta \in R^+$.

$\alpha \in R^+$ is a *simple root* if it cannot be written as the sum of two elements in R^+. Let S denote the set of simple roots.

The elements of the positive root system R^+ are called the *positive roots*. The elements of the set $-R^+ := R \setminus R^+$ are called the *negative roots*.

R is contained in the $\mathbb{Z}$-linear span of S, and it follows that W is generated by the simple reflections, i.e. $W = \langle s_\alpha \mid \alpha \in S \rangle$.

There is a partial ordering on $X(T)$, and consequently on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, defined by:

$$\lambda \leq \mu \Longleftrightarrow \mu - \lambda \in \sum_{\alpha \in S} \mathbb{N}\alpha.$$

It follows that $R^+ = \{\alpha \in R \mid \alpha > 0\}$ and $-R^+ = \{\alpha \in R \mid \alpha < 0\}$.

Definition 2.2.13. A *highest-weight root* is a root that is maximal with respect to this ordering.

Definition 2.2.14. $\lambda \in R$ is *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for each positive root α, and is *antidominant* if $\langle \lambda, \alpha^\vee \rangle \leq 0$ for each positive root α. It is *integral* if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for each positive root α, and it is *regular* if its stabiliser under the W-action is trivial.

Let $\rho = \frac{1}{2} \sum_{\alpha \in S} \alpha$ denote the half-sum of the simple roots. λ is *ρ-dominant* if $\lambda + \rho$ is dominant, and λ is *ρ-regular* if $\lambda + \rho$ is regular.

Since W has a simple transitive action on the positive root systems, there is a unique $w_0 \in W$ such that $w_0(R^+) = -R^+$. Then $w_0^2 = 1$, and $\lambda \leq \mu$ if and only if $w_0\mu \leq w_0\lambda$ for all $\lambda, \mu \in X(T)$.

Let $\rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $2\rho \in \mathbb{Z}R \subseteq X(T)$ and:

$$\langle \rho, \beta^\vee \rangle = 1$$

for all $\beta \in S$. Hence $s_\beta \rho - \rho \in \mathbb{Z}R$, so $w\rho - \rho \in \mathbb{Z}R$ for all $w \in W$.

Definition 2.2.15. The *dot action* of W on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is:

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

This maps $X(T)$ onto itself.

The centre $Z(G)$ of G is equal to the intersection:

$$Z(G) = \bigcap_{\alpha \in R} \ker(\alpha) \in T.$$

If G is a split semisimple algebraic group, then $Z(G)$ is a finite group scheme; equivalently, $(X(T) : \mathbb{Z}R) < \infty$. In this case, S is a basis of the vector space $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $S^\vee := \{\alpha^\vee \mid \alpha \in S\}$ is a basis of $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q} = (X(T) \otimes_{\mathbb{Z}} \mathbb{Q})^*$. Let $\{\omega_\alpha \mid \alpha \in S\}$ be the dual basis of $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.2.16. G is *adjoint* if $X(T) = \mathbb{Z}R$. Equivalently, $Z(G) = 1$.

G is *simply connected* if $Y(T) = \mathbb{Z}R^\vee$. Equivalently, $\omega_\alpha \in X(T)$ for all $\alpha \in S$.

Example 2.2.17. Suppose G has rank 1. Then the root system of G is $R = \{\alpha, -\alpha\}$. G has just one positive root $\alpha/2$. The adjoint group with root system R is PGL_2; the simply connected group with root system R is SL_2.

Definition 2.2.18. A subset R' of R is *closed* if, for $\alpha, \beta \in R'$, $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap R \subseteq R'$.

R' is *unipotent* if $R' \cap (-R') = \emptyset$. It is *symmetric* if $R' = -R'$.

Let $R' \subseteq R$ be unipotent and closed. Denote by $U(R')$ the closed subgroup of G generated by U_α for $\alpha \in R'$.

Proposition 2.2.19. The multiplication on $U(R')$ induces an isomorphism of schemes:

$$\prod_{\alpha \in R'} U_\alpha \to U(R').$$

Proof. This follows from [38, II.1.2(5)]. □

It follows that Lie $U(R') = \bigoplus_{\alpha \in R'} \mathfrak{g}_\alpha$. Each $U(R')$ is a connected unipotent group, isomorphic as a k-scheme to $\mathbb{A}^n$ where $n = |R'|$.

Now suppose $R' \subseteq R$ is symmetric and closed. Denote by $G(R')$ the closed subgroup of G generated by U_α and T for $\alpha \in R'$. Then:

$$\text{Lie } G(R') = \mathfrak{t} \oplus \bigoplus_{\alpha \in R'} \mathfrak{g}_\alpha.$$

$G(R')$ is a reductive, split and connected k-group, containing T as a maximal torus. The root system of $G(R')$ is R', and its Weyl group can be identified with $\langle s_\alpha \mid \alpha \in R' \rangle \subseteq W$.

Both R^+ and $-R^+$ are unipotent and closed subsets of R. Applying the above argument to these sets, define $U^+ := U(R^+)$, $U := U(-R^+)$. Set $B^+ := U^+T$, $B := UT$.

Definition 2.2.20. A *Borel subgroup* of G is a maximal closed connected solvable algebraic subgroup of G.

Proposition 2.2.21. B^+ and B are Borel subgroups of G. Furthermore, all Borel subgroups of G are conjugate under the adjoint action.

Proof. This is [11, Theorem 11.1]. □

Definition 2.2.22. The *flag variety* of G is the quotient G/B.

Proposition 2.2.23. The flag variety is projective.

Proof. This is [11, Theorem 11.1]. □

Let $\mathfrak{n}^+ := \text{Lie } U(R^+)$, $\mathfrak{n}^- := \text{Lie } U(-R^+)$. These are nilpotent subalgebras of $\mathfrak{g}$. The adjoint action of T on $\mathfrak{g}$ yields a triangular decomposition:

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^+$$

Definition 2.2.24. A *Borel subalgebra* of $\mathfrak{g}$ is $\mathfrak{b} := \text{Lie } B$ for some Borel subgroup B of G.

Over the algebraically closed field K, all Borel subalgebras are conjugate. We will usually take $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$. Written in this way, $\mathfrak{n}$ is the nilradical of $\mathfrak{b}$.

Example 2.2.25. We illustrate the above constructions by calculating these objects in the case $G = SL_n$, the simply connected group of type A_{n-1}. Then $\mathfrak{g} = \mathfrak{sl}_n$ by [34], and we take the Cartan subalgebra $\mathfrak{h}$ to be the subalgebra of diagonal elements of $\mathfrak{sl}_n$.

Write E_{ij} to denote the elementary matrix with 1 in the (i, j)th entry and 0 otherwise. Then the set $\{E_{ii} - E_{nn}\}_{1 \leq i \leq n-1}$ forms a basis for $\mathfrak{h}$. Let $\{\epsilon_i\}_{1 \leq i \leq n-1}$ be the corresponding dual basis of $\mathfrak{h}^*$. Then a basis for the simple roots S of G is:

$$S = \{\epsilon_i - \epsilon_{i+1}\}_{1 \leq i \leq n-1},$$

and the set of positive roots with respect to this basis is:

$$R^+ = \{\epsilon_i - \epsilon_j \mid i < j\}$$

The unipotent subgroup U^+ (resp. U) is the group of upper (resp. lower) triangular matrices with all diagonal entries equal to 1. The maximal torus T can be identified with the scalar matrices of determinant 1 by Example 2.1.10, so B^+ (resp. B) is the group of upper (resp. lower) triangular matrices.

The corresponding Lie algebra to U^+ (resp. U) is $\mathfrak{n}^+$ (resp. $\mathfrak{n}$), the Lie algebra consisting of strictly upper (resp. lower) triangular matrices. Then the Borel subalgebra $\mathfrak{b}^+ = \text{Lie } B^+$ (resp. $\mathfrak{b} = \text{Lie } B$) is the Lie algebra consisting of upper (resp. lower) triangular matrices of trace zero. Note that $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ (resp. $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$), and $\mathfrak{n}^+$ is the nilradical of $\mathfrak{b}^+$ (resp. $\mathfrak{n}$ is the nilradical of $\mathfrak{b}$).

The flag variety of SL_2 with respect to the Borel subgroup B is the set of flags in K^2:

$$\{V_0 \subseteq V_1 \subseteq V_2 \mid V_i \text{ i-dimensional-subspace of } K^2\},$$

which can naturally be identified with the projective variety $\mathbb{P}^1$.

Definition 2.2.26. The *root datum* of G is the quadruple $(X(T), R, Y(T), R^\vee)$, together with the pairing of $X(T)$ and $Y(T)$ and the bijection $\alpha \to \alpha^\vee$ from R to $R^\vee$.

Definition 2.2.27. Let G, G' be connected split reductive K-groups. A *homomorphism of root data* is a group homomorphism $f : X(T') \to X(T)$ that maps R' bijectively to R and such that the dual homomorphism $f^\vee : Y(T) \to Y(T')$ maps $f(\beta^\vee)$ to $\beta^\vee$ for each $\beta \in R'$.

Definition 2.2.28. An *isogeny* is a surjective morphism $\phi : G \to G'$ of algebraic groups with finite kernel.

Example 2.2.29. Given a root system R, there is a family of algebraic groups all admitting the same root system. Recall, from Example 2.2.17, that SL_2 and PGL_2 have the same root system. Over the algebraically closed field K, there is an isogeny $\phi : SL_2(K) \to PGL_2(K)$, since there is an isomorphism $PGL_2(K) \cong SL_2(K)/\mathfrak{m}$, where $\mathfrak{m}$ is the subgroup consisting of multiples of the identity matrix.

Proposition 2.2.30. Let R be a root system. There is a unique simply connected group G_{sc} and a unique group of adjoint type G_{ad} with root system R. Moreover, if G' is another group with the same root system, then G' is a homomorphic image of

G_{sc} and admits G_{ad} as a quotient.

Proof. This follows from [58, 9.1, 11.4]. $\square$

We write $\mathfrak{g}_{\mathrm{sc}}$ (resp: $\mathfrak{g}_{\mathrm{ad}}$) to denote the Lie algebra of the simply connected (resp. adjoint) algebraic group G with a fixed root system.

Example 2.2.31. The simply connected group with root system A_1 is SL_2 and the adjoint group with root system A_1 is PGL_2, from Example 2.2.17. It follows that $\mathfrak{g}_{\mathrm{sc}} = \mathfrak{sl}_2$, which we may identify with the Lie algebra of 2x2 matrices over K with trace zero, and $\mathfrak{g}_{\mathrm{ad}} = \mathfrak{pgl}_2$. If the characteristic of K is not 2, then we have a Lie algebra isomorphism $\mathfrak{pgl}_2 \cong \mathfrak{sl}_2$.

However, if the characteristic of K is 2, then $\mathfrak{pgl}_2$ may be identified with the following Lie algebra:

$$\mathfrak{pgl}_2 \cong \frac{K[x, y, z]}{([x, y] = 0, [x, z] = x, [y, z] = y)}$$

which is not isomorphic to $\mathfrak{sl}_2$.

Theorem 2.2.32. *A connected split reductive K-group G is determined up to isomorphism by its root datum. Furthermore, for each possible root datum, there exists a split reductive K-group G.*

Proof. This is [37, Proposition II.1.15 and Section II.1.17]. $\square$

In future, we will use the standard classification of root systems via Dynkin diagrams as in [35, Theorem 2.8], to denote possible root systems.

2.3 Rigid analytic geometry

In this subsection, we give some background on rigid analytic geometry, largely following [13]. In what follows, let R be a commutative discrete valuation ring with uniformiser π, field of fractions K and residue field k.

Definition 2.3.1. A field K is *non-archimedean* if it admits a non-archimedean absolute value, i.e. there is a map $|\cdot| : K \to \mathbb{R}$ such that, for all $a, b \in K$, the following hold:

(a) $|a| = 0$ if and only if $a = 0$,

(b) $|a||b| = |ab|$,

(c) $|a + b| \leq \max\{a, b\}$.

The absolute value gives rise to a distance function $d(a, b) = |a - b|$, and hence we can define a natural topology on K. K is *complete* if every Cauchy sequence converges.

Let K be a complete discretely valued field with a non-archimedean non-trivial valuation.

Definition 2.3.2. The *nth Tate algebra* $T_n(K) = K\langle\zeta_1, \cdots, \zeta_n\rangle$ is the K-algebra consisting of all formal power series which converge on the unit disk $B^n(\overline{K})$, i.e. power series of the form:

$$\sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu \in K[[\zeta_1, \cdots, \zeta_n]]$$

such that $c_\nu \in K$ and $\lim_{|\nu| \to \infty} |c_\nu| = 0$.

Definition 2.3.3. A K-algebra A is an *affinoid K-algebra* if there is an epimorphism $\alpha : T_n \to A$ of K-algebras for some $n \in \mathbb{N}$.

Let A be an affinoid K-algebra. Any $a \in A$ can be viewed as a function on Max A, the maximal spectrum of A, as follows. Given $x \in$ Max A, we set $a(x)$ to be the residue class of a in A/x. Embedding the field A/x into the algebraic closure $\overline{K}$ of K, the value $a(x)$ is determined up to conjugacy. Hence the absolute value $|a(x)|$ is well-defined, as it is independent of the chosen embedding $A/x \to \overline{K}$.

Definition 2.3.4. The *affinoid K-space* associated to A is the set Sp $A :=$ Max A, together with its K-algebra of functions A.

Any affinoid K-space Sp A carries a Zariski topology, as in the algebraic case. A *Zariski closed subset* of Sp A is defined to be:

$$V(\mathfrak{a}) := \{x \in \text{Sp } A \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\}$$

for an ideal $\mathfrak{a} \subseteq A$.

Definition 2.3.5. Let $\mathbf{X} = \mathrm{Sp}\ A$ be an affinoid K-space. A subset $\mathbf{U} \subseteq \mathbf{X}$ is an *affinoid subdomain* of $\mathbf{X}$ if there is a morphism of affinoid K-spaces $\iota : \mathbf{X}' \to \mathbf{X}$ such that $\iota(\mathbf{X}') = \mathbf{U}$, and any morphism $\phi : \mathbf{Y} \to \mathbf{X}$ of affinoid K-spaces with $\phi(\mathbf{Y}) \subseteq \mathbf{U}$ factors through $\mathbf{X}'$.

Example 2.3.6. Let $\mathbf{X} = \mathrm{Sp}\ A$ be an affinoid K-space, and let $f_0, \cdots, f_r, g_1, \cdots, g_s \in A$ be functions on $\mathbf{X}$.

(a) A *Weierstrass domain* in $\mathbf{X}$ is a subset of the form:

$$X(f_1, \cdots, f_r) := \{x \in \mathbf{X} \mid |f_i(x)| \leq 1\}.$$

(b) A *Laurent domain* in $\mathbf{X}$ is a subset of the form:

$$X(f_1, \cdots, f_r, g_1^{-1}, \cdots, g_s^{-1}) = \{x \in \mathbf{X} \mid |f_i(x)| \leq 1, |g_j(x)| \geq 1\}.$$

(c) A *rational domain* in $\mathbf{X}$ is a subset of the form:

$$X\left(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0}\right) = \{x \in \mathbf{X} \mid |f_i(x)| \leq |f_0(x)|\}$$

where the functions $f_0, \cdots, f_r$ have no common zeros.

By [13, Proposition 3.3/11], these are examples of open affinoid subdomains, called the *special affinoid subdomains.*

Theorem 2.3.7. *Let $\boldsymbol{X}$ be an affinoid K-space and $\boldsymbol{U} \subseteq \boldsymbol{X}$ an affinoid subdomain. Then $\boldsymbol{U}$ is a finite union of rational subdomains of $\boldsymbol{X}$.*

Proof. This is [13, Theorem 3.3/20]. □

Definition 2.3.8. Let $\mathbf{X}$ be an affinoid K-space and $\mathbf{U} \subseteq \mathbf{X}$ an affinoid subdomain. Let $\mathcal{O}_{\mathbf{X}}(\mathbf{U})$ denote the affinoid K-algebra corresponding to $\mathbf{U}$. The *presheaf of affinoid functions* is the functor defined by $\mathbf{U} \mapsto \mathcal{O}_{\mathbf{X}}(\mathbf{U})$.

Theorem 2.3.9. *Let $\boldsymbol{X}$ be an affinoid K-space and $\mathcal{U}$ a finite covering of $\boldsymbol{X}$ by affinoid subdomains. Then $\mathcal{U}$ is acyclic with respect to the presheaf $\mathcal{O}_{\boldsymbol{X}}$ of affinoid functions on $\boldsymbol{X}$.*

Proof. This is [13, Theorem 4.3/1]. □

If $\mathbf{X}$ is given the Zariski topology, the presheaf of affinoid functions $\mathcal{O}_{\mathbf{X}}$ on an affinoid K-space $\mathbf{X}$ does not satisfy sheaf properties for all coverings. Instead, we usually endow $\mathbf{X}$ with a *Grothendieck topology.* This is a generalisation of a topological space that can deal with these issues.

Definition 2.3.10. A *Grothendieck topology* $\mathcal{I}$ consists of a category Cat $\mathcal{I}$ and a set Cov $\mathcal{I}$ of families $(U_i \to U)_{i \in \mathcal{I}}$ of morphisms in Cat $\mathcal{I}$, called *coverings*, such that the following hold:

(a) any isomorphism $\phi : U \to V$ in Cat $\mathcal{I}$ has $(\phi) \in$ Cov $\mathcal{I}$,

(b) if $(U_i \to U)_{i \in I}$ and $(V_{ij} \to U_i)_{j \in J}$ belong to Cov $\mathcal{I}$, then so does the composition $(V_{ij} \to U_i \to U)_{i \in I, j \in J}$,

(c) if $(U_i \to U)_{i\in I}$ belongs to Cov $\mathcal{I}$ and $V \to U$ is a morphism in Cat $\mathcal{I}$, then the fiber products $U_i \times_U V$ exist in Cat $\mathcal{I}$ and $(U_i \times_U V \to V)_{i\in I}$ belongs to Cov $\mathcal{I}$.

We often think of the objects of Cat $\mathcal{I}$ as the open sets and the morphisms of Cat $\mathcal{I}$ as inclusions. A family $(U_i \to U)_{i\in I}$ in Cov $\mathcal{I}$ should then be interpreted as a covering of an open set, and the fiber product $U_i \times_U V$ as an intersection of U_i with V. This correspondence allows us to view an ordinary topological space X as a space equipped with a Grothendieck topology, where Cat $\mathcal{I}$ is the category of open sets of X, with the inclusions as morphisms, and Cov $\mathcal{I}$ consists of open covers of open subsets of X. A space X endowed with a Grothendieck topology is a *site*.

Definition 2.3.11. Let X be a site equipped with a Grothendieck topology $\mathcal{I}$, and $\mathcal{C}$ a category which admits cartesian products. A *presheaf* on $\mathcal{I}$ with values in $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \text{Cat } \mathcal{I} \to \mathcal{C}$. $\mathcal{F}$ is a *sheaf* if in addition the sequence:

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is exact for any covering $(U_i \to U)_{i\in I}$ in Cov $\mathcal{I}$.

We will now specialise to the case where Cat $\mathcal{I}$ is a category of subsets of a space X, with inclusions as morphisms. The objects of Cat $\mathcal{I}$ will be the *admissible open subsets*. The elements of Cov $\mathcal{I}$ will be the *admissible coverings.* We would like the presheaf $\mathcal{O}_X$ of affinoid functions to be a sheaf. To this end, we define a Grothendieck topology on X as follows.

Definition 2.3.12. The *weak Grothendieck topology* X_w on X, $\mathcal{I}$, has Cat $\mathcal{I}$ as the category of affinoid subdomains of X with inclusions as morphisms. The set Cov $\mathcal{I}$

is the set of all *finite* families $(U_i \to U)_{i \in I}$ of inclusions of affinoid subdomains such that $U = \bigcup_{i \in I} U_i$.

Let $\mathbf{X}$ be an affinoid K-space. By Theorem 2.3.9, $\mathcal{O}_{\mathbf{X}}$, where $\mathbf{X}$ is given the weak Grothendieck topology, is in fact a sheaf. There is a canonical enlargement of the weak Grothendieck topology, adding more admissible open sets and more admissible coverings, such that morphisms of affinoid K-coverings remain continuous and sheaves in the weak Grothendieck topology extend to sheaves in this new topology. This Grothendieck topology satisfies certain completeness conditions outlined in [13, Proposition 5.1.5].

Definition 2.3.13. The *strong Grothendieck topology* X_{rig} on a site X is given as follows:

(a) a subset $U \subseteq X$ is admissible open if there is a (possibly infinite) covering $U = \bigcup_{i \in I} U_i$ of U by affinoid subdomains $U_i \subseteq X$ such that for all morphisms of affinoid K-spaces $\phi : Z \to X$ with $\phi(Z) \subseteq U$ the covering $(\phi^{-1}(U_i))_{i \in I}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

(b) a covering $V = \bigcup_{j \in J} V_j$ of some admissible open V by admissible open subsets V_j is admissible if for each morphism of affinoid K-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subseteq V$ the covering $(\phi^{-1}(V_j))_{j \in J}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

Recall that a *ringed K-space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ a sheaf of K-algebras on X. This concept can be naturally generalised to a site

X.

Definition 2.3.14. A *G-ringed K-space* is a pair $(X, \mathcal{O}_X)$, where X is a site and $\mathcal{O}_X$ a sheaf of K-algebras on X. It is a *locally G-ringed K-space* if the stalks $\mathcal{O}_{X,x}$ are each local rings.

Definition 2.3.15. A *morphism* of G-ringed K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (ϕ, ϕ^*) where $\phi : X \to Y$ is a map, continuous with respect to the Grothendieck topologies, such that ϕ^* is a system of K-homomorphisms $\phi^*_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(\phi^{-1}(V))$, where V varies over the admissible open subsets of Y, which are compatible with restriction homomorphisms in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{\phi^*_V} & \mathcal{O}_X(\phi^{-1}(V)) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(U) & \xrightarrow{\phi^*_U} & \mathcal{O}_X(\phi^{-1}(U)) \end{array}$$

for any admissible open subsets $U \subseteq V$ of Y.

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally G-ringed K-spaces, a morphism (ϕ, ϕ^*) is a *morphism* of locally G-ringed K-spaces if, additionally, for all $x \in X$ the ring homomorphisms:

$$\phi^*_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$$

induced from the ϕ^*_V are local in the sense that the unique maximal ideal of $\mathcal{O}_{Y,\phi(x)}$ is mapped to the unique maximal ideal of $\mathcal{O}_{X,x}$.

Proposition 2.3.16. Let $\mathbf{X}$ be a rigid K-space. The category of $\mathcal{O}_{\mathbf{X}}$-modules contains enough injectives.

Proof. This is [13, Proposition 6.2/2]. □

Let $\mathcal{F}$ be an $\mathcal{O}_{\mathbf{X}}$-module. Consider the global sections functor $\Gamma(\mathbf{X}, -) : \mathcal{F} \mapsto \Gamma(\mathbf{X}, \mathcal{F}) := \mathcal{F}(\mathbf{X})$. By Proposition 2.3.16, we may form its right derived functors $R^n\Gamma(X, -)$.

Definition 2.3.17. The qth *sheaf cohomology group* of X is $H^q(\mathbf{X}, \mathcal{F}) := R^q\Gamma(\mathbf{X}, \mathcal{F})$.

The global sections functor is left exact, so $R^0\Gamma(\mathbf{X}, \mathcal{F}) = \Gamma(\mathbf{X}, \mathcal{F})$.

Definition 2.3.18. Let $\mathcal{U}$ be an $\mathbf{X}_w$-covering of $\mathbf{X}$, as in Definition 2.3.12. The qth *Čech cohomology group* of a sheaf $\mathcal{F}$ on $\mathbf{X}$ with respect to this covering is the cohomology of the usual Čech complex; it is denoted $\check{H}^q(\mathcal{U}, \mathcal{F})$.

The qth *Čech cohomology group* of $\mathcal{F}$ on $\mathbf{X}$ is:

$$\check{H}^q(\mathbf{X}, \mathcal{F}) = \varinjlim \check{H}^q(\mathcal{U}, \mathcal{F})$$

where the colimit is taken over all $\mathbf{X}_w$-coverings of $\mathbf{X}$ ordered by refinement.

2.4 Fréchet-Stein algebras

Following [55, Section 3], we now define Fréchet-Stein algebras and give some of their basic properties. Many of the constructions in later sections admit the natural structure of a Fréchet-Stein algebra.

Definition 2.4.1. Let X be a topological vector space. X is a *Fréchet space* if it is Hausdorff, its topology may be induced by a countable family of seminorms $||\cdot||_k, k \in \mathbb{N}$ and it is complete with respect to this topology.

More precisely, $U \subseteq X$ is open if for every $u \in U$ there is $K \geq 0$ and $\epsilon > 0$ such that $\{v \mid ||v-u||_k < \epsilon \ \forall k \leq K\}$ is a subset of U.

Now let A be a K-Fréchet algebra, so A has the underlying structure of a K-Fréchet space and the algebra multiplication is continuous. Let $q_1 \leq \cdots \leq q_n \leq \cdots$ be a sequence of algebra seminorms that define the Fréchet topology. Each seminorm induces a norm on the quotient space:

$$\frac{A}{\{a \in A \mid q_i(a) = 0\}}.$$

Taking the completion of this space with respect to q_i gives a K-Banach space, A_{q_i}. The identity map on A induces a continuous linear map $\phi_{q_j}^{q_i} : A_{q_j} \to A_{q_i}$, which we call the *transition map.* In this case, $\phi_{q_j}^{q_i}$ is an algebra homomorphism and we have an isomorphism of Fréchet algebras:

$$A \to \varprojlim A_{q_n}.$$

Definition 2.4.2. A is a *Fréchet-Stein algebra* if there is a sequence $q_1 \leq \cdots \leq q_n \leq \cdots$ of continuous algebra seminorms which define the Fréchet topology on A

such that:

(a) A_{q_n} is left Noetherian,

(b) A_{q_n} is flat as a right $A_{q_{n+1}}$-module, with the module structure induced by the map $\phi^{q_n}_{q_{n+1}} : A_{q_{n+1}} \to A_{q_n}$.

Definition 2.4.3. A *coherent sheaf* for (A, q_n) is a family $(M_n)_{n\in\mathbb{N}}$ of A_{q_n}-modules, together with A_{q_n}-module isomorphisms $A_{q_n} \otimes_{A_{q_{n+1}}} M_{n+1} \to M_n$ for any $n \in \mathbb{N}$.

The coherent sheaves, together with the obvious notion of a morphism, form a category Coh_{A,q_n}. As a consequence of the flatness requirement, one sees that this category is abelian.

Definition 2.4.4. Let $(M_n)_n$ be a coherent sheaf for (A, q_n). The A-module of *global sections* for $(M_n)_n$ is:

$$\Gamma(M_n) := \varprojlim M_n.$$

Definition 2.4.5. A left A-module is *coadmissible* if it is isomorphic to the module of global sections for some coherent sheaf for (A, q_n).

2.5 Crystalline differential operators on homogeneous spaces

In this section, we recall some of the arguments from [4, Section 4], to define the sheaf of enhanced vector fields $\widetilde{\mathcal{T}}$ on a smooth scheme X, and the relative enveloping algebra $\widetilde{\mathcal{D}}$ of an $\mathbf{H}$-torsor $\xi : \widetilde{X} \to X$. We will see these constructions applied to the case where $X = \mathbf{G}/\mathbf{B}$ is the flag variety of a connected split reductive affine algebraic group scheme, as in the sense of Definition 2.2.22, and compute some explicit examples in the case $\mathbf{G} = SL_2$.

Let R be a fixed commutative Noetherian ring, and let X be a smooth separated R-scheme that is locally of finite type. Let $\mathbf{H}$ be a flat affine algebraic group over R of finite type, and let $\widetilde{X}$ be a scheme equipped with an $\mathbf{H}$-action.

Definition 2.5.1. A morphism $\xi : \widetilde{X} \to X$ is an $\mathbf{H} - \textit{torsor}$ if:

(a) ξ is faithfully flat and locally of finite type,

(b) the action of $\mathbf{H}$ respects ξ,

(c) the map $\widetilde{X} \times \mathbf{H} \to \widetilde{X} \times_X \widetilde{X}$, $(x, h) \to (x, hx)$ is an isomorphism.

An open subscheme U of X *trivialises the torsor* ξ if there is an $\mathbf{H}$-invariant isomorphism:

$$U \times \mathbf{H} \to \xi^{-1}(U)$$

where $\mathbf{H}$ acts on $U \times \mathbf{H}$ by left translation on the second factor.

Definition 2.5.2. Let $\mathcal{S}_X$ denote the set of open subschemes U of X such that:

(a) U is affine,

(b) U trivialises ξ,

(c) $\mathcal{O}(U)$ is a finitely generated R-algebra.

ξ is *locally trivial* for the Zariski topology if X can be covered by open sets in $\mathcal{S}_X$.

Lemma 2.5.3. If ξ is locally trivial, then $\mathcal{S}_X$ is a base for X.

Proof. Since X is separated, $\mathcal{S}_X$ is stable under intersections. If $U \in \mathcal{S}_X$ and W is an open affine subscheme of U, then $W \in \mathcal{S}_X$. Hence $\mathcal{S}_X$ is a base for X. □

The action of $\mathbf{H}$ on $\widetilde{X}$ induces a rational action of $\mathbf{H}$ on $\mathcal{O}(V)$ for any $\mathbf{H}$-stable open subscheme $V \subseteq \widetilde{X}$, and therefore induces an action of $\mathbf{H}$ on $\mathcal{T}_{\widetilde{X}}$ via:

$$(h \cdot \partial)(f) = h \cdot \partial(h^{-1} \cdot f)$$

for $\partial \in \mathcal{T}_{\widetilde{X}}, f \in \mathcal{O}_{\widetilde{X}}$ and $h \in \mathbf{H}$. The *sheaf of enhanced vector fields* on X is:

$$\widetilde{\mathcal{T}} := (\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbf{H}}.$$

Differentiating the $\mathbf{H}$-action on $\widetilde{X}$ gives an R-linear Lie algebra homomorphism:

$$j : \mathfrak{h} \to \mathcal{T}_{\widetilde{X}}$$

where $\mathfrak{h}$ is the Lie algebra of $\mathbf{H}$.

Definition 2.5.4. The *enhanced cotangent bundle* is the vector bundle $\tau : \widetilde{T^*X} \to X$ associated to the sheaf $\widetilde{\mathcal{T}}$, which is locally free by [4, Lemma 4.4].

Definition 2.5.5. Let $\xi : \widetilde{X} \to X$ be an $\mathbf{H}$-torsor. Then $\xi_* \mathcal{D}_{\widetilde{X}}$ is a sheaf of R-algebras with an $\mathbf{H}$-action. The *relative enveloping algebra* of the torsor is the sheaf of $\mathbf{H}$-invariants of $\xi_* \mathcal{D}_{\widetilde{X}}$:

$$\widetilde{\mathcal{D}} := (\xi_* \mathcal{D}_{\widetilde{X}})^{\mathbf{H}}.$$

This sheaf has a natural filtration:

$$F_m \widetilde{\mathcal{D}} := (\xi_* F_m \mathcal{D}_{\widetilde{X}})^{\mathbf{H}}$$

induced by the filtration on $\mathcal{D}_{\widetilde{X}}$ by order of differential operator.

Let $\mathbf{G}$ be a split reductive connected algebraic group over a discrete valuation ring R, with uniformiser π, residue field k and field of fractions K. Let $\mathbf{B}$ be a Borel subgroup. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$, and $\mathbf{H} := \mathbf{B}/\mathbf{N}$ the abstract Cartan group. Let $\widetilde{\mathcal{B}}$ denote the homogeneous space $\mathbf{G}/\mathbf{N}$. There is an $\mathbf{H}$-action on $\widetilde{\mathcal{B}}$ defined by:

$$b\mathbf{N} \cdot g\mathbf{N} := gb\mathbf{N}$$

which is well-defined since $[\mathbf{B}, \mathbf{B}]$ is contained in $\mathbf{N}$. $\mathcal{B} := \mathbf{G}/\mathbf{B}$ is the *flag variety* of $\mathbf{G}$, as in Definition 2.2.22. $\widetilde{\mathcal{B}}$ is the *basic affine space* of $\mathbf{G}$.

By the splitting assumption of $\mathbf{G}$, we can find a Cartan subgroup $\mathbf{T}$ of $\mathbf{G}$ complementary to $\mathbf{N}$ in $\mathbf{B}$. This is naturally isomorphic to $\mathbf{H}$, and induces an isomorphism of the corresponding Lie algebras $\mathfrak{t} \to \mathfrak{h}$. The adjoint action of $\mathbf{T}$ on $\mathfrak{g}$ induces a root space decomposition:

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^+$$

where $\mathfrak{n}$ is spanned by the weight spaces of the negative roots of $\mathbf{G}$. This induces an isomorphism of R-modules:

$$U(\mathfrak{g}) \cong U(\mathfrak{n}) \otimes U(\mathfrak{t}) \otimes U(\mathfrak{n}^+)$$

and a direct sum decomposition:

$$U(\mathfrak{g}) = U(\mathfrak{t}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+).$$

Lemma 2.5.6. The natural projection $\xi : \widetilde{\mathcal{B}} \to \mathcal{B}$ is a locally trivial $\mathbf{H}$-torsor.

Proof. This is [4, Lemma 4.7(c)]. □

We may differentiate the natural $\mathbf{G}$-action on $\widetilde{\mathcal{B}}$ to obtain an R-linear Lie homomorphism:

$$\varphi : \mathfrak{g} \to \mathcal{T}_{\widetilde{\mathcal{B}}}.$$

Since the $\mathbf{G}$-action commutes with the $\mathbf{H}$-action on $\widetilde{\mathcal{B}}$, this map descends to an R-linear Lie homomorphism $\varphi : \mathfrak{g} \to \widetilde{\mathcal{T}}_{\mathcal{B}}$ and an $\mathcal{O}_{\mathcal{B}}$-linear morphism:

$$\varphi : \mathcal{O}_{\mathcal{B}} \otimes \mathfrak{g} \to \widetilde{\mathcal{T}}_{\mathcal{B}}$$

of locally free sheaves on $\mathcal{B}$. Dualising, we obtain a morphism of vector bundles over $\mathcal{B}$:

$$\varphi^* : \widetilde{T^*\mathcal{B}} \to \mathcal{B} \times \mathfrak{g}^*$$

from the enhanced cotangent bundle to the trivial vector bundle of rank dim $\mathfrak{g}$.

Definition 2.5.7. The *enhanced moment map* is the composition of φ^* with the projection onto the second coordinate:

$$\beta : \widetilde{T^*\mathcal{B}} \to \mathfrak{g}^*.$$

The adjoint action of $\mathbf{G}$ on $\mathfrak{g}$ induces an action of $\mathbf{G}$ on $U(\mathfrak{g})$ via algebra automorphisms. Composing the inclusion $U(\mathfrak{g})^{\mathbf{G}} \to U(\mathfrak{g})$ with the projection $U(\mathfrak{g}) \to U(\mathfrak{t})$ defined by the direct sum decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ yields the *Harish-Chandra homomorphism*:

$$\phi : U(\mathfrak{g})^{\mathbf{G}} \to U(\mathfrak{t})$$

Proposition 2.5.8. Let R be an integral domain. Then there is a commutative diagram of filtered rings:

$$\begin{array}{ccc} U(\mathfrak{g})^{\mathbf{G}} & \xrightarrow{\phi} & U(\mathfrak{t}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j\circ i} \\ U(\mathfrak{g}) & \xrightarrow{U(\phi)} & \widetilde{\mathcal{D}}, \end{array}$$

where i denotes the inclusion $U(\mathfrak{g})^{\mathbf{G}} \to U(\mathfrak{g})$.

Proof. This is [4, Lemma 4.10]. □

Example 2.5.9. Suppose R is a complete discrete valuation ring and let $\mathbf{G} = SL_2(K)$. As in Example 2.2.25, we take $\mathbf{B}$ to be the subgroup of lower triangular matrices, and $\mathbf{N}$ to be the subgroup consisting of lower triangular matrices with diagonal entries equal to 1. Then $\mathcal{B} = \mathbf{G}/\mathbf{B} \cong \mathbb{P}^1$, and $\widetilde{\mathcal{B}} = \mathbf{G}/\mathbf{N} \cong \mathbb{A}^2/\{0\}$.

Let $\mathbf{H} = \mathbf{B}/\mathbf{N}$ be the abstract Cartan group. This is isomorphic to the multiplicative group $\mathbb{G}_m$. Its associated Lie algebra $\mathfrak{h}$ is isomorphic to the algebra of scalar matrices.

Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$. Then $\mathfrak{t}$ is an abelian Lie algebra generated by the matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $\mathbf{G}$ act on $\mathcal{B}$ by left multiplication, and let $x \in \mathcal{B}$. The differential of the orbit map $\phi_x : \mathbf{G} \to \mathbf{G}x$ at $1 \in \mathbf{G}$ is a linear map $d\phi_x : \mathfrak{g} \mapsto T_x(\mathbf{G}x) \subseteq T_x\mathcal{B}$. These assemble to give a map of vector bundles: $\mathcal{B} \times \mathfrak{g} \to T\mathcal{B}$ defined by $(x, a) \mapsto (x, d\phi_x(a))$. Identifying the vector bundles with their corresponding locally free sheaves, we have a morphism $\phi : \mathcal{O}_{\mathcal{B}} \otimes \mathfrak{g} \to \mathcal{T}_{\mathcal{B}}$. This can be written as:

$$\phi(a)(\beta)(z) = \lim_{\epsilon \to 0} \frac{\beta((1+\epsilon a)^{-1} z) - \beta(z)}{\epsilon}$$

for $a \in \mathfrak{g}, \beta \in \mathcal{O}_X$ and $z \in X$.

Returning to our situation, let SL_2 act on the flag variety $\mathbb{P}^1$. Let z be a local coordinate of $\mathbb{P}^1$ near 0, and let e, f, h be the standard basis for $\mathfrak{sl}_2$. Applying the formula, we see that:

$$\begin{aligned} \phi(e)(x)(z) &= \lim_{\epsilon \to 0} \frac{x((1+\epsilon e)^{-1} z) - x(z)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\begin{bmatrix} 1 & -\epsilon \\ 0 & 1 \end{bmatrix} z - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z}{\epsilon} = -1 \end{aligned}$$

Hence the vector field corresponding to the derivation $\phi(e)$ is the assignment $z \mapsto (z, -1)$, and so $\phi(e) = -\partial_z$. Similarly, we see that:

$$\phi(f) = z^2 \partial_z,$$

$$\phi(h) = -2z\partial_z.$$

Next we compute the R-linear Lie algebra homomorphism $j : \mathfrak{g} \to \mathcal{T}_{\widetilde{\mathcal{B}}}$ differentiating the $\mathbf{G}$-action on $\widetilde{\mathcal{B}}$. Let x, y be the coordinates of $\widetilde{\mathcal{B}} \cong \mathbb{A}^2 \setminus \{0\}$. Substituting into the above formula, we see that:

$$j(e)(x)(z) = \lim_{\epsilon \to 0} \frac{x((1+\epsilon e)^{-1}z) - x(z)}{\epsilon}$$
$$= \lim_{\epsilon \to 0} \frac{\begin{bmatrix} 1 & -\epsilon \\ 0 & 1 \end{bmatrix} z - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z}{\epsilon}$$

It follows that the vector field corresponding to the derivation $j(e)$ is the assignment $z \mapsto ((0,0),(-z,0))$, so $j(e) = -x\partial_y$. Via similar calculations, $j(f) = y\partial_x$ and $j(h) = -x\partial_x + y\partial_y$.

Finally, we verify that the diagram in Lemma 2.5.8 is commutative. Recall that the $\mathbf{G}$-invariants $U(\mathfrak{g})^{\mathbf{G}}$ may be identified with the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. For $\mathfrak{g} = \mathfrak{sl}_2$, we have $Z(\mathfrak{g}) \cong K[C]$, where C denotes the Casimir element $C := \frac{1}{2}h^2 + 2ef - h$. The Harish-Chandra homomorphism $\phi : U(\mathfrak{g})^G \to U(\mathfrak{t})$ sends C to $\frac{1}{2}(h^2 - 2h)$. By the above calculation, $j(\frac{1}{2}(h^2 - 2h)) = \frac{1}{2}(x^2(\partial_x)^2 + y^2(\partial_y)^2 + x\partial_x) - xy\partial_x\partial_y$.

On the other hand, the bottom map $U(\phi) : U(\mathfrak{g}) \to \widetilde{\mathcal{D}}$ sends $C = \frac{1}{2}h^2 + 2ef - h$ to $\frac{1}{2}(x^2(\partial_x)^2 + y^2(\partial_y)^2 + x\partial_x) - xy\partial_x\partial_y$ also. Hence the diagram is commutative.

2.6 Lie-Rinehart algebras and $\widehat{\mathcal{D}}$-modules on rigid analytic spaces

We now begin our study of $\mathcal{D}$-modules on rigid analytic spaces. In this section, we summarise some of the main constructions and results from [5], in preparation for our work on equivariant $\mathcal{D}$-modules on rigid analytic spaces. The main result of this section is the construction of the sheaf of completed differential operators on a rigid analytic space.

In what follows, let R be a commutative discrete valuation ring with uniformiser π, residue field k and field of fractions K, and let $\mathbf{X}$ be a rigid K-space. Write $\mathbf{X}_w$ to denote the site $\mathbf{X}$ equipped with the weak Grothendieck topology in the sense of Definition 2.3.12, and $\mathbf{X}_{\text{rig}}$ to denote the site $\mathbf{X}$ equipped with the strong Grothendieck topology in the sense of Definition 2.3.13.

Proposition 2.6.1. There is a coherent sheaf $\mathcal{T}_{\mathbf{X}}$ of K-Lie algebras on $\mathbf{X}_{\text{rig}}$ such that $\mathcal{T}_{\mathbf{X}}(\mathbf{U}) := \text{Der}_K\mathcal{O}(\mathbf{U})$ for each affinoid subdomain $\mathbf{U}$ of $\mathbf{X}$. This is the *tangent sheaf* of $\mathbf{X}$.

Proof. This is [5, Proposition 9.1]. □

Definition 2.6.2. A *Lie algebroid* on $\mathbf{X}$ is a pair $(\rho, \mathscr{L})$ such that:

(a) $\mathscr{L}$ is a locally free sheaf of $\mathcal{O}$-modules of finite rank on $\mathbf{X}_{\text{rig}}$,

(b) $\mathscr{L}$ has the structure of a sheaf of K-Lie algebras,

(c) $\rho : \mathscr{L} \to \mathcal{T}$ is an $\mathcal{O}$-linear map of sheaves of Lie algebras such that:

$$[x, ay] = a[x, y] + \rho(x)(a)y$$

for $\mathbf{U}$ an admissible open subset of $\mathbf{X}$, $x, y \in \mathscr{L}(\mathbf{U})$ and $a \in \mathcal{O}(\mathbf{U})$.

Example 2.6.3. (a) Let $\mathbf{X}$ be the one-point space. Then a Lie algebroid $(\rho, \mathcal{L})$ on $\mathbf{X}$ is a K-Lie algebra.

(b) Given a general smooth rigid K-space $\mathbf{X}_{\text{rig}}$, the tangent sheaf $\mathcal{T}$ is locally free and hence is an example of a Lie algebroid by Proposition 2.6.1, with $\mathscr{L} = \mathcal{T}$ and ρ the identity map.

Definition 2.6.4. Let S be a commutative ring and A a commutative S-algebra. A *Lie-Rinehart algebra*, or an $(S-A)-$*Lie algebra*, is a pair (L, ρ), where L is an S-Lie algebra and an A-module, and $\rho : L \to \mathrm{Der}_S(A)$ is a Lie algebra homomorphism satisfying:

$$[x, ay] = a[x, y] + \rho(x)(a)y \text{ for all } x, y \in L, a \in A.$$

ρ is the *anchor map*.

This is the global algebraic analogue of a Lie algebroid. Note that it can be defined over any commutative ring S.

Definition 2.6.5. (a) An $(S - A)$-Lie algebra L is *coherent* if it is coherent as an A-module, i.e. L is finitely generated and every finitely generated A-submodule N of L is finitely presented.

(b) L is *smooth* if it is projective as an A-module.

We now outline the construction of the universal enveloping algebra $U(L)$ of a Lie-Rinehart algebra L over A. The left A-module $A \oplus L$ has a natural Lie algebra structure, where the Lie bracket is given by:

$$[(a, x), (b, y)] = (\rho(x)(b) - \rho(y)(a), [x, y])$$

for $a, b \in A$ and $x, y \in L$. Let $U(A \oplus L)$ be its universal enveloping algebra over S. Let $\iota : A \oplus L \to U(A \oplus L)$ be the canonical inclusion and $\widetilde{U}(A \oplus L)$ the image of ι.

Definition 2.6.6. The *universal enveloping algebra* of a Lie-Rinehart algebra (L, ρ) is:

$$U(L) := \frac{\widetilde{U}(A \oplus L)}{(\iota(s, 0) \cdot \iota(r, x) - \iota(sr, sx))}.$$

$U(L)$ may be equipped with canonical homomorphisms $i_A : A \to U(L)$ and $i_L : L \to U(L)$ of R-algebras and R-Lie algebras respectively, satisfying:

$$i_L(ax) = i_A(a) i_L(x),$$
$$[i_L(x), i_A(a)] = i_A(\rho(x)(a)),$$

for $a \in A$ and $x \in L$. Furthermore, $U(L)$ is universal with respect to these properties: for any other pair of homomorphisms $j_A : A \to S', j_L : L \to S'$ satisfying these relations there is a unique S-algebra homomorphism $\phi : U(L) \to S'$ such that $\phi \circ i_A = j_A$ and $\phi \circ i_L = j_L$.

If $(L, \rho), (L', \rho')$ are two $(S - A)$-Lie algebras, a *morphism* of $(S - A)$-Lie algebras is an A-linear map $f : L \to L'$ that is a morphism of S-Lie algebras satisfying $\rho' \circ f = \rho$.

The morphism f induces an S-algebra homomorphism $U(f) : U(L) \to U(L')$ via:

$$U(f)(a) = a,$$

$$U(f)(i_L(x)) = i_{L'}(f(x)),$$

for $a \in A, x \in L$. Hence U is a functor from $(S - A)$-Lie algebras to S-associative algebras.

Definition 2.6.7. Let $\phi : A \to A'$ be an S-algebra homomorphism, let L be an (S, A)-Lie algebra and let L' be an (S, A')-Lie algebra. Then $\widetilde{\phi} : L \to L'$ is a *ϕ-morphism* if:

(a) $\widetilde{\phi}$ is a homomorphism of S-Lie algebras,

(b) $\widetilde{\phi}(a \cdot v) = \phi(a) \cdot \widetilde{\phi}(v)$,

(c) $\widetilde{\phi}(v) \cdot \phi(a) = \phi(v \cdot a)$ for all $a \in A$, $v \in L$.

Lemma 2.6.8. Let $\phi : A \to A'$ be an S-algebra homomorphism, let L be an (S, A)-Lie algebra and let L' be an (S, A')-Lie algebra. Then every ϕ-morphism $\widetilde{\phi} : L \to L'$ extends uniquely to a filtration-preserving S-algebra homomorphism $U(\phi, \widetilde{\phi})$ such that the following diagram of S-modules commutes:

$$\begin{array}{ccc} A \oplus L & \xrightarrow{\phi \oplus \widetilde{\phi}} & A' \oplus L' \\ \downarrow{\scriptstyle i_A \oplus i_L} & & \downarrow{\scriptstyle i_{A'} \oplus i_{L'}} \\ U(L) & \xrightarrow{U(\phi, \widetilde{\phi})} & U(L'). \end{array}$$

Proof. This is [1, Lemma 2.1.7]. □

We now restrict to the case of K-affinoid algebras. Recall the following definitions from [1, Section 3.1]:

Definition 2.6.9. An R-algebra $\mathcal{A}$ is *topologically of finite presentation* if it can be realised as the quotient of the algebra of restricted formal power series in finitely many variables over R by a finitely generated ideal:

$$\mathcal{A} \cong \frac{R\langle x_1, \cdots, x_n \rangle}{\mathfrak{a}}.$$

Definition 2.6.10. (a) $\mathcal{A}$ is *admissible* if it is topologically of finite presentation and flat as an R-module.

(b) An affine formal R-scheme $\mathcal{X}$ is *admissible* if $\mathcal{X} = \mathrm{Spf}\, \mathcal{A}$ for some admissible R-algebra $\mathcal{A}$.

(c) A formal R-scheme $\mathcal{Y}$ is *admissible* if it is locally isomorphic to an admissible affine formal R-scheme.

Definition 2.6.11. Let $\mathcal{X}$ be an admissible formal scheme.

(a) $\mathcal{G}(\mathcal{X}) := \mathrm{Aut}_R(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ denotes the group of R-linear automorphisms of $\mathcal{X}$.

(b) For every $n \geq 0$, let $R_n := R/\pi^n R$ and $\mathcal{X}_n := \mathcal{X} \times_{\mathrm{Spf}\, R} \mathrm{Spf}\, R_n$. The *nth congruence subgroup* of $\mathcal{G}(\mathcal{X})$ is:

$$\mathcal{G}_{\pi^n}(\mathcal{X}) := \ker(\mathcal{G}(\mathcal{X}) \to \mathrm{Aut}_{R_n}(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})).$$

Definition 2.6.12. Let A be an affinoid K-algebra. An admissible R-algebra $\mathcal{A}$ is an *affine formal model* for A if we have an isomorphism of K-affinoid algebras $A \cong \mathcal{A} \otimes_R K$.

A surjection $R\langle x_1, \cdots, x_n\rangle \to \mathcal{A}$ gives rise to a residue norm on A with unit ball $\mathcal{A}$. It follows that an R-subalgebra of A is an affine formal model if and only if it can be realised as the unit ball of some residue norm on A. We note an easy consequence of the definition of affine formal models for later use.

Lemma 2.6.13. Let $\mathcal{A}, \mathcal{B}$ be two affine formal models of the K-affinoid algebra A. The product $\mathcal{A}\mathcal{B}$ is another formal model of A, and $\mathcal{A}\mathcal{B}$ is finitely generated as a module over $\mathcal{A}$ and $\mathcal{B}$.

Proof. This is [5, Lemma 3.1]. □

Definition 2.6.14. Let $L := \mathrm{Der}_K(A)$ and let $\mathcal{L}$ be an $\mathcal{A}$-submodule of L. $\mathcal{L}$ is an $\mathcal{A}$-*Lie lattice* in L if it satisfies the following conditions:

(a) $\mathcal{L}$ is finitely presented as an $\mathcal{A}$-module,

(b) $\mathcal{L}$ spans L as a K-vector space,

(c) $[\mathcal{L}, \mathcal{L}] \subseteq \mathcal{L}$ and $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$.

This is the Lie algebra analogue of the definition of an affine formal model. Just as above, an $(R, \mathcal{A})$-Lie algebra $\mathcal{L}$ is an $\mathcal{A}$-Lie lattice if and only if it can be realised as the intersection of L with the unit ball of some residue norm on A, defined inside L. The canonical example of an $\mathcal{A}$-Lie lattice $\mathcal{L}$ is the $(R, \mathcal{A})$-Lie algebra $\mathrm{Der}_R(\mathcal{A})$.

In what follows, it will be useful to keep this example in mind.

Definition 2.6.15. Let $\mathbf{Y} = \mathrm{Sp}\ A$ be a smooth affinoid rigid space. The *algebra of differential operators* on $\mathbf{Y}$ is $U(\mathrm{Der}_K(A))$.

For a general smooth rigid space $\mathbf{Y}$, the *sheaf of differential operators* on $\mathbf{Y}$ is the sheaf $\mathcal{D}$ which satisfies:

$$\mathcal{D}_{\mathbf{Y}}(\mathbf{Z}) = U(\mathrm{Der}_K(B))$$

for all affinoid subdomains $\mathbf{Z} := \mathrm{Sp}\ B$ of $\mathbf{Y}$, with natural restriction maps.

Definition 2.6.16. Let $\mathcal{L}$ be an $\mathcal{A}$-Lie lattice in L that is also an $(R, \mathcal{A})$-Lie algebra, and let $U(\mathcal{L})$ be its universal enveloping algebra as in Definition 2.6.6. The $\pi - $*adic completion* of $U(\mathcal{L})$ is $\widehat{U(\mathcal{L})} := \varprojlim U(\mathcal{L})/\pi^a U(\mathcal{L})$.

We also set $\widehat{U(\mathcal{L})}_K := \widehat{U(\mathcal{L})} \otimes_{\mathcal{R}} K$.

Definition 2.6.17. The *Fréchet completion* of the $\mathcal{A}$-Lie lattice $\mathcal{L}$ is:

$$\widehat{U(\mathcal{L})}_{\mathcal{A},\mathcal{L}} := \varprojlim \widehat{U(\pi^n \mathcal{L})}_K.$$

This is a countable inverse limit of K-Banach algebras and so is a Fréchet algebra. By [5, Lemma 6.2 and Proposition 6.2], this definition is independent of the choice of $\mathcal{A}$ and $\mathcal{A}$-Lie lattice $\mathcal{L}$ in L. Hence we can unambiguously define the

Fréchet completion $\widehat{U(\mathcal{L})} := \varprojlim \widehat{U(\pi^n\mathcal{L})}_K$.

Theorem 2.6.18. *Let A be a K-affinoid algebra and L a smooth coherent (K, A)-Lie algebra. Suppose L has an $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some affine formal model $\mathcal{A}$ in $\mathcal{O}(\boldsymbol{X})$. Then $\widehat{U(\mathcal{L})}$ is a two-sided Fréchet-Stein algebra.*

Proof. This is [5, Theorem 6.4] and [10, Theorem 1.1(ii)]. □

2.7 Algebraic background for equivariant $\mathcal{D}$-modules

Following [1], we outline the construction of the category $\mathcal{C}_{\mathbf{X}/G}$ of coadmissible G-equivariant $\mathcal{D}$-modules on a smooth rigid analytic space $\mathbf{X}$ equipped with the continuous action of a p-adic Lie group G. To do so, we introduce the completed skew-group algebra $\widehat{\mathcal{D}}(\mathbf{X}, G)$, which is a particular Fréchet completion of the skew-group product $\mathcal{D}(\mathbf{X}) \rtimes G$. Given certain topological conditions on the pair $(\mathbf{X}, G)$, we show that $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is a Fréchet-Stein algebra in the sense of Definition 2.4.2. We then use this construction to form the sheaf $\text{Loc}^A_{\mathbf{X}}$, dependent on some fixed K-algebra A; the construction of $\mathcal{C}_{\mathbf{X}/G}$ follows from this.

We begin with some topological preliminaries on trivialisations of skew-group rings and G-equivariant sheaves of modules.

Definition 2.7.1. Let S be a ring and G a group acting on S via ring automorphisms. The *skew-group ring* $S \rtimes G$ is a free left S-module with basis G, and whose multiplication is given by:

$$(sg) \cdot (th) = (s(g \cdot t))(gh) \text{ for all } s, t \in S, g, h \in G.$$

Definition 2.7.2. A *trivialisation* of the skew-group ring $S \rtimes G$ is a group homomorphism $\beta : G \to S^\times$ such that for all $g \in G$, the conjugation action of $\beta(g) \in S^\times$ coincides with the action of g on S.

Given a trivialisation $\beta : G \to S^\times$, [1, Lemma 2.2.2] guarantees the existence of a ring isomorphism $\widetilde{\beta} : S[G] \to S \rtimes G$. Now let N be a normal subgroup of G and let $\beta : N \to S^\times$ be a trivialisation of the sub-skew-group ring $S \rtimes N$. We set:

$$S \rtimes_N^\beta G := \frac{S \rtimes G}{(S \rtimes G) \cdot (\widetilde{\beta}(N) - 1)}.$$

Definition 2.7.3. The trivialisation β is *G-equivariant* if $\beta(gng^{-1}) = g \cdot \beta(n)$ for all $g \in G$ and $n \in N$.

Let X be a site equipped with a Grothendieck topology, and let $\text{Homeo}(X)$ be the group of continuous bijections with continuous inverse on X. The group G acts on X if there is a group homomorphism $\rho : G \to \text{Homeo}(X)$. Let U be an admissible open subset of X. Then its image under the action of $g \in G$ is denoted by gU. This gives an auto-equivalence:

$$\rho(g)^* =: g^*$$

of the category of sheaves of X, with inverse $g_* := \rho(g^{-1})^*$. Thus, for any sheaf $\mathcal{F}$ on X, we define:

$$(g_*\mathcal{F})(U) := \mathcal{F}(g^{-1}U)$$

$$(g^*\mathcal{F})(U) := \mathcal{F}(gU).$$

Definition 2.7.4. Let R be a commutative ring and let $\mathcal{F}$ be a presheaf of R-modules on X. $\mathcal{F}$ is an *R-linear G-equivariant presheaf* if it is equipped with a natural transformation $g^{\mathcal{F}} : \mathcal{F} \to g^*\mathcal{F}$, for each $g \in G$, which satisfies:

$$(gh)^{\mathcal{F}} = h^*(g^{\mathcal{F}}) \circ h^{\mathcal{F}} \text{ for any } g, h \in G.$$

We write $(\mathcal{F}, g^{\mathcal{F}})$ to denote a presheaf $\mathcal{F}$ equipped with an R-linear G-equivariant structure. A *morphism* of R-linear G-equivariant presheaves $\phi : (\mathcal{F}, g^{\mathcal{F}}) \to (\mathcal{F}', g^{\mathcal{F}'})$ is a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{F}'$ such that, for any $g \in G$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{F}' \\ \downarrow{\scriptstyle g^*\mathcal{F}} & & \downarrow{\scriptstyle g^*\mathcal{F}'} \\ g^*\mathcal{F} & \xrightarrow{g^*(\phi)} & g^*\mathcal{F}'. \end{array}$$

Definition 2.7.5. Let G act on X and let $\mathcal{A}$ be a sheaf of R-algebras on X. $\mathcal{A}$ is a *G-equivariant sheaf of R-algebras* if we have an R-linear G-equivariant structure $\{g^{\mathcal{A}} \mid g \in G\}$ such that each $g^{\mathcal{A}} : \mathcal{A} \to g^*\mathcal{A}$ is a morphism of sheaves of R-algebras.

Given a G-stable admissible open subset U of X, there is a natural G-action on $\mathcal{A}(U)$ by R-algebra automorphisms, given by:

$$g \cdot a = g^{\mathcal{A}}(a) \text{ for all } g \in G, a \in \mathcal{A}.$$

Definition 2.7.6. Let $\mathcal{A}$ be a G-equivariant sheaf of R-algebras on X. A *G-equivariant sheaf of $\mathcal{A}$-modules* on X, or a *$G-\mathcal{A}$-module*, is an R-linear G-equivariant sheaf $\mathcal{M}$ on X such that $\mathcal{M}$ is a sheaf of (left) $\mathcal{A}$-modules and:

$$g^{\mathcal{M}}(a \cdot m) = g^{\mathcal{A}}(a) \cdot g^{\mathcal{M}}(m) \text{ for any } g \in G, a \in \mathcal{A}, m \in \mathcal{M}.$$

A *morphism* of $G - \mathcal{A}$-modules is a morphism of sheaves of $\mathcal{A}$-modules which is also a morphism of R-linear G-equivariant sheaves. The category of $G - \mathcal{A}$-modules is denoted by $G - \mathcal{A}$-mod.

Proposition 2.7.7. Let X be an admissible open in the Grothendieck topology. The global sections functor $\Gamma(X, -)$ is a functor from $G - \mathcal{A}$-modules to $\mathcal{A}(X) \rtimes G$-modules.

Proof. Let M be a $G - \mathcal{A}$-module on X, and define:

$$ag \bullet m = a \cdot g^{\mathcal{M}}(m)$$

for all $a \in \mathcal{A}(X)$, $g \in G$ and $m \in \mathcal{M}(X)$. Then:

$$g \bullet (a \cdot m) = g^{\mathcal{M}}(a \cdot m) = g^{\mathcal{A}}(a) \cdot g^{\mathcal{M}}(m) = (g^{\mathcal{A}}(a)g) \bullet m$$

by Definition 2.7.6. Similarly, $(gh) \bullet m = g \bullet (h \bullet m)$ for all $g, h \in G$ and $m \in \mathcal{M}(X)$. It follows that $\mathcal{M}(X)$ naturally admits a $\mathcal{A}(X) \rtimes G$-module structure via $\bullet$. Furthermore, if $\phi : M \to N$ is a morphism of $G - \mathcal{A}$-modules, then $\phi(X) : \mathcal{M}(X) \to \mathcal{N}(X)$ is $\mathcal{A}(X) \rtimes G$-linear. $\square$

2.8 Constructing the category $\mathcal{C}_{\mathbf{X}/G}$

In this section, we outline the arguments in [1, Section 3], to construct the completed skew-group algebra $\widehat{\mathcal{D}}(\mathbf{X}, G)$, and use this to define the category $\mathcal{C}_{\mathbf{X}/G}$ of coadmissible G-equivariant locally Fréchet $\mathcal{D}$-modules. Throughout this section, we suppose K is a field equipped with a complete non-archimedean norm $|\cdot|$, as in Definition 2.3.1, $\mathcal{R} := \{\lambda \in K \mid |\lambda| \leq 1\}$ is the unit ball inside K and $\pi \in \mathcal{R}$ is a fixed non-zero non-unit element.

Definition 2.8.1. Let G be a topological group and $\mathbf{X}$ a rigid analytic variety. G *acts continuously* on $\mathbf{X}$ if there is a group homomorphism $\rho : G \to \mathrm{Aut}_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ such that for every quasicompact quasiseparated admissible open subset $\mathbf{U}$ of $\mathbf{X}$:

(a) the stabiliser $G_{\mathbf{U}}$ of $\mathbf{U}$ in G is open in G,

(b) the induced group homomorphism $\rho_{\mathbf{U}} : G_{\mathbf{U}} \to \mathrm{Aut}_K(\mathbf{U}, \mathcal{O}_{\mathbf{U}})$ is continuous, where $G_{\mathbf{U}}$ is given the subspace topology and $\mathrm{Aut}_K(\mathbf{U}, \mathcal{O}_{\mathbf{U}})$ has a Hausdorff topology with a filter base given by congruence subgroups of $\mathrm{Aut}_{\mathcal{R}}(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$, constructed in [1, Theorem 3.1.5].

Definition 2.8.2. Let $\mathbb{G}$ be an $\mathcal{R}$-group scheme. We equip its group of $\mathcal{R}$-points $\mathbb{G}(\mathcal{R})$ with the topology in which the congruence subgroups:

$$\mathbb{G}_{\pi^n}(\mathcal{R}) := \ker(\mathbb{G}(\mathcal{R}) \to \mathbb{G}(\mathcal{R}/\pi^n\mathcal{R}))$$

form a filter base. This is the *congruence-subgroup topology* on $\mathbb{G}(\mathcal{R})$.

Definition 2.8.3. Let $\mathcal{L}$ be an $\mathcal{A}$-Lie lattice in L, and let $U(\mathcal{L})$ denote its universal enveloping algebra in the sense of Definition 2.6.6.

(a) The $\pi-$ *adic completion* of $U(\mathcal{L})$ is:

$$\widehat{U(\mathcal{L})} := \varprojlim \frac{U(\mathcal{L})}{\pi^a U(\mathcal{L})}.$$

(b) The $\mathcal{R}$-*torsion submodule* of $\widehat{U(\mathcal{L})}$ is denoted by $\widehat{U(\mathcal{L})}_{\text{tors}}$. The $\mathcal{R}$-*torsion-free* part of $\widehat{U(\mathcal{L})}$ is $\overline{\widehat{U(\mathcal{L})}} := \widehat{U(\mathcal{L})}/\widehat{U(\mathcal{L})}_{\text{tors}}$.

