

Flows of G_2 -structures on contact Calabi–Yau 7-manifolds

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Abstract

We study the Laplacian flow and cflow on contact Calabi–Yau 7-manifolds. We show that the natural initial condition leads to an ancient solution of the Laplacian flow with a finite time Type I singularity which is not a soliton, whereas it produces an immortal (though not eternal and not self-similar) solution of the Laplacian cflow which has an infinite time singularity, that is Type IIb unless the transverse Calabi–Yau geometry is flat. The flows in each case collapse (after normalising the volume) to a lower-dimensional limit, which is either \mathbb{R} for the Laplacian flow or standard \mathbb{C}^3 for the Laplacian cflow. We also study the Hitchin flow in this setting, which we show coincides with the Laplacian cflow up to reparametrisation of time, and defines an (incomplete) Calabi–Yau structure on the spacetime track of the flow.

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1 Introduction

Geometric flows have proven to be a powerful analytical tool in a variety of geometric and topological problems. In the case of G_2 -geometry, they provide a method to search for metrics with G_2 holonomy, which are then Ricci-flat, on an oriented and spin 7-manifold M . This corresponds to varying a G_2 -structure, given by a non-degenerate 3-form φ on M , so that it becomes torsion-free. Flows in G_2 -geometry were first introduced by Bryant [Bry06] and have been studied by several authors, see e.g. [BX11, KMT12, Lau17, BF18, Gri19, DGK19, KLL20, LSE19, LW19, LMSES21].

A G_2 -structure φ determines a metric g_φ and orientation with Riemannian volume form vol_φ . The *full torsion tensor* T of φ is a 2-tensor which is equivalent to $\nabla^{g_\varphi} \varphi$, where ∇^{g_φ} is the Levi-Civita connection of g_φ . Pairs (M^7, φ) which satisfy $T \equiv 0$ (i.e. so that φ is torsion-free) are called G_2 -manifolds and are of particular interest since the holonomy group of g_φ in this case is contained in G_2 . However, complete examples of G_2 -manifolds (M, φ) are very difficult to construct, especially when M is required to be compact. Fernandez and Gray [FG82] showed that the torsion-free condition is equivalent to φ being both *closed* and *coclosed*, i.e. $d\varphi = 0$ and $d*\varphi = 0$, where $*$ is the Hodge star defined by g_φ and vol_φ . This alternative viewpoint on the torsion-free condition as a system of nonlinear PDE is fundamental to G_2 geometry and to geometric flows.

1.1 Laplacian coflow

Our first goal is to study the *Laplacian coflow* of G_2 -structures, introduced by Karigiannis, McKay and Tsui [KMT12]¹ which is given by

$$\frac{\partial \psi_t}{\partial t} = \Delta_{\psi_t} \psi_t := (dd^{*t} + d^{*t}d)\psi_t, \quad (1.1)$$

where $\psi_t := *_t \varphi_t$ is the Hodge dual of the G_2 -structure φ_t (here, we write $*_t := *_{\varphi_t}$) and Δ_{ψ_t} is the Hodge Laplacian of $g_t := g_{\varphi_t}$ on 4-forms. If M is compact, critical points of this flow (1.1) are then Hodge duals of non-degenerate 3-forms which are closed and coclosed, i.e. torsion-free.

We restrict the Laplacian coflow (1.1) to the case where ψ_t is closed, i.e. the G_2 -structure φ_t is coclosed. In this context, solutions of (1.1) (if they exist) preserve the cohomology class $[\psi_t] = [\psi_0] \in H^4(M)$, for all t , and the flow seeks critical points in this class. When M is compact, (1.1) can be interpreted as the gradient flow of *Hitchin's volume functional* [Hit01] and so the volume of M increases monotonically along the flow (see [Gri13]).

One immediate problem with the Laplacian coflow is that the 4-form $\psi = *_\varphi \varphi$ is generated by both the 3-forms φ and $-\varphi$: in particular, ψ does not determine the orientation on M . However, it is natural to assume that the orientation on M stays fixed along the flow, which is determined by a choice of G_2 -structure dual to the initial 4-form. Another key problem, which is much more serious, is that it is not known whether solutions to (1.1) actually exist in general, even for an arbitrarily short time. There is a modification of the coflow [Gri13] which does have guaranteed short-time existence, but the critical points are no longer necessarily closed and coclosed 4-forms, and so it does not currently appear to be useful as a tool for studying key problems in G_2 geometry.

Finally, even if one has short-time existence and uniqueness of the flow (1.1), one finds that the torsion tensor is not guaranteed to satisfy a heat-type equation. Thus the Laplacian coflow is not a *reasonable* (sometimes also called *Ricci-like*) flow of G_2 -structures in the sense of [Che18], so the analytic theory developed there does not apply. This issue is linked to the lack of parabolicity of the Laplacian coflow, even in the direction of closed 4-forms (see [Gri13]).

In Theorem 1.1, we will find immortal families of coclosed G_2 -structures solving the Laplacian coflow (1.1).

1.2 Laplacian flow

A geometric flow of G_2 -structures which has received much more attention is the Laplacian flow introduced by Bryant [Bry06]:

$$\frac{\partial \varphi_t}{\partial t} = \Delta_{\varphi_t} \varphi_t := (dd^{*t} + d^{*t}d)\varphi_t, \quad (1.2)$$

where we again write $*_t$ for the Hodge star induced by the G_2 -structure φ_t , and Δ_{φ_t} is the Hodge Laplacian of $g_t := g_{\varphi_t}$ on 3-forms. Again, if M is compact, the Laplacian flow (1.2) has torsion-free G_2 -structures as its critical points.

¹In [KMT12], the flow (1.1) is written with an additional minus sign on the right-hand side, but this seems incorrect given the work on the Laplacian flow [BX11, LW17] and the modified Laplacian coflow [Gri13].

Typically, one restricts the Laplacian flow (1.2) to *closed* G_2 -structures, because in that setting one obtains a good analytic theory, the cohomology class of the flowing G_2 -structure φ_t is preserved (so we are seeking torsion-free G_2 -structures in a given class), and the flow (in the compact setting) can be interpreted as the gradient flow of the Hitchin volume functional on $[\varphi_t] = [\varphi_0]$. See [BX11, KLL20, LW17, LW19] for more details on the Laplacian flow. By contrast, it is not known whether starting (1.2) at a *coclosed* G_2 -structure may preserve coclosedness, however much one might naively expect this from a flow which is initially in the direction of coexact forms. Indeed, to our knowledge, there are no particular examples in the literature of a Laplacian flow actually preserving the coclosed property. We also note that we do not currently have any general analytic theory for the Laplacian flow, except when restricted to closed G_2 -structures.

In Theorem 1.2, we will obtain ancient families of coclosed G_2 -structures solving the Laplacian flow (1.2).

1.3 Singularities

For various flows of G_2 -structures (cf. [LW17, Che18]) it has been shown that blow-up of the following quantity characterises the formation of finite-time singularities:

$$\Lambda(x, t) = (|Rm(x, t)|_{g_t}^2 + |T(x, t)|_{g_t}^4 + |\nabla^{g_t} T(x, t)|_{g_t}^2)^{\frac{1}{2}}$$

for $x \in M$ and time t . We then let

$$\Lambda(t) = \sup_{x \in M} \Lambda(x, t) \tag{1.3}$$

for convenience and introduce the following terminology, motivated by the singularity classification in Ricci flow (cf. [CLN06]) and the work in [LW17, Che18].

Definition 1.1. Suppose that $(M^7, \varphi_t, \psi_t, g_t)$ is a solution to a flow of G_2 -structures on a closed manifold on a maximal time interval $[0, T)$ and let $\Lambda(t)$ be as in (1.3).

If we have a finite-time singularity, i.e. $T < \infty$, we say that the solution forms

- a *Type I singularity* (rapidly forming) if $\sup_{t \in [0, T)} (T - t) \Lambda(t) < \infty$; and otherwise
- a *Type IIa singularity* (slowly forming) if $\sup_{t \in [0, T)} (T - t) \Lambda(t) = \infty$.

If we have an *infinite-time* singularity, where $T = \infty$, then it is

- a *Type IIb singularity* (slowly forming) if $\sup_{t \in [0, \infty)} t \Lambda(t) = \infty$; and otherwise
- a *Type III singularity* (rapidly forming) if $\sup_{t \in [0, \infty)} t \Lambda(t) < \infty$.

In Theorem 1.3, we will apply this singularity classification to our solutions to the flows (1.1) and (1.2).

1.4 Hitchin flow

The *Hitchin flow*² [Hit01] for a family φ_t of G_2 -structures on M^7 with dual 4-form ψ_t is given by solving

$$\frac{\partial \psi_t}{\partial t} = d\varphi_t \quad \text{and} \quad d\psi_t = 0 \tag{1.4}$$

for $t \in I \subseteq \mathbb{R}$. We note that the Hitchin flow (1.4) is an evolution equation for *coclosed* G_2 -structures. The significance of the flow is that a solution defines a 4-form Φ on the product $I \times M$ by

$$\Phi = dt \wedge \varphi_t + \psi_t, \tag{1.5}$$

which is closed and so defines a torsion-free $\text{Spin}(7)$ -structure on $I \times M$ (and thus a Ricci-flat metric with holonomy contained in $\text{Spin}(7)$). Relatively little is known about the Hitchin flow and since the 7-manifolds we are studying naturally admit coclosed G_2 -structures, it is worth examining the Hitchin flow in this context.

In Theorem 1.4, we will obtain immortal solutions to the Hitchin flow (1.4), which bear a close relation to our Laplacian coflow (1.1) solutions, and define an incomplete *Calabi-Yau* structure on the 8-dimensional spacetime track of the flow.

²It should be made clear that the Hitchin flow for G_2 -structures is not a geometric flow in the usual sense, but it is nonetheless a valid evolution equation for G_2 -structures.

1.5 Outline and main results

We will consider the flows described above on a *contact Calabi–Yau (cCY) 7-manifold* (M^7, g, η, Υ) , where (M^7, g) is a Sasakian 7-manifold with Riemannian metric g , contact form η and transverse Kähler form $\omega = d\eta \in \Omega^{1,1}(M)$, and $\Upsilon \in \Omega^{3,0}(M)$ is a transverse holomorphic volume form; here (p, q) denotes basic bidegree with respect to the horizontal distribution $\mathcal{D} = \ker \eta$, cf. §2.2. An important class of compact examples of cCY 7-manifolds are *Calabi–Yau links*, which are total spaces of S^1 -(orbi)bundles over Calabi–Yau 3-orbifolds famously listed by Candelas–Lynker–Schimmrigk [CLS90]; these will be discussed in Example 2.1.

On a cCY 7-manifold there exists a natural 1-parameter family of coclosed G_2 -structures defined, for each $\varepsilon > 0$, by

$$\varphi = \varepsilon \eta \wedge \omega + \operatorname{Re} \Upsilon, \quad (1.6)$$

with induced metric g_φ (which equals g when $\varepsilon = 1$) and corresponding dual 4-form

$$\psi = *_\varphi \varphi = \frac{1}{2} \omega^2 - \varepsilon \eta \wedge \operatorname{Im} \Upsilon. \quad (1.7)$$

In the present paper, we are interested in flows of G_2 -structures on cCY 7-manifolds starting at the natural coclosed G_2 -structure in (1.6) or its dual 4-form in (1.7). All of our results are stated in terms of the following standard data (S):

- $(M^7, g_0, \eta_0, \Upsilon_0)$ a contact Calabi–Yau 7-manifold;
- $\mathcal{D}_0 = \ker \eta_0$ its horizontal distribution, and $\omega_0 = d\eta_0$ its transverse Kähler form;
- for $\varepsilon > 0$, the G_2 -structure defined by (1.6) and (1.7), i.e.

(S)

$$\varphi_0 = \varepsilon \eta_0 \wedge \omega_0 + \operatorname{Re} \Upsilon_0 \quad \text{and} \quad \psi_0 = \frac{1}{2} \omega_0^2 - \varepsilon \eta_0 \wedge \operatorname{Im} \Upsilon_0.$$

In §3, we consider the Laplacian coflow on a cCY 7-manifold and prove the following.

