

Asymptotic Analysis of Statistical Estimators related to MultiGraphex Processes under Misspecification : supplementary material

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This document is supplementary material for the article [4]. It contains the missing proofs. We refer to the main document for all the definitions.

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S1. Asymptotic expansion of the log-likelihood: auxiliary results

S1.1. Proof of the expression of the likelihood in [4, Equation (3)]

[4, Equation (3)] is a consequence of [2, Theorem 6]. Using exchangeability, we assume without loss of generality that the non isolated vertices corresponds to the indices $1, \dots, N_t$. Then, from their theorem, we find that

$$\begin{aligned} e^{L_t(\phi)} &\propto s^{N_t} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+^{N_t}} \exp \left\{ - \left(\sum_{i=1}^{N_t} w_i + w_* \right)^2 \right\} \left\{ \prod_{i=1}^{N_t} w_i^{D_{t,i}} \rho(dw_i) \right\} G_\phi(dw_*) \\ &= s^{N_t} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+^{N_t}} \exp \left\{ - \left(\sum_{i=1}^{N_t} w_i + w_* \right)^2 \right\} \left\{ \prod_{i=1}^{N_t} \frac{w_i^{D_{t,i}-1-\sigma} e^{-\tau w_i} dw_i}{\Gamma(1-\sigma)} \right\} G_\phi(dw_*). \end{aligned}$$

That is, introducing the PDF of the Gamma distribution,

$$\begin{aligned} e^{L_t(\phi)} &\propto s^{N_t} \left\{ \prod_{i=1}^{N_t} \frac{\Gamma(D_{t,i} - \sigma)}{\Gamma(1-\sigma) \tau^{D_{t,i}-\sigma}} \right\} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+^{N_t}} \exp \left\{ - \left(\sum_{i=1}^{N_t} w_i + w_* \right)^2 \right\} \\ &\quad \times \left\{ \prod_{i=1}^{N_t} \frac{\tau^{D_{t,i}-\sigma} w_i^{D_{t,i}-1-\sigma} e^{-\tau w_i} dw_i}{\Gamma(D_{t,i} - \sigma)} \right\} G_\phi(dw_*). \end{aligned}$$

Remark that the inner integral is the expectation of $\exp\{-(\sum_{i=1}^{N_t} W_i + w_*)^2\}$ under $(W_1, \dots, W_{N_t}) \sim \otimes_{i=1}^{N_t} \text{Gamma}(D_{t,i} - \sigma, \tau)$. But under this distribution we have that $\sum_{i=1}^{N_t} W_i$ has a $\text{Gamma}(\sum_{i=1}^{N_t} D_{t,i} - \sigma N_t, \tau)$ distribution. Hence, the conclusion follows.

S1.2. Proof of [4, Lemma 2]

Using the definition of $I(\phi)$, we introduce the PDF of the $\text{Gamma}(D_t^* - \sigma N_t, \tau)$ distribution,

$$I(\phi) = \frac{1}{\tau^{D_t^* - \sigma N_t}} \int_{\mathbb{R}_+^2} e^{-(x+y)^2} \frac{\tau^{D_t^* - \sigma N_t} x^{D_t^* - \sigma N_t - 1} e^{-\tau x}}{\Gamma(D_t^* - \sigma N_t)} G_\phi(dy) dx.$$

We rewrite the previous integral by using that the Fourier transform of a Gaussian is Gaussian. In particular,

$$\frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} \exp\left\{-\frac{\xi^2}{4} + i(x+y)\xi\right\} d\xi = \exp\{-(x+y)^2\}.$$

Inserting the last display in the previous expression for $I(\phi)$, it follows from Fubini's theorem and the expressions for the Fourier transforms of G_ϕ (see [4, Equation (4)]) and the Gamma distribution,

$$\begin{aligned} I(\phi) &= \frac{1}{\tau^{D_t^* - \sigma N_t}} \int_{\mathbb{R}_+^2} \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-\frac{\xi^2}{4} + i(x+y)\xi} d\xi \frac{\tau^{D_t^* - \sigma N_t} x^{D_t^* - \sigma N_t - 1} e^{-\tau x}}{\Gamma(D_t^* - \sigma N_t)} G_\phi(dy) dx \\ &= \frac{1}{2\sqrt{\pi} \tau^{D_t^* - \sigma N_t}} \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} \left\{ \int_{\mathbb{R}_+} e^{ix\tau^{-1}\xi} \frac{x^{D_t^* - \sigma N_t - 1} e^{-\tau x}}{\Gamma(D_t^* - \sigma N_t)} dx \right\} \left\{ \int_{\mathbb{R}_+} e^{i\xi y} G_\phi(dy) \right\} d\xi \\ &= \frac{1}{2\sqrt{\pi} \tau^{D_t^* - \sigma N_t}} \int_{\mathbb{R}} \left(1 - \frac{i\xi}{\tau}\right)^{-(D_t^* - \sigma N_t)} \exp\left\{-\frac{\xi^2}{4} - s\psi(\sigma, \tau; \tau - i\xi)\right\} d\xi \\ &= \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} (\tau - i\xi)^{-(D_t^* - \sigma N_t)} \exp\left\{-\frac{\xi^2}{4} - s\psi(\sigma, \tau; \tau - i\xi)\right\} d\xi. \end{aligned}$$

The conclusion follows because for any $\xi \in \mathbb{R}$, from [4, Equation (6)],

$$-\frac{\xi^2}{4} = -D_t^* \mathcal{A}(\phi; \tau - i\xi) + s\psi(\sigma, \tau; \tau - i\xi) + (D_t^* + \sigma N_t) \log(\tau - i\xi).$$

Now we establish [4, Equation (27)], the uniqueness of $\zeta(\phi)$, and that $\zeta(\phi) > \tau$. Remark that for all σ (including $\sigma = 0$), we have from the definition of ψ that $\partial_z \psi(\sigma, \tau; z) = 1/z^{1-\sigma}$ for all $z \in \mathbb{C} \setminus \mathbb{R}_-$. Then for all ϕ ,

$$\partial_z \mathcal{A}(\phi; z) = -\frac{z}{2D_t^*} + \frac{\tau}{2D_t^*} + \left(1 - \frac{\sigma N_t}{D_t^*}\right) \frac{1}{z} + \frac{s}{D_t^*} z^{-1+\sigma}. \quad (\text{S1.1})$$

Therefore,

$$\partial_z \mathcal{A}(\phi; z) = 0 \iff z^2 = \tau z + 2sz^\sigma + 2D_t^* \left(1 - \frac{\sigma N_t}{D_t^*}\right).$$

Since $D_t^* \geq N_t$ and $\sigma < 1$, the last display implies that any solution of $\partial_z \mathcal{A}(\phi; z) = 0$ must satisfy $z^2 > \tau z$, i.e. $z > \tau$ since we retained the non-negative solution. By differentiating Equation (S1.1) another time with respect to z , we see that $\partial_z^2 \mathcal{A}(\phi; z) < 0$ for $z > 0$, hence $\zeta(\phi)$ must be unique. Further $\lim_{z \rightarrow 0} \partial_z \mathcal{A}(\phi; z) = +\infty$ and $\lim_{z \rightarrow \infty} \partial_z \mathcal{A}(\phi; z) = -\infty$ by Equation (S1.1) again, so $\zeta(\phi)$ must exist.

S1.3. Proof of [4, Lemma 3]

Using the defining equation of $z \mapsto R_1(\phi; z)$, it is clear that for any $z \in \mathbb{R}$,

$$\frac{1}{2} \partial_z^2 \mathcal{A}(\phi; \zeta(\phi)) z^2 - R_1(\phi; -iz) = -\{\mathcal{A}(\phi; \zeta(\phi) - iz) - \mathcal{A}(\phi; \zeta(\phi))\}. \quad (\text{S1.2})$$

From the definition of \mathcal{A} , we have,

$$\begin{aligned} \mathcal{A}(\phi; \zeta(\phi) - iz) - \mathcal{A}(\phi; \zeta(\phi)) &= -\frac{(\zeta(\phi) - iz)^2 - \zeta(\phi)^2}{4D_t^\star} + \frac{\tau(\zeta(\phi) - iz) - \tau\zeta(\phi)}{2D_t^\star} \\ &\quad + \left(1 - \frac{\sigma N_t}{D_t^\star}\right) \left(\log(\zeta(\phi) - iz) - \log(\zeta(\phi))\right) \\ &\quad + \frac{s}{D_t^\star} \left(\psi(\sigma, \tau; \zeta(\phi) - iz) - \psi(\sigma, \tau; \zeta(\phi))\right). \end{aligned}$$

Hence for every $z \in \mathbb{R}$,

$$\begin{aligned} \Re\{\mathcal{A}(\phi; \zeta(\phi) - iz) - \mathcal{A}(\phi; \zeta(\phi))\} &= \frac{z^2}{4D_t^\star} + \left(1 - \frac{\sigma N_t}{D_t^\star}\right) \left(\log \sqrt{\zeta(\phi)^2 + z^2} - \log \zeta(\phi)\right) \\ &\quad + \frac{s}{D_t^\star} \Re\left(\psi(\sigma, \tau; \zeta(\phi) - iz) - \psi(\sigma, \tau; \zeta(\phi))\right). \end{aligned}$$

Therefore, for all $z \in \mathbb{R}$,

$$\Re\{\mathcal{A}(\phi; \zeta(\phi) - iz) - \mathcal{A}(\phi; \zeta(\phi))\} \geq \frac{z^2}{4D_t^\star} + \frac{s}{D_t^\star} \Re\left(\psi(\sigma, \tau; \zeta(\phi) - iz) - \psi(\sigma, \tau; \zeta(\phi))\right). \quad (\text{S1.3})$$

Assume first that $\sigma = 0$. Then, whenever $z \in \mathbb{R}$,

$$\begin{aligned} \Re\left(\psi(\sigma, \tau; \zeta(\phi) - iz) - \psi(\sigma, \tau; \zeta(\phi))\right) &= \Re\left(\log(\zeta(\phi) - iz) - \log(\zeta(\phi))\right) \\ &= \log \sqrt{\zeta(\phi)^2 + z^2} - \log \zeta(\phi) \\ &\geq 0. \end{aligned}$$

The previous display combined with Equations (S1.2) and (S1.3) gives the proof of the proposition in the case $\sigma = 0$.

When $\sigma \neq 0$, because $\zeta(\phi) > 0$ we have $\arg(\zeta(\phi) - iz) = -\arctan(z/\zeta(\phi))$, and then,

$$\begin{aligned} \Re\left(\psi(\sigma, \tau; \zeta(\phi) - iz) - \psi(\sigma, \tau; \zeta(\phi))\right) &= \Re\left(\frac{(\zeta(\phi) - iz)^\sigma - \zeta(\phi)^\sigma}{\sigma}\right) \\ &= \frac{(\zeta(\phi)^2 + z^2)^{\sigma/2} \cos(\sigma \arctan(z/\zeta(\phi))) - \zeta(\phi)^\sigma}{\sigma} \\ &= \zeta(\phi)^\sigma f_\sigma(z/\zeta(\phi)), \end{aligned}$$

where for simplicity, we defined $f_\sigma : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f_\sigma(y) := \frac{(1 + y^2)^{\sigma/2} \cos(\sigma \arctan(y)) - 1}{\sigma}.$$

To finish the proof of the proposition, it suffices to show that f is non-negative. When $\sigma < 0$, we have $\cos(\sigma \arctan(y)) \leq 1$ uniformly, and hence

$$f_\sigma(y) \geq \frac{(1+y^2)^{\sigma/2} - 1}{\sigma} \geq \frac{1}{-\sigma} - \frac{(1+y^2)^{\sigma/2}}{-\sigma} \geq 0.$$

Now we consider the case $0 < \sigma < 1$. Because f is symmetric, it suffices to do the analysis for $y \geq 0$. By differentiation, we get

$$f'_\sigma(y) = (1+y^2)^{-1+\sigma/2} \cos(\sigma \arctan(y)) (y - \tan(\sigma \arctan(y))).$$

But, $y \geq 0 \Rightarrow \arctan(y) \in [0, \pi/2]$, and hence $\cos(\sigma \arctan(y)) \geq 0$ as $0 < \sigma < 1$. Similarly, $y > \tan(\sigma \arctan(y))$ when $0 < \sigma < 1$, and thus $f'_\sigma(y) \geq 0$ when $y \geq 0$ and $0 < \sigma < 1$. This means that f is increasing on \mathbb{R}_+ , and thus $f_\sigma(y) \geq f_\sigma(0) = 0$.

S1.4. Remaining part of the proof of [4, Theorem 7]

To finish the proof of [4, Theorem 7], it remains to prove the Item (3). The proof is almost identical to the proof of the Item (2) established in [4, Section 5.1], and consists mostly on obtaining a refinement in the bound for $\Delta(\phi)$. In particular, the starting point to the proof is [4, Equation (24)]. To ease the notations, we write $a(\phi) := (-D_t^* \partial_z^2 \mathcal{A}(\phi; \zeta(\phi)))^{1/2}$. Then, it follows from [4, Equations (23) and (25)] and a suitable change of variables that for some $C > 0$ to be chosen accordingly

$$1 + \Delta(\phi) = \int_{[-Ca(\phi), Ca(\phi)]} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} e^{-D_t^* R_1(\phi, -iu/a(\phi))} du + \int_{[-C, C]^c} \frac{1}{\sqrt{2\pi}} e^{-\frac{a(\phi)^2}{2} z^2 - D_t^* R_1(\phi; -iz)} dz. \quad (\text{S1.4})$$

Now remark that because of [4, Lemma 3] the second integral in the last display is a $O(e^{-C^2/4})$ as $C \rightarrow \infty$, where we have used the well-known tail bound for the Normal distribution. On the other hand, it is seen that $D_t^* \partial_z^2 \mathcal{A}(\phi, z) \leq -1/2$ for all admissible ϕ and for all z , since $\sigma < 1$, $s > 0$ and $N_t \leq D_t^*$. This entails that $a(\phi) \geq 1/2$ for any ϕ , and thus $\int_{[-Ca(\phi), Ca(\phi)]^c} e^{-u^2/2} du = O(e^{-C^2/8})$. By Equation (S1.4), it follows as $C \rightarrow \infty$

$$\Delta(\phi) = \frac{1}{\sqrt{2\pi}} \int_{[-Ca(\phi), Ca(\phi)]} e^{-u^2/2} \left(e^{-D_t^* R_1(\phi; \frac{-iu}{a(\phi)})} - 1 \right) du + O(e^{-C^2/8}). \quad (\text{S1.5})$$

In order to control $\Delta(\phi)$, it thus remain to control $x \mapsto D_t^* R_1(\phi; -ix)$ for $x \in [-C, C]$, in virtue of Equation (S1.5). We will proceed using Cauchy's integral formula. In particular, from our choice for the determination of the complex logarithm, the function $z \mapsto \mathcal{A}(\phi; \zeta(\phi) + z)$ is complex-analytic on $\mathbb{C} \setminus \{z \in \mathbb{C} : \Re(z) = 0, \Im(z) \leq -\zeta(\phi)\}$. Then, by Cauchy's integral formula, for any $z \in \mathbb{C}$ with $|z| < \zeta(\phi)/2$, we have

$$R_1(\phi; z) = \frac{1}{2\pi i} \oint_{|\xi|=\zeta(\phi)/2} \frac{\mathcal{A}(\phi; \zeta(\phi) + \xi)}{\xi - z} \frac{z^3}{\xi^3} d\xi.$$

Since $|\xi - z| \geq |\xi| - |z|$, the previous implies for every $z \in \mathbb{C}$ with $|z| \leq \zeta(\phi)/4$ that

$$\begin{aligned} |R_1(\phi; z)| &\leq \frac{1}{2\pi} \frac{4}{\zeta(\phi)} \frac{2^3}{\zeta(\phi)^3} |z|^3 \sup_{|\xi|=\zeta(\phi)/2} |\mathcal{A}(\phi; \zeta(\phi) + \xi)| \times 2\pi \frac{\zeta(\phi)}{2} \\ &= \frac{16|z|^3}{\zeta(\phi)^3} \sup_{\varphi \in [-\pi, \pi]} |\mathcal{A}(\phi; \zeta(\phi)(1 + e^{i\varphi}/2))| \\ &\leq \frac{16|z|^3}{\zeta(\phi)^3} \sup_{1/2 \leq x \leq 2} \sup_{\varphi \in [-\pi, \pi]} |\mathcal{A}(\phi; x\zeta(\phi)e^{i\varphi})|. \end{aligned} \quad (\text{S1.6})$$

But, assuming without loss of generality $\sigma \neq 0$, for all $x > 0$ and all $\varphi \in (-\pi, \pi)$,

$$\begin{aligned} \mathcal{A}(\phi; x\zeta(\phi)e^{i\varphi}) &= -\frac{\tau^2}{4D_t^*} - \frac{x^2\zeta(\phi)^2 e^{2i\varphi}}{4D_t^*} + \frac{\tau\zeta(\phi)x e^{i\varphi}}{2D_t^*} + \left(1 - \frac{\sigma N_t}{D_t^*}\right) \log(x\zeta(\phi)) \\ &\quad + i\left(1 - \frac{\sigma N_t}{D_t^*}\right)\varphi + \frac{s}{D_t^*} \frac{(x\zeta(\phi))^\sigma e^{i\sigma\varphi} - \tau^\sigma}{\sigma}. \end{aligned}$$

Remark that,

$$\begin{aligned} \frac{(x\zeta(\phi))^\sigma e^{i\sigma\varphi} - \tau^\sigma}{\sigma} &= \frac{(x\zeta(\phi))^\sigma - \tau^\sigma}{\sigma} e^{i\sigma\varphi} + \frac{\tau^\sigma}{\sigma} (e^{i\sigma\varphi} - 1) \\ &= \psi(\sigma, \tau; x\zeta(\phi)) e^{i\sigma\varphi} + \frac{\tau^\sigma}{\sigma} (e^{i\sigma\varphi/2} - e^{-i\sigma\varphi/2}) e^{i\sigma\varphi/2} \\ &= \psi(\sigma, \tau; x\zeta(\phi)) e^{i\sigma\varphi} + \varphi \tau^\sigma \frac{\sin(\sigma\varphi/2)}{\sigma\varphi/2} e^{i\sigma\varphi/2}. \end{aligned}$$

Hence,

$$\begin{aligned} |\mathcal{A}(\phi; x\zeta(\phi)e^{i\varphi})| &\leq \frac{\tau^2 + x^2\zeta(\phi)^2 + 2\tau\zeta(\phi)}{4D_t^*} + 2\left(1 - \frac{\sigma N_t}{D_t^*}\right) \left(\pi + |\log(x\zeta(\phi))|\right) \\ &\quad + \frac{s}{D_t^*} \left| \psi(\sigma, \tau; x\zeta(\phi)) + \pi\tau^\sigma \right|. \end{aligned}$$

By [4, Equation (27)], we find that for every ϕ we always have $\zeta(\phi) \geq \tau$, and $\zeta(\phi)^2 \geq 2D_t^*(1 - \sigma N_t/D_t^*)$. Thus, from the previous display we obtain for all $x \in [1/2, 2]$ and all $\varphi \in [-\pi, \pi]$

$$|\mathcal{A}(\phi; x\zeta(\phi)e^{i\varphi})| \lesssim \frac{\zeta(\phi)^2 |\log(\zeta(\phi))|}{D_t^*} + \frac{s}{D_t^*} \left| \psi(\sigma, \tau; x\zeta(\phi)) + \pi\tau^\sigma \right|. \quad (\text{S1.7})$$

Now, consider the case $\sigma \geq 0$. Then assuming without loss of generality $\sigma \neq 0$ (it suffices to extend by continuity), we have for any $x \in [1/2, 2]$ and any $\tau > 0$,

$$\begin{aligned} |s\psi(\sigma, \tau; x\zeta(\phi))| &\leq s \left| \frac{x^\sigma - 1}{\sigma} \zeta(\phi)^\sigma \right| + s \left| \frac{\zeta(\phi)^\sigma - \tau^\sigma}{\sigma} \right| \\ &\leq (1 \vee x^\sigma) s \zeta(\phi)^\sigma \sup_{u \in [1/2, 1]} \left| \frac{1 - u^\sigma}{\sigma} \right| + s \zeta(\phi)^\sigma \frac{1 - (\tau/\zeta(\phi))^\sigma}{\sigma} \end{aligned}$$

$$\leq \left(2\log(2) + \log \frac{\zeta(\phi)}{\tau}\right) s\zeta(\phi)^\sigma, \quad (\text{S1.8})$$

where for the last line we have used that $(1 - z^\sigma)/\sigma \leq \log(1/z)$ for all $z \in (0, 1)$ and all $\sigma \in (0, 1)$ (this can be seen by differentiating with respect to σ). We also have used that $\zeta(\phi) \geq \tau$, which can be deduced from [4, Equation (27)]. From [4, Equation (27)] again, we also see that $\zeta(\phi)^2 \geq s\zeta(\phi)^\sigma$. Also, because $\phi \in \mathcal{S}_K$ we have $s\tau^\sigma = O(\sqrt{2D_t^*}) = O(\zeta(\phi))$ and $\tau \geq K^{-1}$, and thus we obtain from Equation (S1.7) that $|\mathcal{A}(\sigma; x\zeta(\phi)e^{i\varphi})| = O(\zeta(\phi)^2 \log(\zeta(\phi))/D_t^*)$, at least for $\sigma \geq 0$. When $\sigma < 0$, it is easily seen with a similar reasoning that $|s\psi(\sigma, \tau; x\zeta(\phi))| \lesssim \zeta(\phi)^2 + s\tau^\sigma \log(\zeta(\phi))$, and hence, we obtain from Equation (S1.7) that for any $x \in [1/2, 2]$, any $\varphi \in [-\pi, \pi]$ and any $\phi \in \mathcal{S}_K$,

$$|\mathcal{A}(\phi; x\zeta(\phi)e^{i\varphi})| = o(\zeta(\phi)^3/D_t^*).$$

It then follows by combining Equations (S1.5) to (S1.7) that as $D_t^* \rightarrow \infty$

$$\Delta(\phi) = C^3 a(\phi)^3 \times o(1) + O(e^{-C^2/8}), \quad (\text{S1.9})$$

We already have shown that $a(\phi) \geq 1/2$, we now need a more precise estimate. Recalling that $a(\phi)^2 = -D_t^* \partial_z^2 \mathcal{A}(\phi; \zeta(\phi))$, we have under the assumptions of the Lemma,

$$\begin{aligned} a(\phi)^2 &= \frac{1}{2} + \frac{1}{2} \left(1 - \frac{\sigma N_t}{D_t^*}\right) \frac{2D_t^*}{\zeta(\phi)^2} + (1 - \sigma) \frac{s\zeta(\phi)^\sigma}{\zeta(\phi)^2} \\ &= \frac{1}{2} + \frac{1}{2} \frac{1}{\zeta(\phi)^2} \left(2D_t^* \left(1 - \frac{\sigma N_t}{D_t^*}\right) + (1 - \sigma)s\zeta(\phi)^\sigma\right) \\ &= \frac{1}{2} + \frac{1}{2} \frac{1}{\zeta(\phi)^2} \left(\zeta(\phi)^2 - \tau\zeta(\phi) - \sigma s\zeta(\phi)^\sigma\right), \end{aligned}$$

where the last line follows from the definition of $\zeta(\phi)$, in [4, Equation (27)]. Thus, under the assumptions of the Lemma we have $a(\phi)^2 = 1 + o(1)$. Since Equation (S1.9) is true for arbitrary choice of $C > 0$, this indeed shows that $\Delta(\phi) = o(1)$ if the conditions of the Lemma are met. Then by [4, Equation (24)], we have

$$\begin{aligned} \log I(\phi) &= -D_t^* \mathcal{A}(\phi; \zeta(\phi)) - \log \sqrt{2a(\phi)} + \log(1 + \Delta(\phi)) \\ &= -D_t^* \mathcal{A}(\phi; \zeta(\phi)) - \frac{\log(2)}{2} + o(1). \end{aligned}$$

S2. Existence and uniqueness of the MLE, concentration of the likelihood: auxiliary results

S2.1. Proof of [4, Lemma 4]

We will use multiple times that under [4, Assumption 1] we have for any $x \in (0, 1)$ that $\mathcal{C}_t'(x) = O(N_t)$, otherwise it cannot be the case that $\hat{\alpha}_t$ converges in $(0, 1)$.

We first study $\sigma \leq -C$, for $C > 0$. Observe that for any $c_0 \in (0, 1)$, we have $\mathcal{C}_t(c_0) = \mathcal{C}_t(0) + O(N_t)$. By definition $\Psi(\sigma)$ and by Lemma S2.1,

$$\Psi(\sigma) - \Psi(c_0) \leq N_t \log(-\sigma) + \{\mathcal{C}_t(\sigma) - \mathcal{C}_t(0)\} - D_t^* \log \beta_\sigma - \sigma N_t \log \frac{\beta_\sigma - 1}{\beta_\sigma} + O(N_t)$$

$$=: F(\sigma) + O(N_t).$$

We consider two situations here, according to whether $\sigma \leq -bD_t^*/N_t$ or $\sigma > -bD_t^*/N_t$ for some constant $b > 0$. For any $b > 0$, if $\sigma \leq -bD_t^*/N_t$, [Lemma S2.3](#) implies that

$$\Psi(\sigma) - \Psi(c_0) \leq -\frac{1}{2} \frac{D_t^*(1+o(1))}{1 + \frac{1}{2b}} + O(N_t).$$

If $\sigma \in (-bD_t^*/N_t, -C)$, then using $\beta_\sigma = 1 - \sigma N_t/D_t^*$

$$\begin{aligned} F'(\sigma) &= -\frac{N_t}{-\sigma} + \partial_\sigma \zeta(\phi) + N_t \frac{1}{1 - \sigma N_t/D_t^*} + \frac{N_t^2}{D_t^*} \frac{-\sigma}{1 - \sigma N_t/D_t^*} + N_t \log \left(1 - \frac{\sigma N_t}{D_t^*}\right) \\ &\quad - N_t - N_t \log \left(\frac{-\sigma N_t}{D_t^*}\right) \\ &= -\frac{N_t}{-\sigma} + \partial_\sigma \zeta(\phi) + N_t \log \left(1 - \frac{\sigma N_t}{D_t^*}\right) - N_t \log \left(\frac{-\sigma N_t}{D_t^*}\right) \\ &= -\frac{N_t}{-\sigma} + \partial_\sigma \zeta(\phi) + N_t \log \left(1 + \frac{D_t^*}{-\sigma N_t}\right). \end{aligned}$$

Since $\partial_\sigma^2 \zeta(\phi) < 0$ on $(-\infty, 1)$, $\sup_{\sigma \leq -C} \{-C_t'(\sigma)\} \leq -C_t'(\hat{\alpha}_t) = O(N_t)$ under [\[4, Assumption 1\]](#). On the other hand, $N_t \log(1 + \frac{D_t^*}{-\sigma N_t}) \geq N_t \log(1 + \frac{1}{b})$, which can be made arbitrary larger than any multiple constant of N_t by choosing b small enough. Hence $F'(\sigma) > 0$ on $(-bD_t^*/N_t, -C)$ and for all $\sigma \in (-bD_t^*/N_t, -C)$,

$$\Psi(\sigma) - \Psi(c_0) \leq N_t \log(C) + \{\mathcal{C}_t(-C) - \mathcal{C}_t(0)\} - CN_t \log \left\{1 + \frac{D_t^*}{CN_t}\right\} - CN_t + O\left(\frac{N_t^2}{D_t^*}\right).$$

With similar arguments $\mathcal{C}_t(-C) - \mathcal{C}_t(0) = O(N_t)$ under [\[4, Assumption 1\]](#), and [\[4, Equation \(33\)\]](#) holds.

