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Calculus of Variations

Partial regularity for manifold constrained $p(x)$-harmonic maps

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Abstract
We prove that manifold constrained $p(x)$-harmonic maps are locally C^{1,β_0}-regular outside a set of zero n-dimensional Lebesgue's measure, for some $\beta_0 \in (0, 1)$. We also provide an estimate from above of the Hausdorff dimension of the singular set.

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Contents

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Introduction

We prove local C^{1,β_0}-partial regularity for manifold constrained $p(x)$-harmonic maps. More precisely, we consider local minimizers of the functional

$$W^{1,p(\cdot)}(\Omega,\mathcal{M}) \ni w \mapsto \mathcal{E}(w,\Omega) := \int_\Omega k(x)|Dw|^{p(x)}\,dx, \qquad (0.1)$$

where $p(\cdot)$ and $k(\cdot)$ are Hölder continuous functions (see (P1)-(P2) and (K1)-(K2) below for the precise assumptions), $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open, bounded set and $\mathcal{M} \subset \mathbb{R}^N$, $N \geq 3$, is an m-dimensional, compact submanifold endowed with a suitable topology. We refer to Sect. 1 for the precise notation. Our final outcome is that there exists a relatively open set $\Omega_0 \subset \Omega$ of full n-dimensional Lebesgue measure such that $u \in C^{1,\beta_0}_{\mathrm{loc}}(\Omega_0,\mathcal{M})$ for some $\beta_0 \in (0,1)$ and $\Sigma_0(u) := \Omega \setminus \Omega_0$ has Hausdorff dimension at the most equal to $n-\gamma_1$. Moreover, after imposing some extra restrictions on the variable exponent $p(\cdot)$, we are able to provide a further reduction to the Hausdorff dimension of the singular set of $\mathcal{M}$-constrained minimizers of the $p(\cdot)$-energy

$$w \mapsto \int_\Omega |Dw|^{p(x)}\,dx. \qquad (0.2)$$

Let us put our results into the context of the available literature. Functionals with variable growth exponent modelled on the one in (0.2) have been introduced in the setting of Calculus of Variations and Homogenization in the fundamental works of Zhikov [51–54]. Energies as in (0.2) also occur in the modelling of electro-rheological fluids, a class of non-newtonian fluids whose viscosity properties are influenced by the presence of external electromagnetic fields [3,43]. As for regularity, the first result in the vectorial case has been obtained by Coscia and Mingione in [8], where it is shown that local minimizers of energy (0.2) are locally $C^{1,\beta}$-regular in the unconstrained case. This is the optimal generalization of the classical results of Uhlenbeck concerning the standard case when $p(\cdot)$ is a constant. We refer to [32,33,36,39,48,49] for a survey of regularity results in the p-growth case, both for scalar and vector valued minimizers. Subsequently, the regularity theory of functionals with variable growth has been developed in a series of interesting papers by Ragusa, Tachikawa and Usuba [40–42,46,47], where the authors established partial regularity results for unconstrained minimizers that are on the other hand obviously related to the constrained case. Especially, in [46] Tachikawa gives an interesting partial regularity result and some singular set estimates for a class of functionals related to the constrained minimization problem in which minimizers are assumed to take values in a single chart. This generalizes the well-known results of Giaquinta and Giusti [22] valid in the case of quadratic functionals with special structure. In this paper we finally tackle the case of local minimizers with values into a manifold, provided that suitable topological assumptions are considered on the manifold $\mathcal{M}$ and optimal regularity conditions are in force on $p(\cdot)$ and $k(\cdot)$. Our first main result is the following:

Theorem 1 *Let $u \in W^{1,p(\cdot)}(\Omega,\mathcal{M})$ be a local minimizer of the functional in* (0.1)*, where $p(\cdot)$ satisfies assumptions* (P1)-(P2)*, $k(\cdot)$ satisfies* (K1)-(K2) *and $\mathcal{M}$ is as in* ($\mathcal{M}$1)-($\mathcal{M}$2)*. Then there exists a relatively open set $\Omega_0 \subset \Omega$ such that $u \in C^{1,\beta_0}_{\mathrm{loc}}(\Omega_0,\mathcal{M})$ for some $\beta_0 \in (0,1)$, and $\mathcal{H}^{n-\gamma_1}(\Omega \setminus \Omega_0) = 0$.*

By strengthening further the assumptions on the variable exponent $p(\cdot)$, we are then able to provide a better dimension estimate for the singular set. This is in the following:

Theorem 2 *Let $u \in W^{1,p(\cdot)}(\Omega,\mathcal{M})$ be a constrained local minimizer of energy* (0.2)*, where $p(\cdot) \in Lip(\Omega)$, $\gamma_1 \geq 2$ and $\mathcal{M}$ is as in* ($\mathcal{M}$1)-($\mathcal{M}$2)*. Then,*

i. *if* $n \le [\gamma_1] + 1$, *then u can have only isolated singularities;*
ii. *if* $n > [\gamma_1]+1$, *then the Hausdorff dimension of the singular set is at the most* $n-[\gamma_1]-1$.

As they are stated, our results are the natural generalization of the classical ones in [27, 28,34,44] for the case $p(\cdot) \equiv$ constant. For the vectorial quasiconvex case with standard p-growth we refer to the recent work of Hopper [31]. The extension we make here to the variable exponent case requires a number of non-trivial additional ideas and tools, in particular as far as the dimension estimates stated in Theorem 2 are concerned. This is also related to the recent, aforementioned paper of Tachikawa [46], and it is based on the use of a suitable monotonicity formula. We remark that the variable exponent functional in (0.1) is a significant instance of functionals with (p, q)-growth (following the terminology introduced by Marcellini, [37,38]). These are variational integrals of the type $w \mapsto \int F(x, Dw)\, dx$, where the integrand $F(\cdot)$ satisfies

$$|z|^p \lesssim F(x, y, z) \lesssim (1 + |z|^q), \qquad 1 < p \le q.$$

The study of such functionals has undergone an intensive development over the last years, see for instance [5,11,15,35,37–39]. Another prominent model in this class is the so called Double Phase energy, where

$$F(x, z) = |z|^p + a(x)|z|^q, \qquad 0 \le a(x) \le L.$$

This model shares several features with the variable growth exponent and has been again introduced by Zhikov in [53]. Indeed, here once again the growth exponent with respect to the gradient variable is determined by the space variable x, since the ellipticity type changes according to the the positivity of the coefficient $a(\cdot)$. There are several analogies between the variable exponent energy and the double phase one. In particular, one should notice the similarities between the use of the Gehring's Lemma-based reverse Hölder inequalities made here and the reverse Hölder inequality coming from fractional differentiability exploited in [6,7]. Moreover, compare the use of localization methods based on p-harmonic type approximation implemented here and in [4]. Such analogies point to a unified approach to non-autonomous functionals with (p, q)-growth conditions, partially implemented in [9]. We plan to investigate this in the context of constrained minimizers in a forthcoming paper [10].

1 Notation, main assumptions and functional setting

Throughout this paper, Ω denotes an open, bounded subset of $\mathbb{R}^n$ with $n \ge 2$ and the target space will be a submanifold of $\mathbb{R}^N$, $N \ge 3$. As usual, we denote by c a general constant larger than one. Different occurrences from line to line will be still indicated by c and relevant dependencies from certain parameters will be emphasized using brackets, i.e.: $c(n, p)$ means that c depends on n and p. We denote $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball centered in x_0 with radius $r > 0$; when not relevant, or clear from the context, we shall omit indicating the center: $B_r \equiv B_r(x_0)$. Moreover, for integer $k \ge 1$, by ω_k we mean the k-dimensional Lebesgue measure of the unit ball $B_1(0) \subset \mathbb{R}^k$. Along the paper, k will assume values N or m. When referring to balls in $\mathbb{R}^k$, $k \in \{m, N\}$, we will stress it with an apex "k", i.e.: $B^k_r(a_0)$ is the open ball with center $a_0 \in \mathbb{R}^k$ and positive radius r. For $\alpha, \beta \in \{1, \cdots, n\}$ and $i, j \in \{1, \cdots, N\}$, we set $\delta^{\alpha\beta} \equiv 0$, $\delta_{ij} \equiv 0$ if $\alpha \ne \beta$, $i \ne j$ respectively and $\delta^{\alpha\alpha} = \delta_{ii} \equiv 1$. With $U \subset \mathbb{R}^n$ being a measurable subset with positive, finite Lebesgue measure $0 < |U| < \infty$ and with $f : U \to \mathbb{R}^k$ being a measurable map, we shall denote by

$$(f)_U := \fint_U f(x)\,dx = \frac{1}{|U|}\int_U f(x)\,dx$$

its integral average. In particular, when $U \equiv B_r(x_0)$, we will indicate only the radius and, if necessary, the centre of the ball, i.e.: $(f)_r \equiv (f)_{r,x_0} := (f)_{B_r(x_0)}$. For $g\colon \Omega \to \mathbb{R}^k$ and $U \subset \Omega$, with $\gamma \in (0,1]$ being a given number we shall denote

$$[g]_{0,\gamma;U} := \sup_{x,y\in U; x\neq y} \frac{|g(x)-g(y)|}{|x-y|^{\gamma}}, \quad [g]_{0,\gamma} := [g]_{0,\gamma;\Omega}.$$

It is well known that the quantity defined above is a seminorm and when $[g]_{0,\gamma;U} < \infty$, we will say that g belongs to the Hölder space $C^{0,\gamma}(U,\mathbb{R}^k)$. Let us turn to the main assumptions that will characterize our problem. When considering the functional in (0.1), the exponent $p(\cdot)$ will always satisfy

$$\begin{aligned}
&(\mathrm{P1})\ p \in C^{0,\alpha}(\Omega), \quad \alpha \in (0,1],\\
&(\mathrm{P2})\ 1 < \gamma_1 \le p(x) \le \gamma_2 < \infty \ \text{ for all } x \in \Omega,\\
&\gamma_1 := \inf_{x\in\Omega} p(x) \quad \text{and} \quad \gamma_2 := \sup_{x\in\Omega} p(x),
\end{aligned}$$

while the coefficient $k(\cdot)$ is so that

$$\begin{aligned}
&(\mathrm{K1})\ k \in C^{0,\nu}(\Omega), \quad \nu \in (0,1],\\
&(\mathrm{K2})\ 0 < \lambda \le k(x) \le \Lambda < \infty \ \text{ for all } x \in \Omega,
\end{aligned}$$

hold true. Clearly, in hypotheses (P1)-(K1) there is no loss of generality in supposing $\alpha = \nu$, since in the forthcoming estimates only $\min\{\alpha,\nu\}$ will be relevant, so, for simplicity, from now on we will assume $p(\cdot), k(\cdot) \in C^{0,\alpha}(\Omega)$. These assumptions are optimal in order to get local Hölder continuity for the gradient of a minimizer of problem (0.1). This is evident already in the scalar, linear case, (Schauder estimates). For any given ball $B_r \Subset \Omega$, we denote

$$p_1(r) := \inf_{x\in B_r} p(x) \quad \text{and} \quad p_2(r) := \sup_{x\in B_r} p(x). \tag{1.1}$$

Notice that there is no loss of generality in assuming $\gamma_1 < \gamma_2$, otherwise $p(\cdot) \equiv$ const on Ω, and in this case the problem is very well understood, [23,27,28,34,44,45]. Furthermore, we need to impose some topological restriction on the manifold $\mathcal{M}$. Precisely, we ask that

$(\mathcal{M}1)$ $\mathcal{M}$ is a compact, $m-$dimensional, C^3 Riemannian submanifold without boundary of $\mathbb{R}^N$,

$(\mathcal{M}2)$ $\mathcal{M}$ is $[\gamma_2]-1$ connected.

Here $[x]$ denotes the integer part of x. We refer to Sect. 2 for a detailed description of the geometry of $\mathcal{M}$. Finally, for shorten the notation we shall collect the main parameters of the problem in the quantity

$$\texttt{data} := (n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [k]_{0,\alpha}, [p]_{0,\alpha}, \alpha).$$

As to fully clarify the framework we are going to adopt, we need to introduce some basic terminology on the so-called Musielak-Orlicz-Sobolev spaces. Essentially, these are Sobolev spaces defined by the fact that the distributional derivatives lie in a suitable Musielak-Orlicz space, rather than in a Lebesgue space as usual. Classical Sobolev spaces are then a particular case. Such spaces and related variational problems are discussed for instance in [29,54], to which we refer for more details. Here, we will consider spaces related to the variable exponent case in both unconstrained and manifold-constrained settings.

Definition 1 Given an open set $\Omega \subset \mathbb{R}^n$, the Musielak-Orlicz space $L^{p(\cdot)}(\Omega, \mathbb{R}^k)$, $k \geq 1$, with $p(\cdot)$ satisfying (P1)-(P2), is defined as

$$L^{p(\cdot)}(\Omega, \mathbb{R}^k) := \left\{ w \colon \Omega \to \mathbb{R}^k \text{ measurable and } \int_\Omega |w|^{p(x)}\, dx < \infty \right\},$$

and, consequently,

$$W^{1,p(\cdot)}(\Omega, \mathbb{R}^k) := \left\{ w \in W^{1,1}(\Omega, \mathbb{R}^k) \cap L^{p(\cdot)}(\Omega, \mathbb{R}^k) \text{ such that } Dw \in L^{p(\cdot)}(\Omega, \mathbb{R}^{k\times n}) \right\}.$$

The variants $W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^k)$ and $W_{\mathrm{loc}}^{1,p(\cdot)}(\Omega, \mathbb{R}^k)$ are defined in an obvious way.

It is well known that, under assumptions (P1)-(P2), the set of smooth maps is dense in $W^{1,p(\cdot)}(\Omega, \mathbb{R}^k)$, see e.g. [15,51,53,54]. Following [9,31] we also recall the analogous definition of such spaces when mappings take values into $\mathcal{M}$.

Definition 2 Let $\mathcal{M}$ be a compact submanifold of $\mathbb{R}^k$, $k \geq 3$, without boundary and $\Omega \subset \mathbb{R}^n$ an open set. For $p(\cdot)$ satisfying (P1)-(P2), the Musielak-Orlicz-Sobolev space $W^{1,p(\cdot)}(\Omega, \mathcal{M})$ of functions into $\mathcal{M}$ can be defined as

$$W^{1,p(\cdot)}(\Omega, \mathcal{M}) := \left\{ w \in W^{1,p(\cdot)}(\Omega, \mathbb{R}^k) \colon\ w(x) \in \mathcal{M} \text{ for a. e. } x \in \Omega \right\}.$$

The local space $W_{\mathrm{loc}}^{1,p(\cdot)}(\Omega, \mathcal{M})$ consists of maps belonging to $W^{1,p(\cdot)}(U, \mathcal{M})$ for all open sets $U \Subset \Omega$.

When (P1)-(P2) and $(\mathcal{M}1)$-$(\mathcal{M}2)$ are in force, a quick modification of [9, Lemma 6] shows that Lipschitz maps are dense in $W^{1,p(\cdot)}(\Omega, \mathcal{M})$ as well. Of course, when $p(\cdot) \equiv$ const, Definitions 1 and 2 reduce to the classical Sobolev spaces $W^{1,p}(\Omega, \mathbb{R}^k)$ and $W^{1,p}(\Omega, \mathcal{M})$ respectively. Owing to the $p(\cdot)$-growth behavior of the integrand in (0.1), we display our definition of local minimizer.

Definition 3 A map $u \in W_{\mathrm{loc}}^{1,p(\cdot)}(\Omega, \mathcal{M})$ is a constrained local minimizer of the functional $\mathcal{E}(\cdot)$ defined in (0.1) if and only if

$$x \mapsto k(x)|Du(x)|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \int_U k(x)|Du|^{p(x)}\, dx \leq \int_U k(x)|Dw|^{p(x)}\, dx,$$

for all open sets $U \Subset \Omega$ and all $w \in W_u^{1,p(\cdot)}(U, \mathcal{M})$, where

$$W_u^{1,p(\cdot)}(U, \mathcal{M}) := \left(u + W_0^{1,p(\cdot)}(U, \mathbb{R}^N)\right) \cap W^{1,p(\cdot)}(U, \mathcal{M}).$$

In Definition 3, local minimizers are given as maps belonging to the local space $W_{\mathrm{loc}}^{1,p(\cdot)}(\Omega, \mathcal{M})$. We stress that, since all the regularity properties of constrained local minimizers treated in this work are of local nature, there is no loss of generality in assuming that $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ and that $x \mapsto k(x)|Du(x)|^{p(x)} \in L^1(\Omega)$, see the statements of Theorems 1-2.

Remark 1 By continuity, all the constants depending on certain fixed values of the map $p(\cdot)$ are stable when $p(\cdot)$ varies in the interval $[\gamma_1, \gamma_2]$. Thus, whenever a constant depends on some $p \in [\gamma_1, \gamma_2]$, this dependence will be denoted by only mentioning γ_1 and γ_2, i.e.: $c(p) \equiv c(\gamma_1, \gamma_2)$.

2 Preliminaries

We shall split this section into two parts. In the first one, we collect some basic results concerning the regularity of minimizers of certain type of functionals and in the second one we will give a detailed description of the topology of $\mathcal{M}$, together with some extension lemmas, which will turn fundamental in order to construct suitable comparison maps in some steps of the proofs of Theorems 1 and 2.

2.1 Known regularity results

We start by reporting a Lipschitz estimate for the gradient and a decay estimate for the excess functional of unconstrained local minimizers of functionals of the p-laplacean type.

Proposition 1 *[2,20,24] Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set, $p \in (1, \infty)$, $0 < \lambda < \Lambda$ and $0 < \ell < L$ be absolute constants, $g^{\alpha\beta}$ and h_{ij} be constant matrices, uniformly elliptic in the sense that*

$$\ell|\xi|^2 \leq g^{\alpha\beta}\xi_\alpha\xi_\beta \leq L|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \quad \text{and} \quad \ell|\eta|^2 \leq h_{ij}\eta^i\eta^j \leq L|\eta|^2 \text{ for all } \eta \in \mathbb{R}^k,$$

uniformly bounded, $[g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}$; $\alpha, \beta \in \{1, \cdots, n\}$, $i, j \in \{1, \cdots, k\}$. Then, if $v \in W^{1,p}(\Omega, \mathbb{R}^k)$ is a local minimizer of the integral functional

$$W^{1,p}(\Omega, \mathbb{R}^k) \ni w \mapsto \mathcal{F}(w, \Omega) := \int_\Omega k_0 \left(g^{\alpha\beta}h_{ij}D_\alpha w^i D_\beta w^j\right)^{p/2} dx, \tag{2.1}$$

where $k_0 \in [\lambda, \Lambda]$ is a constant, then for all $B_\varrho \subset B_r \Subset \Omega$ the following reference estimates hold:

$$\fint_{B_\varrho} |Dv|^p \, dx \leq c \fint_{B_r} |Dv|^p \, dx \quad \text{and} \quad \fint_{B_\varrho} |Dv - (Dv)_\varrho|^p \, dx \leq c(\varrho/r)^{\mu p} \fint_{B_r} |Dv|^p \, dx, \tag{2.2}$$

for $c = c(n, k, \lambda, \Lambda, \ell, L, p)$ and $\mu = \mu(n, k, \lambda, \Lambda, \ell, L, p)$.

The following result is a p-harmonic approximation lemma, which will play a crucial role in the proof of Theorem 1. We will state it in the form which better fits our necessities.

