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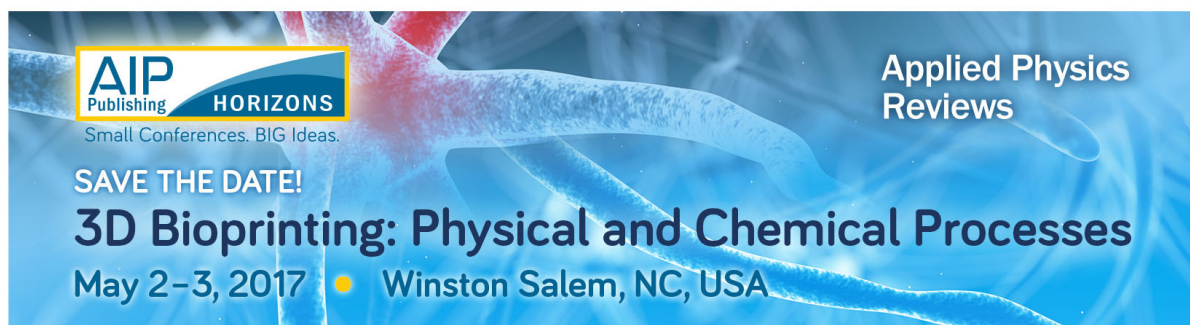
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# The structure of the collisionless transient pinch

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The mechanism of the transient pinch at low densities, outlined in an early paper by Rosenbluth, has been studied in detail. The collapse velocity is comparable to the Alfvén velocity although the theory of magnetohydrodynamics is not applicable to collision-free plasmas. The thickness of the surface current layer is found to be the electron inertial length ( $c/\omega_{pe}$ ). The electron and ion trajectories have been calculated, the latter being essentially due to the electrostatic field which transfers the  $\mathbf{j} \times \mathbf{B}$  force from the electrons to the positive ions. *Published by AIP Publishing.*

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## I. INTRODUCTION

Early work on fusion research included experiments on fast linear pinches or z-pinches.<sup>1</sup> In the present paper, the transient case is considered for the case where collisions may be neglected. At higher densities, shock waves are formed, as described by Allen.<sup>2</sup> The collisionless case has been described, albeit briefly, in a little known paper by Rosenbluth.<sup>3</sup> To analyse this case, it is convenient to move into a reference frame moving with the current layer, plane geometry being employed in the calculation. It is found that the positive ions are reflected in the narrow surface layer by an electrostatic field and return to the field free plasma having moved in trajectories which hardly deviate from straight lines. The electrons perform trajectories within the layer before returning to the field-free plasma. Such electron motion produces the surface current associated with the  $\mathbf{j} \times \mathbf{B}$  force. It is found that this force is transmitted to the positive ions via an electrostatic field as in the case of the steady-state Bennett pinch,<sup>4</sup> which has been revisited by Allen.<sup>5</sup> The characteristic width of the layer was the same as that found by Adlam and Allen<sup>6</sup> in their study of strong collision-free hydrodynamic waves, namely, ( $c/\omega_{pe}$ ), the electron skin depth or electron inertial length.

## II. THEORETICAL MODEL

A schematic diagram of the boundary layer of the plasma is illustrated in Figure 1, in a reference frame moving with the layer. A collisionless plasma is considered, and the theory has been carried out for the plane case, since the layer is shown to be narrow, of the order of the electron collision-free depth or electron inertial length. The calculations have been carried out using the concept of quasineutrality, i.e., the electron and ion densities are taken to be equal but the infinitesimal difference in their densities is sufficient to produce appreciable electric fields. The method has a long history dating back to Tonks and Langmuir.