Theorem 1.1 (Laplacian coflow on contact Calabi–Yau 7-manifolds). *In the setup (S), the Laplacian coflow (1.1) on M^7 , with initial data determined by φ_0 , is solved by the following family of coclosed G_2 -structures φ_t , with associated metric g_t , volume form vol_t and dual 4-form ψ_t :*

$$\begin{aligned} \varphi_t &= \varepsilon p(t)^{-1} \eta_0 \wedge \omega_0 + p(t)^3 \operatorname{Re} \Upsilon_0; \\ \psi_t &= \frac{1}{2} p(t)^4 \omega_0^2 - \varepsilon \eta_0 \wedge \operatorname{Im} \Upsilon_0; \\ g_t &= \varepsilon^2 p(t)^{-6} \eta_0^2 + p(t)^2 g_{\mathcal{D}_0}; \\ \operatorname{vol}_t &= \varepsilon p(t)^3 \eta_0 \wedge \operatorname{vol}_{\mathcal{D}_0}, \end{aligned}$$

where $p(t) = (1 + 10\varepsilon^2 t)^{1/10}$ and $t \in (-\frac{1}{10\varepsilon^2}, \infty)$. Hence, the solution of the Laplacian coflow is immortal, with a finite time singularity (backwards in time) at $t = -\frac{1}{10\varepsilon^2}$.

Remark 1.1. If M^7 is compact then the volume of M determined by the G_2 -structure φ on M is:

$$\mathcal{H}(\varphi) := \operatorname{Vol}(M, \varphi) = \frac{1}{7} \int_M \varphi \wedge \psi.$$

Along the Laplacian coflow solution given in Theorem 1.1 for a compact cCY 7-manifold we have that

$$\mathcal{H}(\varphi_t) = (10t + 1)^{3/10} \mathcal{H}(\varphi_0).$$

Hence, the Hitchin functional on the cohomology class $[\psi_0]$, which is just $\mathcal{H}(\varphi_t)$, tends to infinity as $t \rightarrow \infty$ and tends to 0 as $t \rightarrow -1/10$. In particular, recalling that the Laplacian coflow is the gradient flow of the Hitchin functional on $[\psi_0]$, we observe that the Hitchin functional is unbounded above and does not have a positive lower bound on $[\psi_0]$.

In §4, we switch attention to the Laplacian flow and prove the following result.

Theorem 1.2 (Laplacian flow on contact Calabi–Yau 7-manifolds). *In the setup (S), the Laplacian flow (1.2) on M^7 , with initial data determined by φ_0 , is solved by the following family of coclosed G_2 -structures φ_t on M^7 , with associated metric g_t , volume form vol_t and dual 4-form ψ_t given by*

$$\begin{aligned}\varphi_t &= \varepsilon q(t) \eta_0 \wedge \omega_0 + \text{Re } \Upsilon_0; \\ \psi_t &= \frac{1}{2} \omega_0^2 - \varepsilon q(t) \eta_0 \wedge \text{Im } \Upsilon_0; \\ g_t &= \varepsilon^2 q(t)^2 \eta_0^2 + g_{\mathcal{D}_0}; \\ \text{vol}_t &= \varepsilon q(t) \eta_0 \wedge \text{vol}_{\mathcal{D}_0},\end{aligned}$$

where $q(t) = (1 - 8\varepsilon^2 t)^{-1/2}$ and $t \in (-\infty, \frac{1}{8\varepsilon^2})$. Hence, the solution of the Laplacian flow is ancient, with a finite time singularity (forwards in time) at $t = \frac{1}{8\varepsilon^2}$.

Remark 1.2. Here, we find that the Laplacian flow solution in Theorem 1.2 has the property that $[\psi_t]$ is not constant unless $[\psi_0] = 0$. Moreover, when $[\psi_0] \neq 0$, one can detect the finite time singularity of the flow using the cohomology class $[\psi_t]$, which is somewhat reminiscent of the Kähler–Ricci flow.

Remark 1.3. We can interpret taking $\varepsilon \rightarrow 0$ in Theorems 1.1 and 1.2 as obtaining degenerate eternal solutions to the Laplacian flow or coflow, which is simply given by the Calabi–Yau structure on the horizontal distribution \mathcal{D}_0 . In the quasi-regular setting for the contact Calabi–Yau structure, where M fibres by circles over a Calabi–Yau 3-orbifold Z , sending ε to 0 is equivalent to shrinking the circle fibres and collapsing to Z .

If we instead consider $\varepsilon \rightarrow \infty$ in Theorems 1.1 and 1.2, we appear to obtain a degenerate solution to the flow, which is only defined for non-negative times for the coflow and for non-positive times for the flow. Again, in the quasi-regular setting, this would be equivalent to expanding the circle fibres to infinite size (usually interpreted as becoming lines) over the Calabi–Yau 3-orbifold base.

When M is compact, we can describe the singularities of our flow solutions in terms of Definition 1.1.

Theorem 1.3 (Singularities of the Laplacian flow and coflow). *Suppose we are in the setup (S) and M is compact.*

- (a) *The Laplacian coflow solution in Theorem 1.1 has an infinite-time Type IIb singularity, unless the transverse metric on \mathcal{D}_0 is flat, in which case it has an infinite-time Type III singularity.*
- (b) *The Laplacian flow solution in Theorem 1.2 has a finite-time Type I singularity.*

Remark 1.4. This theorem gives the first examples of compact solutions to the Laplacian coflow which have an infinite-time Type IIb singularity. It also provides the first compact solutions to the Laplacian flow which have a finite-time singularity, and the first Type I singularity which is *not* a soliton. Such finite-time singularities are not yet known to occur in the Laplacian flow for *closed* G_2 -structures on compact manifolds, which moreover are not expected to be Type I because there are no compact shrinking solitons in this setting.

In Section 5 we study the Hitchin flow for the natural coclosed G_2 -structures on cCY 7-manifolds, and obtain the following result.

Theorem 1.4 (Hitchin flow on contact Calabi–Yau 7-manifolds). *In the setup (S), the Hitchin flow (1.4) on M^7 , with initial data determined by φ_0 , is solved by the following family of coclosed G_2 -structures φ_t , with associated metric g_t , volume vol_t and dual 4-form ψ_t given by*

$$\begin{aligned}\varphi_t &= \varepsilon r(t)^{-1} \eta_0 \wedge \omega_0 + r(t)^3 \text{Re } \Upsilon_0; \\ \psi_t &= \frac{1}{2} r(t)^4 \omega_0^2 - \varepsilon \eta_0 \wedge \text{Im } \Upsilon_0; \\ g_t &= \varepsilon^2 r(t)^{-6} \eta_0^2 + r(t)^2 g_{\mathcal{D}_0}; \\ \text{vol}_t &= \varepsilon r(t) \eta_0 \wedge \text{vol}_{\mathcal{D}_0},\end{aligned}$$

where $r(t) = (1 + \frac{5}{2}\varepsilon t)^{1/5}$ and $t \in (-\frac{2}{5\varepsilon}, \infty)$. This Hitchin flow solution coincides with the Laplacian coflow in Theorem 1.1, up to reparametrisation of time, and defines an incomplete Calabi–Yau structure on $(-\frac{2}{5\varepsilon}, \infty) \times M$, with Kähler form $\hat{\omega}$, metric \hat{g} with $\text{Hol}(\hat{g}) \subseteq \text{SU}(4)$, and holomorphic volume form $\hat{\Upsilon}$, as follows:

$$\begin{aligned}\hat{\omega} &= \varepsilon r(t)^{-3} dt \wedge \eta_0 + r(t)^2 \omega_0; \\ \hat{g} &= dt^2 + \varepsilon^2 r(t)^{-6} \eta_0^2 + r(t)^2 g_0; \\ \hat{\Upsilon} &= (r(t)^3 dt + i\varepsilon \eta_0) \wedge \Upsilon_0.\end{aligned}$$

Remark 1.5. The relation in Theorem 1.4 between contact Calabi–Yau 7-manifolds, the Hitchin flow and Calabi–Yau structures on 8-manifolds was previously known, for example, by work in [CF10] and [Fre18]. In the quasi-regular setting, the Calabi–Yau structures in Theorem 1.4 arise via the well-known Calabi ansatz on open subsets of complex line bundles over Calabi–Yau (or, more generally, Kähler–Einstein) orbifolds. It is interesting however to observe the close connection between the Laplacian cflow and the Hitchin flow in this setting, which is a novel aspect of our study.

We also include, in Appendix A, some additional computations which emerged during the course of our research and that we believe will be of further use in the study of flows of coclosed G_2 -structures.

Remark 1.6. A natural question, if perhaps a little bold, is whether flows on cCY 7-manifolds might actually converge to a torsion-free G_2 -structure. While cautioning against excessive optimism, we notice that there are many compact examples with no apparent topological obstructions, on which moreover such a structure would induce a metric with holonomy group G_2 , cf. Remark 2.2 in §2.2. We emphasise that the study in this article does not pertain to this question, because the Ansatz G_2 -structures under consideration admit a non-trivial Killing field and so could not converge under the flow to a torsion-free G_2 -structure.

Acknowledgements: The authors would like to thank Simon Salamon, Mark Haskins and Andrés Moreno for some valuable discussions. JL and HSE were supported by a UK Royal Society Newton Mobility Award [NMG\R1\191068]. JL is also partially supported by the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics (#724071 Jason Lotay). HSE has also been supported by the São Paulo Research Foundation (Fapesp) [2018/21391-1] and the Brazilian National Council for Scientific and Technological Development (CNPq) [311128/2020-3]. JPS was supported by the Coordination for the Improvement of Higher Education Personnel-Brazil (CAPES) [88882.329037/2019-1].

Data availability statement: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

2 Preliminaries on coclosed G_2 -structures

In this section, we collect definitions and important properties regarding (coclosed) G_2 -structures which will be useful in the study of flows on contact Calabi–Yau 7-manifolds.

2.1 G_2 -structures and their torsion forms

A G_2 -structure on a (orientable and spin) 7-manifold M is a reduction of the structure group of TM to G_2 , where G_2 is viewed as the subgroup of $GL(7, \mathbb{R})$ containing all linear maps preserving the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3(\mathbb{R}^7)^*,$$

where $e^i = dx^i$, $e^{ij} = e^i \wedge e^j$ etc. It is equivalently determined by a 3-form φ on M such that (TM, φ) is pointwise isomorphic to $(\mathbb{R}^7, \varphi_0)$. A G_2 -structure φ determines a Riemannian metric g_φ and an orientation given by the Riemannian volume form vol_φ so that

$$6g_\varphi(X, Y)\text{vol}_\varphi = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$$

for any tangent vectors X, Y on M . For simplicity, we write $g = g_\varphi$. The metric and the orientation determine a Hodge star operator $*_\varphi$, so we have the dual 4-form $\psi = *_\varphi \varphi$ of the G_2 -structure φ .

A G_2 -structure gives rise to a decomposition of differential forms on M corresponding to irreducible G_2 representations. In particular, the spaces Ω^2 and Ω^3 of 2-forms and 3-forms decompose as

$$\Omega^2 = \Omega_7^2 \oplus \Omega_{14}^2 \quad \text{and} \quad \Omega^3 = \Omega_1^3 \oplus \Omega_7^3 \oplus \Omega_{27}^3, \quad (2.1)$$

where Ω_l^k has (pointwise) dimension l and this decomposition is orthogonal with respect to the metric g . Via the Hodge star, this defines corresponding decompositions of the 4-forms Ω^4 and 5-forms Ω^5 . If we let S^2 denote the symmetric 2-tensors on M then, as in [Bry06], we define a linear operator $j_\varphi : \Omega^3 \rightarrow S^2$ by

$$j_\varphi(\gamma)(X, Y) = *_\varphi((X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \gamma), \quad (2.2)$$

where X, Y are tangent vectors on M . Then j_φ is surjective with kernel equal to Ω_7^3 (see e.g. [Kar07, Proposition 2.17]). We can also define an injective linear operator $i_\varphi : S^2 \rightarrow \Omega_1^3 \oplus \Omega_{27}^3$ as in [Bry06] which is (up to scaling) a right inverse for j_φ , and is given locally using summation notation by

$$i_\varphi(h) = \frac{1}{2} h_i^l \varphi_{ljk} e^{ijk}, \quad (2.3)$$

where $h \in S^2$ is given locally by $h_{ij} e^i e^j$. We note that $i_\varphi(g) = 3\varphi$, $j_\varphi(\varphi) = 6g$ and

$$j_\varphi \circ i_\varphi(h) = 2(\text{tr } h)g + 4h, \quad (2.4)$$

where $\text{tr } h = g^{ij} h_{ij}$ is the trace of h with respect to g . For later use, we let S_0^2 denote the trace-free symmetric 2-tensors on M and note that $i_\varphi : S_0^2 \rightarrow \Omega_{27}^3$ is an isomorphism.

