Optimal Stopping with Behavioral Objective

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1 Introduction

In this paper, the optimal stopping problem is studied with geometric Brownian motion with probability distortion. As we know, the classical optimal stopping problem is without probability distortion. There have been well-developed methods of solving the classical problem by using the approaches of probability (martingale) or partial differential equations (dynamic programming or variational inequality)[2]. All these approaches are based on the time-consistency of the underlying problem. However, the appearance of the probability distortion breaks the linearity of the original problem and thus leading to the disappearance of the time-consistency feature . As a result, the traditional approaches can not be applied to the problem with probability distortion.

The reason we study the optimal stopping problem with probability distortion is because that in reality, people tend to exaggerate small probabilities. For example, people play lottery games even when they know the probability of winning is extremely small and the risk of playing is huge because they exaggerate the probability of huge gains.

Based on the discovery of Xu and Zhou (2011), the optimal stopping problem with probability distortion can be solved by transforming the problem into a distribution or quantile formulation. Then by solving the new formulation, the optimal stopping state S_{τ^*} can be found which leads to the identification of the optimal stopping time τ^*. However, the optimal stopping time found in Xu and Zhou's paper is subject to the objective determined at time 0. As the problem is defined under time-inconsistency, it is unlike the original optimal stopping problem with time-consistency in which the optimal stopping time $\tau^*$is not changed when we want to find the optimal stopping time at any time t such that $t \leq \tau^*$. Under probability distortion, the optimal stopping time τ^*varies when the decision-making time t of the objective changes. As a decision maker (agent) involved in this case, he only concerns whether $\tau^* = t$ or not.

In this paper, we define a natural extension of the "optimal" stopping time $\tau^*$and solve the stopping time by similar method in Xu and Zhou (2011). We will discuss the different cases when the distortion function or transformed payoff function is convex or concave. Some examples will be given to show how these results will be used in real cases. Also, we will interpret the results in economical ways.

The rest of the paper is organized as follows. In section 2, we formulate the definition of our "optimal" stopping time and transfer our objective into one with martingale underlying process. In section 3, we solve the problem subject to a special case when the transformed payoff function is non-increasing. In section 4, we firstly introduce the transformation of the original objective into its quantile or distribution formulation and then discuss our "optimal" stopping times when we have different shapes of the probability distortion functions and transformed payoff functions. We also interpret the financial or economical implications of the derived results. Finally in section 5, we have the conclusion of the whole paper.

2 Problem setting

2.1 Definition of the problem

Suppose P_t is a stochastic process, $\{P_t, t \geq 0\}$, which follows a Geometric Brownian Motion (GBM),

$$dP_t = \mu P_t dt + \sigma P_t dB_t, P_0 > 0 \tag{2.1}$$

where μ and $\sigma > 0$ are real constants and $\{B_t, t \geq 0\}$ is a standard one-dimensional Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The stochastic process P_t has been interpreted as the price process of an asset in many discussions.

We let $\mathcal{T}_t$ be the set of all stopping times τ with $\mathbb{P}(\tau < +\infty) = 1$ and $\tau \geq t$. A decision maker chooses $\tau \in \mathcal{T}_t$ to stop the process and obtain a payoff $U(P_\tau)$ where $U(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is the so called payoff function. The payoff function is non-decreasing and continuous. Also a probability distortion function $w(\cdot)$ is applied to our objective to represent the feature that people tend to exaggerate probabilities in reality. The probability distortion $w(\cdot) : [0,1] \mapsto [0,1]$ is a strictly increasing, continuous function with $w(0) = 0$ and $w(1) = 1$.

In the classical optimal stopping problem, the objective which the agents try to maximize over$\tau \in \mathcal{T}_t$ is

$$T(\tau; t) = \boldsymbol{E}[U(P_\tau)] = \int_0^\infty \mathbb{P}(U(P_\tau) > x)dx \text{ [2]} \quad \forall t \geq 0 \tag{2.2}$$

This optimal problem can be solved by martingale theories or the dynamic programming methods under time-consistency feature. However, with the function $w(\cdot)$ applied to the probability scale in the above objective, its time-consistency feature will be gone. So new methods are needed to find a solution to our new objective

$$J(\tau; t) = \int_0^\infty w(\mathbb{P}_t(U(P_\tau) > x))dx \quad \forall t \geq 0 \tag{2.3}$$

where the t involved in $J(\tau; t)$ represents the time axes and τ represents all the stopping times belonging to the set $\mathcal{T}_t$. We can see clearly that the objective is changeable with t as the probability of our payoff reaching some specific state can be different at each specific t. Therefore the optimal stopping problem subject to this objective becomes time inconsistent. At different time t, we may have different optimal stopping time. So it is of no use to consider the optimal stopping time which is larger that t, we only concern about whether the optimal stopping time equals t or not. As in Xu and Zhou's paper [1], the

objective is $J(\tau)$ indicating the process is determined at time 0, i.e. the value of P_0 is known, and their problem is to find an optimal stopping time $\tau^* \geq 0$ such that $J(\tau^*)$ maximizes $J(\tau)$. However, our problem is to find a smallest optimal stopping time τ^*such that $J(\tau^*;\tau^*)$ maximizes $J(\tau;\tau^*)$. To make the problem more clear, the definition of our "optimal" stopping time is given below.

Definition 1. τ^* is the solution subject to the objective (2.3) we want to find satisfying two conditions:

1. $J(\tau^*;\tau^*) \geq J(\tau;\tau^*)$, $\forall \tau \geq \tau^*$;
2. For any $t < \tau^*$, there exists $\tau > t$ such that $J(t;t) < J(\tau;t)$.

The definition of our "optimal" stopping time simply means that we want to find a τ^* such that $\tau^* = inf\{t \geq 0 : J(t;t) \geq J(\tau;t)\, \forall \tau \in \mathcal{T}_t\}$. So now our problem is to find the conditions such that when first time the immediate stop is optimal.

2.2 Transformation of the objective

For the following analysis we need to transform the objective (2.3) into one in which the underlying process is a martingale. Let $\beta := \frac{-2\mu+\sigma^2}{\sigma^2} \neq 0, S_t = P_t^\beta$, we can find that $dS_s = \beta\sigma S_s dB_s, S_t = P_t^\beta := m > 0$ where t is the determination time of the system by using Itô's rules. Now we define that $u(x) = U(x^{\frac{1}{\beta}})$, $\forall x \in (0, +\infty)$.

Then we can see that the objective (2.3) can be modified like below:

$$J(\tau;t) = \int_0^\infty w(\mathbb{P}_t(U(P_\tau) > x))dx = \int_0^\infty w(\mathbb{P}_t(u(S_\tau) > x))dx \quad \forall t \geq 0 \quad (2.4)$$

in which $u(\cdot)$ may now have very different shape than $U(\cdot)$. Depending on the discussion in the paper of Xu and Zhou (2011) [1], for a general non-decreasing function $U(\cdot)$, $u(x) = U(x^{\frac{1}{\beta}})$ is non-increasing if $\beta < 0$, and non-decreasing if $\beta > 0$. Also based on the paper of Shiryayev, Xu and Zhou (2008) [4], non-increasing $u(\cdot)$ means the asset is good while non-decreasing $u(\cdot)$ indicates that the asset is bad. We will use these descriptions to interpret our results in an economical way later on.

NOTE : We will assume that $S_t = s$ throughout this paper.

3 Problem solving when $\beta < 0$

Based on the previous section, $u(\cdot)$ is non-increasing when $\beta < 0$.