(c) Set $\widehat{U(\mathcal{L})_K} := \widehat{U(\mathcal{L})} \otimes_R K$.

From now on, we assume that $\mathbf{X}$ is a rigid analytic variety, G is a p-adic Lie group acting continuously on $\mathbf{X}$, and $\mathcal{A}$ is a G-stable formal model in $A := \mathcal{O}(\mathbf{X})$.

Definition 2.8.4. Let $\mathcal{A}$ be an affine formal model in A and $\mathcal{L}$ an $\mathcal{A}$-lattice in L. $\mathcal{L}$ is $G-$ *stable* if the affine formal model $\mathcal{A}$ is G-stable, and $\mathcal{L}$ is invariant under the natural action of G on L.

Let $\mathcal{L}$ be a G-stable $\mathcal{A}$-Lie lattice in $\mathcal{T}(\mathbf{X})$. Recall from [1, Lemma 3.2.4], that any affine formal model $\mathcal{A}$ is contained in a G-stable affine formal model. We set:

$$G_{\mathcal{L}} := \rho^{-1}(\exp(p^e\mathcal{L})),$$

where $e := 1$ if $p > 2$ and $e := 2$ if $p = 2$. By [1, Theorem 3.2.12], this is an open normal subgroup of G.

Let $\mathcal{E} := \mathrm{End}_{\mathcal{R}}(\mathcal{A})$, and $\mathcal{U} := \overline{\widehat{U(\mathcal{L})}}$. Denote by ι the natural map $\overline{\widehat{i_{\mathcal{A}} \oplus i_{\mathcal{L}}}} : \mathcal{A} \oplus \mathcal{L} \to \mathcal{U}$. By [1, Lemma 3.2.10(a)], there is a unique $\mathcal{R}$-algebra homomorphism $\psi_{\mathcal{L}} : \mathcal{U} \to \mathcal{E}$ such that $\psi_{\mathcal{L}}(\iota(a)) = i_{\mathcal{A}}(a)$ and $\psi_{\mathcal{L}}(\iota(v)) = v$ for all $a \in \mathcal{A}$ and $v \in \mathcal{L}$.

Proposition 2.8.5. The map $\beta_{\mathcal{L}} := (\psi_{\mathcal{L}}^{\times})^{-1} \circ \rho : G_{\mathcal{L}} \to \mathcal{U}^{\times}$ is a G-equivariant trivialisation of the $G_{\mathcal{L}}$-actions on $\overline{\widehat{U(\mathcal{L})}}$ and $\widehat{U(\mathcal{L})_K}$, fitting into the commutative diagram:

$$\begin{array}{ccccc} & & G_{\mathcal{L}} & \longrightarrow & G \\ & {\scriptstyle \rho|_{G_{\mathcal{L}}}} \swarrow & \downarrow {\scriptstyle \beta_{\mathcal{L}}} & & \downarrow {\scriptstyle \rho} \\ \exp(p^e \mathcal{L}) & \longrightarrow & \mathcal{U}^{\times} & \xrightarrow{\psi_{\mathcal{L}}^{\times}} & \mathcal{E}^{\times}. \end{array}$$

Proof. This is [1, Theorem 3.2.12(b)]. □

Definition 2.8.6. Let $\mathcal{A}$ be a G-stable affine formal model in A. $(\mathcal{L}, N)$ is an $\mathcal{A}$-*trivialising pair* if $\mathcal{L}$ is a G-stable $\mathcal{A}$-Lie lattice in $\mathrm{Der}_K(A)$ and N is an open normal subgroup of G contained in $G_{\mathcal{L}}$. The set of all $\mathcal{A}$-trivialising pairs is denoted by $\mathcal{I}(G)$.

The set $\mathcal{I}(G)$ becomes directed when ordered by component-wise reverse inclusion, i.e. $(\mathcal{L}_1, N_1) \leq (\mathcal{L}_2, N_2)$ if and only if $\mathcal{L}_2 \subseteq \mathcal{L}_1$ and $N_2 \subseteq N_1$. One can define canonical connecting homomorphisms:

$$\overline{\widehat{U(\mathcal{L}_2)}} \rtimes_{N_2} G \to \overline{\widehat{U(\mathcal{L}_1)}} \rtimes_{N_1} G,$$

$$\widehat{U(\mathcal{L}_2)_K} \rtimes_{N_2} G \to \widehat{U(\mathcal{L}_1)_K} \rtimes_{N_1} G.$$

Definition 2.8.7. The *completed skew-group algebra* is:

$$\widehat{\mathcal{D}}(\mathbf{X}, G)_{\mathcal{A}} := \varprojlim \widehat{U(\mathcal{L})_K} \rtimes_N G$$

and the *integral completed skew-group ring* is:

$$\mathcal{A}\widehat{\rtimes}G := \varprojlim \overline{\widehat{U(\mathcal{L})}} \rtimes_N G.$$

Definition 2.8.8. Let $(N_\bullet) = N_0 \geq N_1 \geq \cdots$ be a separated chain of open normal subgroups of G, and let $\mathcal{L}$ be a G-stable $\mathcal{A}$-Lie lattice in $\mathrm{Der}_K(\mathcal{A})$. $(N_\bullet)$ is a *good chain* for $\mathcal{L}$ if $(\pi^n\mathcal{L}, N_n) \in \mathcal{I}(G)$ for all $n \geq 0$.

This notion allows us to simplify the definition of $\widehat{\mathcal{D}}(\mathbf{X}, G)_{\mathcal{A}}$.

Lemma 2.8.9. For every good chain $(N_\bullet)$ for $\mathcal{L}$, there is a K-algebra isomorphism:

$$\widehat{\mathcal{D}}(\mathbf{X}, G)_{\mathcal{A}} \cong \varprojlim \widehat{U(\pi^n\mathcal{L})} \rtimes_{N_n} G.$$

Proof. This is [1, Lemma 3.3.4]. □

Proposition 2.8.10. $\widehat{\mathcal{D}}(\mathbf{X}, G)_{\mathcal{A}}$ is independent of the choice of $\mathcal{A}$.

Proof. This is [1, Proposition 3.3.8]. □

In what follows, we will drop the subscript $\mathcal{A}$ from $\widehat{\mathcal{D}}(\mathbf{X}, G)$.

Corollary 2.8.11. $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is a K-Fréchet algebra.

Proof. This is [1, Corollary 3.3.9]. □

Let $\mathbf{X}_w/G$ denote the set of G-stable affinoid subdomains of $\mathbf{X}$.

Lemma 2.8.12. $\widehat{\mathcal{D}}(-,G)$ is a presheaf of K-Fréchet algebras on $\mathbf{X}_w(G)$.

Proof. This is [1, Lemma 3.4.2]. □

Definition 2.8.13. Let $\mathbf{U}$ be an affinoid subdomain of $\mathbf{X}$ and H a compact open subgroup of $G_{\mathbf{U}}$. $(\mathbf{U}, H)$ is *small* if $\mathcal{T}(\mathbf{U})$ has an H-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some H-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{U})$.

Definition 2.8.14. Let $\mathbf{X}_w(\mathcal{T})$ denote the set of affinoid subdomains $\mathbf{U}$ of $\mathbf{X}$ such that $\mathcal{T}(\mathbf{U})$ has a free $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{U})$.

The notion of smallness enjoys nice closure properties, as described in the following lemma.

Lemma 2.8.15. (a) Let $(\mathbf{X}, G)$ be small. Then $(\mathbf{U}, H)$ is small for every affinoid subdomain $\mathbf{U}$ of $\mathbf{X}$ and every compact open subgroup H of $G_{\mathbf{U}}$.

(b) For every $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, there is a $\mathbf{U}$-small subgroup H.

Proof. This is [1, Lemma 3.4.5 and Lemma 3.4.7]. □

We now come to a fundamental theorem that relates the construction of $\widehat{\mathcal{D}}(\mathbf{X}, G)$ to the theory of p-adic distribution algebras.

Theorem 2.8.16. *Suppose that $(\boldsymbol{U}, H)$ is small. Then $\widehat{\mathcal{D}}(\boldsymbol{U}, H)$ is a two-sided Fréchet-Stein algebra.*

Proof. This is [1, Lemma 3.4.8]. □

Definition 2.8.17. Let G be a p-adic Lie group, acting continuously on a smooth rigid analytic variety $\mathbf{X}$, and let A be a K-algebra. A acts on $\mathbf{X}$ *compatibly with* G if there is:

(a) a group homomorphism $\eta : G \to A^\times$,

(b) a Fréchet-Stein subalgebra A_H of A for every compact open subgroup H of G,

(c) a continuous homomorphism $\phi^H : A_H \to \widehat{\mathcal{D}}(-, H)$ of presheaves of K-Fréchet algebras on $\mathbf{X}_w/H$ for every compact open subgroup H of G, where A_H is viewed as a constant presheaf.

Furthermore, for any pair $H \leq N$ of compact open subgroups of G:

(i) $A_H \leq A_N$, $\eta(H) \subseteq A_H^\times$ and the canonical map:

$$A_H \otimes_{K[H]} K[N] \to A_N$$

is a bijection.

(ii) the following diagram of presheaves is commutative:

$$\begin{array}{ccc} A_H & \xrightarrow{\phi^H} & \widehat{\mathcal{D}}(-, H) \\ \downarrow & & \downarrow \\ A_N & \xrightarrow{\phi^N} & \widehat{\mathcal{D}}(-, N). \end{array}$$

(iii) for every $g \in G$, the map $\mathrm{Ad}_{\eta(g)} : A \to A$, obtained by conjugation by $\eta(g)$, sends A_H into $A_{gHg^{-1}}$, and for every $\mathbf{U} \in \mathbf{X}_w/H$, the following diagram commutes:

$$\begin{array}{ccc} A_H & \xrightarrow{\mathrm{Ad}_{\eta(g)}} & \widehat{\mathcal{D}}(-,H) \\ \downarrow{\scriptstyle \phi^{\mathbf{U}}(H)} & & \downarrow{\scriptstyle \phi^{gHg^{-1}}(g\mathbf{U})} \\ \widehat{\mathcal{D}}(\mathbf{U},H) & \xrightarrow{\widehat{g_{\mathbf{U},H}}} & \widehat{\mathcal{D}}(g\mathbf{U},gHg^{-1}). \end{array}$$

(iv) $\phi^H \circ \eta \mid_H = \gamma^H$.

Definition 2.8.18. Suppose A acts on $\mathbf{X}$ compatibly with G. The A-module M is *coadmissible* if it is coadmissible as an A_H-module for some compact open subgroup H of G, in the sense of Definition 2.4.5. The full subcategory of coadmissible A-modules is denoted by $\mathcal{C}_A$.

Definition 2.8.19. Let C, D be Fréchet-Stein algebras. A Fréchet space P is a *C-coadmissible (C, D)-bimodule* if P is a coadmissible left C-module equipped with a continuous homomorphism $D^{\mathrm{op}} \to \mathrm{End}_C(P)$, where $\mathrm{End}_C(P)$ is a K-Fréchet space via the inverse limit topology in the category of locally convex vector spaces, via:

$$\mathrm{End}_C(P) \cong \varprojlim \mathrm{Hom}_{C_n}(C_n \otimes_C P, C_n \otimes_C P)$$

Suppose $(\mathbf{U}, H)$ is small and M is a coadmissible A-module. Then $\phi^H(\mathbf{U}) : A_H \to \widehat{\mathcal{D}}(\mathbf{U}, H)$ is a continuous homomorphism between two Fréchet-Stein algebras by Theorem 2.8.16, $\widehat{\mathcal{D}}(\mathbf{U}, H)$ is a $\widehat{\mathcal{D}}(\mathbf{U}, H)$-coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H) - A_H$-bimodule, and so we can form the coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module:

$$\widehat{\mathcal{D}}(\mathbf{U}, H)\widehat{\otimes}_{A_H} M := M(\mathbf{U}, H).$$

Definition 2.8.20. For $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, we set:

$$\mathcal{P}_X^A(M)(\mathbf{U}) := \varprojlim M(\mathbf{U}, H)$$

where H runs over the $\mathbf{U}$-small subgroups of G.

It turns out that each morphism in the inverse system defining $\mathcal{P}_X^A(M)(\mathbf{U})$ is an isomorphism, and so we have a bijection between $\mathcal{P}_X^A(M)(\mathbf{U})$ and $M(\mathbf{U}, H)$ for each H.

Theorem 2.8.21. *Let $\boldsymbol{U} \in \boldsymbol{X}_w(\mathcal{T})$. Then $\mathcal{P}_{\boldsymbol{X}}^A(M) \mid_{\boldsymbol{U}_w}$ is a sheaf on $\boldsymbol{U}_w$ with vanishing higher Čech cohomology, for any coadmissible A-module M.*

Proof. This is [1, Theorem 3.5.11]. □

Definition 2.8.22. $\mathrm{Loc}_{\mathbf{X}}^A(M)$ is the unique sheaf on $\mathbf{X}_{\mathrm{rig}}$ whose restriction to $\mathbf{X}_w$ is the presheaf $\mathcal{P}_{\mathbf{X}}^A(M)$.

Definition 2.8.23. A G-equivariant $\mathcal{D}$-module $\mathcal{M}$ on $\mathbf{U}_{\mathrm{rig}}$ is *locally Fréchet* if:

(a) $\mathcal{M}(\mathbf{U})$ is equipped with a Fréchet topology for every $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$,

(b) the maps $g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{M}(g\mathbf{U})$ are continuous for all $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and $g \in G$.

A *morphism* of G-equivariant locally Fréchet $\mathcal{D}$-modules is a morphism $f : \mathcal{M} \to \mathcal{N}$ of G-equivariant $\mathcal{D}$-modules, such that the induced maps $f(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{N}(\mathbf{U})$ are continuous for every $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$.

The category whose objects are G-equivariant locally Fréchet $\mathcal{D}$-modules and whose morphisms are continuous maps between them is denoted by $\mathrm{Frech}(G - \mathcal{D}) - \mathrm{mod}$.

Definition 2.8.24. Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering. A G-equivariant locally Fréchet $\mathcal{D}$-module $\mathcal{M}$ is $\mathcal{U}$ – *coadmissible* if, for all $\mathbf{U} \in \mathcal{U}$, there is a $\mathbf{U}$-small subgroup H, a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module M, and an isomorphism:

$$\mathrm{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H)}(M) \cong \mathcal{M}\mid_{\mathbf{U}\mathrm{rig}}$$

of H-equivariant locally Fréchet $\mathcal{D}$-modules on $\mathbf{U}$.

$\mathcal{M}$ is *coadmissible* if it is $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$.

The category of coadmissible G-equivariant $\mathcal{D}$-modules is the full subcategory of $\mathrm{Frech}(G - \mathcal{D}) - \mathrm{mod}$ whose objects are coadmissible G-equivariant $\mathcal{D}$-modules. It is denoted by $\mathcal{C}_{\mathbf{X}/G}$.

Theorem 2.8.25. *The category $\mathcal{C}_{X/G}$ is abelian.*

Proof. This is [1, Theorem 3.7.7]. □

Chapter 3

The nilpotent cone and the Springer resolution

3.1 Characteristic

In this chapter, we study the geometric structure of the nilpotent cone $\mathcal{N}$ of the Lie algebra $\mathfrak{g}$ of a reductive algebraic group G in arbitrary characteristic. We begin with a discussion of the ordinary nilpotent cone, defined as a subvariety of $\mathfrak{g}$, and then give a characterisation of the dual nilpotent cone $\mathcal{N}^*$.

Our treatment of the material on $\mathcal{N}$ is based on that of Jantzen in [39]. We generalise some of his arguments which are dependent on certain restrictions on the characteristic. Later, we will specialise further to the case $G = PGL_n$ and $p|n$ at certain points of the argument. The last section of the chapter discusses analogues of the results presented here when we consider a more general algebraic group G.

Let $\mathbf{G}$ be a split reductive algebraic group scheme, defined over $\mathbb{Z}$, and K an algebraically closed field of characteristic $p > 0$. Let $G := \mathbf{G}(K)$. Let $\mathfrak{g}$ denote the Lie algebra of G and $W(G)$ the Weyl group of G. When G is clear from context, we will abbreviate $W(G)$ to W. Since G is a linear algebraic group, we fix an embedding

$G \subseteq GL(V)$ for some n-dimensional K-vector space V.

Definition 3.1.1. Let α_i be the simple roots of the root system R of G, and let β be the highest-weight root. Writing $\beta = \sum_i m_i \alpha_i$, p is *bad* for G if $p = m_i$ for some i. p is *good* if p is not bad.

The prime p is *very good* if one of the following conditions hold:

(a) G is not of type A and p is good,

(b) G is of type A_n and p does not divide $n+1$.

In practice, we have the following classification. In types B, C and D, the only bad prime is 2. For the exceptional Lie algebras, the bad primes are 2 and 3 for types E_6, E_7, F_4 and G_2, and 2,3 and 5 for type E_8. In type A, there are no bad primes. For more details of this classification, see [59, I.4.3].

Definition 3.1.2. A prime p is *special* for G if the pair (Dynkin diagram of G, p) lies in the following list:

(a) (B, 2),

(b) (C, 2),

(c) (F_4, 2),

(d) (G_2, 3).

A prime p is *nonspecial* for G if it is not special.

This definition, and material on the importance of nonspecial primes, can be found in [50, Section 5.6].

3.2 The W-invariants of $S(\mathfrak{h})$

Let $G = PGL_n$ and suppose $p|n$. This short section investigates the structure of the invariants of the Weyl group action on the symmetric algebra $S(\mathfrak{h})$.

Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. Since G is of type A and the prime p is always good for G, there is a G-equivariant isomorphism $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ by the argument in [39, Section 6.5]. Since $\mathfrak{g}$ is a finite-dimensional vector space, we naturally identify the symmetric algebra $S(\mathfrak{g})$ and the algebra of polynomial functions $K[\mathfrak{g}^*]$.

Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. The Weyl group W has a natural action on $\mathfrak{h}$, which can be extended linearly to an action of W on the symmetric algebra $S(\mathfrak{h})$. The identification $S(\mathfrak{h}) \cong K[\mathfrak{h}^*]$ is compatible with the W-action. We begin this section by studying the W-invariants under this action.

Theorem 3.2.1. *Suppose $G = PGL_n$ and $p|n$. Then $S(\mathfrak{h})^W$ is a polynomial ring.*

Proof. Recall the Weyl group W is isomorphic to S_n, and let $\mathfrak{t}$ be the image of the diagonal matrices in $\mathfrak{g} = \mathfrak{pgl}_n$. Then $\mathfrak{t}$ is the quotient of the natural S_n-module V with basis $\{e_1, \cdots, e_n\}$, permuted by S_n, by the trivial submodule $U := K(\sum_{i=1}^n e_i)$. Let $X = V/U$. The quotient map $V \to X$ induces a surjective map $S(V) \to S(X)$.

Suppose $p = n = 2$ and let $\{\overline{e_1}, \overline{e_2}\}$ be the images of the vector space basis $\{e_1, e_2\}$ of V inside X. Let σ denote the non-identity element of S_2. Then $\sigma \cdot \overline{e_1} = \overline{e_2}$ and $\sigma \cdot \overline{e_2} = \overline{e_1}$. Since $\overline{e_1} + \overline{e_2} = 0$, it follows that $\overline{e_1} = \overline{e_2}$. Hence $S(X)^{S_2} = S(X)$, which is a polynomial ring.

Now suppose $n > 2$ and $p|n$. We claim that the S_n-action on V and on X is faithful. The S_n-action on V is by permutation and therefore is faithful. To see the claim for the S_n-action on X, let $N := \{g \in S_n \mid g \cdot x = x \ \forall x \in S(X)\}$ denote the kernel of the natural map $S_n \to S(X)$.

Suppose g is some non-identity element of N. Then, relabelling the elements $\overline{e_i}$ if necessary, $g \cdot \overline{e_1} = \overline{e_2}$. Hence it suffices to show that $\overline{e_1} \neq \overline{e_2}$. If $\overline{e_1} = \overline{e_2}$, then since $\sum_{i=1}^{n} \overline{e_i} = 0$, $\sum_{i=2}^{n} \overline{e_i} = \overline{e_1}$ and $e_1 + \sum_{i=3}^{n} \overline{e_i} = \overline{e_2}$. Rearranging, $\sum_{i=3}^{n} \overline{e_i} = (p-2)\overline{e_1}$. Hence the set $\{\overline{e_1}, \overline{e_3}, \cdots, \overline{e_{n-1}}\}$ spans X, but X is an $(n-1)$-dimensional vector space, a contradiction. It follows that the S_n-action on X is faithful.

The ring of invariants $S(V)^{S_n}$ is generated by the elementary symmetric polynomials $s_1(e_1, \cdots, e_n), \cdots, s_n(e_1, \cdots, e_n)$, which are algebraically independent by [14, Section 6, Theorem 1]. Applying [48, Proposition 4.1], we see that $S(X)^{S_n}$ is also a polynomial ring. The proof of [42, Proposition 5.1] also demonstrates that $S(X)^{S_n}$ is generated by the images of $s_2(e_1, \cdots, e_n), \cdots, s_n(e_1, \cdots, e_n)$ under the map $S(V) \to S(X)$.

To finish the proof, it suffices to note that we may identify $\mathfrak{t} \cong \mathfrak{h}$ and that $\mathfrak{h} \cong$

$V/U = X$. □

We state a version of Kostant's freeness theorem that will be useful for our applications.

Theorem 3.2.2. *$S(\mathfrak{h})$ is a free $S(\mathfrak{h})^W$-module if and only if $S(\mathfrak{h})^W$ is a polynomial ring.*

Proof. See [56, Corollary 6.7.13]. □

3.3 Properties of the nilpotent cone

We now outline some general preliminaries on the structure theory of groups acting on varieties. At first, we do not impose any restriction on the characteristic.

Let M be a variety which admits an algebraic group action by G, and let $x \in M$. The closure $\overline{Gx}$ of the orbit Gx of x is a closed subvariety of M. By [36, Proposition 8.3], Gx is open in $\overline{Gx}$ and so Gx has the structure of an algebraic variety.

The orbit map $\pi_x : G \to Gx$, $\pi_x(g) = gx$, is a surjective morphism of varieties. The stabiliser $G_x := \{g \in G \mid gx = x\}$ is a closed subgroup of G, and π_x induces a bijective morphism:

$$\overline{\pi_x} : G/G_x \to Gx$$

by [36, Section 12].

We now specialise to the case where $M = \mathfrak{g}$ and G acts on $\mathfrak{g}$ via the adjoint action. Let $X \in \mathfrak{g}$ and let GX denote the G-orbit of X under the adjoint action $\mathrm{Ad} : G \to \mathrm{Ad}(\mathfrak{g})$.

Recall that an element $g \in \mathfrak{g}$ is *nilpotent* if the operator $\mathrm{ad}_g(y)$ is nilpotent for each $y \in \mathfrak{g}$. The set of nilpotent elements is denoted $\mathcal{N}$.

Since G is a linear algebraic group, fix an embedding $G \subseteq GL(V)$ for some n-dimensional K-vector space V. Then $\mathcal{N} = \mathfrak{g} \cap \mathcal{N}(\mathfrak{gl}(V))$, where $\mathcal{N}(\mathfrak{gl}(V))$ denotes the set of nilpotent elements of the Lie algebra of $GL(V)$. It follows that $\mathcal{N}$ is closed in $\mathfrak{g}$, and hence $\mathcal{N}$ has the structure of a subvariety of the algebraic variety $\mathfrak{g}$.

Let:

$$P_X(t) := \det(tI - X)$$

denote the characteristic polynomial of X in the variable t. Then:

$$P_X(t) := t^n + \sum_{i=1}^{n} (-1)^i s_i(X) t^{n-i}$$

where each s_i is a homogeneous polynomial of degree i in the entries of X. If $a_1, \cdots, a_n$ are the eigenvalues of X, counted with algebraic multiplicity, then, since K is algebraically closed, $P_X(t) = \prod_{i=1}^{n}(t - a_i)$, and so $s_i(X)$ can be identified with the ith elementary symmetric function in the a_j. It follows that X is nilpotent if and only if $P_X(t) = t^n$ if and only if $s_i(X) = 0$ for each i:

$$\mathcal{N}(\mathfrak{gl}(V)) = \{X \in \mathfrak{gl}(V) \mid s_i(X) = 0 \text{ for all } i\}.$$

Let $S(V)$ denote the algebra of polynomial functions on V. This has a natural grading by degree, with $S(V) = \bigoplus_{i \geq 0} S^i(V)$. Set $S^+(V) := \bigoplus_{i \geq 1} S^i(V)$.

Now the restrictions of the s_i to $\mathfrak{g}$ are G-invariant polynomial functions on $\mathfrak{g}$, and so $s_{i|\mathfrak{g}} \in S^i(\mathfrak{g}^*)^G$. It follows that there exist $f_1, \cdots, f_n \in S^+(\mathfrak{g}^*)^G$ such that:

$$\mathcal{N} = \{X \in \mathfrak{g} \mid f_i(X) = 0 \text{ for all } i\}.$$

Proposition 3.3.1. The nilpotent cone $\mathcal{N}$ may be realised as:

$$\mathcal{N} = \{X \in \mathfrak{g} \mid f(X) = 0 \text{ for all } f \in S^+(\mathfrak{g}^*)^G\}.$$

Hence $\mathcal{N} = V(S^+(\mathfrak{g}^*)^G)$ is an affine variety.

Proof. It is clear that $\{X \in \mathfrak{g} \mid f(X) = 0 \text{ for all } f \in S^+(\mathfrak{g}^*)^G\} \subseteq \mathcal{N}$ by the above discussion. Conversely, given $f \in S^+(\mathfrak{g}^*)^G$, $f(0) = 0$ and f is constant on the closure of the orbits under the adjoint action. Then f is constant on $\overline{GX}$, the closure of the regular orbit under the adjoint action, and $0 \in \overline{GX}$ by [39, Proposition 2.11(1)]. □

Lemma 3.3.2. Let $\mathcal{B}$ be the set of all Borel subalgebras of $\mathfrak{g}$. Then there is a bijection $G/B \leftrightarrow \mathcal{B}$.

Proof. $\mathcal{B}$ is the closed subvariety of the Grassmannian of dim $\mathfrak{b}$-dimensional subspaces in $\mathfrak{g}$ formed by solvable Lie algebras. Hence $\mathcal{B}$ is a projective variety. All Borel subalgebras are conjugate under the adjoint action of G, and the stabiliser subgroup $G_{\mathfrak{b}}$ of $\mathfrak{b}$ in G is equal to B by [11, Theorem 11.16]. Hence the claimed bijection follows via the assignment $g \mapsto g \cdot \mathfrak{b} \cdot g^{-1}$. □

Definition 3.3.3. Set $\widetilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$, and let $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the projection onto the first coordinate. The *enhanced nilpotent cone* is the preimage of $\mathcal{N}$ under the map μ:

$$\widetilde{\mathcal{N}} := \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

Lemma 3.3.4. $\widetilde{\mathcal{N}}$ is a smooth irreducible variety.

Proof. Let $\mathfrak{b} \in \mathcal{B}$ be a fixed Borel subalgebra. The fibre over $\mathfrak{b}$ of the second projection $\pi : \widetilde{\mathcal{N}} \to \mathcal{B}$ is the set of nilpotent elements of $\mathfrak{b}$. Decomposing $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$, an element $x \in \mathfrak{b}$ is nilpotent if and only if it is has no component in the Cartan subalgebra $\mathfrak{h}$. Hence π makes $\widetilde{\mathcal{N}}$ a vector bundle over $\mathcal{B}$ with fibre $\mathfrak{n}$.

The canonical map $G \to G/B$ is locally trivial by [38, II.1.10(2)], so the set of B-orbits on $G \times \mathfrak{n}$ has a natural structure of a variety, denoted $G \times_B \mathfrak{n}$. The above construction yields a G-equivariant vector bundle isomorphism:

$$\widetilde{\mathcal{N}} \cong G \times_B \mathfrak{n},$$

where B is the Borel subgroup of G corresponding to $\mathfrak{b}$. It follows that we may view $\widetilde{\mathcal{N}}$ as a vector bundle over the smooth variety G/B, and so $\widetilde{\mathcal{N}}$ is smooth.

Using Lemma 3.3.2, identify $\mathcal{B}$ with G/B and consider the morphism $f : \mathfrak{g} \times G \to \mathfrak{g} \times \mathcal{B}$ defined by $f(x, g) = (x, gB)$. The inverse image:

$$f^{-1}(\widetilde{\mathcal{N}}) = \{(x, g) \in \mathcal{N} \times G \mid \mathrm{Ad}(g^{-1})(x) \in \mathfrak{n}\}$$

is closed in $\mathfrak{g} \times G$ since it is the inverse image of $\mathfrak{n}$ under the natural map $\mathfrak{g} \times G \to \mathfrak{g}$, $(x, g) \mapsto \mathrm{Ad}(g^{-1})(x)$. Since f is an open map and $f^{-1}(\widetilde{\mathcal{N}})$ is closed, $\widetilde{\mathcal{N}}$ is a closed subvariety of $\mathcal{N} \times \mathcal{B}$.

The morphism $\mathfrak{n} \times G \to \widetilde{\mathcal{N}}$, $(x, g) \mapsto (\mathrm{ad}(g)(x), gB)$ is surjective by definition. Hence $\widetilde{\mathcal{N}}$ is irreducible. □

By [39, Theorem 2.8(1)], there are only finitely many orbits for the G-action in the nilpotent cone $\mathcal{N}$. Let $X_1, \cdots, X_r$ be representatives for these orbits. Then:

$$\mathcal{N} = \bigcup_{i=1}^{r} \overline{\mathcal{O}_{X_i}}$$

Since $\mathcal{N}$ is irreducible by Lemma 3.3.4, one of these closed subsets must be all of $\mathcal{N}$: let $\overline{GZ} = \mathcal{N}$. Then, by [60, 1.13, Corollary 1], this orbit is open in $\mathcal{N}$ and $\dim(GZ) = \dim(\mathcal{N})$, while $\dim(GY) < \dim(GZ)$ for any $GY \neq GZ$. Hence GZ is unique with respect to this property.

Definition 3.3.5. An element $X \in \mathfrak{g}$ is *regular* if it lies in GZ, the unique open dense G-orbit of $\mathcal{N}$.

We now specialise to the case where $G = PGL_n$ and $p|n$.

Lemma 3.3.6. There is a natural G-equivariant vector bundle isomorphism:

$$\widetilde{\mathcal{N}} \cong T^*\mathcal{B}.$$

Proof. This follows from [39, Section 6.5]. □

Definition 3.3.7. The map $\mu : T^*\mathcal{B} \to \mathcal{N}$ is the *Springer resolution* for the nilpotent cone $\mathcal{N}$.

Lemma 3.3.8. Let $\mathcal{N}_s$ denote the set of smooth points of $\mathcal{N}$. Then $\mu^{-1}(\mathcal{N}_s)$ is dense in $\widetilde{\mathcal{N}}$.

Proof. $\mathcal{N}_s$ is an open and non-empty subset of $\mathcal{N}$. Hence it is dense, and its preimage is open and non-empty in $\widetilde{\mathcal{N}}$. By Lemma 3.3.4, $\widetilde{\mathcal{N}}$ is irreducible and so $\mu^{-1}(\mathcal{N}_s)$ is dense. □

Lemma 3.3.9. Let GZ denote the orbit of all regular nilpotent elements. The morphism $\mu^{-1}(GZ) \to GZ$ is an isomorphism of varieties.

Proof. By [39, Corollary 6.8], GZ is an open subset of $\mathcal{N}$, and $|\mu^{-1}(X)| = 1$ for $X \in GZ$. Hence μ induces a bijection $\mu^{-1}(GZ) \to GZ$. Since the morphism $\mu : T^*\mathcal{B} \to \mathcal{N}$ is given by projection onto the first coordinate, from Definition 3.3.3, it is a morphism of varieties and hence so is the restriction $\mu\mid_{\mu^{-1}(GZ)}: \mu^{-1}(GZ) \to GZ$. The result follows. □

Recall from Theorem 3.2.1 that $S(\mathfrak{h})^W$ is a polynomial ring, with algebraically independent generators $f_1, \cdots, f_n$.

Theorem 3.3.10. *Let G be a simple algebraic group, and suppose $(G, p) \neq (B, 2)$. There is a projection map $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n} \to \mathfrak{h}$, which induces a map $S(\mathfrak{g}) \to S(\mathfrak{h})$.*

This map induces a map $\eta : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$, which is an isomorphism.

Proof. This is [40, Theorem 4]. □

When $G = PGL_n$ and $p|n$, the hypotheses of Theorem 3.3.10 are satisfied. This allows us to make sense of the following definition.

Definition 3.3.11. The *Steinberg quotient* is the map $\chi : \mathfrak{g} \to K^n$ defined by $\chi(Z) = (\eta^{-1}(f_1)(Z), \cdots, \eta^{-1}(f_n)(Z))$. Note that the nilpotent cone $\mathcal{N} = \chi^{-1}(0)$.

Lemma 3.3.12. The smooth points of $\mathcal{N}$ are precisely the regular nilpotent elements.

Proof. By the assumptions on the prime p, applying Theorem 3.2.1 and Theorem 3.2.2 shows that $S(\mathfrak{h})$ is a free $S(\mathfrak{h})^W$-module and $S(\mathfrak{h})^W$ is a polynomial ring, with generators $f_1, \cdots, f_n$. Hence the argument for [17, Claim 6.7.10] applies and the Steinberg quotient χ satisfies, for $Z \in \mathfrak{g}$, the condition that $(d\chi)_Z$ is surjective if and only if Z is regular. By [39, Proposition 7.11], for each $b = (b_1, \cdots, b_n) \in K^n$, the ideal of $\chi^{-1}(b)$ is generated by all $\eta^{-1}(f_i) - b_i$.

By [29, I.5], $Z \in \chi^{-1}(b)$ is a smooth point if and only if the $d(\eta(f_i) - b_i)$ are linearly independent at Z if and only if the map $(d\chi)_Z$ is surjective. Let $b = 0$. Then the smooth points in $\chi^{-1}(0)$ are the regular elements contained in $\chi^{-1}(0)$, and so the smooth points of $\mathcal{N}$ are precisely the regular nilpotent elements. □

Theorem 3.3.13. *$\mu : T^*\mathcal{B} \to \mathcal{N}$ is a resolution of singularities for $\mathcal{N}$.*

Proof. By Lemma 3.3.4 and Lemma 3.3.6, $\widetilde{\mathcal{N}}$ is a smooth irreducible variety. Furthermore, μ is proper by [39, Lemma 6.10(1)]. By Lemma 3.3.8, $\mu^{-1}(\mathcal{N}_s)$ is dense in $\widetilde{\mathcal{N}}$, and by Lemma 3.3.9, μ is a birational morphism between $\mu^{-1}(\mathcal{N}_s)$ and $\mathcal{N}_s$. Hence μ is a resolution of singularities. □

3.4 The dual nilpotent cone is a normal variety

In this section, we demonstrate that the dual nilcone $\mathcal{N}^*$ is a normal variety in the case $G = PGL_n$, $p|n$.

Definition 3.4.1. Since we have a G-equivariant isomorphism $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ by [39, Section 6.5], the *dual nilcone* $\mathcal{N}^*$ may be defined as:

$$\mathcal{N}^* = \{X \in \mathfrak{g}^* \mid f(X) = 0 \text{ for all } f \in S^+(\mathfrak{g})^G\}.$$

The same argument as in Proposition 3.3.1 shows that $\mathcal{N} = V(S^+(\mathfrak{g})^G)$ is an affine variety.

We next review some basic properties of normal rings and varieties.

Definition 3.4.2. [17, Definition 2.2.12] A finitely generated commutative K-algebra A is *Cohen-Macaulay* if it contains a subalgebra of the form $\mathcal{O}(V)$ such that A is a free $\mathcal{O}(V)$-module of finite rank, and V is a smooth affine scheme.

A scheme X defined over K is *Cohen-Macaulay* if, at each point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a Cohen-Macaulay ring.

Definition 3.4.3. A commutative ring A is *normal* if the localization $A_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ is an integrally closed domain.

A variety V is *normal* if, for any $x \in V$, the local ring $\mathcal{O}_{V,x}$ is a normal ring.

We now begin the proof of the normality of the dual nilpotent cone $\mathcal{N}^*$. We adapt the arguments in [9] to our situation.

Theorem 3.4.4. *Let X be an irreducible affine Cohen-Macaulay scheme defined over K and $U \subseteq X$ an open subscheme. Suppose dim $(X/U) \leq$ dim $X - 2$ and that the scheme U is normal. Then the scheme X is normal.*

Proof. This is [9, Corollary 2.3]. □

We aim to apply Theorem 3.4.4 to our situation. We begin with the following lemma, a variant on Hartogs' lemma.

Lemma 3.4.5. Let Y be an affine normal variety and $Z \subseteq Y$ be a subvariety of codimension at least 2. Then any rational function on Y which is regular on $Y \setminus Z$ can be extended to a regular function on Y.

Proof. Write $Y = \text{Spec } B$, where B is a normal domain. Set $Z := V(I)$ for some ideal I, and write $U := Y \setminus Z$. Then $U = \bigcup_{f \in I} D(f)$, where $D(f)$ denotes the basic open sets in the Zariski topology.

Let $\mathfrak{p}$ be a prime ideal of height 1. By assumption, ht $I \geq 2$, and so there exists some $f \in I$ with $f \notin \mathfrak{p}$. It follows that $B_f \subseteq B_{\mathfrak{p}}$.

Let a/b be a regular function on U, with $a/b \in \mathrm{Frac}B$, the field of fractions of B. Since $\mathfrak{p}$ has height 1, we can find $f \in I \setminus \mathfrak{p}$. Then a/b is regular on $D(f)$, and so $a/b \in \mathcal{O}(D(f)) = B_f \subseteq B_{\mathfrak{p}}$. As $\mathfrak{p}$ was arbitrary, $a/b \in \bigcap_{\mathrm{ht}\mathfrak{p}=1} B_{\mathfrak{p}} = B$. Hence a/b can be extended to a regular function on Y. □

Lemma 3.4.6. Let X be an affine Cohen-Macaulay scheme with an open subscheme U. Let $r : \mathcal{O}(X) \to \mathcal{O}(U)$ be the restriction morphism. Then:

(a) if dim $(X \setminus U) <$ dim X, then r is injective,

(b) if dim $(X \setminus U) \leq$ dim $X - 2$, then r is an isomorphism.

Proof. We expand on the proof given in [9, Lemma 2.2]. For ease of notation, we suppose $\mathcal{O}(X)$ is a finitely generated $\mathcal{O}(Y)$-module for some smooth affine scheme Y. Now the projection map $p : X \to Y$ is a finite morphism and hence is closed. Without loss of generality, we can shrink U, replacing it by a smaller open subset $p^{-1}(W)$, where $W = Y \setminus p(X \setminus U)$ is an open subset of Y.

Let $F := p_*(\mathcal{O}_X)$. This is a free $\mathcal{O}_Y$-module and we clearly have $\Gamma(Y, F) = p_*(\mathcal{O}_X)(Y) = \mathcal{O}_X(p^{-1}(Y)) = \mathcal{O}_X(X)$, and similarly $\Gamma(W, F) = p_*(\mathcal{O}_X)(W) = \mathcal{O}_X(p^{-1}(W)) = \mathcal{O}_X(U)$. Hence the restriction morphism r agrees with the natural restriction map $r : \Gamma(Y, F) \to \Gamma(W, F)$.

If dim $(X \setminus U) <$ dim X, then dim $(p(U \setminus X)) <$ dim $(p(X))$, so dim $(Y \setminus W) <$ dim Y, and so r is injective.

Similarly, if dim $(X \setminus U) \leq$ dim $X - 2$, then dim $(Y \setminus W) \leq$ dim $Y - 2$. Hence, by Lemma 3.4.5, any regular function on W can be extended to a regular function on Y. Furthermore, F is a free $\mathcal{O}_Y$-module; it follows that r is surjective. $\square$

As an immediate consequence, we see that if the scheme U is reduced and normal, then so is X.

We now demonstrate that the hypotheses in Theorem 3.4.4 are satisfied in our situation. Recall that $\mathcal{N}^*$ is an affine variety. It suffices to show that $\mathcal{N}^*$ is irreducible and Cohen-Macaulay.

Definition 3.4.7. $\lambda \in \mathfrak{h}^*$ is *regular* if its centraliser in $\mathfrak{g}$ under the natural $\mathfrak{g}$-action on $\mathfrak{g}^*$ coincides with the Cartan subalgebra $\mathfrak{h}$. A general $\lambda \in \mathfrak{g}^*$ is *regular* if its coadjoint orbit contains a regular element of $\mathfrak{h}^*$.

The subvariety U in Lemma 3.4.6 will be taken to be the subset of regular nilpotent elements.

Proposition 3.4.8. Suppose p is nonspecial for G. Then:

(a) the dual nilcone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a closed irreducible subvariety of $\mathfrak{g}^*$, and it has codimension r in G, where r is the rank of G.

(b) Let U denote the set of regular elements of $\mathcal{N}^*$. Then U is a single coadjoint orbit, which is open in $\mathcal{N}^*$, and its complement has codimension ≥ 2.

Proof. (a) We define an auxiliary variety S via:

$$S := \{(gB, \zeta) \in G/B \times \mathfrak{g}^* \mid g \cdot \zeta \in \mathfrak{b}^\perp\}.$$

This subset of $G/B \times \mathfrak{g}^*$ is closed. Define a map $\phi : G \times \mathfrak{b}^\perp \to G/B \times \mathfrak{g}^*$ by $\phi(g, \zeta) = (gB, g^{-1} \cdot \zeta)$. Now the image of ϕ is contained in S, and we can also see that $\mathrm{im}(\phi) \cong S$ since we have a linear isomorphism $\mathfrak{b}^\perp \to g^{-1} \cdot \mathfrak{b}^\perp$. Hence the image of ϕ coincides with S. It follows that S is a morphic image of an irreducible variety, and hence S is itself an irreducible subvariety.

Let $p_1 : G/B \times \mathfrak{g}^* \to G/B$ and $p_2 : G/B \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the obvious projection maps. Clearly $p_1(S) = G/B$. The fiber of gB under the map p_1 is $g^{-1} \cdot \mathfrak{n}$, which is isomorphic to $\mathfrak{n}$. Hence the fibers are equidimensional, and we have:

$$\dim\, S = \dim\, G/B + \dim\, \mathfrak{n},$$

$$\dim\, S = \dim\, G/B + \dim\, U$$

$$= \dim\, G - r.$$

Using the second projection, $\dim\, p_2(S) \leq \dim\, G - r$, with equality if some fibre is finite (as a set). First notice that:

$$p_2(S) = \{\zeta \in \mathfrak{g}^* \mid \exists g \in G \text{ s.t. } g \cdot \zeta \in \mathfrak{b}^\perp\} = \mathcal{N}^*.$$

Hence $\mathcal{N}^*$ is irreducible, and, since the flag variety G/B is complete by [11], $\mathcal{N}^*$ is closed. We show that there exists some $\zeta \in \mathfrak{g}^*$ with:

$$|\{gB \mid g \cdot \zeta \in \mathfrak{b}^\perp\}| < \infty,$$
$$|\{gB \mid \zeta(\mathrm{Ad}_g^{-1}(\mathfrak{b})) = 0\}| < \infty.$$

By [31, Proposition 2], we have the following dimension formula:

$$\dim\, p_1(p_2^{-1}(\zeta)) = \frac{\dim\, Z_G(\zeta) - r}{2}.$$

Since p is nonspecial for G, the set of regular nilpotent elements U in $\mathcal{N}^*$ is nonempty, by [26, Section 6.4], and thus we can always pick some $\zeta \in \mathfrak{g}^*$ such that $\dim\, Z_G(\zeta) - r = 0$. Thus there exists ζ with $|\{p_1(p_2^{-1}(\zeta))\}| < \infty$.

Now consider two points $(gB, \zeta), (hB, \zeta) \in S$. By definition, $g{\cdot}\zeta \in \mathfrak{b}^\perp$ and $h{\cdot}\zeta \in \mathfrak{b}^\perp$. The coadjoint action then gives $\zeta(\mathrm{ad}_g^{-1}(\mathfrak{b})) = \zeta(\mathrm{ad}_h^{-1}(\mathfrak{b})) = 0$. It follows that $gB = hB$, and so p_1 is injective when restricted to the fibre $p_2^{-1}(\zeta)$. It follows that there is a fibre of p_2 which is finite as a set.

Given the existence of a finite fibre of p_2, we have $\dim\, S = \dim\, p_2(S) = \dim\, \mathcal{N}^* = \dim\, G - r$.

(b) Now $\mathcal{N}^*$ has only finitely many G-orbits by [63] and [62, Proposition 7.1], so the dimension of $\mathcal{N}^*$ is equal to the dimension of at least one of these orbits. Since dim $\mathcal{N}^* = \dim G - r$, some orbit in $\mathcal{N}^*$ also has dimension equal to dim $G - r$. This orbit is regular and its closure is all of $\mathcal{N}^*$, since the dimensions are equal and $\mathcal{N}^*$ is irreducible. Since any G-orbit is open in its closure, by [60, 1.13, Corollary 1], this class is open in $\mathcal{N}^*$ and thus is dense.

Let R be the root system of G and fix a subset of positive roots $R^+ \subseteq R$. Let α_i be a simple root, X_α the corresponding root subgroup, and set $U_i := \prod_{\alpha \in R^+, \alpha \neq \alpha_i} X_\alpha$. Let T be the maximal torus of G defined by this root system and let $P_i := T \cdot \langle X_{\alpha_i}, X_{-\alpha_i} \rangle \cdot U_i$. Since both T and $\langle X_{\alpha_i}, X_{-\alpha_i} \rangle$ normalise U_i by the commutation formulae in [60, 3.7], we see that P_i is a rank 1 parabolic subgroup of G, U_i is its unipotent radical and $T \cdot \langle X_{\alpha_i}, X_{-\alpha_i} \rangle$ is a Levi subgroup of P_i.

Note that dim $T \cdot \langle X_{\alpha_i}, X_{-\alpha_i} \rangle = r + 2$ and so dim $P_i -$ dim $U_i = r + 2$.

Parallel to the definition of the variety S, we set:

$$S_i := \{(gP_i, \zeta) \in G/P_i \times \mathfrak{g}^* \mid g \cdot \zeta \in \mathfrak{b}_i^\perp\}$$

where $\mathfrak{b}_i^\perp = \{\zeta \in \mathfrak{g}^* \mid \zeta(\mathrm{Lie}(U_i T)) = 0\}$. Then S_i is a closed and irreducible variety and, by the same argument as in part (a) of the proposition:

$$\dim S_i = \dim G/P_i + \dim U_i$$
$$= \dim G - (r+2).$$

Projecting onto the second factor, we see that:

$$\dim p_2(S_i) \leq \dim S_i = \dim G - (r+2).$$

But an element $\zeta \in \mathcal{N}^*$ fails to be regular if and only if $G \cdot \zeta \cap \mathfrak{h}^*_{\text{reg}} = \emptyset$. By the decomposition in [26, Section 6.4], this occurs precisely when the centraliser of each $\xi \in G \cdot \zeta \cap \mathfrak{h}^*$ contains some non-zero root α such that $\xi(\alpha^\vee(1)) = 0$, where $\alpha^\vee$ is the coroot corresponding to α. It follows that $\zeta \in \mathcal{N}^*$ fails to be regular if and only if it lies in $p_2(S_i)$ for some i. Then:

$$\dim (\mathcal{N}^* \setminus U) = \sup_i \dim p_2(S_i) \leq \dim G - (r+2).$$

□

Lemma 3.4.9. Let $r : S(\mathfrak{g}) \to S(\mathfrak{h})$ be the natural map, and r' its restriction to the graded subalgebra $S(\mathfrak{g})^G$. Suppose that r' is an isomorphism onto its image $S(\mathfrak{h})^W$ and $S(\mathfrak{h})$ is a free $S(\mathfrak{h})^W$-module. Then $S(\mathfrak{g})$ is a free R-module, where $R := S(\mathfrak{g}/\mathfrak{h}) \otimes S(\mathfrak{g})^G$, and hence is a free $S(\mathfrak{g})^G$-module.

Proof. The argument is similar to that which is set out in [17, 2.2.12] and the following discussion. Consider the projection map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. This makes $\mathfrak{g}$ a vector bundle over $\mathfrak{g}/\mathfrak{h}$, and defines a natural increasing filtration on $S(\mathfrak{g})$ via:

$$F_pS(\mathfrak{g}) = \{P \in S(\mathfrak{g}) \mid P \text{ has degree } \leq p \text{ along the fibers}\}.$$

Let $\mathrm{gr}_F(S(\mathfrak{g}))$ denote the associated graded ring corresponding to this filtration, and set $S(\mathfrak{g})(p)$ to denote the p-th graded component. Clearly $S(\mathfrak{g})(0) = S(\mathfrak{g}/\mathfrak{h})$, and each graded component is an infinite-dimensional free $S(\mathfrak{g}/\mathfrak{h})$-module. There is a K-algebra isomorphism:

$$S(\mathfrak{g})(p) \cong S(\mathfrak{g}/\mathfrak{h}) \otimes_K S^p(\mathfrak{h}),$$

where $S^p(\mathfrak{h})$ denotes the space of degree p homogeneous polynomials on $\mathfrak{h}$.

Let $\sigma_p : F_pS(\mathfrak{g}) \to S(\mathfrak{g})(p)$ be the principal symbol map. Suppose $f \in F_pS(\mathfrak{g})$ is a homogeneous degree p polynomial whose restriction $r(f)$ to $\mathfrak{h}$ is non-zero. Then $\sigma_p(f)$ equals the image of the element $1 \otimes_k r(f)$ under the above isomorphism, and so is non-zero in $S(\mathfrak{g})(p)$.

To see this, choose a vector subspace $\mathfrak{j}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$. This yields a graded algebra isomorphism $S(\mathfrak{g}) = S(\mathfrak{h}) \otimes S(\mathfrak{j})$, and so one writes $F_pS(\mathfrak{g}) = \sum_{i \leq p} S^i(\mathfrak{h}) \otimes S^{p-i}(\mathfrak{j})$. Hence $f \in F_pS(\mathfrak{g})$ has the form:

$$f = e_p \otimes 1 + \sum_{i \leq p} e_i \otimes w_{p-i},$$

where $e_i \in S^i(\mathfrak{h})$ and $w_{p-i} \in S^{p-i}(\mathfrak{j})$. Hence $r(f) = e_p$ and $\sigma_p(f) = e_p \otimes 1$, as required.

Given this claim, consider the filtration $F^pS(\mathfrak{g})^G$ in $S(\mathfrak{g})$. For any homogeneous element $f \in S(\mathfrak{g})^G$, its symbol $\sigma_p(f)$ coincides with $r(f) \in S(\mathfrak{h}) \subseteq \mathrm{gr}_\Phi(S(\mathfrak{g}))$. Hence the subalgebra $\sigma_\Phi(f) \subseteq \mathrm{gr}_\Phi(S(\mathfrak{g}))$ coincides with $r(S(\mathfrak{g})^G) = S(\mathfrak{h})^W$.

Let $\{a_k\}$ be a free basis for the $S(\mathfrak{h})^W$-module $S(\mathfrak{h})$, and fix $b_k \in S(\mathfrak{g})$ with $r(b_k) = a_k$. Then $\sigma_p(b_k) = a_k$. The a_k form a free basis of the $\mathrm{gr}_p R$-module $\mathrm{gr}_p(S(\mathfrak{g})) = S(\mathfrak{h}) \otimes S(\mathfrak{g}/\mathfrak{h})$, via tensoring on the right and applying the second part of the claim. It follows that the $\{b_k\}$ form a free basis of the R-module $S(\mathfrak{g})$. ☐

Theorem 3.4.10. *Let $G = PGL_n$ and suppose $p|n$. Then the dual nilpotent cone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a normal variety.*

Proof. Recall that $\mathcal{N}^*$ is an affine variety with defining ideal $J := V(S^+(\mathfrak{g})^G))$. It follows that its algebra of global functions $\mathcal{O}(\mathcal{N}^*) = S(\mathfrak{g})/J$. Consider $Y := \mathfrak{g}/\mathfrak{h}$ as an affine variety. Then Lemma 3.4.9 implies that $\mathcal{O}(\mathcal{N}^*)$ is a free finitely generated module over the polynomial algebra $S(Y)$. Hence $\mathcal{N}^*$ is a Cohen-Macaulay variety.

By Proposition 3.4.8, $\mathcal{N}^*$ is a closed irreducible subvariety of $\mathfrak{g}^*$, and the complement of the set of regular elements U in $\mathcal{N}^*$ has codimension ≥ 2. Hence all conditions in the statement of Theorem 3.4.4 are satisfied, and so $\mathcal{N}^*$ is normal. ☐

Proof of Theorem A: This is immediate from Theorem 3.4.10.

We conclude this section with an application of this result, which will be used in later sections.

Corollary 3.4.11. We have an isomorphism $\mu^* : \mathcal{O}(\mathcal{N}^*) \to \mathcal{O}(T^*\mathcal{B})$.

Proof. The map $\mu : T^*\mathcal{B} \to \mathcal{N}$ is a resolution of singularities by Theorem 3.3.13. Let $\tau : T^*\mathcal{B} \to \mathcal{N}^*$ be the composition of μ with the G-equivariant isomorphism $\kappa : \mathcal{N} \to \mathcal{N}^*$ from [39, Section 6.5]. This induces an isomorphism $\tau^s : \tau^{-1}((\mathcal{N}^*)^s) \to (\mathcal{N}^*)^s$ on the smooth points. These are non-empty open subsets of $T^*\mathcal{B}$ and $\mathcal{N}^*$ respectively, and so $T^*\mathcal{B}$ and $\mathcal{N}^*$ are birationally equivalent.

Let $\mathcal{Q}(A)$ denote the field of fractions of an integral domain A. By [29, I.4.5], μ induces an isomorphism $\mathcal{Q}(\mathcal{O}(\mathcal{N}^*)) \to \mathcal{Q}(\mathcal{O}(T^*\mathcal{B}))$, and so $\mathcal{O}(T^*\mathcal{B})$ can be considered as a subring of $\mathcal{Q}(\mathcal{O}(\mathcal{N}^*))$.

Since the map $T^*\mathcal{B} \to \mathcal{N}^*$ is surjective, and $\mathcal{O}(T^*\mathcal{B})$, $\mathcal{O}(\mathcal{N}^*)$ are integral domains, there is an inclusion $\mathcal{O}(\mathcal{N}^*) \to \mathcal{O}(T^*\mathcal{B})$. The map τ is proper, and so the direct image sheaf $\tau_*\mathcal{O}_{T^*\mathcal{B}}$ is a coherent $\mathcal{O}_{\mathcal{N}^*}$-module. In particular, taking global sections, we have that $\Gamma(\mathcal{N}^*, \tau_*\mathcal{O}_{T^*\mathcal{B}})$ is a finitely generated $\mathcal{O}(\mathcal{N}^*)$-module. By definition, $\Gamma(\mathcal{N}^*, \tau_*\mathcal{O}_{T^*\mathcal{B}}) = \mathcal{O}(T^*\mathcal{B})$, so $\mathcal{O}(T^*\mathcal{B})$ is a finitely generated $\mathcal{O}(\mathcal{N}^*)$-module.

The variety $\mathcal{N}^*$ is normal, and so $\mathcal{O}(\mathcal{N}^*)$ is an integrally closed domain. Let $b \in \mathcal{O}(T^*\mathcal{B})$. Then clearly $\mathcal{O}(T^*\mathcal{B})b \subseteq \mathcal{O}(T^*\mathcal{B})$, and hence b is integral over $\mathcal{O}(\mathcal{N}^*)$. Hence, by integral closure, $b \in \mathcal{N}^*$ and there is an isomorphism $\mathcal{O}(\mathcal{N}^*) \to \mathcal{O}(T^*\mathcal{B})$.

□

3.5 Analogous results when G is not of type A

The restriction that $G = PGL_n$, $p|n$ plays a role in only a few places in the argument that $\mathcal{N}^*$ is a normal variety. In this section, we indicate some of the issues that arise when we replace PGL_n by a more general simple algebraic group of adjoint type.

Theorem 3.2.1 demonstrated that, in case $G = PGL_n$, $p|n$, the Weyl group invariants $S(\mathfrak{h})^W$ is a polynomial ring. This result is usually false in bad characteristic. In case the W-action on $\mathfrak{h}$ is irreducible, [16, Theorem 3] gives a full classification of the types in which this result holds, drawing on [42, Theorem 7.2].

Proposition 3.5.1. Suppose the pair (Dynkin diagram of G, p) lies in the following list:

(a) $(E_7, 3)$,

(b) $(E_8, 2)$,

(c) $(E_8, 3)$,

(d) $(E_8, 5)$,

(e) $(F_4, 3)$,

(f) $(G_2, 2)$.

Then the W-action on $\mathfrak{h}$ is irreducible.

Proof. In all of these cases, the argument in [39, Section 6.5] demonstrates that there is a G-equivariant bijection $\kappa : \mathfrak{g} \to \mathfrak{g}^*$, which restricts to a G-equivariant bijection $\mathfrak{h} \to \mathfrak{h}^*$. Furthermore, the classification in [36, Section 0.13] demonstrates that $\mathfrak{g}$ is simple. Given these two statements, we may apply the same proof as that given in [25, Proposition 14.31] to obtain the result. □

Theorem 3.5.2. *Suppose G is of type G_2 and $p = 2$. Then the invariant ring $S(\mathfrak{h})^W$ is polynomial.*

Proof. This follows from the calculations in [42, Theorem 7.2]. □

In case G is of type G_2 and $p = 2$, we may apply the same argument as for $G = PGL_n$ to obtain the following result.

Theorem 3.5.3. *Let $(G, p) = (G_2, 2)$. Then the dual nilpotent cone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a normal variety.*

Proof of Theorem B: This is immediate from Theorem 3.5.3.

If $S(\mathfrak{h})^W$ is not polynomial, there are significant obstacles to generalising the result that $\mathcal{N}^*$ is a normal variety. In particular, the following behaviour may be observed.

- Kostant's freeness theorem, stated as Theorem 3.2.2, fails. This means that $S(\mathfrak{h})$ is not free as an $S(\mathfrak{h})^W$-module, meaning that we cannot apply the argument in Lemma 3.4.9 to show that $\mathcal{N}^*$ is a Cohen-Macaulay variety.

- The Steinberg quotient $\chi : \mathfrak{g} \to K^n$, defined in Definition 3.3.11, makes sense as an abstract function, but since the generators $\{f_1, \cdots, f_n\}$ of $S(\mathfrak{h})^W$ are not algebraically independent, we cannot apply the argument in Lemma 3.3.12 to show that the smooth elements of $\mathcal{N}$ coincide with the regular elements, which is a key step in the proof that the Springer resolution $\mu : T^*\mathcal{B} \to \mathcal{N}$ is a resolution of singularities for $\mathcal{N}$.

Calculations in [16, Section 3.2] show that, in the following cases (Dynkin diagram of G, p), the invariant ring $S(\mathfrak{h})^W$ is not even Cohen-Macaulay.

(a) $(E_7, 3)$,

(b) $(E_8, 3)$,

(c) $(E_8, 5)$.

Conjecture 3.5.4. In case the invariant ring $S(\mathfrak{h})^W$ is not Cohen-Macaulay, is it true that the dual nilpotent cone $\mathcal{N}^*$ is not a normal variety?

Chapter 4

Applications to representations of p-adic Lie groups

4.1 Generalising the Beilinson-Bernstein theorem for $\widehat{\mathcal{D}^{\lambda}_{n,K}}$

In this chapter, we apply the results of Chapter 3 to the constructions given in [4]. This allows us to weaken the restrictions on the characteristic of the base field given in [4, Section 6.8], thereby providing us with generalisations of their results.

Throughout Chapter 4, we suppose R is a fixed complete discrete valuation ring with uniformiser π, residue field k and field of fractions K. Assume throughout this section that K has characteristic 0 and k is algebraically closed.

We recall the constructions from Section 2.5. In particular, let X be a smooth separated R-scheme that is locally of finite type, let $\mathbf{H}$ be a flat affine algebraic group defined over R of finite type, and let $\widetilde{X}$ be a scheme equipped with an $\mathbf{H}$-action. Given an $\mathbf{H}$-torsor $\xi : \widetilde{X} \to X$ as defined in Definition 2.5.1, recall that $\xi_*\mathcal{D}_{\widetilde{X}}$ is a sheaf of R-algebras with an $\mathbf{H}$-action. Then the relative enveloping algebra of the torsor is the sheaf of $\mathbf{H}$-invariants of $\xi_*\mathcal{D}_{\widetilde{X}}$:

$$\widetilde{\mathcal{D}} := (\xi_*\mathcal{D}_{\widetilde{X}})^{\mathbf{H}}.$$

Let $\mathbf{G}$ be a split reductive connected algebraic group defined over a discrete valuation ring R, with uniformiser π, residue field k and field of fractions K. Let $\mathbf{B}$ be a Borel subgroup. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$, and $\mathbf{H} := \mathbf{B}/\mathbf{N}$ the abstract Cartan group. Let $\widetilde{\mathcal{B}}$ denote the homogeneous space $\mathbf{G}/\mathbf{N}$. There is an $\mathbf{H}$-action on $\widetilde{\mathcal{B}}$ defined by:

$$b\mathbf{N} \cdot g\mathbf{N} := gb\mathbf{N}$$

which is well-defined since $[\mathbf{B}, \mathbf{B}]$ is contained in $\mathbf{N}$. $\mathcal{B} := \mathbf{G}/\mathbf{B}$ is the flag scheme of $\mathbf{G}$, as in Definition 2.2.22. $\widetilde{\mathcal{B}}$ is the basic affine space of $\mathbf{G}$.

By the splitting assumption of $\mathbf{G}$, we can find a Cartan subgroup $\mathbf{T}$ of $\mathbf{G}$ complementary to $\mathbf{N}$ in $\mathbf{B}$. This is naturally isomorphic to $\mathbf{H}$, and induces an isomorphism of the corresponding Lie algebras $\mathfrak{t} \to \mathfrak{h}$.

Let $\mathbf{W}$ denote the Weyl group of $\mathbf{G}$. From now on, we assume that $\mathbf{G} = PGL_n$, $p|n$, and $n > 2$.

Let $\mathfrak{h}$ be the Lie algebra of $\mathbf{H}$. We may apply the deformation functor ([4, Section 3.5]) to the map $j : U(\mathfrak{h}) \to \widetilde{\mathcal{D}}$, defined above Definition 2.5.4, to obtain a central embedding of the constant sheaf $U(\mathfrak{h})_n$ into $\widetilde{\mathcal{D}}_n$. This gives $\widetilde{\mathcal{D}}_n$ the structure of a $U(\mathfrak{h})_n$-module.

Let $\lambda \in \mathrm{Hom}_R(\pi^n\mathfrak{h}, R)$ be a linear functional. This extends to an R-algebra homomorphism $U(\mathfrak{h})_n \to R$, which gives R the structure of a $U(\mathfrak{h})_n$-module, denoted R_λ.

Definition 4.1.1. The *sheaf of deformed twisted differential operators* $\mathcal{D}_n^\lambda$ on $\mathcal{B}$ is the sheaf:

$$\mathcal{D}_n^\lambda := \widetilde{\mathcal{D}_n} \otimes_{U(\mathfrak{h})_n} R_\lambda$$

By [4, Lemma 6.4(b)], this is a sheaf of deformable R-algebras.

Definition 4.1.2. The *π-adic completion* of $\mathcal{D}_n^\lambda$ is $\widehat{\mathcal{D}_n^\lambda} := \varprojlim \mathcal{D}_n^\lambda / \pi^a \mathcal{D}_n^\lambda$. Furthermore, set $\widehat{\mathcal{D}_{n,K}^\lambda} := \widehat{\mathcal{D}_n^\lambda} \otimes_R K$.