Lemma 1 *[12] Let $\Omega \subset \mathbb{R}^n$ be an open subset and $p \in (1, \infty)$. For every $\tilde{\theta} > 0$ and $\tilde{d} \in (0, 1)$ there exists $\tilde{\delta} > 0$ depending only on $\tilde{\theta}$, $\tilde{d}$, p, such that the following holds. Let $B_r \subset \mathbb{R}^n$ be a ball and $\tilde{B}_r$ denote either B_r or B_{2r}. If $v \in W^{1,p}(\tilde{B}_r, \mathbb{R}^k)$ is almost p-harmonic in the sense that*

$$\fint_{B_r} p|Dv|^{p-1}\frac{Dv}{|Dv|} \cdot D\varphi \, dx \leq \tilde{\delta} \fint_{\tilde{B}_r} \left(|Dv|^p + \|D\varphi\|^p_{L^\infty(B_r)}\right) dx, \tag{2.3}$$

for all $\varphi \in C_c^\infty(B_r, \mathbb{R}^k)$, then the unique map $\tilde{h} \in W^{1,p}(B_r, \mathbb{R}^k)$, solution to the Dirichlet Problem

$$v + W_0^{1,p}(B_r, \mathbb{R}^k) \ni w \mapsto \min \int_{B_r} |Dw|^p \, dx \tag{2.4}$$

satisfies

$$\left(\fint_{B_r} |Dv - D\tilde{h}|^{p\tilde{d}} \, dx\right)^{\frac{1}{\tilde{d}}} \leq \tilde{\theta} \fint_{\tilde{B}_r} |Dv|^p \, dx. \tag{2.5}$$

The next are a couple of simple inequalities which will be used several times throughout the paper. They are elementary, see e.g.: [8,40,46].

Lemma 2 *The following inequalities hold true.*

i. *For any $\varepsilon_0 > 0$, there exists a constant $c = c(\varepsilon_0)$ such that for all $t \geq 0$, $l \geq m \geq 1$ there holds $|t^l - t^m| \leq c(l-m)\left(1 + t^{(1+\varepsilon_0)l}\right)$.*
ii. *For $t \in (0, 1]$, consider the function $g(t) := t^{-ct^\gamma}$, where $c > 0$ is an absolute constant and $\gamma \in (0, 1]$. Then $\lim_{t\to 0} g(t) = 1$ and $\sup_{t\in(0,1]} g(t) \leq \exp(c/\gamma)$.*

We conclude this section by recalling another fundamental tool in regularity theory, which will help establishing the behavior of certain quantities.

Lemma 3 *[21] Let $h\colon [\varrho, R_0] \to \mathbb{R}$ be a non-negative, bounded function and $0 < \theta < 1$, $0 \leq A$, $0 < \beta$. Assume that $h(r) \leq A(d-r)^{-\beta} + \theta h(d)$, for $\varrho \leq r < d \leq R_0$. Then $h(\varrho) \leq cA/(R_0 - \varrho)^{-\beta}$ holds, where $c = c(\theta, \beta) > 0$.*

2.2 Some extension results

We report some results concerning locally Lipschitz retractions. They have been extensively used in the realm of functionals with p-growth, see [27,28,31]. For integrands exihibiting (p, q)-growth they were used for the first time in [9], to prove that if the Lavrentiev phenomenon does not occur in the unconstrained case, then it is absent also in presence of a geometric constraint. According to assumptions $(\mathcal{M}1)$-$(\mathcal{M}2)$, $\mathcal{M} \subset \mathbb{R}^N$ is a compact, m-dimensional C^3 Riemannian submanifold, $\partial\mathcal{M} = \emptyset$ and, in particular, $\mathcal{M}$ is $[\gamma_2] - 1$ connected. Let us clarify this concept.

Definition 4 [31] Given an integer $j \geq 0$, a manifold $\mathcal{M}$ is said to be j-connected if its first j homotopy groups vanish identically, that is $\pi_0(\mathcal{M}) = \pi_1(\mathcal{M}) = \cdots = \pi_{j-1}(\mathcal{M}) = \pi_j(\mathcal{M}) = 0$.

It is reasonable to expect some good properties in terms of retractions for this kind of manifolds endowed with a relatively simple topology, as the following lemma shows.

Lemma 4 *Let $\mathcal{M} \subset \mathbb{R}^N$, $N \geq 3$ be a compact, j-connected submanifold for some integer $j \in \{1, \cdots, N-2\}$ contained in an N-dimensional cube Q. Then there exists a closed $(N-j-2)$-dimensional Lipschitz polyhedron $X \subset Q \setminus \mathcal{M}$ and a locally Lipschitz retraction $\psi\colon Q \setminus X \to \mathcal{M}$ such that for any $x \in Q \setminus X$, $|D\psi(x)| \leq c/\operatorname{dist}(x, X)$ holds, for some positive $c = c(N, j, \mathcal{M})$.*

Proof See e.g., [28, Lemma 6.1] for the original proof, or [31, Lemma 4.5] for a simplified version relying on some Lipschitz extension properties of maps between Riemannian manifolds. □

A major technical obstruction one can face when dealing with manifold-constrained variational problems is finding comparison maps which satisfy the constraint (notice that, without further regularity details on minimizers, we cannot localize in the image). Precisely, we are no longer allowed to use convex combinations of a minimizer with a suitable cutoff function as to realize valid competitors for the problem. Hence, given any $w \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, we overcome this issue by applying Lemma 4 to assure a local control on the $L^{p(\cdot)}$-norm of the gradient of a suitable projected image of w in terms of the $L^{p(\cdot)}$-norm of w itself. This is the content of the next lemma.

Lemma 5 (Finite energy extension.) *Let $\mathcal{M}$ be as in ($\mathcal{M}$1)-($\mathcal{M}$2) and $U \Subset \Omega$ an open subset of Ω with Lipschitz boundary. Given $w \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $w(\partial U) \subset \mathcal{M}$, there exists $\tilde{w} \in W^{1,p(\cdot)}_{w}(U, \mathcal{M})$ satisfying $\int_U |D\tilde{w}|^{p(x)}\, dx \le c \int_U |Dw|^{p(x)}\, dx$, where $c = c(N, \mathcal{M}, \gamma_2)$.*

Proof Following [31, Section 2.2], we define $\mathrm{Unp}(\mathcal{M})$ as the set of all $x \in \mathbb{R}^N$ for which there exists a unique point of $\mathcal{M}$ nearest to x and, for $a \in \mathcal{M}$, we denote by $\mathrm{reach}(\mathcal{M}, a)$ the supremum of the set of all numbers $r > 0$ for which $\{x \in \mathbb{R}^N : |x - a| < r\} \subset \mathrm{Unp}(A)$. Then, we can set $\mathrm{reach}(\mathcal{M}) := \inf_{a \in \mathcal{M}} \mathrm{Reach}(\mathcal{M}, a)$. Notice that, by assumptions ($\mathcal{M}$1)-($\mathcal{M}$2), $\mathrm{reach}(\mathcal{M}) > 0$, see [17,28,31]. Now, if for some $0 < \sigma < \mathrm{reach}(\mathcal{M})$, $V := \left\{a \in \mathbb{R}^N : \mathrm{dist}(a, \mathcal{M}) < \sigma\right\}$ is a neighbourhood with the nearest point property, then the metric projection $\Pi : \overline{V} \to \mathcal{M}$ associating to any $a \in \overline{V}$ the unique $a_0 \in \mathcal{M}$ such that $\mathrm{dist}(a, \mathcal{M}) = |a - a_0|$, is Lipschitz continuous and $\overline{V}$ and $\mathcal{M}$ are homotopy equivalent spaces with $\pi_i(\overline{V}) = \pi_i(\mathcal{M})$ for all $i \in \{0, \cdots, [\gamma_2] - 1\}$, see e.g.: [30, Proposition 1.17] for more details on this matter. Since $\mathcal{M}$ is compact and w is bounded, there exists an N-dimensional cube Q such that $\mathcal{M} \subset \overline{V} \subset Q$ and $\mathrm{dist}(w, \mathcal{M}) \le \frac{1}{2}\mathrm{dist}(\mathcal{M}, \partial Q)$ almost everywhere. By Lemma 4 with $j \equiv [\gamma_2] - 1$, there exists a locally Lipschitz retraction $\psi : Q \setminus X \to \overline{V}$ for some $(N - [\gamma_2] - 1)$-dimensional Lipschitz polyhedron $X \subset Q \setminus \overline{V}$, which, by construction stands strictly away from $\mathcal{M}$. Thus we have a map $P := \Pi \circ \psi : Q \setminus X \to \mathcal{M}$, satisfying

$$|\nabla P(a)| \le \frac{c}{\mathrm{dist}(a, X)}, \tag{2.6}$$

for $c = c(N, \mathcal{M})$. By a change of variables, the definition of the dual skeleton, the fact that $\mathcal{M}$ is $([\gamma_2] - 1)$-connected and $\dim(X) \le N - [\gamma_2] - 1$, there holds:

$$\int_Q \frac{1}{\mathrm{dist}(a, X)^{p(x)}}\, da \le \int_Q \left(1 + \frac{1}{\mathrm{dist}(a, X)^{\gamma_2}}\right) da < c, \tag{2.7}$$

for a finite, positive constant $c = c(N, \mathcal{M}, \gamma_2)$. Now, for a sufficiently small $0 < \varrho < \min\left\{\frac{\sigma}{2}, \frac{\mathrm{dist}(\mathcal{M}, \partial Q)}{2}\right\}$ and a point $a \in B^N_{\varrho}$, denote the translations $Q_a := \{b + a : b \in Q\}$ and $X_a := \{b + a : b \in X\}$, so that one can define the retraction $P_a : Q_a \setminus X_a \to \mathcal{M}$ given by $P_a(b) := P(b - a)$. Then, by the chain rule, Fubini's theorem, (2.6) and (2.7) we obtain

$$\begin{aligned}\int_{B^N_{\varrho}} \int_U |D(P_a(w))|^{p(x)}\, dx da &\le \int_U |Dw|^{p(x)} \left(\int_{B^N_{\varrho}} |\nabla P(w - a)|^{p(x)}\, da\right) dx \\ &\le \int_U |Dw|^{p(x)} \left(\int_Q |\nabla P(b)|^{p(x)}\, db\right) dx \le c \int_U |Dw|^{p(x)}\, dx,\end{aligned} \tag{2.8}$$

where $c = c(N, \mathcal{M}, \gamma_2)$. Estimate (2.8) and Markov's inequality then render the existence of a positive $c = c(N, \mathcal{M}, \gamma_2)$ and a $\tilde{a} \in B^N_{\varrho}$ so that

$$\int_U |D(P_{\tilde{a}}(w))|^{p(x)}\, dx \le c \int_U |Dw|^{p(x)}\, dx, \tag{2.9}$$

where again $c = c(N, \mathcal{M}, \gamma_2)$. Since $w(\partial U) \subset \mathcal{M}$, the map $\tilde{w} := (P_{\tilde{a}}|_{\mathcal{M}})^{-1} \circ P_{\tilde{a}} \circ w$ is well defined and given that the inverse map $P^{-1}_{\tilde{a}}$ is Lipschitz on $\mathcal{M}$, from (2.9) we conclude that $\int_U |D\tilde{w}|^{p(x)}\, dx \le c \int_U |Dw|^{p(x)}\, dx$, with $c = c(N, \mathcal{M}, \gamma_2)$. Moreover, since $w(\partial U) \subset \mathcal{M}$, by construction we have that $\tilde{w}|_{\partial U} = w|_{\partial U}$ and this concludes the proof. □

Lemma 5 will be particularly helpful when U is a ball B_r or an annulus $B_r \setminus B_\varrho$ for a proper choice of r and ϱ.

3 Partial regularity

In this section we first collect a couple of essential inequalities, some basic regularity results stemming only from the minimality condition and then carry out the proof of Theorem 1.

3.1 Basic regularity results

The first result is Poincaré's inequality, well known in the unconstrained case, see [14, Theorem 3.1], and since it is valid for any map $w \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, it transfers verbatim for functions in $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega, \mathcal{M})$. However, given that we are dealing with bounded maps ($\mathcal{M}$ is compact), we present a simplified proof, including also the case in which the domain is an annulus $A_{r\theta} := B_r \setminus B_{r(1-\theta)}$ for some $0 < \theta < 1$.

Lemma 6 (Poincaré's inequality) *Let $w \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, with $p(\cdot)$ satisfying (P1)-(P2) and $B_r \Subset \Omega$, $0 < r \le 1$. Then, there holds*

$$\int_{B_r} \left| \frac{w - (w)_r}{r} \right|^{p(x)} dx \le c \left(\int_{B_r} |Dw|^{p(x)}\, dx + |B_r| \right), \tag{3.1}$$

with $c = c(n, N, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha, \|w\|_{L^\infty(B_r)})$. Furthermore, if for some $0 < \theta < 1$, $w \in L^\infty(A_{r\theta}, \mathbb{R}^N) \cap W^{1,p(\cdot)}(A_{r\theta}, \mathbb{R}^N)$ is such that $w|_{\partial B_r} = 0$, then

$$\int_{A_{r\theta}} |w/(r\theta)|^{p(x)}\, dx \le c \left(\int_{A_{r\theta}} |Dw|^{p(x)}\, dx + |A_\theta| \right), \tag{3.2}$$

for $c = c(n, N, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha, \|w\|_{L^\infty(A_{r\theta})})$.

Proof Fix $B_r \Subset \Omega$, $0 < r \le 1$. From assumptions (P1)-(P2), Lemma 2 (*ii.*), (1.1) and the standard Poincaré's inequality holding for $p \equiv p_1(r)$ we obtain

$$\begin{aligned}\int_{B_r} \left| \frac{w - (w)_r}{r} \right|^{p(x)} dx \le& c r^{p_1(r) - p_2(r)} \max\left\{1, 2\|w\|_{L^\infty(B_r)}\right\}^{\gamma_2 - \gamma_1} \int_{B_r} \left| \frac{w - (w)_r}{r} \right|^{p_1(r)} dx \\ \le& c \int_{B_r} |Dw|^{p_1(r)}\, dx \le c \int_{B_r} (|Du|^{p(x)} + 1)\, dx,\end{aligned}$$

with $c = c(n, N, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha, \|w\|_{L^\infty(B_r)})$. In the same way, for $w \in L^\infty(A_{r\theta}, \mathbb{R}^N) \cap W^{1,p(\cdot)}(A_{r\theta}, \mathbb{R}^N)$ such that $w|_{\partial B_r} = 0$, we have

$$\int_{A_{r\theta}} |w/(r\theta)|^{p(x)}\, dx \le c(r\theta)^{p_1(r\theta) - p_2(r\theta)} \int_{A_{r\theta}} |Dw|^{p_1(r\theta)}\, dx \le c \int_{A_{r\theta}} (|Dw|^{p(x)} + 1)\, dx,$$

for $c(n, N, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha, \|w\|_{L^\infty(A_{r\theta})})$. Here we denoted $p_1(r\theta) := \inf_{x \in A_{r\theta}} p(x)$ and $p_2(r\theta) := \sup_{x \in A_{r\theta}} p(x)$. □

As to successfully implement Lemma 1, we also need an intrinsic version of Sobolev-Poincaré's inequality.

Lemma 7 (Intrinsic Sobolev-Poincaré's inequality) *Let $p \in (1, \infty)$ and $w \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Then, there exist a positive $c = c(n, N, p)$ and exponents $d_1 > 1$ and $0 < d_2 < 1$ such that*

$$\left(\fint_{B_r} \left| \frac{w - (w)_r}{r} \right|^{pd_1} dx \right)^{\frac{1}{d_1}} \leq c \left(\fint_{B_r} |Dw|^{pd_2}\, dx \right)^{\frac{1}{d_2}}$$

holds whenever $B_r \Subset \Omega$ is such that $0 < r \leq 1$. Here, $d_1 = d_1(n, N, p)$ and $d_2 = d_2(n, N, p)$.

Proof We start by considering the case $1 < p \leq n$. Fix any $\gamma \in \left(\max\left\{\frac{1}{p}, \frac{n}{n+p}\right\}, 1\right)$ and notice that $w \in W^{1,\gamma p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for all such γ. From the standard Sobolev-Poincaré's inequality we obtain

$$\left(\fint_{B_r} \left| \frac{w - (w)_r}{r} \right|^{\frac{n\gamma p}{n-\gamma p}} dx \right)^{\frac{n-\gamma p}{n\gamma p}} \leq c \left(\fint_{B_r} |Dw|^{\gamma p}\, dx \right)^{\frac{1}{\gamma p}},$$

for $c = c(n, N, p, \gamma)$, but, being γ ultimately influenced only by n and p, we can conclude that $c = c(n, N, p)$. Choosing $d_1 := \frac{n\gamma}{n-\gamma p} > 1$ since $\gamma > \frac{n}{n+p}$, and $d_2 := \gamma < 1$ we obtain the thesis. Now, if $p > n$, then there exists $\gamma \in (n/p, 1)$ so that $p\gamma > n$. Let $\kappa := 1 - n/(\gamma p)$. From Morrey's embedding theorem we then have

$$\left(\fint_{B_r} \left| \frac{w - (w_r)}{r} \right|^{\frac{p}{\gamma}} dx \right)^{\frac{\gamma}{p}} \leq c[w]_{0,\kappa;B_r} r^{\kappa - 1} \leq c \left(\fint_{B_r} |Dw|^{p\gamma}\, dx \right)^{\frac{1}{p\gamma}},$$

for $c = c(n, N, p)$. Fixing $d_1 := \gamma^{-1}$ and $d_2 := \gamma$ we can conclude.

Remark 2 Since $\mathcal{M}$ is compact, for a function w taking values in $\mathcal{M}$ the dependence of the constants appearing in the inequalities in Lemma 6 on the L^∞-norm of w will be expressed as a dependence on $\mathcal{M}$.

In the following lemma, we present a Caccioppoli-type inequality, which is fundamental for regularity.

Lemma 8 (Caccioppoli-type inequality) *Let $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ be a constrained local minimizers of (0.1). Then, for any ball $B_r \Subset \Omega$ there holds*

$$\fint_{B_{r/2}} |Du|^{p(x)}\, dx \leq c \fint_{B_r} \left| \frac{u - (u)_r}{r} \right|^{p(x)} dx,$$

for $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$.

Proof With $0 < r/2 \leq s < t \leq r \leq 1$ we determine a cutoff function $\eta \in C^1_c(B_r)$ such that $\chi_{B_s} \leq \eta \leq \chi_{B_t}$ and $|D\eta| \leq 2(t-s)^{-1}$ and define the map $w := u - \eta(u - (u)_r)$. By construction, $w \in W^{1,p(\cdot)}(B_t, \mathbb{R}^N) \cap L^\infty(B_t, \mathbb{R}^N)$ and $w|_{\partial B_t} = u|_{\partial B_t}$, so Lemma 5 renders a map $\tilde{w} \in W^{1,p(\cdot)}_u(B_t, \mathcal{M})$ which is an admissible competitor for u in problem (0.1) and satisfies

$$\int_{B_t} |D\tilde{w}|^{p(x)}\, dx \leq c \int_{B_t} |Dw|^{p(x)}\, dx \leq c \left(\int_{B_t \setminus B_s} |Du|^{p(x)}\, dx + \int_{B_r} \left| \frac{u - (u)_r}{t - s} \right|^{p(x)} dx \right), \tag{3.3}$$

for $c = c(N, \mathcal{M}, \gamma_1, \gamma_2)$. The minimality of u, (K2), the features of η, (3.3) and (1.1) give

$$\begin{aligned}\int_{B_s} |Du|^{p(x)}\,dx \leq& \frac{\Lambda}{\lambda}\int_{B_t} |D\tilde{w}|^{p(x)}\,dx\\ \leq& c\int_{B_t\setminus B_s} |Du|^{p(x)}\,dx + c\int_{B_r}\left|\frac{u-(u)_r}{t-s}\right|^{p(x)}\,dx\\ \leq& c\int_{B_t\setminus B_s} |Du|^{p(x)}\,dx + c(t-s)^{-p_2(r)}\int_{B_r}|u-(u)_r|^{p(x)}\,dx,\end{aligned}$$

with $c = c(N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. Now we are in position to apply Widman's hole filling technique and Lemma 3 to conclude that

$$\begin{aligned}\int_{B_{r/2}} |Du|^{p(x)}\,dx \leq& cr^{-p_2(r)}\int_{B_r}|u-(u)_r|^{p(x)}\,dx\\ =& cr^{p_1(r)-p_2(r)}\int_{B_r} r^{-p_1(r)}|u-(u)_r|^{p(x)}\,dx \leq c\int_{B_r}\left|\frac{u-(u)_r}{r}\right|^{p(x)}\,dx,\end{aligned}$$

for $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$. Here we also used assumption (P1), definition (1.1) and Lemma 2 (*ii.*). □

The next step consists in proving an interior higher integrability result for local minimizers of (0.1).