We now study $\sigma > c_2$. Since $u > 0$ and $g(\sigma, \varepsilon) > 0$, we have

$$H_\sigma(\varepsilon, u) \leq N_t \log u - \frac{D_t^*}{2} \log(1+u) + \frac{D_t^*}{2} \log(1-\varepsilon) + \frac{D_t^*}{2} (1-\varepsilon) \beta_\sigma.$$

the right hand side is maximized in $u = 2N_t/D_t^*(1 + O(N_t/D_t^*))$ and $\varepsilon = 0$ which leads to

$$H(\sigma, \varepsilon, u) \leq N_t \log \frac{N_t}{D_t^*} - N_t + N_t \log(2) + \frac{D_t^*}{2} \beta_\sigma + O\left(\frac{N_t^2}{D_t^*}\right), \quad \forall u, \varepsilon, \sigma.$$

Moreover let $c_0 > 0$ then $\Psi(c_0) = K(c_0) + \sup_{\varepsilon, u} H(c_0, \varepsilon, u) = \mathcal{C}_t(c_0) + \sup_{\varepsilon, u} H(0, \varepsilon, u) + O(N_t)$. Choosing $\varepsilon_* = \frac{c_0 N_t}{D_t^*} = o(1)$, we have at $\sigma = c_0$, $\frac{1}{2} + \beta_\sigma g(\sigma, \varepsilon) = (1 + o(1))c_0^{-1}$ which combined with [Lemma S2.2](#), leads to

$$\sup_{\varepsilon, u} H(c_0, \varepsilon, u) \geq \sup_u H(c_0, \varepsilon_*, u) = -N_t \log \frac{N_t}{D_t^*} + \frac{D_t^*}{2} + O(N_t).$$

Hence, as soon as $N_t = o(D_t^*)$,

$$\Psi(c_0) \geq \mathcal{C}_t(c_0) + N_t \log \frac{N_t}{D_t^*} + \frac{D_t^*}{2} + O(N_t). \quad (\text{S2.10})$$

Using Equation (S2.10), we then obtain that for all $\sigma > c_2$

$$\begin{aligned}
\Psi(\sigma) - \Psi(c_0) &\leq \{\mathcal{C}_t(\sigma) - \mathcal{C}_t(c_0)\} + O(N_t) \\
&= \sum_{j \geq 2} N_{t,j} \sum_{k=1}^{j-1} \log \frac{k - \sigma}{k - c_0} + O(N_t) \\
&\leq \sum_{j \geq 2} N_{t,j} \sum_{k=1}^{j-1} \log \frac{k - c_2}{k - c_0} + O(N_t) \\
&\leq \sum_{j \geq 2} N_{t,j} \log \frac{1 - c_2}{1 - c_0} + O(N_t) \\
&\leq -K(N_t - N_{t,1})
\end{aligned}$$

for any $K > 0$, by choosing c_2 sufficiently close to 1. Furthermore, [4, Assumption 1] implies that $N_t - N_{t,1} \asymp N_t$, otherwise $\hat{\alpha}_t$ would converge to $1 > \alpha_0$. Hence [4, Equation (34)] is proved.

Lemma S2.1. *Let $\sigma \leq -C$ for some $C > 0$. Then,*

$$\begin{aligned}
\sup_{\varepsilon, u} H(\sigma, \varepsilon, u) &= N_t \log \frac{N_t}{D_t^*} + N_t \log(-\sigma) - N_t \log \beta_\sigma - \sigma N_t \log \frac{\beta_\sigma - 1}{\beta_\sigma} \\
&\quad + \frac{D_t^*}{2} - \frac{D_t^*}{2} \log \beta_\sigma + O(N_t).
\end{aligned}$$

Proof. The starting point is Lemma S2.2, and in particular the Equation (S2.12). The term $-\log(\frac{1}{2} + \beta_\sigma g(\sigma, \varepsilon))$ is not trivial to apprehend. We split the analysis into two scenarios, according to whether $1 - \varepsilon \leq \frac{a}{\beta_\sigma}$ or not, for some $a \in (0, 1)$ to be determined. Under the scenario $1 - \varepsilon \leq \frac{a}{\beta_\sigma}$, we note that $\beta_\sigma g(\sigma, \varepsilon) \geq 0$, from Equation (S2.12),

$$H(\sigma, \varepsilon, u) \leq -\frac{D_t^*}{2} \left\{ \log \frac{1}{a} - a \right\} + N_t \log \frac{N_t}{D_t^*} + O(N_t). \quad (\text{S2.11})$$

We claim that the previous implies that the supremum of $(\varepsilon, u) \mapsto H(\sigma, \varepsilon, u)$ has to be achieved for $1 - \varepsilon > \frac{a}{\beta_\sigma}$. We keep that claim in mind, and we now analyse $H(\sigma, \varepsilon, u)$ for $1 - \varepsilon > \frac{a}{\beta_\sigma}$. Then, we can simplify things a bit. Indeed, by a Taylor expansion of $f(\sigma, \cdot)$ near $\varepsilon \approx 1$, we can obtain that $f(\sigma, \varepsilon) \geq 1 - \varepsilon$ for all $\varepsilon \in (0, 1)$. That means that $g(\sigma, \varepsilon) \geq \frac{1}{2} f(\sigma, \varepsilon)$. Thus, $N_t \log(\frac{1}{2} + \beta_\sigma g(\sigma, \varepsilon)) \geq N_t \log(\frac{1}{2} + \frac{\beta_\sigma}{2} f(\sigma, \varepsilon)) \geq N_t \log \beta_\sigma + N_t \log f(\sigma, \varepsilon) + O(N_t)$. But, $-N_t \log f(\sigma, \varepsilon) = N_t \log(-\sigma) + N_t(-\sigma) \log \varepsilon - N_t \log(1 - \varepsilon^{-\sigma})$. Since we assume that $\varepsilon < 1 - \frac{a}{\beta_\sigma}$, and $-\sigma > 0$, then $\varepsilon^{-\sigma} \leq \exp\{-(-\sigma)(-\log(1 - \frac{a}{1 - \sigma N_t/D_t^*}))\} \leq \exp\{-\frac{(-\sigma)a}{1 - \sigma N_t/D_t^*}\}$. Because $-\sigma \geq C$, this means that $\varepsilon^{-\sigma} \leq \exp\{-\frac{Ca}{1 + CN_t/D_t^*}\}$ uniformly, which is always bounded away from 1. Consequently, $-N_t \log(1 - \varepsilon^{-\sigma}) = O(N_t)$, uniformly, and in the second scenario Equation (S2.12) becomes

$$\sup_{u > 0} H(\sigma, \varepsilon, u) = N_t \log \frac{N_t}{D_t^*} + N_t \log(-\sigma) - N_t \log \beta_\sigma$$

$$-\sigma N_t \log \varepsilon + \frac{D_t^*}{2}(1-\varepsilon)\beta_\sigma + \frac{D_t^*}{2} \log(1-\varepsilon) + O(N_t).$$

That is, it is enough to maximize $G_\sigma(\varepsilon) := -\sigma N_t \log \varepsilon + \frac{D_t^*}{2} \log(1-\varepsilon) + \frac{D_t^*}{2}(1-\varepsilon)\beta_\sigma$. We note that $G'_\sigma(\varepsilon) = \frac{-\sigma N_t}{\varepsilon} - \frac{D_t^*}{2} \frac{1}{1-\varepsilon} - \frac{D_t^*}{2} \beta_\sigma$, and clearly $G''_\sigma(\varepsilon) < 0$ for all $\varepsilon > 0$, whence G_σ admits a unique maximizer solution to

$$\begin{aligned} \frac{-\sigma N_t}{\varepsilon} &= \frac{D_t^*}{2} \frac{1}{1-\varepsilon} + \frac{D_t^*}{2} \left(1 - \frac{\sigma N_t}{D_t^*}\right) \iff \frac{-\sigma N_t}{\varepsilon} = \frac{D_t^*}{2} \frac{1}{1-\varepsilon} + \frac{D_t^*}{2} + \frac{-\sigma N_t}{2} \\ &\iff -\sigma N_t(1-\varepsilon) = \frac{D_t^*}{2} \varepsilon + \frac{D_t^*}{2} (1-\varepsilon) \varepsilon + \frac{-\sigma N_t}{2} (1-\varepsilon) \varepsilon \\ &\iff -\frac{1}{2} (D_t^* - \sigma N_t) \varepsilon^2 + \left(D_t^* - \frac{3\sigma N_t}{2}\right) \varepsilon + \sigma N_t = 0. \end{aligned}$$

The previous has two easy solutions, one is seen to be $\varepsilon = 2$ so it is outside the domain of G , and the other one, of interest, is

$$\varepsilon_* := \frac{-\sigma N_t}{D_t^* - \sigma N_t} = \frac{-\sigma N_t}{D_t^*} \cdot \frac{1}{1 - \sigma N_t/D_t^*} = \frac{\beta_\sigma - 1}{\beta_\sigma}.$$

It follows, for all ε such that $1 - \varepsilon > \frac{a}{\beta_\sigma}$,

$$\begin{aligned} \sup_{u>0} H(\sigma, \varepsilon, u) &\leq N_t \log \frac{N_t}{D_t^*} + N_t \log(-\sigma) - N_t \log \beta_\sigma \\ &\quad - \sigma N_t \log \frac{\beta_\sigma - 1}{\beta_\sigma} + \frac{D_t^*}{2} - \frac{D_t^*}{2} \log \beta_\sigma + O(N_t). \end{aligned}$$

It is clear that the previous is indeed achieved at ε_* , and for $a > 0$ small enough (but constant), it is also bigger than the bound in Equation (S2.11) when $\sigma \leq -C \leq 0$. Hence the conclusion. \square

Lemma S2.2. *If $N_t = o(D_t^*)$, for any fixed value of (σ, ε) , the function $u \mapsto H(\sigma, \varepsilon, u)$ admits a unique maximizer $\bar{u}(\sigma, \varepsilon) > 0$. Furthermore, this maximizer has the asymptotic expansion*

$$\bar{u}(\sigma, \varepsilon) = \frac{N_t}{D_t^*} \cdot \frac{1 + O(N_t/D_t^*)}{\frac{1}{2} + \beta_\sigma g(\sigma, \varepsilon)},$$

and,

$$\begin{aligned} H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) &= -N_t + N_t \log \frac{N_t}{D_t^*} - N_t \log \left(\frac{1}{2} + \beta_\sigma g(\sigma, \varepsilon) \right) \\ &\quad + \frac{D_t^*}{2} \log(1-\varepsilon) + \frac{D_t^*}{2} (1-\varepsilon) \beta_\sigma + O\left(\frac{N_t^2}{D_t^*}\right). \end{aligned} \quad (\text{S2.12})$$

Proof. We first establish that for fixed values of (σ, ε) the function $u \mapsto H(\sigma, \varepsilon, u)$ has a maximum. Indeed, any extremum of $H(\sigma, \varepsilon, \cdot)$ must be solution to

$$\frac{N_t}{u} - \frac{D_t^*}{2} \frac{1}{1+u} - D_t^* \beta_\sigma g(\sigma, \varepsilon) = 0. \quad (\text{S2.13})$$

The limit as $u \rightarrow 0$ of the lhs of Equation (S2.13) is $+\infty$, and the limit as $u \rightarrow \infty$ is $-D_t^* \beta_\sigma g(\sigma, \varepsilon) < 0$, and it is a continuous function of u . Hence, it is the case that $\partial_u H(\sigma, \varepsilon, u) = 0$ has solutions. Furthermore, it is clear that any solution also satisfies $\frac{N_t}{u} \geq \frac{D_t^*}{2} \frac{1}{1+u}$, i.e. $\frac{u}{1+u} \leq \frac{2N_t}{D_t^*}$, whence $u \leq \frac{2N_t}{D_t^*} (1 + o(1))$. Hence, it is enough to look for solutions in $(0, \frac{3N_t}{D_t^*})$. On that interval, $\partial_u^2 H(\sigma, \varepsilon, u) = -\frac{N_t}{u^2} + \frac{D_t^*}{2} \frac{1}{(1+u)^2} < 0$. Then, Equation (S2.13) has a unique solution, and it is a maximum of $H(\sigma, \varepsilon, \cdot)$. Regarding the asymptotic form of $\bar{u}(\sigma, \varepsilon)$, let $u = \frac{N_t}{D_t^*} \frac{1}{\frac{1}{2} + \beta_\sigma g(\sigma, \varepsilon)}$. Then, u is the solution to $\frac{N_t}{u} = \frac{D_t^*}{2} + D_t^* \beta_\sigma g(\sigma, \varepsilon)$. It then follows from Equation (S2.13) that $\frac{N_t}{u} = \frac{N_t}{\bar{u}(\sigma, \varepsilon)} + \frac{D_t^*}{2} \frac{\bar{u}(\sigma, \varepsilon)}{1 + \bar{u}(\sigma, \varepsilon)}$. The first claim follows. The second claim follows because $\log(1+x) = x + O(x^2)$ for all $x > -1$. \square

Lemma S2.3. Under [4, Assumption 1], for all $\sigma < 0$,

$$\begin{aligned} \mathcal{C}_t(\sigma) - \mathcal{C}_t(0) &\leq -N_t \log(1 - \sigma) + D_t^* \log \left(1 - \frac{\sigma N_t}{D_t^*} \right) \\ &\quad - \sigma N_t \log \left(1 - \frac{D_t^*}{\sigma N_t} \right) - \frac{1}{2} \frac{D_t^* (1 + o(1))}{1 + \frac{1}{2} \frac{D_t^*}{-\sigma N_t}} + O(N_t). \end{aligned}$$

Proof. For $-\sigma \geq 0$ the function $k \mapsto \log(k - \sigma)$ is non-negative and monotone increasing on $(1, \infty)$, and hence we can bound

$$\begin{aligned} \mathcal{C}_t(\sigma) &= \sum_{j \geq 2} N_{t,j} \sum_{k=1}^{j-1} \log(k - \sigma) \\ &\leq \sum_{j \geq 2} N_{t,j} \int_1^j \log(x - \sigma) dx \\ &= \sum_{j \geq 2} N_{t,j} \left\{ 1 - j - (1 - \sigma) \log(1 - \sigma) + (j - \sigma) \log(j - \sigma) \right\} \\ &= -D_t^* - N_t (1 - \sigma) \log(1 - \sigma) + \sum_{j \geq 1} N_{t,j} (j - \sigma) \log(j - \sigma). \end{aligned}$$

That is,

$$\mathcal{C}_t(\sigma) \leq -D_t^* - N_t \log(1 - \sigma) - \sigma \sum_{j \geq 1} N_{t,j} \log \left\{ 1 + \frac{j-1}{1-\sigma} \right\} + \sum_{j \geq 1} N_{t,j} \cdot j \log(j - \sigma).$$

By assumption, $-\sigma > 0$, and then we remark that $x \mapsto \log(1 + \frac{x}{-\sigma})$ is concave on $(1, \infty)$, hence by Jensen's inequality,

$$\begin{aligned} \sum_{j \geq 1} N_{t,j} \log \left\{ 1 + \frac{j-1}{1-\sigma} \right\} &\leq N_t \sum_{j \geq 1} \frac{N_{t,j}}{N_t} \log \left\{ 1 + \frac{j}{-\sigma} \right\} \\ &\leq N_t \log \left\{ 1 + \frac{\sum_{j \geq 1} N_{t,j} \cdot j}{-\sigma} \right\} \end{aligned}$$

$$= N_t \log \left\{ 1 + \frac{D_t^*}{-\sigma N_t} \right\}.$$

So,

$$C_t(\sigma) \leq -D_t^* - N_t \log(1 - \sigma) - \sigma N_t \log \left(1 + \frac{D_t^*}{-\sigma N_t} \right) + \sum_{j \geq 1} N_{t,j} \cdot j \log(j - \sigma). \quad (\text{S2.14})$$

On the other hand, the function $x \mapsto \log(x)$ is non-negative and monotone increasing on $(1, \infty)$. Hence we can bound,

$$C_t(0) = \sum_{j \geq 2} N_{t,j} \sum_{k=1}^{j-1} \log(k) = \sum_{j \geq 3} N_{t,j} \sum_{k=2}^{j-1} \log(k) \geq \sum_{j \geq 3} N_{t,j} \int_1^{j-1} \log(x) dx.$$

Thus,

$$C_t(0) \geq \sum_{j \geq 3} N_{t,j} \left\{ 2 - j + (j-1) \log(j-1) \right\}$$

That is,

$$\begin{aligned} C_t(0) &\geq -D_t^* + \sum_{j \geq 1} N_{t,j} \cdot j \log(j) + O(1) \cdot \sum_{j \geq 2} N_{t,j} \log(j) \\ &= -D_t^* + \sum_{j \geq 1} N_{t,j} \cdot j \log(j) + O(N_t), \end{aligned} \quad (\text{S2.15})$$

where the second line is true under [4, Assumption 1], because $-C_t'(\alpha_t) \asymp \sum_{j \geq 2} N_{t,j} \log(j)$, and $-C_t'(\alpha_t) = \frac{N_t}{\alpha_t} = O(N_t)$. Hence, to finish the proof it is enough to understand,

$$\sum_{j \geq 1} N_{t,j} \cdot j \left\{ \log(j - \sigma) - \log(j) \right\} = N_t \sum_{j \geq 1} \frac{N_{t,j}}{N_t} \cdot j \log \left\{ 1 + \frac{-\sigma}{j} \right\}. \quad (\text{S2.16})$$

Let $p_{t,j} := N_{t,j}/N_t$, $\mathbf{p}_t = (p_{t,1}, p_{t,2}, \dots)$, and $\Phi(x) := x \log(1 + \frac{-\sigma}{x})$. Then we may see the rhs of the last display as $N_t \cdot \mathbb{E}_{\mathbf{p}_t}[\Phi(J)]$. We note that Φ is concave on $(1, \infty)$, so we can use Jensen's inequality to obtain a bound, but we actually need a finer estimate. To get the next order term, we remark that $\Phi'(x) = -\log(\frac{x}{-\sigma}) - 1 + \frac{x}{1+\frac{x}{-\sigma}} + \log(1 + \frac{x}{-\sigma}) = -\log(\frac{x}{-\sigma}) + \log(1 + \frac{x}{-\sigma}) - \frac{1}{1+\frac{x}{-\sigma}}$, and then $\Phi''(x) = -\frac{1}{x} + \frac{1}{-\sigma} \frac{1}{1+\frac{x}{-\sigma}} + \frac{1}{-\sigma} \frac{1}{(1+\frac{x}{-\sigma})^2} = \frac{1}{-\sigma} \frac{1}{x(1+\frac{x}{-\sigma})^2} \{x + x(1 + \frac{x}{-\sigma}) - (-\sigma)(1 + \frac{x}{-\sigma})^2\} = -\frac{1}{x(1+\frac{x}{-\sigma})^2}$, and thus Φ is concave as claimed. Furthermore, $\mathbb{E}_{\mathbf{p}_t}[J] = \frac{D_t^*}{N_t}$, and thus by a Taylor expansion of Φ near $\mathbb{E}_{\mathbf{p}_t}[J]$, we find that there is some J_* in the line segment between J and J_* such that,

$$\begin{aligned} \mathbb{E}_{\mathbf{p}_t}[\Phi(J)] &= \Phi(\mathbb{E}_{\mathbf{p}_t}[J]) + \mathbb{E}[\Phi''(\mathbb{E}_{\mathbf{p}_t}[J])(J - \mathbb{E}_{\mathbf{p}_t}[J])] + \frac{1}{2} \mathbb{E}[\Phi''(J_*)(J - \mathbb{E}_{\mathbf{p}_t}[J])^2] \\ &= \Phi(\mathbb{E}_{\mathbf{p}_t}[J]) + \frac{1}{2} \mathbb{E}[\Phi''(J_*)(J - \mathbb{E}_{\mathbf{p}_t}[J])^2]. \end{aligned}$$

Now we just need a tight enough upper bound on the second term of the last display. Since $\Phi' < 0$, we can upper-bound as follows

$$\begin{aligned}
\mathbb{E}_{\mathbf{p}_t}[\Phi''(J_*)(J - \mathbb{E}_{\mathbf{p}_t}[J])^2] &\leq \mathbb{E}_{\mathbf{p}_t}[\Phi''(J_*)(J - \mathbb{E}_{\mathbf{p}_t}[J])^2 \mathbf{1}_{\{J \leq \frac{1}{2}\mathbb{E}_{\mathbf{p}_t}[J]\}}] \\
&\leq -\frac{\mathbb{E}_{\mathbf{p}_t}[J]^2}{4} \cdot \min_{J \leq \frac{1}{2}\mathbb{E}_{\mathbf{p}_t}[J]} \{-\Phi''(J)\} \cdot \mathbb{E}_{\mathbf{p}_t}[\mathbf{1}_{\{J \leq \frac{1}{2}\mathbb{E}_{\mathbf{p}_t}[J]\}}] \\
&\leq -\frac{\mathbb{E}_{\mathbf{p}_t}[J]^2}{4} \cdot \frac{1}{\frac{1}{2}\mathbb{E}_{\mathbf{p}_t}[J](1 + \frac{\frac{1}{2}\mathbb{E}_{\mathbf{p}_t}[J]}{-\sigma})} \cdot \mathbb{E}_{\mathbf{p}_t}[\mathbf{1}_{\{J \leq \frac{1}{2}\mathbb{E}_{\mathbf{p}_t}[J]\}}] \\
&= -\frac{1}{2} \frac{D_t^*}{N_t} \frac{1}{1 + \frac{1}{2} \frac{D_t^*}{-\sigma N_t}} \sum_{j \geq 1} \frac{N_{t,j}}{N_t} \mathbf{1}_{\{j \leq \frac{1}{2} \frac{D_t^*}{N_t}\}}.
\end{aligned}$$

So the rhs of Equation (S2.16) is no more than,

$$N_t \mathbb{E}_{\mathbf{p}_t}[\Phi(J)] \leq D_t^* \log \left(1 + \frac{-\sigma N_t}{D_t^*} \right) - \frac{1}{2} \frac{D_t^*}{1 + \frac{1}{2} \frac{D_t^*}{-\sigma N_t}} \sum_{j \geq 1} \frac{N_{t,j}}{N_t} \mathbf{1}_{\{j \leq \frac{1}{2} \frac{D_t^*}{N_t}\}}. \quad (\text{S2.17})$$

We finish the proof by noting that $j > \frac{D_t^*}{2N_t} \Leftrightarrow \log(j) > \log(\frac{D_t^*}{2N_t})$, and thus by Markov's inequality, we have $\sum_{j \geq 1} N_{t,j} \mathbf{1}_{\{j > \frac{D_t^*}{2N_t}\}} \leq \frac{1}{\log(D_t^*/(2N_t))} \sum_{j \geq 1} N_{t,j} \log(j) = o(N_t)$ by [4, Assumption 1]. Consequently, $\sum_{j \geq 1} N_{t,j} \mathbf{1}_{\{j \leq \frac{D_t^*}{2N_t}\}} = N_t(1 + o(1))$. Then, the conclusion follows by combining Equations (S2.14) to (S2.17). \square

S2.2. Proof of [4, Lemma 5]

We first establish the existence and uniqueness of the maximizer of $(\varepsilon, u) \mapsto H(\sigma, \varepsilon, u)$ when $\sigma \in (-C, c_2)$.