The adjoint action of $\mathbf{G}$ on $\mathfrak{g}$ extends to an action on $U(\mathfrak{g})$ by ring automorphisms, which is filtration-preserving and so descends to an action on $\mathrm{gr}\ U(\mathfrak{g}) \cong S(\mathfrak{g})$. Let:

$$\psi : S(\mathfrak{g})^{\mathbf{G}} \to S(\mathfrak{t})$$

denote the composition of the inclusion $S(\mathfrak{g})^{\mathbf{G}} \to S(\mathfrak{g})$ with the projection $S(\mathfrak{g}) \to S(\mathfrak{t})$. By [21, Theorem 7.3.7], the image of ψ is contained in $S(\mathfrak{t})^{\mathbf{W}}$, and ψ is injective.

Since taking **G**-invariants is left exact, we have an inclusion gr $(U(\mathfrak{g})^{\mathbf{G}}) \to S(\mathfrak{g})^{\mathbf{G}}$. Our next proposition gives a description of the associated graded ring of $U(\mathfrak{g})^{\mathbf{G}}$.

Proposition 4.1.3. The rows of the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}(U(\mathfrak{g})^{\mathbf{G}}) & \xrightarrow{\pi} & \mathrm{gr}(U(\mathfrak{g})^{\mathbf{G}}) & \longrightarrow & \mathrm{gr}(U(\mathfrak{g}_k)^{\mathbf{G}_k}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_k} & & \\
0 & \longrightarrow & S(\mathfrak{g})^{\mathbf{G}} & \xrightarrow{\pi} & S(\mathfrak{g})^{\mathbf{G}} & \longrightarrow & S(\mathfrak{g}_k)^{\mathbf{G}_k} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_k} & & \\
0 & \longrightarrow & S(\mathfrak{t})^{\mathbf{W}} & \xrightarrow{\pi} & S(\mathfrak{t})^{\mathbf{W}} & \longrightarrow & S(\mathfrak{t}_k)^{\mathbf{W}_k} & \longrightarrow & 0
\end{array}$$

are exact, and each vertical map is an isomorphism.

Proof. View the diagram as a sequence of complexes $C^\bullet \to D^\bullet \to E^\bullet$. Since π generates the maximal ideal $\mathfrak{m}$ of R by definition, and $R/\mathfrak{m} = k$, it is clear that each complex is exact in the left and in the middle. The exactness of $E^\bullet$ follows from the fact that $S(\mathfrak{t}_k)^{\mathbf{W}_k}$ is a polynomial ring by Theorem 3.2.1: since $n > 2$ we may fix homogeneous generators $s_1, \cdots, s_l$ and lift these generators to homogeneous generators $S_1, \cdots, S_l$ of the ring $S(\mathfrak{t})^{\mathbf{W}}$ with $s_i = S_i (\mathrm{mod}\ \mathfrak{m})$ by the proof of [42, Proposition 5.1]. Hence the map $S(\mathfrak{t})^{\mathbf{W}} \to S(\mathfrak{t}_k)^{\mathbf{W}_k}$ is surjective, and the complex $E^\bullet$ is exact.

By [21, Theorem 7.3.7], ψ is injective, and since p is nonspecial from Definition 3.1.2, ψ_k is an isomorphism by Theorem 3.3.10. Thus the composite map of complexes $\psi^\bullet \circ \iota^\bullet$ is injective. Set $F^\bullet := \mathrm{coker}(\psi^\bullet \circ \iota^\bullet)$: by definition, the sequence of complexes $0 \to C^\bullet \to E^\bullet \to F^\bullet \to 0$ is exact.

Since $C^\bullet$ is exact in the left and in the middle, $H^0(C^\bullet) = H^1(C^\bullet) = 0$. As $E^\bullet$ is exact, taking the long exact sequence of cohomology shows that $H^0(F^\bullet) = H^2(F^\bullet) = 0$ and yields an isomorphism $H^1(F^\bullet) \cong H^2(C^\bullet)$.

Since K is a field of characteristic zero, the map $\psi_K \circ \iota_K : \mathrm{gr}(U(\mathfrak{g}_K)^{\mathbf{G}_K}) \to S(\mathfrak{t}_K)^{\mathbf{W}_K}$ is an isomorphism by [21, Theorem 7.3.7]. Hence $F^0 = F^1 = \mathrm{coker}(\psi \circ \iota)$ is π-torsion. Now $H^0(F^\bullet) = 0$, and so we have an exact sequence $0 \to F^0 \to F^1$. So $F^0 = F^1 = 0$, and hence $H^1(F^\bullet) = H^2(C^\bullet) = 0$. It follows that the top row $C^\bullet$ is exact.

Hence $\psi^\bullet \circ \iota^\bullet : C^\bullet \to E^\bullet$ is an isomorphism in all degrees except possibly 2, and so is an isomorphism via the Five Lemma. The result follows from the fact that $\psi^\bullet$ and $\iota^\bullet$ are both injections. $\square$

It follows that, since $\psi \circ \iota$ is a graded isomorphism and p is nonspecial, $\mathrm{gr}(U(\mathfrak{g})^{\mathbf{G}})$ is isomorphic to a commutative polynomial algebra over R in l variables by Theorem 3.2.1. The commutative polynomial algebra $R[x_1, \cdots, x_l]$ is a free R-module and hence is flat, and so $(U(\mathfrak{g})^{\mathbf{G}})$ is a deformable R-algebra by [4, Definition 3.5]. Furthermore, $\widehat{U(\mathfrak{g})^{\mathbf{G}}_{n,K}}$ is also a commutative polynomial algebra over R in l variables, so the π-adic completion $\widehat{U(\mathfrak{g})^{\mathbf{G}}_{n,K}}$ is a commutative Tate algebra.

By [4, Proposition 4.10], we have a commutative square consisting of deformable R-algebras:

$$\begin{array}{ccc} U((\mathfrak{g})^{\mathbf{G}})_n & \xrightarrow{\phi_n} & U(\mathfrak{t})_n \\ \big\downarrow{\scriptstyle i_n} & & \big\downarrow{\scriptstyle (j\circ i)_n} \\ U(\mathfrak{g})_n & \xrightarrow{U(\phi)_n} & \widetilde{\mathcal{D}_n}, \end{array}$$

We set:

$$\mathcal{U}_n^\lambda := U(\mathfrak{g}) \otimes_{(U(\mathfrak{g})^{\mathbf{G}})_n} R_\lambda,$$
$$\widehat{\mathcal{U}_n^\lambda} := \varprojlim \frac{\mathcal{U}_n^\lambda}{\pi^a \mathcal{U}_n^\lambda},$$
$$\widehat{\mathcal{U}_{n,K}^\lambda} := \widehat{\mathcal{U}_n^\lambda} \otimes_R K.$$

By commutativity of the diagram, the map:

$$U(\phi)_n \otimes (j \circ i)_n : U(\mathfrak{g})_n \otimes U(\mathfrak{t})_n \to \widetilde{\mathcal{D}_n}$$

factors through $U((\mathfrak{g})^{\mathbf{G}})_n$, and we obtain the algebra homomorphisms:

$$\phi_n^\lambda : \mathcal{U}_n^\lambda \to \mathcal{D}_n^\lambda,$$
$$\widehat{\phi_n^\lambda} : \widehat{\mathcal{U}_n^\lambda} \to \widehat{\mathcal{D}_n^\lambda},$$
$$\widehat{\phi_{n,K}^\lambda} : \widehat{\mathcal{U}_{n,K}^\lambda} \to \widehat{\mathcal{D}_{n,K}^\lambda}.$$

Theorem 4.1.4. *(a)* $\widehat{\mathcal{U}_{n,K}^\lambda} \cong \widehat{U(\mathfrak{g})_{n,K}} \otimes_{\widehat{U(\mathfrak{g})^G_{n,K}}} K_\lambda$ *is an almost commutative affinoid K-algebra.*

(b) The map $\widehat{\phi^{\lambda}_{n,K}} : \widehat{\mathcal{U}^{\lambda}_{n,K}} \to \Gamma(\mathcal{B}, \widehat{\mathcal{D}^{\lambda}_{n,K}})$ *is an isomorphism of complete doubly filtered* K*-algebras.*

(c) There is an isomorphism $S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{G_k}} k \cong Gr\ (\widehat{\mathcal{U}^{\lambda}_{n,K}})$*.*

Proof. (a): This is identical to the proof given in [4, Theorem 6.10(a)].

(b): Let $\{U_1, \cdots, U_m\}$ be an open cover of $\mathcal{B}$ by open affines that trivialise the torsor ξ, which exists by [4, Lemma 4.7(c)]. The special fibre $\mathcal{B}_k$ is covered by the special fibres $U_{i,k}$. It suffices to show that the complex:

$$C^{\bullet} : 0 \to \widehat{\mathcal{U}_{n,K}} \to \bigoplus_{i=1}^{m} \widehat{\mathcal{D}^{\lambda}_{n,K}}(U_i) \to \bigoplus_{i<j} \widehat{\mathcal{D}^{\lambda}_{n,K}}(U_i \cap U_j)$$

is exact.

Clearly, $C^{\bullet}$ is a complex in the category of complete doubly-filtered K-algebras, and so it suffices to show that the associated graded complex $\mathrm{Gr}(C^{\bullet})$ is exact. By [4, Corollary 3.7], there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{gr}(U(\mathfrak{g})^{\mathbf{G}}) & \xrightarrow{\pi} & \mathrm{gr}(U(\mathfrak{g})^{\mathbf{G}}) & \longrightarrow & \mathrm{Gr}(\widehat{U(\mathfrak{g})^{\mathbf{G}}_K}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{gr}(U(\mathfrak{g})) & \xrightarrow{\pi} & \mathrm{gr}(U(\mathfrak{g})) & \longrightarrow & \mathrm{Gr}(\widehat{U(\mathfrak{g})_{n,K}}) & \longrightarrow & 0. \end{array}$$

Via the identification $\mathrm{gr}(U(\mathfrak{g})) = S(\mathfrak{g})$, Proposition 4.1.3 induces a commutative square:

$$
\begin{array}{ccc}
\mathrm{Gr}(\widehat{U(\mathfrak{g})^{\mathbf{G}}_{n,K}}) & \longrightarrow & S(\mathfrak{g}_k)^{\mathbf{G}_k} \\
\downarrow & & \downarrow \\
\mathrm{Gr}(\widehat{U(\mathfrak{g})_{n,K}}) & \longrightarrow & S(\mathfrak{g}_k)
\end{array}
$$

where the horizontal maps are isomorphisms and the vertical maps are inclusions. Since $\mathrm{Gr}(K_\lambda)$ is the trivial $\mathrm{Gr}(\widehat{U(\mathfrak{g})^{\mathbf{G}}_{n,K}})$-module k, we have a natural surjection:

$$S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{\mathbf{G}_k}} k \cong \mathrm{Gr}(\widehat{U(\mathfrak{g})_{n,K}} \otimes_{\mathrm{Gr}(\widehat{U(\mathfrak{g})^{\mathbf{G}}_{n,K}})} \mathrm{Gr}(K_\lambda)) \to \mathrm{Gr}(\widehat{\mathcal{U}^\lambda_{n,K}}).$$

This surjection fits into the commutative diagram:

$$
\begin{array}{ccccccc}
0 \longrightarrow & S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{\mathbf{G}_k}} k & \longrightarrow & \bigoplus_{i=1}^m \mathcal{O}(T^*U_{i,k}) & \longrightarrow & \bigoplus_{i<j} \mathcal{O}(T^*(U_{i,k} \cap U_{j,k})) \\
& \downarrow & & \uparrow & & \uparrow \\
0 \longrightarrow & \mathrm{Gr}(\widehat{\mathcal{U}^\lambda_{n,K}}) & \longrightarrow & \bigoplus_{i=1}^m \mathrm{Gr}(\widehat{\mathcal{D}^\lambda_{n,K}}(U_i)) & \longrightarrow & \bigoplus_{i<j} \mathrm{Gr}(\widehat{\mathcal{D}^\lambda_{n,K}}(U_i \cap U_j)).
\end{array}
$$

The bottom row is $\mathrm{Gr}(C^\bullet)$ by definition, and the top row is induced by the moment map $T^*\mathcal{B}_k \to \mathfrak{g}_k^*$. To see this, note that by Lemma 3.3.6, we have an identification $\widetilde{\mathcal{N}^*} \to T^*\mathcal{B}$ under our assumptions on p, and so exactness of the top row is equivalent to the existence of an isomorphism:

$$S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^{\mathbf{G}}} k \cong \Gamma(\widetilde{\mathcal{N}^*}, \mathcal{O}_{\widetilde{\mathcal{N}^*}}).$$

By Theorem 3.4.10, $\mathcal{N}^*$ is a normal variety and, by Theorem 3.3.13, the map $\gamma : T^*\mathcal{B} \to \mathcal{N}^*$ is a resolution of singularities. It follows, by Corollary 3.4.11, that there is an isomorphism of global sections:

$$\gamma^* : \Gamma(\mathcal{N}^*, \mathcal{O}_{\mathcal{N}^*}) \to \Gamma(T^*\mathcal{B}, \mathcal{O}_{T^*\mathcal{B}}).$$

Recall from the proof of Theorem 3.4.10 that $\mathcal{O}(\mathcal{N}^*) = S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^{\mathbf{G}}} k$. Putting these isomorphisms together, we see that $S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^{\mathbf{G}}} k \cong \Gamma(\widetilde{\mathcal{N}^*}, \mathcal{O}_{\widetilde{\mathcal{N}^*}})$.

Now the second and third vertical arrows are isomorphisms by [4, Proposition 6.5(a)], which shows that $\mathrm{Gr}(C^\bullet)$ is exact.

(c) This is immediate, since one can also show that the first vertical arrow in the above diagram is an isomorphism via the Five Lemma. □

Definition 4.1.5. For each $\lambda \in \mathrm{Hom}_R(\pi^n\mathfrak{h}, R)$, we define a functor:

$$\mathrm{Loc}^\lambda : \widehat{U(\mathfrak{g})^\lambda_{n,K}} - \mathrm{mod} \to \widehat{\mathcal{D}^\lambda_{n,K}} - \mathrm{mod}$$

given by $M \mapsto \widehat{\mathcal{D}^\lambda_{n,K}} \otimes_{\widehat{\mathcal{U}^\lambda_{n,K}}} M$.

4.2 Modules over completed enveloping algebras

The adjoint action of $\mathbf{G}$ on $\mathfrak{g}$ induces an action of $\mathbf{G}$ on $U(\mathfrak{g})$ by algebra automorphisms. Composing the inclusion $U(\mathfrak{g})^{\mathbf{G}} \to U(\mathfrak{g})$ with the projection $U(\mathfrak{g}) \to U(\mathfrak{t})$ defined by the direct sum decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ yields the *Harish-Chandra homomorphism*:

$$\phi : U(\mathfrak{g})^{\mathbf{G}} \to U(\mathfrak{t})$$

This is a morphism of deformable R-algebras, so by applying the deformation and π-adic completion functors, one obtains the *deformed Harish-Chandra homomorphism*:

$$\widehat{\phi_{n,K}} : \widehat{U(\mathfrak{g})^{\mathbf{G}}_{n,K}} \to \widehat{U(\mathfrak{t})_{n,K}}$$

which we will denote via the shorthand $\widehat{\phi} : Z \to \widetilde{Z}$. We have an action of the Weyl group $\mathbf{W}$ on the dual Cartan subalgebra $\mathfrak{t}^*_K$ via the shifted dot-action:

$$w \bullet \lambda = w(\lambda + \rho') - \rho'$$

where ρ' is equal the half-sum of the T-roots on $\mathfrak{n}^+$. Viewing $U(\mathfrak{t})_K$ as an algebra of polynomial functions on $\mathfrak{t}^*_K$, we obtain a dot-action of $\mathbf{W}$ on $U(\mathfrak{t})_K$. This action preserves the R-subalgebra $U(\mathfrak{t})_n$ of $U(\mathfrak{t})_K$ and so extends naturally to an action of $\mathbf{W}$ on $\widetilde{Z}$.

Theorem 4.2.1. *Suppose that that $\boldsymbol{G} = PGL_n$, $p|n$, and $n > 2$. Then:*

(a) set $A := \widehat{U(\mathfrak{g})_{n,K}}$. The algebra Z is contained in the centre of A.

(b) the map $\widehat{\phi}$ is injective, and its image is the ring of invariants $\widetilde{Z}^{\boldsymbol{W}}$.

(c) the algebra $\widetilde{Z}$ is free of rank $|\boldsymbol{W}|$ as a module over $\widetilde{Z}^{\boldsymbol{W}}$.

(d) $\widetilde{Z}^{\mathbf{W}}$ is isomorphic to a Tate algebra $K\langle S_1, \cdots, S_l\rangle$ as complete doubly filtered K-algebras.

Proof. (a): The algebra $U(\mathfrak{g})_K^{\mathbf{G}}$ is central in $U(\mathfrak{g})_K$ via [33, Lemma 23.2]. Since $U(\mathfrak{g})_K$ is dense in A, it is also contained in the centre of A. But $U(\mathfrak{g})_K^{\mathbf{G}}$ is also dense in Z, and so Z is central in A.

(b): By the Harish-Chandra homomorphism (see [21, Theorem 7.4.5]), ϕ sends $U(\mathfrak{g})_K^{\mathbf{G}}$ onto $U(\mathfrak{t})_K^{\mathbf{W}}$, and so $\widehat{\phi}(Z)$ is contained in $\widetilde{Z}^{\mathbf{W}}$. This is a complete doubly filtered algebra whose associated graded ring $\mathrm{Gr}(\widetilde{Z}^{\mathbf{W}})$ can be identified with $S(\mathfrak{t}_k)^{\mathbf{W}_k}$. This induces a morphism of complete doubly filtered K-algebras $\alpha : Z \to \widetilde{Z}^{\mathbf{W}}$. Its associated graded map $\mathrm{Gr}(\alpha) : \mathrm{Gr}(Z) \to \mathrm{Gr}(\widetilde{Z}^{\mathbf{W}})$ can be identified with the isomorphism $\psi_k : S(\mathfrak{g}_k)^{\mathbf{G}_k} \to S(\mathfrak{t}_k)^{\mathbf{W}_k}$ by Proposition 4.1.3. Hence $\mathrm{Gr}(\alpha)$ is an isomorphism, and so α is an isomorphism by completeness.

(c): By Theorem 3.2.1 and Theorem 3.5.2, $S(\mathfrak{t}_k)$ is a free graded $S(\mathfrak{t}_k)^{\mathbf{W}_k}$-module of rank $|\mathbf{W}|$. Hence, by [4, Lemma 3.2(a)], $\widetilde{Z}$ is finitely generated over Z, and in fact is free of rank $|\mathbf{W}|$.

(d): By Theorem 3.2.1 and Theorem 3.5.2, $S(\mathfrak{t}_k)^{\mathbf{W}_k}$ is a polynomial algebra in l variables. Fix double lifts $s_1, \cdots, s_l \in U(\mathfrak{t})^{\mathbf{W}}$ of these generators, as in the proof of Proposition 4.1.3. Define an R-algebra homomorphism $R[S_1, \cdots, S_l] \to \widetilde{Z}^{\mathbf{W}}$ which sends S_i to s_i. This extends to an isomorphism $K\langle S_1, \cdots, S_l\rangle \to \widetilde{Z}^{\mathbf{W}}$ of complete doubly filtered K-algebras. □

We identify the k-points of the scheme $\mathfrak{g}^* := \mathrm{Spec}(\mathrm{Sym}_R \mathfrak{g})$ with the dual of the k-vector space $\mathfrak{g}$, so $\mathfrak{g}^*(k) = \mathfrak{g}_k^*$. Let G denote the k-points of the algebraic group scheme $\mathbf{G}$. G acts on $\mathfrak{g}_k$ and $\mathfrak{g}_k^*$ via the adjoint and coadjoint action respectively.

Recall the definition of the enhanced moment map $\beta : \widetilde{T^*\mathcal{B}}(k) \to \mathfrak{g}_k^*$ from Definition 2.5.7. Given $y \in \mathfrak{g}_k^*$, write $G.y$ to denote the G-orbit of y under the coadjoint action. We write $\mathcal{N}$ (resp. $\mathcal{N}^*$) to denote the nilpotent cone (resp. dual nilpotent cone) of the k-vector spaces $\mathfrak{g}_k$ and $\mathfrak{g}_k^*$.

Proposition 4.2.2. Suppose p is nonspecial for G. For any $y \in \mathcal{N}^*$, we have $\dim \beta^{-1}(y) = \dim \mathcal{B} - \frac{1}{2}\dim G.y$.

Proof. This is stated for $\mathcal{N}$ as [39, Theorem 10.11]. The result follows by applying the G-equivariant bijection $\kappa : \mathcal{N} \to \mathcal{N}^*$ from [39, Section 6.5]. □

We now let $\mathfrak{g}_{\mathbb{C}}$ denote the complex semisimple Lie algebra with the same root system as G, and let $G_{\mathbb{C}}$ be the corresponding adjoint algebraic group. By [18, Remark 4.3.4], there is a unique non-zero nilpotent $G_{\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathbb{C}}^*$, under the coadjoint action, of minimal dimension. Since each coadjoint $G_{\mathbb{C}}$-orbit is a symplectic manifold, it follows that each of these dimensions is an even integer. We set:

$$r := \frac{1}{2}\min \{\dim G_{\mathbb{C}} \cdot y \mid 0 \neq y \in \mathfrak{g}_{\mathbb{C}}\}$$

Proposition 4.2.3. For any non-zero $y \in \mathcal{N}^*$, $\frac{1}{2}\dim G \cdot y \geq r$, with no restrictions on (G, p).

Proof. We will demonstrate that this inequality holds for all split semisimple algebraic groups G defined over an algebraically closed field k of positive characteristic.

When the characteristic p is small, we will proceed via a case-by-case calculation of the maximal dimension of the centraliser $Z_G(y)$ of $y \in \mathcal{N}^*$.

By Proposition 4.2.2, dim $\beta^{-1}(y) = \dim \mathcal{B} - \frac{1}{2}\dim G \cdot y$. We may assume $y \in \mathcal{N}$ and G acts on $\mathfrak{g}$ via the adjoint action by [39, Section 6.5]. By [31, Theorem 2], we see that:

$$\dim \beta^{-1}(y) = \frac{1}{2}(\dim Z_G(y) - \mathrm{rk}\ (G))$$

where $Z_G(y)$ denotes the centraliser of y in G. Hence it suffices to demonstrate that the following inequality:

$$\dim \mathcal{B} - \frac{1}{2}(\dim Z_G(y) - \mathrm{rk}\ (G)) \geq r$$

holds in all types. We evaluate on a case-by-case basis, aiming to find the maximal dimension of the centraliser. We first note that, using the work of [57, 1.6], we have the following table:

Type	dim $\mathcal{B}$	r
A_n	$1/2n(n+1)$	n
B_n	n^2	$2n-2$
C_n	n^2	n
D_n	n^2-n	$2n-3$
E_6	36	11
E_7	63	17
E_8	120	29
F_4	24	8
G_2	6	3

By [44, Theorem 2.33], when p is nonspecial, the dimension of the centraliser is independent of the isogeny type of G.

Since p is always nonspecial for a group of type A, it therefore suffices to consider $Z_{\mathfrak{sl}_n}(y)$. Since p is good and SL_n is a simply connected algebraic group, by [44, Lemma 2.15], it suffices to consider the centraliser of a non-identity unipotent element in SL_n. Via the identification $GL_n(k) = SL_n(k)Z(GL_n(k))$, it is sufficient to compute $Z_{GL_n(k)}(u)$, for some unipotent matrix u. This dimension is bounded above by n^2, the dimension of $GL_n(k)$ as an algebraic group, and so we have the expression:

$$\begin{aligned} &\dim \mathcal{B} - \frac{1}{2}(\dim Z_G(y) - \text{rk } (G)) \\ &\geq \frac{1}{2}n(n+1) - \frac{1}{2}(n^2 - n) \geq n. \end{aligned}$$

Hence the inequality is verified in type A.

For the remaining classical groups, view $y \in \mathcal{N}$ as a nilpotent matrix, which without loss of generality may be taken to be in Jordan normal form. Let $m_1 \geq \cdots \geq m_r$ be the sizes of the Jordan blocks, with $\sum_{i=1}^{r} m_i = n$, the rank of the group. By [30, Theorem 4.4], we have:

$$\dim Z_G(y) = \sum_{i=1}^{r}(im_i - \chi_V(m_i))$$

where χ_V is a function $\chi_V : \mathbb{N} \to \mathbb{N}$. It follows that:

$$\dim\, Z_G(y) \leq \sum_{i=1}^{r} i m_i = \sum_{j=1}^{n} \sum_{i=j}^{r} m_i.$$

Since $m_1 \geq \cdots \geq m_r$ by construction, the maximum value of this sum is attained when $m_k = 1$ for all k. Hence we obtain the inequality $\dim\, Z_G(y) \leq \frac{1}{2}n(n+1)$. Using this, it is easy to see that the required inequality holds except possibly in the cases B_2, B_3, D_4 and D_5.

For these cases, along with all exceptional cases, we directly verify that the inequality holds using the calculations on dimensions of centralisers in [44, Chapter 8 and Chapter 22].

□

This allows us to prove our generalisation of [4, Theorem 9.10]; a result on the minimal dimension of finitely generated modules over π-adically completed enveloping algebras.

Definition 4.2.4. Let A be a Noetherian ring. A is *Auslander-Gorenstein* if the left and right self-injective dimension of A is finite and every finitely generated left or right A-module M satisfies, for $i \geq 0$ and every submodule N of $\mathrm{Ext}_A^i(M, A)$, $\mathrm{Ext}_A^j(N, A) = 0$ for $j < i$.

In this case, the *grade* of M is given by:

$$j_A(M) := \inf\{j \mid \mathrm{Ext}_A^j(M, A) \neq 0\}$$

and the *canonical dimension* of M is given by:

$$d_A(M) := \text{inj.dim}_A(A) - j_A(M).$$

By the discussion in [4, Section 9.1], the ring $\widehat{U(\mathfrak{g})_{n,K}}$ is Auslander-Gorenstein and so it makes sense to define the canonical dimension function:

$$d : \{\text{finitely generated } \widehat{U(\mathfrak{g})_{n,K}} - \text{modules}\} \to \mathbb{N}.$$

Theorem 4.2.5. *Suppose $n > 0$ and let M be a finitely generated $\widehat{U(\mathfrak{g})_{n,K}}$-module with $d(M) \geq 1$. Then $d(M) \geq r$.*

Proof. By [4, Proposition 9.4], we may assume that M is Z-locally finite. We may also assume that M is a $\widehat{\mathcal{U}^{\lambda}_{n,K}}$-module for some $\lambda \in \mathfrak{h}^*_K$, by passing to a finite field extension if necessary and applying [4, Theorem 9.5].

By Proposition 4.2.1(b), $\lambda \circ (i \circ \widehat{\phi}) = (w \bullet \lambda) \circ (i \circ \widehat{\phi})$ for any $w \in \mathbf{W}$. Hence we may assume λ is ρ-dominant by [4, Lemma 9.6]. Hence Gr (M) is a Gr $(\widehat{\mathcal{U}^{\lambda}_{n.K}}) \cong S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{\mathbf{G}_k}} k$-module by Theorem 4.1.4. If $\mathcal{M} := \text{Loc}^{\lambda}(M)$ is the corresponding coherent $\widehat{\mathcal{D}^{\lambda}_{n,K}}$-module in the sense of Definition 4.1.5, then $\beta(\text{Ch}(\mathcal{M})) = \text{Ch}(M)$ via [4, Corollary 6.12].

Let X and Y denote the k-points of the characteristic varieties $\text{Ch}(\mathcal{M})$ and $\text{Ch}(M)$ respectively. Now $\text{Gr}(M)$ is annihilated by $S^+(\mathfrak{g}_k)^{\mathbf{G}_k}$, and so $Y \subseteq \mathcal{N}^*$. We see that the map $\beta : T^*\mathcal{B} \to \mathfrak{g}$ maps X onto Y.

Let $f : X \to Y$ be the restriction of β to X. By [4, Corollary 9.1], since dim $Y = d(M) \geq 1$ we can find a non-zero smooth point $y \in Y$. By surjectivity, we have a smooth point $x \in f^{-1}(y)$. The induced differential $df_x : T_{X,x} \to T_{Y,y}$ on Zariski tangent spaces yields the inequality:

$$\dim Y + \dim f^{-1}(y) \geq \dim T_{X,x}$$

By [4, Theorem 7.5], dim $T_{X,x} \geq \dim \mathcal{B}$. Hence:

$$d(M) = \dim Y \geq \dim \mathcal{B} - \dim \beta^{-1}(y)$$

By Proposition 4.2.2 and Proposition 4.2.3, the RHS equals r. □

Proof of Theorem C: This follows from Theorem 4.2.5 and [4, Section 10] in the split semisimple case. We may then apply the same argument as in [3] to remove the split hypothesis on the Lie algebra.

Chapter 5

The Beilinson-Bernstein localisation theorem: integral case

5.1 Classifying sheaves of twisted differential operators

This chapter marks the beginning of the second half of the thesis. The overall goal is to prove a locally analytic Beilinson-Bernstein localisation theorem for twisted equivariant $\mathcal{D}$-modules on rigid analytic spaces. In this chapter, we will prove the theorem for the case where the central character λ is a ρ-dominant ρ-regular integral weight.

In this section, we give a classification of sheaves of twisted differential operators on a smooth proper rigid variety $\mathbf{X}$ and discuss some of their main properties.

Definition 5.1.1. Let $\mathbb{X}$ be a K-scheme of locally finite type. A *rigid analytification* of $\mathbb{X}$ is a rigid K-space $\mathbf{X} =: \mathbb{X}^{\mathrm{an}}$ together with a morphism of locally G-ringed K-spaces $\iota : \mathbf{X} \to \mathbb{X}$, in the sense of Definition 2.3.15, such that for any rigid K-space $\mathbf{Y}$ and morphism of locally G-ringed K-spaces $\mathbf{Y} \to \mathbb{X}$, the latter morphism factors through ι via a unique morphism of rigid K-spaces $\mathbf{Y} \to \mathbf{X}$; i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\iota} & \mathbb{X} \\ & \nwarrow^{\exists!} & \uparrow \\ & & \mathbf{Y}. \end{array}$$

By [13, Corollary 5.4/5], the assignment $\mathbb{X} \mapsto \mathbf{X}$ defines a functor $(-)^{\mathrm{an}}$ from the category of K-schemes of locally finite type to the category of rigid analytic K-spaces, the *rigid analytification functor*.

Definition 5.1.2. A *sheaf of twisted differential operators* is a sheaf $\mathcal{A}$ of K-algebras on $\mathbf{X}$ such that there exists an $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$ of $\mathbf{X}$ and, for all $\mathbf{U} \in \mathcal{U}$, $\mathcal{A}_{\mathbf{U}}$ has an $\mathcal{O}_{\mathbf{U}}$-module structure, equipped with $\mathcal{O}_{\mathbf{U}}$-linear K-algebra isomorphisms:

$$\theta_{\mathbf{U}} : \mathcal{A}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}} \text{ for all } \mathbf{U} \in \mathcal{U}.$$

Two sheaves of twisted differential operators $\mathcal{A}$ and $\mathcal{B}$ lie in the same isomorphism class if there exists a global $\mathcal{O}$-linear sheaf isomorphism $\mathcal{A} \to \mathcal{B}$.

Note that, since $\mathbb{X}$ is a K-scheme of finite type, there is a canonical isomorphism:

$$H^*_{dR}(\mathbb{X}) \to H^*_{dR}(\mathbf{X})$$

by [27, Theorem 2.3].

Lemma 5.1.3. Let ϕ be an $\mathcal{O}_{\mathbf{X}}$-linear ring endomorphism of $\mathcal{D}_{\mathbf{X}}$. Then there exists a closed 1-form ω on $\mathbf{X}$ such that:

$$\phi(\xi) = \xi - \omega(\xi)$$

for any element $\xi \in \mathcal{T}_{\mathbf{X}}$, and ϕ is completely determined by ω. In particular, ϕ is an $\mathcal{O}_{\mathbf{X}}$-linear automorphism.

Proof. We can adapt the proof of [46, Lemma 1.1]. Let $f \in \mathcal{O}_{\mathbf{X}}$ and $\xi \in \mathcal{T}_{\mathbf{X}}$. Then:

$$[\phi(\xi), f] = [\phi(\xi), \phi(f)] = \phi([\xi, f]) = \phi(\xi(f)) = \xi(f).$$

Evaluating this on the constant function 1, we have an equality of functions $\phi(\xi)(f) = \xi(f) + f\phi(\xi)(1)$. Hence we set $\omega(\xi) := -\phi(\xi)(1)$. Then ω is a 1-form on $\mathbf{X}$ by construction. Also, note that:

$$\begin{aligned}\omega([\xi,\eta]) = -\phi([\xi,\eta])(1) &= -(\phi(\xi)\phi(\eta) - \phi(\eta)\phi(\xi))(1)\\ &= \phi(\xi)(\omega(\eta)) - \phi(\eta)(\omega(\xi))\\ &= \xi(\omega(\eta)) - \eta(\omega(\xi))\end{aligned}$$

for $\xi, \eta \in \mathcal{T}_{\mathbf{X}}$. It follows that:

$$d\omega(\xi \wedge \eta) = \xi(\omega(\eta)) - \eta(\omega(\xi)) - \omega([\xi,\eta]) = 0.$$

Therefore ω is a closed 1-form, completely determined by ϕ. Finally, ϕ is filtration-preserving and the induced endomorphism gr ϕ of gr $\mathcal{D}_{\mathbf{X}}$ is the identity automorphism. Hence ϕ is itself an automorphism. □

Lemma 5.1.4. Let $\mathcal{Z}^1(\mathbf{X})$ denote the additive group of closed 1-forms on $\mathbf{X}$. The natural morphism:

$$\alpha : \{\mathcal{O}_{\mathbf{X}} - \text{ linear automorphisms of } \mathcal{D}_{\mathbf{X}}\} \to \mathcal{Z}^1(\mathbf{X})$$

is an isomorphism.

Proof. The proof of Lemma 5.1.3 shows that α is injective. To show that α is surjective, let ω be a closed 1-form on $\mathbf{X}$, and define $\phi : \mathcal{T}_{\mathbf{X}} \to \mathcal{D}_{\mathbf{X}}$ by $\phi(\xi) = \xi - \omega(\xi)$ for $\xi \in \mathcal{T}_{\mathbf{X}}$. Then ϕ extends to an $\mathcal{O}_{\mathbf{X}}$-linear ring endomorphism $\overline{\phi} : \mathcal{D}_{\mathbf{X}} \to \mathcal{D}_{\mathbf{X}}$ if the following three conditions are satisfied:

(a) $\phi(f\xi) = f\phi(\xi)$,

(b) $\phi(\xi)(f) = [\phi(\xi), f](1)$,

(c) $\phi([\xi, \eta]) = [\phi(\xi), \phi(\eta)]$

for $f \in \mathcal{O}_{\mathbf{X}}$ and $\xi, \eta \in \mathcal{T}_{\mathbf{X}}$.

Condition (a) is immediate, since ϕ is an $\mathcal{O}_{\mathbf{X}}$-linear morphism. Condition (b) follows from the observation that $\phi(\xi)(f) = \xi(f) + f\phi(\xi)(1)$, as in the proof of Lemma 5.1.3. Finally, condition (c) is satisfied since, for $\xi, \eta \in \mathcal{T}_{\mathbf{X}}$:

$$\phi([\xi,\eta]) = [\xi,\eta] - \omega([\xi,\eta]) = [\xi,\eta] - \xi(\omega(\eta)) + \eta(\omega(\xi))$$

$$= [\xi - \omega(\xi), \eta - \omega(\eta)] = [\phi(\xi), \phi(\eta)].$$

Hence ϕ extends to an $\mathcal{O}_{\mathbf{X}}$-linear endomorphism of $\mathcal{D}_{\mathbf{X}}$. By Lemma 5.1.3, ϕ is an automorphism. □

For future reference, for any morphism of sheaves ϕ on a rigid analytic variety $\mathbf{Y}$, and $\mathbf{U}, \mathbf{V}$ admissible open affinoid subspaces of $\mathbf{Y}$, set $\phi_{\mathbf{UV}} := \phi_{\mathbf{V}} \mid_{\mathbf{U}\cap\mathbf{V}}$.

Let $\mathcal{A}$ be a sheaf of twisted differential operators on $\mathbf{X}$, and let $\mathcal{U}$ be an $\mathbf{X}_\omega(\mathcal{T})$-covering. For each $\mathbf{U} \in \mathcal{U}$, fix a local isomorphism $\theta_{\mathbf{U}} : \mathcal{A}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}}$, which exists by Definition 5.1.2. Then there exists an automorphism $\phi_{\mathbf{UV}}$ of $\mathcal{D}_{\mathbf{UV}}$ on $\mathbf{U} \cap \mathbf{V}$ such that $\theta_{\mathbf{U}} = \phi_{\mathbf{UV}} \circ \theta_{\mathbf{V}}$. By Lemma 5.1.4, there is a closed 1-form $\omega_{\mathbf{UV}}$ on $\mathbf{U}\cap\mathbf{V}$ which determines $\phi_{\mathbf{UV}}$.

If $\mathbf{U}, \mathbf{V}, \mathbf{W} \in \mathcal{U}$ such that $\mathbf{U} \cap \mathbf{V} \cap \mathbf{W} \neq \emptyset$, then on this set we have:

$$\theta_{\mathbf{U}} = \phi_{\mathbf{UV}} \circ \theta_{\mathbf{V}} = \phi_{\mathbf{UV}} \circ \phi_{\mathbf{VW}} \circ \theta_{\mathbf{W}}.$$

Therefore $\phi_{\mathbf{UW}} = \phi_{\mathbf{UV}} \circ \phi_{\mathbf{VW}}$. Evaluating this equation at ξ, it follows that:

$$\phi_{\mathbf{UW}}(\xi) = \xi - \omega_{\mathbf{UW}}(\xi) = (\phi_{\mathbf{UV}} \circ \phi_{\mathbf{VW}})(\xi) = \phi_{\mathbf{UV}}(\xi - \omega_{\mathbf{VW}}(\xi))$$

$$= \xi - \omega_{\mathbf{UV}}(\xi) - \omega_{\mathbf{VW}}(\xi)$$

and so ω satisfies the cocycle condition $\omega_{\mathbf{UW}} = \omega_{\mathbf{UV}} + \omega_{\mathbf{VW}}$ on $\mathbf{U} \cap \mathbf{V} \cap \mathbf{W}$.

Let $\mathcal{Z}^1_{\mathbf{X}}$ denote the sheaf of closed 1-forms on $\mathbf{X}$, and let $C^\bullet(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$ denote the Čech complex of $\mathcal{Z}^1_{\mathbf{X}}$ corresponding to the covering $\mathcal{U}$. Then, for $\mathbf{V} \in \mathcal{U}$, $\omega = (\omega_{\mathbf{UV}} \mid \mathbf{U}, \mathbf{V} \in \mathcal{U})$ is an element of $C^1(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$ and $d\omega = 0$, so $\omega \in Z^1(\mathcal{U}, \mathcal{Z}^1_X)$.

Now take another set of local isomorphisms $\{\theta'_{\mathbf{U}} : \mathcal{A}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}} \mid \mathbf{U} \in \mathcal{U}\}$. Arguing as above, we obtain another set $\{\phi'_{\mathbf{UV}} \mid \mathbf{U}, \mathbf{V} \in \mathcal{U}\}$ and another closed 1-form $\omega' \in Z^1(\mathcal{U}, \mathcal{Z}^1_X)$. Applying Lemma 5.1.4, we obtain $\mathcal{O}_{\mathbf{U}}$-linear automorphisms $\sigma_{\mathbf{U}}$ of $\mathcal{D}_{\mathbf{U}}$ such that $\theta'_{\mathbf{U}} = \sigma_{\mathbf{U}} \circ \theta_{\mathbf{U}}$, and associated closed 1-forms $\rho_{\mathbf{U}}$ associated to them. The set $\rho := \{\rho_{\mathbf{U}} \mid \mathbf{U} \in \mathcal{U}\}$ is an element of $C^0(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$. On $\mathbf{U} \cap \mathbf{V}$, we have:

$$\sigma_{\mathbf{U}} \circ \phi_{\mathbf{UV}} \circ \theta_{\mathbf{V}} = \sigma_{\mathbf{U}} \circ \theta_{\mathbf{U}} = \theta'_{\mathbf{U}} = \phi'_{\mathbf{UV}} \circ \theta'_{\mathbf{V}} = \phi'_{\mathbf{UV}} \circ \sigma_{\mathbf{V}} \circ \theta_{\mathbf{V}}.$$

Hence $\sigma_{\mathbf{U}} \circ \phi_{\mathbf{UV}} = \phi'_{\mathbf{UV}} \circ \sigma_{\mathbf{V}}$. Taking the corresponding closed 1-forms, it follows that $\rho_{\mathbf{U}} + \omega_{\mathbf{UV}} = \omega'_{\mathbf{UV}} + \rho_{\mathbf{V}}$ on $\mathbf{U} \cap \mathbf{V}$. Hence $\omega' - \omega = d\rho$, and so the twisted sheaf of differential operators $\mathcal{A}$ determines an element of $\check{H}^1(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$. Passing to the colimit, we have a well-defined map between isomorphism classes of sheaves of twisted differential operators on $\mathbf{X}$ and $\check{H}^1(\mathbf{X}, \mathcal{Z}^1_{\mathbf{X}})$, denoted by t.

Theorem 5.1.5. *t is a bijection, i.e. there is a bijection between isomorphism classes of sheaves of twisted differential operators on $\boldsymbol{X}$ and $\check{H}^1(\boldsymbol{X}, \mathcal{Z}^1_{\mathbf{X}})$.*

Proof. We first check that this map is injective. Let $\mathcal{A}$, $\mathcal{B}$ be sheaves of twisted differential operators with $t(\mathcal{A}) = t(\mathcal{B})$. Then there is an $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$ and elements $\omega, \omega' \in Z^1(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$ which define the same element of $\check{H}^1(\mathbf{X}, \mathcal{Z}^1_{\mathbf{X}})$, and families of local isomorphisms:

$$\theta_{\mathbf{U}} : \mathcal{A}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}},$$

$$\eta_{\mathbf{U}} : \mathcal{B}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}}.$$

These families then define families of automorphisms $\{\phi_{\mathbf{UV}} \mid \mathbf{U}, \mathbf{V} \in \mathcal{U}\}$ and $\{\chi_{\mathbf{UV}} \mid \mathbf{U}, \mathbf{V} \in \mathcal{U}\}$ such that, for all $\mathbf{U}, \mathbf{V} \in \mathcal{U}$, we have:

$$\theta_{\mathbf{U}} = \phi_{\mathbf{UV}} \circ \theta_{\mathbf{V}},$$

$$\eta_{\mathbf{U}} = \chi_{\mathbf{UV}} \circ \eta_{\mathbf{V}}.$$

Since ω, ω' define the same element of $\check{H}^1(\mathbf{X}, \mathcal{Z}^1_{\mathbf{X}})$, it follows that, by refining the cover $\mathcal{U}$ if necessary, ω, ω' differ by a coboundary, and so we may assume $\omega - \omega' = d\rho$ for some $\rho := \{\rho_{\mathbf{U}} \mid \mathbf{U} \in \mathcal{U}\} \in C^0(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$. The map $\rho_{\mathbf{U}}$ determines an automorphism $\sigma_{\mathbf{U}} : \mathcal{A}_{\mathbf{U}} \to \mathcal{A}_{\mathbf{U}}$. Then $\eta'_{\mathbf{U}} := \sigma_{\mathbf{U}} \circ \eta_{\mathbf{U}}$ is a family of local isomorphisms $\eta'_{\mathbf{U}} : \mathcal{B}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}}$ satisfying:

$$\sigma_{\mathbf{U}} \circ \chi_{\mathbf{UV}} \circ \eta_{\mathbf{V}} = \sigma_{\mathbf{U}} \circ \eta_{\mathbf{U}} = \eta'_{\mathbf{U}} = \chi'_{\mathbf{UV}} \circ \eta'_{\mathbf{UV}} = \chi'_{\mathbf{UV}} \circ \sigma_{\mathbf{V}} \circ \eta_{\mathbf{V}}$$

where the family of local isomorphisms $\sigma_{\mathbf{U}} \circ \eta_{\mathbf{U}}$ defines the family of automorphisms $\{\chi'_{\mathbf{UV}} \mid \mathbf{U}, \mathbf{V} \in \mathcal{U}\}$, with corresponding 1-form ω'', as explained above.

Therefore $\sigma_{\mathbf{U}} \circ \chi_{\mathbf{UV}} = \chi'_{\mathbf{UV}} \circ \sigma_{\mathbf{V}}$, so $\chi'_{\mathbf{UV}} = \sigma_{\mathbf{U}} \circ \chi_{\mathbf{UV}} \circ \sigma^{-1}_{\mathbf{V}}$ on $\mathbf{U} \cap \mathbf{V}$.

Hence $\omega''_{\mathbf{UV}} = \omega'_{\mathbf{UV}} + \rho_{\mathbf{U}} - \rho_{\mathbf{V}}$ on $\mathbf{U} \cap \mathbf{V}$, and so $\omega'' = \omega' + d\rho = \omega$. Therefore $\chi'_{\mathbf{UV}} = \phi_{\mathbf{UV}}$ on $\mathbf{U} \cap \mathbf{V}$. We define local isomorphisms $\beta_{\mathbf{U}} : \mathcal{A}_{\mathbf{U}} \to \mathcal{B}_{\mathbf{U}}$ by:

$$\beta_{\mathbf{U}} = (\eta'_{\mathbf{U}})^{-1} \circ \theta_{\mathbf{U}}.$$

Then, on $\mathbf{U} \cap \mathbf{V}$, one obtains:

$$\beta_{\mathbf{U}} = (\eta'_{\mathbf{U}})^{-1} \circ \theta_{\mathbf{U}} = (\phi_{\mathbf{UV}} \circ \eta'_{\mathbf{V}})^{-1} \circ \phi_{\mathbf{UV}} \circ \theta_{\mathbf{V}} = (\eta'_{\mathbf{V}})^{-1} \circ \theta_{\mathbf{V}} = \beta_{\mathbf{V}}$$

and so the $\beta_{\mathbf{U}}$ extend to a global isomorphism of sheaves $\mathcal{A} \to \mathcal{B}$. Hence t is injective.

To show t is surjective, let $\lambda \in \check{H}^1(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$. Then λ determines a closed 1-form $\omega \in Z^1(\mathcal{U}, \mathcal{Z}^1_{\mathbf{X}})$ with $d\omega = 0$. Write $\omega = \{\omega_{\mathbf{UV}} \mid \mathbf{U}, \mathbf{V} \in \mathcal{U}\}$. By Lemma 5.1.4, each $\omega_{\mathbf{UV}}$ corresponds to a local automorphism $\phi_{\mathbf{UV}} : \mathcal{D}_{\mathbf{U}\cap\mathbf{V}} \to \mathcal{D}_{\mathbf{U}\cap\mathbf{V}}$: since ω is closed $\phi_{\mathbf{UW}} = \phi_{\mathbf{VW}} \circ \phi_{\mathbf{UV}}$.

Hence the family $(\mathcal{D}_{\mathbf{U}}, \phi_{\mathbf{UV}})_{\mathbf{U},\mathbf{V}\in\mathcal{U}}$ is gluing data for sheaves of K-algebras with respect to the covering $\mathcal{U}$. Hence, by [6, Lemma 6.33.2 and Lemma 6.33.3], there exists a sheaf $\mathcal{A}$ of K-algebras with isomorphisms $\theta_{\mathbf{U}} : \mathcal{A}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}}$. The sheaf $\mathcal{A}$ satisfies $t(\mathcal{A}) = \lambda$ by construction: hence t is surjective. □

Theorem 5.1.5 demonstrates that we can parametrise the set of isomorphism classes of twisted differential operators by the space $\check{H}^1(\mathbf{X}, \mathcal{Z}^1_{\mathbf{X}})$.

5.2 The category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$

In this section, we construct the category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ of coadmissible G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules on $\mathbf{X}$, in the case where λ is integral. We also show that there is an equivalence of abelian categories $\mathcal{C}_{\mathbf{X}/G} \to \mathcal{C}^{\lambda}_{\mathbf{X}/G}$.

Let K be a field equipped with a complete non-archimedean norm $|\cdot|$, $\mathcal{R} := \{\lambda \in K \mid |\lambda| \leq 1\}$ and $\pi \in \mathcal{R}$ a fixed non-zero non-unit element. Throughout, assume G is a compact p-adic Lie group and $\mathbb{G}$ is a connected split reductive affine algebraic group scheme over K, equipped with a continuous group homomorphism $\sigma : G \to \mathbb{G}(K)$. Let $\mathbb{B}$ be a choice of Borel subgroup of $\mathbb{G}$. $\mathbb{G}$ naturally acts on the flag scheme $\mathbb{X} = \mathbb{G}/\mathbb{B}$ by left translation. Further fix the unipotent radical $\mathbb{N}$ of $\mathbb{B}$, and set $\mathbb{H} := \mathbb{B}/\mathbb{N}$ to be the *abstract Cartan group*. Let $\widetilde{\mathbb{X}} = \mathbb{G}/\mathbb{N}$ be the basic affine space.

Let $\mathbf{H}, \mathbf{X}, \widetilde{\mathbf{X}}$ be the analytifications of the abstract Cartan group $\mathbb{H}$, the flag scheme $\mathbb{X}$, and the basic affine space $\widetilde{\mathbb{X}}$ respectively. Similarly to the above, there is a left G-action on $\widetilde{\mathbb{X}}$, and applying the rigid analytification functor gives an action of G on $\widetilde{\mathbf{X}}$.

Write $\rho : G \to \mathrm{Aut}_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ to denote the continuous group homomorphism arising from the G-action on $\mathbf{X}$. The G-action on $\mathbf{X}$ is faithful and so $\ker\rho$ is trivial.

Since $[\mathbb{B}, \mathbb{B}]$ is contained in $\mathbb{N}$, there is an $\mathbb{H}$-action on $\widetilde{\mathbb{X}}$ defined by the formula:

$$b\mathbb{N} \cdot g\mathbb{N} := gb\mathbb{N},$$

for $b \in \mathbb{B}, g \in \mathbb{G}$.

Lemma 5.2.1. Let $\mathcal{F}$ be a K-linear G-equivariant sheaf of $\mathcal{O}_{\mathbb{X}}$-modules on $\mathbb{X}$. Applying the rigid analytification functor, $\mathcal{F}^{\mathrm{an}}$ is a K-linear G-equivariant sheaf of $\mathcal{O}_{\mathbf{X}}$-modules on $\mathbf{X}$.

Proof. By Definition 2.7.4, for each $g \in G$, there is a natural transformation $g^{\mathcal{F}} : \mathcal{F} \to g^*\mathcal{F}$ which satisfies:

$$(gh)^{\mathcal{F}} = h^*(g^{\mathcal{F}}) \circ h^{\mathcal{F}} \text{ for any } g, h \in G.$$

Applying the rigid analytification functor, there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad g^{\mathcal{F}} \quad} & g^*\mathcal{F} \\ \big\downarrow {\scriptstyle (-)^{\mathrm{an}}} & & \big\downarrow {\scriptstyle (-)^{\mathrm{an}}} \\ \mathcal{F}^{\mathrm{an}} & \xrightarrow{g^{\mathcal{F}^{\mathrm{an}}}} & g^*\mathcal{F}^{\mathrm{an}} = (g^*\mathcal{F})^{\mathrm{an}}, \end{array}$$

and so, for each $g \in G$, there is a natural transformation $g^{\mathcal{F}^{\mathrm{an}}} : \mathcal{F}^{\mathrm{an}} \to g^*\mathcal{F}^{\mathrm{an}}$ which satisfies:

$$(gh)^{\mathcal{F}^{\mathrm{an}}} = ((gh)^{\mathcal{F}})^{\mathrm{an}} = (h^*(g^{\mathcal{F}}) \circ h^{\mathcal{F}})^{\mathrm{an}} = (h^*(g^{\mathcal{F}^{\mathrm{an}}})) \circ h^{\mathcal{F}^{\mathrm{an}}}.$$

Hence $\mathcal{F}^{\mathrm{an}}$ is a G-equivariant sheaf of $\mathcal{O}_{\mathbf{X}}$-modules on $\mathbf{X}$. □

Definition 5.2.2. Let $\xi : \widetilde{\mathbb{X}} \to \mathbb{X}$ be the canonical $\mathbb{H}$-torsor, in the sense of Lemma 2.5.6. Let $\mathcal{T}_{\widetilde{\mathbb{X}}}$ be the tangent sheaf on $\widetilde{\mathbb{X}}$. Set:

$$\widetilde{\mathcal{T}} := (\xi_* \mathcal{T}_{\widetilde{\mathbb{X}}})^{\mathbb{H}}.$$

This is a coherent $\mathcal{O}_{\mathbb{X}}$-module. Since $\widetilde{\mathcal{T}}$ is a coherent sheaf over a projective scheme by Proposition 2.2.23, applying the rigid analytification functor yields a coherent sheaf $\widetilde{\mathcal{T}}^{\text{an}}$ on $\mathbf{X}$ by [13, Theorem 6.3/13]. We set:

$$\widetilde{\mathcal{D}} := \mathcal{U}_{\mathcal{O}_{\mathbf{X}}}(\widetilde{\mathcal{T}}^{\text{an}}).$$

Let $\mathfrak{h}_K$ be the K-Lie algebra of $\mathbb{H}$. An element $\lambda \in \mathfrak{h}_K^*$ gives K the structure of a $U(\mathfrak{h})_K$-module via:

$$h \cdot a = \lambda(h)a$$

for $a \in K, h \in U(\mathfrak{h})_K$. Denote this $U(\mathfrak{h})_K$-module by K_λ.

Note that the right $\mathbf{H}$-action on $\widetilde{\mathbf{X}}$ yields a central embedding $j : U(\mathfrak{h})_K \to \widetilde{\mathcal{D}}$, as in the discussion above Definition 2.5.4. This allows us to make sense of the following definition.

Definition 5.2.3. The *sheaf of twisted differential operators* on $\mathbf{X}$ is the sheaf of K-algebras:

$$\mathcal{D}^\lambda := \widetilde{\mathcal{D}} \otimes_{U(\mathfrak{h})_K} K_\lambda.$$

Lemma 5.2.4. There exists an $\mathbf{X}_w(\mathcal{T})$-covering of $\mathbf{X}, \mathcal{U}$, such that for all $\mathbf{U} \in \mathcal{U}$, there is an isomorphism of sheaves of K-algebras:

$$\mathcal{D}^{\lambda}_{\mathbf{U}} \to \mathcal{D}_{\mathbf{U}}.$$

Hence $\mathcal{D}^{\lambda}$ is a sheaf of twisted differential operators in the sense of Definition 5.1.2. Furthermore, we may view λ as an element of $\check{H}^1(\mathbf{X}, \mathcal{Z}^1_{\mathbf{X}})$, and so $\mathfrak{h}^*_K \subseteq \check{H}^1(\mathbf{X}, \mathcal{Z}^1_{\mathbf{X}})$.

Proof. By [1, Lemma 5.3.1(a)], there is an open cover $\{\mathbb{U}_i\}$ of $\mathbb{X}$ such that there is an isomorphism of sheaves of K-algebras:

$$\phi^{\lambda}_{\mathbb{U}_i} : \mathcal{D}^{\lambda}_{\mathbb{U}_i} \to \mathcal{D}_{\mathbb{U}_i}.$$

Let $\mathbf{U}_i := \widehat{\mathbb{U}_{\mathrm{rig},i}}$. The set $\{\mathbf{U}_i\}$ forms an $\mathbf{X}_w(\mathcal{T})$-covering of $\mathbf{X}$ by [1, Notation 5.4.8]. The functor $\mathcal{D}_{\mathbb{U}_i} \mapsto \mathcal{D}_{\mathbf{U}_i}$ sends isomorphisms to isomorphisms, and so we have an isomorphism of sheaves of K-algebras:

$$\phi^{\lambda}_{\mathbf{U}_i} : \mathcal{D}^{\lambda}_{\mathbf{U}_i} \to \mathcal{D}_{\mathbf{U}_i}.$$

Furthermore, if $\lambda, \mu \in \mathfrak{h}^*_K$ with $\lambda \neq \mu$, then K_λ and K_μ are non-isomorphic as $U(\mathfrak{h})_K$-modules. Since K_λ is a free and hence faithfully flat $U(\mathfrak{h})_K$-module, it follows that the sheaves $\mathcal{D}^{\lambda}$ and $\mathcal{D}^{\mu}$ are non-isomorphic. Applying Theorem 5.1.5, we have an injective map $\mathfrak{h}^*_K \to \check{H}^1(\mathbf{X}, \mathcal{Z}^1_{\mathbf{X}})$.

□

Lemma 5.2.5. The sheaf $\mathcal{D}^{\lambda}$ is a G-equivariant sheaf of K-algebras.

Proof. The group G acts on $\widetilde{\mathbb{X}}$ and $\mathbb{X}$ by left translation, and the $\mathbb{H}$-torsor ξ is G-equivariant. By the argument given in [1, Lemma 3.4.3], the sheaves $\mathcal{O}_{\widetilde{\mathbb{X}}}, \mathcal{T}_{\widetilde{\mathbb{X}}}$ are G-equivariant. Since the right H-action on $\widetilde{\mathbb{X}}$ commutes with the left G-action it follows that the sheaf $\widetilde{\mathcal{T}}$ is G-equivariant. By Lemma 5.2.1, $\widetilde{\mathcal{T}}^{\mathrm{an}}$ is G-equivariant, and so the universal enveloping algebra $\widetilde{\mathcal{D}}$ is G-equivariant.

Recall the map $j : U(\mathfrak{h})_K \to \widetilde{\mathcal{D}}$ from the discussion above Definition 2.5.4. The induced G-action on $\mathfrak{h}_K$ is by the adjoint action, which is trivial on $\mathfrak{h}_K$ since $\mathfrak{h}_K$ is commutative. Hence the image of $j : U(\mathfrak{h})_K \to \widetilde{\mathcal{D}}$ lies in the G-invariants of $\widetilde{\mathcal{D}}$. We obtain a G-equivariant structure on the central reduction:

$$\mathcal{D}^{\lambda} = \widetilde{\mathcal{D}} \otimes_{U(\mathfrak{h})_K} K_{\lambda},$$

where G acts on the left factor only. □

This allows us to make sense of the category $(G - \mathcal{D}^{\lambda}) - \mathrm{mod}$, in the sense of Definition 2.7.6.

Definition 5.2.6. A G-equivariant $\mathcal{D}^{\lambda}$-module $\mathcal{M}$ on $\mathbf{U}_{\mathrm{rig}}$ is *locally Fréchet* if:

(a) $\mathcal{M}(\mathbf{U})$ is equipped with a Fréchet topology for every $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$,

(b) the maps $g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{M}(g\mathbf{U})$ are continuous for all $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and $g \in G$.

A *morphism* of G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules is a morphism $f : \mathcal{M} \to \mathcal{N}$ of G-equivariant $\mathcal{D}^{\lambda}$-modules, such that the induced maps $f(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{N}(\mathbf{U})$

are continuous for every $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$.

The category whose objects are G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules and whose morphisms are continuous maps between them is denoted by $\text{Frech}(G - \mathcal{D}^\lambda)$. There is a natural forgetful functor to G-equivariant $\mathcal{D}^\lambda$-modules on $\mathbf{X}$:

$$\Phi : \text{Frech}(G - \mathcal{D}^\lambda) \to G - \mathcal{D}^\lambda - \text{mod}.$$

From this point on, we suppose λ is an integral weight, in the sense of Definition 2.2.14.

There is an $\mathbb{H}$-action on $\mathcal{O}_{\widetilde{\mathbb{X}}}$, and hence there is an $\mathbb{H}$-action on $\xi_*\mathcal{O}_{\widetilde{\mathbb{X}}}$ obtained by pushing forward the $\mathbb{H}$-action on $\mathcal{O}_{\widetilde{\mathbb{X}}}$. Given some element $\mathbb{U}$ of a trivialising open cover $\{\mathbb{U}_i\}$ of $\mathbb{X}$, the $\mathbb{H}$-action on $\xi_*\mathcal{O}_{\widetilde{\mathbb{X}}}$ yields a comorphism:

$$\rho_{\mathbb{U}} : \mathcal{O}(\mathbb{U}) \to \mathcal{O}(\mathbb{U}) \otimes \mathcal{O}(\mathbb{H}).$$

We set $\mathcal{O}^\lambda(\mathbb{U}) := \{a \in \mathcal{O}(\mathbb{U}) \mid \rho_{\mathbb{U}}(a) = a \otimes \lambda\ \}$ to denote the subsheaf of $\xi_*\mathcal{O}_{\widetilde{\mathbb{X}}}$ where the $\mathbb{H}$-action agrees with the action by the character λ. Set $\mathcal{O}^\lambda_{\mathbf{X}} := (\mathcal{O}^\lambda)^{\text{an}}$.

Lemma 5.2.7. $\mathcal{O}^\lambda_{\mathbf{X}}$ is a locally free sheaf of coherent $\mathcal{O}_{\mathbf{X}}$-modules of rank 1.

Proof. By [32, Section 9.11], we may write $\mathcal{O}^\lambda = \mathcal{L}(\lambda)$, which corresponds to a $\mathbb{G}$-equivariant line bundle $\Lambda(\lambda)$ on $\mathbb{X}$. This further corresponds to a 1-dimensional

$\mathbb{B}$-module V_λ. By [38, Proposition I.5.9(c)], it follows that $\mathcal{O}^\lambda$ is a coherent $\mathcal{O}_{\mathbb{X}}$-module. Applying [13, Theorem 6.3/13], $\mathcal{O}^\lambda_{\mathbf{X}}$ is a coherent $\mathcal{O}_{\mathbf{X}}$-module.

Since the canonical map $\mathbb{G} \to \mathbb{X}$ is locally trivial by [38, II.1.10.2], and V_λ is a free and hence projective $\mathbb{B}$-module, applying [38, I.5.16.2] shows that $\mathcal{O}^\lambda$ is a locally free $\mathcal{O}_{\mathbb{X}}$-module of rank 1. Applying the rigid analytification functor, it follows that $\mathcal{O}^\lambda_{\mathbf{X}}$ is a locally free $\mathcal{O}_{\mathbf{X}}$-module of rank 1. □

Lemma 5.2.8. $\mathcal{O}^\lambda_{\mathbf{X}}$ is a G-equivariant $\mathcal{D}^\lambda$-module.

Proof. There is a natural $\mathcal{D}_{\widetilde{\mathbb{X}}}$-action on $\mathcal{O}_{\widetilde{\mathbb{X}}}$ given by $\partial \cdot a = \partial(a), \partial \in \mathcal{D}_{\widetilde{\mathbb{X}}}, a \in \mathcal{O}_{\widetilde{\mathbb{X}}}$. It follows that there is a $\xi_*\mathcal{D}_{\widetilde{\mathbb{X}}}$-action on $\xi_*\mathcal{O}_{\widetilde{\mathbb{X}}}$ given by the same formula. Let $\partial \in \widetilde{\mathcal{T}}$, $a \in \mathcal{O}^\lambda$. Then, for $h \in \mathbb{H}$, we see that:

$$\begin{aligned} h \cdot \partial(a) &= (h \cdot \partial)(h \cdot a) \\ &= \partial(h \cdot a) \text{ since } h \cdot \partial = \partial \\ &= \partial(\lambda(h)a) \text{ since } a \in \mathcal{O}^\lambda \\ &= \lambda(h)\partial(a). \end{aligned}$$

Hence we have an action of $\widetilde{\mathcal{D}}$ on $\mathcal{O}^\lambda$. By Definition 5.2.3, we may write $\mathcal{D}^\lambda = \frac{\widetilde{\mathcal{D}}}{(\mathfrak{h}-\lambda(\mathfrak{h}))\cdot\widetilde{\mathcal{D}}}$. Since $\mathbb{H}$ acts as the character λ on $\mathcal{O}^\lambda$, it follows that, for $\partial \in (\mathfrak{h} - \lambda(\mathfrak{h})) \cdot \widetilde{\mathcal{D}}, a \in \mathcal{O}^\lambda$, $\partial \cdot a = 0$. Hence the action of $\widetilde{\mathcal{D}}$ on $\mathcal{O}^\lambda$ descends to a well-defined action of $\mathcal{D}^\lambda$ on $\mathcal{O}^\lambda$.