Lemma 9 *Let $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ be a constrained local minimizer of (0.1). Then there esists a positive integrability threshold $\tilde{\delta}_0 = \tilde{\delta}_0(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$ such that*

$$|Du|^{(1+\delta)p(\cdot)} \in L^1_{\mathrm{loc}}(\Omega) \quad for\ all\ \ \delta \in [0, \tilde{\delta}_0)$$

and, for any $B_r \Subset \Omega$

$$\left(\fint_{B_{r/2}} |Du|^{(1+\delta)p(x)}\,dx\right)^{\frac{1}{1+\delta}} \leq c\fint_{B_r}(1+|Du|^2)^{p(x)/2}\,dx \quad for\ all\ \delta \in [0, \tilde{\delta}_0),$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$.

Proof For a fixed $B_r \Subset \Omega$, combining Lemmas 8 and 7 with $p \equiv p_1(r)$, we end up with

$$\begin{aligned}\fint_{B_{r/2}} |Du|^{p(x)}\,dx \leq& c\fint_{B_r}\left|\frac{u-(u)_r}{r}\right|^{p(x)}\,dx\\ \leq& c\max\left\{1, 2\|u\|_{L^\infty(B_r)}\right\}^{\gamma_2-\gamma_1} r^{p_1(r)-p_2(r)}\fint_{B_r}\left|\frac{u-(u)_r}{r}\right|^{p_1(r)}\,dx\\ \leq& c\left(\fint_{B_r}\left|\frac{u-(u)_r}{r}\right|^{p_1(r)d_1}\,dx\right)^{\frac{1}{d_1}} \leq c\left(\fint_{B_r}|Du|^{p_1(r)d_2}\,dx\right)^{\frac{1}{d_2}}\\ \leq& c\left(\fint_{B_r}|Du|^{p(x)d_2}\,dx\right)^{\frac{1}{d_2}} + c,\end{aligned}$$

for $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$. Here we also used (P1)-(P2), Lemma 2 (*ii.*) and Hölder's inequality. Now, an application of Gehring-Giaquinta-Modica's lemma, [26, Chapter 6] renders the existence of a positive $\tilde{\delta}_0 = \tilde{\delta}_0(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$ so that

$$\left(\fint_{B_{r/2}} |Du|^{(1+\delta)p(x)}\,dx \right)^{\frac{1}{1+\delta}} \le c \fint_{B_r} (1 + |Du|^2)^{p(x)/2}\,dx,$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$, for all $\delta \in [0, \tilde{\delta}_0)$. Finally, after a standard covering argument, we obtain that $|Du|^{(1+\delta)p(\cdot)} \in L^1_{\text{loc}}(\Omega)$ for all $\delta \in [0, \tilde{\delta}_0)$. □

Remark 3 Before proceding further we need to stress that, if $B_r \Subset \Omega$ and $w \in W^{1,p}(B_r, \mathbb{R}^N)$ is such that $w \equiv 0$ on $U \subset B_r$ with $|U| > \tilde{c}|B_r|$ for some positive, absolute $\tilde{c}$, then Sobolev-Poincaré's inequality gives

$$\int_{B_r} |w/r|^p\,dx \le c r^{-n(p/p_*-1)} \left(\int_{B_r} |Dw|^{p_*}\,dx \right)^{\frac{p}{p_*}}, \tag{3.4}$$

for $c = c(n, N, p, \tilde{c})$. Here $p_* := \max\left\{1, \frac{np}{n+p}\right\}$, as usual.

The following lemma is an up to the boundary higher integrability result. The argument is well-known to specialists, see [1,40], and it essentially relies on the fact that Caccioppoli's inequality can be carried up to the boundary. However, we did not manage to find in the literature a proof for the manifold-constrained case, so we shall report it here.

Lemma 10 *Let $p \in [\gamma_1, \gamma_2]$, $u \in W^{1,p}_{\text{loc}}(\Omega, \mathcal{M})$ be such that $|Du|^{p(1+\delta_1)} \in L^1_{\text{loc}}(\Omega)$ for some $\delta_1 > 0$ and let $v \in W^{1,p}_u(B_r, \mathcal{M})$ be a solution to the Dirichlet problem*

$$W^{1,p}_u(B_r, \mathcal{M}) \ni w \mapsto \min \int_{B_r} k_0 |Dw|^p\,dx,$$

where $k_0 \in [\lambda, \Lambda]$ is a positive constant and $B_r \Subset \Omega$ is any ball with $r \in (0, 1]$. Then there exists a positive integrability threshold $\tilde{\sigma}_0 \in (0, \delta_1)$ such that

$$\fint_{B_r} (1 + |Dv|^2)^{(1+\sigma)p/2}\,dx \le c \fint_{B_r} (1 + |Du|^2)^{(1+\sigma)p/2}\,dx \quad \text{for all } \sigma \in [0, \tilde{\sigma}_0).$$

Here $\tilde{\sigma}_0 = \tilde{\sigma}_0(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$ and $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$.

Proof With $x_0 \in B_r$, let us fix a ball $B_\varrho(x_0) \subset \mathbb{R}^n$, $0 < \varrho \le 1$. We start with the case in which $|B_\varrho(x_0) \setminus B_r| > |B_\varrho(x_0)|/10$. Let us fix parameters $0 < \varrho/2 \le s < t \le \varrho$ and consider $\eta \in C^1_c(B_t(x_0))$ such that $\chi_{B_s(x_0)} \le \eta \le \chi_{B_t(x_0)}$ and $|D\eta| \le 2(t-s)^{-1}$. The function $w := v - \eta(v - u)$ coincides with v on ∂B_r and on $\partial(B_r \cap B_t(x_0))$ in the sense of traces, so Lemma 5 with $p(\cdot)$ equal to constant p, provides us with a map $\tilde{w} \in W^{1,p}_v(B_r \cap B_t(x_0), \mathcal{M})$ such that

$$\begin{aligned}\int_{B_r \cap B_t(x_0)} |D\tilde{w}|^p\,dx \le& c \int_{B_r \cap B_t(x_0)} |Dw|^p\,dx \\ \le& c \int_{B_r \cap (B_t(x_0) \setminus B_s(x_0))} |Dv|^p\,dx + c \int_{B_r \cap B_\varrho(x_0)} |Du|^p + \left| \frac{v-u}{t-s} \right|^p\,dx,\end{aligned} \tag{3.5}$$

for $c = c(N, \mathcal{M}, \gamma_1, \gamma_2)$. The minimality of v in the Dirichlet class $W_v^{1,p}(B_r \cap B_t(x_0), \mathcal{M})$ and (3.5) render

$$\begin{aligned}\int_{B_r\cap B_s(x_0)} |Dv|^p \, dx \leq & \frac{\Lambda}{\lambda} \int_{B_r\cap B_t(x_0)} |D\tilde{w}|^p \, dx \\ \leq & c \int_{B_r\cap(B_t(x_0)\setminus B_s(x_0))} |Dv|^p \, dx + c \int_{B_r\cap B_\varrho(x_0)} |Du|^p + \left|\frac{v-u}{t-s}\right|^p \, dx,\end{aligned}$$

for $c = c(N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. By filling the hole and applying Lemma 3, we get

$$\int_{B_r\cap B_{\varrho/2}(x_0)} |Dv|^p \, dx \leq c \int_{B_r\cap B_\varrho(x_0)} |Du|^p + \left|\frac{u-v}{\varrho}\right|^p \, dx, \tag{3.6}$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. Notice that we can extend $v - u \equiv 0$ outside B_r, since $u - v \in W_0^{1,p}(B_r, \mathbb{R}^N)$, so there are no discontinuities on $\partial(B_r \cap B_\varrho(x_0))$. Recalling also that $|B_\varrho(x_0) \setminus B_r| > |B_\varrho(x_0)|/10$, from (3.4) we have that

$$\begin{aligned}\int_{B_r\cap B_\varrho(x_0)} \left|\frac{u-v}{\varrho}\right|^p \, dx = & \int_{B_\varrho(x_0)} \left|\frac{u-v}{\varrho}\right|^p \, dx \\ \leq & c\varrho^{-n(p/p_*-1)} \left(\int_{B_\varrho(x_0)} |Du - Dv|^{p_*} \, dx\right)^{\frac{p}{p_*}} \\ = & c\varrho^{-n(p/p_*-1)} \left(\int_{B_r\cap B_\varrho(x_0)} |Du - Dv|^{p_*} \, dx\right)^{\frac{p}{p_*}},\end{aligned}$$

for $c = c(n, N, \gamma_1, \gamma_2)$. Averaging in the previous display and keeping in mind that $|B_\varrho(x_0) \cap B_r| \leq |B_\varrho(x_0)|$, we obtain

$$\fint_{B_r\cap B_\varrho(x_0)} \left|\frac{u-v}{\varrho}\right|^p \, dx \leq c \left(\fint_{B_r\cap B_\varrho(x_0)} |Du - Dv|^{p_*} \, dx\right)^{\frac{p}{p_*}}, \tag{3.7}$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. Coupling (3.6) and (3.7) we get, by triangle and Hölder's inequalities,

$$\fint_{B_r\cap B_{\varrho/2}(x_0)} |Dv|^p \, dx \leq c \left\{\fint_{B_r\cap B_\varrho(x_0)} |Du|^p \, dx + \left(\fint_{B_r\cap B_\varrho(x_0)} |Dv|^{p_*} \, dx\right)^{\frac{p}{p_*}}\right\},$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. We next consider the situation when it is $B_\varrho(x_0) \Subset B_r$. In this case, we can apply the standard Sobolev-Poincaré's inequality, thus getting, as in the interior case, (Lemma 9 with $p(\cdot)$ equal to constant p),

$$\fint_{B_{\varrho/2}(x_0)} |Dv|^p \, dx \leq c \left(\fint_{B_\varrho(x_0)} |Dv|^{p_*} \, dx\right)^{\frac{p}{p_*}},$$

for $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. The two cases can be combined via a standard covering argument. Precisely, upon defining

$$V(x) := \begin{cases} |Dv(x)|^{p_*} & x \in B_\varrho(x_0) \\ 0 & x \in \mathbb{R}^n \setminus B_\varrho(x_0) \end{cases} \quad \text{and} \quad U(x) := \begin{cases} |Du(x)|^p & x \in B_\varrho(x_0) \\ 0 & x \in \mathbb{R}^n \setminus B_\varrho(x_0) \end{cases},$$

we get

$$\fint_{B_{\varrho/2}(x_0)} V(x)^{\frac{p}{p_*}}\,dx \le c\left\{\fint_{B_\varrho(x_0)} U(x)\,dx + \left(\fint_{B_\varrho(x_0)} V(x)\,dx\right)^{\frac{p}{p_*}}\right\},$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. At this point, by a variant of Gehring's Lemma, we obtain that there exists a positive $\tilde{\sigma}_0 = \tilde{\sigma}_0(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$ such that $0 < \tilde{\sigma}_0 < \delta_1$ and

$$\left(\fint_{B_r} |Dv|^{p(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}} \le c\left\{\fint_{B_r} |Dv|^p\,dx + \left(\fint_{B_r} |Du|^{(1+\sigma)p}\,dx\right)^{\frac{1}{1+\sigma}}\right\}, \tag{3.8}$$

for all $\sigma \in [0, \tilde{\sigma}_0)$ where $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$, see [25, Theorem 3 and Proposition 1, Chapter 2]. From (3.8) and the minimality of v within the Dirichlet class $W^{1,p}_u(B_r, \mathcal{M})$ we can conclude that

$$\fint_{B_r} |Dv|^{p(1+\sigma)}\,dx \le c\fint_{B_r} |Du|^{p(1+\sigma)}\,dx \quad \text{for all} \quad \sigma \in [0, \tilde{\sigma}_0),$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$. □

The next corollary allows recovering some useful estimates for the average of the gradient of solutions to problem (0.1).

Corollary 3 *Let $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ be a constrained local minimizer of (0.1). Then, for any $B_r \subset \Omega$ with $r \in (0, 1]$, such that $B_{4r} \Subset \Omega$ there holds*

$$\fint_{B_r} |Du|^{p(x)}\,dx \le cr^{-p_2(2r)}, \tag{3.9}$$

$$\fint_{B_r} |Du|^{p(x)(1+\delta)}\,dx \le cr^{-p_2(4r)(1+\delta)}, \tag{3.10}$$

where $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$ and $\delta \in (0, \tilde{\delta}_0)$, where $\tilde{\delta}_0$ is the higher integrability threshold given by Lemma 9.

Proof Inequality (3.9) comes from an application of Lemma 8 and the boundedness of u. In fact we have

$$\fint_{B_r} |Du|^{p(x)}\,dx \le c\fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^{p(x)}\,dx \le c\max\left\{1, 2\|u\|_{L^\infty(B_{2r})}\right\}^{\gamma_2} r^{-p_2(2r)},$$

for $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$. On the other hand, combining Lemmas 9 and 8, we have

$$\begin{aligned}\fint_{B_r} |Du|^{(1+\delta)p(x)}\,dx &\le c\left(\fint_{B_{2r}} (1+|Du|^2)^{p(x)/2}\,dx\right)^{1+\delta}\\ &\le c\left(1+\fint_{B_{4r}} \left|\frac{u-(u)_{4r}}{r}\right|^{p(x)}\,dx\right)^{1+\delta} \le cr^{-p_2(r)(1+\delta)},\end{aligned}$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$, for any $\delta \in (0, \tilde{\delta}_0)$.

3.2 Proof of Theorem 1

Now we are ready to prove Theorem 1. For the reader's convenience, we shall split the proof in seven steps.

Step 1: comparison, first time. We define $\delta_0 := \frac{1}{2}\min\left\{\tilde{\delta}_0, 1\right\}$, where $\tilde{\delta}_0$ is the higher integrability threshold from Lemma 9. Notice that by (P1), the set

$$\Omega^+ := \left\{x \in \Omega\colon\ p(x) > n - \frac{\delta_0}{2}\right\}, \tag{3.11}$$

is open and by Lemma 9, $|Du|^{(1+\delta_0)p(\cdot)} \in L^1(\Omega^+)$, thus $u \in W^{1,n+\frac{\delta_0}{4}}(\Omega^+, \mathcal{M})$. An application of Morrey's embedding theorem then renders that $u \in C^{0,\beta'}(\Omega^+, \mathcal{M})$ with $\beta' := \frac{\delta_0}{4n+\delta_0}$. We will treat this case in *Step 7*, so, from now on, $\gamma_2 < n$ holds. We set

$$\sigma_0 := \frac{1}{2}\min\left\{\delta_0, \frac{\alpha}{2\max\{\gamma_2, n\}}\right\} \tag{3.12}$$

and fix a $\tilde{R}_* \in (0, 1]$ so small that

$$[p]_{0,\alpha} 4\tilde{R}_*^{\alpha} \le \frac{\sigma_0\gamma_1}{\sigma_0 + 2} \tag{3.13}$$

is satisfied on $\bar{B}_{\tilde{R}_*} \equiv \bar{B}_{\tilde{R}_*}(x_0) \Subset \Omega$. Clearly this condition transfers on any ball $B_r(x_1) \subset B_{\tilde{R}_*}$. We select also an $R_* \in (0, \tilde{R}_*/2)$, whose size will be specified along the proof. Now notice that, since Lemma 9 holds true for all balls $B_{4r} \subset B_{R_*} \subset B_{\tilde{R}_*}$, $|Du|^{(1+\delta)p_1(2r)} \in L^1(B_{2r})$ for all $\delta \in (0, \delta_0]$. Therefore, by (3.13) and assumption (P1) it easily follows that

$$p_2(2r) < \left(1 + \frac{\sigma_0}{2}\right) p_2(2r) \le (1 + \sigma_0) p_1(2r), \tag{3.14}$$

so, recalling that $\sigma_0 < \delta_0$, we get

$$|Du|^{(1+\sigma/2)p_2(2r)} \in L^1(B_{2r}) \quad \text{for all } \sigma \in (0, \sigma_0]. \tag{3.15}$$

On such a ball we impose the following smallness condition on the energy: there exists an $\varepsilon \in (0, 1)$, whose size will be fixed later on, such that

$$\left((2r)^{p_2(2r)-n}\int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2}\, dx\right)^{\frac{1}{p_2(2r)}} < \varepsilon. \tag{3.16}$$