As already established in Lemma S2.2, the equation has a unique solution $\bar{u}(\sigma, \varepsilon) \in (0, 3N_t/D_t^*)$, and $\bar{u}(\sigma, \varepsilon) = \frac{N_t}{D_t^*} \frac{1+O(N_t/D_t^*)}{\frac{1}{2}+\beta_\sigma g(\sigma, \varepsilon)}$. Similarly, if there is a solution $\tilde{\varepsilon}$ to $\partial_\varepsilon H(\sigma, \varepsilon, u) = 0$, it must be the case that

$$D_t^* \beta_\sigma u \tilde{\varepsilon}^{-1+\sigma} \geq D_t^* \beta_\sigma u \left(\tilde{\varepsilon}^{-1+\sigma} - \frac{1}{2} \right) = \frac{D_t^*}{2} \frac{1}{1-\tilde{\varepsilon}} + \frac{D_t^*}{2} \beta_\sigma \geq \frac{D_t^*}{2} (1 + \beta_\sigma).$$

Since $\sigma \geq -C$, we have $\beta_\sigma = 1 + O(N_t/D_t^*)$, and thus any solution $\tilde{\varepsilon}$ to $\partial_\varepsilon H(\sigma, \varepsilon, u) = 0$ must satisfy $\tilde{\varepsilon}^{-1+\sigma} \geq u^{-1}(1 + O(N_t/D_t^*))$. In particular, if $(\tilde{\varepsilon}, \tilde{u})$ is solution to $\partial_\varepsilon H(\sigma, \varepsilon, u) = 0$ and $\partial_u H(\sigma, \varepsilon, u) = 0$, then it has to be the case that

$$\tilde{\varepsilon}^{-1+\sigma} \geq \frac{1 + O(\frac{N_t}{D_t^*})}{\tilde{u}} = \frac{D_t^*}{N_t} \left(\frac{1}{2} + \beta_\sigma g(\sigma, \tilde{\varepsilon}) \right) \left(1 + O\left(\frac{N_t}{D_t^*}\right) \right) \geq \frac{D_t^*}{N_t} g(\sigma, \tilde{\varepsilon}) \left(1 + O\left(\frac{N_t}{D_t^*}\right) \right).$$

Now we remark that $g(\sigma, \varepsilon) = f(\sigma, \varepsilon) - \frac{1-\varepsilon}{2} \geq \frac{1}{2} f(\sigma, \varepsilon) = \frac{1}{2} \frac{1-\varepsilon^\sigma}{\sigma}$, where the last equality is assuming without loss of generality that $\sigma \neq 0$. It follows, whenever $\sigma \geq -C$, because the mapping $\sigma \mapsto \frac{\tilde{\varepsilon}^{-\sigma}-1}{\sigma}$

is monotone increasing (recall $\tilde{\varepsilon} \in (0, 1)$),

$$\tilde{\varepsilon}^{-1} \geq \frac{1}{2} \frac{D_t^*}{N_t} \frac{\tilde{\varepsilon}^{-\sigma} - 1}{\sigma} \left(1 + O\left(\frac{N_t}{D_t^*}\right)\right) \geq \frac{1}{2} \frac{D_t^*}{N_t} \frac{1 - \tilde{\varepsilon}^C}{C} \left(1 + O\left(\frac{N_t}{D_t^*}\right)\right).$$

We have proved that $\tilde{\varepsilon}(\sigma)$, if it exists, must satisfies $\tilde{\varepsilon}(\sigma) \leq 3CN_t/D_t^*$ for all $\sigma \geq -C$, at least when t is large enough.

To prove that $(\tilde{\varepsilon}(\sigma), \tilde{u}(\sigma))$ exists uniquely, it is enough to establish that $\varepsilon \mapsto H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon))$ has a unique maximum at $\tilde{\varepsilon}(\sigma)$, and then $\tilde{u}(\sigma) = \bar{u}(\sigma, \tilde{\varepsilon}(\sigma))$. Let $\tilde{H}_\sigma(\varepsilon) := H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon))$. Clearly, $\tilde{H}'_\sigma(\varepsilon) = \partial_\varepsilon H_\sigma(\varepsilon, \bar{u}(\sigma, \varepsilon))$ since $\partial_u H_\sigma(\varepsilon, \bar{u}(\sigma, \varepsilon)) = 0$. Then, for any $\varepsilon \leq 3CN_t/D_t^*$, recalling that $\bar{u}(\sigma, \varepsilon) \leq 3N_t/D_t^*$,

$$\begin{aligned} \tilde{H}'_\sigma(\varepsilon) &= D_t^* \beta_\sigma \bar{u}(\sigma, \varepsilon) \left(\varepsilon^{-1+\sigma} - \frac{1}{2} \right) - \frac{D_t^*}{2} \frac{1}{1-\varepsilon} - \frac{D_t^*}{2} \beta_\sigma \\ &= D_t^* \bar{u}(\sigma, \varepsilon) (1 + o(1)) \varepsilon^{-1+\sigma} - D_t^* (1 + o(1)). \end{aligned}$$

By [Lemma S2.2](#), we obtain easily that when $\varepsilon \rightarrow 0$ we have $\bar{u}(\sigma, \varepsilon) \asymp \sigma N_t/D_t^*$ if $\sigma > 0$, $\bar{u}(\sigma, \varepsilon) \asymp -\sigma \varepsilon^{-\sigma} N_t/D_t^*$ if $\sigma < 0$, and $\bar{u}(\sigma, \varepsilon) \asymp \frac{N_t}{D_t^*} \frac{1}{\log(1/\varepsilon)}$ if $\sigma = 0$. So $\lim_{\varepsilon \rightarrow 0} \tilde{H}'_\sigma(\varepsilon) = +\infty$. We already know from the above that $\tilde{H}'_\sigma(\varepsilon) < 0$ when $\varepsilon > 3CN_t/D_t^*$. Since \tilde{H}'_σ is a continuous function of ε , there are solutions to $\tilde{H}'_\sigma(\varepsilon) = 0$ in $(0, 3CN_t/D_t^*)$, and only in this interval. To prove the uniqueness, it is enough to show that $\tilde{H}''_\sigma(\varepsilon) < 0$ for all $\varepsilon \in (0, 3N_t/D_t^*)$. We have,

$$\tilde{H}''_\sigma(\varepsilon) = \partial_\varepsilon^2 H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) + \partial_u \partial_\varepsilon H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) \partial_\varepsilon \bar{u}(\sigma, \varepsilon).$$

Using $\partial_u H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) = 0$, we find that $\partial_\varepsilon \partial_u H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) + \partial_u^2 H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) \partial_\varepsilon \bar{u}(\sigma, \varepsilon) = 0$, *i.e.*

$$\partial_\varepsilon \bar{u}(\sigma, \varepsilon) = \frac{\partial_\varepsilon \partial_u H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon))}{-\partial_u^2 H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon))}.$$

Then,

$$\begin{aligned} \tilde{H}''_\sigma(\varepsilon) &= \partial_\varepsilon^2 H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) + \frac{\{\partial_\varepsilon \partial_u H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon))\}^2}{-\partial_u^2 H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon))} \\ &= -\frac{D_t^*}{2} \frac{1}{(1-\varepsilon)^2} - (1-\sigma) D_t^* \beta_\sigma \bar{u}(\sigma, \varepsilon) \varepsilon^{-2+\sigma} + \frac{\bar{u}(\sigma, \varepsilon)^2 D_t^{*2}}{N_t} \left(\varepsilon^{-1+\sigma} - \frac{1}{2} \right)^2 (1 + o(1)) \\ &\leq -\frac{D_t^*}{2} - D_t^* \bar{u}(\sigma, \varepsilon) \varepsilon^{-2+\sigma} \left((1-\sigma)(1 + o(1)) - \frac{\bar{u}(\sigma, \varepsilon) D_t^* \varepsilon^\sigma}{N_t} \left(1 - \frac{\varepsilon^{1-\sigma}}{2} \right)^2 (1 + o(1)) \right) \\ &\leq -\frac{D_t^*}{2} - D_t^* \bar{u}(\sigma, \varepsilon) \varepsilon^{-2+\sigma} \left((1-\sigma)(1 + o(1)) - \frac{\bar{u}(\sigma, \varepsilon) D_t^* \varepsilon^\sigma}{4N_t} (1 + o(1)) \right). \end{aligned}$$

Recall that

$$\bar{u}(\sigma, \varepsilon) = \frac{N_t}{D_t^*} \frac{1 + O(N_t/D_t^*)}{\frac{1}{2} + \beta_\sigma g(\sigma, \varepsilon)} = \frac{N_t}{D_t^*} \frac{1 + O(N_t/D_t^*)}{f(\sigma, \varepsilon)}, \quad \text{if } \varepsilon \leq 3CN_t/D_t^*.$$

Assuming without loss of generality that $\sigma \neq 0$, we have $\frac{\bar{u}(\sigma, \varepsilon) D_t^* \varepsilon^\sigma}{4N_t} = \frac{1+o(1)}{4} \frac{\sigma \varepsilon^\sigma}{1-\varepsilon^\sigma}$ and remark that

$$\begin{aligned} 1 - \sigma - \frac{\bar{u}(\sigma, \varepsilon) D_t^* \varepsilon^\sigma}{4N_t} &= 1 - \sigma - \frac{\sigma \varepsilon^\sigma (1+o(1))}{4(1-\varepsilon^\sigma)} = 1 - \sigma + o(1) \quad \text{if } \sigma > 0 \\ &= 1 - \frac{3\sigma}{4} + o(1) \quad \text{if } \sigma < 0, \end{aligned}$$

and in all cases it is large than $1 - c_2$. This result extends to $\sigma = 0$ by continuity. Therefore, $\tilde{H}_\sigma''(\varepsilon) \leq -D_t^*/2 < 0$ for all $\varepsilon \in (0, 3CN_t/D_t^*)$ and $(\tilde{u}(\sigma), \tilde{\varepsilon}(\sigma))$ exists and is unique.

We now prove [4, Equation (35)]. We have $\Psi(\sigma) = K(\sigma) + H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma))$, and, $\partial_\varepsilon H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma)) = 0$ and $\partial_u H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma)) = 0$. Then, $\Psi'(\sigma) = K'(\sigma) + \partial_\sigma H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma))$. Recall that $\beta_\sigma = 1 - \sigma N_t/D_t^*$, and $\tilde{\varepsilon}(\sigma) \leq 3CN_t/D_t^*$. then

$$\begin{aligned} \Psi'(\sigma) &= -\frac{N_t^2}{D_t^*} \frac{1}{\beta_\sigma} + \mathcal{C}_t'(\sigma) + \frac{N_t}{2} \frac{1}{\beta_\sigma} - D_t^* \beta_\sigma \tilde{u}(\sigma) \partial_\sigma g(\sigma, \tilde{\varepsilon}(\sigma)) + N_t \tilde{u}(\sigma) g(\sigma, \tilde{\varepsilon}(\sigma)) - \frac{N_t}{2} (1 - \tilde{\varepsilon}(\sigma)) \\ &= \partial_\sigma \zeta(\phi) - D_t^* \beta_\sigma \tilde{u}(\sigma) \partial_\sigma g(\sigma, \tilde{\varepsilon}(\sigma)) + N_t \tilde{u}(\sigma) g(\sigma, \tilde{\varepsilon}(\sigma)) + O\left(\frac{N_t^2}{D_t^*}\right). \end{aligned}$$

Since $\tilde{u}(\sigma) = \frac{N_t}{D_t^*} \frac{1+O(N_t/D_t^*)}{f(\sigma, \tilde{\varepsilon}(\sigma))}$, so that $N_t \tilde{u}(\sigma) g(\sigma, \tilde{\varepsilon}(\sigma)) = O(N_t^2/D_t^*)$. Hence, [4, Equation (35)] follows because $\partial_\sigma g(\sigma, \tilde{\varepsilon}(\sigma)) = \partial_\sigma f(\sigma, \tilde{\varepsilon}(\sigma))$.

Finally, assuming $\sigma \neq 0$ we have,

$$\begin{aligned} \frac{\partial_\sigma f(\sigma, \varepsilon)}{f(\sigma, \varepsilon)} &= \frac{\sigma}{1-\varepsilon^\sigma} \left\{ -\frac{1-\varepsilon^\sigma}{\sigma^2} - \frac{\varepsilon^\sigma \log(\varepsilon)}{\sigma} \right\} \\ &= -\frac{1}{\sigma} \frac{\varepsilon^{-\sigma} - 1 - \sigma \log(1/\varepsilon)}{\varepsilon^{-\sigma} - 1}, \end{aligned}$$

which can be extended by continuity at $\sigma = 0$. Since $\tilde{\varepsilon}(\sigma) = o(1)$, this establishes that for any $K > 0$ we can choose $c_1 > 0$ such that $-N_t \frac{\partial_\sigma f(\sigma, \tilde{\varepsilon}(\sigma))}{f(\sigma, \tilde{\varepsilon}(\sigma))} \geq KN_t$ for all $\sigma \geq c_1$. But on the other hand, $\partial_\sigma^2 \zeta(\phi) < 0$, meaning that $\mathcal{C}_t'(\sigma) \geq \mathcal{C}_t'(c_1)$ for all $\sigma \in (-C, c_1)$. But for $c_1 \in (0, 1)$, $\mathcal{C}_t'(c_1) = -\sum_{j \geq 2} N_{t,j} \sum_{k=1}^{j-1} \frac{1}{k-c_1} \asymp \sum_{j \geq 1} N_{t,j} \cdot \log(j) = O(N_t)$, by [4, Assumption 1]. It follows that $\Psi'(\sigma) > 0$ for all $\sigma < c_1$ if $c_1 > 0$ is small enough. With similar arguments, if c_2 is close enough to one, it must be the case that $\Psi'(c_2) < 0$, by [4, Equation (35)].

S2.3. Proof of [4, Lemma 6]

By [4, Lemma 5], $\Psi(\sigma) = K(\sigma) + H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma))$, $\partial_\varepsilon H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma)) = 0$ and $\partial_u H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma)) = 0$. It follows that $\Psi'(\sigma) = K'(\sigma) + \partial_\sigma H(\sigma, \tilde{\varepsilon}(\sigma), \tilde{u}(\sigma))$. We now write $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(\sigma)$ and $\tilde{u} \equiv \tilde{u}(\sigma)$ to ease the notations. Then,

$$\Psi''(\sigma) = K''(\sigma) + \partial_\sigma^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) + \partial_\varepsilon \partial_\sigma H(\sigma, \tilde{\varepsilon}, \tilde{u}) \tilde{\varepsilon}' + \partial_u \partial_\sigma H(\sigma, \tilde{\varepsilon}, \tilde{u}) \tilde{u}'.$$

By definition, $\tilde{u} = \bar{u}(\sigma, \tilde{\varepsilon})$, so that $\tilde{u}' = \partial_\sigma \bar{u}(\sigma, \tilde{\varepsilon}) + \partial_\varepsilon \bar{u}(\sigma, \tilde{\varepsilon}) \tilde{\varepsilon}'$. From the fact that $\partial_u H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) = 0$, we deduce that $\partial_\sigma \partial_u H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) + \partial_u^2 H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) \partial_\sigma \bar{u}(\sigma, \varepsilon) = 0$, and $\partial_\varepsilon \partial_u H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) +$

$\partial_u^2 H(\sigma, \varepsilon, \tilde{u}(\sigma, \varepsilon)) \partial_\varepsilon \tilde{u}(\sigma, \varepsilon) = 0$; that is,

$$\tilde{u}' = \frac{\partial_\sigma \partial_u H(\sigma, \tilde{\varepsilon}, \tilde{u})}{-\partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u})} + \frac{\partial_\varepsilon \partial_u H(\sigma, \tilde{\varepsilon}, \tilde{u})}{-\partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u})} \tilde{\varepsilon}'. \quad (\text{S2.18})$$

It follows,

$$\begin{aligned} \Psi''(\sigma) = K''(\sigma) + \partial_\sigma^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) + \frac{\{\partial_\sigma \partial_u H(\sigma, \tilde{\varepsilon}, \tilde{u})\}^2}{-\partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u})} \\ + \left\{ \partial_\varepsilon \partial_\sigma H(\sigma, \tilde{\varepsilon}, \tilde{u}) + \frac{\partial_u \partial_\sigma H(\sigma, \tilde{\varepsilon}, \tilde{u}) \cdot \partial_\varepsilon \partial_u H(\sigma, \tilde{\varepsilon}, \tilde{u})}{-\partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u})} \right\} \tilde{\varepsilon}'. \end{aligned}$$

By definition $K'(\sigma) = -\frac{N_t^2}{D_t^*} \frac{1}{\beta_\sigma} + \mathcal{C}_t'(\sigma) + \frac{N_t}{2} \frac{1}{\beta_\sigma}$, and hence $K''(\sigma) = \partial_\sigma^2 \zeta(\phi) + o(N_t)$. It follows using the estimates established in [Lemma S2.4](#),

$$\Psi''(\sigma) = \mathcal{C}_t''(\sigma) - \frac{N_t(1+o(1))}{\sigma^2} - \frac{D_t^*(1+o(1)) \log \frac{D_t^*}{N_t}}{1-\sigma} \tilde{\varepsilon}' + o(N_t).$$

We obtain an estimate on $\tilde{\varepsilon}'$ by using that $\partial_\varepsilon H(\sigma, \tilde{\varepsilon}, \tilde{u}) = 0$. Differentiating both sides of the equation with respect to σ gives $\partial_\sigma \partial_\varepsilon H(\sigma, \tilde{\varepsilon}, \tilde{u}) + \partial_\varepsilon^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) \tilde{\varepsilon}' + \partial_u \partial_\varepsilon H(\sigma, \tilde{\varepsilon}, \tilde{u}) \tilde{u}' = 0$. Hence, by [Equation \(S2.18\)](#), and then by [Lemma S2.4](#), since $\tilde{\varepsilon} \lesssim [N_t/D_t^*]^{1/(1-\sigma)}$ and $0 < \sigma < 1$,

$$\tilde{\varepsilon}' = - \frac{\partial_\sigma \partial_\varepsilon H(\sigma, \tilde{\varepsilon}, \tilde{u}) + \frac{\partial_u \partial_\varepsilon H(\sigma, \tilde{\varepsilon}, \tilde{u}) \cdot \partial_\sigma \partial_u H(\sigma, \tilde{\varepsilon}, \tilde{u})}{-\partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u})}}{\partial_\varepsilon^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) + \frac{\{\partial_u \partial_\varepsilon H(\sigma, \tilde{\varepsilon}, \tilde{u})\}^2}{-\partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u})}} = - \frac{\tilde{\varepsilon}(1+o(1)) \log \frac{D_t^*}{N_t}}{(1-\sigma)^2}. \quad (\text{S2.19})$$

It follows,

$$\Psi''(\sigma) = \partial_\sigma^2 \zeta(\phi) - \frac{N_t(1+o(1))}{\sigma^2} + N_t \frac{\tilde{\varepsilon} \frac{D_t^*}{N_t} \log^2 \frac{D_t^*}{N_t}}{(1-\sigma)^3} + o(N_t) = \partial_\sigma^2 \zeta(\phi) - \frac{N_t(1+o(1))}{\sigma^2} < 0$$

since $\mathcal{C}_t''(\sigma) < 0$.

Lemma S2.4. *Let $\sigma \in [c_1, c_2]$, and let $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(\sigma)$ and $\tilde{u} \equiv \tilde{u}(\sigma)$ as given in [\[4, Lemma 5\]](#). Then, the following estimates are true.*

1. $\partial_\sigma \log f(\sigma, \tilde{\varepsilon}) = -\frac{1+o(1)}{\sigma}$; and $\partial_\sigma^2 \log f(\sigma, \tilde{\varepsilon}) = \frac{1+o(1)}{\sigma^2}$.
2. $\tilde{u} = \frac{\sigma N_t}{D_t^*} (1+o(1))$; and $\tilde{\varepsilon}^{1-\sigma} = \frac{\sigma N_t}{D_t^*} (1+o(1))$.
- 3.

$$\partial_\sigma^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) \sim -\frac{2N_t}{\sigma^2}, \quad \partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) \sim -\frac{D_t^{*2}}{\sigma^2 N_t}, \quad \partial_\varepsilon^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) \sim -(1-\sigma) D_t^* \tilde{\varepsilon}^{-1}.$$

4. $\partial_\sigma \partial_u H(\sigma, \tilde{\varepsilon}, \tilde{u}) = \frac{D_t^*(1+o(1))}{\sigma^2}$.
5. $\partial_\varepsilon \partial_\sigma H(\sigma, \tilde{\varepsilon}, \tilde{u}) = -\frac{D_t^*(1+o(1)) \log \frac{D_t^*}{N_t}}{1-\sigma}$.
6. $\partial_\varepsilon \partial_u H(\sigma, \tilde{\varepsilon}, \tilde{u}) = \frac{D_t^{*2}(1+o(1))}{\sigma N_t}$.

Proof. [Item \(1\)](#). It follows by definition of f , whenever $\varepsilon = o(1)$, and because $\sigma \in (c_2, c_1)$,

$$\partial_\sigma \log f(\sigma, \varepsilon) = \frac{\partial_\sigma f(\sigma, \tilde{\varepsilon})}{\partial_\sigma f(\sigma, \tilde{\varepsilon})} = -\frac{1}{\sigma} - \frac{\tilde{\varepsilon}^\sigma \log(\tilde{\varepsilon})}{1 - \varepsilon^\sigma} = -\frac{1 + o(1)}{\sigma}.$$

From the previous computation,

$$\partial_\sigma^2 \log f(\sigma, \varepsilon) = \frac{1}{\sigma^2} - \frac{\varepsilon^\sigma \log^2(\varepsilon)}{(1 - \varepsilon^\sigma)^2} = \frac{1 + o(1)}{\sigma^2}.$$

[Item \(2\)](#). We already know that $\tilde{u} = \frac{N_t}{D_t^*} \frac{1+o(1)}{f(\sigma, \tilde{\varepsilon})}$, which follows for instance from [Lemma S2.2](#), as $\tilde{\varepsilon} = o(1)$ and $\frac{1}{2} + \beta_\sigma g(\sigma, \tilde{\varepsilon}) = x_1 f(\sigma, \tilde{\varepsilon})(1 + o(1))$. Then it is obvious that $f(\sigma, \tilde{\varepsilon}) = (1/\sigma)(1 + o(1))$ as $\sigma \in (c_2, c_1)$. The second claim follows from $\partial_\varepsilon H(\sigma, \tilde{\varepsilon}, \tilde{u}) = 0$. [Item \(3\)](#). By definition of H , $\partial_\sigma^2 H(\sigma, \varepsilon, u) = -D_t^* \beta_\sigma u \partial_\sigma^2 g(\sigma, \varepsilon) + 2N_t u \partial_\sigma g(\sigma, \varepsilon)$. Clearly $\partial_\sigma g(\sigma, \varepsilon) = \partial_\sigma f(\sigma, \varepsilon)$, and $\partial_\sigma^2 \log f(\sigma, \varepsilon) = \frac{\partial_\sigma^2 f(\sigma, \varepsilon)}{f(\sigma, \varepsilon)} - \frac{\{\partial_\sigma f(\sigma, \varepsilon)\}^2}{f(\sigma, \varepsilon)^2}$. It follows,

$$\begin{aligned} \partial_\sigma^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) &= -D_t^* \beta_\sigma \tilde{u} f(\sigma, \tilde{\varepsilon}) \partial_\sigma^2 \log f(\sigma, \tilde{\varepsilon}) - D_t^* \beta_\sigma \tilde{u} \frac{\{\partial_\sigma f(\sigma, \tilde{\varepsilon})\}^2}{f(\sigma, \tilde{\varepsilon})} + o(N_t) \\ &= -D_t^* \beta_\sigma u f(\sigma, \varepsilon) \partial_\sigma^2 \log f(\sigma, \varepsilon) - u f(\sigma, \varepsilon) D_t^* \beta_\sigma \{\partial_\sigma \log f(\sigma, \varepsilon)\}^2 + o(N_t). \end{aligned}$$

Now, we have that $\tilde{u} f(\sigma, \tilde{\varepsilon}) = \frac{N_t}{D_t^*} (1 + o(1))$, and then $\partial_\sigma^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) = -\frac{2N_t(1+o(1))}{\sigma^2}$ by [Item \(1\)](#). Since, $\tilde{u} = \frac{N_t}{D_t^*} \frac{1+o(1)}{f(\sigma, \tilde{\varepsilon})} = \frac{\sigma N_t}{D_t^*} (1 + o(1))$, and $\partial_u^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) = -\frac{N_t}{\tilde{u}^2} + \frac{D_t^*}{2} \frac{1}{(1+\tilde{u})^2} = -\frac{N_t}{\tilde{u}^2} (1 + o(1))$ we obtain the second term of [Item \(3\)](#). Moreover $\partial_\varepsilon^2 H(\sigma, \varepsilon, u) = -(1 - \sigma) D_t^* \beta_\sigma u \varepsilon^{-2+\sigma} - \frac{D_t^*}{2} \frac{1}{(1-\varepsilon)^2}$. But we know that $\tilde{\varepsilon} = o(1)$ and that $\tilde{\varepsilon}^{-1+\sigma} = \frac{1}{\tilde{u}} (1 + o(1))$. Then,

$$\partial_\varepsilon^2 H(\sigma, \tilde{\varepsilon}, \tilde{u}) = -(1 - \sigma) D_t^* \tilde{\varepsilon}^{-1} (1 + o(1)) - \frac{D_t^* (1 + o(1))}{2}.$$

Since $\tilde{\varepsilon} = o(1)$ and $\sigma \leq c_1 < 1$, the result follows.

[Item \(4\)](#), we have that $\partial_u \partial_\sigma H(\sigma, \varepsilon, u) = -D_t^* \beta_\sigma \partial_\sigma g(\sigma, \varepsilon) + N_t g(\sigma, \varepsilon)$, and since $\partial_\sigma g(\sigma, \varepsilon) = \partial_\sigma f(\sigma, \varepsilon)$, $f(\sigma, \tilde{\varepsilon}) = (1 + o(1))/\sigma$ and $g(\sigma, \tilde{\varepsilon}) = (\frac{1}{\sigma} - \frac{1}{2})(1 + o(1))$, we deduce that

$$\begin{aligned} \partial_u \partial_\sigma H(\sigma, \tilde{\varepsilon}, \tilde{u}) &= -D_t^* (1 + o(1)) f(\sigma, \tilde{\varepsilon}) \partial_\sigma \log f(\sigma, \tilde{\varepsilon}) + \frac{N_t}{\sigma} (1 + o(1)) - \frac{N_t}{2} (1 + o(1)) \\ &= \frac{D_t^* (1 + o(1))}{\sigma^2}. \end{aligned}$$

[Item \(5\)](#). By definition of H , $\partial_\varepsilon \partial_\sigma H(\sigma, \varepsilon, u) = D_t^* \beta_\sigma u \varepsilon^{-1+\sigma} \log(\varepsilon) - N_t u \varepsilon^{-1+\sigma} + \frac{N_t}{2}$. But $\tilde{\varepsilon}^{-1+\sigma} = \tilde{u}^{-1} (1 + o(1))$ by [Item \(2\)](#), so that,

$$\begin{aligned} \partial_\varepsilon \partial_\sigma H(\sigma, \tilde{\varepsilon}, \tilde{u}) &= D_t^* (1 + o(1)) \log(\tilde{\varepsilon}) - \frac{N_t (1 + o(1))}{2} \\ &= \frac{D_t^* (1 + o(1))}{1 - \sigma} \log \tilde{u} - \frac{N_t (1 + o(1))}{2} \end{aligned}$$

$$= -\frac{D_t^*(1+o(1))\log \frac{D_t^*}{N_t}}{1-\sigma}.$$

Item (6). We have $\partial_\varepsilon \partial_u H(\sigma, \varepsilon, u) = D_t^* \beta_\sigma(\varepsilon^{-1+\sigma} - \frac{1}{2})$. We already know that $\tilde{\varepsilon}^{-1+\sigma} = \frac{1}{u}(1+o(1)) = \frac{D_t^*}{\sigma N_t}(1+o(1))$. Hence the result. \square

S2.4. Proof of [4, Lemma 7]

We first prove **Item (1)**. From [4, Lemma 4], we need only prove

$$\Psi(\sigma) - \sup \Psi \leq -K \log D_t^*, \quad \text{for } \sigma \in [-C, c_2]^c$$

and from [4, Lemma 6], the function Ψ is concave on $[c_1, c_2]$, $\Psi(\sigma) \leq \Psi(c_1)$ if $\sigma \leq c_1$ and for any $\eta = o(1)$, and any $\sigma \in [c_1, c_2]$ such that $|\sigma - \hat{\sigma}_t| > \eta$,

$$\begin{aligned} \Psi(\sigma) - \sup \Psi &\leq \max\{\Psi(\hat{\sigma}_t + \eta) - \Psi(\hat{\sigma}_t), \Psi(\hat{\sigma}_t - \eta) - \Psi(\hat{\sigma}_t)\} \\ &= \frac{1}{2} \Psi''(\hat{\sigma}_t)(1+o(1))\eta^2 \\ &= -\frac{1+o(1)}{2} \left(-\mathcal{C}_t''(\hat{\sigma}_t) + \frac{N_t}{\hat{\sigma}_t^2} \right) \eta^2 \\ &\leq -K \log(D_t^*), \end{aligned}$$

choosing $\eta^2 = 4K\alpha_0 \log(D_t^*)/N_t$ since $-\mathcal{C}_t''(\hat{\sigma}_t) > 0$, when t is large enough.

