

Connectivity for bridge-alterable graph classes

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Abstract

A collection \mathcal{A} of graphs is called bridge-alterable if, for each graph G with a bridge e , G is in \mathcal{A} if and only if $G - e$ is. For example the class \mathcal{F} of forests is bridge-alterable. For a random forest F_n sampled uniformly from the set \mathcal{F}_n of forests on vertex set $\{1, \dots, n\}$, a classical result of Rényi (1959) shows that the probability that F_n is connected is $e^{-\frac{1}{2} + o(1)}$.

Recently Addario-Berry, McDiarmid and Reed (2012) and Kang and Panagiotou (2013) independently proved that, given a bridge-alterable class \mathcal{A} , for a random graph R_n sampled uniformly from the graphs in \mathcal{A} on $\{1, \dots, n\}$, the probability that R_n is connected is at least $e^{-\frac{1}{2} + o(1)}$. Here we give a more straightforward proof, and obtain a stronger non-asymptotic form of this result, which compares the probability to that for a random forest. We see that the probability that R_n is connected is at least the minimum over $\frac{2}{5}n < t \leq n$ of the probability that F_t is connected.

Keywords: random graph, connectivity, bridge-addable, bridge-alterable

1 Introduction

A collection \mathcal{A} of graphs is *bridge-addable* if for each graph G in \mathcal{A} and pair of vertices u and v in different components, the graph $G + uv$ obtained by adding the edge (bridge) uv is also in \mathcal{A} ; that is, if \mathcal{A} is closed under adding bridges. This property was introduced in [9] (under the name ‘weakly addable’). If also \mathcal{A} is closed under deleting bridges we call \mathcal{A} *bridge-alterable*. Thus \mathcal{A} is bridge-alterable exactly when, for each graph G with a bridge e , G is in \mathcal{A} if and only if $G - e$ is in \mathcal{A} . The class \mathcal{F} of forests

is bridge-alterable, as for example is the class of series-parallel graphs, the class of planar graphs, and indeed the class of graphs embeddable in any given surface. All natural examples of bridge-addable classes seem to satisfy the stronger condition of being bridge-alterable.

Given a class \mathcal{A} of graphs we let \mathcal{A}_n denote the set of graphs in \mathcal{A} on vertex set $[n] := \{1, \dots, n\}$. Also, we use the notation $R_n \in_u \mathcal{A}$ to mean that R_n is a random graph sampled uniformly from \mathcal{A}_n (where we assume implicitly that \mathcal{A}_n is non-empty).

For a random forest $F_n \in_u \mathcal{F}$, a classical result of Rényi [13] from 1959 shows that, as $n \rightarrow \infty$

$$\mathbb{P}(F_n \text{ is connected}) = e^{-\frac{1}{2} + o(1)}. \quad (1)$$

In their investigations on random planar graphs, McDiarmid, Steger and Welsh [9] showed that, when \mathcal{A} is bridge-addable, for $R_n \in_u \mathcal{A}$

$$\mathbb{P}(R_n \text{ is connected}) \geq e^{-1}. \quad (2)$$

It was observed by the same authors [10] in 2006 that the class of forests seems to be the ‘least connected’ bridge-addable class of graphs, and they made the following conjecture.

Conjecture 1.1. *When \mathcal{A} is bridge-addable, for $R_n \in_u \mathcal{A}$*

$$\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{1}{2} + o(1)}.$$

This conjecture was then strengthened (see Conjecture 1.2 of [3], Conjecture 5.1 of [1], or Conjecture 6.2 of [8]) to the following non-asymptotic form.

Conjecture 1.2. *When \mathcal{A} is bridge-addable, for $R_n \in_u \mathcal{A}$*

$$\mathbb{P}(R_n \text{ is connected}) \geq \mathbb{P}(F_n \text{ is connected}).$$

Early progress was made on Conjecture 1.1 by Balister, Bollobás and Gerke [2, 3]; and recently Norin [12] made further progress, showing that $\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{2}{3} + o(1)}$. Addario-Berry, McDiarmid and Reed (2012) and Kang and Panagiotou (2013) independently proved the following theorem, which establishes the special case of Conjecture 1.1 when \mathcal{A} is bridge-alterable.