Theorem 2. *If u(·) is non-increasing, then the objective (2.3) has the optimal value u(0+) and if u(l)=u(0+) for some l>0, then our optimal stopping time subject to definition1 is of the form*

$$\tau^* = inf\{t \geq 0 : s \leq l\}.$$

If u(l)<u(0+) for every l>0, then we have our τ^* doesn't exist.

Proof. As $u(\cdot)$ is non-increasing, so the maximal value of $u(\cdot)$ is $u(0+)$.We have

$$\begin{aligned} \sup_{\tau\in\mathcal{T}_t} J(\tau;t) &= \sup_{\tau\in\mathcal{T}_t} \int_0^{u(0+)} w(\mathbb{P}_t(u(S_\tau) > x))dx \\ &\leq \sup_{\tau\in\mathcal{T}_t} \int_0^{u(0+)} w(1)dx = u(0+). \end{aligned}$$

Also we have

$$\begin{aligned} \sup_{\tau\in\mathcal{T}_t} J(\tau;t) &\geq \limsup_{T\to\infty} J(T) \geq \liminf_{T\to\infty} J(T) \\ &= \liminf_{T\to\infty} \int_0^\infty w(\mathbb{P}_t(u(S_T) > x))dx \\ &\geq \int_0^\infty \liminf_{T\to\infty} w(\mathbb{P}_t(u(S_T) > x))dx \\ &\geq \int_0^\infty w(\liminf_{T\to\infty} \mathbb{P}_t(u(S_T) > x))dx \\ &= \int_0^\infty w(\mathbb{P}_t(u(0+) > x))dx \\ &= u(0+) \end{aligned}$$

where as S_m is a martingale, we use the fact that $\lim_{m\to\infty} S_m = 0$ almost surely. These inequalities prove that $\sup_{\tau\in\mathcal{T}_t} J(\tau;t) = u(0+)$. Also as S_m is an exponential martingale, so $S_m > 0$. If there exists $l > 0$ such that $u(l) = u(0+)$, then we have $u(x) = u(0+)$ for each $x \in (0, l)$ as $u(\cdot)$ is non-increasing. As a result, we have that $\sup_{\tau\in\mathcal{T}_t} J(\tau;t) = u(x)\ \forall x \in (0, l]$. So the optimal stopping time of definition 1 occurs when first time we have $J(t;t) = u(x)\ \forall x \in (0, l]$, which means

$$\tau^* = inf\{t \geq 0 : s \leq l\}.$$

If there is no $l > 0$ such that $u(l) = u(0+)$, the optimal stopping time τ^* will not occur as the value $s > 0$ which means $u(s) < u(0+)$. The optimal value of $J(\tau;t)$ will never be reached for all $t \geq 0$. Thus no optimal stopping time τ^*can be achieved. □

4 Problem solving when $\beta > 0$

As discussed in previous sections, $\beta > 0$ indicates that $u(\cdot)$ is non-decreasing. Following the methods described on Xu and Zhou's paper (2011), we need to use distribution and quantile formulation to solve the problem in this case. This section will firstly give you an introduction on the distribution and quantile functions and then solve the problem by using these new formulations.

4.1 Distribution and Quantile formulations

Firstly, the assumption imposed is specified as below.

Assumption.[1]

1. $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is non-decreasing, absolutely continuous with $u(0) = 0$;
2. $w(\cdot) : [0,1] \mapsto [0,1]$ is strictly increasing, absolutely continuous with $w(0) = 0$ and$w(1) = 1$.

Remark. $u(0) = 0$ is just for convenience, we can take $\hat{u}(\cdot) = u(\cdot) - u(0)$ if $u(0) \neq 0$.

Also we say $F : \mathbb{R}^+ \mapsto [0,1]$ is a cumulative distribution function if $F(0) = 0$, $F(+\infty) = 1$ and F is non-decreasing and cadlág; $G : [0,1] \mapsto \mathbb{R}^+$ is a quantile function if $G(0) = 0$, $G(x) > 0 \ \forall x \in (0,1)$, G is non-decreasing and left-continuous.

Following the definitions and theorems described in Xu and Zhou's paper, we can define the constraint distribution set $\mathcal{D}$ and the quantile set $\mathcal{Q}$ of S_τ as follows, [1]

$$\mathcal{D} := \{F : \mathbb{R}^+ \mapsto [0,1] \mid F \text{ is the CDF of } S_\tau \text{and } \int_0^\infty (1 - F(x))dx \leq s\},$$
$$\mathcal{Q} := \{G : [0,1) \mapsto \mathbb{R}^+ \mid G = F^{-1} \text{ for some } F \in \mathcal{D} \text{and } \int_0^1 G(x)dx \leq s\}.$$

By setting all these definitions ready, we have the following transformations from the objective (2.3) by using the distribution and quantile functions based on the Lemma 4.1 in Xu and Zhou'paper (2011). The prove of these transformations can be found in page 11 of their paper. Here we just write down the new formulations.

Formulation.[1]

$$J(\tau; t) = J_D(F; t) := \int_0^\infty w(1 - F(x))u^{'}(x)dx \tag{4.1}$$

$$J(\tau; t) = J_Q(G; t) := \int_0^1 u(G(x))w^{'}(1 - x)dx \tag{4.2}$$

where F and G are the CDF and the quantile function of S_τ respectively for $\tau \in \mathcal{T}_t$.
From the new formulation, we can see clearly that

$$\sup_{\tau \in \mathcal{T}_t} J(\tau; t) = \sup_{F \in \mathcal{D}} J_D(F; t) = \sup_{G \in \mathcal{Q}} J_Q(G; t).$$

So we can find the optimal F or G first, then find the corresponding S_{τ^*}. Finally the "optimal" stopping time τ^*can be achieved. With the objective reformulated, we will obtain the optimal stopping time subject to definition 1 by considering different shapes of the functions $u(\cdot)$ and $w(\cdot)$.

Remark. An important feature has to be noticed is that both the sets $\mathcal{D}$ and $\mathcal{Q}$ are convex.

4.2 Case with convex $w(\cdot)$

In this subsection, we assume that the probability distortion function $w(\cdot)$ is convex under the circumstance when $u(\cdot)$ is non-decreasing. In this case, the distribution formulation (4.1) is easier to use than the quantile formulation (4.2) as the shape of $w(\cdot)$ is known but the shape of $u(\cdot)$ is unknown. Based on the discovery of Xu and Zhou, the maximum of the objective occurs just at the "corners" of the constraint set $\mathcal{D}$ [1]. They found that these corners must be step functions having at most two jumps.

Define the set $\mathcal{D}_2$as below,

$$\mathcal{D}_2 := \{F \in \mathcal{D} : F = c(s)\mathbf{1}_{[a(s),b(s))} + \mathbf{1}_{[b(s),+\infty)}, 0 < c \leq 1, 0 < a(s) \leq b(s)\},$$

where $c(s)$, $b(s)$, $a(s)$ are functions depending on the value of s. Xu and Zhou found that if $w(\cdot)$ is convex, then we will have $\sup_{F\in\mathcal{D}} J_D(F;t) = \sup_{F\in\mathcal{D}_2} J_D(F;t)$. The prove can be found in page 17 of their paper [1]. Based on the their discoveries, we formulate the theorem for obtaining τ^* as below.