Consider now $|\sigma - \hat{\sigma}_t| \leq C\sqrt{\log(D_t^*)/N_t}$ and $|\varepsilon - \hat{\varepsilon}_t| \leq C\sqrt{\log(D_t^*)/(D_t^*)^{3/4}}$. We know from **Lemma S2.2** that $\tilde{H}_\sigma(\varepsilon) := \sup_u H(\sigma, u, \varepsilon) = H(\sigma, \varepsilon, \tilde{u}(\sigma, \varepsilon))$. Furthermore, we proved in [4, Lemma 5] that \tilde{H}_σ attains its unique maximum at $\tilde{\varepsilon}(\sigma)$, and that $\tilde{H}_\sigma''(\varepsilon) < 0$ for all $\varepsilon \in (0, 1)$ whenever $\sigma \approx \hat{\sigma}_t \in [c_1, c_2]$. Concretely, $\sup_u H(\sigma, \varepsilon, u)$ is a concave function attaining its maximum at $\tilde{\varepsilon}(\sigma)$, and thus for all $\varepsilon \in (0, 1)$ such that $|\varepsilon - \tilde{\varepsilon}(\sigma)| > (C/2)\sqrt{\log D_t^*/(D_t^*)^{3/4}} =: \eta_t$

$$\tilde{H}_\sigma(\varepsilon) - \tilde{H}_\sigma(\tilde{\varepsilon}(\sigma)) \leq \frac{1}{2} \sup_{|x| \leq \eta_t} \tilde{H}_\sigma''(\tilde{\varepsilon}(\sigma) + x) \frac{4C^2 \log D_t^*}{(D_t^*)^{3/2}},$$

and,

$$\sup_{|x| \leq \eta_t} \tilde{H}_\sigma''(\tilde{\varepsilon}(\sigma) + x) \lesssim -D_t^* \cdot \frac{N_t}{D_t^*} \tilde{\varepsilon}(\sigma)^{-2+\sigma} \asymp -D_t^* \cdot \frac{1}{\tilde{\varepsilon}(\sigma)}.$$

Since $|\sigma - \hat{\sigma}_t| \leq C\sqrt{\log(D_t^*)/N_t}$, the **Lemma S2.4** together with [4, Equation (30)] imply

$$\tilde{\varepsilon}(\sigma) \asymp [N_t/D_t^*]^{1/(1-\hat{\sigma}_t)} \asymp [N_t/D_t^*]^{1/(1-\alpha_0)} \asymp 1/\sqrt{D_t^*}$$

so that $\sup_{|x| \leq \eta_t} \tilde{H}_\sigma''(\tilde{\varepsilon}(\sigma) + x) \lesssim -\sqrt{D_t^*}$, which implies that

$$\tilde{H}_\sigma(\varepsilon) - \tilde{H}_\sigma(\tilde{\varepsilon}(\sigma)) \lesssim -C \log D_t^*.$$

Moreover using a Taylor expansion of $\tilde{\varepsilon}(\sigma)$, there exists $\bar{\sigma} \in (\sigma, \hat{\sigma}_t)$ such that $\tilde{\varepsilon}(\sigma) - \tilde{\varepsilon}(\hat{\sigma}_t) = \tilde{\varepsilon}'(\bar{\sigma})(\sigma - \hat{\sigma}_t)$. Using the approximation of $\tilde{\varepsilon}'(\sigma)$ in Equation (S2.19), we obtain

$$|\tilde{\varepsilon}(\sigma) - \tilde{\varepsilon}(\hat{\sigma}_t)| = \frac{\tilde{\varepsilon}(\bar{\sigma})(1 + o(1)) \log \frac{D_t^*}{N_t}}{(1 - \bar{\sigma})^2} |\sigma - \hat{\sigma}_t| \lesssim \frac{1}{\sqrt{D_t^*}} \cdot \log \frac{D_t^*}{N_t} \cdot \sqrt{\frac{\log(D_t^*)}{N_t}} = o((D_t^*)^{-3/2}).$$

Hence, if $|\varepsilon - \tilde{\varepsilon}(\hat{\sigma}_t)| \geq C\sqrt{\log D_t^*}/(D_t^*)^{3/4}$ then $|\varepsilon - \tilde{\varepsilon}(\hat{\sigma}_t)| \geq C\sqrt{\log D_t^*}/(D_t^*)^{3/4}$ and for all $|\sigma - \hat{\sigma}_t| \leq C\sqrt{\log(D_t^*)/N_t}$ and all $u > 0$,

$$\sup_{|\sigma - \hat{\sigma}_t| \leq C\sqrt{\log(D_t^*)/N_t}, u > 0} \mathcal{Q}_t^*(\sigma, \varepsilon, u) - \sup \Psi \leq -K \log D_t^*.$$

Finally we consider $|\sigma - \hat{\sigma}_t| \leq C\sqrt{\log(D_t^*)/N_t}$, $|\varepsilon - \tilde{\varepsilon}(\hat{\sigma}_t)| \leq C\sqrt{\log D_t^*}/(D_t^*)^{3/4}$ and $|u - \hat{u}_t| > C\sqrt{N_t \log D_t^*}/D_t^*$. Recall that $\sup_u H(\sigma, u, \varepsilon) = H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon))$. Furthermore, from the proof of Lemma S2.2 we know that $\partial_u H(\sigma, \varepsilon, u) < 0$ if $u > 3N_t/D_t^*$, and that $\partial_u^2 H(\sigma, \varepsilon, u) = -\frac{N_t}{u^2}(1 + o(1))$ for all $0 < u \leq 3N_t/D_t^*$. Therefore, when $u > 3N_t/D_t^*$,

$$\begin{aligned} H(\sigma, \varepsilon, u) - \sup_u H(\sigma, \varepsilon, u) &\leq H(\sigma, \varepsilon, 3N_t/D_t^*) - \sup_u H(\sigma, \varepsilon, u) \\ &= \frac{\partial_u^2 H(\sigma, \varepsilon, \bar{u}(\sigma, \varepsilon)) \left(\frac{3N_t}{D_t^*} - \bar{u}(\sigma, \varepsilon) \right)^2}{2} \\ &\lesssim -\frac{N_t}{(N_t/D_t^*)^2} \frac{N_t^2}{D_t^{*2}} \lesssim -N_t, \end{aligned}$$

and whenever $0 < u \leq 3N_t/D_t^*$,

$$\begin{aligned} H(\sigma, \varepsilon, u) - \sup_u H(\sigma, \varepsilon, u) &\leq -\frac{1}{2} \frac{N_t}{(3N_t/D_t^*)^2} (u - \bar{u}(\sigma, \varepsilon))^2 \\ &\leq -\frac{1}{18} \frac{D_t^{*2}}{N_t} (u - \bar{u}(\sigma, \varepsilon))^2. \end{aligned}$$

To conclude we thus only need to prove that $|u - \hat{u}_t| > C\sqrt{N_t \log D_t^*}/D_t^* \implies |u - \bar{u}(\sigma, \varepsilon)| > (C/2)\sqrt{N_t \log D_t^*}/D_t^*$. It is enough to show that $|\hat{u}_t - \bar{u}(\sigma, \varepsilon)| \leq (C/2)\sqrt{N_t \log D_t^*}/D_t^*$. But $\hat{u}_t = \bar{u}(\hat{\sigma}_t, \hat{\varepsilon}_t)$, and thus by a Taylor expansion, we can find $\bar{\sigma} \in (\sigma, \hat{\sigma}_t)$ and $\bar{\varepsilon} \in (\varepsilon, \hat{\varepsilon}_t)$ such that $\bar{u}(\sigma, \varepsilon) - \bar{u}(\hat{\sigma}_t, \hat{\varepsilon}_t) = \partial_{\sigma} \bar{u}(\bar{\sigma}, \bar{\varepsilon})(\sigma - \hat{\sigma}_t) + \partial_{\varepsilon} \bar{u}(\bar{\sigma}, \bar{\varepsilon})(\varepsilon - \hat{\varepsilon}_t)$. That is (see for instance the proof of [4, Lemma 6] for details, and also Lemma S2.4),

$$\begin{aligned} \bar{u}(\sigma, \varepsilon) - \bar{u}(\hat{\sigma}_t, \hat{\varepsilon}_t) &= \frac{\partial_{\sigma} \partial_u H(\bar{\sigma}, \bar{\varepsilon}, \bar{u}(\bar{\sigma}, \bar{\varepsilon}))}{-\partial_u^2 H(\bar{\sigma}, \bar{\varepsilon}, \bar{u}(\bar{\sigma}, \bar{\varepsilon}))} (\sigma - \hat{\sigma}_t) + \frac{\partial_{\varepsilon} \partial_u H(\bar{\sigma}, \bar{\varepsilon}, \bar{u}(\bar{\sigma}, \bar{\varepsilon}))}{-\partial_u^2 H(\bar{\sigma}, \bar{\varepsilon}, \bar{u}(\bar{\sigma}, \bar{\varepsilon}))} (\varepsilon - \hat{\varepsilon}_t) \\ &= O\left(\frac{N_t}{D_t^*} \sqrt{\frac{\log(D_t^*)}{N_t}}\right) + O\left(\sqrt{\frac{\log(D_t^*)}{(D_t^*)^{3/2}}}\right) \\ &= O\left(\frac{\sqrt{N_t \log(D_t^*)}}{D_t^*}\right), \end{aligned}$$

which concludes the proof of Item (1) by choosing C large enough.

We now prove [Item \(2\)](#). Recall that $\varphi(\sigma, \tau, s) = (\sigma, \varepsilon(\sigma, \tau, s), u(\sigma, \tau, s))$ and

$$U_t(C) = \left\{ \phi : |\sigma - \hat{\sigma}_t|^2 \leq \frac{C \log(D_t^*)}{N_t}, |\varepsilon(\phi) - \hat{\varepsilon}_t|^2 \leq \frac{C \log(D_t^*)}{(D_t^*)^{3/2}}, |u(\phi) - \hat{u}_t|^2 \leq \frac{CN_t \log(D_t^*)}{D_t^{*2}} \right\}$$

The maximum belongs to $U_t(C)$ from [Item \(1\)](#). We now prove that φ is invertible on U_t . To do so we compute the Jacobian of the transformation. We first compute $\nabla \zeta(\phi)$. Using [\[4, Equation \(27\)\]](#) which defined ζ ,

$$\begin{aligned} \nabla \zeta(\phi) &= \frac{2\zeta(\phi)}{2\zeta(\phi)^2 - \tau\zeta(\phi) - 2\sigma s\zeta(\phi)^\sigma} \begin{pmatrix} s\zeta(\phi)^\sigma \log \zeta(\phi) - N_t \\ \zeta(\phi)/2 \\ \zeta(\phi)^\sigma \end{pmatrix} \\ &= \frac{1}{\zeta(\phi)} \begin{pmatrix} s\zeta(\phi)^\sigma \log \zeta(\phi) - N_t \\ \zeta(\phi)/2 \\ \zeta(\phi)^\sigma \end{pmatrix} (1 + o(1)), \quad \forall \phi \in U_t(C). \end{aligned} \quad (\text{S2.20})$$

Next, since $\tau = \zeta(\phi)\varepsilon(\phi)$,

$$\nabla \varepsilon(\phi) = \frac{1}{\zeta(\phi)} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\varepsilon(\phi)}{\zeta(\phi)} \nabla \zeta(\phi), \quad (\text{S2.21})$$

and, from the fact that $D_t^*(1 - \sigma N_t/D_t^*)u(\phi) = s\zeta(\phi)^\sigma$, when $\phi \in U_t$,

$$\nabla u(\phi) = \frac{1}{D_t^*(1 - \sigma N_t/D_t^*)} \left\{ \begin{pmatrix} N_t u(\phi) + s\zeta(\phi)^\sigma \log \zeta(\phi) \\ 0 \\ \zeta(\phi)^\sigma \end{pmatrix} + \sigma s\zeta(\phi)^{-1+\sigma} \nabla \zeta(\phi) \right\}. \quad (\text{S2.22})$$

Now we define,

$$J_*(\phi) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\zeta(\phi)} & 0 \\ \frac{N_t u(\phi) + s\zeta(\phi)^\sigma \log \zeta(\phi)}{D_t^* \beta_\sigma} & 0 & \frac{\zeta(\phi)^\sigma}{D_t^* \beta_\sigma} \end{pmatrix}, \quad (\text{S2.23})$$

as well as,

$$E(\phi) := \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\varepsilon(\phi)}{\zeta(\phi)^2} (s\zeta(\phi)^\sigma \log \zeta(\phi) - N_t) & -\frac{\varepsilon(\phi)}{\zeta(\phi)^2} \frac{\zeta(\phi)}{2} & -\frac{\varepsilon(\phi)}{\zeta(\phi)^2} \zeta(\phi)^\sigma \\ \frac{\sigma s\zeta(\phi)^{-1+\sigma}}{D_t^* \beta_\sigma \zeta(\phi)} (s\zeta(\phi)^\sigma \log \zeta(\phi) - N_t) & \frac{\sigma s\zeta(\phi)^{-1+\sigma}}{D_t^* \beta_\sigma} \frac{1}{2} & \frac{\sigma s\zeta(\phi)^{-1+\sigma}}{D_t^* \beta_\sigma \zeta(\phi)} \zeta(\phi)^\sigma \end{pmatrix}. \quad (\text{S2.24})$$

Likewise, by [Equations \(S2.20\) to \(S2.24\)](#), we have that the Jacobian matrix of φ is given by $J(\phi) \sim J_*(\phi) + E(\phi)$ as $t \rightarrow \infty$, at least when $\phi \in U_t$. We further remark that

$$J_*(\phi)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta(\phi) & 0 \\ -\frac{N_t u(\phi)}{\zeta(\phi)^\sigma} + s \log \zeta(\phi) & 0 & \frac{D_t^* \beta_\sigma}{\zeta(\phi)^\sigma} \end{pmatrix}$$

and, When $\phi \in U_t$, we can find constants $c_1, \dots, c_6 \in \mathbb{R}$, such that asymptotically as $t \rightarrow \infty$,

$$E(\phi) \sim \begin{pmatrix} 0 & 0 & 0 \\ c_1 \frac{N_t \log(D_t^*)}{(2D_t^*)^{3/2}} & c_2 \frac{1}{2D_t^*} & c_3 \frac{N_t}{4D_t^{*2}} \\ c_4 \frac{N_t^2 \log(D_t^*)}{4D_t^{*2}} & c_5 \frac{N_t}{(2D_t^*)^{3/2}} & c_6 \frac{N_t^2}{(2D_t^*)^{5/2}} \end{pmatrix}.$$

The constants c_1, \dots, c_6 depend uniquely on α_0 and τ_* and can be made explicit. We choose to not do it since this is not needed for our purpose. Similarly, we have $\zeta(\phi) \sim \sqrt{2D_t^*}$, and for some constant $c_7 \in \mathbb{R}$, we also have that $\frac{D_t^* \beta_\sigma}{\zeta(\phi)^\sigma} \sim c_7 \frac{(2D_t^*)^{3/2}}{N_t}$. Thus,

$$J_*(\phi)^{-1} E(\phi) \sim \begin{pmatrix} 0 & 0 & 0 \\ c_1 \frac{N_t \log(D_t^*)}{2D_t^*} & \frac{c_2}{\sqrt{2D_t^*}} & c_3 \frac{N_t}{(2D_t^*)^{3/2}} \\ c_4 c_7 \frac{N_t \log(D_t^*)}{\sqrt{2D_t^*}} & c_5 c_7 & c_6 c_7 \frac{N_t}{2D_t^*} \end{pmatrix}.$$

The eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of $J_*(\phi)^{-1} E(\phi)$ go to zero with t under [4, Assumption 1] and $\lambda_1 = 0$. Then, by a Neumann series expansion of the inverse of $J(\phi) = J_*(\phi)(I + J_*(\phi)^{-1} E(\phi))$, we obtain that whenever $\phi \in U_t$,

$$J(\phi)^{-1} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2D_t^*} & 0 \\ c_8 \sqrt{2D_t^*} \log(2D_t^*) & 0 & c_7 \frac{(2D_t^*)^{3/2}}{N_t} \end{pmatrix}.$$

S3. Local analysis of the likelihood

S3.1. Computation of $\hat{\Sigma}_t$

In the whole Section S3, we define for convenience $\mathcal{Q}_t^*(\sigma, \tau, u) = \mathcal{Q}_t(\sigma, \tau, s_{*,t}u)$. With this definition in mind, we recall that by definition $\Sigma_t(\sigma, \tau, u)^{-1} = -D^2 \mathcal{Q}_t^*(\sigma, \tau, u)$. From [4, Equation (29) and Assumption 1], $|\hat{\sigma}_t - \alpha_0| = o(1/\log N_t)$, $|\hat{\tau}_t - \tau_*| = o(1)$ and $|\hat{s}_t - s_{*,t}| = o(s_{*,t})$. Now let $|\phi_u - \hat{\phi}_u| \leq N_t^{-\delta}$ with $\delta > 0$, Lemma S3.5 implies that $\Sigma_t(\sigma, \tau, s/s_{*,t})^{-1}$ can be written as

$$\left(-C_t''(\sigma) + \frac{s_{*,t}(2D_t^*)^{\sigma/2}}{\alpha_0^3} (1 + o(1)) \right) \text{diag}(1, 0, 0) + \begin{pmatrix} M_1(\phi) & O(s_{*,t}) & M_2(\phi) \\ O(s_{*,t}) & s(1-\sigma)\tau^{\sigma-2} + O(1) & O(s_{*,t}) \\ M_2(\phi) & O(s_{*,t}) & \frac{N_t}{u^2} + O(s_{*,t}^{2\alpha_0+2\epsilon}) \end{pmatrix}$$

where

$$M_1(\phi) := u s_{*,t} \frac{\zeta(\phi)^\sigma \log^2(\zeta(\phi))}{2\sigma} \left(1 - \frac{1}{\sigma \log(\zeta(\phi))} \right)^2,$$

and

$$M_2(\phi) := s_{*,t} \frac{\zeta(\phi)^\sigma \log(\zeta(\phi))}{\sigma} \left(1 - \frac{1}{\sigma \log(\zeta(\phi))} \right) (1 + O(s_{*,t}^{-\delta_1})).$$

Let $X = (x_1, x_2, x_3)^t$. We bound $X^t \Sigma_t(\sigma, \tau, s/s_{*,t})^{-1} X$ using that for $i \neq j$ it holds $2|a_{ij}x_i x_j| = \inf_{b>0} \{a_{ij}^2 x_i^2 b + x_j^2/b\}$. In particular we apply the above $i, j = 1, 2$ with $a_{12} = O(s_{*,t})$ and $b = s_{*,t}^{a-1}$ for $a < \alpha_0$ so that

$$2|s_{*,t}x_1x_2| \leq x_1^2 s_{*,t}^{1+a} + x_2^2 s_{*,t}^{1-a}$$

and similarly for $i, j = 2, 3$

$$2|s_{*,t}x_2x_3| \leq x_3^2 s_{*,t}^{1+a} + x_2^2 s_{*,t}^{1-a}.$$

which implies that

$$\begin{aligned} X^t \Sigma_t(\sigma, \tau, s/s_{*,t})^{-1} X &\leq x_1^2 \left[-C_t''(\sigma) + \frac{s_{*,t}(2D_t^*)^{\sigma/2}}{\alpha_0^3} (1 + o(1)) \right] \\ &\quad + x_2^2 s_{*,t} (1 - \alpha_0) \tau_*^{\alpha_0-2} (1 + o(1)) + 2x_3^2 s_{*,t}^{1+a} \\ &\quad + (x_1, x_3) \begin{pmatrix} M_1(\hat{\phi}_t) & M_2(\hat{\phi}_t) \\ M_2(\hat{\phi}) & \frac{N_t}{u^2} \end{pmatrix} (x_1, x_3)^t \end{aligned}$$

and similarly

$$\begin{aligned} X^t \Sigma_t(\sigma, \tau, s/s_{*,t})^{-1} X &\geq x_1^2 \left[-C_t''(\sigma) + \frac{s_{*,t}(2D_t^*)^{\sigma/2}}{\alpha_0^3} (1 + o(1)) \right] \\ &\quad + x_2^2 s_{*,t} (1 - \alpha_0) \tau_*^{\alpha_0-2} (1 + o(1)) - 2x_3^2 s_{*,t}^{1+a} \\ &\quad + (x_1, x_3) \begin{pmatrix} M_1(\hat{\phi}_t) & M_2(\hat{\phi}_t) \\ M_2(\hat{\phi}_t) & \frac{N_t}{u^2} \end{pmatrix} (x_1, x_3)^t \end{aligned}$$

From [4, Equation (28)], we have that $\hat{s}_t \zeta(\phi)^{\hat{\sigma}_t} = \hat{\sigma}_t N_t (1 + O(s_{*,t}^{-\delta_1}))$ for some $\delta_1 > 0$ which in turns implies that $M_1(\hat{\phi}_t) \frac{N_t}{u^2} - M_2(\hat{\phi}_t)^2 = O(s_{*,t}^{-\delta_1} M_1(\hat{\phi}_t) N_t)$. It thus implies that if $|\phi - \hat{\phi}_{t,u}| \leq s_{*,t}^{-\delta}$ for some $\delta > 0$ then

$$X^t \Sigma_t(\sigma, \tau, s/s_{*,t})^{-1} X \gtrsim x_1^2 s_{*,t}^{1+\alpha_0} + x_2^2 s_{*,t} + x_3^2 \frac{N_t}{\log^2 D_t^*}$$

and

$$X^t \Sigma_t(\sigma, \tau, s/s_{*,t})^{-1} X \lesssim x_1^2 s_{*,t}^{1+\alpha_0} \log^2(D_t^*) + x_2^2 s_{*,t} + x_3^2 N_t.$$

Lemma S3.5. *Let $\epsilon > 0$, and*

$$B_\epsilon = \{ \phi : |\sigma - \sigma_*| \leq \epsilon, |\tau - \tau_*| \leq \epsilon, |s - s_{*,t}| \leq \epsilon s_{*,t} \}.$$

Then, there exists $\delta_1 > 0$ such that for all $\phi \in B_\epsilon$ and ϵ small enough

$$\begin{aligned} \partial_\sigma^2 \mathcal{Q}_t^*(\sigma, \tau, u) &= C_t''(\sigma) - s \frac{\zeta(\phi)^\sigma \log^2 \zeta(\phi)}{\sigma} \left[1 - \frac{1}{\sigma \log \zeta(\phi)} \right]^2 (1 + O(s_{*,t}^{-\delta_1})), \\ \partial_\tau^2 \mathcal{Q}_t^*(\sigma, \tau, u) &= -s(1 - \sigma) \tau_*^{\sigma-2} + O(1), \quad \partial_u^2 \mathcal{Q}_t^*(\sigma, \tau, u) = -\frac{N_t}{u^2} (1 + s_{*,t}^{-\delta_1}), \end{aligned}$$

and

$$\begin{aligned}\partial_\sigma \partial_u \mathcal{Q}_t^*(\sigma, \tau, u) &= -s_{*,t} \frac{\zeta(\phi)^\sigma \log^2 \zeta(\phi)}{\sigma} \left[1 - \frac{1}{\sigma \log \zeta(\phi)} \right] (1 + O(s_{*,t}^{-\delta_1})), \\ \partial_\tau \partial_\sigma \mathcal{Q}_t^*(\sigma, \tau, u) &= -s\tau_*^{\sigma-1} \log(\tau) (1 + O(s_{*,t}^{-\delta_1})), \\ \partial_u \partial_\tau \mathcal{Q}_t^*(\sigma, \tau, u) &= s_{*,t} \tau^{\sigma-1} (1 + O(s_{*,t}^{-\delta_1})).\end{aligned}$$

Proof. We recall the convention in the main paper that $\phi = (\sigma, \tau, s)$ and $\phi_u = (\sigma, \tau, u)$ with $u = s/s_{*,t}$. Recall that

$$\mathcal{Q}_t^*(\phi_u) = \mathcal{C}_t(\sigma) + N_t \log s - D_t^* \mathcal{A}(\phi, \zeta(\phi)) - \log 2/2.$$

