
MATHEMATICAL MODELLING OF FLUID FLOWS DURING URETEROSCOPIC KIDNEY STONE REMOVAL

Jessica G. Williams

Mansfield College

PREFACE

This research describes a body of work performed at the Mathematical Institute at the University of Oxford between May 2016 and September 2019 under the supervision of Professor Sarah L. Waters, Professor Derek E. Moulton, and Mr. Benjamin W. Turney. We are grateful for funding provided by Boston Scientific Corporation in partnership with the InFoMM CDT. No part of this thesis has been submitted for any qualification other than the degree of Doctor of Philosophy at the University of Oxford.

This thesis is my own work including the bench-top experiments described in Sections 2.2.1 and 4.2.1. The particle image velocimetry experiments in Section 5.2 were also performed by me, although under the guidance of Alfonso A. Castrejon-Pita who also built the experimental apparatus. The *ex-vivo* experiments in Section 2.3.1, however, were performed by urologist Dr. Louise Rouse, with subsequent data analysis and interpretation executed by me.

Presently, three papers based on work in this thesis have been submitted to peer-reviewed academic journals and are in varying stages along the process spanning submission to publication. A paper based on the work in Section 2.2 has been published in the *Journal of Endourology*, co-authored by B. W. Turney, S. L. Waters, and D. E. Moulton, along with N. P. Rauniyar and T. P. Harrah who are Boston Scientific Corporation affiliates. A paper based on Chapter 3 has been submitted to the *Journal of Fluid Mechanics* co-authored by S. L. Waters, D. E. Moulton, and B. W. Turney and is currently undergoing minor adjustments following an initial review. A third paper based on the work in Section 2.3 and co-authored by D. E. Moulton, S. L. Waters, B. W. Turney, and L. Rouse has been submitted to the *IMA Journal of Applied Mathematics* and is awaiting a reviewer response. In all three publications I performed the analysis under the mathematical supervision of S. L. Waters and D. E. Moulton and clinical guidance of B. W. Turney. I performed the bench-top experiments in the *J. Endourol.* publication and L. Rouse performed the *ex-vivo* experiments in the *IMA J. Appl. Math.* article.

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A final, very important, acknowledgement is to my loving family. Thank you to Mum, Dad, Barnaby, and Sebastian, who have supported and encouraged me, always and in all ways.

ABSTRACT

A *ureteroscope* is a hand-held surgical device. The scope consists of a handle and a long, thin shaft which is inserted through the urethra of the anaesthetised patient; visualisation is provided by a microscopic camera embedded in the scope tip. To prevent stone debris and blood from obscuring the image, a weak saline solution continuously flows from a suspended saline bag, through a long, thin cylindrical channel within the ureteroscope shaft – the *working channel* – and flushes out the kidney. The fluid returns through an *access sheath*, a rigid tube surrounding the scope shaft. This process is termed *irrigation*. Remote areas of the kidney are accessed by utilising the flexible scope tip and kidney stones are removed via auxiliary *working tools* which are passed through the working channel.

This thesis is concerned with the **mathematics** and **fluid mechanics** of ureteroscopy with an aim to improve surgical visualisation and safety through a mechanistic understanding of irrigation flow rate, kidney pressures, and flow patterns within the kidney as functions of relevant clinical parameters.

We commence by modelling flow through an isolated scope, incorporating the effects of saline bag height, scope tip deflection, and working tools. We use a **hydraulic network** approach to equate irrigation inflow through the working channel with a return flow through the access sheath. We validate mathematical predictions with the results of bench-top and *ex-vivo* experiments.

Flow through the working channel containing a working tool, and flow through the access sheath containing the scope shaft, are modelled as flow through coaxial cylinders, where the position of the inner cylinder is unknown. We determine positions for the inner cylinder and **elliptical cross-sections** for the ureteroscopic instruments to minimise axial flow resistance, thereby maximising irrigation flow rate for fixed saline bag height. To aid ureteroscope design, we also consider the **Stokes resistance** to movement of the inner cylinder, highlighting the effects of ellipticity and inner cylinder position.

We then characterise the influence of inertial pressures losses (**minor losses**) – due to connections between tubular components of ureteroscope irrigation systems – as quadratic

corrections to unidirectional, fully-developed theory. We explore the effect of minor losses with numerical simulations and bench-top experiments.

Finally, we turn to modelling flow patterns within the kidney through a combination of numerical simulations and **particle image velocimetry experiments**. Flow **symmetry-breaking** is observed and quantified both theoretically and experimentally as a function of Reynolds number and domain length. We use the computed flow fields to simulate the flushing of kidney-stone dust through solutions to an **advection-diffusion** equation describing the dynamics of a passive tracer. The time required to irrigate the kidney depends on the Péclet number, with relevant scaling law determined by the position and size of flow recirculation zones.

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CHAPTER 1

Introduction

I will not cut persons labouring under the stone, but will leave this to be done by men who are practitioners of this work.

(Hippocrates)

1.1 Motivation

The likelihood of developing kidney stones is on the rise (Stamatelou *et al.*, 2003), and recent data reports a nine percent lifetime affliction rate in the United States (Scales *et al.*, 2012). Although risk factors, such as soda consumption (Ferraro *et al.*, 2013), often correlate with a modern lifestyle, solid calcifications in the urinary tract are by no means a recent peril; the oldest known urinary stone was, in fact, discovered in the bladder of an Egyptian mummy (Shattock, 1905). Left untreated, large stones may lead to infection and eventual loss of kidney function, and can, on occasion, be life threatening (Kum *et al.*, 2016). Urinary stones have even claimed the lives of notable historical figures, such as Sir Isaac Newton in 1727 (Ostad & Wise, 2005). In early times, stone-removal procedures were primitive, with bleeding, shock, and infection accepted as likely surgical hazards (Costello, 1829). Thankfully, with the invention of modern medicine, successful, minimally invasive procedures, such as *flexible uretero-renoscopy* have emerged. Nonetheless, although associated surgical risks have diminished, modern stone-removal remains a complicated surgical procedure, with much room for improvement through a mechanistic understanding of the physical processes at play.

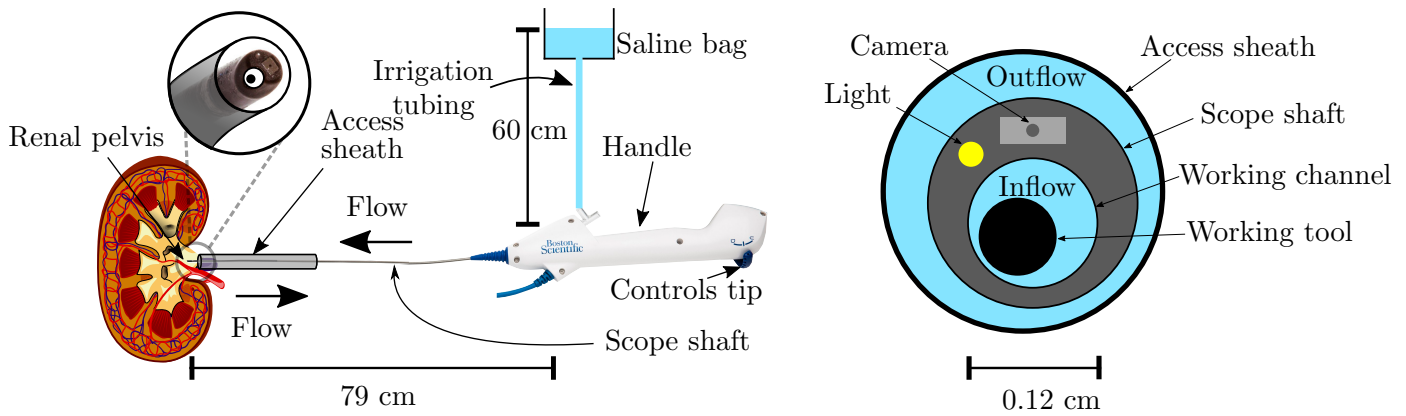


Figure 1.1: A photograph of a Boston Scientific ureterscope (left) with the tip of the scope circled and a zoomed-in schematic provided of the scope tip (right). The scope lies within an access sheath, and a working tool sits inside the working channel. A camera and a light are embedded in the scope wall. Dimensions of the scope shaft and working channel are labelled. A larger schematic of the kidney is provided in Figure 6.2.

Flexible uretero-renaloscopy is performed with a hand-held device, a ureterscope, an endoscope designed to visualise and work within the ureter and kidney (Figure 1.1). The scope shaft is inserted through the urethra of the anaesthetised patient, passing through the bladder and the ureter, until its tip reaches the *renal pelvis*, the hollow cavity within the kidney where stones commonly form. A light and a minuscule camera embedded within the scope tip transmit a live digital image which enables the urological surgeon to locate stones. Flexible, as opposed to rigid ureteroscopes, have an actively deflectable tip; approximately 6 cm at the distal end of the scope is steerable through operation of a small lever at the base of the handle (Figure 1.1). This degree of controllability allows superior access to remote areas of the urinary system (Basillote *et al.*, 2004). Within the shaft of the scope is a hollow cylindrical channel, the *working channel*. The stone-removal procedure is performed by passing auxiliary instruments, *working tools*, through the working channel to break up and capture stone fragments. Stone destruction in conjunction with ureteroscopy is performed via *laser lithotripsy*; a long, thin optical fibre transmits laser power through the ureterscope and laser pulses create shock waves to fragment stones. One of two disparate lasering methodologies is typically adopted to facilitate stone removal: *stone dusting*, in which the laser is fired at sufficiently high frequency to obliterate the stone into “dust-like” fragments that can spontaneously pass; or *fragmentation with extraction* in which lower frequency pulses are delivered to break the stone into larger fragments for subsequent removal with an auxiliary stone basket (Matlaga *et al.*, 2018).

Resulting particles and debris obstructing the field-of-view for the operating surgeon is an undesirable outcome of stone lasering. To mitigate this, a saline bag is hung above the patient and connected to the working channel via vertical tubing. This provides *irrigation*, a continuous fluid flow through the *irrigation tubing*, the working channel, out of the scope

tip, and into the renal pelvis. The fluid exits the body in the opposite direction, along the outside of the scope shaft. There is only a narrow gap between the scope and the ureteral wall, and if too much saline is delivered, overfilling of the kidney (and hence elevated renal pressures) may occur. High kidney pressures during ureteroscopy have been linked to infective post-operative complications, such as sepsis (Wilson & Preminger, 1990). Thus, to prevent kidney pressures from rising unduly, many urologists insert the scope through a ureteral access sheath; a hollow cylinder that surrounds the scope, temporarily widening the ureter to allow irrigation fluid to flow more freely back out of the kidney alongside the shaft of the scope (Figure 1.1). The use of ureteral access sheaths has been shown to decrease the risk of postoperative complications, with no effect on stone-removal success (Traxer *et al.*, 2015).

Good visualisation, as supported by sufficient flow (Bach *et al.*, 2011), is key to a successful stone-removal surgery. Thus, an impaired endoscopic view results in the operating surgeon requesting a procedural adjustment to increase flow rate. However, current knowledge of the effects of clinical parameters on flow rate is fairly limited, and thus, adjustments to enhance visualisation are achieved in an *ad hoc* manner, with a nurse raising the bag of saline solution, without awareness of the direct effect on flow rate or subsequent intrarenal pressures. Our present understanding of procedural modifications in a clinical setting has been informed by various experimental studies, exploring the relationships between saline bag height, instrument size, tip deflection, flow rate, and intrarenal pressure. Flow rate was found to increase, albeit nonlinearly, with height in cystoscopy irrigation for visualisation of the bladder (Chang *et al.*, 2012). Working tool size has been shown to have a considerable effect on the flow of irrigation fluid and subsequent visibility, while deflection of the scope tip does not seem to impact these metrics significantly (Bach *et al.*, 2008; Nagele *et al.*, 2006; Bach *et al.*, 2011; Pasqui *et al.*, 2004; Ng *et al.*, 2010; Kruck *et al.*, 2011). The notable decrease in irrigation flow caused by working tools in particular has led to experimental exploration of ureteroscope design modifications, such as the addition of a second working channel (Haberman *et al.*, 2011) or procedural adjustments such as removing the coating on nitinol stone baskets to reduce their diameters (Landman *et al.*, 2002).

While such experimental studies provide useful empirical evidence, they are limited both by a sparsity of data (due to, e.g., a small number of available tool geometries with which to experiment) and by a lack of theoretical underpinning. In this regard, the potential for mathematical modelling approaches to advance the field is significant. A mechanistic mathematical model can provide fundamental insights into system behaviour. Simulations can be performed across multi-dimensional parameter spaces that are too time-consuming, costly, and/or impractical to access via experiment. The resulting quantitative predictions can be exploited to assess the implications of the geometry of ureteroscopic devices

(e.g. length, diameter, cross-sectional shape) and operating conditions (e.g. saline bag height, irrigation fluid properties) on the resulting fluid flows, and facilitate the optimisation of ureteroscopy design and clinical protocols. As such, a mathematical approach enables the discovery of advantageous clinical methods and potential new products without the time and cost required to perform numerous experiments.

The key challenge of this thesis is to build a physically-motivated, mechanistic understanding of ureteroscopy irrigation. We will determine explicit relationships between operating and design parameters, namely, saline bag height, and the geometry of working tools, the working channel, the scope shaft, and the access sheath, on flow rate and intrarenal pressure. Additionally, we will address a question of optimal device design: whether we can increase flow rates and lower renal pressures by modifying the cross-sectional geometries of relevant ureteroscopic devices. Flow rate, as typically set by adjusting the height of the saline bag, is an intricate function of flow resistance through each component of the irrigation system: irrigation tubing, the working channel, the renal pelvis, and the access sheath. The influence of pressure drops arising from flow separation at the intersection between components provides additional complications which we will quantify to enhance the accuracy of our modelling predictions. We will also consider the complex fluid mechanics within the renal pelvis, and its impact on the clearance of stone dust. Due to the real-world application of this research, it is important to validate our theoretical predictions via physical experiments. Thus, we incorporate validation through bench-top, *ex-vivo*, and particle image velocimetry experiments.

1.2 Key mathematical concepts

The application of a physics-based modelling approach to ureteroscopy irrigation has the potential to revolutionise the field by allowing control over flow rate and the washout of debris through a fundamental understanding of the fluid dynamical processes. Within endourology, similar physics based models have been used, for instance, to guide the design of ureteral stents to minimize urine reflux (Cummings *et al.*, 2004). During the course of the research in this thesis, mathematical modelling was applied to ureteroscopy irrigation in another instance: the development of a lumped-parameter model to predict flow and kidney pressures (Oratis *et al.*, 2018). This model highlights the importance of flow resistance through the working channel and back through the access sheath on intrarenal pressures, and we will revisit and extend this model in Chapter 2.

Although we are driven by applications to ureteroscopy, this thesis comprises a study of several topics fundamental to fluid mechanics. Brief summaries of key questions are provided in Sections 1.2.1-1.2.4. We commence mathematical discussion with an introduction to the governing equations.

Navier-Stokes equations

Irrigation fluid (saline) is well-modelled as an incompressible fluid with constant density ρ and viscosity μ . Thus, the velocity and pressure distribution of the fluid can be predicted by solving the Navier-Stokes equations and incompressibility condition, derived by considering conservation of mass principles and a balance of forces (Batchelor, 1967, p. 147). The flows considered in this thesis are steady, and thus derivatives with respect to time can be ignored. The governing equations are:

$$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}, \quad \nabla \cdot \mathbf{u} = 0, \quad (1.1a,b)$$

where $\mathbf{u} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3$ is the fluid velocity vector field in a coordinate system with directions $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and p is the scalar pressure field. The presence of gravity is included via the $\rho \mathbf{g}$ term in equation (1.1a), where $\mathbf{g}$ is a constant vector with magnitude equal to gravitational acceleration, g . This term is included when modelling flow through the vertically oriented irrigation tubing and otherwise ignored.

Our models of ureteroscopy irrigation will be mostly built upon systematic reductions of equations (1.1), with the exceptions of: a differential equation derived from a conservation of fluid mass and properties of renal tissue compliance in Chapter 2; a set of differential equations describing the motion of a rod within a bounding cylinder, derived from conservation of linear and angular momentum in Chapter 3; and an advection-diffusion equation describing the movement of stone dust in Chapter 5.

The relative effect of fluid inertia to viscosity in equations (1.1) is captured in the corresponding dimensionless system by the Reynolds number $\text{Re} = \rho UL/\mu$ where U and L are a characteristic velocity-scale and length-scale, respectively. If the geometry through which the fluid flows is long and thin and the Reynolds number is calculated by employing the axial length-scale, then the governing dimensionless parameter is the reduced Reynolds number $\epsilon^2 \text{Re}$, where ϵ is the ratio of radius to length. The important parameters for the components of ureteroscopy irrigation are summarised in Table 1.1.

Component	Flow rate (cm^3/s)	Radius (cm)	Length (cm)	Re
Tubing	1.00×10^0	1.25×10^{-1}	6.00×10^1	$\epsilon^2 \text{Re} = \mathcal{O}(10^{-1})$
Working channel	1.00×10^0	6.00×10^{-2}	7.90×10^1	$\epsilon^2 \text{Re} = \mathcal{O}(10^{-1})$
Access sheath	1.00×10^0	1.93×10^{-1}	3.60×10^1	$\epsilon^2 \text{Re} = \mathcal{O}(10^{-1})$
Lengthscale (cm)				
Renal pelvis	1.00×10^0	6.00×10^{-2}		$\text{Re} = \mathcal{O}(10^2)$

Table 1.1: Characteristic dimensional parameters and corresponding key dimensionless parameter.

Henceforth, due to the long, thin geometries of the working channel, access sheath, and irrigation tubing, $\epsilon^2 \text{Re} = \mathcal{O}(10^{-1})$ and the flows considered in Chapter 2 and Chapter 3 are governed purely by viscous effects, with the left-hand-side of equation (1.1a) set to zero.¹ In Chapters, 4 and 5, however, flow within the renal pelvis is characterised by $\text{Re} = \mathcal{O}(10^2)$ and we consider the full non-linear equations to determine the influence of inertial effects on flow through the full irrigation system, and to model the fluid dynamics specifically within the renal pelvis, respectively.

1.2.1 Flow resistance

The existence of a linear relationship between flow rate through a cylindrical tube and the pressure drop across it was first discovered experimentally by Poiseuille nearly two centuries ago (Sutera & Skalak, 1993). The precise formula relating pressure, viscosity, and pipe dimensions, can be derived as a solution to

$$\nabla^2 w = -\frac{\Delta P}{\mu L}, \quad (1.2)$$

where w is the fluid velocity in the axial direction (a function of cross-sectional coordinates), μ is the dynamic viscosity of the fluid, and ΔP is the pressure drop along the pipe of length L (Bazant, 2016). Equation (1.2) can be solved to model flow through a pipe of arbitrary cross-section (e.g. Figure 1.2) subject to Dirichlet (no-slip) conditions at the boundary. The flow rate Q is subsequently obtained by integrating w over the

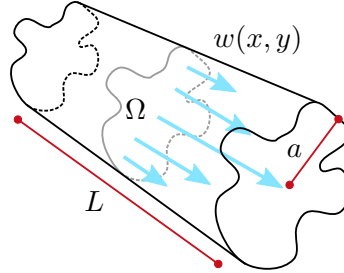


Figure 1.2: A pipe of length L and arbitrary cross-sectional shape that is uniform along its length. The cross-sectional domain is denoted by Ω , and the characteristic length-scale for the pipe cross-section is a .

cross-sectional domain Ω :

$$Q = \iint_{\Omega} w(x, y) d\Omega. \quad (1.3)$$

¹An exception occurs when we consider flows through a deflected working channel, modelled in Section 2.2.4.

Due to the linearity of equation (1.2), Q can equivalently be written as:

$$Q = \frac{\Delta P}{R\mu L}, \quad (1.4)$$

where R is a measure of *geometric resistance* due to viscosity, dependent on the cross-sectional size and shape of the pipe, and can be calculated by solving:

$$\nabla^2 f = -1, \quad R = \left[\iint_{\Omega} f(x, y) d\Omega \right]^{-1}. \quad (1.5)$$

A dimensional analysis of equation (1.4) reveals that $R \sim a^{-4}$, where a is a characteristic length-scale for the cross-section of the pipe. Thus, modest modifications to the size of a pipe's cross-section have vast effects on flow rate for a fixed pressure drop. Even for fixed cross-sectional area, the pipe's cross-sectional shape impacts the flux significantly due to the enforcement of zero velocity on the walls dictating the velocity distribution. For example, it is known that for a pipe of annular cross-section, the resistance decreases monotonically as the core of the annulus is moved away from the centre of the pipe (Piercy *et al.*, 1933). This will be important in Chapters 2 and 3 when we model a working tool within the working channel (or the scope shaft within the access sheath) as a solid rod within a coaxial pipe. The dependence of cross-sectional geometry on flow resistance also motivates us to consider alternate cross-sectional shapes for the ureteroscopic instruments (access sheath, scope shaft, working channel, and working tools). We examine possibilities within the class of elliptical shapes; this is motivated by the ureteral opening resembling an ellipse (Bergman, 1981), and channels with elliptical cross-sections being relatively easy to manufacture. Although several geometric configurations have been previously considered, which will be summarised in Chapter 3, a full characterisation of the dependence of geometric resistance on elliptical eccentricity and core position has not been previously considered. In particular, we will reveal that elliptical geometries can lead to significant flux enhancement (reduction in geometric resistance), and will determine the optimal configuration within the constraint of elliptical geometries of fixed cross-sectional area in Section 3.4.4.

1.2.2 Stokes resistance

The significant variance in resistance to flow through an annular pipe on the position of the core motivates us to consider how a rod will move and rotate if free to do so within a fluid-filled pipe, representing a working tool within the working channel (or scope shaft within the access sheath). Although we are motivated by ureteroscopy, this question has a general relevance within engineering design, with fluid flows in annular geometries prevalent in a

range of apparatuses from oil wells to many surgical tools. If we consider movements of the rod within the cross-section of the bounding cylinder to be small, fluid inertia can be ignored, and equations (1.1) solved within the cross-section, with the left-hand side of equation (1.1a) set to zero. The resulting two-dimensional Stokes equations have many interesting properties that can be exploited when considering the hydrodynamic forces on a submerged object (Happel & Brenner, 1983). For instance: the forces and torque are linearly related to the object’s velocity via a tensor of *resistance coefficients*. An analysis of these scalar quantities as functions of the position of a moving rod, as well as the cross-sectional shapes of the rod and bounding cylinder, will allow insight into whether it is necessary to pin the position of a working tool and the scope shaft, or whether they will be naturally driven by hydrodynamic forces away from the centre of the working channel or access sheath, respectively, to positions of minimal axial flow resistance. Provoked by the revealed relationship between axial flow resistance and ellipticity, we consider the cross-sectional geometries of both the rod and bounding cylinder to be elliptical. Stokes resistance to the translation and rotation of a circular rod within a bounding circular cylinder has been fully characterised (Finn & Cox, 2001) and we contribute to this body of literature with a comprehensive study on the influence of elliptical geometries.

1.2.3 Inertial pressure losses

A classical question for engineering applications is to quantify the relationship between flow rate and pressure in a series of distinct, connected pipes (White, 2008). Applications other than ureteroscopy include agricultural irrigation systems (Juana *et al.*, 2002), plumbing distribution networks (Ladd, 2005), oil drilling (Millheim & Tulga, 1982), and airflow through the lungs (Buess *et al.*, 2019). Combining fully-developed predictions, equation (1.4), with a conservation of flux, yields accurate approximations as long as pipe intersections cause minimal disruption to parallel streamlines (Rouse, 1946). However, abrupt changes in geometry caused by pipe entrances and exits, bends, or sudden expansions and contractions leads to flow separation and regions of fluid recirculation through the influence of fluid inertia. There is an associated dissipation of kinetic energy, which is typically captured via a dimensionless *loss coefficient* K

$$\text{energy loss} = K \frac{1}{2} m V^2, \quad (1.6)$$

where m is the fluid mass and V the average velocity (White, 2008). Values for K can be derived from a conservation of energy and momentum in particular instances, such as flow development from a uniform stream to a fully-developed Poiseuille profile (Lundgren *et al.*, 1964), or flow through a sudden expansion (Batchelor, 1967, p. 372). Although

complicated flow patterns resulting from individual pipe fittings constrain the generalisability of equation (1.6), its simplicity yields practical use. Particularly in ureteroscopy where flow rate is set arbitrarily by the height of the bag of saline, an understanding of the relative contributions of fluid viscosity, equation (1.4), and inertia, equation (1.6), is essential for accurate estimations of irrigation flow rate as a function of system variables. This will be the primary discussion of Chapter 4.

1.2.4 Symmetry-breaking in fluid flows

The efficacy of debris clearance is dependent not only on the flux of saline, but on flow patterns within the kidney. With an aim to explore the complex associations between flow rate, scope position, kidney geometry, flow structure within the renal pelvis, and debris clearance, we consider the jet of saline emerging from the working channel into the renal pelvis, with a return flow through the access sheath. In a highly idealised scenario, with the ureteroscope centred within the access sheath, and the renal pelvis adopting a simple geometric shape, we could envision a domain with perfect symmetry. However, even within this hypothetical example, the nonlinearity of equation (1.1) may render asymmetries in the resulting flow patterns. Symmetry-breaking bifurcations in fluid mechanics are ubiquitous (Crawford & Knobloch, 1991). Stable, asymmetric solutions to equations (1.1) on symmetric domains are observable both mathematically and experimentally in a plethora of geometric configurations, with flow through a two-dimensional channel with a sudden expansion providing a classical example (Fearn *et al.*, 1990). The bifurcation phenomena, in this instance, are associated with the relationship between fluid viscosity and inertia, as well as the expansion ratio of the channel. An exploration of symmetry-breaking and its effect on the advection and diffusion of a passive dust cloud, the topic of Chapter 5, highlights undeniable complexities associated with ureteroscope irrigation and kidney stone clearance.

1.3 Structure of the thesis

In Chapter 2, we begin by modelling three variable factors in ureteroscopy irrigation: saline bag height, tip deflection, and the presence of working tools. We commence, for completeness, with a derivation of Poiseuille flow through the working channel. To model the effects of saline bag height and tip deflection we adopt a pipe network approach. We determine the influence of saline bag height by equating unidirectional flow through the irrigation tubing under the influence of gravity with unidirectional flow through the horizontal working channel. This results in a flow rate that is a nonlinear function of

bag height. We include flow through a deflected tip with algebraic relationships between flow rate and pressure drop through a weakly curved pipe. Two expressions for different regimes of scope tip curvature are extracted from existing literature, originally derived through consideration of the Dean flow equations (Dean, 1928). We show tip deflection to have a negligible impact on flow rate. We include the presence of working tools within the working channel by modelling flow through an annulus, a description that generalises to modelling irrigation outflow through the access sheath, alongside the scope shaft. The interdependence between flow rate and the position of the annulus core yields a quantifiable degree of uncertainty in the model predictions. All component models are validated through comparison with the results of bench-top experiments. The verified models are combined to form a systems model, reliant on a constitutive description of renal pelvis volume as a function of intrarenal pressure. A suitable exponential function is determined from *ex-vivo* data. The systems model comprises an ordinary differential equation to predict kidney pressure (from which flow rate can be algebraically deduced) as a function of time, with the ratio of flow resistance through the working channel to flow resistance through the access sheath a key parameter. The effect of working tools on kidney pressure and total flow rate is theorised through modifying the inflow resistance appropriately. The steady-state systems model predictions are validated through comparison of model predictions with *ex-vivo* measurements, with the exception of a discrepancy between measured and predicted intrarenal pressures. This will be revisited in Chapter 5.

Motivated by the work in Chapter 2, in Chapter 3, we aim to optimise the cross-sectional shape of a working channel containing a working tool (similarly an access sheath containing the scope shaft), by considering the class of elliptical shapes with fixed cross-sectional area. As the resistance is strongly tied to the position of the tool within the working channel or scope within the sheath, we also address the question of where a rod will sit within a channel filled with an axially flowing viscous fluid. Considering a perturbation to the position of the rod, our model decouples into unidirectional flow down the axis and Stokes flow in the cross-section. We explore the complicated and often counter-intuitive effects of varying elliptical eccentricity on axial flow resistance and Stokes resistance. We obtain the optimal geometry to minimise axial flow resistance within the class of elliptical shapes, and determine cross-sectional trajectories of the rod by considering a conservation of linear and angular momentum coupled to the hydrodynamic forces and torque within the fluid. We conclude Chapter 3 by discussing the influence of a geometrically optimised scope shaft/access sheath configuration on the prediction of the systems model from Chapter 2.

The results of Chapter 3 provide candidate elliptical configurations for the working tool, working channel, scope shaft, and access sheath, theorised to maximise axial flow rate and minimise kidney pressure. This provokes an attempt to provide experimental validation. Intriguingly, the measured flow rates as functions of the driving pressure head are

significantly lower than the predictions for fully-developed flow. This discrepancy reveals the importance of accounting for inertial adjustments due to entrance effects and sudden geometric expansions which are present within the experimental set-up. A description of the experiments and accompanying theory comprise the first half of Chapter 4. The second half constitutes a numerical study to determine the relative influence of inertia, as opposed to viscosity, on flow predictions through each component of ureteroscopy irrigation, namely, irrigation tubing, the working channel, the renal pelvis, and the access sheath. The inclusion of inertial adjustments to the linear relationships between flow and pressure explored in Chapters 2 and 3 allows for more accurate estimations of flow rate as a function of relevant ureteroscopic parameters.

Finally, in Chapter 5, we turn to modelling the emerging jet of irrigation fluid into the kidney. We model the renal pelvis as a two-dimensional rectangular cavity, prescribing a parabolic inlet profile and zero-stress outflow conditions. We simulate the steady Navier-Stokes equations, and determine the existence of multiple solutions which vary with the Reynolds number of the flow (set here by the flow rate) and the length of the cavity. We calculate bifurcation diagrams with deflated numerical continuation, and validate these, along with the numerical flow profiles, with particle image velocimetry experiments. Flows in the cavity contain multiple vortices, and we turn to considering the effect of these on the advection and diffusion of a cloud of debris within the kidney. We find different scaling laws between the diffusion coefficient of the debris and washout time, depending on the flow structure within the cavity and the initial position of the debris cloud. We conclude by resolving the discrepancy between the kidney pressure prediction of the systems model and the results of *ex-vivo* experiments by introducing the potential for pressure variation within the renal pelvis.

In Chapter 6 we summarise and discuss a few of the myriad avenues for future research.

CHAPTER 2

Building a systems model of irrigation

The whole is greater than the sum of its parts.

(Aristotle)

Synopsis

In this chapter, we derive simple mathematical models for irrigation flow during ureteroscopy, which we validate via comparison of model predictions with data from bench-top and *ex-vivo* experiments. We begin by considering flow through an isolated working channel, and determine the effects of bag height, tip deflection, and working tools on flow rate. We uncover a nonlinear relationship between flow rate and bag height due to viscous pressure losses through the irrigation tubing. We derive an expression for flow rate as a function of tip curvature which reveals that although flux decreases monotonically with tip deflection, the influence is negligible. Flow through the working channel with a working tool is modelled as flow through an annulus. Thus, tools cause a large impedance to flow, the magnitude of which depends on tool position. The models for the effects of saline bag height, tip curvature, and working tools agree with the results of bench-top experiments. We incorporate the presence of a kidney with exponential compliance, and a return flow through an access sheath, deriving a systems model for flow rate and kidney pressure as functions of time. We validate the steady-state solutions of the systems model with the results of *ex-vivo* experiments, and determine the impact of working tools.

A paper based on the isolated working channel models has been published in *J. Endo.* (Williams *et al.*, 2019a), and a paper on the systems model submitted to the *IMA J. Appl. Math* (Williams *et al.*, 2019b).

2.1 Introduction

Understanding flow through a rigid, long and thin cylindrical pipe, is key to modelling ureteroscopy irrigation through the working channel and access sheath. The linearity between driving pressure, pipe length, cross-sectional resistance, viscosity, and flow rate through a pipe, equation (1.4), enables simple quantifications of irrigation flow as a function of system variables. Ureteroscopy irrigation has previously been modelled as a series of distinct, connected tubes, resulting in a single, ordinary differential equation describing kidney pressure as a function of time (Oratis *et al.*, 2018). As the flow is assumed to be steady, time-dependence enters only through modelling the kidney as compliant; its volume changes as a function of its internal pressure. In Oratis *et al.* (2018), kidney compliance is taken to be linear. However, this approximation is somewhat unphysical; biological tissue is typically shown to be poorly compliant, represented by an increase in the slope of the stress-strain curve at higher strains (Holzapfel, 2001; Chagnon *et al.*, 2018). This is often accounted for by constitutive laws that include exponential terms, ensuring that tissue stiffens as it stretches (Cruces *et al.*, 2014). The systems model described in Oratis *et al.* (2018) highlights the importance of low outflow resistance to maintain low intrarenal pressures by increasing access sheath size or intermittently withdrawing the scope. However, the strong dependence of resistance through the access sheath on the position of the scope is not discussed, and the cross-section of the access sheath containing the scope shaft is modelled as a concentric annulus. In actuality hydrodynamic forces may drive the scope to sit offset from the centre of the sheath (a question addressed in more detail in Chapter 3). Additionally, the likely presence of working tools within the working channel (again of unknown position) is neglected in the Oratis *et al.* (2018) model.

As successful ureteroscopy hinges on good visualisation, provided by sufficient flow, factors that influence driving pressure or flow resistance are of particular interest to the clinical community. Consequently, a variety of experimental studies have been conducted, focussing, in particular, on the effects of three procedural variables:

- I) the height of the saline bag to vary the flow rate,
- II) the deflection of the scope tip to access areas of the kidney,
- III) the selection of working tools to destroy and remove stones,

where I) relates to changes in driving pressure and II) and III) to variation in flow resistance through the working channel. Experimental studies of irrigation flow in cystoscopy for bladder visualisation suggest a nonlinear dependence between saline bag height and flow rate (Chang *et al.*, 2012), although an appropriate explanation for this phenomenon has not been provided. Studies measuring flow rate as a function of the degree of tip deflection

have determined a maximum of 8% loss in flow rate with 180° deflection when compared to a straight scope (Nagele *et al.*, 2006; Bach *et al.*, 2008). Experimental research on the impact of working tools on flow rate report between 62% and 99% losses in flow rate, depending on the diameter of the tool inserted (Bach *et al.*, 2008; Nagele *et al.*, 2006; Bach *et al.*, 2011; Pasqui *et al.*, 2004; Ng *et al.*, 2010; Kruck *et al.*, 2011). Although these experiments shed light on the relationship between I) bag height, II) tip deflection, III) working tools, and flow rate, they are limited by a sparsity of data and lack of theoretical underpinning.

2.1.1 Chapter outline

In this chapter, we first consider the simplest description of fluid flow during ureteroscopy: flow through an isolated working channel, which can be modelled as flow through a long, thin, cylindrical pipe. We incorporate further complexities to the model to analyse the effects of: I) bag height, II) scope tip deflection, and III) working tools. We validate these models with the results of bench-top experiments.

We then incorporate the presence of the kidney and access sheath into the modelling framework, to construct a *systems model*. This is a pipe network description of irrigation, similar to that presented by Oratis *et al.* (2018). The kidney, in this description, is represented by a single pressure point, and the material properties encapsulated through an expression for kidney compliance; the relationship between fluid pressure and volume within the kidney. We use the results of *ex-vivo* experiments to motivate an exponential, as opposed to linear, kidney compliance, and to validate the systems model. We conclude by integrating the effect of working tools into the systems model, and determining the subsequent reduction in flow rate and effect on intrarenal pressure.

2.2 Flow through an isolated working channel

2.2.1 Bench-top experiments

A series of bench-top experiments were performed to explore the effects of I) bag height, II) tip deflection, and III) working tools on flow rate. The basic experimental set-up is depicted in Figure 2.1a; a LithoVue™ ureteroscope was clamped in a horizontal position, and tubing, connected to the scope, was fed from a water-filled bucket (simulating the saline bag), within which the fluid was maintained at a constant level by manual correction. The tubing was connected to the bucket via a valve in order to switch on and off the flow of water (Figure 2.1b). The tip of the scope was situated directly above a container (to

collect the water flowing from the scope) on a mass balance, which was connected to a laptop computer to record the mass of fluid in the container at specified intervals. At the start of each experiment, the valve was turned to allow water to flow through the channel, and mass measurements were recorded at one-second intervals.

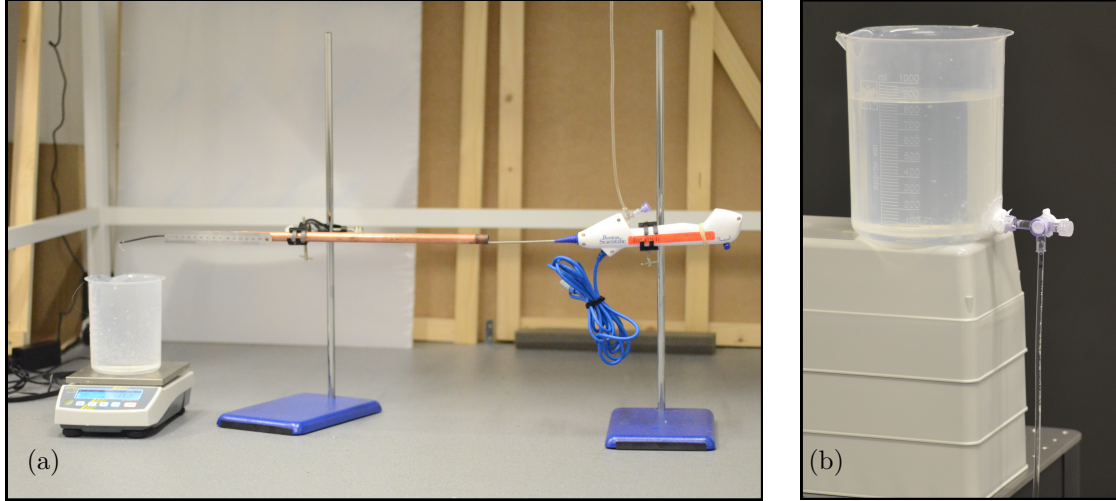


Figure 2.1: (a) A photo of the experimental setup. The LithoVue™ ureterscope is held in position by passing it through a horizontally clamped metal tube. The inlet to the scope is attached via vertical tubing to a bucket, the height of which is raised or lowered accordingly. The distal end of the scope is open to the air, and the fluid flows from this to a second bucket on a mass balance.

To test the relationship between bag height and flow rate, experiments to measure flow through a straight ureterscope without working tools were performed at five pressure heads. To adjust the height of the suspended reservoir, the bucket was raised and longer tubing attached, so that the tubing remained vertical without slack for each experiment (Figure 2.1b).

For the deflected scope experiments, the head height was fixed, and experiments were performed for five different deflections. The radius of curvature for each deflection was estimated by fitting a circle to the photo image of the deflected scope tip, taken next to a standard ruler for scale (see, for example, Figures 2.2a-c).

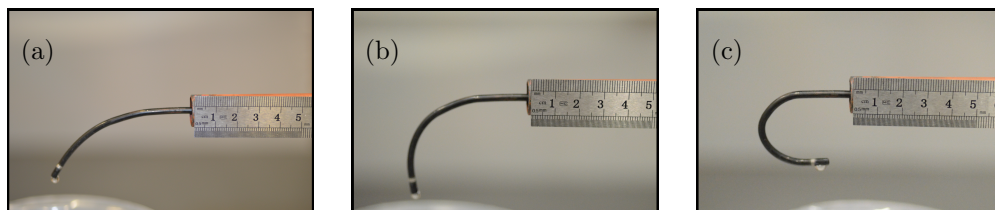


Figure 2.2: Photos used to measure the radius of curvature for the deflected scope. (a) $\gamma = 5.4$ cm, (b) $\gamma = 2.8$ cm, (c) $\gamma = 1.3$ cm.

	Working tool	Radius ($\times 10^{-2}$ cm)
(a)	No tool	
(b)	OptiFlex™ basket	2.17
(c)	Flexiva™ 200 Laser Fibre	2.21
(d)	Flexiva™ 365 Laser Fibre	3.02
(e)	Zero Tip™ basket	3.17
(f)	ZIPwire™	4.83

Table 2.1: Table of tools used in Figure 2.8. All manufactured by Boston Scientific Corporation.

For the working tool experiments, the head height was again fixed, and different working tools were inserted into the working channel before starting the experimental procedure. Table 2.1 summarises the tools used in the experiments. The experiments are described in more detail in Appendix 2.A, along with the procedure for data analysis.

2.2.2 Modelling flow through a straight channel

The working channel that runs through the ureteroscope shaft only comprises a small portion of the scope, as space on the tip is required for auxiliary features, such as the light and camera necessary for visualisation. Although the scope shaft has a typical radius of approximately 0.2 cm, the working channel radius is only 0.06 cm. The channel runs through the entire 79 cm length, and hence is well modelled by a hollow cylinder, as pictured in Figure 2.3. We model the flow of fluid through the working channel of

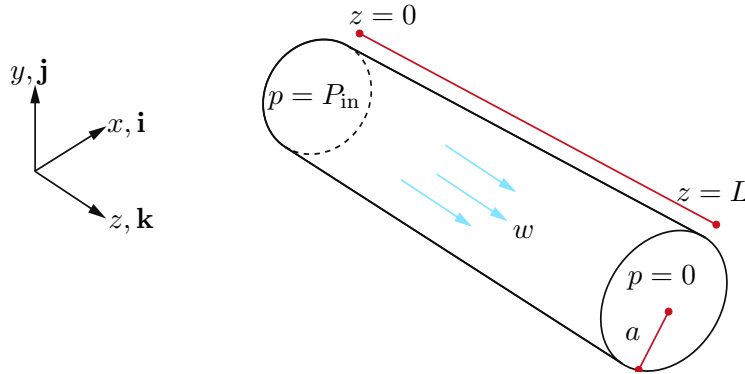


Figure 2.3: The ureteroscope working channel (isolated from the scope shaft) depicted in a Cartesian coordinate system as a hollow cylinder, with length aligned with the z -axis.

radius a and length L within the scope with the steady Navier-Stokes equations and incompressibility, equations (1.1). To enable modifications to the channel geometry, we adopt a Cartesian coordinate system (x, y, z) , with corresponding coordinate directions $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, where $\mathbf{k}$ is oriented along the cylinder's axis, $z = 0$ is at the working channel entrance, and the cross-section is centred at $(x, y) = (0, 0)$. We assume the elevated saline

bag generates an inlet pressure, $p = P_{\text{in}}$, and take the pressure at the distal end of the scope to be $p = 0$, without loss of generality. Continuity of normal stress at the entrance and exit to the working channel, along with the assumption of a no-slip condition at the channel walls, provides boundary conditions

$$(\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \mathbf{n}|_{z=0} = P_{\text{in}}, \quad (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \mathbf{n}|_{z=L} = 0, \quad \mathbf{u}|_{\text{walls}} = \mathbf{0}, \quad (2.1\text{a,b,c})$$

where $\boldsymbol{\sigma} = [-p\mathbf{I} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)]$ is the viscous stress tensor for the fluid, and $\mathbf{n}$ is the outward-pointing normal vector; $\mathbf{n} = (0, 0, -1)$ at $z = 0$ and $\mathbf{n} = (0, 0, 1)$ at $z = L$.

We introduce the aspect ratio of the working channel

$$\epsilon = a/L = \mathcal{O}(10^{-3}), \quad (2.2)$$

in order to non-dimensionalise appropriately, where the value is approximated using the parameters in Table 2.2 (Appendix 2.A). We adopt the following scalings

$$\mathbf{x} = L(\epsilon\hat{x}, \epsilon\hat{y}, \hat{z}), \quad \mathbf{u} = W(\epsilon\hat{u}, \epsilon\hat{v}, \hat{w}), \quad p = P_{\text{in}}\hat{p}, \quad (2.3\text{a,b,c})$$

where hats denote dimensionless variables, and the scaling for velocity is chosen to balance the pressure term in the Navier-Stokes equations (1.1a)

$$W = \epsilon^2 P_{\text{in}} L / \mu. \quad (2.4)$$

Inserting equations (2.3), (2.4) into the Navier-Stokes equations and incompressibility condition (1.1), without the influence of gravity, we recover the Reynolds number (Batchelor, 1967, p. 213), a key dimensionless parameter relating the viscous and inertial forces of the fluid

$$\text{Re} = \rho W L / \mu = \mathcal{O}(10^4), \quad (2.5)$$

where the magnitude is estimated using the measured flux values in Figure 2.4 and the parameter values in Table 2.2 (Appendix 2.A). The reduced Reynolds number, $\epsilon^2 \text{Re} = \mathcal{O}(10^{-2})$, appears in the dimensionless equations, and taking

$$\epsilon^2 \ll 1, \quad \epsilon^2 \text{Re} \ll 1, \quad (2.6\text{a,b})$$

we obtain the leading order system

$$\hat{\nabla}_{\perp}^2 \hat{w} = \frac{d\hat{p}}{d\hat{z}}, \quad \hat{\nabla} \cdot \hat{\mathbf{u}}_{\perp} + \hat{w}_{\hat{z}} = 0, \quad (2.7\text{a,b})$$

where $\hat{\nabla}_\perp = \left(\frac{\partial}{\partial \hat{x}}, \frac{\partial}{\partial \hat{y}} \right)$ and $\hat{\mathbf{u}}_\perp = (\hat{u}, \hat{v})$. Boundary conditions (2.1) at leading order become

$$\hat{p}(\hat{z} = 0) = 1, \quad \hat{p}(\hat{z} = 1) = 0, \quad \hat{\mathbf{u}}|_{\text{walls}} = \mathbf{0}. \quad (2.8a,b,c)$$

We seek a separable solution of the form

$$\hat{w}(\hat{x}, \hat{y}) = \frac{d\hat{p}}{d\hat{z}} f(\hat{x}, \hat{y}). \quad (2.9)$$

Integrating equation (2.7b) over a cross-section of the fluid domain, using equations (2.9), (2.8c), and applying the divergence theorem provides

$$\hat{w}_{\hat{z}} = 0, \quad (2.10)$$

and hence, by the ansatz (2.9), $\frac{d\hat{p}}{d\hat{z}}$ is a constant. With pressure subject to boundary conditions (2.8a,b), we must have

$$\frac{d\hat{p}}{d\hat{z}} = -1. \quad (2.11)$$

Therefore, the governing dimensionless system is

$$\hat{\nabla}_\perp^2 \hat{w} = -1 \quad (2.12)$$

subject to

$$\hat{w} = 0 \quad (2.13)$$

on the channel walls. We determine $\hat{u} = \hat{v} = 0$ by equating terms of $\mathcal{O}(\epsilon^2)$ in equation (1.1a) scaled by equations (2.3); Poisson equations for $\hat{u} = 0$ and $\hat{v} = 0$ subject to no-slip conditions at the channel walls require the cross-sectional velocity components to be identically zero.