Theorem 3. *If $w(\cdot)$ is convex, then we have that*

$$\begin{aligned}\sup_{\tau\in\mathcal{T}_t} J(\tau;t) &= \sup_{F\in\mathcal{D}_2} J(F;t) \\ &= \sup_{0<a(s)\leq s\leq b(s)} [(1 - w(\tfrac{s-a(s)}{b(s)-a(s)}))u(a(s)) + w(\tfrac{s-a(s)}{b(s)-a(s)})u(b(s))]\end{aligned} \tag{4.3}$$

Moreover, if

$$(a^*(s), b^*(s)) = \operatorname*{argmax}_{0<a(s)\leq s\leq b(s)} [(1-w(\frac{s-a(s)}{b(s)-a(s)}))u(a(s))+w(\frac{s-a(s)}{b(s)-a(s)})u(b(s))] \tag{4.4}$$

then we have

$$\tau^* = inf\{t \geq 0 : s \in \{s : a^*(s) = s \text{ or } b^*(s) = s\}\}. \tag{4.5}$$

is an optimal stopping time subject to definition 1.
And we have if τ^* exists

$$\sup_{\tau\in\mathcal{T}_t} J(\tau;t) = J(t;t) = J(\tau^*;\tau^*) = u(s).$$

Proof. [1] Based on the discoveries of Xu and Zhou (2011), we need to find the optimal distribution function in $\mathcal{D}_2$ to maximize the objective (2.3). The form of the distribution function F can be expressed as:

$$F(x) = c(s)\mathbf{1}_{[a(s),b(s))} + \mathbf{1}_{[b(s),+\infty)}, x \in [0, +\infty).$$

Thus we have that the objective (2.3) is transformed to

$$\begin{aligned} J(\tau;t) = J_D(F;t) &= \int_0^\infty w(1-F(x))u'(x)dx \\ &= u(a(s)) + w(1-c(s))(u(b(s)) - u(a(s))) \\ &= (1-w(1-c(s)))u(a(s)) + w(1-c(s))u(b(s)), \end{aligned}$$

and the constraint for the distribution function becomes

$$\begin{aligned} \int_0^\infty (1-F(x))dx &= a(s) + (1-c(s))(b(s)-a(s)) \\ &= a(s)c(s) + b(s)(1-c(s)). \end{aligned}$$

Now, the problem becomes that we need to find a τ^* from definition 1 such that it maximizes:

$$J(a(s), b(s), c(s); t) := (1-w(1-c(s)))u(a(s)) + w(1-c(s))u(b(s)) \tag{4.6}$$

with the constraints $a(s)c(s) + b(s)(1-c(s)) \leq s$,$0 < a(s) \leq b(s), 0 \leq c(s) \leq 1$.

We can deduce that $a(s) \leq s$, otherwise the first constraint can be written as $a(s)c(s) + b(s)(1-c(s)) \leq s < a$, which leads to the inequality $c(s) > 1$ which contradicts to the definition of $F(x)$ where $0 \leq c \leq 1$. Also, we can deduce that the maximum of $J(a(s), b(s), c(s); t)$ should occur in the range when $s \leq b(s)$, since when $a(s)$ and $c(s)$ are fixed, we should choose $b(s)$ as large as possible based on the non-decreasing feature of $u(\cdot)$. So the range we need to consider is $0 < a(s) \leq s \leq b(s)$ to maximize (4.6). Also we notice that $J(a(s), b(s), c(s); t)$ can also be written as $u(a(s)) + w(1-c(s))(u(b(s)) - u(a(s)))$ where we find it is non-increasing in $c(s)$ when we set $a(s)$ and $b(s)$ to be fixed. From the first constraint, we have $c(s) \geq \frac{b(s)-s}{b(s)-a(s)}$, so we have the maximum when $c(s) = \frac{b(s)-s}{b(s)-a(s)}$ subject to $a < b$, while c can be any arbitrary belonging to the set $[0, 1]$ when $a(s) = b(s)$. The reason that c can be any arbitrary is because that the $a(s)c(s)$ and $b(s)c(s)$ terms in the first constraint cancel out and left with $b(s) \leq s$. As we just discussed before $b(s) \geq s$, so we must have that $b(s) = a(s) = s$ and the value of $J(a(s), b(s), c(s))$ equals $u(s)$. Therefore,

$$\begin{aligned} \sup_{\tau \in \mathcal{T}_t} J(\tau;t) &= \sup_{F \in \mathcal{D}_2} J_D(F;t) \\ &= \sup_{0 < a(s) \leq s \leq b(s)} [(1 - w(\tfrac{s-a(s)}{b(s)-a(s)}))u(a(s)) + w(\tfrac{s-a(s)}{b(s)-a(s)})u(b(s))]. \end{aligned}$$

If we obtain the $a^*(s)$ and $b^*(s)$ with $0 < a^*(s) < b^*(s)$ from (4.4), then we

can see clearly that S_{τ^*} has a two-point distribution with $\mathbb{P}(S_{\tau^*} = a^*(s)) = c^*$ and $\mathbb{P}(S_{\tau^*} = b^*(s)) = 1 - c^*$. So we can see clearly that our "optimal" stopping time occurs if and only if when first time s belongs to the set $\{s : s = a^*(s)$ or $s = b^*(s)\}$. When $a^*(s) = b*(s)$, just as we discussed above, we must have that $a^*(s) = b^*(s) = s$ for all possible values of s which indicates $\tau^* = 0$. So (4.5) defines correctly the optimal stopping time τ^* subject to definition 1.
Further more, when $0 < a^*(s) < b^*(s)$, if $s = a^*(s)$, substitute it into (4.3), we get the maximum of the objective is $u(s)$. Same thing happens when $s = b^*(s)$ as well, we get the maximum value $u(s)$. When $a^*(s) = b*(s)$, just as we have discussed above, the value is also $u(s)$. □

We can interpret the results in an economical way. According to Yarri (1987) [5], a convex probability distortion captures the risk aversion of an agent. The theorem suggests that a risk-averse agent's optimal stopping strategy is to stop at one of the two thresholds, $a*(s)$ or $b^*(s)$. In the context of stock selling, this corresponds to the "stop gain and cut loss" strategy [1]. So one should sell the stock either when it reaches a gaining level $b*(s)$ or reaches a loss level $a^*(s)$.
Note : We will use the symbol J to short for the objective $J(\tau; t)$ through all my examples.

Example 4. Suppose we have convex $w(\cdot)$ and concave $u(\cdot)$ to have the form as below.

$$\begin{cases} w(x) = x^2, x \in [0, 1] \\ u(x) = x^{\frac{1}{2}}, x \in \mathbb{R}^+ \end{cases}$$

By following the theorem 3, we have our objective of the form:

$$J = (1 - (\frac{s - a(s)}{b(s) - a(s)})^2)a(s)^{\frac{1}{2}} + (\frac{s - a(s)}{b(s) - a(s)})^2 b(s)^{\frac{1}{2}}. \tag{4.7}$$

Next step is that we have to find a pair of $(a^*(s), b^*(s))$ such that it maximize J. The method we use is we differentiate J with respect to $a(s)$ and $b(s)$ separately, by equating the equations to 0 then we can find the pair.

$$\begin{aligned} \frac{\partial J}{\partial b(s)} &= 2(\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-a(s)}{(b(s)-a(s))^2}a(s)^{\frac{1}{2}} - 2(\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-a(s)}{(b(s)-a(s))^2}b(s)^{\frac{1}{2}} \\ &+\tfrac{1}{2}(\tfrac{s-a(s)}{b(s)-a(s)})^2 b(s)^{-\frac{1}{2}} = 0 \end{aligned} \tag{4.8}$$