There is a natural $\mathbb{G}$-action on $\xi_*\mathcal{D}_{\widetilde{\mathbb{X}}}$ given by $(g \cdot \partial)(f) = g \cdot \partial(g^{-1} \cdot f)$, where

$g \in \mathbb{G}, \partial \in \xi_* \mathcal{D}_{\widetilde{\mathbb{X}}}$ and $f \in \mathcal{O}_{\mathbb{X}}$, and the $\mathbb{G}$-action on $\mathbb{X}$ is given by left translation. It is clear that this restricts to a $\mathbb{G}$-action on $\xi_* \mathcal{O}_{\widetilde{\mathbb{X}}}$.

Applying the group homomorphism $\sigma : G \to \mathbb{G}(K)$, there is a G-action on $\xi_* \mathcal{O}_{\widetilde{\mathbb{X}}}$. It follows that:

$$\begin{aligned}(g \cdot (\partial \cdot a))(f) &= g \cdot \partial(a)(g^{-1} \cdot f)\\ ((g \cdot \partial) \cdot (g \cdot a))(f) &= (g \cdot \partial) \cdot (g \cdot a(g^{-1} \cdot f))\\ &= g \cdot \partial(g^{-1} \cdot g \cdot (a(g^{-1} \cdot f)))\\ &= g \cdot \partial(a)(g^{-1} \cdot f),\end{aligned}$$

and hence $\xi_* \mathcal{O}_{\widetilde{\mathbb{X}}}$ is a $\mathbb{G}$-equivariant $\xi_* \mathcal{D}_{\widetilde{\mathbb{X}}}$-module. The same calculation demonstrates that $\mathcal{O}^{\lambda}$ is a $\mathbb{G}$-equivariant $\mathcal{D}^{\lambda}$-module, in the sense of Definition 2.7.6. By Lemma 5.2.1, $\mathcal{O}^{\lambda}_{\mathbf{X}}$ is a G-equivariant $\mathcal{D}^{\lambda}$-module.

□

Lemma 5.2.9. There is an isomorphism of sheaves of rings:

$$\mathcal{D}^{\lambda}_{\mathbb{X}} \cong \mathcal{O}^{\lambda}_{\mathbb{X}} \otimes \mathcal{D}_{\mathbb{X}} \otimes \mathcal{O}^{-\lambda}_{\mathbb{X}}.$$

Proof. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules, and let $\mathbb{U} \subseteq \mathbb{X}$ be an affine open subset. We define the *endomorphism sheaf* $\mathscr{E}nd(\mathcal{F})$ to be the sheaf:

$$\mathbb{U} \mapsto \mathscr{H}om_K(\mathcal{F} |_{\mathbb{U}}, \mathcal{F} |_{\mathbb{U}}).$$

Recall that there is a $\mathcal{D}^\lambda$-action on $\mathcal{O}^\lambda$, and there is an $\mathcal{O}^\lambda \otimes \mathcal{D} \otimes \mathcal{O}^{-\lambda}$-action on $\mathcal{O}^\lambda$. Hence there are globally defined sheaf morphisms:

$$\rho_\lambda : \mathcal{D}^\lambda \to \mathscr{E}nd(\mathcal{O}^\lambda),$$

$$\sigma_\lambda : \mathcal{O}^\lambda \otimes \mathcal{D} \otimes \mathcal{O}^{-\lambda} \to \mathscr{E}nd(\mathcal{O}^\lambda).$$

There is an action of $\mathcal{D}$ on $\mathcal{O}$ which induces a sheaf morphism $\rho : \mathcal{D} \to \mathscr{E}nd(\mathcal{O})$. It is clear that this morphism agrees with the morphism $\sigma : \mathcal{O} \otimes \mathcal{D} \otimes \mathcal{O} \to \mathscr{E}nd(\mathcal{O})$.

Let $\{\mathbb{U}_i\}$ be a trivialising cover of $\mathbb{X}$, in the sense of Lemma 5.2.4. This allows us to form the following diagram of sheaves of K-algebras:

$$\begin{array}{ccccc}
(\mathcal{O}^\lambda \otimes \mathcal{D} \otimes \mathcal{O}^{-\lambda}) \mid_{\mathbb{U}_i} & \xrightarrow{\sigma_\lambda|_{\mathbb{U}_i}} & (\mathscr{E}nd(\mathcal{O}^\lambda)) \mid_{\mathbb{U}_i} & \xleftarrow[\rho_\lambda|_{\mathbb{U}_i}]{} & (\mathcal{D}^\lambda) \mid_{\mathbb{U}_i} \\
\downarrow{\scriptstyle \alpha^\lambda} & & \downarrow{\scriptstyle \beta^\lambda} & & \downarrow{\scriptstyle \phi^\lambda} \\
\mathcal{D} \mid_{\mathbb{U}_i} & \xrightarrow{\sigma|_{\mathbb{U}_i}} & \mathscr{E}nd(\mathcal{O}) \mid_{\mathbb{U}_i} & \xleftarrow[\rho|_{\mathbb{U}_i}]{} & \mathcal{D} \mid_{\mathbb{U}_i}
\end{array}$$

where the vertical arrows are each sheaf isomorphisms.

We verify that this diagram is commutative. Since all morphisms in the diagram are compatible with restrictions to Zariski open subschemes $\mathbb{V}_{ij}$ contained in $\mathbb{U}_i$, it suffices to show the corresponding diagram:

$$\begin{array}{ccccc}
(\mathcal{O}^\lambda \otimes \mathcal{D} \otimes \mathcal{O}^{-\lambda})(\mathbb{U}_i) & \xrightarrow{\sigma_\lambda(\mathbb{U}_i)} & (\mathscr{E}nd(\mathcal{O}^\lambda))(\mathbb{U}_i) & \xleftarrow[\rho_\lambda(\mathbb{U}_i)]{} & \mathcal{D}^\lambda(\mathbb{U}_i) \\
\downarrow{\scriptstyle \alpha^\lambda(\mathbb{U}_i)} & & \downarrow{\scriptstyle \beta^\lambda(\mathbb{U}_i)} & & \downarrow{\scriptstyle \phi^\lambda(\mathbb{U}_i)} \\
\mathcal{D}(\mathbb{U}_i) & \xrightarrow{\sigma(\mathbb{U}_i)} & \mathscr{E}nd(\mathcal{O})(\mathbb{U}_i) & \xleftarrow[\rho(\mathbb{U}_i)]{} & \mathcal{D}(\mathbb{U}_i)
\end{array}$$

is commutative.

First consider the left-hand square. Let $\gamma^\lambda(\mathbb{U}_i) : \mathcal{O}^\lambda(\mathbb{U}_i) \to \mathcal{O}(\mathbb{U}_i)$ be the local untwisting automorphism of $\mathcal{O}^\lambda$ on a piece of the trivialising cover $\{\mathbb{U}_i\}$. Note that:

$$\sigma_\lambda(\mathbb{U}_i)(a \otimes \partial \otimes a') = (m \mapsto a \otimes \partial(a' \otimes m))$$

for $a, m \in \mathcal{O}^\lambda(\mathbb{U}_i), a' \in \mathcal{O}^{-\lambda}(\mathbb{U}_i), \partial \in \mathcal{D}(\mathbb{U}_i)$. It follows that:

$$(\beta^\lambda \circ \sigma_\lambda)(\mathbb{U}_i) = \beta^\lambda(\mathbb{U}_i)(m \mapsto a \otimes \partial(a' \otimes m))$$

$$(\sigma \circ \alpha^\lambda)(\mathbb{U}_i) = \sigma(\mathbb{U}_i)(a\partial a') = (m \mapsto a\partial(a'm))$$

where a, a', m in the second equation are interpreted as the corresponding elements of $\mathcal{O}(\mathbb{U}_i)$ under the local untwisting automorphism $\gamma^\lambda(\mathbb{U}_i)$. Hence the left-hand square is commutative.

For the right-hand square, we have:

$$(\beta^\lambda \circ \rho_\lambda(\mathbb{U}_i)) = \beta^\lambda(m \mapsto \partial(m))$$

$$(\rho \circ \phi^\lambda)(\mathbb{U}_i)(m) = \rho(\mathbb{U}_i)(\phi^\lambda(\partial)) = (m \mapsto \partial(m))$$

where ∂, m in the second equation are the corresponding elements of $\mathcal{D}(\mathbb{U}_i), \mathcal{O}(\mathbb{U}_i)$ respectively under the local untwisting automorphisms $\phi^\lambda(\mathbb{U}_i), \gamma^\lambda(\mathbb{U}_i)$. Hence the right-hand square is commutative.

By [38, II.1.10(2)], we may assume $\mathbb{U}_i \cong \mathbb{A}^{\dim \widetilde{\mathbb{X}}}$. By [19, Theorem 2.1], the ring $\mathcal{D}(\mathbb{U}_i)$ is simple and so the morphisms $\rho(\mathbb{U}_i), \sigma(\mathbb{U}_i)$ are injective, and they are equal by construction. Commutativity of the diagram implies that $\rho_\lambda(\mathbb{U}_i)$, $\sigma_\lambda(\mathbb{U}_i)$ are injective and their images agree. All morphisms in the diagram are global sheaf morphisms, and so the global morphisms $\rho_\lambda, \sigma_\lambda$ are injective and their images agree. The claimed isomorphism follows. □

Lemma 5.2.10. Let $\mathcal{M} \in (G - \mathcal{D}) - \text{mod}$. Then $\mathcal{O}^\lambda_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M} \in (G - \mathcal{D}^\lambda) - \text{mod}$.

Proof. By the isomorphism $\mathcal{D}^\lambda \cong \mathcal{O}^\lambda \otimes \mathcal{D} \otimes \mathcal{O}^{-\lambda}$ from Lemma 5.2.9, there is a $\mathcal{D}^\lambda$-action on $\mathcal{O}^\lambda \otimes \mathcal{M}$, given by the formula:

$$(a \otimes \partial \otimes a') \cdot (b \otimes m) = a \otimes \partial(m)$$

for $a, b \in \mathcal{O}^\lambda, a' \in \mathcal{O}^{-\lambda}, \partial \in \mathcal{D}$ and $m \in \mathcal{M}$. Let $g \in G$. We compute:

$$\begin{aligned} g \cdot ((a \otimes \partial \otimes a') \cdot (b \otimes m)) = g \cdot (a \otimes \partial(m)) = g \cdot a \otimes g \cdot \partial(m) \\ = g \cdot a \otimes (g \cdot \partial) \cdot (g \cdot m) \\ = (g \cdot (a \otimes \partial \otimes a')) \cdot (g \cdot (b \otimes m)), \end{aligned}$$

since $\mathcal{M} \in (G - \mathcal{D}) - \text{mod}$. Hence $\mathcal{O}^\lambda_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M} \in (G - \mathcal{D}^\lambda) - \text{mod}$ by definition. □

Lemma 5.2.11. Let $\mathcal{M} \in \text{Frech}(G - \mathcal{D}_{\mathbf{X}})$. Then $\mathcal{O}^\lambda_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M} \in \text{Frech}(G - \mathcal{D}^\lambda_{\mathbf{X}})$.

Proof. By Lemma 5.2.10, $\mathcal{O}_{\mathbf{X}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M} \in (G - \mathcal{D}^{\lambda}) - \text{mod}$. Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and suppose $\mathcal{M}_{\mathbf{U}} \in \text{Frech}(G_{\mathbf{U}} - \mathcal{D}_{\mathbf{U}})$, so, for $\mathbf{V} \in \mathbf{U}_w(\mathcal{T})$, the K-algebra $\mathcal{M}_{\mathbf{U}}(\mathbf{V})$ is equipped with a Fréchet topology and the K-algebra morphisms $g^{\mathcal{M}_{\mathbf{U}}(\mathbf{V})} : \mathcal{M}_{\mathbf{U}}(\mathbf{V}) \to \mathcal{M}_{\mathbf{U}}(g\mathbf{V})$ are continuous. The $\mathcal{O}(\mathbf{V})$-module $\mathcal{O}^{\lambda}(\mathbf{V})$ is a section of the sheaf $\mathcal{O}^{\lambda}$ over an affinoid subdomain $\mathbf{V} \in \mathbf{X}_w(\mathcal{T})$ and hence carries a Fréchet topology. Hence the tensor product $\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{M}_{\mathbf{U}}(\mathbf{V})$ carries a Fréchet topology. Furthermore, for all $g \in G$, the algebra morphisms $\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{M}_{\mathbf{U}}(\mathbf{V}) \to \mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{M}_{\mathbf{U}}(g\mathbf{V})$ are continuous by construction. It follows that $\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{M}_{\mathbf{U}} \in \text{Frech}(G_{\mathbf{U}} - \mathcal{D}_{\mathbf{U}}^{\lambda})$. □

The functor $\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \text{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}$ defined by, for $\mathbf{V} \in \mathbf{U}_w(\mathcal{T})$:

$$(\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \text{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})})(M_{\mathbf{U}})(\mathbf{V}) = \mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} (\text{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})})(M_{\mathbf{U}})(\mathbf{V})$$

is a functor from coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-modules to G-equivariant locally Fréchet $\mathcal{D}_{\mathbf{U}}^{\lambda}$-modules on $\mathbf{U}$, by applying [1, Proposition 3.6.6].

Lemma 5.2.12. Let $\mathcal{M} \in \text{Frech}(G - \mathcal{D})$. There is an isomorphism of locally Fréchet $\mathcal{D}$-modules:

$$\mathcal{O}^{-\lambda} \otimes_{\mathcal{O}} \mathcal{O}^{\lambda} \otimes_{\mathcal{O}} \mathcal{M} \cong \mathcal{M}.$$

Proof. By Lemma 5.2.11, $\mathcal{O}^{-\lambda} \otimes_{\mathcal{O}} \mathcal{O}^{\lambda} \otimes_{\mathcal{O}} \mathcal{M} \in \text{Frech}(G - \mathcal{D})$. Since $\mathcal{O}^{\lambda}$ is a locally free sheaf of rank 1 by Lemma 5.2.7, it is invertible by definition, and $\mathcal{O}^{-\lambda}$ is its inverse. Hence there is an isomorphism of $\mathcal{D}$-modules:

$$\mathcal{O}^{-\lambda} \otimes_{\mathcal{O}} \mathcal{O}^{\lambda} \otimes_{\mathcal{O}} \mathcal{M} \cong \mathcal{M}.$$

For each $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, the induced morphism $\mathcal{O}^{-\lambda} \otimes_{\mathcal{O}} \mathcal{O}^{\lambda} \otimes_{\mathcal{O}} \mathcal{M}(\mathbf{U}) \cong \mathcal{M}(\mathbf{U})$ is continuous. Hence we have an isomorphism of locally Fréchet $\mathcal{D}$-modules. $\square$

Definition 5.2.13. Let $\mathcal{M}$ be a G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-module on $\mathbf{X}$.

(a) Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering. $\mathcal{M}$ is $\mathcal{U} -$ *coadmissible* if, for each $\mathbf{U} \in \mathcal{U}$, there is a $\mathbf{U}$-small subgroup $H_{\mathbf{U}}$ of G, a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module $M_{\mathbf{U}}$, and an isomorphism:

$$\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \operatorname{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}) \to \mathcal{M}_{\mathbf{U}_{\text{rig}}}$$

of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules on $\mathbf{U}$.

(b) $\mathcal{M}$ is *coadmissible* if it is $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$.

(c) The full subcategory of $\text{Frech}(G - \mathcal{D}_{\mathbf{X}}^{\lambda})$ consisting of coadmissible G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules is denoted by:

$$\mathcal{C}_{\mathbf{X}/G}^{\lambda}.$$

It may seem reasonable to expect that the twisted category $\mathcal{C}_{\mathbf{X}/G}^{\lambda}$ could be defined in the following way. Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering and, for each $\mathbf{U} \in \mathcal{U}$, let $\theta_{\mathbf{U}} : \mathcal{D}_{\mathbf{U}}^{\lambda} \to \mathcal{D}_{\mathbf{U}}$ be an $\mathcal{O}_{\mathbf{U}}$-linear isomorphism in the sense of Definition 5.1.2.

We may define the category $G-\mathcal{D}^\lambda$-mod as in Definition 2.7.6. There is a natural functor $[\theta_{\mathbf{U}}] : G-\mathcal{D}^\lambda_{\mathbf{U}}$-mod $\to G-\mathcal{D}_{\mathbf{U}}$-mod. As sheaves of K-vector spaces, $[\theta_{\mathbf{U}}](\mathcal{M}_{\mathbf{U}}) = \mathcal{M}_{\mathbf{U}}$, but the $\mathcal{D}_{\mathbf{U}}$-action on $[\theta_{\mathbf{U}}](\mathcal{M}_{\mathbf{U}})$ is given by:

$$a \cdot_{\mathcal{D}^\lambda_{\mathbf{U}}} ([\theta_{\mathbf{U}}]m) = [\theta_{\mathbf{U}}](\theta_{\mathbf{U}}^{-1}(a) \cdot_{\mathcal{D}_{\mathbf{U}}} m) \text{ for all } a \in \mathcal{D}_{\mathbf{U}}, m \in \mathcal{M}_{\mathbf{U}}.$$

If $f : \mathcal{M}_{\mathbf{U}} \to \mathcal{N}_{\mathbf{U}}$ is a morphism in $G-\mathcal{D}^\lambda_{\mathbf{U}}$-mod, then we define $[\theta_{\mathbf{U}}]f : [\theta_{\mathbf{U}}](\mathcal{M}_{\mathbf{U}}) \to [\theta_{\mathbf{U}}](\mathcal{N}_{\mathbf{U}})$ to be the unique morphism that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{M}_{\mathbf{U}} & \xrightarrow{f} & \mathcal{N}_{\mathbf{U}} \\ \downarrow{\scriptstyle [\theta_{\mathbf{U}}]} & & \downarrow{\scriptstyle [\theta_{\mathbf{U}}]} \\ [\theta_{\mathbf{U}}]\left(\mathcal{M}_{\mathbf{U}}\right) & \xrightarrow{[\theta_{\mathbf{U}}]f} & [\theta_{\mathbf{U}}]\left(\mathcal{N}_{\mathbf{U}}\right) \end{array}$$

We then say that $\mathcal{C}^\lambda_{\mathbf{X}/G}$ is the full subcategory of $\text{Frech}(G-\mathcal{D}^\lambda_{\mathbf{X}})$ consisting of $\mathcal{D}^\lambda$-modules $\mathcal{M}$ such that, for each $\mathbf{U} \in \mathcal{U}$, $[\theta_{\mathbf{U}}](\mathcal{M}_{\mathbf{U}_{\text{rig}}}) \cong \text{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}})$. The following example demonstrates why this cannot be the case in general, since the local isomorphisms $\theta_{\mathbf{U}}$ are not necessarily $H_{\mathbf{U}}$-equivariant for any open subgroup $H_{\mathbf{U}}$ of G.

Example 5.2.14. We illustrate these constructions by studying the case $\mathbb{G} = SL_2$, $G = SL_2(\mathbb{Q}_p)$. Let $\mathbb{B}$ denote the Borel subgroup consisting of lower triangular matrices with unipotent radical $\mathbb{N}$. Let $\mathbb{H} = \mathbb{B}/\mathbb{N}$ denote the abstract Cartan group. This is isomorphic to $\mathbb{G}_m$.

The flag variety of $\mathbb{G}$, $\mathbb{X}$, is isomorphic to $\mathbb{P}^1$, and the basic affine space $\mathbb{G}/\mathbb{N}$ is isomorphic to $\mathbb{A}^2 \setminus \{0\}$. We have an $\mathbb{H}$-torsor ξ given by:

$$\xi(a,b) = [a:b].$$

Let $\{\mathbb{U}, \mathbb{V}\}$ be an open affine covering for $\mathbb{P}^1$. Suppose $\mathbb{U} \cong \mathbb{A}^1$ with local coordinate $z := xy^{-1}$, and $\mathbb{V} \cong \mathbb{A}^1$ with local coordinate z^{-1}. Then there is an $\mathbb{H}$-invariant isomorphism:

$$\mathbb{U} \times \mathbb{H} \to \xi^{-1}(\mathbb{U})$$

$$(z,t) \mapsto (tz,t)$$

Hence $\mathbb{U}$ trivialises the torsor ξ in the sense of Definition 2.5.2, and we may identify $\xi^{-1}(\mathbb{U})$ with the open subscheme $\mathbb{A}^2 \setminus \{y=0\}$. Since the same is true for $\mathbb{V}$, ξ is a locally trivial $\mathbb{H}$-torsor.

By [4, Proposition 4.6], there is an isomorphism of sheaves of K-algebras:

$$\widetilde{\mathcal{D}}_{\mathbb{U}} \to \mathcal{D}_{\mathbb{U}} \otimes_K U(\mathfrak{h}).$$

It follows that $\widetilde{\mathcal{D}}(\mathbb{U}) \cong \frac{K[z,\partial_z,h]}{([\partial_z,z]-1)}$. Applying the isomorphism $\mathcal{D}^\lambda_{\mathbb{U}} \cong \widetilde{\mathcal{D}}_{\mathbb{U}} \otimes_{U(\mathfrak{h})} K_\lambda$, we see that:

$$\mathcal{D}^\lambda(\mathbb{U}) \cong \frac{K[z,\partial_z,h]}{([\partial_z,z]-1, h-\lambda(h))}.$$

Let $J \leq G_{\mathbb{U}}$ be an open subgroup of G and let $g \in J$. View $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in J$ as a matrix. Explicitly, the J-action on $\mathcal{D}^\lambda(\mathbb{U})$ is given by:

$$g \cdot z = \frac{dz - b}{-cz + a},$$
$$g \cdot \partial_z = c(a - cz)(h - 2z\partial_z) + (a - cz)^2\partial_z,$$
$$g \cdot h = h.$$

Give $\mathcal{D}(\mathbb{U})$ the filtration by order of differential operator from [32, Proposition 1.1.3], and let $u \in F_1\mathcal{D}(\mathbb{U})$. Let $\phi : \mathcal{D}^\lambda(\mathbb{U}) \to \mathcal{D}(\mathbb{U})$ be a filtered $\mathcal{O}(\mathbb{U})$-linear local ring automorphism. By assumption, $\phi(g) = g$ for all $g \in F_0\mathcal{D}(\mathbb{U})$ and so we see that:

$$[\phi(u), g] = \phi([u, g]) = [u, g]$$

since $[u, g] \in F_0\mathcal{D}(\mathbb{U})$. Hence $[\phi(u) - u, g] = 0$ for all $g \in F_0\mathcal{D}(\mathbb{U})$ and so $\phi(u) - u \in F_0\mathcal{D}(\mathbb{U})$.

It follows that the filtered local isomorphism $\phi : \mathcal{D}^\lambda(\mathbb{U}) \to \mathcal{D}(\mathbb{U})$ is obtained by a function of the form:

$$z \mapsto z, \partial_z \mapsto \partial_z - f(z), h \mapsto \lambda(h).$$

for some $f \in \mathcal{O}(\mathbb{U})$. Since ϕ is filtered, f has degree 1 and so we may write $f(z) = \alpha z + \beta$.

Suppose that ϕ were a J-equivariant isomorphism. In this case, $\phi(g \cdot \partial_z) = g \cdot \phi(\partial_z)$. Then we see that:

$$\phi(g\cdot\partial_z) = \phi(c(a-cz)(h-2z\partial_z)+(a-cz)^2\partial_z)$$

$$= c(a-cz)(\lambda(h)-2z(\partial_z-\alpha z-\beta))+(a-cz)^2(\partial_z-\alpha z-\beta),$$

$$g\cdot\phi(\partial_z) = c(a-cz)(-2z\partial_z)+(a-cz)^2\partial_z+g\cdot(-\alpha z-\beta).$$

Hence $c(a-cz)(\lambda(h)-2z(\alpha z+\beta))-(a-cz)^2(\alpha z+\beta) = -\alpha(\frac{dz-b}{-cz+a})-\beta$. Provided $\lambda(h)\neq 0$, for any fixed values of α and β, this is only true in general if $c=0$. It follows that J must be contained in the Borel subgroup B consisting of upper triangular matrices in G. But there are no open subgroups of G contained in B, and so ϕ cannot be J-equivariant.

This calculation demonstrates the difficulties with simplifying Definition 5.2.13 by imposing the condition that $\mathcal{M}\in\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ if the untwist $\mathcal{O}^{-\lambda}_{\mathbf{X}}\otimes_{\mathcal{O}_{\mathbf{X}}}\mathcal{M}\in\mathcal{C}_{\mathbf{X}/G}$. In this case, the corresponding $G_{\mathbf{U}}$-action on $\mathcal{M}(\mathbf{U})$ and $\mathcal{O}^{-\lambda}_{\mathbf{X}}(\mathbf{U})\otimes_{\mathcal{O}_{\mathbf{X}}(\mathbf{U})}\mathcal{M}(\mathbf{U})$ cannot be identified in a way that is compatible with the local isomorphism by the above calculation. More explicitly, $\mathcal{O}^{\lambda}_{\mathbf{X}}(\mathbf{U})$ and $\mathcal{O}_{\mathbf{X}}(\mathbf{U})$ are not isomorphic as $\widehat{\mathcal{D}}(\mathbf{U},G_{\mathbf{U}})$-modules, since the $G_{\mathbf{U}}$-action is shifted. We keep track of this shift by keeping the $\mathcal{O}^{\lambda}_{\mathbf{X}}$-term in the definition of $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$.

The $\mathcal{O}(\mathbb{U})$-module $\mathcal{O}^{\lambda}(\mathbb{U})$ may be identified with the following set:

$$\mathcal{O}^{\lambda}(\mathbb{U})\cong\{\sum_{i,j\in\mathbb{Z},i\geq 0,j\leq\lambda(h)} a_{ij}x^iy^j \mid a_{ij}\in K, i+j=\lambda(h)\}.$$

This is a free $\mathcal{O}(\mathbb{U})$-module on the generator $e:=y^{\lambda(h)}$.

Let $f \in \mathcal{O}(\mathbb{U})$. The $\mathcal{D}^\lambda(\mathbb{U})$-action given by $\partial_z \cdot e = 0, \partial_z \cdot (fe) = \partial_z(f) \cdot e$. The G-action on $\mathcal{O}^\lambda(\mathbb{U})$ is given by:

$$g \cdot (x^i y^j) = (dx - by)^i(-cx + ay)^j.$$

Note that $g \cdot (x^k y^{\lambda(h)-k}) = (dx - by)^k(-cx + ay)^{\lambda(h)-k}$. It follows that, for $f \in \mathcal{O}(\mathbb{U})$, $g \cdot (f \cdot e) = (g \cdot f) \cdot (g \cdot e)$. Hence $\mathcal{O}^\lambda(\mathbb{U})$ is a $G_{\mathbb{U}}$-equivariant $\mathcal{D}^\lambda(\mathbb{U})$-module in the sense of Definition 2.7.6.

The next part of this section is devoted to establishing a technical result on extending continuous $\mathcal{A}$-linear K-automorphisms of $\mathcal{D}(\mathbf{U})$ to continuous $\mathcal{A}$-linear K-automorphisms of $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$. This result turns out to be a key ingredient in demonstrating the existence of a natural functor $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) - \text{mod} \to \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) - \text{mod}$, which we will eventually show to define an equivalence of categories.

Let $\mathbf{X} := \widehat{\mathbb{X}_{\text{rig}}}$ denote the rigid analytic flag variety. By [38, II.1.10(2)], we may assume $\mathbf{X}$ admits a finite covering $\mathcal{U}$, where each $\mathbf{U} \in \mathcal{U}$ is isomorphic to $(\widehat{\mathbb{A}^n_{\mathcal{R}}})_{\text{rig}}$, where n is the dimension of $\mathbf{X}$, and so $(\widehat{\mathbb{A}^n_{\mathcal{R}}})_{\text{rig}}$ is an affinoid subvariety of $\mathbf{X}$. Let $A = \mathcal{O}(\mathbf{U})$ be a K-affinoid algebra, and let $\mathcal{A}$ be a G-stable affine formal model in A, in the sense of Definition 2.8.4. Let $L = \text{Der}_K(A) = \mathcal{T}(\mathbf{U})$ and $\mathcal{L}$ an $\mathcal{A}$-Lie lattice inside L. Note that $\pi^n\mathcal{L}$ is a G-stable $\mathcal{A}$-Lie lattice for all $n \in \mathbb{N}$, by [5, Lemma 6.1(a)]. Recall also that we have a continuous injective group homomorphism $\rho : G_{\mathbf{U}} \to \text{Aut}_K(\mathbf{U}, \mathcal{O}_{\mathbf{U}})$, obtained by restricting the map $\rho : G \to \text{Aut}_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ to the stabiliser $G_{\mathbf{U}}$ of $\mathbf{U}$ under the G-action.

We recall the construction from Theorem 2.8.5. Let $\mathcal{E} := \mathrm{End}_{\mathcal{R}}(\mathcal{A})$, and $\mathcal{U} := \overline{\widehat{U(\mathcal{L})}}$. Denote by ι the natural map $\overline{\widehat{i_{\mathcal{A}} \oplus i_{\mathcal{L}}}} : \mathcal{A} \oplus \mathcal{L} \to \mathcal{U}$. By [1, Lemma 3.2.10(a)], there is a unique $\mathcal{R}$-algebra homomorphism $\psi_{\mathcal{L}} : \mathcal{U} \to \mathcal{E}$ such that $\psi_{\mathcal{L}}(\iota(a)) = i_{\mathcal{A}}(a)$ and $\psi_{\mathcal{L}}(\iota(v)) = v$ for all $a \in \mathcal{A}$ and $v \in \mathcal{L}$. The map $\beta_{\mathcal{L}} := (\psi_{\mathcal{L}}^{\times})^{-1} \circ \rho : G_{\mathcal{L}} \to \mathcal{U}^{\times}$ is a G-equivariant trivialisation of the $G_{\mathcal{L}}$-actions on $\overline{\widehat{U(\mathcal{L})}}$ and $\widehat{U(\mathcal{L})_K}$, fitting into the commutative diagram:

$$\begin{array}{ccccc} & & G_{\mathcal{L}} & \longrightarrow & G \\ & \swarrow^{\rho|_{G_{\mathcal{L}}}} & \big\downarrow{\scriptstyle \beta_{\mathcal{L}}} & & \big\downarrow{\scriptstyle \rho} \\ \exp(p^e\mathcal{L}) & \longrightarrow & \mathcal{U}^{\times} & \xrightarrow{\psi_{\mathcal{L}}^{\times}} & \mathcal{E}^{\times}. \end{array}$$

Theorem 5.2.15. *Let $\phi^{\lambda}_{\boldsymbol{U}} : \mathcal{D}(\boldsymbol{U}) \to \mathcal{D}(\boldsymbol{U})$ be a continuous $\mathcal{A}$-linear ring automorphism which is bounded with respect to the norm on $\mathcal{D}(\boldsymbol{U})$ induced by the $\mathcal{R}$-subalgebra $U(\mathcal{L})$ of $U(\mathcal{L})_K$. There is a compact open subgroup $H_{\boldsymbol{U}}$ of G and a unique $\mathcal{A}$-linear ring automorphism $\widehat{\phi}^{\lambda}_{\boldsymbol{U}} := \widehat{\mathcal{D}}(\boldsymbol{U}, H_{\boldsymbol{U}}) \to \widehat{\mathcal{D}}(\boldsymbol{U}, H_{\boldsymbol{U}})$ extending $\phi^{\lambda}_{\boldsymbol{U}}$.*

The next part of the thesis works towards the proof of Theorem 5.2.15.

By definition, we may write $\mathcal{D}(\mathbf{U}) = U(\mathcal{L})_K$. This is a dense subalgebra of the K-algebra $\widehat{U(\mathcal{L})}_K$: it follows that $\phi^{\lambda}_{\mathbf{U}}$ has a unique extension to a continuous $\mathcal{A}$-linear K-automorphism $\widehat{\phi}^{\lambda}_{\mathbf{U}} : \widehat{U(\mathcal{L})}_K \to \widehat{U(\mathcal{L})}_K$ by continuity.

For clarity, write $\widehat{\phi}^{\lambda} := \widehat{\phi}^{\lambda}_{\mathbf{U}}$. By Theorem 2.8.5, there is a G-equivariant trivialisation $\beta = \beta_{\mathcal{L}} : G_{\mathcal{L}} \to \widehat{U(\mathcal{L})}_K^{\times}$. Write $U_n := \widehat{U(\pi^n\mathcal{L})}_K$, and for each $n \in \mathbb{N}$, set:

$$H_n := \beta^{-1}(U_n^{\times}).$$

Lemma 5.2.16. The natural map $U_n \to U_0$ is injective.

Proof. First note that we may choose $\mathcal{L}$ and $\pi^n\mathcal{L}$ both to be smooth $\mathcal{A}$-Lie lattices in L inside the smooth rigid affinoid variety $\mathbf{X}$. In this case, the natural maps $i_{\mathcal{L}} : \mathcal{L} \to U(\mathcal{L})$ and $i_{\pi^n\mathcal{L}} : \pi^n\mathcal{L} \to U(\pi^n\mathcal{L})$ are injective by [52, Theorem 3.1].

We have an injective map $\iota : \pi^n\mathcal{L} \to \mathcal{L}$ of $\mathcal{A}$-Lie lattices. This morphism induces a morphism $U(\iota) : U(\pi^n\mathcal{L}) \to U(\mathcal{L})$ by [5, Section 2.1], which is injective since the maps $i_{\mathcal{L}}$ and $i_{\pi^n\mathcal{L}}$ are both injective.

The functor $\widehat{(-)}$ is left exact since it may be realised as an inverse limit, and the functor $(-)_K$ is left exact since K is a field and hence a flat $\mathcal{R}$-module. The result follows. □

Lemma 5.2.17. Let $H := H_0 = G_{\mathcal{L}}$ and define $\beta_n : H_n \to U_n \subseteq U_0$ by $\beta |_{H_n} = \beta_n$. Then β_n is a G-equivariant trivialisation of $U_n \rtimes H$.

Proof. Given $h \in H_n$, the conjugation-by-$\beta_n(h)$-action on U_0 agrees with the h-action on U_0. Applying Lemma 5.2.16, U_n is an h-stable subring of U_0, it is preserved under conjugation by $\beta_n(h)$ and so the conjugation-by-$\beta_n(h)$-action on U_n agrees with the h-action on U_n. □

Let $\overline{\beta} : U_n[H] \to U_n \rtimes H$ be the ring isomorphism provided by Lemma 5.2.17. This allows us to form the ring $U_n \rtimes_{H_n} H$ for any $n \in \mathbb{N}$ as a quotient of $U_n \rtimes H$ by the two-sided ideal $(U_n \rtimes H) \cdot (\overline{\beta}(H_n) - 1)$.

Let $\widehat{\mathcal{A}_K}$ denote the Tate-Weyl algebra:

$$\widehat{\mathcal{A}_K} := \{ \sum_{\alpha,\beta\in\mathbb{N}^n} \lambda_{\alpha\beta} x^\alpha \partial^\beta \mid \lambda_{\alpha\beta} \in K \text{ and } |\lambda_{\alpha\beta}| \to 0 \text{ as } |\alpha| + |\beta| \to \infty\}.$$

This should be viewed as the p-adic completion of the Weyl algebra generated by $\{x_1, \cdots, x_n, \partial_1, \cdots, \partial_n\}$ with coefficients in K. Here, x^α is understood to be the product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, and similarly for ∂^β.

Lemma 5.2.18. $\widehat{\mathcal{A}_K}$ is a simple algebra with centre $Z(\widehat{\mathcal{A}_K}) = K$.

Proof. The claims follows from [49, Corollary 1.4.5 and Proposition 1.4.6]. □

Lemma 5.2.19. The group action of H on U_n induces a well-defined injective group homomorphism $\frac{H}{H_n} \to \mathrm{Out}(U_n)$. It follows that the ring $U_n \rtimes_{H_n} H$ is simple.

Proof. There is a G-equivariant trivialisation $\beta_n : H_n \to U_n^\times$ by Lemma 5.2.17, and so the action of $h \in H_n$ on U_n coincides with the conjugation action of $\beta_n(h)$. It follows that, if $h \in H$, then h acts on U_n by an inner automorphism. It suffices to show that, given $h \in H$ which acts by an inner automorphism on U_n, then $h \in H_n$.

By assumption, the conjugation action by $\beta_0(h)$ agrees with the conjugation action by α for some $\alpha \in U_n^\times$. The action of U_0 on itself by conjugation defines a group homomorphism $\gamma : U_0^\times \to \mathrm{Aut}\ U_0$, with $\ker \gamma = Z(U_0) = K$ by Lemma 5.2.18. By construction, $\beta_0(h)\alpha^{-1}$ is a unit in U_0 which commutes with each element of U_n. Since multiplication is continuous and U_n is a dense subring of U_0, it follows that $\beta_0(h)\alpha^{-1}$ is contained in $Z(U_0) = K$. Hence $\beta_0(h)$ is a non-zero scalar multiple of α and so lies inside $U_n^\times$.

By [1, Proposition 2.2.4(b)], $U_n \rtimes_{H_n} H$ is isomorphic to a crossed product $U_n * \frac{H}{H_n}$, which is a crossed product of a simple ring with a group of outer automorphisms acting on that ring. We may apply the same argument as in [45, Proposition 7.8.12] to show that the ring $U_n \rtimes_{H_n} H$ is simple. The proof in *loc. cit.* is only stated for a skew-group ring S, but it it also valid in the situation of a general crossed product.

□

Lemma 5.2.20. For each $n \in \mathbb{N}$, there is a natural injective ring homomorphism $i_n : U_n \rtimes_{H_n} H \to U_0$.

Proof. Recall from Lemma 5.2.16 that we have a natural injective map $f : U_n \to U_0$. It follows that we also have an injective group homomorphism $\beta^{-1} \circ f \circ \beta : H_n \to H_0$. Hence we may apply [1, Lemma 2.2.7], with the τ in this lemma as the identity map, to deduce the existence of a ring homomorphism $i_n : U_n \rtimes_{H_n} H \to U_0$. This map is non-zero and therefore is injective since the ring $U_n \rtimes_{H_n} H$ is simple by Lemma 5.2.19. □

The map $\widehat{\phi}_n := \widehat{\phi} \mid_{U_n \rtimes_{H_n} H}: U_n \rtimes_{H_n} H \to U_0$ is defined by $\widehat{\phi}_n(u, h) = (\widehat{\phi} \circ i_n)(u, h)$. Since $i_n : U_n \rtimes_{H_n} H \to U_0$ is a well-defined ring homomorphism, so is $\widehat{\phi}_n$.

Let $\{x_1, \cdots x_n, \partial_1, \cdots, \partial_n\}$ be a set of algebra generators for the algebra of finite-order differential operators $\mathcal{D}(\mathbf{U})$ on $\mathbf{U}$, and let $\gamma : \mathcal{D}(\mathbf{U}) \to \mathcal{D}(\mathbf{U})$ be a ring automorphism such that $\gamma(x_i) = x_i$ for each x_i. We may find an increasing filtration $F_i\mathcal{D}(\mathbf{U})$ of $\mathcal{D}(\mathbf{U})$ such that $\mathcal{D}(\mathbf{U}) = \bigcup_{i \geq 0} F_i\mathcal{D}(\mathbf{U})$, $F_0\mathcal{D}(\mathbf{U}) = T_n = K\langle x_1, \cdots, x_n\rangle$ and, given $P \in F_l\mathcal{D}(\mathbf{U})$ and $Q \in F_m\mathcal{D}(\mathbf{U})$, $[P, Q] \in F_{l+m-1}\mathcal{D}(\mathbf{U})$.

Lemma 5.2.21. For all $u \in F_1\mathcal{D}(\mathbf{U})$, $\gamma(u) = u - f$ for some $f \in K\langle x_1, \cdots x_n\rangle$, dependent on u.

Proof. Let $u \in F_1\mathcal{D}(\mathbf{U})$. By assumption, $\gamma(g) = g$ for all $g \in F_0\mathcal{D}(\mathbf{U})$ and so we see that:

$$[\gamma(u), g] = \gamma([u, g]) = [u, g]$$

since $[u, g] \in F_0\mathcal{D}(\mathbf{U})$. Hence $[\gamma(u) - u, g] = 0$ for all $g \in F_0\mathcal{D}(\mathbf{U})$ and so $\gamma(u) - u \in F_0\mathcal{D}(\mathbf{U})$.

□

Proposition 5.2.22. The image of $\widehat{\phi}_n^\lambda$ is contained in $U_n \rtimes_{H_n} H$.

Proof. Note that the image of the map $\beta : H \to U_0$ is contained in $\exp p^e\mathcal{L}$, by construction. Since $\mathcal{L} \subseteq \mathcal{T}(\mathbf{U}) = F_1\mathcal{D}(\mathbf{U})$, it follows that, for $h \in H$, we may identify $\beta(h) = \exp(\partial)$, where $\partial \in p^e\mathcal{L} \subseteq F_1\mathcal{D}(\mathbf{U})$.

Let $(u, h) \in U_n \rtimes_{H_n} H$. It follows that:

$$\widehat{\phi}_n^\lambda(u, h) = \widehat{\phi}^\lambda(u\beta(h))$$
$$= \widehat{\phi}^\lambda(u\exp(\partial)) = \widehat{\phi}^\lambda(u)\exp(\widehat{\phi}^\lambda(\partial))$$
$$= \widehat{\phi}^\lambda(u)\exp(\partial - f)$$

where $\widehat{\phi}^\lambda(u\exp(\partial)) = \widehat{\phi}^\lambda(u)\exp(\widehat{\phi}^\lambda(\partial))$ since $\widehat{\phi}^\lambda$ is a continuous $\mathcal{A}$-linear ring automorphism by assumption, and $f \in \mathcal{O}(\mathbf{U})$.

By definition of the map i_n, it suffices to show that $\exp(\partial - f)$ is contained in $U_n \cdot \mathrm{im}\beta$.

Applying the Baker-Campbell-Hausdorff formula as in [43, Equation 12], we see that, for general variables A, B:

$$\exp(X)\exp(Y) = \exp(X)(1 + \sum_{t=1}^{\infty} \sum_{n_1,\cdots,n_t=1}^{\infty} \frac{(-1)^{(\sum_{i=1}^{t} n_i)-t} \prod_{i=1}^{t} n_i}{\prod_{j=0}^{t-1} (\sum_{k=t}^{t-j} n_k)} (\prod_{i=1}^{t} \mathcal{B}_{n_i})).$$

where the $\mathcal{B}_{n_i}$ are defined as in [43, Equation 8]. In this case, the only non-zero terms occur when $t = 1$, since a general nested commutator containing k Xs and l Ys is non-zero if and only if $l = 1$ and the commutator is of the form $[X, [X, [X, \cdots, [X, Y]], \cdots]$. Then, writing $\partial = X, f = Y$, $[X, [X, [X, \cdots, [X, Y]], \cdots] = \partial^{k-1}(f)$, it follows that this sum simplifies to:

$$\exp(\partial - f) = \exp(\partial)\exp(-f + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k!} \partial^{(k-1)} f),$$

We verify that $\exp(-f + \sum_{k\geq 2} \frac{(-1)^{k-1}}{k!} \partial^{(k-1)} f) \in A$. Recall that the map $\widehat{\phi}^{\lambda} : U_0 \to U_0$ is a continuous ring automorphism . We claim that $\widehat{\phi}^{\lambda}$ is an isometry. To see this, let $\partial \in \mathcal{L}$ such that $|\partial| = 1$, let $\widehat{\phi}^{\lambda}(\partial) = \partial - f$ and suppose that $|f| > 1$. By the strong triangle inequality, $|\widehat{\phi}^{\lambda}(\partial)| = |f|$.

Expanding $(\partial - f)^m$, we see that $\widehat{\phi}^{\lambda}(\partial^m) = \widehat{\phi}^{\lambda}(\partial)^m$ can be written as the sum of f^m and terms with norm strictly less than $|f^m|$. As A is power-multiplicative, $|f^m| = |f|^m$ and so $|\widehat{\phi}^{\lambda}(\partial^m)| = |f|^m$. Hence:

$$\frac{|\widehat{\phi}^{\lambda}(\partial^m)|}{|\partial^m|} \geq (\frac{|f|}{|\partial|})^m \to \infty$$

as $m \to \infty$. Hence $\widehat{\phi}^{\lambda}$ is not bounded in norm on $U(\mathcal{L})_K$, which contradicts the assumption that $\widehat{\phi}^{\lambda}$ is continuous. Hence $|\widehat{\phi}^{\lambda}(x)| \leq |x|$ for any $x \in U(\mathcal{L})_K$. Since $\widehat{\phi}^{\lambda}$ is an automorphism, applying the same argument to its inverse shows that $|\widehat{\phi}^{\lambda}(x)| = |x|$. Hence $\widehat{\phi}^{\lambda}$ is an isometry.

It follows that, for $\partial \in F_1\mathcal{D}(\mathbf{U})$, $|\widehat{\phi}^{\lambda}(\partial)| = |\partial - f| = |\partial|$. Hence $|f| \leq |\partial|$.

Given $\partial \in p^e\mathcal{L}$, $f \in p^e\mathcal{A}$, and since $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ by definition, $\partial(f) \in p^{2e}\mathcal{A}$. Inductively, we see that $\partial^{k-1}(f) \in p^{ke}\mathcal{A}$. Hence:

$$|\frac{(-1)^{k-1}}{k!}\partial^{(k-1)}f| \leq |p|^{ke-\frac{k}{p-1}}.$$

As $k \to \infty$, $|p|^{ke-\frac{k}{p-1}} \to 0$ and so the sum $\sum_{k\geq 2} \frac{1}{k!}\partial^{(k-1)}f$ is convergent.

It suffices to verify that $|-f + \sum_{k\geq 2} \frac{(-1)^{k-1}}{k!}\partial^{(k-1)}f| < p^{\frac{1}{p-1}}$, which demonstrates that $-f + \sum_{k\geq 2} \frac{(-1)^{k-1}}{k!}\partial^{(k-1)}f$ lies inside the radius of convergence of the function $\exp : A \to A$ by [41, Example 0.4.1]. Since $|f| \leq |\partial|$ by continuity, and $|\partial| = p^{-e}$ by construction, it is clear that $|f| < p^{\frac{1}{p-1}}$. Since $|\frac{(-1)^{k-1}}{k!}\partial^{(k-1)}f| \leq |p|^{ke-\frac{k}{p-1}}$, it suffices to check that $ke - \frac{k}{p-1} > \frac{1}{p-1}$. Equivalently, $ke > \frac{k+1}{p-1}$.

If $p = 2, e = 2$ and $2k > k + 1$ for $k \geq 2$. If $p > 2, e = 1$ and $k > \frac{k+1}{p-1}$ for $k \geq 2$.

Hence the inequality is verified in all cases.

Hence we may write $\exp(\partial - f) = \exp(\partial)\exp(\widetilde{f})$ for some function $\widetilde{f} \in \mathcal{O}(\mathbf{U})$. It follows that:

$$\exp(\partial - f) = \exp(\partial)\exp(\widetilde{f})\exp(-\partial)\exp(\partial)$$
$$= (\beta(h) \cdot \exp(\widetilde{f}))\exp(\partial).$$

Now $\exp(\partial)$ lies in im β by construction, and $\beta(h) \cdot \exp(\widetilde{f}) \in \mathcal{O}(\mathbf{U})$, which is a subalgebra of U_n by definition. It follows that $\exp(\partial - f) \in U_n \cdot \mathrm{im}\beta$.

□

Proposition 5.2.23. $\widehat{\mathcal{D}}(\mathbf{U}, H) \cong \bigcap_{n\in\mathbb{N}} U_n \rtimes_{H_n} H$ as subalgebras of U_0.

Proof. By Lemma 5.2.20, we have an isomorphism $\varprojlim U_n \rtimes_{H_n} H \cong \bigcap_{n\in\mathbb{N}} U_n \rtimes_{H_n} H$. Hence it suffices to show that $\widehat{\mathcal{D}}(\mathbf{U}, H) \cong \varprojlim U_n \rtimes_{H_n} H$.

Let $(N_\bullet) := N_0 \geq N_1 \geq N_2 \geq \cdots$ be a good chain for $\mathcal{L}$ in the sense of [1, Definition 3.3.3]. By [1, Lemma 3.3.4], we have an isomorphism:

$$\widehat{\mathcal{D}}(\mathbf{U}, H) \cong \varprojlim U_n \rtimes_{N_n} H$$

of K-algebras. We claim that for each $n \in \mathbb{N}$ there is $m \geq n$ such that $H_m \subseteq N_n$. We first show that $\bigcap_{m\in\mathbb{N}} H_m = \{e\}$. We claim that, given $h \in \bigcap_{m\in\mathbb{N}} H_m$, then $\beta(h)$ lies in the unit ball of U_m for each m.

Write $|-|$ for the norm on U_n and $|-|_0$ for the norm on U_0. Since any $h \in H_n$ is contained in $G_{\mathcal{L}}$, we can write $\beta(h) = \exp(p^e \xi)$ for some $\xi \in \mathcal{L}$. Furthermore, $\beta(h) \in U_n$, so we may write $\beta(h)$ uniquely as:

$$\beta(h) = \sum_{i,j} a_{ij} x^i \partial^j, |a_{ij}||\pi|^{-|j|n} \to 0.$$

Since $\beta(h) \in \exp(p^e \mathcal{L})$, it follows that $\beta(h)(1) = 1$ and so we may suppose that $a_{00} = 1, a_{i0} = 0$ for $i > 0$. Hence, when filtering by ordering of differential operator, the degree zero part of $\beta(h)$ is 1.

Furthermore, $\beta(h^{p^m}) \in \exp(p^{m+e}\mathcal{L})$ for $m \geq 0$. Write $\beta(h) = \sum_k \frac{p^{ek}}{k!} \xi^k$. Since $|p^e| < |p|^{\frac{1}{p-1}}$, the radius of convergence of the p-adic exponential function by [41, Example 0.4.1], the coefficients $\frac{p^{ek}}{k!} \to 0$ and hence are bounded in norm. Also $\frac{p^{ek}}{k!} \in \mathbb{Z}_p$. Hence we can write:

$$\beta(h^{p^m}) = \sum_k (\frac{p^{ek}}{k!} (p^m \xi)^k)$$

Since $\frac{p^{ek}}{k!} \to 0$ and the coefficients are elements of $\mathbb{Z}_p$, it follows that all summands in the expansion of $\beta(h^{p^m})$, except the constant 1, tend to 0 with respect to the U_n-norm as $m \to \infty$. Hence $|\beta(h^{p^m})| \to 1$ as $m \to \infty$. Since the norm on U_n is submultiplicative, if $|\beta(h)| < 1$, $|\beta(h^{p^m})| \to 0$ as $m \to \infty$, a contradiction. Applying the same argument to $\beta(h^{-1})$ forces $|\beta(h)| = 1$.

By the above argument, for each $m \in \mathbb{N}$ $\beta(h)$ lies inside the unit ball U_m° of U_m.

But then $\beta(h) \in \bigcap_{m\in\mathbb{N}} U_m^\circ = \mathcal{A}$. On the other hand, using the previous expansion, we can write:

$$\beta(h) = \sum_k \frac{p^{ek}}{k!}\xi^k$$

and so, since $(p^e\mathcal{L})^k \cap \mathcal{A} = \{0\}$ for all $k \neq 0$, it follows that $\beta(h) = 1$. It follows that $\rho(h)$ is the identity automorphism on A. Since ρ is faithful, it follows that $\bigcap_{m\in\mathbb{N}} H_m$ is trivial.

Since H_m is an open subgroup of G, $G \setminus N_n$ is closed and therefore compact. From the above, $\bigcap_{m\in\mathbb{N}} H_m = \{e\}$, and so $G \setminus N_n \subseteq \bigcup_{m\in\mathbb{N}} G \setminus H_m$. The right-hand side is an open covering of G, so by compactness, $G \setminus N_n \subseteq G \setminus H_k$ for some fixed subgroup H_k. Hence $H_k \subseteq N_n$, which proves the claim.

On the other hand, since the N_n form a good chain for $\mathcal{L}$, we see that $\bigcap_{n\in\mathbb{N}} N_n$ is trivial. Hence the same argument as in the previous paragraph allows us to find some H_l with $N_n \subseteq H_l$, and the result follows. $\square$

Proof of Theorem 5.2.15: By Proposition 5.2.23, $\widehat{\mathcal{D}}(\mathbf{U}, H) \cong \bigcap_{n\in\mathbb{N}} U_n \rtimes_{H_n} H$. Hence the required automorphism $\widehat{\phi}^\lambda_{\mathbf{U}} : \widehat{\mathcal{D}}(\mathbf{U}, H) \to \widehat{\mathcal{D}}(\mathbf{U}, H)$ coincides with $\widehat{\phi}^\lambda_0 \mid_{\widehat{\mathcal{D}}(\mathbf{U},H)}$. Furthermore, it is clear by construction that the map $\widehat{\phi}^\lambda_{\mathbf{U}}$ is a restriction of the map $\widehat{\phi}^\lambda_{\mathbf{U}}$. Since $\widehat{\mathcal{D}}(\mathbf{U}, H)$ is a dense subalgebra of U_0, and the map $\widehat{\phi}^\lambda_{\mathbf{U}}$ is unique, it follows that the map $\widehat{\phi}^\lambda_{\mathbf{U}}$ must be the unique such continuous extension.

$\square$

Proof of Theorem D: This follows from Proposition 5.2.23.

Recall that, for $\mathbb{X}$ the scheme-theoretic flag variety and $\mathbb{U} \subseteq \mathbb{X}$ open affine, we may find a local coordinate system $\{x_i, \partial_i\}$ satisfying:

$$[\partial_i, \partial_j] = 0, \partial_i(x_j) = \delta_{ij}, \mathcal{T}_{\mathbb{U}} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{U}}(\partial_i)$$

by [32, Theorem A.5.1]. Our next lemma shows that the same is true for $\mathbf{X}$. We state this lemma in a more general setting, in the case where $\mathbf{Y}$ is a general smooth affinoid variety, not necessarily the rigid analytic flag variety.

Lemma 5.2.24. Let $\mathbf{Y}$ be a smooth affinoid variety of dimension n. For each point $p \in \mathbf{Y}$, there exists an admissible open neighbourhood $\mathbf{V}_p$ of p, regular functions $x_i \in \mathcal{O}_{\mathbf{Y}}(\mathbf{V}_p)$ and vector fields $\partial_i \in \mathcal{T}_{\mathbf{Y}}(\mathbf{V}_p)$ satisfying the conditions:

$$[\partial_i, \partial_j] = 0, \partial_i(x_j) = \delta_{ij}, \mathcal{T}_{\mathbf{V}_p} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbf{V}_p}(\partial_i).$$

Moreover, we may choose the x_i such that they generate the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathcal{O}_{\mathbf{V}_p,p}$ at p. Hence $\widehat{\mathcal{D}_{\mathbf{Y}}}(\mathbf{V}_p) \cong \widehat{\mathcal{A}_K} \widehat{\otimes}_{\mathcal{O}(\mathbb{A}^n_{\mathcal{R}})_{\text{rig}}} \mathcal{O}_{\mathbf{Y}}(\mathbf{V}_p)$.

Proof. Since $\mathbf{Y}$ is smooth, by [24, Theorem 3.6.3(2)], the stalk $\mathcal{O}_{\mathbf{Y},p}$ is a regular local ring. Hence there exist n functions $x_1, \cdots, x_n \in \mathfrak{m}_p$ generating the unique maximal ideal $\mathfrak{m}_p$. By definition, $dx_1, \cdots, dx_n$ is a basis for the free $\mathcal{O}_{\mathbf{Y},p}$-module $\Omega^1_{\mathbf{Y},p}$. It follows that we can take an admissible open neighbourhood $\mathbf{V}_p$ of p such that $\Omega^1_{\mathbf{Y}}(\mathbf{V}_p)$ is a free module with basis $dx_1, \cdots dx_n$ over $\mathcal{O}_{\mathbf{Y}}(\mathbf{V}_p)$. Let $\partial_1, \cdots, \partial_n$

be the dual basis of $\mathcal{T}_{\mathbf{Y}}(\mathbf{V}_p)$. Then $\partial_i(x_j) = \delta_{ij}$. Writing $[\partial_i, \partial_j] = \sum_{l=1}^n g_{ij}^l \partial_l$, where $g_{ij}^l \in \mathcal{O}_{\mathbf{Y}}(\mathbf{V}_p)$, it follows that $g_{ij}^l = [\partial_i, \partial_j]x_l = \partial_i\partial_j x_l - \partial_j\partial_i x_l = 0$. Hence $[\partial_i, \partial_j] = 0$.

For the last part, note by [2, Proposition 4.15] that $\mathbf{Y}$ admits an admissible covering by a collection of affinoids $\mathbf{Y}_i$ such that there exist étale morphisms $g_i : \mathbf{Y}_i \to \mathbb{D}^{n_i}$, where $n_i \in \mathbb{N}$ and $\mathbb{D}^{n_i}$ denotes the n_i-dimensional polydisc.

Let $\mathbf{Y} :=$ Sp A, $\mathbb{D} =$ Sp T_{n_i}, where T_{n_i} is the Tate algebra in n_i variables. Since g_i is étale, it follows that we may write $\mathcal{T}_{\mathbf{Y}}(\mathbf{Y}_i) = A \otimes_{T_{n_i}} \mathcal{T}(\mathbb{D}^{n_i})$. Since $\mathcal{D}_{\mathbf{Y}}(\mathbf{Y}_i) = U(\mathcal{T}_{\mathbf{Y}}(\mathbf{Y}_i))$ by definition, applying [5, Corollary 2.4] and a completed version of [5, Proposition 2.3] shows us that $\widehat{\mathcal{D}}_{\mathbf{Y}}(\mathbf{V}_p) \cong \widehat{\mathcal{A}_K} \widehat{\otimes}_{\widehat{\mathcal{O}(\mathbb{A}^n_{\mathcal{R}})_{\text{rig}}}} \mathcal{O}_{\mathbf{Y}}(\mathbf{V}_p)$. □

We now apply Theorem 5.2.15 to our situation. Recall that $\mathbf{X}$ is the rigid analytic flag variety. By [38, II.1.10(2)], we may assume $\mathbf{U}$ is isomorphic to $(\widehat{\mathbb{A}^n_{\mathcal{R}}})_{\text{rig}}$, where n is the dimension of $\mathbf{X}$. Let $\{x_i, \partial_i\}$ be a local coordinate system for $\mathcal{D}(\mathbf{U})$, as in Lemma 5.2.24, and let $\{h_i\}$ be a basis for the Cartan subalgebra $\mathfrak{h}$. Since λ is integral, we may find an $\mathcal{A}$-linear isomorphism $\phi^\lambda_{\mathbf{U}} = \mathcal{D}(\mathbf{U}) \to \mathcal{D}^\lambda(\mathbf{U})$ given by $\phi^\lambda(\partial_i) = \partial_i - \lambda(h_i)x_i$. Extending this definition multiplicatively, we may suppose that ϕ^λ is a ring homomorphism. Hence we may find a continuous $\mathcal{A}$-linear homomorphism $\phi^\lambda_{\mathbf{U}} : \mathcal{D}(\mathbf{U}) \to \mathcal{D}^\lambda(\mathbf{U})$ of the form $\phi^\lambda(u) = u - f$ for $f \in \mathcal{O}(\mathbf{U})$, as in the statement of Theorem 5.2.15.

Let $[\phi^\lambda_{\mathbf{U}}] : \mathcal{D}(\mathbf{U}) - \text{mod} \to \mathcal{D}^\lambda(\mathbf{U}) - \text{mod}$ be the functor that views a $\mathcal{D}(\mathbf{U})$-module as a $\mathcal{D}^\lambda(\mathbf{U})$-module, with the $\mathcal{D}^\lambda(\mathbf{U})$-action given by:

$$a \cdot_{\mathcal{D}^\lambda_{\mathbf{U}}} ([\phi^\lambda_{\mathbf{U}}]m) = [\phi^\lambda_{\mathbf{U}}]((\phi^\lambda)^{-1}_{\mathbf{U}}(a) \cdot_{\mathcal{D}_{\mathbf{U}}} m) \text{ for all } a \in \mathcal{D}^\lambda_{\mathbf{U}}, m \in \mathcal{M}_{\mathbf{U}}.$$

Note that the functor $[\phi^\lambda_{\mathbf{U}}]$ defines an equivalence of categories $\mathcal{D}_{\mathbf{U}} - \text{mod} \to \mathcal{D}^\lambda_{\mathbf{U}} - \text{mod}$. Theorem 5.2.15 implies that we may similarly define a functor $[\widehat{\phi}^\lambda_{\mathbf{U}}] : \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) - \text{mod} \to \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}}) - \text{mod}$, which is also an equivalence of categories.

Lemma 5.2.25. Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering. Let $\mathbf{U} \in \mathcal{U}$, let $H_{\mathbf{U}}$ be a $\mathbf{U}$-small subgroup of G, and let $M_{\mathbf{U}}$ be a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module. Then $\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}}$ is also a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module.

Proof. Since $M_{\mathbf{U}} \in \mathcal{D}_{\mathbf{U}}(\mathbf{U}) - \text{mod}$, $\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}} \in \mathcal{D}^\lambda_{\mathbf{U}}(\mathbf{U}) - \text{mod}$ by Lemma 5.2.10. Hence we may write $\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}} = [\phi^\lambda_{\mathbf{U}}](M_{\mathbf{U}})$ for some continuous $\mathcal{A}$-linear ring isomorphism $\phi^\lambda_{\mathbf{U}}$.

There is a corresponding functor $[\widehat{\phi}^\lambda_{\mathbf{U}}] : \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) - \text{mod} \to \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) - \text{mod}$ extending $[\phi^\lambda_{\mathbf{U}}]$, and, by construction, $[\widehat{\phi}^\lambda_{\mathbf{U}}](M_{\mathbf{U}}) = \mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}}$. It follows that $\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}} \in \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) - \text{mod}$.

Since $M_{\mathbf{U}}$ is a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module, we may write $M_{\mathbf{U}} = \varprojlim (M_{\mathbf{U}})_n$, where the $(M_{\mathbf{U}})_n$ are finitely generated $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_n$-modules for some Noetherian K-Banach algebras $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_n$ satisfying $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) = \varprojlim \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_n$. We verify the $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module $[\widehat{\phi}^\lambda_{\mathbf{U}}](M_{\mathbf{U}})$ is coadmissible. Note that:

$$[\widehat{\phi}^\lambda_{\mathbf{U}}](M) = [\widehat{\phi}^\lambda_{\mathbf{U}}](\varprojlim(M_{\mathbf{U}})_n)$$
$$= \varprojlim[\widehat{\phi}^\lambda_{\mathbf{U}}]((M_{\mathbf{U}})_n)$$

where the inverse limit commutes with the functor $[\widehat{\phi}^\lambda_{\mathbf{U}}]$ since it is an auto-equivalence of the category $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) - \text{mod}$. Furthermore, it is clear that the $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_n$-modules $[\widehat{\phi}^\lambda_{\mathbf{U}}]((M_{\mathbf{U}})_n)$ are finitely generated.