Let $v \in W_u^{1,p_2(2r)}(B_r, \mathcal{M})$ be a solution to the frozen Dirichlet problem

$$\inf_{w\in W_u^{1,p_2(2r)}(B_r,\mathcal{M})} \mathcal{G}(w, B_r) := \inf_{w\in W_u^{1,p_2(2r)}(B_r,\mathcal{M})} \int_{B_r} k_0|Dw|^{p_2(2r)}\, dx, \tag{3.17}$$

where $k_0 := k(x_0)$ is the value the coefficient $k(\cdot)$ attains in the centre of B_r. Needless to say, being (K2) in force, $k(\cdot)$ ranges between two positive, absolute constants λ and Λ, so none of the estimates we will provide is going to depend on x_0. By minimality, v solves the Euler-Lagrange equation

$$0 = \int_{B_r} k_0 p_2(2r)|Dv|^{p_2(2r)-2} \left(Dv \cdot D\varphi - A_v(Dv, Dv)\varphi\right)\, dx, \tag{3.18}$$

for any $\varphi \in W_0^{1,p_2(2r)}(B_r,\mathbb{R}^N)\cap L^\infty(B_r,\mathbb{R}^N)$, where, for $y\in\mathcal{M}$, $A_y\colon T_y\mathcal{M}\times T_y\mathcal{M}\to(T_y\mathcal{M})^\perp$ denotes the second fundamental form of $\mathcal{M}$. In particular, by tangentiality,

$$\nabla^2\Pi(v)(Dv,Dv)=-A_v(Dv,Dv)\quad\text{and}\quad |A_v(Dv,Dv)|\le c_{\mathcal{M}}|Dv|^2, \tag{3.19}$$

where $c_{\mathcal{M}}$ depends only on the geometry of $\mathcal{M}$, see [45, Appendix to Chapter 2]. In all the forthcoming estimates, any dependency on $c_{\mathcal{M}}$ of the constants will always be denoted as a dependency on $\mathcal{M}$, i.e.: $c(c_{\mathcal{M}})\equiv c(\mathcal{M})$. From (3.15), the compactness of $\mathcal{M}$ and the fact that $v|_{\partial B_r}=u|_{\partial B_r}$, we see that the map $\varphi:=u-v$ is admissible in (3.18). Let us define

$$\sigma:=\frac{1}{2}\min\{\tilde\sigma_0,\sigma_0\}, \tag{3.20}$$

where $\tilde\sigma_0$ is the boundary higher integrability threshold given by Lemma 10. Now, exploiting assumptions $(P2)$-$(K2)$, $(3.19)_2$ and Hölder's inequality we estimate

$$\begin{aligned}
&\left|\int_{B_r}k_0p_2(2r)|Dv|^{p_2(2r)-2}A_v(Dv,Dv)(u-v)\,dx\right|\\
&\le c\int_{B_r}|Dv|^{p_2(2r)}|u-v|\,dx\\
&\le cr^n\left(\fint_{B_r}|Dv|^{(1+\sigma)p_2(2r)}\,dx\right)^{\frac{1}{1+\sigma}}\left(\fint_{B_r}|u-v|^{p_2(2r)}\,dx\right)^{\frac{\sigma}{1+\sigma}}=:cr^n\,[(\mathrm{I})\cdot(\mathrm{II})],
\end{aligned}\tag{3.21}$$

where $c=c(n,N,\mathcal{M},\lambda,\Lambda,\gamma_1,\gamma_2)$. Using Lemma 10, (3.20), Hölder's inequality, (1.1), assumptions (P1)-(P2), (3.14), Lemma 9, (3.16) and Lemma 2 (*ii.*) we have

$$\begin{aligned}
(\mathrm{I})\le&c\left(\fint_{B_r}|Du|^{(1+\sigma)p_2(2r)}\,dx\right)^{\frac{1}{1+\sigma}}\le c\left(\fint_{B_r}|Du|^{(1+\sigma_0)p(x)}\,dx\right)^{\frac{p_2(2r)}{(1+\sigma_0)p_1(r)}}\\
\le&c\left(\fint_{B_{2r}}(1+|Du|^2)^{p(x)/2}\,dx\right)^{\frac{p_2(2r)}{p_1(2r)}-1}\fint_{B_{2r}}(1+|Du|^2)^{p_2(2r)/2}\,dx\\
\le&cr^{-4^\alpha r^\alpha[p]_{0,\alpha}\frac{\gamma_2}{\gamma_1}}\left((2r)^{p_2(2r)-n}\int_{B_{2r}}(1+|Du|^2)^{p_2(2r)/2}\,dx\right)^{\frac{p_2(2r)-p_1(2r)}{p_1(2r)}}\\
&\fint_{B_{2r}}(1+|Du|^2)^{p_2(2r)/2}\,dx\\
\le&c\varepsilon^{\frac{p_2(2r)(p_2(2r)-p_1(2r))}{p_1(2r)}}\fint_{B_{2r}}(1+|Du|^2)^{p_2(2r)/2}\,dx\le c\fint_{B_{2r}}(1+|Du|^2)^{p_2(2r)/2}\,dx,
\end{aligned}$$

with $c=c(n,N,\mathcal{M},\lambda,\Lambda,\gamma_1,\gamma_2,[p]_{0,\alpha},\alpha)$. On the other hand, by Poincaré's inequality, the minimality of v in class $W_u^{1,p_2(2r)}(B_r,\mathcal{M})$ and (3.16) we bound

$$\begin{aligned}
(\mathrm{II})\le&c\left(r^{p_2(2r)}\fint_{B_r}|Du-Dv|^{p_2(2r)}\,dx\right)^{\frac{\sigma}{1+\sigma}}\\
\le&c\left((2r)^{p_2(2r)-n}\int_{B_{2r}}(1+|Du|^2)^{p_2(2r)/2}\,dx\right)^{\frac{\sigma}{1+\sigma}}\le c\varepsilon^{\frac{\gamma_1\sigma}{1+\sigma}},
\end{aligned}$$

for $c = c(n, N, \lambda, \Lambda, \gamma_1, \gamma_2)$. Inserting the content of the previous two displays in (3.21) we obtain

$$\left| \int_{B_r} k_0 p_2(2r)|Dv|^{p_2(2r)-2} A_v(Dv, Dv)(u - v)\, dx \right| \le c\varepsilon^{\frac{\gamma_1\sigma}{1+\sigma}} \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2}\, dx, \tag{3.22}$$

for $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$. For the ease of notation, if $z \in \mathbb{R}^{N\times n}$, let us name $g(z) := k_0|z|^{p_2(2r)}$. The convexity of $g(\cdot)$ and (3.18) then render

$$\begin{aligned}
&\mathcal{G}(u, B_r) - \mathcal{G}(v, B_r)\\
&\quad= \int_{B_r} \partial g(Dv)(Du - Dv)\, dx\\
&\qquad+ \int_{B_r} \left(\int_0^1 (1 - t)\partial^2 g(tDu + (1 - t)Dv)\, dt\right)(Du - Dv)(Du - Dv)\, dx\\
&\quad\ge -\left| \int_{B_r} k_0 p_2(2r)|Dv|^{p_2(2r)-2} A_v(Dv, Dv)(u - v)\, dx \right|\\
&\qquad+ c\int_{B_r} \left(\int_0^1 (1 - t)|tDu + (1 - t)Dv|^{p_2(2r)-2}\, dt\right)|Du - Dv|^2\, dx,
\end{aligned}$$

with $c = c(n, N, \lambda, \gamma_1, \gamma_2)$. From this and (3.22) we obtain

$$\begin{aligned}
c\int_{B_r} (|Dv|^2 + |Du|^2)^{\frac{p_2(2r)-2}{2}}|Du - Dv|^2\, dx \le \quad & c\varepsilon^{\frac{\gamma_1\sigma}{1+\sigma}} \int_{B_{2r}} (1 + |Du|^2)^{\frac{p_2(2r)}{2}}\, dx\\
+ \mathcal{G}(u, B_r) - \mathcal{G}(v, B_r), &
\end{aligned} \tag{3.23}$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, [p]_{0,\alpha}, \alpha)$. Using this time the minimality of u, we see that

$$\begin{aligned}
&\mathcal{G}(u, B_r) - \mathcal{G}(v, B_r)\\
&\quad\le \mathcal{G}(u, B_r) - \mathcal{G}(v, B_r) + \mathcal{E}(v, B_r) - \mathcal{E}(u, B_r)\\
&\quad\le |\mathcal{G}(u, B_r) - \mathcal{E}(u, B_r)| + |\mathcal{E}(v, B_r) - \mathcal{G}(v, B_r)|.
\end{aligned}$$

Recall the definitions of σ_0 and of k_0. From assumptions (K1)-(K2) and (P1)-(P2), Lemma 2 (*i.*) with $\varepsilon_0 \equiv \sigma_0/2$, (3.14), Lemma 9 and (3.16) we obtain

$$\begin{aligned}
&|\mathcal{G}(u, B_r) - \mathcal{E}(u, B_r)|\\
&\quad\le \int_{B_r} |k_0 - k(x)||Du|^{p_2(2r)}\, dx + \Lambda \int_{B_r} \left||Du|^{p_2(2r)} - |Du|^{p(x)}\right|\, dx\\
&\quad\le cr^{\alpha+n} \fint_{B_r} 1 + |Du|^{p_2(2r)(1+\sigma_0/2)}\, dx \le cr^{\alpha+n} \fint_{B_r} 1 + |Du|^{p_1(2r)(1+\sigma_0)}\, dx\\
&\quad\le cr^{\alpha+n} \left(\fint_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2}\, dx\right)^{1+\sigma_0}\\
&\quad\le cr^{\alpha+n-\gamma_2\sigma_0} \left((2r)^{p_2(2r)-n} \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2}\, dx\right)^{\sigma_0} \fint_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2}\, dx\\
&\quad\le cr^{\kappa}\varepsilon^{\gamma_1\sigma_0} \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2}\, dx,
\end{aligned}$$

where $c = c(\mathtt{data})$ and $\kappa := \alpha - \gamma_2\sigma_0 > 0$ because of (3.12). Choosing now σ as in (3.20), using also Lemma 10 and (3.14), in a totally similar way we get

$$
\begin{aligned}
&|\mathcal{E}(v, B_r) - \mathcal{G}(v, B_r)| \\
&\le cr^{\alpha+n} \fint_{B_r} 1 + |Dv|^{(1+\sigma)p_2(2r)}\,dx \le cr^{\alpha+n} \fint_{B_r} 1 + |Du|^{(1+\sigma)p_2(2r)}\,dx \\
&\le cr^{\alpha+n} \fint_{B_r} 1 + |Du|^{(1+\sigma_0)p(x)}\,dx \le cr^{\alpha+n} \left(\fint_{B_{2r}} (1 + |Du|)^{p(x)/2}\,dx\right)^{1+\sigma_0} \\
&\le cr^{\kappa}\varepsilon^{\gamma_1\sigma_0} \int_{B_{2r}} (1+|Du|^2)^{p_2(2r)/2}\,dx,
\end{aligned}
$$

where we set $\varepsilon_0 \equiv \sigma$ while applying Lemma 2 (*i.*). Here $c = c(\mathtt{data})$ and $\kappa > 0$ is as before. All in all, remembering that, by definition, $0 < \sigma < \sigma_0$, we can conclude

$$
\begin{aligned}
&\int_{B_r} (|Du|^2 + |Dv|^2)^{\frac{p_2(2r)-2}{2}} |Du - Dv|^2\,dx \\
&\quad\le c\left[\varepsilon^{\frac{\gamma_1\sigma}{1+\sigma}} + r^{\kappa}\varepsilon^{\gamma_1\sigma}\right] \int_{B_{2r}} (1+|Du|^2)^{p_2(2r)/2}\,dx.
\end{aligned} \tag{3.24}
$$

Since the next estimates will be slightly different for the cases $p_2(2r) \ge 2$ or $1 < p_2(2r) < 2$, we introduce the quantities

$$
\kappa_1 := \begin{cases} \kappa & 2 \le p_2(2r) \\ \frac{\kappa p_2(2r)}{2} & 1 < p_2(2r) < 2 \end{cases}, \quad \kappa_2 := \begin{cases} \gamma_1\sigma & 2 \le p_2(2r) \\ \frac{\gamma_1\sigma p_2(2r)}{2} & 1 < p_2(2r) < 2 \end{cases},
$$

$$
\kappa_3 := \begin{cases} \frac{\gamma_1\sigma}{1+\sigma} & 2 \le p_2(2r) \\ \frac{\gamma_1 p_2(2r)\sigma}{2(1+\sigma)} & 1 < p_2(2r) < 2 \end{cases}.
$$

Now, if $p_2(2r) \ge 2$, then we directly have

$$
\begin{aligned}
\int_{B_r} |Du - Dv|^{p_2(2r)}\,dx \le& c\int_{B_r} (|Du|^2 + |Dv|^2)^{\frac{p_2(2r)-2}{2}} |Du - Dv|^2\,dx \\
\le& c\left[\varepsilon^{\frac{\gamma_1\sigma}{1+\sigma}} + r^{\kappa}\varepsilon^{\gamma_1\sigma}\right] \int_{B_{2r}} (1+|Du|^2)^{p_2(2r)/2}\,dx,
\end{aligned} \tag{3.25}
$$

while, if $1 < p_2(2r) < 2$, by Hölder's inequality and the minimality of v we obtain

$$
\begin{aligned}
\int_{B_r} |Du - Dv|^{p_2(2r)}\,dx \le& \left(\int_{B_r} (|Du|^2 + |Dv|^2)^{\frac{p_2(2r)-2}{2}} |Du - Dv|^2\,dx\right)^{\frac{p_2(2r)}{2}} \\
& \left(\int_{B_r} (|Du|^2 + |Dv|^2)^{\frac{p_2(2r)}{2}}\,dx\right)^{\frac{2-p_2(2r)}{2}} \\
\le& c\left[\varepsilon^{\frac{\gamma_1\sigma p_2(2r)}{2(1+\sigma)}} + r^{\frac{\kappa p_2(2r)}{2}}\varepsilon^{\frac{\gamma_1\sigma p_2(2r)}{2}}\right] \int_{B_{2r}} (1+|Du|^2)^{p_2(2r)/2}\,dx.
\end{aligned} \tag{3.26}
$$

Coupling estimates (3.25) and (3.26), we can conclude in any case that

$$
\int_{B_r} |Du - Dv|^{p_2(2r)}\,dx \le c\left[\varepsilon^{\kappa_3} + r^{\kappa_1}\varepsilon^{\kappa_2}\right] \int_{B_{2r}} (1+|Du|^2)^{\frac{p_2(2r)}{2}}\,dx, \tag{3.27}
$$

where $c = c(\texttt{data})$. As mentioned before, our choice of σ_0 assures the positivity of κ_1 as well.

Step 2: harmonic approximation. We aim to apply Lemma 1 in order to obtain an unconstrained $p_2(2r)$-harmonic map suitably close to v. Hence, we need to transfer condition (3.16) from u to v. From the minimality of v in class $W_u^{1,p_2(2r)}(B_r, \mathcal{M})$ and (3.16) we see that

$$\begin{aligned} E(B_r) :=& \fint_{B_r} |Dv|^{p_2(2r)}\, dx \le \lambda^{-1} \fint_{B_r} k_0 |Du|^{p_2(2r)}\, dx \\ \le& \frac{2^{n-\gamma_1}\Lambda}{\lambda\omega_n} r^{-p_2(2r)} (2r)^{p_2(2r)-n} \int_{B_{2r}} (1+|Du|^2)^{p_2(2r)/2}\, dx \le c_*(\varepsilon/r)^{p_2(2r)}, \end{aligned} \tag{3.28}$$

where we set $c_* := \frac{2^{n-\gamma_1}\Lambda}{\lambda\omega_n} + 1$. Now we claim that v is approximately $p_2(2r)$-harmonic in the sense of (2.3). This is actually the case: in fact, with reference to the terminology used in Lemma 1, let $\tilde{d} \equiv d_2$, where $d_2 \in (0,1)$ is the exponent given by Lemma 7, pick any $\tilde{\theta} \in (0,1)$ and let $\tilde{\delta} = \tilde{\delta}(\tilde{\theta}, \tilde{d}, p_2(2r))$ be the "closeness" parameter appearing in (2.3). Moreover, for reasons that will be clear in a few lines, we also impose a first restriction on the size of ε. Precisely, keeping in mind the definition of c_*, we ask that

$$\varepsilon \le \min\left\{ \frac{\lambda\omega_n}{2^{n-\gamma_1}\Lambda + \lambda\omega_n}, \left(\frac{\tilde{\delta}\lambda}{\Lambda\gamma_2 c_{\mathcal{M}} + \lambda} \right)^{\frac{\gamma_1}{\gamma_1 - 1}} \right\}. \tag{3.29}$$

By (3.18), we estimate

$$\begin{aligned} &\left| \fint_{B_r} p_2(2r)|Dv|^{p_2(2r)-2} Dv \cdot D\varphi\, dx \right| \\ &= k_0^{-1} \left| \fint_{B_r} k_0 p_2(2r) |Dv|^{p_2(2r)-2} A_v(Dv, Dv)\varphi\, dx \right| \\ &\overset{(3.19)}{\le} \frac{\Lambda\gamma_2 c_{\mathcal{M}}}{\lambda} \fint_{B_r} |Dv|^{p_2(2r)} |\varphi|\, dx \le \tilde{c} \|D\varphi\|_{L^\infty(B_r)} r E(B_r), \end{aligned}$$

for all $\varphi \in C_c^\infty(B_r, \mathbb{R}^N)$, where $\tilde{c} := 1 + \frac{\Lambda\gamma_2 c_{\mathcal{M}}}{\lambda}$. For $\delta \in (0,1)$, by Young's inequality: $ab \le \delta a^p + \delta^{-\frac{1}{p-1}} b^{p'}$, with exponents $p_2(2r)$ and $p_2'(2r) := \frac{p_2(2r)}{p_2(2r)-1}$ we get

$$\begin{aligned} \tilde{c} E(B_r) r \|D\varphi\|_{L^\infty(B_r)} \le& \delta^{-\frac{1}{p_2(2r)-1}} \tilde{c}^{p'(2r)} (rE(B_r))^{p_2'(2r)} + \delta \|D\varphi\|_{L^\infty(B_r)}^{p_2(2r)} \\ \le& \delta^{-\frac{1}{\gamma_1 - 1}} \tilde{c}^{\gamma_1'} (rE(B_r))^{p_2'(2r)-1} (rE(B_r)) + \delta \|D\varphi\|_{L^\infty(B_r)}^{p_2(2r)} \\ \overset{(3.28)}{\le}& \delta^{-\frac{1}{\gamma_1 - 1}} \tilde{c}^{\gamma_1'} \left[r c_* (\varepsilon/r)^{p_2(2r)} \right]^{p_2'(2r)-1} (rE(B_r)) + \delta \|D\varphi\|_{L^\infty(B_r)}^{p_2(2r)} \\ \le& \delta^{-\frac{1}{\gamma_1 - 1}} \tilde{c}^{\gamma_1'} \left[c_* \varepsilon (\varepsilon/r)^{p_2(2r)-1} \right]^{p_2'(2r)-1} (rE(B_r)) + \delta \|D\varphi\|_{L^\infty(B_r)}^{p_2(2r)} \\ \overset{(3.29)}{\le}& \delta^{-\frac{1}{\gamma_1 - 1}} \tilde{c}^{\gamma_1'} (\varepsilon/r) r E(B_r) + \delta \|D\varphi\|_{L^\infty(B_r)}^{p_2(2r)} \\ \le& \delta^{-\frac{1}{\gamma_1 - 1}} \tilde{c}^{\gamma_1'} \varepsilon E(B_r) + \delta \|D\varphi\|_{L^\infty(B_r)}^{p_2(2r)}. \end{aligned}$$

Choosing $\delta \equiv \tilde{\delta}$ in the previous display, we can conclude that

$$\left| \fint_{B_r} p_2(2r)|Dv|^{p_2(2r)-2} Dv \cdot D\varphi \, dx \right| \overset{(3.29)}{\leq} \tilde{\delta} \fint_{B_r} \left(|Dv|^{p_2(2r)} + \|D\varphi\|_{L^\infty(B_r)}^{p_2(2r)} \right) dx,$$

thus Lemma 1 renders a map $\tilde{h} \in v + W_0^{1,p_2(2r)}(B_r, \mathbb{R}^N)$ solution to (2.4) with $p \equiv p_2(2r)$ and satisfying

$$\left(\fint_{B_r} |Dv - D\tilde{h}|^{p_2(2r)d_2} \, dx \right)^{\frac{1}{d_2}} \leq \tilde{\theta} \fint_{B_r} |Dv|^{p_2(2r)} \, dx. \tag{3.30}$$

Before going on, we would like to stress that ε depends on $\tilde{\theta}$ as well, since by looking at the dependencies of d_2, it is evident that $\tilde{\delta} = \tilde{\delta}(n, \gamma_1, \gamma_2, \tilde{\theta})$, and (3.29) yields in particular that $\varepsilon = \varepsilon(\tilde{\delta})$. This is not an obstruction, since the value of $\tilde{\theta}$ will be fixed in the next step as a function of $(\texttt{data})$.