Let us consider the electrons and ions moving from left to right in the diagram. The Eulerian description will be employed (in which the coordinates are fixed in space). The

$x$  and  $y$  components of velocity are denoted by  $u$  and  $v$ , respectively. The equations of motion for the electrons are

$$u \frac{du}{dx} = -\frac{e}{m_e} (E_x + v_e B_z), \quad (1)$$

$$u \frac{dv_e}{dx} = \frac{e}{m_e} u B_z. \quad (2)$$

The corresponding equations for the ions are

$$u \frac{du}{dx} = \frac{e}{m_i} (E_x + v_i B_z), \quad (3)$$

$$u \frac{dv_i}{dx} = -\frac{e}{m_i} u B_z. \quad (4)$$

The velocity components in the  $x$  direction are the same for electrons and ions since quasineutrality has been assumed.

In this study, only one of Maxwell's equations is employed, namely, Ampère's Law. The current involved, however, is that due to particles both entering and leaving the current layer. Hence, Ampère's Law becomes the following, containing a factor of two:

$$\frac{dB_z}{dx} = 2\mu_0 ne(v_e - v_i). \quad (5)$$

The final equation is the equation of continuity

$$n_0 u_0 = nu. \quad (6)$$

Elimination of  $E_x$  between (1) and (3) gives

$$(m_e + m_i)u \frac{du}{dx} = -evB_z, \quad (7)$$

where we have introduced the relative velocity  $v = v_e - v_i$ .

Equations (2) and (4) yield

$$\frac{dv}{dx} = \left( \frac{m_e + m_i}{m_e m_i} \right) e B_z. \quad (8)$$

Equations (6) and (5) give

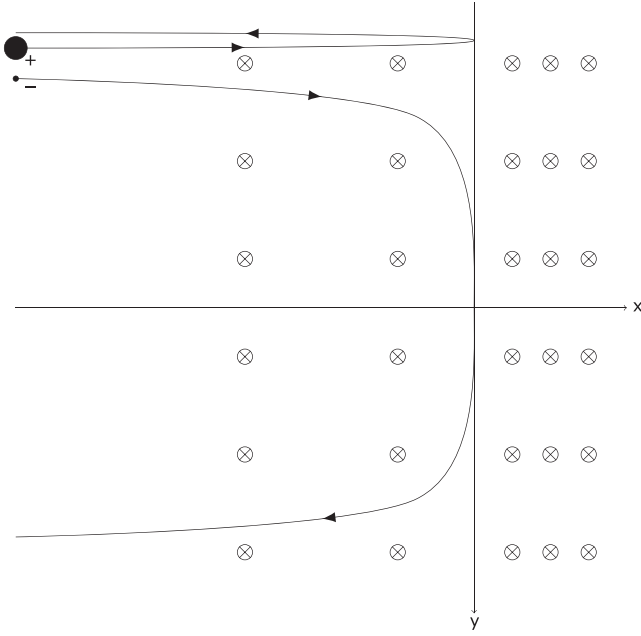


FIG. 1. Illustration of electron and ion trajectories in the rest frame of the plasma boundary. On this scale, the  $z$ -pinch boundary ( $x=0$ ) appears planar and moves in the negative  $x$  direction in the laboratory frame with speed  $u_0$ . The “azimuthal” magnetic field is now in the  $z$  direction, and the skin current is in the negative  $y$  direction.

$$u \frac{dB_z}{dx} = 2\mu_0 n_0 u_0 e v. \quad (9)$$

We now have three simultaneous equations for  $u$ ,  $v$ , and  $B_z$ .

### III. SOLUTION OF THE EQUATIONS

The following quantities will be used to normalize the equations:

$$d = \sqrt{\frac{m_e m_i}{n_0 \mu_0 e^2 (m_e + m_i)}}, \quad (10)$$

$$v_A = \sqrt{\frac{B_0^2}{n_0 \mu_0 (m_e + m_i)}}, \quad (11)$$

$$v_* = \sqrt{\frac{B_0^2 (m_e + m_i)}{n_0 \mu_0 m_e m_i}}, \quad (12)$$

where we take  $V \equiv \frac{v}{v_*}$ ,  $U \equiv \frac{u}{v_A}$ ,  $X \equiv \frac{x}{d}$ , and  $\beta \equiv \frac{B_z}{B_0}$ .  $B_0$  is the magnetic field at the edge of the boundary layer, which can be seen at  $x=0$  in Figure 1 and  $d \sim c/\omega_{pe}$ , the aforementioned electron skin depth.