For flow through a cylindrical channel of circular cross-section, equations (2.12), subject to (2.13) can be solved analytically to obtain the well-known Poiseuille formula

$$\hat{w}(r) = \frac{1}{4}(1 - r^2), \quad (2.14)$$

where $r = \sqrt{x^2 + y^2}$ (Lamb, 1916, p. 578). Re-dimensionalising equation (2.14) using equations (2.3), (2.4), and integrating over the cross-sectional domain, we obtain the dimensional flux

$$Q = \frac{P_{\text{in}}}{R\mu L}, \quad R = \frac{8}{\pi a^4}. \quad (2.15)$$

In the following sections, we consider the effects of I) saline bag height, II) scope tip deflection, and III) the addition of working tools on flow rate, and compare with flow rate through an empty working channel with equation (2.15).

2.2.3 Effect of saline bag height

The saline bag is typically elevated approximately 60 cm above the height of the scope (Figure 1.1). If the fluid in the irrigation bag and tubing were purely hydrostatic, the inlet pressure would be

$$P_{\text{in}} = \rho g H, \quad (2.16)$$

where H is the vertical distance between the scope inlet and the fluid level in the saline bag, regardless of the radius of the irrigation tubing and bucket, or any bends in the irrigation tubing. Thus, combining equations (2.15) and (2.16), we would anticipate a linear relationship between Q and bag height, H . This prediction is plotted as the dashed black line in Figure 2.4, along with experimental data (red marks). We observe a sys-

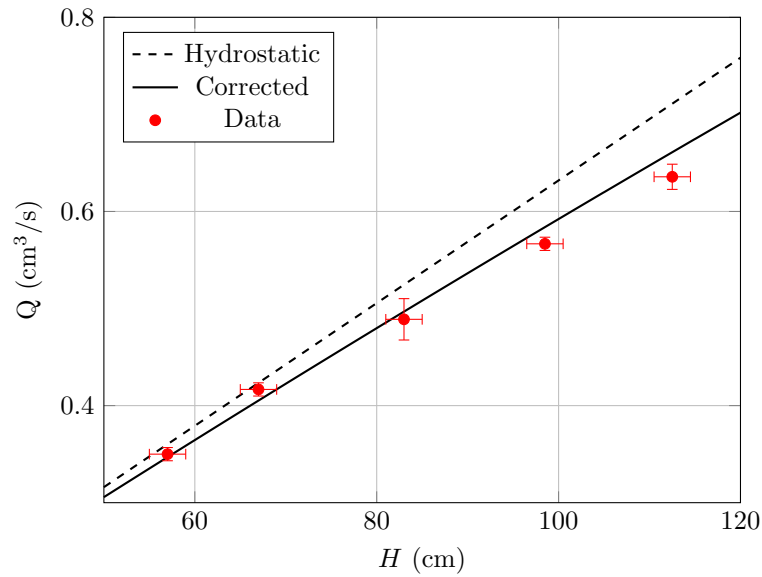


Figure 2.4: Experimental and theoretical results for the flow of water through an open ureterscope working channel of internal radius 0.06 cm. The dashed line equation (2.16) assumes a hydrostatic inlet pressure. The solid line equation (2.18) accounts for viscous dissipation. Horizontal lines on the data points depict error in the height measurements, and vertical lines depict error in measured flow rates.

tematic deviation between this linear prediction and experiment, which becomes more pronounced at higher head heights. This deviation from the linearity predicted by the classic Poiseuille result was also observed in a previous study on irrigation in cystoscopy for bladder visualisation (Chang *et al.*, 2012). In Chang *et al.* (2012), the non-linear relationship between bag height and flow rate was justified by applying the Bernoulli equation for an inviscid fluid to flow through the irrigation tubing and the cystoscope. However, our simple dimensional analysis demonstrates a small reduced Reynolds number, equation (2.6b), and hence reveals a flow regime dominated by viscous forces. Thus, we propose

an alternative explanation for the non-linear data trend in Figure 2.4; that the deviation from equations (2.15), (2.16) mostly arises from neglecting the vertical flow that occurs from the suspended reservoir into the scope. We model the tubing as a cylindrical channel of radius ℓ , oriented vertically, and we assume that the column of fluid, flowing under gravitational forces, is maintained at height H . We assume that the top of the column of fluid is at the same pressure as the scope outlet (zero), and that the bottom of the column is at pressure P_{in} . As the fluid is incompressible, and assuming that both the tubing and working channel are rigid, flow rate through the irrigation tubing must be the same as flow rate through the working channel. The ratio of radius to length of the tubing is, once again, small, $\ell/H = \mathcal{O}(10^{-3})$; thus, the methodology in Section 2.2.2 is applicable for deriving an expression relating flow rate through the tubing to the pressure drop across it. As gravity acts parallel to the length of the tubing, the governing momentum equation is (1.1a) including the gravitational force term, which reduces to (Lamb, 1916)

$$Q = \frac{(\rho g H - P_{\text{in}})}{R_t \mu H}, \quad R_t = \frac{8}{\pi \ell^4}, \quad (2.17)$$

where R_t is the geometric resistance through the tubing. Thus, solving for P_{in} using equations (2.15) and (2.17) we obtain

$$P_{\text{in}} = \frac{\rho g H}{1 + \sigma}, \quad (2.18)$$

where $\sigma = (H/L)(a/\ell)^4$ characterises the relative importance of viscous dissipation through the vertical tubing. As H increases (or ℓ decreases), σ will increase and the imposed pressure difference is increasingly smaller than $\rho g H$. In Figure 2.4, the model prediction with P_{in} specified by equation (2.18) is shown as the solid line, which indeed shows superior agreement with the experiment. We note there still appears to be a small deviation between the solid black lines and the final data point in Figure 2.4; we hypothesise this is due to neglecting minor pressure losses due to inertial effects at the irrigation tubing entrance and exit. We will revisit this at the end of Chapter 4.

For the parameters explored here, viscous dissipation becomes increasingly significant as the height of the irrigation fluid, increases (Figure 2.4). We thus conclude that even in this simple setting, classic results may yield inaccurate predictions. In a clinical setting, the dependence of the volumetric flow rate on both height and tube radius should be considered when determining how high to hang the irrigation bag.

We note that equation (2.18) can also be applied for unidirectional flow with general geometric resistance, R , as in equation (1.4) with $\sigma = (H/L)(R_t/R)$.

2.2.4 Effect of tip deflection

We next consider the effect of ureterscope tip deflection on the flow rate. The ureterscope shaft is comprised of a straight channel (73.3 cm long) connected to a shorter deflectable tip (5.7 cm long). When the scope is deflected during ureteroscopic procedures, the majority of the scope remains straight, while the tip can bend up to 270° in both directions. We model the deflected scope as a straight pipe of length L_s , connected to a shorter pipe with constant radius of curvature γ and arc length L_c (Figure 2.5). We adopt a hydraulics network approach to modelling flow through this geometry equating Poiseuille flow through the straight portion of the shaft with relationships between flow rate and pressure through the deflected tip derived from the Dean flow equations which maintain the leading-order curvature effects for flow through curved pipes (Dean, 1928). As before, the radius of the working channel is a , and $L_c + L_s = L$. As in the previous section, the straight

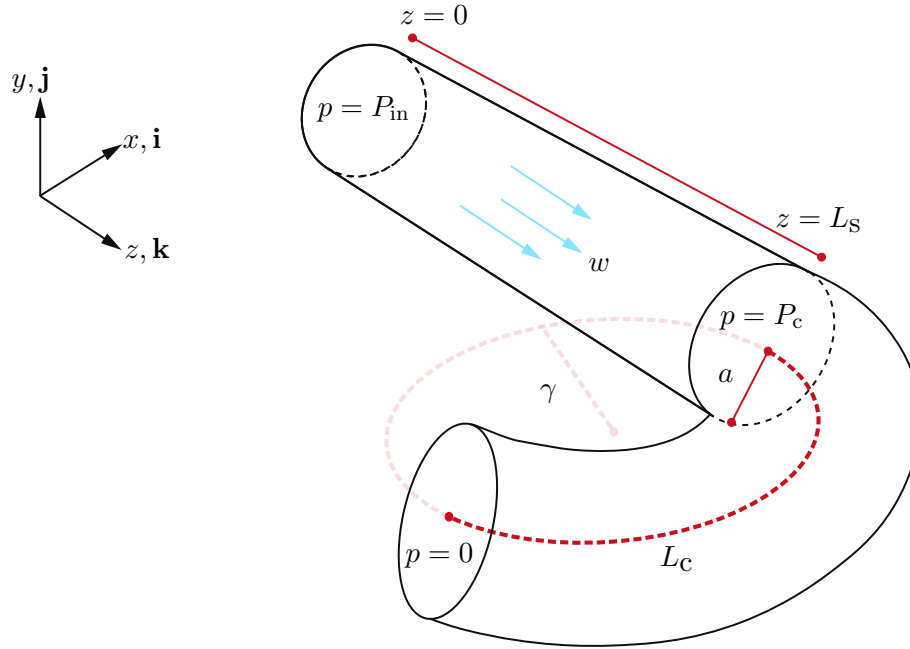


Figure 2.5: The deflected ureterscope working channel of radius a , comprised of straight section of length L_s , aligned with the z -axis, and bent portion of length L_c and radius of curvature, γ . The pressures at the entrance to the working channel, the cross-section connecting the straight and curved segments, and the exit are P_{in} , P_c , and zero, respectively.

portion of the channel is oriented along the z -axis, with $z = 0$ at the working channel inlet. We denote the pressure where the scope transitions from straight to curved as P_c , with pressures P_{in} at the working channel inlet and zero pressure at the outlet (Figure 2.5). In the straight section, the volumetric flow rate is related to the pressure drop by a Poiseuille law

$$Q_s = \frac{P_{in} - P_c}{RL_s}, \quad R = \frac{8}{\pi a^4}, \quad (2.19)$$

while in the curved section we determine the relationship between flow rate and pressure drop using the Dean flow equations, detailed in Dean (1928). The Dean flow equations are derived based on the assumptions of steady flow, and of weak curvature

$$\delta = a/\gamma \ll 1. \quad (2.20)$$

Equation (2.20) is consistent with the values considered, as the minimum value for $\gamma \approx 1$ cm (fully deflected scope), corresponds to a maximum $\delta = 0.06$. Following the formulation in Dean (1928), flow through the curved section of the working channel involves a single dimensionless parameter, the Dean number

$$D = P_c \rho a^3 \sqrt{2\delta} / (L_c \mu^2). \quad (2.21)$$

For D small the volumetric flow rate through the curved section is approximately (Dean, 1928)

$$Q_c \approx \frac{\pi D a \nu}{8\sqrt{2\delta}} \left[1 - 0.0306 \left(\frac{D}{96} \right)^4 \right], \quad (2.22)$$

an expression that has been shown to be valid for Dean numbers up to $D = 96$. Using a combination of asymptotic and numerical methods, Collins & Dennis (1975) computed a relationship between flux and D that is valid in the range $D = 96$ to $D = 5000$, which is

$$Q_c \approx \frac{\pi D a \nu}{8\sqrt{2\delta}} \left[8.12 D^{-1/3} - 16.7 D^{-2/3} \right]. \quad (2.23)$$

To apply (2.22) or (2.23) to compute the flux through the curved portion of the scope, we must first calculate the Dean number via equation (2.21). This involves knowing P_c , which we calculate as a function of P_{in} by equating the fluxes through the straight and curved portions of the scope, Q_s and Q_c . Equating equation (2.19) with (2.22) or (2.23) we have the relations

$$\frac{P_{in} - P_c}{L_s} - \frac{P_c}{L_c} \left[1 - 0.0306 \left(\frac{D}{96} \right)^4 \right] = 0, \quad \text{for } D < 96, \quad (2.24a)$$

$$\frac{P_{in} - P_c}{L_s} - \frac{P_c}{L_c} \left[8.12 D^{-1/3} - 16.7 D^{-2/3} \right] = 0, \quad \text{for } D \geq 96, \quad (2.24b)$$

which, because D depends linearly on P_c , are each algebraic equations for P_c . However, because we require P_c to compute D , it is not known whether to use (2.24a) or (2.24b) beforehand. Thus, to solve for the flux for a given curvature ratio, δ , we solve both equations (2.24a) and (2.24b) for P_c , and then, after computing the Dean number, select the appropriate result. As described in Section 2.2.3, a more accurate estimation of the

flow rate is obtained by including the effects of viscous dissipation in addition to the hydrostatic head. Here, this is achieved by equating the flow through the vertical tubing, equation (2.17), with the flow through the straight portion of the scope, equation (2.19). We thus have a system of two equations for the unknown pressures P_{in} and P_c , which we solve using MATLAB's `fzero` solver. Figure 2.6 plots flow rate as a function of tip

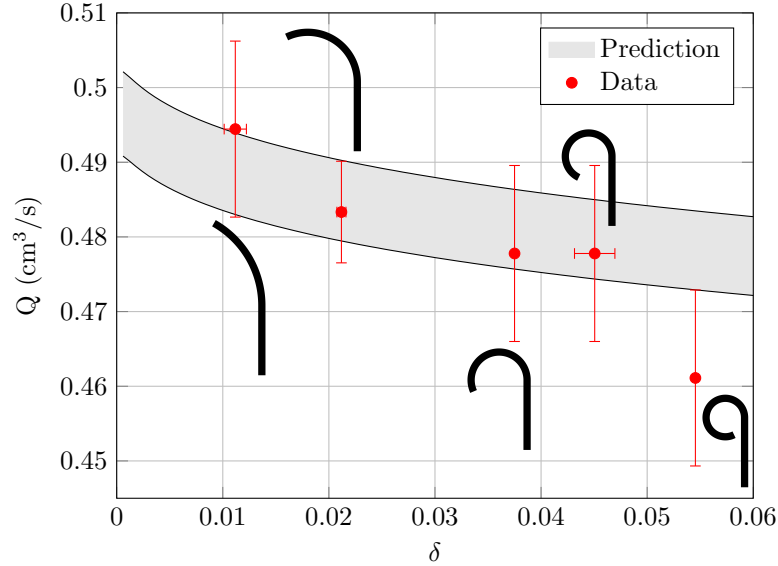


Figure 2.6: Experimental and theoretical results for flow through an open channel with a deflected tip. The horizontal axis shows the ratio of the radius of the working channel (0.06 cm) to the radius of curvature of the curved tip. Inset diagrams demonstrate the extent of tip deflection for each data point. Experiments performed at $H = 83$ cm (uncertainty in this measurement applied to theory provides grey region). Vertical error bars on experiments depict error in flow measurements, and horizontal error bars provide uncertainty in radius of curvature measurements.

curvature. The theoretical prediction appears as the grey region, and incorporates the measurement uncertainty in the height of the fluid above the scope, which dictates the inlet pressure. The horizontal axis is the dimensionless curvature; a larger value corresponds to greater deflection. The decrease in flow rate with tip deflection is relatively minimal, e.g., the highest possible LithoVue™ curvature corresponds to only a 5% decrease in flow rate compared to that of a straight scope. This confirms previous observations of minimal flow decrease with deflection from previous studies (Bach *et al.*, 2008; Nagele *et al.*, 2006; Ng *et al.*, 2010), while providing a theoretical underpinning. When the tip is maximally curved, the channel outlet is partially blocked by the scope shaft, which may explain the flow rate falling below predictions in the final data point in Figure 2.6.

In contrast to tip deflection, working tools have been shown experimentally to cause large disruptions to flow rate (Bach *et al.*, 2008; Nagele *et al.*, 2006; Bach *et al.*, 2011; Pasqui *et al.*, 2004; Ng *et al.*, 2010; Kruck *et al.*, 2011). We turn to modelling their influence in the following section.

2.2.5 Effect of working tools

We model the addition of a working tool within the working channel as a coaxial, cylindrical rod of radius b and length L . This modelling framework also generalises to considering the solid scope shaft within an access sheath. If the tool is concentric within the channel,

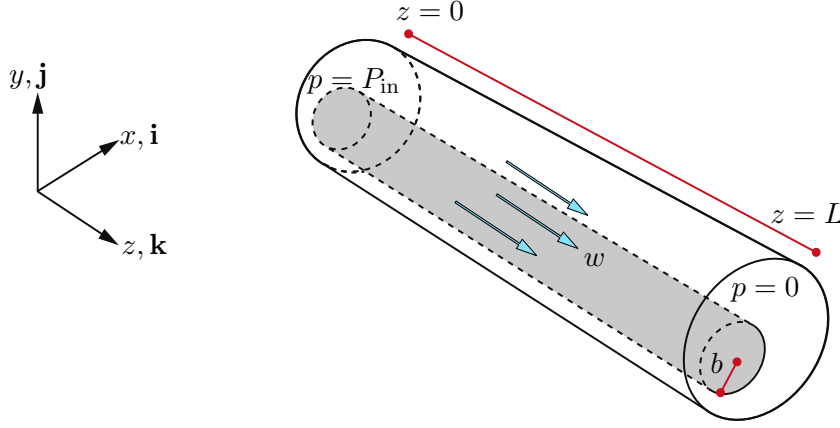


Figure 2.7: Tool of radius b , coaxial within a cylindrical working channel of radius a . The length of both tool and channel is L , aligned with the z -axis. Fluid of velocity w flows in the axial direction subject to pressure difference P_{in} .

we can apply an exact solution for fully-developed laminar flow in an annular duct (Lamb, 1916)

$$Q = \frac{P_{in}}{RL\mu}, \quad R = \frac{8}{\pi(b^2 - a^2)} \left[a^2 + b^2 - \frac{(a^2 - b^2)}{\log\left(\frac{a}{b}\right)} \right]^{-1}. \quad (2.25)$$

If the tool is offset within the channel, a bipolar coordinate system can be adopted which maps the non-coincident circles to a rectangular domain. An analytical solution in this coordinate system exists for flow rate through an offset annulus, which contains an infinite sum (Piercy *et al.*, 1933).¹ Through conversion to a bipolar coordinate system, there is the natural emergence of a parameter, ϕ , which denotes the degree of offset; $\phi = 0$ for a tool in the channel centre, and $\phi = 1$ for a tool at the edge. Due to the linearity of the governing Poisson equation (1.2), the flux is always related to the pressure drop via a resistance $R(\phi)$. It has been previously determined that R decreases (and correspondingly flux increases) monotonically with ϕ , i.e., with tool distance from the centre of the working channel (Piercy *et al.*, 1933; Redberger, 1962; Ebrahim *et al.*, 2013; Snyder & Goldstein, 1965). The inset plot in Figure 2.8 shows Q as a function of ϕ for a tool of radius 0.03 cm within the working channel.

In practice, the position of the tool within the working channel is unknown, so we expect flow measurements to lie between minimum and maximum values corresponding to $\phi = 0$

¹More details in Appendix 3.B.1 of Chapter 3.

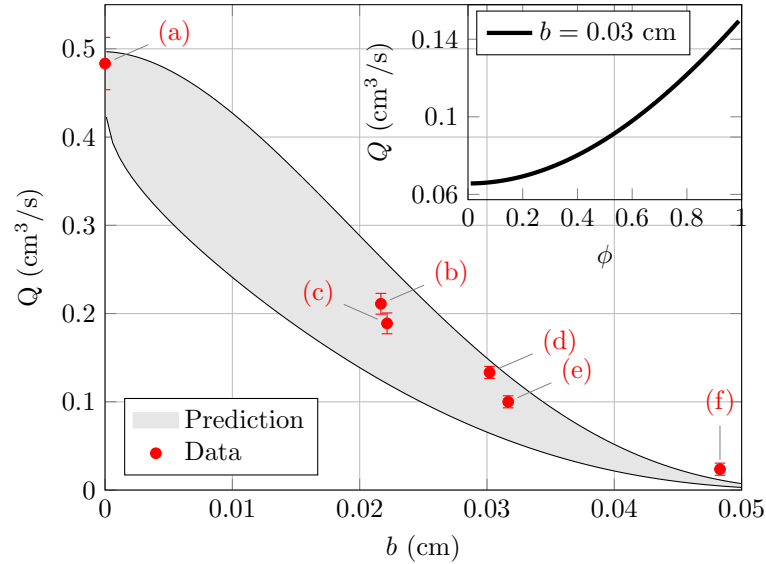


Figure 2.8: Experimental and theoretical results for flow through a working channel containing a working tool with the column of irrigation fluid at height $h = 83 \text{ cm}$. The grey region indicates flow predictions for a circular working channel; the lower line for a tool concentric within the working channel, equation (2.25), and the upper line for a tool at the edge. The tools used are listed in Table 2.1.

and $\phi = 1$, respectively. This is shown in Figure 2.8; the lower black line is the prediction for a concentric tool, and the upper black line for a fully offset tool, with the grey region between indicating flow predictions for all possible positions of the working tool. The data points in Figure 2.8 are labelled with letters corresponding to the tools listed in Table 2.1.

The thickness of the grey region in Figure 2.8 demonstrates the basic physical principle that flow resistance created by the presence of a working tool is strongly tied to the geometry of the space available for fluid flow. This motivates the question of whether a more optimal channel configuration could be realised with non-circular cross-sections for the working channel and working tool. This will be discussed in Chapter 3.

The results in this section illuminate the effects of three key variables in ureteroscopy irrigation on flow through an isolated channel: I) a nonlinear effect of bag height on flow due to viscous dissipation through irrigation tubing; II) a negligible decrease in flow due to tip deflection; III) a severe diminishment in flow with the addition of working tools, with the degree of flow reduction varying with tool position.

Although these results provide valuable insight, it is important to recognise that we have ignored significant complexities; in particular, the effect of the kidney and outflow through the access sheath. Thus far, we have assumed a known pressure at the distal end of the working channel, and hence fixed pressure drop driving flow through the working channel. In actuality, the distal pressure is the intrarenal pressure, which will vary for

fixed inlet pressure, depending on flow resistance through the working channel, flow resistance through the access sheath, and material properties of the renal tissue. Modelling the influence of the kidney and access sheath allows, not only more accurate determination of flow rate for fixed inlet pressure, but also an estimation of kidney pressure as a function of system parameters. This is of particular importance in ureteroscopy, where it is paramount to maintain renal pressures below biophysical thresholds to reduce the risk of post-operative infection (Wilson & Preminger, 1990). We consider a systems model for ureteroscopy irrigation in the following section, with model validation and determination of an appropriate function for kidney compliance provided through the results of *ex-vivo* experiments.

2.3 A systems model of ureteroscopy irrigation

2.3.1 Ex-vivo experiments

A suite of *ex-vivo* experiments were performed on porcine kidneys, with a laboratory configuration shown in Figure 2.9. Irrigation fluid flowed from a saline bag 60 cm above the height of the kidney, when measured from the base of the bag. A pressure cuff surrounded the bag, allowing the inlet pressure to be increased above hydrostatic, and a pressure probe was placed at the inlet to the scope. A 36 cm long 11/13 F Navigator™ access sheath – inner diameter approximately 0.4 cm – was inserted through the ureter, with a pressure probe passed between the ureter and the wall of the sheath. A second pressure probe recorded the inlet pressure. A LithoVue™ ureteroscope was placed within the sheath, and the tips of the sheath, scope, and pressure probe were all within the renal pelvis. The pressure sensors were calibrated to atmospheric pressure. The kidney and a dish to collect the outflow from the sheath were on mass balances, which collected measurements at half-second intervals. Multiple experimental runs were performed on three different kidneys, termed kidneys A, B, and C. The kidneys were drained prior to the experiments, so the initial renal pressure before the insertion of irrigation fluid was approximately atmospheric. Further experimental details are provided in Appendix 2.B, including the data analysis and methodology for the estimation of kidney compliance.

2.3.2 Theoretical formulation

We consider a theoretical depiction of irrigation in an idealised laboratory setting, replicating the configuration in Figure 2.9. A scope of length L lies horizontally, with fluid of density ρ and viscosity μ entering the working channel of radius a at fixed pressure

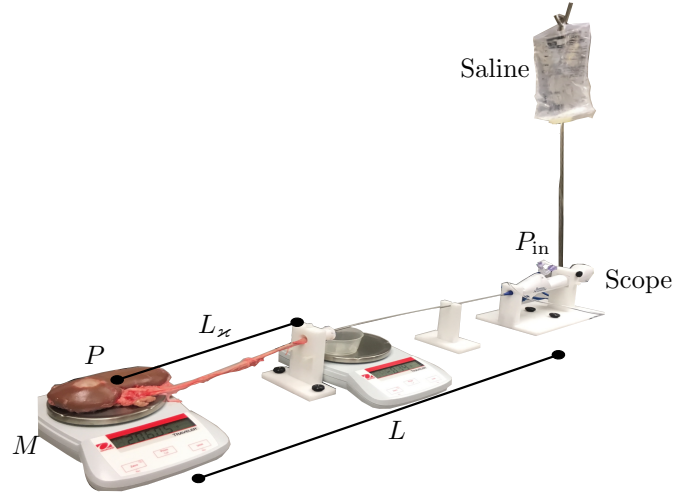


Figure 2.9: A photograph of the experimental set-up, with experimentally measured variables labelled, along with the lengths of the scope and the access sheath.

P_{in} . We allow for the possibility of a working tool of radius b to lie within the working channel. The tip of the scope is inside the kidney which contains irrigation fluid of mass M , volume V , and pressure P , all functions of time. Irrigation fluid flows back from the kidney through an access sheath of radius a_x and length L_x which surrounds the scope shaft of radius b_x . The use of subscript x will differentiate variables relevant to flow ‘out of’ the kidney, from inward flow (the Greek preposition, $\acute{\epsilon}x$, means ‘out of’). One end of the scope shaft is within the kidney, and the other exposed to atmospheric pressure, which, without loss of generality, we take to be zero. We term the fluid volumetric flux into the kidney as Q and back out through the sheath as Q_x .

Conservation of mass provides

$$\frac{dM}{dt} = \rho(Q - Q_x). \quad (2.26)$$

Due to incompressibility of the irrigation fluid increases in fluid mass and volume within the kidney are correlated directly

$$M = \rho V. \quad (2.27)$$

To model kidney pressure as a function of time with equation (2.26), it remains to determine the kidney compliance; a constitutive relationship between changes in pressure and changes in volume. We take

$$P = P_0(\exp(CV) - 1), \quad (2.28)$$

where P_0 is a characteristic pressure scaling and C is a constant with units of inverse volume. The use of equation (2.28) is motivated by a fit to experimental data for a renal pressure-volume curve in Cruces *et al.* (2014), and validated by a fit to our experimental

data in Appendix 2.B. Values for P_0 and C are determined through the fit to experimental data to be

$$P_0 = 5.24 \text{ cm H}_2\text{O}, \quad C = 0.43 \text{ mL}^{-1}. \quad (2.29\text{a,b})$$

Differentiating equation (2.28) with respect to time we obtain

$$\frac{dP}{dt} = P_0 C \exp(CV) \frac{dV}{dt}, \quad (2.30)$$

and using equations (2.27) and (2.26) we obtain the following nonlinear differential equation

$$\frac{dP}{dt} = C(P_0 + P)(Q - Q_\infty). \quad (2.31)$$

As the kidneys are initially drained in the experiments we take

$$P(0) = 0. \quad (2.32)$$

as an initial condition for equation (2.31).

It now remains to determine expressions for Q and Q_∞ as functions of P in equation (2.31). We consider flows through the working channel and access sheath to be steady and unidirectional. We justify this separation of time-scales; i.e., ignoring time-dependent inertia in the Navier-Stokes equations whilst retaining time-dependent changes in kidney pressure in equation (2.31), by considering flow through the working channel (with a similar discussion for flow through the access sheath). Non-dimensionalising the time-dependent Navier-Stokes equations with (2.3) and arbitrary time-scale T , the key dimensionless parameters are ϵ^2 , $\epsilon^2 \text{Re}$, and $\epsilon^2 L^2 \rho / \mu T$. The latter parameter captures the ratio of time-dependent fluid inertia to viscous effects and we recall that $\epsilon = a/L \ll 1$ is the aspect ratio of the working channel. Here, the relevant time-scale for equation (2.31) is the rise-time for kidney pressure (or flow rate) to adjust to steady-state after changing the inlet pressure. This can be estimated from the *ex-vivo* data to be $T = \mathcal{O}(10^1 \text{ s})$. With values for a , L , ρ , and μ taken from Table 2.4, and $W = Q/\pi a^2$ with $Q = \mathcal{O}(10^0 \text{ cm}^3/\text{s})$ providing a typical velocity scaling, we determine $\epsilon^2 = \mathcal{O}(10^{-6})$, $\epsilon^2 \text{Re} = \mathcal{O}(10^{-1})$, and $a^2 \rho / \mu T = \mathcal{O}(10^{-2})$. Thus, inertial and time-dependent effects within the fluid can be neglected, and we model flow through the working channel and back through the access sheath as steady, unidirectional flow through pipes of arbitrary cross-sections.

Without the influence of tip deflection, (which was shown to be negligible in Section 2.2.4), expressions for flow through the working channel and back through the access

sheath are provided by the following formulae:

$$Q = \frac{1}{\mu} \left(\frac{P_{\text{in}} - P}{L} \right) \frac{1}{R}, \quad Q_{\text{sc}} = \frac{1}{\mu} \frac{P}{L_{\text{sc}}} \frac{1}{R_{\text{sc}}}, \quad (2.33\text{a,b})$$

where R is the resistance in through the working channel and R_{sc} the resistance out through the access sheath. The parameter R captures the size and position of working tools within the working channel, and R_{sc} the position of the scope shaft within the access sheath, as discussed in Section 2.2.5. Combining equations (2.31), (2.33a), and (2.33b) we obtain

$$\frac{dP}{dt} = (C/\mu)(P_0 + P) \left(\frac{P_{\text{in}} - P}{LR} - \frac{P}{L_{\text{sc}}R_{\text{sc}}} \right). \quad (2.34)$$

2.3.3 Nondimensionalisation and analytical solution

We now make the following non-dimensionalisations

$$P = P_0 \hat{P}, \quad t = (LR\mu/CP_0)\hat{t}, \quad (2.35)$$

where hats denote dimensionless variables. The dimensionless form of equation (2.34) is then

$$\frac{d\hat{P}}{d\hat{t}} = (1 + \hat{P})(\hat{P}_{\text{in}} - (1 + \mathcal{R})\hat{P}), \quad (2.36)$$

where we define the dimensionless parameters

$$\hat{P}_{\text{in}} = P_{\text{in}}/P_0, \quad \mathcal{R} = (LR)/(L_{\text{sc}}R_{\text{sc}}). \quad (2.37)$$

$\hat{P}_{\text{in}}$ is the dimensionless inlet pressure and $\mathcal{R}$ is the ratio of resistance in through the working channel to resistance out through the access sheath. The dimensionless initial condition, equation (2.32) is

$$\hat{P}(0) = 0. \quad (2.38)$$

Equation (2.36) with initial condition (2.38) can be solved by separating variables to obtain the analytical solution

$$\hat{P} = \frac{(e^{\iota\hat{t}} - 1)\hat{P}_{\text{in}}}{\hat{P}_{\text{in}} + e^{\iota\hat{t}}(1 + \mathcal{R})}, \quad (2.39)$$

where $\iota = 1 + \hat{P}_{\text{in}} + \mathcal{R}$. The steady-state intrarenal pressure can be easily found by setting the time derivative in equation (2.36) to zero. We note that the presence of $(1 + \hat{P})$ in equation (2.36) is due to the chosen exponential function for kidney compliance, $\frac{dP}{dV} = C(P + P_0)$, equation (2.28). Thus we can exclude the non-physical steady-state

solution $\hat{P} = -1$ which corresponds to zero compliance. The dimensional relevant steady-state solution is thus

$$P^* = \frac{P_{\text{in}}}{1 + \mathcal{R}}. \quad (2.40)$$

If $\mathcal{R} \ll 1$, and hence resistance through the sheath dominates, we see from equation (2.40) that $P^* \approx P_{\text{in}}$. Conversely if $\mathcal{R} \gg 1$, and there is negligible resistance through the sheath, $P^* \approx 0$. We note that the steady state is not affected by the inclusion of a nonlinear kidney compliance, equation (2.28), and thus the parameters P_0 and C determine only the time required to reach equilibrium pressure. The steady-state will be reached when $Q = Q_{\infty} = Q^*$, and the dimensional value of the steady-state flow rate can be immediately determined from equations (2.40) and one of either (2.33a) or (2.33b)

$$Q^* = \frac{1}{\mu} \left(\frac{P_{\text{in}}}{LR + L_{\infty}R_{\infty}} \right). \quad (2.41)$$

2.3.4 Model validation

We now turn to validating the steady-state model predictions for kidney pressure and flow rate, equations (2.40) and (2.41), with the results of the *ex-vivo* experiments, detailed in Section 2.3.1 and Appendix 2.B. There is no fitting required between the model and the data, as the steady state predictions are independent of kidney compliance.

As discussed in Section 2.2.5, R_{∞} will vary with the placement of the scope shaft within the access sheath, decreasing monotonically with distance between the centres of the scope shaft and sheath. As the position of the scope is unknown in practice, in Figure 2.10 we consider both the minimum and maximum values for R_{∞} , when the scope is fully offset and centred within the sheath, respectively (solid black lines), and shade the area between the two curves, corresponding to the predicted Q^* and P^* over the range of feasible R_{∞} values. The experimental flow-rate data for the three different kidneys, A, B, and C, as calculated from the outflow mass values over time, is shown to lie within the predicted range (Figure 2.10a).

In Figure 2.10b, we plot the predictions of equation (2.40) for the steady state intrarenal pressure, along with experimental data. We observe that although the measured pressures for kidneys B and C appear to lie within the predictive ‘wedge’, the data from kidney A falls systematically below the estimated range. One possible explanation for this discrepancy is the position of the pressure probe within the kidney in the experiments; although it was placed proximal to the end of the scope, its position was likely to vary. A limitation of the model presented in this chapter is that the intrarenal pressure is assumed constant. We will discuss the possibility of pressure variation within the kidney in Chapter 5, and will revisit the results of Figure 2.10b.

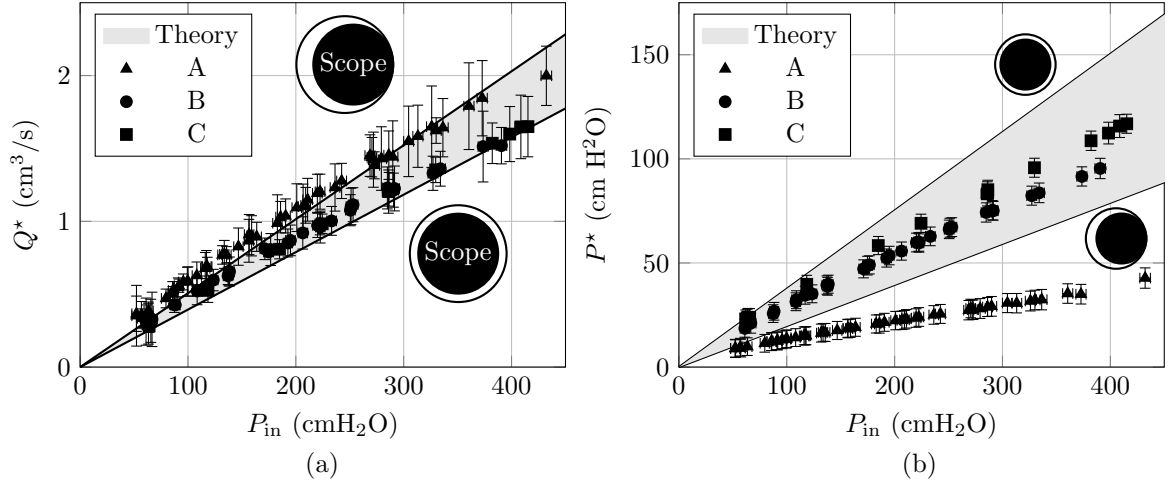


Figure 2.10: (a) Equation (2.41) as the black lines compared to data collected from three different kidneys labelled A, B, and C. (b) Equation (2.40). The resistance ratio corresponding to a centred scope is $\mathcal{R} = 1.7$, and to an offset scope is $\mathcal{R} = 4.1$.

The results in Figure 2.10 provide strong validation of the mathematical systems model through agreement between the predicted steady-states of the model and measured values.² The model solution, equation (2.39), can now be used to interrogate the dynamics of the system. Additionally, we turn to harnessing the model's predictive power by considering the effect of working tools of variable position through changes in R in the following section.

2.3.5 Effect of working tools and scope position

In Section 2.2.5, we determined that the effect of working tools, such as laser fibres or stone baskets, can be incorporated by increasing R according to the size, geometry, and placement of the working tool within the working channel. Hence, for fixed P_{in} , adding a working tool to the channel will increase R , thus decreasing P^* and Q^* by equations (2.40) and (2.41), respectively. Considering fixed inlet pressure $P_{in} = 60$ cm H₂O and a fully offset working tool and scope shaft $\phi = \phi_{\infty} = 1$, the predicted steady-state kidney pressure as a function of working tool radius (where we recall that the working channel has radius 0.06 cm) is plotted in Figure 2.11a as the dashed black line. As a minimum flow rate is often required to maintain a clear view-field during ureteroscopic surgery, the reduction in flow caused by working tools may motivate a manual increase in inlet pressure to maintain flow and subsequent visualisation. From equation (2.41), the inlet pressure required to maintain the steady-state flow rate at Q^* is $P_{in} = Q^* \mu (RL + R_{\infty} L_{\infty})$. The

²With the exception of the pressure measurements of kidney A.

corresponding steady-state kidney pressure, equation (2.40), will simply be

$$P^* = Q^* \mu L_{\mathcal{K}} R_{\mathcal{K}}, \quad (2.42)$$

which is independent of R . This indicates that if a fixed flow rate is required, the kidney pressure will be independent of the size of the working tools and their position within the working channel. Equation (2.42) is plotted as the constant solid black line in Figure 2.11a.

The influence of scope shaft position within the access sheath (variable $R_{\mathcal{K}}$) on both flow rate and kidney pressure for a range of inlet pressures is demonstrated by the thickness of the wedges in Figure 2.10. In Figure 2.11b we consider fixed $P_{\text{in}} = 60 \text{ cm H}_2\text{O}$ and plot the time-dependent kidney pressure, equation (2.39), for both a centred and fully offset scope shaft. Here we have taken R for an empty working channel (without a working tool). The rise-time to steady-state is shorter for the offset scope than for the centred scope, and again the decrease in renal pressure with a reduction in $R_{\mathcal{K}}$ is observed. Notably, the steady-state kidney pressure for the offset scope ($\phi_{\mathcal{K}} = 1$) is nearly 50% lower than for the centred scope ($\phi_{\mathcal{K}} = 0$) with corresponding resistance ratios $\mathcal{R} = 4.1$ and $\mathcal{R} = 1.7$, respectively.

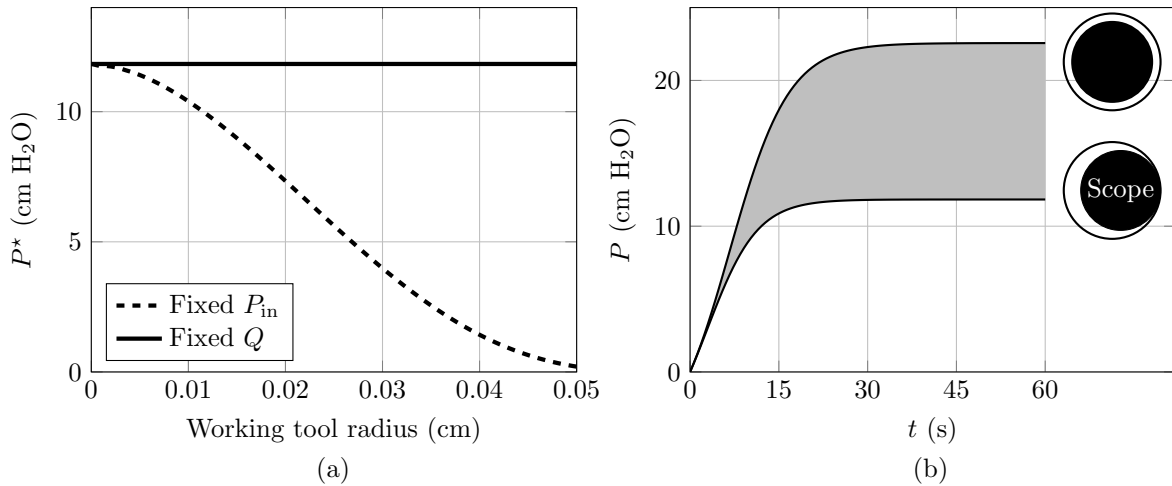


Figure 2.11: Figure (a) plots P^* for a fully offset working tool as a function of working tool radius. The value of $R_{\mathcal{K}}$ is taken to be for a fully offset scope shaft. Plot (b) shows the effect of $R_{\mathcal{K}}$ on P as a function of time; geometries corresponding to the two different values of $R_{\mathcal{K}}$ are indicated.

2.4 Summary and discussion

We commenced this chapter with a discussion of the simplest model of irrigation in ureteroscopy: flow through an isolated working channel. Even within this idealised frame-

work, however, complexities are to be considered, such as how bag height affects inlet pressure and subsequent flow rate, and how tip deflection and working tools may impede flux for a given pressure drop.

If we were to model flow within the irrigation tubing connecting the saline bag to the working channel as hydrostatic, we would predict flow to scale linearly with bag height. However, through comparison with bench-top experiments, we determined this to be an overestimation, and that viscous dissipation through the irrigation tubing leads to bag height increasing flow rate sublinearly.

To model flow through a deflected scope, we patched together classic results on Dean flow through the curved tip with a Poiseuille description through the straight portion of the shaft. We consequently obtained flow rate as a function of tip curvature, and found tip deflection to cause only a minimal (at most 5%) reduction in flow rate. This was confirmed with the results of bench-top experiments.

Under the assumption that working tools are coaxial with the working channel, we modelled their effect by considering flow through an annulus. As the position of the working tool within the channel is unknown, and flow resistance is maximum for a concentric annulus and minimum for one where the core is fully offset, we obtained a predictive range for flow rate as a function of tool size. The data from bench-top experiments fell within this range.

In this chapter we have only considered the effects of working tools and tip deflection separately; including both within our modelling framework would require consideration of slightly curved flow in an annular pipe (Karahalios & Petrakis, 1991), complicated by uncertainty in tool position. However, experimental results indicate a negligible influence of deflection, even with the addition of working tools (Bach *et al.*, 2008), so we choose not to include this in our study.

We then turned our discussion to consider a systems model, bolting together flow through the working channel with flow through the access sheath, connected by a pressure point representing a zero-dimensional kidney. This enabled predictions of kidney pressure as a function of flow resistance parameters and kidney compliance. The results of *ex-vivo* experiments motivated us to consider an exponential renal stiffness (described in Appendix 2.B.1), distinguishing our work from a previous model for ureteroscopy irrigation (Oratis *et al.*, 2018). The governing nonlinear ordinary differential equation can be solved analytically, with kidney pressure and flow rate as functions of time rising to steady-state values.

We also improve on the model by Oratis *et al.* (2018), by modelling the effects of variable resistance through the working channel and access sheath, depending on the addition of

working tools, and their position, as well as the placement of the scope shaft within the access sheath. An unknown scope position within the sheath incorporates a degree of uncertainty into the theoretical predictions even for an empty working channel, and thus we obtain a predicted range for flow rate and kidney pressure as functions of inlet pressure. The flow rate data from the *ex-vivo* experiments agreed with the steady-state theoretical prediction, but there was a deviation between the steady-state pressure estimation and the results from one kidney. We hypothesised that this could be attributed to pressure variation in the kidney, not accounted for in the model. This will be discussed in Chapter 5. We found that adding a working tool to the working channel decreases both the steady-state flow rate and kidney pressure for fixed inlet pressure. However, if a particular flow rate is sought during ureteroscopy, and the inlet pressure is adjusted accordingly to compensate for the presence of working tools, then the kidney pressure is unaffected by working tools and their size. Minimising resistance through the access sheath (here, considered by moving the scope shaft from the centre to the edge of the sheath) both increases flow and decreases kidney pressure for fixed inlet pressure.

Minimising both inflow resistance when a working tool is present, and outflow resistance through the access sheath containing the scope shaft, will maximise flow rate. While minimising inflow resistance also causes a concurrent, undesirable, increase in kidney pressure, minimising outflow resistance decreases kidney pressure. Thus, we speculate that decreasing both resistances simultaneously may improve visualisation while maintaining kidney pressures below a physically-determined safety threshold. This motivates the question of whether more optimal annular geometries can be achieved, as well as the question of where a solid rod is likely to sit inside a bounding channel through which there is an axially flowing viscous fluid. We focus on these questions in Chapter 3.

Appendix 2.A Bench-top experiments

The basic experimental set-up is depicted in Figure 2.1a. Fluid flows from a bucket (Figure 2.1b), through the LithoVue™ into a second bucket on a Kern PCB™ 6000-0 balance which records mass measurements at one-second intervals.

At the start of each experiment, the valve (pictured in Figure 2.1b) was turned to allow water to flow through the irrigation tubing and working channel, and mass measurements were recorded at one-second intervals. The water level in the bucket was maintained by continuously refilling to a marked height. Each experiment was performed for at least sixty seconds, with mass measurements taken at one second intervals, and the volumetric flow rate (in g/s) approximated from this data by taking the final mass measurement divided by the total time.³ Each data point was computed by performing the same experiment at least three times; the mean flux value is plotted in Figures 2.4, 2.6, and 2.8, with the vertical error for each data point taken to be the standard deviation of the three results, combined with the systematic error caused by the accuracy of the mass balance. The parameters involved in these experiments can be found in Table 2.2.

	Symbol	Value	Unit
Working channel radius	a	6.00×10^{-2}	cm
Working channel length	L	7.90×10^1	cm
Fluid density	ρ	1.00×10^0	g/cm ³
Fluid viscosity	μ	1.00×10^{-2}	g/cm s
Gravitational acceleration	g	9.81×10^2	cm/s ²
Flexible tip length	L_c	5.70×10^0	cm
Irrigation tubing radius	ℓ	1.25×10^{-1}	cm

Table 2.2: *Experimental parameters for the bench-top experiments.*

We consider three distinct experiments:

- I) a straight scope,
- II) a scope with a deflected tip, and
- III) a straight scope with a working tool inserted,

with specific details in the following sections.

³The flux through the working channel with a ZIPwire™ inserted (data point (f) in Figure 2.8) was calculated differently (see Section 2.A.5).

2.A.1 Bag height experiments

The experiments on the straight scope were performed at five different head heights: (a) 57 cm, (b) 67 cm, (c) 83 cm, (d) 98.5 cm, and (e) 112.5 cm. To adjust the head height, the bucket was raised and longer tubing was attached, so that the tubing remained vertical without slack for each experiment. Head heights were estimated using both a standard measuring tape and by analysing photos of the set-up, with a measurement error of ± 1 cm.