Finally, we have the chain of isomorphisms:

$$\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_n \otimes_{\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_{n+1}} [\widehat{\phi}^\lambda_{\mathbf{U}}]((M_{\mathbf{U}})_{n+1})$$
$$= [\widehat{\phi}^\lambda_{\mathbf{U}}](\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_n \otimes_{\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})_{n+1}} ((M_{\mathbf{U}})_{n+1})$$
$$= [\widehat{\phi}^\lambda_{\mathbf{U}}](M_{\mathbf{U}})_n$$

where the last line follows since $M_{\mathbf{U}}$ is a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module. Hence, by Definition 2.4.5, $\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}}$ is a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module. $\square$

Lemma 5.2.26. Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering. Let $\mathbf{U} \in \mathcal{U}$, let $H_{\mathbf{U}}$ be a $\mathbf{U}$-small subgroup of G, and let $M_{\mathbf{U}}$ be a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module. By Lemma 5.2.25, $\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}}$ is also a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module.

There is an isomorphism of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules:

$$\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathrm{Loc}^{\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})}_{\mathbf{U}}(M_{\mathbf{U}}) \to \mathrm{Loc}^{\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})}_{\mathbf{U}}(\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}}).$$

Proof. By [5, Theorem 9.1], it suffices to check that we have an isomorphism of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules:

$$\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{P}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}) \to \mathcal{P}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(\mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}}).$$

Let $\mathbf{V} \in \mathbf{X}_w(\mathcal{T})$, $\mathbf{V} \subseteq \mathbf{U}$, $H_{\mathbf{V}}$ a $\mathbf{V}$-small subgroup of G. It follows that:

$$\begin{aligned}
\mathcal{P}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(\mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}})(\mathbf{V}) &= \varprojlim (\mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}})(\mathbf{V}, H_{\mathbf{V}}) \\
&= \varprojlim \widehat{\mathcal{D}}(\mathbf{V}, H_{\mathbf{V}}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{V}})} \mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}} \\
&\cong \varprojlim \mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} \widehat{\mathcal{D}}(\mathbf{V}, H_{\mathbf{V}}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{V}})} M_{\mathbf{U}},
\end{aligned}$$

where the inverse limit is taken over the $\mathbf{V}$-small subgroups of G.

By [1, Corollary 3.5.6], whenever $(\mathbf{V}, H_{\mathbf{V}})$ is small, the canonical map $\mathcal{P}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}})(\mathbf{V}) \to M(\mathbf{V}, H_{\mathbf{V}})$ is a bijection. It follows that the inverse system $M(\mathbf{V}, H_{\mathbf{V}})_{H_{\mathbf{V}}}$ satisfies the Mittag-Leffler property, and so, by [61, Proposition 3.5.7], we have the following chain of isomorphisms:

$$\begin{aligned}
&\varprojlim \mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} \widehat{\mathcal{D}}(\mathbf{V}, H_{\mathbf{V}}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{V}})} M_{\mathbf{U}} \\
&\cong \mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} \varprojlim \widehat{\mathcal{D}}(\mathbf{V}, H_{\mathbf{V}}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{V}})} M_{\mathbf{U}} \\
&= \mathcal{O}_{\mathbf{U}}^{\lambda}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} \mathcal{P}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}})(\mathbf{V}).
\end{aligned}$$

□

Proposition 5.2.27. The functor $\mathcal{O}^\lambda \otimes_{\mathcal{O}} - : \mathrm{Frech}(G - \mathcal{D}) \to \mathrm{Frech}(G - \mathcal{D}^\lambda)$ restricts to an equivalence of categories $\mathcal{C}_{\mathbf{X}/G} \to \mathcal{C}^\lambda_{\mathbf{X}/G}$.

In particular, $\mathcal{C}^\lambda_{\mathbf{X}/G}$ is an abelian category.

Proof. Let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$. Then, by [1, Definition 3.6.1], there exists a $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$ such that for each $\mathbf{U} \in \mathcal{U}$, there is a $\mathbf{U}$-small subgroup $H_{\mathbf{U}}$ of G, a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module $M_{\mathbf{U}}$, and an isomorphism:

$$\mathrm{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}) \to \mathcal{M}_{\mathbf{U}_{\mathrm{rig}}}$$

of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{U}$.

By Lemma 5.2.11, for each $\mathbf{U} \in \mathcal{U}$, the $\mathcal{D}^\lambda_{\mathbf{U}}$-module $\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{M}_{\mathbf{U}} \in \mathrm{Frech}(H_{\mathbf{U}} - \mathcal{D}^\lambda_{\mathbf{U}})$.

We check that $\mathcal{O}^\lambda \otimes_{\mathcal{O}} \mathcal{M}$ is an object in the full subcategory $\mathcal{C}^\lambda_{\mathbf{X}/G}$. Applying the functor $\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} -$, we have an isomorphism of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules:

$$\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathrm{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}) \to (\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{M})_{\mathbf{U}_{\mathrm{rig}}}.$$

By Lemma 5.2.26, we have an isomorphism:

$$\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathrm{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}) \to \mathrm{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(\mathcal{O}^\lambda_{\mathbf{U}}(\mathbf{U}) \otimes_{\mathcal{O}_{\mathbf{U}}(\mathbf{U})} M_{\mathbf{U}}).$$

By Definition 6.4.1, $\mathcal{O}^\lambda \otimes_{\mathcal{O}} \mathcal{M}$ is an object in $\mathcal{C}^\lambda_{\mathbf{X}/G}$.

By Lemma 5.2.7, there is an isomorphism of locally Fréchet $\mathcal{D}$-modules:

$$\mathcal{O}^{-\lambda} \otimes_{\mathcal{O}} \mathcal{O}^\lambda \otimes_{\mathcal{O}} \mathcal{M} \cong \mathcal{M}.$$

Hence the functor $\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} -$ has a natural quasi-inverse $\mathcal{O}^{-\lambda}_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} -$. The desired equivalence of categories follows. By Theorem 2.8.25, it follows that $\mathcal{C}^\lambda_{\mathbf{X}/G}$ is an abelian category. □

5.3 Beilinson-Bernstein equivalence for integral λ

In this section, we prove the main result of this chapter: the Beilinson-Bernstein-type equivalence for coadmissible G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules where λ is integral. We begin by constructing the module category which the category $\mathcal{C}^\lambda_{\mathbf{X}/G}$ should be equivalent to. We retain the notation and assumptions of the preceding sections. In particular, we assume λ is a ρ-dominant ρ-regular integral weight of $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$, W the Weyl group of $\mathfrak{g}$.

Definition 5.3.1. Suppose H is a compact open subgroup of G. We let $\mathcal{J}(H)$ denote the set of all pairs $(\mathcal{L}, N)$, where $\mathcal{L}$ is an H-stable Lie lattice in $\mathfrak{g}$ and N is an open subgroup of $H_{\mathcal{L}}$ which is normal in H. By [1, Proposition 6.2.6(b)], we may find an H-equivariant trivialisation of the N-action on $\widehat{U(\mathcal{L})_K}$, allowing us to define the algebra:

$$\widehat{U(\mathcal{L})_K} \rtimes_N H.$$

We set $\widehat{U}(\mathfrak{g}, H) := \varprojlim_{(\mathcal{L},N)\in\mathcal{J}(H)} \widehat{U(\mathcal{L})_K} \rtimes_N H$.

By [1, Theorem 6.2.9], $\widehat{U}(\mathfrak{g}, H)$ is Fréchet-Stein for every compact subgroup H of G.

Definition 5.3.2. $\widehat{U}(\mathfrak{g}, G) := \widehat{U}(\mathfrak{g}, H) \otimes_{K[H]} K[G]$, for any choice of compact open subgroup $H \leq G$. This definition is independent of the choice of H by [1, Theorem 6.1.15].

A $\widehat{U}(\mathfrak{g}, G)$-module is *coadmissible* if it is coadmissible as a $\widehat{U}(\mathfrak{g}, H)$-module for some compact open subgroup H of G.

Let $\phi : U(\mathfrak{g})^G \to U(\mathfrak{t})$ be the Harish-Chandra homomorphism, as in the discussion above Proposition 2.5.8. Given $\lambda \in \mathfrak{h}^*$, we define a K-algebra homomorphism $\lambda \circ \phi : Z(\mathfrak{g}) \to K$, and denote its kernel by $\mathfrak{m}_\lambda$.

Theorem 5.3.3. *The localisation functor:*

$$Loc_{\boldsymbol{X}}^{\widehat{U}(\mathfrak{g},G)} : \{M \in \mathcal{C}_{\widehat{U}(\mathfrak{g},G)} \mid \mathfrak{m}_0 \cdot M = 0\} \to \mathcal{C}_{\boldsymbol{X}/G}$$

is an equivalence of categories.

Proof. This is [1, Theorem 6.4.9]. □

Let:

$$\mathcal{C}^{\lambda}_{\widehat{U}(\mathfrak{g},G)} := \{M \in \mathcal{C}_{\widehat{U}(\mathfrak{g},G)} \mid \mathfrak{m}_\lambda \cdot M = 0\}.$$

Lemma 5.3.4. Let $\mathcal{A}$ be a Fréchet-Stein algebra, let $M \in \mathcal{C}_{\mathcal{A}}$ and suppose $M = \bigoplus_{i \in I} M_i$, where the indexing set I is finite and the M_i are each $\mathcal{A}$-modules. Then each $M_i \in \mathcal{C}_{\mathcal{A}}$.

Proof. Since $\mathcal{A}$ is Fréchet-Stein, we may write $\mathcal{A} = \varprojlim \mathcal{A}_n$, $M = \varprojlim M_n$, where the $\mathcal{A}_n$ are each Noetherian K-Banach algebras and the M_n are each finitely generated $\mathcal{A}_n$-modules. Then each M_n has a direct sum decomposition of $\mathcal{A}_n$-modules as $M_n = \bigoplus_{i \in I} (M_n)_i$ and:

$$\bigoplus_{i \in I} M_i = M = \varprojlim M_n = \varprojlim (\bigoplus_{i \in I} (M_n)_i) = \bigoplus_{i \in I} (\varprojlim (M_n)_i)$$

since finite direct sums commute with taking limits. Furthermore, each $(M_n)_i$ is an $\mathcal{A}_n$-submodule of the finitely generated $\mathcal{A}_n$-module M_n and so is finitely generated since $\mathcal{A}_n$ is Noetherian.

Furthermore, suppose that we have an isomorphism of $\mathcal{A}_n$-modules:

$$A_n \otimes_{A_{n+1}} M_{n+1} \to M_n.$$

Then:

$$\bigoplus_{i \in I} (A_n \otimes_{A_{n+1}} (M_{n+1})_i) = (A_n \otimes_{A_{n+1}} \bigoplus_{i \in I} (M_{n+1})_i) \to \bigoplus_{i \in I} (M_n)_i$$

since finite direct sums commute with taking tensor products. It follows that there is an $\mathcal{A}_n$-module isomorphism:

$$A_n \otimes_{A_{n+1}} (M_{n+1})_i \to (M_n)_i$$

for each $i \in I$. Hence each M_i is a coadmissible $\mathcal{A}$-module by Definition 2.4.5. $\square$

Lemma 5.3.5. Let E be a finite-dimensional $\mathfrak{g}$-module and M a coadmissible $\widehat{U}(\mathfrak{g}, G)$-module. Then $E \otimes M$ is a coadmissible $\widehat{U}(\mathfrak{g}, G)$-module.

Proof. By [1, Theorem 6.5.1], there is a continuous K-algebra isomorphism:

$$\eta_G : D(G, K) \to \widehat{U}(\mathfrak{g}, G)$$

where $D(G, K)$ denotes the K-algebra of locally L-analytic distributions on G. Hence we may view M as a coadmissible $D(G, K)$-module. By [55, Definition p.33], M may be realised as the strong dual of an admissible locally analytic G-representation N.

Since E is a finite-dimensional $\mathfrak{g}$-module, it may be viewed as a finite-dimensional $D(G, K)$-module and so may be realised as the strong dual of a finite-dimensional admissible locally analytic G-representation F.

Without loss of generality, we may replace G by some compact open subgroup H: then N, F are admissible locally analytic H-representations of compact type.

By [23, Proposition 6.1.5], the tensor product $F \otimes N$ is an admissible locally analytic H-representation.

Applying [53, Proposition 20.13], there is an isomorphism of admissible locally analytic H-representations $(F \otimes N)'_b \cong F'_b \otimes N'_b = E \otimes M$. Hence the strong dual $E \otimes M$ is a coadmissible $D(G,K)$-module, by definition. Applying the isomorphism η_G, $E \otimes M$ is a coadmissible $\widehat{U}(\mathfrak{g}, G)$-module. □

Proposition 5.3.6. Let E be a finite-dimensional $\mathfrak{g}$-module, and $M \in \mathcal{C}^{\lambda}_{\widehat{U}(\mathfrak{g},G)}$. The coadmissible $\widehat{U}(\mathfrak{g}, G)$-module $E \otimes M$ has a finite direct sum decomposition:

$$E \otimes M = \bigoplus_{\mu \in \mathfrak{h}^*/W} \mathrm{pr}_\mu(E \otimes M),$$

where $\mathrm{pr}_\mu(E \otimes M) \in \mathcal{C}^{\mu}_{\widehat{U}(\mathfrak{g},G)}$.

Proof. Note that $Z(\mathfrak{g})$ acts on $E \otimes M$ via the diagonal action. Since E is finite-dimensional, by $\mathfrak{h}$-semisimplicity there exists a basis $\{e_1, \cdots, e_n\}$ of E such that $z \cdot e_i = \lambda_i(z)e_i$ for $z \in Z(\mathfrak{g})$ and $\lambda_i \in \mathfrak{h}^*$.

Let $a_i \in K$. It follows that:

$$z \cdot (a_i e_i \otimes m) = a_i \lambda_i(z) e_i \otimes m + a_i e_i \otimes \lambda(z) m$$

$$= (\lambda(z) + \lambda_i(z))(a_i e_i \otimes m).$$

Hence, as a $Z(\mathfrak{g})$-module, $E \otimes M$ has a finite direct sum decomposition:

$$E \otimes M = \bigoplus_{\mu \in \mathfrak{h}^*/W} \mathrm{pr}_\mu(E \otimes M).$$

We verify that each direct summand $\mathrm{pr}_\mu(E \otimes M)$ is a $\widehat{U}(\mathfrak{g}, G)$-submodule of $E \otimes M$. This will demonstrate that the above direct sum decomposition is in fact a decomposition into $\widehat{U}(\mathfrak{g}, G)$-submodules.

Given $v \in \mathrm{pr}_\mu(E \otimes M)$, it follows by definition that $(z - \mu(z)) \cdot v = 0$, where $z \in Z(\mathfrak{g})$. It is then immediate that $\mathrm{pr}_\mu(E \otimes M)$ is a $\widehat{U}(\mathfrak{g})$-submodule of $E \otimes M$.

Let H be a compact open subgroup of G: the H-action on $\mathrm{pr}_\mu(E \otimes M)$ is given by the adjoint action of $\mathbb{G}(K)$ on $\widehat{U}(\mathfrak{g})$. Hence H fixes $Z(\mathfrak{g})$ pointwise under this action and so we see that:

$$0 = h \cdot (z - \mu(z)) \cdot v) = (h \cdot (z - \mu(z))) \cdot (h \cdot v) = (z - \mu(z)) \cdot (h \cdot v),$$

for $h \in H$. Hence there is a $\widehat{U}(\mathfrak{g}) \rtimes H$-action on $\mathrm{pr}_\mu(E \otimes M)$: since $\widehat{U}(\mathfrak{g}) \rtimes H$ is dense in $\widehat{U}(\mathfrak{g}, H)$ by construction it follows that $\mathrm{pr}_\mu(E \otimes M)$ is a $\widehat{U}(\mathfrak{g}, G)$-submodule of $E \otimes M$.

$\square$

Given a regular integral weight $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ denote the irreducible highest-weight $\mathfrak{g}$-module with weight λ. We also let $\overline{\lambda}$ denote the unique ρ-dominant integral weight in the W-orbit of λ. Recall that, when μ is ρ-dominant and integral, $L(\mu)$ is a finite-dimensional $\mathfrak{g}$-module.

Proposition 5.3.7. The *translation functors* $\theta^\lambda : \mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)} \to \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)}$ and $\theta_\lambda : \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)} \to \mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)}$ are defined by:

$$\theta^\lambda(M) := \mathrm{pr}_\lambda(L(\overline{\lambda}) \otimes M),$$

$$\theta_\lambda(N) := \mathrm{pr}_0(L(\overline{-\lambda}) \otimes N).$$

Proof. We verify that the essential image of θ_λ is contained in $\mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)}$. The second case will follow by symmetry.

Let $N \in \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)}$. By Lemma 5.3.5, the tensor product $L(\overline{\lambda}) \otimes N$ is a coadmissible $\widehat{U}(\mathfrak{g}, G)$-module, and it has a direct sum decomposition into $\widehat{U}(\mathfrak{g}, G)$-submodules:

$$L(\overline{\lambda}) \otimes N = \bigoplus_{\mu \in \mathfrak{h}^*/W} \mathrm{pr}_\mu(L(\overline{\lambda}) \otimes N)$$

by Proposition 5.3.6.

Applying Lemma 5.3.4, $\theta_\lambda(N) := \mathrm{pr}_0(L(\overline{\lambda}) \otimes N)$ is a coadmissible $\widehat{U}(\mathfrak{g}, G)$-module, and it lies in $\mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)}$ by definition. □

Proposition 5.3.8. Set $\gamma := \mathcal{O}^{-\lambda}_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} -$. Up to canonical equivalence of functors, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}^\lambda_{\mathbf{X}/G} & \xrightarrow{\Gamma} & \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)} \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \theta_\lambda} \\ \mathcal{C}^0_{\mathbf{X}/G} & \xrightarrow{\Gamma} & \mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)}. \end{array}$$

In particular, the essential image of the global sections functor $\Gamma : C^0_{\mathbf{X}/G} \to \widehat{U}(\mathfrak{g}, G)-$ mod is contained in $\mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)}$, and the essential image of $\Gamma : C^\lambda_{\mathbf{X}/G} \to \widehat{U}(\mathfrak{g}, G) -$ mod is contained in $\mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)}$.

Proof. First note that the functor $\Gamma : C^0_{\mathbf{X}/G} \to \widehat{U}(\mathfrak{g}, G)-$mod satisfies $\mathrm{Loc}^{\widehat{U}(\mathfrak{g},G)}_{\mathbf{X}} \cdot \Gamma \cong \mathrm{id}_{C^0_{\mathbf{X}/G}}$ by [1, Theorem 6.4.7]. Since $\mathrm{Loc}^{\widehat{U}(\mathfrak{g},G)}_{\mathbf{X}} : \mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)} \to \mathcal{C}^0_{\mathbf{X}/G}$ is an equivalence of categories by Theorem 5.3.3, the functor $\Gamma : C^0_{\mathbf{X}/G} \to \widehat{U}(\mathfrak{g}, G) -$ mod is its quasi-inverse and so the essential image of Γ is contained in $\mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)}$.

Let E be a finite-dimensional $\mathfrak{g}$-module and let $E_{\mathbb{X}} := E \otimes \mathcal{O}_{\mathbb{X}}$ be the trivial sheaf of E-valued regular functions on $\mathbb{X}$. Given such a function f, we define a function $\varphi : \mathbb{G} \to E$ by $\varphi_f(g) := g^{-1} \cdot f(g)$. The assignment $f \mapsto \varphi_f$ identifies $E_{\mathbb{X}}$ with the induced sheaf $\mathrm{Ind}^{\mathbb{G}}_{\mathbb{X}} E$:

$$\{\varphi : \mathbb{G} \to E \mid \varphi(g \cdot b) = b^{-1} \cdot \varphi(g) \forall b \in \mathbb{B}\}.$$

Applying the rigid analytification functor to the coherent sheaf $E_{\mathbb{X}}$ on the projective scheme $\mathbb{X}$, and appealing to [13, Theorem 6.3/13], we may identify $E_{\mathbf{X}} := E \otimes \mathcal{O}_{\mathbf{X}}$ with the analytification $(\mathrm{Ind}^{\mathbb{G}}_{\mathbb{X}} E)^{\mathrm{an}}$. $(\mathrm{Ind}^{\mathbb{G}}_{\mathbb{X}} E)^{\mathrm{an}}$ is a left $\mathcal{O}_{\mathbf{X}} \otimes \mathfrak{g}$-module via the infinitesimal action of $\mathfrak{g}$ on $\mathbb{G}$ by left translation.

By the Lie-Kolchin theorem, [11, Corollary 10.5], there is a $\mathbb{B}$-stable filtration $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ such that $\dim(\frac{E_i}{E_{i-1}}) = 1$ for each i. This defines a filtration on $\mathrm{Ind}^{\mathbb{G}}_{\mathbb{X}}$ by the $\mathfrak{g}$-stable coherent subsheaves $\mathrm{Ind}^{\mathbb{G}}_{\mathbb{X}} E_i$, defined by replacing E with E_i in the above equation. Furthermore, we have:

$$\frac{\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E_i}{\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E_{i-1}} = \mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}}\Big(\frac{E_i}{E_{i-1}}\Big) = \mathcal{O}_{\mathbb{X}}^{\nu_i}.$$

where ν_i is the character of $\mathbb{X}$ corresponding to the 1-dimensional $\mathbb{B}$-module $\frac{E_i}{E_{i-1}}$.

Applying the rigid analytification functor to the coherent sheaves $(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E_i)$, it follows that there is a filtration of $(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E)^{\mathrm{an}}$ by the $\mathcal{O}_{\mathbf{X}} \otimes \mathfrak{g}$-stable coherent subsheaves $(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E_i)^{\mathrm{an}}$, and we have:

$$\frac{(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E_i)^{\mathrm{an}}}{(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E_{i-1})^{\mathrm{an}}} = \Big(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}}\Big(\frac{E_i}{E_{i-1}}\Big)\Big)^{\mathrm{an}} = \mathcal{O}_{\mathbf{X}}^{\nu_i}.$$

Now set $E := L(\lambda)$ to be the irreducible highest-weight $\mathfrak{g}$-module with weight λ, and let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}^{\lambda}$. The sheaf $(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E)^{\mathrm{an}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}$ may be equipped with the tensor product $\mathcal{O}_{\mathbf{X}} \otimes \mathfrak{g}$-module structure. The filtration $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ induces a $\mathcal{O}_{\mathbf{X}} \otimes \mathfrak{g}$-stable filtration on $(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E)^{\mathrm{an}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}$ by the subsheaves $(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E_i)^{\mathrm{an}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}$, and the quotients are of the form $\mathcal{O}_{\mathbf{X}}^{\mu} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}$, where μ is a weight of $L(\lambda)$.

By [8, Lemma 1.5(iii)], for any weight μ of $L(\overline{-\lambda})$ not equal to λ, the point μ is not W-conjugate to λ. It follows that the subquotient sheaf $\mathcal{O}_{\mathbf{X}}^{\overline{(-\lambda)}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}$ splits off from $(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} L(\overline{(-\lambda)})^{\mathrm{an}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}$ as a sheaf of $Z(\mathfrak{g})$-modules:

$$\mathrm{pr}_0((\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} L(\overline{-\lambda})^{\mathrm{an}}) \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}) = \mathcal{O}_{\mathbf{X}}^{\overline{(-\lambda)}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}.$$

Observe that, for any E, we have:

$$E \otimes \Gamma(\mathcal{M}) = \Gamma(E_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}) = \Gamma((\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} E)^{\mathrm{an}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}).$$

Then:

$$\begin{aligned}
\theta_\lambda \cdot \Gamma(\mathcal{M}) &= \mathrm{pr}_0(L(\overline{-\lambda}) \otimes \Gamma(\mathcal{M})) \\
&= \mathrm{pr}_0(\Gamma((\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} L(\overline{-\lambda})^{\mathrm{an}}) \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M})) \\
&= \Gamma(\mathrm{pr}_0((\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} L(\overline{-\lambda})^{\mathrm{an}}) \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M})) \\
&= \Gamma(\mathcal{O}_{\mathbf{X}}^{\overline{(-\lambda)}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}) \\
&= \Gamma(\mathcal{O}_{\mathbf{X}}^{-\lambda} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}) = \Gamma \cdot \gamma(\mathcal{M}),
\end{aligned}$$

where the third line follows from the second since sheaf cohomology commutes with taking finite direct sums.

To show that the essential image of $\Gamma : \mathcal{C}_{\mathbf{X}/G}^{\lambda} \to \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}$ is contained in $\mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{\lambda}$, we first verify that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{C}_{\mathbf{X}/G}^{\lambda} & \xrightarrow{\Gamma} & \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{\lambda} \\
\downarrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \theta^{\lambda}} \\
\mathcal{C}_{\mathbf{X}/G}^{0} & \xrightarrow{\Gamma} & \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{0}.
\end{array}$$

Let $\mathcal{M} \in \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{\lambda}$. Then $\gamma(\mathcal{M}) \in \mathcal{C}_{\mathbf{X}/G}^{0}$ and we see that:

$$
\begin{aligned}
\theta^\lambda \cdot \Gamma \cdot \gamma(\mathcal{M}) &= \theta^\lambda \cdot \Gamma(\mathcal{O}_{\mathbf{X}}^{(\overline{-\lambda})} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M}) \\
&= \mathrm{pr}_\lambda(L(\overline{\lambda}) \otimes \Gamma(\mathcal{O}_{\mathbf{X}}^{(\overline{-\lambda})} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M})) \\
&= \mathrm{pr}_\lambda(\Gamma(L(\overline{\lambda})_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{O}_{\mathbf{X}}^{(\overline{-\lambda})} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M})) \\
&= \mathrm{pr}_\lambda(\Gamma(\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} L(\overline{\lambda})^{\mathrm{an}}) \otimes_{\mathcal{O}_{\mathbf{X}}} (\mathrm{Ind}_{\mathbb{X}}^{\mathbb{G}} L(\overline{-\lambda})^{\mathrm{an}}) \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M})) \\
&= \mathrm{pr}_\lambda(\Gamma(\mathcal{O}_{\mathbf{X}}^{(\overline{\lambda})} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{O}_{\mathbf{X}}^{(\overline{-\lambda})} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M})) \\
&= \mathrm{pr}_\lambda(\Gamma(\mathcal{O}_{\mathbf{X}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{O}_{\mathbf{X}}^{-\lambda} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{M})) \\
&= \mathrm{pr}_\lambda(\Gamma(\mathcal{M})) = \Gamma(\mathcal{M}).
\end{aligned}
$$

By Proposition 5.3.7, it follows that the essential image of $\Gamma : \mathcal{C}_{\mathbf{X}/G}^{\lambda} \to \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}$ is contained in $\mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{\lambda}$. ☐

For a fixed integral weight $\mu \in \mathfrak{h}^*$, let $U^\mu - \mathrm{mod}$ denote the category of $U(\mathfrak{g})$-modules with central character μ. There are natural forgetful functors $F^0 : \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{0} \to U^0 - \mathrm{mod}$, $F^\lambda : \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{\lambda} \to U^\lambda - \mathrm{mod}$, obtained by forgetting the G-action and by restricting the $\widehat{U}(\mathfrak{g})$-action to the $U(\mathfrak{g})$-action. There are functors $\widetilde{\theta^\lambda} : U^0 - \mathrm{mod} \to U^\lambda - \mathrm{mod}$ and $\widetilde{\theta_\lambda} : U^\lambda - \mathrm{mod} \to U^0 - \mathrm{mod}$ obtained by viewing the translation functors $\theta^\lambda, \theta_\lambda$ from Proposition 5.3.7 as functors between $U^0 - \mathrm{mod}$ and $U^\lambda - \mathrm{mod}$.

Proposition 5.3.9. The following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{\lambda} & \xrightarrow{F^\lambda} & U^\lambda - \mathrm{mod} \\
\downarrow{\scriptstyle \theta_\lambda} & & \downarrow{\scriptstyle \widetilde{\theta_\lambda}} \\
\mathcal{C}_{\widehat{U}(\mathfrak{g},G)}^{0} & \xrightarrow{F^0} & U^0 - \mathrm{mod}.
\end{array}
$$

Proof. This is immediate by construction of the functor $\widetilde{\theta_\lambda}$, since forgetful functors commute with taking projections and tensor products. $\square$

Proposition 5.3.10. The functors $\widetilde{\theta^\lambda} : U^0-\text{mod} \to U^\lambda-\text{mod}$ and $\widetilde{\theta_\lambda} : U^\lambda-\text{mod} \to U^0-\text{mod}$ are both faithful.

Proof. Let $\overline{K}$ denote the algebraic closure of K. For a K-algebra A, write $\overline{A} := A \otimes_K \overline{K}$. Consider the following diagram:

$$\begin{array}{ccc} U^\lambda - \text{mod} & \xrightarrow{\widetilde{\theta_\lambda}} & U^0 - \text{mod} \\ \downarrow{\scriptstyle -\otimes_K \overline{K}} & & \downarrow{\scriptstyle -\otimes_K \overline{K}} \\ \overline{U^\lambda} - \text{mod} & \xrightarrow{\overline{\widetilde{\theta_\lambda}}} & \overline{U^0} - \text{mod}, \end{array}$$

where $\overline{\widetilde{\theta_\lambda}} : \overline{U^\lambda} - \text{mod} \to \overline{U^0} - \text{mod}$ agrees with the functor $\widetilde{\theta_\lambda} \otimes_K \overline{K}$ defined by $(\widetilde{\theta_\lambda} \otimes_K \overline{K})(a \otimes m) = \widetilde{\theta_\lambda}(a) \otimes m$, for $a \in M$, $M \in U^\lambda - \text{mod}$, $m \in \overline{K}$. It is immediate from its construction that the diagram commutes.

Since $\overline{K}$ is faithfully flat over K, the functor $-\otimes_K \overline{K} : A-\text{mod} \to \overline{A}-\text{mod}$ is faithful for any K-algebra A. The functor $\overline{\widetilde{\theta_\lambda}} : \overline{U^\lambda} - \text{mod} \to \overline{U^0} - \text{mod}$ is an equivalence of categories by [8, Theorem 4.1], and in particular is faithful.

By commutativity of the diagram, the composition $(- \otimes_K \overline{K}) \cdot \widetilde{\theta_\lambda} : U^\lambda - \text{mod} \to \overline{U^0}-\text{mod}$ is faithful, and so the first factor $\widetilde{\theta_\lambda}$ is faithful. An exactly similar argument demonstrates that $\widetilde{\theta^\lambda}$ is faithful. $\square$

Theorem 5.3.11. *Let λ be an ρ-dominant ρ-regular integral weight of G. There is an equivalence of categories:*

$$\theta_\lambda^{-1} \cdot \Gamma \cdot \gamma : \mathcal{C}^\lambda_{\mathbf{X}/G} \to \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)}.$$

Proof. Applying Proposition 5.3.8 and Proposition 5.3.9, there is a commutative diagram:

$$\begin{array}{ccccc}
\mathcal{C}^\lambda_{\mathbf{X}/G} & \xrightarrow{\Gamma} & \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)} & \xrightarrow{F^\lambda} & U^\lambda - \text{mod} \\
\downarrow{\gamma} & & \downarrow{\theta_\lambda} & & \downarrow{\widetilde{\theta_\lambda}} \\
\mathcal{C}^0_{\mathbf{X}/G} & \xrightarrow{\Gamma} & \mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)} & \xrightarrow{F^0} & U^0 - \text{mod}.
\end{array}$$

Since the forgetful functor F^λ is faithful by construction, and the functor $\widetilde{\theta}_\lambda$: $U^\lambda - \text{mod} \to U^0 - \text{mod}$ is faithful by Proposition 5.3.10, it follows that the composition $\widetilde{\theta}_\lambda \cdot F^\lambda : \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)} \to U^0 - \text{mod}$ is faithful. By commutativity of the diagram, the composition $F^0 \cdot \theta_\lambda$ is faithful and so θ_λ is faithful.

The composition $\Gamma \cdot \gamma : \mathcal{C}^\lambda_{\mathbf{X}/G} \to \mathcal{C}^0_{\widehat{U}(\mathfrak{g},G)}$ is an equivalence of categories by Theorem 5.2.27 and [1, Theorem 6.4.7]. By commutativity of the diagram, the composition $\theta_\lambda \cdot \Gamma$ is also an equivalence of categories, and so the functor θ_λ is full and essentially surjective. It follows that θ_λ is an equivalence of categories. The result follows. □

Proof of Theorem E: This is immediate from Theorem 5.3.11.

Chapter 6

The Beilinson-Bernstein localisation theorem: non-integral case

6.1 Constructing $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$

In this chapter, we extend the results of Chapter 5 to prove a twisted version of locally analytic Beilinson-Bernstein localisation for equivariant $\mathcal{D}$-modules for a general ρ-dominant ρ-regular weight, which is not necessarily integral. Complications arise because there is no geometric analogue to the idea of twisting by the line bundle $\mathcal{O}^{\lambda}$ as in the integral case. Furthermore, in general, we should not expect the resulting category $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ to be equivalent to the category $\mathcal{C}_{\mathbf{X}/G}$.

This section is devoted to defining the algebra $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$ in the case where λ is not necessarily integral. We demonstrate that the basic properties of this construction agrees with the untwisted version, following [1, Section 3].

We fix some notation. Let K be a field equipped with a complete non-archimedean norm $|\cdot|$, $\mathcal{R} := \{a \in K \mid |a| \leq 1\}$ the unit ball inside K and $\pi \in \mathcal{R}$ a fixed non-zero non-unit element. Let p be a prime number: the integer e is defined by $e = 1$ for $p > 2$ and $e = 2$ for $p = 2$.

Let $\mathbb{Y}$ be a smooth algebraic variety of finite type. Let G be a compact p-adic Lie group and $\mathbb{G}$ a connected split reductive affine algebraic group scheme over $\mathcal{R}$, equipped with a continuous group homomorphism $\sigma : G \to \mathbb{G}(\mathcal{R})$. Let $\mathbb{H}$ be a flat affine algebraic group over $\mathcal{R}$ of finite type, let $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$ and $\mathfrak{h} := \mathrm{Lie}(\mathbb{H})$. Set $\mathfrak{g}_K := \mathfrak{g} \otimes_{\mathcal{R}} K$, $\mathfrak{h}_K := \mathfrak{h} \otimes_{\mathcal{R}} K$.

Let $\xi : \widetilde{\mathbb{Y}} \to \mathbb{Y}$ be a locally trivial $\mathbb{H}$-torsor as in Definition 2.5.2. Let $\mathcal{S}_{\mathbb{Y}}$ be a base for $\mathbb{Y}$ consisting of affine subschemes $\mathbb{X}$ of finite type that trivialise the torsor ξ in the sense of Definition 2.5.2.

Since $\mathbb{G}$ is smooth and of finite type, $\mathcal{O}(\mathbb{G})$ is a finitely generated $\mathcal{R}$-algebra that is flat as an $\mathcal{R}$-module, and so is a finitely presented $\mathcal{R}$-algebra by [51, Theorem 3.4.6]. Taking the π-adic completion, $\mathcal{A} := \widehat{\mathcal{O}(\mathbb{G})}$ is an admissible $\mathcal{R}$-algebra, so $\widehat{\mathbb{G}} := \mathrm{Spf}\, \widehat{\mathcal{O}(\mathbb{G})}$ is an affine formal scheme of topologically finite presentation, and its rigid generic fibre $\mathbf{G} := \widehat{\mathbb{G}_{\mathrm{rig}}}$ is an affinoid rigid analytic variety over K. We may apply the same argument to define the affinoid rigid analytic varieties $\mathbf{H}$ and $\mathbf{X}$, and the rigid analytic varieties $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$, which are not in general affinoid.

Since $\mathbb{G}$ and $\mathbb{H}$ are smooth affine algebraic groups of finite type over $\mathcal{R}$, [28, Corollaire 19.5.4] implies that they are both infinitesimally flat group schemes.

We suppose G acts continuously on $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ in the sense of Definition 2.8.1. Applying the π-adic completion functor and taking the generic fibre, we obtain an $\mathbf{H}$-torsor:

$$\widehat{\xi_{\mathrm{rig}}} : \widetilde{\mathbf{Y}} \to \mathbf{Y}.$$

Since ξ is locally trivial, there is a base $\mathcal{S}_{\mathbf{Y}}$ of $\mathbf{Y}$ consisting of affinoid subdomains $\mathbf{X}$ such that:

$$\mathbf{X} \times \mathbf{H} \cong \widehat{\xi_{\mathrm{rig}}}^{-1}(\mathbf{X}).$$

Write $\widetilde{\mathbf{X}} := \mathbf{X} \times \mathbf{H}$. Since $\mathbf{X}$ is affinoid and $\mathbb{H}$ is an affine algebraic group, it follows that $\widetilde{\mathbf{X}}$ is an affinoid subdomain of $\widetilde{\mathbf{Y}}$.

From now on, we will assume that $\mathbf{X}$ is a G-stable affinoid subdomain of $\mathbf{Y}$ which trivialises the torsor $\widehat{\xi_{\mathrm{rig}}}$; so there is an isomorphism $\mathbf{X} \times \mathbf{H} \cong \widehat{\xi_{\mathrm{rig}}}^{-1}(\mathbf{X})$. Set:

$$\widetilde{\mathcal{T}} := ((\widehat{\xi_{\mathrm{rig}}})_* \mathcal{T}_{\widetilde{\mathbf{X}}})^{\mathbb{H}}$$

to be the *sheaf of enhanced vector fields* on $\mathbf{X}$.

By assumption, there is a G-action on $\widetilde{\mathbf{Y}}$. Since the G-action on $\widetilde{\mathbf{X}}$ is given by the diagonal G-action on $\mathbf{X} \times \mathbf{H}$, where the G-action on $\mathbf{H}$ is trivial, it follows that $\widetilde{\mathbf{X}}$

is a G-stable affinoid subdomain of $\widetilde{\mathbf{Y}}$ whenever $\mathbf{X}$ is a G-stable affinoid subdomain of $\mathbf{Y}$. Hence there is a continuous group homomorphism:

$$\rho : G \to \mathrm{Aut}_K(\widetilde{\mathbf{X}}, \mathcal{O}_{\widetilde{\mathbf{X}}}),$$

and so we may form the completed skew-group algebra:

$$\widehat{\mathcal{D}}(\widetilde{\mathbf{X}}, G).$$

as in Definition 2.8.7.

Recall, by [4, Lemma 4.3], that there is a sheaf isomorphism:

$$\mathcal{O}_{\mathbb{X}} \to (\xi_* \mathcal{O}_{\widetilde{\mathbb{X}}})^{\mathbb{H}}.$$

Applying the formal completion functor and taking the generic fibre, it follows that there is a sheaf isomorphism:

$$\mathcal{O}_{\mathbf{X}} \to ((\widehat{\xi_{\mathrm{rig}}})_* \mathcal{O}_{\widetilde{\mathbf{X}}})^{\mathbb{H}}.$$

Let $A := \mathcal{O}(\mathbf{X}) = ((\widehat{\xi_{\mathrm{rig}}})_* \mathcal{O}_{\widetilde{\mathbf{X}}})(\widetilde{\mathbf{X}})^{\mathbb{H}}$, $\mathcal{A}$ be an affine formal model in A, $\mathcal{L}$ an $\mathcal{A}$-Lie lattice in $L := \widetilde{\mathcal{T}}(\mathbf{X})$. Finally, let $\mathcal{E} = \mathrm{Aut}_K(\mathcal{O}(\widetilde{\mathbf{X}}))$.

Lemma 6.1.1. Let $\mathcal{A}$ be a G-stable affine formal model in A. Then there exists a G-stable $\mathcal{A}$-Lie lattice in L.

Proof. Since G acts continuously on $\mathbf{X}$, there is a continuous group homomorphism:

$$\rho : G \to \mathrm{Aut}_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}}).$$

Since $\widetilde{\mathcal{T}}(\mathbf{X})$ is a finitely generated A-module, we see that there is a continuous action of G on L, where L is endowed with the Banach topology it has as a finitely generated A-module. Hence there is a continuous group homomorphism:

$$\sigma : G \to \mathrm{Aut}_K(\mathbf{X}, \widetilde{\mathcal{T}}_{\mathbf{X}}).$$

Let $\mathcal{L}$ be an $\mathcal{A}$-Lie lattice in L. Then the associated gauge norm, [53, Section 2.1], induces the Banach topology on L, and so $\mathcal{L}$ is open in L. By continuity, the stabiliser $H := \mathrm{Stab}_G(\mathcal{L})$ is open in G: since G is a compact p-adic Lie group, it follows that H has finite index in G.

Write $\{g_i\}_{i \in I}$ to denote a finite set of coset representatives of H in G. We claim that some suitably large π-power multiple of $\mathcal{M} := \sum_{i \in I} g_i \mathcal{L}$ is a G-stable $\mathcal{A}$-Lie lattice in L.

Since $\mathcal{L}$ is a finitely generated $\mathcal{A}$-submodule of A, it is clear that $\mathcal{M}$ is too. It also spans L as a K-vector space, since $\mathcal{L}$ is spanning. Let $\{m_j\}_{j \in J}$ be a finite generating set for $\mathcal{M}$ as an $\mathcal{A}$-module. Then $[m_i, m_j] = \sum_{ijk} x_{ijk} m_k$ for $x_{ijk} \in A$. By definition, there exists $n_0 \geq 0$ such that $\pi^n x_{ijk} \in \mathcal{A}$ for $n \geq n_0$. It follows that:

$$[\pi^n m_i, \pi^n m_j] = \sum_{ijk} \pi^{2n} x_{ijk} m_k$$

and $\pi^n x_{ijk} \in \mathcal{A}$. Hence $\pi^n \mathcal{M}$ is an $\mathcal{A}$-Lie lattice in L.

Finally, it is G-stable: let $g \in G$ and write $g = g_k h$, where $h \in H$. Then $g_k h g_i = g_j h_{ijk}$ for $i, j, k \in I$ and it follows that:

$$g \cdot g_i \mathcal{L} = g_k h g_i \mathcal{L} = g_j h_{ijk} \mathcal{L} \in g_j \mathcal{L}$$

since $H = \mathrm{Stab}_G(\mathcal{L})$ by definition. It follows that the $\mathcal{A}$-Lie lattice $\mathcal{M}$ is G-stable.

$\square$

Lemma 6.1.2. Let:

$$\iota : \mathcal{A} \oplus \mathcal{L} \to \widehat{U(\mathcal{L})} =: \mathcal{U}$$

denote the natural map, constructed below Definition 2.8.4.

(a) There is a unique $\mathcal{R}$-algebra homomorphism $\psi_{\mathcal{L}} : \mathcal{U} \to \mathcal{E}$ such that $\psi_{\mathcal{L}}(\iota(a)) = l(a)$ and $\psi_{\mathcal{L}}(\iota(v)) = v$ for all $a \in \mathcal{A}$ and $v \in \mathcal{L}$.

(b) The restriction of $\psi_{\mathcal{L}}$ to $\iota(\mathcal{A} \oplus \mathcal{L})$ is injective.

(c) The restriction of $\psi_{\mathcal{L}}^{\times}$ to $\exp(\iota(p^e\mathcal{L}))$ is injective with image $\exp(p^e\mathcal{L})$.

(d) If $\mathcal{A}$ and $\mathcal{L}$ are G-stable, then:

$$\psi_{\mathcal{L}}(g \cdot s) = \rho(g)\psi_{\mathcal{L}}(s)\rho(g)^{-1}$$

for all $g \in G$ and $s \in \mathcal{U}$.

Proof. (a) Since we have a surjective morphism of varieties $\widetilde{\mathbf{X}} \to \mathbf{X}$, there is a chain of inclusions $\mathcal{A} \to A \to \mathcal{O}(\widetilde{\mathbf{X}})$. Define a map $f : \mathcal{O}(\widetilde{\mathbf{X}}) \to \mathcal{E}$ by $f(a)(b) = ab$ for $a, b \in \mathcal{O}(\widetilde{\mathbf{X}})$. Let $l : \mathcal{A} \to \mathcal{E}$ be the composition of the inclusions $\mathcal{A} \to A \to \mathcal{O}(\widetilde{\mathbf{X}})$ with f.

Let $j : \mathcal{L} \to \mathcal{E}$ be the natural inclusion. It is clear that l is an $\mathcal{R}$-algebra homomorphism, j is an $\mathcal{R}$-Lie algebra homomorphism, $j(av) = l(a)j(v)$ for $a \in \mathcal{A}$, $v \in \mathcal{L}$, and:

$$[j(v), l(a)](b) = v(ab) - av(b) = v(a)b = l(v(a))(b)$$

for $a, b \in \mathcal{A}$. Hence $[j(v), l(a)] = l(v \cdot a)$ for $v \in \mathcal{L}, a \in \mathcal{A}$ and so, applying [1, Lemma 2.1.2], there is an $\mathcal{R}$-algebra homomorphism $\psi_{\mathcal{L}} : U(\mathcal{L}) \to \mathcal{E}$ such that $\psi_{\mathcal{L}}(\iota(a)) = l(a)$ and $\psi_{\mathcal{L}}(\iota(v)) = v$ for all $a \in \mathcal{A}$ and $v \in \mathcal{L}$.

Since $\mathcal{A}$ is π-adically complete and $\mathcal{R}$-flat, the same is true for $\mathcal{E}$. Hence ψ extends to an $\mathcal{R}$-algebra homomorphism $\psi_{\mathcal{L}} : \mathcal{U} \to \mathcal{E}$ with the required properties. Since the $\mathcal{R}$-subalgebra of $\mathcal{U}$ generated by $\iota(\mathcal{A} \oplus \mathcal{L})$ is dense in $\mathcal{U}$, and any $\mathcal{R}$-algebra homomorphism between two π-adically complete $\mathcal{R}$-algebras is continuous, it follows that this homomorphism is unique.

(b) Since the map $l : \mathcal{A} \to \mathcal{E}$ is injective, this follows immediately from part (a).

(c) This follows from part (b) since the maps exp and log are bijections over a field of characteristic 0.

(d) Let $g \in G, a \in \mathcal{A}$. Then:

$$\rho(g)l(a)\rho(g)^{-1} = l(g \cdot a)$$

since $\rho(g)$ is an $\mathcal{R}$-algebra automorphism of $\mathcal{A}$ and so:

$$(\rho(g)l(a)\rho(g)^{-1})(b) = g \cdot (a(g^{-1} \cdot b)) = (g \cdot a)(b) = l(g \cdot a)(b)$$

for all $b \in \mathcal{A}$. We define two maps $\alpha : \mathcal{U} \to \mathcal{E}$ and $\alpha' : \mathcal{U} \to \mathcal{E}$ by:

$$\alpha(s) = \psi_{\mathcal{L}}(g \cdot s)$$
$$\alpha'(s) = \rho(g)\psi_{\mathcal{L}}(s)\rho(g)^{-1}$$

for $s \in \mathcal{U}$. Then, for $a \in \mathcal{A}, v \in \mathcal{L}$, we see that:

$$\alpha(\iota(a)) = \psi_{\mathcal{L}}(g \cdot \iota(a)) = \psi_{\mathcal{L}}(\iota(g \cdot a)) = l(g \cdot a)$$

$$= \rho(g)l(a)\rho(g)^{-1} = \rho(g)\psi_{\mathcal{L}}(\iota(a))\rho(g)^{-1} = \alpha'(\iota(a))$$

and similarly,

$$\alpha(\iota(v)) = \psi_{\mathcal{L}}(g \cdot \iota(v)) = \psi_{\mathcal{L}}(\iota(g \cdot v)) = l(g \cdot v)$$

$$= \rho(g)l(v)\rho(g)^{-1} = \rho(g)\psi_{\mathcal{L}}(\iota(v))\rho(g)^{-1} = \alpha'(\iota(v))$$

Hence α, α' are $\mathcal{R}$-algebra homomorphisms that agree on the dense subalgebra $\iota(\mathcal{A} \oplus \mathcal{L})$, and so are equal by the argument in part (a). $\square$

Definition 6.1.3. Let $\mathcal{A}$ be a G-stable affine formal model in A, $\mathcal{L}$ an $\mathcal{A}$-Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$. Set:

$$G_{\mathcal{L}} := \rho^{-1}(\exp(p^e \mathcal{L}))$$

$$\beta_{\mathcal{L}} := (\psi_{\mathcal{L}}^{\times})^{-1} \circ \rho : G_{\mathcal{L}} \to \mathcal{U}^{\times}.$$

These maps fit into the commutative diagram:

$$\begin{array}{ccccc} & & G_{\mathcal{L}} & \longrightarrow & G \\ & \swarrow^{\rho|_{G_{\mathcal{L}}}} & \downarrow{\scriptstyle \beta_{\mathcal{L}}} & & \downarrow{\scriptstyle \rho} \\ \exp(p^{\epsilon}\mathcal{L}) & \longrightarrow & \mathcal{U}^{\times} & \xrightarrow{\psi_{\mathcal{L}}^{\times}} & \mathcal{E}^{\times}. \end{array}$$

Theorem 6.1.4. *Let $\mathcal{A}$ be a G-stable affine formal model in A, $\mathcal{L}$ an $\mathcal{A}$-Lie lattice in $\widetilde{\mathcal{T}}(\boldsymbol{X})$. Then:*

(a) $G_{\mathcal{L}}$ is an open normal subgroup of G.

(b) $\beta_{\mathcal{L}}$ is a G-equivariant trivialisation of the $G_{\mathcal{L}}$-actions on $\widehat{U(\mathcal{L})}$ and $\widehat{U(\mathcal{L})_K}$.

Proof. (a) Since $\mathcal{E}$ is a K-Banach algebra, the Baker-Campbell-Hausdorff series:

$$\Phi(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots$$

converges at $(X,Y) = (p^e u, p^e v)$ for any $u, v \in \mathcal{L}$, and:

$$\exp(p^e u)\exp(p^e v) = \exp(\Phi(p^e u, p^e v))$$

by [22, Proposition 6.27]. Furthermore, since $[\mathcal{L},\mathcal{L}] \subseteq \mathcal{L}$ and $\mathcal{L}$ is π-adically complete, it follows that $\Phi(p^e u, p^e v) \in p^e\mathcal{L}$. Hence $\exp(p^e\mathcal{L})$ is a subgroup of $\mathcal{E}^\times$.

Let $g \in G_{\mathcal{L}}$; by definition, $\rho(g) = \exp(u)$ for some $u \in p^e\mathcal{L}$. If $x \in G$ then $x \cdot u = \rho(x)u\rho(x)^{-1}$ by definition, and so:

$$\rho(xgx^{-1}) = \rho(x)\exp(u)\rho(x)^{-1} = \exp(x \cdot u) \in \exp(p^e\mathcal{L})$$

since $\mathcal{L}$ is G-stable. Hence $xgx^{-1} \in G_{\mathcal{L}}$ and so $G_{\mathcal{L}}$ is normal in G.

To see that $G_{\mathcal{L}}$ is open in G, we may choose an open uniform pro-p subgroup $N \leq G$ contained in $\rho^{-1}(\mathcal{G}_{p^e}(\mathcal{O}(\widetilde{\mathbf{X}}))$, defined in Definition 2.6.11, as in the proof of [1, Proposition 3.2.5]. Then $\log\rho(N)$ is a finitely generated $\mathbb{Z}_p$-submodule of $\mathcal{T}(\widetilde{\mathbf{X}})$, so $\log\rho(N)\cap\widetilde{\mathcal{T}}(\mathbf{X})$ is a finitely generated $\mathbb{Z}_p$-submodule of $\widetilde{\mathcal{T}}(\mathbf{X})$. Since $\mathcal{L}$ is an $\mathcal{A}$-lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$, we can find some $n\geq 0$ such that $p^n\log\rho(N)\subseteq p^e\mathcal{L}$. Hence $N^{p^n}\leq G_{\mathcal{L}}$ and so $G_{\mathcal{L}}$ is open in G.

(b) By Lemma 6.1.2(c), $\beta_{\mathcal{L}}$ is a well-defined group homomorphism, and using Lemma 6.1.2(d), we see that:

$$\psi_{\mathcal{L}}(\beta_{\mathcal{L}}(g)s\beta_{\mathcal{L}}(g)^{-1})=\rho(g)\psi_{\mathcal{L}}(s)\rho(g)^{-1}=\psi_{\mathcal{L}}(g\cdot s)$$

for all $g\in G, s\in\mathcal{U}$. Applying Lemma 6.1.2(b), it follows that $\beta_{\mathcal{L}}(g)s\beta_{\mathcal{L}}(g)^{-1}=g\cdot s$ for all $g\in G_{\mathcal{L}}, s\in\iota(\mathcal{A}\oplus\mathcal{L})$. Since $\iota(\mathcal{A}\oplus\mathcal{L})$ generates $\mathcal{U}$ as a topological $\mathcal{R}$-algebra, this equation holds for all $s\in\mathcal{U}$. Hence $\beta_{\mathcal{L}}$ is a trivialisation of the $G_{\mathcal{L}}$-action on $\mathcal{U}$.

Applying Lemma 6.1.2(d), we see that:

$$\psi_{\mathcal{L}}(\beta_{\mathcal{L}}(xgx^{-1}))=\rho(xgx^{-1})=\rho(x)\psi_{\mathcal{L}}(\beta_{\mathcal{L}}(g))\rho(x)^{-1}=\psi_{\mathcal{L}}(x\cdot\beta_{\mathcal{L}}(g))$$

for all $x\in G$ and $g\in G_{\mathcal{L}}$. The map $\exp: p^e\mathcal{U}\to\mathcal{U}^\times$ is G-equivariant, and $\beta_{\mathcal{L}}(g)\in\exp(\iota(p^e\mathcal{L}))$ by construction. Hence $x\cdot\beta_{\mathcal{L}}(g)\in\exp(\iota(p^e\mathcal{L}))$, so $\beta_{\mathcal{L}}(xgx^{-1})=x\cdot\beta_{\mathcal{L}}(g)$ by Lemma 6.1.2(c).

This shows that $\beta_{\mathcal{L}}$ is a G-equivariant trivialisation of the $G_{\mathcal{L}}$-action on $\mathcal{U}$. When regarded as a map $G_{\mathcal{L}}\to\widehat{U(\mathcal{L})_K}^\times=(\mathcal{U}\otimes_{\mathcal{R}}K)^\times$, it follows that $\beta_{\mathcal{L}}$ is a G-equivariant trivialisation of the $G_{\mathcal{L}}$-action on $\widehat{U(\mathcal{L})_K}$. $\square$

Hence, for any open normal subgroup N of G contained in $G_{\mathcal{L}}$, we may form the skew-group algebra $\widehat{U(\mathcal{L})_K} \rtimes_N G$.

Definition 6.1.5. The pair $(\mathcal{L}, N)$ is an $\mathcal{A}-$ *trivialising pair* if $\mathcal{L}$ is a G-stable Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$ and N is an open normal subgroup of G contained in $G_{\mathcal{L}}$. The set of all $\mathcal{A}$-trivialising pairs is denoted by $I(G)$.

Definition 6.1.6. The *enhanced completed skew-group algebra* is:

$$\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G) := \varprojlim_{(\mathcal{L},N)\in I(G)} \widehat{U(\mathcal{L})_K} \rtimes_N G.$$

As written, this definition is dependent on the choice of affine formal model $\mathcal{A}$ in A. However, the same argument as that given in [1, Proposition 3.3.8] demonstrates that, for any two such affine formal models $\mathcal{A}, \mathcal{B}$, there is a canonical K-algebra isomorphism:

$$\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)_{\mathcal{A}} \cong \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)_{\mathcal{B}}.$$

The next part of the thesis is devoted to deriving some basic properties arising from the construction of $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$.

Remark 6.1.7. We have a canonical group homomorphism:

$$\gamma : G \to \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)^{\times}$$

and a canonical K-algebra homomorphism:

$$i : \widetilde{\mathcal{D}}(\mathbf{X}) \to \widetilde{\widehat{\mathcal{D}}}(\mathbf{X}, G).$$

These extend to a canonical K-algebra homomorphism:

$$i \rtimes \gamma : \widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G \to \widetilde{\widehat{\mathcal{D}}}(\mathbf{X}, G).$$

It follows that we may view $\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G$ as a dense K-subalgebra of $\widetilde{\widehat{\mathcal{D}}}(\mathbf{X}, G)$.

Definition 6.1.8. Let $(N_\bullet) = N_0 \geq N_1 \geq \cdots$ be a separated chain of open normal subgroups of G, and let $\mathcal{L}$ be a G-stable $\mathcal{A}$-Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$. $(N_\bullet)$ is a *good chain* for $\mathcal{L}$ if $(\pi^n\mathcal{L}, N_n) \in \mathcal{I}(G)$ for all $n \geq 0$.

Lemma 6.1.9. For every good chain $(N_\bullet)$ for $\mathcal{L}$, there is a K-algebra isomorphism:

$$\widetilde{\widehat{\mathcal{D}}}(\mathbf{X}, G) \cong \varprojlim \widehat{U(\pi^n\mathcal{L})} \rtimes_{N_n} G.$$

Proof. Apply the same proof as in [1, Lemma 3.3.4]. □

In some cases, Lemma 6.1.9 provides an alternative definition of the enhanced completed skew-group algebra $\widetilde{\widehat{\mathcal{D}}}(\mathbf{X}, G)$. In particular, Theorem 6.1.4(a) and Definition 6.1.5 show that $(\mathcal{L}, G_{\mathcal{L}})$ is always an $\mathcal{A}$-trivialising pair, so if the G-action is faithful, the set $\{(\pi^n\mathcal{L}, G_{\pi^n\mathcal{L}}) \mid n \geq 0\}$ is a good chain for $\mathcal{L}$. However, in general, $\ker \rho \subseteq \bigcap_{n\in\mathbb{N}} G_{\pi^n\mathcal{L}}$, and hence the chain $(N_\bullet)$ is not separated when the G-action is not faithful. We give a more general construction to account for this possibility.

We next discuss the functoriality of the construction $\wideparen{\widetilde{\mathcal{D}}}(\mathbf{X}, G)$.

Proposition 6.1.10. Let $\phi : A \to A'$ be an étale morphism of K-affinoid algebras, set $\mathbf{X} := \mathrm{Sp}(A), \mathbf{X}' := \mathrm{Sp}(A')$, and let G, G' be compact p-adic Lie groups acting continuously on A and A' respectively, in the sense of Definition 2.8.1. Let $L := \widetilde{\mathcal{T}}(\mathbf{X}), L' := \widetilde{\mathcal{T}}(\mathbf{X}')$. Suppose $\tau : G \to G'$ is a group homomorphism such that:

$$\phi(g \cdot a) = \tau(g) \cdot \phi(a) \text{ for all } g \in G, a \in A.$$

Then there is a unique continuous K-algebra homomorphism:

$$\wideparen{\phi \rtimes \tau} : \wideparen{\widetilde{\mathcal{D}}}(\mathbf{X}, G) \to \wideparen{\widetilde{\mathcal{D}}}(\mathbf{X}', G')$$

which makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G & \xrightarrow{U(\phi,\widetilde{\phi}) \rtimes \tau} & \widetilde{\mathcal{D}}(\mathbf{X}') \rtimes G' \\ \big\downarrow{\scriptstyle i \rtimes \gamma} & & \big\downarrow{\scriptstyle i' \rtimes \gamma'} \\ \wideparen{\widetilde{\mathcal{D}}}(\mathbf{X}, G) & \xrightarrow{\wideparen{\phi \rtimes \tau}} & \wideparen{\widetilde{\mathcal{D}}}(\mathbf{X}', G'). \end{array}$$

Here, $U(\phi, \widetilde{\phi})$ is the unique extension of $\widetilde{\phi} : L \to L'$ to $U(L) \to U(L')$ given in Lemma 2.6.8.

Proof. Let $\mathcal{A} \subseteq A$ and $\mathcal{A}' \subseteq A'$ be G-stable (respectively G'-stable) formal models. Then $\phi(\mathcal{A}) \cdot A'$ is another affine formal model in A' containing $\phi(\mathcal{A}')$, so by applying [1, Lemma 3.2.4], we may find a G'-stable formal model $\mathcal{A}''$ containing $\phi(\mathcal{A}) \cdot A'$.

Replacing $\mathcal{A}'$ with $\mathcal{A}''$, we may assume $\phi(\mathcal{A}) \subseteq \mathcal{A}'$.

Using Lemma 6.1.1, we may choose a G-stable $\mathcal{A}$-Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$ and a G'-stable $\mathcal{A}'$-Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X}')$. Since $\mathcal{L}$ is a finitely generated $\mathcal{A}$-module and $\widetilde{\phi}(av) = \phi(a)\widetilde{\phi}(v)$ for any $a \in \mathcal{A}$ and $v \in \widetilde{\mathcal{T}}(\mathbf{X})$, we see that $\pi^m\widetilde{\phi}(\mathcal{L}) \subseteq \mathcal{L}'$ for some $m \geq 0$. Rescaling $\mathcal{L}$, we may assume $m = 0$ and so $\widetilde{\phi}(\mathcal{L}) \subseteq \mathcal{L}'$.

Choose a good chain $(N_\bullet)$ in G for $\mathcal{L}$, and a good chain $(N'_\bullet)$ in G' for $\mathcal{L}'$ by Lemma 6.1.9. Applying [22, Corollary 8.34 and Corollary 1.21(i)], the group homomorphism τ is continuous, and so $\tau^{-1}(N'_n)$ is open in G for each $n \geq 0$. Applying [1, Lemma 3.3.6] to the open subgroups $N_n \cap \tau^{-1}(N'_n)$, we may suppose that $\tau(N_n) \leq N'_n$ for each $n \geq 0$. The same argument as in [1, Proposition 3.2.15] yields the compatible sequence of commutative diagrams:

$$\begin{array}{ccc}
\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G & \xrightarrow{U(\phi,\widetilde{\phi})\rtimes\tau} & \widetilde{\mathcal{D}}(\mathbf{X}') \rtimes G' \\
\downarrow & & \downarrow \\
\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{N_n} G & \xrightarrow{\widehat{\phi_{n,K}}} & \widehat{U(\pi^n\mathcal{L}')}_K \rtimes_{N'_n} G'.
\end{array}$$

Passing to the limit and applying Lemma 6.1.9 produces the required map:

$$\widehat{\phi \rtimes \tau} : \widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, G) \to \widehat{\widetilde{\mathcal{D}}}(\mathbf{X}', G')$$

which fits into the commutative diagram:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G & & \xrightarrow{U(\widetilde{\phi},\phi)\rtimes\tau} & & \widetilde{\mathcal{D}}(\mathbf{X}') \rtimes G' \\
\downarrow & \searrow^{i\rtimes\gamma} & & \swarrow^{i'\rtimes\gamma'} & \downarrow \\
& \widetilde{\widetilde{\mathcal{D}}}(\mathbf{X}, G) & \xrightarrow{\widehat{\phi\rtimes\tau}} & \widetilde{\widetilde{\mathcal{D}}}(\mathbf{X}', G') & \\
& \swarrow_{\cong} & & \searrow^{\cong} & \\
\varprojlim \widehat{U(\pi^n\mathcal{L})_K} \rtimes_{N_n} G & & \xrightarrow{\varprojlim \widehat{\phi_{n,K}}\rtimes\tau} & & \varprojlim \widehat{U(\pi^n\mathcal{L}')_K} \rtimes_{N_n'} G'.
\end{array}
$$

To show uniqueness, suppose $\psi : \widetilde{\widetilde{\mathcal{D}}}(\mathbf{X}, G) \to \widetilde{\widetilde{\mathcal{D}}}(\mathbf{X}', G')$ is another continuous K-algebra map such that $\psi \circ (i \rtimes \gamma) = (i' \rtimes \gamma') \circ U(\phi, \widetilde{\phi})$. Then ψ agrees with $\widehat{\phi \rtimes \tau}$ on the dense image of $\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G$ inside $\widetilde{\widetilde{\mathcal{D}}}(\mathbf{X}, G)$, and so the two maps are equal by continuity. $\square$

Let $\mathbf{X}_w/G$ denote the set of G-stable affinoid subdomains of $\mathbf{X}$.