Different units of normalization are used for the velocities in the  $x$  and  $y$  directions, since the important  $y$  motion is performed by the electrons and is much faster than the  $x$  motion. After normalization, the Equations (7)–(9) become

$$U \frac{dU}{dX} = -V\beta, \quad (13)$$

$$\frac{dV}{dX} = \beta, \quad (14)$$

$$U \frac{d\beta}{dX} = 2\alpha V. \quad (15)$$

The quantity  $\alpha = \frac{u_0}{v_A}$ , and it will be demonstrated that  $\alpha = \frac{1}{2}$ . Eliminating  $V$  between (13) and (15) and integrating the subsequent equation (with the initial condition  $U = \alpha$ ,  $\beta = 0$ ) gives us

$$U = \alpha - \frac{\beta^2}{4\alpha}. \quad (16)$$

The particles turn back at  $\beta = 1$ , and at this point  $U = 0$ ; thus, substituting these values into (16) yields  $\alpha = \frac{1}{2}$ . Eliminating  $X$  between (13) and (14) and integrating gives

$$V = \beta \sqrt{\frac{1}{2} \left( 1 - \frac{\beta^2}{2} \right)}. \quad (17)$$

Substituting (16) and (17) into (15) yields

$$\frac{d\beta}{dX} = \frac{\beta \sqrt{2 - \beta^2}}{1 - \beta^2}. \quad (18)$$

Inversion and integration then gives an equation describing the magnetic field profile, albeit in an implicit form

$$X = \sqrt{2 - \beta^2} - 1 + \frac{1}{\sqrt{2}} \ln \left( \frac{(2 + \sqrt{2})\beta}{2 + \sqrt{4 - 2\beta^2}} \right). \quad (19)$$

The resulting magnetic field is plotted in Figure 2, together with the relative velocity, electric field, and particle density.

### IV. PARTICLE TRAJECTORIES

To fully characterise the problem, we now calculate the particle trajectories. Since the  $x$  velocity is the same for both species and the  $y$  velocity differs only by a scaling

$$V_e = -\frac{m_i}{m_e} V_i, \quad (20)$$

we choose to solve for the electron trajectories, from which the ion trajectories will follow. We note now that due to the large difference in particle masses, the ion deflection is negligible when compared with that of the electrons. The electrons gain appreciable energy in the layer from their movement in the electrostatic field, and this energy is derived from the kinetic energy of the ions as they are slowed down.

#### A. Boundary layer frame

Normalizing  $y_e$  with  $d$  so that it is on the same scale as the  $x$  axis, our first step to finding the electron trajectory,  $Y_e(X)$ , is

$$\frac{dY_e}{dX} = \frac{(m_e + m_i)}{\sqrt{m_e m_i}} \frac{V_e}{U}. \quad (21)$$