Given any G_2 -structure φ , there exist unique differential forms $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega_{14}^2$ and $\tau_3 \in \Omega_{27}^3$, using the decomposition of forms in (2.1), such that (see e.g. [Bry06, Proposition 1])

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + * \tau_3, \quad (2.5)$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi. \quad (2.6)$$

Together, the forms $\{\tau_0, \tau_1, \tau_2, \tau_3\}$ are called the *intrinsic torsion forms* of the G_2 -structure φ . The *full torsion tensor* T is defined locally by the formula

$$\nabla_i \varphi_{jkl} = T_i^m \psi_{m jkl} \quad (2.7)$$

and may be expressed using the torsion forms as (see e.g. [Kar07, Theorem 2.27])

$$T = \frac{\tau_0}{4} g - \tau_1^\sharp \lrcorner \varphi - \frac{1}{2} \tau_2 - \frac{1}{4} j_\varphi(\tau_3). \quad (2.8)$$

By (2.6), a G_2 -structure is coclosed if and only if $\tau_1 = 0$ and $\tau_2 = 0$. Hence, the full torsion tensor of a coclosed G_2 -structure simplifies to the symmetric 2-tensor

$$T = \frac{\tau_0}{4} g - \frac{1}{4} j_\varphi(\tau_3) \in S^2. \quad (2.9)$$

2.2 Contact Calabi–Yau manifolds

In this section we review the properties of contact Calabi–Yau (cCY) 7-manifolds and the natural coclosed G_2 -structures on them. The original sources for cCY manifolds are [TV08, HV15]; for a more detailed exposition and study of G_2 -geometry on cCY 7-manifolds, see [CARSE20].

Definition 2.1. A *contact Calabi–Yau* (cCY) manifold is a quadruple $(M^{2n+1}, g, \eta, \Upsilon)$ such that

- (M, g) is a $(2n+1)$ -dimensional Sasakian manifold with contact form η ;
- Υ is a nowhere vanishing transversal form on $\mathcal{D} = \ker \eta$ of type $(n, 0)$ with

$$\Upsilon \wedge \bar{\Upsilon} = c_n \omega^n, \quad d\Upsilon = 0,$$

where $c_n = (-1)^{\frac{n(n+1)}{2}} i^n$ and $\omega = d\eta$. We shall write:

$$\text{Re } \Upsilon := \frac{\Upsilon + \bar{\Upsilon}}{2}, \quad \text{Im } \Upsilon := \frac{\Upsilon - \bar{\Upsilon}}{2i}.$$

Remark 2.1. A contact Calabi–Yau manifold (M, g, η, Υ) has a transverse Calabi–Yau geometry on $\mathcal{D} = \ker \eta$, given by $g|_{\mathcal{D}}$, ω and Υ . When the Sasakian geometry is regular or quasi-regular, M is an S^1 -(orbi)bundle over a Calabi–Yau orbifold, but the Sasakian geometry can also be irregular, and then there is no S^1 -fibration structure on M compatible with the contact Calabi–Yau geometry.

We now see how to relate the contact Calabi–Yau geometry in 7 dimensions to G_2 geometry (cf. [HV15, Corollary 6.8] and [LSE21]).

Proposition 2.1. *Let (M^7, g, η, Υ) be a contact Calabi–Yau 7-manifold. Then M carries a 1-parameter of coclosed G_2 -structures defined by (1.6), for $\varepsilon > 0$, where $\omega = d\eta$. Furthermore, $d\varphi = \varepsilon\omega \wedge \omega$ and its corresponding dual 4-form is given by (1.7).*

Notice that ψ is manifestly closed, as ω and Υ are closed and $\omega \wedge \Upsilon = 0$ by bidegree considerations, i.e. φ is coclosed, and its torsion is encoded in $d\varphi = \varepsilon\omega \wedge \omega$. Various other properties of the ε -family (1.6) are studied in [LSE21].

We now provide a concrete means for finding examples of contact Calabi–Yau 7-manifolds, which are circle (orbi)bundles over Calabi–Yau 3-orbifolds.

Example 2.1 (Calabi–Yau links). Given $w = (w_0, \dots, w_4) \in \mathbb{Q}^5$, the zero set of a w -weighted homogeneous polynomial $f \in \mathbb{C}[z_0, \dots, z_4]$ of degree $d = \sum_{i=0}^4 w_i$ is an affine hypersurface in \mathbb{C}^5 with an isolated singularity at 0.

Its link M_f on a local 9-sphere is a compact and 2-connected smooth cCY 7-manifold, fibering by circles over a Calabi–Yau 3-orbifold $Z \subseteq \mathbb{P}^4(w)$ by the weighted Hopf fibration [CARSE20, Theorem 1.1]:

$$\begin{array}{ccc} M_f^7 & \hookrightarrow & S^9 \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & \mathbb{P}^4(w) \end{array}$$

In particular, Z can be taken to be any of the weighted Calabi–Yau 3-folds listed in [CLS90]. For a detailed survey on Calabi–Yau links, see [CARSE20, §2]. The \mathbb{C} -family of Fermat quintics yields the simplest examples, and indeed the only ones for which the base Z is smooth.

Remark 2.2. Regarding the possibility of a torsion-free G_2 -structure on a Calabi–Yau link, we observe a number of favourable topological circumstances, in the terms of [Joy00, §10.2].

Since it admits natural G_2 -structures such as in (S), every cCY 7-manifold is obviously orientable and spin. In particular, CY links are 2-connected, so trivially $\pi_1(M_f)$ is finite and there is no obstruction coming from an intersection form on $H^2(M_f)$. Moreover, CY links are compact, so if a torsion-free G_2 -structure exists at all, then its induced Riemannian metric will have holonomy group precisely G_2 . Finally, since every such CY link $M_f \rightarrow Z \subseteq \mathbb{P}^4(w)$ admits solutions to the heterotic Bianchi identity [LSE21, Theorem 1], its first Pontryagin class $p_1(M_f)$ coincides with the pullback of the second Chern class $c_2(Z)$ of the weighted projective 3-orbifold, which is certainly not trivial in general, e.g. for the Fermat quintic itself.

3 The Laplacian coflow solution

In this section we solve the Laplacian coflow (1.1) of G_2 -structures on a contact Calabi–Yau 7-manifold by an explicit ansatz, and we study the behaviour of the metric and torsion along the flow.

3.1 Solving the flow

We want to consider the Laplacian coflow starting at the natural coclosed G_2 -structure defined by (1.6) and (1.7) on a cCY setup (S):

$$\varphi_0 = \varepsilon\eta_0 \wedge \omega_0 + \operatorname{Re} \Upsilon_0 \quad \text{and} \quad \psi_0 = \frac{1}{2}\omega_0^2 - \varepsilon\eta_0 \wedge \operatorname{Im} \Upsilon_0. \quad (3.1)$$

To this end, we consider the family of G_2 -structures given by

$$\varphi_t = f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \operatorname{Re} \Upsilon_0, \quad (3.2)$$

for functions f_t, h_t depending only on time, with

$$f_0 = \varepsilon \quad \text{and} \quad h_0 = 1. \quad (3.3)$$

The induced metric and associated volume form are then given by

$$g_t = f_t^2 \eta_0^2 + h_t^2 g_{\mathcal{D}_0} \quad \text{and} \quad \operatorname{vol}_t = f_t h_t^6 \eta_0 \wedge \operatorname{vol}_{\mathcal{D}_0}, \quad (3.4)$$

with

$$\text{vol}_{\mathcal{D}} = \frac{1}{3!} \omega_0^3 = \frac{i}{8} \Upsilon_0 \wedge \bar{\Upsilon}_0 = \frac{1}{4} \text{Re } \Upsilon_0 \wedge \text{Im } \bar{\Upsilon}_0. \quad (3.5)$$

It follows from (3.2), (3.4), and (3.5) that

$$\psi_t = \frac{1}{2} h_t^4 \omega_0^2 - f_t h_t^3 \eta_0 \wedge \text{Im } \Upsilon_0. \quad (3.6)$$

We observe that (3.1) is indeed the instance at $t = 0$ of (3.2) and (3.6).

Lemma 3.1. *Let φ_t be a G_2 -structure as in (3.2). Then we have*

$$d\varphi_t = f_t h_t^2 \omega_0^2, \quad d\psi_t = 0 \quad \text{and} \quad *_t d\varphi_t = 2f_t^2 \eta_0 \wedge \omega_0. \quad (3.7)$$

Hence, the torsion forms of φ_t as in (2.5) are

$$(\tau_0)_t = \frac{6f_t}{7h_t^2}, \quad (\tau_1)_t = 0, \quad (\tau_2)_t = 0, \quad (\tau_3)_t = \frac{8}{7} f_t^2 \eta_0 \wedge \omega_0 - \frac{6}{7} f_t h_t \text{Re } \Upsilon_0. \quad (3.8)$$

Proof. From (3.2), (3.6), and Definition 2.1, we easily see that

$$d\varphi_t = f_t h_t^2 \omega_0^2 \quad \text{and} \quad d\psi_t = 0, \quad (3.9)$$

since $d\eta_0 = \omega_0$ and $\omega_0 \wedge \Upsilon_0 = 0$. We also note from (3.4) and (3.9) that

$$\begin{aligned} *_t d\varphi_t &= *_t (f_t h_t^2 \omega_0 \wedge \omega_0) = *_t (2f_t h_t^{-2} \cdot \frac{1}{2} h_t^4 \omega_0^2) = 2f_t h_t^{-2} \cdot f_t h_t^2 \eta_0 \wedge \omega_0 \\ &= 2f_t^2 \eta_0 \wedge \omega_0, \end{aligned}$$

which then yields (3.7).

We can now compute the torsion forms of φ_t as follows. First,

$$\begin{aligned} (\tau_0)_t &= \frac{1}{7} *_t (\varphi_t \wedge d\varphi_t) = \frac{1}{7} *_t ((f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \text{Re } \Upsilon_0) \wedge f_t h_t^2 \omega_0^2) \\ &= \frac{1}{7} *_t f_t^2 h_t^4 \eta_0 \wedge \omega_0^3 = \frac{6f_t}{7h_t^2}. \end{aligned} \quad (3.10)$$

We know that $(\tau_1)_t = 0$ and $(\tau_2)_t = 0$ since φ_t is coclosed. Furthermore, from (3.9) and (3.10), we have

$$\begin{aligned} (\tau_3)_t &= *_t d\varphi_t - (\tau_0)_t \varphi_t \\ &= 2f_t^2 \eta_0 \wedge \omega_0 - \frac{6f_t}{7h_t^2} (f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \text{Re } \Upsilon_0) \\ &= \frac{8}{7} f_t^2 \eta_0 \wedge \omega_0 - \frac{6}{7} f_t h_t \text{Re } \Upsilon_0. \end{aligned}$$

Equation (3.8) then follows. \square

We can now prove Theorem 1.1, by finding f_t, h_t so that (3.6) is a solution of the Laplacian coflow (1.1).

Proof of Theorem 1.1. On the cCY setup (S), consider the family of G_2 -structures given by (3.2) and (3.6). We may then compute the Laplacian of ψ_t using (3.9) and Definition 2.1:

$$\Delta_t \psi_t = d *_t d\varphi_t = d(2f_t^2 \eta_0 \wedge \omega_0) = 2f_t^2 \omega_0 \wedge \omega_0. \quad (3.11)$$

Differentiating (3.6) with respect to t and using (3.11), we can compute $\frac{\partial \psi}{\partial t} = \Delta_t \psi_t$ and then equate the coefficients of $\eta_0 \wedge \text{Im } \Upsilon_0$ and ω_0^2 (since the flow preserves the class of (3.6)) to obtain

$$\frac{d}{dt} h_t^4 = 4f_t^2, \quad \frac{d}{dt} f_t h_t^3 = 0. \quad (3.12)$$

Therefore, from (3.3) and the second equation in (3.12) we conclude that

$$f_t = \varepsilon h_t^{-3}. \quad (3.13)$$

Substituting (3.13) into (3.12), we have

$$\frac{d}{dt}h_t^4 = 4\varepsilon^2 h_t^{-6}. \quad (3.14)$$

The ODE (3.14) can be easily solved and so, together with (3.3) and (3.13), we find that

$$f_t = \varepsilon(1 + 10\varepsilon^2 t)^{-3/10} \quad \text{and} \quad h_t = (1 + 10\varepsilon^2 t)^{1/10}. \quad (3.15)$$

In conclusion, we have found a solution to the Laplacian coflow (1.1) in the cCY setup (S), with initial condition (3.1), which induces the following family of G_2 -structures and their duals, metrics and volume forms:

$$\varphi_t = \varepsilon(1 + 10\varepsilon^2 t)^{-1/10} \eta_0 \wedge \omega_0 + (1 + 10\varepsilon^2 t)^{3/10} \operatorname{Re} \Upsilon_0; \quad (3.16)$$

$$\psi_t = \frac{1}{2}(1 + 10\varepsilon^2 t)^{2/5} \omega_0^2 - \varepsilon \eta_0 \wedge \operatorname{Im} \Upsilon_0; \quad (3.17)$$

$$g_t = \varepsilon^2(1 + 10\varepsilon^2 t)^{-3/5} \eta_0^2 + (1 + 10\varepsilon^2 t)^{1/5} g_{\mathcal{D}_0}; \quad (3.18)$$

$$\operatorname{vol}_t = \varepsilon(1 + 10\varepsilon^2 t)^{3/10} \eta_0 \wedge \operatorname{vol}_{\mathcal{D}_0}, \quad (3.19)$$

where $\mathcal{D}_0 = \ker \eta_0$, defined for all $t \in (-\frac{1}{10\varepsilon^2}, \infty)$. \square

Remark 3.1. We notice that the solution to the Laplacian coflow we have obtained is *immortal* (i.e. exists for all positive time), but it is not eternal, since it fails to exist for $t \leq -\frac{1}{10\varepsilon^2}$. If M is compact, this will be the unique solution of the form (3.2) satisfying (3.3), and we expect that it will be the unique solution starting at (3.1) in general, but such a general theory is lacking as mentioned in §1.1.