Step 3: comparison, second time. First notice that, since v is a solution to the frozen Dirichlet problem (3.17), a Caccioppoli-type inequality holds. In fact, with the same strategy adopted for the proof of Lemma 8, we have

$$\fint_{B_\varrho} |Dv|^{p_2(2r)} \leq c \fint_{B_{2\varrho}} \left| \frac{v - (v)_{2\varrho}}{\varrho} \right|^{p_2(2r)} dx, \tag{3.31}$$

with $c = c(n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2)$ for all balls $B_{2\varrho} \Subset B_r$. Now fix any $\varrho \in (0, r/4)$. According to the previous estimates we can proceed in the following way

$$\begin{aligned}
&\int_{B_\varrho} (1 + |Du|^2)^{p_2(2r)/2} \\
&\leq c\varrho^n + c \left\{ \int_{B_\varrho} |Du - Dv|^{p_2(2r)} \, dx + \varrho^n \fint_{B_\varrho} |Dv|^{p_2(2r)} \, dx \right\} \\
&\overset{(3.31)}{\leq} c\varrho^n + c \left\{ \int_{B_r} |Du - Dv|^{p_2(2r)} \, dx + \varrho^n \fint_{B_{2\varrho}} \left| \frac{v - (v)_{2\varrho}}{\varrho} \right|^{p_2(2r)} dx \right\} \\
&\overset{(3.27)}{\leq} c\varrho^n + c \left\{ \left[\varepsilon^{\kappa_3} + r^{\kappa_1} \varepsilon^{\kappa_2} \right] \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2} \, dx + \varrho^n \left(\fint_{B_{2\varrho}} |Dv|^{p_2(2r)d_2} \, dx \right)^{\frac{1}{d_2}} \right\} \\
&\leq c\varrho^n + c \left[\varepsilon^{\kappa_3} + r^{\kappa_1} \varepsilon^{\kappa_2} \right] \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2} \, dx \\
&\quad + c\varrho^n \left\{ \left(\fint_{B_{2\varrho}} |Dv - D\tilde{h}|^{p_2(2r)d_2} \, dx \right)^{\frac{1}{d_2}} + \varrho^n \fint_{B_{2\varrho}} |D\tilde{h}|^{p_2(2r)} \, dx \right\} \\
&\overset{(3.30)}{\leq} c\varrho^n + c \left[\varepsilon^{\kappa_3} + r^{\kappa_1} \varepsilon^{\kappa_2} \right] \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2} \, dx \\
&\quad + c(r/\varrho)^{n(d_2^{-1}-1)} \tilde{\theta} \int_{B_r} |Dv|^{p_2(2r)} \, dx + c\varrho^n \fint_{B_{2\varrho}} |D\tilde{h}|^{p_2(2r)} \, dx \\
&\overset{(2.2)}{\leq} c\varrho^n + c \left[\varepsilon^{\kappa_3} + r^{\kappa_1} \varepsilon^{\kappa_2} + (r/\varrho)^{n(d_2^{-1}-1)} \tilde{\theta} \right] \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2} \, dx \\
&\quad + c(\varrho/r)^n \int_{B_r} |Dv|^{p_2(2r)} \, dx \\
&\leq c\varrho^n + c \left[\varepsilon^{\kappa_3} + r^{\kappa_1} \varepsilon^{\kappa_2} + (r/\varrho)^{n(d_2^{-1}-1)} \tilde{\theta} + (\varrho/r)^n \right] \int_{B_{2r}} (1 + |Du|^2)^{p_2(2r)/2} \, dx,
\end{aligned}$$

where we also used Lemma 7, the minimality of v in the Dirichlet class $W_u^{1,p_2(2r)}(B_r, \mathcal{M})$ and of $\tilde{h}$ in $v + W_0^{1,p_2(2r)}(B_r, \mathbb{R}^N)$ and the reference estimate in (2.2)$_1$ which holds for $\tilde{h}$ since $\tilde{h}$ is a solution to (2.4) with $p \equiv p_2(2r)$, (set $k_0 \equiv 1$, $p \equiv p_2(2r)$, $g^{\alpha\beta} \equiv \delta^{\alpha\beta}$ and $h_{ij} \equiv \delta_{ij}$ for $\alpha, \beta \in \{1, \cdots, n\}$ and $i, j \in \{1, \cdots, N\}$ in (2.1)). Here $c = c(\texttt{data})$. For the ease of exposition, let us set $s \equiv 2r$. Hence we can rewrite the previous estimate as

$$\int_{B_\varrho} (1 + |Du|^2)^{p_2(s)/2}\, dx \le c_0 \varrho^n + c_1 \left[\varepsilon^{\kappa_3} + s^{\kappa_1}\varepsilon^{\kappa_2} + (s/\varrho)^{n(d_2^{-1}-1)}\tilde{\theta} + (\varrho/s)^n\right] \int_{B_s} (1 + |Du|^2)^{p_2(s)/2}\, dx, \tag{3.32}$$

for $c_0 = c_0(n, \gamma_1, \gamma_2)$ and $c_1 = c_1(\texttt{data})$.

Step 4: Morrey-type estimates. Our goal now is to prove a Morrey type estimate for the energy which will eventually lead to the continuity of solutions. For $\tau \in \left(0, \frac{1}{4}\right)$, we let $\varrho \equiv \tau s$ in (3.32) and multiply both sides of it by $(\tau s)^{p_2(s)-n}$. We then have

$$\begin{aligned}(\tau s)^{p_2(s)-n} &\int_{B_{\tau s}} (1 + |Du|^2)^{p_2(s)/2}\, dx \\ &\le c_1 \tau^{p_2(s)} \left\{\tau^{-n}(\varepsilon^{\kappa_3} + s^{\kappa_1}\varepsilon^{\kappa_2}) + \tau^{-nd_2^{-1}}\tilde{\theta} + 1\right\} s^{p_2(s)-n} \\ &\quad \int_{B_s} (1 + |Du|^2)^{p_2(s)/2}\, dx + c_0(\tau s)^{p_2(s)}.\end{aligned} \tag{3.33}$$

Adopting the notation introduced in [42], we consider the following quantities:

$$\begin{aligned}\phi(r, p) &:= \left(r^p \fint_{B_r} (1 + |Du|^2)^{p/2}\, dx\right)^{\frac{1}{p}} = \omega_n^{-\frac{1}{p}} \left(r^{p-n} \int_{B_r} (1 + |Du|^2)^{p/2}\, dx\right)^{\frac{1}{p}}, \\ \psi(r) &:= \phi(r, p_2(r)).\end{aligned}$$

In these terms, (3.33) reads as

$$\phi^{p_2(s)}(\tau s, p_2(s)) \le c_1 \tau^{p_2(s)} \left\{\tau^{-n}(\varepsilon^{\kappa_3} + s^{\kappa_1}\varepsilon^{\kappa_2}) + \tau^{-nd_2^{-1}}\tilde{\theta} + 1\right\} \psi^{p_2(s)}(s) + c_0(\tau s)^{p_2(s)}, \tag{3.34}$$

so recalling that

$$\phi(r, p) \le \phi(r, q) \quad \text{for} \quad p \le q, \tag{3.35}$$

we obtain from (3.34)

$$\psi(\tau s) \le c_2 \tau \left\{\tau^{-\frac{n}{p_2(s)}} \left[\varepsilon^{\frac{\kappa_3}{p_2(s)}} + s^{\frac{\kappa_1}{p_2(s)}} \varepsilon^{\frac{\kappa_2}{p_2(s)}}\right] + \tau^{-\frac{n}{d_2 p_2(s)}} \tilde{\theta}^{\frac{1}{p_2(s)}} + 1\right\} \psi(s) + c_3(\tau s), \tag{3.36}$$

where $c_2 = c_2(\texttt{data})$ and $c_3 = c_3(n, \gamma_1, \gamma_2)$. Since $\tau, s, \varepsilon, \tilde{\theta} \in (0, 1)$, from the definitions of κ_1, κ_2 and κ_3 we have

$$\begin{aligned}&\tau^{-\frac{n}{p_2(s)}} \le \tau^{-\frac{n}{\gamma_1}}, \quad \tau^{-\frac{n}{d_2 p_2(s)}} \le \tau^{-\frac{n}{d_2\gamma_1}}, \quad s^{\frac{\kappa_1}{p_2(s)}} \le s^{\tilde{\kappa}_1}, \quad \varepsilon^{\frac{\kappa_2}{p_2(s)}} \le \varepsilon^{\tilde{\kappa}_2}, \\ &\varepsilon^{\frac{\kappa_3}{p_2(s)}} \le \varepsilon^{\tilde{\kappa}_3}, \quad \tilde{\theta}^{\frac{1}{p_2(s)}} \le \tilde{\theta}^{\frac{1}{\gamma_2}},\end{aligned}$$

where

$$\tilde{\kappa}_1 := \begin{cases} \frac{\kappa}{\gamma_2} & 2 \le p_2(s) \\ \frac{\kappa}{2} & 1 < p_2(s) < 2 \end{cases}, \quad \tilde{\kappa}_2 := \begin{cases} \frac{\gamma_1\sigma}{\gamma_2} & 2 \le p_2(s) \\ \frac{\gamma_1\sigma}{2} & 1 < p_2(s) < 2 \end{cases},$$

$$\tilde{\kappa}_3 := \begin{cases} \frac{\gamma_1\sigma}{\gamma_2(1+\sigma)} & 2 \le p_2(s) \\ \frac{\gamma_1\sigma}{2(1+\sigma)} & 1 < p_2(s) < 2 \end{cases},$$

thus (3.36) becomes

$$\psi(\tau s) \le c_2\tau \left\{ \tau^{-\frac{n}{\gamma_1}} \left(\varepsilon^{\tilde{\kappa}_3} + s^{\tilde{\kappa}_1}\varepsilon^{\tilde{\kappa}_2} \right) + \tau^{-\frac{n}{d_2\gamma_1}} \tilde{\theta}^{\frac{1}{\gamma_2}} + 1 \right\} \psi(s) + c_3(\tau s). \tag{3.37}$$

Now we need to make a proper choice of the parameters appearing in (3.37). We first select any $\beta \in (0, 1)$ and an $\eta \in (\beta, 1)$ and ask that $c_2\tau \le \tau^\eta/5$, thus obtaining

$$\psi(\tau s) \le (\tau^\eta/5) \left\{ \tau^{-\frac{n}{\gamma_1}} \left(\varepsilon^{\tilde{\kappa}_3} + s^{\tilde{\kappa}_1}\varepsilon^{\tilde{\kappa}_2} \right) + \tau^{-\frac{n}{d_2\gamma_1}} \tilde{\theta}^{\frac{1}{\gamma_2}} + 1 \right\} \psi(s) + c_3(\tau s)^\beta.$$

Moreover, we require that the threshold radius R_* introduced at the beginning of *Step 1* satisfies $c_3(\tau R_*)^\beta \le (\varepsilon/5)$. Finally we recall that $\tilde{\theta}$ is arbitrary, therefore we fix $\tilde{\theta} = 2^{-\gamma_2}\tau^{\frac{n\gamma_2}{d_2\gamma_1}}$ and, since $\tilde{\kappa}_2 \ge \tilde{\kappa}_3$ renders $\varepsilon^{\tilde{\kappa}_2} \le \varepsilon^{\tilde{\kappa}_3}$, the choice

$$\varepsilon \le \left\{ \left(\frac{2^{n-\gamma_1}\Lambda}{\lambda\omega_n} + 1 \right)^{-1}, \left(\frac{\tilde{\delta}\lambda}{\Lambda\gamma_2 c m + \lambda} \right)^{\frac{\gamma_1}{\gamma_1 - 1}}, \tau^{\frac{n}{\gamma_1\tilde{\kappa}}} \right\}$$

and (3.16) allow concluding that

$$\psi(\tau s) \le \tau^\eta\psi(s) + c_3(\tau s)^\beta \quad \text{and} \quad \psi(\tau s) \le \frac{4}{5}\tau^\eta\psi(s) + \frac{\varepsilon}{5} < \varepsilon. \tag{3.38}$$

We remark that, since η is ultimately influenced only by the choice of β, we can incorporate the dependency from η in the one from β, so the above procedure defines the following dependencies: $\tau = \tau(\texttt{data}, \beta)$, $\varepsilon = (\texttt{data}, \beta)$ and $R_* = R_*(\texttt{data}, \beta)$. Estimate $(3.38)_2$ legalizes iterations, so we can repeat $(3.38)_1$ replacing s by $\tau s, \tau^2 s, \tau^3 s, \cdots$ to get

$$\psi(\tau^{j+1}s) \le \tau^{\eta(j+1)}\psi(s) + c_3 s^\beta \tau^{\beta(j+1)} \sum_{i=0}^{j} \tau^{i(\eta-\beta)} \le \tau^{(j+1)\eta}\psi(s) + c_4 s^\beta \tau^{(j+1)\beta}, \tag{3.39}$$

for $c_4 = c_4(\texttt{data}, \beta)$. Now, for any $\varsigma \in (0, s/8)$ we can find an integer $j \ge 1$ with $\tau^{j+1}s < \varsigma \le \tau^j s$, so, from (3.39) we obtain

$$\begin{aligned} \psi(\varsigma) \le& \tau^{1-\frac{n}{\gamma_1}}\psi(\tau^j s) \le \tau^{1-\frac{n}{\gamma_1}} \left\{ \tau^{\eta j}\psi(s) + c_4 s^\beta \tau^{\beta j} \right\} \\ \le& \tau^{-\frac{n}{\gamma_1}} \left\{ (\varsigma/s)^\eta \psi(s) + c_4 s^\beta (\varsigma/s)^\beta \right\} \\ \overset{(3.16)}{\le}& \tau^{-\frac{n}{\gamma_1}} \left\{ (\varsigma/s)^\beta \varepsilon + c_4 \varsigma^\beta \right\} \le c_5(\varsigma/s)^\beta, \end{aligned} \tag{3.40}$$

with $c_5 = c_5(\texttt{data}, \beta)$, while if $\varsigma \in [s/8, s)$, since $(s/\varsigma) \le 8$, there obviously holds

$$\psi(\varsigma) \le 8^{\frac{n}{\gamma_1}+\beta-1}(\varsigma/s)^\beta \left\{ \psi(s) + s^\beta \right\} \le 8^{\frac{n}{\gamma_1}+\beta-1}(\varsigma/s)^\beta(\varepsilon + s^\beta) \le c(\varsigma/s)^\beta, \tag{3.41}$$

again for $c = c(\mathtt{data}, \beta)$. We can actually improve estimates (3.40)-(3.41) by getting rid of the restriction $s \le R_*$; we shall only retain $s \le \tilde{R}_*/2$. In fact, in case $0 < \varsigma < R_* \le s \le \tilde{R}_*/2$, we see that

$$\psi(\varsigma) \overset{(3.40)}{\le} c_5(\varsigma/R_*)^\beta \le c(\varsigma/s)^\beta, \tag{3.42}$$

with $c = c(\mathtt{data}, \beta)$, while, if $0 < R_* < \varsigma \le s \le \tilde{R}_*/2$ we have

$$\psi(\varsigma) \le (s/\varsigma)^{\frac{n}{\gamma_1}-1}\psi(s) \le R_*^{1-\frac{n}{\gamma_1}}(R_*/\varsigma)^{\frac{n}{\gamma_1}-1+\beta}(\varsigma/s)^\beta\psi(s) \le c(\varsigma/s)^\beta, \tag{3.43}$$

again for $c = c(\mathtt{data}, \beta)$. Now, by the continuity of Lebesgue's integral and of the mapping $y \mapsto p_2(y, s) := \sup_{x \in B_s(y)} p(x)$, we can conclude that if (3.16) holds for $B_s(x_0)$ then it holds also on $B_s(y)$ for all y belonging to a sufficiently small neighborhood of x_0. Hence, if we let

$$D_0 := \left\{ y \in B_{\tilde{R}_*}(x_0) \colon \left(s^{p_2(y,s)-n} \int_{B_s(y)} (1 + |Du|^2)^{p_2(y,s)/2} \, dx \right)^{\frac{1}{p_2(y,s)}} < \varepsilon \text{ for some } s \le \tilde{R}_*/2;\ B_s(y) \Subset B_{\tilde{R}_*}(x_0) \right\},$$

we see that it is open and, taking radii $0 < \varsigma < s$, for $y \in D_0$ we have

$$\varsigma^{-n+\gamma_1-\beta\gamma_1} \int_{B_\varsigma(y)} |Du|^{\gamma_1} \, dx = \omega_n \varsigma^{-\beta\gamma_1}\phi^{\gamma_1}(\varsigma, \gamma_1) \overset{(3.35)}{\le} \omega_n \varsigma^{-\beta\gamma_1}\psi^{\gamma_1}(\varsigma) \overset{(3.40)}{\le} c_6 s^{-\beta\gamma_1}, \tag{3.44}$$

for $c_6 = c_6(\mathtt{data}, \beta)$, so, from Morrey's growth theorem, $u \in C^{0,\beta}(D_0, \mathcal{M})$. We would like to stress that β is an arbitrary number in $(0, 1)$ and, being the interval open it is always possible to find $\eta \in (0, 1)$ so that $\beta < \eta < 1$. Hence we can take any $\beta \in (0, 1)$ in the above estimates and deduce from (3.44) that actually $u \in C^{0,\beta}(D_0, \mathcal{M})$ for all $\beta \in (0, 1)$. Of course, the values of all the parameters involved will change accordingly to the one of β (and consequently of η) we choose. After a standard covering argument, we obtain that $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega_0, \mathcal{M})$ for any $\beta \in (0, 1)$. Now consider any open subset $\tilde{\Omega} \Subset \Omega_0$. From (3.40)-(3.43) and a standard covering argument, we also obtain the Morrey type estimate

$$\fint_{B_\varsigma} |Du|^{p_2(\varsigma)} \, dx \le c_7 \varsigma^{-\gamma_2(1-\beta)}, \tag{3.45}$$

for all $B_\varsigma \Subset \tilde{\Omega}$, $\varsigma \le \tilde{R}_*/2$ and any $\beta \in (0, 1)$. Here $c_7 = c_7(\mathtt{data}, \beta)$.

Step 5: Hausdorff dimension of the Singular Set. Given the characterization of D_0, we easily see that, if $\Sigma_0(u, B_{\tilde{R}_*}(x_0)) := B_{\tilde{R}_*}(x_0) \setminus D_0$, then

$$\Sigma_0(u, B_{\tilde{R}_*}(x_0)) \subset \left\{ y \in B_{\tilde{R}_*}(x_0) \colon \limsup_{s \to 0} \left(s^{p_2(y,s)-n} \int_{B_s(y)} |Du|^{p_2(y,s)} \, dy \right)^{\frac{1}{p_2(y,s)}} > 0 \right\}.$$

Now, if $p_m(x_0, \tilde{R}_*) := \inf_{x \in B_{\tilde{R}_*}(x_0)} p(x)$, then, as in (3.14),

$$p_2(y, s) \le (1 + \sigma_0) p_m(x_0, \tilde{R}_*) \ \text{ for all } \ 0 < s \le \tilde{R}_*/2, \quad B_s(y) \Subset B_{\tilde{R}_*}(x_0), \tag{3.46}$$

so we obtain,

$$\begin{aligned}&\left(s^{p_2(y,s)} \fint_{B_s(y)} |Du|^{p_2(y,s)}\, dx\right)^{\frac{1}{p_2(y,s)}}\\ &\overset{(3.46)}{\leq} \left(s^{p_m(x_0,\tilde{R}_*)(1+\sigma_0)} \fint_{B_s(y)} |Du|^{p_m(x_0,\tilde{R}_*)(1+\sigma_0)}\, dx\right)^{\frac{1}{p_m(x_0,\tilde{R}_*)(1+\sigma_0)}}\\ &\leq \left(s^{p_m(x_0,\tilde{R}_*)(1+\delta_0)} \fint_{B_s(y)} |Du|^{p_m(x_0,\tilde{R}_*)(1+\delta_0)}\, dx\right)^{\frac{1}{p_m(x_0,\tilde{R}_*)(1+\delta_0)}},\end{aligned}$$

where we also used that $\sigma_0 < \delta_0$. Hence, if $y \in B_{\tilde{R}_*}(x_0)$ is such that

$$0 < \limsup_{s\to 0} \left(s^{p_2(y,s)-n} \int_{B_s(y)} |Du|^{p_2(y,s)}\, dx\right)^{\frac{1}{p_2(y,s)}},$$

then

$$0 < \limsup_{s\to 0} \left(s^{p_m(x_0,\tilde{R}_*)(1+\delta_0)-n} \int_{B_s(y)} |Du|^{p_m(x_0,\tilde{R}_*)(1+\delta_0)}\, dx\right)^{\frac{1}{p_m(x_0,\tilde{R}_*)(1+\delta_0)}}.$$

This allows to conclude that

$$\begin{aligned}\Sigma_0(u, B_{\tilde{R}_*}(x_0)) \subset \Bigg\{ y \in B_{\tilde{R}_*(x_0)} \colon \limsup_{s\to 0} \Bigg(& s^{p_m(x_0,\tilde{R}_*)(1+\delta_0)-n}\\ & \int_{B_s(y)} |Du|^{p_m(x_0,\tilde{R}_*)(1+\delta_0)}\, dx\Bigg)^{\frac{1}{p_m(x_0,\tilde{R}_*)(1+\delta_0)}} > 0\Bigg\} =: D_1.\end{aligned}$$

By [26, Proposition 2.7] it follows that $\dim_{\mathcal{H}}(D_1) \leq n - p_m(x_0, \tilde{R}_*)(1+\delta_0)$. Now, covering Ω with balls having the same features of $B_{\tilde{R}_*}(x_0)$ and remembering that $p_m(x_0, \tilde{R}_*) \geq \gamma_1$, we obtain that $\dim_{\mathcal{H}}(\Sigma_0(u)) \leq n - \gamma_1(1+\delta_0) < n - \gamma_1$, and so $\dim_{\mathcal{H}}(\Sigma_0(u)) < n - \gamma_1$.