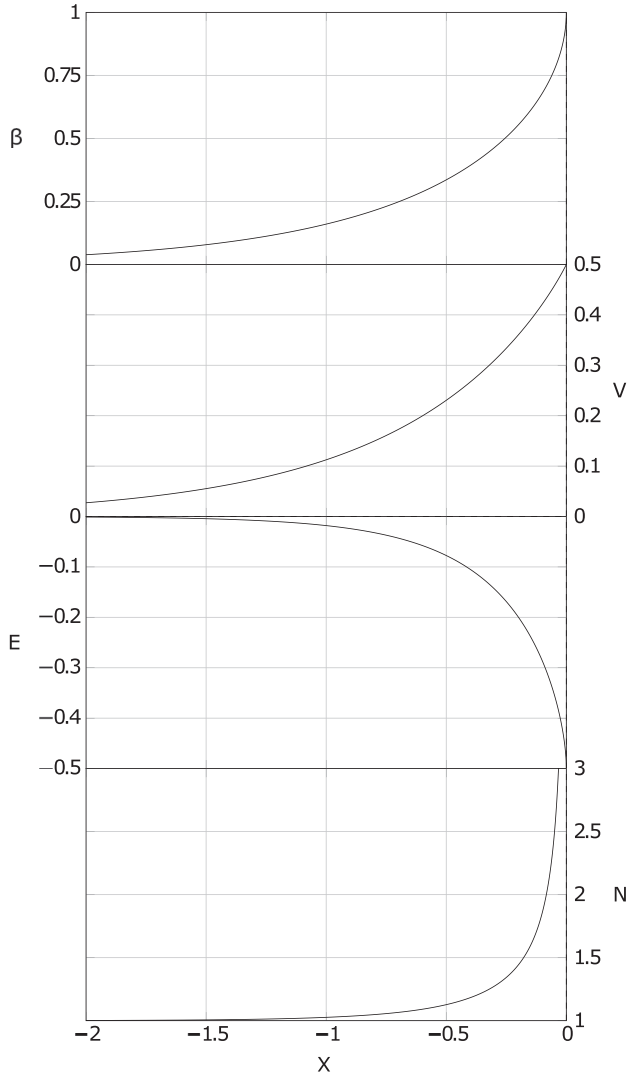


FIG. 2. Profiles of normalized magnetic field, relative transverse velocity, electric field, and density in the boundary layer. The electric field is normalized by  $v \times B_0$  and the density by  $n_0$ .

We use Equation (20) to write (15) in terms of the electron transverse motion, instead of the relative motion

$$V_e = \frac{m_i U}{2\alpha(m_i + m_e)} \frac{d\beta}{dX}, \quad (22)$$

so (21) and (22) give

$$Y_e = \sqrt{\frac{m_i}{m_e}} \beta \quad (23)$$

for incoming electrons starting out along the  $x$  axis and

$$Y_e = \sqrt{\frac{m_i}{m_e}} (2 - \beta) \quad (24)$$

for outgoing electrons. An electron trajectory is plotted in Figure 3.

## B. Laboratory frame

For completeness, we derive the laboratory frame trajectories, to start with, the lab and boundary layer coordinates are related by

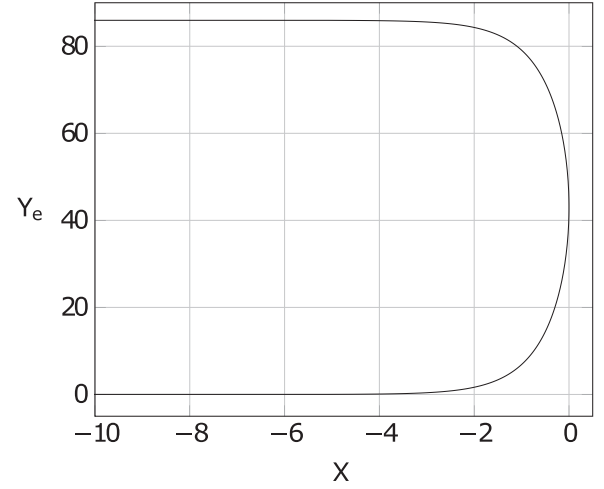


FIG. 3. An electron trajectory in the rest frame of the boundary layer. The plot is for an electron-proton plasma,  $m_i = 1836 m_e$ .

$$\frac{dX_L}{dX} = \frac{U_L}{U} = \frac{U - \frac{1}{2}}{U}, \quad (25)$$

so that

$$\frac{dX_L}{d\beta} = \frac{dX_L}{dX} \frac{dX}{d\beta} = \frac{U - \frac{1}{2}}{U} \frac{1 - \beta^2}{\beta \sqrt{2 - \beta^2}}, \quad (26)$$

where we have used (18). Now,

$$U = \pm \left( \frac{1}{2} - \frac{\beta^2}{2} \right) \quad (27)$$

depending on whether the particles are inbound (+) or outbound (−) in the boundary layer rest frame. For inbound particles