By simply re-arranging the above equation, we get

$$(b(s)^{\frac{1}{2}} - a(s)^{\frac{1}{2}})(b(s)^{\frac{1}{2}} + a(s)^{\frac{1}{2}}) = 4b(s)^{\frac{1}{2}}(b(s)^{\frac{1}{2}} - a(s)^{\frac{1}{2}}) \tag{4.9}$$

which indicates that $a(s) = b(s)$. Also by differentiating with respect to $a(s)$, we have

$$\begin{aligned}\frac{\partial J}{\partial a(s)} &= -2(\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-b(s)}{(b(s)-a(s))^2}a(s)^{\frac{1}{2}} + \tfrac{1}{2}(1-(\tfrac{s-a(s)}{b(s)-a(s)})^2)a(s)^{-\frac{1}{2}} \\ &+2(\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-b(s)}{(b(s)-a(s))^2}b(s)^{\frac{1}{2}} = 0.\end{aligned} \tag{4.10}$$

By simply re-arranging the equation, we can get that

$$4(s-a(s))(s-b(s))a(s)+(b(s)-a(s))^2(a(s)^{\frac{1}{2}}b(s)^{\frac{1}{2}}+a(s)) = (s-a(s))^2 \tag{4.11}$$

Then we substitute the results $a(s) = b(s)$ obtained from (4.9), we can get that

$$4(s-a(s))^2a(s) = (s-a(s))^2,$$

which indicates that $a^*(s) = b^*(s) = s$.
Following the theorem, we can conclude that s can be any arbitrary to make the immediate stop optimal which implies that

$$\tau^* = 0.$$

This indicates that the optimal stopping strategy of an agent with such distortion and transformed payoff functions should be stopped immediately.

Example 5. Suppose we have convex $w(\cdot)$ and convex $u(\cdot)$ to have the form as below.

$$\begin{cases} w(x) = x^2, x \in [0,1] \\ u(x) = \begin{cases} \frac{1}{2}x, x \in [0,2) \\ 3x-5, x \in [2,+\infty) \end{cases} \end{cases}$$

In this problem, the form of the objective depends on the range of $a(s)$ and $b(s)$:

First case: $a(s)$ and $b(s)$ are all in the range $[0,2)$
We have the form of the objective as

$$J = (1-(\frac{s-a(s)}{b(s)-a(s)})^2)\frac{1}{2}a(s) + (\frac{s-a(s)}{b(s)-a(s)})^2\frac{1}{2}b(s). \tag{4.12}$$

By differentiating J with respect to $a(s)$ and $b(s)$, we will have

$$\begin{aligned}\frac{\partial J}{\partial b(s)} &= (\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-a(s)}{(b(s)-a(s))^2}a(s) - (\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-a(s)}{(b(s)-a(s))^2}b(s) \\ &+\tfrac{1}{2}(\tfrac{s-a(s)}{b(s)-a(s)})^2 = 0.\end{aligned} \tag{4.13}$$

Simply re-arranging the equation, we have

$$(b(s) - a(s)) = 2(b(s) - a(s)),$$

which indicates that $a(s) = b(s)$.
By differentiating with respect to $a(s)$, we have

$$\begin{aligned} \frac{\partial J}{\partial a(s)} &= -(\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-b(s)}{(b(s)-a(s))^2}a(s) + \tfrac{1}{2}(1 - (\tfrac{s-a(s)}{b(s)-a(s)})^2) \\ &+(\tfrac{s-a(s)}{b(s)-a(s)})\tfrac{s-b(s)}{(b(s)-a(s))^2}b(s) = 0 \end{aligned} \tag{4.14}$$

By re-arranging the equation, we get

$$2(s - a(s))(s - b(s))a(s) + (b(s) - a(s))^2 = (s - a(s))^2$$

Substituting $a(s) = b(s)$ into (4.15), we have

$$2(s - a(s))^2 a(s) = (s - a(s))^2.$$

The only possible choice is that $a(s) = s$, which combined with the previous result, we have to have $a(s)^* = b(s)^* = s$. So when $a(s)$ and $b(s)$ are all in the range $[0, 2)$, we have our optimal stopping time

$$\tau^* = 0.$$

Second case: $a(s)$ and $b(s)$ are all in the range $[2, +\infty)$
Follow nearly the same lines of prove just like in case 1, we will find that our optimal stopping time

$$\tau^* = 0.$$

Third case: As we have $a(s) \leq b(s)$, so this is the case when $a(s)$ is in range $[0, 2)$ while $b(s)$ is in range $[2, +\infty)$
We have the form of the objective

$$J = (1 - (\frac{s - a(s)}{b(s) - a(s)})^2)\frac{1}{2}a(s) + (\frac{s - a(s)}{b(s) - a(s)})^2(3b(s) - 5). \tag{4.15}$$

By differentiating with respect to $a(s)$ and $b(s)$, we have that

$$\frac{\partial J}{\partial b(s)} = (\frac{s-a(s)}{b(s)-a(s)})\frac{s-a(s)}{(b(s)-a(s))^2}a(s) - 2(\frac{s-a(s)}{b(s)-a(s)})\frac{s-a(s)}{(b(s)-a(s))^2}(3b(s)-5) \quad (4.16)$$
$$+3(\frac{s-a(s)}{b(s)-a(s)})^2 = 0.$$

$$\frac{\partial J}{\partial a(s)} = -(\frac{s-a(s)}{b(s)-a(s)})\frac{s-b(s)}{(b(s)-a(s))^2}a(s) + \frac{1}{2}(1-(\frac{s-a(s)}{b(s)-a(s)})^2) \quad (4.17)$$
$$+2(\frac{s-a(s)}{b(s)-a(s)})\frac{s-b(s)}{(b(s)-a(s))^2}(3b(s)-5) = 0.$$

From equation (4.16), we can get that

$$3b(s) = 10 - 2a(s). \quad (4.18)$$

From equation (4.17), we can get that

$$6(s-a(s))(s-b(s)) + (b(s)-a(s))^2 = (s-a(s))^2. \quad (4.19)$$

Substitute (4.18) to (4.19), we can get that

$$4(2-a(s))^2 = (3s-6)^2. \quad (4.20)$$

As the range of s can be in either $(0,2]$ or $(2,+\infty)$, so we have to split the problem into 2 cases, and find the union of either result of s:
First case when s$\in (2,+\infty)$,we have by (4.20)

$$a^*(s) = \frac{10-3s}{2}, \quad (4.21)$$

Combined with (4.18), we will have that

$$b^*(s) = s \quad (4.22)$$

When we equate (4.21) and (4.22) to s according to the theorem, we find that

$$\tau^* = inf\{t \geq 0 : s \in \{s : s > 2\}\}. \quad (4.23)$$

Second case when $s \in (0,2]$, we have by (4.20)

$$a^*(s) = \frac{3s-2}{2}, \quad (4.24)$$

combining with (4.18), we have that

$$b^*(s) = 4 - s. \tag{4.25}$$

Equating (4.24) and (4.25) to s, we have

$$s = 2.$$

We then find that in this case

$$\tau^* = inf\{t \geq 0 : s = 2\}. \tag{4.26}$$

So by combining (4.23) and (4.26), we have that

$$\tau^* = inf\{t \geq 0 : s \in \{s : s \geq 2\}\}.$$

Finally, we find that the intersection of three cases is

$$\tau^* = inf\{t \geq 0 : s \in \{s : s \geq 2\}\}.$$

This indicates that the optimal stopping strategy for an agent at time t to stop immediately occurs when $s \geq 2$.