Step 6: partial C^{1,β_0}-regularity. In this part we follow the approach of [28, Theorem 3.1]. So far we know that the regular set $\Omega_0 \subset \Omega$ is a relatively open set of full n-dimensional Lebesgue measure and $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega_0, \mathcal{M})$ for all $\beta \in (0,1)$. For reasons that will be clear in a few lines, we fix

$$\tilde{\beta} := \max\left\{\frac{1}{2}, 1 - \frac{1}{4\gamma_2} \min\left\{\frac{1}{2}, \alpha - n\sigma_0, \frac{\gamma_1\sigma}{2(1+\sigma)}\right\}\right\} \in (0,1), \tag{3.47}$$

where σ_0, σ are as in (3.12)-(3.20) respectively, and two open subsets $\tilde{\Omega} \Subset \Omega' \Subset \Omega_0$. Given the expression of $\tilde{\beta}$, we shall incorporate any dependency from $\tilde{\beta}$ of the constants appearing in the forthcoming estimates into the one from (`data`). We cover $\tilde{\Omega}$ with finitely many balls contained in Ω', (with size and number depending only on $\mathcal{M}$, $[u]_{0,\tilde{\beta};\Omega'}$ and on $\mathrm{diam}(\Omega')$), whose image lies in small coordinate neighborhoods of $\mathcal{M}$. Precisely, by the continuity of u and up to scaling, rotating and translating $\mathcal{M}$ we can now assume that $u(\tilde{\Omega})$ is contained into the image of a single chart $f(B^m_1)$, so we can find an $\omega\colon \tilde{\Omega} \to \mathbb{R}^m$ such that $u = f(\omega)$ and $|\omega| \leq 1$. Here $f : \mathbb{R}^m \mapsto \mathcal{M}$ is such that

$$\|\nabla f\|_{L^\infty(B^m_{4m})} \leq c(\mathcal{M}), \quad \|\nabla^2 f\|_{L^\infty(B^m_{4m})} \leq c(\mathcal{M}) \quad \text{and} \quad |\nabla(f^{-1})(u)| \leq c(\mathcal{M}). \tag{3.48}$$

The above conditions are for instance satisfied by the inverse of the stereographic projection

$$S\colon \mathbb{R}^{N-1} \ni y \mapsto \left(\frac{|y|^2-1}{|y|^2+1}, \frac{2y}{|y|^2+1}\right) \in \mathbb{S}^{N-1},$$

see [10,28]. From $(3.48)_3$ we get that

$$\int_U |D\omega|^{p(x)}\, dx \le c \int_U |Du|^{p(x)}\, dx, \tag{3.49}$$

for any $U \subseteq \tilde{\Omega}$, with $c = c(\mathcal{M}, \gamma_1, \gamma_2)$. Since u is an $\mathcal{M}$-constrained local minimizer of (0.1), then ω minimizes the variational integral

$$W^{1,p(\cdot)}(\tilde{\Omega}, \mathbb{R}^m) \ni \zeta \mapsto \mathcal{H}(\zeta, \tilde{\Omega}) := \int_{\tilde{\Omega}} k(x)(\delta^{\alpha\beta} h_{ij}(\zeta) D_\alpha \zeta^i D_\beta \zeta^j)^{p(x)/2}\, dx, \tag{3.50}$$

where $(\delta^{\alpha\beta})_{\alpha\beta}$ is the $n \times n$ identity matrix and $(h_{ij})_{ij}$ is the $m \times m$ symmetric matrix $((\nabla f)^T \nabla f)_{ij}$. From (3.48) and being f a chart, $(h_{ij})_{ij}$ is uniformly elliptic and uniformly bounded, in the sense that

$$\sup_{i,j \in \{1,\cdots,m\}} \|h_{ij}\|_{L^\infty(B^m_{4m})} < c \quad \text{and} \quad c_1|\zeta|^2 \le h_{ij}(y)\zeta^i\zeta^j \le c_2|\zeta|^2$$

for all $\zeta \in \mathbb{R}^{m\times m}$, whenever $|y| \le 4m$. Here c, c_1, c_2 depend only on $\mathcal{M}$. Given the previous considerations, it is easy to see that the integrand

$$H(x, y, z) := k(x)(\delta^{\alpha\beta} h_{ij}(y) z^i_\alpha z^j_\beta)^{p(x)/2}$$

satisfies the following set of conditions:

$$\begin{cases} c_1|z|^{p(x)} \le H(x, y, z) \le c_2|z|^{p(x)} \\ |H(x_1, y, z) - H(x_2, y, z)| \le c_{\varepsilon_0}|x_1 - x_2|^\alpha \left(1 + |z|^{(1+\varepsilon_0)\max\{p(x_1), p(x_2)\}}\right) \quad \text{for any } \varepsilon_0 > 0 \\ |H(x, y_1, z) - H(x, y_2, z)| \le c|y_1 - y_2||z|^{p(x)} \\ |\partial H(x, y, z)||z| + |\partial^2 H(x, y, z)||z|^2 \le c|z|^{p(x)} \\ \langle \partial^2 H(x, y, z)\xi, \xi\rangle \ge c|z|^{p(x)-2}|\xi|^2, \end{cases} \tag{3.51}$$

for $|y| \le 4m$. Here, all the constants depend only on m, $\mathcal{M}$, λ, Λ, γ_1, γ_2, $[k]_{0,\alpha}$, $[p]_{0,\alpha}$ and α, except for c_{ε_0}, which, in addition, depends also from ε_0. In particular, from $(3.51)_1$, we see that ω minimizes a functional controlled from below and above by the $p(\cdot)$-laplacean energy, so there is no loss of generality in assuming that Lemmas 8 and 9 (and 10 for the associated frozen problem) hold true with the same parameters as before. Moreover, $(3.48)_3$, (3.49) and (3.45) allow transferring regularity from u to ω. In fact we have

$$\omega \in C^{0,\beta}(\tilde{\Omega}, \mathbb{R}^m), \quad [\omega]_{0,\beta;\tilde{\Omega}} \le c(\mathcal{M})[u]_{0,\beta;\tilde{\Omega}}, \quad \fint_{B_\varrho} |D\omega|^{p_2(\varrho)}\, dx \le c\varrho^{-\gamma_2(1-\beta)}, \tag{3.52}$$

for any $\beta \in (0, 1)$ and all $B_\varrho \Subset \tilde{\Omega}$. Notice that, by (3.49) and $(3.52)_2$ we can incorporate any dependency from $\|(|D\omega|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}$ or from $[\omega]_{0,\beta;\tilde{\Omega}}$ of the constants in the forthcoming estimates into the one from $\|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}$ or from $[u]_{0,\beta;\tilde{\Omega}}$. In (3.52) we are going to choose $\beta \equiv \tilde{\beta}$, where $\tilde{\beta}$ is as in (3.47). Let σ_0, $\tilde{R}_*$ and σ be as in (3.12), (3.13) and (3.20) respectively and fix any ball $B_\varrho \Subset B_{\tilde{R}_*} \Subset \tilde{\Omega}$, $\varrho \le \tilde{R}_*/2$ and let $\vartheta \in W^{1,p_2(\varrho)}(B_{\varrho/4},\mathbb{R}^m)$ be the solution to the frozen Dirichlet problem

$$\omega + W_0^{1,p_2(\varrho)}(B_{\varrho/4},\mathbb{R}^m) \ni \zeta \mapsto \min \int_{B_{\varrho/4}} k_0(\delta^{\alpha\beta}h_{ij}((\omega)_{\varrho/4})D_\alpha\zeta^i D_\beta\zeta^j)^{p(\varrho)/2}\,dx, \tag{3.53}$$

where k_0 is the value of $k(\cdot)$ in the centre of $B_{\varrho/4}$. For simplicity, define $H_0(y,z) := k_0(\delta^{\alpha\beta}h_{ij}(y)z^i_\alpha z^j_\beta)^{p_2(\varrho)/2}$, and notice that, since $|(\omega)_{\varrho/4}| \le 1$, then the integrand $H_0((\omega)_{\varrho/4},z)$ is of the type covered by Proposition 1, see [2,20,24]. Furthermore, given the specific structure of the integrand, the Maximum principle in [13] applies, thus $\sup_{x\in B_{\varrho/4}}|\vartheta(x)| \le m$. By (3.14), ω is an admissible competitor for ϑ in problem (3.53) and, as a consequence

$$\int_{B_{\varrho/4}} \partial H_0((\omega)_{\varrho/4},D\vartheta)(D\omega - D\vartheta)\,dx = 0. \tag{3.54}$$

Taking into account $(3.51)_5$ (with $p_2(\varrho)$ instead of $p(x)$) and (3.54) we then estimate

$$\begin{aligned}
c\int_{B_{\varrho/4}} &(|D\omega|^2+|D\vartheta|^2)^{\frac{p_2(\varrho)-2}{2}}|D\omega-D\vartheta|^2\,dx + c\int_{B_{\varrho/4}} \partial H_0((\omega)_{\varrho/4},D\vartheta)(D\omega-D\vartheta)\,dx\\
&= c\int_{B_{\varrho/4}} (|D\omega|^2+|D\vartheta|^2)^{\frac{p_2(\varrho)-2}{2}}|D\omega-D\vartheta|^2\,dx\\
&\le \int_{B_{\varrho/4}} H_0((\omega)_{\varrho/4},D\omega) - H_0((\omega)_{\varrho/4},D\vartheta)\,dx\\
&= \int_{B_{\varrho/4}} H_0((\omega)_{\varrho/4},D\omega) - H(x,(\omega)_{\varrho/4},D\omega)\,dx\\
&\quad + \int_{B_{\varrho/4}} H(x,(\omega)_{\varrho/4},D\omega) - H(x,\omega,D\omega)\,dx\\
&\quad + \int_{B_{\varrho/4}} H(x,\omega,D\omega) - H(x,\vartheta,D\vartheta)\,dx\\
&\quad + \int_{B_{\varrho/4}} H(x,\vartheta,D\vartheta) - H(x,(\vartheta)_{\varrho/4},D\vartheta)\,dx\\
&\quad + \int_{B_{\varrho/4}} H(x,(\vartheta)_{\rho/4},D\vartheta) - H_0((\vartheta)_{\varrho/4},D\vartheta)\,dx\\
&\quad + \int_{B_{\varrho/4}} H_0((\vartheta)_{\varrho/4},D\vartheta) - H_0((\omega)_{\varrho/4},D\vartheta)\,dx = \sum_{i=1}^{6}(\mathrm{I})_i,
\end{aligned}$$

where $c = c(m,\mathcal{M},\lambda,\Lambda,\gamma_1,\gamma_2,[k]_{0,\alpha},[p]_{0,\alpha})$. Before estimating terms $(\mathrm{I})_1$-$(\mathrm{I})_6$, let us take care of some quantities which will be recurrent in the forthcoming estimates. By $(3.52)_{1,2}$, we easily have

$$\sup_{x\in B_{\varrho/4}} |\omega(x)-(\omega)_{\varrho/4}| \le c\varrho^{\tilde{\beta}}, \tag{3.55}$$

with $c = c(\mathcal{M},[u]_{0,\tilde{\beta};\tilde{\Omega}})$. Moreover, it follows from the convex-hull property in [13] that

$$\sup_{x,y\in B_{\varrho/4}} |\vartheta(x)-\vartheta(y)| \le \sup_{x,y\in\partial B_{\varrho/4}} |\omega(x)-\omega(y)| \overset{(3.52)_2}{\le} c\varrho^{\tilde{\beta}},$$

for $c = c(\mathcal{M}, [u]_{0,\tilde{\beta};\tilde{\Omega}})$, therefore

$$\sup_{x\in B_{\varrho/4}} |\vartheta(x) - (\vartheta)_{\varrho/4}| \le c\varrho^{\tilde{\beta}}. \tag{3.56}$$

Finally, from Poincaré's and Hölder's inequalities, Lemmas 9, 2 (*ii.*), 8 and by the minimality of ϑ we see that

$$\begin{aligned}
&\fint_{B_{\varrho/4}} |(\omega)_{\varrho/4} - (\vartheta)_{\varrho/4}|^{p_2(\varrho)}\, dx \\
&\le c\varrho^{p_2(\varrho)} \fint_{B_{\varrho}/4} |D\omega|^{p_2(\varrho)}\, dx \overset{(3.14)}{\le} c\varrho^{p_2(\varrho)} \left(\fint_{B_{\varrho/4}} |D\omega|^{(1+\sigma_0)p(x)}\, dx\right)^{\frac{p_2(\varrho)}{(1+\sigma_0)p_1(\varrho)}} \\
&\le c\varrho^{p_2(\varrho)} \left(\fint_{B_{\varrho/2}} (1+|D\omega|^2)^{p(x)/2}\, dx\right)^{\frac{p_2(\varrho)-p_1(\varrho)}{p_1(\varrho)}} \fint_{B_{\varrho/2}} (1+|D\omega|^2)^{p(x)/2}\, dx \\
&\le c\varrho^{p_2(\varrho)} \fint_{B_\varrho} 1 + \left|\frac{\omega - (\omega)_\varrho}{\varrho}\right|^{p(x)} dx \overset{(3.52)_{1,2}}{\le} c\varrho^{p_2(\varrho)+(\tilde{\beta}-1)p_2(\varrho)} = c\varrho^{\tilde{\beta}p_2(\varrho)},
\end{aligned} \tag{3.57}$$

with $c = c(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\tilde{\beta};\tilde{\Omega}})$. From $(3.51)_2$ with $\varepsilon_0 \equiv \sigma_0/2$ and Lemma 9 we get

$$\begin{aligned}
|(\mathrm{I})_1| \le& c\varrho^{\alpha+n} \fint_{B_{\varrho/4}} 1 + |D\omega|^{(1+\varepsilon_0)p_2(\varrho)}\, dx \overset{(3.14)}{\le} c\varrho^{\alpha+n} \fint_{B_{\varrho/4}} (1+|D\omega|^2)^{(1+\sigma_0)p(x)/2}\, dx \\
\le& c\varrho^{\alpha+n} \left(\fint_{B_{\varrho/2}} (1+|D\omega|)^{p(x)/2}\, dx\right)^{\sigma_0} \fint_{B_\varrho} (1+|D\omega|^2)^{p_2(\varrho)/2}\, dx \\
\le& c\varrho^{\bar{\kappa}_1} \int_{B_\varrho} (1+|D\omega|^2)^{p_2(\varrho)/2}\, dx,
\end{aligned} \tag{3.58}$$

where $c = c(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})})$ and $\bar{\kappa}_1 := \alpha - n\sigma_0 > 0$ by (3.12). Now, from $(3.51)_3$ and (3.55) we have

$$|(\mathrm{I})_2| \le c \int_{B_{\varrho/4}} |\omega - (\omega)_{\varrho/4}||D\omega|^{p(x)}\, dx \le c\varrho^{\tilde{\beta}} \int_{B_\varrho} (1+|D\omega|^2)^{p_2(\varrho)/2}\, dx, \tag{3.59}$$

for $c = c(\texttt{data}, [u]_{0,\tilde{\beta};\tilde{\Omega}})$. Since ω is a local minimizer of (3.50), then

$$(\mathrm{I})_3 \le 0. \tag{3.60}$$

Concerning term $(\mathrm{I})_4$, we use $(3.51)_3$, (3.56) and the minimality of ϑ to bound

$$\begin{aligned}
|(\mathrm{I})_4| \le& c \int_{B_{\varrho/4}} |\vartheta - (\vartheta)_{\varrho/4}||D\vartheta|^{p(x)}\, dx \\
\le& c\varrho^{\tilde{\beta}} \int_{B_{\varrho/4}} (1+|D\vartheta|^2)^{p_2(\varrho)/2}\, dx \le c\varrho^{\tilde{\beta}} \int_{B_\varrho} (1+|D\omega|^2)^{p_2(\varrho)/2}\, dx,
\end{aligned} \tag{3.61}$$

with $c=c(\texttt{data},[u]_{0,\tilde{\beta};\tilde{\Omega}})$. To take care of term $(\mathrm{I})_5$, we use (3.51)$_2$ with $\varepsilon_0\equiv\sigma$ again together with the minimality of ϑ and Lemmas 10-9 to obtain

$$
\begin{aligned}
|(\mathrm{I})_5| \le & c\varrho^{\alpha+n}\fint_{B_{\varrho/4}} 1+|D\vartheta|^{(1+\sigma)p_2(\varrho)}\,dx \le c\varrho^{\alpha+n}\fint_{B_{\varrho/4}} 1+|D\omega|^{(1+\sigma)p_2(\varrho)}\,dx\\
\le & c\varrho^{\alpha+n}\fint_{B_{\varrho/4}} 1+|D\omega|^{(1+\sigma_0/2)p_2(\varrho)}\,dx \le cr^{\alpha+n}\left(\fint_{B_{\varrho/2}}(1+|D\omega|^2)^{p(x)/2}\,dx\right)^{1+\sigma_0}\\
\le & c\varrho^{\bar{\kappa}_1}\int_{B_\varrho}(1+|D\omega|^2)^{p_2(\varrho)/2}\,dx, \qquad (3.62)
\end{aligned}
$$

with $c=c(\texttt{data},\|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})})$. Finally, by (3.51)$_3$, Lemmas 10-9 and (3.57) we obtain

$$
\begin{aligned}
|(\mathrm{I})_6| \le & c\varrho^{n}\fint_{B_{\varrho/4}} |(\omega)_{\varrho/4}-(\vartheta)_{\varrho/4}||D\vartheta|^{p_2(\varrho)}\,dx\\
\le & c\varrho^{n}\left(\fint_{B_{\varrho/4}} |(\omega)_{\varrho/4}-(\vartheta)_{\varrho/4}|^{p_2(\varrho)}\,dx\right)^{\frac{\sigma}{1+\sigma}}\left(\fint_{B_{\varrho/4}}|D\vartheta|^{(1+\sigma)p_2(\varrho)}\,dx\right)^{\frac{1}{1+\sigma}}\\
\le & c\varrho^{n+\frac{\tilde{\beta}\gamma_1\sigma}{1+\sigma}}\left(\fint_{B_{\varrho/4}}|D\omega|^{1+\sigma_0 p(x)}\,dx\right)^{\frac{p_2(\varrho)}{(1+\sigma_0)p_1(\varrho)}}\\
\le & c\varrho^{n+\frac{\tilde{\beta}\gamma_1\sigma}{1+\sigma}}\left(\fint_{B_{\varrho/2}}(1+|D\omega|^2)^{p(x)/2}\right)^{\frac{p_2(\varrho)-p_1(\varrho)}{p_1(\varrho)}}\fint_{B_\varrho}(1+|D\omega|^2)^{p_2(\varrho)/2}\,dx\\
\le & c\varrho^{\frac{\tilde{\beta}\gamma_1\sigma}{1+\sigma}}\fint_{B_\varrho}(1+|D\omega|^2)^{p_2(\varrho)/2}\,dx, \qquad (3.63)
\end{aligned}
$$

where $c=c(\texttt{data},\|(|Du|)\|_{L^1(\tilde{\Omega})},[u]_{0,\tilde{\beta};\tilde{\Omega}})$. Collecting estimates (3.58)-(3.63) we can conclude that

$$
\int_{B_{\varrho/4}}(|D\omega|^2+|D\vartheta|^2)^{\frac{p_2(\varrho)-2}{2}}|D\omega-D\vartheta|^2\,dx \le c\left(\varrho^{\frac{\tilde{\beta}\gamma_1\sigma}{1+\sigma}}+\varrho^{\bar{\kappa}_1}+\varrho^{\tilde{\beta}}\right)\int_{B_\varrho}(1+|D\omega|^2)^{p_2(\varrho)/2}\,dx,
$$

with $c=c(\texttt{data},\|(|Du|)\|_{L^1(\tilde{\Omega})},[u]_{0,\tilde{\beta};\tilde{\Omega}})$. Manipulating the content of the previous display as we did in *Step 1*, estimates (3.25)-(3.26) we can conclude that

$$
\int_{B_{\varrho/4}}|D\omega-D\vartheta|^{p_2(\varrho)}\,dx \le c\left(\varrho^{\frac{\tilde{\beta}\gamma_1\sigma}{2(1+\sigma)}}+\varrho^{\frac{\bar{\kappa}_1}{2}}+\varrho^{\frac{\tilde{\beta}}{2}}\right)\int_{B_\varrho}(1+|D\omega|^2)^{p_2(\varrho)/2}\,dx. \qquad (3.64)
$$