$$\frac{dX_L}{d\beta} = - \frac{\beta}{\sqrt{2 - \beta^2}}, \quad (28)$$

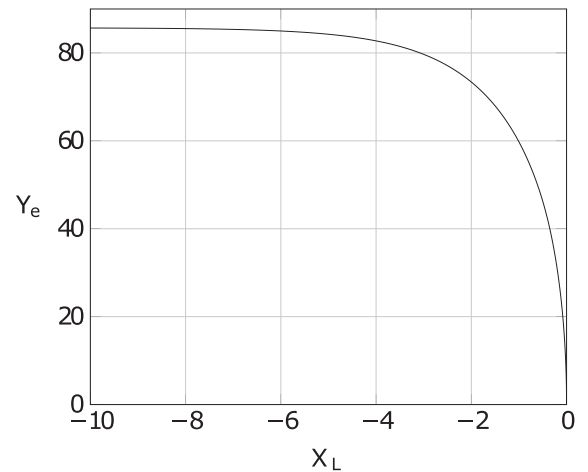


FIG. 4. An electron trajectory in the laboratory frame;  $X_L$  is measured from the original position of the electron. The plot is for an electron-proton plasma,  $m_i = 1836 m_e$ .

and so, integrating this and eliminating  $\beta$  between the result and (23) gives

$$Y_e = \sqrt{\frac{m_i}{m_e}} \sqrt{2 - (X_L + \sqrt{2})^2}, \quad (29)$$

which applies to the range  $1 - \sqrt{2} < X_L < 0$ . For outbound particles

$$\frac{dX_L}{d\beta} = \frac{\sqrt{2 - \beta^2}}{\beta}. \quad (30)$$

Integration, followed by substitution from (24), gives an implicit form of the trajectory

$$X_L = \sqrt{2 - \left(2 - \sqrt{\frac{m_e}{m_i}} Y_e\right)^2} - \sqrt{2} + \sqrt{2} \log \left( \frac{(\sqrt{2} + 2) \left(2 - \sqrt{\frac{m_e}{m_i}} Y_e\right)}{\sqrt{4 - 2 \left(2 - \sqrt{\frac{m_e}{m_i}} Y_e\right)^2} + 2} \right), \quad (31)$$

which is valid for  $X_L < 1 - \sqrt{2}$ . Joining the functions (29) and (31) together, an electron trajectory in the laboratory frame of reference is plotted in Figure 4.

## V. CONCLUSIONS

The basic mechanism of the transient pinch, as suggested by Rosenbluth in a conference paper,<sup>3</sup> has been examined in detail. The calculation is for a cold collision-free plasma, and the theory of magnetohydrodynamics (MHD) is not applicable, although the collapse velocity is of the order of the Alfvén velocity. The equations have been solved in a coordinate system moving with the plasma boundary. The particle trajectories have been calculated first in this frame of reference and then in the laboratory frame. The magnetic field is screened out within a narrow surface layer of width  $(c/\omega_{pe})$ , the electron skin depth, or electron inertial length. This characteristic length is the same as that found by Adlam and Allen<sup>6</sup> in their study of strong solitary hydrodynamic waves propagating across a magnetic field. The width of the current carrying layer is much smaller than the ion Larmor radius (initial value on entry to the layer).

<sup>1</sup>C. Braams and P. Stott, *Nuclear Fusion: Half a Century of Magnetic Confinement Research* (IOP Publishing, 2002), Chap. 2.

<sup>2</sup>J. E. Allen, *Proc. Phys. Soc., Sect. B* **70**, 24 (1957).

<sup>3</sup>M. Rosenbluth, *Magnetohydrodynamics* (Stanford University Press, 1957), p. 57.

<sup>4</sup>W. H. Bennett, *Phys. Rev.* **45**, 890 (1934).

<sup>5</sup>J. E. Allen, in *37th E.P.S. Conference on Plasma Physics, E.C.A.* (2010), Vol. 34A, p. 4.181.

<sup>6</sup>J. H. Adlam and J. E. Allen, *Philos. Mag.* **3**, 448 (1958).