2.A.2 Deflected tip experiments

For the deflected scope experiments, the head height was set at 83 ± 1 cm, and experiments were performed for five different deflections. Images of the scope at various degrees of deflection are shown in Figure 2.2. The radius of curvature for each deflection was estimated by fitting a circle to a photo image of the deflected scope tip, which taken next to a standard ruler for scale. The circle fitting was performed three times, and the standard deviation of the three radius values was taken to be the error on the radius of curvature measurement.

2.A.3 Working tool experiments

For the working tool experiments, the head height was set at 83 ± 1 cm, and different Boston Scientific working tools – (b) OptiFlex™ basket, (c) Flexiva™ 200 laser fibre, (d) Flexiva™ 365 laser fibre, Zero Tip™ basket, and (f) ZIPwire™ – were inserted into the working channel before starting the experiment procedure. The radii of the tools are listed in Table 2.1, and the data points in Figure 2.8 correspond to the labelled lines in Figure 2.12c.

Compared with the case of no tool, the largest tool used, the ZIPwire™, caused a 93% reduction in flow; the Flexiva™ 200 and 365 laser fibres led to 60% and 72% decreases in flow, respectively; and the OptiFlex™ and Zero Tip™ baskets caused 56% and 79% reductions, respectively (Figure 2.8).

2.A.4 Data Analysis

The raw data from the experiments, obtained as mass measurements at one second intervals are displayed in Figure 2.12.

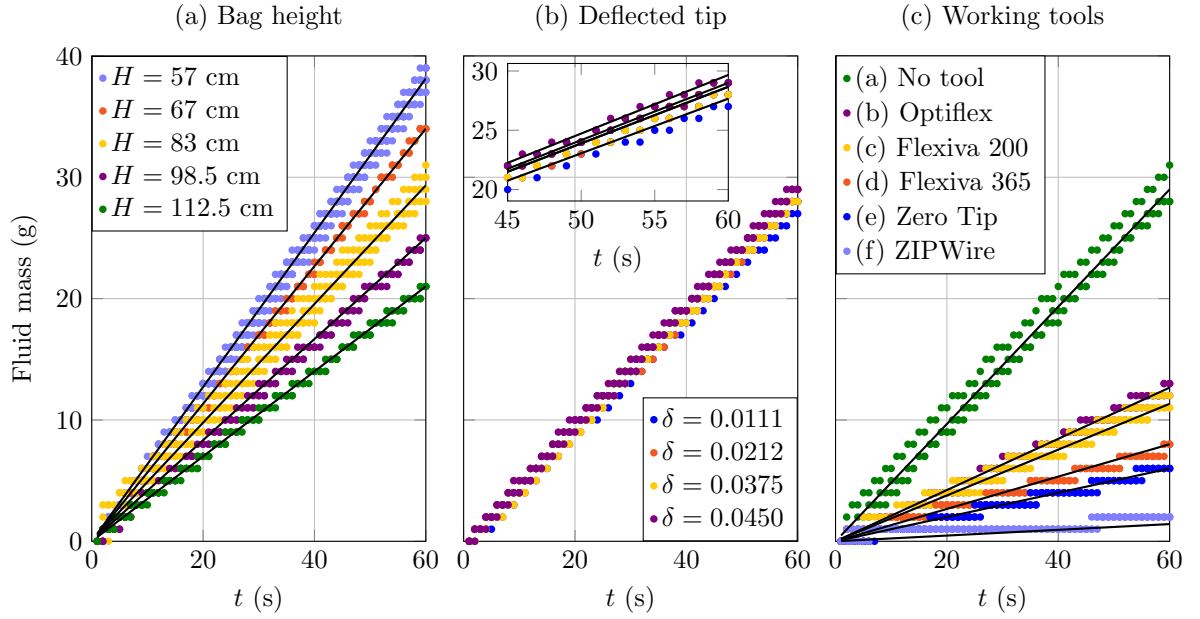


Figure 2.12: The data collected from the experiments. The vertical axis gives the mass of water in the collecting bucket in grams (measurement readability of 1 g), and the horizontal axis is time in units of seconds. Solid black lines have slope equal to the estimated flux for each data set. The slope lines would be indistinguishable in Figure (b), so are only plotted in the inset figure for a subset of the data points.

The flow rate was approximated from the data by taking the final mass measurement for each run and dividing it by the total time that the run was performed for. For the working tool and deflected scope experiments, the total time was always sixty seconds and three runs were performed for each working tool or radius of curvature, respectively. For the straight scope experiments at least three runs of at least sixty seconds each were performed at each height. The mean flow rates, Q , and standard deviations, ς , in the estimated flow rates were recorded for every experimental set-up.

2.A.5 Error estimation

The mass balance used was the Kern PCB™ 6000-0, which has a readability of 1 g. Therefore, the error in the final mass measurement was taken to be ± 0.5 g, and the error in the estimated flow rate for each run to be $\sqrt{2}(0.5 \text{ g})/T$, where T is the total time of the experiment run. The $\sqrt{2}$ appears as subtractions of two mass measurements is required for flow-rate estimation. Therefore, the systematic error in the mean flow rate is $\Delta = \sqrt{\sqrt{2} \sum_{i=1}^n (0.05 \text{ g}/T_i)^2 / n}$, where n is the number of runs for each configuration (ranging from three to fourteen). We assume that the standard deviation in the flow rates and the measurement errors due to the readability of the mass balance are independent

and thus our total error estimation is

$$T.E. = \sqrt{\zeta^2 + \Delta^2}, \quad (2.43)$$

which is taken as the vertical error bar on the experimental data in Figures 2.4, 2.6, and 2.8.

Data Point (*f*) in Figure 5

The raw data for the mass of fluid exiting the scope with a ZIPwire™ inserted is shown as the light-blue lines in Figure 2.12c. As the flow is so low, the mass only increases by 2 g over the sixty second interval. In this slow-flow region, we approximated flow rate by taking $Q = 1/T_{1,2}$ g/s, where $T_{1,2}$ is the time in seconds for the mass to increase from one to two grams.

Appendix 2.B Ex-vivo experiments

A suite of *ex-vivo* experiments were performed on porcine kidneys, with a laboratory configuration shown in Figure 2.9 and general experiment description in Section 2.3.1. The kidney and a dish to collect the outflow from the sheath were on OHAUS Traveler™ mass balances, and FISO Fiber Optic Pressure Sensors™ (calibrated to atmospheric pressure) recorded the inlet and intrarenal pressures.

Inflation of the pressure cuff surrounding the irrigation bag enabled control of the inlet pressure, and the resulting inlet pressure, kidney mass, kidney pressure, and outflow mass were all measured over time. Pressure data was collected at 8 ms intervals, and mass data collected at approximately 500 ms intervals.

Experiments were performed on three different kidneys (A, B and C) which were drained prior to the experiments, so the initial renal pressure before the insertion of irrigation fluid was approximately atmospheric. The inlet pressure was increased over time in two ways: I) continuously (by inflating the pressure cuff at a constant rate), and II) step-wise, maintaining the pressure of the cuff at arbitrary values for approximately 30 seconds before rapid further inflation. The number of data sets I and II collected from each kidney are summarised in Table 2.3, and the inlet pressure for all data sets are plotted over time in Figures 2.13a,b.

The data from set I enabled extraction of a continuous curve (parametrised by time) relating fluid mass and pressure within the kidney. The data from set II was used to test

	A	B	C
Data set I	1	0	3
Data set II	4	2	1

Table 2.3: Number of experiment runs on each kidney, for each type of data set.

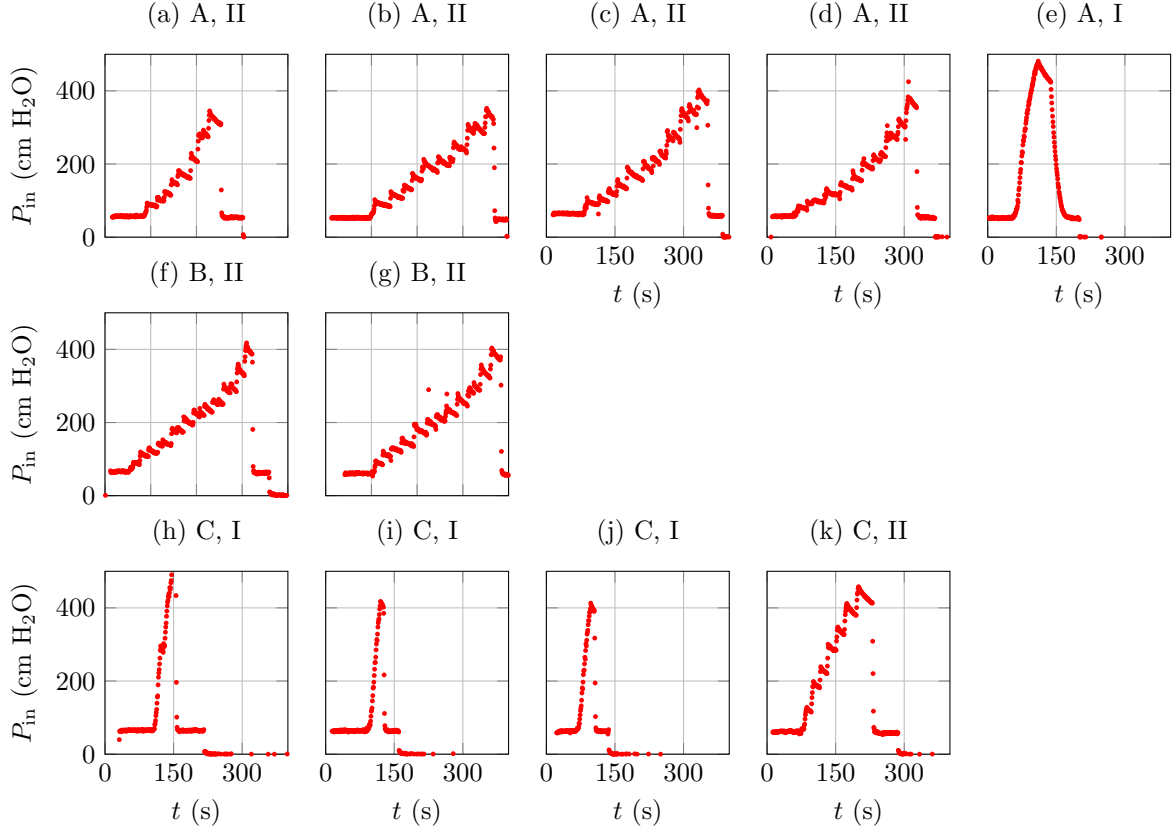


Figure 2.13: Inlet pressure (P_{in}) over time. All vertical axes go from 0-500 cm H₂O and horizontal axes from 0-400 s. Pressure data is recorded at 8 ms intervals, but plotted here at 800 ms intervals for visualisation purposes.

the steady-state flow rate and kidney pressure predictions of the model as functions of inlet pressure. As the inlet pressure was held constant at particular values we assumed the kidney pressure and flow rate had equilibrated to steady-state. We then extracted five-seconds of data from each interval of constant inlet pressure, which we used to determine the steady-state kidney pressure and flow rate as functions of inlet pressure.⁴ We were also able to extract two steady-state data points from the set I data. For example, in Figure 2.13e we extracted five-second intervals starting at $t = 40$ s and $t = 130$ s. As the

⁴Due to deflation of the pressure cuff, the inlet pressure slowly decreased at each ‘step’ (Figure 2.13). However, the time-scale was sufficiently slow that the inlet pressure could be well-approximated as constant over each five-second interval.

steady-state predictions are independent of compliance parameters (2.29) (these only set the time required to reach steady-state), no model fitting is required for the data validation in Figures 2.10.

Experimental parameters are given in Table 2.4.

	Symbol	Value	Unit
Working channel radius	a	6.00×10^{-2}	cm
Scope shaft radius	a_{scope}	1.58×10^{-1}	cm
Access sheath radius	b_{scope}	1.93×10^{-1}	cm
Working channel length	L	7.90×10^1	cm
Access sheath length	L_{scope}	3.60×10^1	cm
Fluid density	ρ	1.00×10^0	g/cm ³
Fluid viscosity	μ	1.00×10^{-2}	g/cm s

Table 2.4: *Experimental parameters for the ex-vivo experiments.*

2.B.1 Compliance experiments

It is well-established that biological tissue displays properties of viscoelastic solids (Humphrey, 2003). Thus, stress depends not only on the applied strain, but on the rate of strain. This leads to viscoelastic features such as hysteresis (difference in loading and unloading stress-strain curves), and stress relaxation, where if a material is suddenly strained and the strain is maintained constant, the stress in the material will decrease over time (Gregersen, 1996). Additionally, with repeated loading and unloading cycles, the load-deformation curves will shift to the right, and the hysteretic effects will diminish. By repeated cycling, eventually a steady-state is reached at which no further change will occur unless the cycling routine is changed. In this state the tissue is said to be preconditioned (Kauer, 2001). These effects are seen in the curves of intrarenal pressure, plotted against mass for the continuous loading data sets. Figure 2.14a shows data from both the increasing and decreasing P_{in} times in Figures 2.13 (e), (h), (i), and (j). The initial mass reading for the first run is subtracted from the data sets, and the data interpolated so the mass and pressure readings have the same time step and can be plotted against each other. By looking at the plots of inlet pressure over time (Figure 2.13), we see that Figures (h)-(j) will lend themselves best to tissue preconditioning. This is reflected in Figure 2.14a, where we see that the corresponding curves shift to the right, and curves (i) and (j) appear to be nearly indistinguishable.

We observe that all loading curves fit the exponential trend of equation (2.28), shown in Figure 2.14b by the solid red lines of best fit, although the values of P_0 and C will differ. This further validates the use of an exponential kidney compliance, where P_0 and C may depend on the kidney, loading rate, and any previous loading.

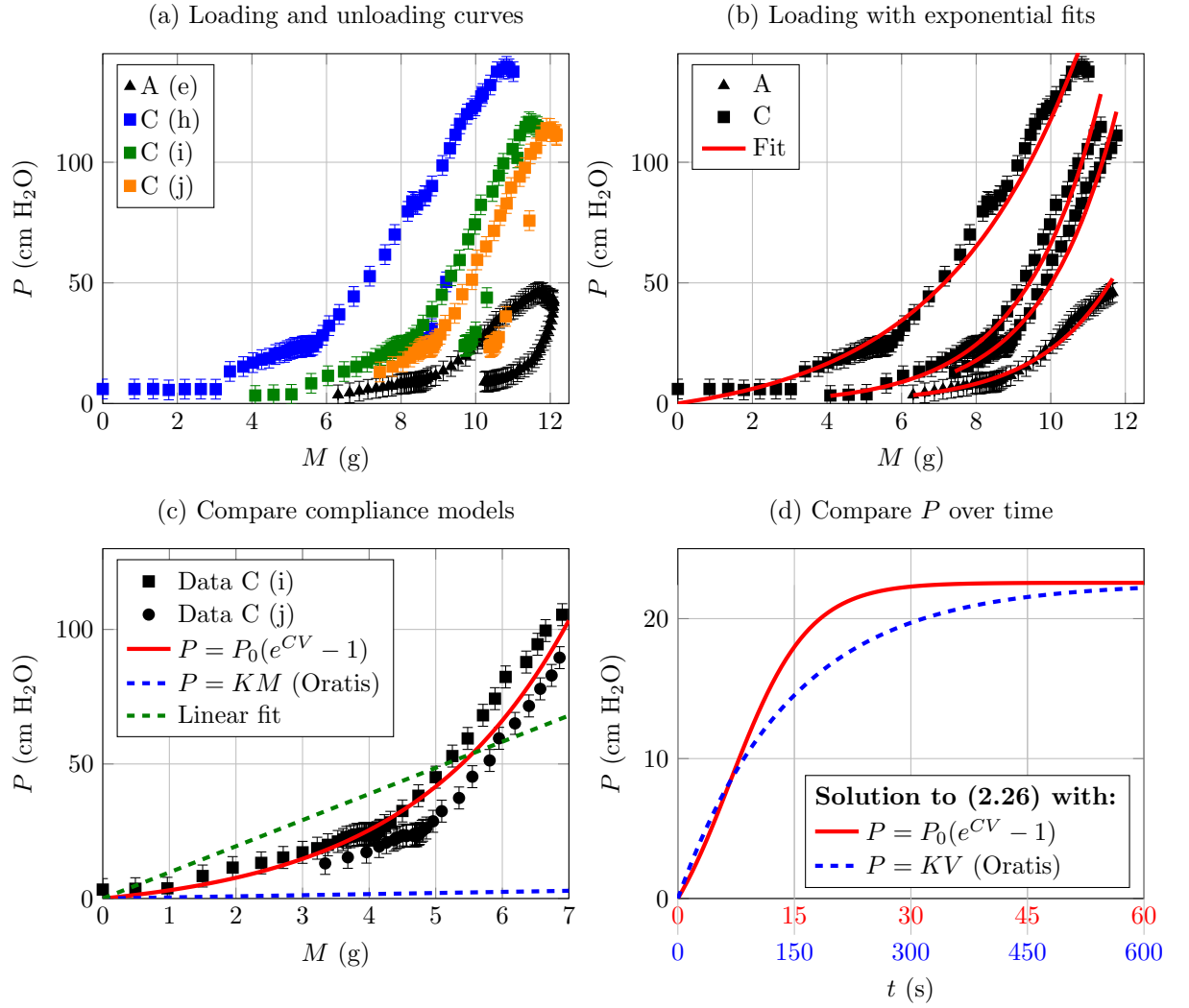


Figure 2.14: All continuous-loading data sets. (a) P is plotted against M for the loading and unloading curves for (e), (h), (i), and (j) in Figure 2.13. (b) The loading section of the curve is plotted, with red lines providing exponential fits. (c) Data from kidney C, runs (i) and (j), with exponential fit (red line), linear fit (dashed green line), and linear model from Oratis et al. (2018). (d) A plot of the predicted $P(t)$ – solution to equation (2.26) – for an exponential compliance (red line) against the model solution with a linear compliance (Oratis et al., 2018) (dashed blue line). The flow resistance values R and R_{∞} are taken as those for an empty working channel and a scope centred within the access sheath, respectively. The two sets of horizontal axis labels indicate the different time-scales for the lines of corresponding colours.

To obtain the particular parameters in equation (2.29) utilised for the full-system results in Figure 2.11b, we use the loading curves of the ‘preconditioned’ data sets (i) and (j). These two are shown, after subtracting the initial mass value from curve (i), in Figure 2.14c, with the fit with parameters (2.29) shown in red. This is compared to a linear fit to the experimental data (dashed green line). In Oratis *et al.* (2018), a linear renal stiffness relation was assumed, $P = KV$, with mean value $K = 0.41$ cm H₂O mL⁻¹. This is plotted

as the dashed blue line in Figure 2.14c, where we use $M = \rho V$ from equation (2.27). The shallow slope models the kidney as significantly more compliant than the data suggests.

For the red line in Figure 2.14d, we consider a fixed inlet pressure of $P_{\text{in}} = 60 \text{ cm H}_2\text{O}$, and plot equation (2.39) with values for P_0 and C from equations (2.29). The value for R_{sc} is taken for a scope sitting concentrically in the access sheath. This is compared against the analytical solution to

$$\frac{dP}{dt} = (K/\mu) \left(\frac{P_{\text{in}} - P}{LR} - \frac{P}{L_{\text{sc}}R_{\text{sc}}} \right), \quad (2.44)$$

with the same values for P_{in} , R , and R_{sc} , plotted as the dashed blue line. Equation (2.44) is the model used by Oratis *et al.* (2018), derived from equations (2.26) and (2.27), assuming a linear renal stiffness relation $P = KV$ with $K = 0.41 \text{ cm H}_2\text{O mL}^{-1}$. The different time-scales for equation (2.39) and the solution to equation (2.44) are indicated by the corresponding colours of the horizontal axis labels in Figure 2.14d. Thus, we note that not only do the solution curves in Figure 2.14d have slightly different shapes, but the time scales required to reach steady-state differ significantly. This result is intuitive from the two different compliance relationships, as a stiffer kidney (red line in Figure 2.14c) will allow the system to reach steady-state more quickly than a more compliant kidney (dashed blue line in Figure 2.14c). Oratis *et al.* (2018) predicted a characteristic rise time of the order of several minutes, which does not correlate with experimental and clinical observations of $\mathcal{O}(10^1 \text{ s})$. Thus, including a stiffer exponential compliance, rather than a linear compliant relationship between renal pressure and volume within the modelling framework, may allow for more accurate predictions of the time required to reach steady-state intrarenal pressure for a given inlet pressure.

2.B.2 Steady-state experiments

The data in set II, with inlet pressure shown in Figures 2.13 (a)-(d), (f), (g), (j), and (k), is used to estimate steady-state flow rate and pressure data. Multiple five-second intervals, where the inlet pressure is approximately constant, are extracted from each data set. The flow rate as a function of time is estimated by taking the difference between successive outflow mass measurements, divided by the time step. A plot of the mean flow rate and mean kidney pressure against the mean inlet pressure over each five-second interval provides the data points in Figures 2.10a and 2.10b, respectively. The error bars in Figures 2.10a,b combine the standard deviation and the systematic measurement error, assuming the two sources to be independent. The error bars in Figures 2.14a-c show the systematic pressure measurement error.

2.B.3 Data analysis

‘Steady-state’ data was approximated by taking five-second intervals when the inlet pressure, and kidney pressure were approximately constant. The start of the five second intervals for each data run are given in Table 2.5. Flow rate data is taken as the difference

Experiment	Start of 5-second ‘steady’ intervals
(a)	75, 107, 123, 138, 153, 180, 197, 209, 220, 245
(b)	85, 135, 162, 182, 203, 238, 260, 289, 303, 317, 341, 360
(c)	70, 105, 127, 147, 169, 204, 225, 241, 255, 271, 285, 305, 323, 345
(d)	50, 80, 93, 119, 153, 180, 201, 223, 240, 253, 279, 297, 321
(e)	40, 130
(f)	50, 70, 89, 109, 120, 139, 151, 165, 187, 201, 211, 227, 242, 252, 267, 281, 299, 316
(g)	90, 118, 137, 159, 187, 215, 235, 255, 281, 304, 331, 353, 377
(h)	95, 122, 150
(i)	75, 121
(j)	55, 99
(k)	60, 91, 110, 125, 141, 163, 190, 225

Table 2.5: *Start of steady intervals.*

in successive outflow mass measurements divided by the difference in time. The mean flow rates and mean kidney pressures, along with their standard deviations, are recorded for each five-second interval.

2.B.4 Error estimation

The mass balance used was the OHAUS Traveler™, which has a readability of 0.01 g. Therefore, the error in mass measurements was taken to be ± 0.005 g. As flow rate is estimated by subtracting successive mass measurements, the total systematic error in mean flow rate was $\Delta = \sqrt{2}(0.005 \text{ g})/5 \text{ s}$. A FISO Fiber Optic Pressure Sensor™ FOP-M260-21 was used to record pressure measurements, and has a readability of 0.01 mmHg. Therefore, the systematic error in pressure measurements was taken to be $\Delta = \pm 0.005$ mmHg. Conversion between mmHg and H₂O requires multiplication by a factor of approximately 1.36.

As with the *ex-vivo* experiments, we assumed that the standard deviation in the measurements and the systematic errors are independent and apply formula (2.43) for the total error. This is taken as the vertical error bar on the experimental data in Figures 2.10.

CHAPTER 3

Resistance to ellipticity

Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

(Isaac Newton)

Synopsis

Motivated by flow through the working channel containing a working tool and through the access sheath containing the scope shaft, this chapter considers the significant role of cross-sectional geometry on resistance in co-axial pipe flows. We consider an axially flowing viscous fluid in between two long and thin coaxial cylinders, one inside the other. The outer cylinder is stationary, while the inner cylinder (or rod) is free to move. The inner rod thus poses a resistance to the axial flow, while the viscous fluid poses a resistance to any motion of the inner rod. The objective of this chapter is to characterise the nature of these resistances and how they depend on the cross-sectional geometry. We consider three particular cases: I) a circle inside an ellipse, II) an ellipse inside a circle, and III) an ellipse inside an ellipse of equal eccentricity and orientation. We show that the equations for flow in the axial direction – driven by a prescribed flux – and for flow within the cross-section of the domain – driven by the motion of the rod – decouple in the asymptotic limit of small cylinder aspect ratio. The governing fluid equations reduce to axial Poiseuille flow and transverse Stokes flow, respectively. We reveal interesting effects of elliptical eccentricity on both axial flow and Stokes resistances. We also determine trajectories of the inner rod after an initial instantaneous velocity.

3.1 Introduction

Fluid flows in annular geometries between a co-axial rod and outer cylinder motivate fundamental design questions, such as:

1. How to position the inner cylindrical rod to maximise axial flow?
2. How will the inner rod move and rotate if free to do so?
3. How do the answers to 1 and 2 depend on the cylinders' cross-sectional shapes?

Our interest in this area is motivated by flow through the working channel containing a working tool, or flow back through the access sheath containing the scope shaft. Assuming the instruments remain coaxial, both these scenarios reduce to flow through an annulus. We determined, in Chapter 2, that minimising the flow resistance posed by working tools in the working channel increases flow rate, and that minimising flow resistance through the access sheath increases flow rate and decreases kidney pressure.

Thus, as high flow rate and renal pressures below a biophysical threshold are desirable to enhance visualisation and lower the risk of post-operative complications, respectively, there is potential to improve ureteroscopy by minimising flow resistance through annular geometries. We address questions of flow optimality in the sense of achieving the maximum flow rate for a prescribed axial pressure drop and fixed cross-sectional area available for fluid flow. We consider the possibility of elliptical cross-sections for the access sheath, scope shaft, working channel, and working tools. This is motivated by the ureteral opening resembling an ellipse (Bergman, 1981) and channels with elliptical cross-sections being relatively easy to manufacture.

When designing an optimal device the ideal scenario would be to place the inner cylindrical rod in the position that minimises axial flow resistance. A key consideration is whether the surrounding fluid would then resist changes to the inner rod position, or whether the device must be designed to constrain the inner rod to the position that minimises axial flow resistance.

Our modelling framework comprises a solid cylindrical rod of mass m and effective radius b , rotating about, and moving in a direction perpendicular to, its longitudinal axis, within a coaxial outer cylinder of equal length, L , and comparable radius, a , filled with an axially flowing viscous fluid of typical flow rate Q , density ρ , and dynamic viscosity, μ . We take the cross-sections of the rod and outer cylinder to be ellipses of varying eccentricities; this is both appropriate for our uretero-rensoscopy application and allows us to investigate the effects of non-axisymmetry.

	Symbol	Value	Unit
Scope in sheath	$a_{\times}$	1.9×10^{-1}	cm
	$b_{\times}$	1.6×10^{-1}	cm
	$L_{\times}$	3.6×10^1	cm
	$m_{\times}$	6.1×10^0	g
Tool in working channel	a	6.0×10^{-2}	cm
	b	2.2×10^{-2}	cm
	L	7.9×10^1	cm
	m	1.1×10^0	g

Table 3.1: Summary of typical parameter values for ureteroscopy irrigation. Table of typical parameter values for ureteroscopy irrigation: $a_{\times}$ is the inner radius of an access sheath (11/13 F NavigatorTM); $b_{\times}$ and $m_{\times}$ are the radius and mass of a scope shaft (LithoVueTM); b and m are the radius and mass of a working tool (ZeroTipTM). Mass values are estimated by measuring the mass-per-unit length and multiplying by the relevant length.

Defining the radius-to-length ratio of the rod as $\epsilon = b/L$ and the Reynolds number of the flow as $\text{Re} = QL\rho/\mu a^2$, the key parameters governing the behaviour of the fluid and the inner rod are ϵ^2 , $\epsilon^2\text{Re}$, and the ratio of fluid-to-rod mass multiplied by the reduced Reynolds number, $\varrho^{-1} = (m/\rho a^2 L)\epsilon^2\text{Re}$. Using the summarised parameter values in Table 3.1, we find $\epsilon^2 = \mathcal{O}(10^{-5} - 10^{-6})$, $\epsilon^2\text{Re} = \mathcal{O}(10^{-1})$, and $\varrho^{-1} = (10^0)$. These suggest that fluid inertia is negligible while rod inertia is not. In this regime, we show that the axial fluid flow through the annular region and the cross-sectional flows induced by the perpendicular movement and rotation of the rod can be decoupled into axial Poiseuille flow and Stokes flow in the cross-section. The motion of the rod is determined by conservation of linear and angular momentum, incorporating the hydrodynamic resistance exerted by the surrounding viscous fluid. Within this set-up, our objective is to investigate the nature of the resistance, both the cross-sectional resistance to the motion of the inner rod and resistance to the axial flow, in terms of the geometry of the two cylinders. To illustrate characteristic features of the flow and to investigate geometric effects, we focus primarily on three reduced parameter spaces, namely:

- I) a circular rod inside an elliptical cylinder,
- II) an elliptical rod inside a circular cylinder, and
- III) an elliptical rod inside an elliptical cylinder of the same eccentricity and orientation.

We also briefly consider the generic ellipse in an ellipse case when determining the optimal geometry to minimise axial flow resistance. The literature on viscous-dominated fluid flows through annular pipes is vast. A description of relevant literature is given in the following section and summarised in Table 3.2. As our primary concern is the effect of cross-sectional geometry on resistance, we have grouped references by geometry in Table 3.2, and have further distinguished which previous works have studied axial versus cross-sectional flow.

3.2 Comprehensive literature review

3.2.1 Axial flow resistance

An analytical solution exists for the steady, fully-developed flow of a viscous fluid through an annular region formed by concentric circles (Lamb, 1916), equation (2.25). As discussed in Chapter 2, by employing conformal maps, analytical solutions have also been obtained in a domain bounded by non-concentric circles (Piercy *et al.*, 1933; Heyda, 1959; Sastry, 1964; Shivakumar & Chuanxiang, 1993; MacDonald, 1982; Redberger, 1962; Ebrahim *et al.*, 2013; Snyder & Goldstein, 1965), which demonstrate that axial flow resistance decreases with increasing distance between the centres of the bounding circles. Analytical solutions for steady, viscous flow through annular ducts bounded by confocal ellipses (Piercy *et al.*, 1933; Sastry, 1964), and externally by an ellipse and internally by a circle (with coincident centres) have also been determined by employing conformal maps (Sastry, 1964), although the dependence of the flow rate on the geometry of the domain was not discussed. Shivakumar & Chuanxiang (1993) also considered a region bounded internally by a circle and externally by a concentric ellipse and noted a flux enhancement compared to cross-sections bounded by two concentric circles or two confocal ellipses of the same cross-sectional area. Flow through an annular region bounded by non-concentric ellipses or by an ellipse in a circle, which form key components of our analysis, have not been previously considered (see Table 3.2).

3.2.2 Cross-sectional flow resistance

Two-dimensional Stokes flow between two circular cylinders has been well-studied (Jeffrey, 1922; Jeffrey & Onishi, 1981; Frazer, 1926; Chwang & Wu, 1975; Wannier, 1950; Slezkin, 1955). The dynamics of a viscous fluid confined in the gap between rotating cylinders, i.e. Taylor-Couette flow, has many mechanical applications, e.g. to the lubrication of rotary bearing systems. Stokes flow due to a line rotlet (a rotating circular cylinder of infinitesimal radius) inside an elliptic cylinder was solved analytically by Hackborn (1991). The results focussed on the resulting flow structure and it was found that the number of eddies in the cross-section produced by the line rotlet increased approximately linearly with the ratio of length to width of the outer elliptical cross-section. Hackborn (1991) postulated that the flow features generated by a line rotlet inside a fixed elliptic cylinder are expected to persist when the line rotlet is replaced by a rotating circular cylinder. Stokes flow between rotating confocal ellipses has also been considered (Saatdjian *et al.*, 1994), and an analytical solution for the stream function obtained using elliptical cylindrical coordinates. It was shown that for counter-rotating ellipses, two hyperbolic points appear in the flow.

Fixing the gap between the rod and the cylinder, the domain approaches one of a rod between parallel plates as the elliptical eccentricity of the outer cylinder cross-section tends to one. The motion of a rod of circular cross-section rotating and translating in Stokes flow between parallel plates has been studied numerically (Dvinsky & Popel, 1987). The authors computed the position between the centreline and the wall where the rod experienced the minimum translational drag. It was also found that the torque on a cylinder rotating between parallel plates is minimised when the cylinder is centred between the two walls. An asymptotic solution (for small gap between the cylinder and the walls compared to the cylinder radius) for a circular cylinder rotating between parallel plates was obtained by Yang *et al.* (2013). It was shown that if the cylinder is centred between the two plates, rotation will only induce an opposing torque, whereas if the cylinder is offset from the centreline, there is an additional force parallel to the walls.

Two-dimensional Stokes flow in a bounded annular domain with a translating and rotating inner rod has also been considered. Finn & Cox (2001) presented an analytical solution for the stream function for such a flow when both the inner rod and outer cylinder have circular cross-sections. The biharmonic equation for the stream function was solved with complex variable methods. The motion of the cylindrical rod was prescribed as a function of time, and the energy required to maintain the system in equilibrium was determined. It was shown that the power input depends upon the position of the rod and the prescribed motion. As the two cylinders approach each other, the power input required to maintain all motions diverges. Cox & Finn (2007) considered multiple rods with elliptical cross-sections moving inside a circular cylinder via numerical methods. In both Finn & Cox (2001) and Cox & Finn (2007) the fluid flow is quasi-static, with temporal variation in the velocity field only occurring due to changes in domain geometry.

While the above works demonstrate the strong effect of geometry on rod motion, to our knowledge the effect of geometry on an elliptical cylindrical rod translating and rotating inside an elliptical cylinder has not been previously investigated.

Due to the linearity of Stokes equations, the velocity field resulting from any prescribed translational motion (in any bounded or unbounded two or three-dimensional domain) can be calculated by considering component motions directed along orthogonal axes and summing the component solutions. The velocity field resulting from prescribed rotational motion in these domains can also be considered separately, and for combined translational and rotational motions the solution is obtained by adding the relative contributions. By implementing the Lorentz reciprocal theorem, it can be shown that the magnitudes of the hydrodynamic forces and torques on a particle moving in Stokes flow vary linearly with the imposed translational and rotational velocities, respectively, and these relationships can be captured by the coefficients of two symmetric *resistance tensors* (Brenner, 1962c;

1963; Hinch, 1972). An additional tensor characterises the interactions between translation and rotation that can occur when particle or domain symmetry is broken (Brenner, 1963; Hinch, 1972). The general theory for the effect of finite domain boundaries on the resistance tensors has been considered for the case where the particle is small in comparison to its distance from the boundary (Brenner, 1962*a*; *b*). In two-dimensions, the three resistance tensors can be formulated as a single *resistance matrix*. To illustrate this, consider a Cartesian coordinate system with orthogonal directions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. For a given prescribed translational velocity, $\frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j}$, and a prescribed angular velocity $\frac{d\theta}{dt} \mathbf{k}$, the resistance matrix provides the hydrodynamic forces $F_x \mathbf{i}$, $F_y \mathbf{j}$ and hydrodynamic torque $\tau_z \mathbf{k}$ via

$$\begin{bmatrix} F_x \\ F_y \\ \tau_z \end{bmatrix} = - \begin{bmatrix} K_{xx} & K_{xy} & C_x \\ K_{xy} & K_{yy} & C_y \\ C_x & C_y & A_{zz} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{d\theta}{dt} \end{bmatrix} \quad (3.1)$$

The scalar matrix coefficients K_{ij} for $i, j = x, y$ in equation (3.1) characterise the resistive force in the i -direction due to motion in the j -direction, and equivalently, as the matrix is symmetric, the resistance in the j -direction due to motion in the i -direction. Coefficient A_{zz} provides the linear relationship between a rotational motion in the cross-section and the resistive torque. Finally, coefficients C_x and C_y describe the coupling between translational and rotational motions, i.e., the rotation induced by translation (and vice versa). The coefficients are all functions of the geometry of the domain (and scale linearly with viscosity).

When considering translation without rotation, equation (3.1) reduces to

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = - \begin{bmatrix} K_{xx} & K_{xy} \\ K_{xy} & K_{yy} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}. \quad (3.2)$$

The eigenvectors and corresponding eigenvalues of the matrix in equation (3.2) are the directions and magnitudes of the minimum and maximum resistance, respectively. Using principles of energy dissipation, it can be shown that the eigenvalues must be positive. This is proved for a three-dimensional particle of arbitrary shape moving in Stokes flow in Brenner (1962*c*).

As a simple example of the resistance matrix for two dimensional flows, consider the case of concentric circles of inner radius b and outer radius a . Stokes equations can be solved analytically in this geometry for prescribed translational and rotational velocities of the rod (Slezkin, 1955, pp. 135, 163). The resulting resistance matrix components, K_{xx} ,

K_{yy} non-dimensionalised by μ and A_{zz} by μb^2 , are

$$K_{xy} = C_x = C_y = 0, \quad (3.3a,b)$$

$$K_{xx} = K_{yy} = 4\pi(1 + r^2)/(1 - r^2 + (1 + r^2) \log r), \quad (3.3a,b)$$

$$A_{zz} = 4\pi/(1 - (1/r)^2), \quad (3.3a,b)$$

where $r = a/b$. Due to the symmetry of the domain there is no coupling between translation and rotation. Moreover, the two eigenvalues of the matrix in equation (3.2) are given by $K_{xx} = K_{yy}$, i.e. there is equivalent translational resistance in all directions. Both the translational and rotational resistances decrease monotonically as the ratio r increases. A resistance matrix of the form given in equation (3.1) has been used to describe the motion of a cylindrical rod of circular cross-section translating and rotating in Stokes flow between parallel plates (Dvinsky & Popel, 1987). More recently, resistance matrices have been used to understand the interactions between swimming micro-organisms (Ishikawa *et al.*, 2006) and to control the movement of particles in suspension (Btait *et al.*, 2019).

3.2.3 The ‘most stable’ rod position – touching the wall?

Although somewhat beyond the scope of this chapter, for the cross-sectional flow problem it is important to mention a complexity that arises if we are to consider the limiting behaviour that occurs when the rod touches the bounding cylinder. At this point, an interesting paradox arises: the resistance to motion in all directions tends to infinity; i.e., it will require infinite force to move the rod in any direction, including away from, or along, the contacting boundary (Jeffrey & Onishi, 1981). To address this paradoxical behaviour, it is worthwhile to consider the problem of a rod rolling or sliding along a plane wall in Stokes flow, as this will approximate the behaviour of a moving rod touching a bounding cylinder close to the point of contact. Even in this simplified scenario, theory detects a pathologic problem with solutions to the incompressible Stokes equations at the contact point, indicating the necessity for fluid compressibility or cavitation to limit the pressure drop across the point of contact to a physically acceptable value. This results in an infinite lift force, opening an interstice between the wall and the cylinder; the fluid dynamics through this narrow gap can be studied through lubrication analysis (Merlen & Frankiewicz, 2011). The creation of cavitation bubbles for a rotating cylinder near a proximal wall has been confirmed experimentally (Seddon & Mullin, 2006), and these findings indicate a physical necessity for a small gap between a moving rod and bounding cylinder. Thus, in Section 3.5.3, where we discuss rod positions that maximise cross-sectional resistance, we limit to configurations where the rod is located a finite distance

from the bounding cylinder. This also prevents divergence of numerical code, as mesh elements will become degenerate at the point of contact. Within this framework, we define a ‘most stable’ configuration as one that maximises the minimum eigenvalue of the Stokes resistance matrix. It is important to note that in the classical sense of linear stability, every rod position within the domain is stable, as imposed rod motion will always be dissipated by the hydrodynamic forces of the surrounding fluid.

3.2.4 Chapter outline

This chapter is organised as follows. In Section 3.3 we describe the model set-up. In the regime in which both the aspect ratio of the cylinders and the reduced Reynolds number are small, we show that at leading order the axial and cross-sectional flows can be decoupled into Poiseuille flow and Stokes flow, respectively. In Section 3.4 we solve the axial flow equations and compute the flux for a given pressure drop as a function of rod position and the cross-sectional shapes of the rod and bounding cylinder, with an aim to determine configurations that minimise axial flow resistance. We will first consider the three special cases: I) a circular rod inside an elliptical cylinder, II) an elliptical rod inside a circular cylinder, and III) an elliptical rod inside an elliptical cylinder of the same eccentricity and orientation, before extending, in Section 3.4.4, our discussion of optimal axial flow to the full geometric parameter space of an elliptical rod in an elliptical cylinder with fixed area available for the fluid. In Section 3.5 we solve the cross-sectional Stokes flow equations and calculate the forces and torque exerted on the rod by the surrounding viscous fluid when the rod undergoes a prescribed motion. We calculate the resistance matrix coefficients as functions of cross-sectional geometry, focussing on configurations I, II, and III. For ellipses in ellipses of the same eccentricity and orientation, we find that the position of the rod that maximises the axial flow rate is also the ‘most stable’ position. In Section 3.6 we turn to rod trajectories of an elliptical rod inside an elliptical cylinder of the same eccentricity, starting from a position where the axes are aligned, assuming that the motion of the rod is subject to an initial instantaneous prescribed velocity in the direction of minimum resistance. As the rod moves through the domain, we allow the relative orientation of the two ellipses to change and we also compute the corresponding axial flow rate, assuming a fixed axial pressure-drop. We show that the axial flow rate increases as a function of time, which demonstrates that the axial flow rate for ellipses in ellipses might increase naturally if the inner ellipse is free to move. We conclude in Section 3.7.

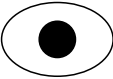
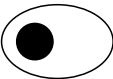
Description	Example Geometry	Poiseuille flow	Stokes flow (rotation)	Stokes flow (translation)
Centred circle in circle		Lamb (1916)	Lamb (1916)	Frazer (1926); Chwang & Wu (1975)
Offset circle in circle		Piercy <i>et al.</i> (1933); Heyda (1959); Redberger (1962)*; Ebrahim <i>et al.</i> (2013)*; Snyder & Goldstein (1965)	Jeffrey (1922); Jeffrey & Onishi (1981); Wannier (1950); Slezkin (1955); Finn & Cox (2001)	Finn & Cox (2001); Secomb & El-Kareh (1994)
Fully offset circle in circle		MacDonald (1982)	N/A	N/A
Centred circle in ellipse		Sastry (1964); Shivakumar (1973)	Hackborn (1991) (rod of infinitesimal radius)	
Offset circle in ellipse			Hackborn (1991) (rod of infinitesimal radius)	
Centred ellipse in circle				
Centred ellipse in ellipse		Piercy <i>et al.</i> (1933); Sastry (1964) (confocal ellipses)	Saatdjian <i>et al.</i> (1994) (confocal ellipses)	
Offset ellipse in circle			Cox & Finn (2007)* (including multiple ellipse case)	Cox & Finn (2007)* (including multiple ellipse case)
Offset ellipse in ellipse				
Circle between parallel plates		N/A	Dvinsky & Popel (1987)*; Yang <i>et al.</i> (2013)	Dvinsky & Popel (1987)*

Table 3.2: Summary of existing literature for relevant geometries. References marked with an asterisk are primarily numerical solutions; all others contain significant analytical results.

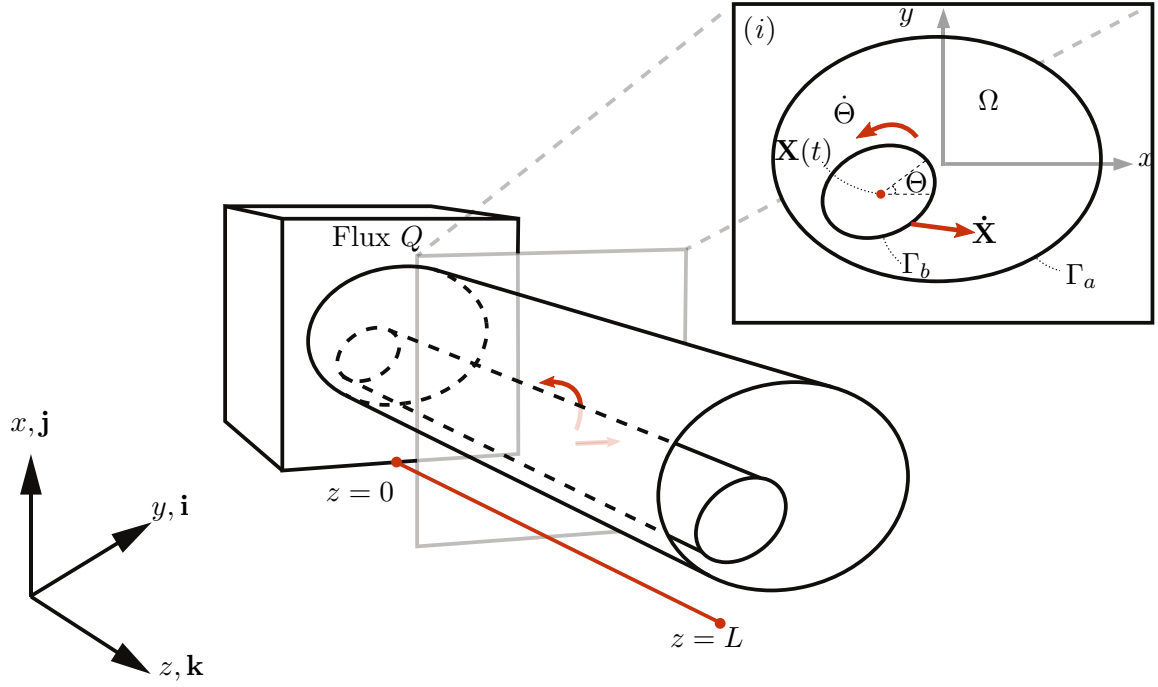


Figure 3.1: A schematic of the set-up in a Cartesian coordinate system (x, y, z) with corresponding coordinate directions $\mathbf{i}, \mathbf{j}, \mathbf{k}$, where $\mathbf{k}$ is oriented along the common axis of the cylinders of length L . At $z = 0$ the flow is driven by a flux, Q . The boundaries of a cross-sectional slice of the cylinders are denoted Γ_b (inner) and Γ_a (outer), and the fluid-filled area between them is Ω . The inset figure shows a cross-sectional slice, where the position and orientation of the rod are given by $\mathbf{X}$ and Θ , respectively. The rod has translational velocity $\dot{\mathbf{X}}$ and angular velocity $\dot{\Theta}$.

