Perverse coherent sheaves and algebraic K-theory of Kac-Moody flag varieties

Tom Zielinski
Pembroke College
University of Oxford

DPhil

Hilary 2022

No one will ask me to leave, there is nothing
very intimidating about me here, not now
that they‘ve taken the windows away
and let the outside go everywhere.
Seven AM in April- Sally Rooney

Acknowledgements

Firstly, I would like to thank my advisor Kevin McGerty for much helpful and generous advice. Many ideas in this thesis arose out of conversations with Ian Grojnowski, who I would like to thank for his continuing support in my development as a mathematician. I would also like to thank Constantin Teleman for introducing me to the world of loop groups and for inviting me to Berkeley for a semester.

During my time as a DPhil, I have received countless support. In the Oxford maths community, I would especially like to thank Christoph Weis, for innumerable conversations about maths and everything else, Richard Mathers, for making me feel welcome and valued at the Mathematical Institute, Kieran Calvert, for being the best officemate a first year DPhil could wish for, André Henriques and Marcelo DeMartino for always making time to chat and teaching me so much about root systems, and Jacob Gross for reminding me not to take things too seriously.

Outside of Oxford, I would like to thank Kiran Luecke for all the help when I was in Berkeley, Eric Ahlqvist for some wonderful lunches at Yali's, Leo Lai for inviting me to Princeton, and Karl Winsor for helping develop my understanding of Lie algebras and for hosting me (a few times!) in Boston.

Outside of mathematics, I'd like to thank my friends and family: Gab for more than ten years of friendship, Ben A. for always listening to my rants, my housemates in Oxford throughout the DPhil: Constantin, Dominik, Toby, Scott, Brennan and Neil for providing a great place to live in and my former housemates in Berkeley: Clarissa, Dan, Dave, Kristina, Özge, Mattie, Julia & Zhen, for making me feel at home halfway around the world. I would also like thank my parents and my sister Estelle for supporting me from the start with regular visits and phone calls.

Last but not least, this thesis would not have been possible without the support of my partner Claire, who has always believed in me even when I didn't.

Abstract

This thesis studies derived categories of coherent sheaves on affine Kac-Moody flag varieties and their associated Grothendieck groups. In chapter 2, we equip such categories with a rigid monoidal structure, and compute explicitly the ring obtained for the affine flag variety. In chapter 3, we study P. Achar's theory of staggered sheaves in the context of affine flag varieties. In particular, we construct non-standard t-structures on Schubert varieties on both the thin and thick affine flag variety. In chapter 4, we study a conjecture of Cautis and Williams describing the equivariant K-theory of the affine Grassmannian. More specifically, we prove a special case of their conjecture in rank 1 by an explicit computation, and provide a strategy for a proof in type A. We also provide further evidence for the conjecture to hold in type ADE by relating simple perverse coherent sheaves under convolution with fusion products of Demazure modules.

Contents

Chapter 0

Introduction

This thesis focuses on the study of sheaves on certain infinite-dimensional spaces called affine Kac-Moody flag varieties. Affine Kac-Moody Lie algebras, on which the corresponding groups and flag varieties are based, arise naturally from physics (2-dimensional conformal field theory, string theory) and have found many applications in pure mathematics such as combinatorics (Macdonald identities) and finite group theory (the Monstrous Moonshine conjecture). Enhancing the study from Lie algebras and representations to sheaves on flag varieties allows us to construct new interesting algebras and provide additional insights into old results.

0.1 Geometric Representation Theory in infinite dimensions

The field of geometric representation theory is concerned with constructing algebras and representations out of geometric data on spaces. A simple example of this procedure is to fix an algebraic group G, and to consider the category of G-equivariant coherent sheaves on a point $\mathrm{Coh}^G(pt)$. We can see that this category is equivalent to the category of finite-dimensional representations of the group G. More generally, one usually studies categories of sheaves on spaces equipped with an algebraic group action. Depending on the context, these sheaves might be constructible, coherent, perverse, Mixed Hodge modules etc.

One fundamental insight of the theory is that certain properties of algebras and their representations arise out of properties of the corresponding geometric categories considered. For instance, if when applying this procedure we obtain a monoidal abelian category $\mathcal{C}$, its Grothendieck group $K_0(\mathcal{C})$ will naturally possess an algebra structure. Furthermore, one might hope that simple objects in $\mathcal{C}$ will provide an interesting basis of the algebra $K_0(\mathcal{C})$.

An early application of this philosophy appears in the work of Kazhdan and Lusztig in the 1980s. In [KL87], [Lus85], the authors construct the affine Hecke algebra and some of its representations by considering $G \times \mathbb{C}^\times$-equivariant coherent sheaves on the Steinberg variety. The monoidal structure arises from convolution, of which the following is a typical example.

Let G be a complex reductive group with a choice of a Borel subgroup B and a torus T and associate to this data the flag variety G/B and Weyl group W. We can equip the (derived) category $\mathrm{Coh}^B(G/B)$ with a monoidal structure by considering the following diagram:

$$\begin{tikzcd}
G \times G/B \arrow[r, "q"] \arrow[d, "p"] & G \times_B G/B \arrow[d, "m"] \\
G/B \times G/B & G/B.
\end{tikzcd}$$

Here $G \times_B G/B$ is the quotient of $G \times G/B$ by the diagonal B actions $b(g, hB) = (gb^{-1}, bhB)$, m is the multiplication map and p, q are the natural quotient maps.

Taking two sheaves $\mathcal{F}$,$\mathcal{G}$, their convolution product is then defined as

$$\mathcal{F} \star \mathcal{G} = m_*(q^*)^{-1}p^*(\mathcal{F} \boxtimes \mathcal{G}).$$

Note that equivariance here is essential, as q^* induces an equivalence

$$\mathrm{Coh}^{B\times B}(G \times G/B) \cong \mathrm{Coh}^{B}(G \times_B G/B).$$

Therefore we obtain an algebra structure on the Grothendieck group $K^B(G/B)$. More insight can be gained into this algebra by fixing a basis. A natural choice is to take the classes of structure sheaves Schubert varieties $X_w = \overline{BwB/B}$ for $w \in W$, which we will denote by $\mathcal{O}_w$. We will show in Chapter 2 that if $l(ws) > l(w)$ for $w \in W$ and s a simple reflection, then $[\mathcal{O}_w] \star [\mathcal{O}_s] = [\mathcal{O}_{ws}]$ and else $[\mathcal{O}_w] \star [\mathcal{O}_s] = [\mathcal{O}_w]$, which combined with the natural $R(T)$-module structure describes the algebra $K^B(G/B)$ completely (we refer to [CG10] for more examples of this kind).

The first two chapters of this thesis study generalisations of the previous example to the infinite-dimensional setting. More precisely, we will replace the complex reductive group G by the loop group $G_{\mathcal{K}} = G(\mathbb{C}((t)))$ and the Borel subgroup by parahoric subgroups $\mathcal{P}$ of $G_{\mathcal{K}}$. We then obtain the *affine Kac-Moody flag varieties* $\mathfrak{X}^G(\mathcal{P}) := G_{\mathcal{K}}/\mathcal{P}$. These spaces are not schemes in general and are acted on by pro-linear groups, therefore some care will be needed to set-up categories of coherent sheaves. Following [Kum02] and [Zhu17], in Chapter 1, we equip affine Kac-Moody flag varieties with an ind-scheme structure and prove some key geometric properties. In Chapter 2, we construct the $\mathcal{P}$-equivariant derived category of coherent sheaves on $\mathfrak{X}^G(\mathcal{P})$ and equip it with a rigid monoidal structure arising from convolution. Furthermore, in the special case of the affine flag variety Fl_G obtained taking $\mathcal{P}$ to be the Iwahori subgroup, we obtain the following result:

Theorem 0.1.0.1 (2.2.0.1 in the text)**.** *Let G be a simply-connected almost simple complex algebraic group, and let $[\mathcal{O}_w]$ be the class of the structure sheaf of the Schubert variety X_w for $w \in W^{aff}$, viewed as an element of $K^I(\mathrm{Fl}_G)$. Then the set $\{[\mathcal{O}_w] \mid w \in W^{aff}\}$ forms a generating set of $K^I(\mathrm{Fl}_G)$ as an $R(T)$-module. Furthermore, if s is a simple reflection in W^{aff}, we have*

$$[O_w] \star_I [O_s] = \begin{cases} [O_{ws}] \text{ if } l(ws) > l(w) \\ \\ [O_w] \text{ else.} \end{cases}$$

We also obtain an explicit description of duals in terms of the classes of affine Schubert varieties in $K^I(\mathrm{Fl}_G)$.

Perverse (constructible) sheaves have become ubiquitous in modern geometric representation theory, as the spaces one is interested in are usually very singular. In the finite-dimensional case, they have been central in proving the Kazhdan-Lusztig conjectures, and in the infinite-dimensional set-up play a fundamental role in the geometric Satake equivalence of Lusztig-Ginzburg-Mirkovic-Vilonen (see [BR18] for details). In analogy with the constructible case, Bezrukavnikov (after Deligne) defined in [AB10] a category of perverse coherent sheaves on certain schemes with group actions. This construction was later generalized by P. Achar in [Ach09], to the theory of staggered sheaves which depends on a stratifcation of the category called an *s-structure.* In Chapter 3, we present an overview of the theory of perverse coherent and staggered sheaves. Taking an affine Schubert variety $\overline{X_w}$, our main result is the following.

Theorem 0.1.0.2 (3.2.1.1 in the text)**.** *With respect to a suitable s-structure and dualizing complex, $D^b(\mathrm{Coh}^I(\overline{X_w}))$ admits an Artinian staggered t-structure. The simple objects in the heart are the staggered IC-sheaves $\mathcal{IC}(v, \chi)$ for χ a character of T and*

$v \leq w$.

We also apply similar techniques in the case of the thick flag variety defined by Kashiwara in [Kas89]. Taking a suitable open cover Ω_S, we construct a staggered t-structure on $D^b(\mathrm{Coh}^I(\Omega_S))$.

Theorem 0.1.0.3 (3.2.3.1 in the text)**.** *With respect to a suitable s-structure and dualizing complex, $D^b(\mathrm{Coh}^I(\Omega_S))$ admits an Artinian staggered t-structure. The simple objects in the heart are the staggered IC-sheaves $\mathcal{IC}(v, \chi)$ for χ a character of T and $v \in S$.*

0.2 Coulomb branches and the Coherent Satake Category

In chapter 4, we turn our focus to a special affine Kac-Moody flag variety, the affine Grassmannian Gr_G obtained by taking $\mathcal{P} = G_{\mathcal{O}} := G(\mathbb{C}[[t]])$ and to its equivariant K-theory $K^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$. Whilst a description of its spectrum as a geometric space has been known since the appearance of [BFM05], it has been of renewed interest as a special intance of a Coulomb branch for 3-dimensional $\mathcal{N} = 4$ SUSY Gauge theory, of which a rigorous construction was provided in [BFN18] and subsequent work. Motivated by this, Cautis and Williams in [CW19] have recently equipped the algebra with a cluster structure in the case of $G = \mathrm{GL}_n$ using the Beilinson-Drinfeld Grassmannian. More precisely, they show that the category of perverse coherent sheaves $\mathcal{P}_{coh}^{\mathrm{GL}_n(\mathbb{C}[[t]])}(\mathrm{Gr}_{\mathrm{GL}_n})$ is a cluster categorification of a specified cluster algebra. Furthermore, they conjecture that such a categorification should exist in all types.

In semisimple rank 1, we use the results of [BFM05] to prove the Cautis-Williams conjecture:

Proposition 0.2.0.1 (4.2.2.1 in the text)**.** *There is an algebra isomorphism*

$$K^{\mathrm{SL}_2}(\mathrm{Gr}_{\mathrm{PGL}_2}) \cong \mathcal{A}(2,2),$$

where the perverse sheaves $\mathcal{P}_{1,0}, \mathcal{P}_{1,1}$ *are mapped to the initial cluster. Furthermore, under the isomorphism, all cluster variables are classes of simple perverse sheaves supported on the orbit* Gr_1*.*

We also explain work in progress to prove the conjecture in type A using the results of [CW19].

In the final part of the chapter, we provide further evidence of the Cautis-Williams conjecture. In particular, we show that in the simply-laced case the global sections of certain perverse coherent sheaves are Demazure modules, and show that for these, global section of convolution goes to fusion. We also provide some open questions and directions for further work.

Chapter 1

The geometry of affine Kac-Moody flag varieties

Introduction

The goal of this chapter is to provide a self-contained exposition to the theory of affine Kac-Moody flag varieties, in particular avoiding as much as possible the use of affine buidlings. Affine Kac-Moody flag varieties are ind-schemes which can be understood as certain homogeneous spaces for infinite-dimensional groups. For the topology-inclined reader, they closely resemble based loop groups in topology. In the first section, we will work in the special case of GL_n and then proceed to the general case using an embedding argument. Finally, we construct stratifications and realise them as singular projective varieties. As this will be used later on in the text, we conclude the chapter by explaining a different approach to affine Kac-Moody flag varieties pionneered by Kashiwara ([Kas89]). Although none of the results presented in this chapter are new, the use of the Bialynicki-Birula stratification to construct the Cartan decomposition does not seem to appear written down in the litterature and some of the proofs are new. Our approach will at times be close to [Kum02], [Sor00] and [Zhu17].

1.1 Affine Grassmannians over non-constant group schemes

1.1.1 The construction of the affine Grassmannian over GL_n

Definition 1.1.1.1. Let R be a commutative $\mathbb{C}$-algebra, and let $R[[t]] := R \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ the R-valued Taylor series and $R((t)) := R \otimes_{\mathbb{C}} \mathbb{C}((t))$ the R-valued Laurent series. An *R-lattice* Λ is a finitely generated projective $R[[t]]$-module such that as an $R[[t]]$-module,

- $\Lambda \subset R((t))^n$
- $\Lambda \otimes_{R[[t]]} R((t)) \cong R((t))^n$

Definition 1.1.1.2. The affine Grassmannian for GL_n is the $\mathbb{C}$-space $\mathrm{Gr}_{\mathrm{GL}_n}$ given by

$$\begin{aligned} \text{Commutative } \mathbb{C}\text{-algebras}^{op} &\to \text{Sets} \\ R &\mapsto \{\Lambda \subset R((t))^n \mid \Lambda \text{ a lattice}\} \end{aligned}$$

Example 1.1.1.1. *We let $\Lambda_{0,R}$ be the standard lattice $R[[t]]^n$. In general, we can obtain a lattice by picking n linearly independent elements of $R((t))^n$.*

Theorem 1.1.1.1. *The $\mathbb{C}$-space $\mathrm{Gr}_{\mathrm{GL}_n}$ can be represented by an ind-projective ind-scheme of ind-finite type.*

The proof will take the remainder of this section. While this is standard (see [BL94a], [Kum02], [Zhu17] for instance), the arguments will be useful for computations in chapter 4. To prove the theorem, we want to realise $\mathrm{Gr}_{\mathrm{GL}_n}$ as a union of finite dimensional projective schemes. These are obtained as follows:

Definition 1.1.1.3. For $N \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{Gr}_N(R) = \{\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}(R) \mid t^N \Lambda_{0,R} \subset \Lambda \subset t^{-N} \Lambda_{0,R} \}$$

As any projective module is finitely generated, this defines a filtration of $\mathrm{Gr}_{\mathrm{GL}_n}$ by subfunctors. To deduce the theorem, we will show the following:

Proposition 1.1.1.1. Gr_N *can be represented by a proper scheme over* $\mathbb{C}$.

Proof. We follow [Zhu17].

We will exhibit $\mathrm{Gr}_N(R)$ as a closed subscheme of the classical Grassmannian. Let $V_{N,R}$ be the free R-module $t^{-N}\Lambda_{0,R}/t^N\Lambda_{0,R}$

$$\phi_N : \mathrm{Gr}_N(R) \to Grass(V_{N,R})$$

given by $\Lambda \mapsto t^{-N}\Lambda_{0,R}/\Lambda$.

We claim that this is a closed immersion, with image the $\bar{t}$-stable subspaces of $V_{N,R}$. Here $\bar{t}$ is the induced endomorphism coming from the action of $t \in R[[t]]$. This will be broken down into several steps:

(1) First we need to show that the quotient $t^{-N}\Lambda_{0,R}/\Lambda$ is projective. As $\Lambda/t\Lambda = \Lambda \otimes_{R[[t]]} R$ is projective by assumption, note that $R((t))^n/\Lambda$ is also projective (as it is isomorphic to a direct sum of copies of $\Lambda/t\Lambda$). Furthermore, there is a natural injection $t^{-N}R[[t]]^n/\Lambda \to R((t))^n/\Lambda$ whose cokernel is the free module $R((t))/t^{-N}R[[t]]$. Therefore, $t^{-N}R[[t]]/\Lambda$ is projective.

(2) Consider the set $Grass(V_{0,R})^t := \{W \in Grass(V_{N,R}) \mid tW \subset W \}$. Using Plücker coordinates, we can see that it defines a closed subscheme of $Grass(V_{N,R})$, and that ϕ_N factors through $Grass(V_{0,R})^t$.

(3) It remains to show that the induced map $\overline{\phi_N} : \mathrm{Gr}_N(R) \to Grass(V_{0,R})^t$ is surjective. We only need to check this for finitely generated $\mathbb{C}$-algebras R as any

$\mathbb{C}$-algebra is the limit of finitely generated subalgebras and $Grass(V_{N,R})$ is finite-dimensional. Take $P \in Grass(V_{0,R})^t$ and define Λ_P as $\ker(t^{-N}\Lambda_{0,R} \to P)$. Note that $V_{N,R} \cong t^{-N}R[t]/t^N R[t]$ and thus we can construct an $R[t]$-module Λ_P^{fin} such that $\Lambda_P^{fin} \otimes_{R[t]} R[[t]] \cong \Lambda_P$. Thus we only need to show that Λ_P^{fin} is projective as the embedding $R[t] \to R[[t]]$ is flat. We can then reduce by localisation to the case that R is a local ring, and then conclude using Nakayama's lemma.

□

1.1.2 Affine Grassmannians: the general case

Let $\mathcal{O} := \mathbb{C}[[t]]$ and $\mathcal{K} := \mathbb{C}((t))$.

Definition 1.1.2.1. (1) Let X be a scheme over $\mathbb{C}$. The *positive loop space* is the functor

$$\begin{aligned} X_{\mathcal{O}} :\ & \mathrm{AffSch}_{\mathcal{O}}^{op} \to \mathrm{Sets} \\ & \mathrm{Spec}(R) \mapsto X(R[[t]]) \end{aligned}$$

(2) Let Y be a scheme over C. The *loop space* is the functor

$$\begin{aligned} Y_{\mathcal{K}} :\ & \mathrm{AffSch}_{\mathcal{K}}^{op} \to \mathrm{Sets} \\ & \mathrm{Spec}(R) \mapsto Y(R((t))) \end{aligned}$$

The following lemma is well-known and its proof can be found in [Zhu17].

Lemma 1.1.2.1. *(i) If X is a scheme of finite type over $\mathcal{O}$, $X_{\mathcal{O}}$ can be represented by a scheme. Furthermore, if X is affine, so is $X_{\mathcal{O}}$.*

(ii) If Y is an affine scheme of finite type over $\mathcal{K}$, $Y_{\mathcal{K}}$ can be represented by an ind-affine ind-scheme. Furthermore, if $Y = X \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{K}$ where X is affine

of finite type over $\mathcal{O}$, $X_{\mathcal{O}}$ is a closed subscheme of $Y_{\mathcal{K}}$

Using this lemma, we can define our main object of study:

Definition 1.1.2.2. Let $\underline{G}$ be a smooth affine group scheme over $\mathcal{O}$. The *affine Grassmannian* for $\underline{G}$ is the fpqc quotient

$$\mathrm{Gr}_{\underline{G}} := \underline{G}_{\mathcal{K}}/\underline{G}_{\mathcal{O}}$$

Lemma 1.1.2.2. *Taking $G = \mathrm{GL}_n$, the two definitions of the affine Grassmannian agree.*

Proof. Let $\Lambda_0 := \mathbb{C}[[t]]^n$. Note that $\mathrm{GL}_n(\mathcal{K})$ acts transitively on the set of lattices (taking $g \in \mathrm{GL}_n(\mathcal{K})$ to $g\Lambda_0$), and that $\mathrm{GL}_n(\mathcal{O})$ is the stabiliser of the standard lattice Λ_0. Thus it remains to show that fpqc-locally every lattice Λ admits a basis. Indeed, picking one such basis would give an element $g \in \mathrm{GL}_n(\mathcal{K})$ such that $g\Lambda_0 = \Lambda$.
The statement follows from noting that finitely generated projective modules over a Noetherian ring are locally free (see [G1̈0] for details). □

Remark 1.1.2.1. Yet another definition exists in the literature (see [Zhu17]): one can define the affine Grassmannian of $\underline{G}$ as the $\mathbb{C}$-space given by

$$R \mapsto \left\{ (\mathcal{E}, \beta) \mid \begin{cases} \mathcal{E} \text{ a } \underline{G}\text{-bundle over } \operatorname{Spec}(R[[t]]) \\ \beta \text{ a trivialisation } \mathcal{E} \times_{\operatorname{Spec} R[[t]]} \operatorname{Spec} R((t)) \cong \mathcal{E}^0 \times_{\operatorname{Spec} R[[t]]} \operatorname{Spec} R((t)) \end{cases} \right\}.$$

We will not use this definition here, but one can show the various definitions are equivalent using the Beauville-Laszlo glueing theorem ([BL95]).

To obtain an ind-scheme structure on $\mathrm{Gr}_{\underline{G}}$, we will use embeddings $\underline{G} \hookrightarrow \mathrm{GL}_n$ together with the ind-scheme structure from the previous section. The main result in that direction is the following:

Theorem 1.1.2.1. *[BD91] If* $\rho : \underline{G} \to \mathrm{GL}_n$ *is a linear representation such that the quotient* $\underline{G}\backslash\mathrm{GL}_n$ *is affine, the induced map* $f_\rho : \mathrm{Gr}_{\underline{G}} \to \mathrm{Gr}_{\mathrm{GL}_n}$ *is a closed embedding.*

Remark 1.1.2.2. The theorem leaves open the question of determining which group schemes satisfy the assumption. A large class of such group schemes are called *parahoric* subgroups. These can be described using the affine building of the group $\underline{G}$. In the remainder of the text, we will tacitly use the fact that all group schemes considered are parahoric.

A stronger result due to T. Richarz ([Ric16]) states that the affine Grassmannian is ind-proper if and only if the group scheme $\underline{G}$ is a parahoric model associated with a connected reductive group over $\mathcal{K}$. This closely resembles the finite-dimensional case: homogeneous spaces for reductive groups over $\mathbb{C}$ are projective if and only if they are of the form G/P where P is a parabolic subgroup.

Proof. We provide a sketch of the arguments in [PR08]. Let Z be the quotient $\underline{G}\backslash\mathrm{GL}_n$, and consider the natural map $f :$

$GL_{n,\mathcal{K}} \to Z_{\mathcal{K}}$. By the previous lemma, we know that $Z_{\mathcal{O}} \subset Z_{\mathcal{K}}$ is closed. Therefore $Y := f^{-1}(Z_{\mathcal{O}})$ is a closed subscheme of $\mathrm{GL}_{n,\mathcal{K}}$. Furthermore, note that by construction, Y is stable under the action of $\mathrm{GL}_n(\mathcal{O})$. Therefore, we can find a closed $Y' \subset \mathrm{Gr}_{\mathrm{GL}_n}$ such that $Y = q^{-1}(Y')$ under the quotient map $q : \mathrm{GL}_n(\mathcal{K}) \to \mathrm{Gr}_{\mathrm{GL}_n}$. Finally, note that the map $\underline{G}_{\mathcal{K}} \to Y'$ induces an isomorphism between Y' and $\mathrm{Gr}_{\underline{G}}$. □

Corollary 1.1.2.1. *For* G *a reductive group scheme over* $\mathbb{C}$, Gr_G *can be represented by an ind-projective ind-scheme of ind-finite type.*

More generally, for any parahoric group scheme $\underline{G}$ *over* $\mathcal{O}$, $\mathrm{Gr}_{\underline{G}}$ *can be represented by an ind-projective ind-scheme of ind-finite type.*

Remark 1.1.2.3. We will describe explicitly the parahoric group schemes we will use in the next section.

1.2 Orbit decompositions

1.2.1 Fixing the root datum

In this section, we will take G to be a connected complex reductive group together with a choice of maximal torus and Borel subgroup $T \subset B$. The Weyl group W is given as the quotient $N_G(T)/T$. To this data we can associate the root lattice Q, choose positive roots Δ and simple roots Δ_s. We let $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ be the cocharacter lattice, which contains the coroot lattice $Q^\vee$ and the dominant cocharacters $X_*^+(T)$.

Definition 1.2.1.1. (1) The *affine Weyl group* W^{aff} is the semidirect product $W \ltimes Q^\vee$

(2) The *extended affine Weyl group* $\tilde{W}^{aff}$ is the semidirect product $W \ltimes X_*(T)$.

The natural length function on the finite Weyl group can be extended to the (extended) affine Weyl group as follows: for $w \in W$, $\lambda \in X_*(T)$,

$$l(w\lambda) := \sum_{\alpha \in \Delta \cap w^{-1}\Delta} |\langle \lambda, \alpha \rangle| + \sum_{\alpha \in \Delta \cap w^{-1}(-\Delta)} |\langle \lambda, \alpha \rangle + 1|$$

The affine Weyl group is a Coxeter group, generated by simple reflections $s_0, \ldots, s_n$ which depend on our choice of simple roots. s_i for $i = 1, \ldots, n$ can be defined as the reflection through the i-th simple root, and $s_0(\lambda) = \lambda - (\langle \lambda, \theta \rangle - 1)\theta^\vee$ where θ is the highest root. Furthermore, this choice induces a partial ordering $\geq$ which restricts to the standard Bruhat ordering on W.
To any element $w \in \tilde{W}^{aff}$, we can associate (non-uniquely) an element $\overline{w} \in G_{\mathcal{K}}$ as follows: Write $w = \dot{w}\lambda$ and pick a lift $\tilde{w} \in N_G(T)$ of $\dot{w}$. Furthermore $\lambda : \mathbb{G}_m \to T$ induces a group homomorphism $\lambda_{\mathcal{K}} : \mathbb{G}_m(\mathcal{K}) \to T_{\mathcal{K}}$ and let $t^\lambda \in G_{\mathcal{K}}$ be given by $\lambda_{\mathcal{K}}(t)$.

Then we let $\overline{w} = \tilde{w}t^{\lambda}$.

Similarly, we can write $\tilde{W}^{aff}$ as the quotient $\tilde{N}/T$ where $\tilde{N} = N_G(T) \ltimes X_*(T) \subset G_{\mathcal{K}}$. As with the finite-dimensional case, to any pair (α, m) where α is a root and $m \in \mathbb{Z}$, we have a root-group homomorphism

$$\phi_{\alpha,m} : \mathrm{SL}_2(\mathbb{C}) \to G_{\mathcal{K}},$$

given as the composition of the root map $\phi_{\alpha,\mathcal{K}}$ with the homomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & t^m b \\ t^{-m} c & d \end{pmatrix}$$

We let $\mathcal{U}_{\alpha,m}$ be the unipotent group in $G_{\mathcal{K}}$ given by the image of the standard unipotent subgroup $U \subset \mathrm{SL}_2(\mathbb{C})$

Definition 1.2.1.2. The *Iwahori subgroup* $I \subset G_{\mathcal{O}}$ is the preimage $ev^{-1}(B)$ under the natural map $ev : G_{\mathcal{O}} \to G$ obtained by setting t to 0.

For any set of simple reflections J in W^{aff}, the *parahoric subgroup* $\mathcal{P}_J$ is given by $IW_JI \subset G_{\mathcal{K}}$, where $W_J \subset W^{aff}$ is the subgroup generated by the simple reflections $s_i, i \in J$.

Remark 1.2.1.1. Note that I is generated by $\mathcal{U}_{\alpha_i,m}$ for the simple reflections α_i and $m \geq 0$, and the root subgroups $\phi_{\alpha_i,0}(\mathrm{SL}_2(\mathbb{C})) \subset G_{\mathcal{O}}$.

Lemma 1.2.1.1. *$\mathcal{P}_J$ is indeed a subgroup of $G_{\mathcal{K}}$.*

Proof. We prove this for G simply-connected (i.e. $\tilde{W}^{aff} = W^{aff}$), but the result holds more generally (see [Kum02]). This reduces to showing that if s is a simple reflection and $w \in \tilde{W}^{aff}$, $IsIwI \in IwI \cup IswI$. We will use the unipotents $\mathcal{U}_\alpha$ and follow the argument of [Kum02, 6.2 E3.(c)].

First note that $Is_iI = Is_i\mathcal{U}_{\alpha_i}$: indeed, we have $w\mathcal{U}_\alpha w^{-1} = \mathcal{U}_{w\alpha}$ and $\mathcal{U}_{w\alpha} \subset I$ for

$w\alpha > 0$. Hence, as $s_j(\alpha_i) > 0$ for $i \neq j$, the claim follows.

Now $Is_iIwI = Is_i\mathcal{U}_{\alpha_i}wI$. For $w^{-1}\alpha_i > 0$, so repeating the previous argument for w leads to the result.

If $w^{-1}\alpha_i < 0$, we can note that $\mathcal{U}_{\alpha_i}s_i \subset T\mathcal{U}_{\alpha_i}\mathcal{U}_{-\alpha_i} \cup T\mathcal{U}_{\alpha_i}s_i$ which can be checked directly on SL_2. The result then follows. □

Example 1.2.1.1. *For $SL_2(\mathcal{K})$ under the standard choice of simple root, the Iwahori subgroup is given by matrices in* $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix} \cap \mathrm{SL}_2(\mathcal{K})$. *There are two other maximal parahoric subgroups:* $\mathrm{SL}_2(\mathcal{O})$ *and* $\begin{pmatrix} \mathcal{O} & t^{-1}\mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix} \cap \mathrm{SL}_2(\mathcal{K})$.

In $\mathrm{SL}_3(\mathcal{K})$, *the situation is more complicated. We still obtain the maximal parahoric* $\mathrm{SL}_3(\mathcal{O})$ *and the Iwahori I, but also* $\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \mathrm{SL}_3(\mathcal{K})$.

Remark 1.2.1.2. As mentioned in the previous section, we assume without proof that these groups are indeed parahoric in the sense of Corollary 1.1.2.1.

Definition 1.2.1.3. The Kac-Moody flag variety flag variety associated with a parahoric subgroup $\mathcal{P}$ is denoted $\mathfrak{X}^G(\mathcal{P}) := \mathcal{G}_{\mathcal{K}}/\mathcal{P}$. Taking $\mathcal{P} = I$ the Iwahori subgroup we obtain the affine flag variety, usually denoted $\mathrm{Fl}_G = \mathfrak{X}^G(I)$.

By definition, each of the parahoric subgroups we use contain I, and therefore admit a map $\mathrm{Fl}_G \to \mathfrak{X}^G(\mathcal{P})$. The main property of this map is as follows.

Proposition 1.2.1.1. *Let $\mathcal{P} \subset G_{\mathcal{K}}$ be a parahoric subgroup. Then the natural quotient map $\pi : \mathrm{Fl}_G \to G_{\mathcal{K}}/\mathcal{P}$ is a locally trivial fibration with projective fibres.*

For the proof, see [PR04, 8.e.1].

1.2.2 Bruhat-Cartan-Iwahori-Matsumoto decompositions

In this section and for the remainder of this text, unless specified otherwise, all spaces will be understood as their $\mathbb{C}$-points equipped with the reduced (ind-)scheme structure, and G will be as in the previous section.

The Cartan decomposition of Gr_G

The main result in this section is the following:

Theorem 1.2.2.1 (Cartan decomposition)**.** *The affine Grassmannian decomposes as a disjoint union*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in X_*^+(T)} \mathrm{Gr}_G^\lambda$$

where $\mathrm{Gr}_G^\lambda = (G_\mathcal{O}\ t^\lambda\ G_\mathcal{O})/G_\mathcal{O}$ *such that:*

(1) Each Gr^λ *is a single* $G_\mathcal{O}$*-orbit.*

(2) Let $\varrho := 1/2 \sum_{\alpha \in \Delta} \alpha$*. Then* Gr^λ *is a smooth quasi projective scheme with*

$$\dim \mathrm{Gr}_G^\lambda = \langle 2\varrho, \lambda \rangle$$

(3)

$$\overline{\mathrm{Gr}_G^\lambda} = \bigsqcup_{\mu \leq \lambda} \mathrm{Gr}_G^\mu$$

(4) Gr_G^λ *is an affine bundle over the generalised flag variety* G/P_λ*.*

Remark 1.2.2.1. Note that the theorem provides an explicit stratification of Gr_G by finite-dimensional projective schemes, which will allow us to study the geometry of the affine Grassmannian more explicitly. Furthermore, the objects we will consider in the next chapter are $G_\mathcal{O}$-equivariant and therefore will naturally be supported on unions of such varieties. This theorem will follow from identifying the T-fixed points using an argument of Zhu, then using the Bialynicki-Birula stratification.

First, note that for a torus T, Gr_T is naturally isomorphic to the cocharacter lattice $X_*(T)$.

Proposition 1.2.2.1. *The embedding $T \hookrightarrow G$ induces an identification*

$$(\mathrm{Gr}_G)^T = \mathrm{Gr}_T$$

where $(\mathrm{Gr}_G)^T$ is the set of fixed points under the left T-action.

To prove this, we will use the following lemma:

Lemma 1.2.2.1 ([Zhu09])**.** *There is a decomposition*

$$G_{\mathcal{K}} = B_{\mathcal{K}} G_{\mathcal{O}}$$

Proof of Proposition 1.2.2.1. Note that for any cocharacter λ, $t^{\lambda} G_{\mathcal{O}}/G_{\mathcal{O}}$ is a fixed point of Gr_G under the left T-action.
Conversely, take $g \in (\mathrm{Gr}_G)^T$. By the lemma, we can take $g \in B(\mathcal{K})$. Now as $\mathbb{C}$-varieties, $B \cong T \times U$ where U is the unipotent radical. Therefore, we can write $g = su$ where $s \in T_{\mathcal{K}}$ and $u \in U_{\mathcal{K}}$. Furthermore, as U is unipotent, we can write $u = u_- u_+$ where $u_- \in U(t^{-1}\mathbb{C}[t^{-1}])$ and $u_+ \in U_{\mathcal{O}}$ (indeed the result holds for $\mathbb{G}_a$ and U is generated by a product of such).
Therefore, for any $t \in T$, we have $tgG_{\mathcal{O}} = gG_{\mathcal{O}} \iff Ad_{g^{-1}} t \in G_{\mathcal{O}} \cap B_{\mathcal{K}} \cong B_{\mathcal{O}}$. Writing $g = su_-u_+$ as above, we see that this reduces to $u_-^{-1} t u_- \in B_{\mathcal{O}}$, which is equivalent to $u_-^{-1} t u_- = t$. As this holds for all $t \in T$, we see that $u_- = 1$. Therefore, as $u_+ \in G_{\mathcal{O}}$, $g \in T_{\mathcal{K}} G_{\mathcal{O}}/G_{\mathcal{O}}$ as required. $\square$

Now, consider the $T \rtimes \mathbb{G}_m$ action on Gr_G where $\mathbb{G}_m$ acts by loop rotation $zP(t) = P(zt)$.