Lemma 6.1.11. If G is compact, then $\widetilde{\widetilde{\mathcal{D}}}(-, G)$ is a presheaf of K-Fréchet algebras on $\mathbf{X}_w(G)$.

Proof. The same argument as in [1, Corollary 3.3.9], demonstrates that $\widetilde{\widetilde{\mathcal{D}}}(\mathbf{U}, G)$ is a K-Fréchet algebra for each $\mathbf{U} \in \mathbf{X}_w/G$. Given $\mathbf{V} \subseteq \mathbf{U}$, the restriction map $\phi : \mathcal{O}(\mathbf{U}) \to \mathcal{O}(\mathbf{V})$ is étale, and it is G-equivariant by [1, Remark 2.2.3(b)]. Applying Proposition 6.1.10, there is a unique continuous K-algebra homomorphism $\tau_{\mathbf{V}}^{\mathbf{U}} := \widehat{\phi \rtimes 1_G} : \widetilde{\widetilde{\mathcal{D}}}(\mathbf{U}, G) \to \widetilde{\widetilde{\mathcal{D}}}(\mathbf{V}, G)$ that extends $U(\phi, \widetilde{\phi}) \rtimes 1_G : \widetilde{\mathcal{D}}(\mathbf{U}) \rtimes G \to \widetilde{\mathcal{D}}(\mathbf{V}) \rtimes G$.

If $\mathbf{W} \subseteq \mathbf{V}$ is another object of $\mathbf{X}_w/G$, the functoriality of $\widetilde{\mathcal{D}}(-)$ ensures that $\tau_{\mathbf{W}}^{\mathbf{V}} \circ \tau_{\mathbf{V}}^{\mathbf{U}}$ and $\tau_{\mathbf{W}}^{\mathbf{U}}$ both extend the restriction map $\widetilde{\mathcal{D}}(\mathbf{U}) \to \widetilde{\mathcal{D}}(\mathbf{W})$, and are therefore equal by the uniqueness part of Proposition 6.1.10. $\square$

The next part of the thesis specialises the definition of $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$ to a central reduction, allowing us to define the twisted completed skew-group algebra $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$.

Let $\lambda : \mathfrak{h}_K \to K$ be a character of $\mathfrak{h}_K$ with the property that $\lambda(\pi^m \mathfrak{h}) \subseteq p^e \mathcal{R}$. Since $\mathfrak{h}$ is finite-dimensional, there exists some $n \geq 0$ such that this property holds for any $m \geq n$.

Lemma 6.1.12. There is a natural $\widehat{U(\mathfrak{h})_K}$-module structure on $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$.

Proof. Recall, by [4, Section 4.10], that there is a continuous central embedding of $U(\mathfrak{h})_K$ into the algebra $\widetilde{\mathcal{D}}(\mathbf{X})$, and so there is a continuous central embedding of $U(\mathfrak{h})_K$ into $\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G$. It follows that we may view $\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G$ as a $U(\mathfrak{h})_K$-module. Since $\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G$ is dense in $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$, we may extend the $U(\mathfrak{h})_K$-module structure on $\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G$ to obtain a $\widehat{U(\mathfrak{h})_K}$-module structure on $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$. □

Given $\lambda \in \mathfrak{h}_K^*$ as above, the map $\lambda : \mathfrak{h}_K \to K$ extends to a K-algebra homomorphism $U(\mathfrak{h})_K \to K$. This gives K the structure of a $U(\mathfrak{h})_K$-module, denoted by K_λ. Extending the $U(\mathfrak{h})_K$-action by continuity, we further see that K_λ has a $\widehat{U(\mathfrak{h})_K}$-module structure. This allows us to make sense of the following definition.

Definition 6.1.13. Let $\lambda \in \mathfrak{h}_K^*$. We set:

$$\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G) := \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G) \otimes_{\widehat{U(\mathfrak{h})_K}} K_\lambda.$$

Lemma 6.1.14. The algebra $\mathcal{D}^{\lambda}(\mathbf{X}) \rtimes G$ is dense in $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$.

Proof. We may identify $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$ as:

$$\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G) = \frac{\widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, G)}{(h - \lambda(h) \mid h \in \mathfrak{h}_K) \cdot \widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, G)}$$

equipped with the quotient topology. We construct the following commutative diagram:

$$\begin{array}{ccc}
\widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, G) & \longrightarrow & \widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G) \\
\uparrow & & \uparrow \\
\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G & \longrightarrow & \mathcal{D}^{\lambda}(\mathbf{X}) \rtimes G
\end{array}$$

where the horizontal arrows are strict surjections by construction. The K-algebra $\widetilde{\mathcal{D}}(\mathbf{X}) \rtimes G$ is dense in $\widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, G)$ by Remark 6.1.7. Commutativity of the diagram then implies that $\mathcal{D}^{\lambda}(\mathbf{X}) \rtimes G$ is dense in $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$. □

Theorem 6.1.15. *Suppose $(\boldsymbol{U}, H)$ is small, in the sense of Definition 2.8.13. Then $\widehat{\widetilde{\mathcal{D}}}(\boldsymbol{U}, H)$ and $\widehat{\mathcal{D}}^{\lambda}(\boldsymbol{U}, H)$ are both Fréchet-Stein.*

Proof. To show $\widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)$ is Fréchet-Stein, we may apply the proof given in [1, Theorem 3.4.8], since this is valid for any choice of Lie lattice $\mathcal{L}$.

By definition, we may write:

$$\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H) := \frac{\widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)}{(h - \lambda(h) \mid h \in \mathfrak{h}_K) \cdot \widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)}.$$

Applying [55, Corollary 3.4(iv) and Lemma 3.6], the ideal $(h - \lambda(h) \mid h \in \mathfrak{h}_K) \cdot \widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)$ is closed in $\widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)$, since it is finitely generated. By [55, Proposition 3.7], it follows that the quotient $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H)$ is a Fréchet-Stein K-algebra. □

Proposition 6.1.16. Let H be a compact open subgroup of G and let $\mathbf{U} \in \mathbf{X}_w/H$. For every $g \in G$, there is a continuous K-algebra isomorphism:

$$\widehat{g_{\mathbf{U},H}} : \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) \to \widehat{\mathcal{D}}^\lambda(g\mathbf{U}, gHg^{-1}).$$

Proof. By Lemma 5.2.5, the sheaf $\mathcal{D}^\lambda$ is G-equivariant. Using this, the proof is the same as that given for $\mathcal{D}$ in [1, Lemma 3.4.3], replacing $\mathcal{D}$ by $\mathcal{D}^\lambda$ and applying Proposition 6.1.10. □

We next define an important compatibility condition for the action of a K-algebra A on the rigid analytic variety $\mathbf{X}$.

Definition 6.1.17. Let A be a K-algebra. A acts on $\mathbf{X}$ λ-*compatibly with* G if there is:

(a) a group homomorphism $\eta : G \to A^\times$,

(b) a Fréchet-Stein subalgebra A_H of A for every compact open subgroup H of G,

(c) a continuous homomorphism $\phi^H : A_H \to \widehat{\mathcal{D}}^\lambda(-, H)$ of presheaves of K-Fréchet algebras on $\mathbf{X}_w/H$ for every compact open subgroup H of G.

Furthermore, for any pair $H \leq N$ of compact open subgroups of G, the following compatibility conditions are satisfied:

(i) $A_H \leq A_N$, $\eta(H) \subseteq A_H^\times$ and the canonical map:

$$\nu : A_H \otimes_{K[H]} K[N] \to A_N$$

is a bijection.

(ii) the following diagram of presheaves is commutative:

$$\begin{array}{ccc} A_H & \xrightarrow{\phi^H} & \widehat{\mathcal{D}}^\lambda(-, H) \\ \downarrow & & \downarrow \\ A_N & \xrightarrow{\phi^N} & \widehat{\mathcal{D}}^\lambda(-, N). \end{array}$$

(iii) for every $g \in G$, the map $\mathrm{Ad}_{\eta(g)} : A \to A$, obtained by conjugation by $\eta(g)$, sends A_H into $A_{gHg^{-1}}$, and for every $\mathbf{U} \in \mathbf{X}_w/H$, the following diagram commutes:

$$\begin{array}{ccc} A_H & \xrightarrow{\mathrm{Ad}_{\eta(g)}} & A_{gHg^{-1}} \\ \downarrow{\scriptstyle \phi^{\mathbf{U}}(H)} & & \downarrow{\scriptstyle \phi^{gHg^{-1}}(g\mathbf{U})} \\ \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) & \xrightarrow{\widehat{g_{\mathbf{U},H}}} & \widehat{\mathcal{D}}^\lambda(g\mathbf{U}, gHg^{-1}). \end{array}$$

(iv) let $\gamma^G : G \to \widehat{\mathcal{D}}^\lambda(\mathbf{U}, G)^\times$ be the composition of the canonical map $G \to \widehat{\widehat{\mathcal{D}}}(\mathbf{U}, G)^\times$ from Remark 6.1.7 with the strict surjection $\widehat{\widehat{\mathcal{D}}}(\mathbf{U}, G)^\times \to \widehat{\mathcal{D}}^\lambda(\mathbf{U}, G)^\times$ from Lemma 6.1.14. Then $\phi^H \circ \eta \mid_H = \gamma^H$.

Lemma 6.1.18. Suppose G is a p-adic Lie group acting continuously on $\mathbf{X}$. Suppose A is a K-Fréchet algebra and I is a closed two-sided ideal of A. Write, for every compact open subgroup $H \leq G$, $(A/I)_H := A_H/(I \cap A_H)$. Suppose there is a continuous homomorphism $\phi^H : A_H \to \widehat{\mathcal{D}}^\lambda(-, H)$ of presheaves of K-Fréchet algebras on $\mathbf{X}_w/H$ for every compact open subgroup H of G, such that $\phi^H(I \cap A_H) = 0$. Further suppose that there is a natural map $\psi : (I \cap A_H) \otimes_{K[H]} K[N] \to A_H \otimes_{K[H]} K[N]$, with image $I \cap A_N$.

If A acts on $\mathbf{X}$ λ-compatibly with G, then so does the quotient A/I.

Proof. By assumption, there is a group homomorphism $\eta : G \to A^\times$, and so there is a group homomorphism $\zeta : G \to A^\times \to (A/I)^\times$ given by composition.

Since the two-sided ideal $I \cap A_H$ is closed in A_H, the quotient $A_H/(I \cap A_H)$ is also a Fréchet-Stein algebra by [55, Corollary 3.4(iv) and Lemma 3.6]. Via the isomorphism $A_H/(I \cap A_H) \cong (A/I)_H$, we see that $(A/I)_H$ is a Fréchet-Stein algebra for each compact open subgroup $H \leq G$.

By assumption, we have a map $\phi^H : (A/I)_H = A_H/(I \cap A_H) \to \widehat{\mathcal{D}}^\lambda(-, H)$. We now verify that this data satisfies the axioms required for the action to be compatible. We fix a pair $H \leq N$ of compact open subgroups of G.

(i) By assumption, $A_H \leq A_N$. It is immediate that $A_H/(I \cap A_H) \leq A_N/(I \cap A_N)$. Also $\eta(H) \subseteq A_H^\times$, so $\zeta(H) \subseteq (A_H/(I \cap A_H))^\times$ by construction.

There is a natural map $\psi : I \cap A_H \otimes_{K[H]} K[N] \to A_H \otimes_{K[H]} K[N]$, with image $I \cap A_N$ by assumption. It follows that we have a canonical bijection:

$$A_H/(I \cap A_H) \otimes_{K[H]} K[N] \to A_N/(I \cap A_N).$$

(ii) Consider the diagram of presheaves:

$$\begin{array}{ccc} A_H/(I \cap A_H) & \xrightarrow{\phi^H} & \widehat{\mathcal{D}}^\lambda(-, H) \\ \downarrow & & \downarrow \\ A_N/(I \cap A_N) & \xrightarrow{\phi^N} & \widehat{\mathcal{D}}^\lambda(-, N). \end{array}$$

Since the map of presheaves $\phi^H : A_H \to \widehat{\mathcal{D}}^\lambda(-, H)$ factors through the two-sided ideal $I \cap A_H$ by definition, it is immediate that the given diagram is commutative, since the diagram given in part (ii) of Definition 6.1.17 is commutative.

(iii) Note that, for each $g \in G$, the map $\mathrm{Ad}_{\zeta(g)} : A/I \to A/I$ given by conjugation with $\zeta(g)$ sends $A_H/(I \cap A_H) \to A_{gHg^{-1}}/(I \cap A_{gHg^{-1}})$ by construction. Furthermore, we see that the following diagram is commutative:

$$\begin{array}{ccc} A_H/(I \cap A_H) & \xrightarrow{\mathrm{Ad}_{\zeta(g)}} & A_{gHg^{-1}} \\ \downarrow{\scriptstyle \phi^{\mathbf{U}}(H)} & & \downarrow{\scriptstyle \phi^{gHg^{-1}}(g\mathbf{U})} \\ \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) & \xrightarrow{\widehat{g_{\mathbf{U},H}}} & \widehat{\mathcal{D}}^\lambda(g\mathbf{U}, gHg^{-1}). \end{array}$$

(iv) Since the map of presheaves $\phi^H : A_H \to \widehat{\mathcal{D}}^\lambda(-, H)$ factors through the two-sided ideal $I \cap A_H$ by definition, it follows that we have the composition:

$$H \to A^\times \to (A/(I \cap A_H))^\times \to \widehat{\mathcal{D}}^\lambda(-, H)$$

which is equal to the map $\phi^H \circ \eta \mid_H$ by construction. Hence, since A acts on $\mathbf{X}$ λ-compatibly with G, it follows that we have an equality of functions:

$$\phi^H \circ \zeta \mid_H = \gamma^H.$$

$\square$

Proposition 6.1.19. Suppose that $(\mathbf{X}, G)$ is small. Then both $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$ and $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$ act on $\mathbf{X}$ λ-compatibly with G, in the sense of Definition 6.1.17.

Proof. As $\mathbf{X}$ is a G-stable affinoid variety, we may define the group homomorphism:

$$\gamma : G \to \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)^{\times}.$$

For every compact open subgroup $H \subseteq G$, set:

$$A_H := \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, H).$$

By Lemma 2.8.15 and Theorem 6.1.15, this is a Fréchet-Stein subalgebra of $A := \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$. For each $\mathbf{U} \in \mathbf{X}_w/H$, let:

$$\phi^H(\mathbf{U}) : A_H \to \widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H)$$

be the composition of the restriction map $\tau^{\mathbf{X}}_{\mathbf{U}}$ in the presheaf $\widehat{\widehat{\mathcal{D}}}(-, H)$ on $\mathbf{X}_w/H$ constructed in the proof of Lemma 6.1.11 with the quotient map $\widehat{\widehat{\mathcal{D}}}(-, H) \to \widehat{\mathcal{D}}^{\lambda}(-, H)$. Viewing A_H as a constant sheaf on $\mathbf{X}_w/H$, we see that $\phi^H(\mathbf{U}) : A_H \to \widehat{\widehat{\mathcal{D}}}(\mathbf{U}, H)$ is a continuous morphism of presheaves.

This defines the data in Definition 6.1.17. In view of [1, Proposition 3.3.10], and Proposition 6.1.10, this data satisfies the requisite compatibility conditions for $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$

to act on $\mathbf{X}$ λ-compatibly with G. The verification is very similar to the proof of [1, Proposition 3.4.9].

Set $I_G := (h - \lambda(h) \mid h \in \mathfrak{h}_K) \cdot \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$ and $I_H := (h - \lambda(h) \mid h \in \mathfrak{h}_K) \cdot \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, H)$. We verify that $I_G \cap A_H = I_H$. Note that I_G is a finitely generated ideal of the Fréchet-Stein algebra $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$. Applying [55, Corollary 3.4(iv) and Lemma 3.6], we see that I_G is a coadmissible $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$-module. In particular, $I_G \cap \widehat{\widehat{\mathcal{D}}}(\mathbf{X}, H)$ is closed in $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, H)$, and so is a coadmissible $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, H)$-module.

Via the same argument as in the previous paragraph, I_H is a coadmissible $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, H)$-module. Hence it suffices to show equality on the Banach level.

Following Definition 6.1.6, write $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G) := \varprojlim_{(\mathcal{L}, N) \in I(G)} \widehat{U(\mathcal{L})_K} \rtimes_N G$. Write $(h - \lambda(h)) := (h - \lambda(h) \mid h \in \mathfrak{h}_K)$, fix $(\mathcal{L}, P) \in I(G)$ and set $I_{\mathcal{L},P} := (h - \lambda(h)) \cdot (\widehat{U(\mathcal{L})_K} \rtimes_P G)$, $I_{\mathcal{L},P\cap H} := (h - \lambda(h)) \cdot (\widehat{U(\mathcal{L})_K} \rtimes_{P\cap H} H)$. Set $I_{\mathcal{L}} := (h - \lambda(h)) \cdot \widehat{U(\mathcal{L})_K}$. Now [1, Lemma 2.2.4(b)] implies that we have an isomorphism $\widehat{U(\mathcal{L})_K} \rtimes_P G \cong \widehat{U(\mathcal{L})_K} * (G/P)$. Since the G-action on $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$ fixes the image of $h \in \mathfrak{h}_K$ pointwise, it follows that we have a vector space isomorphism $I_{\mathcal{L},P} \cong I_{\mathcal{L}} * [G/P]$. This implies that we have the following chain of isomorphisms.

$$\begin{aligned} I_{\mathcal{L},P} \cap (\widehat{U(\mathcal{L})_K} \rtimes_{P\cap H} H) &\cong (I_{\mathcal{L}} * [G/P]) \cap (\widehat{U(\mathcal{L})_K} * (H/P \cap H)) \\ &\cong I_{\mathcal{L}} * (H/P \cap H) \cong I_{\mathcal{L},P\cap H}. \end{aligned}$$

Taking the inverse limit over the subsystem $I(H)$ of $I(G)$ consisting of pairs

$(\mathcal{L}, P \cap H)$, where $\mathcal{L}$ is a G-stable Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$ and N is an open normal subgroup of G contained in $G_{\mathcal{L}}$, we see that $I \cap A_H = I \cdot A_H$. This proves the claim.

Hence $I \cap A_H = (h - \lambda(h)) \cdot \widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, H)$, which is a Fréchet-Stein algebra by the proof of Theorem 6.1.15. We set:

$$(A/I)_H := \frac{\widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, H)}{(h - \lambda(h) \mid h \in \mathfrak{h}_K) \cdot \widehat{\widetilde{\mathcal{D}}}(\mathbf{X}, H)}$$

and so there is a K-algebra isomorphism $A_H / I \cap A_H \cong (A/I)_H$.

Furthermore, note that $\phi^H(\mathbf{U}) \mid_{I \cap A_H} = 0$ by construction. We verify that the image of the natural map $\psi : (I \cap A_H) \otimes_{K[H]} K[N] \to A_H \otimes_{K[H]} K[N]$ is $I \cap A_N$. Since $\widehat{U(\mathcal{L})_K} \rtimes_P N$ is a crossed product of $\widehat{U(\mathcal{L})_K}$ with N/P by [1, Lemma 2.2.4(b)], it follows that the canonical map:

$$(h - \lambda(h)) \cap ((\widehat{U(\mathcal{L})_K} \rtimes_P H) \otimes_{K[H]} K[N]) \to (h - \lambda(h)) \cap (\widehat{U(\mathcal{L})_K} \rtimes_P N)$$

is a bijection. Taking limits on both sides, we see that $(I \cap A_H) \otimes_{K[H]} K[N] \cong I \cap A_N \cong I \cap (A_H \otimes_{K[H]} K[N])$. This is sufficient to prove the claim. It follows that we may apply Lemma 6.1.18 to deduce that $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$ also acts on $\mathbf{X}$ λ-compatibly with G. $\square$

6.2 The twisted localisation functor $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}$

In this section, we construct the twisted localisation functor $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}} : \mathcal{C}_A \to \mathrm{Sh}(\mathbf{X}_{\mathrm{rig}})$. This is analogous to the localisation functor $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}$, generalising Definition 2.8.22. For Sections 6.2 and 6.3, we only assume that $\mathbf{X}$ is a smooth rigid analytic variety, not necessarily affinoid.

Definition 6.2.1. Suppose that A acts on $\mathbf{X}$ λ-compatibly with G. The A-module M is *coadmissible* if it is coadmissible as an A_H-module for some compact open subgroup H of G.

Applying [1, Lemma 3.4.11(a)], M is coadmissible if and only if it is coadmissible as an A_H-module for every compact open subgroup H of G.

Throughout, we continue to assume that G is a compact p-adic Lie group, A acts on $\mathbf{X}$ λ-compatibly with G in the sense of Definition 6.1.17, and M is a coadmissible A-module. Recall that because we are assuming throughout Section 6.2 that G is acting continuously on $\mathbf{X}$ in the sense of Definition 2.8.1, the stabiliser $G_{\mathbf{U}}$ in G of every affinoid subdomain $\mathbf{U}$ of $\mathbf{X}$ is an open subgroup of G.

Suppose $(\mathbf{U}, H)$ is small. Then there is a continuous homomorphism $\phi^H(\mathbf{U}) : A_H \to \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$ between two Fréchet-Stein algebras, and so we may view $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$ as a $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) - A_H$-bimodule in the sense of [5, Definition 7.3]. By [5, Lemma 7.3], we may form the coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-module:

$$\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)\widehat{\otimes}_{A_H} M.$$

Definition 6.2.2. Whenever $(\mathbf{U}, H)$ is small, we define:

$$M(\mathbf{U}, H) := \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)\widehat{\otimes}_{A_H} M.$$

Definition 6.2.3. For each $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, we set:

$$(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)(\mathbf{U}) := \varprojlim M(\mathbf{U}, H)$$

where the limit is taken over all $\mathbf{U}$-small subgroups H of G.

In fact, each arrow in the inverse system defining $(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)$ is an isomorphism. This follows from our next result.

Proposition 6.2.4. Let $H \leq N$ be compact open subgroups of G, and let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})/N$. Then for every coadmissible A-module M, the natural map

$$M(\mathbf{U}, H) \to M(\mathbf{U}, N)$$

from [1, Proposition 3.5.2(a)] is an isomorphism of coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-modules.

Proof. The same argument as in [1, Lemma 3.4.11(b)] applies to show that the natural map

$$\alpha : \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) \otimes_{A_H} A_N \to \widehat{\mathcal{D}}^\lambda(\mathbf{U}, N)$$

is an isomorphism of $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-$A_N$-bimodules. On the other hand, there is an isomorphism $\beta : A_N \underset{A_N}{\widehat{\otimes}} M \to M$ of coadmissible left A_H-modules, and A_N is a coadmissible $A_H - A_N$-bimodule in the sense of [5, Definition 7.3]. These maps combine to produce a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) \underset{A_H}{\widehat{\otimes}} (A_N \underset{A_N}{\widehat{\otimes}} M) & \xrightarrow{1\widehat{\otimes}\beta} & \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) \underset{A_H}{\widehat{\otimes}} M \\ \downarrow \cong & & \downarrow \\ (\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H) \underset{A_H}{\widehat{\otimes}} A_N) \underset{A_N}{\widehat{\otimes}} M & \xrightarrow{\alpha\widehat{\otimes}1} & \widehat{\mathcal{D}}^\lambda(\mathbf{U}, N) \underset{A_N}{\widehat{\otimes}} M. \end{array}$$

where the vertical map on the left is the canonical associativity isomorphism given by [5, Proposition 7.4]. The result follows, because α and β are isomorphisms. □

Corollary 6.2.5. For each $\mathbf{U}$-small subgroup H of G, the canonical map $(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)(\mathbf{U}) \to M(\mathbf{U}, H)$ is a bijection.

Proof. This is immediate from Proposition 6.2.4. □

Proposition 6.2.6. Suppose $(\mathbf{U}, J)$ is small. Set $B := \widehat{\mathcal{D}}^\lambda(\mathbf{U}, J)$ and set $N := B\widehat{\otimes}_{A_J} M$. Then N is a coadmissible B-module, and there is a natural isomorphism:

$$(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M) \mid_{\mathbf{U}_w} \cong (\mathcal{P}^\lambda)^B_{\mathbf{U}}(N)$$

of J-equivariant presheaves of $\mathcal{D}^\lambda$-modules on $\mathbf{U}_w$.

Proof. In view of Proposition 6.1.19, the proof is identical to that of [1, Proposition 3.5.9]. □

Theorem 6.2.7. *Let $\boldsymbol{U} \in \boldsymbol{X}_w(\mathcal{T})$. Then $(\mathcal{P}^\lambda)^A_{\boldsymbol{X}}(M) \mid_{\boldsymbol{U}_w}$ is a sheaf on $\boldsymbol{U}_w$ with vanishing higher Čech cohomology, for every coadmissible A-module M.*

Proof. By Lemma 6.1.1, we may choose a G-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{X})$ and a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\widetilde{\mathcal{T}}(\mathbf{X})$. We may then apply the same proof as in [1, Theorem 3.5.11]. $\square$

Since the rigid analytic variety $\widetilde{\mathbf{X}}$ is smooth, the tangent sheaf $\mathcal{T}$ is locally free. Hence $\mathbf{X}_w(\mathcal{T})$ forms a basis for $\mathbf{X}$, and so Proposition 6.2.7 implies that $(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)$ is a sheaf on $\mathbf{X}_w(\mathcal{T})$. Applying [5, Theorem 9.1], $(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)$ extends uniquely to a sheaf on $\mathbf{X}_{\text{rig}}$, the strong G-topology on $\mathbf{X}$.

Definition 6.2.8. Set $(\text{Loc}^\lambda)^A_{\mathbf{X}}(M)$ to be the unique sheaf on $\mathbf{X}_{\text{rig}}$ whose restriction to $\mathbf{X}_w(\mathcal{T})$ is the presheaf $(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)$.

6.3 The category $\mathcal{C}^\lambda_{\mathbf{X}/G}$

In this section, we define the category $\mathcal{C}^\lambda_{\mathbf{X}/G}$ of coadmissible equivariant $\mathcal{D}^\lambda$-modules, using the theory developed in Sections 6.1 and 6.2, and show that this category is abelian.

Definition 6.3.1. A G-equivariant $\mathcal{D}^\lambda$-module $\mathcal{M}$ on $\mathbf{U}_{\text{rig}}$ is *locally Fréchet* if:

(a) $\mathcal{M}(\mathbf{U})$ is equipped with a Fréchet topology for every $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$,

(b) the maps $g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{M}(g\mathbf{U})$ are continuous for all $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and $g \in G$.

A *morphism* of G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules is a morphism $f : \mathcal{M} \to \mathcal{N}$ of G-equivariant $\mathcal{D}^\lambda$-modules, such that the induced maps $f(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{N}(\mathbf{U})$ are continuous for every $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$.

The category whose objects are G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules and whose morphisms are continuous maps between them is denoted by $\mathrm{Frech}(G-\mathcal{D}^\lambda)$. There is a natural forgetful functor to G-equivariant $\mathcal{D}^\lambda$-modules on $\mathbf{X}$:

$$\Phi : \mathrm{Frech}(G-\mathcal{D}^\lambda) \to G-\mathcal{D}^\lambda - \mathrm{mod}.$$

Proposition 6.3.2. Suppose that A acts on $\mathbf{X}$ λ-compatibly with G. Then $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}$ is a functor from coadmissible A-modules to G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$.

Proof. Fix $M \in \mathcal{C}_A$, and set $\mathcal{M} := (\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)$. This is a G-equivariant presheaf of $\mathcal{D}^\lambda$-modules on $\mathbf{X}_w(\mathcal{T})$ by the same argument as in [1, Theorem 3.5.8]. Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and choose some $\mathbf{U}$-small subgroup J of G by Lemma 2.8.15. By Proposition 6.2.5, there is a canonical isomorphism:

$$\mathcal{M}(\mathbf{U}) \cong M(\mathbf{U}, J) = \widehat{\mathcal{D}^\lambda}(\mathbf{U}, J)\widehat{\otimes}_{A_J} M,$$

and so $\mathcal{M}(\mathbf{U})$ is a coadmissible $\widehat{\mathcal{D}^\lambda}(\mathbf{U}, J)$-module. Hence it carries a canonical Fréchet topology.

Given $g \in G$, the map:

$$g^M_{\mathbf{U},J} : M(\mathbf{U}, J) \to M(g\mathbf{U}, gJg^{-1})$$

is continuous since it is a $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, J)$-linear map between two coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, J)$-modules by [1, Lemma 3.6.5]. Furthermore, by the same argument as in [1, Proposition 3.5.7], this map satisfies:

$$g^M_{\mathbf{U},J}(a \cdot m) = \widehat{g_{\mathbf{U},J}}(a) \cdot (g \cdot m)$$

for all $a \in \widehat{\mathcal{D}}^\lambda(\mathbf{U}, J)$ and $m \in M$, where $\widehat{g_{\mathbf{U},J}}$ is the continuous ring isomorphism from Proposition 6.1.16.

Since the restriction of the functor $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}$ to $\mathbf{X}_w(\mathcal{T})$ is $\mathcal{M}$ by Theorem 6.2.7, it follows that $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}$ is a G-equivariant locally Fréchet $\mathcal{D}^\lambda$-module on $\mathbf{X}$.

Given $f : M \to N$ an A-linear map between two coadmissible A-modules, then for any $\mathbf{V} \in \mathbf{X}_w(\mathcal{T})$ and $\mathbf{V}$-small subgroup H of G, functoriality of $\widehat{\otimes}$ induces a $\widehat{\mathcal{D}}^\lambda(\mathbf{V}, H)$-linear map $1\widehat{\otimes}f : M(\mathbf{V}, H) \to N(\mathbf{V}, H)$, which is continuous by [1, Lemma 3.6.5]. This induces a G-equivariant morphism of presheaves $(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M) \to (\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)$ on $\mathbf{X}_w(\mathcal{T})$ whose local sections are continuous, and after applying [5, Theorem 9.1], it follows that there is a morphism $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(M) \to (\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(N)$ of G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules. Finally, we see that $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(g \circ f) = (\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(g) \circ (\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(f)$ whenever $g : N \to N'$ is another A-linear map to a coadmissible A-module N'. □

Definition 6.3.3. Let $\mathcal{M}$ be a G-equivariant locally Fréchet $\mathcal{D}^\lambda$-module on $\mathbf{X}$.

(a) Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering. $\mathcal{M}$ is $\mathcal{U} -$ *coadmissible* if, for each $\mathbf{U} \in \mathcal{U}$, there is a $\mathbf{U}$-small subgroup $H_{\mathbf{U}}$ of G, a coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-module $M_{\mathbf{U}}$, and an isomorphism:

$$(\mathrm{Loc}^{\lambda})_{\mathbf{U}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}) \to \mathcal{M}_{\mathbf{U}_{\mathrm{rig}}}$$

of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules on $\mathbf{U}$, which makes sense by Proposition 6.3.2.

(b) $\mathcal{M}$ is *coadmissible* if it is $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$.

(c) The full subcategory of $\mathrm{Frech}(G - \mathcal{D}^{\lambda}_{\mathbf{X}})$ consisting of coadmissible G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules is denoted by:

$$\mathcal{C}^{\lambda}_{\mathbf{X}/G}.$$

We next show that the twisted localisation functor $(\mathrm{Loc}^{\lambda})^{A}_{\mathbf{X}}$ constructed in Section 6.2, which, *a priori*, is a functor to $\mathrm{Frech}(G - \mathcal{D}^{\lambda})$ by Proposition 6.3.2, has its essential image contained in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$.

Proposition 6.3.4. Suppose that A acts on $\mathbf{X}$ λ-compatibly with G. The functor $(\mathrm{Loc}^{\lambda})^{A}_{\mathbf{X}}$ from coadmissible A-modules to $\mathrm{Frech}(G - \mathcal{D}^{\lambda})$ takes values in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$.

Proof. Choose an $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$ of $\mathbf{X}$, fix $\mathbf{U} \in \mathcal{U}$ and choose a $\mathbf{U}$-small subgroup J of G by Lemma 2.8.15. Let M be a coadmissible A-module and let N be the coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, J)$-module:

$$N := \widehat{\mathcal{D}}^\lambda(\mathbf{X}, J)\widehat{\otimes}_{A_J} M.$$

Applying [5, Theorem 9.1], the isomorphism:

$$(\mathcal{P}^\lambda)^A_{\mathbf{X}}(M) \mid_{\mathbf{U}_w} \cong (\mathcal{P}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},J)}_{\mathbf{U}}(N)$$

of J-equivariant presheaves of $\mathcal{D}^\lambda$-modules on $\mathbf{U}_w$ extends uniquely to an isomorphism:

$$(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(M) \mid_{\mathbf{U}_w} \cong (\mathrm{Loc}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},J)}_{\mathbf{U}}(N)$$

of J-equivariant $\mathcal{D}^\lambda$-modules on $\mathbf{U}$. By [1, Lemma 3.6.5], this isomorphism is continuous, and so $(\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(M)$ is $\mathcal{U}$-coadmissible by Definition 6.4.2. □

Theorem 6.3.5. *Suppose* $(\boldsymbol{X}, G)$ *is small. Then the localisation functor:*

$$(Loc^\lambda)^{\widehat{\mathcal{D}}^\lambda(\boldsymbol{X},G)}_{\boldsymbol{X}} : \mathcal{C}_{\widehat{\mathcal{D}}^\lambda(\boldsymbol{X},G)} \to \mathcal{C}^\lambda_{\boldsymbol{X}/G}$$

is an equivalence of categories.

The proof of essential surjectivity in Theorem 6.3.5 is very lengthy. We will postpone it until Section 6.5. The remainder of this section is devoted to proving that the category $\mathcal{C}^\lambda_{\mathbf{X}/G}$ is abelian. By [6, Definition 12.5.1], it suffices to show that it contains all kernels and cokernels, that every monomorphism is the kernel of its cokernel, and that every epimorphism is the cokernel of its kernel. We begin with the following technical proposition.

Proposition 6.3.6. If $(\mathbf{X}, G)$ is small then:

$$\Phi \circ (\mathrm{Loc}^{\lambda})_{\mathbf{X}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},G)} : \mathcal{C}_{\mathcal{D}^{\lambda}(\mathbf{X},G)} \to (G - \mathcal{D}_{\mathbf{X}}^{\lambda}) - \mathrm{mod}$$

is an exact functor.

Proof. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$-modules, let $\mathbf{U} \in \mathbf{X}_w$ and let H be an open subgroup of G. Then Lemma 2.8.15(a) demonstrates that $(\mathbf{U}, H)$ is small, so the sequence of $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H)$-modules:

$$0 \to M_1(\mathbf{U}, H) \to M_2(\mathbf{U}, H) \to M_3(\mathbf{U}, H) \to 0$$

is exact since $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H)$ is a c-flat right $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, H)$-module by the same argument as in [1, Theorem 3.7.1], and by applying [5, Proposition 7.5(a)]. Applying Corollary 6.2.5 and Theorem 6.2.7, we see that $(\mathcal{P}^{\lambda})_{\mathbf{X}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},G)}$ is an exact functor from $\mathcal{C}_{\mathcal{D}^{\lambda}(\mathbf{X},G)}$ to G-equivariant $\mathcal{D}^{\lambda}$-modules on $\mathbf{X}_w$. Since the extension functor from sheaves on $\mathbf{X}_w$ to sheaves on $\mathbf{X}_{\mathrm{rig}}$ is an equivalence of categories by [5, Theorem 9.1], it follows that $\Phi \circ (\mathrm{Loc}^{\lambda})_{\mathbf{X}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},G)}$ is also exact. □

Lemma 6.3.7. Let $\mathcal{M}$ be a G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-module on $\mathbf{X}$, let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering of $\mathbf{X}$ and let $\mathcal{V}$ be an $\mathbf{X}_w(\mathcal{T})$-refinement of $\mathcal{U}$. If $\mathcal{M}$ is $\mathcal{U}$-coadmissible, then $\mathcal{M}$ is also $\mathcal{V}$-coadmissible.

Proof. Apply the same argument as that in [1, Lemma 3.6.9] to $\mathcal{D}^{\lambda}$. □

Proposition 6.3.8. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$. Then the G-equivariant $\mathcal{D}^{\lambda}$-module $\ker\alpha$ is coadmissible, and the canonical morphism $\ker\alpha \to \mathcal{M}$ is continuous.

Proof. For any $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, $(\ker\alpha)$ is the kernel of the continuous map $\alpha(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{N}(\mathbf{U})$ between two Fréchet spaces. Therefore it is closed, and may be equipped with the subspace Fréchet topology from $\mathcal{M}(\mathbf{U})$. Given $g \in G$, the map:

$$g^{\ker\alpha}(\mathbf{U}) : (\ker\alpha)(\mathbf{U}) \to (\ker\alpha)(g\mathbf{U})$$

is the restriction of $g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{M}(g\mathbf{U})$, which is continuous. Hence the G-equivariant $\mathcal{D}^{\lambda}$-module $\ker\alpha$ is locally Fréchet.

By choosing a common refinement and applying Lemma 6.3.7, we may assume $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$. Fix $\mathbf{U} \in \mathcal{U}$. By Definition 6.4.2, we can find a $\mathbf{U}$-small subgroup $H_{\mathbf{U}}$ of G, a morphism $f : M \to N$ of coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-modules and a commutative diagram of $H_{\mathbf{U}}$-equivariant $\mathcal{D}^{\lambda}_{\mathbf{U}}$-modules:

$$\begin{array}{ccc} \mathrm{Loc}^{\lambda}(M) & \xrightarrow{\mathrm{Loc}^{\lambda}(f)} & \mathrm{Loc}^{\lambda}(N) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} \\ \mathcal{M}_{\mathbf{U}} & \xrightarrow{\alpha_{\mathbf{U}}} & \mathcal{N}_{\mathbf{U}}, \end{array}$$

where $\mathrm{Loc}^{\lambda} := (\mathrm{Loc}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}.H_{\mathbf{U}})}_{\mathbf{U}}$ and the vertical maps μ, ν are continuous isomorphisms. By [55, Corollary 3.4(ii)], the $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-module $\ker f$ is coadmissible. Since Loc^{λ} is exact by Proposition 6.3.6, we obtain a commutative diagram of $H_{\mathbf{U}}$-equivariant $\mathcal{D}^{\lambda}_{\mathbf{U}}$-modules:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Loc}^\lambda(\ker f) & \longrightarrow & \mathrm{Loc}^\lambda(M) & \xrightarrow{\mathrm{Loc}^\lambda(f)} & \mathrm{Loc}^\lambda(N) \\
 & & \big\downarrow \psi & & \big\downarrow \mu & & \big\downarrow \nu \\
0 & \longrightarrow & \ker\alpha & \longrightarrow & \mathcal{M}_{\mathbf{U}} & \xrightarrow{\alpha_{\mathbf{U}}} & \mathcal{N}_{\mathbf{U}},
\end{array}$$

where the rows are exact. Since $(H_{\mathbf{U}} - \mathcal{D}^\lambda_{\mathbf{U}}) - \mathrm{mod}$ is an abelian category, the Five Lemma yields an isomorphism of $H_{\mathbf{U}}$-equivariant $\mathcal{D}^\lambda_{\mathbf{U}}$-modules ψ completing the diagram. For any $\mathbf{V} \in \mathbf{U}_w$, $\psi(\mathbf{V})$ is the restriction of the continuous map $\mu(\mathbf{V})$ to $\mathrm{Loc}^\lambda(\ker f)(\mathbf{V})$, and hence is continuous. It follows that $\ker\alpha$ is $\mathcal{U}$-coadmissible, and the canonical morphism $\ker\alpha \to \mathcal{M}$ is continuous by construction. □

Lemma 6.3.9. Let $\mathcal{M} \in \mathcal{C}^\lambda_{\mathbf{X}/G}$ and let $\mathbf{Y}$ be an affinoid subdomain of $\mathbf{X}$. Then $\mathcal{M}_{\mathbf{Y}} \in \mathcal{C}^\lambda_{\mathbf{Y}/G_{\mathbf{Y}}}$.

Proof. We may view $\mathcal{M}_{\mathbf{Y}}$ as a $G_{\mathbf{Y}}$-equivariant locally Fréchet $\mathcal{D}^\lambda$-module on $\mathbf{X}$. Since $\mathbf{Y}$ is admissible open in $\mathbf{X}$, $\mathbf{Y}_w(\mathcal{T}) \subseteq \mathbf{X}_w(\mathcal{T})$ and so $\mathcal{M}_{\mathbf{Y}}$ is a $G_{\mathbf{Y}}$-equivariant locally Fréchet $\mathcal{D}^\lambda$-module on $\mathbf{Y}$.

It suffices to show $\mathcal{M}_{\mathbf{Y}}$ is coadmissible. Suppose $\mathcal{M}$ is $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$. For each $\mathbf{U} \in \mathcal{U}$, choose an admissible affinoid covering $\mathcal{V}_{\mathbf{U}}$ of $\mathbf{Y} \cap \mathbf{U}$ and set $\mathcal{V} := \bigcup_{\mathbf{U} \in \mathcal{U}} \mathcal{V}_{\mathbf{U}}$. Then $\mathcal{V}$ is a $\mathbf{Y}_w(\mathcal{T})$-covering of $\mathbf{Y}$ which refines $\mathcal{U}$.

This allows us to reduce the problem to the case where $\mathbf{Y}$ and $\mathbf{X}$ are both affinoid, $(\mathbf{X}, G)$ is small, and $\mathcal{M} \cong (\mathrm{Loc}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},G)}_{\mathbf{X}}(M)$ for some coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-module M. By Proposition 6.2.6, it follows that:

$$\mathcal{M}_{\mathbf{Y}} \cong (\mathrm{Loc}^{\lambda})_{\mathbf{X}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},G)}(M)_{\mathbf{Y}} \cong (\mathrm{Loc}^{\lambda})_{\mathbf{Y}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},G_{\mathbf{Y}})}(\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},G_{\mathbf{Y}})\widehat{\otimes}_{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},G_{\mathbf{Y}})}M)$$

as $G_{\mathbf{Y}}$-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules on $\mathbf{Y}$. □

Lemma 6.3.10. Let $\mathcal{M} \in \mathcal{C}^{\lambda}_{\mathbf{X}/G}$, $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and $i > 0$. Then $H^i(\mathbf{U}, \mathcal{M}) = 0$.

Proof. Let $G_{\mathbf{U}}$ be a $\mathbf{U}$-small subgroup of G. By Lemma 6.3.9, $\mathcal{M}_{\mathbf{U}} \in \mathcal{C}^{\lambda}_{\mathbf{U}/G_{\mathbf{U}}}$ and $H^i(\mathbf{U}, \mathcal{M}) = H^i(\mathbf{U}, \mathcal{M}_{\mathbf{U}})$. Hence we may assume $(\mathbf{X}, G)$ is small. Then, by Theorem 6.3.5, $\mathcal{M} \cong (\mathrm{Loc}^{\lambda})_{\mathbf{X}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},G)}(M)$ for some coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{X}, G)$-module M, so $\check{H}^i(\mathcal{U}, \mathcal{M}_{\mathbf{V}}) = 0$ for any finite affinoid covering $\mathcal{U}$ of any affinoid subdomain $\mathbf{V}$ of $\mathcal{U}$ by Proposition 6.2.6 and Proposition 6.2.7.

It follows that $\check{H}^i(\mathcal{U}, \mathcal{M}_{\mathbf{V}}) = 0$ and the result is immediate by applying [6, Lemma 21.11.9]. □

Proposition 6.3.11. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$. Then the G-equivariant $\mathcal{D}^{\lambda}$-module $\mathrm{coker}\alpha$ is coadmissible, and the canonical morphism $\mathcal{N} \to \mathrm{coker}\alpha$ is continuous.

Proof. First suppose $\ker\alpha = 0$, so there is a short exact sequence of G-equivariant $\mathcal{D}^{\lambda}$-modules:

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathrm{coker}\alpha \to 0.$$

For any $\mathbf{U} \in \mathcal{U}$, the sequence:

$$0 \to \mathcal{M}(\mathbf{U}) \to \mathcal{N}(\mathbf{U}) \to \mathrm{coker}(\alpha(\mathbf{U})) \to 0$$

is exact by Lemma 6.3.10, and so there is an isomorphism $(\mathrm{coker}\alpha)(\mathbf{U}) \cong \mathrm{coker}(\alpha(\mathbf{U}))$.

Now $\mathcal{M}(\mathbf{U})$ and $\mathcal{N}(\mathbf{U})$ are coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$-modules for any $\mathbf{U}$-small subgroup $H_\mathbf{U}$ of G, by Lemma 6.3.9, and the map $\alpha(\mathbf{U})$ is $\mathcal{D}^\lambda(\mathbf{U}) \rtimes H_\mathbf{U}$-linear as well as continuous, by Proposition 2.7.7, so it is $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$-linear. Since $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$ is Fréchet-Stein by Theorem 6.1.15, $(\mathrm{coker}\alpha)(\mathbf{U}) \cong \mathrm{coker}(\alpha(\mathbf{U}))$ is a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$-module by [55, Corollary 3.4(ii)], and so we may equip it with the canonical Fréchet topology.

Let $g \in G$. Since the map $g^{\mathrm{coker}\alpha}(\mathbf{U}) : (\mathrm{coker}\alpha)(\mathbf{U}) \to \mathrm{coker}\alpha)(g\mathbf{U})$ is induced by the $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$-linear map $g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{M}(g\mathbf{U})$, it is also $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$-linear and continuous. Hence the G-equivariant $\mathcal{D}^\lambda$-module $\mathrm{coker}\alpha$ is locally Fréchet.

By choosing a common refinement and applying Lemma 6.3.7, we may assume $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$. Fix $\mathbf{U} \in \mathcal{U}$. By Definition 6.4.2 and Proposition 6.3.6, we can find a $\mathbf{U}$-small subgroup $H_\mathbf{U}$ of G, a morphism of coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$-modules $f : M \to N$ and a diagram of $H_\mathbf{U}$-equivariant $\mathcal{D}^\lambda_\mathbf{U}$-modules:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Loc}^\lambda(M) & \xrightarrow{\mathrm{Loc}^\lambda(f)} & \mathrm{Loc}^\lambda(N) & \longrightarrow & \mathrm{Loc}^\lambda(\mathrm{coker} f) \\
& \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \psi} \\
0 \longrightarrow & \mathcal{M}_\mathbf{U} & \xrightarrow{\alpha_\mathbf{U}} & \mathcal{N}_\mathbf{U} & \longrightarrow & \mathrm{coker}\alpha,
\end{array}$$

where the rows are exact and μ, ν are continuous isomorphisms. Let $\mathbf{V} \in \mathbf{U}_w$, and let $H_\mathbf{V}$ be a $\mathbf{V}$-small subgroup of G. By Lemma 6.3.10 and Theorem 6.3.5, applying the global sections functor $\Gamma(\mathbf{V}, -)$ keeps the rows exact and sends all objects to coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{V}, H_\mathbf{V})$-modules. By Proposition 2.7.7, $\lambda(\mathbf{V})$ and $\mu(\mathbf{V})$ are continuous $\mathcal{D}^\lambda(\mathbf{V}) \rtimes H_\mathbf{V}$-linear maps and hence are $\widehat{\mathcal{D}}^\lambda(\mathbf{V}, H_\mathbf{V})$-linear. Hence $\nu(\mathbf{V})$ is also $\widehat{\mathcal{D}}^\lambda(\mathbf{V}, H_\mathbf{V})$-linear by exactness of the diagram, and so is continuous by [1, Lemma 3.6.5]. Hence ψ is continuous and, by definition, cokerα is $\mathcal{U}$-coadmissible. Furthermore, the map $\mathcal{N}(\mathbf{U}) \to (\text{coker}\alpha)(\mathbf{U})$ is $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_\mathbf{U})$-linear by construction, and so is continuous. Hence the canonical morphism $\mathcal{N} \to \text{coker}\alpha$ is continuous.

Now suppose that kerα is not necessarily zero. We have two short exact sequences of G-equivariant $\mathcal{D}^\lambda$-modules:

$$0 \to \ker\alpha \to \mathcal{M} \to \text{im}\alpha \to 0,$$

$$0 \to \text{im}\alpha \to \mathcal{N} \to \text{coker}\alpha \to 0.$$

By Proposition 6.3.8, $\ker\alpha \to \mathcal{M}$ is a morphism in $\mathcal{C}^\lambda_{\mathbf{X}/G}$, and so the morphism $\mathcal{M} \to \text{im}\alpha$ lies in $\mathcal{C}^\lambda_{\mathbf{X}/G}$ by the special case applied to the first sequence.

Finally, the canonical map $\text{im}\alpha \to \mathcal{N}$ is continuous, since the canonical topology on $\text{im}\alpha(\mathbf{U})$ agrees with the subspace topology induced from $\mathcal{N}(\mathbf{U})$. Hence, applying the special case to the second sequence, we have a continuous map $\mathcal{N} \to \text{coker}\alpha$. □

Theorem 6.3.12. *The category $\mathcal{C}^\lambda_{\mathbf{X}/G}$ is abelian.*

Proof. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathcal{C}^\lambda_{\mathbf{X}/G}$. By Proposition 6.3.8, the canonical map $i : \ker\alpha \to \mathcal{M}$ is a morphism in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$. Suppose that $u : \mathcal{P} \to \mathcal{M}$ is another morphism in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ such that $\alpha \circ u = 0$. Since i is the kernel of α in $G-\mathcal{D}^{\lambda}$-mod, there is a unique morphism of G-equivariant $\mathcal{D}^{\lambda}$-modules $j : \mathcal{P} \to \ker\alpha$ such that $u = i \circ j$.

Let $\mathbf{U} \in \mathcal{U}$ and let $H_{\mathbf{U}}$ be a $\mathbf{U}$-small subgroup of G. Then all objects in the commutative diagram:

$$\begin{array}{ccccccc} & & \mathcal{P}(\mathbf{U}) & & & & \\ & & \downarrow{\scriptstyle j(\mathbf{U})} & \searrow^{u(\mathbf{U})} & & & \\ 0 & \longrightarrow & (\ker\alpha)(\mathbf{U}) & \xrightarrow[i(\mathbf{U})]{} & \mathcal{M}(\mathbf{U}) & \xrightarrow[\alpha(\mathbf{U})]{} & \mathcal{N}(\mathbf{U}) \end{array}$$

are coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-modules by Theorem 6.3.5, and so by [1, Lemma 3.7.6], the map $j(\mathbf{U})$ is continuous. Hence $j : \mathcal{P} \to \ker\alpha$ is a morphism in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$, and if $j' : \mathcal{P} \to \ker\alpha$ is another such map with $u = i \circ j'$ then $j = j'$ since the forgetful functor Φ is faithful. Hence i is the kernel of α in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$. A similar argument using Lemma 6.3.10 and Proposition 6.3.11 shows that $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$ has cokernels.

Finally, we need to show that every monomorphism $\alpha : \mathcal{M} \to \mathcal{N}$ is the kernel of its cokernel $q : \mathcal{N} \to \text{coker}\alpha$. We can argue on the dual side to then show that every epimorphism is the cokernel of its kernel. Consider the commutative diagram in $\mathcal{C}^{\lambda}_{\mathbf{X}/G}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}(\mathbf{U}) & \xrightarrow{\alpha} & \mathcal{N}(\mathbf{U}) & \xrightarrow{q} & (\text{coker}\alpha)(\mathbf{U}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\ker q)(\mathbf{U}) & \longrightarrow & \mathcal{N}(\mathbf{U}) & \xrightarrow{q} & (\text{coker}\alpha)(\mathbf{U}) & \longrightarrow & 0 \end{array}$$

where j is induced by the universal property of $\ker q$, and the rows are exact in $G-\mathcal{D}^{\lambda}$-

mod. In the abelian category $G - \mathcal{D}^\lambda$-mod, the arrow $\Phi(j)$ is an isomorphism by the Five Lemma, and so the local sections $j(\mathbf{U})$ are continuous bijections for each $\mathbf{U} \in \mathcal{U}$. By the Open Mapping Theorem, the inverses are also continuous and hence j is an isomorphism in $\mathcal{C}^\lambda_{\mathbf{X}/G}$. It follows that j is an isomorphism in $\mathcal{C}^\lambda_{\mathbf{X}/G}$. A similar argument shows that every epimorphism in $\mathcal{C}^\lambda_{\mathbf{X}/G}$ is the cokernel of its kernel. Hence $\mathcal{C}^\lambda_{\mathbf{X}/G}$ is an abelian category. □

6.4 Discussion of the constructions in Sections 5.2 and 6.3

In this section, we assume that $\mathbf{X} = (\mathbb{G}/\mathbb{B})^{\text{an}}$ is the rigid analytic flag variety of $\mathbb{G}$. We have now constructed two different categories of coadmissible G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules, in Sections 5.2 and 6.3, both denoted $\mathcal{C}^\lambda_{\mathbf{X}/G}$. The construction in Section 6.3 is more general since it does not require the weight λ to be integral. However, we should expect this construction to reduce to the construction of $\mathcal{C}^\lambda_{\mathbf{X}/G}$ in Section 5.2 in case λ is integral. In this short section we briefly discuss how we might check that these two definitions agree.

We restate the two definitions below for the convenience of the reader.

Definition 6.4.1. Let $\mathcal{M}$ be a G-equivariant locally Fréchet $\mathcal{D}^\lambda$-module on $\mathbf{X}$ and suppose $\lambda \in \mathfrak{h}^*_K$ is an integral weight.

(a) Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering. $\mathcal{M}$ is $\mathcal{U} -$ *coadmissible* if, for each $\mathbf{U} \in \mathcal{U}$, there is a $\mathbf{U}$-small subgroup $H_\mathbf{U}$ of G, a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_\mathbf{U})$-module $M_\mathbf{U}$, and an isomorphism:

$$\mathcal{O}_{\mathbf{U}}^{\lambda} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathrm{Loc}_{\mathbf{U}}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}) \to \mathcal{M}_{\mathbf{U}_{\mathrm{rig}}}$$

of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules on $\mathbf{U}$.

(b) $\mathcal{M}$ is *coadmissible* if it is $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$.

(c) The full subcategory of $\mathrm{Frech}(G - \mathcal{D}_{\mathbf{X}}^{\lambda})$ consisting of coadmissible G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules is denoted by:

$$\mathcal{C}_{\mathbf{X}/G}^{\lambda}.$$

Definition 6.4.2. Let $\mathcal{M}$ be a G-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-module on $\mathbf{X}$.

(a) Let $\mathcal{U}$ be an $\mathbf{X}_w(\mathcal{T})$-covering. $\mathcal{M}$ is $\mathcal{U}$ *– coadmissible* if, for each $\mathbf{U} \in \mathcal{U}$, there is a $\mathbf{U}$-small subgroup $H_{\mathbf{U}}$ of G, a coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-module $M_{\mathbf{U}}^{\lambda}$, and an isomorphism:

$$(\mathrm{Loc}^{\lambda})_{\mathbf{U}}^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{U},H_{\mathbf{U}})}(M_{\mathbf{U}}^{\lambda}) \to \mathcal{M}_{\mathbf{U}_{\mathrm{rig}}}$$

of $H_{\mathbf{U}}$-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules on $\mathbf{U}$, which makes sense by Proposition 6.3.2.

(b) $\mathcal{M}$ is *coadmissible* if it is $\mathcal{U}$-coadmissible for some $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$.

(c) The full subcategory of $\text{Frech}(G - \mathcal{D}^\lambda_{\mathbf{X}})$ consisting of coadmissible G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules is denoted by:

$$\mathcal{C}^\lambda_{\mathbf{X}/G}.$$

By inspection of the definitions, it suffices to check that $\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} \text{Loc}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}_{\mathbf{U}}(M_{\mathbf{U}}) \cong (\text{Loc}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{U},H_{\mathbf{U}})}_{\mathbf{U}}(M^\lambda_{\mathbf{U}})$ for all $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. Applying [5, Theorem 9.1], we see that it suffices to check that $\mathcal{O}^\lambda_{\mathbf{U}} \otimes_{\mathcal{O}_{\mathbf{U}}} \mathcal{P}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}_{\mathbf{U}}(M_{\mathbf{U}}) \cong (\mathcal{P}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{U},H_{\mathbf{U}})}_{\mathbf{U}}(M^\lambda_{\mathbf{U}})$ for all $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. Writing:

$$\begin{aligned}\mathcal{P}^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}_{\mathbf{U}}(M_{\mathbf{U}}) &= \varprojlim M_{\mathbf{U}}(\mathbf{U}, H_{\mathbf{U}}) = \varprojlim \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H_{\mathbf{U}})} M_{\mathbf{U}} \\ &\cong \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H_{\mathbf{U}})} M_{\mathbf{U}}\end{aligned}$$

by Definition 2.8.20 and the note immediately following it, and

$$\begin{aligned}(\mathcal{P}^\lambda)^{\widehat{\mathcal{D}}(\mathbf{U},H_{\mathbf{U}})}_{\mathbf{U}}(M^\lambda_{\mathbf{U}}) &= \varprojlim M^\lambda_{\mathbf{U}}(\mathbf{U}, H_{\mathbf{U}}) = \varprojlim \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}}) \widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H_{\mathbf{U}})} M^\lambda_{\mathbf{U}} \\ &\cong \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}}) \widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H_{\mathbf{U}})} M^\lambda_{\mathbf{U}}\end{aligned}$$

by Definition 6.2.2 and Corollary 6.2.5, it suffices to check that the following result holds:

Conjecture 6.4.3. For all $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, there exists a $\mathbf{U}$-small subgroup $H_{\mathbf{U}}$ of G, a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}})$-module $M_{\mathbf{U}}$ and a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}})$-module $M_{\mathbf{U}}^\lambda$ such that:

$$\mathcal{O}_{\mathbf{U}}^\lambda \otimes_{\mathcal{O}_{\mathbf{U}}} \widehat{\mathcal{D}}(\mathbf{U}, H_{\mathbf{U}}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X}, H_{\mathbf{U}})} M_{\mathbf{U}} \cong \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}}) \widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H_{\mathbf{U}})} M_{\mathbf{U}}^\lambda.$$

6.5 Characterising the essential image of the localisation functor on global sections

The rest of the thesis is devoted to proving a twisted equivariant version of locally analytic Beilinson-Bernstein localisation. We begin by giving a characterisation of the essential image of $(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},G)}$ on the global sections of a coadmissible G-equivariant $\mathcal{D}^\lambda$-module $\mathcal{M}$ on $\mathbf{X}$, and also prove Theorem 6.3.5.

We continue to work under the same assumptions as stated at the beginning of Section 6.1. In particular, we assume that $\mathbb{H}$ is a flat affine algebraic group, $\widetilde{\mathbf{X}}$ and $\mathbf{X}$ are K-affinoid varieties and $\xi : \widetilde{\mathbf{X}} \to \mathbf{X}$ is a locally trivial $\mathbb{H}$-torsor.

In this section, we assume that G is a compact p-adic Lie group acting continuously on $\mathbf{X}$, $\mathcal{A}$ is a G-stable affine formal model in $A := \mathcal{O}(\mathbf{X})$, and $(\mathcal{L}, N)$ is an $\mathcal{A}$-trivialising pair in the sense of Definition 6.1.5. That is, $\mathcal{L}$ is a G-stable Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$ and N is an open normal subgroup of G contained in $G_{\mathcal{L}}$. We begin by recalling the definitions of certain covers of $\mathbf{X}$ which satisfy nice topological properties, generalising those given in [5, Sections 3-5] in the case where the ground ring $\mathcal{R}$ is not necessarily Noetherian.

Definition 6.5.1. Let $\mathcal{L}$ be an $(\mathcal{R}, \mathcal{A})$-Lie algebra for some affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{X})$. Let $\mathbf{Y}$ be an affinoid subdomain of $\mathbf{X}$ and let $\sigma : \mathcal{O}(\mathbf{X}) \to \mathcal{O}(\mathbf{Y})$ be the pullback map on functions.

(a) An affine formal model $\mathcal{B}$ in $\mathcal{O}(\mathbf{Y})$ is $\mathcal{L}$-*stable* if $\sigma(\mathcal{A}) \subseteq \mathcal{B}$ and the $\mathcal{L}$-action on $\mathcal{A}$ lifts to $\mathcal{B}$.

(b) $\mathbf{Y}$ is $\mathcal{L}$-*admissible* if it admits an $\mathcal{L}$-stable affine formal model. The full subcategory of $\mathbf{X}_w$ consisting of the $\mathcal{L}$-admissible affinoid subdomains of $\mathbf{X}$ is denoted by $\mathbf{X}_w(\mathcal{L})$.

(c) An $\mathcal{L}$-*admissible covering* of an $\mathcal{L}$-admissible affinoid subdomain of $\mathbf{X}$ is a finite covering of $\mathbf{X}$ by objects in $\mathbf{X}_w(\mathcal{L})$.

(d) Write $\mathbf{X}_w/G$ to denote the set of G-stable affinoid subdomains $\mathbf{Y}$ of $\mathbf{X}$. We set $\mathbf{X}_w(\mathcal{L}, G) := \mathbf{X}_w(\mathcal{L}) \cap \mathbf{X}_w/G$ to be the set of G-stable affinoid subdomains $\mathbf{Y}$ of $\mathbf{X}$ which are also $\mathcal{L}$-admissible.

Definition 6.5.2. (a) Let $\mathbf{Y} \subseteq \mathbf{X}$ be a rational subdomain. If $\mathbf{Y} = \mathbf{X}$, it is $\mathcal{L}$-*accessible in 0 steps*. Inductively, if $n \geq 1$ we say that $\mathbf{Y}$ is *accessible in n steps* if there is a chain $\mathbf{Y} \subseteq \mathbf{Z} \subseteq \mathbf{X}$ such that:

- $\mathbf{Z} \subseteq \mathbf{X}$ is accessible in $n-1$ steps,
- $\mathbf{Y} = \mathbf{Z}(f)$ or $\mathbf{Z}(1/f)$ for some non-zero $f \in \mathcal{O}(\mathbf{Z})$,
- there is an $\mathcal{L}$-stable affine formal model $\mathcal{C} \in \mathcal{O}(\mathbf{Z})$ such that $\mathcal{L} \cdot f \subseteq \pi C$.

(b) A rational subdomain $\mathbf{Y} \subseteq \mathbf{X}$ is $\mathcal{L}$-*accessible* if it is $\mathcal{L}$-accessible in n steps for some $n \in \mathbb{N}$.

(c) An affinoid subdomain $\mathbf{Y} \subseteq \mathbf{X}$ is $\mathcal{L}$-*accessible* if it is $\mathcal{L}$-admissible and there is a finite covering $\mathbf{Y} = \bigcup_{j=1}^{r} \mathbf{X}_j$, where each $\mathbf{X}_j$ is an $\mathcal{L}$-accessible rational subdomain of $\mathbf{X}$.

(d) A finite affinoid covering $\{\mathbf{X}_j\}$ of $\mathbf{X}$ is $\mathcal{L}$-*accessible* if each $\mathbf{X}_j$ is an $\mathcal{L}$-accessible affinoid subdomain of $\mathbf{X}$.

(e) $\mathbf{X}_{\mathrm{ac}}(\mathcal{L}, G)$ denotes the set of G-stable $\mathcal{L}$-accessible affinoid subdomains of $\mathbf{X}$.

It follows from [1, Lemma 4.3.2] and [5, Lemma 4.8(a)] that $\mathbf{X}_{\mathrm{ac}}(\mathcal{L}, G)$ is a G-topology on $\mathbf{X}$.