3.3 Theoretical formulation

We consider an inner cylindrical rod which is free to move within a fluid-filled outer cylinder. The rod is prescribed a small, instantaneous, translational motion in a direction perpendicular to, and a rotation about, its longitudinal axis. The subsequent motion of the rod is driven by the hydrodynamic forces exerted on the rod by the fluid. As the rod moves, we assume that it remains coaxial with the outer cylinder. While the rod is able to move in the direction perpendicular to its longitudinal axis, we assume that it does not move axially. It is also able to rotate about its longitudinal axis. Both the translational and angular velocities of the inner rod are constant along its length. Both cylinders are of length L and have uniform elliptical cross-sections. The space between the inner and outer cylinders is filled with an incompressible, Newtonian fluid of viscosity μ and density ρ . Flow is driven by both an applied non-zero constant axial flux and the motion of the cylindrical rod. We adopt a Cartesian coordinate system (x, y, z) with corresponding coordinate directions $\mathbf{i}, \mathbf{j}, \mathbf{k}$, where $\mathbf{k}$ is oriented along the common axis of the cylinders and $z = 0$ is at the entrance to the annular region (See Figure 3.1). We take

the characteristic radii of the cylinders to be

$$b = \sqrt{A_b/\pi}, \quad a = \sqrt{A_a/\pi}, \quad (3.4a,b)$$

where A_b and A_a are the respective cross-sectional areas. We assume that b and a are comparable and much smaller than L . We denote the annular boundaries of a cross-sectional slice of the cylinders as Γ_b and Γ_a , respectively. The coordinate position of the geometric centre of the inner rod is given by $(X(t), Y(t), L/2)$, where X and Y are functions of time, and thus the position of the inner rod within the cross-section is $\mathbf{X} = (X(t), Y(t))$. The orientation of the major axis of the rod's cross-section, with respect to the x -axis, is given by $\Theta(t)$, also a function of time (Figure 3.1).

3.3.1 Dimensionless system

As motivated in Section 3.1, we are interested in a regime where fluid inertia is negligible. Thus, the fluid flow is governed by the Stokes and continuity equations. We define the aspect ratio of the rod to be $\epsilon = b/L \ll 1$ and non-dimensionalise the axial coordinate by L and the cross-sectional coordinates by $b = \epsilon L$. We assume a non-zero constant flux Q is applied at $z = 0$ and thus, non-dimensionalise the axial velocity by characteristic velocity scale $W = Q/b^2$. We choose the pressure scaling $P = W\mu/(L\epsilon^2)$ to balance the viscous terms in the Stokes equations and the cross-sectional force and torque are scaled by $\epsilon L\mu W$ and $\epsilon^2 L^2 \mu W$, respectively. The dimensionless Stokes equations are given in component form by

$$-p_x + \epsilon^2(u_{xx} + u_{yy}) + \epsilon^4 u_{zz} = 0, \quad (3.5a)$$

$$-p_y + \epsilon^2(v_{xx} + v_{yy}) + \epsilon^4 v_{zz} = 0, \quad (3.5b)$$

$$-p_z + w_{xx} + w_{yy} + \epsilon^2 w_{zz} = 0, \quad (3.5c)$$

$$u_x + v_y + w_z = 0, \quad (3.5d)$$

where we have adopted the subscript notation for partial derivatives. Due to the scaling for velocity, the dimensionless flux scales to

$$\iint_{\Omega} w \, d\Omega = 1, \quad (3.6)$$

where Ω is the two-dimensional region of the cross-section between the cylinders.

No-slip conditions at the surfaces of the inner and outer cylinder give the dimensionless boundary conditions

$$\left. \begin{aligned} u &= \dot{X} - \dot{\Theta}(y - Y), \\ v &= \dot{Y} + \dot{\Theta}(x - X), \\ w &= 0, \end{aligned} \right\} \quad \text{on } \Gamma_b \quad (3.7\text{a-c})$$

and

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma_a. \quad (3.8)$$

To calculate $\dot{\mathbf{X}}$ and $\dot{\Theta}$, where dots denote derivatives with respect to time, we consider conservation of linear and angular momentum for the inner rod which gives, in dimensionless form

$$\ddot{\mathbf{X}} = \varrho \mathbf{F}(\mathbf{X}, \Theta, \dot{\mathbf{X}}, \dot{\Theta}), \quad \ddot{\Theta} = \varrho' \tau(\mathbf{X}, \Theta, \dot{\mathbf{X}}, \dot{\Theta}), \quad (3.9\text{a,b})$$

where $\mathbf{F}$ and τ are the hydrodynamic force in the cross-section and z -component of the torque exerted on the rod due to the motion of the fluid, obtained by integrating contributions from the viscous pressure forces. Because the cylinders remain coaxial, τ and $\mathbf{F}$ are the only non-zero force and torque components. The dimensionless constants in equations (3.9a,b) are defined as

$$\varrho = \mu L^2 / mW, \quad \varrho' = b^2 \mu L^2 / IW, \quad (3.10\text{a,b})$$

where m and I are the rod mass and the moment of inertia about the rod axis, respectively, and ϱ' is the same size as ϱ as I scales with $b^2 m$. We note that ϱ can be rearranged to relate the fluid density to the material density of the inner rod multiplied by the reduced Reynolds number. As discussed in the introduction, we will consider the regime where this parameter is $\mathcal{O}(1)$. As dimensionless initial conditions for equations (3.9) we prescribe

$$\mathbf{X}(0) = \mathcal{X}, \quad \Theta(0) = \Phi, \quad \dot{\mathbf{X}}(0) = U\boldsymbol{\ell}, \quad \dot{\Theta}(0) = \omega, \quad (3.11\text{a-d})$$

where $\mathcal{X}$ is the initial location of the rod's axis in the (x, y) plane, Φ is its initial orientation angle, U is its initial velocity in a direction given by $\boldsymbol{\ell}$, a unit vector in the (x, y) -plane, and ω is its initial angular velocity about the z -axis. We assume $\mathcal{X}$, U , Φ , and ω are all $\mathcal{O}(1)$; due to the chosen non-dimensionalisations, this requires the initial imposed rod velocities to be ϵ times smaller than the axial velocity.

3.3.2 Asymptotic analysis

We take $\epsilon^2 \ll 1$ and seek expansions to our dimensionless variables of the forms

$$\mathbf{u} = \mathbf{u}_0 + \epsilon^2 \mathbf{u}_1 + \cdots, \quad p = p_0 + \epsilon^2 p_1 + \cdots \quad (3.12a,b)$$

$$\mathbf{X} = \mathbf{X}_0 + \epsilon^2 \mathbf{X}_1 + \cdots, \quad \Theta = \Theta_0 + \epsilon^2 \Theta_1 + \cdots. \quad (3.12c,d)$$

Inserting (3.12a,b) into equations (3.5c), (3.5d) and equating $\mathcal{O}(1)$ terms gives

$$\nabla_{\perp}^2 w_0 = \frac{dp_0}{dz}, \quad \nabla_{\perp} \cdot \mathbf{u}_0^{\perp} + w_{0z} = 0, \quad (3.13a,b)$$

where $\nabla_{\perp} = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j}$ and $\mathbf{u}_0^{\perp} = u_0 \mathbf{i} + v_0 \mathbf{j}$. To determine the cross-sectional velocity components, u_0 and v_0 , we consider equations (3.5a) and (3.5b) at $\mathcal{O}(\epsilon^2)$, and obtain the two-dimensional Stokes equations

$$\nabla_{\perp}^2 u_0 = p_{1x}, \quad \nabla_{\perp}^2 v_0 = p_{1y}. \quad (3.14a,b)$$

The leading-order flux condition is

$$\iint_{\Omega} w_0 \, d\Omega = 1, \quad (3.15)$$

and leading-order boundary conditions are

$$\left. \begin{aligned} u_0 &= \dot{X}_0 - \dot{\Theta}_0(y - Y_0), \\ v_0 &= \dot{Y}_0 + \dot{\Theta}_0(x - X_0), \\ w_0 &= 0, \end{aligned} \right\} \quad \text{on } \Gamma_b \quad (3.16a-c)$$

and

$$\mathbf{u}_0^{\perp} = \mathbf{0} \quad \text{and} \quad w_0 = 0, \quad \text{on } \Gamma_a. \quad (3.17a,b)$$

The equations of motion for the rod, (3.9), read at leading order

$$\ddot{\mathbf{X}}_0 = \varrho \mathbf{F}_0(\mathbf{X}_0, \Theta_0, \dot{\mathbf{X}}_0, \dot{\Theta}_0), \quad \ddot{\Theta}_0 = \varrho' \tau_0(\mathbf{X}_0, \Theta_0, \dot{\mathbf{X}}_0, \dot{\Theta}_0), \quad (3.18a,b)$$

where

$$\mathbf{F}_0 = \int_0^1 \left[\oint_{\Gamma_b} \boldsymbol{\sigma}_0 \mathbf{n} \, ds \right] dz, \quad (3.19a)$$

$$\tau_0 = \int_0^1 \left[\oint_{\Gamma_b} (x - X_0)(\boldsymbol{\sigma}_0 \mathbf{n}) \cdot \mathbf{j} - (y - Y_0)(\boldsymbol{\sigma}_0 \mathbf{n}) \cdot \mathbf{i} \, ds \right] dz, \quad (3.19b)$$

for

$$\boldsymbol{\sigma}_0 = \left[-(p_0/\epsilon^2 + p_1)\mathbf{I} + \nabla \mathbf{u}_0^\perp + (\nabla \mathbf{u}_0^\perp)^T \right]. \quad (3.20)$$

The leading-order initial conditions for the rod motion are

$$\mathbf{X}_0(0) = \boldsymbol{\mathcal{X}}, \quad \Theta_0(0) = \Phi, \quad \dot{\mathbf{X}}(0) = \mathbf{U}, \quad \dot{\Theta} = \omega. \quad (3.21)$$

We seek a separable solution for w_0 of the form

$$w_0 = f(x, y) \frac{dp_0}{dz}. \quad (3.22)$$

Integrating equation (3.13b) over a cross-section of the fluid domain between the two cylinders, using equations (3.16) and (3.17), and applying the divergence theorem gives

$$\iint_{\Omega} w_{0z} d\Omega = 0. \quad (3.23)$$

Inserting the form (3.22) and using the imposed flux condition (3.6), we conclude that

$$\frac{d^2 p_0}{dz^2} = 0 \rightarrow \frac{dp_0}{dz} = \text{const.}, \quad (3.24)$$

and it follows that

$$w_{0z} = 0. \quad (3.25)$$

Hence equation (3.13b) becomes

$$\nabla_\perp \cdot \mathbf{u}_0^\perp = 0, \quad (3.26)$$

and the governing equation for the axial fluid flow, (3.13a), is

$$\nabla_\perp^2 f(x, y) = 1, \quad (3.27)$$

with no-slip boundary conditions, equations (3.16c) and (3.17b),

$$f = 0, \text{ on } \Gamma_b, \Gamma_a. \quad (3.28)$$

Additionally, using the flux condition (3.6) we can solve for the constant pressure gradient

$$\frac{dp_0}{dz} = \left[\iint_{\Omega} f(x, y) d\Omega \right]^{-1}, \quad (3.29)$$

and thus

$$w_0 = f(x, y) \left[\iint_{\Omega} f(x, y) d\Omega \right]^{-1}, \quad (3.30)$$

combining (3.22) and (3.29). As p_0 is constant within a cross-section it has no contribution to the leading-order hydrodynamic force and torque given by equations (3.19). Additionally, the terms in square brackets in equations (3.19) are independent of z , and hence equations (3.19) and (3.20) to calculate the force and torque reduce to

$$\mathbf{F}_0 = \oint_{\Gamma_b} \boldsymbol{\sigma}_0 \mathbf{n} ds, \quad (3.31a)$$

$$\tau_0 = \oint_{\Gamma_b} (x - X_0)(\boldsymbol{\sigma}_0 \mathbf{n}) \cdot \mathbf{j} - (y - Y_0)(\boldsymbol{\sigma}_0 \mathbf{n}) \cdot \mathbf{i} ds, \quad (3.31b)$$

for

$$\boldsymbol{\sigma}_0 = \left[-p_1 \mathbf{I} + \nabla \mathbf{u}_0^\perp + (\nabla \mathbf{u}_0^\perp)^T \right]. \quad (3.32)$$

Henceforth, we will drop leading-order subscripts. We will retain the subscript on the only first-order term that appears in our leading-order system, p_1 , to differentiate the first-order pressure that drives flow in the cross-section, equations (3.14), from the leading-order pressure that drives the axial flow, equation (3.13a).

3.3.3 Model summary and computational approach

Equations (3.27), (3.28), and (3.30) give rise to Poiseuille-flow, where the axial flow resistance, $R(\mathbf{X}, \Theta)$, is equal to the constant pressure gradient required to maintain the axial flow through the annular domain at unit flux. This is given by

$$R(\mathbf{X}, \Theta) = \left[\iint_{\Omega} f(x, y) d\Omega \right]^{-1}, \quad (3.33)$$

a function of domain geometry. We note that for any given flux,

$$Q = \frac{1}{R(\mathbf{X}, \Theta)} \frac{dp}{dz}, \quad (3.34)$$

and so, once $R(\mathbf{X}, \Theta)$ is determined via equation (3.33), we can compute $Q(\mathbf{X}, \Theta)$ for any specified pressure gradient. Motivated by applications to ureteroscopy, in which it is most natural to consider optimising flow rate for a given pressure drop, our approach in Section 3.4 will be to fix $\frac{dp}{dz} = -1$ as a model input and compute $Q(\mathbf{X}, \Theta)$ and the associated flow profile as primary outputs. In this view, minimising the resistance is equivalent to maximising the flux for fixed pressure drop.

The equations governing the cross-sectional flow are the Stokes equations (3.14) and incompressibility (3.26), with no-slip boundary conditions (3.16a,b) and (3.17a). As discussed in Section 3.1, the forces and torque on the inner rod, equations (3.31a,b), are linearly related to the imposed velocities via resistance coefficients

$$\mathbf{F}(\mathbf{X}, \Theta, \dot{\mathbf{X}}, \dot{\Theta}) = -[\mathbf{K}(\mathbf{X}, \Theta)\dot{\mathbf{X}} + \mathbf{C}(\mathbf{X}, \Theta)\dot{\Theta}], \quad (3.35a)$$

$$\tau(\mathbf{X}, \Theta, \dot{\mathbf{X}}, \dot{\Theta}) = -[A_{zz}(\mathbf{X}, \Theta)\dot{\Theta} + C_x(\mathbf{X}, \Theta)\dot{X} + C_y(\mathbf{X}, \Theta)\dot{Y}], \quad (3.35b)$$

where $\mathbf{K}(\mathbf{X}, \Theta)$ is the two dimensional translation matrix and $\mathbf{C}(\mathbf{X}, \Theta)$ describes the coupling between translation and rotation

$$\mathbf{K}(\mathbf{X}, \Theta) = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{xy} & K_{yy} \end{bmatrix}, \quad \mathbf{C}(\mathbf{X}, \Theta) = \begin{bmatrix} C_x \\ C_y \end{bmatrix}. \quad (3.36a,b)$$

The resistance coefficients in equations (3.36a,b) are also functions of domain geometry. We will compute these in Section 3.5.

To explore the effect of geometry on both the axial and cross-sectional resistances we consider configurations I, II, and III, as introduced in Section 3.1. In each configuration, we will fix the cross-sectional areas of the cylinders to constrain the space available for fluid. We do this by setting the characteristic radius of the outer cylinder (scaled by b) which we denote a . This reduces the parameter space, allowing for full interrogation of the axial and transverse flows on $\mathbf{X}$, Θ , and the shape of the elliptical cross-sections of the inner and outer cylinders. The governing equations for the fluid flow (3.27), (3.14), and (3.26) are quasi-static, with time entering only via the time-dependence of the rod position, $\mathbf{X} = (X, Y)$ and orientation Θ . In Sections 3.4 and 3.5 we restrict to static calculations, computing the axial resistance for a fixed position of inner rod, and the instantaneous resistance to cross-sectional motion. In Section 3.6 we then compute rod trajectories by solving the equations of motion (3.18) while (quasi-statically) updating the cross-sectional fluid velocities. In this approach we can also monitor changes to the associated instantaneous axial flow.

3.4 Axial flow

We begin by considering the effect of cross-sectional geometry on axial flow by solving equation (3.27) subject to conditions (3.28) for varying ellipse cross-sectional geometries. Results were calculated numerically using an open-source finite element library, `oomph-lib` (Heil & Hazel, 2006). Details on the numerical elements and mesh are provided in Section 3.A. In Section 3.4.3, when we consider the cylinders' cross-sections to be of the same eccentricity and orientation, we also utilise a bipolar coordinate transformation to map the annular geometry to a rectangular domain and solve the transformed equations and boundary conditions using a finite difference method. This alternative approach enabled us to validate the finite element solutions (discussed in more detail in Appendix 3.D).

Our objective is to explore how the flux is affected by the position and orientation of the inner rod, $\mathbf{X}$ and Θ , respectively, and the elliptical eccentricity of the cross-sections. The eccentricity is defined as

$$e_{a,b} = \sqrt{1 - (\beta_{a,b}/\alpha_{a,b})^2}, \quad (3.37)$$

where $\alpha_{a,b}$ and $\beta_{a,b}$ are the major and minor axes of the inner and outer cylinder, respectively. A zero eccentricity value therefore corresponds to a circle and $e = 1$ is a slit of zero width and infinite length. For fixed characteristic radii, see equations (3.4), and given eccentricity, $e_{a,b}$, we can calculate $\alpha_{a,b}$ and $\beta_{a,b}$

$$\alpha_b = (1 - e_b^2)^{-1/4}, \quad \beta_b = (1 - e_b^2)^{1/4}, \quad (3.38a,b)$$

$$\alpha_a = a(1 - e_a^2)^{-1/4}, \quad \beta_a = a(1 - e_a^2)^{1/4}, \quad (3.38c,d)$$

3.4.1 I) Circle in ellipse

In Figure 3.2 we consider a circular rod (of dimensionless unit radius) inside a cylinder of elliptical cross-section (with characteristic radius $a = 2$). We vary the eccentricity of the outer cylinder's cross-section while maintaining the cross-sectional area, using equations (3.38) to determine the corresponding lengths of the major and minor axes.

When, for each value of e_a , the rod is located at the position that maximises flux (see Appendix 3.C), we find that the flux initially increases with e_a . The maximum flux over all e_a (data point *ii*) is nearly 50% higher than the flux for a circular outer cylinder of the same cross-sectional area (data point *i*). The eccentricity at which the maximum flux is achieved is $e_a \approx 0.84$ for $a = 2$. We might hypothesise that this coincides with the

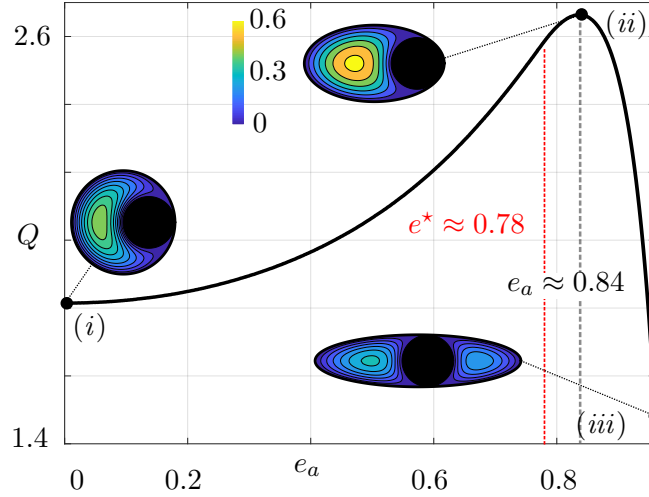


Figure 3.2: Dimensionless flow rate, Q , as a function of e_a , for a circle of unit radius inside an ellipse of characteristic radius $a = 2$. The eccentricity is changed while maintaining the cross-sectional area. For each e_a the inner circle is located at the position that maximises the axial flux (see Appendix 3.C). When $e = e^*$, the ellipse matches curvature with the circle at the ellipse vertex. The colourbars provide the magnitude of the axial velocity.

value for e_a where the rod and bounding cylinder match curvature at the cylinder's vertex, which we denote e^* (Appendix 3.C). However, for $a = 2$, $e^* \approx 0.78$, which is less than the e_a value where maximum flux is achieved. In fact, in configuration *ii* in Figure 3.2, although impossible to discern from the colourmap, the circular rod tangentially touches the elliptic cylinder in two locations. The colourbar in Figure 3.2 gives the magnitude of the axial velocity, and demonstrates that the maximum velocity within the cross-section is also larger in configuration *ii*, than in either *i* or *iii*.

3.4.2 II) Ellipse in circle

When considering an elliptical inner rod and circular outer rod ($e_a = 0$), we can, due to the rotational symmetry of the outer domain, fix $\mathbf{X} = (X, 0)$ and vary the inner rod's position along the x -axis only, without loss of generality. Here, we fix $a = 3$, a larger cross-section to allow for a wider range of feasible positions and orientations for the inner rod, and vary X , Θ , and e_b . We find, as anticipated, that Q increases with X for all values of e_b and Θ (Figure 3.3a-d). Increasing e_b can either increase or decrease Q depending on the position and orientation of the inner rod. For example, when the rod is centred in the outer cylinder – and the flux is independent of Θ due to the geometry of the domain – the largest flux is obtained when the rod is circular, and decreases with e_b . This effect can be seen for $e_b = 0$, $e_b = 0.7$, and $e_i = 0.9$ oriented at $\Theta = \pi/2$ in the inset plot in Figure 3.3d. In contrast, when the rod is sufficiently offset, the more eccentric rods at $\Theta = \pi/2$

cause less obstruction near the centre of the channel, allowing for higher flow (see Figure 3.3d).

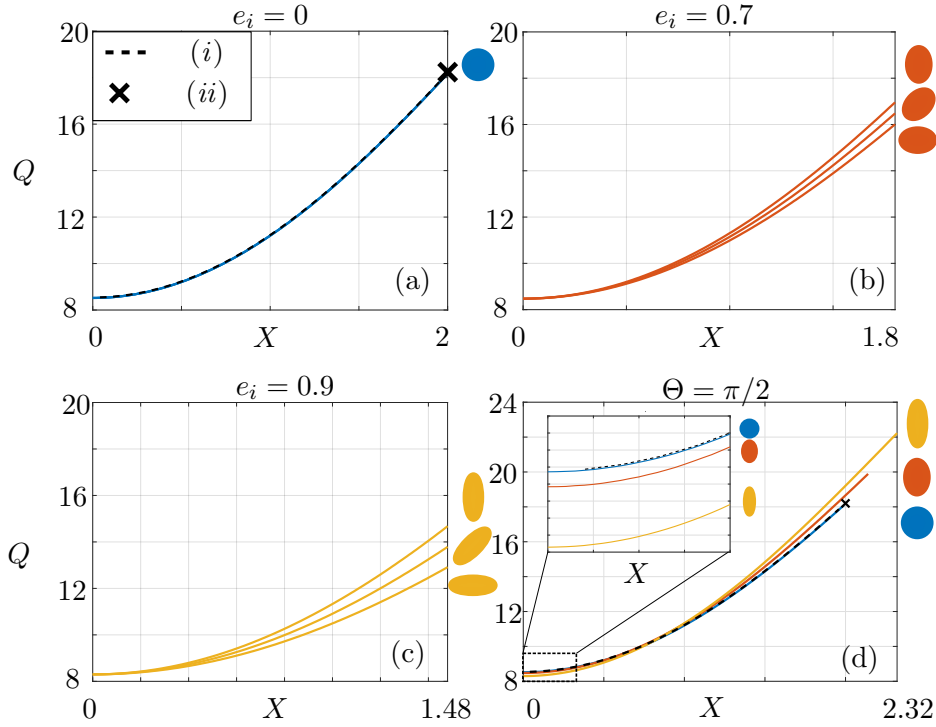


Figure 3.3: Dimensionless flow rate, Q , as a function of X for an ellipse of characteristic radius of $a = 3$. We vary the eccentricity of the inner ellipse (a) $e_b = 0$, (b) $e_b = 0.7$, (c) $e_b = 0.9$. In plots (a)-(c), three orientations $\Theta = 0, \pi/4$, and $\pi/2$ are presented. (d) plots Q as a function of X for $e_b = 0$ (blue) $e_b = 0.7$ (red) $e_b = 0.9$ (yellow) oriented at $\pi/2$. The inset plot provides a zoomed-in view from $X = 0$ to $X = 0.2$. In Figures (a) and (d), the solutions of (i) Piercy *et al.* (1933) and (ii) MacDonald (1982) are included (see Table 3.2 for more details).

For a circular rod ($e_b = 0$), we validate our numerical solution against the analytical solution for offset circles (Piercy *et al.*, 1933)¹ (dashed black lines in Figures 3.3a,d) in addition to confirming that our numerical solution approaches the limiting case of touching circles (MacDonald, 1982) as X approaches 2 (black cross in Figures 3.3a,d).

In Sections 3.4.1 and 3.4.2, we have considered the particular cases where one of the cylinders has a circular cross-section. As anticipated, in both configurations, the maximum flux was achieved for a rod tangent to the outer boundary. Perhaps unexpectedly, Figure 3.2 demonstrated that the optimal configuration may not be one where the inner rod is tangent at a single point. We will explore this in more detail in Section 3.4.4, where we will determine the true optimal geometry that maximises axial flux (for fixed area), relaxing all previous geometric restrictions. However, we will first consider another reduced

¹As Piercy *et al.* (1933) presents a solution for the flow rate in terms of an infinite sum, we truncate this sum to determine the flow rate (see Appendix 3.B.1).

parameter space, namely elliptical rods in elliptical cylinders of the same eccentricity and orientation; a configuration that allows for clear specification of the inner rod's position, and a discussion of the effects of eccentricity as a single parameter.

3.4.3 III) Ellipse in ellipse

We now consider cylinders with elliptical cross-sections of the same eccentricities $e = e_a = e_b$ and orientation ($\Theta = 0$). We fix $a = 2$ and again seek configurations that maximise the flux.² Under the constraint that the inner and outer cross-sections have the same elliptical eccentricity and orientation, we can map a domain of non-concentric ellipses to non-concentric circles and formulate the problem in a bipolar coordinate system (see Appendix 3.D).

To describe the position of the inner ellipse relative to the outer ellipse we first define θ to be the angle between the major axis of the outer ellipse and a line connecting the centres of the two ellipses, see Figure 3.4. The effective radii of the inner and outer ellipses are then defined as

$$r_b = \alpha_b \beta_b (\alpha_b^2 \sin^2 \theta + \beta_b^2 \cos^2 \theta)^{-1/2}, \quad r_a = \alpha_a \beta_a (\alpha_a^2 \sin^2 \theta + \beta_a^2 \cos^2 \theta)^{-1/2}. \quad (3.39)$$

With these definitions, the maximum distance, $d_{\max}$ between the centres at an angle θ (when the rod and the outer cylinder touch) is given by

$$d_{\max} = r_a - r_b. \quad (3.40)$$

From this we define an offset parameter, ϕ , analogous to the parameter introduced in Section 2.2.5 for circular geometries, that is the ratio of the distance d between the centres of the two ellipses at an angle θ and $d_{\max}$,

$$\phi = d/d_{\max}. \quad (3.41)$$

Thus, the domain is characterised by three parameters; the eccentricity, $e \in [0, 1]$, the angle of offset, $\theta \in [0, \pi/2]$, and the relative offset, $\phi \in [0, 1]$ (see Figure 3.4). We note that when using a bipolar coordinate system, outlined in Appendix 3.D, $\phi = 0$ and $\phi = 1$ (when the rod is perfectly centred and when it touches the outer boundary, respectively) are not possible. This is purely an artefact of this mathematical technique, and solving the equations with finite elements enables us to consider all configurations, including touching

²It is worthwhile to recall, when comparing flux values between different geometries, that in this section and in Section 3.4.4, $a = 2$, so the space available for fluid flow is the same as in Figure 3.2 (a circular rod in an elliptical cylinder) but less than in Figure 3.3 (an elliptical rod in a circular cylinder) where $a = 3$.

boundaries. However, to use the method in Appendix 3.D for numerical validation, we restrict to the range $\phi \in [0.01, 0.99]$.

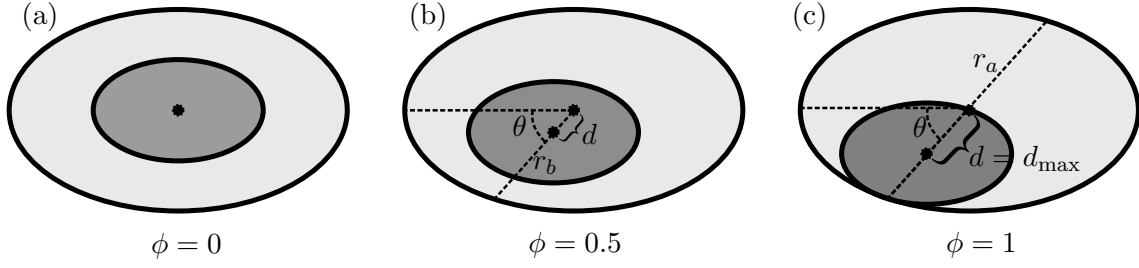


Figure 3.4: Schematic showing a sample geometry for $e = 0.8$, $\theta = \pi/4$ and (a) $\phi = 0$, (b) $\phi = 0.5$, (c) $\phi = 1$. The shaded ellipse is the cross-section of the rod.

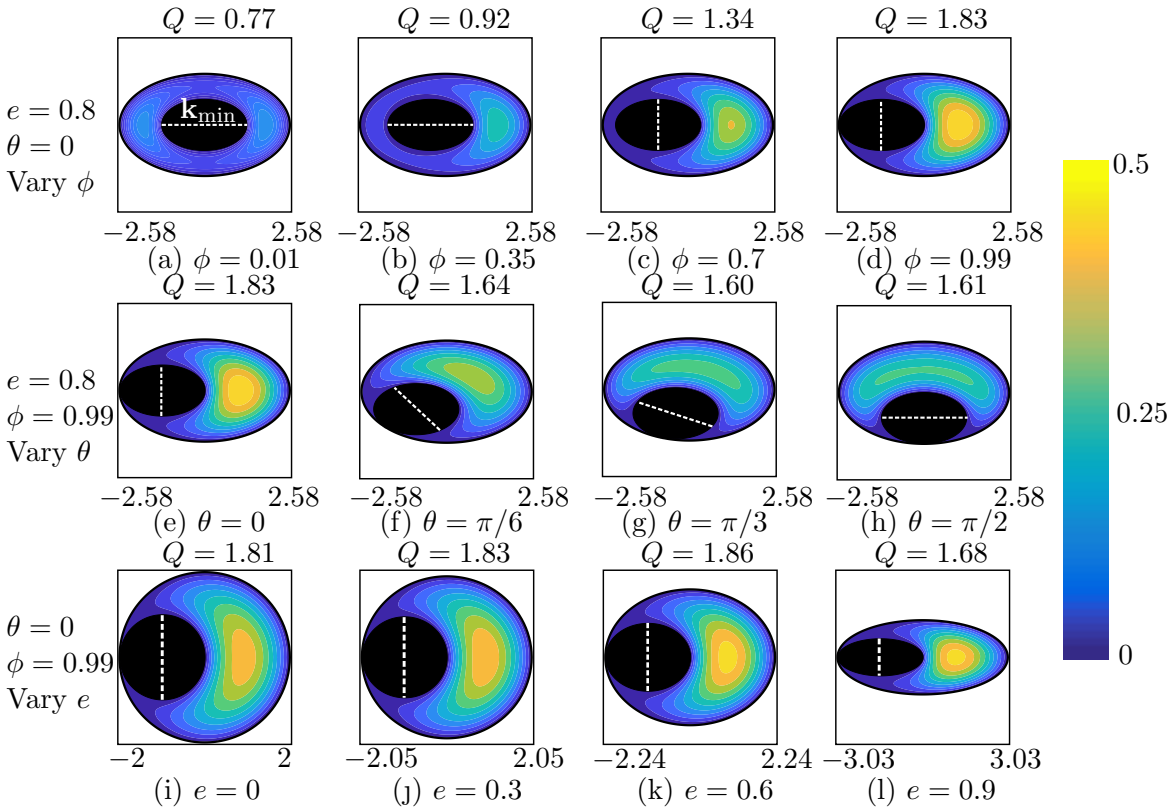


Figure 3.5: Dimensionless velocity colourmaps with flux values, Q , for ellipses of equal eccentricities and orientations. Here $a = 2$. Axes are in (x, y) coordinates and these vary with eccentricity so that the available space for fluid flow is constant. Colourbars reflect different velocity values within the domains. The direction of the axis of minimal resistance, $\mathbf{k}_{\min}$, is indicated by the dashed white line on each diagram. This will be discussed in Section 3.5.

Plots (a)-(d) show the effect of offset, ϕ , for $e = 0.8$ and $\theta = 0$. Offset values are (a) $\phi = 0.01$, (b) $\phi = 0.35$, (c) $\phi = 0.7$, and (d) $\phi = 0.99$.

Plots (e)-(h) show the effect of angular position, θ , for $e = 0.8$ and $\phi = 0.99$. Angular position values are (a) $\theta_e = 0$, (b) $\theta = \pi/6$, (c) $\theta = \pi/3$, and (d) $\theta = \pi/2$.

Plots (i)-(l) show the effect of eccentricity, e , for $\phi = 0.99$ and $\theta = 0$. Eccentricity values are (a) $e = 0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$.

In Figure 3.5a-d we fix $e = 0.8$ and $\theta = 0$, and plot the velocity profiles (with corresponding flux values indicated) for $\phi = 0.01$, $\phi = 0.35$, $\phi = 0.7$, and $\phi = 0.99$. We observe that Q increases with ϕ , a result previously known for $e = 0$ (Piercy *et al.*, 1933; Redberger, 1962). The effects of θ and e on Q are less intuitive. In Figures 3.5e-h we fix $e = 0.8$ and $\phi = 0.99$ and vary θ , plotting the associated velocity colourmaps for $\theta = 0$, $\theta = \pi/6$, $\theta = \pi/3$, and $\theta = \pi/2$. The flux is largest for $\theta = 0$ and smallest for $\theta = \pi/3$ demonstrating, for the chosen parameters, non-monotonicity of Q with θ . We vary e in Figures 3.5i-l for fixed $\theta = 0$ and $\phi = 0.99$. The flux is largest for $e = 0.6$ and smallest for $e = 0.9$, also demonstrating, for the chosen parameters, non-monotonicity of Q with e . For completeness, line-plots of Q as a function of independent variation of e , ϕ , and θ can be found in Appendix 3.E.

In Figures 3.6a-c, we show surface plots of Q in (ϕ, θ) space for three different eccentricities: $e = 0$, $e = 0.7$, and $e = 0.9$, respectively. Note that, as Q varies most significantly with ϕ , we have restricted to $0.9 \leq \phi \leq 0.99$ to isolate the effects of θ and e . When $e = 0$ (Figure 3.6a), Q is independent of θ due to the rotational symmetry of the domain, and increases monotonically with ϕ . For eccentric domains, $e = 0.7$ and $e = 0.9$ (Figures 3.6b and 3.6c, respectively), Q is minimal at an intermediate value of θ and $\phi = 0.9$ (the smallest ϕ value plotted). The maximum Q is seen in Figure 3.6b for $e = 0.7$, further validating the existence of a non-zero eccentricity value that maximises flux, $e^{\max}$. This existence of a non-zero eccentricity at which flux is maximised may be of particular importance to ureteroscopy. However, it is noteworthy that the optimal eccentricity itself will change based on the outer geometry. We plot $e^{\max}$ as a function of a in Figure 3.6d. As $a \rightarrow \infty$, $e^{\max} \rightarrow 0$. As a characterises the ratio between outer and inner cross-sectional areas, the limit as $a \rightarrow \infty$ corresponds to a negligible obstruction within the channel, and hence, the maximum flux will be attained for a circular cross-section.

In this section we have determined that for equal eccentricities and orientations of the outer and inner ellipses, there is a position for the inner ellipse and a non-zero eccentricity value that maximises flux. We note that this may not be the configuration that maximises flux for all values of e_b , e_a , X , Y , and Θ , and we consider this global optimisation problem in more detail in the following section.

3.4.4 Optimal geometry for axial flow

When solving for axial flow in Section 3.4 we considered three reduced geometric parameter spaces: I) circle in ellipse, II) ellipse in circle, and III) ellipse in ellipse of the same eccentricity and orientation. We found, in each case, eccentricity values and positions for the inner rod that maximised Q . In this section, we will relax all previous assumptions

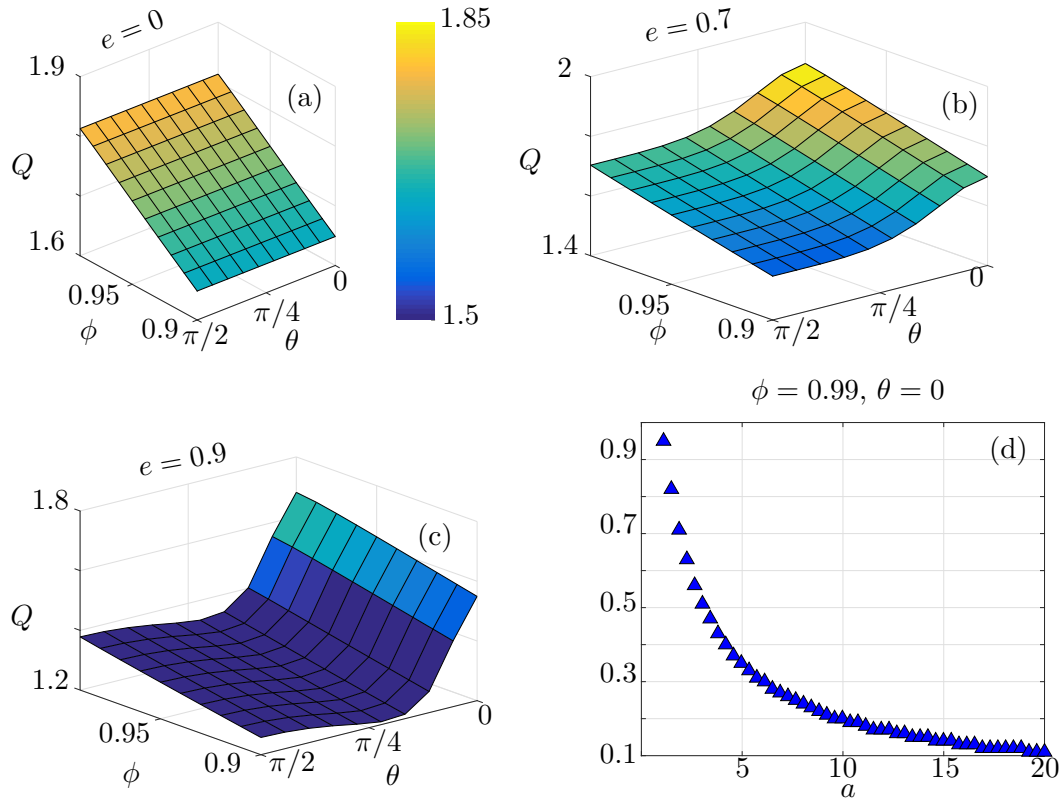


Figure 3.6: Surface plots (a)-(c) give the dimensionless flow rate, Q , as a function of both offset, ϕ , and angular position θ of the inner ellipse for three different eccentricity values: (a) $e = 0$, (b) $e = 0.7$, and (c) $e = 0.9$. Here $a = 2$.

Plot (d) gives e^{max} as a function of the characteristic radius a . The e^{max} value is calculated as the one that produces maximum flow over 100 values from $0.01 \leq e \leq 0.99$ for each radius ratio.

on ellipse eccentricity and explore finding a global optimal geometry for ellipses of fixed cross sectional area, $a = 2$.

For all configurations previously considered, we demonstrated that axial flow resistance was lowered by increasing the distance between the centres of the inner and outer ellipses. Intuitively, we might expect that the optimal configuration is one where the inner ellipse touches the boundary of the outer ellipse and with matching curvature, i.e. when the inner ellipse ‘hugs’ the outer.³ This assumption enables a reduced parameter space that can be approached analytically (see Appendix 3.F). In Figure 3.7a we show a surface plot of Q as a function of e_a and θ – where θ is defined as the angle between the major axis of the outer ellipse and a line connecting the centres of the two ellipses – produced by this calculation. We see that a global maximum occurs at $\theta = 0$ and $e_a \approx 0.71$ with corresponding $Q = 2.62$ (indicated by the red circle and corresponding flow profile Figure 3.7b).⁴

³Although this was shown not to be the case for configuration I in Section 3.4.1

⁴At this maximum, we also have $e_b \approx 0.45$, $\Theta = 0$.

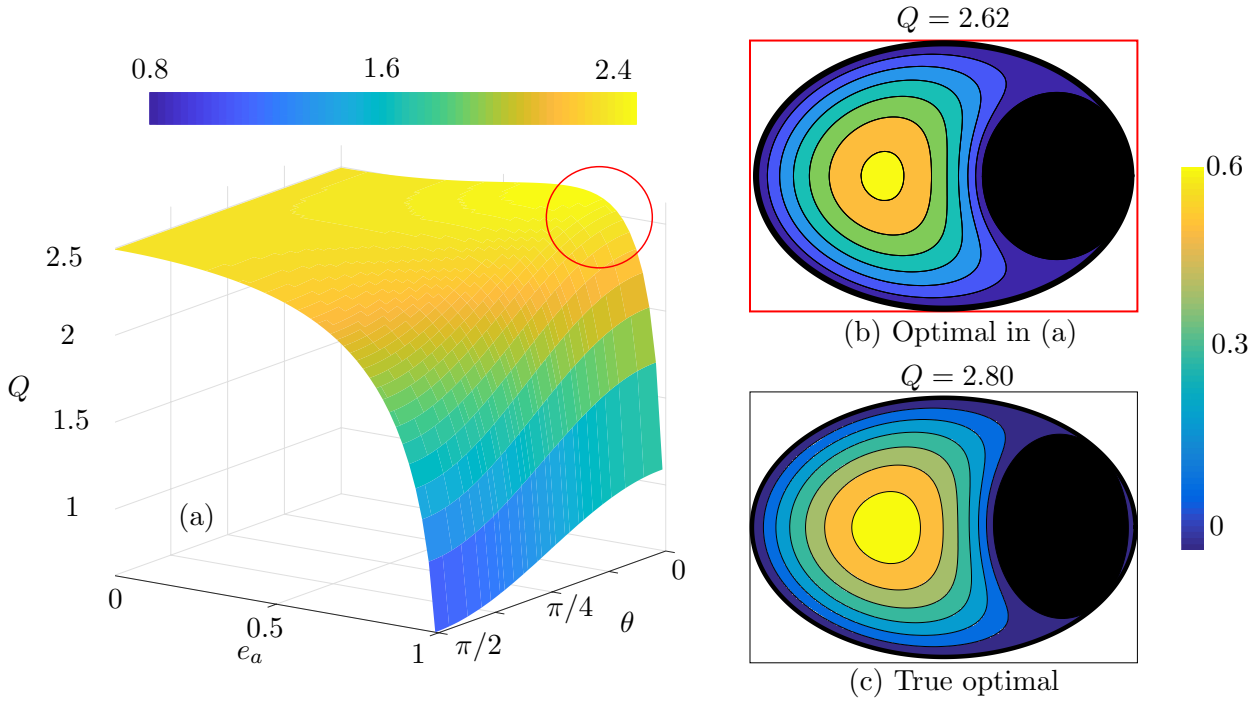


Figure 3.7: A plot of Q as a function of e_a and θ for $a = 2$. Here the inner ellipse touches the outer boundary tangentially at a single point, matching curvature. The red circle indicates the presence of a global maximum.

The optimal configuration in Figure 3.7a shows a nearly 40% improvement in flux over the optimal configuration computed in Section 3.4.3. However, it is still not sufficient to claim this as the global optimum for ellipse geometries. In fact, a parameter sweep over e_b , X , Y , and Θ , for fixed e_a , indicates that a ‘curve-hugging’ configuration may not be optimal (Appendix 3.G).

Motivated by this, we relax assumptions about how the ellipses meet at the boundary and formulate an optimisation problem to traverse the full parameter space. To this end, we define a vector, $\mathbf{g}$, containing the five parameters governing the geometry of the domain

$$\mathbf{g} = (e_b, e_a, X, Y, \Theta)^T. \quad (3.42)$$

The boundaries of the inner and outer ellipses, Γ_b and Γ_a , are

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 0, \quad \mathbf{x}^T \mathbf{B} \mathbf{x} = 0, \quad (3.43a,b)$$

respectively, where $\mathbf{x}^T = (x, y, 1)$ and matrices $\mathbf{A}$ and $\mathbf{B}$ are functions of $\mathbf{g}$. Thus, we seek the solution to the optimisation problem

$$\min_{\mathbf{g}} -Q, \quad \text{s.t. } \mathbf{c} \geq 0, \quad (3.44)$$

where $\mathbf{c}$ is a vector of six non-linear constraints, dependent on $\mathbf{A}$ and $\mathbf{B}$ and hence $\mathbf{g}$, that ensure Γ_b is enclosed by Γ_a (see Appendix 3.H for more details). We solve equation (3.27) in FEniCS (Alnaes *et al.*, 2015), and implement MATLAB’s optimisation routine, `fmincon`, to solve the optimisation problem (3.44) with an interior-point method from eight randomly generated starting points in the feasible five-dimensional parameter space. As the optimisation routine seeks local minima, a multi-start approach allows for global optimisation.

The optimal geometry found through this method is shown in Figure 3.7c with a corresponding flux, $Q \approx 2.80$. The geometric parameters – see equation (3.42) – are $\mathbf{g} \approx (0.68, 0.74, 1.49, 0, \pi/2)^T$. This global optimum gives a non-negligible (nearly 7%) increase in flux compared with the ‘curve hugging’ optimum. Indeed, rather than perfectly matching the geometries at the boundary, the global optimal geometry is tangent to the outer boundary in two locations (this was also true in the results in Section 3.4.1.) It is interesting to note that this divides the fluid domain into two separate regions, which may be connected to previous results (Ranger, 1994; 1996) which demonstrate flux enhancement through creating multiply connected regions. Ranger (1996) provides a short discussion on the mechanism whereby disconnected regions can lead to increased total flux, discussing the distribution of flux-per-unit-area for flow through a circular pipe. As flux-per-unit-area is low at the edge, removing these regions and replacing them with disconnected areas of higher flux may lead to enhanced flow. We discuss this further in Appendix 3.I.

3.5 Cross-sectional flow

We now turn attention to the cross-sectional flow problem, governed by Stokes equations (3.14), (3.26), and the no-slip conditions (3.16a,b) and (3.17a). The objective is again to explore the impact of cross-sectional geometry within the configuration space of I) circular rod in elliptical cylinder, II) elliptical rod in circular cylinder, III) elliptical rod in elliptical cylinder of same eccentricity and orientation; but here the question is how the geometry impacts the resistance coefficients for cross-sectional rod motion.

We solve the equations numerically using a finite element method implemented in `oomph-lib` (Heil & Hazel, 2006) (details in Section 3.A), and subsequently compute the dimensionless forces and torques on the rod due to the surrounding fluid via equations (3.31). We validate our numerical solutions by comparing with analytical solutions for geometries where these are available in the literature.