Theorem 1.3. *[1, 6] Let \mathcal{A} be a bridge-alterable class of graphs, and let $R_n \in_u \mathcal{A}$. Then*

$$\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{1}{2} + o(1)}.$$

Here we give a reasonably short and straightforward proof of the following non-asymptotic form of this result, which together with (1) gives Theorem 1.3. This is a first step towards Conjecture 1.2, at least for a bridge-alterable class.

Theorem 1.4. *Let \mathcal{A} be a bridge-alterable class of graphs, let n be a positive integer, let $R_n \in_u \mathcal{A}$, and let $F_t \in_u \mathcal{F}$ for $t = 1, 2, \dots$. Let $\alpha = 0.4$. Then*

$$\mathbb{P}(R_n \text{ is connected}) \geq \min_{\alpha n \leq t \leq n} \mathbb{P}(F_t \text{ is connected}). \quad (3)$$

The value $\alpha = 0.4$ can be increased towards $\frac{1}{2}$: in the final section of the paper we improve it to $0.48n$, and discuss pushing it up further to $\frac{1}{2}$. Conjecture 1.2 says that we can push α up to 1.

Since this paper was (essentially) completed, the original Conjecture 1.1 (for bridge-addable rather than bridge-alterable classes) has been fully proved by Chapuy and Perarnau, see [4].

2 Proof of Theorem 1.4

We use two lemmas in the proof.

Lemma 2.1. *Let \mathcal{A} be a bridge-alterable class of graphs, let n be a positive integer, let $R_n \in_u \mathcal{A}$, and let $F_t \in_u \mathcal{F}$ for $t = 1, 2, \dots$. Then*

$$\mathbb{P}(R_n \text{ is connected}) \geq \min_{t=1, \dots, n} \max\{e^{-\frac{t}{n}}, \mathbb{P}(F_t \text{ is connected})\}. \quad (4)$$

Lemma 2.2. *Let $\alpha = 0.4$. For each $n = 2, 3, \dots$*

$$\mathbb{P}(F_n \text{ is connected}) < e^{-\alpha}.$$

To deduce Theorem 1.4 from these lemmas, observe that by Lemma 2.2, for each $1 \leq t \leq \alpha n$

$$e^{-\frac{t}{n}} \geq e^{-\alpha} \geq \mathbb{P}(F_n \text{ is connected}),$$

and so the right side in (4) is at least the right side in (3).

Proof of Lemma 2.1 Our proof initially follows the lines of the proofs in [1] and [6] of Theorem 1.3, in that we aim to lower bound the probability of connectedness for the random graph F^n introduced below. Consider a fixed $n \geq 2$.

Given a graph G , let $b(G)$ be the graph obtained by removing all bridges from G . We say G and G' are *equivalent* if $b(G) = b(G')$. This is an

equivalence relation on graphs, and if a graph G is in \mathcal{A}_n then so is the whole equivalence class $[G]$. Thus \mathcal{A}_n is a union of disjoint equivalence classes. To prove the lemma we consider an arbitrary (fixed) equivalence class.

Fix a bridgeless graph G on vertex set $[n]$ and let $\mathcal{B} = [G]$. Let G have t components, with n_1, \dots, n_t vertices, where $n = \sum_{i=1}^t n_i$. We use $\mathbf{n} = (n_1, \dots, n_t)$ to define probabilities. First, given a forest $F \in \mathcal{F}_t$, let

$$\text{mass}(F) = \prod_{i=1}^t n_i^{d_F(i)},$$

where $d_F(i)$ denotes the degree of vertex i in F . For $\mathcal{F}' \subseteq \mathcal{F}_t$ let $\text{mass}(\mathcal{F}') = \sum_{F \in \mathcal{F}'} \text{mass}(F)$. Now let

$$\mathbb{P}(F^{\mathbf{n}} = F) = \frac{\text{mass}(F)}{\text{mass}(\mathcal{F}_t)} \quad \text{for each } F \in \mathcal{F}_t.$$

By Lemma 2.3 of [1], for a uniformly random element $R^{\mathcal{B}}$ of \mathcal{B} ,

$$\mathbb{P}(R^{\mathcal{B}} \text{ is connected}) = \mathbb{P}(F^{\mathbf{n}} \text{ is connected}).$$

Hence to prove the lemma it suffices to consider $F^{\mathbf{n}}$, and show that

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) \geq \max\{e^{-\frac{t}{n}}, \mathbb{P}(F_t \text{ is connected})\}. \quad (5)$$

To see this, observe that then the probability that R_n is connected is an average of values each at least the right side of (5) for some t , and so it is at least the right side in (4).