Proposition 1.2.2.2. *We can choose a cocharacter $\mathbb{G}_m \to T \rtimes \mathbb{G}_m$ such that the* *attracting sets of the Bialynicki-Birula stratification under the induced* $\mathbb{G}_m$*-action are of the form* $It^\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$ *for* $\lambda \in X_*(T)$.

Remark 1.2.2.2. A priori, it is not clear that one can use the Bialynicki-Birula stratification in this context. One approach is to observe that as an ind-scheme the affine Grassmannian is formally smooth, and embeds in an infinite dimensional smooth scheme (the thick affine Grassmannian, which we will discuss in section 4). Furthermore, the T-fixed points are discrete, and we can apply the BB decomposition (see [Gin90] for some more details). For the definition of the tangent space of an ind-scheme and some basic computations, we refer the reader to [HLR18].

Proof. Let X_λ^+ be the attracting set corresponding to the fixed point t^λ. Decompose the Lie algebra $\mathfrak{g}((t))$ as

$$\mathfrak{g}((t)) = \widehat{\mathfrak{n}^-} \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}^+}$$

where $\widehat{\mathfrak{n}^+}$ is $\mathfrak{n}^+ \oplus t\mathfrak{g}[[t]]$, and similarly for $\widehat{\mathfrak{n}^-}$. Note that we can choose the $\mathbb{G}_m$-action such that the above decomposition of $\mathfrak{g}((t))$ corresponds to the plus/minus decomposition [1]. Now, the tangent space $T_e \operatorname{Gr}_G \cong \mathfrak{g}[[t]]/\mathfrak{g}[[t]]$ and therefore $T_{t^\lambda} \operatorname{Gr}_G \cong \mathfrak{g}[[t]]/(Ad_{t^\lambda}\mathfrak{g}[[t]]) \cap \mathfrak{g}[[t]]$. Thus

$$X_\lambda^+ \cong T_\lambda^+ \operatorname{Gr}_G \cong \bigoplus_{\langle \alpha,\lambda\rangle \geq 0, \alpha \in \Delta} g_\alpha(\mathcal{O})/t^{\langle \alpha,\lambda\rangle}\mathfrak{g}_\alpha(\mathcal{O}).$$

Note that this is a finite-dimensional vector space over $\mathbb{C}$, with dimension $\langle 2\rho, \lambda\rangle$. Furthermore, note that $\mathcal{U}t^\lambda \subset X_\lambda^+$ where $\mathcal{U}$ is the pro-unipotent radical of I. Finally, $\mathcal{U}t^\lambda \subset X_\lambda^+$ is both dense open (by dimension checks, and $\mathcal{U}$ unipotent) and closed (as X_λ^+ is affine), so therefore $X_\lambda^+ \cong \mathcal{U}t^\lambda G_{\mathcal{O}}/G_{\mathcal{O}} = It^\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$ as required. □

---

[1] See MathOverflow:The Bialynicki Birula stratification of the affine Grassmannian

Corollary 1.2.2.1. *The affine Grassmannian* Gr_G *decomposes as*

$$\bigsqcup_{\lambda \in X_*(T)} It^\lambda G_\mathcal{O}/G_\mathcal{O}$$

Furthermore, $It^\lambda G_\mathcal{O}/G_\mathcal{O}$ *is an affine space of dimension* $\langle 2\rho, \lambda\rangle$*.*

Note that this immediately implies parts (1) and (2) of Theorem 1.2.2.1 as $G_\mathcal{O} = \bigsqcup_{w\in W} IwI$ and each Weyl group orbit has a unique dominant representative. Part (4) follows from the Bialynicki-Birula decomposition, noting that $Gt^\lambda G_\mathcal{O}/G_\mathcal{O}$ is the fixed set of the loop rotation action on Gr_λ and is isomorphic to G/P_λ. It remains to show part (3):

Lemma 1.2.2.2.

$$\overline{\mathrm{Gr}_\lambda} = \bigcup_{\mu \leq \lambda} \mathrm{Gr}_\mu$$

Remark 1.2.2.3. There are various ways of proving this result, we will present here the approach taken in [BD91] and [Zhu17]. Note that the proof of theorem 1.2.2.2 below provides an alternative proof.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a lowest weight representation of V with lowest weight ω and lowest weight vector v. Consider $V \otimes \mathcal{O}$ as a $G_\mathcal{O}$-module. Then $t^\lambda v = t^{\langle \lambda, \omega\rangle} v$ and therefore, for any $g \in G_\mathcal{O} t^\lambda G_\mathcal{O}$, $\rho(g) \in t^{\langle\lambda,\omega\rangle} \mathrm{End}(V \otimes \mathcal{O})$. Furthermore, the above calculation also implies that $\rho(g) \notin t^{\langle\lambda,\omega\rangle+1} \mathrm{End}(V \otimes \mathcal{O})$. Therefore, if $\mathrm{Gr}_\mu \subset \overline{\mathrm{Gr}_\lambda}$, we deduce that $\langle \mu, \omega\rangle \leq \langle \lambda, \omega\rangle$ for all dominant weights ω and therefore $\mu \leq \lambda$.

Conversely, we will show that for any $\mu \leq \lambda$, we can construct a closed curve in $\overline{\mathrm{Gr}_\lambda}$ containing $t^{\lambda-\alpha}$ for α a positive coroot such that $\mu \leq \lambda - \alpha \leq \lambda$ and $\lambda - \alpha$ dominant (the general result then follows from induction). We proceed as follows: consider the root homomorphism $\phi_\alpha : \mathrm{SL}_2 \to G$ and as before, extend it to a homomorphism $\phi_{\alpha,\mathcal{K}} : \mathrm{SL}_{2,\mathcal{K}} \to G_\mathcal{K}$. Now, define the subgroup $K_m = \{\begin{pmatrix} a & t^m b \\ t^{-m} c & d \end{pmatrix}\} \subset \mathrm{SL}_{2,\mathcal{K}}$ and define $C_\alpha = \phi_{\alpha,\mathcal{K}}(K_m)t^\lambda G_\mathcal{O}/G_\mathcal{O} \subset \mathrm{Gr}_\lambda$ for $m = \langle\alpha,\mu\rangle - 1$.

Note that $K_m^{(1)} := ev^{-1}(Id) \cap K_m$ acts trivially on $t^\lambda G_\mathcal{O}/G_\mathcal{O}$: indeed, and element of $K_m^{(1)}$ can be written as

$$\begin{pmatrix} a & t^{\langle\alpha,\mu\rangle}b \\ t^{-\langle\alpha,\mu\rangle}c & d \end{pmatrix}$$

such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_{2,\mathcal{O}}$. We can then use the identity $Ad_{t^\lambda}\phi_{\alpha,\mathcal{K}}(X) = \phi_{\alpha,K}(Ad_{t^{\langle\lambda,\alpha\rangle}}X)$ to conclude that $K_m^{(1)} \subset t^\lambda G_\mathcal{O} t^{-\lambda}$ and therefore $K_m^{(1)}$ stabilises t^λ.

Now the evaluation map $\mathrm{SL}_{2,\mathcal{O}} \to \mathrm{SL}_2(\mathbb{C})$ gives a natural isomorphism $K_m/K_m^{(1)} \cong \mathrm{SL}_2(\mathbb{C})$ and therefore $\phi_{\alpha,\mathcal{K}}(K_m)t^\lambda$ is isomorphic to $\mathbb{P}^1$ (noting that x_α does not fix t^λ. Furthermore, separating the action of the root subgroups x_α and $x_{-\alpha}$ we see that $G_\mathcal{O} \cap \phi_{\alpha,\mathcal{K}}(K_m)t^\lambda G_\mathcal{O}/G_\mathcal{O} = \mathbb{A}^1 \subset \mathbb{P}^1$. Finally, we can note that the point at infinity is given by the image of $\begin{pmatrix} 0 & -t^m \\ t^{-m} & 0 \end{pmatrix}$. Writing it as a commutator of unipotent matrices (see [Jan03, 2.1.2]), we see that $\phi_{\alpha,\mathcal{K}}(\begin{pmatrix} 0 & -t^m \\ t^{-m} & 0 \end{pmatrix})t^\lambda G_\mathcal{O}/G_\mathcal{O} = t^{\lambda-\alpha}G_\mathcal{O}/G_\mathcal{O}$. □

This concludes the proof of theorem 1.2.2.1.

The Iwahori decomposition of the affine flag variety

Recall that we have a $G_\mathcal{K}$-equivariant map $\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G$ which is a locally trivial fibration with fibres G/B. Using theorem 1.2.2.1 and the map π we obtain the following decomposition of the affine flag variety:

Proposition 1.2.2.3 (The Iwahori Decomposition). Fl_G *admits a decomposition into I-orbits*

$$\mathrm{Fl}_G = \bigsqcup_{w \in \tilde{W}^{aff}} IwI/I.$$

Furthermore, each orbit $X_w := IwI/I$ *is an affine space of dimension* $l(w)$*.*

Proof. We can use the Bialynicki-Birula decomposition again. To obtain the T-fixed points, note that as π is equivariant, all fixed points are contained in $\pi^{-1}(t^\lambda G_{\mathcal{O}}/G_{\mathcal{O}}) = t^\lambda G_{\mathcal{O}}/I$ for some cocharacter λ. Now T commutes with t^λ and $G_{\mathcal{O}}/I \cong G/B$, therefore the problem reduces to finding the T-fixed points in G/B. But these are precisely of the form wB for $w \in W$.
Suppose, without loss of generality, that λ is dominant. Then, the tangent space to $Iwt^\lambda I/I$ is isomorphic to

$$\frac{\mathfrak{b} \oplus t\mathfrak{g}[[t]]}{\mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta\cap w\Delta} t^{\langle\alpha,\lambda\rangle}\mathfrak{g}_{w\alpha}(\mathcal{O})) \oplus (\bigoplus_{\alpha\in\Delta\cap w(-\Delta)} t^{\langle\lambda,\alpha\rangle+1}\mathfrak{g}_{w\alpha}(\mathcal{O})) \oplus (\bigoplus_{\alpha<0}\mathfrak{g}_\alpha[[t]])}$$

which has dimension $l(w)$. □

As with the affine Grassmannian, the main theorem concerns identifying the I-orbit closures in Fl_G.

Theorem 1.2.2.2. *For any* $w \in \tilde{W}^{aff}$*, the closure of* X_w *in* Fl_G *is given as follows.*

$$\overline{X_w} = \bigsqcup_{v\leq w} X_v$$

The techniques of theorem 1.2.2.1 do not apply here, therefore we need a new approach. For simplicity, we will assume that G is simply-connected so that $W^{aff} = \tilde{W}^{aff}$, although the results will readily generalise to the general case (as the affine flag variety decomposes in disconnected blocks labelled by $\pi_1(G)$, see [Fal03]). The main ingredient of the proof is the following:

Definition 1.2.2.1. Let $w \in W^{aff}$, and write it as a reduced decomposition $\underline{w} = s_1 s_2 \dots s_r$ where s_j is a simple reflection. The *Bott-Demazure-Samuelson-Hansen* *resolution* (BDSH) $Z_{\underline{w}}$ of $\overline{X_w}$ is the map

$$\pi_{\underline{w}}: \ D(\underline{w}) := \mathcal{P}_1 \times_I \mathcal{P}_2 \times_I \cdots \times_I \mathcal{P}_r/I \to \mathrm{Fl}_G$$

where $\mathcal{P}_i = I \cup Is_iI$ the minimal parahoric corresponding to s_i.

Proposition 1.2.2.4. *[Fal03, Kum02] $D(\underline{w})$ is smooth, proper and irreducible. Furthermore, the map $\pi_{\underline{w}}$ is a resolution of singularities of $\overline{X_w}$.*

Remark 1.2.2.4. First, note that theorem 1.2.2.2 follows immediately from the proposition: by the proof of lemma 1.2.1.1, the image of $\pi_{\underline{w}}$ consists precisely of the orbits IvI/I for $v \leq w$ (by picking in the product which elements are in I or Is_iI).

Proof. Note that $\mathcal{P}_i/I \cong \mathbb{P}^1$ and therefore smoothness, properness and irreducibility follow by induction on $l(w)$. Now, note that the open dense subvariety $Is_1I \times_I \cdots \times_I Is_rI/I$ maps isomorphically to X_w under $\pi_{\underline{w}}$. Furthermore, the image of a projective variety under a scheme morphism is closed, and therefore the result follows. $\square$

More generally, we can repeat the arguments in the case of parahoric flag varieties $G_{\mathcal{K}}/\mathcal{P}_Y$, and obtain the following:

Corollary 1.2.2.2. *We have the following decompositions of $\mathfrak{X}^G(\mathcal{P})$:*

(i)

$$\mathfrak{X}^G(\mathcal{P}) = \bigsqcup_{w \in \tilde{W}^{aff}/W_{\mathcal{P}}} Iw\mathcal{P}/\mathcal{P}$$

where the I-orbits are affine spaces.

(ii)

$$\mathfrak{X}^G(\mathcal{P}) = \bigsqcup_{w \in W_{\mathcal{P}} \backslash \tilde{W}^{aff}/W_{\mathcal{P}}} \mathcal{P}w\mathcal{P}/\mathcal{P}$$

1.3 Affine Kac-Moody Lie algebras and central extensions

Another approach to defining Kac-Moody flag varieties comes from the work of Kumar and Mathieu. For them, the starting point is the theory of Kac-Moody Lie algebras, and the theory then tries to mimic the finite dimensional case (by defining the loop groups as exponentiated Lie algebras in some precise sense). While the previous sections provided us with the ind-projective ind-scheme structure using embeddings into the affine Grassmannian of GL_n, in this section we will use the representation theory of Kac-Moody Lie algebras to obtain embeddings into projective space. We will first provide a short overview of the theory of Kac-Moody Lie algebras so as to be able to state and prove the results we will need. This follows [Kac90], [Kum02], [Sor00].

1.3.1 Kac-Moody Lie algebras and representations

Definition 1.3.1.1. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Killing form $(-,-)$. The *affine Kac-Moody Lie algebra* (AKLA) $\widehat{\mathfrak{g}}$ is defined as the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ together with Lie bracket

$$[X \otimes f + \mu_1 d, Y \otimes g + \mu_2 d] = [X, Y] \otimes fg + (X, Y) Res(gdf)c + \mu t \frac{dg}{dt} \otimes Y - \mu' t \frac{df}{dt} \otimes X$$

and c central.

As in the finite-dimensional case, associated to an AKLA there is a root system, a set of simple roots, a Killing form, a Weyl group and etc. The following theorem summarises the results we will use later on.

Theorem 1.3.1.1. *(i) The roots of $\widehat{\mathfrak{g}}$ are given by*

$$\Delta = \{j\delta \mid j \in \mathbb{Z}\backslash\{0\}\} \cup \{j\delta + \beta \mid j \in \mathbb{Z} \text{ and } \delta \in \Delta(\mathfrak{g}) \subset (\mathfrak{h}^{fin} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*\}$$

where $\delta|_{\mathfrak{h}^{fin} \oplus \mathbb{C}c} = 0$ and $\delta(d) = 1$ so that $\mathfrak{g}_{j\delta+\beta} = \mathfrak{g}_\beta t^j$.

(ii) The positive roots are

$$\Delta^+ = \{j\delta \mid j > 0\} \cup \{j\delta + \beta \mid j > 0 \text{ and } \delta \in \Delta(\mathfrak{g})\} \cup \{\beta \mid \beta \in \Delta(\mathfrak{g})^+\}.$$

leading to a triangular decomposition $\widehat{\mathfrak{g}} = (\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d) \oplus \widehat{\mathfrak{n}}^+ \oplus \widehat{\mathfrak{n}}^-$.

(iii) The simple roots are given by $\Omega = \{\alpha_0 = \delta - \theta, \alpha_1, \dots, \alpha_l\}$ where θ is the longest root in $\mathfrak{g}$ and α_i are the simple roots of $\mathfrak{g}$. The simple coroots are then $\Omega^\vee = \{c - \theta^\vee, \alpha_1^\vee, \dots, \alpha_l^\vee\}$.

(iv) The associated Weyl group is the affine Weyl group $W^{aff} \cong W^{fin} \ltimes Q^\vee$ where $Q^\vee$ is the coroot lattice of $\mathfrak{g}$.

(v) The Killing form is given by

$$\langle X \otimes f, Y \otimes g \rangle = \langle X, Y \rangle Res(fdg)$$
$$\langle \mathbb{C}c + \mathbb{C}d, \mathfrak{g}[t, t^{-1}] \rangle = \langle c, c \rangle = \langle d, d \rangle = 0$$

Remark 1.3.1.1. We also obtain associated simple roots, raising and lowering operators e_i, f_i and a notion of highest-weight representation.

As AKLAs are infinite-dimensional, it is important to restrict the representations we look at. An especially well-behaved class are the integrable representations.

Definition 1.3.1.2. Let V be a highest-weight representation. V is an *integrable representation* of $\widehat{\mathfrak{g}}$ if f_i acts nilpotently on $\mathbb{C}v$ for all i and $v \in V$.

Finally, we can state a classification of the irreducible representations of $\widehat{\mathfrak{g}}$.

Definition 1.3.1.3. Let $\Lambda \in \widehat{\mathfrak{h}}^*$. The *Verma module* $M(\Lambda)$ is the universal highest-weight module.

Theorem 1.3.1.2. *Let* $P_+ = \{\lambda \in \widehat{\mathfrak{h}}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$.
If $\Lambda \in P_+$, *the Verma module* $M(\Lambda)$ *has a unique irreducible integrable quotient denoted* $L(\Lambda)$.

Remark 1.3.1.2. As c is central in $\widehat{\mathfrak{g}}$, c acts by the integer $\langle c, \Lambda \rangle$ on the irreducible representation $L(\Lambda)$. This integer is called the *level* of the representation. The level 1 representations are parametrised by the fundamental weights $\Lambda_i, i = 0, ..., l$ defined by $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ and $\langle \Lambda_i, d \rangle = 0$ up to a multiple of δ. $L(\Lambda_0)$ is usually called the *basic* level 1 representation.

Let $\mathfrak{I} = \mathfrak{b} \otimes 1 \oplus \mathfrak{g} \otimes t\mathbb{C}[t] \subset \widehat{\mathfrak{g}}$ be the Iwahori subalgebra (note it corresponds up to completion to the Lie algebra of I from the previous section).

Definition 1.3.1.4. Let $w \in W^{aff}$ and $\Lambda \in P_+$. The *Demazure module* $V_w(\Lambda)$ is the cyclic $U(\mathfrak{I})$-submodule of $L(\Lambda)$ generated by a non-zero vector $v_{w\Lambda} \in V(\Lambda)_{w\Lambda}$.

Remark 1.3.1.3. One can check (see [Kac90]) that the subspace $V(\Lambda)_{w\Lambda}$ is 1-dimensional, hence the definition is independent of the choice of $v_{w\Lambda}$. Furthermore, note that we can chose $w \in W^{aff}/W_\Lambda$.

The goal of the next section will be to embed the Schubert varieties $\overline{\mathrm{Gr}_{\lambda^\vee}}$ in $\mathbb{P}(V_w(\Lambda))$ for some choices of w and Λ. For this to be possible, we need to select our Demazure module to be $\mathfrak{g}[t]$-stable, and to have suitable weight vectors. The first step is to note the following: we can extend our construction of Demazure modules in the following way. The quotient P/Q is isomorphic to the group Σ of automorphisms of the affine Dynkin diagram and therefore we can view the affine Weyl group $W^{ext} \cong W \rtimes \Sigma$. Now both W^{aff} and Σ act on the weights, extending the action to any $w \in W^{ext}$.

Lemma 1.3.1.1. *$V_w(\Lambda)$ is a $\mathfrak{g}[t]$-module if and only if $s_i w \leq w \mod W_\Lambda^{ext}$ for all real simple reflections s_i.*

Proof. We know that $V_w(\Lambda)$ is stable under raising operators e_i, so to obtain the result, we need to study the action of f_i for i a real root. Note that for $V_w(\Lambda)$ to be $\mathfrak{g}$-stable, we need the lowering operators to act trivially on $v_{w\Lambda}$ (as all non-zero weight spaces are higher in the Bruhat ordering). Now from [Kac90], we see that $f_i v_{w\Lambda} = 0$ if and only if $\langle w\Lambda, \alpha_i^\vee \rangle \leq 0$. This only holds if $s_i w \leq w$, up to the stabiliser of Λ, as required. Finally, note that the the subalgebra of $U(\widehat{\mathfrak{g}})$ generated by $\mathfrak{I}$ and $\mathfrak{g}$ is precisely $U(\mathfrak{g}[t])$. □

The Demazure modules of most interest to us will be the ones coming from the level 1 representations $L(\Lambda_i)$. The main results used later on is the following:

Proposition 1.3.1.1. *Let i be the isomorphism $\mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathfrak{h}^* \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by the Killing form. For any dominant coweight $\lambda^\vee$, we can find a fundamental level 1 weight Λ_i and $w_{\lambda^\vee} \in W^{ext}$ such that the corresponding Demazure module $V_{w_{\lambda^\vee}}(\Lambda_i)$ is generated by a lowest weight vector of weight $w_0 i(\lambda^\vee)$.*

We will denote the corresponding Demazure module by $D_{1,\lambda^\vee}$.

Proof. Let Λ be a dominant affine weight of level 1. Note that $w\Lambda$ satisfies the conditions of the previous lemma if and only if $\langle w\Lambda, \alpha_i^\vee \rangle \leq 0$ for all real simple roots α_i. Writing $\Lambda = \sum_{i=0}^n p_i \Lambda_i + m\delta$, we see that this means $w\Lambda \in -P_+ + \Lambda_0 + m\delta$ for some integer $m \in \mathbb{Z}$. Conversely, take $\lambda^\vee$ a dominant coweight, and consider the dominant weight $i(\lambda^\vee)$. Then $\Lambda' = i(\lambda^\vee) + \Lambda_0 + m\delta$ is a dominant (affine) weight and therefore $w_0\Lambda'$ is of the required form. Finally Λ' has level 1, therefore by changing m we can assume that $\Lambda' = \Lambda_i$ for some i (see [Kac90, 12.4.4]). This gives the result. □

Remark 1.3.1.4. By the lemma, $D_{1,\lambda^\vee}$ is naturally a $\mathfrak{g}[t]$-module, and we will see in the next section that it integrates to a $G_{\mathcal{O}}$-representation.

1.3.2 Central extensions and integration

Let G be a complex, simply-connected, simple algebraic group with corresponding Lie algebra $\mathfrak{g}$. We can construct an extension of $G_{\mathcal{K}}$ by $\mathbb{G}_m$ as follows: note that there is an action of $\mathbb{G}_m$ on $\mathcal{K}$ by $z.P(t) = P(zt)$. This induces a group homomorphism $\mathbb{G}_m \to Aut(G_{\mathcal{K}})$. Then we let $\widehat{G_{\mathcal{K}}}$ be the corresponding semi-direct product.

Inspired by the finite-dimensional case, one might wonder if it is possible to pass between integrable $\mathfrak{g}$-representations to representations of $\widehat{G_{\mathcal{K}}}$. Let us first do this for the adjoint representation.

Lemma 1.3.2.1. *The completed Lie algebra $\widehat{\mathfrak{g}}_{comp} = \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is naturally a representation of $\widehat{G_{\mathcal{K}}}$, extending the adjoint action of G on $\mathfrak{g}$.*

Proof. First note that as G acts on $\mathfrak{g}$ we obtain a natural adjoint action Ad of $G_{\mathcal{K}}$ on $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{K}$. Choose an embedding of G in $\mathrm{SL}_n(\mathbb{C})$ for some n. Then, for $g \in G_{\mathcal{K}}$, $z \in \mathbb{G}_m$, $X \in \mathfrak{g} \otimes K$ and $\lambda, \mu \in \mathbb{C}$, we let

$$Ad(g)(X+\lambda c+\mu d) = Ad(g)(X)-\mu t(\frac{dg}{dt})g^{-1}+(\lambda-Res_t\langle g^{-1}\frac{dg}{dt}, X-\frac{1}{2}\mu t g^{-1}\frac{dg}{dt}\rangle)c+\mu d$$

$$Ad(z)(X+\lambda c+\mu d) = X(zt)+\lambda c+\mu d$$

Here we've used the embedding to define the derivative dg/dt, and one can check that the image is indeed in $\widehat{\mathfrak{g}}_{comp}$. Taking $\lambda = \mu = 0$ and $g \in G$, we recover the adjoint action on $\mathfrak{g}$ as required. □

Theorem 1.3.2.1 (Faltings,[BL95]). *Let $\overline{\rho}: \ \widehat{\mathfrak{g}} \to End(H)$ be a highest-weight integrable representation. Then there exists a unique algebraic representation $\rho : \widehat{G_{\mathcal{K}}} \to$*

GL(H) *whose derivative corresponds to* $\overline{\rho}$ *up to homothety and such that*

$$\overline{\rho}(Ad(g)X) = Ad(\rho(g))\overline{\rho}(X).$$

Remark 1.3.2.1. Integrability here is essential: it allows us to extend any representation of $\overline{\mathfrak{g}}$ to one of the completion $\overline{\mathfrak{g}}_{comp}$ without the appearance of infinite sums.

We can use the integration theorem to construct a central extension of the loop group $G_{\mathcal{K}}$ as follows: consider the short exact sequence of ind-groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}(L(\Lambda_0)) \longrightarrow \mathrm{PGL}(L(\Lambda_0)) \longrightarrow 1. \tag{A}$$

By the theorem, we have a homomorphism $\phi_0 : \widehat{G_{\mathcal{K}}} \to \mathrm{PGL}(L(\Lambda_0))$.

Definition 1.3.2.1. The Kac-Moody group $\mathcal{G}$ associated with G is the central extension of $\widehat{G_{\mathcal{K}}}$ obtained by pullback of (A) along the homomorphism ϕ_0.

Remark 1.3.2.2. The group $\mathcal{G}$ can also be obtained using a Chevalley-type presentation. This is the approach of the book [Kum02].

Proposition 1.3.2.1. *(1) The Lie algebra of* $\mathcal{G}$ *is isomorphic to the Kac-Moody Lie algebra* $\widehat{\mathfrak{g}}$ *associated with* G*.*

(2) The central extension restricted to $G_{\mathcal{O}} \ltimes \mathbb{G}_m$ *splits.*

Remark 1.3.2.3. The proof of Proposition 1.3.2.1 can be found in [LS97]. The main step is that integrability provides a section $\mathfrak{pgl}(\Lambda_0)|_{Im(\rho)} \to \mathfrak{gl}(\Lambda_0)|_{Im(\rho)}$, which readily induces the Kac-Moody cocycle.

1.3.3 Projective embeddings

The simply-connected case

In this subsection, we keep the assumptions of the previous subsection and assume G is a simple, simply-connected, complex reductive group. Consider the $\mathfrak{g}$-representation $L(\Lambda_0)$. By the previous section, this integrates to a projective representation. Therefore, we obtain an action of $\widehat{G_{\mathcal{K}}}$ on $\mathbb{P}(L(\Lambda_0))$.

Lemma 1.3.3.1. *The stabiliser of the highest-weight line* $[v_0] \in \mathbb{P}(L(\Lambda_0))$ *is exactly* $\widehat{G_{\mathcal{O}}}$.

Proof. Note that by [Kac90, 10.4], v_Λ is annihilated by e_i,f_i for $i > 0$, $\mathfrak{h}$ and d. The Lie subalgebra generated by these is the Lie algebra of $G_{\mathcal{O}} \times \mathbb{G}_m$ and the previous section allows us to conclude. □

Proposition 1.3.3.1. *The action of* $\widehat{G_{\mathcal{K}}}$ *on* $\mathbb{P}(L(\Lambda_0))$ *induces a closed embedding*

$$\Phi: \ \mathrm{Gr}_G \hookrightarrow \mathbb{P}(L(\Lambda_0)),$$

where $\mathbb{P}(L(\Lambda_0))$ *is given the ind-scheme structure by finite-dimensional subspaces.*

Proof. [BL94a, LS97] The map is injective by construction. It preserves the ind-scheme structure by theorem 1.3.2.1. To show that it is an embedding, by $G_{\mathcal{K}}$-equivariance it is enough to show that $d\Phi_e$ is injective. Now, theorem 1.3.2.1 also describes the differential as

$$\begin{cases} d\Phi_e: & \mathfrak{g}((t))/\mathfrak{g}[[t]] \to L(\Lambda_0)/\mathbb{C}^\times \\ & X \mapsto Xv_{\Lambda_0}. \end{cases}$$

But as noted in the previous lemma, the stabiliser of v_{Λ_0} under the action of $\mathfrak{g}((t))$ is precisely $\mathfrak{g}[[t]]$, proving the result. □

By the ind-scheme structure of $\mathbb{P}(L(\Lambda_0))$, the proposition yields an embedding of the Schubert varieties $\overline{\mathrm{Gr}_\lambda}$ into some projective space. This can be described more precisely:

Lemma 1.3.3.2. *The map Φ identifies $\overline{\mathrm{Gr}_{\lambda^\vee}}$ with the closure of $G_{\mathcal{O}}[v_{w_0 i(\lambda^\vee)}] \subset \mathbb{P}(D_{1,\lambda})$, where $v_{w_0 i(\lambda)}$ is a vector of weight $w_0 i(\lambda)$ in $L(\Lambda_0)$.*

Proof. This comes from the adjoint action and Proposition 1.3.1.1, noting that in the simply-connected case the map i induces an isomorphism between the coweight and root lattices. $\square$

Remark 1.3.3.1. As with the definition of $D_{1,\lambda^\vee}$, the lemma does not depend on the choice of weight vector.

Remark 1.3.3.2. The lemma also offers an alternative definition of the Schubert varieties $\overline{\mathrm{Gr}_\lambda}$ which is entirely based on Kac-Moody Lie algebras. This is the approach taken by Mathieu [Mat88].

The non-simply connected case

For G non-simply connected, the situation is somewhat more complicated. First note the following

Proposition 1.3.3.2. *The connected components of Gr_G are labelled by minuscule coweights λ_i. Furthermore, the corresponding connected component Gr_G^i is isomorphic to $G^{sc}_{\mathcal{K}}/\mathcal{P}^i$ where $\mathcal{P}^i$ is the maximal parahoric subgroup obtained by removing the ith simple reflection.*

Remark 1.3.3.3. Removing the affine reflection s_0, the corresponding connected component is isomorphic to the affine Grassmannian $\mathrm{Gr}_{G^{sc}}$.

Proof. This is a slight simplification of the arguments of [Zhu09].

The first part is purely combinatorial: the orbits Gr_λ and Gr_μ corresponding to two dominant coweights λ and μ are in the same connected components if and only if we can find a sequence of dominant coweights λ^i with $\lambda^0 = \lambda$ and $\lambda^n = \mu$ such that there are dominant coweights $\mu_i \leq \lambda^{i-1}$ and $\mu_i \leq \lambda^i$. Taking μ_1 minuscule (including 0) without loss of generality, we see ([Bou05, VIex5]) that Gr_λ and Gr_μ are in the same connected component iff the unique minuscule coweight under λ is the same as the one under μ. This concludes the first part.

To obtain the second part, note that we can write

$$\mathrm{Gr}^i = \bigcup_{\lambda \in X_*^+(T)} G_{\mathcal{O}}^{sc} t^{\omega_i} t^\lambda G_{\mathcal{O}}/G_{\mathcal{O}}.$$

Finally, the orbit of $t^{\omega_i} G_{\mathcal{O}}/G_{\mathcal{O}}$ under $G_{\mathcal{K}}^{sc}$ contains Gr^i and is connected, therefore we can conclude noting that $Ad_{t^{\omega_i}} G_{\mathcal{O}}^{sc} \cong \mathcal{P}_i$. □

By a similar argument as in the simply-connected case, we can embed the various connected components and Schubert varieties in the corresponding projective representations:

Proposition 1.3.3.3. *There is a $G_{\mathcal{K}}^{sc}$-equivariant closed embedding of ind-schemes*

$$\Phi_i : \mathrm{Gr}_G^i \hookrightarrow \mathbb{P}(L(\Lambda_i))$$

obtained by letting $G_{\mathcal{K}}^{sc}$ act on the highest-weight line $[v_{\Lambda_i}]$.

Furthermore, the map Φ_i identifies $\overline{\mathrm{Gr}_{\lambda^\vee}} \subset \mathrm{Gr}_G^i$ with the closure of $G_{\mathcal{O}}^{sc}[v_{w_0 i(\lambda^\vee)}] \subset \mathbb{P}(D_{1,\lambda})$, where $v_{w_0 i(\lambda)}$ is a vector of weight $w_0 i(\lambda)$ in $L(\Lambda_i)$.

1.4 A different approach: Kashiwara's thick flag varieties

A different approach to constructing affine Kac-Moody flag varieties was introduced by Kashiwara in [Kas89]. The difference between his approach and ours so far is that instead of obtaining an ind-scheme of ind-finite type, he constructs a scheme of infinite type.

Definition 1.4.0.1. Let G be a complex reductive group with a choice of Borel subgroup B and opposite Borel $\overline{B}$, and let for any commutative $\mathbb{C}$-algebra R, consider the group scheme $I^{-}(R) = ev^{-1}(\overline{B})$ and $ev : G(R[t]) \to G$. The *thick affine flag variety* is the $\mathbb{C}$-space $\mathfrak{X}_G$ given by

$$R \to G(R((t^{-1}))/I^{-}(R)$$

Remark 1.4.0.1. In the same way as we have defined parahoric flag varieties, we can replace the Borel by any parabolic subgroup of G.