We consider a cross-section of the domain, illustrated in Figure 3.1 (*i*), where the rod is given a prescribed instantaneous velocity, $\dot{\mathbf{X}} = (\dot{X}, \dot{Y})$ in the (x, y) plane and a rotation,

$\dot{\Theta}$ about the z -axis. The resulting forces and torques, $\mathbf{F} = (F_x, F_y)$ and τ , are linearly related to $(\dot{X}, \dot{Y})$ and $\dot{\Theta}$ by a symmetric resistance matrix

$$\begin{bmatrix} F_x \\ F_y \\ \tau_z \end{bmatrix} = - \begin{bmatrix} K_{xx} & K_{xy} & C_x \\ K_{xy} & K_{yy} & C_y \\ C_x & C_y & A_{zz} \end{bmatrix} \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\Theta} \end{bmatrix}, \quad (3.45)$$

as discussed in Sections 3.2.2 and 3.3.3. If $\dot{\Theta} = 0$ then

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = - \begin{bmatrix} K_{xx} & K_{xy} \\ K_{xy} & K_{yy} \end{bmatrix} \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix}. \quad (3.46)$$

The eigenvectors of the 2×2 matrix in equation (3.46) are the directions of minimum and maximum opposing force experienced by the rod due to the flow. We will refer to these as the directions of minimum and maximum resistance. The corresponding eigenvalues give the magnitudes of the opposing forces that result from moving in the directions of minimum and maximum resistance with unit velocity. These values are non-negative (see Section 3.2.2) and are given by

$$K_{\max} = (1/2) \left(K_{xx} + K_{yy} + \sqrt{(K_{xx} - K_{yy})^2 + 4K_{xy}^2} \right), \quad (3.47a)$$

$$K_{\min} = (1/2) \left(K_{xx} + K_{yy} - \sqrt{(K_{xx} - K_{yy})^2 + 4K_{xy}^2} \right), \quad (3.47b)$$

with corresponding directions of minimum and maximum resistance

$$\mathbf{k}_{\max, \min} = (K_{xy}, K_{\max, \min} - K_{xx}). \quad (3.48)$$

The six unique scalars in the resistance matrix (3.45) can be computed by imposing:

$$\dot{\mathbf{X}} = (1, 0), \quad \dot{\Theta} = 0, \quad \rightarrow \quad K_{xx}, K_{xy} \text{ and } C_x, \quad (3.49a)$$

$$\dot{\mathbf{X}} = (0, 1), \quad \dot{\Theta} = 0, \quad \rightarrow \quad K_{yy}, K_{xy} \text{ and } C_y, \quad (3.49b)$$

$$\dot{\mathbf{X}} = (0, 0), \quad \dot{\Theta} = 1, \quad \rightarrow \quad C_x, C_y \text{ and } A_{zz}, \quad (3.49c)$$

where each calculation provides us with the coefficients indicated by the arrows. We note that K_{xy} , C_x , and C_y can each be determined by two separate calculations, and we perform both as validation (to within numerical error).

3.5.1 I) Circle in ellipse

The analytical solution for Stokes flow between concentric and offset circular cylinders demonstrates a dependence of the forces and torque experienced by the rod on the geometry of the domain (Finn & Cox, 2001). We first consider how these results change when the bounding cylinder is elliptical. To limit the candidate geometries, we position the rod along the x -axis, i.e. $\mathbf{X} = (X, 0)$. We fix the cross-sectional area and set $a = 2$.

For a circular rod translating within a circular cylinder, K_{xx} and K_{yy} increase as the rod approaches the edge of the bounding cylinder, and $K_{xy} = 0$, due to the symmetry of the domain (Finn & Cox, 2001). As expected, we find that for $e_a > 0$, the same behaviour occurs. The effect of e_a on the resistance coefficients is not quite so straightforward as the effect of position – as e_a increases for fixed X , the proximity of the rod to the boundary in the vertical direction decreases while the distance to the boundary in the horizontal direction increases, and at nonequal rates. Due to these competing features, K_{yy} exhibits an interesting dependence on e_a . This is illustrated – for $X = 0$ – in Figure 3.8a, where we plot K_{yy} as a function of e_a for a centred rod, observing a non-monotonic behaviour which highlights the complexity of boundary interactions. Comparing resistance coefficients for $X \neq 0$ adds yet another complexity: as the range of feasible X values increases with e_a (as long as the rod can still fit tangent to the vertex of the outer cylinder), we can either compare values at absolute, X , or relative, $X/(\alpha_a - 1)$, distance from the centre, which qualitatively affects the observed trends. This can be seen for K_{xx} by comparing Figure 3.8b, which is plotted as a function of relative offset, to the corresponding inset figure, which is plotted as a function of absolute offset. We find that, depending on the metric used, C_y , and A_{zz} can also be non-monotonically related to e_a when the cylinder is offset.⁵ We note that $C_x = 0$ for a rod centred on the x -axis, due to arguments of symmetry.

We next consider the coupling coefficients C_x and C_y . For concentric circular cylinders, there is no coupling between translation and rotation, i.e., $C_x = C_y = 0$ (see Section 3.2.2). However, coupling does occur for offset circular cylinders; e.g. when the rod is rotating and positioned away from the centre of the outer cylinder, the viscous fluid provides both an opposing torque and a force orthogonal to the minimal spacing. The force parallel to the minimal spacing in this scenario is zero (Slezkin, 1955). We now explore how these results extend for a circular rod in an outer elliptical cylinder. We determine that, as with a circular outer cylinder, if $X \neq 0$, the coupling coefficient $C_x = 0$, while $C_y \neq 0$ for all e_a . In Figure 3.8c we plot C_y as a function of relative offset for $e_a = 0$, $e_a = 0.6$, and $e_a = 0.8$ (lines *i*, *ii*, and *iii*, respectively). As observed by the crossing of the curves, for small relative offset the coupling is highest for the circular rod, while for large offset the most

⁵Results shown in Figure 3.22 (Appendix 3.J).

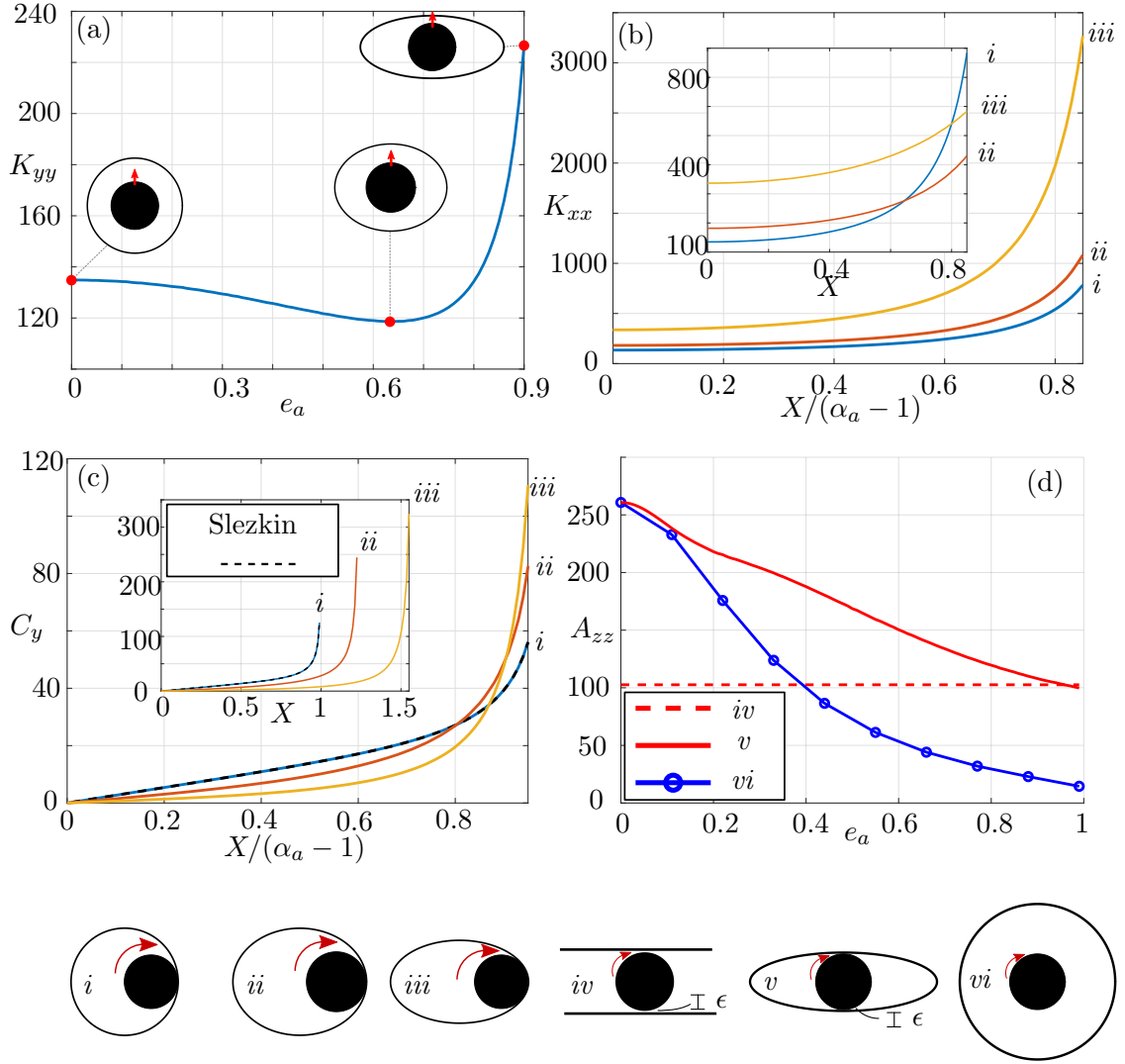


Figure 3.8: Results for $a = 2$. (a) Rod of unit radius centred at $(X, 0)$, translating with unit velocity in the x -direction. K_{xx} is plotted both as a function of relative offset and of X (inset) for three values of e_a : (i) $e_a = 0$ (ii) $e_a = 0.6$ and (iii) $e_a = 0.8$. (b) Rod translating with unit velocity in the y -direction, centred at $(0, 0)$. C_y is plotted as a function of e_a . Inset schematics are for (left-right) $e_a = 0$, $e_a = 0.63$, and $e_a = 0.9$ (c) Rod centred at $(X, 0)$, rotating with unit velocity in the clockwise direction for (i) $e_a = 0$, (ii) $e_a = 0.6$, and (iii) $e_a = 0.8$. The solution for $e_a = 0$ is compared against the analytical solution – (Slezkin, 1955), dashed black line. The main plot is function of relative offset, and inset is a function of X . (d) Plot of A_{zz} for a circular rotating rod centred at $(0, 0)$. The rod is rotating: (iv) between parallel plates with gap size $\epsilon = 0.025$ (Yang et al., 2013) (v) within an outer elliptic cylinder with $\beta_a = 1.025$ (a not fixed) (vi) within an outer circular cylinder of radius $a = 1.025(1 - e_a^2)^{-1/4}$. Solution obtained from equation (3.3c).

eccentric rod has the largest coupling coefficient. That is, as the inner rod approaches the edge of the domain, the fluid couples rotation to vertical force (and likewise vertical velocity to torque) more strongly for increased eccentricity. Again, the picture appears differently when plotted against absolute offset, since a greater total distance is available

with increased eccentricity: for each absolute distance X , C_y decreases with eccentricity (plotted in the inset).

If, instead of fixing cross-sectional area, we fix the gap between the rod and the cylinder, the domain approaches one of a rod between parallel plates as $e_a \rightarrow 1$. An asymptotic approximation to the Stokes force and torque on a cylinder rotating between parallel plates was obtained by Yang *et al.* (2013), where the ratio of gap width to cylinder radius defines the small parameter, ϵ . When the cylinder lies on the centreline between the two plates, $C_x = C_y = 0$; this may explain the behaviour of the lines in the inset of Figure 3.8, which approach zero as e_a increases (for moderate X values). We note that although the asymptotic solution (Yang *et al.*, 2013) is for a rotating cylinder, C_x and C_y still additionally characterise the torque generated via translational motion. For a centred ($X = 0$) circular cylinder, the coefficient A_{zz} , which we recall characterises the resistance to rotation, increases with e_a , but is non-monotonically related to e_a when the cylinder is offset (results not shown). To test whether the resistance coefficients for a rod in an elliptic cylinder approach those for a rotating rod between parallel plates (Yang *et al.*, 2013), we consider a rod at $X = 0$ in an elliptic cylinder of fixed minor axis, $\beta_a = 1.025$, which fixes the gaps between the rod and the cylinder at $x = 0$, to be 0.025. In Figure 3.8d we plot A_{zz} as a function of outer eccentricity (line v), and we see that as $e_a \rightarrow 1$ the resistance approaches the value for parallel plates with $\epsilon = 0.025$ (Yang *et al.*, 2013) (line i). As we increase e_a with fixed minor axis, the cross-sectional area of the elliptic cylinder is also increasing. Thus, as a further geometric comparison, line vi plots A_{zz} for a rod centred in a circular cylinder with increasing and equivalent cross-sectional area (i.e. for each e_a , the bounding circle has cross-sectional area equal to that of an ellipse of eccentricity e_a and minor axis $\beta_a = 1.025$). Although A_{zz} decreases with e_a due to the increase in cross-sectional area available for the fluid, A_{zz} is larger when the outer boundary is an ellipse compared to a circle of the same cross-sectional area.

3.5.2 II) Ellipse in circle

The next domain we consider is an ellipse in a bounding circle. As in Section 3.4.2, due to the rotational symmetry of the outer domain, the number of unique combinations of position and orientation of the rod can be reduced; it is sufficient to consider the rod centred at $\mathbf{X} = (X, 0)$ and to vary X , Θ , and e_b . In Figure 3.9 we investigate how the resistance coefficients are influenced by the shape of the rod's cross-section and its orientation and proximity to the outer boundary. In Figure 3.9a-c, results are plotted as functions of absolute offset, and the proximity of the rod to the boundary is intricately related to its position X , orientation Θ , and shape e_b .

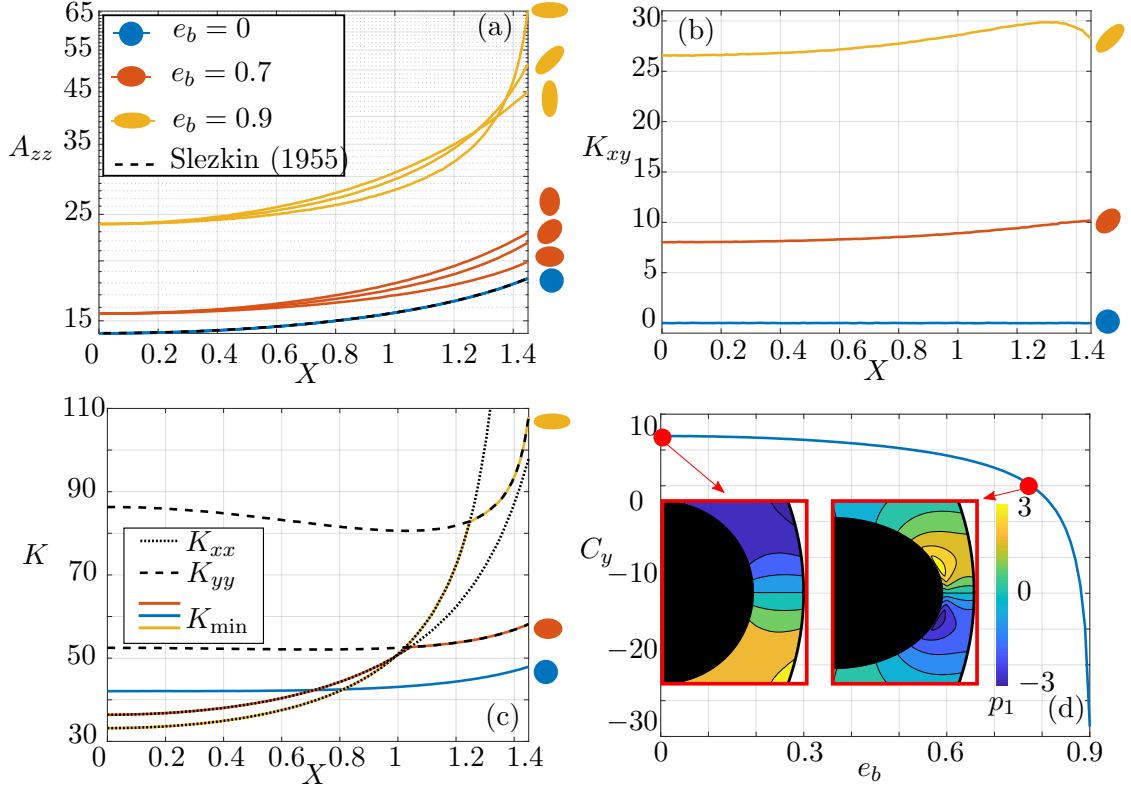


Figure 3.9: Figure (a) and (b) give A_{zz} and K_{xy} , respectively, as functions of X for $e_b = 0$, $e_b = 0.7$, and $e_b = 0.9$ and $\Theta = 0$, $\Theta_0 = \pi/4$, and $\Theta = \pi/2$. In (b), K_{xy} is identically zero for $\Theta = 0$ and $\Theta = \pi/2$, so these are not shown as they coincide with the line for $e_b = 0$. Figure (c) displays K_{min} for $\Theta = 0$ and $e_b = 0$, $e_b = 0.7$, and $e_b = 0.9$. Figure (d) gives C_y as a function of e_b for $\Theta = 0$ and $X = 1.45$. Colourmaps of pressure p_1 near the bounding wall, due to an imposed angular velocity, are shown for $e_b = 0$ and $e_b = 0.7$ with the colourbar providing the magnitude of p_1 , centred around $p_1 = 0$. All plots have a dimensionless outer radius of $a = 3$.

We first consider resistance to rotation. We find that A_{zz} increases as a function of offset for fixed e_b and Θ . Results are shown in Figure 3.9a for three eccentricities, $e_b = 0$, $e_b = 0.7$, and $e_b = 0.9$, oriented at $\Theta = 0$, $\Theta = \pi/4$, and $\Theta = \pi/2$. For fixed position and orientation, rotational resistance increases with eccentricity. Without boundary interactions, A_{zz} would be independent of Θ ; however due to the presence of the bounding circular cylinder, for moderate $X \neq 0$, A_{zz} is largest when $\Theta = \pi/2$ and smallest when $\Theta = 0$. As evidenced by the crossing lines for $e_b = 0.9$ in Figure 3.9a, when X is sufficiently large, a rod oriented at $\Theta = 0$ receives the largest opposing torque, which is likely due to the proximity of the vertex of the major axis to the boundary. We note that, if we increase the X range in Figure 3.9a, this effect is also seen for $e_b = 0.7$.⁵

Turning now to translational resistance, our results validate the intuition that when the major or minor axis of the rod's cross-section is aligned both with the direction of motion and a radial line of the outer cylinder's cross-section, the force on the rod acts parallel to

its direction of motion, and the lift force is zero. Brenner (1962a) proved this to be true for the motion of a bounded arbitrary particle.

Of the three translational resistances, we first consider the lift force, K_{xy} . We find, that for fixed e_b and X away from the boundary, K_{xy} is maximised for $\Theta = \pi/4$. Thus the outer cylinder does not seem to change classic results for an elliptic cylinder in an unbounded fluid (Lee & Leal, 1986). To see how the lift force changes with inner eccentricity, in Figure 3.9b we fix the orientation at the maximal value $\Theta = \pi/4$ for eccentricities $e_b = 0$, $e_b = 0.7$, and $e_b = 0.9$, and plot K_{xy} as a function of X . We find a significant increase in lift force with eccentricity, and also a non-monotonic relation with position in the case of $e_b = 0.9$.

Next, we analyse K_{xx} , K_{yy} , and $K_{\min}$. For relatively small offset, K_{xx} is smallest (and K_{yy} largest) when $\Theta = 0$. We restrict to this orientation, and plot K_{xx} (dotted line) and K_{yy} (dashed line) as functions of X for $e_b = 0.7$ and $e_b = 0.9$ in Figure 3.9c. An interesting feature is the non-monotonic behaviour of K_{yy} with X . The solid coloured lines give the value of the minimum resistance, $K_{\min}$. Looking particularly at $e_b = 0.9$ we see that for small values of X , $K_{\min}$ coincides with K_{xx} (and thus the eigenvector $\mathbf{k}_{\min} = \mathbf{i}$). However, near $X = 1.2$ the lines K_{xx} and K_{yy} cross, and the direction of minimum resistance switches to be aligned with the y -axis. The same behaviour is seen for $e_b = 0.7$, with the switching occurring at a smaller offset (near $X = 1$). The solid lines denoting $K_{\min}$ for each eccentricity also cross. When X is small, $K_{\min}$ is largest for $e_b = 0$ and smallest for $e_b = 0.9$ and vice versa for large X . Again, this can be explained by the significance of boundary interactions: far from the boundary the more eccentric ellipse has a direction of small resistance – its major axis – but closer to the boundary the major vertex is in closer proximity to the boundary, compared to ellipses of smaller eccentricity, hence the resistance in both the x - and y -directions rapidly increases. We have also demonstrated that for $\Theta = \pi/2$, $\mathbf{k}_{\min}$ is aligned with the y -axis for $e_b = 0$, $e_b = 0.7$, and $e_b = 0.9$ (at least for $0 \leq X \leq 1.45$).⁵

Finally, we consider the coupling between translation and rotation. We find, for fixed e_b and X away from the boundary, C_x is largest for $\Theta = \pi/4$ and C_y is largest for $\Theta = \pi/2$.⁵ When the rod is oriented at $\Theta = 0$, there is coupling between rotation and translation in the y -direction ($C_y \neq 0$). This is known analytically for $e_b = 0$ (Slezkin, 1955), and in this classic case, C_y increases monotonically with X . Interestingly, for more eccentric ellipses, this is no longer true; in fact, C_y can change sign as a function of position along the X -axis. Alternatively, this behaviour can be illustrated by plotting C_y as a function of eccentricity for fixed X sufficiently close to the outer boundary. This is shown in Figure 3.9 for $X = 1.45$ (the furthest-right position in Figures 3.9a-c). We observe that C_y decreases as a function of e_b , changing sign from positive to negative. To explain this behaviour,

consider a rotating rod with imposed angular velocity, in which case C_y characterises the induced translational force in the y -direction. Example pressure profiles for $e_b = 0$ and $e_b = 0.775$ near the proximal boundary are shown in the insets, and these show that the pressure drop in the lubrication layer between the rod and the outer boundary changes direction as a function of e_b . Due to the proximity to the boundary, we can appeal to Reynolds' lubrication equation to understand this inversion in terms of the behaviour of the fluid in the lubrication layer: for a circular surface *rotating* near a fixed plane, the pressure drop is in the opposite direction compared to the pressure drop for a highly curved surface *moving parallel* to the plane (Wannier, 1950). For our system, rotating an ellipse of high eccentricity is analogous to moving the vertex (and point closest to the boundary) parallel to the boundary, whereas rotating an ellipse of small eccentricity behaves more like a rotating circle.

3.5.3 III) Ellipse in ellipse

In this section we revisit the configuration considered in Section 3.4.3, with both the rod and bounding cylinder having elliptical cross-sections of the same eccentricity, e , and orientation ($\Theta = 0$). We set $a = 2$, and parametrise $\mathbf{X}$ by offset ϕ and angular position θ , illustrated in Figure 3.4.

Returning to Figure 3.5, the direction of minimum resistance is marked with a dashed white line. We see that when the rod is centred in the outer cylinder, $\mathbf{k}_{\min}$ lies along the major axis of the ellipse. As we move towards the wall, this direction switches to the minor axis, due to the build-up of pressure between the rod and the proximal edge of the outer cylinder. As we break symmetry and vary θ , $\mathbf{k}_{\min}$ no longer lies along an axis of symmetry of the inner ellipse and for $\phi = 0.99$ it varies from vertical for $\theta = 0$ to horizontal for $\theta = \pi/2$, see Figures 3.5e-g.

Figures 3.6a-c showed the axial flow rate in the (ϕ, θ) parameter space for three different eccentricity values, and in Figure 3.10a-c, we plot $K_{\min}$ for the same domains. The minimum resistance is highest at the boundary, $\phi \rightarrow 1$, and at the orientation $\theta = 0$, and is generally larger in magnitude as eccentricity increases. The smallest value of $K_{\min}$ represents the ‘most unstable’ position; for both $e = 0.7$ and $e = 0.9$ this occurs at $\theta = \pi/2$ and an intermediate ϕ value $0 < \phi < 1$. Here $K_{\min}$ is in the x -direction, aligned with the major axis of the rod. This is similar to the behaviour determined by Dvinsky & Popel (1987) of a circular cylinder translating between two plates, parallel to the boundaries, where the lowest force was incurred at a position between the centreline and the wall, with the exact location being dependent upon the ratio of the radius of the cylinder to the distance between the walls.

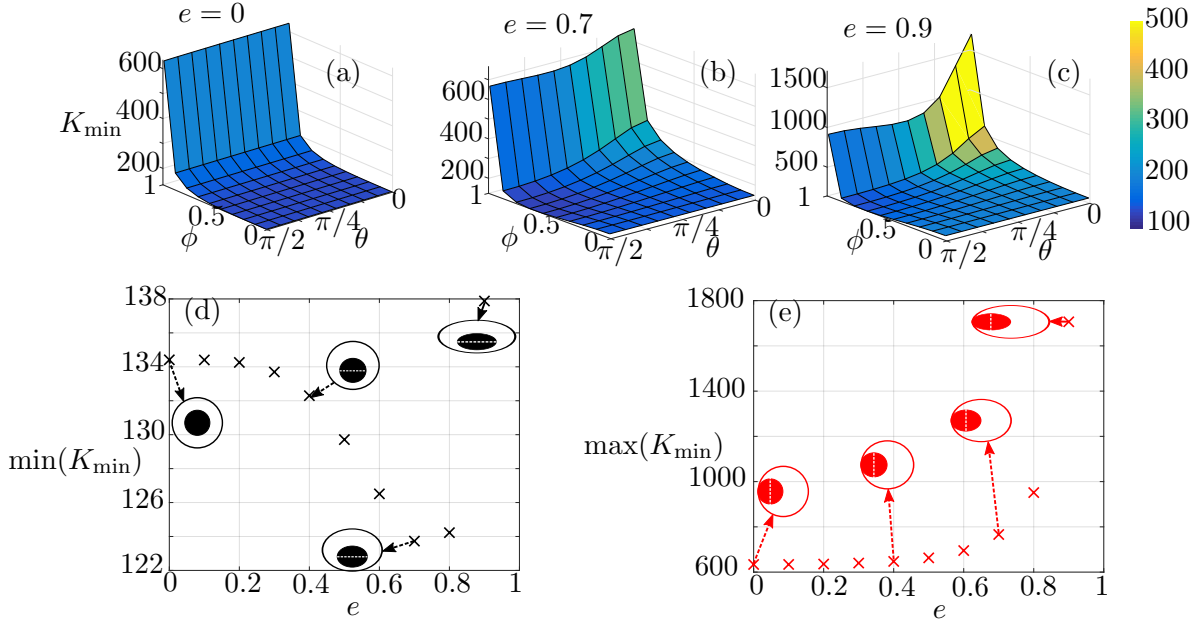


Figure 3.10: Surface plots (a)-(c) give the value of $K_{\min}$ as a function of both offset, ϕ , and angular position θ (where these values are defined as in Section 3.4) of the inner ellipse for three different eccentricity values: (a) $e = 0$, (b) $e = 0.7$, and (c) $e = 0.9$.

Plots (d) and (e), respectively, show the maximum and minimum value of $K_{\min}$ gives the minimum value of $K_{\min}$ for a given equal inner and outer ellipse eccentricity value. The geometries that provide the $\min(K_{\min})$, (d), and $\max(K_{\min})$, (e), for $e = 0$, $e = 0.4$, $e = 0.7$, and $e = 0.9$ are demonstrated by diagrams and the white-dashed line on each of these shows the direction of minimum resistance. $a = 2$.

In Figure 3.10d, the minimum value of $K_{\min}$ over (ϕ, θ) -space is plotted as a function of e . Several points have been illustrated to show the domain that produces the value, and the $\mathbf{k}_{\min}$ axis is demonstrated on these by a dashed white line. We see that $K_{\min}$ has a local minimum near $e = 0.7$. Note also that the position where the minimum $K_{\min}$ is achieved changes with e , moving from the centre of the outer cylinder to the edge with increasing eccentricity. Figure 3.10e shows the maximum value of $K_{\min}$ for increasing eccentricity, and again, the locations, along with the $\mathbf{k}_{\min}$ vector, that produce these values is demonstrated at four points. Here there is a monotonic increase with eccentricity, and the location of the maximum $K_{\min}$ always occurs on the major axis of the outer cylinder, closest to the wall ($\theta = 0$, $\phi = 0.99$).

Connecting these results with those in Section 3.4.3, we find that for each e , the location where $K_{\min}$ is maximised coincides with the position where Q is maximised. This can be seen by comparing Figures 3.10a-c with Figures 3.6a-c. To explore this in more detail we present contour plots of Q , $\log(K_{\min})$, and $\log(K_{\max})$ for $e_a = e_b = 0.7$, $\Theta = 0$ in Figure 3.11 as functions of (X, Y) . The plots are computed for $0 \leq \phi \leq 0.99$, $0 \leq \theta \leq \pi/2$, and the black boundary indicates the edge of the outer ellipse. The coincident positions of maximum Q and $K_{\min}$ are indicated in Figures 3.11a and 3.11b, respectively, by the red

circles. The resistance contours, Figures 3.11b and 3.11c, show increasingly narrow contours with proximity to the edge of the outer ellipse, demonstrating a rapid increase in the gradient of $K_{\min}$ and $K_{\max}$, respectively. In contrast, the distances between neighbouring Q contour lines in Figure 3.11a remain relatively constant. Thus, it is interesting to note that small changes to the position of the inner rod when it is near the outer boundary may incur vast changes to the resistance to cross-sectional motion, but only a small change to the corresponding axial flux. The arrows in Figures 3.11b and 3.11c demonstrate the directions of minimum and maximum resistance, respectively. Along the major axis of the outer ellipse, and close to its centre, the direction of minimum resistance is towards the vertex of the inner rod, demonstrating the effect of the rod's streamlined elliptical cross-section. However, the directions of the arrows change once the rod is sufficiently far from the centre, and the competing effects of the bounding walls become significant. This is in agreement with the results in Figure 3.9c. As anticipated, when the rod is near the outer boundary, the direction of minimum resistance is tangent to the wall and the direction of maximum resistance perpendicular to it. Although similar trends exist between Q , $K_{\min}$, and $K_{\max}$, as shown in Figure 3.11, the non-monotonic contour shapes in Figure 3.11b, as well as the difference in gradients, indicate that the apparent correspondence between axial and cross-sectional resistance is complicated, even in the particular reduced parameter space (of equal ellipse eccentricity and orientation) explored in Figures 3.10 and 3.11. However, since a large value of $K_{\min}$ corresponds to large resistance to the motion of the rod, we conclude that (at least in the case of cylinders of equal eccentricity and orientation) the configuration that provides maximum axial flow is also the 'most stable' to cross-sectional motion.

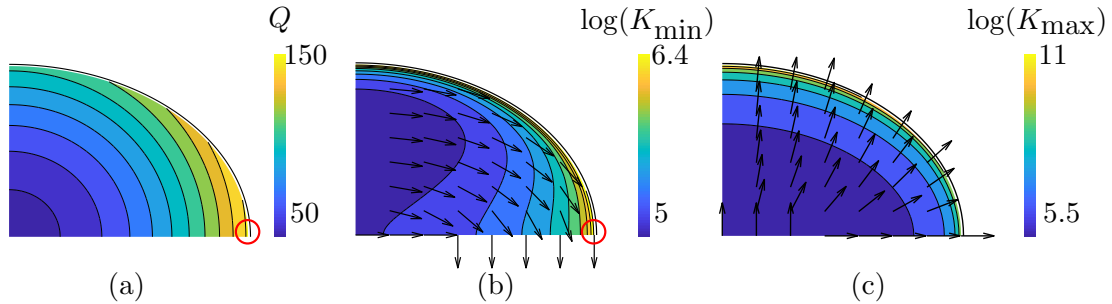


Figure 3.11: Figures (a), (b), and (c), show Q , $\log(K_{\min})$, and $\log(K_{\max})$ as functions of position for ellipses of equal eccentricity $e_a = e_b = 0.7$, and equal orientation ($\Theta = 0$). The arrows in Figures (b) and (c) show the directions of $\mathbf{k}_{\min}$ and $\mathbf{k}_{\max}$, respectively. Red circles in Figures (a) and (b) indicate the coincident locations of maximum Q and maximum $K_{\min}$.

The arrows in Figures 3.11b and 3.11c allow some conjecturing as to local trajectories of the inner rod; however, it is important to recall that the results in Figure 3.11 are computed for fixed orientation of the inner rod ($\Theta = 0$). In actuality, a torque will be induced by the translational motion of the rod, thus changing the resistance contours. We

consider true trajectories of the rod in the following section.

3.6 Trajectories of the rod

In computing resistance coefficients, the rod is prescribed an ‘instantaneous’ motion from which we compute resulting forces and torque. The above result relating maximal flow to ‘most stable’ position holds for cylinders of equal eccentricity and orientation. This suggests intriguing dynamics for a rod that is free to change its cross-sectional position: *will the flow naturally push a rod to its ‘most stable’ position and to where the flow is also optimal?* In this section we briefly explore this by solving the equations of motion (3.18) to determine trajectories of the rod.

We begin by prescribing the initial position, orientation angle, translational velocity and angular velocity of the rod, equations (3.21). We then solve equations the Stokes equations (3.14) and incompressibility condition (3.26) subject to no-slip conditions (3.16a,b) and (3.17a) to calculate u , v , and p_1 . As in Section 3.5, we use these to compute $\mathbf{F}$ and τ via equations (3.31). Using these, we update the velocity of the rod, using the relationships between linear and angular acceleration and force and torque, equations (3.18). We employ a numerical approximation of equations (3.18)

$$\dot{\mathbf{X}}^{i+1} = \varrho \mathbf{F}^i \Delta t + \dot{\mathbf{X}}^i, \quad \dot{\Theta}^{i+1} = \varrho' \tau^i \Delta t + \dot{\Theta}^i, \quad (3.50a,b)$$

where the superscripts denote the iteration number and Δt is the time-step. We then update the position and orientation of the rod using

$$\mathbf{X}^{i+1} = \dot{\mathbf{X}}^{i+1} \Delta t + \mathbf{X}^i, \quad \Theta^{i+1} = \dot{\Theta}^{i+1} \Delta t + \Theta^i. \quad (3.51a,b)$$

We repeat the calculation for updated position and velocity. Recall that in the parameter regime considered, fluid inertia is always negligible, hence the fluid is quasi-statically updated during the rod motion.

We present results for a domain of inner and outer ellipses of equal eccentricities $e = 0.8$, with $a = 2$, where the inner ellipse is given an initial unit translational velocity in the direction of $\mathbf{k}_{\min}$.⁶ In Figure 3.12 we compare the trajectories of a rod starting from its ‘most stable’ (Figure 3.12a) and least stable (Figure 3.12b) positions, with initial velocity in the direction of least resistance, $\mathbf{k}_{\min}$, and with zero initial angular velocity. It is important to note that ‘least/most stable’ refers to our calculation in Section 3.5.3 in a restricted geometrical space ($\Theta = 0$), while in this motion there is no such restriction. In

⁶We take $\varrho = 10^{-2}$ to generate larger motion and easier visualization; while this technically violates our assumed balance $\varrho = O(1)$, the results presented below qualitatively hold for larger ϱ .

each case we show three snapshots of trajectories at times $t = 0$, $t = 1$, and $t = T_{\text{end}}$, where T_{end} is taken to be the time when $\dot{X}$, $\dot{Y}$, and $\dot{\Theta}$ are all less than some small value (we take 10^{-3}) i.e., when the rod has reached an approximately stationary position. From the Stokes descriptions of the forces and torque experienced by the rod, we know they are always negative. Hence, the rod will always come to rest as $t \rightarrow \infty$.

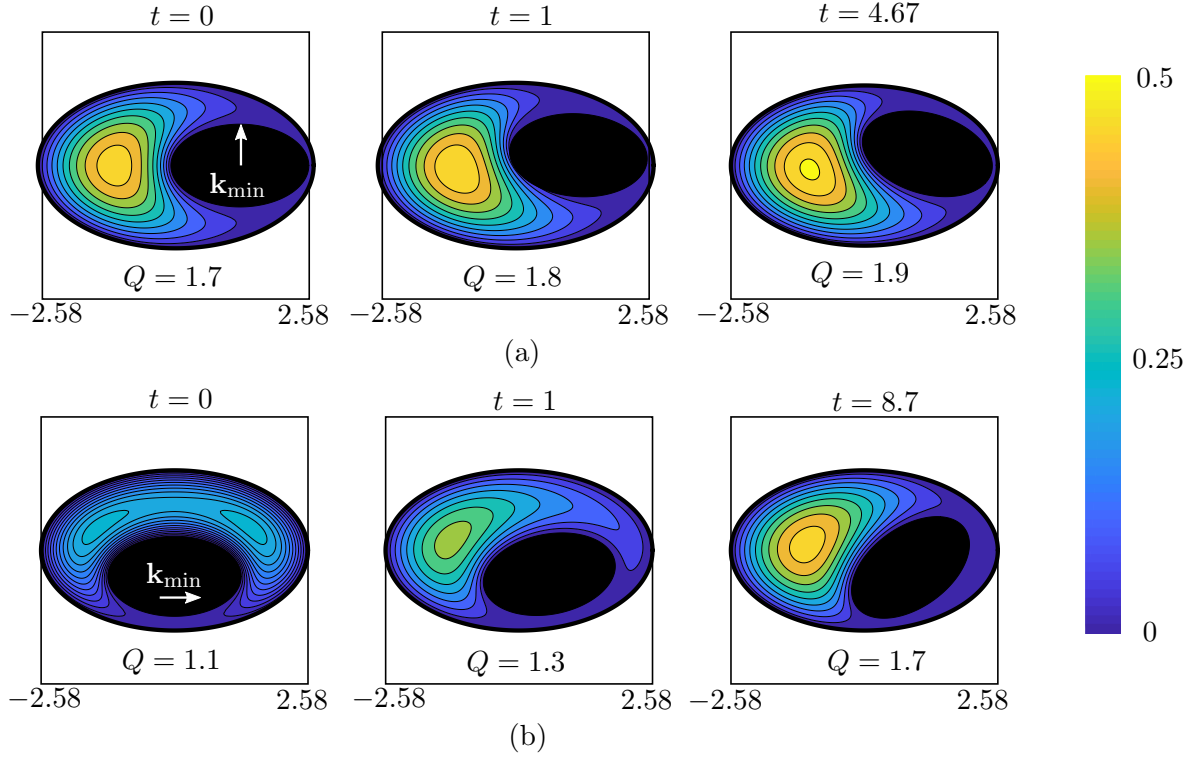


Figure 3.12: Three snapshots of the trajectory for rod subject to an initial unit velocity in the direction of $\mathbf{k}_{\min}$ for $e = 0.8$ and $\Theta = 0$. The $\mathbf{k}_{\min}$ direction is indicated by the white arrow. Here $\varrho = 10^{-2}$, $\varrho' = 4/(\alpha_b^2 + \beta_b^2)^{-1}\varrho$, $\Delta t = 10^{-2}$. In (a) the initial position is where $K_{\min}$ is minimised, and in (b) the initial position is where $K_{\min}$ is maximised for the same domain.

In Figure 3.12a, we see that the inner ellipse initially moves upwards, in the direction of the applied velocity, before rotating, due to torques generated as a result of interactions with the boundary. It reaches the stationary criterion at time $T_{\text{end}} = 4.67$, such that the linear distance between the initial position of the ellipse centre and its final position is 0.2272. In Figure 3.12b, we see similar behaviour, however the rod takes longer to come to rest ($T_{\text{end}} = 8.7$), and its final position is a distance 0.9086 from its initial position. Thus, the elliptical cylinder starting in the ‘least stable’ location of minimum $K_{\min}$ (Figure 3.12b) moved four times as far as the one starting in the ‘most stable’ location of maximum $K_{\min}$ (Figure 3.12a). In Figure 3.12, we have also plotted the axial velocity profiles with colourbar providing velocity magnitude for each domain, with the corresponding flux values denoted. It is interesting to observe that in both simulations, as the position of the inner ellipse evolves, the axial volumetric flow rate increases.

3.7 Summary and discussion

We have considered in this chapter the effect of geometry in an annular region, formed by an elliptic rod and coaxial elliptic cylinder, on two types of resistance: the resistance to axial flow, and the resistance to movement of the rod in the cross-section. This research was prompted by questions of optimal device design: *where to position the rod, and how to shape the cross-sections of the rod and the bounding cylinder, to maximise axial flux and minimise movement of the rod?*

Motivated by the design of urological devices, we have restricted to the parameter regime in which the rod and bounding cylinder are long and thin and the fluid behaviour is dominated by viscous forces, but in which inertia of the inner cylinder is non-negligible. In this regime, we showed that the axial flow decouples from the cross-sectional flow. Therefore, we can separately solve a Poiseuille flow problem for the axial flow and Stokes flow in the cross-section, with the profile of the cross-sectional flow producing a net force and torque that couple to the motion of the inner cylinder.

3.7.1 Axial flow resistance

For all configurations considered, the flux increases monotonically as the rod is positioned further from the centre of the outer cylinder, and the maximum flux is obtained when the rod touches the boundary of the outer cylinder. For a circular rod in a circular cylinder, the maximum flux is achieved when the circle touches the outer boundary at a single point (Snyder & Goldstein, 1965; MacDonald, 1982). When either the outer cylinder or the rod is elliptical, we found that flow can be increased from the circular case by touching the boundary tangentially at two points. Even within the reduced space of elliptical cylinders, the full optimisation problem is five-dimensional and non-trivial, containing many local minima. Along with the global optimum, we have also considered several optimisation problems where we focus on a subset of elliptical shapes; these are summarised in Table 3.3, grouped by geometric constraint. Of particular relevance is the fact that the global optimum gives a 52% increase over the circle in circle case. Also of note is the observation that the true optimum in the case of an ellipse in circle, circle in ellipse, and ellipse in ellipse, all occur at configurations for which the inner cylinder meets the outer cylinder at two tangent points and thus does not correspond to a matching of the curvature of the boundaries.

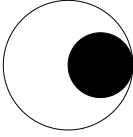
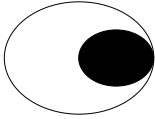
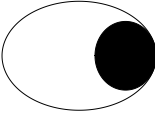
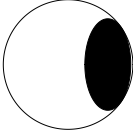
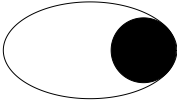
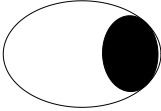
Description	Optimal Configuration	Geometry	Q	Tangent points
Circle in circle		$e_b = 0 \quad e_a = 0$ $X = 1 \quad Y = 0$ $\Theta = 0$	1.84	1
Ellipse in ellipse ($e_b = e_0$, $\Theta = 0$)		$e_b = 0.66 \quad e_a = 0.66$ $X = 1.15 \quad Y = 0$ $\Theta = 0$	1.89	1
Ellipse in ellipse (‘ <i>curve hugging</i> ’)		$e_b = 0.45 \quad e_a = 0.71$ $X = 1.44 \quad Y = 0$ $\Theta = \pi/2$	2.62	1
Ellipse in circle		$e_a = 0 \quad e_b = 0.87$ $X = 1.23 \quad Y = 0$ $\Theta = \pi/2$	2.66	2
Circle in ellipse		$e_b = 0 \quad e_a = 0.83$ $X = 1.65 \quad Y = 0$ $\Theta = 0$	2.69	2
Ellipse in ellipse		$e_b = 0.68 \quad e_a = 0.74$ $X = 1.49 \quad Y = 0$ $\Theta = \pi/2$	2.80	2

Table 3.3: Summary of optimal configurations for $a = 2$.

3.7.2 Cross-sectional flow resistance

The cross-sectional resistance to imposed translational and rotational movement of the rod is fully characterised by six unique scalars. The behaviour of these resistance coefficients is determined both by the shape of the rod's cross-section and its proximity to the bounding cylinder. When varying eccentricity, position, and orientation, and the impact on six scalars, we are thus faced with a very high-dimensional parameter space to investigate. Our results do not represent a complete sweeping of all configurations; rather we have tried to uncover trends, to compare with classic results in particular limits, and to focus on unexpected behaviour. One particularly interesting result was found for an elliptic rod near the edge of a bounding circular cylinder, in which case we found that the sign of the coefficient coupling translational and rotational motion changed as the eccentricity of the inner ellipse increased. This result could be rephrased as a change in motion being generated by a change in shape, and we might speculate whether such a process could be harnessed in a micro-swimming context, where trajectories are determined by principles of Stokes resistance (Lauga & Powers, 2009).

It is important to recall, as discussed in Section 3.2.3 that all resistance coefficients diverge as the rod approaches the outer boundary. Thus, positions for the rod which lead to high axial flux (close to the edge of the bounding cylinder) receive extremely high resistance to translational and rotational motions. We defined the ‘most stable’ configuration as the one where the minimum translational resistance is maximum; i.e. for which the fluid provides the strongest resistance to translational motion in the direction that is easiest to move. Thus, it is impossible to fully connect configurations of maximum axial flux to positions of highest minimal resistance within our modelling framework, as the maximum axial flow occurs when the rod touches the wall, and here the resistance always diverges. Nevertheless, for the particular class of an elliptic rod in an elliptic cylinder of the same eccentricity and orientation, restricting the parameter space of rod positions to those not touching the boundary, we found that the geometry that maximises axial flux is also the ‘most stable’.

3.7.3 Trajectories

Motivated by the coinciding of configurations for optimal axial flow and maximal stability, we also considered rod trajectories by integrating forward the equations of motion for the inner rod. Here we must be careful not to disconnect the results from the parameter regime: rod inertia was included in our calculations while fluid inertia was neglected. In this regime we could quasi-statically update the forces and torques on the rod after updating its position and resolving the Stokes equations in the cross-section.

In the two cases considered (Figure 3.12), we observed an increasing axial flux throughout the motion, as well as a negative correlation between stability of the initial position and magnitude of motion (quantified via distance from initial position). While more work is needed to fully understand how a rod will move in these geometries, we obtain at least an answer to the question of whether the rod will always naturally move to a position of maximum stability and optimal axial flow. The fact that the resting positions of the rods is not the global optimum (which would be against the wall), the answer is *no*. This is not surprising, actually, when one considers that the fluid is not really pushing the rod at all but merely dissipating energy that has to be input to the rod from the external world. Of course, this would cease to be the case if the axial flow were coupled to the cross-sectional flow, which would be true if one breaks the co-axial assumption. It would be interesting to consider how such effects, as well as including fluid inertia, impact on our findings.