By direct computations,

$$\begin{aligned}\partial_\sigma \mathcal{Q}_t^*(\phi_u) &= \mathcal{C}_t'(\sigma) - D_t^* \partial_\sigma \mathcal{A}(\phi, \zeta(\phi)) - D_t^* \partial_\sigma \zeta(\phi) \partial_z \mathcal{A}(\phi, \zeta(\phi)) \\ &= \mathcal{C}_t'(\sigma) - D_t^* \partial_\sigma \mathcal{A}(\phi, \zeta(\phi))\end{aligned}$$

since $\partial_z \mathcal{A}(\phi, \zeta(\phi)) = 0$ for all ϕ . Similarly

$$\partial_\tau \mathcal{Q}_t^*(\phi_u) = -D_t^* \partial_\tau \mathcal{A}(\phi, \zeta(\phi)), \quad \partial_u \mathcal{Q}_t^*(\phi_u) = \frac{N_t}{u} - s_{*,t} D_t^* \partial_s \mathcal{A}(\phi, \zeta(\phi)),$$

so that

$$\begin{aligned}\partial_\sigma^2 \mathcal{Q}_t^*(\phi_u) &= \mathcal{C}_t''(\sigma) - D_t^* \partial_\sigma^2 \mathcal{A}(\phi, \zeta(\phi)) - D_t^* \partial_\sigma \zeta(\phi) \partial_z \partial_\sigma \mathcal{A}(\phi, \zeta(\phi)), \\ \partial_{\sigma,\tau}^2 \mathcal{Q}_t^*(\phi_u) &= -D_t^* \partial_{\sigma,\tau}^2 \mathcal{A}(\phi, \zeta(\phi)) - D_t^* \partial_\tau \zeta(\phi) \partial_z \partial_\sigma \mathcal{A}(\phi, \zeta(\phi)), \\ \partial_\tau^2 \mathcal{Q}_t^*(\phi_u) &= -D_t^* \partial_\tau^2 \mathcal{A}(\phi, \zeta(\phi)) - D_t^* \partial_\tau \zeta(\phi) \partial_z \partial_\tau \mathcal{A}(\phi, \zeta(\phi)), \\ \partial_u^2 \mathcal{Q}_t^*(\phi_u) &= -\frac{N_t}{u^2} - s_{*,t}^2 D_t^* \partial_s^2 \mathcal{A}(\phi, \zeta(\phi)) - s_{*,t}^2 D_t^* \partial_s \zeta(\phi) \partial_z \partial_s \mathcal{A}(\phi, \zeta(\phi)), \\ \partial_{\sigma,u}^2 \mathcal{Q}_t^*(\phi_u) &= -D_t^* s_{*,t} \partial_{\sigma,s}^2 \mathcal{A}(\phi, \zeta(\phi)) - s_{*,t} D_t^* \partial_\sigma \zeta(\phi) \partial_z \partial_s \mathcal{A}(\phi, \zeta(\phi)), \\ \partial_{\tau,u}^2 \mathcal{Q}_t^*(\phi_u) &= -D_t^* s_{*,t} \partial_{\tau,s}^2 \mathcal{A}(\phi, \zeta(\phi)) - s_{*,t} D_t^* \partial_s \zeta(\phi) \partial_z \partial_\tau \mathcal{A}(\phi, \zeta(\phi)),\end{aligned}$$

Moreover

$$\begin{aligned}D_t^* \partial_\sigma \mathcal{A}(\phi, \zeta(\phi)) &= -N_t \log \zeta(\phi) + s \left[\frac{\zeta^\sigma \log \zeta - \tau^\sigma \log \tau}{\sigma} - \frac{\zeta^\sigma - \tau^\sigma}{\sigma^2} \right], \\ D_t^* \partial_\tau \mathcal{A}(\phi, \zeta(\phi)) &= \frac{\zeta(\phi) - \tau}{2} - s\tau^{\sigma-1}, \\ D_t^* \partial_s \mathcal{A}(\phi, \zeta(\phi)) &= \frac{\zeta^\sigma - \tau^\sigma}{\sigma},\end{aligned}$$

which implies that

$$D_t^* \partial_\sigma^2 \mathcal{A}(\phi, \zeta(\phi)) = s \left[\frac{\zeta^\sigma \log^2 \zeta - \tau^\sigma \log^2 \tau}{\sigma} - 2 \frac{\zeta^\sigma \log \zeta - \tau^\sigma \log \tau}{\sigma^2} + 2 \frac{\zeta^\sigma - \tau^\sigma}{\sigma^3} \right]$$

$$\begin{aligned}
D_t^* \partial_\tau^2 \mathcal{A}(\phi, \zeta(\phi)) &= -\frac{1}{2} - s(\sigma - 1)\tau^{\sigma-2}, \\
D_t^* \partial_s^2 \mathcal{A}(\phi, \zeta(\phi)) &= 0, \\
D_t^* \partial_{\sigma, \tau}^2 \mathcal{A}(\phi, \zeta(\phi)) &= -s\tau^{\sigma-1} \log \tau \\
D_t^* \partial_{\sigma s}^2 \mathcal{A}(\phi, \zeta(\phi)) &= \frac{\zeta^\sigma \log \zeta - \tau^\sigma \log \tau}{\sigma} - \frac{\zeta^\sigma - \tau^\sigma}{\sigma^2}, \\
D_t^* \partial_{\tau, s}^2 \mathcal{A}(\phi, \zeta(\phi)) &= -\tau^{\sigma-1},
\end{aligned}$$

and that

$$\begin{aligned}
D_t^* \partial_z \partial_\sigma \mathcal{A}(\phi, \zeta(\phi)) &= -\frac{N_t}{\zeta(\phi)} + s\zeta^{\sigma-1} \log \zeta, \\
D_t^* \partial_z \partial_\tau \mathcal{A}(\phi, \zeta(\phi)) &= \frac{1}{2}, \\
D_t^* \partial_z \partial_s \mathcal{A}(\phi, \zeta(\phi)) &= \zeta^{\sigma-1}.
\end{aligned}$$

Note also that:

$$\partial_s \zeta(\phi) = \sigma \zeta(\phi)^{\sigma-1} \left(1 - \frac{\tau}{2\zeta(\phi)} - s\sigma \zeta(\phi)^{\sigma-3} \right)^{-1} = \alpha_0 (2D_t^*)^{(\sigma-1)/2} (1 + o(1)), \quad (\text{S3.25})$$

and similarly

$$\partial_\tau \zeta(\phi) = \frac{1}{2} \left(1 - \frac{\tau}{2\zeta(\phi)} - s\sigma \zeta(\phi)^{\sigma-3} \right)^{-1} = \frac{1}{2} (1 + o(1)),$$

and,

$$\begin{aligned}
\partial_\sigma \zeta(\phi) &= \left(s\zeta(\phi)^{\sigma-1} \log \zeta(\phi) - \frac{N_t}{\zeta(\phi)} \right) \left(1 - \frac{\tau}{2\zeta(\phi)} - s\sigma \zeta(\phi)^{\sigma-3} \right)^{-1} \\
&= \left(\frac{s_{*,t} (2D_t^*)^{(\sigma-1)/2} \log(2D_t^*)}{2} (1 + o(1)) - \frac{N_t}{\sqrt{2D_t^*}} + O(1) \right) (1 + o(1)). \quad (\text{S3.26})
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\partial_\sigma^2 \mathcal{Q}_t^*(\phi_u) &= \mathcal{C}_t''(\sigma) - s \left[\frac{\zeta^\sigma \log^2 \zeta}{\sigma} - 2 \frac{\zeta^\sigma \log \zeta}{\sigma^2} + 2 \frac{\zeta^\sigma}{\sigma^3} + O(1) \right] + O(s_{*,t}^{2\alpha_0+2\epsilon}) \\
\partial_u^2 \mathcal{Q}_t^*(\phi_u) &= -\frac{N_t}{u^2} + O(s_{*,t}^{2\alpha_0+2\epsilon}), \\
\partial_\tau^2 \mathcal{Q}_t^*(\phi_u) &= \frac{1}{2} - s(1-\sigma)\tau^{\sigma-2} + O(1) = -s(1-\sigma)\tau^{\sigma-2} + O(1), \\
\partial_{\sigma, u}^2 \mathcal{Q}_t^*(\phi_u) &= -s_{*,t} \frac{\zeta^\sigma \log \zeta - \tau^\sigma \log \tau}{\sigma} - s_{*,t} \frac{\zeta^\sigma - \tau^\sigma}{\sigma^2} + O(s_{*,t}^{2\alpha_0+2\epsilon}), \\
\partial_{\tau, u}^2 \mathcal{Q}_t^*(\phi_u) &= s_{*,t} \tau^{\sigma-1} + O(s_{*,t}^{\alpha_0+\epsilon}), \\
\partial_{\tau, \sigma}^2 \mathcal{Q}_t^*(\phi_u) &= s\tau^{\sigma-1} \log \tau + O(s_{*,t}^{\alpha_0+\epsilon}).
\end{aligned}$$

□

S3.2. Proof of [4, Lemma 8]

We write $\tilde{I}_0(\phi) = e^{-D_t^* \mathcal{A}(\phi; \zeta(\phi))} / (2\sqrt{\pi})$, from [4, Equation (22)],

$$\begin{aligned} I(\phi) &= \tilde{I}_0(\phi) \int_{\mathbb{R}} \exp\{-D_t^*[\mathcal{A}(\phi; \zeta(\phi) - iy) - \mathcal{A}(\phi; \zeta(\phi))]\} du; = \tilde{I}_0(\phi) \int_{\mathbb{R}} H_\phi(y) dy \\ I_0(\phi) &= \tilde{I}_0(\phi) \int_{\mathbb{R}} \exp\left\{-\frac{y^2 D_t^*[-\partial_z^2 \mathcal{A}(\phi; \zeta(\phi))]}{2}\right\} dy; = \tilde{I}_0(\phi) \int_{\mathbb{R}} H_{0,\phi}(y) dy \end{aligned} \quad (\text{S3.27})$$

and

$$\nabla[\mathcal{Q}_t - L_t] = -\nabla \log I(\phi) + \nabla \log I_0(\phi) + \frac{1}{2} \nabla \log \left(D_t^* \partial_z^2 \mathcal{A}(\phi; \zeta(\phi)) \right)$$

Then,

$$\begin{aligned} \nabla I(\phi) &= \tilde{I}_0(\phi) \int_{\mathbb{R}} \nabla H_\phi(y) dy + \frac{\nabla \tilde{I}_0(\phi)}{\tilde{I}_0(\phi)} I(\phi) \\ \nabla I_0(\phi) &= -\frac{I_0(\phi)}{2} \nabla \log \left(D_t^* \partial_z^2 \mathcal{A}(\phi; \zeta(\phi)) \right) + \frac{\nabla \tilde{I}_0(\phi)}{\tilde{I}_0(\phi)} I_0(\phi) \end{aligned}$$

and

$$\begin{aligned} \nabla[\mathcal{Q}_t - L_t] &= \frac{-\tilde{I}_0(\phi) \int_{\mathbb{R}} \nabla H_\phi(y) dy}{I(\phi)} - \frac{\nabla \tilde{I}_0(\phi)}{\tilde{I}_0(\phi)} \frac{I(\phi) - I_0(\phi)}{I_0(\phi)} \\ &= \frac{-\int_{\mathbb{R}} \nabla H_\phi(y) dy}{\int_{\mathbb{R}} H_{0,\phi}(y) dy} + \frac{I(\phi) - I_0(\phi)}{I_0(\phi)} \left[\frac{\tilde{I}_0(\phi) \int_{\mathbb{R}} \nabla H_\phi(y) dy}{I(\phi)} - \frac{\nabla \tilde{I}_0(\phi)}{\tilde{I}_0(\phi)} \right] \end{aligned} \quad (\text{S3.28})$$

We have

$$\begin{aligned} \int_{\mathbb{R}} \nabla H_\phi(y) dy &= -D_t^* \int_{\mathbb{R}} e^{-D_t^*[\mathcal{A}(\phi; \zeta(\phi) - iy) - \mathcal{A}(\phi; \zeta(\phi))]} [\nabla_\phi \mathcal{A}(\phi; \zeta(\phi) - iy) - \nabla_\phi \mathcal{A}(\phi; \zeta(\phi))] dy \\ &\quad - D_t^* \nabla \zeta(\phi) \int_{\mathbb{R}} e^{-D_t^*[\mathcal{A}(\phi; \zeta(\phi) - iy) - \mathcal{A}(\phi; \zeta(\phi))]} [\nabla_z \mathcal{A}(\phi; \zeta(\phi) - iy) - \nabla_z \mathcal{A}(\phi; \zeta(\phi))] dy \\ &=: \Delta_1 + \Delta_2 \end{aligned}$$

Recall that $\sqrt{-2D_t^* \partial_z^2 \mathcal{A}(\phi_0; \zeta(\phi_0))} = 2^{-1/2}(1 + o(1))$ so that

$$\int_{\mathbb{R}} H_{0,\phi}(y) dy = 2\sqrt{\pi}(1 + o(1)).$$

Also

$$D_t^* \partial_\sigma \mathcal{A}(\phi; z) = -N_t \log z + \frac{s}{2} \left[\frac{\log(z) z^\sigma - \log(\tau) \tau^\sigma}{\sigma} - \frac{z^\sigma - \tau^\sigma}{\sigma^2} \right] \quad (\text{S3.29})$$

$$D_t^* \partial_\tau \mathcal{A}(\phi; z) = \frac{z - \tau}{2} - s\tau^{\sigma-1}, \quad D_t^* \partial_s \mathcal{A}(\phi; z) = \frac{z^\sigma - \tau^\sigma}{\sigma}$$

In particular when $|y| \leq \sqrt{\zeta(\phi_0)}$, i.e. $|y| \leq c_0 \sqrt{t}$ for some $c_0 > 0$, then

$$\begin{aligned} \log(\zeta(\phi_0) - iy) - \log(\zeta(\phi_0)) &= \frac{\log(\zeta(\phi_0)^2 + u^2) - \log(\zeta(\phi_0)^2)}{2} - \frac{iu}{\zeta(\phi_0)} + O(|y|^2/t^2), \\ (\zeta(\phi_0) - iy)^{\sigma_0} - (\zeta(\phi_0))^{\sigma_0} &= -iy\sigma_0\zeta(\phi_0)^{\sigma_0-1} + O(|y|^2 t^{\sigma_0-2}) \end{aligned}$$

and also, from [Equations \(S1.6\) to \(S1.8\)](#), writing $A_0 = 2[-D_t^* \partial_z^2 \mathcal{A}(\phi; \zeta(\phi_0))]$

$$\begin{aligned} e^{-D_t^* [\mathcal{A}(\phi; \zeta(\phi) - iy) - \mathcal{A}(\phi; \zeta(\phi_0))]} &= e^{-y^2 A_0/4} \times \\ &\left(1 - \frac{iy^3 D_t^*}{2\pi} \left[\int_{|\xi|=\zeta(\phi_0)/2} \frac{\mathcal{A}(\phi_0; \zeta(\phi_0) + \xi)}{\xi^4} + O\left(\frac{\sup_{1/2 \leq x \leq 2} \sup_{\varphi \in [-\pi, \pi]} |\mathcal{A}(\phi_0; x\zeta(\phi_0)e^{i\varphi})|}{\zeta(\phi_0)^4} \right) \right] \right) \\ &= e^{-y^2 A_0/4} \left(1 - \frac{iy^3 D_t^*}{2\pi} \int_{|\xi|=\zeta(\phi_0)/2} \frac{\mathcal{A}(\phi_0; \zeta(\phi_0) + \xi)}{\xi^4} d\xi + O\left(\frac{|y|^4 \log t}{t^2} \right) \right). \end{aligned}$$

We combine this with the arguments as in the proof of [\[4, Theorem 7\]](#) in [Section S1.4](#), but we consider a decomposition into $|y| \leq \zeta(\phi)^{1/2}$ and $|y| > \zeta(\phi)^{1/2}$, instead of $|y| \leq Ca(\sigma)$ and its complement we obtain that

$$\begin{aligned} &\left| D_t^* \int_{|y| \leq \sqrt{\zeta(\phi_0)}} e^{-D_t^* [\mathcal{A}(\phi; \zeta(\phi) - iy) - \mathcal{A}(\phi; \zeta(\phi_0))]} [\nabla_\sigma \mathcal{A}(\phi; \zeta(\phi) - iy) - \nabla_\sigma \mathcal{A}(\phi; \zeta(\phi_0))] dy \right| \\ &\lesssim \left(\int e^{-y^2/4} |y|^6 dy \right) t^{\sigma_0-1} \log t = O(t^{\sigma_0-1} \log t) \\ &\left| D_t^* \int_{|y| \leq \sqrt{\zeta(\phi_0)}} e^{-D_t^* [\mathcal{A}(\phi; \zeta(\phi) - iy) - \mathcal{A}(\phi; \zeta(\phi_0))]} [\nabla_\tau \mathcal{A}(\phi; \zeta(\phi) - iy) - \nabla_\tau \mathcal{A}(\phi; \zeta(\phi_0))] dy \right| \\ &\lesssim t^{-1} \\ &\left| D_t^* \int_{|y| \leq \sqrt{\zeta(\phi_0)}} e^{-D_t^* [\mathcal{A}(\phi; \zeta(\phi) - iy) - \mathcal{A}(\phi; \zeta(\phi_0))]} [\nabla_s \mathcal{A}(\phi; \zeta(\phi) - iy) - \nabla_s \mathcal{A}(\phi; \zeta(\phi_0))] dy \right| \\ &\lesssim t^{\sigma_0-2} \end{aligned}$$

Similarly, using [Equation \(S1.1\)](#), if $|y| \leq c_0 \delta t$

$$\begin{aligned} &D_t^* [\partial_z \mathcal{A}(\phi_0; \zeta(\phi_0) - iy) - \partial_z \mathcal{A}(\phi_0; \zeta(\phi_0))] \\ &= \frac{iy}{2} \left(1 + \frac{2(D_t^* - \sigma_0 N_t)}{\zeta(\phi_0)^2} + 2(1 - \sigma_0)t\zeta(\phi_0)^{\sigma_0-2} \right) + O(|y|^2 t^{-1}) \end{aligned}$$

so that combining with $\nabla \zeta(\phi_0)$ computed in the previous Section,

$$|\Delta_2(1)| \lesssim \frac{\log t}{t^{1-\sigma_0}}, \quad |\Delta_2(2)| \lesssim \frac{\log t}{t}, \quad |\Delta_2(3)| \lesssim \frac{1}{t^{1-\sigma_0}}.$$

Finally, since the integrals over $|y| > \sqrt{\zeta(\phi_0)}$ are exponentially small. we obtain that

$$\int_{\mathbb{R}} \nabla_{\sigma} H_{\phi}(y) dy = O\left(\frac{\log t}{t^{1-\sigma_0}}\right), \quad \int_{\mathbb{R}} \nabla_{\tau} H_{\phi}(y) dy = O\left(\frac{1}{t}\right), \quad \int_{\mathbb{R}} \nabla_u H_{\phi}(y) dy = O\left(\frac{1}{t^{1-\sigma_0}}\right). \quad (\text{S3.30})$$

Combining the above upper bound, with $|I(\phi_0) - I_0(\phi_0)|/I_0(\phi_0) = O(\log t/t^2)$ and [Equation \(S3.29\)](#) to bound $\nabla I_0(\phi_0)/I_0(\phi_0)$ we obtain

$$\nabla_{\sigma}[\mathcal{Q}_t - L_t] \lesssim \frac{(\log t)^2}{t^{1-\sigma_0}}, \quad \nabla_{\tau}[\mathcal{Q}_t - L_t] \lesssim \frac{\log t}{t}, \quad t \nabla_s[\mathcal{Q}_t - L_t] \lesssim \frac{\log t}{t^{1-\sigma_0}}.$$

Note that the above control on $\nabla \mathcal{Q}_t - \nabla L_t$ is valid in expectation since

$$\zeta(\phi)^2 \geq 2D_t^* - 2\sigma_0 N_t \geq 2D_t^*(1 - \sigma_0), \quad \zeta(\phi) \geq (2t)^{1/(2-\sigma_0)}.$$

We now prove that

$$\Delta = \left| \text{Tr} \left(V_0 [D_{\phi_u}^2 L_t - D_{\phi_u}^2 \mathcal{Q}_t^*](\phi_{0,u}) \right) \right| = o(1)$$

with

$$V_0 = \frac{1}{t} \begin{pmatrix} c_1 t^{-\sigma_0} & 0 & c_2 t^{-\sigma_0} \log t \\ 0 & 1 & 0 \\ t^{-\sigma_0} \log t & 0 & t^{-\sigma_0} \log^2 t \end{pmatrix}.$$

This boils down to proving that

$$\begin{aligned} t^{-1-\sigma_0} |D_{\sigma,\sigma}^2 [L_t - \mathcal{Q}_t](\phi_0)| &= o(1) \\ t^{-1-\sigma_0} \log t |D_{\sigma,s}^2 [L_t - \mathcal{Q}_t](\phi_0)| &= o(1) \\ t^{-1} |D_{\tau,\tau}^2 [L_t - \mathcal{Q}_t](\phi_0)| &= o(1) \\ t^{-1-\sigma_0} (\log t)^2 |D_{s,s}^2 [L_t - \mathcal{Q}_t](\sigma_0)| &= o(1) \end{aligned}$$

We use [Equation \(S3.28\)](#) so that

$$\begin{aligned} D_{\phi}^2 [\mathcal{Q}_t - L_t] &= \frac{-\int_{\mathbb{R}} D^2 H_{\phi}(u) du}{\int_{\mathbb{R}} H_{0,\phi}(u) du} + \frac{\int_{\mathbb{R}} \nabla H_{\phi}(u) du \int_{\mathbb{R}} \nabla H_{0,\phi}(u) du}{[\int_{\mathbb{R}} H_{0,\phi}(u) du]^2} \\ &\quad + \frac{I(\phi) - I_0(\phi)}{I_0(\phi)} \left[\frac{\tilde{I}_0(\phi) \int_{\mathbb{R}} \nabla H_{\phi}(u) du}{I(\phi)} - \frac{\nabla \tilde{I}_0(\phi)}{\tilde{I}_0(\phi)} \right]. \end{aligned}$$

First since

$$-D_t^* \partial_z^2 \mathcal{A}(\phi, z) = \frac{D_t^* - \sigma N_t}{z^2} + s(1 - \sigma) z^{-2+\sigma}$$

simple computations imply that

$$\begin{aligned} \nabla_{\sigma} [-D_t^* \partial_z^2 \mathcal{A}(\phi_0, \zeta(\phi_0))] &= O(t^{-1+\sigma_0} \log t), \quad \nabla_{\tau} [-D_t^* \partial_z^2 \mathcal{A}(\phi_0, \zeta(\phi_0))] = O(t^{-1}) \\ t^2 \nabla_s [-D_t^* \partial_z^2 \mathcal{A}(\phi_0, \zeta(\phi_0))] &= O(t^{\sigma_0}) \end{aligned}$$

which implies the same orders for $\int_{\mathbb{R}} \nabla H_{0,\phi}(u) du$. Moreover [Equation \(S3.29\)](#), together with the relation $\sigma_0 N = t\zeta(\phi_0)^{\sigma_0}(1 + O(t^{-\delta}))$ for some $\delta > 0$, imply that

$$|\nabla_{\sigma} \log \tilde{I}_0(\phi_0)| \lesssim t^{1+\sigma_0}, \quad |\nabla_{\tau} \log \tilde{I}_0(\phi_0)| \lesssim t, \quad t|\nabla_s \log \tilde{I}_0(\phi_0)| \lesssim t^{1+\sigma_0}.$$

Combining this with

$$|I(\phi_0) - I_0(\phi_0)|/I_0(\phi_0) = O(\log t/t^2)$$

implies that term by term with $D_{\phi_u}^2$ representing either $D_{\sigma,\sigma}^2$, $D_{\tau,\tau}^2$ or $t^2 D_{s,s}^2$ and ∇_{ϕ_u} representing either ∇_{σ} , ∇_{τ} or $t\nabla_s$

$$\begin{aligned} |D_{\phi_u}^2[Q_t - L_t]| &\lesssim \left| \int_{\mathbb{R}} D_{\phi_u}^2 H_{\phi_u}(x) dx \right| + \left| \int_{\mathbb{R}} \nabla_{\phi_u} H_{\phi_u}(x) dx \right| \left| \int_{\mathbb{R}} \nabla_{\phi_u} H_{0,\phi_u}(x) dx \right| \\ &\quad + \frac{\log t}{t^2} \left| \int_{\mathbb{R}} \nabla_{\phi_u} H_{\phi_u}(x) dx \right| + o(1) \end{aligned}$$

Using [Equation \(S3.30\)](#) we then obtain that

$$|D_{\phi_u}^2[Q_t - L_t]| \lesssim \left| \int_{\mathbb{R}} D_{\phi_u}^2 H_{\phi_u}(x) dx \right| + o(1)$$

We have We now study the second derivatives .

$$\begin{aligned} \int_{\mathbb{R}} D_{\phi_u}^2 H_{\phi_u}(x) dx &= -D_t^* \int_{\mathbb{R}} e^{-D_t^*[\mathcal{A}(\phi;\zeta(\phi)-iu)-\mathcal{A}(\phi;\zeta(\phi))]} \left(D_{\phi_u}^2 \mathcal{A}(\phi;\zeta(\phi)-iu) - D_{\phi_u}^2 \mathcal{A}(\phi;\zeta(\phi)) \right) du \\ &\quad - 2D_t^* \nabla_{\phi_u} \zeta(\phi_u) \int_{\mathbb{R}} e^{-D_t^*[\mathcal{A}(\phi;\zeta(\phi)-iu)-\mathcal{A}(\phi;\zeta(\phi))]} \left(\partial_{\phi_u} \partial_z \mathcal{A}(\phi;\zeta(\phi)-iu) - \partial_{\phi_u} \partial_z \mathcal{A}(\phi;\zeta(\phi)) \right) du \\ &\quad - D_t^* D_{\phi_u}^2 \zeta(\phi) \int_{\mathbb{R}} e^{-D_t^*[\mathcal{A}(\phi;\zeta(\phi)-iu)-\mathcal{A}(\phi;\zeta(\phi))]} \left(\partial_z^2 \mathcal{A}(\phi;\zeta(\phi)-iu) - \partial_z^2 \mathcal{A}(\phi;\zeta(\phi)) \right) du \\ &:= \Delta_1 + \Delta_2 + \Delta_3 \end{aligned}$$

Using [Equations \(S3.25\)](#) and [\(S3.26\)](#), we can bound

$$\partial_{\sigma,\sigma}^2 \zeta(\phi) = O(t^{\sigma_0} \log^2 t), \quad \partial_{\tau,\tau}^2 \zeta(\phi) = O(1/t), \quad t^2 \partial_{s,s}^2 \zeta(\phi) = O(t^{2\sigma_0-1}), \quad t \partial_{\sigma,s}^2 \zeta(\phi) = O(t^{\sigma_0} \log t).$$

Since $D_t^* \partial_z^2 \mathcal{A}(\phi;\zeta(\phi)) = O(1)$,

$$\text{Tr}[V_0 \Delta_3] = o(1).$$

Also using

$$D_t^* \partial_z \mathcal{A}(\phi; z) = -z/2 + \tau/2 + (D_t^* - \sigma N_t)/z + s z^{-1+\sigma}$$

we have that

$$\begin{aligned} |D_t^* \partial_{\sigma} [\partial_z \mathcal{A}(\phi;\zeta(\phi)-iu) - \partial_z \mathcal{A}(\phi;\zeta(\phi))]| &\lesssim |u| t^{\sigma_0-1} \log t, \\ |D_t^* \partial_{\tau} [\partial_z \mathcal{A}(\phi;\zeta(\phi)-iu) - \partial_z \mathcal{A}(\phi;\zeta(\phi))]| &= 0, \\ t |D_t^* \partial_{\sigma} [\partial_z \mathcal{A}(\phi;\zeta(\phi)-iu) - \partial_z \mathcal{A}(\phi;\zeta(\phi))]| &\lesssim |u| t^{\sigma_0-1}, \end{aligned}$$

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which together with

$$\partial_\sigma \zeta(\phi_0) = O(t^{\sigma_0} \log t), \quad \partial_\tau \zeta(\phi_0) = O(1), \quad t \partial_s \zeta(\phi_0) = O(t^{\sigma_0})$$

lead to

$$\text{Tr}[V_0 \Delta_2] = o(1).$$

Finally using [Equation \(S3.29\)](#)

$$\begin{aligned} D_t^\star \partial_{\sigma,\sigma}^2 \mathcal{A}(\phi; z) &= \frac{s}{2} \left[\frac{\log(z)^2 z^\sigma - \log(\tau)^2 \tau^\sigma}{\sigma} - 2 \frac{z^\sigma \log z - \log \tau \tau^\sigma}{\sigma^2} + 2 \frac{z^\sigma - \tau^\sigma}{\sigma^3} \right] \\ D_t^\star \partial_{\tau,\tau}^2 \mathcal{A}(\phi; z) &= \frac{1}{2} - s(\sigma - 1) \tau^{\sigma-1}, \\ D_t^\star \partial_{s,s}^2 \mathcal{A}(\phi; z) &= 0, \\ D_t^\star \partial_{s,\sigma}^2 \mathcal{A}(\phi; z) &= \frac{z^\sigma \log z - \tau^\sigma \log \tau}{\sigma} - \frac{z^\sigma - \tau^\sigma}{\sigma^2}, \end{aligned}$$

which implies that

$$\begin{aligned} D_t^\star |\partial_{\sigma,\sigma}^2 [\mathcal{A}(\phi; z - iu) - \mathcal{A}(\phi; z)]| &\lesssim t^{\sigma_0} |u| \log^2 t \\ |D_t^\star \partial_{\tau,\tau}^2 [\mathcal{A}(\phi; z - iu) - \mathcal{A}(\phi; z)]| &= 0 \\ t |D_t^\star \partial_{s,\sigma}^2 [\mathcal{A}(\phi; z - iu) - \mathcal{A}(\phi; z)]| &\lesssim t^{\sigma_0} \log t |u|. \end{aligned}$$

Hence

$$\text{Tr}[V_0 \Delta_1] = o(1)$$

and

$$\left| \text{Tr} \left(V_0 [D_{\phi_u}^2 L_t - D_{\phi_u}^2 \mathcal{Q}_t^*](\phi_{0,u}) \right) \right| = o(1).$$

S4. Asymptotic properties of multigraphex processes

S4.1. Proof of [\[4, Theorem 5\]](#)

S4.1.1. Number of nodes and number of multiedges

The number of nodes N_t is the same in the multigraph and simple graph. The asymptotic results for the number of nodes N_t then follow directly from Theorems 3 and 4 in [\[3\]](#).