Remark 3.2. We see from (3.19) that the volume form is strictly increasing pointwise in time. We therefore confirm that, if M is compact, then its volume is indeed strictly increasing in time, tending to infinity:

$$\operatorname{Vol}(M, g_t) \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

We also notice that ψ_t does indeed stay in the cohomology class $[\psi_0]$.

However, suppose the contact Calabi–Yau structure is quasi-regular, so that M is an S^1 -(orbi)bundle over a Calabi–Yau 3-orbifold Z . We then observe that, by (3.11),

$$\Delta_t \psi_t = 2f_t^2 \omega_0 \wedge \omega_0,$$

which is a transverse 4-form that will descend to Z and cannot be exact on Z . Hence, the Laplacian coflow does not induce a flow of $SU(3)$ -structures on Z with forms staying in some fixed cohomology class.

3.2 Metric and curvature

It is worth studying aspects of the asymptotic behaviour of the solution to the Laplacian coflow we have found in terms of the metric geometry.

It is straightforward to see from (3.18) that, as $t \rightarrow \infty$, the direction dual to η_0 on M collapses whilst the transverse geometry given by \mathcal{D}_0 expands. On the other hand, the reverse situation occurs as $t \rightarrow -\frac{1}{10\varepsilon^2}$.

Example 3.1. For illustration, let us suppose that M is compact and that the cCY structure is quasi-regular, so that M is an S^1 -(orbi)bundle over a compact Calabi–Yau 3-orbifold Z with metric g_Z . Then, as $t \rightarrow \infty$ the circle fibres collapse, whilst the base Z expands to become \mathbb{C}^3 with the flat metric, since the coefficient of $g_{\mathcal{D}_0} = g_Z$ in (3.18) tends to infinity. As $t \rightarrow -\frac{1}{10\varepsilon^2}$, on the other hand, the base Z collapses to a point and the circles expand to become the real line \mathbb{R} .

In the analysis so far we have neglected the fact that the Laplacian coflow is the gradient flow of the volume functional, and so the volume must always be increasing along the flow. To remedy this, we may rescale the family (M, g_t) so that the volume is fixed and obtain the following.

Lemma 3.2. *Let M be compact. After normalising (M, g_t) to a fixed volume, the Laplacian coflow solution collapses to \mathbb{R} , as $t \rightarrow -\frac{1}{10\varepsilon^2}$, and to \mathbb{C}^3 with the flat metric, as $t \rightarrow \infty$.*

Proof. Fixing the volume to be constant is equivalent to multiplying the metric by $(1 + 10\varepsilon^2 t)^{-3/35}$ by (3.19), which gives

$$(1 + 10\varepsilon^2 t)^{-3/35} g_t = (1 + 10\varepsilon^2 t)^{-24/35} \eta_0^2 + (1 + 10\varepsilon^2 t)^{4/35} g_{\mathcal{D}_0}.$$

The result then follows. \square

We now take the standard approach to understanding the behaviour of volume along a family of compact manifolds, by normalising the diameter. From the formula (3.18) for the metric g_t , we immediately obtain:

Lemma 3.3. *Let M be compact. After normalising (M, g_t) to unit diameter, the Laplacian coflow solution is volume-collapsing to a 1-dimensional space M_- as $t \rightarrow -\frac{1}{10\varepsilon^2}$ and to a 6-dimensional space M_+ as $t \rightarrow \infty$. If the Sasakian structure is quasi-regular, so that M is an S^1 -orbibundle over a Calabi–Yau 3-orbifold Z , then $M_- = S^1$ and $M_+ = Z$.*

In the light of what is known for the Laplacian flow of closed G_2 -structures, cf. [LW19, Theorem 8.1], we might expect that a condition on the uniform continuity of the metrics along the Laplacian coflow of coclosed G_2 -structures, together with a (pointwise) bound on the torsion tensor, would lead to long-time existence results. Since no such general theory currently exists, it is therefore useful to examine the uniform continuity properties of the solution to the Laplacian coflow we have found.

Lemma 3.4. *Let φ_t be the solution to the Laplacian coflow given by (3.16). Then the associated metric g_t is uniformly continuous (in t) on any compact interval contained in $(-\frac{1}{10\varepsilon^2}, \infty)$, but it is not uniformly continuous on $(-\frac{1}{10\varepsilon^2}, T)$ or (T, ∞) for any T .*

Proof. We immediately see from the formula (3.18) for the metrics g_t that, since the 2-tensors η_0^2 and $g_{\mathcal{D}_0}$ are fixed, the uniform continuity in t of g_t on an interval I is equivalent to the uniform continuity of both $(1 + 10\varepsilon^2 t)^{-3/5}$ and $(1 + 10\varepsilon^2 t)^{1/5}$ on I . The result follows. \square

We now study the behaviour of the Riemann curvature tensor along our Laplacian coflow solution, based on some computations of the Riemannian geometry of cCY 7-manifolds carried out in detail in [LSE21, §3.1].

Proposition 3.5. *Let φ_t be the solution to the Laplacian coflow given by (3.16) with associated metric g_t as in (3.18). Let Rm_t denote the Riemann curvature tensor of g_t and let $Rm_0^{\mathcal{D}_0}$ denote the curvature of the transverse connection on \mathcal{D}_0 induced by the Levi-Civita connection of g_0 . Then*

$$|Rm_t|_{g_t}^2 = (1 + 10\varepsilon^2 t)^{-2/5} |Rm_0^{\mathcal{D}_0}|_{g_0}^2 + c_0 \varepsilon^4 (1 + 10\varepsilon^2 t)^{-2}$$

for some constant $c_0 > 0$.

Proof. If we write g_t as in (3.4) for functions f_t, h_t given in (3.15), we may observe that if we let

$$\bar{g}_t = \frac{f_t^2}{h_t^2} \eta_0^2 + g_{\mathcal{D}_0} \tag{3.20}$$

then $g_t = h_t^2 \bar{g}_t$. Hence, if \overline{Rm}_t is the Riemann curvature tensor of \bar{g}_t in (3.20), we deduce that

$$|Rm_t|_{g_t} = \frac{1}{h_t^2} |\overline{Rm}_t|_{\bar{g}_t}. \tag{3.21}$$

It is a direct consequence of [LSE21, Proposition 3.2] that the transverse connection on \mathcal{D}_0 induced by \bar{g}_t is the same as for g_0 , and hence its curvatures are independent of t . It follows from [LSE21, Proposition 3.8] that the remaining components of the Riemann curvature of \bar{g}_t scale by f_t^2/h_t^2 in comparison to the curvature of the standard Sasakian metric g . We deduce that

$$|\overline{Rm}_t|_{\bar{g}_t}^2 = |Rm_0^{\mathcal{D}_0}|_{g_0}^2 + \frac{c_0 f_t^4}{h_t^4} \tag{3.22}$$

for some constant $c_0 > 0$. The assertion now follows from (3.15), (3.21), and (3.22). \square

Remark 3.3. We see from Proposition 3.5 that the curvature of the metric g_t given by the solution to the Laplacian coflow decays to 0 as $t \rightarrow \infty$ but blows up as t approaches the finite negative time singularity at $-\frac{1}{10\varepsilon^2}$. However, if we consider $t|Rm_t|_{g_t}$, this will tend to infinity as $t \rightarrow \infty$ unless the curvature $Rm_0^{\mathcal{D}_0}$ of the transverse connection is zero. However, we see that $(1 + 10\varepsilon^2 t)|Rm_t|_{g_t}$ will remain bounded as $t \rightarrow -\frac{1}{10\varepsilon^2}$. We will return to these observations later.

3.3 Full torsion tensor

We now wish to analyse the behaviour of the torsion along our solution to the Laplacian coflow. We begin by writing down the full torsion tensor for the solution.

Proposition 3.6. *The full torsion tensor T_t of the solution to the Laplacian coflow in (3.16)–(3.19) is*

$$\begin{aligned} T_t &= \frac{1}{4}(\tau_0)_t g_t - \frac{1}{4}j_{\varphi_t}((\tau_3)_t) \\ &= \frac{3\varepsilon}{14}(1 + 10\varepsilon^2 t)^{-1/2} g_t \\ &\quad - \frac{1}{14}j_{\varphi_t} \left(4\varepsilon^2(1 + 10\varepsilon^2 t)^{-3/5} \eta_0 \wedge \omega_0 - 3\varepsilon(1 + 10\varepsilon^2 t)^{-1/5} \operatorname{Re} \Upsilon_0 \right) \\ &= -\frac{3}{2}\varepsilon^3(1 + 10\varepsilon^2 t)^{-11/10} \eta_0^2 + \frac{1}{2}\varepsilon(1 + 10\varepsilon^2 t)^{-3/10} g_{\mathcal{D}_0}. \end{aligned} \quad (3.23)$$

Proof. This first two expressions for T_t follow from (2.9), Lemma 3.1 and (3.15).

For the final expression, we first let ξ_0 be the dual vector field to η_0 , we let $X, Y \in \mathcal{D}_0$ and consider φ_t as in (3.2) for functions f_t and h_t . We recall that, since (ω_0, Υ_0) defines an $\operatorname{SU}(3)$ -structure on \mathcal{D}_0 we have that $\omega_0 \wedge \operatorname{Re} \Upsilon_0 = 0$,

$$(X \lrcorner \operatorname{Re} \Upsilon_0) \wedge (Y \lrcorner \operatorname{Re} \Upsilon_0) \wedge \omega_0 = 2g_{\mathcal{D}_0}(X, Y) \operatorname{vol}_{\mathcal{D}_0} \quad (3.24)$$

$$(X \lrcorner \omega_0) \wedge (Y \lrcorner \operatorname{Re} \Upsilon_0) \wedge \operatorname{Re} \Upsilon_0 = -2g_{\mathcal{D}_0}(X, Y) \operatorname{vol}_{\mathcal{D}_0}. \quad (3.25)$$

We then see from (2.2), (3.2), (3.4), and (3.24) that

$$j_{\varphi_t}(\eta_0 \wedge \omega_0)(\xi_0, \xi_0) = *_t(f_t^2 h_t^4 \eta_0 \wedge \omega_0^3) = 6 \frac{f_t}{h_t^2}, \quad (3.26)$$

$$j_{\varphi_t}(\eta_0 \wedge \omega_0)(\xi_0, X) = 0, \quad (3.27)$$

$$j_{\varphi_t}(\eta_0 \wedge \omega_0)(X, Y) = *_t(2h_t^6 g_{\mathcal{D}_0}(X, Y) \eta_0 \wedge \operatorname{vol}_{\mathcal{D}_0}) = \frac{2}{f_t} g_{\mathcal{D}_0}(X, Y). \quad (3.28)$$

Similarly, using (3.25),

$$j_{\varphi_t}(\operatorname{Re} \Upsilon_0)(\xi_0, \xi_0) = 0, \quad (3.29)$$

$$j_{\varphi_t}(\operatorname{Re} \Upsilon_0)(\xi_0, X) = 0, \quad (3.30)$$

$$j_{\varphi_t}(\operatorname{Re} \Upsilon_0)(X, Y) = *_t(4f_t h_t^5 g_{\mathcal{D}_0}(X, Y) \eta_0 \wedge \operatorname{vol}_{\mathcal{D}_0}) = \frac{4}{h_t} g_{\mathcal{D}_0}(X, Y), \quad (3.31)$$

Applying (3.26)–(3.31) along with the particular expressions for f_t and h_t in (3.15) then gives that

$$\begin{aligned} j_{\varphi_t} \left(4\varepsilon^2(1 + 10\varepsilon^2 t)^{-3/5} \eta_0 \wedge \omega_0 - 3\varepsilon(1 + 10\varepsilon^2 t)^{-1/5} \operatorname{Re} \Upsilon_0 \right) \\ = 24\varepsilon^3(1 + 10\varepsilon^2 t)^{-11/10} \eta_0^2 - 4\varepsilon(1 + 10\varepsilon^2 t)^{-3/10} g_{\mathcal{D}_0}. \end{aligned} \quad (3.32)$$

Substituting (3.32) into (3.23) gives the claimed final expression for T_t . \square

Given the description of the full torsion tensor in Proposition 3.6, we can compute its norm, the norm of its gradient and its divergence as follows.