Theorem 1.4.0.1. *$\mathfrak{X}_G$ can be represented by a scheme of infinite type.*

Proof. We present a quick sketch here. Start with the completed Kac-Moody Lie algebra $\widehat{\mathfrak{g}_{comp}}$ at t^{-1} considered as a Tate vector space (as the completion of finite-dimensional subspaces).

The Sato Grassmannian $\mathrm{Grass}(\widehat{\mathfrak{g}_{comp}})$ is the $\mathbb{C}$-space which assigns to any $\mathbb{C}$-algebra R the set

$$\{\Lambda \text{ an open bounded submodule of } \widehat{\mathfrak{g}_{comp}}\hat{\otimes} R \mid \widehat{\mathfrak{g}_{comp}}\hat{\otimes} R/\Lambda \text{ projective } R-\text{module }\}.$$

One can check that the Sato Grassmannian is a separated scheme. Furthermore, $\mathfrak{X}_G$ embeds naturally in $\mathrm{Grass}(\widehat{\mathfrak{g}_{comp}})$ as the $G((t^{-1}))$ orbit of the Iwahori subalgebra

$\widehat{\mathfrak{b}_-} \subset \widehat{\mathfrak{g}_{comp}}$. One can check that this is an embedding of schemes, giving $\mathfrak{X}_G$ its scheme structure. □

The main theorem concerning thick flag varieties that we will need is the following (it can be proven analogously to the Iwahori decomposition from the previous section).

Proposition 1.4.0.1. *Let W^{aff} be the affine Weyl group associated with the AKLA $\widehat{\mathfrak{g}}$. Let $X^w = IwI^-/I^-$ the I orbit through w and for a set of roots S of $\widehat{\mathfrak{g}}$ let $U(S) = \prod_{\alpha \in S} \widehat{\mathfrak{g}}_\alpha$. Then there is a decomposition*

$$\mathfrak{X}_G = \bigsqcup_{w \in W^{aff}} X^w$$

such that

(i) X^w is locally closed, isomorphic to

$$U(\Delta^- \cap w\Delta^-) \cong X^w$$

thus to $\mathbb{A}^\infty$, and $\operatorname{codim} X^w = l(w)$.

(ii) $\overline{X^w} = \bigcup_{v \geq w} X^v$.

(iii) $X_w \cong U(\Delta^+ \cap w\Delta^-) \subset \bigcup_{v \leq w} X^w$.

Remark 1.4.0.2. Note that the proposition gives an embedding of Fl_G into $\mathfrak{X}_G$.

A technical difficulty in working with the thick flag variety is that the orbits and orbit closures are non-Noetherian. We will see with the following two results that when working I-equivariantly, some of those difficulties disappear. The first one is the following easy corollary of Proposition 1.4.0.1.

Definition 1.4.0.2. Let S be a finite subset of W^{aff}. We say S is *open* if $w \in S$ and $v \leq w$ implies that $v \in S$.

Lemma 1.4.0.1. *If S is an open subset of W^{aff}, then the set*

$$\Omega_S := \bigsqcup_{w \in S} X^w$$

is an I-stable open subscheme of $\mathfrak{X}_G$. Furthermore, the collection of all Ω_S with S open forms an open cover of $\mathfrak{X}_G$

The main result we will be using later on is the following proposition, whose proof can be found in [KT95, VV09a]

Proposition 1.4.0.2. *Let l be a positive integer, and define $U_l \triangleleft I$ to be the subgroup generated by the positive roots of length $\geq l$.*
Then, for any open S, there exists an $l \gg 0$ such that the quotient $U_l \backslash \Omega_S$ is a separated smooth scheme of finite type.

Chapter 2

Coherent sheaves on affine flag varieties and their K-theory

Introduction

The goal of this section is twofold: firstly, to define (equivariant) derived categories of coherent sheaves on the affine Kac-Moody flag varieties constructed in the previous chapter. Secondly, equip these categories with a rigid monoidal structure induced by convolution. These constructions will be computed more explicitly in the case of the affine flag variety Fl_G. In particular, we will show that $K^I(\mathrm{Fl}_G)$ is isomorphic as an algebra under convolution with the Iwahori-Matsumoto Hecke algebra, and provide an algorithm to construct the duals of structure sheaves of Schubert varieties.

2.1 Equivariant coherent sheaves on affine Kac-Moody flag varieties

In this section, we define an equivariant derived category of coherent sheaves and construct a monoidal structure. We mostly follow the first section of [VV09b].

2.1.1 Derived categories of coherent sheaves on ind-schemes

Coherent sheaves on ind-schemes

Let X be an ind-scheme, given as a direct limit of quasi-compact schemes X_α, where $\alpha \in I$ a directed set with closed embeddings $i_{\alpha\beta} : X_\alpha \to X_\beta$. To obtain a well-behaved category of coherent sheaves, we need the following definition:

Definition 2.1.1.1. X is *ind*-coherent if it can be represented as the direct limit of coherent schemes with good embeddings.

Remark 2.1.1.1. Recall that a scheme X is *coherent* if its structure sheaf $\mathcal{O}_X$ is coherent. A closed embedding $Y \hookrightarrow X$ is then good if for any open set U, $\mathcal{I}_Y(U)$ is finitely generated in $\mathcal{O}_X(U)$.

Definition 2.1.1.2. Let $X = \operatorname{colim} X_\alpha$ be an ind-coherent ind-scheme, and consider the associated direct system of triangulated categories $(D^b(\mathrm{Coh}(X_\alpha)), i_{\alpha,\beta*})$. Then, we define the derived category of coherent sheaves on X by

$$D^b(\mathrm{Coh}(X)) := 2\operatorname{colim}_\alpha D^b(\mathrm{Coh}(X_\alpha)).$$

Remark 2.1.1.2. (1) From general considerations, this is a triangulated category.

(2) We will write an object of $D^b(\mathrm{Coh}(X))$ as a pair $(\mathcal{F}, \alpha)$ where $\mathcal{F} \in D^b(\mathrm{Coh}(X_\alpha))$.

Definition 2.1.1.3. The Grothendieck group of an ind-coherent ind-scheme X is defined as

$$K(X) := \mathbf{K}_0(D^b(\mathrm{Coh}(X))).$$

Pro-group actions, equivariant derived categories

Most of the spaces considered will be equipped with a group action. Therefore, we want to consider equivariant derived categories. However, the groups acting will in general be pro-linear groups, so we need to define the precise meaning of those.

Definition 2.1.1.4. Suppose an algebraic group G acts on an ind-coherent ind-scheme X. Suppose further that the ind-scheme X can be represented as the colimit of a system of quasi-compact G-invariant schemes X_α. Then we define the equivariant derived category as

$$D^b(\mathrm{Coh}^G(X)) := 2\,\mathrm{colim}_\alpha\, D^b(\mathrm{Coh}^G(X_\alpha))$$

Remark 2.1.1.3. In general, an ind-scheme with a G-action does not need to be stratified by G-schemes. If this holds, we call the ind-scheme a ind-G-scheme. Note that for instance the affine Grassmannian is a G-ind-scheme when using the orbit closure stratification.

Now, suppose we have a pro-linear group $G = \lim G^n$ where G^n is a finite-dimensional linear algebraic group. We would like to simplify the situation further by letting G act only through a given G^n on each X_α. A set of conditions for which this holds is the following:

Theorem 2.1.1.1. *Let $G = \lim G^n$ be a pro-group and suppose X is a ind-G-scheme. Suppose further that it can be represented by a system of G-invariant schemes of finite type X_α such that the action of G on X_α factors through some G^n for n large enough. Then there is a natural equivalence of triangulated categories*

$$2\,\mathrm{colim}_\alpha\, 2\,\mathrm{colim}_{n\geq n_\alpha}\, D^b(\mathrm{Coh}^{G^n}(X_\alpha)) \cong D^b(\mathrm{Coh}^G(X))$$

Remark 2.1.1.4. (1) This theorem is useful as the categories on the left hand side are all categories of coherent sheaves on a finite-dimensional scheme with finite dimensional group actions and therefore the standard theory of derived functors, coherent duality and so on can be used. As $\mathbf{K}_0$ commutes with colimits, we also get

$$K^G(X) := \mathbf{K}_0(D^b(\mathrm{Coh}^G(X))) \cong \mathrm{colim}_\alpha \, \mathrm{colim}_{n \geq n_\alpha} \mathbf{K}_0(D^b(\mathrm{Coh}^{G^n}(X_\alpha))).$$

(2) If $G_0 = \ker(G \to G^0)$ is pro-unipotent, we further get that $K^G(X) \cong K^{G^0}(X)$, which we will use implicitly throughout the text.

Example 2.1.1.1. *The thin flag varieties* $\mathfrak{X}^G(\mathcal{P})$ *acted on the left by* $\mathcal{P}$ *all satisfy the assumptions of the above theorem. To see this explicitly, we can note that on the affine Grassmannian* $\mathrm{Gr}_{\mathrm{GL}_n}$*, the endomorphism* t *acts nilpotently on each stratum* Gr^N *viewed as a set of lattices. Therefore, the* $\mathrm{GL}_n(\mathcal{O})$*-action factors through a finite dimensional quotient, and the general case then follows using an embedding* f_ρ *as in theorem 1.1.2.1. Furthermore, remark (2) above also applies to* $G_\mathcal{O}$ *and* I *(see [NP01, 2]). In particular, we'll freely use in the remainder of the text the isomorphisms* $K^I(\mathrm{Fl}_G) \cong K^T(\mathrm{Fl}_G)$ *and* $K^{G_\mathcal{O}}(\mathrm{Gr}_G) \cong K^G(\mathrm{Gr}_G)$

Global line bundles on ind-schemes

As the affine Grassmannian is a homogeneous space, one might want to extend certain results such as the Borel-Weil-Bott theorem to the infinite-dimensional case. So far, the sheaves considered on Gr_G have all been supported on finite-dimensional varieties, but the natural line bundles associated with characters are not. For this, we need the following definition ([Kum02]).

Definition 2.1.1.5. Let X be a G-ind-scheme equipped with a G-invariant filtration $X_n, n \in \mathbb{N}$ by finite-type schemes. Then a *G-equivariant vector bundle of rank r on X* is a G-equivariant map of ind-schemes $\pi : E \to X$ such that:

(1) The following diagram of ind-schemes commutes:

$$\begin{tikzcd}
G \times E \arrow[r] \arrow[d] & E \arrow[d] \\
G \times X \arrow[r] & X
\end{tikzcd}$$

and the action is linear on fibres.

(2) $E_n = \pi^{-1}(X_n)$ gives a G-invariant filtration of E.

(3) $E_n \to X_n$ is a G-equivariant vector bundle of rank r.

Example 2.1.1.2. *An important example is the following: consider an affine Kac-Moody flag variety $G_{\mathcal{K}}/\mathcal{P}$ equipped with its ind-scheme structure. To any character $\chi : \mathcal{P} \to \mathbb{C}^\times$ we can associate a line bundle $\mathcal{L}_\chi = G_{\mathcal{K}} \times_{\mathcal{P}} \mathbb{C}_\chi$ where $\mathcal{P}$ acts by right multiplication on $\mathcal{P}$ and by χ on $\mathbb{C}_\chi$.*

As with the finite-dimensional case, one can associate to a vector bundle $E \to X$ its sheaf of sections $\underline{E}$. Note that however $\underline{E} \notin \mathrm{Coh}(X)$ as it is not supported on a finite-dimensional stratum. However, we have the following:

Proposition 2.1.1.1. *Let E be a G-equivariant vector bundle over a G-ind-scheme X. Then, there is a well-defined exact functor*

$$\underline{E} \otimes - : D^{b,G}(\mathrm{Coh}(X)) \to D^{b,G}(\mathrm{Coh}(X)),$$

which is an auto-equivalence when E is a line bundle.

Proof. Let $(\mathcal{F}, n) \in D^G(\mathrm{Coh}(X))$. Then we define $\underline{E} \otimes \mathcal{F} = i_n^*(\underline{E}) \otimes \mathcal{F}$. This is well defined as X_n are finite-dimensional varieties, therefore the left derived tensor product exists and is bounded. If instead we take $m \geq n$ and $(i_{nm*}\mathcal{F}, m)$, the projection formula gives $i_m^*(\underline{E}) \otimes i_{nm*}\mathcal{F} \cong i_{nm*}(i_n^*\underline{E} \otimes \mathcal{F})$, which provides the required compatibilities. Exactness follows from reducing to the case of $\underline{E} = \mathcal{O}_X$.
Finally, if E is a line bundle, tensoring by its dual defined in the standard way provides the required inverse. $\square$

Remark 2.1.1.5. By the previous proposition, for any vector bundle E on X, we obtain an $R(G)$-module endomorphism on $K^G(X)$, which we will denote by $E \otimes -$, slightly abusing notation.

Definition 2.1.1.6. The G-Picard group of an ind-scheme X is the group $\mathrm{Pic}^G(X)$ of G-equivariant line bundles on X up to isomorphism, where the operation is given by tensor product.

Some thoughts on the derived category of $\mathfrak{X}_G$

For the thick flag variety, defining a suitable derived category of coherent sheaves is more difficult. We present some conjectures in this section.

By lemma 1.4.0.2, the categories

$$\mathrm{Coh}^I(\Omega_S) \cong \mathrm{Coh}_{I/U_l}(U_l \backslash \Omega_S)$$

are well-behaved triangulated categories for l large enough. However, by the orbit closure relations of 1.4.0.1, to get a triangulated category out of those, the natural candidate would be to define it as

$$D^b(\mathrm{Coh}^I(\mathfrak{X}_G)_{(1)} := 2\lim_{S} \mathrm{Coh}^I(\Omega_S).$$

However, the $2\lim$ of a system of triangulated categories with exact functors between them does not have to be triangulated. Bypassing conjecturing this, the equivariant K-theory is still well-defined, and has been used in the literature (see [KS09],[VV09b]). However, in chapter 3 we will be concerned with constructing a non-standard coherent t-structure induced from a t-structure on the orbits, which does require an underlying triangulated category.

Another direction one could look at is to take the limit in the ∞-category of *stable ∞-categories*. In this case, the limit is indeed a well-defined stable ∞-category, and

taking its homotopy category leads to a triangulated category.
A final candidate would be to consider the full unbounded derived category $D(\mathrm{QCoh}(\mathcal{X}^G))$ and to look at the triangulated subcategory of compact objects. It is a plausible candidate, as for a finite-dimensional smooth scheme X this gives the category of perfect complexes, which is equivalent to the derived category of coherent sheaves. We intend to explore those questions in further work.

2.1.2 Convolution

What we want to do

As mentioned in the introduction, for an algebraic group G and a closed subgroup H, certain categories of H-equivariant sheaves on the quotient G/H can be equipped with a monoidal structure, obtained by looking at the following diagram:

$$\begin{tikzcd}
G \times G/H \arrow[r, "q"] \arrow[d, "p"] & G \times_H G/H \arrow[d, "m"] \\
G/H \times G/H & G/H
\end{tikzcd}$$

where p is the quotient map on the first factor, q is the quotient by the middle H action (right action on G and left action on G/H), and m is the group multiplication.

For two "nice" sheaves $\mathcal{F}, \mathcal{G}$ on G/H , the convolution product should be given by the sheaf $\mathcal{F} \star \mathcal{G} = m_*(q^*)^{-1}p^*(\mathcal{F} \boxtimes \mathcal{G})$. The goal of this section is to make sense of this for equivariant coherent sheaves on affine flag varieties.

Remark 2.1.2.1. Note that H-equivariant sheaves on G/H are equivalent to sheaves on the stack $H \backslash G/H$. In the terminology of [BD91], this is called a *Hecke stack*, and categories of sheaves on such with the right restrictions will usually possess a monoidal structure.

Definition of the operations

Let $\mathcal{P} := \mathcal{P}_Y \subset G_{\mathcal{K}}$ be a parahoric subgroup corresponding to a subset Y of the simple roots. We will show that the category $D^b(\mathrm{Coh}^{\mathcal{P}}(G_{\mathcal{K}}/\mathcal{P}))$ admits a monoidal structure.

Start with two objects $(\mathcal{F}, w)$ and $(\mathcal{G}, v)$ in $D^b(\mathrm{Coh}^{\mathcal{P}}(G_{\mathcal{K}}/\mathcal{P}))$ and let $w \bullet v$ be the maximal elements in the poset $\{xy \mid x \leq w,\ y \leq v\} \subset W_{\mathcal{P}}^{aff}$, and consider the following diagram:

$$\begin{tikzcd}
\bigsqcup_{u\leq w} \mathcal{P}u\mathcal{P} \times \overline{X_v} \arrow[r, "q"] \arrow[d, "p"] & \overline{X_w}\tilde{\times}\overline{X_v} := \bigsqcup_{u\leq w} \mathcal{P}u\mathcal{P} \times_{\mathcal{P}} \overline{X_v} \arrow[d, "m"] \\
\overline{X_w} \times \overline{X_v} & \overline{X_{w\bullet v}}
\end{tikzcd}$$

where the maps are as in the previous part, and by convention, if $w \bullet v$ consists in more than one element, we take the union of the corresponding closures. Now, note that there is a natural projection

$$p_1 : \overline{X_w}\tilde{\times}\overline{X_v} \to \overline{X_w}.$$

Furthermore, by descent, we have an equivalence of categories

$$D^b(\mathrm{Coh}^{\mathcal{P}}(\overline{X_w}\tilde{\times}\overline{X_v})) \cong D^b(\mathrm{Coh}^{\mathcal{P}\times\mathcal{P}}(\bigsqcup_{u\leq w} \mathcal{P}u\mathcal{P} \times \overline{X_v})),$$

which gives a pullback morphism

$$p_2^* : D^b(\mathrm{Coh}^{\mathcal{P}}(\overline{X_v})) \to D^b(\mathrm{Coh}^{\mathcal{P}}(\overline{X_w}\tilde{\times}\overline{X_v})).$$

Definition 2.1.2.1. For $(\mathcal{F}, w), (\mathcal{G}, v) \in D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$, the convolution product is defined as

$$(\mathcal{F}, w) \star (\mathcal{G}, v) = (m_*(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}), w \bullet v).$$

Remark 2.1.2.2. In the remainder of the text, we will often write the convolution product as

$$\mathcal{F} \star \mathcal{G} = m_*(q^*)^{-1}p^*(\mathcal{F} \boxtimes \mathcal{G}).$$

This has the added advantage of being more canonical for categorical uses (one does not need to look at the stratification directly), but the disadvantage of not being clearly well-defined.

Proposition 2.1.2.1. *The convolution product defined in 2.1.2.1 gives a well-defined monoidal structure on* $D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$.

We will break down the proof in two steps: first we need to show that convolution is well-defined:

Lemma 2.1.2.1. *For* $\mathcal{F}, \mathcal{G} \in D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$, $\mathcal{F} \star \mathcal{G} \in D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$.

Proof. First note that all the maps used to define convolution are equivariant. Thus, we only have to show that m_* and p_i^* preserve coherence to get a well-defined operation.

- p_1 is a map of projective varieties, therefore its pullback preserves coherence.
- $m : \overline{X_w} \tilde{\times} \overline{X_v} \to \overline{X_{w \bullet v}}$ is a locally trivial fibration with projective fibres, thus is proper. Therefore its pushforward preserves coherence.
- $p_2 : \bigsqcup_{u \leq w} \mathcal{P}u\mathcal{P} \times \overline{X_v} \to \overline{X_v}$ is a trivial fibration with Noetherian fibres (the orbits $\mathcal{P}u\mathcal{P}$ are isomorphic to a quotient of $\mathcal{P} \times \mathcal{P}$ which is Noetherian as $\mathbb{C}[[t]]$ is), and therefore the pullback preserves coherence.

□

Lemma 2.1.2.2. *Convolution on* $D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$ *is associative and unital with left and right unit the structure sheaf of the orbit* $e\mathcal{P}$, $\mathcal{O}_e$.

Remark 2.1.2.3. The proof will follow the structure sketched in [BR18] for constructible sheaves on Gr_G.

Proof. First note that the structure sheaf $\mathcal{O}_e$ of the orbit $e\mathcal{P}$ is both the left and right unit.

For associativity, we will construct natural isomorphisms $(\mathcal{F}\star\mathcal{G})\star\mathcal{H} \cong C(\mathcal{F},\mathcal{G},\mathcal{H})$ and $\mathcal{F}\star(\mathcal{G}\star\mathcal{H}) \cong C(\mathcal{F},\mathcal{G},\mathcal{H})$ where $C(\mathcal{F},\mathcal{G},\mathcal{H})$ will be defined during the proof. First, note that the following diagram of ind-schemes is Cartesian:

$$\begin{tikzcd}
G_{\mathcal{K}} \times G_{\mathcal{K}} \times \mathfrak{X}^G \arrow[r, "m_{gp}\times 1"] \arrow[d, "\tilde{q}"] & G_{\mathcal{K}} \arrow[d, "p\times 1"] \\
G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \arrow[r, "m\times 1"] & \mathfrak{X}^G \times \mathfrak{X}^G
\end{tikzcd}$$

But by definition, we can write the convolution of the three (complexes of) sheaves as

$$(\mathcal{F}\star\mathcal{G})\star\mathcal{H} \cong m_*(q^*)^{-1}(p^*m_*(q^*)^{-1}(p^*\mathcal{F}\boxtimes\mathcal{G})\boxtimes\mathcal{H}).$$

Before using flat base change to compute p^*m_*, note that pushing forward under the map $m_{gp}\times 1$ is not a priori well-defined. However, as the sheaves are equivariant, we have the following equivalences:

First, $D^b(\mathrm{Coh}^{\mathcal{P}\times\mathcal{P}\times\mathcal{P}}(G_{\mathcal{K}} \times G_{\mathcal{K}} \times \mathfrak{X}^G)) \cong D^b(\mathrm{Coh}^{\mathcal{P}}(G_{\mathcal{K}} \times_{\mathcal{P}} G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G))$. Similarly, $D^b(\mathrm{Coh}^{\mathcal{P}\times\mathcal{P}}(G_{\mathcal{K}} \times \mathfrak{X}^G)) \cong D^b(\mathrm{Coh}^{\mathcal{P}}(G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G))$.

This leads to the commutative diagram of categories:

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\mathcal{P}}(G_{\mathcal{K}} \times_{\mathcal{P}} G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G)) \arrow[r, "\mu_{1*}"] & D^b(\mathrm{Coh}^{\mathcal{P}}(G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G) \\
D^b(\mathrm{Coh}^{\mathcal{P}\times\mathcal{P}\times\mathcal{P}}(G_{\mathcal{K}} \times G_{\mathcal{K}} \times \mathfrak{X}^G)) \arrow[u, "eq"] & D^b(\mathrm{Coh}^{\mathcal{P}\times\mathcal{P}}(G_{\mathcal{K}} \times \mathfrak{X}^G)) \arrow[u, "eq"] \\
D^b(\mathrm{Coh}^{\mathcal{P}\times\mathcal{P}}(G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \times \mathfrak{X}^G) \arrow[u, "q^*"] \arrow[r, "m_*\times 1"] & D^b(\mathrm{Coh}^{\mathcal{P}\times\mathcal{P}}(\mathfrak{X}^G \times \mathfrak{X}^G) \arrow[u, "p^*\times 1"]
\end{tikzcd}$$

where $\mu_1 : G_{\mathcal{K}} \times_{\mathcal{P}} G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \to G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G$ is the map $[g_1, g_2, g_3] \to [g_1g_2, g_3]$. From the diagram, we deduce that

$$(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong m_* \mu_{1*}(\mathcal{F} \tilde{\boxtimes} \mathcal{G} \tilde{\boxtimes} \mathcal{H})$$

Define $C(\mathcal{F}, \mathcal{G}, \mathcal{H}) = \mathcal{F} \tilde{\boxtimes} \mathcal{G} \tilde{\boxtimes} \mathcal{H}$. Therefore, as the composition $m_3 := m\mu_1$ is the multiplication map $[g_1, g_2, g_3] \mapsto [g_1 g_2 g_3]$, we deduce that

$$(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong m_{3*} C(\mathcal{F}, \mathcal{G}, \mathcal{H})$$

A similar reasoning applied to the map $\mu_2 : [g_1, g_2, g_3] \mapsto [g_1, g_2 g_3]$ leads to a natural isomorphism

$$\mathcal{F} \star (\mathcal{G} \star \mathcal{H}) \cong m_{3*} C(\mathcal{F}, \mathcal{G}, \mathcal{H})$$

which proves associativity. □

2.1.3 Duality

In this section, we generalise the construction of [CW19, 3] and show that the monoidal category $D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$ admits duals.

Motivation: the Geometric Satake equivalence

The definition of duals we will use is inspired by the Geometric Satake equivalence. It says that the category $P_{G_\mathcal{O}}(\mathrm{Gr}_G)$ of $G_\mathcal{O}$-equivariant perverse (constructible) sheaves on the affine Grassmannian under convolution is tensor equivalent to the category of representations of the Langlands dual group G^L. The proof of this equivalence uses Tannakian reconstruction, and in particular the fact that the category $P_{G_\mathcal{O}}(\mathrm{Gr}_G)$ is a rigid symmetric monoidal category. In the constructible case, duals can be constructed in the following way:

Note that inversion $g \mapsto g^{-1}$ induces an involution $i : G_{\mathcal{K}} \to G_{\mathcal{K}}$ and therefore an autoequivalence $i^* : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G) \to P_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$. The dual of a perverse sheaf $\mathcal{A}$ is then defined by $\mathcal{A}^\vee = \mathbb{D}(i^*\mathcal{A})$, where $\mathbb{D}$ is the *constructible* Verdier dual. For more details, see [MV07].

In the coherent case, the situation is more involved: the categories while still monoidal are no longer symmetric, so the Tannakian formalism cannot be applied. This will lead to a different notion of right and left duals. Note that however, the fundamental ingredient remains the inversion map on the group $G_{\mathcal{K}}$.

The Cautis-Williams dual

Theorem 2.1.3.1. *$D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$ is a rigid monoidal category.*

The monoidal structure given by convolution was defined in the previous section. It remains to define left and right duals and check the unit and counit axioms. This is done in [CW19] for the case of the affine Grassmannian, and their proof readily adapts to the affine Kac-Moody case with some minor modifications. We will point those out and define the unit and counit, leaving it to the interested reader to look at [CW19] for the bulk of the technical work. Concrete computations of the duals will be interspersed in the remainder of the text.

Consider the convolution space

$$\begin{tikzcd}
 & G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \arrow[dl, "p_1"'] \arrow[dr, "m"] & \\
\mathfrak{X}^G & & \mathfrak{X}^G
\end{tikzcd}$$

The map p_1 has a section, given by

$$s([g]) = (g, [g^{-1}]).$$

This is well-defined as if $h \in \mathcal{P}$, $s([gh]) = (gh, [h^{-1}g^{-1}]) = s([g])$.

Furthermore, the map s preserves the ind-structure: if we restrict s to an orbit closure $\overline{X_v}$, we obtain

$$\begin{tikzcd}
 & \overline{X_v}\tilde{\times}\overline{X_{v*}} \arrow[dl, "p_1"'] \arrow[dr, "m"] & \\
\overline{X_v} & & \overline{X_{v\bullet v^*}}
\end{tikzcd}$$

where we can chose $v^* \in W$ to be the unique minimal coset representative in $W_{\mathcal{P}}v^{-1}W_{\mathcal{P}}/W_{\mathcal{P}}$ such that $Iv^*\mathcal{P}/\mathcal{P} \hookrightarrow \overline{X_{v^{-1}}}$ is open.

Example 2.1.3.1. *For the affine Grassmannian, taking* $v = \lambda^\vee$ *to be a dominant coweight,* $v^* = -w_0\lambda^\vee$ *where* w_0 *is the longest element of the finite Weyl group* W*.*

Now, define $(-)^* = s^*(\mathcal{O}\tilde{\boxtimes}-)$ on $D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$.

Lemma 2.1.3.1. $(-)^*$ *is an involution on* $D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$*.*

Proof. For any $\mathcal{F} \in D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$, we want to construct an isomorphism $\mathcal{F} \cong s^*(\mathcal{O}\tilde{\boxtimes}s^*(\mathcal{O}\tilde{\boxtimes}\mathcal{F}))$.

Let $\tilde{s} : G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \to G_{\mathcal{K}} \times_{\mathcal{P}} G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G$ be given by $[g, h] \mapsto [g, h, h^{-1}]$.

Claim: $(\mathcal{O}\tilde{\boxtimes}s^*(\mathcal{O}\tilde{\boxtimes}\mathcal{F})) \cong \tilde{s}^*(\mathcal{O}\tilde{\boxtimes}\mathcal{O}\tilde{\boxtimes}\mathcal{F})$.

Indeed, consider the commutative square

$$\begin{tikzcd}
G_{\mathcal{K}} \times G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \arrow[r, "\tilde{q}"] & G_{\mathcal{K}} \times_{\mathcal{P}} G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \\
G_{\mathcal{K}} \times \mathfrak{X}^G \arrow[r, "q"] \arrow[u, "1\times s"] & G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G \arrow[u, "\tilde{s}"]
\end{tikzcd}$$

Then

$$\begin{aligned}
\mathcal{O}\tilde{\boxtimes}s^*(\mathcal{O}\tilde{\boxtimes}\mathcal{F}) &= (q^*)^{-1}(\mathcal{O}\boxtimes s^*(\mathcal{O}\tilde{\boxtimes}\mathcal{F}))\\
&= (q^*)^{-1}(1\times s^*)(\mathcal{O}\boxtimes \mathcal{O}\tilde{\boxtimes}\mathcal{F})\\
&= (q^*)^{-1}(1\times s^*)(\tilde{q}^*)(\mathcal{O}\tilde{\boxtimes}\mathcal{O}\tilde{\boxtimes}\mathcal{F})\\
&= \tilde{s}^*(\mathcal{O}\tilde{\boxtimes}\mathcal{O}\tilde{\boxtimes}\mathcal{F})
\end{aligned}$$

This proves the claim.

Let ϕ be the composition $\tilde{s}s$, which is the map $\mathfrak{X}^G \to G_{\mathcal{K}} \times_{\mathcal{P}} G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G$ given by $g \mapsto [g, g^{-1}, g]$. Then we have the following commutative diagram:

$$\begin{tikzcd}
G_{\mathcal{K}} \arrow[r, "\tilde{\phi}"] \arrow[d, "p"] & G_{\mathcal{K}} \times G_{\mathcal{K}} \times \mathfrak{X}^G \arrow[r, "p_3"] \arrow[d, "\tilde{q}"] & \mathfrak{X}^G \\
\mathfrak{X}^G \arrow[r, "\phi"] & G_{\mathcal{K}} \times_{\mathcal{P}} G_{\mathcal{K}} \times_{\mathcal{P}} \mathfrak{X}^G. &
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
s^*(\mathcal{O}\tilde{\boxtimes}s^*(\mathcal{O}\tilde{\boxtimes}\mathcal{F})) &= \phi^*(\mathcal{O}\tilde{\boxtimes}\mathcal{O}\tilde{\boxtimes}\mathcal{F})\\
&= \phi^*(\tilde{q}^*)^{-1}(\mathcal{O}\boxtimes\mathcal{O}\boxtimes\mathcal{F})\\
&= \phi^*(\tilde{q}^*)^{-1}(p_3^*\mathcal{F}).
\end{aligned}$$

By the diagram, $p\phi = \tilde{q}\tilde{\phi}$ and furthermore $p_3\tilde{\phi} = p$. Therefore

$$p^*(\mathcal{F}) = p^*(\phi^*(\tilde{q}^*)^{-1}(p_3^*(\mathcal{F})).$$

As p^* is fully faithful (p is flat and surjective), we get an isomorphism $\mathcal{F} \cong \phi^*(\tilde{q}^*)^{-1}p_3^*(\mathcal{F})) \cong s^*(\mathcal{O}\tilde{\boxtimes}s^*(\mathcal{O}\tilde{\boxtimes}\mathcal{F}))$. □

Let $\mathbb{D}$ be the coherent duality functor.

Definition 2.1.3.1. For $\mathcal{F} \in D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$, the (CW) left and right duals are defined by

$$\mathcal{F}^L = \mathbb{D}(\mathcal{F}^*)$$
$$\mathcal{F}^R = \mathbb{D}(\mathcal{F})^*.$$

For Theorem 2.1.3.1 to be true, we need to construct maps $\mathcal{F} \star \mathcal{F}^R \to \mathcal{O}_e$ and $\mathcal{O}_e \to \mathcal{F}^R \star \mathcal{F}$ (and similarly for $\mathcal{F}^L$). These are given as follows (see [CW19] for the equivalent maps using left duals):

Consider the composition

$$\mathcal{F}\tilde{\boxtimes}\mathcal{F}^R \to s_*s^*(\mathcal{F}\tilde{\boxtimes}\mathbb{D}(\mathcal{F})^*) \cong s_*(\mathcal{F} \otimes \mathbb{D}(\mathcal{F})) \to s_*\omega_{\overline{X_{v^*}}}$$

where the first map is the adjunction, the isomorphism comes from the previous lemma and the final map is the evaluation. Pushing forward under m, we get the required map

$$\mathcal{F} \star \mathcal{F}^R \to m_*s_*\omega \to \mathcal{O}_e$$

(note that the composition ms is the projection to the base point e).

To apply a similar strategy for the counit map, note that by adjunction, a map $s_*\mathcal{O}_{\overline{X_v}} \to \mathcal{F}^R\tilde{\boxtimes}\mathcal{F}$ is equivalent to a map $\mathcal{O}_{\overline{X_v}} \to s^!(\mathcal{F}^R\tilde{\boxtimes}\mathcal{F})$. But now,

$$s^!(\mathcal{F}^R\tilde{\boxtimes}\mathcal{F}) \cong \mathbb{D}s^*\mathbb{D}(\mathbb{D}(\mathcal{F})^*\tilde{\boxtimes}\mathcal{F}) \cong \mathbb{D}s^*(\mathbb{D}(\mathbb{D}(\mathcal{F})^*)\boxtimes\mathbb{D}(\mathcal{F})) \cong \mathbb{D}(\mathbb{D}(\mathbb{D}(\mathcal{F})^*) \otimes \mathbb{D}(\mathcal{F})^*)$$

Note that $\mathbb{D}(\mathbb{D}(\mathbb{D}(\mathcal{F})^*))\otimes\mathbb{D}(\mathcal{F})^*) \cong \mathit{End}(\mathbb{D}(\mathcal{F})^*)$ and therefore we have a map $\mathcal{O}_{\overline{X_v}} \to \mathbb{D}(\mathbb{D}(\mathbb{D}(\mathcal{F})^* \otimes \mathbb{D}(\mathcal{F})^*) \cong s^!(\mathcal{F}^R\tilde{\boxtimes}\mathcal{F})$. Taking the induced map on adjoints and pushing forward yields the desired map.