Recall that D_t^\star is twice the number of multiedges. We have, using the Slivnyak-Mecke formula,

$$\begin{aligned} \mathbb{E}[D_t^\star] &= \mathbb{E} \left[\sum_{i \geq 1} D_{t,i} \mathbb{1}_{\theta_i \leq t} \right] \\ &= t \int_0^\infty \overline{W}_1(x, x) dx + t^2 \int_{\mathbb{R}_+^2} \overline{W}_1(x, y) dx dy. \end{aligned}$$

Similarly,

$$\begin{aligned}
& \mathbb{E}[(D_t^*)^2] \\
&= \mathbb{E} \left[\sum_{i,j,k,\ell} \tilde{n}_{ik} \tilde{n}_{j\ell} \mathbb{1}_{\theta_i \leq t} \mathbb{1}_{\theta_j \leq t} \mathbb{1}_{\theta_k \leq t} \mathbb{1}_{\theta_\ell \leq t} \right] \\
&= t^4 \int_{\mathbb{R}_+^4} \overline{W}_1(x, u) \overline{W}_1(y, v) dx dy du dv + 4t^3 \int_{\mathbb{R}_+^3} \overline{W}_1(x, u) \overline{W}_1(y, u) dx dy du \\
&\quad + 2t^3 \int_{\mathbb{R}_+^3} \overline{W}_1(x, u) \overline{W}_1(y, y) dx dy du + t^2 \int_{\mathbb{R}_+^2} \overline{W}_1(x, x) \overline{W}_1(y, y) dx dy + 2t^2 \int_{\mathbb{R}_+^2} \overline{W}_2(x, y) dx dy \\
&\quad + 4t^2 \int_{\mathbb{R}_+^2} \overline{W}_1(x, x) \overline{W}_1(x, y) dx dy + t \int_0^\infty \overline{W}_2(x, x) dx \\
&= \mathbb{E}[D_t^*]^2 + 4t^3 \int_{\mathbb{R}_+^3} \overline{W}_1(x, u) \overline{W}_1(y, u) dx dy du \\
&\quad + 2t^2 \int_{\mathbb{R}_+^2} \overline{W}_2(x, y) dx dy + 4t^2 \int_{\mathbb{R}_+^2} \overline{W}_1(x, x) \overline{W}_1(x, y) dx dy + t \int_0^\infty \overline{W}_2(x, x) dx
\end{aligned}$$

hence

$$\begin{aligned}
\text{var}(D_t^*) &= 4t^3 \int_{\mathbb{R}_+^3} \overline{W}_1(x, u) \overline{W}_1(y, u) dx dy du \\
&\quad + 2t^2 \int_{\mathbb{R}_+^2} \overline{W}_2(x, y) dx dy + 4t^2 \int_{\mathbb{R}_+^2} \overline{W}_1(x, x) \overline{W}_1(x, y) dx dy + t \int_0^\infty \overline{W}_2(x, x) dx.
\end{aligned}$$

Note that, for all $x, y > 0$,

$$W_1(x, y)^2 \leq W_2(x, y).$$

Hence, by Cauchy-Schwarz

$$\begin{aligned}
\int_{\mathbb{R}_+^2} \overline{W}_1(x, x) \overline{W}_1(x, y) dx dy &\leq \left(\int \overline{W}_1(x, x)^2 dx \int \left(\int \overline{W}_1(x, y) dy \right)^2 dx \right)^{1/2} \\
&\leq \left(\int \overline{W}_2(x, x) dx \int \left(\int \overline{W}_1(x, y) dy \right)^2 dx \right)^{1/2} < \infty.
\end{aligned}$$

Therefore,

$$\text{var}(D_t^*) = O(t^{-1} \mathbb{E}[D_t^*]^2)$$

and it follows from Lemma S3.1 in [3] that

$$D_t^* \sim \mathbb{E}[D_t^*] \sim t^2 \int_{\mathbb{R}_+^2} \overline{W}_1(x, y) dx dy$$

almost surely as t tends to infinity.

S4.1.2. Number of nodes of degree j .

Case $\alpha_0 = 0$. Let $N_{tj}^{(s)}$ denote the number of nodes of degree j in the simple graph. A node with degree j in the multigraph \mathcal{G}_t has a degree smaller or equal to j in the corresponding simple graph $\mathcal{G}_t^{(s)}$, and we therefore have, for all $j \geq 1$.

$$N_{tj} \leq \sum_{k=1}^j N_{tk}^{(s)}$$

If [4, Assumption 3] holds for $\alpha_0 = 0$ and some slowly varying function ℓ_1 , Theorems 3 and 4 in [3] imply that $N_{tk}^{(s)} = o(t\ell_1(t))$ in mean, and almost surely if [4, Assumption 4] also holds. The result for N_{tj} then follows by comparison.

Case $\alpha_0 \in (0, 1]$. We have

$$N_{tj} = N_j^{(s)} + (N_{tj} - N_{tj}^{(s)})$$

where, using [4, Proposition 1],

$$|N_{tj} - N_{tj}^{(s)}| \leq \tilde{N}_t = o(N_t).$$

and therefore, using the results for the simple graph [3], for $\alpha_0 > 0$, for all $j \geq 1$

$$N_{tj} \sim N_{tj}^{(s)}$$

almost surely, and for $\alpha_0 = 1$ $N_{t1} \sim N_{t1}^{(s)}$ and, for $j \geq 2$, $N_{tj} = o(N_t)$.

S4.2. Second order asymptotics of number of vertices

[4, Theorem 5] gives the first order asymptotic of the number of vertices as $t \rightarrow \infty$. We need, however, in the proof of [4, Theorem 6] the second order asymptotics under the additional [4, Assumption 6]. This is given in the next Lemma.

Lemma S4.6. *Let [4, Assumptions 3 and 6] be satisfied for some $\alpha_0 \in (0, 1)$, and let $\int_{\mathbb{R}_+} W(x, x) dx < \infty$. Then, there exists $\eta > 0$ such that as $t \rightarrow \infty$*

$$\mathbb{E}[N_t] = t\mu^{-1}(t^{-1})\Gamma(1 - \alpha_0)(1 + O(t^{-\eta})). \quad (\text{S4.31})$$

Proof. As for the proof of [4, Theorem 5],

$$\mathbb{E}[N_t] = t \int_{\mathbb{R}_+} (1 - e^{-t\mu(x)}) dx + t \int_{\mathbb{R}_+} W(x, x) e^{-t\mu(x)} dx. \quad (\text{S4.32})$$

The second term in the previous display is $O(t)$ as $x \mapsto W(x, x)$ is integrable. Define $\bar{\mu}$ as

$$\bar{\mu}(x) = c_0^{1/\alpha_0} x^{-1/\alpha_0}, \quad \text{so that } t \int_{\mathbb{R}_+} (1 - e^{-t\bar{\mu}(x)}) dx = t^{1+\alpha_0} c_0 \Gamma(1 - \alpha_0). \quad (\text{S4.33})$$

Then since $|\mu^{-1}(y) - \bar{\mu}^{-1}(y)| \leq Cy^\beta$ when y is close to 0, then there exists x_0 such that for all $x \geq x_0$,

$$|x - c_0\mu(x)^{-\alpha_0}| \leq C\mu(x)^\beta \quad |\mu(x) - \bar{\mu}(x)| \leq C'\bar{\mu}(x)^{\beta+\alpha_0+1} \quad (\text{S4.34})$$

for some $C' > 0$. We split the first integral of Equation (S4.32) between $x \leq x_0$ and $x > x_0$. We have

$$|t \int_0^{x_0} (e^{-t\bar{\mu}(x)} - e^{-t\mu(x)}) dx| \leq t[e^{-t\mu(x_0)} + e^{-t\bar{\mu}(x_0)}]x_0 = o(t)$$

and

$$\begin{aligned} \left| t \int_{x_0}^{\infty} (e^{-t\bar{\mu}(x)} - e^{-t\mu(x)}) dx \right| &\leq t^2 \int_{x_0}^{\infty} |\mu(x) - \bar{\mu}(x)| e^{-t\bar{\mu}(x)} dx \\ &\lesssim t^{1-\beta} = o(t) \end{aligned}$$

so that Equation (S4.31) is proved with $\eta = \alpha_0$. \square

S4.3. Proof of [4, Lemma 9]

The proof first approximates $-\mathcal{C}'_t(\alpha)$ using the subgraph $\mathcal{G}_t^{(s)}$ obtained from \mathcal{G}_t by removing the multiedges and selfedges and [4, Proposition 1]. More precisely define $Z_{i,j} := \mathbf{1}_{\tilde{n}_{i,j} \geq 1}$ if $i \neq j$, and $Z_{i,i} = 0$. The variables $Z_{i,j}$ with $i \neq j$ have a Bernoulli distribution with expectation $W(x, y)$ where $W(x, y) = 1 - W_m(x, y, 0)$ and let $D_{t,i}^{(s)} := \sum_{j \neq i} Z_{i,j} \mathbf{1}_{\theta_j \leq t}$ denote the degrees of node i in the simple graph $\mathcal{G}_t^{(s)}$. We define $X_t(\alpha) := \sum_i \mathbf{1}_{\theta_i \leq t} \sum_{k \geq 1} \frac{1}{k-\alpha} \mathbf{1}_{D_{t,i}^{(s)} > k}$ which is the simple graph analogous of $-\mathcal{C}'_t(\alpha)$.

In a first step, we reduce to the simple graph model by showing that under [4, Assumption 5], we have $-\mathcal{C}'_t(\alpha) - X_t(\alpha) = O((\tilde{N}_t + N_t^{\text{se}}) \log(D_t^*)) = o(t^{1+\alpha_0/2+\delta})$ for any $\delta > 0$, almost-surely, where \tilde{N}_t is as in the [4, Proposition 1] and N_t^{se} is the number of vertices with at least one selfedge in \mathcal{G}_t . This implies that is enough to understand $X_t(\alpha)$. We will proceed by showing in a second time that

$$X_t(\alpha) = U_t(\alpha) + V_t(\alpha),$$

where

$$U_t(\alpha) := \sum_i \mathbf{1}_{\theta_i \leq t} \int_0^1 \frac{1 - (1-u)^{D_{t,i}^{(s)}}}{u(1-u)^\alpha} du,$$

and,

$$V_t(\alpha) := - \sum_i \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{D_{t,i}^{(s)} \geq 1} \int_0^1 \frac{(1-u)^{D_{t,i}^{(s)}}}{(1-u)^{1+\alpha}} du.$$

Finally, the proof is finished by computing the expectations and variances of $U_t(\alpha)$ and $V_t(\alpha)$.

S4.3.1. Control of $-\mathcal{C}'_t(\alpha) - X_t(\alpha)$.

By definition,

$$-\mathcal{C}'_t(\alpha) - X_t(\alpha) = \sum_i \mathbf{1}_{\theta_i \leq t} \sum_{k \geq 1} \frac{1}{k-\alpha} (\mathbf{1}_{D_{t,i} > k} - \mathbf{1}_{D_{t,i}^{(s)} > k}).$$

Since $Z_{i,j} \leq \tilde{n}_{i,j}$, $D_{t,i}^{(s)} \leq D_{t,i}$ and

$$\begin{aligned} 0 \leq -\mathcal{C}'_t(\alpha) - X_t(\alpha) &= \sum_i \mathbf{1}_{\theta_i \leq t} \sum_{k \geq 1} \frac{1}{k - \alpha} \mathbf{1}_{D_{t,i}^{(s)} \leq k} \mathbf{1}_{D_{t,i} > k} \\ &\leq \sum_i \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{D_{t,i} - D_{t,i}^{(s)} \geq 1} \sum_{k \geq 1} \frac{1}{k - \alpha} \mathbf{1}_{D_{t,i} > k}, \end{aligned}$$

where the last line follows because $(D_{t,i}^{(s)} \leq k \text{ and } D_{t,i} > k) \Rightarrow D_{t,i} - D_{t,i}^{(s)} \geq 1$. Now we remark that $\sum_{k \geq 1} \frac{1}{k - \alpha} \mathbf{1}_{D_{t,i} > k} \leq \sum_{k \geq 1} \frac{1}{k - \alpha} \mathbf{1}_{D_t^* > k} = (1 + o(1)) \log(D_t^*)$. In addition,

$$\begin{aligned} \sum_i \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{D_{t,i} - D_{t,i}^{(s)} \geq 1} &= \sum_i \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{D_{t,i} - D_{t,i}^{(s)} \geq 1} \mathbf{1}_{\tilde{n}_{i,i} = 0} + \sum_i \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{D_{t,i} - D_{t,i}^{(s)} \geq 1} \mathbf{1}_{\tilde{n}_{i,i} \geq 1} \\ &\leq \tilde{N}_t + N_t^{\text{se}}. \end{aligned}$$

Hence

$$0 \leq -\mathcal{C}'_t(\alpha) - X_t(\alpha) \leq 2(\tilde{N}_t + N_t^{\text{se}}) \log(D_t^*).$$

Moreover, from [4, Proposition 1], for any $\delta > 0$,

$$\mathbb{E}(\tilde{N}_t) = o(t^{1+\alpha_0/2+\delta})$$

and

$$\mathbb{E}(N_t^{\text{se}}) \leq \mathbb{E}\left(\sum_i \mathbf{1}_{\theta_i \leq t} \tilde{n}_{ii}\right) = t \int_{\mathbb{R}_+} W(x, x) dx$$

so that for any $\delta > 0$,

$$\mathbb{E}[-\mathcal{C}'_t(\alpha)] = \mathbb{E}(X_t(\alpha)) + O(\mathbb{E}(\tilde{N}_t + N_t^{\text{se}}) \log t) = o(t^{1+\alpha_0/2+\delta}). \quad (\text{S4.35})$$

S4.3.2. Proof that $X_t(\alpha) = U_t(\alpha) + V_t(\alpha)$

The decomposition of X_t into $U_t + V_t$ follows by remarking that $\sum_{k=1}^{m-1} \frac{1}{k - \alpha} = \varphi(m - \alpha) - \varphi(1 - \alpha)$, where $\varphi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Using the integral representation of the digamma function, we can rewrite

$$\begin{aligned} \varphi(m - \alpha) - \varphi(1 - \alpha) &= \mathbf{1}_{m \geq 1} \int_0^\infty \frac{e^{-(1-\alpha)u}}{1 - e^{-u}} \left(1 - e^{-(m-1)u}\right) du \\ &= \mathbf{1}_{m \geq 1} \int_0^1 \frac{1}{u(1-u)^\alpha} \left(1 - (1-u)^{m-1}\right) du. \end{aligned}$$

S4.3.3. Computation of $\mathbb{E}[U_t(\alpha) + V_t(\alpha)]$ and proof of [4, Equation (46)]

By the Equation (S4.35) and by the fact that $X_t(\alpha) = U_t(\alpha) + V_t(\alpha)$, it is obvious that the [4, Equation (46)] will follow from the computation of $\mathbb{E}[U_t(\alpha) + V_t(\alpha)]$, which we do now.

Since $\mathbb{E}[(1-u)^{D_{t,i}^{(s)}} | M] = \exp\{\sum_{j \neq i} \mathbf{1}_{\theta_j \leq t} \log(1-uW(\vartheta_i, \vartheta_j))\}$ and by a combination of the Slivnyak-Mecke's formula, Fubini's theorem, and Campbell's formula, we obtain

$$\begin{aligned} \mathbb{E}[U_t(\alpha) + V_t(\alpha)] &= t \int_0^\infty \int_0^1 \left(\frac{1-e^{-ut\mu(x)}}{u(1-u)^\alpha} - \frac{e^{-ut\mu(x)}(1-e^{-(1-u)t\mu(x)})}{(1-u)^{1+\alpha}} \right) du \\ &=: G(t, \mu) \end{aligned}$$

We show below that $G(t, \mu) - G(t, \bar{\mu}) = O(G(t, \bar{\mu})t^{-\eta})$ for some $\eta > 0$ with $\bar{\mu}(x)$ defined in [Equation \(S4.33\)](#), and

$$\begin{aligned} G(t, \bar{\mu}) &= \alpha_0 c_0 t^{1+\alpha_0} \int_0^\infty \int_0^1 \left(\frac{1-e^{-uy}}{u} - \frac{e^{-uy} - e^{-y}}{1-u} \right) (1-u)^{-\alpha} y^{-1-\alpha_0} dy \\ &= \alpha_0 c_0 t^{1+\alpha_0} \left(\int_0^\infty \int_0^1 \frac{y^{-\alpha_0}}{(1-u)^\alpha} e^{-uy} dy du \left[\frac{1}{\alpha_0} - \frac{1}{\alpha} \right] + \int_0^\infty \frac{(1-e^{-y})y^{-\alpha_0-1}}{\alpha} dy \right) \\ &= \frac{\alpha_0 c_0 t^{1+\alpha_0} \Gamma(1-\alpha_0)}{\alpha_0} \left(\frac{1}{\alpha} + \left[1 - \frac{\alpha_0}{\alpha} \right] \int_0^1 u^{\alpha_0-1} (1-u)^\alpha du \right) \\ &= \frac{\alpha_0 c_0 t^{1+\alpha_0} \Gamma(1-\alpha_0)}{\alpha_0} \left(\frac{1}{\alpha} + \left[1 - \frac{\alpha_0}{\alpha} \right] \frac{\Gamma(1-\alpha)\Gamma(\alpha_0)}{\Gamma(1+\alpha_0-\alpha)} \right). \end{aligned}$$

Hence to prove the first part of [\[4, Lemma 9\]](#), it remains to show that $G(t, \mu) - G(t, \bar{\mu}) = O(G(t, \bar{\mu})t^{-\eta})$.

We have, writing $\Delta(x) = \mu(x) - \bar{\mu}(x)$,

$$\begin{aligned} G(t, \mu) - G(t, \bar{\mu}) &= t \int_0^\infty \int_0^1 \left(\frac{e^{-ut\bar{\mu}(x)} - e^{-ut\mu(x)}}{u(1-u)^\alpha} - \frac{[e^{-ut\mu(x)} - e^{-ut\bar{\mu}(x)} - e^{-t\mu(x)} + e^{-t\bar{\mu}(x)}]}{(1-u)^{1+\alpha}} \right) du \\ &= t \int_0^\infty \int_0^1 \left(\frac{e^{-ut\bar{\mu}(x)}(1-e^{ut\Delta(x)})}{u(1-u)^\alpha} - \frac{[e^{-ut\bar{\mu}(x)}(1-e^{ut\Delta(x)}) - e^{-t\bar{\mu}(x)}(1-e^{t\Delta(x)})]}{(1-u)^{1+\alpha}} \right) du. \end{aligned}$$

We split the above integrals into $x \leq x_0$ and $x > x_0$, with x_0 defined by [Equation \(S4.34\)](#). We have

$$\begin{aligned} I_1 &= t \int_0^{x_0} \int_0^1 \left(\frac{e^{-ut\bar{\mu}(x)} - e^{-ut\mu(x)}}{u(1-u)^\alpha} - \frac{[e^{-ut\mu(x)} - e^{-ut\bar{\mu}(x)} - e^{-t\mu(x)} + e^{-t\bar{\mu}(x)}]}{(1-u)^{1+\alpha}} \right) du \\ &\leq x_0 \int_0^1 \frac{e^{-ut\bar{\mu}(x_0)} + e^{-ut\mu(x_0)}}{u(1-u)^\alpha} du + t \int_0^1 \frac{\int_0^{x_0} [\mu(x)e^{-ut\mu(x)} + \bar{\mu}(x)e^{-ut\bar{\mu}(x)}] dx}{(1-u)^\alpha} du \\ &\lesssim e^{-a_0 t} + 1 \end{aligned}$$

for some $a_0 > 0$ when t is large enough and

$$I_2 = t \int_{x_0}^\infty \int_0^1 \left(\frac{e^{-ut\bar{\mu}(x)} - e^{-ut\mu(x)}}{u(1-u)^\alpha} - \frac{[e^{-ut\mu(x)} - e^{-ut\bar{\mu}(x)} - e^{-t\mu(x)} + e^{-t\bar{\mu}(x)}]}{(1-u)^{1+\alpha}} \right) du$$

$$\begin{aligned}
&= t \int_{x_0}^{\infty} \int_0^1 (1 - e^{ut\Delta(x)}) \left(\frac{e^{-ut\bar{\mu}(x)}}{u} - \frac{[e^{-ut\bar{\mu}(x)} - e^{-t\bar{\mu}(x)}]}{1-u} \right) (1-u)^{-\alpha} du \\
&\quad + t \int_{x_0}^{\infty} \int_0^1 (e^{ut\Delta(x)} - e^{t\Delta(x)}) \frac{e^{-t\bar{\mu}(x)}}{(1-u)^{\alpha+1}} du \\
&\leq t^2 \int_{x_0}^{\infty} \bar{\mu}(x)^{\beta+\alpha_0+1} \int_0^1 \frac{e^{-ut\bar{\mu}(x)}}{(1-u)^{\alpha}} du dx + t^2 \int_{x_0}^{\infty} \bar{\mu}(x)^{\beta+\alpha_0+1} \frac{u|e^{-ut\bar{\mu}(x)} - e^{-t\bar{\mu}(x)}|}{(1-u)^{\alpha+1}} du \\
&\quad + t \int_{x_0}^{\infty} \int_0^1 e^{ut\Delta(x)} (1 - e^{(1-u)t\Delta(x)}) \frac{e^{-t\bar{\mu}(x)}}{(1-u)^{\alpha+1}} du \\
&\lesssim t^{1-\beta} = o(t^{1+\alpha_0-\eta}).
\end{aligned}$$

So that we finally obtain that

$$G(t, \mu) = G(t, \bar{\mu})(1 + o(1))$$

and

$$\mathbb{E}[X_t(\alpha)] = t\mu^{-1}(t^{-1})\Gamma(1-\alpha_0)\left(1 + O(t^{-\eta})\right)\left\{\frac{1}{\alpha} + \left(1 - \frac{\alpha_0}{\alpha}\right)\frac{\Gamma(1-\alpha)\Gamma(\alpha_0)}{\Gamma(1+\alpha_0-\alpha)}\right\}.$$

S4.3.4. Proof of [4, Equation (47)]

By the results of

$$\begin{aligned}
&\mathbb{P}(|-\mathcal{C}'_t(\alpha) - \mathbb{E}[-\mathcal{C}'_t(\alpha)]| > t^{1+\alpha_0-\eta+\delta}) \\
&\leq \mathbb{P}(|X(t) - \mathbb{E}[X_t(\alpha)]| > t^{1+\alpha_0-\eta+\delta}/2) + \mathbb{P}([\tilde{N}_t + N_t^{\text{se}}] > t^{1+\alpha_0-\eta+\delta}/2) \\
&\leq \mathbb{P}(|X(t) - \mathbb{E}[X_t(\alpha)]| > t^{1+\alpha_0-\eta+\delta}/2) + t^{-\alpha_0/2+\eta-\delta} \\
&\leq \mathbb{P}(|X(t) - \mathbb{E}[X_t(\alpha)]| > t^{1+\alpha_0-\eta+\delta}/2) + 2t^{-\delta},
\end{aligned}$$

by choosing $\eta \leq \alpha_0$. We control $\mathbb{P}(|X(t) - \mathbb{E}[X_t(\alpha)]| > t^{1+\alpha_0-\eta+\delta}/2)$ by bounding the variance of U_t and of V_t .