Our next goal is to construct a twisted levelwise localisation functor defined on suitable affinoid subdomains of $\mathbf{X}$. We do this by defining a sheaf of rings on $\mathbf{X}_{\mathrm{ac}}(\mathcal{L}, G)$ which is locally isomorphic to one of the factors in the inverse limit defining $\widehat{\widehat{\mathcal{D}}}(\mathbf{X}, G)$ in Definition 6.1.6.

Definition 6.5.3. For any $\mathbf{Y} \in \mathbf{X}_w(\mathcal{L}, G)$ and any $\mathcal{L}$-stable, G-stable affine formal model $\mathcal{B}$ on $\mathcal{O}(\mathbf{Y})$, we define:

$$(\widehat{\mathscr{U}(\mathcal{L})_K} \rtimes_N G)(\mathbf{Y}) := U(\widehat{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}})_K \rtimes_N G.$$

Let $\mathcal{L}' := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$. [1, Corollary 4.3.7] implies that $(\mathcal{L}', N)$ is a $\mathcal{B}$-trivialising pair, and so Definition 6.5.3 makes sense. Furthermore, [1, Corollary 4.3.12], demonstrates that $\widetilde{\mathcal{Q}} := \widehat{\mathscr{U}(\mathcal{L})_K} \rtimes_N G$ is a sheaf of rings on $\mathbf{X}_w(\mathcal{L}, G)$ with vanishing higher Čech cohomology.

Lemma 6.5.4. There is a natural $\widehat{U(\mathfrak{h})_K}$-module structure on $\widehat{\mathscr{U}(\mathcal{L})_K} \rtimes_N G$.

Proof. Fix $\mathbf{Y} \in \mathbf{X}_w(\mathcal{L}, G)$. We may apply the same argument as in Lemma 6.1.12 to see that there is a $\widehat{U(\mathfrak{h})_K}$-module structure on $\widehat{\widehat{\mathcal{D}}}(\mathbf{Y}, G)$.

By definition, we may write:

$$\widehat{\widehat{\mathcal{D}}}(\mathbf{Y}, G) := \varprojlim_{(\mathcal{L}',N)\in I_{\mathbf{Y}}(G)} \widehat{U(\mathcal{L})_K} \rtimes_N G.$$

where $I_{\mathbf{Y}}(G)$ denotes the set of pairs $(\mathcal{L}', N)$ such that $\mathcal{L}'$ is a G-stable Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{Y})$ and N is an open normal subgroup of G contained in $G_{\mathcal{L}'}$.

Fix $(\mathcal{L}', N) \in I_{\mathbf{Y}}(G)$. Via the natural map $\widehat{\widehat{\mathcal{D}}}(\mathbf{Y}, G) \to \widehat{U(\mathcal{L}')_K} \rtimes_N G$, there is a $\widehat{U(\mathfrak{h})_K}$-module structure on $\widehat{U(\mathcal{L}')_K} \rtimes_N G$. ∎

Definition 6.5.5. Let $\lambda \in \mathfrak{h}_K^*$. We set:

$$\mathcal{Q}^\lambda := (\widehat{\mathscr{U}(\mathcal{L})_K} \rtimes_N G) \otimes_{\widehat{U(\mathfrak{h})_K}} K_\lambda.$$

Proposition 6.5.6. Suppose $(\mathbf{X}, G)$ is small. Choose a G-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{X})$ and a G-stable smooth $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\widetilde{\mathcal{T}}(\mathbf{X})$, by Lemma 6.1.1. Let $(N_\bullet)$ be a good chain for $\mathcal{L}$, and set:

$$\mathcal{Q}_n^\lambda := (\widehat{\mathscr{U}(\pi^n\mathcal{L})_K} \rtimes_{N_n} G) \otimes_{\widehat{U(\mathfrak{h})_K}} K_\lambda.$$

viewed as a sheaf of K-Banach algebras on $\mathbf{X}_w(\mathcal{L}, G)$. Then there is a natural isomorphism:

$$\widehat{\mathcal{D}}^\lambda(-, G)\mid_{\mathbf{X}_w(\mathcal{L},G)} \to \varprojlim \mathcal{Q}_n^\lambda$$

of presheaves on $\mathbf{X}_w(\mathcal{L}, G)$.

Proof. This follows from Lemma 6.1.9. □

Definition 6.5.7. The *localisation functor* $\mathrm{Loc}_{\mathcal{Q}^\lambda}$ from finitely generated $\mathcal{Q}^\lambda(\mathbf{X})$-modules to presheaves of $\mathcal{Q}^\lambda$-modules on $\mathbf{X}_{\mathrm{ac}}(\mathcal{L}, G)$ is given by:

$$\mathrm{Loc}_{\mathcal{Q}^\lambda}(M)(\mathbf{Y}) := \mathcal{Q}^\lambda(\mathbf{Y}) \otimes_{\mathcal{Q}^\lambda(\mathbf{X})} M$$

for $\mathbf{Y} \in \mathbf{X}_{\mathrm{ac}}(\mathcal{L}, G)$.

We next isolate a technical calculation from the proof of Theorem 6.3.5. Recall the definition of the map $\gamma^G : G \to \widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$ from Remark 6.1.7 and Lemma 6.1.14.

Proposition 6.5.8. Suppose $(\mathbf{X}, G)$ is small. Let H be an open subgroup of G, let $M \in \mathcal{C}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}$ and suppose there is an isomorphism of H-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$:

$$\alpha : (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}(M) \to \mathcal{M}.$$

(a) For every $\mathbf{U} \in \mathbf{X}_w$, there is a unique coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-module structure on $\mathcal{M}(\mathbf{U})$ such that:

(i) $\gamma^H(g) \cdot m = g^{\mathcal{M}}(m)$ for all $g \in H, m \in \mathcal{M}(\mathbf{U})$.

(ii) the topology on $\mathcal{M}(\mathbf{U})$ induced by the coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-module structure coincides with the given K-Fréchet topology on $\mathcal{M}(\mathbf{U})$.

(iii) the $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-action on $\mathcal{M}(\mathbf{U})$ extends the given $\mathcal{D}^\lambda(\mathbf{U})$-action on $\mathcal{M}(\mathbf{U})$.

(b) The $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-module structure is independent of the choice of α.

(c) There is an isomorphism of H-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$:

$$\theta : (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}(\mathcal{M}(\mathbf{X})) \to \mathcal{M}$$

whose restriction to $\mathbf{X}_w$ is given by:

$$\theta(\mathbf{U})(s\widehat{\otimes}m) = s \cdot (m\mid_{\mathbf{U}})$$

for any $\mathbf{U} \in \mathbf{X}_w, s \in \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$ and $m \in \mathcal{M}(\mathbf{X})$.

Proof. (a) Suppose first that $\mathbf{U} = \mathbf{X}$ and write $\mathcal{N} := (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}^\lambda}(\mathbf{X},H)}(M)$. By Corollary 6.2.5, we may identify $\mathcal{N}(\mathbf{X})$ with $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)\widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}M$, and so there is a canonical map $M \to \mathcal{N}(\mathbf{X})$ given by $n \mapsto 1\widehat{\otimes}n$.

Let $\psi : M \to \mathcal{M}(\mathbf{X})$ be the composition of the canonical map $M \to \mathcal{N}(\mathbf{X})$ and $\alpha(\mathbf{X}) : \mathcal{N}(\mathbf{X}) \to \mathcal{M}(\mathbf{X})$. By Proposition 6.2.7 and the definition of the topology on $\mathcal{N}(\mathbf{X})$, this map is a continuous isomorphism.

By Proposition 2.7.7, $\Gamma(\mathbf{X}, -)$ is a functor from H-equivariant $\mathcal{D}^\lambda$-modules on $\mathbf{X}$ to $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$-modules, and so the continuous map $\alpha(\mathbf{X}) : \mathcal{N}(\mathbf{X}) \to \mathcal{M}(\mathbf{X})$ is $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$-linear. It follows that $\psi : M \to \mathcal{M}(\mathbf{X})$ is also $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$-linear.

Since $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$ is dense in $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$ by Lemma 6.1.14, applying [1, Lemma 4.4.4] to ψ shows that the $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$ on $\mathcal{M}(\mathbf{X})$ extends to an $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$-action which satisfies the required properties. The general case follows by applying the argument in the proof of Proposition 6.3.9.

(b) This is immediate from [1, Lemma 4.4.4(c)].

(c) By part (a), $\mathcal{M}(\mathbf{X})$ is a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$-module, and the map $\psi : M \to \mathcal{M}(\mathbf{X})$ given by $\psi(m) = \alpha(\mathbf{X})(1\widehat{\otimes}m)$ is a $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$-linear isomorphism. Write $\mathcal{P}^\lambda := (\mathcal{P}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}$ and consider the diagram:

$$\begin{array}{ccc} & \mathcal{P}^\lambda(M) & \\ {}^{\mathcal{P}^\lambda(\psi)}\swarrow & & \searrow^{\alpha|_{\mathbf{X}_w}} \\ \mathcal{P}^\lambda(\mathcal{M}(\mathbf{X})) & \xrightarrow{\quad\theta\quad} & \mathcal{M}\ |_{\mathbf{X}_w}, \end{array}$$

where $\theta(\mathbf{U})(s\widehat{\otimes}m) = s \cdot (m\ |_{\mathbf{U}})$ for $\mathbf{U} \in \mathbf{X}_w$, $s \in \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$ and $m \in \mathcal{M}(\mathbf{U})$. By Proposition 6.2.5, we may identify:

$$\mathcal{P}^\lambda(M)(\mathbf{U}) \cong \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)\widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}M$$

for each $\mathbf{U} \in \mathbf{X}_w$. We compute:

$$(\theta \circ \mathcal{P}^\lambda(\psi))(\mathbf{U})(s\widehat{\otimes}m) = \theta(\mathbf{U})(s\widehat{\otimes}\psi(m)) = s \cdot \alpha(\mathbf{X})(1\widehat{\otimes}m)\ |_{\mathbf{U}}$$

$$= s \cdot \alpha(\mathbf{U})(1\widehat{\otimes}m) = \alpha(\mathbf{U})(s \cdot (1\widehat{\otimes}m))$$

$$= \alpha(\mathbf{U})(s\widehat{\otimes}m),$$

since the map $\alpha(\mathbf{U})$ is $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$-linear and α is a morphism of sheaves on $\mathbf{X}_w$.

Since $\mathcal{P}^\lambda(\psi)$ and α are morphisms of sheaves on $\mathbf{X}_w$, and since $\alpha(\mathbf{U})$ is an isomorphism, it follows that θ is also a morphism of sheaves on $\mathbf{X}_w$ and the triangle is commutative. Since ψ is an isomorphism, θ is an isomorphism. By [5, Theorem 9.1], we see that θ extends to the required isomorphism $\theta : (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}(\mathcal{M}(\mathbf{X})) \to \mathcal{M}$ of sheaves on $\mathbf{X}$, which is H-equivariant, $\mathcal{D}^\lambda$-linear and continuous since $\mathcal{P}^\lambda(\psi)$ and $\alpha\,|_{\mathbf{X}_w}$ both have these properties. $\square$

We now make the following assumptions. Suppose $\mathbf{X}$ is an affinoid variety, G is compact, $\mathcal{A}$ is a G-stable affine formal model in $A := \mathcal{O}(\mathbf{X})$, $\mathcal{L}$ is a G-stable free $\mathcal{A}$-Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{X})$. Further suppose that $[\mathcal{L},\mathcal{L}] \subseteq \pi\mathcal{L}$ and $\mathcal{L}(\mathcal{A}) \subseteq \pi\mathcal{A}$.

Let $\mathcal{U}$ be a finite $\mathbf{X}_{\mathrm{ac}}(\mathcal{L})$-covering of $\mathbf{X}$ which admits an $\mathcal{L}$-accessible refinement. Let H be an open normal subgroup of G which stabilises every element of $\mathbf{U}$, $(N_\bullet)$ be a good chain for $\mathcal{L}$ in H, which exists by Lemma 6.1.9, and suppose $\mathcal{M}$ is a $\mathcal{U}$-coadmissible H-equivariant $\mathcal{D}^\lambda$-module on $\mathbf{X}$.

Let $n \geq 0$ and let $\mathbf{Y}$ be an intersection of members of $\mathbf{U}$. We set:

(a) $\mathcal{X}_n := \mathbf{X}_{\mathrm{ac}}(\pi^n\mathcal{L}, H)$, a G-topology on $\mathbf{X}$,

(b) $\mathcal{Y}_n := \mathbf{Y}_{\mathrm{ac}}(\pi^n\mathcal{L}, H) = \mathcal{X}_n \cap \mathbf{Y}_w$, a G-topology on $\mathbf{Y}$,

(c) $\mathcal{Q}_n^\lambda := (\widehat{\mathscr{U}(\pi^n\mathcal{L})}_K \rtimes_{N_n} H) \otimes_{\widehat{U(\mathfrak{h})_K}} K_\lambda$, a sheaf of K-Banach algebras on $\mathcal{X}_n$,

(d) $\mathcal{Q}_\infty^\lambda := \widehat{\mathcal{D}}^\lambda(-, H)$ a sheaf of Fréchet-Stein algebras on $\mathbf{X}_w/H$.

Recall that the G-topologies $\mathcal{X}_n$ become finer as n increases: there is a chain of inclusions:

$$\mathcal{X}_0 \subseteq \mathcal{X}_1 \subseteq \cdots \subseteq \mathcal{X}_n \subseteq \cdots \subseteq \mathbf{X}_w/H.$$

Furthermore, by Proposition 6.5.6, there is an isomorphism of sheaves on $\mathcal{X}_0$:

$$\mathcal{Q}_\infty^\lambda \mid_{\mathcal{X}_0} \to \varprojlim \mathcal{Q}_n^\lambda.$$

Since $\mathcal{M}$ is $\mathcal{U}$-coadmissible, [1, Proposition 3.5.2 and Proposition 6.5.8(b)] demonstrate that $\mathcal{M}_{\mathbf{Y}_w/H}$ is naturally a $\mathcal{Q}_\infty^\lambda \mid_{\mathbf{Y}_w/H}$-module for each $\mathbf{Y} \in \mathcal{U}$.

Lemma 6.5.9. There is a unique structure of a $\mathcal{Q}_\infty^\lambda$-module on $\mathcal{M}_{\mathbf{X}_w/H}$ which extends the $\mathcal{Q}_\infty^\lambda \mid_{\mathbf{Y}_w/H}$-module structure on $\mathcal{M}_{\mathbf{Y}_w/H}$ for each $\mathbf{Y} \in \mathcal{U}$.

Proof. Applying Proposition 6.5.8(b) shows that the action maps:

$$a_{\mathbf{Y}} : \mathcal{Q}_\infty^\lambda \mid_{\mathbf{Y}_w/H} \times \mathcal{M}_{\mathbf{Y}_w/H} \to \mathcal{M}_{\mathbf{Y}_w/H}$$

for each $\mathbf{Y} \in \mathcal{U}$ agree on overlaps: $(a_{\mathbf{Y}}) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = (a_{\mathbf{Y}'}) \mid_{\mathbf{Y}\cap\mathbf{Y}'}$ for each $\mathbf{Y}, \mathbf{Y}' \in \mathcal{U}$. Since $\mathcal{Q}_\infty^\lambda \mid_{\mathbf{Y}_w/H}$ and $\mathcal{M}_{\mathbf{Y}_w/H}$ are sheaves and $a_{\mathbf{Y}}$ is a sheaf morphism, the $a_{\mathbf{Y}}$ patch together uniquely to a sheaf morphism $a : \mathcal{Q}_\infty^\lambda \times \mathcal{M}_{\mathbf{X}_w/H} \to \mathcal{M}_{\mathbf{X}_w/H}$. □

Lemma 6.5.10. Let $n \geq 0$ and $\mathbf{Y} \in \mathcal{U}$. There is a presheaf $\mathcal{P}_n$ on $\mathcal{X}_n$ of $\mathcal{Q}_n^\lambda$-modules given by:

$$\mathcal{P}_n(\mathbf{Z}) = \mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbf{Z})} \mathcal{M}(\mathbf{Z})$$

for $\mathbf{Z} \in \mathcal{X}_n$. Moreover, the canonical map:

$$\mathrm{Loc}_{\mathcal{Q}_n^\lambda}(\mathcal{P}_n(\mathbf{Y})) \to \mathcal{P}_n \mid_{\mathcal{Y}_n}$$

is an isomorphism, and the restriction of $\mathcal{P}_n$ to $\mathcal{Y}_n$ is a sheaf.

Proof. Applying Lemma 6.5.9, $\mathcal{M}(\mathbf{W})$ is a $\mathcal{Q}_\infty^\lambda(\mathbf{W})$-module for each $\mathbf{W} \in \mathcal{X}_n \subseteq \mathbf{X}_w/H$, so $\mathcal{P}_n(\mathbf{W})$ is well-defined. Since $\mathcal{Q}_n^\lambda, \mathcal{Q}_\infty^\lambda \mid_{\mathcal{X}_n}$ and $\mathcal{M} \mid_{\mathcal{X}_n}$ are all well-defined functors on $\mathcal{X}_n$, the functoriality of the tensor product ensures that $\mathcal{P}_n$ is a presheaf.

Since $\mathcal{M}$ is $\mathcal{U}$-coadmissible and $\mathbf{Y} \in \mathcal{U}$, we may apply Proposition 6.5.8(c) to find an isomorphism:

$$\theta : (\mathrm{Loc}^\lambda)_{\mathbf{Y}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{Y},H)}(\mathcal{M}(\mathbf{Y})) \to \mathcal{M} \mid_{\mathbf{Y}} .$$

Let $\mathbf{Z} \in \mathcal{Y}_n$ and consider the diagram:

$$\begin{array}{ccc}
\mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_n^\lambda(\mathbf{Y})} \mathcal{P}_n(\mathbf{Y}) & \longrightarrow & \mathcal{P}_n(\mathbf{Z}) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_n^\lambda(\mathbf{Y})} (\mathcal{Q}_n^\lambda(\mathbf{Y}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbf{Y})} \mathcal{M}(\mathbf{Y})) & & \mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbf{Z})} \mathcal{M}(\mathbf{Z}) \\
\downarrow \cong & & \uparrow \cong \\
\mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbf{Y})} \mathcal{M}(\mathbf{Y}) & \xrightarrow{\cong} & \mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbf{Z})} (\mathcal{Q}_\infty^\lambda(\mathbf{Z}) \widehat{\otimes}_{\mathcal{Q}_\infty^\lambda(\mathbf{Y})} \mathcal{M}(\mathbf{Y})).
\end{array}$$

By definition, if $b \in \mathcal{Q}_n^\lambda(\mathbf{Y})$ and $m \in \mathcal{M}(\mathbf{Y})$ then $(b \otimes m) \mid_{\mathbf{Z}} = b \mid_{\mathbf{Z}} \otimes m \mid_{\mathbf{Z}}$. It follows that the diagram is commutative and so the top horizontal arrow is an isomorphism.

Finally, since $\mathcal{M}(\mathbf{Y})$ is a coadmissible $\mathcal{Q}^{\lambda}_{\infty}(\mathbf{Y})$-module by Proposition 6.5.8(a), it follows that $\mathcal{P}_n(\mathbf{Y})$ is a finitely generated $\mathcal{Q}^{\lambda}_n(\mathbf{Y})$-module. The result follows by applying [1, Corollary 4.3.19]. □

We set $\mathcal{M}_n$ to be the sheafification of the presheaf $\mathcal{P}_n$ on $\mathcal{X}_n$.

Corollary 6.5.11. $\mathcal{M}_n$ is a $\mathcal{U}$-coherent $\mathcal{Q}^{\lambda}_n$-module on $\mathcal{X}_n$. Furthermore, $\mathcal{M}_n(\mathbf{X})$ is a finitely generated $\mathcal{Q}^{\lambda}_n(\mathbf{X})$-module, and the canonical $\mathcal{Q}^{\lambda}_n$-linear morphism:

$$\sigma_n : \mathrm{Loc}_{\mathcal{Q}^{\lambda}_n}(\mathcal{M}_n(\mathbf{X})) \to \mathcal{M}_n$$

is an isomorphism.

Proof. The first statement follows from Lemma 6.5.10, and the rest follows from [1, Theorem 4.3.21], since $\mathcal{U}$ admits a $\mathcal{L}$-accessible refinement by assumption. □

Lemma 6.5.12. For each $n \geq 0$ there is a $\mathcal{Q}^{\lambda}_n$-linear isomorphism:

$$\tau_n : \mathrm{Loc}_{\mathcal{Q}^{\lambda}_n}(\mathcal{Q}^{\lambda}_n(\mathbf{X}) \otimes_{\mathcal{Q}^{\lambda}_{n+1}(\mathbf{X})} \mathcal{M}_{n+1}(\mathbf{X})) \to \mathrm{Loc}_{\mathcal{Q}^{\lambda}_n}(\mathcal{M}_n(\mathbf{X})).$$

Proof. For each $\mathbf{Z} \in \mathcal{X}_n$, there is a canonical functorial isomorphism:

$$\mathcal{Q}^{\lambda}_n(\mathbf{Z}) \otimes_{\mathcal{Q}^{\lambda}_{n+1}(\mathbf{Z})} \mathcal{P}_{n+1}(\mathbf{Z}) \to \mathcal{P}_n(\mathbf{Z})$$

by definition of $\mathcal{P}_n(\mathbf{Z})$. If $\mathbf{Z} \subseteq \mathbf{Y}$ for some $\mathbf{Y} \in \mathcal{U}$, then Lemma 6.5.10 induces a functorial isomorphism:

$$\mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbf{Z})} \mathcal{M}_{n+1}(\mathbf{Z}) \to \mathcal{M}_n(\mathbf{Z})$$

which is independent of the choice of $\mathbf{Y}$. We next consider the commutative diagram:

$$\begin{array}{ccc}
\mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_n^\lambda(\mathbf{X})} (\mathcal{Q}_n^\lambda(\mathbf{X}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbf{X})} \mathcal{M}_{n+1}(\mathbf{X})) & \xrightarrow{\tau_n(\mathbf{Z})} & \mathcal{Q}_n^\lambda(\mathbf{X}) \otimes_{\mathcal{Q}_n^\lambda(\mathbf{X})} \mathcal{M}_n(\mathbf{X}) \\
\downarrow \cong & & \\
\mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbf{Z})} (\mathcal{Q}_{n+1}^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbf{X})} \mathcal{M}_{n+1}(\mathbf{X})) & & \downarrow \sigma_n(\mathbf{Z}) \\
\downarrow 1\otimes\sigma_{n+1}(\mathbf{Z}) & & \\
\mathcal{Q}_n^\lambda(\mathbf{Z}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbf{Z})} \mathcal{M}_{n+1}(\mathbf{Z}) & \xrightarrow{\cong} & \mathcal{M}_n(\mathbf{Z}).
\end{array}$$

A diagram chase shows that the top horizontal arrow is also an isomorphism, which is functorial in $\mathbf{Z}$. Since both $\mathcal{Q}_n^\lambda(\mathbf{X}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbf{X})} \mathcal{M}_{n+1}(\mathbf{X})$ and $\mathcal{M}_n(\mathbf{X})$ are finitely generated $\mathcal{Q}_n^\lambda(\mathbf{X})$-modules by Corollary 6.5.11 and $\mathcal{U}$ is a $\mathcal{X}_n$-covering, the $\tau_n(\mathbf{Z})$ patch together to the required isomorphism by [1, Corollary 4.3.19]. □

Corollary 6.5.13. The $\mathcal{Q}_\infty^\lambda(\mathbf{X})$-module $M_\infty := \varprojlim \mathcal{M}_n(\mathbf{X})$ is coadmissible.

Proof. Applying Lemma 6.5.12, the maps $\tau_n(\mathbf{X})$ induce $\mathcal{Q}_n^\lambda(\mathbf{X})$-linear isomorphisms:

$$\mathcal{Q}_n^\lambda(\mathbf{X}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbf{X})} \mathcal{M}_{n+1}(\mathbf{X}) \to \mathcal{M}_n(\mathbf{X})$$

for each $n \in \mathbb{N}$. Hence, by Definition 2.4.5, M_∞ is a coadmissible $\varprojlim \mathcal{Q}_n^\lambda(\mathbf{X}) = \mathcal{Q}_\infty^\lambda(\mathbf{X})$-module. □

Lemma 6.5.14. For each $\mathbf{Y} \in \mathcal{U}$, there is a $\mathcal{Q}_\infty^\lambda(\mathbf{X})$-linear map:

$$\upsilon_{\mathbf{Y}} : M_\infty \to \mathcal{M}(\mathbf{X})$$

such that $\upsilon_{\mathbf{Y}}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = \upsilon_{\mathbf{Y}'}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'}$ for all $m \in M_\infty$ and $\mathbf{Y}' \in \mathcal{U}$.

Proof. Let $m := (m_n)_n \in M_\infty$, where $m_n \in \mathcal{M}_n(\mathbf{X})$, and define $\upsilon_{\mathbf{Y}}$ by:

$$\upsilon_{\mathbf{Y}}(m) := ((m_n)_{\mathbf{Y}})_n.$$

Since the restriction maps in $\mathcal{M}_n$ are $\mathcal{M}_n(\mathbf{X})$-linear, it follows that the map $\upsilon_{\mathbf{Y}}$ is $\mathcal{Q}^\lambda_\infty(\mathbf{X})$-linear. Furthermore, we see that:

$$\upsilon_{\mathbf{Y}}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}',n} = (m_n \mid_{\mathbf{Y}}) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = m_n \mid_{\mathbf{Y}\cap\mathbf{Y}'} = (m_n \mid_{\mathbf{Y}'}) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = \upsilon_{\mathbf{Y}'}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}',n}$$

for all $n \geq 0$. Hence $\upsilon_{\mathbf{Y}}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = \upsilon_{\mathbf{Y}'}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'}$ for all $m \in M_\infty$. □

Proposition 6.5.15. There is an isomorphism:

$$(\mathrm{Loc}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}_{\mathbf{X}}(M_\infty) \to \mathcal{M}$$

of H-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$.

Proof. By Corollary 6.5.13, we have constructed a coadmissible $\mathcal{Q}^\lambda_\infty(\mathbf{X})$-module M_∞. Since $(\mathrm{Loc}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}_{\mathbf{X}}(M_\infty)$ and $\mathcal{M}$ are sheaves, [5, Theorem 9.1] demonstrates that it suffices to construct an isomorphism of G-equivariant $\mathcal{D}^\lambda$-modules on $\mathcal{U}$:

$$\alpha : (\mathcal{P}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}_{\mathbf{X}}(M_\infty) \mid_{\mathcal{U}} \to \mathcal{M} \mid_{\mathcal{U}} .$$

Let $\mathbb{Y} \in \mathcal{S}$ such that $\mathbf{Y} := \widehat{\mathbb{Y}}_{\mathrm{rig}} \in \mathcal{U}$. By Lemma 6.5.14, we may define:

$$g_{\mathbf{Y}} : \widehat{\mathcal{D}}^{\lambda}(\mathbf{Y}, H)\widehat{\otimes}_{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},H)} M_{\infty} \to \mathcal{M}(\mathbf{Y})$$

via $g_{\mathbf{Y}}(s\widehat{\otimes}m) = s \cdot v_{\mathbf{Y}}(m)$. This is a $\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y}, G)$-linear map, and we construct the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{Q}^{\lambda}_{\infty}(\mathbf{Y})\widehat{\otimes}_{\mathcal{Q}^{\lambda}_{\infty}(\mathbf{X})} M_{\infty} & \xrightarrow{\quad g_{\mathbf{Y}} \quad} & \mathcal{M}(\mathbf{Y}) \\
\downarrow \cong & & \downarrow \cong \\
\varprojlim \mathcal{Q}^{\lambda}_{n}(\mathbf{Y}) \otimes_{\mathcal{Q}^{\lambda}_{n}(\mathbf{X})} \mathcal{M}_n(\mathbf{X}) & & \varprojlim \mathcal{Q}^{\lambda}_{n}(\mathbf{Y}) \otimes_{\mathcal{Q}^{\lambda}_{\infty}(\mathbf{X})} \mathcal{M}(\mathbf{Y}) \\
\downarrow \cong & & \downarrow \cong \\
\varprojlim \mathcal{M}_n(\mathbf{Y}) & \xleftarrow[\cong]{\qquad} & \mathcal{P}_n(\mathbf{Y}).
\end{array}$$

The bottom left vertical arrow in the diagram is an isomorphism by Corollary 6.5.11, and the bottom horizontal arrow is an isomorphism by Lemma 6.5.10. It follows that $g_{\mathbf{Y}}$ is an isomorphism.

Now consider the following diagram:

$$\begin{array}{ccc}
(\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},H)}_{\mathbf{X}}(M_{\infty}) \mid_{\mathbf{Y}_w} & \xrightarrow{\quad \alpha_{\mathbf{Y}} \quad} & \mathcal{M} \mid_{\mathbf{Y}_w} \\
\downarrow \cong & & \theta_{\mathbf{Y}} \uparrow \\
(\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},H)}_{\mathbf{Y}}(\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y}, H)\widehat{\otimes}_{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},H)} M_{\infty}) & \xrightarrow[(\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},H)}_{\mathbf{Y}}(g_{\mathbf{Y}})]{} & (\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},H)}_{\mathbf{Y}}(\mathcal{M}(\mathbf{Y})).
\end{array}$$

The left vertical arrow is an isomorphism by Proposition 6.2.6, and the right vertical arrow is also an isomorphism by Proposition 6.5.8(c). Since $g_{\mathbf{Y}}$ is an isomorphism, the bottom arrow is an isomorphism by functoriality of $(\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},H)}_{\mathbf{Y}}$, using Proposition 6.3.2. Hence we obtain the H-equivariant $\mathcal{D}^{\lambda}$-linear isomorphism:

$$\alpha\mid_{\mathbf{Y}_w} : (\mathcal{P}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}_{\mathbf{X}}(M_\infty)\mid_{\mathbf{Y}_w} \to \mathcal{M}\mid_{\mathbf{Y}_w}$$

which makes the diagram commute.

Applying Proposition 6.5.8(a), $\mathcal{M}(\mathbf{Z})$ is naturally a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{Z}, H_{\mathbf{Z}})$-module for any $\mathbf{Z} \in \mathbf{Y}_w$. Via Corollary 6.2.5, we may identify:

$$(\mathcal{P}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}_{\mathbf{X}}(M_\infty)(\mathbf{Z}) \cong \widehat{\mathcal{D}}^\lambda(\mathbf{Z}, H_{\mathbf{Z}})\widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H_{\mathbf{Z}})} M_\infty,$$

and so it follows that the map

$$\alpha\mid_{\mathbf{Y}}(\mathbf{Z}) : (\mathcal{P}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}_{\mathbf{X}}(M_\infty) \to \mathcal{M}(\mathbf{Z})$$

is given by:

$$\alpha_{\mathbf{Y}}(\mathbf{Z})(s\widehat{\otimes}m) = s \cdot (v_{\mathbf{Y}}(m)\mid_{\mathbf{Z}})$$

for all $s \in \widehat{\mathcal{D}}^\lambda(\mathbf{Z}, H_{\mathbf{Z}})$ and $m \in M_\infty$.

By Lemma 6.5.14, the local isomorphisms $\alpha_{\mathbf{Y}}$ satisfy:

$$\alpha_{\mathbf{Y}}\mid_{\mathbf{Y}\cap\mathbf{Y}'} = \alpha_{\mathbf{Y}'}\mid_{\mathbf{Y}\cap\mathbf{Y}'}$$

for any $\mathbf{Y}, \mathbf{Y}' \in \mathcal{U}$. Since $\mathcal{M} \mid_{\mathbf{X}_w}$ is a sheaf by assumption and since $(\mathcal{P}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}(\mathbf{X},H)}(M_\infty)$ is a sheaf on $\mathbf{X}_w$ by Proposition 6.2.7, the $\alpha_{\mathbf{Y}}$ patch together to give the required isomorphism $\alpha : (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{A}(M_\infty) \to \mathcal{M}$. □

We are finally ready to prove Theorem 6.3.5. For clarity, we restate the theorem below.

Theorem 6.5.16. *Suppose* $(\boldsymbol{X}, G)$ *is small. Then the localisation functor:*

$$(Loc^\lambda)_{\boldsymbol{X}}^{\widehat{\mathcal{D}}^\lambda(\boldsymbol{X},G)} : \mathcal{C}_{\widehat{\mathcal{D}}^\lambda(\boldsymbol{X},G)} \to \mathcal{C}_{\boldsymbol{X}/G}^\lambda$$

is an equivalence of categories.

Proof. Let M, N be coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-modules, and write $\mathrm{Loc}^\lambda := (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},G)}$. By Corollary 6.2.5 and Theorem 6.2.7, we may identify $M(\mathbf{X}, G)$ with $\mathrm{Loc}^\lambda(M)(\mathbf{X})$, functorially in M. Hence, given a $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-linear morphism $f : M \to N$, we have the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ \downarrow & & \downarrow \\ M(\mathbf{X}, G) & \xrightarrow[\mathrm{Loc}^\lambda(f)(\mathbf{X})]{} & N(\mathbf{X}, G). \end{array}$$

Since the map $N \to N(\mathbf{X}, G)$ which sends $n \mapsto 1 \widehat{\otimes} n$ is an isomorphism, it is immediate that Loc^λ is faithful.

Now suppose $\alpha : \mathrm{Loc}^\lambda(M) \to \mathrm{Loc}^\lambda(N)$ is a morphism of G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$. Applying Proposition 2.7.7, we see the morphism α :

$\mathrm{Loc}^\lambda(M)(\mathbf{X}) \to \mathrm{Loc}^\lambda(N)(\mathbf{X})$ is $\mathcal{D}^\lambda(\mathbf{X}) \rtimes G$-linear, and continuous by Definition 6.3.1(b), so it must also be $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-linear since $M(\mathbf{X}, G)$ and $N(\mathbf{X}, G)$ are both coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-modules. Define f by the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ \downarrow & & \downarrow \\ M(\mathbf{X}, G) & \xrightarrow{\alpha(\mathbf{X})} & N(\mathbf{X}, G). \end{array}$$

We see that f is $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-linear.

We claim that $\mathrm{Loc}^\lambda(f) = \alpha$. To see this, let $\mathbf{U} \in \mathbf{X}_w$ and recall that the map $M(\mathbf{X}, G_{\mathbf{U}}) \to M(\mathbf{X}, G)$ is a bijection by Proposition 6.2.4. Consider the commutative diagram:

$$\begin{array}{ccccccc} M(\mathbf{U}, G_{\mathbf{U}}) & & & \xrightarrow{\alpha(\mathbf{U}),\mathrm{Loc}^\lambda(f)(\mathbf{U})} & & & N(\mathbf{U}, G_{\mathbf{U}}) \\ & \nwarrow & & & & \nearrow & \\ \uparrow & & M(\mathbf{X}, G_{\mathbf{U}}) & & N(\mathbf{X}, G_{\mathbf{U}}) & & \uparrow \\ & \swarrow_{\cong} & & & & \searrow^{\cong} & \\ M(\mathbf{X}, G) & & & \xrightarrow{\alpha(\mathbf{X}),\mathrm{Loc}^\lambda(f)(\mathbf{X})} & & & N(\mathbf{X}, G). \end{array}$$

By construction, $\alpha(\mathbf{X}) = \mathrm{Loc}^\lambda(f)(\mathbf{X})$, and the image of $M(\mathbf{X}, G)$ in $M(\mathbf{U}, G_{\mathbf{U}})$ under the vertical map in this diagram generates a dense $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, G_{\mathbf{U}})$-submodule of $M(\mathbf{U}, G_{\mathbf{U}}) = \widehat{\mathcal{D}}^\lambda(\mathbf{U}, G_{\mathbf{U}}) \widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)} M$. Since $\alpha(\mathbf{U})$ and $\mathrm{Loc}^\lambda(f)(\mathbf{U})$ are continuous $\mathcal{D}^\lambda(\mathbf{U}) \rtimes G_{\mathbf{U}}$-linear maps that agree on this submodule, they are equal, and so $\mathrm{Loc}^\lambda(f) = \alpha$ since $\mathbf{X}_w$ is a basis. Hence Loc^λ is full.

To show Loc^λ is essentially surjective, let $\mathcal{U}$ be an admissible affinoid covering of $\mathbf{X}$ and let $\mathcal{M}$ be a $\mathcal{U}$-coadmissible G-equivariant $\mathcal{D}^\lambda$-module on $\mathbf{X}$. Since $(\mathbf{X}, G)$ is small, we can find a G-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{X})$ and a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\widetilde{\mathcal{T}}(\mathbf{X})$ by Lemma 6.1.1.

Since $\mathbf{X}$ is affinoid, we may replace $\mathcal{U}$ by a finite refinement and apply [1, Lemma 3.6.9] to assume $\mathcal{U}$ is finite. Choose a Laurent refinement $\mathcal{V}$ of $\mathcal{U}$ by [12, Lemmas 8.2.2/2-4]. By [5, Proposition 7.6], we may replace $\mathcal{L}$ by $\pi^t\mathcal{L}$, for a sufficiently large integer $t \geq 0$, to ensure that every member of $\mathcal{U}$ and $\mathcal{V}$ is an $\mathcal{L}$-accessible affinoid subdomain of $\mathbf{X}$. Hence $\mathcal{U}$ is $\mathcal{L}$-accessible and admits an $\mathcal{L}$-accessible Laurent refinement. By replacing $\mathcal{L}$ with $\pi\mathcal{L}$ if necessary, we may further assume that $[\mathcal{L}, \mathcal{L}] \subseteq \pi\mathcal{L}$ and $\mathcal{L} \cdot \mathcal{A} \subseteq \pi\mathcal{A}$. Applying [1, Lemma 4.4.1], we may find an open normal subgroup H of G which stabilises $\mathcal{A}$, $\mathcal{L}$ and each member of $\mathcal{U}$. Choosing a good chain $(N_\bullet)$ for $\mathcal{L}$ by [1, Lemma 3.3.6], it follows that all of our assumptions are satisfied.

By Corollary 6.5.13, $M := \mathcal{M}(\mathbf{X})$ is a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$-module, so by Proposition 6.5.8(c) and Proposition 6.5.15, there is an isomorphism of H-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$:

$$\theta : (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}(\mathcal{M}(\mathbf{X})) \to \mathcal{M}$$

whose restriction to $\mathbf{X}_w$ is given by:

$$\theta(\mathbf{U})(s\widehat{\otimes}m) = s \cdot (m \mid_{\mathbf{U}})$$

for any $\mathbf{U} \in \mathbf{X}_w, s \in \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$ and $m \in \mathcal{M}(\mathbf{X})$. Furthermore, M is a $\mathcal{D}^\lambda \rtimes G$-module by Proposition 2.7.7, and by Proposition 6.5.8(a), the $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-action and the $\mathcal{D}^\lambda(\mathbf{X}, G)$-action are compatible in the following sense:

$$\gamma^H(h) \cdot m = h^{\mathcal{M}}(m)$$

for all $h \in H$ and $m \in M$. Hence [1, Proposition 3.3.11] implies the $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-action on M extends to an action of $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H) \rtimes G \cong \widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$. Since the restriction of M back to $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$ is coadmissible, it follows that M is coadmissible as a $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-module.

By construction, we see that $(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},G)}(\mathcal{M}(\mathbf{X})) = (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{X},H)}(\mathcal{M}(\mathbf{X}))$ as H-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$. It then suffices to verify that the isomorphism θ is G-equivariant.

Fix $\mathbf{U} \in \mathbf{X}_w$ and $g \in G$. We consider the diagram:

$$\begin{array}{ccc} M(\mathbf{U}, H_{\mathbf{U}}) & \xrightarrow{\theta(\mathbf{U})} & \mathcal{M}(\mathbf{U}) \\ \downarrow g^M_{\mathbf{U},H_{\mathbf{U}}} & & \downarrow g^{\mathcal{M}}(\mathbf{U}) \\ M(g\mathbf{U}, H_{g\mathbf{U}g^{-1}}) & \xrightarrow{\theta(g\mathbf{U})} & \mathcal{M}(g\mathbf{U}). \end{array}$$

Applying Proposition 6.5.8(a), we see that $\mathcal{M}(\mathbf{U})$ is a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}})$-module and $\mathcal{M}(g\mathbf{U})$ is a coadmissible $\widehat{\mathcal{D}}^\lambda(g\mathbf{U}, gH_{\mathbf{U}}g^{-1})$-module, and so the morphism $\theta(\mathbf{U})$ is $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}})$-linear, and the morphism $\theta(g\mathbf{U})$ is $\widehat{\mathcal{D}}^\lambda(g\mathbf{U}, gH_{\mathbf{U}}g^{-1})$-linear.

Via the isomorphism $\widehat{g_{\mathbf{U},H_{\mathbf{U}}}} : \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H_{\mathbf{U}}) \to \widehat{\mathcal{D}}^\lambda(g\mathbf{U}, gH_{\mathbf{U}}g^{-1})$ from Lemma 6.1.16, we may regard $\mathcal{M}(g\mathbf{U})$ as a coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-module. Since $\mathcal{M}$ is a G-equivariant $\mathcal{D}^{\lambda}$-module, it follows from Definition 2.7.6 that the map:

$$g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{M}(g\mathbf{U})$$

is $\mathcal{D}^{\lambda}(\mathbf{U}) \rtimes H_{\mathbf{U}}$-linear. Since it is also continuous by Definition 6.3.1(a), applying Remark 6.1.7 shows that this map is $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-linear.

Similarly, we may regard $M(g\mathbf{U}, H_{g\mathbf{U}g^{-1}})$ as a coadmissible $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-module via $\widehat{g_{\mathbf{U},H_{\mathbf{U}}}}$, and so the maps $g^{M}_{\mathbf{U},H_{\mathbf{U}}}$ and $\theta(g\mathbf{U})$ are $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-linear.

Since the image of $M \in M(\mathbf{U}, H_{\mathbf{U}})$ generates a dense $\widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H_{\mathbf{U}})$-submodule in $M(\mathbf{U}, H_{\mathbf{U}})$, to show that the diagram commutes it suffices to verify that $g^{\mathcal{M}}(\mathbf{U}) \circ \theta(\mathbf{U})$ and $\theta(g\mathbf{U}) \circ g^{M}_{\mathbf{U},H_{\mathbf{U}}}$ agree on this image by [1, Lemma 3.6.5]. We see that:

$$\begin{aligned}\left[g^{\mathcal{M}}(\mathbf{U}) \circ \theta(\mathbf{U})\right](1\widehat{\otimes}m) &= g^{\mathcal{M}}(\mathbf{U})(m\mid_{\mathbf{U}}) = g^{\mathcal{M}}(\mathbf{X})(m)\mid_{\mathbf{U}} \\ &= \theta(g\mathbf{U})(1\widehat{\otimes}g\cdot m) = \left[\theta(g\mathbf{U}) \circ g^{M}_{\mathbf{U},H_{\mathbf{U}}}\right](1\widehat{\otimes}m).\end{aligned}$$

for all $m \in M$, and the result follows. □

6.6 The localisation functor is essentially surjective

In this section, we show that the twisted localisation functor $(\mathrm{Loc}^{\lambda})^{A}_{\mathbf{X}} : \mathcal{C}_A \to \mathcal{C}^{\lambda}_{\mathbf{X}/G}$ is essentially surjective on objects. This will be a key step in the eventual proof of the twisted equivariant locally analytic Beilinson-Bernstein theorem.

Recall the following notation from the beginning of Section 6.1. Let K be a field equipped with a complete non-archimedean norm $|\cdot|$, $\mathcal{R} := \{a \in K \mid |a| \leq 1\}$ the unit ball inside K and $\pi \in \mathcal{R}$ a fixed non-zero non-unit element. Let $\widetilde{\mathbb{X}}, \mathbb{X}$ be smooth schemes over $\mathcal{R}$. Let G be a compact p-adic Lie group and $\mathbb{G}$ a connected split reductive affine algebraic group scheme over $\mathcal{R}$, equipped with a continuous group homomorphism $\sigma : G \to \mathbb{G}(\mathcal{R})$.

We equip $\mathbb{G}(\mathcal{R})$ with the congruence subgroup topology given in Definition 2.8.2. For example, if $\mathbb{G}$ is defined over the ring of integers $\mathcal{O}_L$ of some finite extension L of $\mathbb{Q}_p$, contained in K, then σ could be the inclusion $\mathbb{G}(\mathcal{O}_L) \to \mathbb{G}(\mathcal{R})$.

Let $\mathbb{H}$ be a smooth affine algebraic group scheme over $\mathcal{R}$ and suppose $\xi : \widetilde{\mathbb{X}} \to \mathbb{X}$ is a locally trivial $\mathbb{H}$-torsor as in Definition 2.5.2. We suppose both $\widetilde{\mathbb{X}}$ and $\mathbb{X}$ are equipped with $\mathbb{G}$-actions that commute with the $\mathbb{H}$-action on $\widetilde{\mathbb{X}}$.

The group $\mathbb{G}$ acts on itself by left and right translations. The corresponding homomorphisms of $\mathcal{R}$-group functors:

$$\gamma : \mathbb{G} \to \mathrm{Aut}(\mathbb{G}), \delta : \mathbb{G} \to \mathrm{Aut}(\mathbb{G})$$

are given by the formulae $\gamma(g)(x) = gx, \delta(g)(x) = xg^{-1}$ for $g, x \in \mathbb{G}$. These actions induce $\mathbb{G}(\mathcal{R})$-actions on $\mathcal{O}(\mathbb{G})$ by $\mathcal{R}$-algebra automorphisms as in [38, I.2.7].

Since $\mathbb{G}$ is smooth and of finite type, $\mathcal{O}(\mathbb{G})$ is a finitely generated $\mathcal{R}$-algebra that is flat as an $\mathcal{R}$-module, and so is a finitely presented $\mathcal{R}$-algebra by [51, Theorem 3.4.6]. Taking the π-adic completion, $\mathcal{A} := \widehat{\mathcal{O}(\mathbb{G})}$ is an admissible $\mathcal{R}$-algebra, so $\widehat{\mathbb{G}} := \operatorname{Spf} \widehat{\mathcal{O}(\mathbb{G})}$ is an affine formal scheme of topologically finite presentation, and its rigid generic fibre $\mathbf{G} := \widehat{\mathbb{G}_{\mathrm{rig}}}$ is an affinoid rigid analytic variety over K. We may apply the same argument to define the affinoid rigid analytic varieties $\mathbf{H}$ and $\mathbf{Y}$, and the rigid analytic varieties $\mathbf{X}$ and $\widetilde{\mathbf{X}}$, which are not in general affinoid.

Since $\mathbb{G}$ and $\mathbb{H}$ are smooth affine algebraic groups of finite type over $\mathcal{R}$, [28, Corollaire 19.5.4] implies that they are both infinitesimally flat group schemes.

The discussion above [1, Definition 5.2.2] allows us to construct the completed skew-group algebra $\widehat{\mathcal{D}}(\mathbf{G}, G)$. We first outline the construction of the algebra of right $\mathbf{G}$-invariants $\widehat{\mathcal{D}}(\mathbf{G}, G)^{\mathbf{G}}$. The overall goal of the first part of this section is to show that the algebra $\widehat{\mathcal{D}}(\mathbf{G}, G)^{\mathbf{G}}$ acts on the rigid analytic flag variety $\mathbf{X}$ λ-compatibly with G. Applying Proposition 6.3.4, we will then be able to define the twisted localisation functor:

$$(\mathrm{Loc}^{\lambda})^{A}_{\mathbf{X}} : \mathcal{C}_A \to \mathcal{C}^{\lambda}_{\mathbf{X}/G}.$$

Definition 6.6.1. Let $\mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be an action of the $\mathcal{R}$-group scheme $\mathbb{G}$ on the $\mathcal{R}$-scheme $\mathbb{X}$, with corresponding homomorphism $\alpha : \mathbb{G} \to \mathrm{Aut}(\mathbb{X})$. The *infinitesimal action* of $\mathfrak{g} := \text{Lie}(\mathbb{G})$ on $\mathbb{X}$ *associated to* α is the $\mathcal{R}$-linear map:

$$\alpha' : \mathfrak{g} \to \mathcal{T}(\mathbb{X})$$

given by $\alpha'(u) := D_\alpha$, where $\alpha := \text{Lie}(\alpha)(\mathcal{R})(u^{-1}) \in \text{Lie}(\text{Aut}(\mathbb{X}))(\mathcal{R})$. Here, D_α is the image of α under the homomorphism:

$$\text{Lie}(\text{Aut}(\mathbb{X}))(\mathcal{R}) \to \text{Der}_{\mathcal{R}}(\mathcal{O}_{\mathbb{X}})$$

provided by [20, Chapter II, Section 4, Proposition 2.4].

Recall that we have a G-action on $\mathfrak{g}$ by $\mathcal{R}$-linear automorphisms via $\text{Ad} \circ \sigma$.

Definition 6.6.2. (a) A *Lie lattice* in $\mathfrak{g}$ is a finitely generated $\mathcal{R}$-submodule $\mathcal{J}$ of $\mathfrak{g}$ which is stable under the Lie bracket on $\mathfrak{g}$ and which contains a π-power multiple of $\mathfrak{g}$.

(b) The Lie lattice $\mathcal{J}$ is *G-stable* if it is preserved by $\text{Ad}(\sigma(g))$ for all $g \in G$.

It is immediate from Definition 6.6.2 that $\mathcal{J} := \pi^n\mathfrak{g}$ is a G-stable Lie lattice in $\mathfrak{g}$ for any $n \in \mathbb{N}$.

From now on, we suppose $\mathcal{R}$ is a complete valuation ring of height one and of mixed characteristic $(0, p)$. We may apply Definition 6.6.1 to the left translation action $\gamma : \mathbb{G} \to \text{Aut}(\mathbb{G})$ to obtain the $\mathcal{R}$-linear map $\gamma' : \mathfrak{g} \to \mathcal{T}(\mathbb{G})$. Let $\mathcal{A} := \widehat{\mathcal{O}(\mathbb{G})}$, let $\mathcal{J}$ be a Lie lattice in $\mathfrak{g}$ and let $\mathcal{L} := \mathcal{A} \cdot \gamma'(\mathcal{J})$. By functoriality, the $\mathcal{R}$-Lie algebra homomorphism $\gamma' : \mathcal{J} \to \mathcal{L}$ extends to an $\mathcal{R}$-algebra homomorphism:

$$\theta := \widehat{U(\gamma')} : \widehat{U(\mathcal{J})} \to \widehat{U(\mathcal{L})}.$$

Definition 6.6.3. The *completed skew-group algebra* is:

$$\widehat{\mathcal{D}}(\mathbf{G}, G)^{\mathbf{G}} = \varprojlim \widehat{U(\mathcal{J})_K} \rtimes_N G.$$

The inverse limit is taken over the set $\mathcal{K}(G)$ of pairs $(\mathcal{J}, N)$ where $\mathcal{J}$ is a G-stable Lie lattice in $\mathfrak{g}$ and N is an open normal subgroup of G contained in $G_{\mathcal{A}\cdot\gamma'(\mathcal{J})}$.

We note that [1, Proposition 5.2.4] demonstrates that this definition makes sense; there is a G-equivariant trivialisation of the G-action on $\widehat{U(\mathcal{J})_K}$.

Lemma 6.6.4. Let H be an open subgroup of G. Then $\mathcal{K}(H) \cap \mathcal{K}(G)$ is cofinal in both $\mathcal{K}(H)$ and $\mathcal{K}(G)$.

Proof. Let $\mathcal{L} := \mathcal{A} \cdot \gamma'(\mathcal{J})$. Whenever $\mathcal{J}$ is G-stable, this is a G-stable Lie lattice in $\mathcal{A}$ by [1, Lemma 5.2.3]. Recall that $H_{\mathcal{L}} := G_{\mathcal{L}} \cap H$ by construction. Given $(\mathcal{J}, N) \in \mathcal{K}(G)$, choose an open normal subgroup $U \leq G$ contained in the open subgroup $N \cap H$. Then $U \leq N \cap H \leq G_{\mathcal{L}} \cap H = H_{\mathcal{L}}$, so $(\mathcal{L}, U) \in \mathcal{K}(H) \cap \mathcal{K}(G)$. Now let $(\mathcal{J}, N) \in \mathcal{K}(H)$ and choose some open normal subgroup $U \leq G$ contained in N. Then $U \leq N \leq H_{\mathcal{L}} \leq G_{\mathcal{L}}$, so $(\mathcal{L}, U) \in \mathcal{K}(H) \cap \mathcal{K}(G)$. $\square$

We now specialise to the following situation. Suppose $\mathbb{B}$ is a closed and flat $\mathcal{R}$-subgroup scheme of $\mathbb{G}$ with unipotent radical $\mathbb{N}$, $\mathbb{X} := \mathbb{G}/\mathbb{B}$, and $\widetilde{\mathbb{X}} := \mathbb{G}/\mathbb{N}$ are both flat $\mathcal{R}$-schemes of finite presentation. Let $\mathbb{H} := \mathbb{B}/\mathbb{N}$. Then the canonical map $\xi : \widetilde{\mathbb{X}} \to \mathbb{X}$ is a locally trivial $\mathbb{H}$-torsor by Lemma 2.5.6. By [1, Proposition 3.1.12], the group G acts continuously on the rigid analytic flag variety $\mathbf{X}$.

We fix an affinoid subdomain $\mathbf{U}$ of $\mathbf{X}$, an open subgroup H of $G_{\mathbf{U}}$, and an H-stable affine formal model $\mathcal{B}$ in $\mathcal{O}(\mathbf{U})$.

Furthermore, the canonical map $\zeta : \mathbb{G} \to \mathbb{X}$ is a locally trivial $\mathbb{B}$-torsor by [38, II.1.10(2)]. This allows us to define the anchor map:

$$\kappa : (\zeta_* \mathcal{T}_{\mathbb{G}})^{\mathbb{B}} \to \mathcal{T}_{\mathbb{X}}.$$

Let:

$$\kappa_{\mathbf{U}} : \mathcal{T}(\mathbb{G})^{\mathbb{G}} \to \mathcal{T}(\mathbf{U})$$

be the composition of the maps:

$$\mathcal{T}(\mathbb{G})^{\mathbb{G}} \to \mathcal{T}(\mathbb{G})^{\mathbb{B}} \to \mathcal{T}(\mathbb{X}) \to \mathcal{T}(\widehat{\mathbb{X}}) \to \mathcal{T}(\mathbf{X}) \to \mathcal{T}(\mathbf{U}).$$

Given an H-action on an affinoid variety $\mathbf{Y}$, write $\rho^{\mathbf{Y}} : H \to \mathrm{Aut}_K(\mathcal{O}(\mathbf{Y}))$ to denote the corresponding action of H on $\mathcal{O}(\mathbf{Y})$.

The actions $\rho^{\mathbf{G}}$ and $\rho^{\mathbf{U}}$ preserve the affine formal models $\mathcal{A} = \widehat{\mathcal{O}(\mathbb{G})}$ and $\mathcal{B}$ respectively. Recall also from the proof of [1, Proposition 3.2.5] that we have the congruence subgroups $\mathcal{G}_{p^e}(\mathcal{A})$ and $\mathcal{G}_{p^e}(\mathcal{B})$.

Lemma 6.6.5. Let $N := (\rho^{\mathbf{G}})^{-1}(\mathcal{G}_{p^e}(\mathcal{A})) \cap (\rho^{\mathbf{U}})^{-1}(\mathcal{G}_{p^e}(\mathcal{B}))$, considered as a subgroup of H. Then $\kappa_{\mathbf{U}} \circ \log \circ \rho^{\mathbf{G}} |_N = \log \circ \rho^{\mathbf{U}} |_N$.

Proof. This is the same proof as in [1, Lemma 5.2.11], replacing $\mathcal{T}$ with $\widetilde{\mathcal{T}}$. □

The $\mathbb{G}$-action on $\widetilde{\mathbb{X}}$ induces a map $\phi : \mathbb{G} \to \mathrm{Aut}(\widetilde{\mathbb{X}})$, along with the corresponding infinitesimal action $\phi' : \mathfrak{g} \to \mathcal{T}(\widetilde{\mathbb{X}})$ parallel to Definition 6.6.1.

Now $\widetilde{\mathcal{T}}(\mathbb{X}) = (\xi_* \mathcal{T}_{\widetilde{\mathbb{X}}})^{\mathbb{H}}(\mathbb{X}) = \mathcal{T}(\widetilde{\mathbb{X}})^{\mathbb{H}}$. Since the $\mathbb{G}$- and $\mathbb{H}$-actions commute, this map descends to define an $\mathcal{R}$-linear map $\phi' : \mathfrak{g} \to \widetilde{\mathcal{T}}(\mathbb{X})$.

Consider the $\mathcal{R}$-Lie algebra map $\phi'_{\mathbf{U}} : \mathfrak{g} \to \widetilde{\mathcal{T}}(\mathbf{U})$ that is the composition of the following natural maps:

$$\phi'_{\mathbf{U}} : \mathfrak{g} \to \widetilde{\mathcal{T}}(\mathbb{X}) \to \widetilde{\mathcal{T}}(\widehat{\mathbb{X}}) \to \widetilde{\mathcal{T}}(\mathbf{X}) \to \widetilde{\mathcal{T}}(\mathbf{U}).$$

Applying [1, Lemma 5.1.3] to the sheaf $\widetilde{\mathcal{T}}$, along with the functoriality of π-adic completion and the rigid generic fibre functor, we see that the map $\phi'_{\mathbf{U}}$ is H-equivariant.

Proposition 6.6.6. Let $(\mathcal{J}, J)$ be a $\mathcal{B}$-trivialising pair. Then there is an H-stable Lie lattice $\mathcal{H}$ in $\mathfrak{g}$ and an open normal subgroup N of $H_{\mathcal{A}\cdot\gamma'(\mathcal{H})}$, contained in J, such that the map:

$$U(\phi'_{\mathbf{U}}) \rtimes H : U(\mathfrak{g}_K) \rtimes H \to \widehat{U(\mathcal{J})_K} \rtimes_J H$$

factors through $\widehat{U(\mathcal{H})_K} \rtimes_N H$.

Proof. Recall that $\mathcal{J}$ is an H-stable $\mathcal{B}$-Lie lattice in $\widetilde{\mathcal{T}}(\mathbf{U})$ and J is an open normal subgroup of H contained in $H_{\mathcal{J}}$, by Definition 6.1.5. Since $\mathfrak{g}$ has finite rank as an $\mathcal{R}$-module and $\phi'_{\mathbf{U}}$ is H-equivariant, we can find a π-power multiple $\mathcal{H}$ of $\mathfrak{g}$ contained in the preimage of $(\phi'_{\mathbf{U}})^{-1}(\mathcal{J})$ of $\mathcal{J}$ in $\mathfrak{g}$. and we have a K-algebra map $\widehat{\phi'_{\mathbf{U},K}} : \widehat{U(\mathcal{H})_K} \to \widehat{U(\mathcal{J})_K}$. Let $\mathcal{A} := \widehat{\mathcal{O}(\mathbb{G})}$, $\mathcal{L} := \mathcal{A} \cdot \gamma'(\mathcal{H}), H_0 := (\rho^{\mathbf{G}})^{-1}(\mathcal{G}_{p^e}(\mathcal{A})) \cap (\rho^{\mathbf{U}})^{-1}(\mathcal{G}_{p^e}(\mathcal{B}))$ and consider the subgroup:

$$N := H_{\mathcal{L}} \cap J \cap H_0$$

of H. It is open and normal in H, contained in J and satisfies:

$$\kappa_{\mathbf{U}} \circ \log \circ \rho^{\mathbf{G}} \mid_N = \log \rho^{\mathbf{U}} \mid_N$$

by Lemma 6.6.5. Recall that $\theta^{-1} \circ \beta_{\mathcal{L}} : N \to \widehat{U(\mathcal{H})_K}^{\times}$ is an H-equivariant trivialisation of the N-action on $\widehat{U(\mathcal{H})_K}$ by [1, Proposition 5.2.4(c)], and $\beta_{\mathcal{J}} : J \to \widehat{U(\mathcal{J})_K}$ is an H-equivariant trivialisation of the J-action on $\widehat{U(\mathcal{J})_K}$ by Theorem 6.1.4(b).

We aim to show that:

$$\widehat{\phi'_{\mathbf{U},K}}^{\times} \circ \theta^{-1} \circ \beta_{\mathcal{L}} = \beta_{\mathcal{J}}.$$

Let $n \in N$ satisfy $\beta_{\mathcal{L}}(n) = \exp(\iota(u))$, where $u := \log\rho^{\mathbf{G}}(n) \in \mathcal{T}(\mathbf{G})$, and ι is the inclusion of $\mathcal{H}$ (resp. $\mathcal{J}, \mathcal{L}$) into $\widehat{U(\mathcal{H})_K}$ (resp. $\widehat{U(\mathcal{J})_K}$, $\widehat{U(\mathcal{L})_K}$). Applying [1, Proposition 5.2.4(b)], we see that $u = \theta(w) = \gamma'(w)$ for some $w \in \mathcal{H}$. Hence:

$$\theta^{-1}(\beta_{\mathcal{L}}(n)) = \theta^{-1}(\exp\iota(\theta(w))) = \exp\iota(w).$$

Similarly $\beta_{\mathcal{J}}(n) = \exp\iota(v)$ where $v := \log\rho^{\mathbf{U}}(n) \in \mathcal{J} \subseteq \widetilde{\mathcal{T}}(\mathbf{U})$. Now:

$$\phi'_{\mathbf{U}}(w) = \kappa_{\mathbf{U}}(\gamma'(w)) = \kappa_{\mathbf{U}}(u) = \kappa_{\mathbf{U}}(\log\rho^{\mathbf{G}}(n)) = \log\rho^{\mathbf{U}}(n) = v$$

by [1, Lemma 5.1.4]. Since $\widehat{\phi'_{\mathbf{U},K}} \circ \iota = \iota \circ \phi'_{\mathbf{U}}$, it follows that:

$$(\widehat{\phi'_{\mathbf{U},K}}^{\times} \circ \theta^{-1} \circ \beta_{\mathcal{L}})(n) = \widehat{\phi'_{\mathbf{U},K}}^{\times}(\exp\iota(w)) = \exp\iota(\phi'_{\mathbf{U}}(w)) = \exp\iota(v) = \beta_{\mathcal{J}}(n).$$

This shows that $\widehat{\phi'_{\mathbf{U},K}}^{\times} \circ \theta^{-1} \circ \beta_{\mathcal{L}} = \beta_{\mathcal{J}}$. We now apply [1, Lemma 2.2.7] to prove the proposition. The first condition holds because $N \leq J$ and the second holds since $\phi'_{\mathbf{U}}$ is H-equivariant. The third condition is satisfied by the above calculation. Hence we obtain a K-algebra homomorphism:

$$\phi'_{\mathbf{U}} \rtimes 1 : U(\mathfrak{g})_K \rtimes H \to \widehat{U(\mathcal{J})_K} \rtimes_J H$$

which extends $\widehat{\phi'_{\mathbf{U},K}} : \widehat{U(\mathcal{H})_K} \to \widehat{U(\mathcal{J})_K}$ and $1 : H \to H$. □

Our next result is a variant of Proposition 6.1.10 which we will need to prove the main result of this section. We give the details of the proof for the convenience of the reader.

Proposition 6.6.7. Let $\phi'_{\mathbf{U}} : \mathfrak{g} \to \widetilde{\mathcal{T}}(\mathbf{U})$ denote the map defined above Proposition 6.6.6. Further let $i \rtimes \nu : U(\mathfrak{g})_K \rtimes H \to \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}}$ denote the natural embedding of $U(\mathfrak{g})_K \rtimes H$ as a dense subalgebra into $\widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}}$, and let $i' \rtimes \nu' : \widetilde{\mathcal{D}}(\mathbf{U}) \rtimes H \to \widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)$ denote the embedding of $\widetilde{\mathcal{D}}(\mathbf{U}) \rtimes H$ as a dense subalgebra into $\widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)$ provided by Remark 6.1.7.