Apparently, for the case when $w(\cdot)$ is convex and $u(\cdot)$ is concave, like in example 3, in which the optimal stopping strategy for an agent is always to stop immediately, the corollary of this result is given below.

Corollary 6. *If $u(\cdot)$ is concave and $w(\cdot)$ is convex, we will always have*

$$\tau^* = 0.$$

which means the optimal stopping strategy for an agent involved in this case is always to stop immediately.

Proof. As we know, a convex function $f(\cdot)$ satisfies

$$f(\lambda x + (1-\lambda)y) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(y),$$

for x, y located in the domain of $f(\cdot)$ and $\lambda \in [0, 1]$.
According to this property, we have

$$w(x) \leq x, x \in [0, 1]$$

as a consequence of $w(0) = 0$ and $w(1) = 1$.
So we have that

$$\begin{aligned}
\sup_{\tau \in \mathcal{T}_t} J(\tau; t) &= \sup_{0<a(s)\leq s\leq b(s)} [(1 - w(\tfrac{s-a(s)}{b(s)-a(s)}))u(a(s)) + w(\tfrac{s-a(s)}{b(s)-a(s)})u(b(s))] \\
&= \sup_{0<a(s)\leq s\leq b(s)} [(u(b(s)) - u(a(s))w(\tfrac{s-a(s)}{b(s)-a(s)}) + u(a(s))] \\
&\leq \sup_{0<a(s)\leq s\leq b(s)} [(u(b(s)) - u(a(s))\tfrac{s-a(s)}{b(s)-a(s)} + u(a(s))] \\
&= \sup_{0<a(s)\leq s\leq b(s)} [u(b(s))\tfrac{s-a(s)}{b(s)-a(s)} + u(a(s))(1 - \tfrac{s-a(s)}{b(s)-a(s)})]
\end{aligned}$$

Also we know a concave function $f(\cdot)$ satisfies

$$f(\lambda x + (1 - \lambda)y) \geq \lambda \cdot f(x) + (1 - \lambda) \cdot f(y),$$

for x, y located in the domain of $f(\cdot)$ and $\lambda \in [0, 1]$. So we have that

$$\sup_{0<a(s)\leq s\leq b(s)} [u(b(s))\frac{s - a(s)}{b(s) - a(s)} + u(a(s))(1 - \frac{s - a(s)}{b(s) - a(s)})] \leq u(s) = J(t; t).$$

So we have that the optimal stopping τ^* of the form

$$\tau^* = 0.$$

which means the optimal stopping strategy for an agent in this case is always to stop immediately.

For some examples illustrated in Xu and Zhou's paper (2011), the $u(\cdot)$ being non-decreasing and concave means that the agent is holding a "bad" asset [1]. And the convexity of $w(\cdot)$ means that the agent is risk-averse as he lowers the probability of the underlying process reaching a high state, so the optimal stopping strategy of a risk-averse agent an time t holding a unfavorable asset is to stop immediately. The result of τ^* obtained from example 4 coincides with the conclusion of this corollary. □

4.3 Case with convex $u(\cdot)$

Now it is time to consider the case when $u(\cdot)$ is convex and the shape of $w(\cdot)$ is unknown. In this case, the quantile formulation (4.2) is easier to be applied. The theorems stated in this section is very similar to the previous case.

Firstly we have to define the set $\mathcal{Q}_2$ as below just like we define $\mathcal{D}_2$,

$$\mathcal{Q}_2 := \{G \in \mathcal{Q} : G = a(s)\mathbf{1}_{(0,c(s)]} + b(s)\mathbf{1}_{(c(s),1)}, 0 < a(s) \leq b(s), 0 < c(s) \leq 1\},$$

where $a(s), b(s), c(s)$ are all functions of s. Xu and Zhou (2011) found that if $u(\cdot)$ is convex, then we must have

$$\sup_{\tau \in \mathcal{T}_t} J(\tau; t) = \sup_{G \in \mathcal{Q}} J_Q(G; t) = \sup_{G \in \mathcal{Q}_2} J_Q(G; t).$$

The prove of this can be found in page 21 of their paper.

Theorem 7. *If $u(\cdot)$ is convex, then we have that*

$$\begin{aligned} \sup_{\tau \in \mathcal{T}_t} J(\tau; t) &= \sup_{G \in \mathcal{Q}_2} J_Q(G; t) \\ &= \sup_{0 < a(s) \leq s \leq b(s)} [(1 - w(\tfrac{s-a(s)}{b(s)-a(s)}))u(a(s)) + w(\tfrac{s-a(s)}{b(s)-a(s)})u(b(s))] \end{aligned} \tag{4.27}$$

Moreover, if

$$(a^*(s), b^*(s)) = \operatorname*{argmax}_{0 < a(s) \leq s \leq b(s)} [(1 - w(\frac{s-a(s)}{b(s)-a(s)}))u(a(s)) + w(\frac{s-a(s)}{b(s)-a(s)})w(b(s))] \tag{4.28}$$

then we have

$$\tau^* = inf\{t \geq 0 : s \in \{s : a^*(s) = s \ \text{or} \ b^*(s) = s\}\} \tag{4.29}$$

is an optimal stopping time subject to definition 1.
And we have if τ^* exists

$$\sup_{\tau \in \mathcal{T}_t} J(\tau; t) = J(t; t) = J(\tau^*; \tau^*) = u(s).$$

Proof. [1] Following the discovery of Xu and Zhou, we only need to find the optimal quantile function in $\mathcal{Q}_2$ to maximize the objective. The form of $G \in \mathcal{Q}_2$ can be expressed as

$$G(x) = a(s)\mathbf{1}_{(0,c(s)]} + b(s)\mathbf{1}_{(c(s),1)}, x \in [0, 1).$$

Then we have that

$$J_Q(G; t) = \int_0^1 u(G(x))w'(1-x)dx = (1-w(1-c(s)))u(a(s)) + w(1-c(s))u(b(s)),$$

and the constraints for the quantile function becomes

$$\begin{aligned}\int_0^1 G(x)dx &= a(s) + (1 - c(s))(b(s) - a(s)) \\ &= a(s)c(s) + b(s)(1 - c(s)).\end{aligned}$$

Now, the problem becomes that we need to find a τ^* from definition 1 such that it maximizes:

$$J(a(s), b(s), c(s); t) := (1 - w(1 - c(s)))u(a(s)) + w(1 - c(s))u(b(s)) \quad (4.30)$$

with the constraints $a(s)c(s) + b(s)(1 - c(s)) \leq s$,$0 < a(s) \leq b(s), 0 \leq c(s) \leq 1$.

which is the same when we prove the theorem when $w(\cdot)$ is convex, so the rest of the proof follows exactly the same lines of proof of Theorem 3 to conclude that (4.29) is the optimal τ^* subject to definition1. □

The economical interpretation is similar as when $w(\cdot)$ is convex. This theorem suggests the "stop gain and cut loss" strategy in context of stock selling as well. Also the example 5 can be seen as a case involved in this situation when we have $u(\cdot)$ to be convex. We can illustrate the results found in example 5 intuitively. As from Xu and Zhou's paper, we have that $u(\cdot)$'s convexity and non-decreasing features indicate that the underlying asset ranges from "intermediate" to "bad" [1]. And in example 5, the shape of $w(\cdot)$ is convex, which implies that the agent is risk-averse. The optimal stopping strategy of a risk-averse agent when holding a bad asset is to stop immediately. And we can see from the actual shape of $u(x)$, the changing point for asset to become intermediate to bad is right at $s = 2$. We can deduce the asset is comparatively bad when $s \geq 2$, so the agent will stop immediately when $s \geq 2$.