Remark 2.1.3.1. As can be inferred from the definition of the unit and the counit, the fact that those maps make $D^b(\mathrm{Coh}^{\mathcal{P}}(\mathfrak{X}^G))$ into a rigid monoidal category is far from obvious. The proof in section 3 of [CW19] is based on a series of technical compatibilities within the 6-functor formalism for quasi-coherent sheaves.

2.2 Equivariant K-theory of Fl_G

In this section, we describe the algebra $K^I(\mathrm{Fl}_G)$ under convolution using Demazure operators. We study further its relationship with the convolution algebra $K^G(\mathrm{Gr}_G)$. Throughout this section, assume G is simply-connected. The main theorem is the following:

Theorem 2.2.0.1. *Let $[\mathcal{O}_w]$ be the class of the structure sheaf of the Schubert variety X_w for $w \in W^{aff}$, viewed as an element of $K^I(\mathrm{Fl}_G)$. Then the set $\{[\mathcal{O}_w] \mid w \in W^{aff}\}$ generates $K^I(\mathrm{Fl}_G)$ as a left $R(T)$-module. Furthermore, if s is a simple reflection in the affine Weyl group W^{aff}, we have*

$$[O_w] \star_I [O_s] = \begin{cases} [O_{ws}] \text{ if } l(ws) > l(w) \\ [O_w] \text{ else.} \end{cases}$$

$K^I(\mathrm{Fl}_G)$ being generated by the classes of Schubert varieties is an application of the excision sequence in equivariant K-theory. The content of the theorem is in computing the convolution. This will follow from Proposition 2.2.2.1 below together with the results of [Sch16].

Remark 2.2.0.1.

(i) This can be viewed as an algebro-geometric analogue of [KK87], working with algebraic instead of topological K-theory.

(ii) This theorem is known to experts (it is stated in [FT17]), but as far as the author can tell no complete proof has appeared in the literature to date.

(iii) Thanks to the work of Schmidt [Sch16], the results in this section make sense over an algebraically closed field of positive characteristic. As this is not the focus of this work, we will remain in characteristic 0, but all proofs readily generalise.

2.2.1 Demazure operators

Warm-up: Demazure operators in the finite-dimensional case

Let (G, B, T) be a simple complex reductive group with a choice of Borus $T \subset B$. Following [Dem74], we can construct operators on $K^T(G/B)$ as follows.
Pick a simple reflection s in the finite Weyl group W associated with (G, B, T) and consider the associated parabolic subgroup $P_s = B \cup BsB$. The natural map $\pi_s : G/B \to G/P_s$ is a $\mathbb{P}^1$-bundle, and we can then define an operator on $K^T(G/B)$ by $D_s = \pi_s^* \pi_{s*}$. Here, we explain this construction in some detail, and will then show how it naturally generalises to affine flag varieties, following the work of [KK87].

Definition 2.2.1.1. Let X, Y be quasi-projective G-varieties and suppose $\pi : X \to Y$ is a G-equivariant $\mathbb{P}^1$-fibration. The operator $D = \pi^*\pi_* \in \mathrm{End}(K^G(X))$ is called a *Demazure operator.*

Let us first compute this for the trivial $\mathbb{P}^1$-bundle over a point in T-equivariant K-theory.

Lemma 2.2.1.1. *Let $G = \mathrm{SL}(2)$ with standard torus T and Borel subgroups B, and consider the Demazure operator associated with the map $\pi : \mathbb{P}^1 \to pt$. Then for any character $\lambda : T \to \mathbb{C}^\times$,*

$$D[\mathcal{L}(\lambda)] = [\mathcal{L}(D_s\lambda)]$$

in $K^T(\mathbb{P}^1)$ where

$$D_s(\lambda) = \frac{e^\lambda - e^{s\lambda}}{e^\alpha - 1},$$

for s the simple reflection generating the Weyl group and α the unique positive simple root.

Proof. Note that by definition $\pi_*[\mathcal{L}(\lambda)] = \sum_i (-1)^i [H^i(\mathbb{P}^1, \mathcal{L}(\lambda))]$. Furthermore, recall that the degree of the line bundle $\mathcal{L}(\lambda)$ is $\langle \lambda, \check{\alpha} \rangle$. Therefore,we can use our knowledge of the cohomology of line bundles on $\mathbb{P}^1$ to obtain the following

- If $\langle \lambda, \check{\alpha} \rangle \geq 0$, then the cohomology is concentrated in degree 0 and thus

$$\pi_* \mathcal{L}(\lambda) = H^0(\mathbb{P}^1, \mathcal{L}(\lambda)) = e^{\lambda} + e^{\lambda - \check{\alpha}} + \cdots + e^{s\lambda}.$$

- If $\langle \lambda, \check{\alpha} \rangle < -1$, we can use Serre duality: $H^1(\mathbb{P}^1, \mathcal{L}(\lambda)) = H^0(\mathbb{P}^1, \mathcal{L}(\alpha - \lambda))$, thus $\pi_*(\mathcal{L}(\lambda)) = -(e^{\lambda+\alpha} + \cdots + e^{s\lambda+\alpha})$.
- Finally if $\langle \lambda, \check{\alpha} \rangle = -1$, we get 0.

By separating cases, we observe that this is precisely $D_s(\lambda)$.

Thus

$$D[\mathcal{L}(\lambda)] = \pi^* D_s(\lambda) = D_s(\lambda) \otimes [\mathcal{O}].$$

Now, recalling that $[S^n\mathbb{C}^2 \otimes \mathcal{O}] = [\mathcal{O}(n)] + [\mathcal{O}(n-2)] + \cdots + [\mathcal{O}(-n)]$, and using the above expansions, we get the required formula. □

We can now state and prove the general case: (see [KL87, (o)]).

Proposition 2.2.1.1. *Let $\mathcal{V}$ be a T-equivariant rank 2 vector bundle over a Noetherian T-scheme Y, and consider the associated projective bundle $X = \mathbb{P}(\mathcal{V})$. Then if*

$\mathcal{L}$ is a line bundle with degree d along the fibres, in $K^T(X)$ we have

$$D\mathcal{L} = \begin{cases} \displaystyle\sum_{i=0}^{d} \mathcal{L} \otimes \Omega_{X/Y}^{\otimes i} & \textit{if } d \geq 0 \\ \displaystyle\sum_{i=1}^{-1-d} \mathcal{L} \otimes \Omega_{X/Y}^{\otimes i} & \textit{if } d < -1 \\ 0 & \textit{if } d = -1. \end{cases}$$

Proof. This follows from standard facts about dualizing sheaves of projective fibrations (see for instance [Har77, III, Ex 8.4]).First note that $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y) \times \mathbb{Z}$, where the isomorphism is given by pullback on the first factor and tensoring by the line bundle $\mathcal{O}_X(d)$ for $d \in \mathbb{Z}$. Now, $\mathcal{L}$ has degree d along the fibre, which means that we can write $\mathcal{L} = \pi^*\mathcal{L}_0 \otimes \mathcal{O}_X(d)$.

First, suppose $d \geq 0$. Then we have

$$\begin{aligned} D\mathcal{L} &= \pi^*\pi_*(\pi^*\mathcal{L}_0 \otimes \mathcal{O}_X(d)) \\ &= \pi^*(\mathcal{L}_0 \otimes \pi_*\mathcal{O}_X(d)) \\ &= \pi^*\mathcal{L}_0 \otimes \pi^* S^d\mathcal{V} \end{aligned}$$

where we have used the projection formula in the first line and loc. cit. in the second. Now, the relative Euler sequence gives a short exact sequence

$$0 \longrightarrow \Omega_{X/Y}(1) \longrightarrow \pi^*\mathcal{V} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0.$$

Therefore, in $K^T(X)$, we have

$$[\pi^* S^d\mathcal{V}] = \sum_{i=0}^{d} [\Omega_{X/Y}(1)^{\otimes i} \otimes \mathcal{O}_X(1)^{\otimes d-i}] = [\mathcal{O}_X(d)] \otimes \sum_{i=0}^{d} [\Omega_{X/Y}^{\otimes i}].$$

Combining the two leads to the first part of the proposition.

For $d \leq -1$, note that the derived pushforward is concentrated in degree 1, and therefore we need to compute the derived pushforward $R^1\pi_*\mathcal{O}(d)$. But by loc. cit. again, this is $\pi_*\mathcal{O}_X(-d-2)^\vee \otimes (\Lambda^2\mathcal{V})^\vee$. But $\Omega_{X/Y} \cong \pi^*(\Lambda^2\mathcal{V})(-2)$, so we can use the previous part. $\square$

Demazure operators in the affine case

We want to apply the previous section to the maps $\pi_i : \mathrm{Fl}_G \to \mathcal{G}/\mathcal{P}_i$ defined in Proposition 1.2.1.1. First note the following:

Proposition 2.2.1.2. *The quotient map* $\pi_i : \mathrm{Fl}_G \to \mathfrak{X}_i^G$ *naturally identifies*

$$\mathrm{Fl}_G \cong G_{\mathcal{K}} \times_{\mathcal{P}_i} \mathbb{P}(V_i)$$

where V_i *is the unique* 2*-dimensional representation of* $\mathcal{P}_i$ *which pulls back to the standard representation of* SL_2 *under the root homomorphism* ϕ_{α_i}.

Proof. Let $v_i \in V_i$ be the highest-weight vector in V_i and consider the map $\phi : \mathrm{Fl}_G \to G_{\mathcal{K}} \times_{\mathcal{P}_i} \mathbb{P}(V_i)$ given by $\phi(gI) = [g, \bar{v}_i]$. This is well-defined: if $g' = g\iota$ with $\iota \in I$, then as $I = \mathrm{Stab}(\bar{v}_i)$, $\iota\bar{v}_i = \bar{v}_i$ and therefore $[g', \bar{v}_i] = [g, \bar{v}_i]$.
Now an easy check shows that the map is injective and surjective. $\square$

Definition 2.2.1.2. For each simple reflection s_i in the affine Weyl group W^{aff}, the Demazure operator $D_i \in \mathrm{End}(K^T(\mathrm{Fl}_G))$ is defined by $D_i = \pi_i^*\pi_{i*}$.

Note that for these to make sense, we need to check compatibility with the ind-scheme structure given by the $\overline{X_w}$. The following lemma and its proof come from [Sch16].

Lemma 2.2.1.2. *The Demazure operators are well-defined.*

Proof. For any minimal parabolic subgroup $\mathcal{P}_i$, define a stratification of $\mathcal{G}/\mathcal{P}_i$ by $S_n(\mathcal{P}_i) = \bigcup_{w|\ l(w)=n} \overline{X_w(\mathcal{P}_i)}$. This induces a filtration on Fl_G by $\pi_i^{-1}(S_n(\mathcal{P}_i))$.

Note that the maps π_i are flat and proper and thus define operators

$$\pi_{i*} : K^T(\pi_i^{-1}(S_n(\mathcal{P}_i))) \to K^T(S_n(\mathcal{P}_i))$$

and similarly

$$\pi_i^* : K^T(S_n(\mathcal{P}_i)) \to K^T(\pi_i^{-1}(S_n(\mathcal{P}_i))).$$

Thus to get a well-defined global operator, we need to check that the Demazure operator commutes with the inclusions $i_{n-1} : \pi^{-1}(S_{n-1}(\mathcal{P}_i)) \hookrightarrow \pi_i^{-1}(S_n(\mathcal{P}_i))$ and $j_{n-1} : S_{n-1}(\mathcal{P}_i) \hookrightarrow S_n(\mathcal{P}_i)$.

Now, we have the Cartesian diagram

$$\begin{tikzcd}
\pi_i^{-1}(S_{n-1}(P_i)) \arrow[r, "\pi_i"] \arrow[d, "i_{n-1}"] & S_{n-1}(P_i) \arrow[d, "j_{n-1}"] \\
\pi_i^{-1}(S_n(P_i)) \arrow[r, "\pi_i"] & S_n(P_i)
\end{tikzcd}.$$

By flatness of π_i, base change gives $\pi_i^* j_{n-1*} = i_{n-1*}\pi^*$ and furthermore commutativity of the diagram leads to $\pi_{i*} i_{n-1*} = j_{n-1*}\pi_{i*}$. Combining the two gives the result. □

Proposition 2.2.1.3. *[KK87, Sch16] Let $w \in W$ and let $O_w = O_{S_w}$. Then we have*

$$D_i[O_w] = \begin{cases} [O_{ws_i}] \text{ if } l(ws_i) > l(w) \\ [O_w] \text{ else.} \end{cases}$$

Proof. There are two cases:

- Either $ws_i < w$. In this case, $\pi_i^{-1}\pi_i(\overline{X}_w) = \overline{X}_w$ and therefore the map π_i restricts to a (finite-dimensional) $\mathbb{P}^1$-bundle $\overline{X}_w \to \overline{X}_w(\mathcal{P}_i)$. By standard results on cohomology of $\mathbb{P}^1$-bundles, we deduce that $\pi_{i*}[\mathcal{O}_w] = [\mathcal{O}_w]$. Therefore, by flatness of π_i, $D_i[\mathcal{O}_w] = [\mathcal{O}_w]$.

- If $ws_i > w$, $\pi_i^{-1}\pi_i(\overline{X}_w) = \overline{X}_{ws_i}$. First notice the result holds trivially if w is the identity, so we consider $w \neq e$. Then the restriction of π_i to the open cells

  $$Iws_iI/I \to Iws_i\mathcal{P}_i/\mathcal{P}_i$$

  is an isomorphism, therefore the induced map $\pi : \overline{X}_{ws_i} \to \overline{X}_w(\mathcal{P}_i)$ is birational, and proper by construction. Therefore, as the varieties are normal, we obtain $\pi_{i*}[\mathcal{O}_w] = [\mathcal{O}_w]$ and therefore invoking flatness of π_i again, we get $D_i[\mathcal{O}_w] = [\mathcal{O}_{ws_i}]$.

□

2.2.2 The affine Hecke algebra and convolution

In this subsection, we relate the Demazure operators to convolution. First, note the following:

Lemma 2.2.2.1. *Let s_i be a simple reflection in W. Then $S_i = \overline{Is_iI/I} \cong \mathbb{P}(V_i)$, equivariantly for the left $\mathcal{P}_i$ action.*

Proof. Let v_i be the highest weight vector in V_i. As noted in Proposition 1.2.1.1, the stabiliser of $[v_i] \in \mathbb{P}(V_i)$ is I. Furthermore, $\overline{Is_iI/I} = (Is_iI \cup I)/I = \mathcal{P}_i/I$. As the action is transitive on the highest-weight vector, we deduce the result. □

Proposition 2.2.2.1. *For any element $F \in K^T(\mathrm{Fl}_G)$, $D_iF = F \star_I [\mathcal{O}_{s_i}]$.*

The main diagram we will use in this proof is the following:

$$\begin{tikzcd}
G_{\mathcal{K}} \times S_i \arrow[r, "q"] \arrow[d, "p"] & G_{\mathcal{K}} \times_I S_i \arrow[r, "q_i"] \arrow[d, "m"] & G_{\mathcal{K}} \times_{\mathcal{P}_i} S_i \arrow[d, "\pi_i"] \\
\mathrm{Fl}_G \times S_i & \mathrm{Fl}_G \arrow[r, "\pi_i"] & G_{\mathcal{K}}/\mathcal{P}_i
\end{tikzcd}$$

Proof. Let $\mathcal{F} \in D^b(\mathrm{Coh}^I(\mathrm{Fl}_G))$. We'll prove the result in two steps: first we identify the twisted product $\mathcal{F}\tilde{\boxtimes}\mathcal{O}_i$ with $q_i^*\mathcal{F}$, then we use flat base change on the right square of the above diagram.

(1) By 2.2.2.1 and 1.2.1.1, we have that $G_{\mathcal{K}} \times_{\mathcal{P}_i} S_i \cong \mathrm{Fl}_G$. Furthermore, the composition $q_i q : G_{\mathcal{K}} \times S_i \to \mathrm{Fl}_G$ under the isomorphism takes $(g, v) \mapsto gI$. Therefore $(q_i q)^*\mathcal{F} = (p\mathrm{pr}_1)^*\mathcal{F} = p^*(\mathcal{F} \boxtimes \mathcal{O}_i)$.

(2) By (1), we have

$$\begin{aligned} \mathcal{F} \star_I [\mathcal{O}_{s_i}] &= m_* q_i^* \mathcal{F} \\ &= \pi_i^* \pi_{i*} \mathcal{F} \\ &= D_i \mathcal{F} \end{aligned}$$

where we have used flat base change for the quotient maps q_i and π_i.

□

Lemma 2.2.2.2. *Convolution on* $K^T(\mathrm{Fl}_G)$ *is* $R(T)$*-linear. That is for* $\mathcal{A}, \mathcal{B} \in K^T(\mathrm{Fl}_G)$, *and* $\chi_1, \chi_2 \in R(T)$,

$$(\chi_1 \mathcal{A}) \star (\chi_2 \mathcal{B}) = \chi_1 \chi_2 (\mathcal{A} \star \mathcal{B}).$$

Proof. Recall that the module structure comes from pulling back along the projection to a point. Let V_χ be the representation of T (viewed as a T-bundle over a point) associated with the character χ. For any $\mathcal{F}, \mathcal{G} \in D^{b,I}(\mathrm{Fl}_G)$, we want to compute $(p_4^* V_\chi \otimes \mathcal{F}) \star \mathcal{G}$, where p_4 is defined in the following diagram.

Now, consider the following diagram, where p_i are the natural projections to a point.

$$\begin{tikzcd}
G_{\mathcal{K}} \times \mathrm{Fl}_G \arrow[rr, "q"] \arrow[dd, "p"'] \arrow[dr, "p_2"] & & G_{\mathcal{K}} \times_I \mathrm{Fl}_G \arrow[dl, "p_3"'] \arrow[dd, "m"] \\
& pt & \\
\mathrm{Fl}_G \times \mathrm{Fl}_G \arrow[ur, "p_1"] & & \mathrm{Fl}_G \arrow[ul, "p_4"]
\end{tikzcd}$$

By definition, we have

$$\begin{aligned}
(p_4^*V_\chi \otimes \mathcal{F}) \star \mathcal{G} &= m_*(q^*)^{-1}(p^*(p_4^*V_\chi \otimes \mathcal{F}) \boxtimes \mathcal{G}) \\
&= m_*(q^*)^{-1}((p^*(\mathcal{F}) \boxtimes \mathcal{G}) \otimes p_2^*V_\chi) \\
&= m_*((q^*)^{-1}((p^*(\mathcal{F}) \boxtimes \mathcal{G})) \otimes p_3^*V_\chi) \\
&= m_*((q^*)^{-1}((p^*(\mathcal{F}) \boxtimes \mathcal{G}) \otimes m^*p_4^*V_\chi) \\
&= m_*(q^*)^{-1}((p^*(\mathcal{F}) \boxtimes \mathcal{G}) \otimes p_4^*V_\chi,
\end{aligned}$$

where we have used the above diagram throughout and the projection formula in the second to last line. $R(T)$-linearity in the right factor follows analogously. ☐

2.2.3 An aside: convolution and structure constants

There are two natural ways of generating $K^T(\mathrm{Fl}_G)$: on the one hand, we can use the structure sheaves of Schubert varieties $[\mathcal{O}_w]$ as an $R(T)$ basis. On the other hand, we can consider the equivariant line bundles $\mathcal{L}(\chi)$ associated with a character χ, and restrict those to Schubert varieties, which form a $\mathbb{C}$-basis. Thus we can write

$$\mathcal{L}(\chi) \otimes [\mathcal{O}_w] = \sum_{v \leq w} c_v^{\chi,w} [\mathcal{O}_v].$$

The main result presented in this section is a combinatorial algorithm to compute the coefficients $c_v^{\chi,w}$. As a consequence, this allows us to refine our description of the convolution product on the affine flag variety. Note that these results appear elsewhere in the literature, starting in the finite dimensional case with Pittie and Ram in '98

[PR04].

Let $L_\lambda \in End_{R(T)}(K^T(\mathrm{Fl}_G))$ be the operator given by tensoring by $\mathcal{L}(\lambda)$ and let

$$\mathbf{D}_i\lambda = \frac{e^\lambda - e^{s_i\lambda}}{1 - e^{-\alpha_i}} \in R(T).$$

Theorem 2.2.3.1. *Let $x \in K^T(\mathrm{Fl}_G)$. Then*

$$L_\lambda(D_i x) = D_i(L_{s_i\lambda} x) + L_{\mathbf{D}_i\lambda}(x).$$

Proof. The class of x is supported on some Schubert variety X_w. We want to use Proposition 2.2.1.1 but the Schubert varieties are not smooth. To remedy this, we can use BDSH resolutions. Take $w \in W^{aff}$ and suppose that $l(ws_i) < l(w)$ (the other case follows analogously). Write $w = w's_i$ where $w' \in W_i^{'}$ and pick a reduced decomposition of w with associated Bott-Samelson varieties $Z_{\underline{w}}$, $Z_{\underline{w'}}$.

Then we have the following diagram, where the left square is Cartesian:

$$\begin{tikzcd}
Z_w \arrow[r, "m_{\underline{w}}"] \arrow[d, "\Pi_i"] & \overline{X_w} \arrow[r] \arrow[d, "\pi_i"] & \mathrm{Fl}_G \arrow[d, "\pi"] \\
Z_{w'} \arrow[r, "m^i_{\underline{w'}}"] & \overline{X^i_{w'}} \arrow[r] & G_{\mathcal{K}}/\mathcal{P}_i.
\end{tikzcd}$$

Here $m_{\underline{w}} : Z_w \to \overline{X_w}$ and $m^i_{\underline{w'}} : Z_{w'} \to \overline{X^i_{w'}}$ are BDSH resolutions in Fl_G and $G_{\mathcal{K}}/\mathcal{P}_i$ respectively, π_i is the restriction of the projection $\pi : \mathrm{Fl}_G \to G_{\mathcal{K}}/\mathcal{P}_i$ to $\overline{X_w}$ and $\Pi_i : Z_w \to Z_{w'}$ is the natural quotient map. By flat base change, we have:

$$\begin{aligned}
\pi^*\pi_*\mathcal{L}(\lambda)_w &= \pi^*\pi_* m_{w*}\mathcal{L}(\lambda)_{\underline{w}} \\
&= \pi^* m^i_{\underline{w'}*}\Pi_{i*}(\mathcal{L}(\lambda)_{\underline{w}})
\end{aligned}$$

$$= m_{\underline{w}*}\Pi_i^*\Pi_{i*}(\mathcal{L}(\lambda)_{\underline{w}})$$
$$= m_{\underline{w}*}\mathcal{L}(D_i\lambda)_{\underline{w}}.$$

□

The combinations of the above theorem with Proposition 2.2.1.3 allows us to compute the change of bases coefficients between the two bases: indeed, we see

Corollary 2.2.3.1. *Let* $w = w_1 s_i \in W$ *be a reduced decomposition where* s_i *is a simple reflection. Then*

$$L_\lambda[\mathcal{O}_w] = D_i(L_{s_i\lambda}[\mathcal{O}_{w'}]) + L_{\mathbf{D}_i\lambda}[\mathcal{O}_{w'}].$$

Thus we get the following algorithm: write

$$[\mathcal{L}(\lambda)] \otimes [\mathcal{O}_w] = \sum_{w' \leq w} c_{w'}^{\lambda,w}[\mathcal{O}_{w'}],$$

where $c_{w'}^{\lambda,w} \in R(T)$. Then we can compute the coefficients $c_{w'}^{\lambda,w}$ recursively:

- $c_e^{\lambda,e} = e^{-\lambda}[\mathcal{O}_e]$
- Fix a simple reflection s such that $l(ws) = l(w) + 1$. Then

$$c_{w'}^{\lambda,ws} = \begin{cases} c_{w''}^{s\lambda,w} + c_{w'}^{D_s(\lambda),w} & \text{where } w''s = w' \text{ and } w'' \leq w \\ c_w^{s\lambda,w} & \text{if } w' = ws. \end{cases}.$$

For example, $c_s^{\lambda,s} = e^{-s\lambda}$, $c_e^{\lambda,s} = e^{-s\lambda} + D_s(\lambda)$.

As a corollary, we obtain an expression for the convolution product in $K^T(\mathrm{Fl}_G)$:

Corollary 2.2.3.2.

$$\mathcal{L}_{\lambda,w} \star \mathcal{L}_{\lambda',w'} = \sum_{\tilde{w} \leq w, \bar{w} \leq w'} c_{\tilde{w}}^{\lambda,w} c_{\bar{w}}^{\lambda',w'} [\mathcal{O}_w] \star [\mathcal{O}_{w'}]$$

where $[\mathcal{O}_w] \star [\mathcal{O}_{w'}] = D_{s_n} \dots D_{s_1}([\mathcal{O}_w])$ *and* $w' = s_1 \dots s_n$ *is a reduced decomposition.*

Remark 2.2.3.1. Note that the above result leaves open the question of interpreting the right hand side in terms of a $\mathbb{C}$-linear combination of classes $[\mathcal{L}_{\lambda,w}]$. As far as we know, this is an open problem.

2.2.4 Duals for the affine flag variety

Now that we have a description of the algebra $K^T(\mathrm{Fl}_G)$, we can ask for a description of the right and left duals of the structure sheaves $\mathcal{O}_w$. First, we need to know what the dualising sheaves of Schubert varieties are.

Lemma 2.2.4.1. *[Kum17]*

$$\omega_{X_w} = e^{-\rho}\mathcal{L}(-\rho)\mathcal{O}_{X_w}(-\partial X_w)[l(w)],$$

where $\mathcal{O}_{X_w}(-\partial X_w)$ *is the ideal sheaf of the boundary of* $\overline{X_w}$.

Proposition 2.2.4.1. *Write* $[\mathcal{O}_{\partial X_w}] = \sum_{v<w} r_{v,w}[\mathcal{O}_v]$ *in* $K^T(\mathrm{Fl}_G)$. *Then we have*

$$\begin{cases} (\mathcal{O}_w)^L & = (-1)^{l(w)} e^{-\rho}([\mathcal{O}_{w^{-1}}] - \sum_{u \leq v < w^{-1}} r_{v,w^{-1}} c_u^{v,-\rho}[\mathcal{O}_u]) \\ (\mathcal{O}_w)^R & = (-1)^{l(w)} e^{-\rho}([\mathcal{O}_{w^{-1}}] - \sum_{u \leq v < w} r_{v,w} c_u^{v,-\rho}[\mathcal{O}_{u^{-1}}]). \end{cases}$$

Remark 2.2.4.1. While we have a recursive formula for $c_v^{u,-\rho}$, we have not yet described how to compute the coefficients $r_{v,w}$. One can check using [Kum17, BK06] that $r_{v,w} = (-1)^{l(w)-l(v)+1}$.

Proof. Note that $(\mathcal{O}_w)^* = \mathcal{O}_{w^{-1}}$ as we are pulling back the structure sheaf along the map $X_{w^{-1}} \to X_{w^{-1}} \tilde{\times} X_w$. Therefore, using the fact that $\mathbb{D}(\mathcal{O}_w) = \omega_{X_w}$, we deduce that

$$(\mathcal{O}_w)^L = [\omega_{X_{w^{-1}}}].$$

Now, the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X_w}(-\partial X_w) \longrightarrow \mathcal{O}_w \longrightarrow \mathcal{O}_{\partial X_w} \longrightarrow 0$$

leads to the equality

$$[\omega_{X_w}] = (-1)^{l(w)} e^{-\rho} \mathcal{L}(-\rho)([\mathcal{O}_w] - \sum_{v<w} r_{v,w}[\mathcal{O}_v]).$$

Finally, using the previous section, we can write $\mathcal{L}(-\rho)[\mathcal{O}_v] = \sum_{u \leq v} c_u^{v,-\rho}[\mathcal{O}_u]$ which allows us to conclude the result for left duals.

The case of right duals follows analogously, where we first use the expansion of the canonical sheaf in the structure sheaf basis and then apply $(-)^*$. □

Example 2.2.4.1. *Applying the above formula for a simple reflection s_i, we see that*

$$[\mathcal{O}_{s_i}^L] = [\mathcal{O}_{s_i}^R] = -[\mathcal{O}_{s_i}] + e^{-2\rho+\alpha_i}[\mathcal{O}_e].$$

2.2.5 Pulling back from the affine Grassmannian

First recall the following corollary of Proposition 1.2.1.1

Corollary 2.2.5.1. *The quotient map $\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G$ is a $G_{\mathcal{K}}$-equivariant locally trivial fibration with fibres G/B.*

Therefore, pulling back gives a functor $\pi^* : \mathcal{D}^b(\mathrm{Coh}^I(\mathrm{Gr}_G)) \to \mathcal{D}^b(\mathrm{Coh}^I(\mathrm{Fl}_G))$ which induces an embedding $\pi^* : K^T(\mathrm{Gr}_G) \to K^T(\mathrm{Fl}_G)$.

A natural question now is to try to relate the convolution algebra structures arising from both sides: note that $K^G(\mathrm{Gr}_G)$ embeds in $K^T(\mathrm{Gr}_G)$ as the Weyl invariants (see [CG10]), and admits a convolution product, but so does $K^T(\mathrm{Fl}_G)$. The precise relationship is as follows:

Proposition 2.2.5.1. *Let* $F : \mathcal{D}^b(\mathrm{Coh}^G(\mathrm{Gr}_G)) \to \mathcal{D}^b(\mathrm{Coh}^I(\mathrm{Gr}_G))$ *be the forgetful functor. Then for any* $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b(\mathrm{Coh}^G(\mathrm{Gr}_G))$*, we have isomorphisms*

$$\pi^* F(\mathcal{F} \star \mathcal{G}) \cong \pi^* F(\mathcal{F}) \star_I \pi^* F(\mathcal{G}).$$

Proof. Firstly, consider the convolution diagrams for the affine Grassmannian and affine flag variety, together with the composition of the pullback map and the forgetful map π_f^*.

$$\begin{tikzcd}
D^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}} \times \mathrm{Gr}_G) & D^{G_{\mathcal{O}}}(G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G) \arrow[l, "q^*"'] \arrow[d, "m_*"] \\
D^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(\mathrm{Gr}_G \times \mathrm{Gr}_G) \arrow[u, "p^*"] \arrow[d, "\pi_f^* \times \pi_f^*"] & D^{G_{\mathcal{O}}}(\mathrm{Gr}_G) \arrow[d, "\pi_f^*"] \\
D^I(\mathrm{Fl}_G \times \mathrm{Fl}_G) \arrow[d, "p_I^*"] & D^I(\mathrm{Fl}_G) \\
D^{I \times I}(G_{\mathcal{K}} \times \mathrm{Fl}_G) & D^I(G_{\mathcal{K}} \times_I \mathrm{Fl}_G) \arrow[l, "q_I^*"'] \arrow[u, "m_{I*}"']
\end{tikzcd}$$

Using the notation from the diagram, we want to show that for $\mathcal{F}, \mathcal{G} \in D^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$, there is an isomorphism

$$m_{I*}(q_I^*)^{-1}(p_I^* \pi_f^* \mathcal{F} \boxtimes \pi_f^* \mathcal{G}) \cong \pi_f^* m_*(q^*)^{-1}(p^* \mathcal{F} \boxtimes \mathcal{G}).$$

First, notice that we can rewrite the left hand side as

$$m_{I*}(q_I^*)^{-1}(F(p^* \mathcal{F}) \boxtimes \pi_f^* \mathcal{G}).$$

Secondly, the following square commutes

$$\begin{tikzcd}
D^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}} \times \mathrm{Gr}_G) \arrow[d, "F \times \pi_f^*"] & D^{G_{\mathcal{O}}}(G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G) \arrow[l, "q^*"'] \arrow[d, "F \circ \tilde{\pi}^*"] \\
D^{I \times I}(G_{\mathcal{K}} \times \mathrm{Fl}_G) & D^I(G_{\mathcal{K}} \times_I \mathrm{Fl}_G). \arrow[l, "q_I^*"']
\end{tikzcd}$$

Therefore, as q^* and q_I^* are equivalences, the above expression becomes

$$m_{I*}(F \circ \tilde{\pi}^*)(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

Thirdly, by flat base change, we obtain the following commutative square:

$$\begin{tikzcd}
D^I(G_{\mathcal{K}} \times_I \mathrm{Fl}_G) \arrow[d, "m_{I*}"] & D^I(G_{\mathcal{K}} \times_I \mathrm{Gr}_G) \arrow[l, "\pi_{II}^*"'] \arrow[d, "\tilde{m}_{I*}"] \\
D^I(\mathrm{Fl}_G) & D^I(\mathrm{Gr}_G). \arrow[l, "\pi^*"']
\end{tikzcd}$$

Letting $u : G_{\mathcal{K}} \times_I \mathrm{Gr}_G \to G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G$ be the natural quotient map, the above diagram implies that

$$m_{I*}(F \circ \tilde{\pi}^*)(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \cong \pi^* \tilde{m}_{I*} u^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

Finally, we can decompose

$$\tilde{m}_{I*} : G_{\mathcal{K}} \times_I \mathrm{Gr}_G \to G_K \times_{G_{\mathcal{O}}} \mathrm{Gr}_G \to \mathrm{Gr}_G .$$

Therefore, as u is a locally trivial fibration with projective fibres G/B_- and therefore is proper, we obtain

$$\pi^* \tilde{m}_{I*} u^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \cong \pi_f^* m_* u_* u^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

Using the projection formula combined with local triviality of u, we can conclude. □

Chapter 3

Perverse and staggered coherent sheaves on Kac-Moody flags

Introduction

In this chapter, we introduce the categories of perverse coherent sheaves and of staggered sheaves on a scheme equipped with an algebraic group action. Provided they exist, these provide interesting non-standard t-structures inside equivariant derived categories. We then specialise to the case of the affine Grassmannian and of the affine flag variety. In particular, we show that perverse coherent sheaves on the affine Grassmannian are stable under convolution, mimicking the classical argument in the constructible case and following [BFM05]. We then construct staggered t-structures on Schubert varieties inside the thick and thin affine flag variety, generalising the techniques of [AS09] to the infinite-dimensional case.

3.1 Perverse and staggered coherent t-structures

In this section, we will describe two similar but distinct ways of equipping the equivariant derived category of coherent sheaves on a finite type scheme with a t-structure.

In both cases, we will study when the corresponding heart is of finite length.