There is a unique continuous K-algebra homomorphism:

$$\widehat{\phi'_{\mathbf{U}} \rtimes 1} : \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \to \widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)$$

which makes the following diagram commute:

$$\begin{array}{ccc} U(\mathfrak{g}_K) \rtimes H & \xrightarrow[U(\phi'_{\mathbf{U}}) \rtimes 1]{} & \widetilde{\mathcal{D}}(\mathbf{U}) \rtimes H \\ \downarrow{\scriptstyle i \rtimes \nu} & & \downarrow{\scriptstyle i' \rtimes \nu'} \\ \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} & \xrightarrow{\widehat{\phi'_{\mathbf{U}} \rtimes 1}} & \widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H). \end{array}$$

Proof. Using Lemma 6.1.1, we may choose an H-stable $\mathcal{B}$-Lie lattice $\mathcal{L}$ in $\widetilde{\mathcal{T}}(\mathbf{U})$. Since $\mathcal{L}$ is a finitely generated $\mathcal{B}$-module and $\phi'_{\mathbf{U}}([v, w]) = [\phi'_{\mathbf{U}}(v), \phi'_{\mathbf{U}}(w)]$ for any $v, w \in \mathfrak{g}$, we see that $\pi^m \phi'_{\mathbf{U}}(\mathfrak{g}) \subseteq \mathcal{L}$ for some $m \geq 0$. Rescaling $\mathfrak{g}$, we may assume $m = 0$ and so $\phi'_{\mathbf{U}}(\mathfrak{g}) \subseteq \mathcal{L}$.

Applying [1, Lemma 3.3.6] to the open subgroups $N_n \cap N'_n$, we may suppose that $N_n \leq N'_n$ for each $n \in \mathbb{N}$.

Set $\mathcal{A} := \widehat{\mathcal{O}(\mathbb{G})}$. Choose a good chain $(N_\bullet)$ in H for the $\mathcal{A}$-Lie lattice $\mathcal{A} \cdot \gamma'(\mathfrak{g})$ in $\mathcal{T}(\widehat{\mathbb{G}})$, by [1, Lemma 3.3.4]. By [1, Proposition 5.2.6], it follows that we have a K-algebra isomorphism:

$$\widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \cong \varprojlim \widehat{U(\pi^n \mathfrak{g})_K} \rtimes_{N_n} H.$$

Furthermore, choose a good chain $(N'_\bullet)$ in H for $\mathcal{L}$ by Lemma 6.1.9. By assumption, $N_n \subseteq N'_n$ for each n, and the ring homomorphism $U(\phi'_{\mathbf{U}})$ is H-equivariant. Via the same argument as in the proof of Proposition 6.6.6, we may use [1, Lemma 2.2.7] to form the compatible sequence of commutative diagrams:

$$\begin{array}{ccc}
U(\mathfrak{g}_K) \rtimes H & \xrightarrow{U(\phi'_{\mathbf{U}}) \rtimes 1} & \widetilde{\mathcal{D}}(\mathbf{U}) \rtimes H \\
\downarrow & & \downarrow \\
\widehat{U(\pi^n \mathfrak{g})_K} \rtimes_{N_n} H & \xrightarrow{\widehat{\phi'_{\mathbf{U},n,K}} \rtimes 1} & \widehat{U(\pi^n \mathcal{L})_K} \rtimes_{N'_n} H.
\end{array}$$

Passing to the limit and applying Lemma 6.1.9 produces the required map:

$$\widehat{\phi'_{\mathbf{U}} \rtimes 1} : \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \to \widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H)$$

which fits into the commutative diagram:

$$\begin{array}{ccccc}
U(\mathfrak{g}_K) \rtimes H & & \xrightarrow{U(\phi'_{\mathbf{U}}) \rtimes 1} & & \widetilde{\mathcal{D}}(\mathbf{U}) \rtimes H \\
\downarrow & \searrow^{i \rtimes \nu} & & \swarrow^{i' \rtimes \nu'} & \downarrow \\
& \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} & \xrightarrow{\widehat{\phi'_{\mathbf{U}} \rtimes 1}} & \widehat{\widetilde{\mathcal{D}}}(\mathbf{U}, H) & \\
\downarrow & \swarrow^{\cong} & & \searrow^{\cong} & \downarrow \\
\varprojlim \widehat{U(\pi^n \mathfrak{g})_K} \rtimes_{N_n} H & & \xrightarrow{\varprojlim \widehat{\phi'_{\mathbf{U},n,K}} \rtimes 1} & & \varprojlim \widehat{U(\pi^n \mathcal{L})_K} \rtimes_{N'_n} H.
\end{array}$$

To show uniqueness, suppose $\psi : \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \to \widehat{\widehat{\mathcal{D}}}(\mathbf{U}, H)$ is another continuous K-algebra map such that $\psi \circ (i \rtimes \nu) = (i' \rtimes \nu') \circ U(\phi'_{\mathbf{U}})$. Then ψ agrees with $\widehat{\phi'_{\mathbf{U}} \rtimes 1}$ on the dense image of $U(\mathfrak{g}_K) \rtimes H$ inside $\widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}}$, and so the two maps are equal by continuity. □

Theorem 6.6.8. *Set $A := \widehat{\mathcal{D}}(\boldsymbol{G}, G)^{\boldsymbol{G}}$. Then A acts on $\boldsymbol{X}$ λ-compatibly with G.*

Proof. There is a canonical group homomorphism $\eta : G \to A^\times$. For every compact open subgroup H of G, set $A_H := \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}}$. This is a K-subalgebra of A by [1, Definition 5.2.5], which is Fréchet-Stein by [1, Theorem 5.2.7].

Fix a compact open subgroup H of G and let $\mathbf{U} \in \mathbf{X}_w/H$. Let $\mathcal{B}$ be an H-stable affine formal model in $\mathcal{O}(\mathbf{U})$ and let $(\mathcal{J}, J)$ be a $\mathcal{B}$-trivialising pair in the sense of Definition 6.1.5. Then Proposition 6.6.6 yields a K-algebra homomorphism:

$$\widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \to \widehat{U(\mathcal{H})_K} \rtimes_N H \to \widehat{U(\mathcal{J})_K} \rtimes_J H$$

where $\mathcal{H} \leq (\phi'_{\mathbf{U}})^{-1}(\mathcal{J})$ is some H-stable Lie lattice in $\mathfrak{g}$ and N is the open normal subgroup:

$$N := H_{\mathcal{A}\cdot\gamma'(\mathcal{H})} \cap J \cap H_0$$

of H, where H_0 is some open normal subgroup which is independent of the choice of $\mathcal{B}$-trivialising pair $(\mathcal{J}, J)$. Choosing a different H-stable Lie lattice $\mathcal{H}'$ with these properties yields the same composite homomorphism $\widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \to \widehat{U(\mathcal{J})_K} \rtimes_J H$ since both candidates factor through $\widehat{U(\mathcal{H}'')_K} \rtimes_{N''} H$ for $\mathcal{H}'' = \mathcal{H} \cap \mathcal{H}'$ and $N'' =$

$N \cap H_{\mathcal{A}\cdot\gamma'(\mathcal{H})}$.

If $(\mathcal{J}', J')$ is another $\mathcal{A}$-trivialising pair such that $\mathcal{J}' \leq \mathcal{J}$ and $J \leq J'$, then we choose the corresponding pair $(\mathcal{H}', N')$ such that $\mathcal{H}' \leq H$ and $N' \leq N$, yielding the commutative diagram:

$$\begin{array}{ccccc}
\widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} & \longrightarrow & \widehat{U(\mathcal{H})_K} \rtimes_N H & \longrightarrow & \widehat{U(\mathcal{J})_K} \rtimes_J H \\
 & \searrow & \uparrow & & \uparrow \\
 & & \widehat{U(\mathcal{H}')_K} \rtimes_N H' & \longrightarrow & \widehat{U(\mathcal{J}')_K} \rtimes_J H'.
\end{array}$$

Passing to the limit over all $\mathcal{B}$-trivialising pairs $(\mathcal{J}, J)$ induces a continuous K-algebra homomorphism:

$$\alpha^H(\mathbf{U}) : \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \to \widehat{\widehat{\mathcal{D}}}(\mathbf{U}, H).$$

We verify that $\alpha^H(-)$ is a morphism of presheaves. Let $\mathbf{V}$ be an H-stable affinoid subdomain of $\mathbf{U}$ and consider the diagram:

$$\begin{array}{ccccc}
\widetilde{\mathcal{D}}(\mathbf{U}) \rtimes H & & \longrightarrow & & \widehat{\widehat{\mathcal{D}}}(\mathbf{U}, H) \\
\downarrow & \nwarrow & & \nearrow^{\alpha^H(\mathbf{U})} & \downarrow \tau^{\mathbf{U}}_{\mathbf{V}} \\
 & U(\mathfrak{g}_K) \rtimes H & \longrightarrow & \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} & \\
 & \swarrow & & \searrow^{\alpha^H(\mathbf{V})} & \\
\widetilde{\mathcal{D}}(\mathbf{V}) \rtimes H & & \longrightarrow & & \widehat{\widehat{\mathcal{D}}}(\mathbf{V}, H).
\end{array}$$

The trapezia commute by definition of $\alpha^H(\mathbf{U})$ and $\alpha^H(\mathbf{V})$. The outer rectangle commutes by definition of the restriction map $\tau^{\mathbf{U}}_{\mathbf{V}} : \widehat{\widehat{\mathcal{D}}}(\mathbf{U}, H) \to \widehat{\widehat{\mathcal{D}}}(\mathbf{V}, H)$ from

Lemma 6.1.11, and the triangle on the left commutes since $\alpha'_{\mathbf{U}}(v) \mid_{\mathbf{V}} = \alpha'_{\mathbf{V}}(v)$ for any $v \in \mathfrak{g}$. It follows that $\tau_{\mathbf{V}}^{\mathbf{U}} \circ \alpha^H(\mathbf{U})$ agrees with $\alpha^H(\mathbf{V})$ on the image of $U(\mathfrak{g}_K) \rtimes H$ in $\widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}}$. Since this image is dense and the maps are continuous, it follows that they are equal. Hence $\alpha^H : A_H \to \widehat{\widehat{\mathcal{D}}}(-, H)$ is a morphism of presheaves of K-Fréchet algebras on $\mathbf{X}_w/H$.

Composing the map $\alpha^H(\mathbf{U})$ with the continuous quotient map $\widehat{\widehat{\mathcal{D}}}(\mathbf{U}, H) \to \widehat{\mathcal{D}}^{\lambda}(\mathbf{U}, H)$ yields a continuous morphism of presheaves $\alpha^{\mathbf{H}} : \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \to \widehat{\mathcal{D}}^{\lambda}(-, H)$.

We now verify that the compatibility conditions (i)-(iv) in Definition 6.1.17 are satisfied.

(i) Let $(\mathcal{J}, P) \in \mathcal{K}(H) \cap \mathcal{K}(N)$. Since $\widehat{U(\mathcal{J})_K} \rtimes_P N$ is a crossed product of $\widehat{U(\mathcal{J})_K}$ with N/P by [1, Lemma 2.2.4(b)], it follows that the canonical map:

$$\widehat{U(\mathcal{J})_K} \rtimes_P H \otimes_{K[H]} K[N] \to \widehat{U(\mathcal{J})_K} \rtimes_P N$$

is a bijection. Consider the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathcal{D}}(\mathbf{G}, H)^{\mathbf{G}} \otimes_{K[H]} K[N] & \longrightarrow & \widehat{\mathcal{D}}(\mathbf{G}, N)^{\mathbf{G}} \\ \downarrow & & \downarrow \\ \varprojlim (\widehat{U(\mathcal{J})_K} \rtimes_P H \otimes_{K[H]} K[N]) & \longrightarrow & \varprojlim \widehat{U(\mathcal{J})_K} \rtimes_P N. \end{array}$$

The bottom horizontal arrow is a bijection since it is the inverse limit over all $(\mathcal{J}, P) \in \mathcal{K}(H) \cap \mathcal{K}(N)$ of the maps considered above. Since H has finite index in

N and inverse limits commute with finite direct sums, Lemma 6.6.4 demonstrates that the left vertical arrow is a bijection. Furthermore, Lemma 6.6.4 shows that the right vertical arrow is also a bijection. It follows that the top horizontal arrow is bijective, as required.

(ii) For each $\mathbf{U} \in \mathbf{X}_w/H \cap N$, we have a diagram of K-algebras:

$$\begin{array}{ccccc} \widehat{\mathcal{D}}(\mathbf{G},H)^{\mathbf{G}} & \xrightarrow{\alpha^H(\mathbf{U})} & \widehat{\widehat{\mathcal{D}}}(\mathbf{U},H) & \longrightarrow & \widehat{\mathcal{D}^\lambda}(\mathbf{U},H) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{D}}(\mathbf{G},N)^{\mathbf{G}} & \xrightarrow{\alpha^N(\mathbf{U})} & \widehat{\widehat{\mathcal{D}}}(\mathbf{U},N) & \longrightarrow & \widehat{\mathcal{D}^\lambda}(\mathbf{U},N) \end{array}$$

where the right-hand square commutes by definition of $\widehat{\mathcal{D}^\lambda}(\mathbf{U},-)$. Furthermore, the left-hand square commutes by the uniqueness part of Proposition 6.6.7. The result follows.

(iii) Consider the diagram:

$$\begin{array}{ccc} \mathcal{D}^\lambda(\mathbf{X}) \rtimes G & \xrightarrow[g^{\mathcal{D}^\lambda}(\mathbf{X})\rtimes \mathrm{Ad}_g]{} & \widehat{\mathcal{D}^\lambda}(\mathbf{X},G) \\ \uparrow & & \uparrow \\ \mathcal{D}^\lambda(\mathbf{X}) \rtimes H & \xrightarrow[g^{\mathcal{D}^\lambda}(\mathbf{X})\rtimes \mathrm{Ad}_g]{} & \widehat{\mathcal{D}^\lambda}(\mathbf{X},gHg^{-1}) \\ \downarrow & & \downarrow \\ \mathcal{D}^\lambda(\mathbf{U}) \rtimes H & \xrightarrow[g^{\mathcal{D}^\lambda}(\mathbf{U})\rtimes \mathrm{Ad}_g]{} & \widehat{\mathcal{D}^\lambda}(g\mathbf{U},gHg^{-1}). \end{array}$$

The top square is induced by the embedding of H into G, and commutes by construction. Since $g^{\mathcal{D}^\lambda} : \mathcal{D}^\lambda \to g^*\mathcal{D}^\lambda$ is a morphism of presheaves, the bottom square is commutative. Now $\mathrm{Ad}_{\eta(g)} = \widehat{g_{\mathbf{X}} \rtimes \mathrm{Ad}_g}$ by definition of the map $\eta : G \to \widehat{\mathcal{D}^\lambda}(\mathbf{G},G)^{\mathbf{G}}$, and it follows that $\mathrm{Ad}_{\eta(g)}$ sends $\widehat{\mathcal{D}^\lambda}(\mathbf{X},H)$ into $\widehat{\mathcal{D}^\lambda}(\mathbf{X},gHg^{-1})$. Furthermore:

$$\widehat{g_{\mathbf{U},H}} \circ \alpha^H(\mathbf{U}) = \alpha^{gHg^{-1}}(g\mathbf{U}) \circ \mathrm{Ad}_{\eta(g)}$$

by the uniqueness part of Proposition 6.6.7.

(iv) We need to show that $\alpha^H \cdot \eta \mid_H = \beta \mid_H$, where β is the natural map $\beta : G \to \widehat{\mathcal{D}}(\mathbf{G}, G)^{\mathbf{G}}$. This follows directly from the definitions. $\square$

The rigid analytic flag variety $\mathbf{X} := \widehat{\mathbb{X}}_{\mathrm{rig}}$ of $\mathbb{G}$ is not in general affinoid: we now extend the sheaves appearing in Definition 6.5.3 to certain affinoid subdomains of $\mathbf{X}$. We will work on the special fibre of the $\mathcal{R}$-scheme $\mathbb{X}$ for simplicity.

By Theorem 6.6.8, the K-algebra $A := \widehat{\mathcal{D}}(\mathbf{G}, G)^{\mathbf{G}}$ acts on $\mathbf{X}$ λ-compatibly with G. Applying Proposition 6.3.4, we have the twisted localisation functor:

$$(\mathrm{Loc}^{\lambda})^A_{\mathbf{X}} : \mathcal{C}_A \to \mathcal{C}^{\lambda}_{\mathbf{X}/G}.$$

Let $\phi : U(\mathfrak{g})^{\mathbb{G}}_K \to U(\mathfrak{h})_K$ denote the Harish-Chandra homomorphism, from Definition 2.5.7. Let $\lambda : \mathfrak{h}_K \to K$ be a character of $\mathfrak{h}_K$ with the property that $\lambda(\pi^n \mathfrak{h}) \subseteq \mathcal{R}$. Since $\mathfrak{h}$ is finite-dimensional, there exists some $m \geq 0$ such that this property holds for any $n \geq m$.

For each such λ, we have a K-algebra homomorphism $\lambda \circ \phi : Z(\mathfrak{g}_K) \to K$. Let $\mathfrak{m}_\lambda$ denote the kernel of this map. From now on, we assume that λ is a ρ-dominant

ρ-regular weight of $\mathfrak{h}_K$, m is large enough such that $\lambda(\pi^m\mathfrak{h}) \subseteq \mathcal{R}$ and $n \geq m$.

Applying the deformation functor from [4, Section 3.5] to the map $j : U(\mathfrak{h}) \to \widetilde{\mathcal{D}}$ defined above Definition 2.5.4 gives us a central embedding $U(\pi^n\mathfrak{h}) \to \widetilde{\mathcal{D}}_n$. Given a linear functional $\lambda : \pi^n\mathfrak{h} \to \mathcal{R}$, this extends to an $\mathcal{R}$-algebra homomorphism $U(\pi^n\mathfrak{h}) \to \mathcal{R}$, which gives $\mathcal{R}$ the structure of a $U(\pi^n\mathfrak{h})$-module, denoted by $\mathcal{R}_\lambda$.

Definition 6.6.9. (a) We set $\mathcal{D}_n^\lambda := \widetilde{\mathcal{D}}_n \otimes_{U(\pi^n\mathfrak{h})} \mathcal{R}_\lambda$.

(b) The π-adic completion of $\mathcal{D}_n^\lambda$ is $\widehat{\mathcal{D}_n^\lambda} := \varprojlim \mathcal{D}_n^\lambda/\pi^a\mathcal{D}_n^\lambda$.

(c) $\widehat{\mathcal{D}_{n,K}^\lambda} := \widehat{\mathcal{D}_n^\lambda} \otimes_{\mathcal{R}} K$.

Recall also the definition of the congruence subgroups $\mathbb{G}_{\pi^n}(\mathcal{R})$ from Definition 2.8.2. For each $n \geq 1$, set $G_n := \sigma^{-1}(\mathbb{G}_{\pi^n}(\mathcal{R}))$. Since the group homomorphism σ is continuous, the G_n form a descending chain of open normal subgroups of G.

Lemma 6.6.10. Let $\mathcal{S}$ denote the set of open subschemes $\mathbb{Y}$ of $\mathbb{X}$ for which the $\mathbb{H}$-torsor ξ is locally trivial, from Definition 2.5.2. Suppose $n \geq m$ and n is large enough such that $\pi^n \in p^e\mathcal{R}$. There is a unique sheaf of K-Banach algebras:

$$\widehat{\mathcal{D}_{n,K}^\lambda} \rtimes_{G_n} G$$

on $\mathbb{X}$ whose value on $\mathbb{Y} \in \mathcal{S}$ is $U(\widehat{\pi^n\mathcal{T}(\mathcal{Y})})_K \rtimes_{G_n} G$, where $\mathcal{Y} := \widehat{\mathbb{Y}}$. It is a finitely generated free $\widehat{\mathcal{D}_{n,K}^\lambda}$-module of rank $[G : G_n]$.

Proof. Via the same argument as in [1, Lemma 5.4.2 and Proposition 3.2.15], this construction is functorial in $\mathbb{Y} \in \mathcal{S}$ and hence defines a presheaf of K-Banach algebras $\widehat{\mathcal{D}^{\lambda}_{n,K}} \rtimes_{G_n} G$ on $\mathcal{S}$. From [1, Lemma 5.3.1(a)] and Definition 6.6.9(c), we have an isomorphism $\widehat{\mathcal{D}^{\lambda}_{n,K}}(\mathbb{Y}) \cong \widehat{\mathcal{D}_{n,K}}(\mathbb{Y})$, and so $U(\widehat{\pi^n\mathcal{T}(\mathcal{Y})})_K = \widehat{\mathcal{D}^{\lambda}_{n,K}}(\mathbb{Y})$. Therefore [1, Lemma 2.2.4(b)] implies that $\widehat{\mathcal{D}^{\lambda}_{n,K}} \rtimes_{G_n} G$ is finitely generated and free of rank $[G : G_n]$ as a presheaf of $\widehat{\mathcal{D}^{\lambda}_{n,K}}$-modules.

Since $\mathcal{S}$ is a base for the Zariski topology on $\mathbb{X}$, it is immediate that $\widehat{\mathcal{D}^{\lambda}_{n,K}} \rtimes_{G_n} G$ extends to a sheaf of K-Banach algebras on $\mathbb{X}$. □

We will write the sheaf $\widehat{\mathcal{D}^{\lambda}_{n,K}} \rtimes_{G_n} G$ as $\mathcal{Q}^{\lambda}_n$ for ease of notation. From now on, we will suppose $n \geq m$ and n is large enough such that $\pi^n \in p^e\mathcal{R}$.

Proposition 6.6.11. $\mathbb{X}$ is coherently $\mathcal{Q}^{\lambda}_n$-affine.

Proof. Since λ is ρ-dominant and ρ-regular, we see that $\mathbb{X}$ is coherently $\widehat{\mathcal{D}^{\lambda}_{n,K}}$-affine by [1, Theorem 5.3.12]. Applying Lemma 6.6.10, $\mathcal{Q}^{\lambda}_n$ is a free $\widehat{\mathcal{D}^{\lambda}_{n,K}}$-module of finite rank, and so, by [4, Definition 5.1], $\mathbb{X}$ is also coherently $\mathcal{Q}^{\lambda}_n$-affine. □

Since $\pi^n \in p^e\mathcal{R}$, we may apply [1, Lemma 5.4.2 and Proposition 5.2.6] to see that the algebra $A := \widehat{\mathcal{D}}(\mathbf{G}, G)^{\mathbf{G}}$ has a presentation of the form $A = \varprojlim A_n$, where:

$$A_n := U\widehat{(\pi^n\mathfrak{g})}_K \rtimes_{G_n} G.$$

By Proposition 6.6.6, there is a K-algebra homomorphism $A_n \to \mathcal{Q}^{\lambda}_n(\mathbb{Y})$ for each $\mathbb{Y} \in \mathcal{S}$.

Let $M \in \mathcal{C}_A$. We may form the finitely generated A_n-module $M_n := A_n \otimes_A M$ and then form the sheaf of $\mathcal{Q}_n^\lambda$-modules on $\mathbb{X}$:

$$\mathcal{M}_n := \mathcal{Q}_n^\lambda \otimes_{A_n} M = \mathcal{Q}_n^\lambda \otimes_A M.$$

Lemma 6.6.12. Let $M \in \mathcal{C}_A$ and set $\mathcal{M} := (\mathrm{Loc}^\lambda)^A_{\mathbf{X}}(M)$. This is a coadmissible G-equivariant $\mathcal{D}^\lambda$-module on $\mathbf{X}$. For each $\mathbb{Y} \in \mathcal{S}$, there is an A-linear isomorphism:

$$\mathcal{M}(\widehat{\mathbb{Y}}_{\mathrm{rig}}) \cong \varprojlim \mathcal{M}_n(\mathbb{Y}).$$

Proof. Since $(\widehat{\mathbb{Y}}_{\mathrm{rig}}, G)$ is small, we may apply Definition 6.2.8 and Proposition 6.2.5 to see that:

$$\mathcal{M}(\widehat{\mathbb{Y}}_{\mathrm{rig}}) \cong (\mathcal{P}^\lambda)^A_{\mathbf{X}}(M)(\widehat{\mathbb{Y}}_{\mathrm{rig}}) \cong M(\widehat{\mathbb{Y}}_{\mathrm{rig}}, G) = \widehat{\mathcal{D}}^\lambda(\widehat{\mathbb{Y}}_{\mathrm{rig}}, G)\widehat{\otimes}_A M.$$

By construction, $\widehat{\mathcal{D}}^\lambda(\widehat{\mathbb{Y}}_{\mathrm{rig}}, G) = \varprojlim \mathcal{Q}_n^\lambda(\mathbb{Y})$ and $A = \varprojlim A_n$, so:

$$\widehat{\mathcal{D}}^\lambda(\widehat{\mathbb{Y}}_{\mathrm{rig}}, G)\widehat{\otimes}_A M = \varprojlim \mathcal{Q}_n^\lambda(\mathbb{Y}) \otimes_A M = \varprojlim \mathcal{M}_n(\mathbb{Y}),$$

and these isomorphisms are functorial in $\mathbb{Y}$. □

Lemma 6.6.13. The $\mathcal{Q}_n^\lambda$-module $\mathcal{M}_n$ is isomorphic to $\widehat{\mathcal{D}^\lambda_{n,K}} \otimes_{U(\widehat{\pi^n \mathfrak{g}})_K} M_n$ as a $\widehat{\mathcal{D}^\lambda_{n,K}}$-module.

Proof. By construction, $\mathcal{M}_n$ is a finitely generated $\mathcal{Q}_n^\lambda$-module, and $\mathcal{Q}_n^\lambda$ is a free $\widehat{\mathcal{D}^\lambda_{n,K}}$-module of finite rank by Lemma 6.6.10. It follows that $\mathcal{M}_n$ is a finitely generated $\widehat{\mathcal{D}^{\lambda}_{n,K}}$-module.

Applying [1, Proposition 4.3.11], there is an isomorphism of presheaves:

$$\widehat{\mathcal{D}^{\lambda}_{n,K}} \otimes_{\widehat{\mathcal{D}^{\lambda}_{n,K}}(\mathbb{X})} \mathcal{Q}^{\lambda}_{n}(\mathbb{X}) \cong \mathcal{Q}^{\lambda}_{n}.$$

Applying the functor $- \otimes_{\mathcal{Q}^{\lambda}_{n}(\mathbb{X})} \mathcal{M}_n$, there is an isomorphism of presheaves of $\widehat{\mathcal{D}^{\lambda}_{n,K}}$-modules:

$$(\widehat{\mathcal{D}^{\lambda}_{n,K}} \otimes_{\widehat{\mathcal{D}^{\lambda}_{n,K}}(\mathbb{X})} \mathcal{Q}^{\lambda}_{n}(\mathbb{X})) \otimes_{\mathcal{Q}^{\lambda}_{n}(\mathbb{X})} \mathcal{M}_n \cong \mathcal{Q}^{\lambda}_{n} \otimes_{\mathcal{Q}^{\lambda}_{n}(\mathbb{X})} \mathcal{M}_n.$$

Contracting the tensor product on the left-hand side of the above isomorphism yields the result. □

These lemmas allow us to give a characterisation of the global sections of the sheaf $(\mathrm{Loc}^{\lambda})^{A}_{\mathbf{X}}(M)$.

Theorem 6.6.14. *Let $M \in \mathcal{C}_A$. There is an A-module isomorphism:*

$$\Gamma(\boldsymbol{X}, (Loc^{\lambda})^{A}_{\boldsymbol{X}}(M)) \cong \frac{M}{\mathfrak{m}_{\lambda} \cdot M},$$

where $\mathfrak{m}_{\lambda}$ denotes the kernel of the map $\lambda \circ \phi : Z(\mathfrak{g}_K) \to K$.

Proof. Choose an $\mathcal{S}$-covering $\mathcal{U} := \{\mathbb{Y}_1, \cdots, \mathbb{Y}_t\}$ of $\mathbb{X}$ and let $\mathbf{Y}_i := \widehat{\mathbb{Y}}_{i,\mathrm{rig}}$. Then $\widehat{\mathcal{U}}_{\mathrm{rig}} := \{\mathbf{Y}_1, \cdots, \mathbf{Y}_t\}$ is an $\mathbf{X}_w(\mathcal{T})$-covering of the rigid analytic flag variety $\mathbf{X}$.

Set $\mathcal{M} := (\mathrm{Loc}^\lambda)^A_{\mathbf{X}}$. Then Lemma 6.6.12 implies that we have the following chain of isomorphisms:

$$\Gamma(\mathbf{X}, \mathcal{M}) = \check{H}^0(\widehat{\mathcal{U}}_{\mathrm{rig}}, \mathcal{M}) \cong \check{H}^0(\mathcal{U}, \varprojlim \mathcal{M}_n) \cong \varprojlim \check{H}^0(\mathcal{U}, \mathcal{M}_n) = \varprojlim \Gamma(\mathbb{X}, \mathcal{M}_n).$$

By Lemma 6.6.13, the $\mathcal{Q}_n^\lambda$-module $\mathcal{M}_n$ is isomorphic to $\widehat{\mathcal{D}^\lambda_{n,K}} \otimes_{\widehat{U(\pi^n\mathfrak{g})_K}} M_n$ as a $\widehat{\mathcal{D}^\lambda_{n,K}}$-module. Since λ is ρ-dominant and ρ-regular, applying [1, Corollary 5.3.13 and Theorem 5.3.5(a)] shows that there is an isomorphism of $\widehat{\mathcal{D}^\lambda_{n,K}}$-modules

$$\Gamma(\mathbb{X}, \mathcal{M}_n) \cong \widehat{\mathcal{D}^\lambda_{n,K}}(\mathbb{X}) \otimes_{\widehat{U(\pi^n\mathfrak{g})_K}} M_n \cong \frac{M_n}{\mathfrak{m}_\lambda \cdot M_n}$$

for each $n \in \mathbb{N}$. Furthermore, the composite isomorphism $\Gamma(\mathbb{X}, \mathcal{M}_n)$ is A_n-linear. By [1, Lemma 5.4.6], $\mathfrak{m}_\lambda A$ is a two-sided ideal in A, and so $\mathfrak{m}_\lambda A$ is an A-submodule of M.

Choosing a finite generating set for the $Z(\mathfrak{g}_K)$-ideal $\mathfrak{m}_\lambda$, we have an exact sequence:

$$M^t \to M \to \frac{M}{\mathfrak{m}_\lambda \cdot M} \to 0$$

of A-modules, and since A is Fréchet-Stein by [1, Theorem 5.2.7], it follows from [55, Corollary 3.4(ii)] that $\frac{M}{\mathfrak{m}_\lambda \cdot M}$ is a coadmissible A-module. Hence:

$$\frac{M}{\mathfrak{m}_\lambda \cdot M} \cong \varprojlim A_n \otimes_A (\frac{M}{\mathfrak{m}_\lambda \cdot M}) \cong \frac{M_n}{\mathfrak{m}_\lambda \cdot M_n}.$$

Finally, we see that:

$$\frac{M}{\mathfrak{m}_\lambda \cdot M} \cong \varprojlim \frac{M_n}{\mathfrak{m}_\lambda \cdot M_n} \cong \varprojlim \Gamma(\mathbb{X}, \mathcal{M}_n) \cong \Gamma(\mathbf{X}, \mathcal{M}).$$

$\square$

We now work towards the proof of essential surjectivity. The arguments here are similar to those given in the second half of Section 6.5; we spell them out in detail here for the convenience of the reader. We fix the following notation:

(a) $\mathcal{U} := \{\widehat{\mathbb{Y}}_{\text{rig}} \mid \mathbb{Y} \in \mathcal{S}\}$, a subset of $\mathbf{X}_w(\mathcal{T})$,

(b) $\mathcal{Q}^\lambda_\infty := \widehat{\mathcal{D}}^\lambda(\widehat{\mathbb{Y}}_{\text{rig}}, G)$ for each $\mathbb{Y} \in \mathcal{S}$,

(c) $\mathcal{M}$ is a coadmissible G-equivariant $\mathcal{D}^\lambda$-module on $\mathbf{X}$,

(d) Let m be an integer large enough such that $\lambda(\pi^m \mathfrak{h}) \subseteq \mathcal{R}$. Then the integer $n \geq m$ satisfies $\pi^n \in p^e\mathcal{R}$.

Let $\mathbf{Y} \in \mathcal{U}$. By Lemma 6.3.9, $\mathcal{M}_\mathbf{Y} \in \mathcal{C}^\lambda_{\mathbf{Y}/G}$, so, by Theorem 6.3.5, $\mathcal{M}(\mathbf{Y})$ is a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{Y}, G)$-module and there is an isomorphism:

$$(\text{Loc}^\lambda)^{\widehat{\mathcal{D}}^\lambda(\mathbf{Y},G)}_\mathbf{Y}(\mathcal{M}(\mathbf{Y})) \cong \mathcal{M}_\mathbf{Y}$$

of G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{Y}$.

Lemma 6.6.15. There is a coherent sheaf $\mathcal{M}_n$ of $\mathcal{Q}^\lambda_n$-modules on $\mathbb{X}$, whose value on $\mathbb{Y} \in \mathcal{S}$ is given by:

$$\mathcal{M}_n(\mathbb{Y}) = \mathcal{Q}_n^\lambda(\mathbb{Y}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbb{Y})} \mathcal{M}(\widehat{\mathbb{Y}}_{\mathrm{rig}}).$$

Proof. The given formula is functorial in $\mathbb{Y}$ and so defines a presheaf $\mathcal{M}_n$ of $\mathcal{Q}_n^\lambda$-modules on $\mathcal{S}$. Fix $\mathbb{U}, \mathbb{Y} \in \mathcal{S}$ with $\mathbb{U} \subseteq \mathbb{Y}$. We see there is an isomorphism:

$$\mathcal{M}(\widehat{\mathbb{U}}_{\mathrm{rig}}) \cong \mathcal{Q}_\infty^\lambda(\mathbb{U}) \widehat{\otimes}_{\mathcal{Q}_\infty^\lambda(\mathbb{Y})} \mathcal{M}(\widehat{\mathbb{Y}}_{\mathrm{rig}}).$$

Therefore:

$$\begin{aligned}\mathcal{M}_n(\mathbb{U}) &= \mathcal{Q}_n^\lambda(\mathbb{U}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbb{U})} \mathcal{M}(\widehat{\mathbb{U}}_{\mathrm{rig}}) \\ &\cong \mathcal{Q}_n^\lambda(\mathbb{U}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbb{U})} (\mathcal{Q}_\infty^\lambda(\mathbb{U}) \widehat{\otimes}_{\mathcal{Q}_\infty^\lambda(\mathbb{U})} \mathcal{M}(\widehat{\mathbb{Y}}_{\mathrm{rig}})) \\ &= \mathcal{Q}_n^\lambda(\mathbb{U}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbb{U})} \mathcal{M}(\widehat{\mathbb{Y}}_{\mathrm{rig}}) \\ &= \mathcal{Q}_n^\lambda(\mathbb{U}) \otimes_{\mathcal{Q}_n^\lambda(\mathbb{Y})} \mathcal{M}_n(\mathbb{Y}).\end{aligned}$$

By [1, Lemma 5.4.2], $\widehat{\mathbb{U}}_{\mathrm{rig}}$ is a G-stable affinoid subdomain of $\mathbf{Y} := \widehat{\mathbb{Y}}_{\mathrm{rig}}$. Since $\mathbb{U}$ is a affine open Zariski subscheme of $\mathbb{Y}$, it is a finite union of basic open subsets, so $\widehat{\mathbb{U}}_{\mathrm{rig}}$ is an $\mathcal{L} := \pi^n \widetilde{\mathcal{T}}(\widehat{\mathbb{Y}})$-accessible affinoid subdomain of $\widehat{\mathbb{Y}}_{\mathrm{rig}}$ by Definition 6.5.2.

Recall the definition of the sheaf $\mathcal{Q}_n^\lambda$ from Proposition 6.5.6 as:

$$\mathcal{Q}_n^\lambda := (\widehat{\mathscr{U}(\pi^n \mathcal{L})}_K \rtimes_{N_n} G) \otimes_{\widehat{U(\mathfrak{h})_K}} K_\lambda.$$

The above calculation demonstrates that the presheaf $\mathcal{N} := \mathrm{Loc}_{\mathcal{Q}^\lambda}(\mathcal{M}_n(\mathbb{Y}))$ on $\mathbf{Y}_{\mathrm{ac}}(\mathcal{L}, G)$ is related to the restriction of $\mathcal{M}_n$ to $\mathbb{Y}$ via:

$$\mathcal{N}(\widehat{\mathbb{U}}_{\text{rig}}) \cong \mathcal{M}_n(\mathbb{U}),$$

for all $\mathbb{U} \in \mathcal{S}$ with $\mathbb{U} \subseteq \mathbb{Y}$. Hence $\mathcal{M}_n$ is a sheaf on $\mathcal{S}$ by [1, Corollary 4.3.19]. Since $\mathcal{S}$ is a basis for $\mathbb{X}$, $\mathcal{M}_n$ extends to a sheaf of $\mathcal{Q}_n^\lambda$-modules on $\mathbb{X}$, which is coherent. $\square$

Lemma 6.6.16. There is a $\mathcal{Q}_n^\lambda$-linear isomorphism:

$$\tau_n : \mathcal{Q}_n^\lambda \otimes_{\mathcal{Q}_n^\lambda(\mathbb{X})} (\mathcal{Q}_n^\lambda(\mathbb{X}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbb{X})} \mathcal{M}_{n+1}(\mathbb{X})) \cong \mathcal{Q}_n^\lambda \otimes_{\mathcal{Q}_n^\lambda(\mathbb{X})} \mathcal{M}_n(\mathbb{X}).$$

Proof. This is the same as the proof of Lemma 6.5.12. $\square$

Corollary 6.6.17. The $\mathcal{Q}_\infty^\lambda(\mathbb{X})$-module $M_\infty := \varprojlim \mathcal{M}_n(\mathbb{X})$ is coadmissible.

Proof. Applying Proposition 6.6.11 and Lemma 6.6.16, the maps $\tau_n(\mathbb{X})$ induce $\mathcal{Q}_n^\lambda(\mathbb{X})$-linear isomorphisms:

$$\mathcal{Q}_n^\lambda(\mathbb{X}) \otimes_{\mathcal{Q}_{n+1}^\lambda(\mathbb{X})} \mathcal{M}_{n+1}(\mathbb{X}) \to \mathcal{M}_n(\mathbb{X})$$

for each $n \in \mathbb{N}$. Hence, by Definition 2.4.5, M_∞ is a coadmissible $\varprojlim \mathcal{Q}_n^\lambda(\mathbb{X}) = \mathcal{Q}_\infty^\lambda(\mathbb{X})$-module. $\square$

If $\mathbb{Y} \in \mathcal{S}$ and $\mathbf{Y} := \widehat{\mathbb{Y}}_{\text{rig}}$, then $\mathcal{M}(\mathbf{Y})$ is a coadmissible $\mathcal{Q}_\infty^\lambda(\mathbb{Y}) \cong \widehat{\mathcal{D}}^\lambda(\widehat{\mathbb{Y}}_{\text{rig}}, G)$-module by Proposition 6.5.8(a). It follows that the canonical map:

$$\mathcal{M}(\mathbf{Y}) \to \varprojlim \mathcal{Q}_n^\lambda(\mathbb{Y}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbb{Y})} \mathcal{M}(\mathbf{Y}),$$

is an isomorphism, and we will identify $\mathcal{M}(\mathbf{Y})$ with $\varprojlim \mathcal{Q}_n^\lambda(\mathbb{Y}) \otimes_{\mathcal{Q}_\infty^\lambda(\mathbb{Y})} \mathcal{M}(\mathbf{Y})$.

Lemma 6.6.18. For each $\mathbf{Y} \in \mathcal{U}$, there is a $\mathcal{Q}_\infty^\lambda(\mathbb{X})$-linear map:

$$v_{\mathbf{Y}} : M_\infty \to \mathcal{M}(\mathbb{X})$$

such that $v_{\mathbf{Y}}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = v_{\mathbf{Y}'}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'}$ for all $m \in M_\infty$ and $\mathbf{Y}' \in \mathcal{U}$.

Proof. Let $m := (m_n)_n \in M_\infty$, where $m_n \in \mathcal{M}_n(\mathbb{X})$, and define $v_{\mathbf{Y}}$ by:

$$v_{\mathbf{Y}}(m) := ((m_n)_{\mathbf{Y}})_n.$$

Since the restriction maps in $\mathcal{M}_n$ are $\mathcal{M}_n(\mathbb{X})$-linear, it follows that the map $v_{\mathbf{Y}}$ is $\mathcal{Q}_\infty^\lambda(\mathbb{X})$-linear. Furthermore, we see that:

$$v_{\mathbf{Y}}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}',n} = (m_n \mid_{\mathbf{Y}}) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = m_n \mid_{\mathbf{Y}\cap\mathbf{Y}'} = (m_n \mid_{\mathbf{Y}'}) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = v_{\mathbf{Y}'}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}',n}$$

for all $n \geq 0$. Hence $v_{\mathbf{Y}}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'} = v_{\mathbf{Y}'}(m) \mid_{\mathbf{Y}\cap\mathbf{Y}'}$ for all $m \in M_\infty$. □

Theorem 6.6.19. *The localisation functor* $(Loc^\lambda)_{\mathbf{X}}^A : \mathcal{C}_A \to \mathcal{C}_{\mathbf{X}/G}^\lambda$ *is essentially surjective on objects.*

Proof. Let $\mathcal{M}$ be a coadmissible G-equivariant $\mathcal{D}^\lambda$-module on $\mathbf{X}$. By Corollary 6.6.17, we have constructed a coadmissible $\mathcal{Q}_\infty^\lambda(\mathbb{X})$-module M_∞. We aim to construct an isomorphism of G-equivariant $\mathcal{D}^\lambda$-modules on $\mathcal{U}$:

$$\alpha : (\mathcal{P}^\lambda)_{\mathbf{X}}^A(M_\infty) \mid_{\mathcal{U}} \to \mathcal{M} \mid_{\mathcal{U}} .$$

Let $\mathbb{Y} \in \mathcal{S}$ such that $\mathbf{Y} := \widehat{\mathbb{Y}}_{\text{rig}} \in \mathcal{U}$. By Lemma 6.6.18, we may define:

$$g_{\mathbf{Y}} : \widehat{\mathcal{D}}^{\lambda}(\mathbf{Y}, G)\widehat{\otimes}_A M_\infty \to \mathcal{M}(\mathbf{Y})$$

via $g_{\mathbf{Y}}(s\widehat{\otimes}m) = s \cdot v_{\mathbf{Y}}(m)$. This is a $\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y}, G)$-linear map, and we construct the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{Q}^{\lambda}_{\infty}(\mathbb{Y})\widehat{\otimes}_{\mathcal{Q}^{\lambda}_{\infty}(\mathbb{X})} M_\infty & \xrightarrow{g_{\mathbf{Y}}} & \mathcal{M}(\mathbf{Y}) \\
\downarrow \cong & & \\
\mathcal{Q}^{\lambda}_n \otimes_{\mathcal{Q}^{\lambda}_n(\mathbb{X})} \mathcal{M}_n(\mathbb{X}) & & \downarrow \cong \\
\downarrow \cong & & \\
\varprojlim \mathcal{M}_n(\mathbb{Y}) & \xleftarrow[\cong]{} & \varprojlim \mathcal{Q}^{\lambda}_n(\mathbb{Y}) \otimes_{\mathcal{Q}^{\lambda}_{\infty}(\mathbb{Y})} \mathcal{M}(\mathbf{Y}).
\end{array}$$

By Lemma 6.6.15, $\mathcal{M}_n$ is a coherent $\mathcal{Q}^{\lambda}_n$-module, so the bottom left vertical arrow in the diagram is an isomorphism by Proposition 6.6.11. By definition of $\mathcal{M}_n$, the bottom horizontal arrow is an isomorphism, and the right vertical arrow is an isomorphism by the remarks made before Lemma 6.6.18. It follows that $g_{\mathbf{Y}}$ is an isomorphism.

Now consider the following diagram:

$$\begin{array}{ccc}
(\mathcal{P}^{\lambda})^{A}_{\mathbf{X}}(M_\infty) \mid_{\mathbf{Y}_w} & \xrightarrow{\alpha_{\mathbf{Y}}} & \mathcal{M} \mid_{\mathbf{Y}_w} \\
\downarrow & & \uparrow \\
(\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},G)}_{\mathbf{Y}}(\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y}, G)\widehat{\otimes}_A M_\infty) & \xrightarrow[(\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},G)}_{\mathbf{Y}}(g_{\mathbf{Y}})]{} & (\mathcal{P}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{Y},G)}_{\mathbf{Y}}(\mathcal{M}(\mathbf{Y})).
\end{array}$$

The left vertical arrow is an isomorphism by [1, Proposition 3.5.9], and the right vertical arrow is also an isomorphism by Proposition 6.5.8(c). Since $g_{\mathbf{Y}}$ is an isomorphism, the bottom arrow is an isomorphism by functoriality of $(\mathcal{P}^\lambda)_{\mathbf{Y}}^{\widehat{\mathcal{D}}^\lambda(\mathbf{Y},G)}$, using Proposition 6.3.2. Hence we obtain the G-equivariant $\mathcal{D}^\lambda$-linear isomorphism:

$$\alpha\mid_{\mathbf{Y}_w}: (\mathcal{P}^\lambda)_{\mathbf{X}}^A(M_\infty)\mid_{\mathbf{Y}_w}\to \mathcal{M}\mid_{\mathbf{Y}_w}$$

which makes the diagram commute.

Applying Proposition 6.5.8(a), $\mathcal{M}(\mathbf{U})$ is naturally a coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U},G_{\mathbf{U}})$-module for any $\mathbf{U}\in\mathbf{Y}_w$. Via Proposition 6.2.5, we may identify $(\mathcal{P}^\lambda)_{\mathbf{X}}^A(M_\infty)(\mathbf{U})$ with $\widehat{\mathcal{D}}^\lambda(\mathbf{U},G_{\mathbf{U}})\widehat{\otimes}_{\widehat{\mathcal{D}}^\lambda(\mathbf{X},G_{\mathbf{U}})}M_\infty$, and so we see that the map

$$\alpha\mid_{\mathbf{Y}}(\mathbf{U}): (\mathcal{P}^\lambda)_{\mathbf{X}}^A(M_\infty)\mid_{\mathbf{Y}}(\mathbf{U})\to \mathcal{M}\mid_{\mathbf{Y}}(\mathbf{U})$$

is given by:

$$\alpha_{\mathbf{Y}}(\mathbf{U})(s\widehat{\otimes}m)=s\cdot(v_{\mathbf{Y}}(m)\mid_{\mathbf{U}})$$

for all $s\in\widehat{\mathcal{D}}^\lambda(\mathbf{U},G_{\mathbf{U}})$ and $m\in M_\infty$.

By Lemma 6.6.18, the local isomorphisms $\alpha_{\mathbf{Y}}$ satisfy:

$$\alpha_{\mathbf{Y}}\mid_{\mathbf{Y}\cap\mathbf{Y}'}=\alpha_{\mathbf{Y}'}\mid_{\mathbf{Y}\cap\mathbf{Y}'}$$

for any $\mathbf{Y}, \mathbf{Y}' \in \mathcal{U}$. Since $(\mathrm{Loc}^{\lambda})^{A}_{\mathbf{X}}(M_{\infty})$ and $\mathcal{M}$ are sheaves on $\mathbf{X}$ and $\mathcal{U}$ is an admissible covering of $\mathbf{X}$ by definition, the $\alpha_{\mathbf{Y}}$ patch together to the required isomorphism $\alpha : (\mathrm{Loc}^{\lambda})^{A}_{\mathbf{X}}(M_{\infty}) \to \mathcal{M}$. $\square$

6.7 The proof of the main result

The final section of the thesis will be devoted to the proof of the main theorem, applying the theory built up across the rest of Chapter 6.

Let $\mathbb{G}_0$ be a connected, simply connected, split semisimple affine algebraic group scheme over $\mathcal{R}$ with generic fibre $\mathbb{G} := \mathbb{G}_0 \otimes_{\mathcal{R}} K$. Let G be a p-adic Lie group and $\sigma : G \to \mathbb{G}(K)$ a continuous group homomorphism.

Since $\mathbb{G}$ is semisimple, the centre $Z(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$ is zero. It follows that we may apply Definition 5.3.2 to define the completed skew-group algebra $\widehat{U}(\mathfrak{g}, G)$. Furthermore, whenever G is a compact open subgroup of $\sigma^{-1}\mathbb{G}_0(\mathcal{R})$, defined in Definition 2.8.2, we may also form the completed skew-group algebra $\widehat{\mathcal{D}}(\mathbf{G}_0, G_0)^{\mathbf{G}_0}$. By [1, Proposition 6.4.2], there is a continuous K-algebra isomorphism:

$$\widehat{U}(\mathfrak{g}, H) \to \widehat{\mathcal{D}}(\mathbf{G}_0, H)^{\mathbf{G}_0}$$

for every compact open subgroup H of $\sigma^{-1}\mathcal{G}_0(\mathcal{R})$.

Theorem 6.7.1. *$\widehat{U}(\mathfrak{g}, H)$ acts on $\boldsymbol{X}$ λ-compatibly with G.*

Proof. By Theorem 6.6.8, $\widehat{\mathcal{D}}(\mathbf{G}_0, G_0)^{\mathbf{G}_0}$ acts on $\mathbf{X}$ λ-compatibly with G. Hence we may apply the same proof as in [1, Theorem 6.4.4]. $\square$

Lemma 6.7.2. Let $\mathbf{Y}$ be a smooth rigid analytic variety equipped with a continuous action of a p-adic Lie group H, let $\mathbf{Z}$ be a quasi-compact open subset of $\mathbf{Y}$ and let $\mathcal{M} \in \mathcal{C}^{\lambda}_{\mathbf{Y}/H}$.

(a) The K-vector space $\mathcal{M}(\mathbf{Z})$ carries a canonical K-Fréchet topology such that the restriction maps $\mathcal{M}(\mathbf{Z}) \to \mathcal{M}(\mathbf{U})$ are continuous for any $\mathbf{U} \in \mathbf{Z}_w$.

(b) For any $f : \mathcal{M} \to \mathcal{N}$ in $\mathcal{C}^{\lambda}_{\mathbf{Y}/H}$, the map $f(\mathbf{Z}) : \mathcal{M}(\mathbf{Z}) \to \mathcal{N}(\mathbf{Z})$ is continuous.

Proof. (a) Since $\mathbf{Z}$ is quasi-compact, we may choose a finite $\mathbf{Z}_w(\mathcal{T})$-covering $\mathcal{V}$ of $\mathbf{Z}$. Applying Theorem 6.3.5, the restriction maps $\mathcal{M}(\mathbf{U}) \to \mathcal{M}(\mathbf{V})$ are continuous for any $\mathbf{U} \in \mathbf{Z}_w(\mathcal{T})$ and $\mathbf{V} \in \mathbf{U}_w$. Since $\mathcal{V}$ is finite and $\mathcal{M}(\mathbf{U})$ is a K-Fréchet space for each $\mathbf{U} \in \mathcal{V}$ by Definition 6.3.1. it follows that $\check{H}^0(\mathcal{V}, M)$ is the kernel of a continuous map between two K-Fréchet spaces. Via the isomorphism $\mathcal{M}(\mathbf{Z}) \cong \check{H}^0(\mathcal{V}, M)$, we may equip $\mathcal{M}(\mathbf{Z})$ with a K-Fréchet topology, which is independent of the choice of $\mathcal{V}$.

(b) This is immediate from the construction of the topologies on $\mathcal{M}(\mathbf{Z})$ and $\mathcal{N}(\mathbf{Z})$ in part (a), along with the fact that $f(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \to \mathcal{N}(\mathbf{U})$ is continuous for each $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ by Definition 6.3.1. $\square$

Proposition 6.7.3. Let H be a compact open subgroup of G, let $M \in \mathcal{C}_{\widehat{U}(\mathfrak{g},H)}$ and suppose there is an isomorphism of H-equivariant locally Fréchet $\mathcal{D}^{\lambda}$-modules on $\mathbf{X}$:

$$\alpha : (\mathrm{Loc}^{\lambda})^{\widehat{\mathcal{D}}^{\lambda}(\mathbf{X},H)}_{\mathbf{X}}(M) \to \mathcal{M}.$$

(a) For every $\mathbf{U} \in \mathbf{X}_w$, there is a unique coadmissible $\widehat{U}(\mathfrak{g}, H)$-module structure on $\mathcal{M}(\mathbf{X})$ such that:

(i) $\gamma^H(g) \cdot m = g^{\mathcal{M}}(m)$ for all $g \in H, m \in \mathcal{M}(\mathbf{X})$.

(ii) the topology on $\mathcal{M}(\mathbf{U})$ induced by the coadmissible $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-module structure coincides with the canonical K-Fréchet topology given by Lemma 6.7.2(i)

(iii) the $\widehat{U}(\mathfrak{g}, H)$-action on $\mathcal{M}(\mathbf{X})$ extends the given $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$-action on $\mathcal{M}(\mathbf{X})$.

(b) The $\widehat{U}(\mathfrak{g}, H)$-module structure is independent of the choice of α.

(c) There is an isomorphism of H-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$:

$$\theta : (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},H)}(\mathcal{M}(\mathbf{X})) \to \mathcal{M}$$

whose restriction to $\mathbf{X}_w$ is given by:

$$\theta(\mathbf{U})(s \widehat{\otimes} m) = s \cdot (m \mid_{\mathbf{U}})$$

for any $\mathbf{U} \in \mathbf{X}_w, s \in \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$ and $m \in \mathcal{M}(\mathbf{X})$.

Proof. (a) Since the rigid analytic flag variety $\mathbf{X}$ is quasi-compact, its global sections $\mathcal{M}(\mathbf{X})$ carries a K-Fréchet topology by Lemma 6.7.2(a). Furthermore, we see that

$\mathcal{M}(\mathbf{X})$ is a $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$-module by Proposition 2.7.7, and so may also be viewed as a $U(\mathfrak{g}) \rtimes H$-module.

Set $\mathcal{N} := (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},H)}(\mathcal{M})$ and let $\psi : M \to \mathcal{M}(\mathbf{X})$ be the composition of the canonical map $M \to \mathcal{N}(\mathbf{X})$ and $\alpha(\mathbf{X}) : \mathcal{N}(\mathbf{X}) \to \mathcal{M}(\mathbf{X})$. Since the maps $M \to \mathcal{M}(\mathbf{U})$ are continuous for each $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, we see that ψ is continuous. It is also $U(\mathfrak{g}) \rtimes H$-linear by construction.

Since $U(\mathfrak{g}) \rtimes H$ is dense in $\widehat{U}(\mathfrak{g}, H)$, we may apply [1, Lemma 4.4.4] to ψ shows that the $U(\mathfrak{g}) \rtimes H$-action on $\mathcal{M}(\mathbf{X})$ extends to a $U(\mathfrak{g}, H)$-action which satisfies the required properties. The general case follows by applying the argument in the proof of Proposition 6.3.9.

(b) This follows from Lemma 6.5.8(b).

(c) After replacing $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, H)$ with $\widehat{U}(\mathfrak{g}, H)$, this is the same proof as that for Lemma 6.5.8(c). □

Proposition 6.7.4. Let $\mathcal{M}$ be a coadmissible G-equivariant $\mathcal{D}^\lambda$-module on $\mathbf{X}$. Then $\mathcal{M}(\mathbf{X})$ is a coadmissible $\widehat{U}(\mathfrak{g}, G)$-module, and there is an isomorphism:

$$(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}(\mathcal{M}(\mathbf{X})) \to \mathcal{M}$$

of G-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$.

Proof. By Theorem 6.7.1, $\widehat{U}(\mathfrak{g}, G)$ acts on $\mathbf{X}$ λ-compatibly with G, so the twisted localisation functor $(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}$ is well-defined by Definition 6.2.8 and sends coadmissible $\widehat{U}(\mathfrak{g}, G)$-modules to coadmissible G-equivariant $\mathcal{D}^\lambda$-modules on $\mathbf{X}$ by Proposition 6.3.4.

Fix a compact open subgroup H of $\sigma^{-1}\mathbb{G}_0(R)$. Then $M \in \mathcal{C}^\lambda_{\mathbf{X}/H}$ by Proposition 6.3.9, and so there is a coadmissible $\widehat{\mathcal{D}}(\mathbf{G}_0, H)^{\mathbf{G}_0}$-module M_∞ and an isomorphism:

$$(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{\mathcal{D}}(\mathbf{G}_0,H)^{\mathbf{G}_0}}(M_\infty) \to \mathcal{M}$$

in $\mathcal{C}^\lambda_{\mathbf{X}/H}$ by Theorem 6.6.19. By [1, Proposition 6.4.2], there is a continuous K-algebra isomorphism:

$$\kappa_H : \widehat{U}(\mathfrak{g}, H) \to \widehat{\mathcal{D}}(\mathbf{G}_0, H)^{\mathbf{G}_0}.$$

Hence we may find a coadmissible $\widehat{U}(\mathfrak{g}, H)$-module M and an isomorphism:

$$(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},H)}(\mathcal{M}(\mathbf{X})) \to \mathcal{M}$$

in $\mathcal{C}^\lambda_{\mathbf{X}/H}$. By Proposition 6.7.3(a), the $U(\mathfrak{g}) \rtimes H$-module structure on $\mathcal{M}(\mathbf{X})$ and the canonical topology on $\mathcal{M}(\mathbf{X})$ given by Lemma 6.7.2(a) extend to a coadmissible $\widehat{U}(\mathfrak{g}, H)$-structure. There is also an isomorphism in $\mathcal{C}^\lambda_{\mathbf{X}/H}$:

$$\theta : (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},H)}(\mathcal{M}(\mathbf{X})) \to \mathcal{M}$$

whose restriction to $\mathbf{X}_w$ is given by:

$$\theta(\mathbf{U})(s\widehat{\otimes}m) = s \cdot (m\mid_{\mathbf{U}})$$

for any $\mathbf{U} \in \mathbf{X}_w$, $s \in \widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$ and $m \in \mathcal{M}(\mathbf{X})$.

By Proposition 2.7.7, $\mathcal{M}(\mathbf{X})$ is a $\mathcal{D}^\lambda(\mathbf{X}) \rtimes H$-module and hence also a $U(\mathfrak{g}) \rtimes H$-module. The restriction of this $U(\mathfrak{g}) \rtimes G$-module back to $U(\mathfrak{g}) \rtimes H$ agrees with the restriction of the $\widehat{U}(\mathfrak{g}, H)$-module to $U(\mathfrak{g}) \rtimes H$. It follows that the $U(\mathfrak{g}) \rtimes H$- and $K[G]$-actions on $\mathcal{M}(\mathbf{X})$ extend to an action of $\widehat{U}(\mathfrak{g}, G)$. Since the restriction of $\mathcal{M}(\mathbf{X})$ back to $\widehat{U}(\mathfrak{g}, H)$ is coadmissible, $\mathcal{M}(\mathbf{X})$ is a coadmissible $\widehat{U}(\mathfrak{g}, G)$-module. By construction, $(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},H)}(M) = (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}(M)$ as H-equivariant locally Fréchet $\mathcal{D}^\lambda$-modules on $\mathbf{X}$. Finally, the same argument as that used in the proof of Theorem 6.3.5 shows that θ is G-equivariant. $\square$

Theorem 6.7.5. *Let $\mathbb{G}$ be a connected, simply connected, split semisimple affine algebraic group scheme over K, and let $\boldsymbol{X}$ be the rigid analytification of the flag variety of $\mathbb{G}$. Let G be a p-adic Lie group and let $\sigma : G \to \mathbb{G}(K)$ be a continuous group homomorphism. Let λ be a ρ-dominant ρ-regular weight of $\mathfrak{h}_K$. Then the localisation functor:*

$$(Loc^\lambda)_{\boldsymbol{X}}^{\widehat{U}(\mathfrak{g},G)} : \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)} \to \mathcal{C}^\lambda_{\boldsymbol{X}/G}$$

is an equivalence of categories.

Proof. By the classification of split semisimple algebraic groups, [38, II.1] implies that $\mathbb{G}$ extends to an affine algebraic group scheme $\mathbb{G}_0$ over $\mathcal{R}$ satisfying the conditions outlined at the beginning of this section, such that $\mathbb{G} = \mathbb{G}_0 \otimes_{\mathcal{R}} K$ is the generic fibre of $\mathbb{G}_0$. Applying Proposition 6.7.4, it suffices to show that the restriction of $(\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}$ to the full subcategory of $\mathcal{C}_{\widehat{U}(\mathfrak{g},G)}$ consisting of objects killed by $\mathfrak{m}_\lambda$ is fully faithful.

Let $M, N \in \mathcal{C}^\lambda_{\widehat{U}(\mathfrak{g},G)}$ and write $\mathrm{Loc}^\lambda := (\mathrm{Loc}^\lambda)_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}$, $\mathcal{M} := \mathrm{Loc}^\lambda(M)$ and $\mathcal{N} := \mathrm{Loc}^\lambda(N)$. Given a $\widehat{U}(\mathfrak{g},G)$-linear morphism $f : M \to N$, we have the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathbf{X}) & \xrightarrow[\mathrm{Loc}^\lambda(f)(\mathbf{X})]{} & \mathcal{N}(\mathbf{X}). \end{array}$$

Since the vertical arrows are bijective by Theorem 6.6.14, it is immediate that Loc^λ is faithful.

Now suppose $\alpha : \mathrm{Loc}^\lambda(M) \to \mathrm{Loc}^\lambda(N)$ is a morphism in $\mathcal{C}^\lambda_{\mathbf{X}/G}$. Since the rigid analytic flag variety $\mathbf{X}$ is quasi-compact, the morphism $\alpha(\mathbf{X}) : \mathcal{M}(\mathbf{X}) \to \mathcal{N}(\mathbf{X})$ is continuous by Lemma 6.7.2(b). Applying Proposition 2.7.7, we further see that $\alpha(\mathbf{X})$ is $U(\mathfrak{g}) \rtimes G$-linear, so it must also be $\widehat{U}(\mathfrak{g},G)$-linear since $\mathcal{M}(\mathbf{X})$ and $\mathcal{N}(\mathbf{X})$ are both coadmissible $\widehat{U}(\mathfrak{g},G)$-modules by Proposition 6.7.4. Hence $\alpha(\mathbf{X})$ is $\widehat{U}(\mathfrak{g},G)$-linear.

Define $f : M \to N$ by the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathbf{X}) & \xrightarrow{\alpha(\mathbf{X})} & \mathcal{N}(\mathbf{X}). \end{array}$$

We see that f is $\widehat{\mathcal{D}}^\lambda(\mathbf{X}, G)$-linear.

We claim that $\mathrm{Loc}^\lambda(f) = \alpha$. To see this, let $\mathbf{U} \in \mathbf{X}_w$ and choose a compact open subgroup H of G such that $(\mathbf{U}, H)$ is small by Lemma 2.8.15. By construction, $\alpha(\mathbf{U})$ and $\mathrm{Loc}^\lambda(f)(\mathbf{U})$ agree on the image of M in $\mathcal{M}(\mathbf{U})$. Since both maps are $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-linear and since this image generates $\mathcal{M}(\mathbf{U})$ as a $\widehat{\mathcal{D}}^\lambda(\mathbf{U}, H)$-module, it follows that $\alpha(\mathbf{U}) = \mathrm{Loc}^\lambda(f)(\mathbf{U})$. Since $\mathcal{M}$ and $\mathcal{N}$ are sheaves and $\mathbf{X}_w(\mathcal{T})$ is a basis for $\mathbf{X}$, it follows that $\mathrm{Loc}^\lambda(f) = \alpha$. □

Proof of Theorem F: This is immediate from Theorem 6.7.5.

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