Example 8. Suppose we have convex $u(\cdot)$ and concave $w(\cdot)$ to have the form as below.

$$\begin{cases} u(x) = x^2, x \in [0, +\infty) \\ w(x) = x^{\frac{1}{2}}, x \in [0, 1]. \end{cases}$$

By following the above theorem, we have our objective as

$$J = (1 - (\frac{s - a(s)}{b(s) - a(s)})^{\frac{1}{2}})a^2(s) + (\frac{s - a(s)}{b(s) - a(s)})^{\frac{1}{2}}b^2(s).$$

By differentiating with respect to $b(s)$, we have the the following equation

$$\begin{aligned}\frac{\partial J}{\partial b(s)} &= \tfrac{1}{2}(\tfrac{s-a(s)}{b(s)-a(s)})^{-\frac{1}{2}}\tfrac{s-a(s)}{(b(s)-a(s))^2}a^2(s) - \tfrac{1}{2}(\tfrac{s-a(s)}{b(s)-a(s)})^{-\frac{1}{2}}\tfrac{s-a(s)}{(b(s)-a(s))^2}b^2(s) \\ &\quad +2(\tfrac{s-a(s)}{b(s)-a(s)})^{\frac{1}{2}}b(s) = 0.\end{aligned} \tag{4.31}$$

By re-arranging the equation (4.31), we have that

$$a(s) = 3b(s), \tag{4.32}$$

which is impossible, as we know from definition that $0 < a(s) \leq b(s)$. This implies that the pair $(a^*(s), b^*(s))$ is not able to achieve. So the optimal stopping strategy of an agent in this case is to hold the underlying asset. We can interpret this result in an economical way, as $u(\cdot)$ being convex and non-decreasing means the underlying asset is "bad" while the concavity of the distortion function $w(\cdot)$ means the agent is risk-seeking as he exaggerates the probability of the underlying asset reaching a high value. This means the risk-seeking agent who is holding a bad asset will continue holding the asset as he expects for extraordinary return.

4.4 Concave $u(\cdot)$

As we have discussed the cases separately when $w(\cdot)$ and $u(\cdot)$ are convex functions, the case when $u(\cdot)$ being concave will be discussed in this section. Also we apply the quantile formulation as the shape of $u(\cdot)$ is known. And we notice that the objective $J_Q(\cdot)$ becomes concave. In sharp contrast to the case when $u(\cdot)$ is convex, in general the maxima of $J_Q(G)$ is now in the interior of the constraint set. In this case, by following the methods listed by Xu and Zhou (2011), we can find the optimal $G^*(\cdot)$ by using Lagrange method.

Firstly, we consider a family of relaxed problems following the methods described by Xu and Zhou [1],

$$\begin{aligned}J_Q^\lambda(G;t) &= J_Q(G;t) - \lambda(\textstyle\int_0^1 G(x)dx - s) \\ &= \textstyle\int_0^1 u(G(x))w^{'}(1-x)dx - \lambda(\int_0^1 G(x)dx - s) \\ &= \textstyle\int_0^1 (u(G(x))w^{'}(1-x) - \lambda G(x))dx + \lambda s \\ &= \textstyle\int_0^1 f^\lambda(x, G(x))dx + \lambda s,\end{aligned} \tag{4.33}$$

in which $\lambda \geq 0$ and we have

$$f^\lambda(x, G(x)) = u(G(x))w^{'}(1-x) - \lambda G(x). \tag{4.34}$$

Assuming $G(x) = y$, we have that

$$f^\lambda(x, y) = u(y)w^{'}(1-x) - \lambda y, \tag{4.35}$$

in which $x \in (0,1)$ and $y \in [0,+\infty)$. Therefore, to maximize $J_Q^\lambda(\cdot)$ it suffices to maximize $f^\lambda(x,\cdot)$ for each x. As the transformed payoff function $u(\cdot)$ needs not to be smooth, we define that

$$\begin{aligned} u'(x) &:= \limsup_{h\to 0+} \tfrac{u(x+h)-u(x)}{h}, \\ (u')_u^{-1}(x) &:= inf\{y \geq 0 : u'(x) < x\}, \\ (u')_l^{-1}(x) &:= inf\{y \geq 0 : u'(x) \leq x\}. \end{aligned}$$

We can see that $(u')_u^{-1}(\cdot)$ is left-continuous while both $(u')_l^{-1}(\cdot)$ and $u'(\cdot)$ are right-continuous. As $f^\lambda(x,\cdot)$ is concave, so y maximizes $f^\lambda(x,\cdot)$ if and only if

$$y \in [(u')_l^{-1}(\frac{\lambda}{w'(1-x)}), (u')_u^{-1}(\frac{\lambda}{w'(1-x)})]. \tag{4.36}$$

When all the above settings are made, we need to further specify the shape of $w(\cdot)$ to solve the problem. As in previous section , we derived the optimal stopping strategy when $u(\cdot)$ is concave and $w(\cdot)$ is convex, the only case we have to consider now is when $w(\cdot)$ is concave.

4.4.1 Case with concave $w(\cdot)$

Let $w(\cdot)$ be a concave function. The following function maximizes $f^\lambda(x,\cdot)$ on$\mathbb{R}^+$ for each $x \in [0,1]$ as we assume the transformed payoff function $u(\cdot)$ is smooth.

$$G_\lambda(x) := (u')_l^{-1}(\frac{\lambda}{w'(1-x)}).$$

By observing the the form of the above function, we can see $G_\lambda(x)$ is non-decreasing as $w'(1-x)$ is non-decreasing in x and $(u')_l^{-1}(\cdot)$ is non-increasing. By defining $G_\lambda(0) = 0$, we can get G_λ is indeed a quantile function provided that $0 < G_\lambda(x) < \infty, \forall x \in (0,1)$.

Theorem 9. *We define*

$$G_{\lambda^*}(x) := (u')_l^{-1}(\frac{\lambda^*}{w'(1-x)})$$

which is the optimal quantile function maximizing the objective $J_Q(G;t)$ where λ^* is a solution to $\int_0^1 (u')_l^{-1}(\frac{\lambda^*}{w'(1-x)})dx = s$.

Proof. As we know from above discussion, $G_\lambda(x)$ maximizes $f^\lambda(x, G(x))$, so it maximizes $J_Q^\lambda(G;t)$.

$$J_Q^\lambda(G;t) \leq J_Q^\lambda(G_\lambda;t).$$

Suppose that there exists $0 \leq \lambda^* < \infty$, such that

$$\int_0^1 (u')_l^{-1}(\frac{\lambda^*}{w'(1-x)})dx = s.$$

Then we have

$$J_Q(G) \leq J_Q^{\lambda^*}(G;t) \leq J_Q^{\lambda^*}(G_{\lambda^*};t) = J_Q(G_{\lambda^*};t).$$

So G_{λ^*} is an optimal quantile function to maximize the objective $J_Q(G)$. □

We can see that in general , the quantile function, G_{λ^*}, of the optimal stopping state s does not correspond to a two-point distribution anymore, however, it is now corresponding to a single point distribution, which means to obtain the τ^* from definition1, we have to have $G_{\lambda^*}(x)$ to be a constant for all $x \in (0,1)$. By observing the form of $G_{\lambda^*}(x)$, the only possible case for $G_{\lambda^*}(x)$ to be a constant is when $w(x) = x$ and $w'(x) = 1$, i.e. without distortion.