3.1.1 Setup

Let X be a finite-type scheme over $\mathbb{C}$ equipped with an action of a complex linear algebraic group G. We will be concerned with the derived category of G-equivariant coherent sheaves on X, $D^{*,G}_{coh}(X)$, defined as the subcategory of $D^{*,G}(QCoh(X))$ consisting of complexes with coherent cohomology, where $* \in \{+,-,b\}$. Any functor will be understood as derived, and any subscheme as G-invariant.

For any closed subscheme $Z \subset X$, we have the following functors $D^{b,G}_{coh}(X) \to D^{b,G}_{coh}(Z)$:

$$i^*_Z = \mathcal{O}_Z \otimes (-)$$
$$i^!_Z = Hom_{\mathcal{O}_X}(\mathcal{O}_Z, -)$$

We will also consider the (equivariant) coherent duality functor $\mathbb{D}$, whose existence is proved in [AB10], associated with a dualizing complex $\mathbb{DC}$.

Finally, we will consider the associated functors of $\mathbb{C}_X$-modules, given by

$$\mathbf{i}^*_Z = \mathbb{C}_Z \otimes (-)$$
$$\mathbf{i}^!_Z = Hom_{\mathbf{C}_X}(\mathbb{C}_Z, -).$$

For any point x, we assume that the complex $\mathbf{i}^!_x \mathbb{DC}$ is concentrated in a single cohomological degree. In Section 3.1.2, this will be normalized to be $-\dim x$.

We write X^{gen} for the set of generic points of G-orbits, and will often interchange between an orbit and its generic point.

3.1.2 The Bezrukavnikov-Deligne theory

The results in this section are due to Bezrukavnikov-Deligne, and are explained in the more general context of stacks in [AB10]. Let X^{top} be topological space underlying X (the reader might like to think of X^{top} as the set of generic points of locally closed subschemes of X). Define an equivalence relation on X^{top} by $x \sim y$ if x, y are in the same G-orbit

Definition 3.1.2.1.

A *perversity* p is a function $p : X^{top} \to \mathbb{Z}$ which is constant on the equivalence classes of $\sim$.

The *dual perversity* associated with p is the perversity $\overline{p}(x) = -dim(x) - p(x)$.

A perversity is

- *monotone* if $x \in \overline{x'} \implies p(x') \geq p(x)$
- *strictly monotone* if $x \in \overline{x'} \implies p(x') > p(x)$
- *comonotone* if the dual perversity is monotone.

Associated to a perversity, define the following subcategories:

Definition 3.1.2.2. Let p be a perversity.

$$D^{p,\leq 0} := \{\mathcal{F} \in D^{-,G}(\mathrm{Coh}(X)) \mid \mathbf{i}_x^* \mathcal{F} \in D^{\leq p(x)}(\mathcal{O}_x)\}$$
$$D^{p,\geq 0} := \{\mathcal{F} \in D^{+,G}_{coh}(\mathrm{QCoh}(X)) \mid \mathbf{i}_x^! \mathcal{F} \in D^{\geq p(x)}(\mathcal{O}_x)\}$$

Theorem 3.1.2.1. *If the perversity p is monotone and co-monotone, then the pair*

$$(D^{p,\leq 0} \cap D^{b,G}_{coh}(X), D^{p,\geq 0} \cap D^{b,G}_{coh}(X))$$

forms a t-structure on $D^{b,G}_{coh}(X)$.

Definition 3.1.2.3. For p a monotone and co-monotone perversity, objects in the heart of the t-structure obtained by Theorem 3.1.2.1 are called *perverse coherent sheaves.*

In this case, we also have a description of the simple objects, called coherent IC-sheaves.

Theorem 3.1.2.2. *Suppose p is strictly monotone and comonotone. Then the following are equivalent:*

(1) $\mathcal{F}$ *is a simple perverse coherent sheaf*

(2) There exists a G-orbit $j : O \to X$ *and a G-equivariant line bundle* $\mathcal{L}$ *on* O *such that* $\mathcal{F}$ *is the unique perverse coherent sheaf with the property* $j^*\mathcal{F} = \mathcal{L}[p(O)]$.

Remark 3.1.2.1. In keeping with the constructible case, the perverse coherent sheaf $\mathcal{F}$ is called a coherent IC-sheaf, and written $\mathcal{IC}(O, \mathcal{L})$.

Corollary 3.1.2.1. *If* G *acts on* X *with finitely many orbits and p is monotone and comonotone, then the category of perverse coherent sheaves* $\mathcal{P}^G$ *is Artinian.*

Example 3.1.2.1. *(1) If we take* $\mathcal{N}$ *to be the nilpotent cone associated with a Lie algebra* $\mathfrak{g}$ *and acted on by an algebraic group* G *with the perversity given by* $p(O) = -\dim(O)/2$, *we get an Artinian heart. This is studied in [Bez03].*

(2) Similarly, in the affine Grassmannian Gr_G, *dimensions of* $G_{\mathcal{O}}$*-orbit closures differ by even numbers, thus the middle perversity* $p(\mathrm{Gr}_\lambda) = -\dim(\mathrm{Gr}_\lambda)/2$ *(allowing for half-integer shifts) gives an Artinian heart on the derived category* $D^{b,G_{\mathcal{O}}}_{coh}(\overline{\mathrm{Gr}_\lambda})$. *The corresponding coherent IC-sheaves will be written* $\mathcal{P}_{\lambda,\chi}$ *for* $\chi : T \to \mathbb{C}^\times$ *a character which is dominant for the Levi factor of the parabolic* P_λ.

In many classical examples, such as flag varieties, there will be no obvious perversity function that is both monotone and comonotone. In the following two subsections, we will present an analogous theory due to P. Achar [Ach09], called *staggered sheaves*. In it, we incorporate weights of the action in the definition of the perversity. This will lead to t-structures with finite length hearts in a wider class of examples.

3.1.3 Staggered coherent sheaves

In this subsection, we review the definitions and main theorems of the theory of staggered sheaves, as introduced by Achar in [Ach09]. Most of the result proved in that paper have been later generalised using *baric structures* in [AT11]. However, we have chosen to stick with the original definitions, as they do not rely on the number of orbits being finite. While this will not be needed in this text, it is our belief that this theory has a better chance of giving a global t-structure on the thick flag variety, which is why we choose it here (see the end of the chapter for a more thorough discussion of this point).

s-structures

An s-structure on X is a way of filtering the category of coherent sheaves by well-behaved subcategories. The definition is very lengthy, so we will only write the few axioms that will be needed in the rest of the text.

Definition 3.1.3.1. An s-structure on X is a collection of full subcategories of $\mathrm{Coh}^G(X)$, denoted

$$(\mathrm{Coh}^G(X)_{\leq w}, \mathrm{Coh}^G(X)_{\geq w})),$$

labelled by integers $w \in \mathbb{Z}$ satisfying the axioms S1-S9 in [Ach09]. In particular

- $\mathrm{Coh}^G(X)_{\leq w}$ is a Serre subcategory, $\mathrm{Coh}^G(X)_{\leq w} \subset \mathrm{Coh}^G(X)_{\leq w+1}$.

- For a fixed $w \in \mathbb{Z}$, $(\mathrm{Coh}^G(X)_{\leq w}, \mathrm{Coh}^G(X)_{\geq w+1})$ is a torsion pair in the abelian category $\mathrm{Coh}^G(X)$.
- For every object $\mathcal{F} \in \mathrm{Coh}^G(X)$ there exists w_1, w_2 such that $\mathcal{F} \in \mathrm{Coh}^G(X)_{\leq w_1}$ and $\mathcal{F} \in \mathrm{Coh}^G(X)_{\geq w_2}$.
- If $\mathcal{F} \in \mathrm{Coh}^G(X)_{\leq w_1}$ and $\mathcal{G} \in \mathrm{Coh}^G(X)_{\leq w_2}$, then $\mathcal{F} \otimes \mathcal{G} \in \mathrm{Coh}^G(X)_{\leq w_1+w_2}$.

Example 3.1.3.1. *If we take $X = pt$, the category $\mathrm{Coh}^G(pt)$ becomes naturally equivalent to the category of finite dimensional representations of G, $Rep(G)$. If G is reductive and (B, T) is a choice of torus and Borel subgroup, one can obtain an s-structure by picking a dominant weight μ and letting*

$$Rep(B)_{\leq n} = \{V \mid \langle \lambda, \mu \rangle \leq n \text{ for all the weights } \lambda \text{ of } V\}.$$

The main theorem we will use to construct s-structures is the following:

Theorem 3.1.3.1. *[AS09] Suppose G acts on X with finitely many orbits. For each orbit $C \subset X$, suppose that the category $\mathrm{Coh}^G(C)$ is equipped with an s-structure. Then there exists a unique s-structure on $\mathrm{Coh}^G(X)$ which restricts to the various s-structures on the orbits provided that*

$$i_C^* \mathcal{I}_C|_C \in \mathrm{Coh}^G(C)_{\leq -1},$$

where $i_C : \overline{C} \to X$ is the closed embedding and $\mathcal{I}_C$ is the ideal sheaf of $\overline{C}$ in X.

Staggered sheaves

The construction of the staggered t-structure will be similar to the one of perverse coherent sheaves, but the perversity function will be modified to incorporate the s-structure. First we need the following definition:

Definition 3.1.3.2. For X irreducible, the *altitude*, denoted by $\operatorname{alt} X$ is the unique integer such that for any open subset $V \subset X$ such that $\mathbb{DC}|_V$ is concentrated in a single degree d, $\mathbb{DC}|_V[d] \in \mathrm{Coh}^G(V)_{\geq \operatorname{alt} X}$ and $\mathbb{DC}|_V[d] \notin \mathrm{Coh}^G(V)_{\geq \operatorname{alt} X+1}$.

Remark 3.1.3.1. By our assumptions, there is an open cover of X such that all open sets in that cover have the property of the definition.

Definition 3.1.3.3. The *staggered codimension* of an irreducible closed G-invariant subscheme $Y \subset X$ is given by

$$\operatorname{scod} Y = \operatorname{codim} Y + \operatorname{alt} Y.$$

Remark 3.1.3.2. The codimension of an orbit closure $\overline{G.y}$ is given by the cohomological degree of $\mathbf{i}_y^! \mathbb{DC}$. For a subscheme Y, we take the minimal codimension over all orbit closures of generic points.

Equipped with our notion of codimension, we can now define the perversities we will work with:

Definition 3.1.3.4. A (staggered) perversity function is a map $p : X^{gen} \to \mathbb{Z}$ which is

- Monotone: for any $x \in \overline{Gy}$, $p(x) \geq p(y)$
- Comonotone: define the dual function $\overline{p}(x) = \operatorname{scod}(\overline{Gx}) - p(x)$. Then we require $\overline{p}$ to be monotone as well.

Now we have all the necessary data to define our t-structure.

Definition 3.1.3.5. Let p be a (staggered) perversity function. Define the subcategory of $D^{-,G}(\mathrm{Coh}(X))$ by

$$D_{-}^{G,p,\leq 0} = \left\{ \mathcal{F} \in D^{-,G}(\mathrm{Coh}(X)) | \begin{array}{c} \text{for any } x \in X^{gen},\ i_Z : Z = \overline{Gx} \hookrightarrow X, \\ Li_Z^* \mathcal{F} \in D^{G,-,\leq p(x)}(\mathrm{Coh}(X)) \end{array} \right\}$$

Now define the categories $D^{G,p,\leq 0} = D^{G,p,\leq 0} = D_-^{G,p,\leq 0} \cap D^{G,b}(\mathrm{Coh}(X))$ and let $D^{G,p,\geq 0} = \mathbb{D}(D_-^{G,\overline{p},\leq 0}) \cap D^{b,G}(\mathrm{Coh}(X))$.

Theorem 3.1.3.2. *The pair* $(D^{G,p,\leq 0}, D^{G,p,\geq 0})$ *forms a non-degenerate bounded t-structure on* $D^{b,G}(\mathrm{Coh}(X))$*.*

Definition 3.1.3.6. Objects of the heart of the t-structure defined in Theorem 3.1.3.2 are called *staggered sheaves.*

Simple objects

In both the perverse coherent t-structure and the staggered t-structure, we can obtain a description of the simple objects provided the perversity function satisfies some constraints.

Remark 3.1.3.3. Supposing that the number of orbits is finite, Achar and Treumann in their paper [AT11] showed that the heart of the staggered t-structure is always of finite length, regardless of the perversity. As mentioned earlier, the results stated in this text however use a different, earlier construction of Achar ([Ach09]), which has the advantage of working even in the case where the number of orbits is infinite, but which imposes restrictions on the staggered codimension. This will require us to choose our perversities more carefully.

Theorem 3.1.3.3. *Suppose the perversity is strictly monotone and comonotone. Then, the category of staggered sheaves has finite length, and there is a bijection*

$$\{\textit{Simple staggered sheaves}\} \leftrightarrow \{(C, \mathcal{L}) \mid C \textit{ an orbit}\,, \mathcal{L} \textit{ an irreducible vector bundle on}\, C\}.$$

Remark 3.1.3.4. The simple staggered sheaf corresponding to $(C, \mathcal{L})$ is denoted by $\mathcal{IC}(C, \mathcal{L})$. It is supported on $\overline{C}$ and has the property that

$$\mathcal{IC}(C, \mathcal{L})|_C = \mathcal{L}[-d + \operatorname{step} \mathcal{L}]$$

where d is given in the definition of altitude, and $\operatorname{step}\mathcal{L}$ is just the s-structure grade of $\mathcal{L}$ in $\mathrm{Coh}^G(C)$.

Remark 3.1.3.5. Note that if $\operatorname{scod}(Gx) - \operatorname{scod}(Gy) > 1$, then there always exists a strictly monotone and comonotone perversity. In the final section of this chapter, we will use this inequality in order to apply the theorem.

3.1.4 Stratified semismall maps

Let X, Y be schemes over $\mathbb{C}$ acted on by a linear algebraic group G and admitting equivariant dualizing complexes. Fix two G-invariant stratifications $\mathcal{X}$ and $\mathcal{Y}$ on X, Y respectively.

Definition 3.1.4.1. A proper G-map $f : (Y, \mathcal{X}) \to (X, \mathcal{Y})$ is *stratified semismall* if for each stratum $T \subset f(U)$ and $x \in T$, we have

$$2 \dim f^{-1}(x) \cap U + \dim T \leq \dim U$$

Pushforward

Denote by $\mathcal{X}_G$ the stratification of X by G-orbits (which we assume are locally closed).

Theorem 3.1.4.1. *Let X, Y be G-schemes with even-dimensional orbits, and let f be a flat, proper stratified semismall map $Y \to X$ with respect to a stratification $(X, \mathcal{X}_G)$, $(Y, \mathcal{Y}_G)$. Then the pushforward f_* maps $P_{coh}(Y) \to P_{coh}(X)$ where both categories are equipped with the middle perversity.*

Proof. We will first show that $f_! = f_*$ is right t-exact, and use duality to conclude. Let $x \in X^{gen}$ and consider the orbit O_x. We want to show that if $F \in D^{p,\leq 0}$, $i^*_{O_x} f_! F \in D^{\leq p(x)}$. We have the following fibre diagram:

$$\begin{tikzcd}
f^{-1}(O_x) \arrow[r, "i_Y"] \arrow[d, "f_x"] & Y \arrow[d, "f"] \\
O_x \arrow[r, "i_x"] & X.
\end{tikzcd}$$

By flat base change, we have

$$i_x^* f_! F \cong f_{x!} i_Y^* F.$$

Now $f^{-1}(O_x)$ is locally closed, so by [AB10, Lemma 3.3], $i_Y^* F \in D^{p_{f^{-1}(O_x)},\leq 0}$. Similarly, if $U \in \mathcal{Y}$, we get $i_U^* F \in D^{p_{f^{-1}(O_x)\cap U},\leq 0}$.

By [AB10, Lemma 2.21], we can find a cover of $f^{-1}(O_x) \cap U$ by open sets U_y with generic points y such that $i_{U_y}^* F \in D^{\leq p(y)}$.

Now by semismallness, $2\dim(f^{-1}(x) \cap U_y) + \dim O_x \leq \dim f^{-1}(O_x) \cap U_y$, which we can rewrite as

$$\dim f^{-1}(x) \cap U_y + p(y) \leq p(x).$$

But $f|_{U_y*}$ has homological dimension $\leq \dim f^{-1}(x) \cap U_y$, therefore

$$f|_{U_y*} F \in D^{\leq p(x)},$$

as required.

To prove that f_* is left t-exact, we use the fact that $\mathbb{D}$ interchanges $D^{p,\leq 0}$ and $D^{p,\geq 0}$ and that $\mathbb{D} f_* \mathbb{D} = f_!$. □

3.1.5 Perverse coherent sheaves on the affine Grassmannian

The goal of this subsection is to give a proof of the following theorem, and of its corollaries.

Theorem 3.1.5.1. *[BFM05] The convolution map m is proper and stratified semismall. Therefore $P_{coh}^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ is a monoidal category.*

First note that the convolution space $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G$ is stratified by the varieties $\tilde{\mathrm{Gr}}_{\lambda,\mu} = \overline{\mathrm{Gr}_\lambda} \tilde{\times} \overline{\mathrm{Gr}_\mu}$ so the statement indeed makes sense.

The proof

We follow the argument from [BR18, MV07], assuming the following technical lemma from [MV07].

Lemma 3.1.5.1. *For any dominant coweight* λ*, and* $X \subset \overline{\mathrm{Gr}_\lambda}$ *a* T*-invariant subvariety,*

$$\dim(X) \leq \max_{\mu \ \mid \ t^\mu \in X} \langle \rho, \lambda + \mu \rangle.$$

Lemma 3.1.5.2. *For any dominant coweights* λ, μ, ν*,*

$$\dim(\tilde{\mathrm{Gr}}_{\lambda,\mu} \cap m^{-1}(t^{-\nu})) \leq \langle \rho, \lambda + \mu - \nu \rangle.$$

Proof. We want to use the previous lemma, so we need to study T-fixed points. Note the map $(p_1, m) : \ G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G$ is an isomorphism, T-equivariant for the left torus action. By Chapter 1, we can deduce that the T-fixed points of the convolution space $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G$ are of the form $[t^\alpha, t^\beta]$ for some coweights α, β. Therefore, the fixed points inside the stratum $\tilde{\mathrm{Gr}}_{\lambda,\mu}$ are of the same form, with the additional constraint that the dominant conjugate of α is $\leq \mu$ and the dominant conjugate of β is $\leq \mu$.

Finally, note that the restriction of the map (p_1, m) to $\tilde{\mathrm{Gr}}_{\lambda,\mu} \cap m^{-1}(t^{-\nu})$ gives an isomorphism between the latter and a subvariety of $\mathrm{Gr}_G \times \{t^{-\nu}\}$. Therefore by the previous lemma, we deduce that

$$\dim(\tilde{\mathrm{Gr}}_{\lambda,\mu} \cap m^{-1}(t^{-\nu})) \leq \max_{(\alpha,\beta) \ \mid \ [t^\alpha, t^\beta] \in \tilde{\mathrm{Gr}}_{\lambda,\mu} \cap m^{-1}(t^{-\nu})} \langle \rho, \lambda + \alpha \rangle,$$

giving the result. ☐

Using this lemma, we can finish the proof of the theorem.

Proof of Theorem 3.1.5.1. As mentioned in 2.1.2, the map m is proper and locally trivial so for $\mathcal{F}, \mathcal{G} \in P_{coh}^{G_{\mathcal{O}}}$, $\mathcal{F}\tilde{\boxtimes}\mathcal{G} \in P_{coh}^{G_{\mathcal{O}}}(G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G)$. Therefore, we can use Theorem 3.1.4.1, i.e. we need to show that m is stratified semismall.
Recalling that the map m is left $G_{\mathcal{O}}$-equivariant, we see that the previous lemma precisely gives the required inequality. □

Some consequences

Now that the category $P_{coh}^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ has been equipped with a monoidal structure, we can look at convolution of simple perverse coherent sheaves.

Proposition 3.1.5.1. *[CW19] Let $\lambda_1^\vee, \lambda_2^\vee$ be dominant coweights and χ_1, χ_2 weights, dominant for the corresponding Levi factors. Then in $K^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$, we have*

$$[\mathcal{P}_{\lambda_1^\vee,\chi_1} \star \mathcal{P}_{\lambda_2^\vee,\chi_2}] = [\mathcal{P}_{\lambda_1^\vee+\lambda_2^\vee,\chi_1+\chi_2}] + \sum_{(\lambda^\vee,\chi)} c_{\lambda^\vee,\chi}[\mathcal{P}_{\lambda^\vee,\chi}]$$

where the sum is taken over dominant pairs where $\lambda^\vee < \lambda_1^\vee + \lambda_2^\vee$ or $||\chi||^2 \leq ||\chi_1||^2 + ||\chi_2||^2$ for some W-invariant bilinear form.

Proof. The main observation is that the map $m : \overline{\mathrm{Gr}_{\lambda_1}} \tilde{\times} \overline{\mathrm{Gr}_{\lambda_2}} \to \overline{\mathrm{Gr}_{\lambda_1+\lambda_2}}$ is birational, and restricts to an isomorphism on the open orbit $\mathrm{Gr}_{\lambda_1+\lambda_2}$.
Now, by Theorem 3.1.5.1, $\mathcal{P}_{\lambda_1,\chi_1} \star \mathcal{P}_{\lambda_2,\chi_2}$ is a perverse coherent sheaf, and therefore its restriction to the open orbit $\mathrm{Gr}_{\lambda_1+\lambda_2}$ is a $G_{\mathcal{O}}$-equivariant vector bundle up to shifts. To determine the corresponding character, we can restrict to the fixed points: note that the restriction of the convolution map is just the multiplication $t^{\lambda_1} \tilde{\times} t^{\lambda_2} G_{\mathcal{O}} \to t^{\lambda_1+\lambda_2} G_{\mathcal{O}}$, therefore the fibre of the vector bundle at the fixed point $t^{\lambda_1+\lambda_2}$ is just $V_{\chi_1} \otimes V_{\chi_2}$, where V_{χ_i} are the corresponding representations. Decomposing this tensor product as a sum of irreducible representations leads to the result. □

In Section 2.1.3 we have constructed duals on the derived category of coherent sheaves. This behaves well under perverse coherence:

Lemma 3.1.5.3. *[CW19] The category $P^{G_\mathcal{O}}_{coh}(\mathrm{Gr}_G)$ is stable under left and right dual. In particular, it forms a rigid monoidal category.*

Proof. From [AB10], we know that the category of perverse coherent sheaves is stable under Verdier duals. It remains to check that it is stable under $(-)^*$. Let $\mathcal{F} \in P^{G_\mathcal{O}}_{coh}(\mathrm{Gr}_G)$. This means that for all dominant coweights λ, $i^*_\lambda\mathcal{F}$ is supported in degrees $\leq 1/2 \dim \mathrm{Gr}_\lambda$ and $i^!_\lambda\mathcal{F}$ is supported in degrees $\geq 1/2 \dim \mathrm{Gr}_\lambda$.
Now, note that $(\mathcal{O}\tilde{\boxtimes}\mathcal{F}) \in \mathrm{Gr}_\lambda \tilde{\times} \mathrm{Gr}_{\lambda^*}$ is supported in degrees $\leq 1/2 \dim \mathrm{Gr}_{\lambda^*}$ as the map $\mathrm{Gr}_\lambda \tilde{\times} \mathrm{Gr}_{\lambda^*} \to \mathrm{Gr}_\lambda$ is a locally trivial fibration. Now the map s is a closed embedding, and therefore the result follows from [Huy06, Lemma 3.29]. The corresponding result for $i^!$ can be obtained by coherent duality. □

Finally, the following lemma, credited to Mirkovic in [CW19] will allow us in Chapter 4 to construct some simple perverse coherent sheaves.

Lemma 3.1.5.4. *Let $\mathcal{L}$ be a $G_\mathcal{O}$-equivariant line bundle on Gr_G. Then for any dominant coweight λ, the restriction*

$$\mathcal{L}|_{\overline{\mathrm{Gr}_\lambda}}[\frac{1}{2}\dim \mathrm{Gr}_\lambda]$$

is a simple perverse coherent sheaf. Furthermore, its square under convolution is also simple.

Remark 3.1.5.1. Simple perverse coherent sheaves $\mathcal{P}$ such that $\mathcal{P} \star \mathcal{P}$ is also simple are called *real*. This will be important when we consider cluster categorifications in chapter 4.

3.2 Staggered sheaves on thin and thick Schubert varieties

In [AS09], Achar and Sage constructed staggered t-structures on the finite-dimensional partial flag varieties and showed that these are all of finite length. In this section, we adapt their arguments to the infinite-dimensional setting. In particular, for any element w in the affine Weyl group we will construct a finite-length staggered t-structure on the category $D^b(\mathrm{Coh}^I(\overline{X_w}))$, where $\overline{X_w}$ is a Schubert variety inside the affine flag variety Fl_G. Similarly, we will construct a finite-length staggered t-structure on the category $D^b(\mathrm{Coh}^I(\Omega_S))$ of coherent sheaves on the open affine Ω_S inside the thick affine flag variety $\mathfrak{X}_G$. We also expect our discussion to generalise easily to the case of parabolic affine flag varieties.

While the thin and thick flag varieties may seem very different from a geometric standpoint, the proofs of these two results are actually very similar, and boil down to the same affine Weyl group combinatorics. Following Theorem 3.1.3.1, once we have defined our s-structure, the main step will be to compute the associated staggered codimension and show that it allows for stricly monotone and comonotone perversities.

3.2.1 Staggered perversities on affine Schubert cells

In this section, as in chapter 1, we use W^{aff} to mean the affine Weyl group associated with an affine Lie algebra. The roots, weights etc. are the ones corresponding to the affine root system. We use angled brackets to denote both the pairing between roots and coroots, and a choice of the normalized invariant form on $\mathfrak{h}^*$ (which exists as the Lie algebras considered are symmetrizable, see [Kum02, Chapter 1]). Let $w \in W^{aff}$ and choose a weight ρ such that $\langle \rho, \alpha_i \rangle = 1$ for all the affine simple roots. Now consider the associated Iwahori orbit closure $\overline{X_w}$. By Chapter 1, we know that we can write

$$\overline{X_w} = \bigsqcup_{v \leq w} X_v.$$

We want to use theorem 3.1.3.1 to construct an s-structure on $D^b(\mathrm{Coh}^I(\overline{X_w}))$. First note that $\mathrm{Coh}^I(X_w) \cong Rep(I^w \cap I)$ which contains $Rep(T)$ where $I^v = wIw^{-1}$. Thus to a character χ of $I^w \cap I$, we get an associated line bundle $\mathcal{L}(\chi)$. We will need the following lemma:

Lemma 3.2.1.1. *(1) For any $v \in W^{aff}$, the dualizing sheaf ω_{X_v} of the orbit X_v is T-equivariantly isomorphic to the line bundle $\mathcal{L}(-\theta(v^{-1}))$, where*

$$\theta(v) = \sum_{\alpha \in \Delta^+ \cap v\Delta^-} \alpha.$$

(2) Furthermore, for $v \leq w$, let $\mathcal{I}_v$ be the ideal sheaf of the orbit $i_v : X_v \hookrightarrow \overline{X_w}$. Then we have

$$i_v^* \mathcal{I}_v \cong \bigoplus_{\alpha \in \Pi(v,w)} \mathcal{L}(\alpha)$$

where

$$\Pi(v,w) = (\Delta^+ \cap w^{-1}\Delta^-) \backslash (\Delta^+ \cap v^{-1}\Delta^-).$$

Proof. Note that we have an isomorphism

$$X_v \cong I/I^v \cap I.$$

Under this isomorphism, the tangent space at the point vI can be identified with the tangent space of the identity of I/I^v. But this is just

$$T_e I/I^v \cap I \cong \frac{\mathfrak{b} + t\mathfrak{g}[[t]]}{w.(\mathfrak{b} + t\mathfrak{g}[[t]]) \cap (\mathfrak{b} + t\mathfrak{g}[[t]])}.$$

Now using the Cartan decomposition of the (affine) Lie algebra, we see that the only weights which are not in the quotient are precisely the $\alpha \in \Delta^+$ such that $w.\alpha < 0$. Taking the top exterior power of the dual gives the result.

Part (2) follows from (1) by noting that $i_v^*\mathcal{I}_v$ is the conormal bundle of X_v in $\overline{X_w}$. $\square$

These two observations allow us to construct an s-structure on $\mathrm{Coh}^I(\overline{X_w})$: for any $w \in W^{aff}$, we let

$$\mathrm{Coh}^I(X_w)_{\leq n} = \{V \in \mathit{Rep}(I^w \cap I) \mid \langle \lambda, -2w^{-1}\rho\rangle \leq n \text{ for all the weights } \lambda \text{ of } V\}$$
$$\mathrm{Coh}^I(X_w)_{\geq n} = \{V \in \mathit{Rep}(I^w \cap I) \mid \langle \lambda, -2w^{-1}\rho\rangle \geq n \text{ for all the weights } \lambda \text{ of } V\}.$$

Corollary 3.2.1.1. *There is a unique s-structure on* $\mathrm{Coh}^I(\overline{X_w})$ *compatible with the one described above.*

Proof. From Theorem 3.1.3.1 this follows provided that $i_v^*(\mathcal{I}_v) \in \mathrm{Coh}^I(X_v)_{\leq -1}$. But if $\alpha \in \Pi(v, w)$, $v\alpha > 0$, so $\langle \alpha, -2v\rho\rangle < 0$ as required. $\square$

Furthermore, we can find the staggered codimension with respect to the dualizing complex $\omega_{\overline{X_w}}$: from the first part of the lemma, we can see that for $v \leq w$,

$$\operatorname{scod} X_v = l(w) - l(v) - \langle \theta(v), 2\rho\rangle$$

(note that $v^{-1}\theta(v) = -\theta(v^{-1})$).

Theorem 3.2.1.1. *With respect to the s-structure and dualizing complex as above,* $D^b(\mathrm{Coh}^I(\overline{X_w}))$ *admits an Artinian staggered t-structure. The simple objects in the heart are the staggered IC-sheaves* $\mathcal{IC}(v, \chi)$ *for* χ *a character of* T *and* $v \leq w$.

Corollary 3.2.1.2. $K^I(\mathrm{Fl}_G)$ *admits a basis as* $\mathbb{C}$*-vector space formed by staggered IC-sheaves.*

3.2.2 Proof of Theorem 3.2.1.1

By Theorem 3.1.3.3, the proof of the theorem reduces to showing that $\operatorname{scod}\overline{X_{v_1}} - \operatorname{scod}\overline{X_{v_2}} > 1$. This reduces further to showing

Proposition 3.2.2.1. *For any $v_1 \leq v_2$ in W^{aff}, we have*

$$\langle \theta(v_2) - \theta(v_1), 2\rho \rangle > 0$$

where θ was defined in Lemma 3.2.1.1.

We will adapt the proof from [AS09] to the affine case. Most of the arguments will remain the same, but there are a few key differences which will arise.
The strategy for the proof is to use induction on the length of the Weyl group.

Lemma 3.2.2.1. *For any $v \in W^{aff}$*

$$\theta(v) = \rho - v\rho.$$

Thus for any simple reflection $s \in W^{aff}$ with respect to a root α, we have

$$\theta(sv) = s\theta(v) + \alpha.$$

Proof. The first part is [Kum02, 1.3.22]. The second part follows from noting that $s\rho = \rho - \alpha$. □

We let $\Theta(v) = \Delta^+ \cap v\Delta^-$ so that $\theta(v) = \sum_{w \in \Theta(v)} w$, $\pi(v) = \rho + v\rho$ and $\Pi(v) = \{\alpha \in \Delta^+ | v\alpha > 0\}$.

Lemma 3.2.2.2. *For any $v \in W^{aff}$, $\langle \theta(v), \pi(v) \rangle = 0$.*

Proof. From Lemma 3.2.2.1 and the definition of π, the inner product is

$$\langle \theta(v), \pi(v) \rangle = \langle \rho - v\rho, \rho + v\rho \rangle.$$

The lemma then follows from Weyl invariance of the inner product. □

Lemma 3.2.2.3. *If $\alpha \in \Pi(v)$ is a simple root, then*

$$\langle \alpha, \theta(v) \rangle \leq 0.$$

Proof. Let s be the simple reflection corresponding to α and $\beta \in \Theta(v)$. There are three cases:

- If $s\beta = \beta$, then $\langle \alpha, \beta \rangle = 0$.
- If $s\beta \neq \beta$ but $s\beta \in \Theta(v)$, then $s(\beta + s\beta) = \beta + s\beta$ and thus from the previous part $\langle \alpha, \beta + s\beta \rangle = 0$.
- Now, if $s\beta \notin \Theta(v)$, we want to show $\langle \alpha, \beta \rangle \leq 0$. Suppose for contradiction that $\langle \alpha, \beta \rangle > 0$. As $\alpha \in \Pi(v)$, we can write $\alpha = v\alpha_+$ where $\alpha_+ \in \Delta^+$. Then we have

  $$\begin{aligned} \langle s\beta, v(2\rho) \rangle &= \langle \beta, v(2\rho) \rangle - \langle \alpha^\vee, \beta \rangle \langle \alpha, v(2\rho) \rangle \\ &= \langle v^{-1}\beta, 2\rho \rangle - \langle \alpha^\vee, \beta \rangle \langle \alpha_+, 2\rho \rangle. \end{aligned}$$

  Note that by assumption $\langle \alpha, \beta \rangle > 0$, thus from the above equation we deduce that $\langle s\beta, v(2\rho) \rangle < 0$. But this means that $s\beta \in v\Delta^-$ as well as $s\beta \in \Delta^+$ (as $\alpha \neq \beta$ is simple). Thus $s\beta \in \Theta(v)$, which contradicts our assumption.

Combining those three cases, we see that for any $\beta \in \Theta(v)$, $\langle \alpha, \beta \rangle \leq 0$ and the result follows. □

Lemma 3.2.2.4. *Let s be a simple reflection corresponding to a simple root α, and suppose $v, w \in W^{aff}$ are such that $l(vsw) = l(v) + l(w) + 1$. Then*

$$\langle \theta(vsw) - \theta(vw), 2\rho \rangle = (1 - \langle \alpha^\vee, \theta(v^{-1}) \rangle) \langle w^{-1}\alpha, 2\rho \rangle > 0.$$

Proof. We proceed by induction on the length of v. For $l(v) = 1$, $v = e$, and thus $\theta(v^{-1}) = 0$. Furthermore using Lemma 3.2.2.1 and the fact that $s(\rho) = \rho - \alpha$

$$\begin{aligned}\langle \theta(sw) - \theta(w), 2\rho\rangle &= \langle \rho - sw\rho - \rho + w\rho, 2\rho\rangle \\ &= \langle w\rho, 2\rho\rangle - \langle sw\rho, 2\rho\rangle \\ &= \langle w\rho, 2\rho\rangle - (\langle w\rho, 2\rho\rangle - \langle 2w\rho, \alpha\rangle) \\ &= \langle w^{-1}\alpha, 2\rho\rangle.\end{aligned}$$

Furthermore $l(sw) > l(w)$, thus $w^{-1}\alpha \in \Delta^+$, and hence this expression is strictly positive.