3.8 Epilogue: optimising the access sheath/scope shaft

The existence of optimally-shaped cross-sections to minimise resistance through a cylinder containing a coaxial rod is of particular interest to the endourological community. It is important to note, however, that in a biomedical context, the value of a varies with the particular medical device under consideration. Even within the category of ureteroscopes, there are a variety of available sizes for working tools and access sheaths, and hence, as the optimal geometry depends on the size of a , there is no single optimal configuration. However, for instruments of a set size, we can determine the optimal shapes through the methodology outlined in Section 3.4.4. To illustrate this, we return now to the systems lumped-parameter model from Chapter 2, and consider minimising R_{Σ} , through determining optimal elliptical cross-sections for the access sheath and scope shaft. Theoretically, we could also optimise the cross-sectional shapes for a working tool and the working channel to minimise R . We consider the scope and sheath radii of 1.58×10^{-1} cm and 1.93×10^{-1} cm, respectively; a dimensionless outer effective radius of $a = 1.22$ (as opposed to $a = 2$ as in Section 3.4.4). The flow profile corresponding to the optimal configuration is shown in Figure 3.13a. We see that, much like for $a = 2$, as considered in Section 3.4.4, the optimal configuration has a small gap between the vertex and the edge of the rod, splitting the domain in two. Interestingly, the optimal orientation for $a = 1.22$ is $\Theta = 0$, rather than $\Theta = \pi/2$ for $a = 2$. For an inlet pressure of $P_{\text{in}} = 60$ cm H₂O, the dimensionless steady-state flux, via equation (2.41), is $0.34 \text{ cm}^3/\text{s}$. This is in comparison to $0.3 \text{ cm}^3/\text{s}$ and $0.24 \text{ cm}^3/\text{s}$ for an offset and centred circular scope in a circular sheath, respectively.

In Figure 3.13b, we plot the time-dependent solution for kidney pressure from the systems model, equation (2.39), assuming an inlet pressure of $P_{\text{in}} = 60$ cm H₂O. This is a

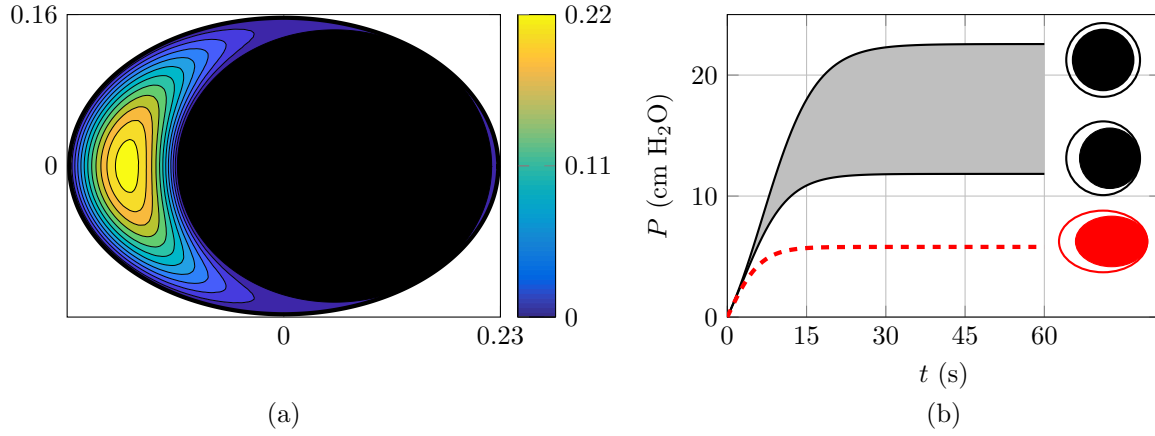


Figure 3.13: (a) The optimal geometry for the scope/sheath configurations with dimensions provided in Table 2.4. Dimensions indicated in cm units, and the colourbar provides velocity (cm/s) for $P_{in} = 60$ cm H₂O. (b) Kidney pressure, P , from equation (2.39) for a centred and offset circular scope in a circular sheath, and the solution for the optimal configuration, dashed red line, from (a).

reproduction of Figure 2.11b, although we add in a dashed red line, corresponding to the solution for the optimal scope/sheath configuration, as shown in Figure 3.13a.

The dimensionless resistance ratio of the optimal configuration is $\mathcal{R} = 9.1$, compared to that of the offset circular scope/sheath, $\mathcal{R} = 4.1$. Notably, the predicted steady-state renal pressure for the optimal configuration is over 50% lower than for the offset circular scope in a circular sheath, demonstrating the potential for significant reductions in renal pressure through optimising the cross-sectional shapes of relevant ureteroscopic devices.

Appendix 3.A Numerical details

3.A.1 Elements

The finite elements are implemented with an open-source finite element library, `oomph-lib` (Heil & Hazel, 2006). In Section 3.4, the elements for solving the Poisson equation are first-order triangular elements. In Section 3.5, the elements for solving the Stokes equations are triangular Taylor-Hood elements (second-order for velocity and first-order for pressure).

3.A.2 Mesh

An unstructured mesh of triangular elements is generated for each domain via the use of the Triangle library within an `oomph-lib` driver code (Shewchuk, 1996). A single, automatic mesh adaptation is enabled, to allow elements to cluster where there is a narrow gap between the rod and the outer cylinder. The maximum allowed element area is set to be 0.1 and the minimum element area to be 10^{-6} . A typical number of elements is 4000.

Appendix 3.B Existing analytical solutions

3.B.1 Piercy et al. (1933)

The flow rate, equation (3.34) for $\frac{dp}{dz} = -1$, through a circular pipe with offset circular core is (Piercy *et al.*, 1933)

$$Q = \frac{\pi}{8} \left[a^4 - b^4 + \frac{4d^2c^2}{\eta_a - \eta_b} - 8d^2c^2 \sum_{n=1}^{\infty} \frac{n \exp[-n(\eta_a + \eta_b)]}{\sinh[n(\eta_b - \eta_a)]} \right], \quad (3.52)$$

where a is the radius of the outer pipe, b the radius of the core, $d = \phi(a - b)$ the distance between the centres of the pipe and the core,

$$\eta_b = \cosh^{-1} [(\gamma(1 + \phi^2) + (1 - \phi^2))/(2\phi\gamma)], \quad \eta_a = \cosh^{-1} [(\gamma(1 - \phi^2) + (1 + \phi^2))/(2\phi)], \quad (3.53a,b)$$

where $\gamma = b/a$, and

$$c = b|\sinh \eta_b| = a|\sinh \eta_a|. \quad (3.54)$$

In Figure 3.14 we plot $Q(N)$ (which we define as equation (3.52) with the sum truncated at $n = N$) as a function of N , for $a = 2$, $b = 1$, and three values of ϕ . We observe that

the rate of convergence decreases monotonically with ϕ . When validating our numerical solution for flow in an offset circular annulus, and when plotting the analytical solution for Q in Figure 3.3, we take $N = 50$. For $\phi = 0.99$ – the largest ϕ -value considered in this chapter, and thus the ϕ -value which will have the largest error induced by truncating (3.52) – the relative error for $N = 50$ is

$$\varepsilon = \frac{Q(50) - Q(49)}{Q(50)} = \mathcal{O}(10^{-8}). \quad (3.55)$$

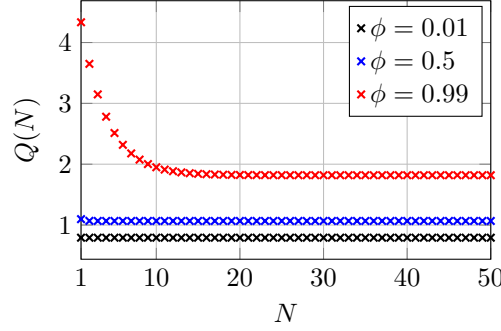


Figure 3.14: Equation (3.52) with the infinite sum truncated at $n = N$ as a function of N for $\phi = 0.01$, $\phi = 0.5$, and $\phi = 0.99$.

3.B.2 Slezkin (1955)

The opposing torque and generated lift force – for a circular rod of radius b , offset in a fluid-filled circular pipe of radius a – have been solved for analytically (Slezkin, 1955). The forms of the Stokes coefficients are thus

$$C_y = 4\pi \frac{\sqrt{k^2 + 1} - 2k \cosh(\sigma)}{\sigma(k^2 + 1) - 2k \sinh(\sigma)}, \quad A_{zz} = 4\pi \frac{\sigma k^2 \coth(\sigma) - 2k \cosh(\sigma) + 1}{\sigma(k^2 + 1) - 2k \sinh(\sigma)}, \quad (3.56a,b)$$

where

$$k = 1 + (a - b)/b, \quad \sigma = \eta_b - \eta_a, \quad (3.57a,b)$$

where η_b and η_a are functions of the offset between the rod and the pipe, ϕ , as defined in equation (3.53). Slezkin (1955) also determined, in this scenario, that $C_x = 0$.

Appendix 3.C Position that maximises flux (Section 3.4.1)

The rod position that maximises axial flux for a circular rod in an elliptic cylinder is the position as close to the vertex of the bounding cylinder as geometrically possible. The rod

will be able to fit tangent to the vertex when the radius of the inner circle is less than the radius of curvature of the ellipse at the vertex, which for a cylinder of major axis α_a and minor axis β_a is β_a^2/α_a .

Thus, with b the radius of the rod, if $\beta_a^2/\alpha_a \geq b$ then the rod position that maximises flux is $\alpha_a - b$ from the centre of the ellipse. If $\beta_a^2/\alpha_a < b$, we can solve for the position of the circle centre where the circle and the ellipse are tangent in two locations. This is $\sqrt{\alpha_a^2 - \beta_a^2} \sqrt{1 - b/\beta_a^2}$ from the centre of the bounding ellipse.

Thus, for our configuration, where the dimensionless rod radius is 1, the rod position that maximises flux is

$$X = \alpha_a - 1, \text{ if } e_a \geq e^*, \quad X = \alpha_a e_a \sqrt{1 - \beta_a^{-2}}, \text{ if } e_a < e^*, \quad (3.58a,b)$$

where

$$e^* = \sqrt{1 - a^{-4/3}}. \quad (3.59)$$

A closer look at the ellipse vertex for configuration (ii) in Figure 3.2 (the configuration that maximises flux over all e_a for $a = 2$) is shown in Figure 3.15. The two points of intersection between the rod and bounding cylinder are indicated by red dots. As the eccentricity of the outer ellipse is greater than e^* , the inner circle touches the outer ellipse in two locations, rather than just at the vertex.

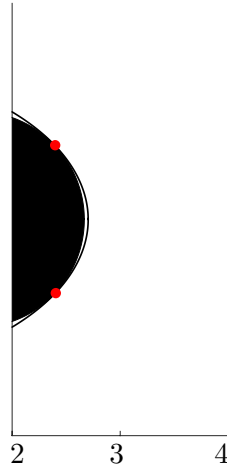


Figure 3.15: The geometry that maximises flux in Figure 3.2. A close-up of configuration (ii) near the vertex of the outer cylinder.

Appendix 3.D Domain transformation (Section 3.4.3)

Here we describe the coordinate transformation used in Section 3.4.1; we map a domain of an ellipse in an ellipse of equal eccentricities and orientations to a rectangle (this facilitates

use of a finite-difference scheme).

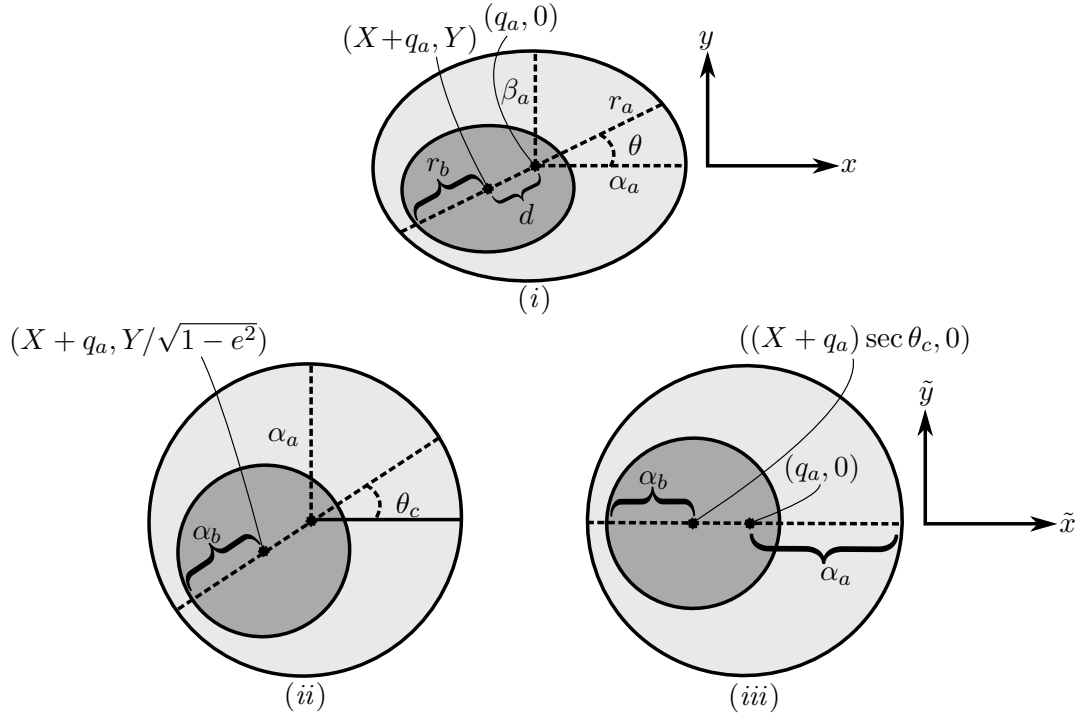


Figure 3.16: A two-step coordinate transformation maps the offset ellipses in Figure (i) to the offset circles in Figure (ii) whose centres both lie along the $\tilde{x}$ -axis. In Figure (i) the ellipses both have eccentricity e with major axes α_b, α_a and minor axes β_b, β_a . The centre of the inner ellipse is offset from the x -axis by an angle θ and the radii of the inner and outer ellipses along a line connecting their centres are r_b and r_a , respectively. The coordinates of the centres of the inner and outer ellipse in the (x, y) coordinate system are $(X + q_a, Y)$ and $(q_a, 0)$, respectively. Figure (ii) depicts the domain after scaling the y -axis by $1/\sqrt{1 - e^2}$ to convert the ellipses to circles. Figure (iii) shows the domain after rotating the coordinates in (ii) so both centres now lie on the same axis in $(\tilde{x}, \tilde{y})$ coordinates.

The centre of the inner ellipse cross-section in the (x, y) plane, $\mathbf{X} = (X, Y)$, is a distance d away from the centre of the outer ellipse, $(0, 0)$ at an angle θ from the x -axis as shown in Figure 3.16 (i) so that

$$\mathbf{X} = (d \cos(\theta + \pi), d \sin(\theta + \pi)). \quad (3.60)$$

We first shift the centres of the circles along the x -axis by q_a , which we define in Section 3.D.2. This is to allow for the subsequent usage of bipolar coordinates. The equations for boundaries of the curved surfaces Γ_b and Γ_a are then given by

$$(x - X - q_a)^2 / \alpha_b^2 + (y - Y)^2 / \beta_b^2 = 1, \quad (x - q_a)^2 / \alpha_a^2 + y^2 / \beta_a^2 = 1, \quad (3.61)$$

respectively. We transform the elliptical geometry in Figure 3.16 (i) to offset circles via the following steps.

3.D.1 Ellipses to circles

We scale the axes to transform the ellipses to circles, see Figure 3.16 (ii), and then rotate this new coordinate system so that the centre of the rod cross-section is located along the horizontal-axis of the new coordinate system, denoted $\tilde{x}$, see Figure 3.16 (iii). We introduce the change of variables

$$\tilde{x} = x \cos \theta_c - (\sin \theta_c / \sqrt{1 - e^2})y - q_a \cos \theta_c + q_a, \quad (3.62a)$$

$$\tilde{y} = (x - q_a) \sin \theta_c + (\cos \theta_c / \sqrt{1 - e^2})y - \sin \theta_c, \quad (3.62b)$$

from (x, y) to $(\tilde{x}, \tilde{y})$. The scaling of the y -axis to transform the ellipses to circles appears as the $1/\sqrt{1 - e^2}$ factor in equation (3.62). The angle θ_c , illustrated in Figure 3.16 (ii), is given by

$$\theta_c = \arctan \left(Y \sqrt{1 - e^2} / X \right). \quad (3.63)$$

In terms of $(\tilde{x}, \tilde{y})$ the expressions for the inner and outer ellipses, equation (3.61), are now

$$(\tilde{x} - (X + q_a) \sec \theta_c)^2 + \tilde{y}^2 = \alpha_b^2, \quad \tilde{x}^2 + \tilde{y}^2 = \alpha_a^2, \quad (3.64)$$

respectively, which correspond to circles of radii α_b and α_a centred along the $\tilde{x}$ -axis, see Figure 3.16 (iii). We find that

$$\phi = d/d_{\max} = X \sec \theta_c / (\alpha_a - \alpha_b), \quad (3.65)$$

so ϕ is conserved through the coordinate transformation and the relative shift in the elliptical domain in Figure 3.16 (i) is the same as the relative shift in the circular domain in Figure 3.16 (iii).

3.D.2 Bipolar coordinates

To convert the offset circles to a rectangular domain we use a bipolar coordinate system defined by

$$\eta = \arctan [2c\tilde{x}/(\tilde{y}^2 + \tilde{x}^2 + c^2)], \quad \xi = \arctan [2c\tilde{y}/(\tilde{y}^2 + \tilde{x}^2 - c^2)], \quad (3.66a,b)$$

where lines of constant η correspond to non-intersecting, non-concentric circles of different radii centred along the x -axis. Constant parameter c is given by equation (3.54) which here is

$$c = \alpha_b |\sinh \eta_b| = \alpha_a |\sinh \eta_a|, \quad (3.67)$$

where η_b and η_a are constant η values parametrising the two circles, given by equation (3.53) for $\gamma = \alpha_b/\alpha_a$. We note from equations (3.53) that η_b and η_a are defined and distinct for $0 < \phi < 1$. The outer circle is centred at x -value

$$q_a = c \coth \eta_a. \quad (3.68)$$

Equation (3.68) provides the initial shift along the x -axis in the domain. Thus, we have the required information to perform the coordinate transformation specified by equation (3.62) and to subsequently convert to bipolar coordinates via equation (3.66). Under the transformations stipulated by equations (3.62) and (3.66), the governing Poisson equation (3.27) becomes

$$A \frac{\partial^2 w}{\partial \eta^2} + 2B \frac{\partial^2 w}{\partial \eta \partial \xi} + C \frac{\partial^2 w}{\partial \xi^2} + D_1 \frac{\partial w}{\partial \eta} + D_2 \frac{\partial w}{\partial \xi} = c^2, \quad (3.69)$$

where A , B , C , D_1 , and D_2 , are functions of η and ξ given by

$$\begin{aligned} A(\eta, \xi) &= (\cos \xi \cosh \eta - 1)^2 (\cos^2 \theta + k^2 \sin^2 \theta) + \sin^2 \xi (k^2 \cos^2 \theta + \sin^2 \theta) \sinh^2 \eta, \\ B(\eta, \xi) &= (k^2 - 1) \cos(2\theta) (\cos \xi \cosh \eta - 1) \sin \xi \sinh \eta, \\ C(\eta, \xi) &= (\cos \xi \cosh \eta - 1)^2 (k^2 \cos^2 \theta + \sin^2 \theta) + \sin^2 \xi (\cos^2 \theta + k^2 \sin^2 \theta) \sinh^2 \eta, \\ D_1(\eta, \xi) &= (k^2 - 1) \cos^2 \theta (\cos \xi - \cos(2\xi) \cosh \eta) \sinh \eta, \\ D_2(\eta, \xi) &= (k^2 - 1) \cos^2 \theta \sin \xi (\cos \xi \cosh^2 \eta + \cos \xi \sinh^2 \eta - \cosh \eta), \end{aligned}$$

where $k = 1/\sqrt{1 - e^2}$. The no-slip boundary conditions, equations (3.28), are

$$f(\eta, \xi) = 0 \text{ on } \eta = \eta_b, \eta_a. \quad (3.71)$$

Additionally, we implement periodic boundary conditions in ξ , given by

$$f(\eta, 0) = f(\eta, 2\pi), \quad \frac{df}{d\xi}(\eta, 0) = \frac{df}{d\xi}(\eta, 2\pi). \quad (3.72)$$

We solve equation (3.69) with boundary conditions (3.71) and (3.72) with a finite difference scheme implemented in MATLAB, using central differences to approximate the derivatives in equation (3.69) and one-sided differences to approximate the periodic boundary conditions in equation (3.72). Finally, we can use the computed velocity profiles to calculate the dimensionless volumetric flow rate, given by

$$Q = \int_0^{2\pi} \int_{\eta_a}^{\eta_b} \left| \frac{\partial(x, y)}{\partial(\eta, \xi)} \right| f d\eta d\xi, \quad (3.73)$$

which, after explicitly calculating the Jacobian of the transformation, becomes

$$Q = c^2 \sqrt{1 - e^2} \int_0^{2\pi} \int_{\eta_a}^{\eta_b} \frac{f}{(\cos \xi - \cosh \eta)^2} d\eta d\xi. \quad (3.74)$$

We approximate this integral numerically in MATLAB with the trapezoidal rule. We validated our numerics by comparing the flux results for circular domains to the analytical solution given by Piercy *et al.* (1933), equation (3.52), and for the elliptical domains we also solved equation (3.27) using finite elements in `oomph-lib` (Heil & Hazel, 2006), demonstrating good agreement.

Appendix 3.E Additional results (Section 3.4.3)

To complement the results in Section 3.4.3, we fix two of our three parameters (e , ϕ , and θ) and plot Q as a function of the third. Figure 3.17a shows a monotonic increase in Q with ϕ for $\theta = 0$ and $e = 0, 0.4, 0.65$, and 0.8 . Figure 3.17b captures a zoomed-in view of Figure 3.17a between $0.9 \leq \phi \leq 0.99$, and shows that over this range of ϕ , for the four eccentricity values plotted, $e = 0.65$ has the highest flux and $e = 0$ has the lowest. Figure 3.17c plots the non-monotonic relationship between θ and Q indicated by Figures 3.5e-h for $e = 0, 0.4, 0.65$, and 0.8 . The flux is largest for $\theta = 0$ and has a minimum between $\theta = \pi/4$ and $\theta = \pi/2$. For $e = 0$, Q is constant with respect to ϕ as expected due to the rotational symmetry of the domain. Figure 3.17d shows Q as a function of e for fixed offset, $\phi = 0.99$, and $\theta = 0, \pi/4$, and $\pi/2$. For $\theta = 0$, the flux is maximum for e between 0.6 and 0.8 . For $\theta = \pi/4$, there is a maximum Q near $e = 0.3$, whereas for $\pi/2$, Q decreases monotonically with e . The results in Figure 3.17 suggest that $\phi = 0.99$, $\theta = 0$, and a non-zero value of e lead to the maximum flux.

Appendix 3.F Curvature matching (Section 3.4.4)

The equations for Γ_b and Γ_a are

$$\frac{((x - X) \cos \Theta + (y - Y) \sin \Theta)^2}{\alpha_b^2} + \frac{((x - X) \sin \Theta + (y - Y) \cos \Theta)^2}{\beta_b^2} = 1, \quad (3.75a)$$

$$\frac{x^2}{\alpha_a^2} + \frac{y^2}{\beta_a^2} = 1, \quad (3.75b)$$

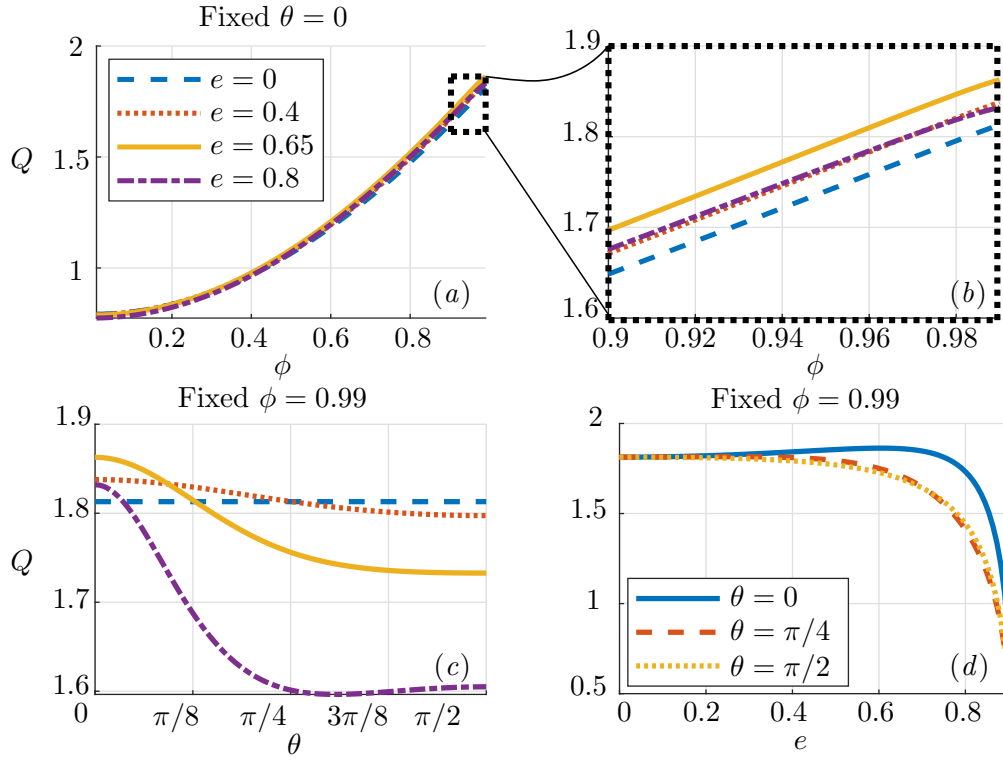


Figure 3.17: Dimensionless flow rate, Q , as a function of (a) and (b) offset, ϕ , (c) eccentricity, e , and (d), angular position, θ . Figure (b) displays the outlined section of (a). Here $a = 2$.

respectively, where $\alpha_{a,b}$ and $\beta_{a,b}$ can be calculated from a and $e_{a,b}$ via equations (3.38). We seek values for X , Y , Θ , and e_b so that the inner ellipse will touch the outer ellipse tangentially with matching curvature at a point

$$(x, y) = (\alpha_a \cos \theta, \beta_a \sin \theta), \quad (3.76)$$

with its centre located furthest from the centre of the outer ellipse.

Transforming coordinates

$$(x, y) \rightarrow \left(\frac{x}{\alpha_a} \cos \theta' + \frac{y}{\beta_a} \sin \theta', \frac{x}{\alpha_a} \sin \theta' - \frac{y}{\beta_a} \cos \theta' \right) := (\tilde{x}, \tilde{y}), \quad (3.77)$$

where $\theta' = \pi/2 - \theta$. The equations for the inner and outer ellipse are then

$$\frac{(\tilde{x} \cos \tilde{\Theta} + (\tilde{y} - \tilde{Y}) \sin \tilde{\Theta})^2}{\tilde{\alpha}_b^2} + \frac{(\tilde{x} \sin \tilde{\Theta} + (\tilde{y} - \tilde{Y}) \cos \tilde{\Theta})^2}{\tilde{\beta}_b^2} = 1, \quad (3.78a)$$

$$\frac{\tilde{x}^2}{\tilde{\alpha}_a^2} + \frac{y^2}{\tilde{\beta}_a^2} = 1, \quad (3.78b)$$

As the outer boundary is now circular, we seek without loss of generality, a geometry for

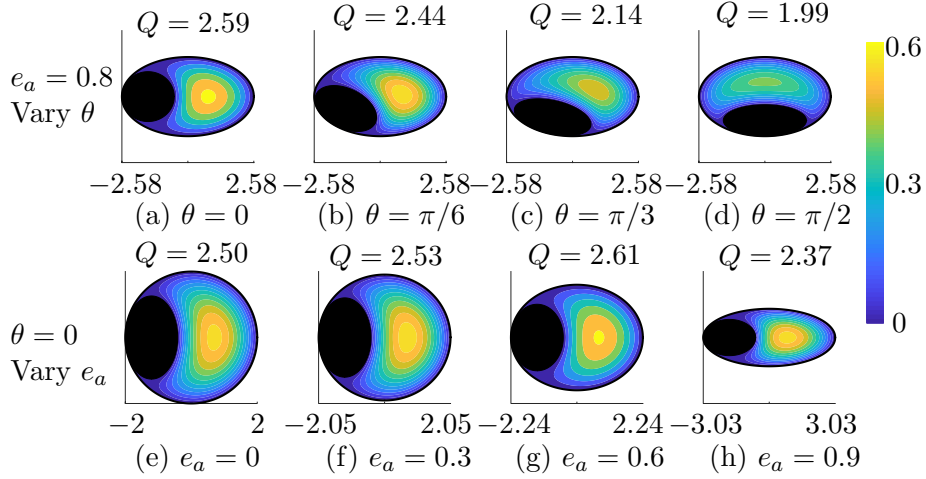


Figure 3.18: Dimensionless velocity colourmaps with dimensionless flow rates, Q . Axes are in (x, y) coordinates and these vary with eccentricity as the available space for fluid flow is constant. Plots (a)-(d) show the effect of the position of the inner ellipse, θ , and plots (e)-(h) show the effect of the outer ellipse eccentricity, e_a . Given θ and e_a , we solve for X , Y , and Θ , so that the ellipses touch at a single point with matching tangent and curvature.

the inner ellipse that touches the outer boundary in a single point, $(0, \tilde{Y})$ with matching tangent and curvature. As a is fixed, the major and minor axes of the ellipses are determined by their eccentricity values. To maximise the distance between the centres of the ellipse we take $\tilde{\Theta} = 0$ and the ellipse centres are a distance $1 - \tilde{\beta}_b$ apart. Hence, we have

$$\tilde{\alpha}_b = a^{-2/3}, \quad \tilde{\beta}_b = a^{-4/3}, \quad \tilde{X} = 0, \quad \tilde{Y} = 1 - a^{-4/3}. \quad (3.79)$$

We can therefore determine the position and orientation angle of the initial ellipse by reversing the transformation (3.77). Some velocity colourmaps are shown in Figure 3.18. The values for θ and e_a in Figures 3.18a-h correspond to those in Figures 3.5e-l, respectively. This comparison shows that Q is larger for e_b , Θ that match the curvature of the outer ellipse (Figure 3.18), compared to $e_b = e_a$ and $\Theta = 0$ (Figure 3.5).

Appendix 3.G Parameter sweep (Section 3.4.4)

Before solving the axial flow optimisation problem (outlined in Section 3.4.4 with more details in Appendix 3.H) we considered whether geometries other than the ‘curve hugging’ configurations would provide maximum flow by performing a parameter sweep for fixed e_a . We set $e_a = 0.6$ (and $a = 2$), and swept through the remaining four geometric parameters, e_b , X , Y , and Θ . For each configuration, we solved equation (3.27) subject to conditions (3.28) in `oomph-lib` (Heil & Hazel, 2006). Due to the symmetry of the domain, it was necessary to only vary the position of the rod in a single (x, y) quadrant. Without loss of generality, we chose (X, Y) in Quadrant I (see Figure 3.19).

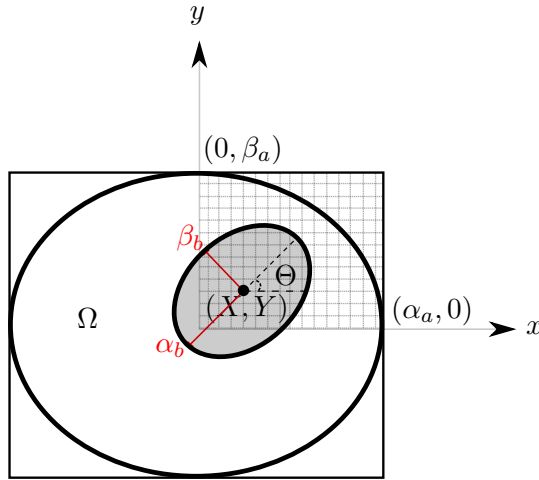


Figure 3.19: The outer ellipse has fixed eccentricity, $e_a = 0.6$, and characteristic radius, $a = 2$. The major and minor axes are $\alpha_a \approx 2.2$ and $\beta_a \approx 1.8$, respectively. The inner ellipse has eccentricity $e_b \in [0, 0.9]$, and is centred at (X, Y) , where $X \in [0, 2.2]$ and $Y \in [0, 1.8]$. The inner ellipse is oriented so that its major axis makes an angle of $\Theta \in [0, 2\pi]$ with the x -axis.

We varied $e_b \in [0, 0.9]$, $X \in [0, \alpha_a]$, $Y \in [0, \beta_a]$, and $\Theta \in [0, 2\pi]$. Not all these configurations are feasible, i.e. the ellipses will often intersect, so we selected candidate geometries as those where the equation of the outer ellipse was negative on a fixed number of uniformly-spaced boundary points (we took $N = 1000$) of the inner ellipse.

3.G.1 Initial sweep

We initially swept through e_b , X , Y , and Θ with the following increments:

- $e_b = [0 : 0.1 : 0.9]$,
- $X = [0 : 0.1 : 2.2]$,
- $Y = [0 : 0.1 : 1.8]$,

- $\Theta = [0 : 2\pi/100 : 2\pi]$.

We calculated the maximum Q for each e_b , and we plot this as a function of e_b in Figure 3.20a. The geometries and corresponding flow profiles are shown as inset diagrams for $e_b = 0$, $e_b = 0.5$, $e_b = 0.8$, and $e_b = 0.9$. This initial sweep allowed us to identify ranges of X , Y , and Θ where the maximum Q occurs. We subsequently narrowed our search to the identified region and reduced the step-sizes in X , Y , and e_b .

3.G.2 Finer sweep

Following the initial sweep, we narrowed our search area to encompass only the region where maximum flow was achieved, and halved the step-sizes in X , Y , and e_b . We considered the following increments:

- $e_b = [0 : 0.05 : 0.9]$,
- $X = [0.9 : 0.05 : 1.4]$,
- $Y = [0 : 0.05 : 1.4]$,
- $\Theta = [2\pi(47/100) : 2\pi/100 : 2\pi(76/100)]$.

3.G.3 Results

In the results of both the initial and finer parameter sweeps, we see that maximum flow is achieved for $e_b = 0.8$ (Figures 3.20a and 3.20b). However, in the initial sweep, the steps in X and Y were too coarse to identify the true optimum geometry, and the geometry that maximised flux in Figure 3.20a was asymmetric. Refinement in the finer sweep allowed us to identify the symmetric optimum (Figure 3.20b) which is symmetric and touches the boundary tangentially in two locations, rather than a single point. These results motivated us to explore the full optimisation problem, beyond the cases that are tangent at a single point and match curvature.

The lowest flux for a given e_b , $\min(Q)$, occurs when $(X, Y) = (0, 0)$ and $\Theta \approx 0$. A plot of $\min(Q)$ vs. e_b is shown with geometries and flow profiles indicated for $e_b = 0$, $e_b = 0.5$, and $e_b = 0.9$.

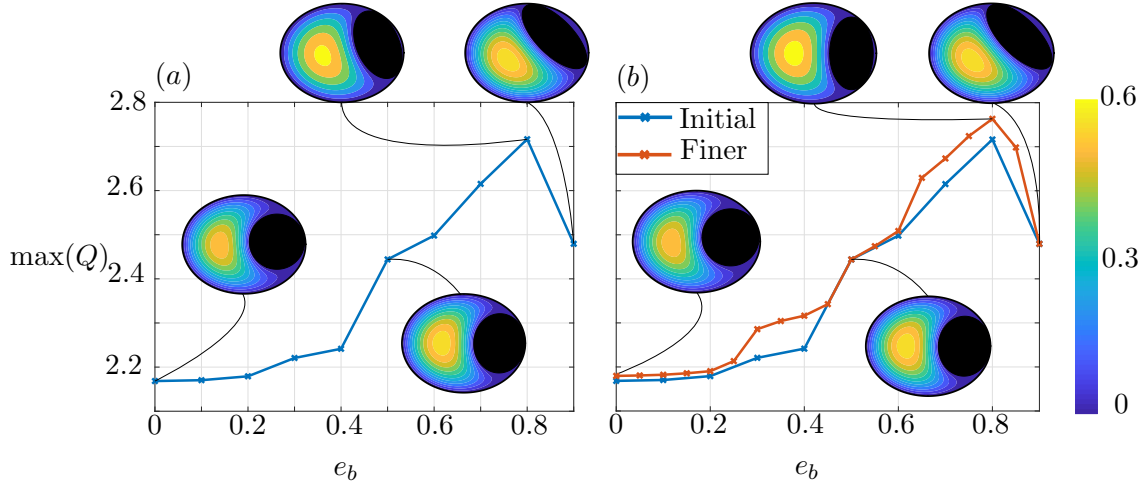


Figure 3.20: Plot of the maximum flux for each e_b value as a function of e_b . The inset flow profiles for $e_b = 0$, $e_b = 0.5$, $e_b = 0.8$, and $e_b = 0.9$ show the corresponding geometries and velocity profiles. The blue line shows results from the initial sweep, and the red line shows the results from the finer sweep.

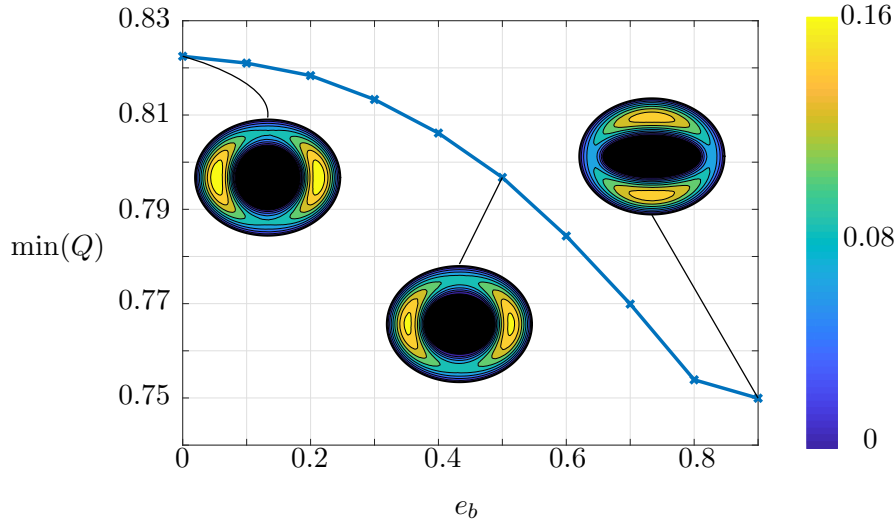


Figure 3.21: Plot of the minimum flux for each e_b value as a function of e_b . The inset flow profiles for $e_b = 0$, $e_b = 0.5$, and $e_b = 0.9$ show the corresponding geometries and velocity profiles.

Appendix 3.H Optimisation constraints (Section 3.4.4)

From equations (3.43), we have that the equations of the outer and inner ellipses can be written respectively as

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 0, \quad \mathbf{x}^T \mathbf{B} \mathbf{x} = 0,$$

where $\mathbf{x} = (x, y, 1)^T$ and $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of coefficients, functions of the components of $\mathbf{g}$, equation (3.42). The intersections of two conics can be determined by considering their pencil, $\lambda \mathbf{A} + \mathbf{B}$ (Casey, 1893). Conditions to determine the relative

position of two ellipses were derived by Etayo & Gonzalez-Vega (2006), which we can then write in terms of the components of $\mathbf{g}$. The characteristic polynomial of the pencil (once turned monic) is

$$f(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c. \quad (3.80)$$

In order for the inner ellipse to be fully enclosed in the outer ellipse the following conditions must hold

$$a > 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 > 0, \quad (3.81a-c)$$

$$-27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0, \quad (3.81d)$$

along with conditions

$$n_{22}\det(N) > 0, \quad \det(N_{11}) > 0, \quad (3.82a,b)$$

where $N = v\mathcal{A} + \mathcal{B} = (n_{ij})$ with $v = (-a + \sqrt{a^2 - 3b})/3$ and N_{11} is the minor of N (Etayo & Gonzalez-Vega, 2006). Equations (3.81) and (3.82) provide a set of six nonlinear constraints on $\mathbf{g}$. Thus, the optimisation problem can be formulated as equation (3.44)

$$\min_{\mathbf{g}} -Q, \quad \text{s.t. } \mathbf{c} \geq 0, \quad (3.83)$$

where

$$\mathbf{c} = \begin{bmatrix} a \\ -3b + a^2 \\ 3ac + ba^2 - 4b^2 \\ -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 \\ n_{22}\det(N) \\ \det(N_{11}) \end{bmatrix}. \quad (3.84)$$

Appendix 3.1 Flux enhancement in multiply connected ellipses

Ranger (1996) suggests that physical insight can be gained into mechanisms of flux enhancement by examining flow through a circular pipe, although considering an elliptical pipe, instead, leads to an identical discussion which we present here. The solution to equation (3.27) subject to conditions (3.28) through an ellipse of major axis α and minor axis β is

$$f(x, y) = \frac{\pi}{2(\alpha^2 + \beta^2)}(\beta^2x^2 + \alpha^2y^2 - \alpha^2\beta^2), \quad (3.85)$$

and hence flux is

$$Q = \frac{\pi(\alpha\beta)^3}{4(\alpha^2 + \beta^2)}. \quad (3.86)$$

Dividing equation (3.86) by the cross-sectional area of the ellipse, $\pi\alpha\beta$, we obtain the flux-per-unit area

$$S = \frac{(\alpha\beta)^2}{4(\alpha^2 + \beta^2)}. \quad (3.87)$$

Now, considering an elliptical sub-region of the pipe cross-section, centred at zero, bounded by major and minor axes $\lambda\alpha$ and $\lambda\beta$, respectively (for $0 \leq \lambda \leq 1$), the flux is

$$Q_\lambda = \frac{(\alpha\beta)^3}{4(\alpha^2 + \beta^2)}(2\lambda^2 - \lambda^4), \quad (3.88)$$

and dividing by the cross-sectional area of the sub-region, $\lambda^2\pi ab$, to again obtain the flux-per-unit-area

$$S_\lambda = \frac{(\alpha\beta)^2}{4(\alpha^2 + \beta^2)}(2 - \lambda^2). \quad (3.89)$$

Now considering the ratio S_λ/S ,

$$\frac{S_\lambda}{S} = 2 - \lambda^2, \quad (3.90)$$

demonstrates that flux-per-unit-area is larger for any region containing the centre, than through the remainder of the domain, away from the central axis. This demonstrates the variation in flux-per-unit area through the cross-section of an elliptical domain, and suggests that flux enhancement can occur with multiply connected regions by removing areas of low flux-per-unit-area, and replacing them with regions of equivalent area with higher flux. Although the question of optimising the axial flow through a region bounded by two ellipses is complicated by fixed geometric constraints, this discussion of how splitting the domain may enhance total flux can be extended to understanding the optimal geometry in Figure 3.7c. In this case, although the cross-sectional area to the left of the rod is smaller than in the case of the ‘curve-hugging’ geometry (Figure 3.7b), the benefit of ‘removing’ some regions of low flux-per-unit-area above and below the rod and replacing them with a disconnected sub-region between the outer ellipse vertex and the edge of the rod, can increase total flux.

Appendix 3.J Additional results (Section 3.5.2)

Figure 3.22a plots A_{zz} as a function of absolute offset, $1.45 \leq X \leq 1.8$, for $e_b = 0.7$, continuing the plot on Figure 3.9a for larger values of X . We observe the same line crossing behaviour noted for $e_b = 0.9$ in Figure 3.9a.

Figure 3.22b plots K_{xx} (dotted), K_{yy} (dashed), and $K_{\min}$ (solid) for ellipses of eccentricities $e_b = 0$, $e_b = 0.7$, and $e_b = 0.9$, all oriented at angle $\Theta = \pi/2$. We see that over this range of offset values ($0 \leq X \leq 1.45$), $K_{\min} = K_{yy}$, and $\mathbf{k}_{\min} = \mathbf{j}$.

Figures 3.22c-h show either coupling or lift coefficients for $e_b = 0.7$ (left column) and $e_b = 0.9$ (right column) as functions of absolute offset, X . The dashed black lines are different Θ values, ranging from $\Theta = 0$ to $\Theta = \pi/2$ in steps of $\pi/40$. In each plot, the Θ value that maximises the force coefficient (over the majority of the X range) is indicated by a solid red line. Schematics indicate orientation for the minimal and maximal lines.

The coupling between rotation and translation in the x -direction is shown in Figures 3.22c-d. We see that for $\Theta = 0$ and $\Theta = \pi/2$, $C_x = 0$. Over the majority of the offset values, C_x is maximal when $\Theta = \pi/4$. However, for $e_b = 0.9$, there is a region close to the outer boundary, where smaller offset angles provide a higher coupling force.

The coupling between rotation and translation in the y -direction is shown in in Figure 3.22e and Figure 3.22f. For both $e_b = 0.7$ and $e_b = 0.9$, C_y increases monotonically with Θ from $\Theta = 0$ to $\Theta = \pi/2$ over the full range of offset values.

The lift force, K_{xy} , is plotted in Figure 3.22g and Figure 3.22h. This is zero for $\Theta = 0$ and $\Theta = \pi/2$ and maximal (over the majority of the offset values) for $\Theta = \pi/4$. However, for $e_b = 0.9$, as with C_x , there is a region close to the boundary, where crossing of the lines occurs.