The proof of (5) breaks into two parts, and the first is standard. Given a graph G , let $\kappa(G)$ denote the number of components. By Lemma 3.2 of [1], for $i = 1, \dots, t-1$

$$\mathbb{P}(\kappa(F^{\mathbf{n}}) = i+1) \leq \frac{1}{i} \frac{t}{n} \mathbb{P}(\kappa(F^{\mathbf{n}}) = i),$$

and thus

$$\mathbb{P}(\kappa(F^{\mathbf{n}}) = i+1) \leq \frac{1}{i!} \left(\frac{t}{n}\right)^i \mathbb{P}(\kappa(F^{\mathbf{n}}) = 1).$$

Hence

$$1 = \sum_{i=0}^{t-1} \mathbb{P}(\kappa(F^{\mathbf{n}}) = i+1) \leq \sum_{i=0}^{t-1} \frac{1}{i!} \left(\frac{t}{n}\right)^i \mathbb{P}(\kappa(F^{\mathbf{n}}) = 1) < e^{\frac{t}{n}} \cdot \mathbb{P}(\kappa(F^{\mathbf{n}}) = 1)$$

and so $\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) > e^{-\frac{t}{n}}$ (as noted at the end of Section 3 of [1]).

It remains to show that

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) \geq \mathbb{P}(F_t \text{ is connected}). \quad (6)$$

We may assume that $t \geq 2$. Let \mathcal{T} be the class of trees. Then

$$\text{mass}(\mathcal{T}_t) = \prod_{i=1}^t n_i \cdot n^{t-2}. \quad (7)$$

This result is proved for example in [1] (see the proof of Lemma 4.2) and in [6], though in fact it has long been known, see Theorem 6.1 of Moon [11] (1970), and see also Problems 5.3 and 5.4 of Lovász [7]. We let $N = \prod_{i=1}^t n_i$ and rewrite (7) as

$$\text{mass}(\mathcal{T}_t) = N \left(\frac{n}{t}\right)^{t-2} \cdot |\mathcal{T}_t|. \quad (8)$$

For the case $t = 2$, $\text{mass}(\mathcal{T}_2) = n_1 n_2$ and $\text{mass}(\mathcal{F}_2) = \text{mass}(\mathcal{T}_2) + 1$, so

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) = \frac{n_1 n_2}{n_1 n_2 + 1} \geq \frac{1}{2} = \mathbb{P}(F_2 \text{ is connected}).$$

Thus we may assume from now on that $t \geq 3$.

For each integer k with $1 \leq k \leq t$ let \mathcal{F}_t^k be the set of forests in \mathcal{F}_t with k components. We shall show that for each such k

$$\text{mass}(\mathcal{F}_t^k) \leq N \left(\frac{n}{t}\right)^{t-2} \cdot |\mathcal{F}_t^k|. \quad (9)$$

Summing over k will then give

$$\text{mass}(\mathcal{F}_t) \leq N \left(\frac{n}{t}\right)^{t-2} \cdot |\mathcal{F}_t|,$$

and so, using also (8)

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) = \frac{\text{mass}(\mathcal{T}_t)}{\text{mass}(\mathcal{F}_t)} \geq \frac{|\mathcal{T}_t|}{|\mathcal{F}_t|} = \mathbb{P}(F_t \text{ is connected}).$$

This will complete the proof of (6) and thus of the lemma. Hence it remains now to prove (9).

Fix an integer k with $1 \leq k \leq t$. Given a partition $\mathbf{U} = (U_1, \dots, U_k)$ of $[t]$ into k unordered sets, let $J = J(\mathbf{U}) = \{i : |U_i| \geq 2\}$, and let $\mathcal{F}(\mathbf{U})$ be the set of forests in \mathcal{F}_t^k such that the U_i are the vertex sets of the k component

trees. For non-empty sets $U \subseteq [t]$, let $p(U) = \prod_{i \in U} n_i$ and $s(U) = \sum_{i \in U} n_i$. Observe that the mass of a forest is the product of the masses of its component trees, and a singleton component just gives a factor 1. Now fix a partition $\mathbf{U} = (U_1, \dots, U_k)$ as above.