Theorem 10. *When $u(\cdot)$ is concave and $w(\cdot)$ is concave, the optimal stopping time with respect to definition 1 will not exist unless the probability distortion function is of the form*

$$w(x) = x.$$

Proof. Done by previous discussion. □

This phenomena can also be explained in an economical way. As described in Xu and Zhou's paper (2011), the concavity of non-decreasing $u(\cdot)$ corresponds to, at least in some interests, an "unfavorable" underlying asset [1]. Also concavity of $w(\cdot)$ suggests that the agent is risk-seeking just as discussed in previous cases. In the context of stock selling, the above results indicate that a risk-seeking agent holding a bad stock will not set any specific stop-gain or cut-loss prices. As in common sense, the bad stock will be sold immediately to avoid further lose, but a risk-seeking agent will still hold it as he hopes for large return.

Example 11. (This example follows the one listed in Xu and Zhou's paper [1].) Consider a model of asset selling with a concave function $u(x)$ and a concave distortion function $w(x)$ in the forms as below.

$$\begin{cases} u(x) = \frac{1}{\gamma}x^\gamma, 0 < \gamma < 1 \\ w(x) = x^\alpha, 0 < \alpha < 1. \end{cases}$$

By simple calculation, we can get that

$$\begin{cases} (u')^{-1}(x) = x^{\frac{1}{\gamma-1}} \\ w'(1-x) = \alpha(1-x)^{\alpha-1}. \end{cases} \tag{4.37}$$

Follow theorem 9, we have that

$$\begin{cases} \int_0^1 (u')_l^{-1}(\frac{\lambda^*}{w'(1-x)})dx = s \\ G_{\lambda^*}(x) := (u')_l^{-1}(\frac{\lambda^*}{w'(1-x)}). \end{cases} \tag{4.38}$$

By substituting the equations of (4.37) into the equations of (4.38), we have the optimal quantile function

$$G^*(x) = G_{\lambda^*}(x) = s\frac{\alpha-\gamma}{1-\gamma}(\frac{1}{1-x})^{\frac{1-\alpha}{1-\gamma}}, x \in (0,1). \tag{4.39}$$

To find the optimal stopping time of definition 1, we have to make $G^*(x)$ a constant, which means we have to eliminate the part with x. By observing the form of it, we can easily see that the only possible way is to make $\alpha = 1$.When $\alpha = 1$, we get that

$$G^*(x) = s,$$

which indicates that the solution τ^* exists and

$$\tau^* = 0.$$

However, we notice that when $\alpha = 1$, the probability distortion function becomes $w(x) = x$, i.e. no probability distortion. So the results we find exactly coincide with the conclusions from theorem 10.

4.4.2 Case with S-shaped $w(\cdot)$ and reverse S-shaped $w(\cdot)$

I put these two problems together because the methods for finding the "optimal" stopping time of definition 1 is similar.

We say the distortion function is S-shaped if it is convex on $[0, 1-p]$ and concave on $[1-p, 1]$. Similarly, the distortion function is reverse S-shaped if it is concave on $[0, 1-p]$ and convex on $[1-p, 1]$. The key idea to these problems is to search among a special class of quantile functions so as to solve the relaxed problem (4.33). Following the results found by Xu and Zhou, the special class of quantile functions we need to search are of the below form (we list them according to the shape of the distortion functions) [1]:

1. When $w(\cdot)$ is reverse S-shaped:

$$G(x) := a\mathbf{1}_{(0,p]}(x) + a \vee (u')_l^{-1}(\frac{\lambda}{w'(1-x)})\mathbf{1}_{(p,1)}(x), \tag{4.40}$$

   where the parameters $a(s)$ and λ are subject to

$$ap + \int_p^1 a \vee (u')_l^{-1}(\tfrac{\lambda}{w'(1-x)})dx = s.\lambda \geq 0, a(s) \geq 0 \tag{4.41}$$

The objective under G becomes

$$\begin{aligned} J(\tau;t) = J_Q(G;t) &= (1-w(1-p))u(a) \\ &+ \textstyle\int_p^1 u(a \vee (u')_l^{-1}(\tfrac{\lambda}{w'(1-x)}))w'(1-x)dx. \end{aligned} \tag{4.42}$$

Thus the optimal quantile function G^* can be achieved by solving a mathematical program, which is to choose (a^*, λ^*) to maximize (4.42) subject to (4.41).

2. When $w(\cdot)$ is S-shaped:

$$G(x) := a \wedge (u')_l^{-1}(\tfrac{\lambda}{w'(1-x)})\mathbf{1}_{(0,p]}(x) + a\mathbf{1}_{(p,1)}(x), \tag{4.43}$$

where the parameters $a(s)$ and λ are subject to

$$\int_0^p a \wedge (u')_l^{-1}(\tfrac{\lambda}{w'(1-x)})dx + a(1-p) = s. \tag{4.44}$$

The objective under G becomes

$$\begin{aligned} J(\tau;t) = J_Q(G;t) &= \textstyle\int_0^p u(a \wedge (u')_l^{-1}(\tfrac{\lambda}{w'(1-x)}))w'(1-x)dx \\ &+ w(1-p)u(a). \end{aligned} \tag{4.45}$$

Thus the optimal quantile function G^* can be achieved by solving a mathematical program, which is to choose (a^*, λ^*) to maximize (4.45) subject to (4.44).

By observing the forms of the quantile functions of these two similar cases and assume $w(x) \neq x$, we can see that the optimal quantile function of optimal stopping state "s" corresponds to a single point distribution. As discussed previously, the term $(u')_l^{-1}(\frac{\lambda}{w'(1-x)})$ can never be a constant which means the probability of s equaling to $(u')_l^{-1}(\frac{\lambda}{w'(1-x)})$ for all x is 0. So to make immediate stopping optimal, we must have $G(x)$ correspond to a single point distribution with only $a^* = s$. Now we need to discuss the cases separately.

Case with reverse S-shaped $w(\cdot)$ As discussed previously, for $G(x)$ (4.40) to be a single point distribution, we need the condition $a \geq (u')_l^{-1}(\frac{\lambda}{w'(1-x)})$ for all $x \in (p, 1)$. Now, suppose

$$\hat{c} = inf\{x \geq p : a(s) \leq (u')_l^{-1}(\tfrac{\lambda}{w'(1-x)})\} \wedge 1.$$

As we see, at the point $\hat{c}$, we have

$$a = (u')_l^{-1}(\tfrac{\lambda}{w'(1-\hat{c})}). \tag{4.46}$$

So we can see that λ can be expressed as a function of a and $\hat{c}$ which is of the form

$$\lambda = u'(a)w'(1-\hat{c}). \tag{4.47}$$

Also by substituting (4.47) into the constraint (4.41), we can get that

$$ap + \int_p^1 a(s) \vee (u')_l^{-1}(\tfrac{u'(a)w'(1-\hat{c})}{w'(1-x)})dx = s. \tag{4.48}$$