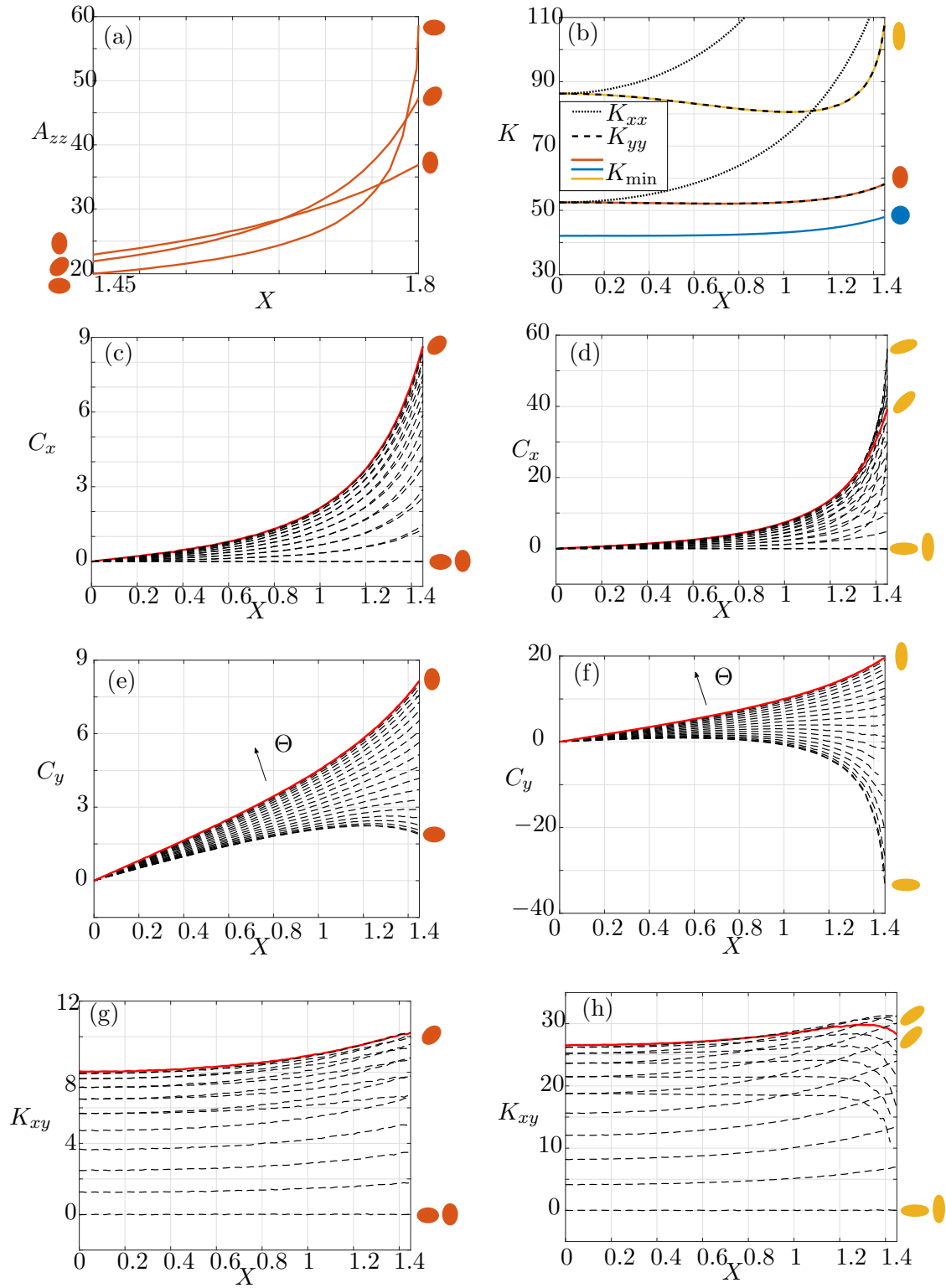


Figure 3.22: Figure (a) shows A_{zz} as a function of offset for $e_b = 0.7$ oriented at $\Theta = 0$, $\Theta = \pi/4$, and $\Theta = \pi/2$. Offset, $X \in [1.45, 1.8]$ (a continuation from Figure 3.9a). Figure (b) shows $K_{\min}$ for $e_b = 0$, $e_b = 0.7$, and $e_b = 0.9$ all oriented at $\Theta = \pi/2$. Figures (c)-(h) show coupling and lift force coefficients for $e_b = 0.7$ (left column) and $e_b = 0.9$ (right column). Orientation angles vary between $\Theta = 0$ and $\Theta = \pi/2$ in steps of $\pi/40$.

CHAPTER 4

On linearity between velocity and pressure: just a pipe dream?

It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong.

(Richard P. Feynman)

Synopsis

In this chapter we address the validity of a fully-developed flow assumption through components of ureteroscopy irrigation systems. We are motivated by a set of experimental results aimed to validate flow through elliptical annular pipes. Comparison of the experiments with fully-developed theory highlights the importance of quadratic inertial adjustments to the estimated pressure drop. We perform a suite of numerical simulations of the steady Navier-Stokes equations on domains representing typical ureteroscopic set-ups. We vary three parameters of interest: saline bag height, renal pelvis size, and scope position, with a view to determining how these factors affect the importance of including inertia in the governing equations for accurate flow rate predictions. We find that fully-developed theory underestimates the flow rate compared to predictions from the numerical simulations. We determine the influence of pressure losses due to geometrical contractions from the saline bag to the irrigation tubing, and from the irrigation tubing to the working channel. We show that a simple inertial adjustment to fully-developed theory is insufficient to account for complex entrance effects in the access sheath which are inextricably linked to flow patterns in the renal pelvis.

4.1 Introduction

Ureteroscopy irrigation relies, in essence, on a system of interconnected pipes; thus, additional fittings and sudden changes in geometry are nearly unavoidable. For instance, fluid flowing from the suspended irrigation bag to the attached tubing undoubtedly undergoes a contraction, and fluid entering the working channel from the irrigation tubing is forced through a combination of a second area contraction and a 90° bend from vertical to horizontal (Figure 1.1). The velocity patterns corresponding to flow through bespoke pipe fittings are unique and intricate involving boundary layer separation, vortices, and fluid recirculation, and hence deriving an exact analytical expression for flow and pressure through each pipe fitting is an unattainable objective.

However, an ability to capture changes in pressure resulting from pipe contractions, expansions, bends, entrances, and exits – essentially any deviation from fully-developed flow that results from the inertial $\mathbf{u} \cdot \nabla \mathbf{u}$ term in the Navier-Stokes equations – is an invaluable tool to estimate flow through hydraulic networks of particular configurations. To this end, constitutive laws have been constructed from empirical measurements, and common practice is to encapsulate the behaviour of the flow and associated dissipated kinetic energy for each sudden change in pipe geometry with a single dimensionless parameter, K

$$\text{energy loss} = K \frac{1}{2} m V^2, \quad (4.1)$$

where m is the fluid mass and V the average velocity of the flow (White, 2008). The value of K in equation (4.1) typically depends on the flow regime (laminar or turbulent), and the cross-sectional dimensions of the pipe. The pressure loss ΔP_m , associated with equation (4.1) is termed a *minor loss* (as opposed to *major losses* due to frictional effects of fluid viscosity)

$$\Delta P_m = K \frac{1}{2} \rho (Q/A)^2. \quad (4.2)$$

The gap between strictly empirical K s and those supported by mathematical analysis has been bridged in particular instances. For example, at the entrance to a pipe where fluid enters as a uniform stream, a viscous boundary layer is formed and grows until its width equals the radius of the pipe. Within the entrance length of the pipe, where the width of the boundary layer is smaller than the pipe radius, there is a core of fluid which is uninfluenced by viscosity. As the flux of fluid across any section is constant, and the boundary layer thickness is increasing, the core is accelerated and there is a corresponding fall of pressure (Tosun *et al.*, 1995; Shah & London, 1978; Crabtree *et al.*, 1988). This additional pressure drop in the entrance region of a pipe takes the form of equation (4.2). Values for K due to developing flow can be estimated analytically for pipes of arbitrary

cross-sectional geometry from the Navier-Stokes equations and incompressibility condition (1.1) under appropriate assumptions as to the relative sizes of velocity gradients (Lundgren *et al.*, 1964). The result is an expression for K that only requires knowledge of the geometry of the pipe and the fully-developed profile

$$K = \frac{2}{A} \iint_{\Omega} \left[\left(\frac{w}{W} \right)^3 - \left(\frac{w}{W} \right)^2 \right] d\Omega, \quad (4.3)$$

where A is the cross-sectional area, W the average velocity, w the fully-developed profile, and Ω the cross-sectional domain.

Another simple example of an analytical estimate for K is obtained through considering a force balance on control surfaces of the fluid before and after transition from one uniform stream to another across the sudden enlargement of a pipe (Batchelor, 1967, p. 372). Consider a pipe of cross-sectional area A_1 and uniform velocity V_1 , transitioning to a

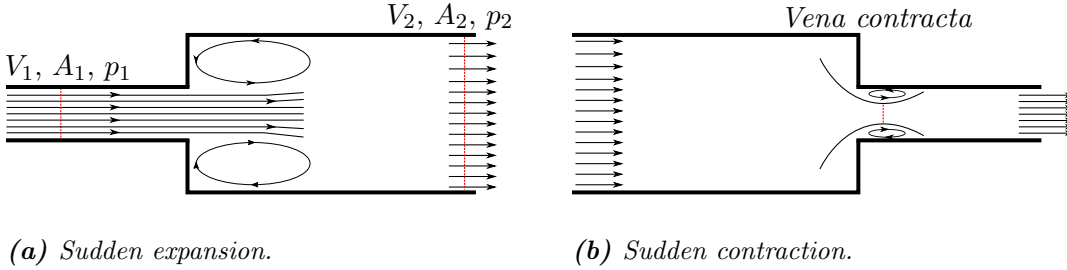


Figure 4.1: (a) Uniform flow with velocity V_1 in a pipe of cross-sectional area A_1 towards a sudden enlargement of the cross-section to a pipe of cross-sectional area A_2 in which velocity is V_2 . The constant pressures within the cross-sections indicated by red lines before and after the enlargement are p_1 and p_2 , respectively. (b) Flow through a sudden contraction. The vena contracta, where the diameter of the flow stream is minimal, is indicated by the dashed red line.

second pipe of area A_2 and uniform velocity V_2 , where $A_2 > A_1$ (Figure 4.1a). We assume constant pressures sufficiently far upstream and downstream of the contraction with values p_1 and p_2 , respectively. Conservation of mass provides $A_1 V_1 = A_2 V_2$, and, assuming viscous effects through the expansion are negligible, the internal fluid stresses are caused only by pressure, and a momentum balance is represented by

$$\rho V_2 A_2 (V_2 - V_1) = p_1 A_1 + p_2 (A_2 - A_1) - p_2 A_2, \quad (4.4)$$

where $p_2 (A_2 - A_1)$ is due to the pressure on the fluid exerted by the expansion's wall, perpendicular to the axis. Conservation of energy provides Bernoulli's principle given by

$$p_2 = p_1 + \frac{1}{2} \rho (V_1^2 - V_2^2). \quad (4.5)$$

The difference between p_2 and p_1 can thus be estimated using equations (4.4) and (4.5)

$$p_2 - p_1 = \frac{1}{2} \rho V_1^2 \left(1 - \frac{A_1}{A_2} \right)^2. \quad (4.6)$$

In equation (4.6), $K = (1 - A_1/A_2)^2$, a value ranging from zero if the cross-sectional areas of the two pipes are equal, to one for an infinitely large expansion. Sudden contractions, as demonstrated in Figure 4.1b, are more relevant to ureteroscopy than sudden expansions; however, the physical intuition for pressure losses through these geometries is particularly weak (Kays, 1950). In fact, fluid undergoing a sudden contraction initially forms a *vena contracta*, where the diameter of the flow stream is minimal, followed by a secondary expansion; theory for pressure loss through a sudden contraction assumes pressure loss due to the initial contraction of the jet to be minimal and attributes all losses to the secondary expansion. In this case, knowledge of the radius of the jet at its minimum diameter is required.

We highlight a number of key simplifications in the preceding argument. Firstly, the assumption of uniform cross-sectional velocities. More commonly, fluid entering a contraction or expansion in a pipe will have a fully-developed parabolic profile upstream of the sudden change in geometry. This can be accounted for through additional contributions to K , evaluated from knowledge of the fully-developed profile, which relate the actual momentum flux to the momentum flux evaluated from the bulk average velocity (Kays, 1950). Additionally, the neglect of frictional forces within the fluid may appear at odds with the consideration of flows governed purely by viscosity, as considered in Chapters 2 and 3. To encapsulate the full pressure drop along the length of the pipe, equation (4.2) is typically taken as the inertial contribution, and added to the pressure loss due to viscosity. Equation (1.4), which relates unidirectional flux to flow via viscous resistance can be rearranged to compute the pressure loss due to frictional effects

$$\Delta P_f = QLR\mu, \quad (4.7)$$

where we recall that R is the geometric resistance due to viscosity. Thus, the total pressure drop through a pipe, accounting for minor inertial losses, can be estimated by summing equations (4.2) and (4.7)

$$\Delta P = \Delta P_m + \Delta P_f = \frac{K\rho}{2A^2} Q^2 + L R \mu Q \quad (4.8)$$

for an appropriate value of K .

4.1.1 Chapter outline

The work in this chapter comprises a study of the influence of minor losses in ureteroscopy systems. The analysis is provoked by attempts to experimentally validate fully-developed flow through elliptical annular pipes, motivated by the work in Chapter 3. We commence this chapter with a discussion of our experiments to validate fully-developed flow through elliptical pipes and the challenges we incurred. We found that observed deviations of nearly an order of magnitude between the experimental results and fully-developed theory could be accounted for by including inertial adjustments to pressure of the form in equation (4.2).

The realisation that minor pressure losses due to inertial effects can be ironically termed, contributing significantly to the total pressure along the length of a pipe, prompts the second half of this chapter. We perform numerical simulations of the steady Navier-Stokes equations, including both viscous and inertial terms, through two-dimensional geometries representing flow from the saline bag, through the tubing, the working channel, the renal pelvis, and back through the access sheath. We consider the effects of bag height, renal pelvis length, and scope position on flow rate, with a view to determine the significance of pressure losses unaccounted for by fluid viscosity.

4.2 Experimental motivation to study entrance and exit effects

In Chapter 3, we determined how the viscous resistance to flow through a cylinder containing a coaxial rod, where cross-sectional area is fixed, varies with the elliptical eccentricities of the rod and bounding cylinder. In particular, Figure 3.2 demonstrated that for an offset circular rod, an increase in the elliptical eccentricity of the outer cylinder lowers geometric resistance. This significant finding urges experimental validation, requiring pipes of custom cross-sectional geometries to be manufactured.

4.2.1 Experimental design

Three primary challenges were encountered during conceptualisation and realisation of the experimental designs. Photographs of earlier iterations are shown in Appendix 4.A. The final 3D-printed design was chosen to overcome three primary engineering challenges:

Pipe dimensions

The length-scales of the working tool/working channel (or even slightly larger dimensions of the access sheath/scope shaft) were small compared to the tolerance of available 3D printers. Thus, it was necessary to up-scale the dimensions of the pipe cross-section. However, we were also limited in the dimension of the axial direction, with typical 3D printers having a maximum build-height of approximately $\mathcal{O}(10\text{ cm})$. We fused together two separate pieces to create a final pipe of length $L \approx 22\text{ cm}$ and chose $a = 0.5\text{ cm}$ as the radius of the outer cylinder, as this still results in a necessary small aspect ratio ($\epsilon \approx 2.0 \times 10^{-2}$). The radius of the rod was taken as $b = 0.4\text{ cm}$; this results in a radius ratio a/b similar to that of a typical scope shaft/access sheath configuration.

Position of the inner rod

We have shown, in Chapters 2 and 3, the strong interdependence between flow resistance and rod position. Thus, it was paramount that the rod position was fixed to be fully offset for experimental validation. We considered the possibility of extraneous “pins” at the entrance and exit of the pipe to hold the rod in place. However, the implementation of this mechanism created flow resistance that was unequal along the length of the pipe, and therefore complicated to account for within our two-dimensional modelling framework. Hence, we fused the rod to the wall of the bounding cylinder and filled a small area close to the point of contact creating ‘blockers’ between the outer cylinder and solid rod along the length of the pipe.

Constant driving pressure

A final complication arose when considering how to drive fluid into the pipe. A prescribed, constant inlet pressure is desirable, as flow rate is easily measurable, and would allow determination of the geometric resistance. In the bench-top experiments discussed in Section 2.2.1 and Appendix 2.A, flow was driven under the influence of gravity from a suspended reservoir. However, as determined in Section 2.2.3, viscous dissipation through irrigation tubing leads to pressure losses which depend on the resistance to flow through the working channel. This will also be true for sources of minor losses in our experiments; their relative influence will depend on resistance through the 3D-printed pipes, leading to unequal inlet pressures between the elliptical and circular geometries. Hence, in an attempt to minimise minor pressure losses, we oriented the system vertically with fluid flowing directly from a 3D printed “bucket”, which was glued to the circular or elliptical pipe. This removed the necessity for additional tubing to connect a suspended reservoir

to the 3D-printed pipes. To minimise pressure losses due to viscous dissipation through the bucket, we chose its radius to be 2.5 cm; significantly larger than the radius of the attached pipe.

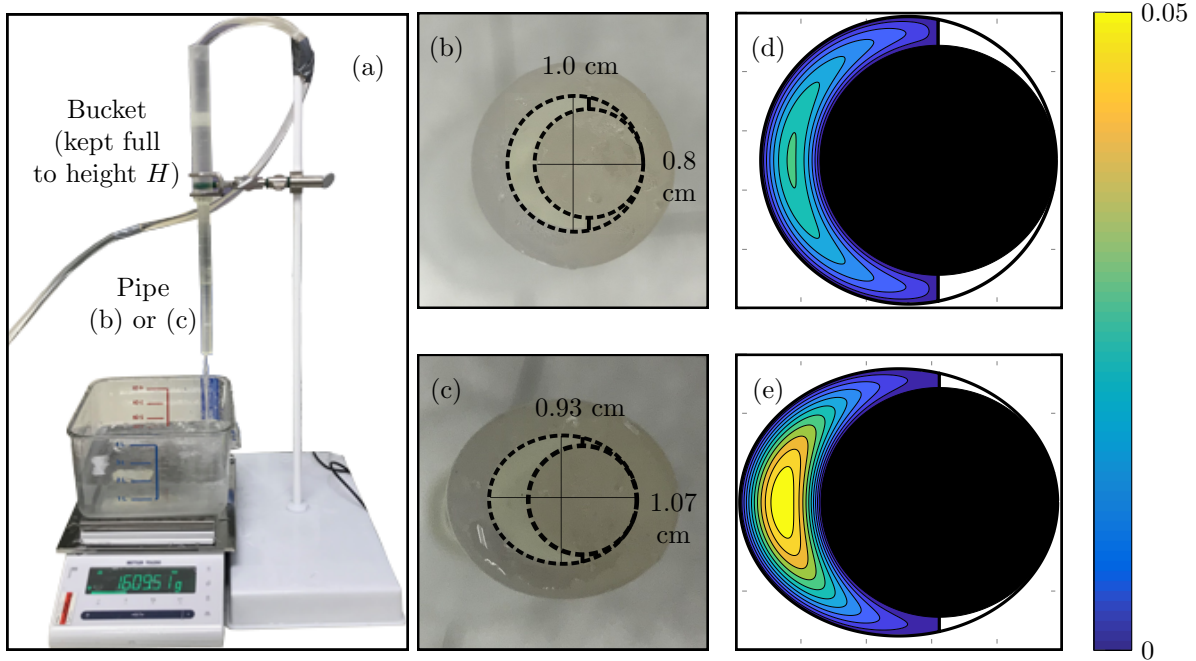


Figure 4.2: The experimental set-up is shown in Figure (a). The top half of the experimental apparatus (above the clamp) is the “bucket”, and the lower half is the 3D-printed pipe. The two pipe cross-sections are shown in (b) and (c); the outer shape is circular and elliptical, respectively. Fully-developed flow predictions in equations (d) and (e) for the two cross-sectional geometries. The white area indicates presence of the ‘blockers’. The corresponding colourbar denotes velocity in dimensionless units.

The experimental set-up is pictured in Figure 4.2a. A “bucket” of radius 2.5 cm was connected to a pipe of custom cross-section (either Figure 4.2b or 4.2c) and length 22.069 cm. The system was oriented vertically, and the bucket continuously filled to one of three marked levels above the entrance to the connected pipe: $H = 9.8$ cm, $H = 18.2$ cm, or $H = 28.2$ cm. Both the top surface of the bucket and the exit of the pipe were open to atmospheric pressure. Fluid flowed into a container on a mass balance with measurements taken at one-second intervals for sixty seconds. More experimental details and a description of the data analysis are provided in Appendix 4.B.

4.2.2 Theoretical model

We now consider a fully-developed pipe network model of the experimental set-up in Figure 4.2a. A schematic of the experimental set-up is shown in Figure 4.3. A fluid-filled “bucket”

of radius ℓ_b and filled to a height H is connected to a pipe of arbitrary cross-sectional geometry and length L . The top surface of the fluid in the bucket and the tube exit are open to atmospheric pressure and the pressure at the entrance to the pipe (base of the bucket) is P_c for the circular configuration and P_e for the elliptical configuration.

Due to the presence of ‘blockers’ which fuse the inner rod to the wall of the outer channel the fully-developed flow predictions are slightly modified from those of an offset annulus. The ‘blockers’ cause additional reductions in cross-sectional area to both the elliptical and circular pipes, with more area blocked for the circular configuration than the elliptical one. However, the removed area is close to the point of contact (between two no-slip boundaries), where the velocity is lowest. Thus, the difference in the estimated geometric resistance is negligible. Even, so, the presence of the blocked area can be accounted for mechanistically, by meshing the precise cross-sectional geometries. The solution to equation (1.5), solved using finite elements in **Firedrake** is shown in Figures 4.2d,e with colours indicating the dimensionless velocity magnitude. The values obtained for the geometric resistances are

$$R_c \approx 1.76 \times 10^3 \text{ cm}^{-4}, \quad R_e \approx 1.01 \times 10^3 \text{ cm}^{-4}, \quad (4.9a,b)$$

where R_c corresponds to Figure 4.2d and R_e to Figure 4.2e. We note there is a predicted 40% increase in flow for fixed pressure drop with the elliptical design. The resistance values for the configurations without the presence of ‘blockers’ are approximately 3% and 0.2% lower for the circular and elliptical geometries, respectively, confirming the minimal effect of ‘blockers’ on the predicted geometric resistance for fully-developed flow.

If the flux through the pipe is fully-developed, the relationship between pressure drop and flux is given by (Lamb, 1916)

$$Q_{c,e} = \frac{P_{c,e} + \rho g L}{\mu L R_{c,e}}, \quad (4.10)$$

where resistance values $R_{c,e}$ are given in equation (4.9), and the addition of $\rho g L$ is due to gravitational acceleration. We can incorporate the effect of viscous dissipation within the “bucket” by adjusting the inlet pressure $P_{c,e}$ from hydrostatic as in Section 2.2.3

$$P_{c,e} = \rho g H \left(1 - \frac{\sigma}{1 + \sigma} \right) \left(1 - \frac{R_b}{R_{c,e}} \right), \quad \sigma = (H/L)(R_b/R_{c,e}). \quad (4.11a,b)$$

Here resistance through the bucket R_b is negligible in comparison to $R_{c,e}$, $R_b = 8/\pi \ell_b^4 \approx 6.52 \times 10^{-2} \text{ cm}^{-4}$, as opposed to $R_{c,e} = \mathcal{O}(10^3 \text{ cm}^{-4})$. Thus, $P_{c,e} = \rho g H$ is a valid approximation.

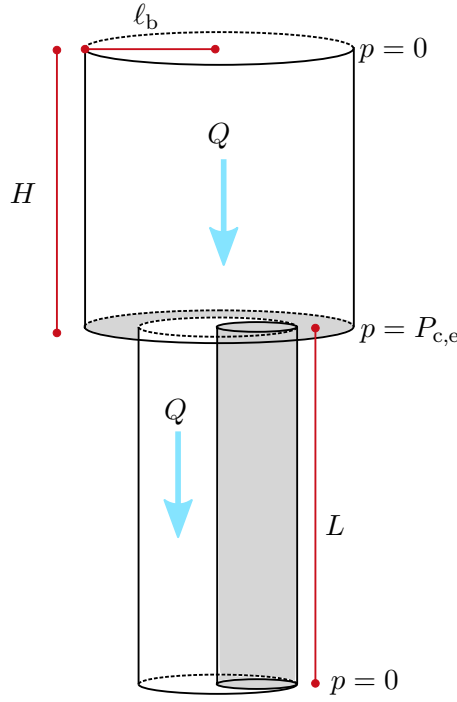


Figure 4.3: A schematic of the experiment set-up. The 3D printed tubes are of length L , connected to the bucket with fluid at variable height, H . The system is oriented vertically with the top of the bucket and tube exit at atmospheric pressure.

4.2.3 Experimental and theoretical results

The flux Q computed via equation (4.10) is plotted as the solid red and solid blue lines for the elliptical and circular configurations, respectively, in Figure 4.4a. These lines are plotted against the data points for the two pipes with corresponding colours. There is a marked discrepancy between theory and data for both sets. Adjusting for viscous dissipation through the bucket via equation (4.11) only provides a negligible decrease in the predicted pressure $P_{c,e}$. These predictions are plotted as the dashed black lines in Figure 4.4a, indiscernible from the corresponding solid lines.

Thus, we turn to inertial pressure losses to determine whether these can account for the severity of the deviation between theory and experiment. To allow for the presence of minor pressure losses, we apply equation (4.8)

$$P_{c,e} = \frac{K_{c,e}\rho}{2A_{c,e}^2}Q_{c,e}^2 + LR_{c,e}\mu Q_{c,e}, \quad (4.12)$$

where A_c and A_e are the fluid-filled cross-sectional areas of the circular and elliptical pipes, respectively, and loss coefficients $K_{c,e}$ capture the contribution of inertial pressure losses. We anticipate two contributions to $K_{c,e}$: the first due to developing flow in the pipe, equation (4.3), and the second due to a complicated geometric contraction between

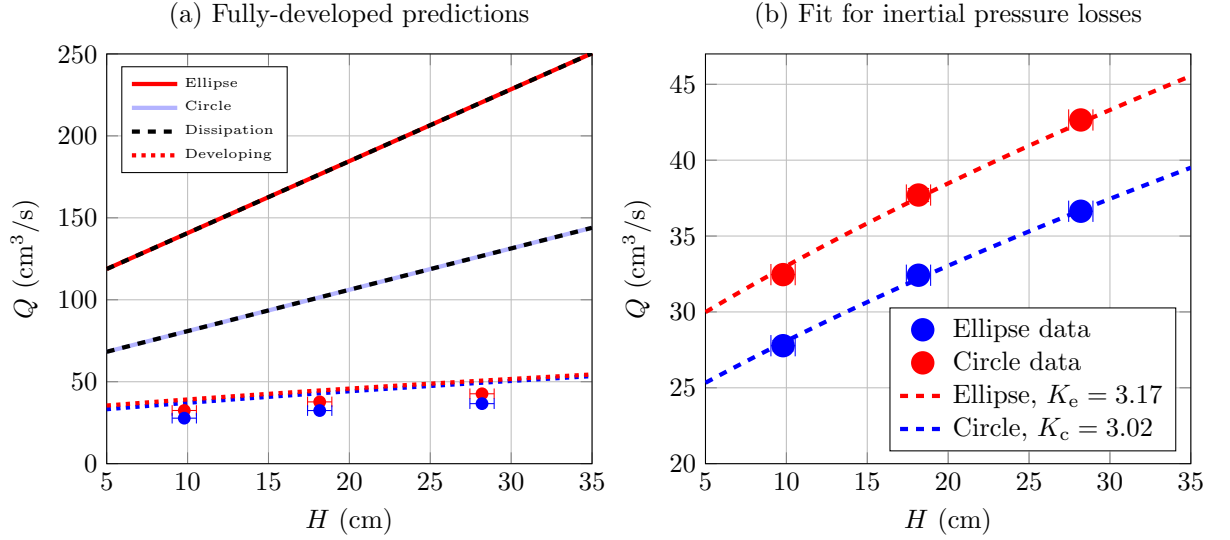


Figure 4.4: (a) A comparison of the fully-developed predictions with experimental data. (b) Fitting for an inertial pressure adjustment due to geometric contraction.

the bucket and the pipe. We thus assume $K_{c,e} = K_{c,e}^{\text{dev}} + K_{c,e}^{\text{con}}$, where $K_{c,e}^{\text{dev}}$ is due to developing flow at the pipe entrance and $K_{c,e}^{\text{con}}$ is due to flow detachment in the region of sudden contraction. A schematic of the two theorised contributions to pressure loss is drawn in Figure 4.5.

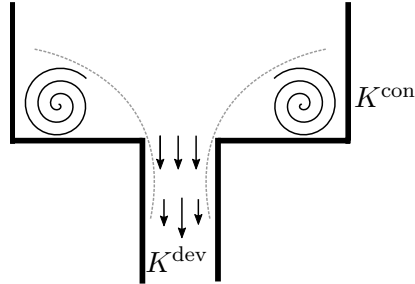


Figure 4.5: Two possible sources of inertial pressure loss.

Using the formula for pressure loss due to developing flow through a pipe of arbitrary cross-section, equation (4.3), we estimate the pressure loss through the developing region to be

$$K_c^{\text{dev}} \approx 1.43, \quad K_e^{\text{dev}} \approx 2.12. \quad (4.13a,b)$$

Letting $K_{c,e} = K_{c,e}^{\text{dev}}$ from equations (4.13) in equation (4.12), we plot the predictions accounting for developing flow in the pipe entrance in Figure 4.4a as the dotted red and blue lines (for the ellipse and circle, respectively). We observe that this adjusted prediction

is much closer to the measured flow rates, although a discrepancy still exists, indicating that perhaps all inertial contributions to the flow have not been accounted for. Pressure loss due to contraction has been estimated for simple geometries such as parallel plates and circular channels. However, for complicated geometries such as those considered in our experiments, pressure loss due to contraction is unknown, without performing three-dimensional numerical simulations. Thus, we use $K_{c,e}^{\text{con}}$ as fitting parameters and obtain best-fit values of

$$K_c^{\text{con}} \approx 1.59, \quad K_e^{\text{con}} \approx 1.05, \quad (4.14a,b)$$

with the corresponding predictions shown as the dashed curves in Figure 4.2b. Equations (4.14) are consistent with K -values in the literature which are typically $\mathcal{O}(1)$.

In summary, the results in this section highlight the potential significance of inertial pressure losses. We determined sources of minor losses in our experiments which led to a decrease in the predicted flux of nearly an order of magnitude.

4.3 The influence of inertial pressure losses in ureteroscopy

We anticipate that the influence of the quadratic term in equation (4.12) is particularly significant in the bench-top experiments due to the large cross-sections and short lengths of the 3D-printed tubes in comparison to the dimensions of the working channel and access sheath; this was necessitated by manufacturing constraints. A larger cross-section allows higher flow rate for the same driving pressure, and consequently, an inflation of the quadratic contribution to equation (4.12). Furthermore, a decrease in the length of the pipe reduces the effect of the linear, viscous term in equation (4.12), again enhancing the relative contribution of fluid inertia. However, although we anticipate inertial pressure losses in ureteroscopy to be less significant than in the experiments of Section 4.2, it is still of interest to quantify their magnitude. Due to the quadratic dependence of inertial terms on Q , the importance of minor losses increases non-linearly with flow rate. Extending this result to flow through the many tubular components in ureteroscopy systems, we anticipate that predictions obtained by bolting together fully-developed approximations may fall short of accurately predicting flow rate as a function of ureteroscopy system parameters, particularly when the flow rate is high. Thus, we aim to determine the validity of fully-developed approximations, when applied to modelling ureteroscopy irrigation, and the relative influence of inertial flow adjustments.

We adopt the complementary approaches of an analytical hydraulic network, accounting for quadratic inertial terms, and numerical simulations of the steady, Navier Stokes equa-

tions along with incompressibility condition (1.1) in characteristic geometries of representative length-scales. To avoid simulating the set-ups in three dimensions, which would significantly increase computational cost, we consider representative two-dimensional ‘slices’ through a ureteroscopy set-up.

We consider a simplified representation of a ureteroscopy system in a Cartesian coordinate system (Figure 4.6). The saline bag, tubing, scope, renal pelvis, and sheath are represented by rectangles of appropriate dimensions; the light-blue area in Figure 4.6a demonstrates the basic fluid-filled domain for a scope centred within the sheath. We define $y = 0$ to be at the base of the tubing, and $x = 0$ to be along the centreline of the sheath. We consider different positions for the scope within the sheath x_s , where

$$x_s = a_\varkappa + \phi_\varkappa(a_\varkappa - b_\varkappa), \quad (4.15)$$

with $\phi_\varkappa \in [0, 1]$. Figure 4.6b demonstrates the configuration for $\phi_\varkappa = 0.5$ and Figure 4.6c for $\phi_\varkappa = 1$.

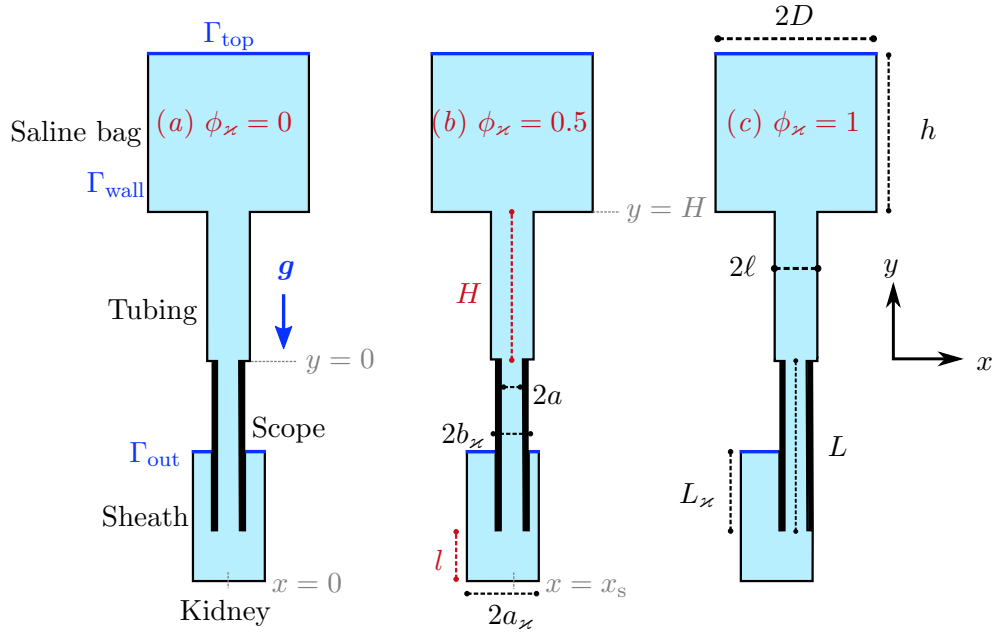


Figure 4.6: A schematic of the considered domain. We adopt a Cartesian coordinate system (x, y) with components exaggerated for visualisation purposes. In (a) the scope is at $\phi_\varkappa = 0$. The physical analogues of each part (scope, tubing, sheath, kidney, and scope) are labelled along with boundaries Γ_{top} and Γ_{out} . The direction of gravity, $\mathbf{g}$ is indicated by the arrow. In (b) the scope is at $\phi_\varkappa = 0.5$. The variable tubing length and kidney depths, H and l , respectively, are labelled in red. Fixed lengths are labelled in black. In (c) the scope is at $\phi_\varkappa = 1$ with fixed measurements provided.

We incorporate the effect of flow through the irrigation tubing draining under gravity, although without the additional complication of flow through 90° bend from the irrigation

tubing to the working channel, as in Figure 1.1, which would add complexity to the analytical approximations. We thus consider a discontinuous influence of gravity on the domain

$$\mathbf{g} = \begin{cases} -g \mathbf{j} & y \geq 0 \\ \mathbf{0} & y < 0, \end{cases} \quad (4.16)$$

where g is gravitational acceleration.

4.3.1 Hydraulic network approach

As in Chapter 2, we first consider a hydraulic network description of the configurations in Figure 4.6. The approach constitutes a system of algebraic expressions – involving geometric parameters and fluid properties – between flow rate and driving pressure drop through each component. It is important to recall that we have deviated from cylindrical geometries to modelling flow through parallel plates for numerical simplicity. The solution to the governing Poisson equation for fully-developed flow (1.2) through a gap between parallel plates of length L and half-width a is (Batchelor, 1967, p. 183)

$$Q = \frac{\Delta P}{R\mu L}, \quad R = \frac{12}{(2a)^3}. \quad (4.17)$$

To include the potential for inertial pressure losses through the irrigation tubing, working channel, and scope shaft we will adopt equation (4.8) through each component, where area corresponds to gap width between parallel plates within our two-dimensional Cartesian framework.

We consider Figure 4.7a, where we have labelled key pressure points through the irrigation system. We label pressures: at the base of the bag as P_0 , at the tubing exit as P_1 , at the working channel exit as P_2 , and at the entrances to the left and right sheath outflows as P_{3L} and P_{3R} , respectively.

The corresponding hydraulic network is indicated in Figure 4.7b, and the system of algebraic equations is thus

$$P_0 - P_1 + \rho g H = H R_1 \mu Q + \frac{K_1 \rho}{2(2\ell)^2} Q^2, \quad (4.18a)$$

$$P_1 - P_2 = L R_2 \mu Q + \frac{K_2 \rho}{2(2a)^2} Q^2, \quad (4.18b)$$

$$P_{3L} = L_{\times} R_{3L} \mu Q_L + \frac{K_{3L} \rho}{2d_L^2} Q_L^2, \quad (4.18c)$$

$$P_{3R} = L_{\times} R_{3R} \mu Q_R + \frac{K_{3R} \rho}{2d_R^2} Q_R^2, \quad (4.18d)$$

$$Q = Q_L + Q_R. \quad (4.18e)$$

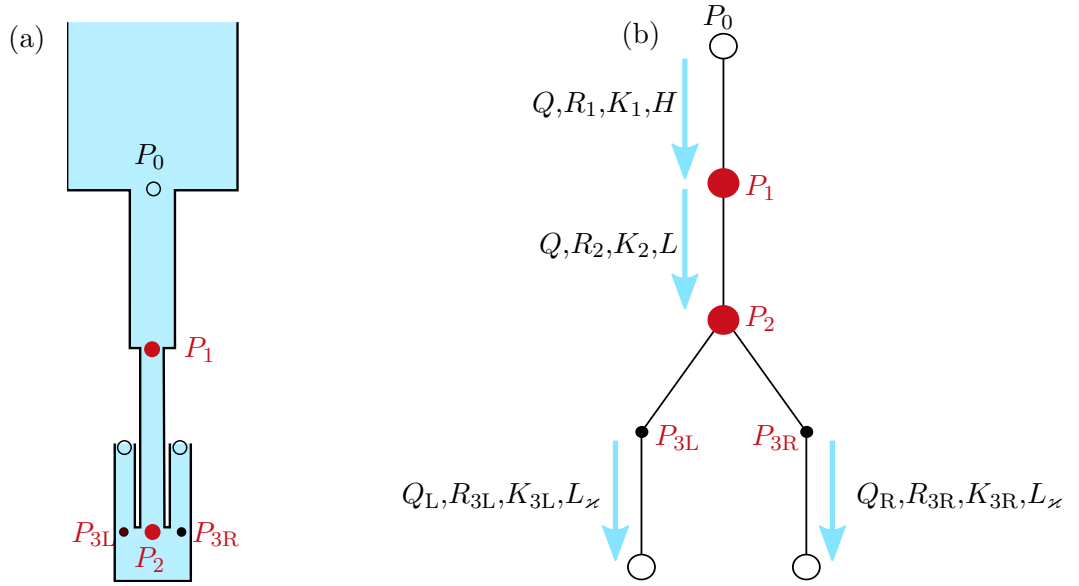


Figure 4.7: (a) A schematic of the basic numerical domain for a centred scope ($\phi_{\varkappa} = 0$). Key pressure points P_0 , P_1 , P_2 , P_{3L} , P_{3R} are labelled. The pressures at the top surface of the saline bag and sheath exits are atmospheric. We will assume a value of $P_0 = \rho gh$, and all other pressures (in red) are assumed unknown. (b) The translation of Figure (a) to a hydraulic network. Flow through each tubular component is demonstrated with a blue arrow, and corresponding flux, resistance, and loss-coefficients are labelled.

for pressure loss coefficients K_1 , K_2 , K_{3L} , and K_{3R} , and resistance values

$$R_1 = 12/(2\ell)^3 \quad R_2 = 12/(2a)^3 \quad R_{3L} = 12/d_L^3, \quad R_{3R} = 12/d_R^3, \quad (4.19a,b,c,d)$$

where

$$d_L = a_{\varkappa} + \phi(a_{\varkappa} - b_{\varkappa}), \quad d_R = a_{\varkappa} - \phi(a_{\varkappa} - b_{\varkappa}). \quad (4.19a,b)$$

We note there is a missing component to Figure 4.7 and hence equations (4.21): a relationship between P_2 , P_{3L} and P_{3R} . We will show through the results of numerical simulations in the following section – and throughout Chapter 5 – that flow patterns in the rectangular cavity representing the renal pelvis are highly complex. It is therefore difficult to estimate a priori how Q will divide into Q_L and Q_R and how this will affect the pressure distribution. For analytical progress, we make the simplifying assumption

$$P_2 = P_{3L} = P_{3R}, \quad (4.20)$$

Thus, equations (4.18c) and (4.18d) become

$$P_2 = L_{\varkappa} R_{3L} \mu Q_L + \frac{K_{3L} \rho}{2d_L^2} Q_L^2, \quad (4.21a)$$

$$P_2 = L_{\varkappa} R_{3R} \mu Q_R + \frac{K_{3R} \rho}{2d_R^2} Q_R^2. \quad (4.21b)$$

To close the system, we will estimate $P_0 = \rho g h$. Ignoring the effect of viscous dissipation through the bag is justified as $D \gg \ell$, and any losses due to contraction from the saline bag to the irrigation tubing can be incorporated through loss coefficient K_1 . We thus have a system of five algebraic equations (4.18a), (4.18b), and (4.21a), (4.21b), and (4.18e) for P_1 , P_2 , Q_R , Q_L , and Q .

In Section 4.3.4 we will demonstrate the difficulties associated with quantifying K_{3R} and K_{3L} . Hence, it is useful to consider predictions with these values equal to zero. Assuming $K_{3R} = K_{3L} = 0$, equations (4.21a) and (4.21b) further reduce to

$$P_2 = L_{\varkappa} R_3 \mu Q, \quad (4.22)$$

where

$$R_3 = (R_{3L}^{-1} + R_{3R}^{-1})^{-1}. \quad (4.23)$$

Hence, under the simplifying assumption that $K_{3R} = K_{3L} = 0$, equations (4.18a), (4.18b), and (4.21a), (4.21b), and (4.18e) reduce to a single quadratic

$$\rho g(h + H) = Q \mu \Sigma R + \frac{\rho \Sigma K}{2} Q^2, \quad (4.24)$$

where

$$\Sigma R = (H R_1 + L R_2 + L_{\varkappa} R_3), \quad \Sigma K = \frac{K_1}{(2\ell)^2} + \frac{K_2}{(2a)^2}. \quad (4.25a,b)$$

Taking the positive root of equation (4.24), we obtain the solution

$$Q = \frac{-\Sigma R \mu + \sqrt{(\Sigma R)^2 \mu^2 + 2g(h + H) \Sigma K \rho^2}}{\Sigma K \rho}. \quad (4.26)$$

From equation (4.24) it is also trivial to determine the solution for $\Sigma K = 0$ i.e., ignoring all inertial effects

$$Q = \frac{\rho g(h + H)}{\mu \Sigma R}. \quad (4.27)$$

4.3.2 Numerical simulations

To understand the influence of inertial pressure contributions, we seek to compare the predictions of equations (4.26) and (4.27) with the results of numerical simulations of the steady, Navier-Stokes equations and incompressibility condition (1.1). We reconsider the set-up in Figure 4.6 in a Cartesian coordinate system with orthogonal directions, $\mathbf{i}$ and $\mathbf{j}$ and corresponding velocity field $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$, and scalar pressure field p .

We adopt the following non-dimensionalisations of the governing Navier-Stokes equations and incompressibility condition (1.1)

$$\mathbf{x} = a\hat{\mathbf{x}}, \quad \mathbf{u} = U\hat{\mathbf{u}}, \quad p = (\mu U/a)\hat{p}, \quad \mathbf{g} = g\hat{\mathbf{g}}, \quad (4.28a-d)$$

where

$$\hat{\mathbf{g}} = \begin{cases} -\mathbf{j} & \hat{y} \geq 0 \\ \mathbf{0} & \hat{y} < 0. \end{cases} \quad (4.29)$$

We take the velocity scaling, U , from the purely viscous hydraulic network approximation, equation (4.27)

$$U = \frac{\rho g(h+H)}{a\mu\Sigma R}. \quad (4.30)$$

With scalings (4.28) and (4.30), the Navier-Stokes equations and incompressibility condition (1.1) become

$$\text{Re} (\hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}) = -\nabla \hat{p} + \nabla^2 \hat{\mathbf{u}} + \mathcal{H}^{-1} \hat{\mathbf{g}}, \quad \nabla \cdot \hat{\mathbf{u}} = 0, \quad (4.31a,b)$$

where Re is the Reynolds number and $\mathcal{H}$ is a geometrical parameter

$$\text{Re} = \frac{\rho^2 g(h+H)}{\mu^2 \Sigma R}, \quad \mathcal{H} = \frac{h+H}{a^3 \Sigma R}. \quad (4.32a,b)$$

At the boundary at the top of the saline bag, we prescribe a zero-stress condition

$$2\hat{v}_{\hat{y}} - \hat{p} = 0, \quad \hat{u}_{\hat{y}} + \hat{v}_{\hat{x}} = 0, \quad \text{on } \Gamma_{\text{top}}. \quad (4.33)$$

At the outflows between the wall of the sheath and scope shaft we prescribe zero normal stress and parallel velocity

$$\hat{v}_{\hat{y}} = 0, \quad \hat{u} = 0, \quad \text{on } \Gamma_{\text{out}}. \quad (4.34a,b)$$

All other boundaries we treat as solid walls, and apply a no-slip condition

$$\hat{\mathbf{u}} = \mathbf{0}, \quad \Gamma_{\text{wall}}. \quad (4.35)$$

The total dimensionless flux

$$\hat{Q} = \frac{\rho g(h + H)}{\mu \Sigma R} Q \quad (4.36)$$

can be extracted from the simulations in two different ways:

$$\hat{Q} = - \int_{\Gamma_{\text{top}}} \hat{v} \, d\hat{x}, \quad \hat{Q} = \int_{\Gamma_{\text{out}}} \hat{v} \, d\hat{x}, \quad (4.37\text{a,b})$$

where the negative sign in equation (4.37a) is due to the orientation of the coordinate axes in Figure 4.6. We calculate both for validation purposes, and these agree to within numerical error.

4.3.3 Computational parameter sweep

In ureteroscopy, the following variables will typically be fixed during a given procedure: the size of the saline bag, the lengths of the sheath and scope, the radii of the irrigation tubing, working channel, scope shaft, and access sheath, and the properties of the irrigation fluid. Thus, we assign these to have fixed typical values. Parameters which are likely to vary during ureteroscopy are: the distance between the tip of the scope and end of the renal pelvis l , the position of the scope in the sheath $\phi_{\varkappa} \in [0, 1]$, and the height of the bag of saline H . We choose values for each of these variables representing the typical range: $l = \{0.25 \text{ cm}, 0.5 \text{ cm}, 1.0 \text{ cm}\}$, $\phi_{\varkappa} = \{0, 0.5, 1.0\}$, and $H = \{30 \text{ cm}, 60 \text{ cm}, 90 \text{ cm}, 120 \text{ cm}\}$. All parameters used for the computations are summarised in Table 4.1, with those in boldface indicating varied values.