S4.3.5. Variance of $U_t(\alpha)$

We split $\mathbb{E}[U_t(\alpha)^2]$ into two terms:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i \in \mathbb{N}} \mathbf{1}_{\theta_i \leq t} \int_{[0,1]^2} \frac{(1 - (1-u)^{D_{t,i}^{(s)}})(1 - (1-v)^{D_{t,i}^{(s)}})}{uv(1-u)^{\alpha}(1-v)^{\alpha}} dudv\right] \\
&\quad + \mathbb{E}\left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \int_{[0,1]^2} \frac{(1 - (1-u)^{D_{t,i}^{(s)}})(1 - (1-v)^{D_{t,j}^{(s)}})}{uv(1-u)^{\alpha}(1-v)^{\alpha}} dudv\right].
\end{aligned}$$

Let's call for simplicity those two terms A_1 and A_2 , respectively. We start with the more delicate one, A_2 . For simplicity from now on we will write $D_i = D_{t,i}^{(s)}$, and we also define $D_i^{-j} = \sum_{k \neq j} Z_{i,k} \mathbf{1}_{\theta_k \leq t}$.

Bound on A_2 Using that $D_i = D_i^{-j} + Z_{i,j}$, we decompose,

$$\begin{aligned} (1 - (1 - u)^{D_i})(1 - (1 - v)^{D_j}) &= (1 - (1 - u)^{D_i^{-j}})(1 - (1 - v)^{D_j^{-i}}) \\ &\quad + (1 - v)^{D_j^{-i}}(1 - (1 - v)^{Z_{i,j}}) \\ &\quad + (1 - u)^{D_i^{-j}}(1 - (1 - u)^{Z_{i,j}}) \\ &\quad - (1 - u)^{D_i^{-j}}(1 - v)^{D_j^{-i}}(1 - (1 - u)^{Z_{i,j}}(1 - v)^{Z_{i,j}}), \end{aligned}$$

which is equal to,

$$\begin{aligned} &(1 - (1 - u)^{D_i^{-j}})(1 - (1 - v)^{D_j^{-i}}) \\ &\quad + (1 - v)^{D_j^{-i}}(1 - (1 - u)^{D_i^{-j}})(1 - (1 - v)^{Z_{i,j}}) \\ &\quad + (1 - u)^{D_i^{-j}}(1 - (1 - v)^{D_j^{-i}})(1 - (1 - u)^{Z_{i,j}}) \\ &\quad + (1 - u)^{D_i^{-j}}(1 - v)^{D_j^{-i}}(1 - (1 - u)^{Z_{i,j}} - (1 - v)^{Z_{i,j}} + (1 - u)^{Z_{i,j}}(1 - v)^{Z_{i,j}}). \end{aligned}$$

Using that $Z_{i,j} \in \{0, 1\}$, we deduce that that,

$$\begin{aligned} (1 - (1 - u)^{D_i})(1 - (1 - v)^{D_j}) &= (1 - (1 - u)^{D_i^{-j}})(1 - (1 - v)^{D_j^{-i}}) \\ &\quad + (1 - v)^{D_j^{-i}}(1 - (1 - u)^{D_i^{-j}})(1 - (1 - v)^{Z_{i,j}}) \\ &\quad + (1 - u)^{D_i^{-j}}(1 - (1 - v)^{D_j^{-i}})(1 - (1 - u)^{Z_{i,j}}) \\ &\quad + (1 - u)^{D_i^{-j}}(1 - v)^{D_j^{-i}}(1 - (1 - uv)^{Z_{i,j}}). \end{aligned}$$

Then by symmetry, $A_2 \leq A_{2,1} + 2A_{2,2} + A_{2,3}$, where,

$$\begin{aligned} A_{2,1} &:= \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \int_{[0,1]^2} \frac{(1 - (1 - u)^{D_i^{-j}})(1 - (1 - v)^{D_j^{-i}})}{uv(1 - u)^\alpha(1 - v)^\alpha} du dv \right], \\ A_{2,2} &:= \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \int_{[0,1]^2} \frac{(1 - u)^{D_i^{-j}}(1 - (1 - v)^{D_j^{-i}})(1 - (1 - u)^{Z_{i,j}})}{uv(1 - u)^\alpha(1 - v)^\alpha} du dv \right], \\ A_{2,3} &:= \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \int_{[0,1]^2} \frac{(1 - u)^{D_i^{-j}}(1 - v)^{D_j^{-i}}(1 - (1 - uv)^{Z_{i,j}})}{uv(1 - u)^\alpha(1 - v)^\alpha} du dv \right] \end{aligned}$$

Bound on $A_{2,1}$ Conditional on M , the variables D_i^{-j} and D_j^{-i} are independent as long as $i \neq j$. Then, we obtain that,

$$\begin{aligned} \mathbb{E}[(1 - (1 - u)^{D_i^{-j}})(1 - (1 - v)^{D_j^{-i}}) \mid M] &= \left(1 - e^{\sum_{k \neq j} \mathbf{1}_{\theta_k \leq t} \log(1 - uW(\vartheta_i, \vartheta_k))}\right) \\ &\quad \times \left(1 - e^{\sum_{k \neq i} \mathbf{1}_{\theta_k \leq t} \log(1 - uW(\vartheta_j, \vartheta_k))}\right). \end{aligned}$$

Write $\lambda_{u,x} := \sum_k \mathbf{1}_{\theta_k \leq t} \log(1 - uW(\vartheta_k, x))$ for simplicity. Recall that by assumption that $W(x, x)$ by assumption. From the last display and the Slivnyak-Mecke formula,

$$A_{2,1} = t^2 \int_{\mathbb{R}_+^2} \int_{[0,1]^2} \frac{\mathbb{E}[(1 - e^{\lambda_{u,x}})(1 - e^{\lambda_{v,y}})]}{u(1-u)^\alpha v(1-v)^\alpha} du dv dx dy.$$

We compute the expectation within the last display using Campbell's formula,

$$\begin{aligned} \mathbb{E}[(1 - e^{\lambda_{u,x}})(1 - e^{\lambda_{v,y}})] &= 1 - e^{-ut\mu(x)} - e^{-vt\mu(y)} + e^{-ut\mu(x)}e^{-vt\mu(y)}e^{uvt\mu(x,y)} \\ &= (1 - e^{-ut\mu(x)}) \left(1 - e^{-vt\mu(y)}\right) + e^{-ut\mu(x)}e^{-vt\mu(y)}(e^{uvt\mu(x,y)} - 1). \end{aligned}$$

From the computations of [Section S4.3.3](#), it is easily deduced that

$$\mathbb{E}[U_t(\alpha)] = t \int_{\mathbb{R}_+} \int_{[0,1]} \frac{1 - e^{-ut\mu(x)}}{u(1-u)^\alpha} du dx.$$

Therefore,

$$A_{2,1} \leq \mathbb{E}[U_t(\alpha)]^2 + t^2 \int_{\mathbb{R}_+^2} \int_{[0,1]^2} \frac{e^{-ut\mu(x)}e^{-vt\mu(y)}(e^{uvt\mu(x,y)} - 1)}{u(1-u)^\alpha v(1-v)^\alpha} du dv dx dy.$$

Since $e^{uvt\mu(x,y)} - 1 \leq uvt\mu(x,y)e^{uvt\mu(x,y)} \leq uvt\mu(x,y)e^{\frac{u}{2}t\mu(x)}e^{\frac{v}{2}t\mu(y)}$, see for instance [\[3, Proof of Lemma S3.7\]](#) then $e^{uvt\mu(x,y)} - 1 \leq \ell_3(x)\ell_3(y)\mu(x)^a\mu(y)^a uvt e^{\frac{u}{2}t\mu(x)}e^{\frac{v}{2}t\mu(y)}$ under the [\[4, Assumption 4\]](#), and thus

$$A_{2,1} \leq \mathbb{E}[U_t(\alpha)]^2 + t^3 \left\{ \int_{\mathbb{R}_+} \int_{[0,1]} \frac{\ell_3(x)\mu(x)^a e^{-\frac{u}{2}t\mu(x)}}{(1-u)^\alpha} du dx \right\}^2$$

But,

$$\begin{aligned} \int_0^1 \frac{e^{-\frac{1}{2}ut\mu(x)} du}{(1-u)^\alpha} &= \int_0^{1/2} \frac{e^{-\frac{1}{2}ut\mu(x)} du}{(1-u)^\alpha} + \int_{1/2}^1 \frac{e^{-\frac{1}{2}ut\mu(x)} du}{(1-u)^\alpha} \\ &\leq 2^\alpha \int_0^{1/2} e^{-\frac{1}{2}ut\mu(x)} du + e^{-\frac{1}{4}t\mu(x)} \int_{1/2}^1 \frac{du}{(1-u)^\alpha} \\ &\leq 2^\alpha \frac{1 - e^{-\frac{1}{4}t\mu(x)}}{t\mu(x)} + \frac{1}{1-\alpha} e^{-\frac{1}{4}t\mu(x)}. \end{aligned}$$

Therefore,

$$\begin{aligned} t^{3/2} \int_{\mathbb{R}_+} \int_{[0,1]} \frac{\ell_3(x)\mu(x)^a e^{-\frac{u}{2}t\mu(x)}}{(1-u)^\alpha} du dx \\ \lesssim t^{3/2} \int_{\mathbb{R}_+} \ell_3(x)\mu(x)^a \frac{(1 - e^{-\frac{1}{4}t\mu(x)})}{t\mu(x)} dx + t^{3/2} \int_{\mathbb{R}_+} \ell_3(x)\mu(x)^a e^{-\frac{1}{4}t\mu(x)} dx. \end{aligned}$$

The second integral in the rhs of the last display is $O(t^{3/2+\alpha_0-a+\delta})$ for every $\delta > 0$ by [3, Lemma S3.4]. Regarding the first integral, we observe that $(1 - e^{-x})/x \leq 1$ for all $x > 0$, so that

$$\begin{aligned} \int_{\mathbb{R}_+} \mathbf{1}_{t\mu(x) \leq 1} \ell_3(x) \mu(x)^a \frac{(1 - e^{-\frac{1}{4}t\mu(x)})}{t\mu(x)/4} dx &\leq \int_{\mathbb{R}_+} \mathbf{1}_{t\mu(x) \leq 1} \ell_3(x) \mu(x)^a dx \\ &\lesssim \int_{\mathbb{R}_+} \ell_3(x) \mu(x)^a e^{-\frac{1}{4}t\mu(x)} dx \\ &= O(t^{\alpha_0-a+\delta}), \end{aligned}$$

by the same argument as above, and also because $a \leq 1$ necessarily,

$$\begin{aligned} \int_{\mathbb{R}_+} \mathbf{1}_{t\mu(x) > 1} \ell_3(x) \mu(x)^a \frac{(1 - e^{-\frac{1}{4}t\mu(x)})}{t\mu(x)} dx &\lesssim \frac{1}{t^a} \int_{\mathbb{R}_+} \mathbf{1}_{t\mu(x) > 1} \ell_3(x) \frac{1 - e^{-\frac{1}{4}t\mu(x)}}{(t\mu(x))^{1-a}} dx \\ &\lesssim \frac{1}{t^a} \int_{\mathbb{R}_+} \ell_3(x) (1 - e^{-\frac{1}{4}t\mu(x)}) dx \\ &= O(t^{\alpha_0-a+\delta}), \end{aligned}$$

where the last equality follows from [3, Lemma S3.5]. We have shown that, for any $\delta > 0$,

$$A_{2,1} \leq \mathbb{E}[U_t(\alpha)]^2 + O(t^{3-2a+2\alpha_0+\delta}). \quad (\text{S4.36})$$

Bound on $A_{2,2}$ Obviously $A_{2,2} \geq 0$. Further, since $(1 - u)^{D_i^{-j}} \leq 1$, we have the upper bound,

$$\begin{aligned} A_{2,2} &\leq \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \int_{[0,1]^2} \frac{(1 - (1 - v)^{D_j^{-i}})(1 - (1 - u)^{Z_{i,j}})}{uv(1 - u)^\alpha(1 - v)^\alpha} dudv \right] \\ &= \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \int_{[0,1]^2} \frac{(1 - e^{\sum_{k \neq i} \mathbf{1}_{\theta_k \leq t} \log(1 - vW(\vartheta_k, \vartheta_j))})}{v(1 - v)^\alpha} \frac{W(\vartheta_i, \vartheta_j)}{(1 - u)^\alpha} dudv \right] \\ &= \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \frac{W(\vartheta_i, \vartheta_j)}{1 - \alpha} \int_{[0,1]} \frac{(1 - e^{\sum_{k \neq i} \mathbf{1}_{\theta_k \leq t} \log(1 - vW(\vartheta_k, \vartheta_j))})}{v(1 - v)^\alpha} dv \right] \end{aligned}$$

where the second line follows by taking the conditional expectation with respect to M . Then, by the Slivnyak-Mecke formula, and then by Campbell's formula,

$$\begin{aligned} A_{2,2} &\leq \frac{t^2}{1 - \alpha} \int_{\mathbb{R}_+^2} W(x, y) \int_{[0,1]} \frac{1 - \mathbb{E}[e^{\sum_k \mathbf{1}_{\theta_k \leq t} \log(1 - vW(\vartheta_k, y))}](1 - vW(y, y))}{v(1 - v)^\alpha} dv dx dy \\ &= \frac{t^2}{1 - \alpha} \int_{\mathbb{R}_+} \mu(y) \int_{[0,1]} \frac{1 - e^{-tv\mu(y)}(1 - vW(y, y))}{v(1 - v)^\alpha} dv dy \\ &= \frac{t^2}{1 - \alpha} \int_{\mathbb{R}_+} \mu(y) \int_{[0,1]} \frac{1 - e^{-tv\mu(y)}}{v(1 - v)^\alpha} dv dy + \frac{t^2}{1 - \alpha} \int_{\mathbb{R}_+} \int_{[0,1]} \frac{\mu(y)W(y, y)e^{-tv\mu(y)}}{(1 - v)^\alpha} dv dy \end{aligned}$$

The second term of the last display is $o(t^2)$ by dominated convergence, so it is enough to bound the first term. For some $q \in (0, 1/2)$, we first rewrite the inner integral as

$$\begin{aligned} \int_{[0,1]} \frac{1 - e^{-ut\mu(x)}}{u(1-u)^\alpha} du &= \int_0^q \frac{1 - e^{-ut\mu(x)}}{u(1-u)^\alpha} du + \int_q^1 \frac{1 - e^{-ut\mu(x)}}{u(1-u)^\alpha} du \\ &\leq \frac{1}{(1-q)^\alpha} \int_0^q \frac{1 - e^{-ut\mu(x)}}{u} du + (1 - e^{-t\mu(x)}) \int_q^1 \frac{du}{u(1-u)^\alpha} \\ &\leq \frac{1}{(1-q)^\alpha} \int_0^q t\mu(x) du + (1 - e^{-t\mu(x)}) \left\{ 2^\alpha \int_q^{1/2} \frac{1}{u} + \int_{1/2}^1 \frac{du}{(1-u)^\alpha} \right\} \\ &\leq \frac{qt\mu(x)}{(1-q)^\alpha} + (1 - e^{-t\mu(x)}) \left(2^\alpha \log \frac{1}{q} + \frac{1}{1-\alpha} \right). \end{aligned}$$

Choosing $q = 1/t$, we obtain that as $t \rightarrow \infty$,

$$\int_{[0,1]} \frac{1 - e^{-ut\mu(x)}}{u(1-u)^\alpha} du \leq \mu(x)(1 + o(1)) + 2^\alpha (1 - e^{-t\mu(x)}) (1 + o(1)) \log(t). \quad (\text{S4.37})$$

Then,

$$\begin{aligned} A_{2,2} &\leq \frac{t^2(1 + o(1))}{1-\alpha} \int_{\mathbb{R}_+} \mu(x)^2 dx + \frac{2^\alpha t^2(1 + o(1))}{(1-\alpha)^2} \int_{\mathbb{R}_+} \mu(x) dx + o(t^2) \\ &= O(t^2 \log(t)), \end{aligned} \quad (\text{S4.38})$$

as both μ and μ^2 are integrable by assumption.

Bound on $A_{2,3}$ Obviously $A_{2,3} \geq 0$. Furthermore, $(1-u)^{D_i^{-j}} \leq 1$ and $(1-v)^{D_j^{-i}} \leq 1$ too, so that we have the upper bound,

$$\begin{aligned} A_{2,3} &\leq \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \int_{[0,1]^2} \frac{1 - (1-uv)^{Z_{i,j}}}{uv(1-u)^\alpha(1-v)^\alpha} dudv \right] \\ &= \mathbb{E} \left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} W(\vartheta_i, \vartheta_j) \left\{ \int_{[0,1]} \frac{1}{(1-u)^\alpha} du \right\}^2 \right] \\ &= \frac{t^2}{(1-\alpha)^2} \int_{\mathbb{R}_+^2} W(x, y) dx dy, \end{aligned} \quad (\text{S4.39})$$

where the second line follows by taking the conditional expectation with respect to M , and third line by the Slivnyak-Mecke formula.

Bound on A_1 Remark that we have,

$$\begin{aligned} \mathbb{E}[(1 - (1-u)^{D_{i,i}^{(s)}})(1 - (1-v)^{D_{i,i}^{(s)}}) \mid M] &= 1 - e^{\sum_j \mathbf{1}_{\theta_j \leq t} \log(1-uW(\vartheta_i, \vartheta_j))} \\ &\quad - e^{\sum_j \mathbf{1}_{\theta_j \leq t} \log(1-vW(\vartheta_i, \vartheta_j))} \end{aligned}$$

$$+ e^{\sum_j \mathbf{1}_{\theta_j \leq t} \log(1-(u+v-uv)W(\vartheta_i, \vartheta_j))}.$$

Recall that $W(x, x) = 0$ by assumption. Hence, by the Slivnyak-Mecke formula and by Campbell's theorem,

$$A_1 = t \int_{\mathbb{R}_+} \int_{[0,1]^2} \frac{F(x, u, v)}{uv(1-u)^\alpha(1-v)^\alpha} du dv dx,$$

where,

$$\begin{aligned} F(x, u, v) &:= 1 - e^{-ut\mu(x)} - e^{-vt\mu(x)} + e^{-(u+v-uv)t\mu(x)} \\ &= (1 - e^{-ut\mu(x)})(1 - e^{-vt\mu(x)}) + e^{-(u+v-uv)t\mu(x)}(1 - e^{-uvt\mu(x)}). \end{aligned}$$

Then,

$$\begin{aligned} A_1 &\leq t \int_{\mathbb{R}_+} \left\{ \int_{[0,1]} \frac{1 - e^{-ut\mu(x)}}{u(1-u)^\alpha} du \right\}^2 dx \\ &\quad + t \int_{\mathbb{R}_+} \int_{[0,1]^2} \frac{e^{-(u+v-uv)t\mu(x)}(1 - e^{-uvt\mu(x)})}{uv(1-u)^\alpha(1-v)^\alpha} du dv dx. \end{aligned}$$

Using that $e^{-(u+v-uv)t\mu(x)} \leq 1$ and $1 - e^{-uvt\mu(x)} \leq uvt\mu(x)$, it is immediately seen that the second term is no more than $\frac{t^2}{(1-\alpha)^2} \int_{\mathbb{R}_+} \mu(x) dx$. So it is enough to bound the first term. We remark that the integral within brackets has already been bounded in Equation (S4.37) and is no more than a $1 + o(1)$ times $\mu(x) + 2^\alpha(1 - e^{-t\mu(x)}) \log(t)$. Then,

$$A_1 \lesssim t \int_{\mathbb{R}_+} \mu(x)^2 dx + t \log^2(t) \int_{\mathbb{R}_+} (1 - e^{-t\mu(x)})^2 dx + O(t^2) = O(t^2), \quad (\text{S4.40})$$

as μ^2 is integrable by assumption, and as $\int_{\mathbb{R}_+} (1 - e^{-t\mu(x)})^2 dx \leq \int_{\mathbb{R}_+} (1 - e^{-t\mu(x)}) dx = O(t^\alpha)$; see for instance [3, Lemma S3.5].

Conclusion Gathering Equations (S4.36) and (S4.38) to (S4.40), as $t \rightarrow \infty$, for any $\delta > 0$,

$$\mathbb{E}[U_t(\alpha)^2] \leq \mathbb{E}[U_t(\alpha)]^2 + O\left(t^2 \log(t) \bigvee t^{3-2a+2\alpha_0+\delta}\right).$$

S4.3.6. Variance of $V_t(\alpha)$

Here again we write $D_i \equiv D_{t,i}^{(s)}$ and $D_i^{-j} \equiv \sum_{k \neq j} Z_{i,k} \mathbf{1}_{\theta_k \leq t}$ for simplicity. We split $\mathbb{E}[V_t(\alpha)^2]$ into two terms

$$\begin{aligned} &\mathbb{E}\left[\sum_{i \in \mathbb{N}} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{D_i \geq 1} \int_{[0,1]^2} \frac{(1-u)^{D_i}(1-v)^{D_i}}{(1-u)^{1+\alpha}(1-v)^{1+\alpha}} du dv\right] \\ &\quad + \mathbb{E}\left[\sum_{i \in \mathbb{N}} \sum_{j \neq i} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{\theta_j \leq t} \mathbf{1}_{D_i \geq 1} \mathbf{1}_{D_j \geq 1} \int_{[0,1]^2} \frac{(1-u)^{D_i}(1-v)^{D_j}}{(1-u)^{1+\alpha}(1-v)^{1+\alpha}} du dv\right]. \end{aligned}$$

Let call these two terms A'_1 and A'_2 , respectively. We start by bounding A'_2 .