We can see clearly that a can be expressed as function of $\hat{c}$ and s. Finally we substitute this form of $a(s, \hat{c})$ into the objective (4.42), we will have our objective to be a function of s and $\hat{c}$. We then maximize the objective with respect to $\hat{c}$ by fixing s. We can see clearly that the optimal value of $\hat{c}$ that maximizes the objective (4.42) is a function of s, say $\hat{c}^*(s)$. Thus the values of optimal $a^*(s) = a(s, \hat{c}^*(s))$ and $\lambda^*(s) = u'(a^*(s))w'(1-\hat{c}^*(s))$ can be easily found. The form of the optimal quantile function can be written as

$$G^*(x) := a^*(s)\mathbf{1}_{(0,p]}(x) + a^*(s) \vee (u')_l^{-1}(\tfrac{\lambda^*(s)}{w'(1-x)})\mathbf{1}_{(p,1)}(x). \tag{4.49}$$

By previous discussion, for (4.49) to be the optimal quantile function subject to the state "s", it has to correspond to a single point distribution. The only way for it to be done is when $\hat{c}^*(s) = 1$. If $\hat{c}^*(s) = 1$, then we have the constraint (4.41) become

$$a^*(s) = s.$$

And the objective (4.42) becomes

$$J(\tau; t) = u(s).$$

These equalities can be easily achieved by taking the condition

$$a^*(s) \geq (u')_l^{-1}(\tfrac{\lambda^*(s)}{w'(1-x)})$$

into account. So we can see clearly that the value of the objective becomes a constant, then the "optimal" stopping time subject to definition 1 is obviously

$$\tau^* = inf\{t \geq 0 : s \in \{s : \hat{c}^*(s) = 1\}\}.$$

If $\hat{c}^*(s) < 1$, then we must have the part $(u')_l^{-1}(\frac{\lambda}{w'(1-x)})$ still left which means the single point distribution of the optimal state "s" does not exist. So our "optimal" stopping time can not be achieved.
As we can see, for a reverse S-shaped distortion $w(\cdot)$, $w'(x) > 1$ around both $x = 0$ and $x = 1$. This means the distortion puts higher weights on both very bad or very good cases. As an agent in this case, he exaggerates the probabilities of of both very good and very bad situations. As in He and Zhou (2009b), this is to model the emotion of fear and hope respectively [3]. In context of stocking selling subject to our discussions, the situation of $\hat{c}^*(s) = 1$ corresponds to the very bad scenarios, the agent stop immediately because the he is afraid of further loss. The situation of $\hat{c}^*(s) < 1$ corresponds to the good scenarios, the agent still holds the stock as he hopes for higher return.

Case with S-shaped $w(\cdot)$ This case is very similar to the above case. The condition for (4.43) to be a single point distribution is when $a \leq (u')_l^{-1}(\frac{\lambda}{w'(1-x)})$ for all $x \in (0, p)$. Now suppose

$$\hat{c} = inf\{x > 0 : a(s) \leq (u')_l^{-1}(\tfrac{\lambda}{w'(1-x)})\} \vee 0.$$

By following nearly the same method as in above case, we have the optimal quantile function corresponding to state "s" exists only when $\hat{c}^*(s) = 0$. If $\hat{c}^*(s) = 0$, then we have the constraint (4.44) become

$$a^*(s) = s.$$

And the objective (4.45) becomes

$$J(\tau; t) = u(s).$$

These equalities can be easily achieved by taking the condition

$$a^*(s) \leq (u')_l^{-1}(\tfrac{\lambda^*(s)}{w'(1-x)})$$

into account. So we can see clearly that the value of the objective becomes a constant, then the "optimal" stopping time subject to definition 1 is obviously

$$\tau^* = inf\{t \geq 0 : s \in \{s : \hat{c}^*(s) = 0\}\}.$$

If $\hat{c}^*(s) > 0$, then we must have the part $(u')_l^{-1}(\frac{\lambda^*(s)}{w'(1-x)})$ still left which means the single point distribution of the optimal state "s" does not exist. So our "optimal" stopping time can not be achieved.

In this case, the economical interpretation is just opposite to the reverse S-shaped case. The S-shaped probability distortion suggests that the agent underweighs probability of both very good and very bad situations. In context of stock selling subject to our case here, the case when $\hat{c}^*(s) = 0$ corresponds to the very good situation, the agent stop immediately as he is not hopeful for very high price. The case $\hat{c}^*(s) > 0$ corresponds to the very bad situation, the agent still hold the stock as he believes that the price will not go terribly wrong.

Example 12. (This example follows the one listed in Xu and Zhou's paper on page 31. [1]) Suppose we have a concave transformed payoff function $u(x) = \frac{10}{3}x^{\frac{3}{10}}$ and a reverse S-shaped distortion function

$$w(x) = \begin{cases} 2x - 2x^2, 0 \leq x \leq \frac{1}{2} \\ 2x^2 - 2x + 1, \frac{1}{2} < x \leq 1. \end{cases}$$

By using the methods described above, we can see the constraint (4.41) becomes

$$\frac{1}{2}a + \int_{\frac{1}{2}}^{1} a \vee (\frac{\lambda}{4x-2})^{-\frac{10}{7}} dx = s. \tag{4.50}$$

The objective function (4.42) becomes

$$J(\tau; t) = J(a, \lambda) = \frac{5}{3}a^{\frac{3}{10}} + \frac{10}{3}\int_{\frac{1}{2}}^{1} (a^{\frac{3}{10}} \vee (\frac{\lambda}{4x-2})^{-\frac{3}{7}})(4x-2)dx \tag{4.51}$$

Suppose we have

$$\hat{c} = inf\{x \geq \frac{1}{2} : a \leq (\frac{\lambda}{4x-2})^{-\frac{10}{7}}\} \wedge 1. \tag{4.52}$$

We can get

$$a(s, \hat{c}) = \frac{s}{\hat{c} + \frac{7}{34}((2\hat{c}-1)^{-\frac{10}{7}} - (2\hat{c}-1))} \tag{4.53}$$

and

$$\lambda(s, \hat{c}) = (4\hat{c} - 2)a(s, \hat{c})^{-\frac{7}{10}} \tag{4.54}$$

And thus by substituting (4.52) and (4.53) into (4.51), our objective can be written as

$$J(\tau; t) = J(a, \lambda) = \frac{10}{3}s^{\frac{3}{10}} f(\hat{c}) \tag{4.55}$$

where

$$f(\hat{c}) = (\frac{1}{\hat{c} + \frac{7}{34}((2\hat{c}-1)^{-\frac{10}{7}} - (2\hat{c}-1))})^{\frac{3}{10}}(1 - 2\hat{c} + 2\hat{c}^2 + \frac{7}{17}((2\hat{c}-1)^{-\frac{3}{7}} - (2\hat{c}-1)^2)). \tag{4.56}$$

We can see in this case things become simple as we can separate $\hat{c}$ from s. Following the result in Xu and Zhou's paper, we have that the optimal $\hat{c}^*(s) = 0.70 \neq 1$ [1]. So the our "optimal" stopping time does not exist.

5 Concluding Remarks

In this paper, we formulate the "optimal" stopping problem with time inconsistency which is to find the conditions such that when first time the immediate stop is optimal with probability distortion. The methods we use here are based on the discoveries of Xu and Zhou's paper (2011). By transforming the original objective into its distribution or quantile formulations, we can solve the problem by considering different shapes of the transformed payoff function and probability distortion function. Also in each case, we interpret the result in an economical way to give the readers an intuitive idea of understanding the mathematical frameworks. However, there still exist some interesting cases waiting to be solved such as when the transformed payoff function is S-shaped or reverse S-shaped.

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