Proposition 3.7. *Let T_t be the full torsion tensor of the solution to the Laplacian coflow in (3.16)–(3.19). Then*

$$\begin{aligned} |T_t|_{g_t}^2 &= \frac{15}{4}\varepsilon^2(1 + 10\varepsilon^2 t)^{-1}, \\ |\nabla_t T_t|_{g_t}^2 &= c_0 \varepsilon^4(1 + 10\varepsilon^2 t)^{-2}, \\ \operatorname{div}_t T_t &= 0, \end{aligned}$$

where $c_0 > 0$ is a constant, ∇_t is the Levi-Civita connection of g_t and div_t is the divergence with respect to the metric g_t .

Proof. We know from Proposition 3.6 that

$$T_t = -2\varepsilon^3(1 + 10\varepsilon^2t)^{-11/10}\eta_0^2 + \frac{1}{2}\varepsilon(1 + 10\varepsilon^2t)^{-1/2}g_t. \quad (3.33)$$

On the other hand, from (3.18),

$$|\varepsilon^2\eta_0^2|_{g_t}^2 = (1 + 10\varepsilon^2t)^{6/5} \quad \text{and} \quad |g_{\mathcal{D}_0}|_{g_t}^2 = 6(1 + 10\varepsilon^2t)^{-2/5}.$$

We then deduce the first claimed equation:

$$|T_t|_{g_t}^2 = \frac{9}{4}\varepsilon^2(1 + 10\varepsilon^2t)^{-1} + \frac{3}{2}\varepsilon^2(1 + 10\varepsilon^2t)^{-1} = \frac{15}{4}\varepsilon^2(1 + 10\varepsilon^2t)^{-1}.$$

For the remaining claimed equations, we know that $\nabla_t g_t = 0$ and that $(\nabla_t)_{\xi_0}\xi_0 = 0$ where ξ_0 is the vector field dual to η_0 , since ξ_0 generates a local symmetry of the G_2 -structure φ_t , and hence the metric g_t , for all t . We deduce from (3.33) that $\operatorname{div}_t T_t = 0$ as claimed and that the only non-zero terms in $\nabla_t T_t$ arise from $(\nabla_t)_X \eta_0^2$ for $X \in \mathcal{D}_0$.

We now see from [LSE21, Proposition 3.2] that, for a metric of the form $g_t = f_t^2\eta_0^2 + h_t^2g_{\mathcal{D}_0}$, and $X \in \mathcal{D}_0$ so that $g_t(X, X) = 1$, we have $((\nabla_t)_X \eta_0)^\sharp \in \mathcal{D}_0$, and

$$|(\nabla_t)_X(f_t^2\eta_0^2)|_{g_t}^2 = \frac{c_0 f_t^2}{h_t^4},$$

for some constant $c_0 > 0$. This is because the component of the connection form relevant to $(\nabla_t)_X \eta_0$ scales by f_t/h_t and the metric on \mathcal{D}_0 scales by h_t^2 . Choosing f_t and h_t as in (3.15), we finally obtain

$$\begin{aligned} |(\nabla_t)_X T_t|_{g_t}^2 &= |2\varepsilon(1 + 10\varepsilon^2t)^{-1/2}(\nabla_t)_X(\varepsilon^2(1 + 10\varepsilon^2t)^{-3/5}\eta_0^2)|_{g_t}^2 \\ &= 4\varepsilon^2(1 + 10\varepsilon^2t)^{-1} \cdot c_0\varepsilon^2(1 + 10\varepsilon^2t)^{-1} \\ &= 4c_0\varepsilon^4(1 + 10\varepsilon^2t)^{-2}. \end{aligned} \quad \square$$

Remark 3.4. We see from Proposition 3.7 that $|T_t|_{g_t}^2$ and $|\nabla_t T_t|_{g_t}$ have the same dependence on t , as we would expect from the general theory in the study of flows of G_2 -structures, and that there is a component of $|Rm_t|_{g_t}$ with this same dependence on t , by Proposition 3.5. However, there is a component of $|Rm_t|_{g_t}$ which has a different dependence on t , as long as the transverse geometry on the contact Calabi–Yau is not flat: this is unsurprising because, in particular, having small torsion does not imply that the metric is close to flat.

3.4 Singularity analysis

We can now combine all of the estimates we have obtained to describe the infinite time singularity for the Laplacian coflow in terms of the types in Definition 1.1, which proves Theorem 1.3(a).

Proposition 3.8. *In a cCY setup (S), suppose moreover M^7 is compact, and let $K := \sup_M |Rm_0^{\mathcal{D}_0}|_{g_0}$. Then there is a constant $c_0 > 0$, independent of ε , such that the quantity $\Lambda(t)$ in (1.3), along the Laplacian coflow solution given by Theorem 1.1, is given by*

$$\Lambda(t) = (K^2(1 + 10\varepsilon^2t)^{-2/5} + c_0\varepsilon^4(1 + 10\varepsilon^2t)^{-2})^{1/2}.$$

Hence, the Laplacian coflow has a Type IIb infinite time singularity, unless $g_{\mathcal{D}_0}$ is flat, in which case it has a Type III infinite time singularity.

Proof. The formula for $\Lambda(t)$ follows immediately from Proposition 3.5 and Proposition 3.7. We then see that $t\Lambda(t) \rightarrow \infty$ as $t \rightarrow \infty$ unless $K = 0$, from which the result then follows by Definition 1.1. \square

Remark 3.5. As remarked earlier, these are the first examples of Laplacian coflow solutions which have Type IIb singularities. The other examples are all homogeneous, arising from nilmanifold and solvmanifold constructions, and in those cases one always has a Type III singularity if it does not converge to a flat torsion-free G_2 -structure.

4 The Laplacian flow solution

We now want to consider the Laplacian flow (1.2) starting at one of the natural coclosed G_2 -structures on a contact Calabi–Yau 7-manifold. Whilst the class of coclosed G_2 -structures is not expected to be preserved, in general, along the Laplacian flow, we will see that it is in our setting and that we can solve the flow explicitly. By again studying the evolution of the metric and torsion along an ansatz for the flow, we will see that the behaviour is rather different from the Laplacian coflow, having a finite time singularity but existing for all negative times.

4.1 Solving the flow

Again, we choose the natural coclosed G_2 -structure φ_0 on a cCY setup (S), cf. (3.1), as the initial condition for our flow. We are therefore again led to consider the G_2 -structures φ_t in (3.2) depending on functions f_t, h_t (which only depend on time) satisfying the initial conditions (3.3) to prove Theorem 1.2 as follows.

Proof of Theorem 1.2. We compute the Laplacian of φ_t using (3.4) and (3.11):

$$\begin{aligned}\Delta_{\varphi_t}\varphi_t &= *_{\varphi_t}(\Delta_{\psi_t}\psi_t) = *_{\varphi_t}(2f_t^2\omega_0^2) = *_{\varphi_t}\left(4\frac{f_t^2}{h_t^4} \cdot \frac{h_t^4\omega_0^2}{2}\right) = 4\frac{f_t^2}{h_t^4} \cdot f_th_t^2\eta_0 \wedge \omega_0 \\ &= \frac{4f_t^3}{h_t^2}\eta_0 \wedge \omega_0.\end{aligned}\tag{4.1}$$

We deduce from (3.2) and (4.1) that the Laplacian flow (1.2) preserves the class of coclosed G_2 -structures in (3.2). Moreover, the Laplacian flow is equivalent to the following pair of ODEs:

$$\frac{d}{dt}(f_th_t^2) = \frac{4f_t^3}{h_t^2}, \quad \frac{d}{dt}(h_t^3) = 0.\tag{4.2}$$

We deduce from the second equation in (4.2) and (3.3) that $h_t = 1$. We therefore see that the first equation in (4.2) becomes

$$\frac{d}{dt}f_t = 4f_t^3.\tag{4.3}$$

From (4.3) and (3.3) we conclude that

$$f_t = \varepsilon(1 - 8\varepsilon^2 t)^{-1/2} \quad \text{and} \quad h_t = 1.\tag{4.4}$$

We conclude from (3.2), (3.4), and (3.6) that we have a solution φ_t to the Laplacian flow starting at φ_0 in (3.1) which induces the following data:

$$\varphi_t = \varepsilon(1 - 8\varepsilon^2 t)^{-1/2}\eta_0 \wedge \omega_0 + \operatorname{Re} \Upsilon_0,\tag{4.5}$$

$$\psi_t = \frac{1}{2}\omega_0^2 - \varepsilon(1 - 8\varepsilon^2 t)^{-1/2}\eta_0 \wedge \operatorname{Im} \Upsilon_0,\tag{4.6}$$

$$g_t = \varepsilon^2(1 - 8\varepsilon^2 t)^{-1}\eta_0^2 + g_{\mathcal{D}_0},\tag{4.7}$$

$$\operatorname{vol}_t = \varepsilon(1 - 8\varepsilon^2 t)^{-1/2}\eta_0 \wedge \operatorname{vol}_{\mathcal{D}_0},\tag{4.8}$$

for $t \in (-\infty, \frac{1}{8\varepsilon^2})$. □

Remark 4.1. Our solution to the Laplacian flow is *ancient* (i.e. exists for all negative times), but not eternal, forming a finite-time singularity at $t = \frac{1}{8\varepsilon^2} > 0$. Again, we cannot guarantee that this is the unique solution to the Laplacian flow starting at (3.1), even if we assume M is compact, because we do not have general analytic theory for the Laplacian flow when not restricted to closed G_2 -structures.

Remark 4.2. Even though the Laplacian flow no longer has the interpretation as the gradient flow of the volume functional in this setting, we observe from (4.8) that the volume form of M in the metric g_t is still pointwise strictly increasing in time, just as for the Laplacian coflow. Moreover, if M is compact, we conclude that the volume is strictly increasing in time, but now tending to infinity in *finite* time:

$$\operatorname{Vol}(M, g_t) \rightarrow \infty \quad \text{as } t \rightarrow \frac{1}{8\varepsilon^2}.$$

We also notice from (4.7) that the cohomology class of ψ_t satisfies

$$[\psi_t] = (1 - 8\varepsilon^2 t)^{-1/2} [\psi_0]$$

and so it is not constant, unless ψ_0 is exact. Moreover, the cohomology class $[\psi_t]$ degenerates precisely at the singular time $t = \frac{1}{8\varepsilon^2}$, whenever $[\psi_0] \neq 0$.

4.2 Metric and curvature

Here, we see from (4.7) that the transverse geometry stays constant along the Laplacian flow, and the direction dual to η_0 in M expands as t increases to $\frac{1}{8\varepsilon^2}$ and collapses as $t \rightarrow -\infty$.

Example 4.1. Let M^7 be a compact quasi-regular cCY 7-manifold, so that it is a circle bundle over a compact Calabi–Yau 3-orbifold Z with metric g_Z . Then, for our Laplacian flow solution, we see from (4.7) that the induced metric on Z is fixed along the flow. However, the circle fibres expand to infinite size (and so become copies of \mathbb{R}) as $t \rightarrow \frac{1}{8\varepsilon^2}$, and that M collapses to Z as $t \rightarrow -\infty$.

As in the Laplacian coflow case, we can account for the volume increasing along the flow by normalising the volume of the evolving metrics to be the same as g_0 , yielding

$$(1 - 8\varepsilon^2 t)^{1/7} g_t = \varepsilon^2 (1 - 8\varepsilon^2 t)^{-6/7} \eta_0^2 + (1 - 8\varepsilon^2 t)^{1/7} g_Z.$$

This immediately gives the following result.