Recalling also that, by (3.47), $\tilde{\beta}\ge 1/2$, we can rewrite (3.64) as

$$
\int_{B_{\varrho/4}}|D\omega-D\vartheta|^{p_2(\varrho)}\,dx \le c\varrho^{\kappa_2}\int_{B_\varrho}(1+|D\omega|^2)^{p_2(\varrho)/2}\,dx, \qquad (3.65)
$$

with $\kappa_2:=\frac{1}{2}\min\left\{\frac{1}{2},\bar{\kappa}_1,\frac{\gamma_1\sigma}{2(1+\sigma)}\right\}$. Averaging in (3.65) and using (3.47) again, we readily see that

$$
\fint_{B_{\varrho/4}}|D\omega-D\vartheta|^{p_2(\varrho)}\,dx \overset{(3.52)_3}{\le} c\varrho^{\hat{\kappa}}, \qquad (3.66)
$$

for $c = c(\mathtt{data}, \|(|Du|^{p(\cdot)})\|_{L^1(\tilde{\Omega})}, [u]_{0,\tilde{\beta};\tilde{\Omega}})$. Here $\hat{\kappa} := \kappa_2 - \gamma_2(1-\tilde{\beta}) \geq \kappa_2/2 > 0$. Now fix any $0 < \varsigma < \varrho/8$ and notice that, being ϑ a solution to (3.53), the decay estimate $(2.2)_2$ holds true. So we estimate

$$
\begin{aligned}
&\fint_{B_\varsigma} |D\omega - (D\omega)_\varsigma|^{p_2(\varrho)}\,dx\\
&\quad\leq c\left\{(\varrho/\varsigma)^n \fint_{B_{\varrho/4}} |D\omega - D\vartheta|^{p_2(\varrho)}\,dx + \fint_{B_\varsigma} |D\vartheta - (D\vartheta)_\varsigma|^{p_2(\varrho)}\,dx\right\}\\
&\quad\overset{(3.66)}{\leq} c\left\{(\varrho/\varsigma)^n \varrho^{\hat{\kappa}} + (\varsigma/\varrho)^{\mu p_2(\varrho)} \fint_{B_{\varrho/4}} |D\omega|^{p_2(\varrho)}\,dx\right\}\\
&\quad\overset{(3.52)}{\leq} c\left\{(\varrho/\varsigma)^n \varrho^{\hat{\kappa}} + (\varsigma/\varrho)^{\mu p_2(\varrho)} \varrho^{-\gamma_2(1-\beta)}\right\},
\end{aligned}
\tag{3.67}
$$

with $\beta \in (0,1)$ still to be fixed and $c = c(\mathtt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\tilde{\beta};\tilde{\Omega}}, \beta)$. Set $\beta := 1 - \frac{\mu\gamma_1\hat{\kappa}}{2n\gamma_2}$ in (3.67) and pick $\varsigma = \varrho^{1+a}/2$ with $a := \frac{\hat{\kappa}(2n+\mu\gamma_1)}{2n(n+\mu p_2(\varrho))}$. In these terms, (3.67) reads as

$$
\fint_{B_\varsigma} |D\omega - (D\omega)_\varsigma|^{p_2(\varrho)}\,dx \leq c\left\{\varsigma^{\frac{-an+\hat{\kappa}}{1+a}} + \varsigma^{\frac{a\mu p_2(\varrho)-\gamma_2(1-\beta)}{1+a}}\right\} \leq c\varsigma^{\frac{n\mu\hat{\kappa}\gamma_1}{2n(n+\mu\gamma_2)+\hat{\kappa}(2n+\mu\gamma_1)}} = c\varsigma^{\beta_0\gamma_2},
$$

where we also denoted

$$
\beta_0 := \frac{n\mu\hat{\kappa}\gamma_1}{2n\gamma_2(n+\mu\gamma_2) + \hat{\kappa}\gamma_2(2n+\mu\gamma_1)}. \tag{3.68}
$$

From the content of the previous display and Hölder inequality we finally get

$$
\left(\fint_{B_\varsigma} |D\omega - (D\omega)_\varsigma|^{\gamma_1}\,dx\right)^{\frac{1}{\gamma_1}} \leq \left(\fint_{B_\varsigma} |D\omega - (D\omega)_\varsigma|^{p_2(\varrho)}\,dx\right)^{\frac{1}{p_2(\varrho)}} \leq c\varsigma^{\beta_0},
$$

thus

$$
\fint_{B_\varsigma} |D\omega - (D\omega)_\varsigma|^{\gamma_1}\,dx \leq c\varsigma^{\gamma_1\beta_0},
$$

with $c = c(\mathtt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\tilde{\beta},\tilde{\Omega}})$, so, after covering, we can conclude that $D\omega \in C^{0,\beta_0}_{\mathrm{loc}}(\tilde{\Omega}, \mathbb{R}^{m\times n})$ because of Morrey's growth theorem. By (3.68), it is evident that $\beta_0 = \beta_0(\mathtt{data})$ does not depend on $\tilde{\Omega}$, thus $(3.48)_{2,3}$ and a standard covening argument render that $Du \in C^{0,\beta_0}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{N\times n})$.

Step 7: the case $p(\cdot) > n - \delta_0/2$. As mentioned in *Step 1*, $u \in C^{0,\beta'}(\Omega^+, \mathcal{M})$, with $\beta' := \frac{\delta_0}{4n+\delta_0}$, so we no longer need to impose a smallness condition like (3.16). Being $p(\cdot)$ continuous, Ω^+ is open, so we can fix a ball $B_{\tilde{R}_*} \equiv B_{\tilde{R}_*}(x_0) \Subset \Omega^+$ with $\tilde{R}_*$ satisfying (3.13). Let σ_0 be as in (3.12), so (3.14) is matched on all balls $B_{4\varrho} \subset B_{R_*} \subset B_{\tilde{R}_*}$, where the size of $R_* \leq \tilde{R}_*/2$ will be specified later on. As we did in *Step 6*, we fix open subsets $\tilde{\Omega} \Subset \Omega' \Subset \Omega^+$ and cover $\tilde{\Omega}$ with a finite number of balls contained inside Ω' whose size and number will now depend on $\mathcal{M}$, on $[u]_{0,\beta';\tilde{\Omega}}$ and on $\mathrm{diam}(\Omega')$, having images contained in small coordinate neighborhoods of $\mathcal{M}$. Again we can find $\omega \in W^{1,p(\cdot)}(\tilde{\Omega}, \mathbb{R}^m) \cap C^{0,\beta'}(\tilde{\Omega}, \mathbb{R}^m)$, unconstrained local minimizer of the variational integral (3.50) with integrand $H(\cdot)$ matching (3.51), such that $|\omega| \leq 1$, $u = f(\omega)$ where f is as in (3.48). Our goal is to show the validity of a Morrey decay estimate like (3.52)$_3$. To do so, fix $B_{4\varrho} \Subset B_{R_*}$ and let $\vartheta \in W^{1,p_2(\varrho)}(B_{\varrho/4}, \mathbb{R}^m)$ be a solution to the frozen Dirichlet problem (3.53). Notice that the estimates obtained in *Step 6* till (3.64) do not require any specific value of β, therefore, by (3.64) with $\tilde{\beta}$ replaced by β' we immediately have

$$\int_{B_{\varrho/4}} |D\omega - D\vartheta|^{p_2(\varrho)}\, dx \le c\varrho^{\kappa'} \int_{B_\varrho} (1+|D\omega|^2)^{p_2(\varrho)}\, dx, \tag{3.69}$$

with $c = c(\texttt{data}, \|(|Du|^{p(\cdot)})\|_{L^1(\tilde{\Omega})}, [u]_{0,\beta';\tilde{\Omega}})$ and $\kappa' := \frac{1}{2}\min\left\{\frac{\beta'\gamma_1\sigma}{1+\sigma}, \bar{\kappa}_1, \beta'\right\}$. Now fix $\tau \in \left(0, \frac{1}{8}\right)$ and recall that, being ϑ a solution of (3.53), inequality (2.2)$_1$ holds for all $B_{\varsigma_1} \subset B_{\varsigma_2} \subset B_{\varrho/4}$. Adopting the same terminology appearing in *Step 4*, clearly with ω instead of u, we readily have

$$\begin{aligned}
\psi(\tau\varrho) &\overset{(3.35)}{\le} \phi(\tau\varrho, \varrho)\\
&\le c\left\{(\tau\varrho)^{p_2(\varrho)-n}\left[(\tau\varrho)^n + \int_{B_{\tau\varrho}} |D\omega - D\vartheta|^{p_2(\varrho)}\, dx + \int_{B_{\tau\varrho}} |D\vartheta|^{p_2(\varrho)}\, dx\right]\right\}^{\frac{1}{p_2(\varrho)}}\\
&\overset{(3.69)}{\le} c\left\{\tau^{p_2(\varrho)-n}\left[\tau^n + \varrho^{\kappa'}\right]\varrho^{p_2(\varrho)-n}\int_{B_\varrho}(1+|D\omega|^2)^{p_2(\varrho)}\, dx\right\}^{\frac{1}{p_2(\varrho)}}\\
&\le \tau^\beta\left[c\tau^{1-\beta} + c\varrho^{\frac{\kappa'}{\gamma_2}}\tau^{-\beta-\frac{\delta_0}{2n-\delta_0}}\right]\psi(\varrho),
\end{aligned} \tag{3.70}$$

where we also used $p(\cdot) > n - \frac{\delta_0}{2}$. Here $\beta \in (0,1)$ is arbitrary and $c = c(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\beta';\tilde{\Omega}}, \beta)$. Choosing in (3.70) $\tau \le (2c)^{-1/(1-\beta)}$ and $R_* \le c^{-\frac{\gamma_2}{\kappa'}}2^{-\frac{\gamma_2}{\kappa'}}\tau^{\frac{2n\gamma_2}{(2n-\delta_0)\kappa'}}$ we end up with $\psi(\tau\varrho) \le \tau^\beta\psi(\varrho)$, by remembering also that $\varrho \le R_*$. Notice that our previous decisions fixed the following dependencies: $\tau = \tau(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\beta';\tilde{\Omega}}, \beta)$ and $R_* = R_*(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\beta';\tilde{\Omega}}, \beta)$. By induction, it is easy to see that for any integer j there holds

$$\psi(\tau^j\varrho) \le \tau^{j\beta}\psi(\varrho). \tag{3.71}$$

Now, if $\varsigma \in (0, \varrho/8)$, there exists an integer $j \ge 1$ such that $\tau^{j+1}\varrho < \varsigma \le \tau^j\varrho$. Therefore, proceeding as we did for (3.40), using (3.71) we get

$$\psi(\varsigma) \le \tau^{1-\frac{n}{\gamma_1}}\tau^{j\beta}\psi(\varrho) \le c(\varsigma/\varrho)^\beta\psi(\varrho), \tag{3.72}$$

with $c = c(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\beta';\tilde{\Omega}}, \beta)$. This is the estimate we were looking for. In fact, as in *Step 4*, (3.41) we can extend (3.72) to the full range $0 < \varsigma < \varrho$ and, proceeding as in estimates (3.42)-(3.43) we can get rid of the restriction $s \le R_*$; as already mentioned, we shall only retain $s \le \tilde{R}_*/2$. Furthermore, it directly implies that

$$\begin{aligned}
\varsigma^{\gamma_1-n-\beta\gamma_1}\int_{B_\varsigma}|D\omega|^{\gamma_1}\, dx \le& c\tilde{R}_*^{\gamma_1(1-\beta)}\left(\fint_{B_{\tilde{R}_*/2}}(1+|D\omega|^2)^{p_2(\tilde{R}_*/2)/2}\, dx\right)^{\frac{\gamma_1}{p_2(\tilde{R}_*/2)}}\\
\le& c\tilde{R}_*^{\gamma_1(1-\beta)}\left(\fint_{B_{\tilde{R}_*/2}}(1+|D\omega|^2)^{(1+\sigma_0)p(x)}\, dx\right)^{\frac{\gamma_1}{(1+\sigma_0)p_1(\tilde{R}_*/2)}}
\end{aligned}$$

$$\leq c\tilde{R}_*^{\gamma_1(1-\beta)}\left(\fint_{B_{\tilde{R}_*}}(1+|D\omega|^2)^{p(x)/2}\,dx\right)^{\frac{\gamma_1}{p_1(\tilde{R}_*/2)}}\leq c,$$

for $c = c(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\beta';\tilde{\Omega}}, \beta)$ and therefore, being $\beta \in (0,1)$ arbitrary, by Morrey's growth theorem and a standard covering argument, we can conclude that $\omega \in C^{0,\beta}_{\mathrm{loc}}(\Omega^+, \mathbb{R}^m)$ for any $\beta \in (0,1)$. Now, for all $B_{4\varsigma} \Subset \Omega^+$ such that $0 < \varsigma \leq \tilde{R}_*/2$, by Lemmas 9, 8 and 2 (*ii.*), we obtain

$$\begin{aligned}\fint_{B_\varsigma}|D\omega|^{p_2(\varsigma)}\,dx \leq& c\left(\fint_{B_{2\varsigma}}(1+|D\omega|^2)^{p(x)/2}\,dx\right)^{\frac{p_2(\varsigma)-p_1(\varsigma)}{p_1(\varsigma)}}\fint_{B_{2\varsigma}}(1+|D\omega|^2)^{p(x)/2}\,dx\\ \leq& c\fint_{B_{4\varsigma}}1+\left|\frac{\omega-(\omega)_{4\varsigma}}{\varsigma}\right|^{p(x)}\,dx \leq c\varsigma^{-\gamma_2(1-\beta)},\end{aligned} \tag{3.73}$$

where $c = c(\texttt{data}, \|(|Du|)^{p(\cdot)}\|_{L^1(\tilde{\Omega})}, [u]_{0,\beta';\tilde{\Omega}}, \beta)$. Once (3.73) is available, we can conclude as in *Step 6*.

4 Dimension reduction

In this section we obtain a further reduction of the dimension of the singular set of $p(x)$-harmonic maps, for $p(\cdot) \geq 2$ Lipschitz continuous, thus improving, at least in this case, the result given in Theorem 1, *Step 5*.

4.1 Compactness of minimizers and Monotonicity formula

The proof of Theorem 2 essentially needs two components to be carried out. The first is the compactness of sequences of minimizers of (0.1) under uniform assumptions, while the second is the monotonicity along solutions to (0.1) of a certain quantity strictly related to the $p(x)$-energy. Those arguments are quite classical, see e. g. [23,27,46].

Lemma 11 (Compactness) *Let $(k_j)_{j\in\mathbb{N}}$, $(p_j)_{j\in\mathbb{N}}$ be two sequences of α-Hölder continuous functions, $\alpha \in (0,1]$, satisfying*

$$\begin{cases}\sup_{j\in\mathbb{N}}[k_j]_{0,\alpha} < c_k\\ \lambda \leq k_j(x) \leq \Lambda \text{ for all } x \in B_1\\ \|k_j - k\|_{L^\infty(B_1)} \to 0,\ k(\cdot) \in C^{0,\alpha}(B_1)\end{cases} \quad \textit{and}$$

$$\begin{cases}\sup_{j\in\mathbb{N}}[p]_{0,\alpha} < c_p\\ p_j(x) \geq \gamma_1 > 1 \text{ for all } x \in B_1,\ j \in \mathbb{N}\\ \|p_j - p_0\|_{L^\infty(B_1)} \to 0,\ p_0 \geq \gamma_1 > 1 \text{ constant},\end{cases} \tag{4.1}$$

respectively. For each $j \in \mathbb{N}$, let $u_j \in W^{1,p_j(\cdot)}(B_1, \mathcal{M})$ be a constrained local minimizer of

$$\mathcal{E}_j(w, B_1) := \int_{B_1} k_j(x)|Dw|^{p_j(x)}\,dx,$$

where $\mathcal{M}$ is as in ($\mathcal{M}$1)-($\mathcal{M}$2). Then, there exists a subsequence, still denoted by $(u_j)_{j\in\mathbb{N}}$, such that

$$u_j \rightharpoonup v \quad \text{weakly in } W^{1,(1+\tilde{\sigma})p_0}(B_r, \mathcal{M}) \tag{4.2}$$

for some $\tilde{\sigma} > 0$ *and any* $r \in (0,1)$ *and* v *is a constrained local minimizer of the functional*

$$\mathcal{E}_0(w, B_1) := \int_{B_1} k(x)|Dw|^{p_0}\, dx.$$

Moreover, $\mathcal{E}_j(u_j, B_r) \to \mathcal{E}_0(v, B_r)$ *for all* $r \in (0,1)$*. Finally, if* x_j *is a singular point of* u_j *and* $x_j \to \bar{x}$*, then* $\bar{x}$ *is a singular point for* v*.*

Proof The proof is divided into three steps.