Now, take a general $v \in W^{aff}$ such that $l(v) \geq 1$ and write it as tx where t is a simple reflection with respect to β, and x is such that $l(tx) = l(t) + l(x)$. Using Lemma 3.2.2.1 again, we find

$$\begin{aligned}\langle \theta(vsw) - \theta(vw), 2\rho\rangle &= \langle \theta(txsw) - \theta(txw), 2\rho\rangle \\ &= \langle t\theta(xsw) - t\theta(xw), 2\rho\rangle \\ &= \langle \theta(xsw) - \theta(xw), 2\rho - 2\beta\rangle.\end{aligned}$$

The first term of the expansion is given from the induction hypothesis, thus it remains to calculate

$$\begin{aligned}\langle \theta(xsw) - \theta(xw), \beta\rangle &= \langle xw\rho - xsw\rho, \beta\rangle \\ &= \langle w\rho - sw\rho, x^{-1}\beta\rangle \\ &= \langle w\rho, x^{-1}\beta\rangle - \langle w\rho, x^{-1}\beta\rangle + \langle \alpha^\vee, x^{-1}\beta\rangle\langle w\rho, \alpha\rangle \\ &= \langle \alpha^\vee, x^{-1}\beta\rangle\langle w\rho, \alpha\rangle.\end{aligned}$$

Combining these two, we get that

$$\begin{aligned}\langle \theta(vsw)-\theta(vw), 2\rho\rangle &= (1-\langle \alpha^\vee, \theta(x^{-1})\rangle)\langle w^{-1}\alpha, 2\rho\rangle - \langle \alpha^\vee, x^{-1}\beta\rangle\langle w(2\rho), \alpha\rangle \\ &= (1-\langle \alpha^\vee, \theta(x^{-1})\rangle - \rangle \alpha^\vee, x^{-1}\beta\rangle)\langle w^{-1}\alpha, 2\rho\rangle \\ &= (1-\langle \alpha^\vee, \theta((tx)^{-1})\rangle)\langle w^{-1}\alpha, 2\rho\rangle.\end{aligned}$$

This gives the required formula. To show that it is positive, note that Lemma 3.2.2.2 gives that $\langle \alpha^\vee, \theta(v^{-1})\rangle \leq 0$ and that $\langle w^{-1}\alpha, 2\rho\rangle > 0$. Thus the result follows. $\square$

Using the lemma inductively, we deduce Proposition 3.2.2.1.

3.2.3 The thick flag variety case

We first want to define the appropriate notion of coherent sheaves on the thick flag variety. Recall the following from [Gro60].

Definition 3.2.3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *coherent* if

(1) $\mathcal{F}$ is of finite type.

(2) For any open $U \subset X$, $n > 0$, and homomorphism $\phi : \mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U$, the kernel $\ker \phi$ is of finite type.

We will make use of the following lemma from loc.cit.

Lemma 3.2.3.1. *Let $f : X \to Y$ be a morphism of $\mathcal{O}$-modules. If $\mathcal{O}_X$ is coherent, then $f^*\mathcal{G}$ is also coherent for any coherent $\mathcal{O}_Y$-module $\mathcal{G}$.*

Let S be a finite subset of the affine Weyl group W^{aff}. Recall that S is *open* if for any $w \in S$, $v \leq w$ implies $v \in S$. In particular $e \in S$ for any open S.

Lemma 3.2.3.2. *The structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is coherent.*

Proof. This follows from the open cover of $\mathfrak{X}$ by $U_w = wII^-/I^-$ and noting that U_w is isomorphic to $\mathbb{A}^{\mathbb{N}}$, which has a coherent structure sheaf (as any ideal in $\mathbb{C}[x_1, x_2, \dots]$ is finitely generated). $\square$

Proposition 3.2.3.1. *Fix an open set S, and let $l = \max_{v \in S} l(v) + 1$. Then U_l acts freely on Ω_S and the natural quotient map $p : \Omega_S \to U_l \backslash \Omega_S$ induces an equivalence of triangulated categories*

$$p^* : \; D^b(\mathrm{Coh}^{I/U_l}(U_l \backslash \Omega_S)) \longrightarrow D^b(\mathrm{Coh}^I(\Omega_S)).$$

Proof. The structure sheaf of Ω_S is coherent, and therefore by the previous lemma, pulling back via p preserves coherence. Furthermore, the map p is flat, which follows from [Kum17]. Therefore the pullback is exact, and we only need to show the equivalence of categories at the abelian level. But this follows from equivariant descent:

$$\begin{aligned} \mathrm{Coh}^I(\Omega_S) &\cong \mathrm{Coh}^{I/U_l}(I/U_l \times_I \Omega_S) \\ &\cong \mathrm{Coh}^{I/U_l}(U_l \backslash \Omega_S), \end{aligned}$$

where we have used that U_l is normal in I. $\square$

Note that the left-hand side of the equivalence is a finite dimensional smooth scheme, and therefore we can take the structure sheaf $\mathcal{O}_{U_l \backslash \Omega_S}$ as dualizing complex, which allows us to consider the structure sheaf of Ω_S as an equivariant dualizing complex (pulling back by p preserves internal homs as we are working with finitely presented modules and p is flat). This is useful as Ω_S is non-Noetherian, and therefore a dualizing complex is not guaranteed to exist a priori.

We now want to define an s-structure, and compute the associated staggered codimension. First note the following:

Lemma 3.2.3.3. *The I/U_l orbits in $U_l\backslash\Omega_S$ are given by*

$$X^{v,l} = U_l\backslash IvI^-/I^- \cong I/(I\cap vIv^{-1})U_l.$$

Furthermore, the finite-dimensional torus embeds $T \subset \mathrm{Stab}_{I/U_l}(U_lvI^-/I^-)$.

Proof. This follows from the Cartan decomposition, noting that

$$T \subset I\cap vI^-v^{-1} \cong \mathrm{Stab}_{I/U_l}(U_lvI^-/I^-).$$

□

By an s-structure on $\mathrm{Coh}^I(\Omega_S)$, we will mean an s-structure on the equivalent category $\mathrm{Coh}^{I/U_l}(U_l\backslash\Omega_S)$. The lemma and its proof say that this will be independent of l.

As with the thin case, we can use this to get an s-structure on $\mathrm{Coh}^B(\Omega_S)$: for $w \in W^{aff}$, define an s-structure on $\mathrm{Coh}^I(X^w)$ by

$$\mathrm{Coh}^I(X^w)_{\leq n} = \{V \in \mathit{Rep}(wI^-w^{-1}\cap I) \mid \langle\lambda, -2\rho\rangle \leq n \ \text{ for all the weights } \lambda \text{ of } V\}$$
$$\mathrm{Coh}^I(X^w)_{\geq n} = \{V \in \mathit{Rep}(wI^-w^{-1}\cap I) \mid \langle\lambda, -2\rho\rangle \geq n \ \text{ for all the weights } \lambda \text{ of } V\}.$$

Proposition 3.2.3.2. *There is a unique s-structure on $\mathrm{Coh}^I(\Omega_S)$ which restricts to the s-structure given above on the open orbits.*

This follows from Theorem 3.1.3.1, provided that we can check the following:

Lemma 3.2.3.4. *For any $v \in S$,*

$$i_v^*\mathcal{I}_{\overline{X^v}\cap\Omega_S}|_{X^v} \in \mathrm{Coh}^I(X^v)_{\leq -1}.$$

Proof. For $v \in W^{aff}$, let $V^v = vII^-/I^-$. By [Kas89], $V^v \subset \Omega_S$ is an open subset and $X^v \subset V^v$. Furthermore, the embedding $j_v : X^v \hookrightarrow V^v$ can be identified T-equivariantly with the embedding

$$\prod_{\alpha \in \Delta^+ \cap v\Delta^+} \mathfrak{g}_\alpha \hookrightarrow \prod_{\alpha \in v\Delta^+} \mathfrak{g}_\alpha.$$

Therefore the computation reduces to computing the pullback of the ideal $I = (x_\beta \mid \beta \in v\Delta^+ \cap \Delta^-)$ under the natural quotient homomorphism $\mathbb{C}[x_\alpha | \alpha \in v\Delta^+] \to \mathbb{C}[x_\alpha | \alpha \in \Delta^+ \cap v\Delta^+]$. But this can be identified with I/I^2, which is T-equivariantly isomorphic to $\bigoplus_{\alpha \in v\Delta^+ \cap \Delta^-} \mathcal{O}(-\alpha)$, giving the required result. $\square$

Finally, we want to obtain the staggered codimension, which requires us to find the *altitude* of the open orbits. As we have chosen $\mathcal{O}_{\Omega_S}$ as our dualizing complex, this will reduce to the following computation:

Lemma 3.2.3.5. *Let i_v be as before. Then*

$$\operatorname{alt} \overline{X^v} \cap \Omega_S = \langle \sum_{\alpha \in \Delta^- \cap v\Delta^+} \alpha, -2\rho \rangle.$$

Proof. By assumption, the dualizing sheaf of the orbit closure is given by the pullback $i_v^! \mathcal{O}_{\overline{X^v} \cap \Omega_S}$. To make sense of this, note that i_v is I-equivariant, and thus we can pass to the finite dimensional quotient. Then, we want to find in which staggered degree $i_v^! \mathcal{O}_{\overline{X^v} \cap \Omega_S}|_{X^v}$ lives. But a similar argument to the previous lemma tells us that this reduces to computing $j^! \mathcal{O}_{V^{v,l}}$, where

$$j : X^{v,l} = \prod_{\substack{\alpha \in \Delta^+ \cap v\Delta^+ \\ ht(\alpha) \leq l}} \mathfrak{g}_\alpha \hookrightarrow \prod_{\substack{\alpha \in v\Delta^+ \\ ht(\alpha) \leq l}} \mathfrak{g}_\alpha = V^{v,l}.$$

This follows by induction on $l(v)$. $\square$

From these two observations, we obtain that the staggered codimension is given by

$$\operatorname{scod} \overline{X^v} \cap \Omega_S = l(v) + \langle \theta(v), 2\rho \rangle.$$

By the same proof as the thin case, we deduce that

Theorem 3.2.3.1. *With respect to the s-structure and dualizing complex as above, $D^b(\mathrm{Coh}^I(\Omega_S))$ admits an Artinian staggered t-structure. The simple objects in the heart are the staggered IC-sheaves $\mathcal{IC}(v, \chi)$ for χ a character of T and $v \in S$.*

Remark 3.2.3.1. This raises the natural question of the existence of a staggered t-structure on the whole derived category $\mathcal{D} = D^I_{coh}(\mathfrak{X}_G)$. As mentioned in chapter 2, a well-behaved definition of such a category is not clear to us at this time. But once that is set, the t-structure could be obtained by letting $\mathcal{D}^{\geq 0} = \{\mathcal{F} \in \mathcal{D} \mid i_S^* \mathcal{F} \in D^b(\mathrm{Coh}^I(\Omega_S))^{\geq 0}$ for all open $S\}$ and similarly for $\mathcal{D}^{\leq 0}$.

Chapter 4

Cluster algebras and the Cautis-Williams conjecture

Introduction

In this chapter, we generalise some of the results of [CW19]. In particular, after explaining the construction of the determinant bundle on the affine Grassmannian, we construct a categorification of the bar involution on $K^{G_{\mathcal{O}} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ in simply-connected cases, and provide an explicit construction of dualizing sheaves for quasi-minuscule Schubert varieties. We then state and prove a conjecture of [CW19] in rank 1, and provide a strategy for a proof in type A. Finally, we provide further evidence of the conjecture by relating perverse coherent sheaves and Demazure modules.

4.1 Determinant bundles and perverse coherent sheaves on the affine Grassmannian

4.1.1 Global line bundles on Gr_G

Define the line bundle $\mathcal{L}(\omega_0)$ on Gr_G as follows: using the Kac-Moody central extension, we realise Gr_G as $\mathcal{G}/\mathcal{G}^+$ and let $\mathcal{L}(\omega_0) = \mathcal{G} \times_{\mathcal{G}^+} \mathbb{C}_{\chi_0}$. Here χ_0 is the third projection $\mathcal{G}^+ \cong G_{\mathcal{O}} \rtimes \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$.

Theorem 4.1.1.1. *Suppose G is almost simple and simply-connected. Then*

$$\mathrm{Pic}(\mathrm{Gr}_G) \cong \mathrm{Pic}^{G_{\mathcal{O}}}(\mathrm{Gr}_G) \cong \mathbb{Z}$$

and is generated by $\mathcal{L}(\omega_0)$.

Remark 4.1.1.1. We follow the proof from [Kum02], noting that a very similar argument can also be used to compute the Picard group of the affine flag variety (see [Fal03] for details).

Proof. We will work using the Iwahori orbit stratification. First note that for $w \in W^{aff}$, the closed Schubert cell $\overline{X_w} \subset \mathrm{Gr}_G$ has rational singularities and therefore $H^i(\overline{X_w}, \mathcal{O}_w) = 0$ for $i > 0$. Using the exponential sequence and GAGA, we deduce that $\mathrm{Pic}(\overline{X_w}) \cong H^2(\overline{X_w}, \mathbb{Z})$. Now, the Schubert varieties are paved by affine spaces, therefore taking $w \geq s_0$ the affine reflection, we deduce that $H^2(\overline{X_w}, \mathbb{Z}) \cong H^2(\overline{X_{s_0}}, \mathbb{Z})$. Now, observe $\overline{X_{s_0}} \cong \mathbb{P}^1$ and $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}\mathcal{O}(1)$ and that $\mathcal{O}(1) \cong \mathcal{L}(\omega_0)|_{\overline{X_{s_0}}}$. Therefore, $\mathrm{Pic}(\overline{X_w}) \cong \mathbb{Z}\mathcal{L}(\omega_0)|_{\overline{X_w}}$.

Finally, we need to show that these isomorphisms glue globally, i.e. that there is an isomorphism

$$\mathrm{Pic}(\mathrm{Gr}_G) \cong \lim \mathrm{Pic}(\overline{X_w}).$$

Surjectivity is clear as $\mathcal{L}(\omega_0)$ is defined globally. To prove injectivity, suppose we have a line bundle $\mathcal{L}$ such that $\mathcal{L}|_{\overline{X_w}}$ is trivial on all Schubert varieties. We want to show that $\mathcal{L}$ is isomorphic to the trivial line bundle, i.e. that it admits a nowhere vanishing section. To this aim, fix $v \in \mathcal{L}_e$ a non-zero element. For any w, pick a non-vanishing section σ_w of $\mathcal{L}_{\overline{X_w}}$ such that $\sigma_w(e) = v$. We claim that these sections glue: indeed, note that the restriction $\sigma_w|_{\overline{X_v}}$ is nowhere vanishing and sends e to v. Therefore, we can find $g : \overline{X_v} \to \mathbb{C}^\times$ such that $\sigma_w|_v = g\sigma_v$. As $\overline{X_v}$ is projective, the only such functions are constant, and fixing the base point implies that $\sigma_w|_v = \sigma_v$.

Finally, it remains to show that $\mathrm{Pic}^{G_\mathcal{O}}(\mathrm{Gr}_G) \cong \mathrm{Pic}(\mathrm{Gr}_G)$. There is a natural forgetful map $\mathrm{Pic}^{G_\mathcal{O}}(\mathrm{Gr}_G) \to \mathrm{Pic}(\mathrm{Gr}_G)$, which by the previous part is surjective. Thus it remains to show that the map is injective, i.e. that $\mathcal{L}(\omega_0)$ has a unique $G_\mathcal{O}$-equivariant structure. This follows from noting that $G_\mathcal{O} \times \mathrm{Gr}_G$ is stratified by integral projective varieties: two trivialisations differ by an invertible function $f : G_\mathcal{O} \times \mathrm{Gr}_G \to \mathbb{C}^\times$. As Gr_G is ind-projective, f is uniquely determined by its value on $G_\mathcal{O}$. By the cocycle condition, f has to be a character of $G_\mathcal{O}$ but any such has to be uniformly 1. $\square$

4.1.2 The Determinant bundle

An important motivation for the study of the affine Grassmannian comes from the uniformisation of G-bundles on a curve X. This result states that there is a natural fibration $\mathrm{Gr}_G \to \mathrm{Bun}_G(X)$ (see [BL95, LS97, Zhu17]). Moduli spaces of vector bundles are equipped with a canonical line bundle called the *determinant bundle*, which we can pull back to the affine Grassmannian. As the study of G-bundles will not be needed for this work, we present here, following [Zhu17] and [Sor00], a construction of this line bundle, starting with an explicit description in the case of $G = \mathrm{GL}(n)$.

Definition 4.1.2.1. The *relative determinant bundle* $\mathcal{L}_{det}$ on $\mathrm{Gr}_{\mathrm{GL}(n)} \times \mathrm{Gr}_{\mathrm{GL}(n)}$ is the bundle whose fibres over a pair (Λ_1, Λ_2) is given by

$$\det(\Lambda_1/\Lambda_1 \cap \Lambda_2) \otimes \det(\Lambda_2/\Lambda_1 \cap \Lambda_2)^{-1}$$

The determinant bundle on $\mathrm{Gr}_{\mathrm{GL}(n)}$ is given as the restriction $\mathcal{L}_{det}|_{\{\Lambda_0\}\times\mathrm{Gr}_{\mathrm{GL}(n)}}$.

In general, consider the adjoint representation $\rho : G \to \mathrm{GL}(\mathfrak{g})$. By slight abuse of notation, we define the *determinant bundle* $\mathcal{L}_{det}$ to be the pullback $f_\rho^*\mathcal{L}_{det}$ under the adjoint map $f_\rho : \mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}(\mathfrak{g})}$.

Theorem 4.1.2.1. *[LS97] In* $\mathrm{Pic}(\mathrm{Gr}_G)$, $\mathcal{L}_{det} = \mathcal{L}(2h^\vee\omega_0)$ *where* $h^\vee$ *is the dual Coxeter number.*

To prove the theorem, we will need a slightly different perspective on central extensions:

Definition 4.1.2.2. Let $\mathcal{L}$ be a $\widehat{G_\mathcal{O}}$ equivariant line bundle on Gr_G. The *Mumford group* $M_{\widehat{G_\mathcal{K}}}(\mathcal{L})$ is the group of pairs (f, g) where $g \in \widehat{G_\mathcal{K}}$ and $f : g^*\mathcal{L} \to \mathcal{L}$ is an isomorphism.

Note that the group structure is provided by composing the isomorphisms.

Lemma 4.1.2.1. *The Mumford group* $M_{\widehat{G_\mathcal{K}}}(\mathcal{L}(\omega_0))$ *is a central extension*

$$1 \to \mathbb{G}_m \to M_{\widehat{G_\mathcal{K}}}(\mathcal{L}(\omega_0)) \to \widehat{G_\mathcal{K}} \to 1$$

which identifies $M_{\widehat{G_\mathcal{K}}}(\mathcal{L}(\omega_0))$ *with* $\mathcal{G}$.

Proof. The Mumford group being a central extension follows from noting that $Aut(\mathcal{L}) \cong \mathbb{G}_m$. To prove the lemma, observe that for any $g \in \widehat{G_\mathcal{K}}$, a lift $\gamma \in \mathcal{G}$ with $\overline{\gamma} = g$ provides an isomorphism $g^*\mathcal{L}(\omega_0) \cong \mathcal{L}(\omega_0)$ and that the group operations agree. Therefore, $\mathcal{G} \subset M_{\widehat{G_\mathcal{K}}}(\mathcal{L}(\omega_0))$ but as both of those fit in the same short exact sequence they have to agree. □

Remark 4.1.2.1. An alternative way to see the result is to use the short exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}(\Lambda_0) \to \mathrm{PGL}(\Lambda_0) \to 1$. The embedding from Proposition 1.3.3.1 and ampleness imply that $i^*\mathcal{O}(1) \cong \mathcal{L}(\omega_0)$ and therefore this reduces to show that $\mathrm{GL}(\Lambda_0) = M_{\mathrm{PGL}(\Lambda_0)}(\mathcal{O}_{\mathbb{P}(\Lambda_0)}(1))$.

Proof of Theorem 4.1.2.1. As $\mathcal{L}_{det}$ is a line bundle on Gr_G, $\mathrm{Pic}(\mathrm{Gr}_G) \cong \mathbb{Z}$ implies that $\mathcal{L}_{det} = \mathcal{L}(n_{ad}\omega_0)$. It remains to determine n_{ad}. Now note that the map f_{ad} induces a map of central extensions

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d] & M_{\widehat{G_{\mathcal{K}}}}(\mathcal{L}(n_{ad}\omega_0)) \arrow[r] \arrow[d] & \widehat{G_{\mathcal{K}}} \arrow[r] \arrow[d, "f_{ad}"] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & M_{\widehat{\mathrm{SL}(\mathfrak{g}\otimes\mathcal{K})}}(\mathcal{L}(\omega_0)) \arrow[r] & \widehat{\mathrm{SL}(\mathfrak{g}\otimes\mathcal{K})} \arrow[r] & 1
\end{tikzcd}$$

By the above, $M_{\widehat{\mathrm{SL}(\mathfrak{g}\otimes\mathcal{K})}}(\mathcal{L}(\omega_0))$ is $\mathcal{SL}$ the Kac-Moody extension, and therefore at the Lie algebra level, we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \mathfrak{g}\,\tilde{\otimes}\,\mathcal{K} \arrow[r] \arrow[d] & \mathfrak{g}\otimes\mathcal{K} \arrow[r] \arrow[d, "Ad"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \widehat{\mathfrak{sl}(\mathfrak{g})} \arrow[r] & \mathfrak{sl}(\mathfrak{g}\otimes\mathcal{K}) \arrow[r] & 0
\end{tikzcd}$$

By the above lemma, the cocycle corresponding to the top extension is then $n_{ad}\psi$ where ψ is the Kac-Moody cocyle on $\mathfrak{g}\otimes\mathcal{K}$ (by [Kac90], all central extensions are integer multiples of ψ). The result then follows from the next lemma. □

Lemma 4.1.2.2. $n_{ad} = 2h^\vee$

Proof. Let θ be the highest root. By the above, $n_{ad} = \mathrm{Tr}(\rho(X_\theta), \rho(X_{-\theta}))$ where ρ is the adjoint representation. But under $\mathfrak{sl}_2(\theta)$, $\mathfrak{g}$ decomposes as $\bigoplus_i L(h_i)$ where $L(h_i)$ is the irreducible $\mathfrak{sl}_2$ representation of dimension h_i, with highest weight $\omega_i = \frac{h_i}{2}\theta$. Now, using the standard basis of $L(h_i)$, we get that

$$n_{ad} = \sum_i \sum_{k=0}^{h_i} k(h_i + 1 - k) = \sum_i \frac{1}{6} h_i(h_i+1)(h_i+2).$$

Now, note that

$$\frac{1}{6}h_i(h_i+1)(h_i+2) = 1/2\sum_{j=0}^{m_i}\langle m_i\theta/2 - j\theta, \theta^\vee\rangle^2.$$

Therefore, noting that the $\mathfrak{sl}_2$-strings provide the root space decomposition of $\mathfrak{g}$,we can rewrite

$$n_{ad} = \sum_{\alpha\in\Delta^+}\langle\alpha, \theta^\vee\rangle^2.$$

Finally, as θ is the longest root, $\langle\alpha, \theta^\vee\rangle \in \{0, 1\}$ for $\alpha \neq \theta$. Therefore, we get

$$\begin{aligned} n_{ad} &= 4 + \sum_{\alpha\in\Delta^+\setminus\{\theta\}}\langle\alpha, \theta^\vee\rangle, \\ &= 4 + \langle 2\rho - \theta, \theta^\vee\rangle, \\ &= 2(1 + \langle\rho, \theta^\vee\rangle). \end{aligned}$$

This proves the result. □

Corollary 4.1.2.1. *The determinant bundle admits a square root, unique up to isomorphism.*

The inverse of this square root bundle is traditionally called the *critical line bundle* and denoted $\mathcal{L}_{crit}$.

4.1.3 Canonical bundle of Affine Schubert varieties

An important part in understanding the perverse coherent t-structure on the affine Grassmannian is to describe the dualizing complex of the orbit closures $\overline{\mathrm{Gr}_\lambda}$ explicitly. The importance of $\mathcal{L}_{crit}$ is the following theorem, due to [BD91] and [BK06]. Assume for the rest of the section that G is simply-connected.

Theorem 4.1.3.1. *Let $\lambda \in X^{*,+}(T)$ be a dominant coweight. Then*

- *The affine Schubert variety* $\overline{\mathrm{Gr}_\lambda}$ *is Gorenstein and normal.*

- *Its canonical bundle is given by the restriction*

$$\Omega_{\overline{\mathrm{Gr}_\lambda}} = \mathcal{L}_{crit}|_{\overline{\mathrm{Gr}_\lambda}}.$$

- *Furthermore,* $\overline{\mathrm{Gr}_\lambda}$ *has canonical singularities.*

To prove this, we will need a description of the fibres of $\mathcal{L}_{crit}$ at a coweight t^λ. Let us first describe this for the determinant bundle:

Lemma 4.1.3.1. *[[Zhu14]] As* $G_\mathcal{O} \cap Ad_{t^\lambda} G_\mathcal{O}$*-modules,*

$$\mathcal{L}_{det}|_{t^\lambda} \cong \det\left(\frac{\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}\right)^2.$$

In particular

$$\mathcal{L}_{crit}|_{\mathrm{Gr}_\lambda} \cong \omega_{\mathrm{Gr}_\lambda}.$$

Proof. By the definition of the determinant bundle, we know that

$$\mathcal{L}_{det}|_{t^\lambda} = \det\left(\frac{\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}\right) \otimes \det\left(\frac{Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}\right)^{-1}.$$

Thus it remains to show that

$$\det\left(\frac{Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}\right)^{-1} \cong \det\left(\frac{\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}\right)$$

as $G_\mathcal{O} \cap Ad_{t^\lambda} G_\mathcal{O}$-modules.

Note that

$$\frac{\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})} \cong \bigoplus_{\alpha \in \Delta^+, \langle \alpha, \lambda \rangle > 0} \mathfrak{g}_\alpha(\mathcal{O})/t^{\langle \alpha, \lambda \rangle}\mathfrak{g}_\alpha(\mathcal{O})$$

and similarly

$$\frac{Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})} \cong \bigoplus_{\alpha\in\Delta, \langle\alpha,\lambda\rangle>0} t^{-\langle\alpha,\lambda\rangle}\mathfrak{g}_\alpha(\mathcal{O})/\mathfrak{g}_\alpha(\mathcal{O}).$$

Finally, note that we only need to show the final isomorphism as $T_\mathcal{O}$-modules given that $G_\mathcal{O}$ has no non-trivial characters. This follows from direct inspection. □

Now we can finish the proof of Theorem 4.1.3.1 following [BK06].

Proof of Theorem 4.1.3.1. Recall that $\overline{\mathrm{Gr}_\lambda}$ is normal and Cohen-Macaulay. Therefore, its dualizing complex $\omega_{\overline{\mathrm{Gr}_\lambda}}$ is a sheaf placed in appropriate cohomological degree. By normality and the fact that $\overline{\mathrm{Gr}_\lambda}\backslash \mathrm{Gr}_\lambda$ has codimension 2, we deduce that $\omega_{\overline{\mathrm{Gr}_\lambda}} = j_*\omega_{\mathrm{Gr}_\lambda}$ up to shifts, where $j : \mathrm{Gr}_\lambda \to \overline{\mathrm{Gr}_\lambda}$ is the inclusion. Therefore, $\omega_{\overline{\mathrm{Gr}_\lambda}} \cong j_*j^*(\mathcal{L}_{crit}|_{\overline{\mathrm{Gr}_\lambda}})$ and the result follows. □

Categorifying the bar involution

In [CW19], the authors construct a functor

$$D^{b,\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{G}_m}_{coh}(\mathrm{Gr}_{\mathrm{GL}_n}) \to D^{b,\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{G}_m}_{coh}(\mathrm{Gr}_{\mathrm{GL}_n})$$

which categorifies the involution $q \to q^{-1}$ in $K^{\mathrm{GL}_n\ltimes\mathbb{G}_m}(\mathrm{Gr}_{\mathrm{GL}_n})$. A key observation is that while $\mathrm{GL}_n(\mathcal{O})$-equivariantly the canonical sheaf of affine Schubert varieties is global, this is no longer the case when incorporating loop rotation. For this, we need the following refinement of Lemma 4.1.3.1, whose proof follows directly from the proof of the lemma.

Corollary 4.1.3.1. *As $(G_\mathcal{O} \rtimes \mathbb{G}_m) \cap Ad_{t^\lambda}(G_\mathcal{O} \rtimes \mathbb{G}_m)$-representations, we have*

$$\mathcal{L}_{det}|_{t^\lambda} \cong \det\left(\frac{\mathfrak{g}(\mathcal{O})}{\mathfrak{g}(\mathcal{O}) \cap Ad_{t^\lambda}\mathfrak{g}(\mathcal{O})}\right)^2 \{\langle 2\rho, \lambda\rangle\}.$$

Here $\mathcal{F}\langle n\rangle = \mathcal{F}[n]$ *with the* n*-th power of the* $\mathbb{G}_m$ *acting.*

Remark 4.1.3.1. In order to use the perverse t-structure on the affine Grassmannian, we need to fix the $\mathbb{G}_m$-equivariance on the perverse coherent sheaves $\mathcal{P}_{\lambda^\vee,\chi}$ appropriately. By convention, in the rest of this section, we let $\mathbb{G}_m$ act as the *double cover* of the loop rotation.We then let $\mathbb{G}_m$ act on the fibers of $\mathcal{L}_\chi$ (where $\mathcal{P}_{\lambda^\vee,\chi} = j_{!*}\mathcal{L}_\chi[p(\lambda^\vee)]$) by $z \mapsto z^{2\langle\lambda^\vee,\chi\rangle + \dim \mathrm{Gr}_{\lambda^\vee}}$ (see [CW19, 2.4] or [BFM05] for details).

Now, fix an *opposition* σ of G, i.e. an automorphism $\sigma : G \to G$ which sends $B^- \to B^+$ and $T \to T$ with $\sigma|_T : t \mapsto t^{-1}$, which exists by [Jan03, II.1.16]. Twisting the $G_\mathcal{O}$-action by σ induces an autoequivalence of $D^{G_\mathcal{O}\rtimes\mathbb{G}_m}_{coh}(\mathrm{Gr}_G))$ which, following [CW19], we denote by $\mathcal{F} \mapsto \mathcal{F}'$.
Further, define $\mathbb{L}$ to be the action of tensoring by $\mathcal{L}_{crit}$ and as above $\mathbb{D}$ denotes the coherent Verdier dual.

Proposition 4.1.3.1. *The involution* $\overline{\bullet} : D^{G_\mathcal{O}\rtimes\mathbb{G}_m}_{coh}(\mathrm{Gr}_G)) \to D^{G_\mathcal{O}\rtimes\mathbb{G}_m}_{coh}(\mathrm{Gr}_G))$ *given by the composition*

$$\mathcal{F} \mapsto \mathbb{D} \circ \mathbb{L}((\mathcal{F}^*)')$$

is contravariant with respect to convolution and Homs. Furthermore, $\overline{\mathcal{F}\{k\}} = \overline{\mathcal{F}}\{-k\}$ *and* $\overline{\mathcal{P}_{\lambda^\vee,\chi}} \cong \mathcal{P}_{\lambda^\vee,\chi}$.

Proof. This is just a slight generalisation of the arguments of [CW19, 6.22]. □

Remark 4.1.3.2. The involution in [CW19] is defined for GL_n which is not simply-connected. In that case, one needs to incorporate a twisting when tensoring by the critical line bundle. Our results generalise with minor modification to the non-simply connected case: using Proposition 1.3.3.2, we must replace the Lie algebra $\mathfrak{g}(\mathcal{O})$ by $Ad_{t^{\omega_i}}\mathfrak{g}(\mathcal{O})$ when working over the i-th connected component.

The quasi-minuscule case

In this section, we study the dualizing complex of certain orbit closures in more detail, and formulate a conjecture. Recall that a dominant coweight λ is *quasi-minuscule* if the only dominant coweight $\nu < \lambda$ under the Bruhat ordering is 0. If λ is quasi-minuscule, Ngo-Polo in [NP01] have constructed a canonical equivariant resolution of singularities of $\overline{\mathrm{Gr}_\lambda}$. We will use this to give an alternative description of Theorem 4.1.3.1, and formulate a conjecture about coherent IC-sheaves. This lemma is proven in [NP01].

Lemma 4.1.3.2. *Let $\lambda = \gamma^\vee$ be a quasi-minuscule coweight, associated with a maximal root γ. Then for all roots $\alpha \in R\backslash\{\pm\gamma\}$, we have $\langle \alpha, \lambda\rangle \in \{0, \pm 1\}$.*

Now, define P to be the parabolic subgroup of G generated by T and U_α, for $\alpha \in R$, such that $\langle \alpha, \gamma\rangle \leq 0$.
Further, define

$$V = \mathfrak{h} \oplus \bigoplus_{\alpha\in R\backslash\{\gamma\}} \mathfrak{g}_\alpha.$$

By the lemma, V is P-stable, so we can define a line bundle over G/P by

$$\mathfrak{L}_\gamma := G \times^P \mathfrak{g}/V.$$

The Cartan decomposition of $\mathfrak{g}$ gives a canonical isomorphism $\mathfrak{g}/V \cong \mathfrak{g}_\gamma$.

Theorem 4.1.3.2. *[NP01] There is an isomorphism $\mathrm{Gr}_\lambda \cong \mathfrak{L}_\gamma$, which extends to a $G_\mathcal{O}$-equivariant resolution of singularities*

$$\pi_\gamma : \mathbb{P}(\mathcal{O}_{G/P} \oplus \mathfrak{L}_\gamma) \to \overline{\mathrm{Gr}_\lambda}.$$

We will use this theorem to give a construction of the canonical bundle of $\overline{\mathrm{Gr}_\lambda}$.