Lemma 4.1. *Let M be compact. After normalising the volume of (M, g_t) to be fixed, the Laplacian flow solution collapses to \mathbb{R} as $t \rightarrow \frac{1}{8\varepsilon^2}$ and collapses to \mathbb{C}^3 with the flat metric as $t \rightarrow -\infty$.*

We can also normalise the diameter and immediately obtain the following from (4.7).

Lemma 4.2. *Let M be compact. Then, after normalising (M, g_t) to unit diameter, the Laplacian flow is volume-collapsing to a 6-dimensional space M_- as $t \rightarrow -\infty$ and to a 1-dimensional space M_+ as $t \rightarrow \frac{1}{8\varepsilon^2}$. If the Sasakian structure is quasi-regular, so that M is an S^1 -orbibundle over a Calabi–Yau 3-orbifold Z , then $M_- = Z$ and $M_+ = S^1$.*

We see that there is a finite-time singularity for our flow at $t = \frac{1}{8\varepsilon^2}$. As we know from the study of the Laplacian flow for closed G_2 -structures, the formation of singularities should be related to the lack of uniform continuity of the evolving metrics and the blow-up of curvature, so we examine these properties for our solution.

Proposition 4.3. *Let φ_t be the solution to the Laplacian flow given by (4.5) with associated metric g_t as in (4.7). Then g_t is uniformly continuous (in t) on $(-\infty, T]$, for any $T < \frac{1}{8\varepsilon^2}$, but it is not uniformly continuous on $[T, \frac{1}{8\varepsilon^2})$, for any T .*

Proof. This is immediate from the formula (4.7) for the evolving metric g_t , which shows that its uniform continuity on an interval I is equivalent to the uniform continuity of $(1 - 8\varepsilon^2 t)^{-1}$ on I . \square

Moreover, one can immediately express the norm of curvature along the flow, from the proof of Proposition 3.5 and (4.4), as follows.

Proposition 4.4. *Let φ_t be the solution to the Laplacian flow given by (4.5) with associated metric g_t as in (4.7). Let Rm_t denote the Riemann curvature tensor of g_t and let $Rm_0^{\mathcal{D}_0}$ denote the curvature of the transverse connection on \mathcal{D}_0 induced by the Levi-Civita connection of g_0 . Then*

$$|Rm_t|_{g_t}^2 = |Rm_0^{\mathcal{D}_0}|_{g_0}^2 + c_0 \varepsilon^4 (1 - 8\varepsilon^2 t)^{-2}$$

for some constant $c_0 > 0$.

Remark 4.3. We observe, by Proposition 4.4, that $|Rm_t|_{g_t}$ blows up as $t \rightarrow \frac{1}{8\varepsilon^2}$ (the finite-time singularity), but that $(1 - 8\varepsilon^2 t)|Rm_t|_{g_t}$ remains bounded as $t \rightarrow \frac{1}{8\varepsilon^2}$. We also notice that $|Rm_t|_{g_t}$ remains bounded as $t \rightarrow -\infty$, but that $-t|Rm_t|_{g_t}$ always blows up at $t \rightarrow -\infty$, unless the transverse Calabi–Yau geometry is flat.

4.3 Full torsion tensor

We now study the behaviour of the full torsion tensor along our solution of the Laplacian flow.

Proposition 4.5. *The full torsion tensor T_t of the solution to the Laplacian flow in (4.5)–(4.8) is*

$$\begin{aligned} T_t &= \frac{1}{4}(\tau_0)_t g_t - \frac{1}{4}j_{\varphi_t}((\tau_3)_t) \\ &= \frac{3}{14}\varepsilon(1 - 8\varepsilon^2 t)^{-1/2} g_t \\ &\quad - \frac{1}{14}j_{\varphi_t}(4\varepsilon^2(1 - 8\varepsilon^2 t)^{-1}\eta_0 \wedge \omega_0 - 3\varepsilon(1 - 8\varepsilon^2 t)^{-1/2} \operatorname{Re} \Upsilon_0) \\ &= -\frac{3}{2}\varepsilon^3(1 - 8\varepsilon^2 t)^{-3/2}\eta_0^2 + \frac{1}{2}\varepsilon(1 - 8\varepsilon^2 t)^{-1/2}g_{\mathcal{D}_0}. \end{aligned} \tag{4.9}$$

Proof. The first two descriptions of T_t in (4.9) follow from (2.9), Lemma 3.1 and (4.4).

We then see from (3.26)–(3.28) that

$$j_{\varphi_t}(\eta_0 \wedge \omega_0) = 6\varepsilon(1 - 8\varepsilon^2 t)^{-1/2}\eta_0^2 + \frac{2}{\varepsilon}(1 - 8\varepsilon^2 t)^{1/2}g_{\mathcal{D}_0}, \tag{4.10}$$

and applying (3.29)–(3.31) yields

$$j_{\varphi_t}(\operatorname{Re} \Upsilon_0) = 4g_{\mathcal{D}_0}. \tag{4.11}$$

Combining (4.10) and (4.11) shows that

$$\begin{aligned} j_{\varphi_t}(4\varepsilon^2(1 - 8\varepsilon^2 t)^{-1}\eta_0 \wedge \omega_0 - 3\varepsilon(1 - 8\varepsilon^2 t)^{-1/2} \operatorname{Re} \Upsilon_0) \\ = 24\varepsilon^3(1 - 8\varepsilon^2 t)^{-3/2}\eta_0^2 - 4\varepsilon(1 - 8\varepsilon^2 t)^{-1/2}g_{\mathcal{D}_0}. \end{aligned} \tag{4.12}$$

Substituting (4.7) and (4.12) into (4.9) gives the result. \square

Our next result describes the norm of the full torsion tensor and its gradient along the flow.

Proposition 4.6. *Let T_t be the full torsion tensor of the solution to the Laplacian flow in (4.5)–(4.8). Then*

$$\begin{aligned} |T_t|_{g_t}^2 &= \frac{15}{4}\varepsilon^2(1 - 8\varepsilon^2 t)^{-1}, \\ |\nabla_t T_t|_{g_t}^2 &= c_0\varepsilon^4(1 - 8\varepsilon^2 t)^{-2}, \\ \operatorname{div}_t T_t &= 0, \end{aligned}$$

where $c_0 > 0$ is a constant, ∇_t is the Levi-Civita connection of g_t and div_t is the divergence with respect to the metric g_t .

Proof. The proof is almost identical to that of Proposition 3.7, so we only prove the formula for $|T_t|_{g_t}^2$. We observe from the formula (4.7) for the metric g_t that the transverse metric is not changing along the flow, but that

$$|\varepsilon^2\eta_0^2|_{g_t}^2 = (1 - 8\varepsilon^2 t)^2.$$

Hence, we see from Proposition 4.5 that

$$|T_t|_{g_t}^2 = \frac{9}{4}\varepsilon^2(1 - 8\varepsilon^2 t)^{-1} + \frac{3}{2}\varepsilon^2(1 - 8\varepsilon^2 t)^{-1} = \frac{15}{4}\varepsilon^2(1 - 8\varepsilon^2 t)^{-1}$$

as claimed. The equation for $|\nabla_t T_t|_{g_t}^2$ and the vanishing of $\operatorname{div}_t T_t$ follow exactly as in the proof of Proposition 3.7. \square

Remark 4.4. We again see that $|T_t|_{g_t}^2$ and $|\nabla_t T_t|_{g_t}^2$ blow up at the same rate as we approach the finite time singularity at $t = \frac{1}{8\varepsilon^2}$, as we would expect, and again that there is a component of $|Rm_t|_{g_t}$ that instead stays bounded as $t \rightarrow \frac{1}{8\varepsilon^2}$. Moreover, multiplying each of those quantities by $(1 - 8\varepsilon^2 t)$ leads to functions that are bounded as $t \rightarrow \frac{1}{8\varepsilon^2}$, which will be significant later.

4.4 Singularity analysis

As for the Laplacian coflow, we can now put our estimates together to obtain a description of the finite time singularity in the language of Definition 1.1, which proves Theorem 1.3(b).

Proposition 4.7. *In a cCY setup (S), suppose moreover M^7 is compact, and let $K := \sum_M |Rm_0^{\mathcal{D}_0}|_{g_0}$. Then there is a constant $c_0 > 0$, independent of ε , such that the quantity $\Lambda(t)$ in (1.3), along the Laplacian flow solution given by Theorem 1.2, is given by*

$$\Lambda(t) = (K^2 + c_0 \varepsilon^4 (1 - 8\varepsilon^2 t)^{-2})^{1/2}.$$

Hence, the Laplacian flow has a Type I finite-time singularity at $t = \frac{1}{8\varepsilon^2}$.

Proof. The formula for $\Lambda(t)$ follows from Propositions 4.4 and 4.5. We then see that $(\frac{1}{8\varepsilon^2} - t)\Lambda(t)$ remains bounded as $t \rightarrow \frac{1}{8\varepsilon^2}$, and thus the result follows from Definition 1.1. \square

5 The Hitchin flow solution

In this section we solve the Hitchin flow (1.4) for the natural coclosed G_2 -structures on a contact Calabi–Yau 7-manifold and show a relationship to Calabi–Yau structures in (real) dimension 8.

5.1 Solving the flow

As the initial condition for the Hitchin flow, we again choose the natural coclosed G_2 -structure φ_0 as in (3.1) on a contact Calabi–Yau 7-manifold.

Proposition 5.1. *On a cCY setup (S), the following family of coclosed G_2 -structures φ_t with associated 4-form ψ_t , metric g_t , and volume vol_t , satisfies the Hitchin flow (1.4), with initial condition (3.1):*

$$\varphi_t = \varepsilon(1 + \frac{5}{2}\varepsilon t)^{-1/5} \eta_0 \wedge \omega_0 + (1 + \frac{5}{2}\varepsilon t)^{3/5} \text{Re } \Upsilon_0; \quad (5.1)$$

$$\psi_t = \frac{1}{2}(1 + \frac{5}{2}\varepsilon t)^{4/5} \omega_0^2 - \varepsilon \eta_0 \wedge \text{Im } \Upsilon_0; \quad (5.2)$$

$$g_t = \varepsilon^2(1 + \frac{5}{2}\varepsilon t)^{-6/5} \eta_0^2 + (1 + \frac{5}{2}\varepsilon t)^{2/5} g_{\mathcal{D}_0}; \quad (5.3)$$

$$\text{vol}_t = \varepsilon(1 + \frac{5}{2}\varepsilon t)^{3/5} \eta_0 \wedge \text{vol}_{\mathcal{D}_0}, \quad (5.4)$$

defined for all $t \in (-\frac{2}{5\varepsilon}, \infty)$.

Proof. We consider the ansatz for our G_2 -structures φ_t solving the Hitchin flow as in (3.2), depending on functions f_t, h_t of time only, with dual 4-forms ψ_t as in (3.6), with the constraints on f_0, h_0 as in (3.3).

Under this assumption, we can compute

$$d\varphi_t = f_t h_t^2 \omega_0^2 \quad \text{and} \quad d\psi_t = 0,$$

so that the Hitchin flow (1.4) is then equivalent to solving

$$\frac{d}{dt}(h_t^4) = 2f_t h_t^2 \quad \text{and} \quad \frac{d}{dt}(f_t h_t^3) = 0 \quad (5.5)$$

using (3.6). The second equation in (5.5), together with (3.3), yields

$$f_t = \varepsilon h_t^{-3}. \quad (5.6)$$

Substituting (5.6) into the first equation in (5.5) then gives

$$\frac{d}{dt}(h_t^4) = 2\varepsilon h_t^{-1}. \quad (5.7)$$

The ODE (5.7) can be solved using (3.3) and thus, using (5.6), we can conclude that the solution to (5.5) is

$$f_t = \varepsilon(1 + \frac{5}{2}\varepsilon t)^{-3/5} \quad \text{and} \quad h_t = (1 + \frac{5}{2}\varepsilon t)^{1/5}. \quad \square$$

5.2 Calabi–Yau structure

We can now show that the solution to the Hitchin flow in Proposition 5.1 leads to a Calabi–Yau structure on the spacetime track of the flow.