Bound on A'_2 Using that $Z_{i,j} \in \{0, 1\}$, we decompose

$$\begin{aligned} (1-u)^{D_i}(1-v)^{D_j} \mathbf{1}_{D_i \geq 1} \mathbf{1}_{D_j \geq 1} &= (1-u)^{D_i^{-j}} (1-v)^{D_j^{-i}} (1-u)^{Z_{i,j}} (1-v)^{Z_{i,j}} \mathbf{1}_{D_i \geq 1} \mathbf{1}_{D_j \geq 1} \\ &= (1-u)^{D_i^{-j}} (1-v)^{D_j^{-i}} \mathbf{1}_{D_i \geq 1} \mathbf{1}_{D_j \geq 1} \mathbf{1}_{Z_{i,j}=0} \\ &\quad + (1-u)^{D_i^{-j}+1} (1-v)^{D_j^{-i}+1} \mathbf{1}_{D_i \geq 1} \mathbf{1}_{D_j \geq 1} \mathbf{1}_{Z_{i,j}=1}. \end{aligned}$$

Then,

$$(1-u)^{D_i}(1-v)^{D_j} \mathbf{1}_{D_i \geq 1} \mathbf{1}_{D_j \geq 1} \leq (1-u)^{D_i^{-j}} (1-v)^{D_j^{-i}} \mathbf{1}_{D_i^{-j} \geq 1} \mathbf{1}_{D_j^{-i} \geq 1} + (1-u)(1-v) \mathbf{1}_{Z_{i,j}=1}.$$

Since D_i^{-j} , D_j^{-i} and $Z_{i,j}$ are independent conditional on M , we obtain,

$$\begin{aligned} \mathbb{E}[(1-u)^{D_i}(1-v)^{D_j} \mathbf{1}_{D_i \geq 1} \mathbf{1}_{D_j \geq 1} \mid M] &\leq e^{\sum_{k \neq j} \mathbf{1}_{\theta_k \leq t} \log(1-uW(\vartheta_i, \vartheta_k))} e^{\sum_{k \neq i} \mathbf{1}_{\theta_k \leq t} \log(1-vW(\vartheta_j, \vartheta_k))} \\ &\quad - e^{\sum_{k \neq j} \mathbf{1}_{\theta_k \leq t} \log(1-uW(\vartheta_i, \vartheta_k))} e^{\sum_{k \neq i} \mathbf{1}_{\theta_k \leq t} \log(1-W(\vartheta_j, \vartheta_k))} \\ &\quad - e^{\sum_{k \neq j} \mathbf{1}_{\theta_k \leq t} \log(1-W(\vartheta_i, \vartheta_k))} e^{\sum_{k \neq i} \mathbf{1}_{\theta_k \leq t} \log(1-vW(\vartheta_j, \vartheta_k))} \\ &\quad + e^{\sum_{k \neq j} \mathbf{1}_{\theta_k \leq t} \log(1-W(\vartheta_i, \vartheta_k))} e^{\sum_{k \neq i} \mathbf{1}_{\theta_k \leq t} \log(1-W(\vartheta_j, \vartheta_k))} \\ &\quad + (1-u)(1-v)W(\vartheta_i, \vartheta_j). \end{aligned}$$

Recall that $W(x, x) = 0$ by assumption. Then, by Slivnyak-Mecke's formula and by Campbell's theorem,

$$A'_2 \leq t^2 \int_{\mathbb{R}_+^2} \int_{[0,1]^2} \frac{F(x, y, u, v) \, du \, dv \, dx \, dy}{(1-u)^{1+\alpha} (1-v)^{1+\alpha}} + \frac{t^2}{(1-\alpha)^2} \int_{\mathbb{R}_+^2} W(x, y) \, dx \, dy,$$

where,

$$\begin{aligned} F(x, y, u, v) &= e^{-ut\mu(x)} e^{-vt\mu(y)} e^{utv\nu(x,y)} - e^{-ut\mu(x)} e^{-t\mu(y)} e^{ut\nu(x,y)} \\ &\quad - e^{-t\mu(x)} e^{-vt\mu(y)} e^{vt\nu(x,y)} + e^{-t\mu(x)} e^{-t\mu(y)} e^{t\nu(x,y)}. \end{aligned}$$

In view of the computations made in [Section S4.3.3](#), we have

$$\mathbb{E}[V_t(\alpha)] = - \int_{\mathbb{R}_+} \int_{[0,1]} \frac{e^{-ut\mu(x)} (1 - e^{-(1-u)t\mu(x)})}{(1+u)^\alpha} \, du \, dx,$$

so the previous rewrites as

$$A'_2 \leq \mathbb{E}[V_t(\alpha)]^2 + t^2 \int_{\mathbb{R}_+^2} \int_{[0,1]^2} \frac{\tilde{F}(x, y, u, v) \, du \, dv \, dx \, dy}{(1-u)^{1+\alpha} (1-v)^{1+\alpha}} + O(t^2), \quad (\text{S4.41})$$

where,

$$\begin{aligned}\tilde{F}(x, y, u, v) &= e^{-ut\mu(x)}e^{-vt\mu(y)}(e^{uvt\mu(x,y)} - 1) - e^{-ut\mu(x)}e^{-t\mu(y)}(e^{ut\mu(x,y)} - 1) \\ &\quad - e^{-t\mu(x)}e^{-vt\mu(y)}(e^{vt\mu(x,y)} - 1) + e^{-t\mu(x)}e^{-t\mu(y)}(e^{t\mu(x,y)} - 1).\end{aligned}$$

The main difficulty here is that none of term composing $\tilde{F}(u, v; x, y)$ is integrable with respect to the measure $\frac{dudv}{(1-u)^{1+\alpha}(1-v)^{1+\alpha}}$, though their sum is. We bypass the difficulty by decomposing the region of integration into four subdomains. For some $q \in [0, 1]$ to be chosen accordingly later, we let $D_1 := \{(u, v) \in [0, 1]^2 : 0 \leq u \leq q, 0 \leq v \leq q\}$, $D_2 := \{(u, v) \in [0, 1]^2 : q < u \leq 1, 0 \leq v \leq q\}$, $D_3 := \{(u, v) \in [0, 1]^2 : q < u \leq 1, q < v \leq 1\}$, and $D_4 := \{(u, v) \in [0, 1]^2 : 0 \leq u \leq q, q < v \leq 1\}$. By symmetry, the integral of $(u, v) \mapsto \tilde{F}(u, v; x, y)$ over D_4 is the same as the integral over D_2 , and thus

$$A'_{2,1} \leq \mathbb{E}[V_t(\alpha)]^2 + A'_{2,1} + 2A'_{2,2} + A'_{2,3},$$

where, for $j = 1, \dots, 3$,

$$A'_{2,j} := t^2 \int_{\mathbb{R}_+^2} \int_{D_j} \frac{\tilde{F}(u, v; x, y)}{(1-u)^{1+\alpha}(1-v)^{1+\alpha}} dudvdxdy.$$

Bound on $A'_{2,1}$ Over D_1 , we can bound rather quickly the integral of $(u, v) \mapsto \tilde{F}(u, v; x, y)$ as there is no convergence issue. Indeed, it is enough to keep the non-negative terms,

$$\begin{aligned}\tilde{F}(u, v; x, y) &\leq e^{-ut\mu(x)}e^{-vt\mu(y)}(e^{uvt\mu(x,y)} - 1) + e^{-t\mu(x)}e^{-t\mu(y)}(e^{t\mu(x,y)} - 1) \\ &\leq 2te^{-\frac{u}{2}t\mu(x)}e^{-\frac{v}{2}t\mu(y)}\ell_3(x)\ell_3(y)\mu(x)^a\mu(y)^a,\end{aligned}$$

where the second line follows from the same arguments as in [3, Lemma S3.7]. Therefore,

$$\begin{aligned}A'_{2,1} &\leq 2t^3 \left\{ \int_{\mathbb{R}_+} \ell_3(x)\mu(x)^a \int_0^q \frac{e^{-\frac{u}{2}t\mu(x)}}{(1-u)^{1+\alpha}} dudx \right\}^2 \\ &\leq \frac{8t}{(1-q)^{2+2\alpha}} \left\{ \int_{\mathbb{R}_+} \ell_3(x)\mu(x)^{a-1}(1-e^{-\frac{q}{2}t\mu(x)})dx \right\}^2 \\ &= O(t^{3-2a+2\alpha_0+\delta}).\end{aligned}\tag{S4.42}$$

Bound on $A'_{2,2}$ The challenge here is to reorganize the terms in $I(u, v; x, y)$ such that we can obtain bounds and all the integrals still converge. We rewrite,

$$\begin{aligned}\tilde{F}(u, v; x, y) &= e^{-vt\mu(y)}(e^{-ut\mu(x)} - e^{-t\mu(x)})(e^{uvt\mu(x,y)} - 1) \\ &\quad + e^{-t\mu(x)}e^{-vt\mu(y)}(e^{uvt\mu(x,y)} - e^{vt\mu(x,y)}) \\ &\quad + e^{-t\mu(x)}e^{-t\mu(y)}(e^{t\mu(x,y)} - e^{ut\mu(x,y)}) \\ &\quad - e^{-t\mu(y)}(e^{-ut\mu(x)} - e^{-t\mu(x)})(e^{ut\mu(x,y)} - 1).\end{aligned}$$

As we only need an upper bound, we keep only the non-negative terms,

$$\begin{aligned}\tilde{F}(u, v; x, y) &\leq e^{-vt\mu(y)}(e^{-ut\mu(x)} - e^{-t\mu(x)})(e^{uvt\nu(x,y)} - 1) \\ &\quad + e^{-t\mu(x)}e^{-t\mu(y)}(e^{t\nu(x,y)} - e^{ut\nu(x,y)})\end{aligned}$$

With the same arguments as usual [see 3, Lemma S3.7]

$$\begin{aligned}\tilde{F}(u, v; x, y) &\leq t^2 uv(1-u)\ell_3(x)\ell_3(y)\mu(x)^{1+a}\mu(y)^a e^{-\frac{v}{2}t\mu(x)}e^{-\frac{u}{2}t\mu(y)} \\ &\quad + t(1-u)\ell_3(x)\ell_3(y)\mu(x)^a\mu(y)^a e^{-\frac{1}{2}t\mu(x)}e^{-\frac{1}{2}t\mu(y)}.\end{aligned}$$

Then,

$$\begin{aligned}A'_{2,2} &\leq t^4 \left\{ \int_{\mathbb{R}_+} \int_q^1 \frac{u\ell_3(x)\mu(x)^{1+a}e^{-\frac{u}{2}t\mu(x)} du dx}{(1-u)^\alpha} \right\} \left\{ \int_{\mathbb{R}_+} \int_0^q \frac{v\ell_3(y)\mu(y)^a e^{-\frac{v}{2}t\mu(y)} dv dy}{(1-v)^{1+\alpha}} \right\} \\ &\quad + t^3 \left\{ \int_{\mathbb{R}_+} \int_q^1 \frac{\ell_3(x)\mu(x)^a e^{-\frac{1}{2}t\mu(x)} du dx}{(1-u)^\alpha} \right\} \left\{ \int_{\mathbb{R}_+} \int_0^q \frac{\ell_3(y)\mu(y)^a e^{-\frac{v}{2}t\mu(y)} dv dy}{(1-v)^{1+\alpha}} \right\}.\end{aligned}$$

Again with the usual arguments [see 3, Lemma S3.4 and Lemma S3.5]

$$A'_{2,2} \leq O(t^{3-2a+2\alpha_0+\delta}). \quad (\text{S4.43})$$

Bound on $A'_{2,3}$ The main challenge is to reorganize the terms in a way such that we can get sharp upper-bounds and such that the integrals still converges. Using that $e^{-ut\mu(x)} = e^{-t\mu(x)} + (e^{-ut\mu(x)} - e^{-t\mu(x)})$, similarly for $e^{-vt\mu(y)}$, we can rewrite that

$$\begin{aligned}\tilde{F}(u, v; x, y) &= e^{-t\mu(x)}e^{-t\mu(y)}(e^{uvt\nu(x,y)} - 1) + e^{-t\mu(x)}(e^{-vt\mu(y)} - e^{-t\mu(y)})(e^{uvt\nu(x,y)} - 1) \\ &\quad + (e^{-ut\mu(x)} - e^{-t\mu(x)})(e^{-vt\mu(y)} - e^{-t\mu(y)})(e^{uvt\nu(x,y)} - 1) \\ &\quad - e^{-t\mu(x)}e^{-t\mu(y)}(e^{ut\nu(x,y)} - 1) - (e^{-ut\mu(x)} - e^{-t\mu(x)})e^{-t\mu(y)}(e^{ut\nu(x,y)} - 1) \\ &\quad - e^{-t\mu(x)}e^{-t\mu(y)}(e^{vt\nu(x,y)} - 1) - e^{-t\mu(x)}(e^{-vt\mu(y)} - e^{-t\mu(y)})(e^{vt\nu(x,y)} - 1) \\ &\quad + (e^{-ut\mu(x)} - e^{-t\mu(x)})e^{-t\mu(y)}(e^{uvt\nu(x,y)} - 1) + e^{-t\mu(x)}e^{-t\mu(y)}(e^{t\nu(x,y)} - 1).\end{aligned}$$

Reorganizing the previous, we find that,

$$\begin{aligned}\tilde{F}(u, v; x, y) &= e^{-t\mu(x)}e^{-t\mu(y)}(e^{uvt\nu(x,y)} + e^{t\nu(x,y)} - e^{ut\nu(x,y)} - e^{-vt\nu(x,y)}) \\ &\quad + e^{-t\mu(x)}(e^{-vt\mu(y)} - e^{-t\mu(y)})(e^{uvt\nu(x,y)} - e^{vt\nu(x,y)}) \\ &\quad + e^{-t\mu(y)}(e^{-ut\mu(x)} - e^{-t\mu(x)})(e^{uvt\nu(x,y)} - e^{ut\nu(x,y)}) \\ &\quad + (e^{-ut\mu(x)} - e^{-t\mu(x)})(e^{-vt\mu(y)} - e^{-t\mu(y)})(e^{uvt\nu(x,y)} - 1).\end{aligned}$$

As we are only interested in an upper bound, it is enough to keep only the non-negative terms. That is,

$$\tilde{F}(u, v; x, y) \leq e^{-t\mu(x)}e^{-t\mu(y)}(e^{uvt\nu(x,y)} + e^{t\nu(x,y)} - e^{ut\nu(x,y)} - e^{-vt\nu(x,y)}) \quad (\text{S4.44})$$

$$+ (e^{-ut\mu(x)} - e^{-t\mu(x)})(e^{-vt\mu(y)} - e^{-t\mu(y)})(e^{uvt\nu(x,y)} - 1). \quad (\text{S4.45})$$

We now bound each of the terms in the last display. For fixed (x, y) , let define $\phi(u, v) := e^{-t(1-uv)\nu(x, y)}$, so that the term Equation (S4.44) can be rewritten as $e^{-t\mu(x)}e^{-t\mu(y)}e^{t\nu(x, y)}(\phi(1, 1) + \phi(u, v) - \phi(u, 1) - \phi(1, v))$. By a Taylor expansion of ϕ , for any $(u, v) \in D_3$,

$$\phi(1, 1) - \phi(u, 1) + \phi(u, v) - \phi(1, v) \leq (1-u)(1-v) \sup_{(\bar{u}, \bar{v}) \in D_3} \partial_u \partial_v \phi(\bar{u}, \bar{v}).$$

It is clear that,

$$\partial_u \partial_v \phi(u, v) = t\nu(x, y)\phi(u, v) + t^2 uv\nu(x, y)^2 \phi(u, v)$$

Therefore, $\phi(1, 1) - \phi(u, 1) + \phi(u, v) - \phi(1, v) \leq (1-u)(1-v)(t\nu(x, y) + t^2\nu(x, y)^2)$, at least when $(u, v) \in D_3$. By the usual arguments, it is rapidly seen that the term in Equation (S4.45) is bounded by $t^3 uv(1-u)(1-v)\mu(x)\mu(y)\nu(x, y)e^{-ut\mu(x)}e^{-vt\mu(y)}e^{uv t\nu(x, y)}$, and then when $(u, v) \in D_3$,

$$\begin{aligned} \tilde{F}(u, v; x, y) &\leq t(1-u)(1-v)\ell_3(x)\ell_3(y)\mu(x)^a\mu(y)^a e^{-\frac{1}{2}t\mu(x)}e^{-\frac{1}{2}t\mu(y)} \\ &\quad + t^2(1-u)(1-v)\ell_3(x)^2\ell_3(y)^2\mu(x)^{2a}\mu(y)^{2a} e^{-\frac{1}{2}t\mu(x)}e^{-\frac{1}{2}t\mu(y)} \\ &\quad + t^3 uv(1-u)(1-v)\ell_3(x)\ell_3(y)\mu(x)^{1+a}\mu(y)^{1+a} e^{-\frac{u}{2}t\mu(x)}e^{-\frac{v}{2}t\mu(y)}. \end{aligned}$$

Therefore,

$$\begin{aligned} A'_{2,3} &\leq t^3 \left\{ \int_{\mathbb{R}_+} \int_q \frac{\ell_3(x)\mu(x)^a e^{-\frac{1}{2}t\mu(x)} du dx}{(1-u)^\alpha} \right\}^2 \\ &\quad + t^4 \left\{ \int_{\mathbb{R}_+} \int_q \frac{\ell_3(x)^2\mu(x)^{2a} e^{-\frac{1}{2}t\mu(x)} du dx}{(1-u)^\alpha} \right\}^2 \\ &\quad + t^5 \left\{ \int_{\mathbb{R}_+} \int_q \frac{\ell_3(x)\mu(x)^{1+a} e^{-\frac{u}{2}t\mu(x)} du dx}{(1-u)^\alpha} \right\}^2. \end{aligned}$$

Thus, by [3, Lemma S3.4 and Lemma S3.5] for any $\delta > 0$,

$$A'_{2,3} \leq O(t^{3-2a+2\alpha_0+\delta}). \quad (\text{S4.46})$$

Bound on A'_1 We first remark that, for any $D_i \geq 1$,

$$\int_{[0,1]^2} \frac{(1-u)^{D_i}(1-v)^{D_i}}{(1-u)^{1+\alpha}} (1-v)^{1-\alpha} = \left\{ \int_{[0,1]} \frac{(1-u)^{D_i}}{(1-u)^{1+\alpha}} du \right\}^2 = \frac{1}{(D_i - \alpha)^2} \leq \frac{1}{(1 - \alpha)^2}.$$

Therefore,

$$A'_1 \leq \frac{1}{(1-\alpha)^2} \mathbb{E} \left[\sum_{i \in \mathbb{N}} \mathbf{1}_{\theta_i \leq t} \mathbf{1}_{D_i \geq 1} \right] = \frac{\mathbb{E}[N_t]}{(1-\alpha)^2} = O(t^{1+\alpha_0} \ell_1(t)), \quad (\text{S4.47})$$

where the last estimate follows by [4, Theorem 5].

Conclusion Gathering the Equations (S4.41) to (S4.43), (S4.46) and (S4.47), we have as $t \rightarrow \infty$, for any $\delta > 0$

$$\mathbb{E}[V_t(\alpha)^2] \leq \mathbb{E}[V_t(\alpha)]^2 + O\left(t^2 \bigvee t^{3-2a+2\alpha_0+\delta}\right).$$

S5. Proofs of [4, Lemma 1]

Let define on $(0, 1)$ the function $F^*(\alpha) := \sum_{j \geq 2} f_j \sum_{k=1}^{j-1} \frac{\alpha}{k-\alpha} - 1$; so that α_0 is a solution of $F^*(\alpha) = 0$. In the conditions of the lemma, it is the case that $f_1 < 1$. Then, the function ℓ is monotonic with $\lim_{\alpha \rightarrow 0} F^*(\alpha) = -1$ and $\lim_{\alpha \rightarrow 1} F^*(\alpha) = \infty$. This establishes existence and uniqueness of α_0 .

We now prove that the second part of condition (12) of Assumption 1 is satisfied (in probability). We define for simplicity $F_t(\alpha) = \frac{-\alpha C'_t(\alpha)}{t} - 1$. Then,

$$\begin{aligned} F_t(\hat{\alpha}_t) - F_t(\alpha_0) &= \sum_{j \geq 2} \frac{N_{t,j}}{t} \sum_{k=1}^{j-1} \left(\frac{\hat{\alpha}_t}{k - \hat{\alpha}_t} - \frac{\alpha_0}{k - \alpha_0} \right) \\ &= (\hat{\alpha}_t - \alpha_0) \sum_{j \geq 2} \frac{N_{t,j}}{t} \sum_{k=1}^{j-1} \frac{k}{(k - \alpha_0)(k - \hat{\alpha}_t)} \\ &\geq (\hat{\alpha}_t - \alpha_0) \sum_{j \geq 2} \frac{N_{t,j}}{t} \sum_{k=1}^{j-1} \frac{k}{(k - \alpha_0)^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} F_t(\hat{\alpha}_t) - F_t(\alpha_0) &= (\hat{\alpha}_t - \alpha_0) \sum_{j \geq 2} \frac{N_{t,j}}{t} \sum_{k=1}^{j-1} \frac{k}{(k - \alpha_0)^2} \frac{1}{1 - \frac{\hat{\alpha}_t - \alpha_0}{k - \alpha_0}} \\ &\leq \frac{\hat{\alpha}_t - \alpha_0}{1 - \frac{\hat{\alpha}_t - \alpha_0}{1 - \alpha_0}} \sum_{j \geq 2} \frac{N_{t,j}}{t} \sum_{k=1}^{j-1} \frac{k}{(k - \alpha_0)^2}, \end{aligned}$$

Since $F_t(\hat{\alpha}_t) = 0$ by definition of $\hat{\alpha}_t$, it follows

$$\hat{\alpha}_t - \alpha_0 \leq \frac{-F_t(\alpha_0)}{\underbrace{\sum_{j \geq 2} \frac{N_{t,j}}{t} \sum_{k=1}^{j-1} \frac{k}{(k - \alpha_0)^2}}_{=: Z_t}} \leq \frac{\hat{\alpha}_t - \alpha_0}{1 - \frac{\hat{\alpha}_t - \alpha_0}{1 - \alpha_0}}$$

or in other words,

$$\frac{Z_t}{1 + \frac{Z_t}{1 - \alpha_0}} \leq \hat{\alpha}_t - \alpha_0 \leq Z_t. \quad (\text{S5.48})$$

Hence to prove that the second part of condition (12) of Assumption 1 is satisfied (in probability) it is enough to show that $\log(t)|Z_t| \rightarrow 0$ (in probability). First,

$$|Z_t| \leq \frac{(1 - \alpha_0)^2 |F_t(\alpha_0)|}{\sum_{j \geq 2} \frac{N_{t,j}}{t}} = \frac{(1 - \alpha_0)^2 |F_t(\alpha_0)|}{1 - N_{t,1}/t}. \quad (\text{S5.49})$$

Second, Recall that $N_{t,j} = \sum_{i=1}^t \mathbf{1}_{\tilde{D}_{t,i}=j}$. Hence,

$$\begin{aligned} |F_t(\alpha_0)| &= \left| \frac{1}{t} \sum_{i=1}^t \sum_{j \geq 2} \mathbf{1}_{\tilde{D}_{t,i}=j} \sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} - 1 \right| \\ &\leq \left| \frac{1}{t} \sum_{i=1}^t \sum_{j \geq 2} \mathbf{1}_{D_{t,i}=j} \sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} - 1 \right| + \frac{1}{t} \sum_{j \geq 2} |\mathbf{1}_{\tilde{D}_{t,t}=j} - \mathbf{1}_{D_{t,t}=j}| \sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} \\ &\leq \left| \frac{1}{t} \sum_{i=1}^t \sum_{j \geq 2} \mathbf{1}_{D_{t,i}=j} \sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} - 1 \right| + \frac{1}{t} \sum_{k=1}^{D_{\max,t}-1} \frac{\alpha_0}{k - \alpha_0}. \end{aligned}$$

[Note that the previous estimate will cost a $\log(D_{\max,t})/t$ term that can be improved to $1/t$ at the price of longer computations, which is not worth for our purpose]. But, by the fact that $F^*(\alpha_0) = 0$,

$$\begin{aligned} \mathbb{E} \left[\frac{1}{t} \sum_{i=1}^t \sum_{j \geq 2} \mathbf{1}_{D_{t,i}=j} \sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} \right] &= \frac{1}{\sum_{k=1}^{D_{\max,t}} f_k} \sum_{j \geq 2} f_j \sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} \\ &= \frac{1}{1 - \sum_{k > D_{\max,t}} f_k} \\ &= 1 + \frac{\sum_{k > D_{\max,t}} f_k}{1 - \sum_{k > D_{\max,t}} f_k} \end{aligned}$$

Also,

$$\text{var} \left(\frac{1}{t} \sum_{i=1}^t \sum_{j \geq 2} \mathbf{1}_{D_{t,i}=j} \sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} \right) \leq \frac{1}{t} \sum_{j \geq 2} \frac{f_j}{1 - \sum_{k > D_{\max,t}} f_k} \left(\sum_{k=1}^{j-1} \frac{\alpha_0}{k - \alpha_0} \right)^2.$$

It follows from these estimates that and the fact that $D_{\max,t}$ cannot exceed some power of t

$$|F_t(\alpha_0)| = O_p \left(D_{\max,t}^{-\alpha_1} \sqrt{\frac{\log(t)^2}{t}} \right) = o \left(\frac{1}{\log(t)} \right). \quad (\text{S5.50})$$

With a similar reasoning, it is easily seen that $1 - \frac{N_{t,1}}{t} = 1 - f_1 + o_p(1)$, so that by combining the equations (S5.48), (S5.49), and (S5.50) we obtain that $\log(t)|\hat{\alpha}_t - \alpha_0| = o_p(1)$.

We now prove that the first part of condition (12) of Assumption 1 is satisfied (in probability). We see that $D_t^* = \sum_{i=1}^t \tilde{D}_{t,i} = \sum_{i=1}^t D_{t,i} + O(1)$ almost-surely, and

$$\mathbb{E} \left[\sum_{i=1}^t D_{t,i} \right] = t \frac{\sum_{j \geq 1} j f_j \mathbf{1}_{j \leq D_{\max,t}}}{\sum_{j \geq 1} f_j \mathbf{1}_{j \leq D_{\max,t}}} \sim \frac{\alpha_1 L}{1 - \alpha_1} \cdot t D_{\max,t}^{1-\alpha_1} =: \bar{D}_t^*$$

as $t \rightarrow \infty$ by Lemma S5.7. Also, by the same Lemma, as $t \rightarrow \infty$

$$\text{var} \left(\sum_{i=1}^t D_{t,i} \right) \leq t \frac{\sum_{j \geq 1} j^2 f_j \mathbf{1}_{j \leq D_{\max,t}}}{\sum_{j \geq 1} f_j \mathbf{1}_{j \leq D_{\max,t}}} \sim \frac{\alpha_1 L}{2 - \alpha_1} \cdot t D_{\max,t}^{2-\alpha_1}.$$

Therefore,

$$\frac{D_t^* - \bar{D}_t^*}{\bar{D}_t^*} = O_p\left(\sqrt{\frac{D_{\max,t}^{\alpha_1}}{t}}\right).$$

It follows that if $D_{\max,t} \sim A \cdot t^{\frac{1-\alpha_0}{(1+\alpha_0)(1-\alpha_1)}}$ for some constant $A > 0$, we have

$$D_t^* = \frac{\alpha_1 L A^{1-\alpha_1}}{1-\alpha_1} t^{\frac{2}{1+\alpha_0}} (1 + o_p(1)),$$

and,

$$\sqrt{2D_t^*} \left(\frac{\alpha_0 N_t}{D_t^*} \right)^{1-\alpha_0} = \sqrt{2}\alpha_0^{1-\alpha_0} \left(\frac{\alpha_1 L A^{1-\alpha_1}}{1-\alpha_1} \right)^{\frac{1}{2}-(1-\alpha_0)} (1 + o_p(1)).$$

This concludes the proof.

Lemma S5.7. *Let $(f_j)_{j \geq 1}$ be a probability mass function on $\{1, 2, \dots\}$ such that $1 - \sum_{k=1}^j f_k \sim Lj^{-\alpha_1}$ as $j \rightarrow \infty$, for some $\alpha_1 \in (0, 1)$ and some $L > 0$. Then, as $D \rightarrow \infty$*

$$\sum_{j=1}^D j f_j \sim \frac{\alpha_1 L D^{1-\alpha_1}}{1-\alpha_1}, \quad \sum_{j=1}^D j^2 f_j \sim \frac{\alpha_1 L D^{2-\alpha_1}}{2-\alpha_1}.$$

Proof. These are famous results about regular variations [1], we briefly sketch a proof for completeness. Let $F_j := \sum_{k=1}^j f_k$. By standard computations it is seen that $\sum_{j=1}^D (1 - F_j) = \sum_{j=1}^D j f_j + D(1 - F_D)$. By assumption, for all $\varepsilon > 0$ we can find $K > 0$ such that $1 - \varepsilon \leq \frac{1 - F_j}{Lj^{-\alpha_1}} \leq 1 + \varepsilon$ for all $j \geq K$. It follows, since $0 \leq 1 - F_j \leq 1$ for all $j \geq 1$, that when $D \gg K$,

$$\sum_{j=1}^D (1 - F_j) \leq \sum_{j=1}^{K-1} (1 - F_j) + \sum_{k=K}^D \frac{1 - F_j}{Lj^{-\alpha_1}} \cdot Lj^{-\alpha_1} \leq K + (1 + \varepsilon) \sum_{k=K}^D Lj^{-\alpha_1}$$

and,

$$\sum_{j=1}^D (1 - F_j) \geq \sum_{k=K}^D \frac{1 - F_j}{Lj^{-\alpha_1}} \cdot Lj^{-\alpha_1} \geq (1 - \varepsilon) \sum_{k=K}^D Lj^{-\alpha_1}.$$

But for any $K > 0$, it can be shown that $\sum_{k=K}^D j^{-\alpha_1} \sim \frac{D^{1-\alpha_1}}{1-\alpha_1}$, and since the last two estimates are true for any $\varepsilon > 0$, we deduce the first result. The other result is proved similarly. \square

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