If $J = \emptyset$ then $\text{mass}(\mathcal{F}(\mathbf{U})) = 1 = |\mathcal{F}(\mathbf{U})|$. Now suppose that $J \neq \emptyset$. Then by (8)

$$\begin{aligned} \text{mass}(\mathcal{F}(\mathbf{U})) &= \prod_{i \in J} p(U_i) \left(\frac{s(U_i)}{|U_i|} \right)^{|U_i|-2} |U_i|^{|U_i|-2} \\ &\leq N \cdot \prod_{i \in J} \left(\frac{s(U_i)}{|U_i|} \right)^{|U_i|-2} \cdot \prod_{i \in J} |U_i|^{|U_i|-2} \\ &= N \cdot \prod_{i \in J} \left(\frac{s(U_i)}{|U_i|} \right)^{|U_i|-2} \cdot |\mathcal{F}(\mathbf{U})|. \end{aligned}$$

To handle the middle factor here, we can use Jensen's inequality, since $\log(x)$ is concave: we have

$$\begin{aligned} &\log \prod_{i \in J} \left(\frac{s(U_i)}{|U_i|} \right)^{|U_i|-2} \\ &= (t-2) \sum_{i \in J} \frac{|U_i|-2}{t-2} \log \frac{s(U_i)}{|U_i|} \\ &\leq (t-2) \sum_{i \in J} \frac{|U_i|}{t} \log \frac{s(U_i)}{|U_i|} \quad \text{since } |U_i| \leq t \\ &\leq (t-2) \sum_{i=1}^k \frac{|U_i|}{t} \log \frac{s(U_i)}{|U_i|} \\ &\leq (t-2) \log \left(\sum_{i=1}^k \frac{|U_i|}{t} \frac{s(U_i)}{|U_i|} \right) \quad \text{since } \log \text{ is concave} \\ &= (t-2) \log \frac{n}{t}. \end{aligned}$$

Hence in each case

$$\text{mass}(\mathcal{F}(\mathbf{U})) \leq N \left(\frac{n}{t} \right)^{t-2} |\mathcal{F}(\mathbf{U})|.$$

So, summing over partitions $\mathbf{U} = (U_1, \dots, U_k)$ of $[t]$,

$$\begin{aligned} \text{mass}(\mathcal{F}_t^k) &= \sum_{\mathbf{U}=(U_1, \dots, U_k)} \text{mass}(\mathcal{F}(\mathbf{U})) \\ &\leq \sum_{\mathbf{U}=(U_1, \dots, U_k)} N\left(\frac{n}{t}\right)^{t-2} |\mathcal{F}(\mathbf{U})| \\ &= N\left(\frac{n}{t}\right)^{t-2} |\mathcal{F}_t^k|. \end{aligned}$$

This completes the proof of (9), and thus the proof of Lemma 2.1. \square

To prove Lemma 2.2 we will use the standard inequality

$$\left(1 - \frac{j}{n}\right)^{n-j} \geq e^{-j} \quad \text{for } 1 \leq j < n. \quad (10)$$

[To see this, fix j and let $g(x) = (x-j) \log(1 - \frac{j}{x})$ for $x > j$. Then

$$g'(x) = (x-j) \left(\frac{1}{x-j} - \frac{1}{x} \right) + \log(1 - \frac{j}{x}) = \frac{j}{x} + \log(1 - \frac{j}{x}) < 0,$$

and so $g(n)$ is decreasing for $n > j$. But $g(n) \rightarrow e^{-j}$ as $n \rightarrow \infty$, so $g(n) > e^{-j}$ for each $n > j$.]

Proof of Lemma 2.2 For a graph G let $\text{frag}(G)$ be the number of vertices in G less the number of vertices in a largest component; and for integers n and j with $1 \leq j < n$ let $f(n, j)$ be the number of forests F on $[n]$ with $\text{frag}(F) = j$. By (10), for $1 \leq j < n/2$

$$\begin{aligned} f(n, j) &= \binom{n}{j} |\mathcal{F}_j| (n-j)^{n-j-2} \\ &= n^{n-2} \cdot \frac{|\mathcal{F}_j|}{j!} \cdot \frac{(n)_j}{n^j} \left(1 - \frac{j}{n}\right)^{n-j-2} \\ &\geq n^{n-2} \cdot \frac{|\mathcal{F}_j|}{j! e^j} \cdot \frac{(n)_j}{n^j} \left(1 - \frac{j}{n}\right)^{-2}. \end{aligned}$$