Lemma 5.2. *The following 2-form $\hat{\omega}$, metric \hat{g} and complex 4-form $\hat{\Upsilon}$ define a Calabi–Yau structure on $(-\frac{3}{5\varepsilon}, \infty) \times M^7$:*

$$\hat{\omega} = \varepsilon(1 + \frac{5}{2}\varepsilon t)^{-3/5} dt \wedge \eta_0 + (1 + \frac{5}{2}\varepsilon t)^{2/5} \omega_0; \quad (5.8)$$

$$\hat{g} = dt^2 + \varepsilon^2(1 + \frac{5}{2}\varepsilon t)^{-6/5} \eta_0^2 + (1 + \frac{5}{2}\varepsilon t)^{2/5} g_0; \quad (5.9)$$

$$\hat{\Upsilon} = ((1 + \frac{5}{2}\varepsilon t)^{3/5} dt + i\varepsilon \eta_0) \wedge \Upsilon_0. \quad (5.10)$$

The metric \hat{g} is incomplete and has holonomy contained in $SU(4)$.

Proof. It is straightforward to check that $\hat{\omega}$ in (5.8) and $\hat{\Upsilon}$ in (5.10) induce the metric \hat{g} in (5.9), they satisfy the algebraic conditions to define an $SU(4)$ structure and they are both closed, and thus define a Calabi–Yau structure as claimed. The incompleteness of the metric is clear. \square

Remark 5.1. We know that the metric induced by the Hitchin flow is given by $\hat{g} = dt^2 + g_t$, where g_t is given in (5.3), which agrees with the formula in (5.9).

We also know by the definition of the Hitchin flow that the induced torsion-free $\text{Spin}(7)$ -structure on $(-\frac{5}{2\varepsilon}, \infty)$ is given by the 4-form

$$\Phi = dt \wedge \varphi_t + \psi_t,$$

where φ_t, ψ_t are given in (5.1)–(5.2). It is straightforward to check from (5.8) and (5.10) that

$$\Phi = \frac{1}{2} \hat{\omega}^2 + \text{Re } \hat{\Upsilon}.$$

Thus the Hitchin flow solution in Proposition 5.1 does indeed induce the Calabi–Yau structure in Lemma 5.2, which then completes the proof of Theorem 1.4.

A Additional properties of coclosed G_2 -structures

We gather in this Appendix some basic formulae, as well as some original material on the Hodge Laplacian and the Lie derivative of coclosed G_2 -structures which emerged over the course of research leading to this paper. The latter computations, albeit not essential to our main matter, might nonetheless be useful to readers interested e.g. in soliton formation along flows of coclosed G_2 -structures.

A.1 Contraction identities

Contractions of φ with φ :

$$\varphi_{abc} \varphi^{abc} = 42, \quad (A.1)$$

$$\varphi_{abj} \varphi^{ab}_k = 6g_{jk}, \quad (A.2)$$

$$\varphi_{apq} \varphi^a_{jk} = g_{pj} g_{qk} - g_{pk} g_{qj} + \psi_{pqjk}. \quad (A.3)$$

Contractions of φ with ψ :

$$\begin{aligned} \varphi_{ijk} \psi^i_{jk} &= 0, \\ \varphi_{ijq} \psi^{ij}_{kl} &= 4\varphi_{qkl}, \end{aligned} \quad (A.4)$$

$$\begin{aligned} \varphi_{ipq} \psi^i_{jkl} &= g_{pj} \varphi_{qkl} - g_{jq} \psi_{pkl} + g_{pk} \psi_{jql} \\ &\quad + g_{kq} \psi_{jpl} + g_{pl} \psi_{jkq} - g_{lq} \psi_{jpk}. \end{aligned} \quad (A.5)$$

Contractions of ψ with ψ :

$$\psi_{abcd}\psi^{ab}_{mn} = 4g_{cm}g_{dn} - 4g_{cn}g_{dm} + 2\psi_{abmn}, \quad (\text{A.6})$$

$$\psi_{abcd}\psi^{bcd}_m = 24g_{am}, \quad (\text{A.7})$$

$$\psi_{abcd}\psi^{abcd} = 168$$

The full torsion tensor is a 2-tensor T satisfying

$$\begin{aligned} \nabla_i \varphi_{jkl} &= T_i^m \psi_{mjkl}, \\ T_i^j &= \frac{1}{24} \nabla_i \varphi_{lmn} \psi^{jlmn}, \end{aligned} \quad (\text{A.8})$$

and

$$\nabla_m \psi_{ijkl} = -(T_{mi} \varphi_{jkl} - T_{mj} \varphi_{ikl} - T_{mk} \varphi_{jil} - T_{ml} \varphi_{jki}). \quad (\text{A.9})$$

A.2 Hodge Laplacian

To study the Laplacian coflow, we naturally need to analyse the Hodge Laplacian Δ_ψ of the dual $\psi = *\varphi$ of a coclosed G_2 -structure φ . Then $d\varphi = \tau_0\psi + *\tau_3$ by (2.5)–(2.6), so

$$\begin{aligned} \Delta_\psi \psi &= dd^*\psi + d^*d\psi = d * d\varphi \\ &= d(\tau_0\varphi + \tau_3) = d\tau_0 \wedge \varphi + \tau_0^2 \psi + \tau_0 * \tau_3 + d\tau_3. \end{aligned} \quad (\text{A.10})$$

The following proposition, adapted from [Gri13, Proposition 2.3, Lemma 4.5, (4.30), (5.8)], provides useful properties of a coclosed G_2 -structure and its full torsion tensor T . In the statement, for $h, k \in S^2$ given locally by $h = h_{ij}dx^i dx^j$ and $k = k_{ij}dx^i dx^j$, we define the inner product $\langle h, k \rangle$ and the circ product $h \circ k \in S^2$ by

$$\langle h, k \rangle = h_{ij}k_{ab}g^{ia}g^{jb} \quad \text{and} \quad (h \circ k)_{ab} = \varphi_{amn}\varphi_{bpq}h^{mp}k^{nq}. \quad (\text{A.11})$$

We define the divergence $\text{div}h \in \Omega^1$ and curl $\text{Curl}h$ (which is a 2-tensor) of $h \in S^2$ by

$$(\text{div}h)_a = \nabla^b h_{ba} \quad \text{and} \quad (\text{Curl}h)_{ab} = (\nabla_m h_{an})\varphi_b^{mn}. \quad (\text{A.12})$$

We also let S^t denote the transpose of a 2-tensor S .

Proposition A.1. *Suppose we have a coclosed G_2 -structure φ on a manifold M and recall the notation in (2.3), (A.11) and (A.12).*

(a) *If $\mu = i_\varphi(h) \in \Omega_{27}^3$ with $h \in S_0^2$ then*

$$d\mu = -\frac{1}{2}(\text{div}h)^b \wedge \varphi + *i_\varphi(k)$$

where

$$k = \frac{1}{2}(\text{Curl}h + (\text{Curl}h)^t) + \frac{1}{2}T \circ h + \frac{1}{2}(Th + (Th)^t) - \frac{1}{2}(\text{tr}T)h - \frac{1}{6}\langle T, h \rangle g.$$

(b) *The full torsion tensor T satisfies the following identities:*

$$\begin{aligned} \text{div}T &= \nabla(\text{tr}T), & \text{Curl}T &= (\text{Curl}T)^t, \\ \text{Ric} &= \text{Curl}T - T^2 + (\text{tr}T)T, & R &= (\text{tr}T)^2 - |T|^2. \end{aligned} \quad (\text{A.13})$$

The following lemma, which decomposes the Hodge Laplacian of ψ , is originally proved in [Gri13, Proposition 4.6], although with a different orientation for the G_2 -structure. Since signs are important for applications, we work out the proof again, in our convention.

Lemma A.2. *Let φ be a coclosed G_2 -structure on a manifold M with associated metric g . Then,*

$$\begin{aligned} \Delta_\psi \psi &= \left(\frac{2}{3}R + \frac{4}{3}|T|^2\right)\psi \oplus (d \text{tr}T) \wedge \varphi \\ &\oplus *i_\varphi \left(-\text{Ric} + \frac{1}{14}(R - 2|T|^2)g + \text{tr}(T)T - 2T^2 - \frac{1}{2}T \circ T \right). \end{aligned}$$

To prove this lemma, we will require the following elementary facts relating $\tau_3 = i_\varphi(h)$ and T . By (2.4) and (2.9),

$$h = \frac{1}{7}(\text{tr } T)g - T \quad (\text{A.14})$$

and, applying (A.11) to T and g ,

$$T \circ g = (\text{tr } T)g - T. \quad (\text{A.15})$$

We also use the following elementary result.

Lemma A.3 ([Mor19, Lemma 25]). *Let φ be a coclosed G_2 -structure, let $\tau_3 = i_\varphi(h)$, using the map in (2.3), and recall (A.11)–(A.12).*

- (a) $\text{div } h = \frac{1}{7}\nabla(\text{tr } T) - \text{div } T = -\frac{6}{7}\nabla(\text{tr } T)$.
- (b) $\frac{1}{2}(\text{Curl } h + (\text{Curl } h)^t) = -\text{Curl } T$ and $\text{tr}(\text{Curl } T) = 0$.
- (c) $T \circ h = \frac{1}{7}((\text{tr } T)^2 g - (\text{tr } T)T) - T \circ T$ and $\text{tr}(T \circ T) = (\text{tr } T)^2 - |T|^2$.

Proof of Lemma A.2. We can decompose $\Delta_\psi \psi$ as:

$$\Delta_\psi \psi = Y^\flat \wedge \varphi \oplus *i_\varphi(s) = \frac{3}{7}(\text{tr } s)\psi \oplus Y^\flat \wedge \varphi \oplus *i_\varphi(\bar{s}),$$

where Y is a vector field and \bar{s} is the trace-free part of $s \in S^2$. Now, we apply Proposition A.1(a) to $\tau_3 = i_\varphi(h)$ and use Lemma A.3 to obtain

$$\begin{aligned} d\tau_3 &= \frac{3}{7}d(\text{tr } T) \wedge \varphi \\ &+ *i_\varphi\left(-\text{Curl } T + \frac{1}{2}T \circ h + \frac{1}{2}(Th + (Th)^t) - \frac{1}{2}(\text{tr } T)h - \frac{1}{6}\langle T, h \rangle g\right). \end{aligned} \quad (\text{A.16})$$

Using the identities $\tau_0 = \frac{4}{7}\text{tr}(T)$, $3\psi = *i_\varphi(g)$, and (A.16) in (A.10), we obtain

$$\begin{aligned} \Delta_\psi \psi &= d(\text{tr } T) \wedge \varphi \\ &+ *i_\varphi\left(-\text{Curl } T + \frac{1}{2}(T \circ h) + \frac{1}{2}(Th + (Th)^t) + \frac{1}{14}(\text{tr } T)h + \frac{16}{147}(\text{tr } T)^2 g - \frac{1}{6}\langle T, h \rangle g\right). \end{aligned}$$

Now, replacing h using (A.14), as well as using (A.15), Proposition A.1 and Lemma A.3, we have

$$\Delta_\psi \psi = d(\nabla \text{tr } T)^\flat \wedge \varphi + *i_\varphi\left(-\text{Curl } T - \frac{1}{2}T \circ T - T^2 + \frac{1}{6}(\text{tr } T)^2 g + \frac{1}{6}|T|^2 g\right).$$

Therefore

$$\begin{aligned} \text{tr}(s) &= \frac{2}{3}((\text{tr } T)^2 + |T|^2) = \frac{2}{3}R + \frac{4}{3}|T|^2, \\ \bar{s} &= -\text{Ric} + \frac{1}{14}(R - 2|T|^2)g + \text{tr}(T)T - 2T^2 - \frac{1}{2}T \circ T, \\ Y &= \nabla \text{tr } T. \end{aligned}$$

The result then follows. \square

A.3 Lie derivative

The Lie derivative of the Hodge dual of a coclosed G_2 -structure naturally appears when studying geometric flows of coclosed G_2 -structures in the context of “gauge-fixing” the action by diffeomorphisms (as in DeTurck’s trick) and in the study of solitons. An analogous expression for the Lie derivative of a closed G_2 -structure can be found in [LW17, Lemma 9.7] and proved useful for the study of solitons in that setting. In the statement of the formula we derive below, for a vector field X on M , we let $\text{div } X$ denote its divergence and let $\text{Curl } X$ be the vector field given locally by

$$(\text{Curl } X)^l = \varphi_{ijk}(\nabla_i X^j)g^{kl}.$$

Theorem A.1. Let φ be a coclosed G_2 -structure on M^7 , with associated metric g , and let X be a vector field on M . Then, if $\psi = *\varphi$,

$$\mathcal{L}_X\psi = \frac{4}{7}(\operatorname{div} X)\psi \oplus (\operatorname{Curl} X)^b \wedge \varphi \oplus *i_\varphi \left(\frac{3}{49}(\operatorname{div} X)g - \frac{3}{14}(\mathcal{L}_X g) \right) \in \Omega_1^4 \oplus \Omega_7^4 \oplus \Omega_{27}^4, \quad (\text{A.17})$$

where i_φ is given by (2.3). In particular, any infinitesimal symmetry X of ψ must be a Killing vector field of g with no curl, i.e.,

$$\mathcal{L}_X\psi = 0 \iff \mathcal{L}_X g = 0 \quad \text{and} \quad \operatorname{Curl}(X) = 0.$$