Proposition 4.1.3.2. *Let $\pi : \mathbb{P}(\mathcal{O}_{G/P} \oplus \mathfrak{L}_\gamma) \to \overline{\mathrm{Gr}_\lambda}$ be the bundle map, and define*

$\rho_\gamma = \frac{1}{2}\sum_{\langle\alpha,\gamma\rangle>0}\alpha$. *We have an isomorphism*

$$\Omega_{\overline{\mathrm{Gr}_\lambda}} \cong \pi_{\gamma,*}(\mathcal{O}(-2)\otimes\pi^*(\mathcal{L}(-2\rho_\gamma))).$$

Proof. First note that by Theorem 4.1.3.1, as $\overline{\mathrm{Gr}_\lambda}$ is Cohen-Macaulay and has rational singularities, $\pi_{\gamma,*}\Omega_{\mathbb{P}(\mathcal{O}_{G/P}\oplus\mathfrak{L}_\gamma)} = \Omega_{\overline{\mathrm{Gr}_\lambda}}$. Thus we only need to find the canonical bundle of $\mathbb{P}(\mathcal{O}_{G/P}\oplus\mathfrak{L}_\gamma)$. But now, by [Har77, III.9, exercise 8.4.], the canonical bundle of $\mathbb{P}(\mathcal{O}_{G/P}\oplus\mathfrak{L}_\gamma)$ is given by $\mathcal{O}(-2)\otimes\pi^*\Omega_{G/P}$. Finally, the isomorphism

$$\mathfrak{g}/\mathfrak{p} \cong \bigoplus_{\langle\alpha,\gamma^\vee\rangle>0}\mathfrak{g}_\alpha$$

gives that the dual of the top exterior power of T_eG/P has character $-2\rho_\gamma$, and thus $\Omega_{G/P}\cong\mathcal{L}(-2\rho_\gamma)$. □

A very similar situation occurs when looking at the closure of the middle nilpotent orbit in $SL(3)$ as studied by Achar-Hardesty [AH19]. In this case, they construct the coherent IC-sheaves explicitly. This leads us to the following conjecture:

Conjecture 4.1.3.1. *The coherent IC sheaves on* $\overline{\mathrm{Gr}_\lambda}$ *are given by*

$$\mathcal{IC}(\chi,\lambda) = \pi_{\gamma,*}(\pi^*(\mathcal{L}(\chi))).$$

We suspect that for this conjecture to be true, one might need to further truncate the derived pushforward to live in the correct cohomological degrees. This will be explored in further work.

4.1.4 Computations for GL_n

Recall that a dominant coweight for GL_n is given by an n-tuple of non-negative integers. Denote the standard lattice $\Lambda_0 = \mathbb{C}[[t]]^n \subset \mathbb{C}((t))^n$. We understand all inclusions are inclusions of lattices. Let us first rewrite the Cartan decomposition explicitly for the general linear group:

Lemma 4.1.4.1. *For any dominant coweight $\lambda^\vee$ of GL_n, the corresponding orbit is given by*

$$\mathrm{Gr}_{\lambda^\vee} = \{\Lambda \subset \Lambda_0 \mid t|_{\Lambda_0/\Lambda} \text{ has Jordan type } \lambda^\vee\}.$$

Proof. We will show that the set on the right-hand side is $GL_n(\mathcal{O})$-stable and that the only element of $T_{\mathcal{K}}$ it contains is $t^{\lambda^\vee}$. Suppose $\Lambda \subset \Lambda_0$ is such that $t|_{\Lambda_0/\Lambda}$ has Jordan type $\lambda^\vee$ and let $g_0 \in GL_n(\mathcal{O})$. Then $g_0\Lambda \subset \Lambda_0$. Furthermore, we have the following commutative square:

$$\begin{tikzcd}
\Lambda_0/\Lambda \arrow[r, "t"] \arrow[d, phantom] & \Lambda_0/\Lambda \arrow[d, "g_0"] \\
\Lambda_0/g_0\Lambda \arrow[r, "t"] \arrow[u, "g_0^{-1}"] & \Lambda_0/g_0\Lambda.
\end{tikzcd}$$

Therefore, as the two maps are conjugate, $t|_{\Lambda/g_0\Lambda_0}$ also has Jordan type $\lambda^\vee$.
Finally, note that $t^{\lambda^\vee}$ is the only diagonal matrix in $GL_n(\mathcal{K})$ such that the quotient has the required Jordan type. □

Example 4.1.4.1. *In particular, for the fundamental coweights $\omega_k^\vee$, we get the closed orbits*

$$\mathrm{Gr}_k = \{\Lambda \subset \Lambda_0 \mid \dim \Lambda_0/\Lambda = k,\ t\Lambda_0 \subset \Lambda\}.$$

Another useful class of $GL_n(\mathcal{O})$-stable subvariety are the opposite

$$\mathrm{Gr}^k = \{\Lambda_0 \subset \Lambda \mid \dim \Lambda/\Lambda_0 = k,\ t\Lambda \subset \Lambda_0\}.$$

To obtain simple perverse coherent sheaves, we can use Lemma 3.1.5.4 and restrict the determinant bundle to Gr_k. Following [CW19], we write the restriction as $\det(\Lambda_0/\Lambda)$.

Lemma 4.1.4.2. *[CW19] The object of $D^b_{coh}(\mathrm{Gr}_{GL_n})$ given by*

$$\mathcal{P}_{k,l} = i_{k,*}(\mathcal{O}_{\mathrm{Gr}_k} \otimes \det(\Lambda_0/\Lambda)^l)[\frac{k(n-k)}{2}],$$
$$\mathcal{P}_{-k,l} = i^k_*(\mathcal{O}_{\mathrm{Gr}^k} \otimes \det(\Lambda/\Lambda_0)^l)[\frac{k(n-k)}{2}],$$

where $i_k : \mathrm{Gr}_k \to \mathrm{Gr}_{GL_n}$ and $i^k : \mathrm{Gr}^k \to \mathrm{Gr}_{GL_n}$ are the natural inclusions and $l \in \mathbb{Z}$ are simple perverse coherent sheaves. Furthermore, they generate $K^{GL_n}(\mathrm{Gr}_{GL_n})$ as a $\mathbb{Z}$-algebra under convolution.

4.2 The Cautis-Williams conjecture in type A

The Cautis-Williams conjecture relates the equivariant K-theory of the affine Grassmannian with cluster algebras.

4.2.1 Cluster algebras and categorifications

In this section, we provide a quick introduction to cluster algebras and cluster categorifications, mostly to set up notations and definitions that will be used later on. Our approach will closely follow [FWZ16].

Cluster algebras

Cluster algebras are a certain class of finitely generated commutative algebras possessing a (usually infinite) distinguished set of elements classed cluster variables. Those cluster variables will generate the algebra, and furthermore all cluster variables can be obtained using a recursive formula from a finite set called the initial cluster. The full definition will therefore be broken into multiple parts. Let $m \geq n$ integers and $F = \mathbb{C}(y_1, \ldots, y_m)$ the field of rational functions in m variables.

Definition 4.2.1.1. A *labelled seed* in F is a pair $(\mathbf{x}, \tilde{B})$ where

(1) $\mathbf{x} = (x_1, \ldots, x_m) \in F$ is such that $x_1, \ldots, x_m$ are algebraically independent and $F = \mathbb{C}(x_1, \ldots, x_m)$.

(2) $\tilde{B} \in Mat_{m\times n}(\mathbb{Z})$ can be transformed into a matrix which has a skew-symmetric top $n \times n$ block by positive integer rescaling of the rows.

The elements $x_1, \ldots, x_n$ of a seed are called *cluster variables.*

Definition 4.2.1.2. Let $1 \leq k \leq n$ and consider a labelled seed $(\mathbf{x}, \tilde{B})$. The seed mutation $\mu_k(\mathbf{x}, \tilde{B})$ is a new labelled seed $(\mathbf{x}^{'}, \tilde{B}')$ such that

(1) the matrix B' is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{else.} \end{cases}$$

(2) $\mathbf{x}' = (x'_1, \ldots, x'_m)$ where $x'_i = x_i$ for $i \neq k$ and

$$x'_k x_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}.$$

Remark 4.2.1.1. For the definition to make sense, one needs to check that $\tilde{B}'$ is satisfies the requirements of the definition.

Definition 4.2.1.3. Let $(\mathbf{x}, \tilde{B})$ be a labelled seed. The *cluster algebra* $\mathcal{A}_{\tilde{B}}$ is the $\mathbb{C}$-algebra generated by the cluster variables appearing in seeds obtained from $(\mathbf{x}, \tilde{B})$ by all possible iterated mutations.

Remark 4.2.1.2. (1) Note that there is a slight abuse of notation as we do not write the dependence on $\mathbf{x}$. This should always be clear from context.

(2) The variables $x_{n+1}, \dots x_m$ are invariant under mutations, and are therefore called *frozen*. We can see that the cluster algebra $\mathcal{A}_{\tilde{B}}$ is naturally an algebra over $\mathbb{C}[x_{n+1}, \dots, x_m]$.

Example 4.2.1.1. *Suppose we pick* $n = m = 2$*. Then by definition, we get that*

$$\tilde{B} = \pm \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

where b, c *are both positive (or both zero). Therefore, we can see that starting with the initial cluster* x_1, x_2*, the algebra* $\mathcal{A}_{\tilde{B}} := \mathcal{A}(b, c)$ *is generated by* $x_m, m \in \mathbb{Z}$ *subject to the relations*

$$\begin{cases} x_m x_{m+2} = x_{m+1}^b + 1 & m \text{ odd} \\ x_m x_{m+2} = x_{m+1}^c + 1 & m \text{ even.} \end{cases}$$

Remark 4.2.1.3. Whilst the definition of cluster algebras might seem somewhat convoluted, a broad class of rings possess cluster algebra structures. For instance, the coordinate rings of homogeneous spaces of reductive groups are cluster algebras, which encompasses projective spaces, Grassmannians etc. For example, in the case of the Grassmannian, the Plucker coordinates provide the cluster variables, and the cluster relations are given by the Plucker relations.

Cluster categorification

Let $\mathcal{C}$ be a tensor category. The goal of this section is to find a structure on $\mathcal{C}$ which makes the Grothendieck ring $K(\mathcal{C})$ a cluster algebra.

Definition 4.2.1.4. A finite-length monoidal category $\mathcal{C}$ is a *monoidal categorification* of the cluster algebra $\mathcal{A}_B$ if there is a ring isomorphism $K(\mathcal{C}) \cong \mathcal{A}_B$ which identifies cluster monomials with classes of simple objects.

Remark 4.2.1.4. It is also standard to require that classes of simple objects corresponding to cluster monomials are *real*, that is whose square is simple. Proposition 3.1.5.4 states that appropriately shifted restrictions of the determinant bundle to $\overline{\mathrm{Gr}_\lambda}$ provide simple real objects in $\mathcal{P}^{G_\mathcal{O}}_{coh}(\mathrm{Gr}_G)$.

4.2.2 The conjecture

One way to interpret the results of [BFM05] comes from the recent work of Cautis-Williams [CW19]. In it, they show that for $G = \mathrm{GL}_n$, the equivariant K-theory of the affine Grassmannian forms a cluster algebra, where simple perverse coherent sheaves give a cluster basis. They conjecture the following:

Conjecture 4.2.2.1. *Let C be a finite-type Cartan matrix, let G be the corresponding simply connected simple algebraic group, and let G_{Ad} the adjoint group. Then the monoidal category $\mathcal{P}^{G_\mathcal{O}}_{coh}(\mathrm{Gr}_{G_{Ad}})$ forms a monoidal categorification of the cluster algebra $\mathcal{A}_{B_C}$, where*

$$B_C = \begin{pmatrix} C^T - C & -C^T \\ C & 0 \end{pmatrix}$$

and the initial monoidal cluster is given by the simple perverse coherent sheaves $\mathcal{P}_{\omega_i^\vee,0}$ and $\mathcal{P}_{\omega_i^\vee,\omega_i}$.

Remark 4.2.2.1. Here the algebra structure is induced by convolution, although one might note that we are not in the Hecke stack case treated in Chapter 2, as we are looking at the stack $G_\mathcal{O}\backslash G_{Ad,\mathcal{K}}/G_{Ad,\mathcal{O}}$. However, convolution is still well-defined as before on the stack $G_\mathcal{O}\backslash G_{Ad,\mathcal{K}}/G_\mathcal{O}$. Now we can notice that

$$G_\mathcal{O}\backslash G_{Ad,\mathcal{K}}/G_{Ad,\mathcal{O}} \cong (G_\mathcal{O}\backslash G_{Ad,\mathcal{K}}/G_\mathcal{O})/Z(G),$$

and therefore realise $K^{G_{\mathcal{O}}}(\mathrm{Gr}_{G_{Ad}})$ as the subalgebra of

$$K(G_{\mathcal{O}} \backslash G_{Ad,\mathcal{K}}/G_{\mathcal{O}}) \cong K^{G_{\mathcal{O}} \ltimes Z(G)}(\mathrm{Gr}_{G_{Ad}})$$

where the left $Z(G) \subset G_{\mathcal{O}}$-equivariance matches with the right $Z(G)$-equivariance. See [FT17] for details.

In rank 1, the CW conjecture becomes the following:

Proposition 4.2.2.1. *There is an algebra isomorphism*

$$K^{\mathrm{SL}_2}(\mathrm{Gr}_{\mathrm{PGL}_2}) \cong \mathcal{A}(2,2),$$

where the perverse sheaves $\mathcal{P}_{1,0}, \mathcal{P}_{1,1}$ are mapped to the initial cluster. Furthermore, under the isomorphism, all cluster variables are classes of simple perverse sheaves supported on the orbit Gr_1.

4.2.3 Proof in rank 1

Some convolution computations

In this section, we will compute the algebra $K^{SL_2}(\mathrm{Gr}_{PGL_2})$ under convolution following [BFM05]. We will show the following proposition:

Proposition 4.2.3.1. *[BFM05] There is an algebra isomorphism*

$$\phi : K^{\mathrm{SL}_2}(\mathrm{Gr}_{\mathrm{PGL}_2}) \cong \frac{\mathbb{C}[a,b,c]}{(abc - b^2 - c^2 - 1)}$$

where $a = \mathcal{P}_{0,1}, b = \mathcal{P}_{1,0}, c = \mathcal{P}_{1,1}$.

To be able to compute convolutions using the lattice model, we will need a description of the convolution space $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G := G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G$ in terms of lattices. Let $S_n = \overline{\mathrm{Gr}_{n\omega}}$ where ω is the fundamental weight of PGL_2.

Lemma 4.2.3.1. *We have the isomorphism*

$$S_n \tilde{\times} S_m \cong \{\Lambda_2 \subset \Lambda_1 \subset \Lambda_{\mathbb{C}} |\ \dim_{\mathbb{C}} \Lambda_1/\Lambda_2 = n,\ \dim_{\mathbb{C}} \Lambda_{\mathbb{C}}/\Lambda_1 = m\}$$

where multiplication is given by forgetting Λ_1.

Proof. The isomorphism is given by taking a pair $[g, \Lambda]$ and mapping it to $g\Lambda \subset g\Lambda_0 \subset \Lambda_{\mathbb{C}}$. □

Lemma 4.2.3.2. *The stratum* S_2 *of the affine Grassmannian for* $PGL(2)$ *is isomorphic to the projective variety*

$$S_2 \cong \mathrm{Proj}(\frac{\mathbb{C}[x, y, z, T]}{(x^2 - yz)}).$$

Proof. Note from the appendix that S_2 is isomorphic to the closed subspace of the (standard) Grassmannian $Gr(2, 4)$ formed by the t-stable subspaces, where t acts on $\mathbb{C}^4$ by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now, we can use Plücker coordinates to embed the Grassmannian into projective space $\mathbb{P}(\Lambda^2\mathbb{C}^4)$. Under this embedding,

$$Gr(2, 4) \cong \mathrm{Proj}\, \frac{\mathbb{C}[W_{12}, W_{13}, W_{14}, W_{23}, W_{24}, W_{34}]}{(W_{12}W_{34} + W_{13}W_{24} + W_{14}W_{23})},$$

where W_{ij} is the ijth minor of the matrix $(\mathbf{w}_1\ \mathbf{w}_2)$ picking two basis vectors in $\mathbb{C}^4$(which are the coordinates of the wedge product).

To determine the closed subvariety given by t-stable subspaces, take a 2 dimensional subspace $V = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$. V being t-stable means that we can find $\lambda, \lambda', \mu, \mu' \in \mathbb{C}$ such that

$$\begin{cases} t\mathbf{v}_1 = \lambda \mathbf{v}_1 + \mu \mathbf{v}_2 \\ t\mathbf{v}_2 = \lambda' \mathbf{v}_1 + \mu' \mathbf{v}_2. \end{cases}$$

This implies that $W_{13} = (\mu\lambda' - \lambda\mu')W_{24}$. Furthermore $t^2\mathbf{v}_1 = 0$ and $t^2\mathbf{v}_2 = 0$, therefore

$$\begin{cases} \lambda^2 + \mu\lambda' = 0 \\ \mu(\lambda + \mu') = 0 \\ \lambda'(\lambda + \mu') = 0 \\ \lambda'\mu + \mu'^2 = 0. \end{cases}$$

Combining these two equations, we see that $W_{13} = 0$.
By the same argument, we deduce that $W_{14} + W_{23} = (\lambda + \mu')W_{24} = 0$.

It remains to show that these are the only additional equations. Let V be a 2-dimensional subspace of $\mathbb{C}^4$ such that $V = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ where $\mathbf{v}_1, \mathbf{v}_2$ satisfy $W_{13} = 0$ and $W_{14} + W_{23} = 0$. In the standard basis, we can write

$$\mathbf{v}_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

and

$$\mathbf{v}_2 = \begin{pmatrix} \lambda\alpha_1 \\ \beta_2 \\ \lambda\alpha_3 \\ \beta_4 \end{pmatrix}$$

with the additional constraint that $\lambda(\alpha_2\alpha_3 - \alpha_4\alpha_1) + \alpha_1\beta_4 - \beta_2\alpha_3 = 0$. If $W_{24} = 0$, we can write

$$\mathbf{v}_2 = \begin{pmatrix} \lambda\alpha_1 \\ \mu\alpha_2 \\ \lambda\alpha_3 \\ \mu\alpha_4 \end{pmatrix}.$$

The constraint $W_{14} + W_{23} = 0$ gives that $(\lambda - \mu)(\alpha_2\alpha_3 - \alpha_1\alpha_4) = 0$. Therefore, $\det(t\mathbf{v}_1 \mid \lambda\mathbf{v}_1 - \mathbf{v}_2) = 0$ and therefore $t\mathbf{v}_1 \in V$. We can then conclude as $t\mathbf{v}_2 = \lambda t\mathbf{v}_1$.

If $W_{24} \neq 0$, note that $t\mathbf{v}_1 = \mu\mathbf{v}_1 + \nu\mathbf{v}_2$, where

$$\begin{cases} \nu & = (\alpha_2\alpha_3 - \alpha_4\alpha_1)W_{24}^{-1} \\ \mu & = (\alpha_1\beta_4 - \beta_2\alpha_3)W_{24}^{-1}. \end{cases}$$

□

We can now describe the convolution diagram geometrically:

Lemma 4.2.3.3 ([BFM05]). *The convolution diagram for the two 1-dimensional strata* $\mathrm{Gr}_{PGL(2),1}$ *can be identified with the following:*

$$\begin{tikzcd} PGL(2)_{K,1} \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \arrow[d] \\ \mathbb{P}^1 \times \mathbb{P}^1 & \overline{\mathrm{Gr}_{PGL(2),2}} \cong \mathrm{Proj}(\mathbb{C}[x,y,z,T]/(x^2 - yz)) \end{tikzcd}$$

where multiplication is identified with contraction of the (-2)*-section.*

Proof. As the weight 1 is minuscule, $\mathrm{Gr}_{PGL(2),1}$ is isomorphic to the flag variety $PGL(2)/B$, which is isomorphic to $\mathbb{P}^1$. The orbit closure $\overline{\mathrm{Gr}_{PGL(2),2}}$ is given by Theorem 1.2.2.1. It remains to show that the multiplication is the blowup at the cone point of $\overline{\mathrm{Gr}_{PGL(2),2}}$. For this, we can look at Lemmas 4.2.3.2 and 4.2.3.1. The map m is an isomorphism for $\Lambda_2 \neq t\Lambda_{\mathbb{C}}$, and the fibre over the cone point is isomorphic to

$\mathbb{P}^1$. By [Har77, Example 5.2.11.4] this means that the convolution space is isomorphic to $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(1))$ where C is the quadratic curve $x^2 - yz = 0$ inside $\mathbb{P}^2$. $\square$

Now, let $\mathbf{v}(n)_m$ be the class of the simple perverse sheaf $\mathcal{P}_{n,m}$ in $K^{SL_2}(\mathrm{Gr}_{PGL_2})$. From Chapter 3, we know the following:

Lemma 4.2.3.4. *$K^{SL_2}(Gr_{PGL_2})$ has a $\mathbb{C}$-basis given by $\mathbf{v}(n)_m$ for $(n,m) \in \mathbb{Z} \times \mathbb{Z}_{>0} \cup \mathbb{Z}_{\geq 0} \times \{0\}$. The subspace generated by taking m even (respectively n even) can be identified with $K^{SL_2}(\mathrm{Gr}_{SL_2})$ (respectively $K^{PGL_2}(\mathrm{Gr}_{PGL_2})$).*

To compute convolutions of such sheaves, we have the following rewriting of proposition 3.1.5.1.

Lemma 4.2.3.5. *There exist integers such that*

$$\mathbf{v}(n_1)_{m_1} \star \mathbf{v}_{n_2}(m_2) = \mathbf{v}(m_1 + m_2)_{n_1+n_2} + K,$$

where K can be written as a linear combination of simple perverse sheaves supported on smaller dimensional orbits.

Proposition 4.2.3.2. *$K^{SL_2}(\mathrm{Gr}_{PGL_2})$ is generated as an algebra under convolution by $\mathbf{v}(1)_0, \mathbf{v}(0)_1, \mathbf{v}(1)_1$. Furthermore, we have the relations*

(1) $\mathbf{v}(n)_1 \star \mathbf{v}(n)_1 = \mathbf{v}(2n)_2$,

(2) $\mathbf{v}(1)_1 \star \mathbf{v}(-1)_1 = \mathbf{v}(0)_2 + 1$,

(3) $\mathbf{v}(0)_1 \star \mathbf{v}(1)_0 = \mathbf{v}(1)_1 + \mathbf{v}(-1)_1$.

Proof. We first note that relations (1),(2) and (3) combined with Lemma 4.2.3.5 imply the generation. We can reason inductively on the support: in dimension 0, this just says that $\mathbb{C}^2$ is a tensor generator of the category of representations of SL_2. In dimension 1, this follows from (3).

If we know that all perverse coherent sheaves can be obtained this way up to the orbit n, then note that $\mathbf{v}(m)_{n+1} = \mathbf{v}(m)_n \star \mathbf{v}(0)_1 + K$, where the K can be obtained from our induction hypothesis by the previous lemma.

Now for the relations:

(1) We will work in terms of lattices. As we are working with PGL_2, lattices are considered equivalent when multiplied with an element of $\mathbb{C}((t))^\times$.

The multiplication map can be written as

$$m : \{\Lambda_2 \subset \Lambda_1 \subset \Lambda_0 \mid \dim \Lambda_i/\Lambda_{i+1} = 1\} \longrightarrow \{\Lambda_2 \subset \Lambda_0 \mid \dim \Lambda_0/\Lambda_2 = 2\} \cup \{t\Lambda_0 \subset \Lambda_0\},$$

given by forgetting Λ_1. Using the notation from the previous section, we are looking to compute the (derived) pushforward

$$m_*(\mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \otimes (\Lambda_0/\Lambda_1)^n \otimes (\Lambda_1/\Lambda_2)^n).$$

But note that $m^*(\det(\Lambda_0/\Lambda_2)) = \Lambda_0/\Lambda_1 \otimes \Lambda_1/\Lambda_2$. Thus,

$$\begin{aligned} m_*(\mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1}(\Lambda_0/\Lambda_1)^n \otimes (\Lambda_1/\Lambda_2)^n) &= m_*(\mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \otimes m^*(\det(\Lambda_0/\Lambda_2)^n) \\ &= m_*(\mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1}) \otimes \mathcal{P}_{2,n}, \\ &= \mathcal{P}_{2,n} \end{aligned}$$

where we have used the projection formula and the fact that m is a blowup, and thus the (derived) pushforward of the structure sheaf is the structure sheaf.

(2) Now, we seek to compute

$$m_*(\mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \otimes \Lambda_0/\Lambda_1 \otimes (\Lambda_1/\Lambda_2)^{-1}).$$

But note the following: the fibre of the map m at the point $\{t\Lambda_0 \subset \Lambda_0\}$ is a divisor D given as the degeneracy locus of the map $t : \Lambda_0/\Lambda_1 \to \Lambda_1/\Lambda_2$. Thus we can write the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \otimes \Lambda_0/\Lambda_1 \otimes (\Lambda_1/\Lambda_2)^{-1} \longrightarrow \mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

When taking the derived pushforward with respect to m, $\mathcal{O}_D$ is sent to $\mathcal{O}_e = 1$, and the result follows.

(3) The convolution product in this case is the sheaf $\mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^2$ equivariantly. Note that this is the sheaf of sections associated with the equivariant vector bundle $V = SL_2 \times_B \mathbb{C}^2$. Now V admits a natural subbundle $\mathcal{L}(1)$, and the quotient is naturally isomorphic to $\mathcal{L}(-1)$. Thus the result follows.

□

Proposition 4.2.3.1 follows immediately.

The final step

Equipped with those computations, it remains to construct an algebra isomorphism.

Proof of Proposition 4.2.2.1. By Proposition 4.2.3.1, we only need to show that there is an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{A}(2,2) \cong \frac{\mathbb{C}[a,b,c]}{(abc - b^2 - c^2 - 1)} =: R$$

mapping the initial cluster x_1, x_2 to c, b respectively. To do this, note that both sides are $\mathbb{Z}$-graded:

- On R, we have an automorphism $\sigma : R \to R$ given by $\sigma(a) = a, \sigma(b) = ab - c, \sigma(c) = b$. One can check that this is an isomorphism with $\sigma^{-1}(c) = ac - b$.

- On $\mathcal{A}(2,2)$, we have an automorphism $s : \mathcal{A}(2,2) \to \mathcal{A}(2,2)$ given by $s(x_m) = x_{m+1}$ for all $m \geq 1$. This is also an isomorphism where $s^{-1}(x_1) = (x_1x_4 - x_2x_3)x_1 - x_2$.

Now, consider the algebra homomorphism

$$\Phi : \mathbb{C}[x_1, x_2, \ldots] \longrightarrow R$$

given by $\Phi(x_m) = \sigma^{m-1}(c)$. Then for $m > 1$

$$\begin{aligned}\Phi(x_{m-1}x_{m+1} - (x_m^2 + 1)) &= \sigma^{m-1}(c\sigma^2(c) - \sigma(c)^2 - 1)\\ &= \sigma^{m-1}(abc - c^2 - b^2 - 1)\\ &= 0.\end{aligned}$$

Thus we get an induced homomorphism $\bar{\Phi} : \mathcal{A}(2,2) \to R$.

We claim that $\bar{\Phi}$ is an isomorphism of algebras with inverse $\Psi : \mathcal{A}(2,2) \to R$ given by

$$\begin{aligned}&\Psi(a) = x_1x_4 - x_2x_3\\ &\Psi(b) = x_2\\ &\Psi(c) = x_1.\end{aligned}$$

Indeed, both $\bar{\Phi}$ and Ψ are $\mathbb{Z}$-equivariant, and furthermore we have

- $\bar{\Phi} \circ \Psi(a) = a$. Indeed,

$$\begin{aligned} \bar{\Phi} \circ \Psi(a) &= \bar{\Phi}(x_1 x_4 - x_2 x_3) \\ &= c\sigma^3(c) - \sigma(c)\sigma^2(c) \\ &= c(a^2 b - ac - b) - c(ab - c) \\ &= a. \end{aligned}$$

- Similarly, $\bar{\Phi} \circ \Psi(b) = b$ and $\bar{\Phi} \circ \Psi(c) = c$.
- $\Psi \circ \bar{\Phi}(x_m) = x_m$ as

$$\begin{aligned} \Psi \circ \bar{\Phi}(x_m) &= \Psi(\sigma^{m-1}(c) \\ &= s^{m-1}\Psi(c) \\ &= s^{m-1}(x_1) \\ &= x_m, \end{aligned}$$

  where we have used equivariance of Ψ.

Thus we get that the maps are mutual inverses.

To conclude, recall that from 4.2.3.2 we have the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \otimes \Lambda_0/\Lambda_1 \otimes (\Lambda_1/\Lambda_2)^{-1} \longrightarrow \mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Tensoring by $\mathcal{Q}_n := (\Lambda_0/\Lambda_1)^{\otimes n} \otimes (\Lambda_1/\Lambda_2)^{\otimes n}$ we obtain that

$$0 \to \mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \otimes (\Lambda_0/\Lambda_1)^{n+1} \otimes (\Lambda_1/\Lambda_2)^{n-1} \to \mathcal{O}_{\mathrm{Gr}_1 \tilde{\times} \mathrm{Gr}_1} \otimes \mathcal{Q}_n \to \mathcal{O}_D \otimes \mathcal{Q}_n \to 0.$$

Pushing forward under m and noting that $(\Lambda_0/\Lambda_1)^{\otimes n} \otimes (\Lambda_1/\Lambda_2)^{\otimes n} \cong m^*(\Lambda_0/\Lambda_1)^{\otimes n}$, we obtain the categorified relation:

$$\mathcal{P}_{1,n-1} \star \mathcal{P}_{1,n+1} \longrightarrow \mathcal{P}_{1,n}^2 \longrightarrow \mathcal{O}_e \longrightarrow \mathcal{P}_{1,n-1} \star \mathcal{P}_{1,n+1}[1].$$

□

4.2.4 A proof strategy in type A

In this section, we provide a strategy to prove the following:

Theorem 4.2.4.1. *The isomorphism*

$$\Phi_{CW} : K^{\mathrm{GL}_n}(\mathrm{Gr}_{\mathrm{GL}_n}) \to \mathcal{A}_{\tilde{B}_n}$$

from [CW19] induces an isomorphism

$$K^{\mathrm{SL}_n}(\mathrm{Gr}_{\mathrm{PGL}_n}) \to \mathcal{A}_{B_n}$$

where

$$B_n = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$$

for C the type A Cartan matrix.
Furthermore, this makes the Coherent Satake category into a monoidal cluster categorification of the algebra $\mathcal{A}_{B_n}$.

In particular, we will define the corresponding map Φ and show that it provides the required isomorphism on the level of perverse coherent sheaves. The one missing ingredient is to show that the map we have defined is an algebra morphism, which is currently work in progress.

Changing equivariance

The first step is to understand the difference between GL_n and SL_n equivariance. By [BL94b], there is a functor

$$res_{SL_n}^{GL_n} : D_{coh}^{\mathrm{GL}_n}(\mathrm{Gr}_{\mathrm{GL}_n}) \to D_{coh}^{SL_n}(\mathrm{Gr}_{\mathrm{GL}_n}).$$

Lemma 4.2.4.1. *The functor $res_{SL_n}^{GL_n}$ is t-exact with respect to the perverse coherent t-structure.*

Proof. This follows from the fact that $SL_n(\mathcal{O})$ and $\mathrm{GL}_n(\mathcal{O})$-orbits on $\mathrm{Gr}_{\mathrm{GL}_n}$ coincide, and therefore so do their generic points. Commutativity with shriek and star pullbacks is explained in [BL94b]. □

Now note the following lemma from [FF21]

Lemma 4.2.4.2. *Let $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n)$ be a dominant coweight for GL_n and let $m_i = |\{k \ |\nu_k = \nu_i\}|$. Then there is an equivalence of categories*

$$\mathrm{Coh}^{\mathrm{GL}_n(\mathcal{O})}(\mathrm{Gr}_\nu) \tilde{\to} \mathrm{Rep}(\prod \mathrm{GL}_{m_i}(\mathbb{C})).$$

Proposition 4.2.4.1. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{P}^{\mathrm{GL}_n(\mathcal{O})}(\mathrm{Gr}_{\mathrm{GL}_n}) \arrow[d, "\mathrm{Res}"] & \mathrm{Coh}^{\mathrm{GL}_n(\mathcal{O})}(\mathrm{Gr}_{\mathrm{GL}_n}) \arrow[l, "j_{!*}"'] \arrow[r] \arrow[d, "\mathrm{Res}"] & \mathrm{Rep}(\prod \mathrm{GL}_{m_i}(\mathbb{C})) \arrow[d, "\mathrm{Res}"] \\
\mathcal{P}^{\mathrm{SL}_n(\mathcal{O})}(\mathrm{Gr}_{\mathrm{GL}_n}) & \mathrm{Coh}^{\mathrm{SL}_n(\mathcal{O})}(\mathrm{Gr}_{\mathrm{GL}_n}) \arrow[l, "j_{!*}"'] \arrow[r] & \mathrm{Rep}((\prod \mathrm{GL}_{m_i}(\mathbb{C})) \cap \mathrm{SL}_n(\mathbb{C})).
\end{tikzcd}$$

Proof. Commutativity of the right-hand square is immediate, so it remains to show that the intermediate extensions commute with the restriction functor.

This in turns follows from the construction of intermediate extension as the inverse of j^* (see [AB10] for details), and Lemma 4.2.4.1. □

Corollary 4.2.4.1. *The induced map on K-theory $\mathrm{Res}_{\mathrm{SL}_n}^{\mathrm{GL}_n} : K^{\mathrm{GL}_n}(\mathrm{Gr}_{\mathrm{GL}_n}) \to K^{\mathrm{SL}_n}(\mathrm{Gr}_{\mathrm{GL}_n})$ sends the class of the simple perverse coherent sheaf $[\mathcal{P}_{\lambda^\vee,\mu}]$ to $[\mathcal{P}_{\lambda^\vee,\overline{\mu}}]$.*

Projecting to $\mathrm{Gr}_{\mathrm{PGL_n}}$

Recall that the affine Grassmannian for PGL_n can be realised as the space of lattices up to scaling by elements of $\mathcal{K}^\times$. This defines a map $\pi : \mathrm{Gr}_{\mathrm{GL}_n} \to \mathrm{Gr}_{\mathrm{PGL}_n}$.

Lemma 4.2.4.3. *The map π is a $\mathbb{Z}$-torsor and is ind-finite.*

Proof. The first statement follows from the fact that $\mathcal{O}^\times$ preserves any lattice, and therefore the fibres of π are isomorphic to $\mathcal{K}^\times/\mathcal{O}^\times \cong \mathbb{Z}$.