Now consider just $j \leq 2$ and let $n \geq 5$. Then $\frac{(n)_j}{n^j} \left(1 - \frac{j}{n}\right)^{-2} \geq 1$, so

$$\frac{|\mathcal{F}_n|}{n^{n-2}} > \sum_{j=0}^2 \frac{|\mathcal{F}_j|}{j! e^j} = 1 + \frac{1}{e} + \frac{2}{2! e^2} \approx 1.5032 \approx e^{0.4076}.$$

It is easy to check that this holds also for $n = 2, 3$ and 4 ; so

$$\mathbb{P}(F_n \text{ is connected}) < e^{-2/5} \quad \text{for each } n \geq 2,$$

as required. \square

3 Concluding Remarks

We can easily improve on Lemma 2.2 by pushing the proof further and doing some checking.

Lemma 3.1. *If we set $\alpha = 0.48$ then for each $n = 2, 3, \dots$*

$$\mathbb{P}(F_n \text{ is connected}) < e^{-\alpha}.$$

Proof. It is straightforward to check that $\frac{(n)_j}{n^j} (1 - \frac{j}{n})^{-2} \geq 1$ for each $j \leq 6$ and $n > 12$. Hence, arguing as in the proof of Lemma 2.2, for $n > 12$

$$\frac{|\mathcal{F}_n|}{n^{n-2}} > \sum_{j=0}^6 \frac{|\mathcal{F}_j|}{j! e^j} \approx 1.6167 \approx e^{0.4804} > e^{0.48}.$$

This holds also for $2 \leq n \leq 12$: to check this we may for example use [14] for the values $|\mathcal{F}_j|$ for $j \leq 12$. \square

Lemma 3.1 allows us to strengthen Theorem 1.4 as follows: with the same premises, if we set $\alpha = 0.48$ then

$$\mathbb{P}(R_n \text{ is connected}) \geq \min_{\alpha n \leq t \leq n} \mathbb{P}(F_t \text{ is connected}). \quad (11)$$

It is well known (see for example Flajolet and Sedgewick [5] Section II.5.3) that $\sum_{j \geq 1} \frac{|\mathcal{T}_j|}{j! e^j} = \frac{1}{2}$ and so by the exponential formula $\sum_{j \geq 0} \frac{|\mathcal{F}_j|}{j! e^j} = e^{\frac{1}{2}}$. We could expect with more work to increase the value $\alpha = 0.48$ in (11) to nearer $\frac{1}{2}$ – but can we go all the way to $\frac{1}{2}$?

Perhaps $\mathbb{P}(F_n \text{ is connected})$ is increasing from $n = 4$ onwards? (For $n = 1, \dots, 6$ the values of the probability are $1, \frac{1}{2}, \frac{3}{7} \approx 0.4286, \frac{8}{19} \approx 0.4211, \frac{125}{291} \approx 0.4295, \frac{1296}{2932} \approx 0.4420$ (to 4 decimal places), with minimum at $n = 4$.) In that case, we would have $\mathbb{P}(F_n \text{ is connected}) \leq e^{-\frac{1}{2}}$ for each $n \geq 2$; and we could improve the bounds in Theorem 1.4 and in (11) to

$$\mathbb{P}(R_n \text{ is connected}) \geq \mathbb{P}(F_{\lceil n/2 \rceil} \text{ is connected}) \quad \text{for all } n \geq 7, \quad (12)$$

which is getting closer to Conjecture (1.2). Let us re-state the above question as a final conjecture.

Conjecture 3.2. *$\mathbb{P}(F_n \text{ is connected})$ is increasing for $n \geq 4$.*

In work in progress jointly with Xena Cologne-Brookes, we have shown using standard analytic methods (following a suggestion from a referee) that $\mathbb{P}(F_n \text{ is connected})$ is strictly increasing for n sufficiently large, which shows that the inequality (12) holds for n sufficiently large. The aim is to establish the full Conjecture 3.2, and thus the full inequality (12), though the proof seems to depend on careful analytic estimates together with checking for many small values of n (and thus to be of a different nature from the combinatorial proofs in this paper).

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