Proof. Since φ is coclosed, i.e. $d\psi = 0$, we have

$$\mathcal{L}_X\psi = d(X \lrcorner \psi) + X \lrcorner d\psi = d(X \lrcorner \psi).$$

Let $\alpha = X \lrcorner \psi$, so that locally $\alpha_{ijk} = X^l \psi_{lijk}$ and

$$(\mathcal{L}_X\psi)_{ijkl} = (d\alpha)_{ijkl} = \nabla_i \alpha_{jkl} - \nabla_j \alpha_{ikl} + \nabla_k \alpha_{jil} - \nabla_l \alpha_{jki}.$$

Denoting by $\pi_l^k : \Omega^k \rightarrow \Omega_l^k$ the orthogonal projections, we decompose $\mathcal{L}_X\psi$ as

$$\mathcal{L}_X\psi = \pi_1^4(\mathcal{L}_X\psi) + \pi_7^4(\mathcal{L}_X\psi) + \pi_{27}^4(\mathcal{L}_X\psi) = a\psi + W^b \wedge \varphi + *i_\varphi(h), \quad (\text{A.18})$$

where $a \in \Omega^0$, and h is a trace-free symmetric 2-tensor on M . We compute a as follows:

$$\begin{aligned} a &= \frac{1}{7} \langle \mathcal{L}_X\psi, \psi \rangle = \frac{1}{168} (\nabla_i \alpha_{jkl} - \nabla_j \alpha_{ikl} + \nabla_k \alpha_{jil} - \nabla_l \alpha_{jki}) \psi^{ijkl} \\ &= \frac{1}{42} \nabla_i \alpha_{jkl} \psi^{ijkl} = \frac{1}{42} \nabla_i (\alpha_{jkl} \psi^{ijkl}) - \frac{1}{42} \alpha_{jkl} \nabla_i \psi^{ijkl} \\ &= \frac{24}{42} \nabla_i (X^m g_{mi}) - \frac{1}{42} X^m \psi_{mjkl} (\nabla_i \psi^{ijkl}) = \frac{4}{7} \nabla_i X_i \\ &= \frac{4}{7} \operatorname{div} X, \end{aligned} \quad (\text{A.19})$$

where we used (A.4) and that T is symmetric. To compute W^b , note that

$$\langle *((\mathcal{L}_X\psi) \wedge \varphi), e^m \rangle = 4 \langle W^b, e^m \rangle,$$

thus

$$4W^m = *((\mathcal{L}_X\psi) \wedge \varphi \wedge e^m) = *(\langle \varphi \wedge e^m, \mathcal{L}_X\psi \rangle \operatorname{vol}) = \frac{1}{3!} (\mathcal{L}_X\psi)^{ijkm} \varphi_{ijk}.$$

Therefore, we obtain

$$\begin{aligned} W^m &= \frac{1}{4!} (\mathcal{L}_X\psi)^{ijkm} \varphi_{ijk} = \frac{1}{3!} (g^{si} \nabla_s \alpha^{jkm} \varphi_{ijk}) \\ &= \frac{1}{3!} (g^{si} \nabla_s (X^l \psi_l^{jkm} \varphi_{ijk}) - X^l \psi_l^{jkm} g^{si} \nabla_s \varphi_{ijk}) \\ &= \frac{1}{3!} (4g^{si} \nabla_s (X^l \varphi_{il}^m) - X^l \psi_l^{jkm} g^{si} T_{sa} g^{an} \psi_{nijk}) \\ &= \frac{4}{3!} g^{si} \nabla_s (X^l \varphi_{il}^m) = \frac{4}{3!} g^{si} \nabla_s X^l \varphi_{il}^m + \frac{4}{3!} X^l g^{si} \nabla_s \varphi_{ir}^m \\ &= \frac{4}{3!} \varphi_{ir}^m \nabla^i X^l + \frac{4}{3!} X^l g^{si} \nabla_s \varphi_{ir}^m \\ &= \frac{4}{3!} \varphi_{ila} (\nabla^i X^l) g^{am} + \frac{4}{3!} X^l g^{si} T_{sn} g^{nl} \psi_{lirb} g^{bm} \\ &= \frac{4}{3!} (\operatorname{Curl} X)^m. \end{aligned} \quad (\text{A.20})$$

Finally, to compute h , observe that

$$\begin{aligned} &(\mathcal{L}_X\psi)_{mnpj} \psi_j^{mnp} + (\mathcal{L}_X\psi)_{mnpj} \psi_i^{mnp} \\ &= a(\psi_{mnpj} \psi_j^{mnp} + \psi_{mnpj} \psi_i^{mnp}) + (*i_\varphi(h))_{mnpj} \psi_j^{mnp} + (*i_\varphi(h))_{mnpj} \psi_i^{mnp}, \end{aligned} \quad (\text{A.21})$$

where

$$(*i_\varphi(h))_{mnp} = h_m^q \psi_{qnp} - h_n^q \psi_{mqp} + h_p^q \psi_{mnq} - h_i^q \psi_{mnpq}.$$

Equation (A.21) becomes

$$\begin{aligned} & a\psi_{mnp} \psi_j^{mnp} + (h_m^q \psi_{qnp} - h_n^q \psi_{mqp} + h_p^q \psi_{mnq} - h_i^q \psi_{mnpq}) \psi_j^{mnp} \\ & + a\psi_{mnp} \psi_i^{mnp} + (h_m^q \psi_{qnp} - h_n^q \psi_{mqp} + h_p^q \psi_{mnq} - h_j^q \psi_{mnpq}) \psi_i^{mnp} \\ & = -48ag_{ij} + h_m^q (4g_{qj}g_{im} - 4g_{qm}g_{ij} + 4g_{qi}g_{jm} - 4g_{qm}g_{ij}) \\ & \quad - h_n^q (4g_{qj}g_{in} - 4g_{qn}g_{ij} + 4g_{qi}g_{jn} - 4g_{qn}g_{ij}) + 24h_i^q g_{qj} \\ & \quad + h_p^q (4g_{qj}g_{ip} - 4g_{qp}g_{ji} + 4g_{qi}g_{jp} - 4g_{qp}g_{ij}) + 24h_j^q g_{qi} \\ & = -48ag_{ij} + 56h_{ij}. \end{aligned} \tag{A.22}$$

We can calculate the left-hand side of (A.22) as follows:

$$\begin{aligned} & (\mathcal{L}_X \psi)_{mnp} \psi_j^{mnp} + (\mathcal{L}_X \psi)_{mnp} \psi_i^{mnp} \\ & = (\nabla_m \alpha_{npi} - \nabla_n \alpha_{mpi} + \nabla_p \alpha_{nmi} - \nabla_i \alpha_{npm}) \psi_j^{mnp} \\ & \quad + (\nabla_m \alpha_{npj} - \nabla_n \alpha_{mpj} + \nabla_p \alpha_{nmj} - \nabla_j \alpha_{npm}) \psi_i^{mnp} \\ & = 3(\nabla_m \alpha_{npj} \psi_i^{mnp} + \nabla_m \alpha_{npi} \psi_j^{mnp}) - \nabla_i \alpha_{npm} \psi_j^{mnp} - \nabla_j \alpha_{npm} \psi_i^{mnp} \\ & = \frac{3\nabla_m (\alpha_{npi} \psi_j^{mnp})}{I(i)} - \frac{3\alpha_{npi} \nabla_m \psi_j^{mnp}}{II(i)} - \frac{\nabla_i (\alpha_{mnp} \psi_j^{mnp})}{III(i)} + \frac{\alpha_{mnp} \nabla_i \psi_j^{mnp}}{IV(i)} \\ & \quad + \frac{3\nabla_m (\alpha_{npj} \psi_i^{mnp})}{I(j)} - \frac{3\alpha_{npj} \nabla_m \psi_i^{mnp}}{II(j)} - \frac{\nabla_j (\alpha_{mnp} \psi_i^{mnp})}{III(j)} + \frac{\alpha_{mnp} \nabla_j \psi_i^{mnp}}{IV(j)}. \end{aligned} \tag{A.23}$$

For a coclosed G_2 -structure, T is symmetric, so we have the following expression in terms of the ‘matrix product’ of 2-tensors:

$$X^l T_i^m \varphi_{ljm} = (X \lrcorner \varphi)_{jm} T_i^m = (T \cdot (X \lrcorner \varphi))_{ji} = -((X \lrcorner \varphi) \cdot T)_{ij}. \tag{A.24}$$

Using moreover (A.6), (A.7), (A.9), we obtain

$$\begin{aligned} I(i) & = \nabla_m (\alpha_{npi} \psi_j^{mnp}) = \nabla_m (X^l \psi_{lnpi} \psi_j^{mnp}) \\ & = \nabla_m (X^l (4g_{lj}g_i^m - 4g_l^m g_{ij} + 2\psi_{lij}^m)) \\ & = 4\nabla_i X_j - 4(\operatorname{div} X)g_{ij} + 2\nabla_m (X^l \psi_{lij}^m), \\ I(j) + I(i) & = 4\nabla_j X_i - 4(\operatorname{div} X)g_{ij} + \nabla_m (X^l \psi_{lij}^m) + 4\nabla_i X_j - 4(\operatorname{div} X)g_{ji} + \nabla_m (X^l \psi_{lij}^m) \\ & = 4(\nabla_i X_j + \nabla_j X_i) - 8(\operatorname{div} X)g_{ij}. \end{aligned}$$

Furthermore,

$$\begin{aligned} II(i) + II(j) & = \alpha_{npi} \nabla_m \psi_j^{mnp} + \alpha_{npj} \nabla_m \psi_i^{mnp} = X^l \psi_{lnpi} \nabla_m \psi_j^{mnp} + X^l \psi_{lnpj} \nabla_m \psi_i^{mnp} \\ & = X^l \psi_{lnpi} (-T_{mj} \varphi^{mnp} + T_m^m \varphi_j^{np} - T_m^n \varphi_j^{mp} + T_m^p \varphi_j^{mn}) \\ & \quad + X^l \psi_{lnpj} (-T_{mi} \varphi^{mnp} + T_m^m \varphi_i^{np} - T_m^n \varphi_i^{mp} + T_m^p \varphi_i^{mn}) \\ & = 4(X \lrcorner \varphi \cdot T)_{ij} - 4(T \cdot X \lrcorner \varphi)_{ij} = 4[X \lrcorner \varphi, T]_{ij}, \\ III(i) + III(j) & = \nabla_i (X^l \psi_{lmnp} \psi_j^{mnp}) + \nabla_j (X^l \psi_{lmnp} \psi_i^{mnp}) \\ & = 24(\nabla_i X_j + \nabla_j X_i), \\ IV(i) & = \alpha_{mnp} \nabla_i \psi_j^{mnp} = X^l \psi_{lmnp} \nabla_i \psi_j^{mnp} \\ & = X^l \psi_{lmnp} (-T_{ij} \varphi^{mnp} + T_i^m \varphi_j^{np} - T_i^n \varphi_j^{mp} + T_i^p \varphi_j^{mn}) = 12(X \lrcorner \varphi \cdot T)_{ij}, \\ IV(i) + IV(j) & = 12(X \lrcorner \varphi \cdot T)_{ij} + 12(X \lrcorner \varphi \cdot T)_{ji} = 12[X \lrcorner \varphi, T]_{ij}. \end{aligned}$$

So, using (A.19), (A.22) and the above expressions, we obtain

$$-48ag_{ij} + 56h_{ij} = -12(\nabla_i X_j + \nabla_j X_i) - 24(\operatorname{div} X)g_{ij}$$

which, upon re-arranging, gives

$$\begin{aligned} h_{ij} &= \frac{3}{49} \operatorname{div}(X) g_{ij} - \frac{3}{14} (\nabla_i X_j + \nabla_j X_i) \\ &= \frac{3}{49} (\operatorname{div} X) g_{ij} - \frac{3}{14} (\mathcal{L}_X g)_{ij}. \end{aligned} \tag{A.25}$$

Hence, substituting (A.19), (A.20) and (A.25) into (A.18) we obtain (A.17). \square

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