For the second statement, consider the Cartan decomposition of the affine Grassmannians: we write

$$\begin{aligned}\mathrm{Gr}_{\mathrm{GL}_n} &= \bigcup_{\lambda^\vee \in X^{*,+}} \mathrm{Gr}^{\lambda^\vee}_{\mathrm{GL}_n} \\ \mathrm{Gr}_{\mathrm{PGL}_n} &= \bigcup_{\lambda^\vee \in \tilde{X}^{*,+}} \mathrm{Gr}^{\lambda^\vee}_{\mathrm{PGL}_n},\end{aligned}$$

where we have a natural map $\overline{\bullet} : X \to \tilde{X} = X/(\sum_i \omega_i)$.

Then π induces an isomorphism $\pi|_{\mathrm{Gr}^{\lambda^\vee}} : \mathrm{Gr}^{\lambda^\vee} \to \mathrm{Gr}^{\overline{\lambda}^\vee}$ and therefore

$$\pi^{-1}(\mathrm{Gr}_{\lambda^\vee}) = \bigcup_{l \in \mathbb{Z}} \mathrm{Gr}_{\lambda^\vee + l\omega_n^\vee}.$$

To conclude, we can use the ind-scheme structure by the stratification $S_n = \bigcup_{\langle 2\rho, \lambda^\vee \rangle \leq n} \overline{\mathrm{Gr}_{\lambda^\vee}}$ and note that

$$\pi^{-1}(\mathrm{Gr}_{\lambda^\vee}) \cap S_n$$

is a finite union of Schubert varieties. □

As finite maps are proper, this allows us to define the pushforward

$$\pi_* : D^{\mathrm{SL}_n(\mathcal{O})}_{coh}(\mathrm{Gr}_{\mathrm{GL}_n}) \to D^{\mathrm{SL}_n(\mathcal{O})}_{coh}(\mathrm{Gr}_{\mathrm{PGL}_n}).$$

As the map π restrict to an isomorphism $\mathrm{Gr}_{\lambda^\vee} \to \mathrm{Gr}_{\overline{\lambda^\vee}}$, the previous lemma implies the following.

Corollary 4.2.4.2. *In K-theory,* $\pi_*[\mathcal{P}_{\lambda^\vee,\overline{\mu}}] = [\mathcal{P}_{\overline{\lambda^\vee},\overline{\mu}}]$.

The algebra structure

Define $\Phi : K^{\mathrm{GL}_n}(\mathrm{Gr}_{\mathrm{GL}_n}) \to K^{\mathrm{SL}_n}(\mathrm{Gr}_{PGL_n})$ as the composition $\pi_* \circ \mathrm{Res}^{\mathrm{GL}_n}_{\mathrm{SL}_n}$. The goal of this section is to identify the kernel of Φ and show that it is an algebra morphism. First note the following corollary of the previous section.

Lemma 4.2.4.4. *The map Φ is surjective.*

The following theorem is still work in progress at the time of writing.

Theorem 4.2.4.2. *The map Φ is an algebra morphism.*

Remark 4.2.4.1. Note that this theorem is somewhat similar to Proposition 2.2.5.1. The main subtlety is that $[\mathrm{SL}_n(\mathcal{O})\backslash \mathrm{PGL}_n(\mathcal{K})/\mathrm{PGL}_n(\mathcal{O})]$ is not a Hecke stack, and therefore the proof requires modification.

Lemma 4.2.4.5. *Let $r, r' \in \mathbb{Z}$ with $r \geq 0$. In $K^{\mathrm{GL}_n}(\mathrm{Gr}_{\mathrm{GL}_n})$,*

$$[P_{\lambda^\vee+r\omega_n^\vee,\mu+r'\omega_n}] = [P_{\lambda^\vee,\mu}] \star [P_{\omega_n^\vee,\omega_n}]^{r'} \star [P_{\omega_n^\vee,0}]^{r-r'}.$$

Remark 4.2.4.2. Note that $P_{\omega_n^\vee,0}$ is invertible, so the formula makes sense even if $r' \geq r$.

Proof. As $\mathrm{Gr}_{r\omega_n^\vee}$ is a point, the pushworward map m_* in the convolution diagram is an isomorphism. Furthermore, both $\mathcal{P}_{\omega_n^\vee,0}$ and $\mathcal{P}_{\omega_n^\vee,\omega_n}$ are invertible, therefore the convolution product from the statement is a simple perverse coherent sheaf supported on $\overline{\mathrm{Gr}}_{\lambda^\vee+r\omega_n^\vee}$, thus of the form $P_{\lambda^\vee+r\omega_n^\vee,\nu}$ for some ν. Now, by a similar argument to the proof of Proposition 3.1.5.1, ν can be obtained by looking at the fibres over the T-fixed points. □

Corollary 4.2.4.3. *The map Φ identifies $K^{\mathrm{SL}_n}(\mathrm{Gr}_{\mathrm{PGL}_n})$ with $\mathcal{A}_{\tilde{B}_n}/J$ where J is the ideal generated by setting the frozen variables to* 1.

By the definition of a cluster algebra, $\mathcal{A}_{\tilde{B}_n}/J$ is isomorphic to $\mathcal{A}_{B_n}$, and therefore the corollary proves the Cautis-Williams conjecture in type A.

4.3 Beyond type A: KR-modules and the Q-system

In this final section, which will mostly be a literature survey, we provide some evidence of the Cautis-Williams conjecture in type ADE. This can be seen as an expansion on the remarks after Conjecture 1.10 in [CW19].

4.3.1 Kirillov-Reshetikhin modules and the Q-system

Let $\mathfrak{g}$ be a Lie algebra.

Definition 4.3.1.1. The *current algebra* $\mathcal{C}\mathfrak{g}$ is the Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$

Write the generators $x \otimes t^n$ as $x[n]$, and consider the natural shifted algebra $\mathcal{C}\mathfrak{g}_\xi$ generated by $x[n]_\xi = x \otimes (t-\xi)^n$.
If $V(\xi)$ is a $\mathcal{C}\mathfrak{g}_\xi$-module, we also obtain a $\mathcal{C}\mathfrak{g}$-module structure by expanding

$$x[n].v = x \otimes (t-\xi+\xi)^n.v$$

We can naturally put a grading on the universal enveloping algebra $U(\mathcal{C}\mathfrak{g})$ by letting t be of degree 1. In general, a cyclic module $(v, V(\xi))$ will not be graded. However, taking $\mathcal{F}^n = U^{\leq n}v$, we obtain a filtration on $V(\xi)$, and we let the associated graded be denoted by $\bar{V}(\xi)$. This depends on the choice of cyclic vector in general.
The fusion product takes as input $\xi_i \in \mathbb{C}, i = 1, ..., n$ and n cyclic $\mathcal{C}\mathfrak{g}_{\xi_i}$-modules $(v_1, V_1(\xi_1)), \ldots, (v_n, V_n(\xi_n))$.

Definition 4.3.1.2. The fusion product associated with the above data is given by

$$V_1 \star \cdots \star V_n(\xi_1, \ldots, \xi_n) = Gr(U(\mathcal{C}\mathfrak{g})v_1 \otimes \cdots \otimes v_n).$$

Here the graded module is associated with the grading on $U(\mathcal{C}\mathfrak{g})$. Note that the fusion naturally has the structure of a graded $\mathfrak{g}$-module. Conjecturally, the fusion should be independent of ξ_i provided they are distinct. This is known for some cases but not in general.
Suppose from now on that $\mathfrak{g}$ is simple with r simple roots. An interesting class of $\mathcal{C}\mathfrak{g}$-modules is given by the Kirillov-Reshetikhin modules.

Definition 4.3.1.3. Let $(\xi, a, i) \in \mathbb{C}^\times \times \mathbb{Z}_{\geq 0} \times \{1, 2, \ldots, r\}$. The Kirillov-Reshetikhin module $KR_{a\omega_i}(\xi)$ is the cyclic $\mathcal{C}\mathfrak{g}$-module with cyclic vector v and relations

$$\begin{aligned} &x[n]_\xi v = 0 && \text{if } n \geq 0\ x \in \mathfrak{n}_+ \\ &f_j[n]_\xi v = 0 && \text{for } n \geq \delta_{ij} \\ &f_i[0]_\xi^{a+1} v = 0 && \\ &h_j[n]_\xi v = \delta_{n0}\delta_{ij} a v. && \end{aligned}$$

Remark 4.3.1.1. (i) As before, we can take the graded version $\overline{KR}_{a\omega_i}(\xi)$.

(ii) We can use fusion to define KR modules in general: for ω a dominant weight, write $\omega = \sum_i n_i \omega_i$ and take $KR_\omega = KR_{n_1\omega_1} \star \cdots \star KR_{n_k\omega_k}$.

(iii) In the case of KR-modules, fusion will no longer depend on the parameters ξ [AK07], so we shall omit this.

The more natural $\mathcal{C}\mathfrak{g}$-modules which have appeared in connection with the affine Grassmannians where the Demazure modules $D_{l,\lambda^\vee}$. These are connected to KR-modules in the following way:

Theorem 4.3.1.1. *[FL07] Take a fundamental coweight $\omega_i^\vee$ and suppose that $d_i \in \mathbb{Z}$ is such that $i(\omega_i^\vee) = d_i\omega_i$. Then as $\mathcal{C}\mathfrak{g}$-modules, we have an isomorphism $KR_{d_ik\omega_i} \cong D_{k,\omega_i^\vee}$.*

Furthermore, this isomorphism is compatible with fusion: if $\lambda^\vee = \sum_{i=1}^s \lambda_i^\vee$ *a sum of dominant coweights, then*

$$D_{k,\lambda^\vee} \cong D_{k,\lambda_1^\vee} \star \cdots \star D_{k,\lambda_s^\vee} \cong KR_{a\lambda^\vee},$$

where a *is obtained from the* d_i*.*

Remark 4.3.1.2. Note that in the simply-laced case, $d_i = 1$ and therefore all KR-modules are Demazure modules.

Let $Q_{a,k}$ be the character of $KR_{a\omega_k}$ as a $\mathfrak{g}$-module.

Theorem 4.3.1.2. *[KR87, Nak04, Her10] The polynomials* $Q_{a,k}$ *satisfy the relations* $Q_{a,0} = 1$ *for* $1 \leq a \leq r$ *and*

$$Q_{a,k+1}Q_{a,k-1} = Q_{a,k}^2 - K_{a,k}$$

where $K_{a,k}$ *is a monomial in the* Q *depending on the Dynkin diagram. Conversely, fixing* $Q_{a,0} = 1$ *and* $Q_{a,1} = \operatorname{ch} KR_{\omega_a}$*, the solutions are the characters of the KR-modules.*

The relation of the theorem defines the Q-system.

Example 4.3.1.1. *For* $\mathfrak{sl}_2$*, the Q-system becomes the simple recursion relation*

$$Q_{k-1}Q_{k+1} = Q_k^2 - 1.$$

In this case we can see this explicitly: $KR_k \cong L(k)$ *and so* $Q_k = chL(k) = z^k + z^{k-2} + \cdots + z^{-k}$*.*

Definition 4.3.1.4. Let $\mathfrak{g}$ as above. The Q-system algebra $\mathcal{A}_\mathfrak{g}$ is defined as

$$\mathcal{A}_\mathfrak{g} = \mathbb{C}[Q_{a,i} \mid a \in \{1,\ldots,r\}\ i \in \mathbb{Z}]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by the Q-system relations.

Proposition 4.3.1.1. *[DFK09] The algebra $\mathcal{A}$ is naturally a cluster algebra with initial cluster $\{Q_{a,0}, Q_{a,1} \mid a \in \{1, \dots, r\}\}$ and exchange matrix given by*

$$\tilde{B} = \begin{pmatrix} C^t - C & -C^t \\ C & 0 \end{pmatrix}$$

Combining the above proposition with the results of the previous section, we see that

Corollary 4.3.1.1. *$P_{coh}^{\mathrm{SL}_2(\mathcal{O})}(\mathrm{Gr}_{\mathrm{PGL}_2})$ is a monoidal cluster categorification of the cluster algebra $\mathcal{A}_{\mathfrak{sl}_2}$.*

Remark 4.3.1.3. In the same way, the Cautis-Williams conjecture can be rewritten as stating that $P_{coh}^{G_{sc}(\mathcal{O})}(\mathrm{Gr}_{G_{ad}})$ is a monoidal categorification of $\mathcal{A}_{\mathfrak{g}}$, which we assume was the motivation for the statement. To provide evidence, we will see in the next section that global sections of certain simple perverse coherent sheaves are Demazure modules, and that in certain cases convolution is sent to fusion.

4.3.2 Convolution and fusion

The Beilinson-Drinfeld Grassmannian

Beilinson and Drinfeld made the important observation that the G-bundle description of the affine Grassmannian can be made relative over a curve X. This leads to the following definition

Definition 4.3.2.1. Let X be a smooth curve over $\mathbb{C}$. The *Beilinson-Drinfeld Grassmannian* Gr_{G,X^n} is the ind-scheme representing the functor

$$R \mapsto \{(\mathcal{F}, \nu, \{x_i\}) \,|\{x_i\} \in X(R)^n,\ \mathcal{F} \in \mathrm{Bun}_G(X_R),\ \nu \text{ a trivialisation on } X_R \backslash \cup \{x_i\}\}/\cong$$

Taking the fibre of the natural map $\mathrm{Gr}_{G,X} \to X$ over a closed point $p \in X$ we get an ind-scheme $\mathrm{Gr}_{G,p}$ isomorphic to the affine Grassmannian Gr_G.

Lemma 4.3.2.1. *For a closed point $(x_1, x_2) \in X^2$, we have*

$$\mathrm{Gr}_{G,(x_1,x_2)} \cong \begin{cases} \mathrm{Gr}_{G,x_1} \times \mathrm{Gr}_{G,x_2} & \text{if } x_1 \neq x_2 \\ \mathrm{Gr}_{G,x_1} & \text{if } x_1 = x_2. \end{cases}$$

The most important application of this lemma for us will be the following corollary, obtained by restricting to orbit closures:

Corollary 4.3.2.1. *Let $\lambda^\vee, \mu^\vee$ be dominant coweights and fix $x \in X$ a closed point. Then there is a flat family $\mathcal{GR}_{\lambda^\vee,\mu^\vee} \to X$ such that the fibre away from x is isomorphic to $\overline{\mathrm{Gr}_{\lambda^\vee}} \times \overline{\mathrm{Gr}_{\mu^\vee}}$ and the fiber over x is isomorphic to $\overline{\mathrm{Gr}_{\lambda^\vee+\mu^\vee}}$.*

Proof. The map is given by taking the restriction of $\mathrm{Gr}_{G,X^2} \to X^2$ to $\{x\} \times X$. $\square$

Suppose G is simply connected and let $\mathcal{L}$ be the (ample) generator of the Picard group of Gr_G. Using a relative notion of the determinant bundle, we can see the following:

Proposition 4.3.2.1. *There is an isomorphism of $G_{\mathcal{O}}$-modules*

$$\Gamma(\overline{\mathrm{Gr}_{\lambda^\vee+\mu^\vee}}, \mathcal{L}^{\otimes k}|_{\lambda^\vee} \star \mathcal{L}^{\otimes k}|_{\mu^\vee}) \cong \Gamma(\overline{\mathrm{Gr}_{\lambda^\vee+\mu^\vee}}, \mathcal{L}^{\otimes k}|_{\lambda^\vee+\mu^\vee}).$$

Proof. First note that $\mathcal{L}^{\otimes k}|_\lambda \tilde{\times} \mathcal{L}^{\otimes k}|_\mu \cong m^* \mathcal{L}^{\otimes k}|_{\lambda+\mu}$ where $m : \overline{\mathrm{Gr}_\lambda} \tilde{\times} \overline{\mathrm{Gr}_\mu} \to \overline{\mathrm{Gr}_{\lambda+\mu}}$ is the convolution map. Therefore, we want to show that $\Gamma(m_* m^* \mathcal{L}^{\otimes k}|_{\lambda+\mu}) \cong \Gamma(\mathcal{L}^{\otimes k}|_\lambda \star \mathcal{L}^{\otimes k}|_\mu)$. The derived projection formula states that

$$m_* m^* \mathcal{L}^{\otimes k}|_{\lambda+\mu} \cong m_* \mathcal{O}_{\lambda \tilde{\times} \mu} \otimes \mathcal{L}_{\lambda+\mu}$$

so this reduces to showing that the derived pushworward of the structure sheaf is given by $m_*\mathcal{O}_{\lambda\tilde{\times}\mu} \cong \mathcal{O}_{\overline{\mathrm{Gr}_{\lambda+\mu}}}$. As $\overline{\mathrm{Gr}_{\lambda+\mu}}$ has rational singularities (see [BK06]), any resolution of singularities $\pi : X_{\lambda,\mu} \to \overline{\mathrm{Gr}_{\lambda+\mu}}$ will satisfy $R\pi_*\mathcal{O}_{\mathcal{X}_{\lambda,\mu}} = \mathcal{O}_{\overline{\mathrm{Gr}_{\lambda+\mu}}}$. Now, note that taking $X_{\lambda,\mu}$ to be the BDSH resolution, the map π factors as $X_{\lambda,\mu} \to \overline{\mathrm{Gr}_\lambda}\tilde{\times}\overline{\mathrm{Gr}_\mu} \to \overline{\mathrm{Gr}_{\lambda+\mu}}$. Now, the map $\rho : X_{\lambda,\mu} \to \overline{\mathrm{Gr}_\lambda}\tilde{\times}\overline{\mathrm{Gr}_\mu}$ is also a resolution of singularities of a space with rational singularities (see [Elk78] and [Vie77]) and therefore

$$m_*\rho_*\mathcal{O}_{\lambda,\mu} = \mathcal{O}_{\overline{\mathrm{Gr}_{\lambda+\mu}}}$$

and $\rho_*\mathcal{O}_{\lambda,\mu} = \mathcal{O}_{\lambda\tilde{\times}\mu}$ which allows us to conclude. □

Perverse coherent sheaves and Demazure modules

As above, we assume $\mathfrak{g}$ simply-laced and G simply connected.

Proposition 4.3.2.2. *As $G_{\mathcal{O}}$-modules and for any $k \geq 0$,*

$$\Gamma(\overline{\mathrm{Gr}_{\lambda^\vee}}, \mathcal{L}^{\otimes k}|_{\lambda^\vee})^\vee \cong D_{k,\lambda^\vee}.$$

Proof. We need to show that the cohomology is concentrated in degree 0 and that its dual is isomorphic to the corresponding Demazure module. For the first, note that $\mathcal{L}$ on Gr_G is given as $i^*\mathcal{O}(1)$ under the embedding $i : \mathrm{Gr}_G \to \mathbb{P}(L(\Lambda_0))$ and therefore the restrictions $\mathcal{L}^{\otimes k}|_\lambda$ are ample for all $k \geq 0$. Finally, $\overline{\mathrm{Gr}_\lambda}$ has rational singularities (see [BK06]) and therefore we can use Kodaira vanishing: note that

$$H^i(\overline{\mathrm{Gr}_\lambda}, \mathcal{L}^{\otimes k}|_\lambda \otimes \omega_\lambda) = 0, \quad i > 0.$$

But by Theorem 4.1.3.1, we know that $\omega_\lambda = \mathcal{L}^{-h^\vee}|_\lambda$ and therefore the result follows. For the global section, we use Lemma 1.3.3.2 together with the standard fact that on $\mathbb{P}(V)$, $H^0(\mathbb{P}(V), \mathcal{O}(1)) \cong V^\vee$. □

Remark 4.3.2.1. There is a version of the Borel-Weil-Bott theorem in the Kac-Moody setting due to Kumar (see [Kum02, Chapter VIII] and references within) which also yields the result. In Kumar's proof, he uses the BDSH resolution explicitly to reduce computations to the smooth case. Those same BDSH resolutions also imply rational singularity of affine Schubert varieties, which is what we use instead.

Combining the proposition with Proposition 4.3.2.1 and Theorem 4.3.1.1, we obtain the following:

Corollary 4.3.2.2.

$$\Gamma(\mathcal{P}_{\omega_i^\vee, k\omega_i} \star \mathcal{P}_{\omega_j^\vee, k\omega_j}) \cong \Gamma(\mathcal{P}_{\omega_i^\vee, k\omega_i}) \star \Gamma(\mathcal{P}_{\omega_j^\vee, k\omega_j}).$$

Note that we have only considered convolution between restrictions of the same power of the determinant bundle. The following proposition, which generalises [CW19, 6.31], can be seen as the next simplest case.

Proposition 4.3.2.3. *As complexes of G-modules,*

$$\Gamma(\mathcal{P}_{\omega_i^\vee,(k+1)\omega_i} \star \mathcal{P}_{\omega_i^\vee,(k-1)\omega_i})^\vee \cong D_{k+1,\omega_i^\vee} \otimes D_{k-1,\omega_i^\vee}[-\dim \mathrm{Gr}_{\omega_i^\vee}].$$

Proof. We follow the argument of [CW19]. Take $m : \overline{\mathrm{Gr}_{\omega_i}} \tilde{\times} \overline{\mathrm{Gr}_{\omega_i}} \to \overline{\mathrm{Gr}_{2\omega_i}}$. Note that $\mathcal{P}_{\omega_i^\vee,(k+1)\omega_i} \star \mathcal{P}_{\omega_i^\vee,(k-1)\omega_i} \cong m_*(m^*(\mathcal{L}^{\otimes k-1+h^\vee}) \otimes \pi_1^*\mathcal{L}^{\otimes 2} \otimes m^*\mathcal{L}^{-h^\vee})$. Now, let $\mathcal{G} = m^*(\mathcal{L}^{\otimes k-1+h^\vee}) \otimes \pi_1^*\mathcal{L}^{\otimes 2}$. $\mathcal{G}$ is nef as it is the tensor product of an ample line bundle and a nef line bundle. We can use the deformation from 4.3.2.1 to see that over a curve X, $\mathcal{G}$ deforms to $\tilde{\mathcal{G}} = \mathcal{L}^{k-1+h^\vee} \boxtimes \mathcal{L}^{k-1+h^\vee}$ away from the base point $x \in X$. Therefore $\mathcal{G}$ is big as $\tilde{\mathcal{G}}$ is and bigness is stable in families. Thus, the Kamata-Viehweg vanishing theorem for spaces with rational singularities allows us to conclude that the complex is concentrated in a single degree. Furthermore, using the flat family as before, we see that the non-trivial degree is isomorphic to

$$\Gamma(\mathcal{P}_{\omega_i^\vee,(k+1)\omega_i} \star \mathcal{P}_{\omega_i^\vee,(k-1)\omega_i}) \cong D^\vee_{k+1,\omega_i^\vee} \otimes D^\vee_{k-1,\omega_i^\vee},$$

as required. $\square$

Open questions and further directions

As a first observation, Cautis and Williams conjecture the stronger result that the algebra $K^{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ is a quantum cluster algebra. Once finished, we expect our type A proof to naturally lift to the quantized case.

The proof of the Cautis-Williams conjecture is still open in general. Whilst this work has provided some additional evidence for its validity, it is not clear to us how one should proceed in general. We believe a generalisation of the techniques of [CW19] might yield the desired short exact sequences in classical types (taking for instance orthogonal or symplectic lattices) but one would then run into the issue of generation, as the fundamental coweights are no longer minuscule.

Another approach is to use the embedding $K^{G_\mathcal{O}}(\mathrm{Gr}_G) \to K^I(Fl_G)$ and work instead on the level of the Iwahori-Hecke algebra. The fundamental question in this approach then becomes describing the pullback of coherent IC-sheaves in the Schubert cell basis we worked on in chapter 2. This is where we believe working with staggered sheaves might be necessary as there is no perverse coherent t-structure on the affine flag variety. Furthermore, working with the affine flag variety has the added advantage of allowing us to use BDSH resolutions of orbit closures, as those are I-equivariant but not $G_\mathcal{O}$-equivariant in general.

Underlying the equivalence, we believe that the Cautis-Williams conjecture is a shadow of an equivalence between $P_{coh}(\mathrm{Gr}_G)$ with suitable equivariance and a certain category of representations of some quantum loop algebra. Indeed, both of these admit very similar categorical properties, such as the existence of renormalised R-matrices.

Bibliography

[AB10] Dmitry Arinkin and Roman Bezrukavnikov. Perverse coherent sheaves. *Mosc. Math. J.*, 10(1):3–29, 271, 2010. 4, 66, 67, 74, 77, 115

[Ach09] Pramod N. Achar. Staggered t-structures on derived categories of equivariant coherent sheaves. *Int. Math. Res. Not. IMRN*, (20):3843–3900, 2009. 4, 69, 72

[AH19] Pramod N. Achar and William D. Hardesty. Calculations with graded perverse-coherent sheaves. *Q. J. Math.*, 70(4):1327–1352, 2019. 99

[AK07] Eddy Ardonne and Rinat Kedem. Fusion products of Kirillov–Reshetikhin modules and fermionic multiplicity formulas. *Journal of algebra*, 308(1):270–294, 2007. 119

[AS09] Pramod N. Achar and Daniel S. Sage. Staggered sheaves on partial flag varieties. *Comptes Rendus Mathematique*, 347(3):139 – 142, 2009. 65, 70, 78, 81

[AT11] Pramod N. Achar and David Treumann. Baric structures on triangulated categories and coherent sheaves. *Int. Math. Res. Not. IMRN*, (16):3688–3743, 2011. 69, 72

[BD91] A. Beilinson and V. Drinfeld. Quantization of the Hitchin integrable system. *unpublished*, 1991. 12, 19, 41, 94

[Bez03] Roman Bezrukavnikov. Quasi-exceptional sets and equivariant coherent sheaves on the nilpotent cone. *Representation theory*, 7(1):1–18, 2003. 68

[BFM05] Roman Bezrukavnikov, Michael Finkelberg, and Ivan Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. *Compos. Math.*, 141(3):746–768, 2005. 5, 65, 74, 97, 104, 105, 108

[BFN18] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. *Adv. Theor. Math. Phys.*, 22(5):1071–1147, 2018. 5

[BK06] Alexander Braverman and David Kazhdan. Some examples of Hecke algebras for two-dimensional local fields. *Nagoya Math. J.*, 184:57–84, 2006. 61, 94, 96, 123

[BL94a] Arnaud Beauville and Yves Laszlo. Conformal blocks and generalized theta functions. *Comm. Math. Phys.*, 164(2):385–419, 1994. 8, 29

[BL94b] Joseph Bernstein and Valery Lunts. *Equivariant sheaves and functors.* Lecture notes in mathematics (Springer-Verlag) ; 1578. Springer-Verlag, Berlin ; London, 1994. 115

[BL95] Arnaud Beauville and Yves Laszlo. Un lemme de descente. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(3):335–340, 1995. 11, 27, 91

[Bou05] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7-9.* Bourbaki, Nicolas. Elements of mathematics. Springer, Berlin, 2005. 31

[BR18] P. Baumann and S. Riche. Notes on the geometric Satake equivalence. *arXiv:1703.07288v3*, 2018. 4, 44, 75

[CG10] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition. 3, 63

[CW19] Sabin Cautis and Harold Williams. Cluster theory of the coherent Satake category. *Journal of the American Mathematical Society*, 32(3):709–778, 2019. 5, 6, 45, 46, 49, 50, 76, 77, 89, 96, 97, 100, 101, 104, 114, 118, 124, 125

[Dem74] Michel Demazure. Désingularisation des variétés de Schubert généralisées. *Annales scientifiques de l'École normale supérieure*, 7(1):53–88, 1974. 51

[DFK09] Philippe Di Francesco and Rinat Kedem. Q-systems as Cluster Algebras ii: Cartan Matrix of Finite Type and the Polynomial Property. *Letters in mathematical physics*, 89(3):183–216, 2009. 121

[Elk78] Renée Elkik. Singularités rationnelles et déformations. *Invent. Math.*, 47(2):139–147, 1978. 123

[Fal03] Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *Journal of the European Mathematical Society : JEMS*, 5(1):41–68, 2003. 21, 22, 90

[FF21] Michael Finkelberg and Ryo Fujita. Coherent IC-sheaves on type A_n affine grassmannians and dual canonical basis of affine type A_1. *Representation theory*, 25(3):67, 2021. 115

[FL07] G Fourier and P Littelmann. Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. *Advances in mathematics (New York. 1965)*, 211(2):566–593, 2007. 119

[FT17] Michael Finkelberg and Alexander Tsymbaliuk. Multiplicative slices, relativistic Toda and shifted quantum affine algebras. 2017. 50, 105

[FWZ16] S. Fomin, Lauren Williams, and A. Zelevinsky. Introduction to cluster algebras. chapters 1-3. *arXiv: Combinatorics*, 2016. 101

[GÏ0] Ulrich Görtz. Affine Springer fibers and affine Deligne-Lusztig varieties. In *Affine flag manifolds and principal bundles*, Trends Math., pages 1–50. Birkhäuser/Springer Basel AG, Basel, 2010. 11

[Gin90] V. A. Ginzburg. Sheaves on a loop group, and Langlands duality. *Funktsional. Anal. i Prilozhen.*, 24(4):76–77, 1990. 18

[Gro60] A Grothendieck. *Eléments de géométrie algébrique.* Publications mathématiques (Institut des hautes études scientifiques) ; no. 4, 8, 11, 17, 20, 24, 28, 32. [Presses Universitaires de France], Paris, 1960. 84

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. 53, 99, 109

[Her10] David Hernandez. Kirillov–Reshetikhin Conjecture: The General Case. *International mathematics research notices*, 2010(1):149–193, 2010. 120

[HLR18] Thomas J Haines, João Lourenço, and Timo Richarz. On the normality of schubert varieties: remaining cases in positive characteristic. 2018. 18

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry.* Oxford mathematical monographs. Clarendon, Oxford, 2006. 77

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups.* Mathematical surveys and monographs ; no. 107. American Mathematical Society, Providence, RI, 2nd ed. edition, 2003. 20, 97

[Kac90] Victor G. Kac. *Infinite dimensional Lie algebras [electronic resource].* Cambridge core. Cambridge, third edition. edition, 1990. 23, 25, 26, 29, 93

[Kas89] M. Kashiwara. *The flag manifold of Kac-Moody Lie algebra.* Johns Hopkins Univ. Press, Baltimore, MD, 1989. 5, 7, 32, 86

[KK87] Bertram Kostant and Shrawan Kumar. T-equivariant K-theory of generalized flag varieties. *Proceedings of the National Academy of Sciences of the United States of America*, 84(13):4351, 1987. 50, 51, 55

[KL87] David Kazhdan and George Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Inventiones mathematicae*, 87(1):153–215, 1987. 2, 52

[KR87] A. N. Kirillov and N. Yu. Reshetikhin. Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 160(Anal. Teor. Chisel i Teor. Funktsiĭ. 8):211–221, 301, 1987. 120

[KS09] Masaki Kashiwara and Mark Shimozono. Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials. *Duke Math. J.*, 148(3):501–538, 2009. 40

[KT95] Masaki Kashiwara and Toshiyuki Tanisaki. Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. *Duke mathematical journal*, 77(1):21–62, 1995. 34

[Kum02] S Kumar. *Kac-Moody groups, their flag varieties, and representation theory.* Progress in mathematics ; v. 204. Birkhäuser, Boston, 2002. 3, 7, 8, 14, 22, 23, 28, 38, 78, 81, 90, 124

[Kum17] Shrawan Kumar. Positivity in T-equivariant K-theory of flag varieties associated to Kac-Moody groups. *J. Eur. Math. Soc. (JEMS)*, 19(8):2469–2519, 2017. With an appendix by Masaki Kashiwara. 61, 85

[LS97] Yves Laszlo and Christoph Sorger. The line bundles on the moduli of parabolic G-bundles over curves and their sections. *Annales scientifiques de l'École normale supérieure*, 30(4):499–525, 1997. 28, 29, 91, 92

[Lus85] George Lusztig. Equivariant K-theory and representations of Hecke algebras. *Proc. Amer. Math. Soc.*, 94(2):337–342, 1985. 2

[Mat88] Olivier Mathieu. *Formules de caractères pour les algèbres de Kac-Moody générales*. Astérisque. 159-160. Société mathématique de France, Paris, 1988. 30

[MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007. 46, 75

[Nak04] Hiraku Nakajima. Quiver varieties and t-analogs of q-characters of quantum affine algebras. *Ann. of Math. (2)*, 160(3):1057–1097, 2004. 120

[NP01] B. C. Ngô and P. Polo. Résolutions de Demazure affines et formule de Casselman-Shalika géométrique. *J. Algebraic Geom.*, 10(3):515–547, 2001. 38, 98

[PR04] Harsh Pittie and Arun Ram. A Pieri-Chevalley formula for k(g/b). 2004. 15, 59

[PR08] G Pappas and M Rapoport. Twisted loop groups and their affine flag varieties. *Advances in mathematics (New York. 1965)*, 219(1):118–198, 2008. 12

[Ric16] Timo Richarz. Affine grassmannians and geometric Satake equivalences. *International mathematics research notices*, 2016(12):3717–3767, 2016. 12

[Sch16] Tobias Schmidt. Hecke algebras and affine flag varieties in characteristic p. *Journal of Pure and Applied Algebra*, 220(9):3233–3247, 2016. 50, 51, 54, 55

[Sor00] Christoph Sorger. Lectures on moduli of principal G-bundles over algebraic curves. In *School on Algebraic Geometry (Trieste, 1999)*, volume 1 of *ICTP Lect. Notes*, pages 1–57. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000. 7, 23, 91

[Vie77] Eckart Viehweg. Rational singularities of higher dimensional schemes. *Proc. Amer. Math. Soc.*, 63(1):6–8, 1977. 123

[VV09a] M Varagnolo and E Vasserot. Finite-dimensional representations of DAHA and affine Springer fibers: The spherical case. *Duke mathematical journal*, 147(3):439–540, 2009. 34

[VV09b] Michaela Varagnolo and Eric Vasserot. Double affine Hecke algebras and Affine Flag Manifolds i. *arXiv:0911.5328*, 2009. 35, 40

[Zhu09] Xinwen Zhu. Affine demazure modules and t-fixed point subschemes in the affine grassmannian. *Advances in mathematics (New York. 1965)*, 221(2):570–600, 2009. 17, 30

[Zhu14] Xinwen Zhu. On the coherence conjecture of pappas and rapoport. *Annals of mathematics*, 180(1):1–85, 2014. 95

[Zhu17] Xinwen Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 59–154. Amer. Math. Soc., Providence, RI, 2017. 3, 8, 9, 10, 11, 19, 91