Step 1: weak $W^{(1+\tilde{\sigma})p_0}$*-convergence.* Since $\mathcal{M}$ is compact, $\sup_{j\in\mathbb{N}}\|u_j\|_{L^\infty(B_1)} \le c(\mathcal{M})$, and given that $\gamma_1 > 1$, we obtain, up to (non relabelled) subsequences,

$$u_j \rightharpoonup v \text{ weakly in } L^{\gamma_1}(B_1, \mathbb{R}^N). \tag{4.3}$$

Moreover, being the bounds in (4.1) uniform in $j \in \mathbb{N}$, Lemma 9 and Corollary 3 (and Lemma 10 for the associated frozen problems) hold for all the $\mathcal{E}_j$'s with parameters independent of j. By Lemma 9, we know that $(u_j)_{j\in\mathbb{N}} \subset W^{1,(1+\delta)p(\cdot)}(B_1, \mathcal{M})$ for all $\delta \in (0, \tilde{\delta}_0)$. Let $\delta_2 := \frac{1}{4}\min\{\tilde{\sigma}_0, \tilde{\delta}_0\}$, where $\tilde{\sigma}_0$ is the higher integrability threshold given by Lemma 10 and pick any $\delta \in (0, \delta_2)$. Because of the uniform convergence of the p_j's to the constant p_0, taking j sufficiently large we can find positive constants $\gamma_1 \le q_1 \le q_2 \le \gamma_2$ such that

$$1 < q_1 \le p_j(\cdot) \le q_2 < \infty \text{ on } B_1, \quad q_2\left(1+\frac{\delta}{2}\right) \le q_1(1+\delta), \quad q_2 \le p_0\left(1+\frac{\delta}{2}\right). \tag{4.4}$$

For any $B_\varrho(x_0) \equiv B_\varrho \subset B_1$, Corollary 3 yields that

$$\int_{B_{\varrho/4}} |Du_j|^{(1+\delta)p_j(x)}\, dx \le c$$

for all $j \in \mathbb{N}$, with $c = c(\varrho, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. This last estimate and $(4.4)_{1,2}$ imply that

$$\int_{B_{\varrho/4}} |Du_j|^{(1+\delta/2)q_2}\, dx \le c, \tag{4.5}$$

for $c = c(\varrho, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. Now, for any fixed $r \in (0,1)$, we can cover $B_r \equiv B_r(0)$ by a finite number of balls $B_{(1-\varrho)/4}(x_0)$ with $x_0 \in B_r$, use (4.5) on each ball and then sum them all to get

$$\int_{B_r} |Du_j|^{(1+\delta/2)q_2}\, dx \le c \tag{4.6}$$

for large $j \in \mathbb{N}$. Here $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. From the compactness of $\mathcal{M}$ and (4.6), we derive the uniform boundedness of the u_j's in $W^{1,(1+\delta/2)q_2}(B_r, \mathcal{M})$, so, up to extract a (non relabelled) subsequence, we obtain that $u_j \rightharpoonup \bar{v}$ weakly in $W^{1,(1+\delta/2)q_2}(B_r, \mathcal{M})$, for some $\bar{v} \in W^{1,(1+\delta/2)q_2}(B_r, \mathcal{M})$. Anyway, by (4.3), $\bar{v}(x) = v(x)$, $v(x) \in \mathcal{M}$ for a.e. $x \in B_r$ and, by Rellich's theorem,

$$u_j \to v \;\text{ strongly in }\; L^{(1+\delta/2)q_2}(B_r, \mathcal{M}), \tag{4.7}$$

$$Du_j \to Dv \;\text{ weakly in }\; L^{(1+\delta/2)q_2}(B_r, \mathbb{R}^{N\times n}). \tag{4.8}$$

From $(4.4)_1$ and $(4.1)_2$ we see that $q_2 \geq p_0$, so (4.2) is proved with $\tilde{\sigma} \equiv \delta/2$. In particular, the weak lower semicontinuity of the norm renders that

$$\int_{B_r} |Dv|^{(1+\delta/2)q_2} \, dx \leq c, \tag{4.9}$$

with $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$.

Step 2: compactness. We aim to show that v is an $\mathcal{M}$-constrained local minimizer of $\mathcal{E}_0$. To do so, we first claim that

$$\mathcal{E}_0(v, B_r) \leq \liminf_{j\to\infty} \mathcal{E}_j(u_j, B_r), \tag{4.10}$$

for all $r \in (0, 1)$. Let us rewrite $\mathcal{E}_j(u_j, B_r) = \big(\mathcal{E}_j(u_j, B_r) - \mathcal{E}_0(u_j, B_r)\big) + \mathcal{E}_0(u_j, B_r)$. From (4.2) and weak lower semicontinuity we have

$$\mathcal{E}_0(v, B_r) \leq \liminf_{j\to\infty} \mathcal{E}_0(u_j, B_r). \tag{4.11}$$

On the other hand, from $(4.4)_1$, Lemma 2 (*i.*) with $\varepsilon_0 \equiv \delta/2$ and (4.1) we have

$$\begin{aligned} \big|\, \mathcal{E}_j(u_j, B_r) - \mathcal{E}_0(u_j, B_r) \,\big| &\leq c\|p_j - p_0\|_{L^\infty(B_1)} \int_{B_r} 1 + |Du_j|^{(1+\delta/2)q_2} \, dx \\ &\quad + \|k_j - k\|_{L^\infty(B_1)} \int_{B_r} |Du_j|^{p_0} \, dx \\ &\overset{(4.6)}{\leq} c\left(\|p_j - p_0\|_{L^\infty(B_1)} + \|k_j - k\|_{L^\infty(B_1)}\right) \to 0, \end{aligned} \tag{4.12}$$

where $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. Combining (4.12) and (4.11) we obtain (4.10).

Let $\tilde{v} \in W^{1,p_0}(B_r, \mathcal{M})$ be a solution to the Dirichlet problem

$$W^{1,p_0}_v(B_r, \mathcal{M}) \ni w \mapsto \min \mathcal{E}_0(w, B_r),$$

and extend it to be equal to v outside B_r. In this way, $\tilde{v} \in W^{1,p_0}_{\mathrm{loc}}(B_1, \mathcal{M}) \cap W^{1,p_0}_v(B_r, \mathcal{M})$. Since we are assuming that $(p_j)_{j\in\mathbb{N}}$ converges uniformly to p_0 on B_1, we can take $j \in \mathbb{N}$ so large that

$$\|p_j\|_{L^\infty(B_1)}\left(1 + \frac{\delta}{4}\right) \leq p_0\left(1 + \frac{\delta}{2}\right) \tag{4.13}$$

holds. Moreover, by (4.9) and $(4.4)_1$ we have that $v \in W^{1,(1+\delta/2)q_2}(B_r, \mathcal{M}) \subset W^{1,(1+\delta/2)p_0}(B_r, \mathcal{M})$, so, from Lemma 10 with $p \equiv p_0$ we obtain that $\tilde{v} \in W^{1,(1+\delta/2)p_0}(B_r, \mathcal{M}) \subset W^{1,(1+\delta/4)p_j(\cdot)}(B_r, \mathcal{M}) \cap W^{1,q_2}(B_r, \mathcal{M})$, where the last inclusion is due to (4.13) and $(4.4)_3$. From (4.13), Lemma 10 and (4.9) we get

$$\int_{B_r} |D\tilde{v}|^{(1+\delta/4)p_j(x)} \, dx \leq \int_{B_r} 1 + |D\tilde{v}|^{(1+\delta/2)p_0} \, dx \leq c \int_{B_r} 1 + |Dv|^{(1+\delta/2)p_0} \, dx \leq c, \tag{4.14}$$

where $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. Let $\theta \in (0, 1)$ be a small parameter to be fixed and η a cut-off function with the following specifics

$$\eta \in C^1_c(B_r), \quad \chi_{B_{r(1-\theta)}} \leq \eta \leq \chi_{B_r}, \quad |D\eta| \leq (r\theta)^{-1} \text{ on } A_{r\theta}. \tag{4.15}$$

In correspondence of such a choice of η, we define the comparison map $w_j := (1-\eta)u_j + \eta\tilde{v}$ and notice that $w_j\big|_{\partial B_{r(1-\theta)}} = \tilde{v}|_{\partial B_{r(1-\theta)}}$ and $w_j\big|_{\partial B_r} = u_j\big|_{\partial B_r}$. So Lemma 5 applies to w_j on $A_{r\theta}$ thus rendering a map $w_j' \in W^{1,p_j(\cdot)}(A_{r\theta}, \mathcal{M})$ such that

$$w_j'\Big|_{\partial B_{r(1-\theta)}} = \tilde{v}|_{\partial B_{r(1-\theta)}}, \; w_j'\Big|_{\partial B_r} = u_j\big|_{\partial B_r}, \quad \int_{A_{r\theta}} |Dw_j'|^{p_j(x)}\, dx \le c \int_{A_{r\theta}} |Dw_j|^{p_j(x)}\, dx, \tag{4.16}$$

with $c = c(N, \mathcal{M}, \gamma_2)$. Finally we define

$$\tilde{w}_j := \begin{cases} \tilde{v} & \text{on } B_{r(1-\theta)} \\ w_j' & \text{on } A_{r\theta}, \end{cases} \tag{4.17}$$

which, by $(4.16)_{2,3}$ and (4.14) is an admissible competitor for u_j on B_r. From the minimality of u_j and $(4.16)_{2,3}$ we have

$$\begin{aligned} \mathcal{E}_j(u_j, B_r) \le& \mathcal{E}_j(\tilde{w}_j, B_r) = \mathcal{E}_j(\tilde{v}, B_{r(1-\theta)}) + \mathcal{E}_j(w_j', A_{r\theta}) \\ &\le \int_{B_r} k_j(x)|D\tilde{v}|^{p_j(x)}\, dx + c\int_{A_{r\theta}} k_j(x)|Dw_j|^{p_j(x)}\, dx := (\mathrm{I})_j + (\mathrm{II})_j, \end{aligned}$$

for $c = c(N, \mathcal{M}, \lambda, \Lambda, \gamma_2)$. By (4.1), (4.13), $(4.4)_1$, (4.9) and Lemma 2 $(i.)$ with $\varepsilon_0 \equiv \delta/4$ we see that

$$\begin{aligned} &\left| \int_{B_r} k_j(x)|D\tilde{v}|^{p_j(x)}\, dx - \int_{B_r} k(x)|D\tilde{v}|^{p_0}\, dx \right| \\ &\qquad \overset{(4.14)}{\le} c\left(\|k_j - k\|_{L^\infty(B_1)} + \|p_j - p_0\|_{L^\infty(B_1)}\right) \int_{B_r} 1 \\ &\qquad\qquad + |Dv|^{(1+\delta/2)p_0}\, dx \to 0, \end{aligned}$$

where $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. So we get that

$$(\mathrm{I})_j \to \mathcal{E}_0(\tilde{v}, B_r). \tag{4.18}$$

Exploiting the very definition of the w_j's and $(4.15)_3$ we have

$$\begin{aligned} (\mathrm{II})_j \le& c\int_{A_{r\theta}} k_j(x)\left[|Du_j|^{p_j(x)} + |D\tilde{v}|^{p_j(x)} + \left| \frac{u_j - \tilde{v}}{r\theta} \right|^{p_j(x)} \right] dx \\ \le& c\int_{A_{r\theta}} k_j(x)\left[|Du_j|^{p_j(x)} + |D\tilde{v}|^{p_j(x)} + \left| \frac{u_j - v}{r\theta} \right|^{p_j(x)} + \left| \frac{\tilde{v} - v}{r\theta} \right|^{p_j(x)} \right] dx \\ =&: (\mathrm{II})_j^1 + (\mathrm{II})_j^2 + (\mathrm{II})_j^3, \end{aligned}$$

with $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. Using $(4.4)_1$, (4.5), (4.14) and (4.1) we get

$$(\mathrm{II})_j^1 \le \Lambda \int_{A_{r\theta}} |Du_j|^{p_j(x)} + |D\tilde{v}|^{p_j(x)}\, dx \le c, \tag{4.19}$$

where $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. By $(4.4)_1$ and (4.7), a well known variation on Lebesgue's dominated convergence theorem allows concluding that

$$(\mathrm{II})_j^2 \to 0. \tag{4.20}$$

Finally, by (3.2) with $p(\cdot) \equiv p_j(\cdot)$ and (4.14) we obtain

$$(\mathrm{II})_j^3 \le c \int_{A_{r\theta}} |Dv - D\tilde{v}|^{p_j(x)}\, dx + c|A_{r\theta}| \le c \int_{A_{r\theta}} 1 + |Dv|^{(1+\delta/2)p_0}\, dx + c|A_{r\theta}| \le c, \tag{4.21}$$

for $c = c(r, c_p, n, N, \mathcal{M}, \lambda, \Lambda, \gamma_1, \gamma_2, \alpha)$. By the absolute continuity of Lebesgue's integral, (4.19) and (4.21), given $\sigma > 0$ we can always choose θ sufficiently small in such a way that

$$(\mathrm{II})_j^1 + (\mathrm{II})_j^3 \le \frac{\sigma}{2}, \tag{4.22}$$

and, by (4.20), j large enough such that

$$(\mathrm{II})_j^2 \le \frac{\sigma}{2}. \tag{4.23}$$

All in all, collecting (4.10), (4.18), (4.22) and (4.23) we can conclude that

$$\begin{aligned}\mathcal{E}_0(v, B_r) &\le \liminf_{j\to\infty} \mathcal{E}_j(u_j, B_r) \le \limsup_{j\to\infty} \mathcal{E}_j(u_j, B_r) \\ &\le \limsup_{j\to\infty} \mathcal{E}_j(\tilde{v}, B_r) + \sigma = \mathcal{E}_0(\tilde{v}, B_r) + \sigma,\end{aligned}$$

so, by the arbitrariety of σ and the minimality of $\tilde{v}$, we can conclude that $\mathcal{E}_0(\tilde{v}, B_r) = \mathcal{E}_0(v, B_r)$. Being this true for any $r \in (0, 1)$, v is an $\mathcal{M}$-constrained local minimizer of $\mathcal{E}_0$ and, as a direct consequence of the last chain of inequalities, $\mathcal{E}_j(u_j, B_r) \to \mathcal{E}_0(v, B_r)$.

Step 3: singular points. Once we have the results contained in *Steps 1-2* by hand, the proof of *Step 3* goes as the one in [46, Lemma 3.1] and we shall omit it. □

We stress that Lemma 11 holds with $p(\cdot) \ge \gamma_1 > 1$ Hölder continuous rather than Lipschitz. We need stronger assumptions only to prove a suitable monotonicity formula.

Lemma 12 (Monotonicity formula) *Let* $k(\cdot) \in C^{0,\alpha}(\Omega)$, $\alpha \in (0, 1]$ *be such that* $k(0) = 1$, $p(\cdot) \in Lip(\Omega)$ *and* $n > \gamma_2 \ge p(x) \ge 2$ *for all* $x \in \Omega$. *If* $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ *is a constrained local minimizer of* (0.1) *on* B_1, *then for any* $\gamma \in (0, 1)$ *there exist a positive* $c = c(\mathtt{data}, \gamma)$ *and* $T \in (0, 1)$ *such that for all* $0 < r < R < T$, *we have*

$$\begin{aligned}\int_{\partial B_1} |u(Rx) - u(rx)|^{p_2(r)}\, d\mathcal{H}^{n-1}(x) &\le c r^{p_2(r)-p_2(R)} \left(\log \frac{R}{r}\right)^{p_2(r)-1} \\ &\quad \left((\Phi(R) - \Phi(r)) + (R^\gamma - r^\gamma)\right),\end{aligned}$$

where

$$\Phi(t) := t^{p_2(t)-n} \exp(A t^\alpha) \int_{B_t} k(x)|Du|^{p_2(t)}\, dx,$$

with $A = A(n, [k]_{0,\alpha}, [p]_{0,1}, \alpha) > 0$.

Proof The proof is actually the same as the one given in [46, Lemma 4.1]. There is only one small detail to change: the map v introduced during the proof of Lemma 4.1 to obtain estimate (4.17) must be replaced by a solution to the Dirichlet problem

$$W_u^{1,p_2(t)}(B_t, \mathcal{M}) \ni w \mapsto \inf \int_{B_t} k(x)|Dw|^{p_2(t)}\, dx.$$

The rest stays unchanged. □

4.2 Proof of Theorem 2

Combining the compactness Lemma 11 and the monotonicity formula obtained in Lemma 12, we are ready to prove Theorem 2. If Ω^+ is as in (3.11), then $u \in W^{1,n+\delta_0/4}(\Omega^+, \mathcal{M})$. So, by Morrey's embedding theorem, $u \in C^{0,\beta'}(\Omega^+, \mathcal{M})$ for $\beta' := \delta_0/(4n+\delta_0)$ and, by *Step 7* of Theorem 1, we can conclude that Du is locally β_0-Hölder continuous on Ω^+ for some $\beta_0 \in (0,1)$. This observation shows that, to prove Theorem 2 it is enough to assume that $\gamma_2 < n$, and this condition assures the applicability of Lemma 12.

Case 1: $n \le [\gamma_1]+1$. Since $t \mapsto \Phi(t)$ can be seen as a difference between an increasing function of t and ct^γ for some $\gamma \in (0,1)$ and a positive constant c, it admits a finite limit as $t \to 0$. Assume that u has a singular point at $\bar{x} = 0$ which is not isolated. Then we can find a sequence of singular points $(x_j)_{j\in\mathbb{N}}$ such that $x_j \to 0$. Setting $R_j := 2|x_j| < T < 1$ we see that, for any j the scaled function $u_j(x) := u(R_j x)$ is a constrained local minimizer of the functional

$$\mathcal{E}_j(w, B_1) := \int_{B_1} R_j^{p(0)-p_j(x)} |Dw|^{p_j(x)}\, dx, \quad p_j(x) := p(R_j x)$$

and each u_j has a singular point $y_j := R_j^{-1} x_j$ with $|y_j| = 1/2$. Now we notice that the sequences $(R_j^{p(0)-p_j(\cdot)})_{j\in\mathbb{N}}$ and $(p_j(\cdot))_{j\in\mathbb{N}}$ satisfy (4.1), so by Lemma 11 we get, up to extract a subsequence that the u_j's L^2-weakly converge to a function v, constrained local minimizer of $\mathcal{E}_0(w, B_1) := \int_{B_1} |Dw|^{p(0)}\, dx$ and that the y_j's converge to $\bar{y}$, singular point of v with $|\bar{y}| = 1/2$. Now pick two constants $0 < \lambda < \mu < 1$ and apply Lemma 12 with $r \equiv \lambda R_j$ and $R \equiv \mu R_j$ to get

$$\begin{aligned}\int_{\partial B_1} |u_j(\mu x) - u_j(\lambda x)|^{p_2(\lambda R_j)} d\mathcal{H}^{n-1}(x) &= \int_{\partial B_1} |u(\mu R_j x) - u(\lambda R_j x)|^{p_2(\lambda R_j)} d\mathcal{H}^{n-1}(x) \\ &\le c(\lambda R_j)^{p_2(\lambda R_j)-p_2(\mu R_j)} (\log(\mu/\lambda))^{p_2(\lambda R_j)-1} \left((\Phi(\mu R_j) - \Phi(\lambda R_j)) + (\mu^\gamma - \lambda^\gamma) R_j^\gamma \right) \\ &\to 0. \end{aligned} \tag{4.24}$$

Moreover, Lemma 11 also says that $u_j \to v$ in $L^{(1+\tilde{\sigma})p(0)}(B_r, \mathcal{M})$ for all $r \in (0,1)$ and this leads to

$$|u_j(\mu x) - u_j(\lambda x)|^{p_2(\lambda R_j)} \to |v(\mu x) - v(\lambda x)|^{p(0)} \ \text{ a.e. in } B_1. \tag{4.25}$$

Finally, the compactness of $\mathcal{M}$ renders the u_j's uniformly bounded, so, by the dominated convergence theorem, (4.24) and (4.25) we deduce that

$$\int_{\partial B_1} |v(\mu x) - v(\lambda x)|^{p(0)} d\mathcal{H}^{n-1}(x) = 0,$$

for a.e. λ and μ. This means that v is homogeneous of degree 0, so the whole segment joining $\bar{x}$ and $\bar{y}$ is made of singular points of v, but, since we are assuming $n \le [\gamma_1]+1 \le [p(0)]+1$, we obtain a contradiction to [28, Theorem 4.5], which states that, under these conditions, v can have only isolated singularities.

Case 2: $n > [\gamma_1]+1$. Let us assume that for some $l > 0$, $\mathcal{H}^l(\Sigma_0(u)) > 0$. Then, by blowing up, we obtain a constrained local minimizer v of $\mathcal{E}_0$ with $\mathcal{H}^l(\Sigma_0(v)) > 0$, (see [23], Chapter 10). On the other hand, by [28, Theorem 4.5], $l < n - [p(0)] - 1 \le n - [\gamma_1] - 1$ and this concludes the proof.

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