We solve equations (4.31) subject to boundary conditions (4.33), (4.34), and (4.35), using the open source finite element library `oomph-lib` (Heil & Hazel, 2006), with numerical details provided in Appendix 4.C.

4.3.4 Results

As detailed in Appendix 4.C, the numerical simulations were conducted using Reynolds number continuation for fixed $\mathcal{H}^{-1}$, which is equivalent to varying

$$\mathcal{H}^{-1} \text{Re} = \rho^2 g a^3 / \mu^2. \quad (4.38)$$

Equation (4.38) is termed the Galilei number and is a dimensionless quantity relating gravitational to viscous forces (Kuneš, 2012). The dimensionless flux $\hat{Q}$ is plotted as a function of $\mathcal{H}^{-1} \text{Re}$ in Figure 4.8a,b,c. Numerical data corresponding to each dimensional H value is plotted with a different color: $H = 30 \text{ cm}$ (blue), $H = 60 \text{ cm}$ (purple), $H = 90$

	Component	Symbol	Value	Unit
Radii	Working channel	a	6.00×10^{-2}	cm
	Scope	$b_{\varkappa}$	3.17×10^{-1}	cm
	Sheath	$a_{\varkappa}$	4.33×10^{-1}	cm
	Tubing	ℓ	1.25×10^{-1}	cm
	Saline bag	D	1.00×10^1	cm
Lengths	Scope	L	7.90×10^1	cm
	Sheath	$L_{\varkappa}$	2.20×10^1	cm
	Saline bag	h	1.00×10^1	cm
Physical properties	Density	ρ	1.00×10^1	g/cm ³
	Dynamic viscosity	μ	1.00×10^{-2}	g/cm s
	Gravitational acceleration	g	9.81×10^2	g/cm ²
Varied values	Tubing length	H	30, 60, 90, 120	cm
	Cavity depth	l	0.25, 0.5, 1.0	cm
	Scope position	$\phi_{\varkappa}$	0, 0.5, 1.0	

Table 4.1: Radii and lengths for the scope and sheath are taken for a Boston Scientific LithoVue™ and 13/15 F Navigator™ access sheath.

cm (green), and $H = 120$ cm (yellow). Each scope position $\phi_{\varkappa}$ is differentiated with a different symbol: $\phi_{\varkappa} = 0$ centred (circle), $\phi_{\varkappa} = 0.5$ (triangle), and $\phi_{\varkappa} = 1$ (square). The corresponding values of $\mathcal{H}$ are given in the legend of Figure 4.8. We determine the influence of renal pelvis length l on Q to be effectively negligible (see Appendix 4.E) and thus present results only for the intermediate depth value $l = 0.5$ cm. We do note, however, that the depth parameter influences intriguing symmetry-breaking behaviour in the cavity when $\phi_{\varkappa} = 0$, and we will focus on this in detail in Chapter 5. Dashed black lines in Figures 4.8a,b,c denote $\hat{Q} = 1$ which corresponds to the purely-viscous approximation (4.27). Thus, in Figures 4.8a,b,c the decrease in $\hat{Q}$ from $\hat{Q} = 1$ with $\mathcal{H}^{-1}\text{Re}$ highlights the influence of fluid inertia. The results indicate, as expected, an increase in inertial effects with both H and $\phi_{\varkappa}$ (and consequently $\mathcal{H}$).

The dimensionless form of the analytical quadratic approximation for the flux, equation (4.24), is

$$1 = \hat{Q} + \text{Re} \frac{\Sigma K}{2\Sigma R} \hat{Q}^2. \quad (4.39)$$

To highlight the quadratic inertial corrections, in Figure 4.8d we plot $1 - \hat{Q}$ from the numerical simulations as a function of $\text{Re}(a^2\Sigma R)^{-1}\hat{Q}^2$; from equation (4.39) we expect these to obey a linear relationship with coefficient $a^2\Sigma K/2$. We note the pre-factor of a^2 to ΣR and ΣK serves as a non-dimensionalisation. The data for $\phi_{\varkappa} = 0.5$ and $\phi_{\varkappa} = 1$ for all values of H collapses and follows the anticipated linear trend. The data for $\phi_{\varkappa} = 0$ (a centred scope), however, displays a different trend. We postulate that this is due to a

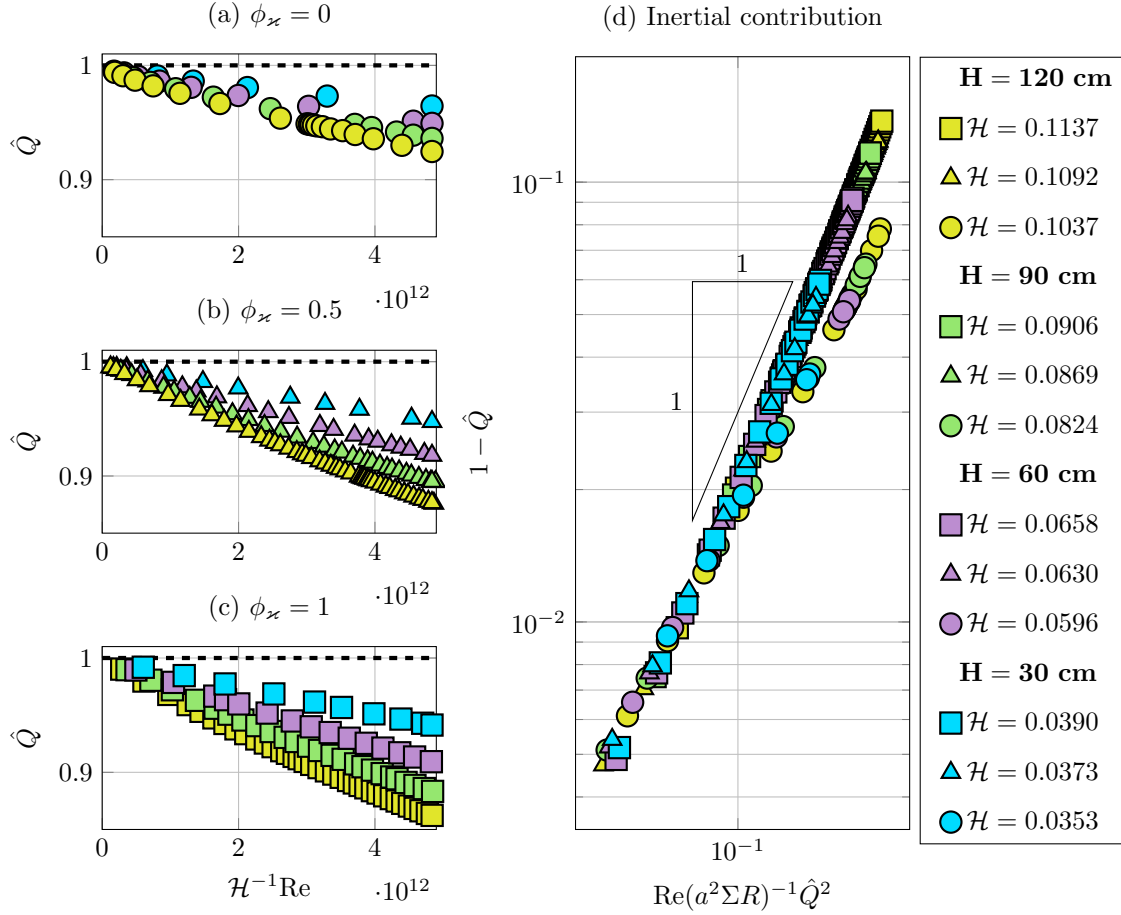


Figure 4.8: The results of the numerical continuation for $l = 0.5$ cm. Colours differentiate H (blue: $H = 30$ cm, purple: $H = 60$ cm, green: $H = 90$ cm, yellow: $H = 120$ cm) and shapes differentiate $\phi_{\mathcal{K}}$ (circles: $\phi_{\mathcal{K}} = 0$, triangles: $\phi_{\mathcal{K}} = 0.5$, and squares: $\phi_{\mathcal{K}} = 1$).

dependence of ΣK on Re . The flow structure for $\phi_{\mathcal{K}} = 0$ is particularly dependent on Re as the symmetric flow pattern impinges on the bottom cavity wall as Re increases. For $\phi_{\mathcal{K}} = 0.5$ or $\phi_{\mathcal{K}} = 1$, in contrast, the jet exits through the larger outflow without hitting the cavity wall (see Figure 4.12b).

The results corresponding to the physical properties of ureteroscopy irrigation in Table 4.1 are the final data points in Figures 4.8a,b,c. To maintain relevance to ureteroscopy, we now re-dimensionalise flux and rotate these twelve data points, plotting Q in Figure 4.9, as a function of H for each value of $\phi_{\mathcal{K}}$. We retain the symbols summarised in the legend of Figure 4.8. As an initial analytical comparison, we plot the hydraulic network approximation assuming no inertial pressure losses, equation (4.27), as the black lines in Figure 4.9 with symbols circles, triangles, and squares corresponding to $\phi_{\mathcal{K}} = 0$, $\phi_{\mathcal{K}} = 0.5$, and $\phi_{\mathcal{K}} = 1$, respectively. We observe a consistent over-prediction of the estimated flux by the fully-developed predictions, as we would expect due to unaccounted for inertial

contributions to the total pressure drop.

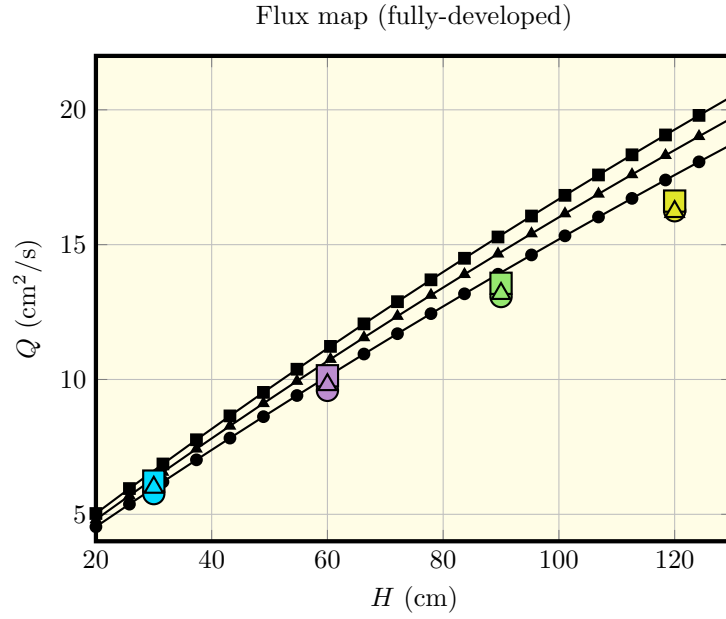


Figure 4.9: A plot of Q as a function of H . Coloured symbols represent data from the numerical simulations and correspond to the legend in Figure 4.8. The black lines obtained from equation (4.27) with markers circles, triangles, and squares for $\phi_{\mathcal{X}} = 0$, $\phi_{\mathcal{X}} = 0.5$, and $\phi_{\mathcal{X}} = 1$, respectively.

To determine the main sources of additional pressure loss, we turn to considering the pressure drops through the tubing and working channel in Figure 4.10, estimating area-averaged pressures for P_1 , and P_2 from the numerical simulations

$$P_1 = \frac{1}{2a} \int_{-a+x_s}^{a+x_s} p|_{y=0} dx, \quad P_2 = \frac{1}{2a} \int_{-a+x_s}^{a+x_s} p|_{y=-L} dx. \quad (4.40a,b)$$

In Figure 4.10a, we plot

$$\rho gh - P_1 + \rho gH - HR_1\mu Q \quad (4.41)$$

as a function of Q on a logarithmic scale. From equation (4.18a) with $P_0 \approx \rho gh$, equation (4.41) should provide the inertial adjustment to the pressure drop. We see, indeed, that equation (4.41) scales quadratically with Q . Additionally, it appears that the inertial pressure loss through the tubing is independent of $\phi_{\mathcal{X}}$. As we anticipate the contributions to K_1 will be due to geometric contraction from the saline bag, and from the tubing to the working channel, this finding is consistent. Fitting a line to the log-log plot in Figure 4.10a, the intercept provides an estimate of $\log(K_1\rho/2(2\ell)^2)$ via equation (4.18a), and subsequently we obtain a best-fit value of

$$K_1 \approx 3.87. \quad (4.42)$$

We can estimate the contraction loss through parallel plates as a function of area ratio from semi-empirical studies (Kays, 1950), see Appendix 4.D. Using equation (4.52) in Appendix 4.D, we approximate the K -value for the contraction from the saline bag to the tubing to be $K \approx 1.17$ and from the tubing to the working channel to be $K \approx 0.36$. It is interesting that equation (4.42) is over twice as large as the sum of these two contributions. We hypothesise that this is due to the additional influence of the discontinuous gravitational force imposed in the numerical simulations, which stops at the base of the tubing. This undoubtedly affects the size of the *vena contracta* through the tubing to the working channel, upon which the analytical estimates of K -values for contraction losses rely.

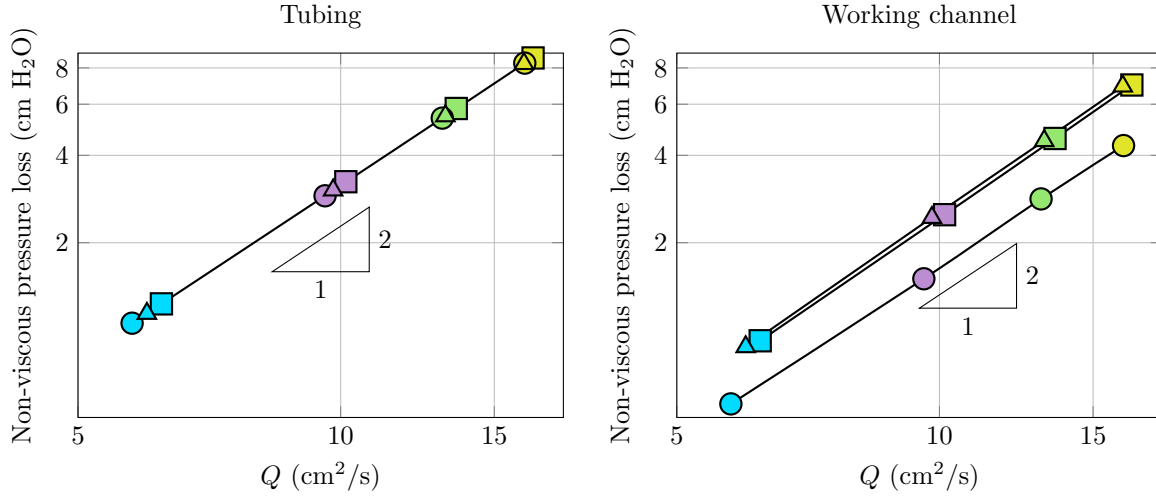


Figure 4.10: Plots of equations (a) (4.41) and (b) (4.43) as functions of Q . Symbols and colours correspond to the legend in Figure 4.8: circles are $\phi_x = 0$, triangles are $\phi_x = 0.5$, and squares are $\phi_x = 1$.

Similarly, in Figure 4.10b we consider the estimated inertial adjustment to the pressure drop through the working channel as a function of Q by plotting

$$P_1 - P_2 - LR_2\mu Q. \quad (4.43)$$

This is the inertial adjustment to the pressure drop through the tubing from equation (4.18b). As anticipated, equation (4.43) scales quadratically with Q . However, in contrast to the inertial pressure adjustment through the irrigation tubing, the inertial pressure adjustment through the working channel varies with ϕ_x . This is to be expected – and was also seen in Figure 4.8d – as the magnitude of the exit effects depends on the outflow geometry, which is influenced significantly by ϕ_x . The fitted loss coefficients are

$$K_2(\phi_x = 0) \approx 0.47, \quad K_2(\phi_x = 0.5) \approx 0.72, \quad K_2(\phi_x = 1) \approx 0.69. \quad (4.44a,b,c)$$

It is impossible to compare equations (4.44a,b,c) with values in the literature, due to the unique exit geometries from the working channel into the simulated renal cavity, although their magnitude is typical.

Finally, we turn to considering the distribution of pressures within the cavity to determine the validity of the simplification $P_2 = P_{3L} = P_{3R}$ used in the analytical approximations. We present P_2 , P_{3L} , and P_{3R} in Figures 4.11a,b,c, respectively, as functions of H . The plots show the dependence of the three pressure values on ϕ_κ . For $\phi_\kappa = 0$, $P_2 \approx P_{3L} \approx P_{3R}$, whereas for asymmetric configurations ($\phi_\kappa = 0.5$ and $\phi_\kappa = 1$) there is large pressure variation within the cavity. Interestingly, negative pressures at the entrance to the left outflow are demonstrated in Figure 4.11b. This indicates the strong influence of inertia with fluid necessarily flowing against an adverse pressure gradient towards the outflow where pressure is set to zero.

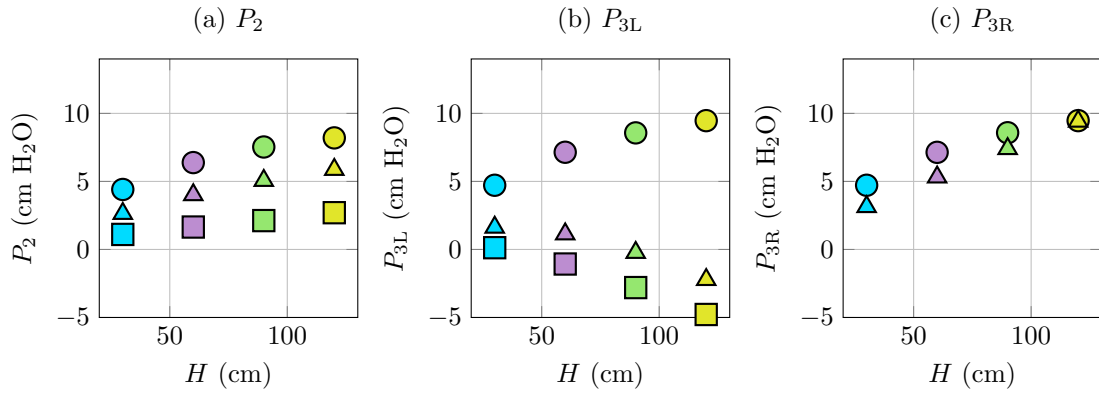


Figure 4.11: Pressure values at three points in the cavity: (a) exit from the working channel, (b) left-hand entrance to the sheath, and (c) right-hand entrance to the sheath. Symbols and colours correspond to the legend in Figure 4.8: circles are $\phi_\kappa = 0$, triangles are $\phi_\kappa = 0.5$, and squares are $\phi_\kappa = 1$.

To investigate this in more detail we plot the area-averaged left outflow pressure

$$P_L(y) = \frac{1}{d_L} \int_{-a_\kappa}^{-b_\kappa + x_s} p \, dx, \quad (4.45)$$

for $H = 60$ cm and each value of ϕ_κ as functions of distance along the length of the sheath in Figure 4.12a. We see that for all configurations, there is an initial pressure rise as the fluid enters the sheath, followed by a constant negative pressure gradient characteristic of fully-developed flow, starting at a location along the sheath. The larger the value of ϕ_κ , i.e., the more asymmetric the outflow from the working channel to the sheath, the longer the sheath length required to establish fully-developed flow. These results highlight that inertial contributions to pressure changes cannot always be so simply captured by a positive

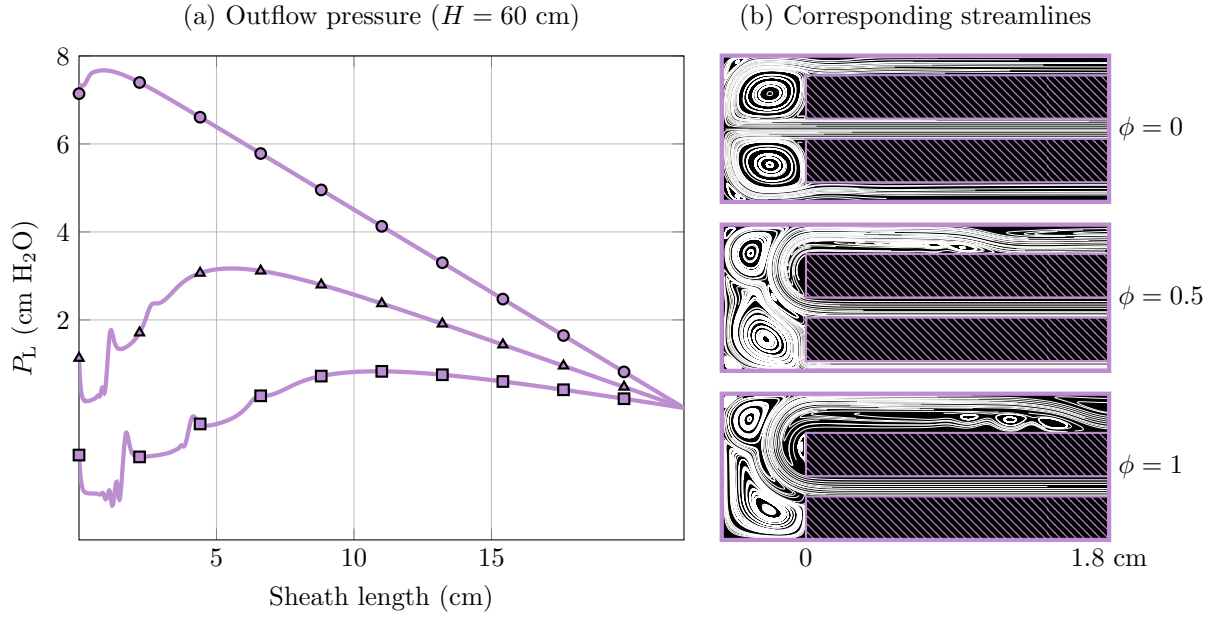


Figure 4.12: Figure (a) Equation (4.45) for $H = 60$ cm; the three ϕ_x values are: $\phi_x = 0$ (circles), $\phi_x = 0.5$ (triangles), and $\phi_x = 1$ (squares). Figure (b) shows corresponding streamlines for the three geometries.

loss coefficient, K . For instance for $\phi_x = 0.5$ and $\phi_x = 1$, the calculated K_{3L} values would be negative, as the pressure at the entrance to the sheath is lower than the atmospheric pressure at the exit (see Figure 4.12a). The associated velocity streamlines with each value of ϕ_x are shown in Figure 4.12b. The presence of multiple vortical structures is striking. These complicated flow patterns are important, not only due to their potential for dissipating the fluid's kinetic energy, but also for their possible effect on the clearance of stone dust in the renal pelvis. We will analyse this in detail in Chapter 5.

We return to considering Figure 4.9. Simple laws predicting flow rate as a function of saline bag height would be of significant assistance to ureteroscopy, and we determine whether this is now possible, accounting for the fitted K -values provided by equations (4.42) and (4.44a,b,c). We highlighted complexities around estimating the relationship between flow rate and pressure drop through the sheath, and hence we determine the effectiveness of just incorporating pressure losses through the tubing and working channel, i.e., assuming $K_{3L} = K_{3R} = 0$ and using equation (4.26). We plot the adjusted analytical approximations, equation (4.26), as the solid black lines in Figure 4.13, which show much better agreement than the initial estimations of equation (4.27) in Figure 4.9 due to the inertial adjustments. We recall that circles represent $\phi_x = 0$, triangles $\phi_x = 0.5$, and squares $\phi_x = 1$. These results indicate that accounting for the pressure losses due to contraction from the saline bag to the tubing, and from the tubing to the working channel are sufficient to obtain good agreement with the numerical simulations. This suggests that,

perhaps, inertial flow patterns in the renal pelvis and back through the access sheath do not have to be accounted for to accurately predict flow rate as a function of saline bag height and scope position, despite the observed fluid dynamical complexities. The closeness of the lines corresponding to the three scope positions indicates that an expression for flux as a function of saline bag height may even be sufficient, with an envelope of uncertainty encapsulating scope position. In fact, the inclusion of inertial adjustments to a fully-developed flow approximation even appears to shrink the width of the predictive ‘wedge’ (seen by comparing the solid black lines in Figure 4.9 to those in Figure 4.13).

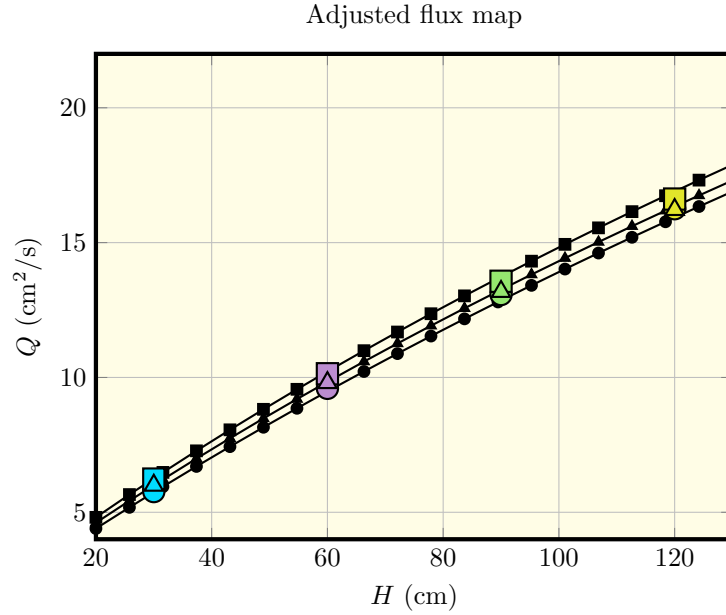


Figure 4.13: A plot of Q as a function of H . Coloured symbols represent data from the numerical simulations and correspond to the legend in Figure 4.8. The black lines are obtained from (4.26) with K_1 from (4.42) and K_2 for each value of ϕ_κ from equation (4.44a,b,c). Marker symbols circles, triangles, and squares for $\phi_\kappa = 0$, $\phi_\kappa = 0.5$, and $\phi_\kappa = 1$, respectively.

4.4 Turning focus to flow in the renal pelvis

The velocity streamlines in Figure 4.12b highlight particularly rich fluid dynamics within the representative renal pelvis. The flow structure is characterised by competing vortices, the number, sizes, and shapes of which are intricately linked to the Reynolds number of the flow, the position of the emerging jet from the working channel, and the renal pelvis length. Furthermore, the latter parameter was observed to control a symmetry breaking behaviour on the symmetric domain, which will be investigated in detail in the following chapter. Due to these compelling complexities, we turn, in Chapter 5, to study flow within the renal pelvis through complementary numerical simulations and particle image velocimetry

experiments. However, before we begin, we encounter a pertinent question: *how to map from a given ureteroscopic configuration to an accurately formulated flow problem in the renal pelvis?* Videlicet, is it necessary to incorporate the full two-dimensional representation of a ureteroscopy set-up, or can we significantly reduce computational expense by providing appropriate boundary conditions on a truncated domain? With regards to an inlet condition, we could impose a parabolic velocity profile with flux informed by a ‘map’ between the desired ureteroscopic set-up and subsequent flow rate. This ‘map’ is provided, in effect, by equation (4.26) with appropriate values for K_1 and K_2 ; i.e., the solid black lines in Figure 4.13. In the middle row of Figure 4.14, we plot streamlines within the renal pelvis, imposing a fully-developed flow profile at the inlet to the cavity, with flux provided by the aforementioned ‘map’. The corresponding Reynolds numbers, calculated with characteristic velocity taken to be the peak of the parabolic flow profile (Appendix 4.C) are $Re = 711.27$, $Re = 737.79$, and $Re = 766.46$, for $\phi_{\varkappa} = 0$, $\phi_{\varkappa} = 0.5$, and $\phi_{\varkappa} = 1$, respectively. As a qualitative comparison, we plot zoomed-in images of the streamlines within the renal pelvis for $H = 60$ cm from the full simulations described in Section 4.3.2 in the top row of Figure 4.14. The observed flow structures appear identical between the two sets of images (I) and (II), justifying truncating the saline bag, irrigation tubing, and working channel from the numerical domain. We note, as indicated in Figure 4.14, the number of elements in the numerical mesh is reduced by an order of magnitude, resulting in a significant reduction in computational expense. However, as the length of the access sheath is over 200 times the width of the working channel, meshing the full outflow channels still results in a large domain, much of which is ignored when focussing on flow within the renal pelvis. To determine whether truncating the outflow channels is also permissible, we display streamlines in the third row of Figure 4.14 from simulations with the prescribed inflow informed, as before, from equation (4.26), and zero-stress outflow conditions at the point where fluid would enter the access sheath (the locations of pressure points P_{3L} and P_{3R} in Figure 4.7a). We see, that for $\phi_{\varkappa} = 0$ and $\phi_{\varkappa} = 1$, the flow patterns are identical to those in the upper two rows. However, for $\phi_{\varkappa} = 0.5$, which is highlighted by a red box in Figure 4.13, the flow patterns resulting from the reduced formulation are markedly different. This is justifiable, however, as a zero-stress boundary condition equates to setting equal pressure just outside the numerical domain. In actuality, with the presence of outflow tubes, we determined in Figures 4.11b,c, that the pressures P_{3L} and P_{3R} for $\phi_{\varkappa} = 0.5$ are not equal. Thus, we conclude, that for symmetric flows, or for domains with a single outflow channel it appears sufficient to both replace the inlet geometry with a prescribed parabolic flow profile of appropriate flux, and to truncate the outflow channels and prescribe a zero-stress condition. This results in a simple, computationally tractable rectangular domain representing the renal pelvis.

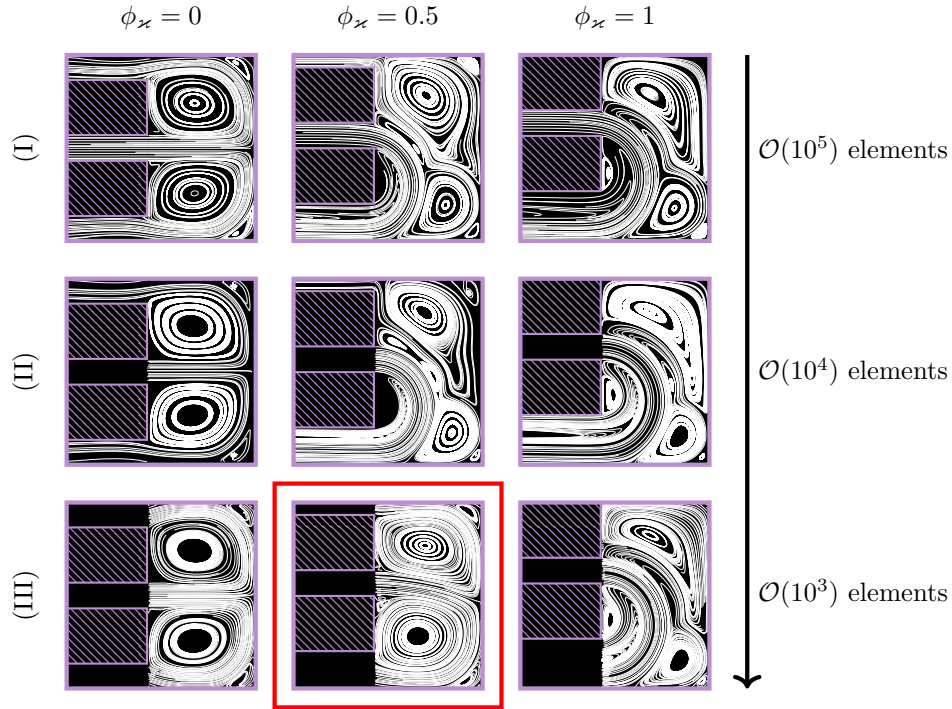


Figure 4.14: Numerical simulations for $l = 0.5$ cm and flow rate corresponding to $H = 60$ cm. (I) Zoomed-in images from the full simulations in Section 4.3.2. (II) Simulations with full outflow channels, but prescribed parabolic velocity profile with flux informed from Figure 4.13. (III) Simulations with the same prescribed inflow condition, and zero-stress outflow conditions. The red box indicates the one anomalous scenario for a partially offset scope ($\phi_x = 0.5$) where the results do not agree with the upper two rows.

4.5 Summary and discussion

This chapter began with an attempt to validate the predictions of Chapter 3 through measuring flux for fixed pressure drop along pipes of elliptical annular cross-section. We found nearly an order of magnitude difference between the experimental measurements and the predictions based on the assumption of fully-developed flow. This led us to investigate the influence of inertial effects due to developing flow and a sudden geometric contraction in the experiments. We found good agreement between experiment and theory by using an analytical estimate for the pressure drop due to developing flow and by fitting for the pressure loss coefficient due to the contraction.

The enormous effect of the inertial pressure adjustment on the predicted flow rate led us to consider numerical simulations of the steady Navier-Stokes equations on simplified two-dimensional geometries, varying parameters representing the saline bag height, the length of the renal pelvis, and the position of the scope within the access sheath. We determined the influence of renal pelvis length on flow rate to be negligible, although we did observe a correlation between the length of the cavity and symmetry-breaking within

the numerical solutions. We will explore this in Chapter 5. We found, as expected from Section 2.2.3 that flux increases sublinearly with saline bag height, and, as expected from Section 2.3, that flux increases as the scope is moved from the centre to the edge of the sheath.

We compared the results of the numerical simulations against analytical approximations obtained by summing contributions due to the viscous and inertial pressure drops through each component of the irrigation system. Initially taking the inertial contributions to be zero, we determined a fairly significant deviation between the results of the simulations and the analytical approximations, that became more pronounced both with increasing bag height and scope offset.

By plotting the differences between the simulated pressure drops through the irrigation tubing and working channel, subtracting the contribution from the viscous pressure drops, we obtained estimated values for the inertial pressure loss coefficients through each of these components. Through the tubing, the inertial pressure adjustment was independent of scope position, and it was larger than what is estimated from the literature by calculating the contributions from the two contractions (from the saline bag to the tubing and from the tubing to the working channel). We hypothesised this was caused by gravity terminating at the base of the tubing in our numerical simulations. The inertial adjustment to the pressure drop through the working channel was found to depend on the position of the scope in the sheath, as this influences the geometric exit effects.

The pressure drop through the sheath was harder to estimate using a simple K -value. We found when the scope was offset within the sheath that pressures in parts of the renal pelvis were lower than atmospheric (at the sheath outlet), which highlights that fluid inertia drives flow against an adverse pressure gradient. We also observed complicated flow patterns in the cavity that were dependent on the position of the scope in the sheath.

Accounting just for the inertial adjustments to the pressure drops through the tubing and working channel, however, we were able to obtain a much more accurate approximation of the flux as a function of saline bag height. This indicates that these two components may cause the most significant deviations to linear predictions between flow and pressure, and that perhaps, inertial pressure losses through the sheath can be ignored when mapping from irrigation set-up to flow rate.

Finally, we discussed how the results in this chapter can be used to inform a truncated numerical domain to focus on flow within the renal pelvis. We determined that replacing the inflow condition with a flux informed by analytical approximations which account for inertial pressure losses results in qualitatively identical flow structures within the renal pelvis. Additionally, the outflow channels could be ignored in domains with symmetric

flow patterns and with only a single outflow channel. However, particular attention may be required when truncating the outflow channels from domains with two outflow channels through which the pressure distributions are unequal.

The results in this chapter reveal that care must be taken when bolting together fully-developed predictions, and that, even through a system of long and thin pipes, the influence of fluid inertia is non-negligible. A valuable tool for the urological community would be a look-up table relating saline bag height to flow rate. However, we have highlighted the strong interdependence between appropriate inertial adjustments to flow rate and the geometry of pipe contractions, entrances, and exits, that may be influenced by uncontrollable factors such as the position of the scope shaft in the access sheath. Thus, there will inherently be a degree of uncertainty in estimating flux purely using a hydraulic network approach, even if inertial adjustments to pressure drops through each component are incorporated. One possibility to mitigate this ambiguity is through the use of gradual geometric changes, rather than abrupt contractions (White, 2008). This allows streamlines to adhere to the changing geometry, avoiding fluid detachment and recirculation that is induced with abrupt geometric adjustments. Appropriately accounting for pressure losses due to each component of a ureteroscopy irrigation system is imperative. If large reductions in flow rate from fully-developed predictions are observed, design alterations to lessen the flow disruptions caused by connectors and fittings may be initiated.

4.6 Epilogue: the effect of inertial adjustments on Chapter 2

We demonstrated in Section 2.3 that a systems model for ureteroscopy irrigation, derived from fully-developed laws for flow through the working channel and access sheath, could accurately predict flow rates when compared to *ex-vivo* data. Contrastingly, in this chapter, we highlighted inadequacies of fully-developed predictions due to effects of fluid inertia at the entrances and exits to the irrigation tubing, working channel, and access sheath (albeit via numerical simulations on two-dimensional parallel plate idealisations). To rectify this modelling dichotomy it would be of interest to incorporate inertial pressure losses to the systems model in Section 2.3 to determine how adjustments from fully-developed flow affect the ‘wedge’ predictions arising due to uncertainty in the position of the scope shaft within the access sheath. However, this is non-trivial, as inertial pressure losses due to flow exiting the working channel and entering the access sheath will require experimental evaluation. In Section 4.3.4, inertial effects were shown to shrink the influence of scope position, as an offset scope incurred a larger pressure loss due to exit effects from the working channel to the renal pelvis than a centred scope. If this finding translates to cylindrical geometries in the systems model, then the necessity for the scope shaft to

reside at the edge of the access sheath – or similarly the working tool at the edge of the working channel – to minimise axial flow resistance may not be so imperative.

We must emphasise, however, the distinction between the models in Chapter 2 and those in this chapter: the geometric setting. In Chapter 2 the geometries were three-dimensional and cylindrical, whereas in this chapter the cylindrical domains were replaced with two-dimensional channels between parallel plates for computational tractability. Sharp corners between each channel in the two-dimensional idealisations induces sequences of eddies, the number and intensities of which depends on the angle of the corner (Moffatt, 1964). For this reason, is likely that three-dimensional flow patterns through abrupt geometrical contractions between cylindrical tubes will incur comparatively lower viscous pressure losses than flow through the two-dimensional ‘slices’ which contain artificial sharp corners.

We consider a simple example for including inertial adjustments to relationships between flow and pressure in Chapter 2 by revisiting the comparison between theory and the results of the bench-top experiment on the effect of saline bag height on flow rate (Section 2.2.3). Figure 2.4, reproduced as Figure 4.15, plots the experimental data (red marks) against fully-developed predictions. The dashed black line assumes the pressure at the inlet to the scope is $P_{\text{in}} = \rho g H$, where H is the vertical distance between the scope inlet and the fluid level in the suspended reservoir, and the solid black line is adjusted to consider fully-developed flow through the irrigation tubing, rather than a hydrostatic inlet pressure. Although the solid black line provides superior agreement with experiment, the theoretical curve still deviates from the physical observations. In the experimental set-up described in Section 2.2.1, fluid flowed from a suspended bucket, through an area contraction and 90° bend from vertical to horizontal into the irrigation tubing and through a second area contraction and 90° bend from horizontal to vertical into the working channel (see Figure 2.1). Therefore, minor pressure losses due to fluid inertia are distinctly possible. We assume an inertial adjustment to equation (2.17) for flow through the irrigation tubing of the form in equation (4.8)

$$\rho g H - P_{\text{in}} = R_t \mu H Q + \frac{K \rho}{2(\pi \ell^2)^2} Q^2, \quad R_t = \frac{8}{\pi \ell^4}, \quad (4.46)$$

where K captures any inertial adjustments to the pressure drop through the irrigation tubing. We could, similarly, assume an inertial adjustment to flow through the working channel, but this would incur a second unknown K -value. Moreover, in the bench-top experiments in Section 2.2, fluid flowed from the working channel to atmospheric pressure, so we speculate that exit effects in this scenario may be less severe than from the working channel to the renal pelvis as simulated in this chapter. Thus we assume flow through the

working channel is governed by

$$Q = \frac{P_{\text{in}}}{R\mu L}, \quad R = \frac{8}{\pi a^4}. \quad (4.47)$$

Equations (4.46) and (4.47) provide an algebraic system for Q and P_{in} . The solution is similar to equation (4.26)

$$Q = \frac{-\Sigma R\mu + \sqrt{(\Sigma R)^2\mu^2 + 2gH\Sigma K\rho^2}}{\Sigma K\rho}, \quad (4.48)$$

where

$$\Sigma R = HR_t + LR, \quad \Sigma K = K/(\pi\ell^2)^2. \quad (4.49)$$

With K describing the inertial loss through the irrigation tubing as a fitting parameter, we obtain a best-fit value to the data from the bench-top experiments on the effect of saline bag height:

$$K \approx 1.77. \quad (4.50)$$

The associated dotted black line in Figure 4.15 plots the solution for Q from equation (4.48) with K provided by (4.50). We observe superior agreement from the dashed and dotted black lines, which assume $K = 0$ with $P_{\text{in}} = \rho gH$ and $P_{\text{in}} = \rho gH/(1 + \sigma)$ with $\sigma = (H/L)(R_t/R)$, respectively.

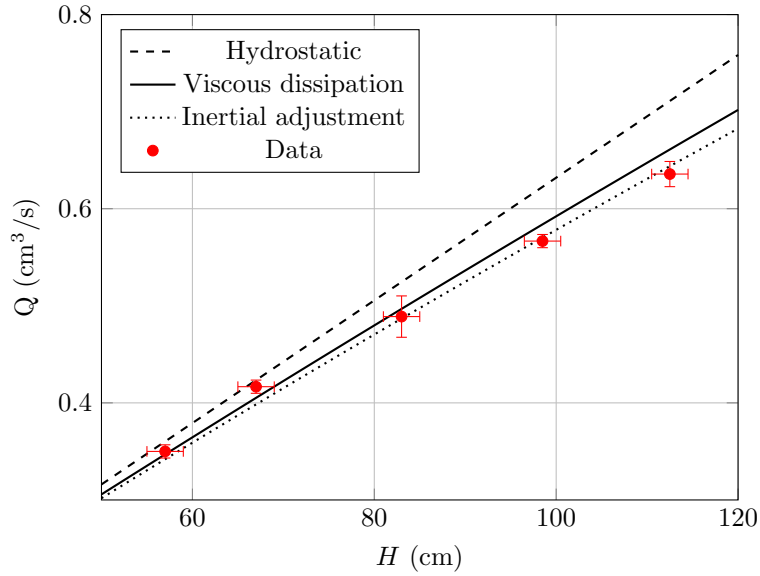


Figure 4.15: Experimental and theoretical results for the flow of water through an open ureteroscopy working channel of internal radius 0.06 cm. The dashed line equation (2.16) assumes a hydrostatic inlet pressure. The solid line equation (2.18) accounts for viscous dissipation. The dotted black line equation (4.48) accounts for inertial pressure losses through the irrigation tubing. Horizontal lines on the data points depict error in the height measurements, and vertical lines depict error in measured flow rates.

Appendix 4.A Other considered elliptical annular pipe designs

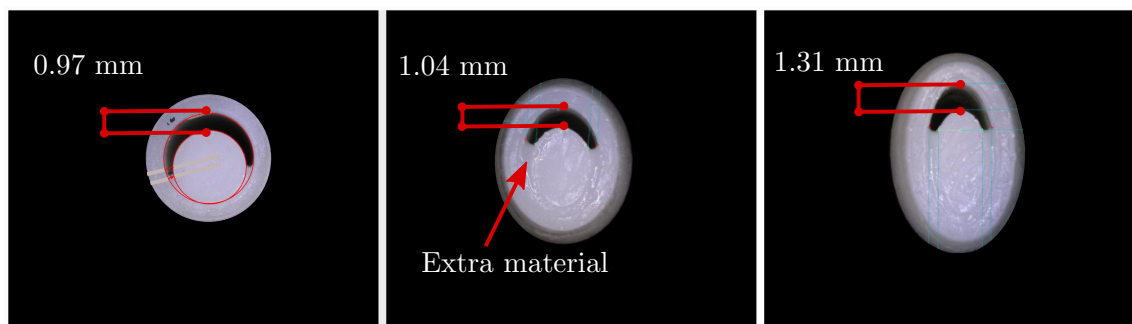
Over the course of this research, several design iterations have been considered for elliptical annular pipes.

The first, pictured in Figure 4.16a, was manufactured using a 3D printer at Boston Scientific Corporation in Marlborough, and designed on the scale of working channel/working tool dimensions. These proved too small for the 3D printer to accurately extrude. We tried both a separate rod (which was difficult to pin to the correct location), and one fused to the channel, shown in Figure 4.16a. The latter designs incurred extra material which was unequally distributed along the length of the pipe, and hence impossible to include in a fully-developed flow model.

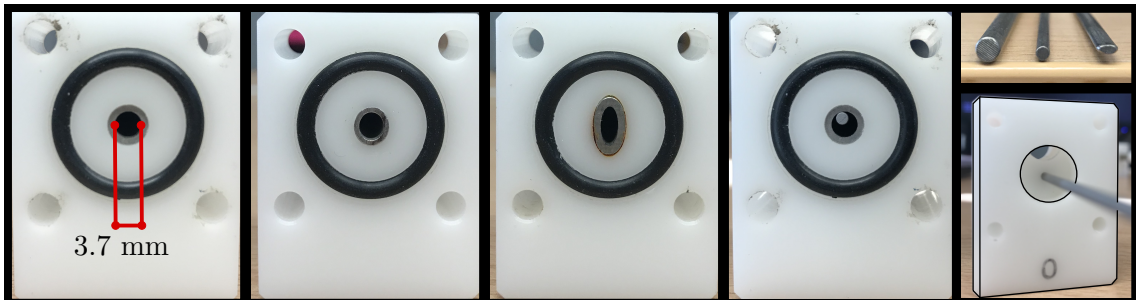
We then moved up to dimensions of the access sheath/scope shaft, and machined tubes out of metal at Boston Scientific Corporation, Minnetonka (Figure 4.16b). We designed a mechanism to place rods inside at specified positions, which consisted of clamps that screwed on both ends of the pipe (Figure 4.16b, bottom right). However, we speculated that the presence of the clamps to position the rod in place incurred extra flow resistance which was complicated to account for within our modelling framework. It was impossible to discern statistically different flow measurements between the different designs.

Finally, we increased the design measurements again to an outer cylinder of characteristic radius 0.5 cm and rod of radius 0.4 cm (Figure 4.16c). These were built using a 3D printer in the Mathematical Institute, Oxford. These first iterations involved ‘pins’ to maintain the position of the inner rod which was printed as an extra piece. However, again, we were concerned that the auxiliary ‘pins’ added extra resistance to the flow which was not uniform along the length of the pipe.

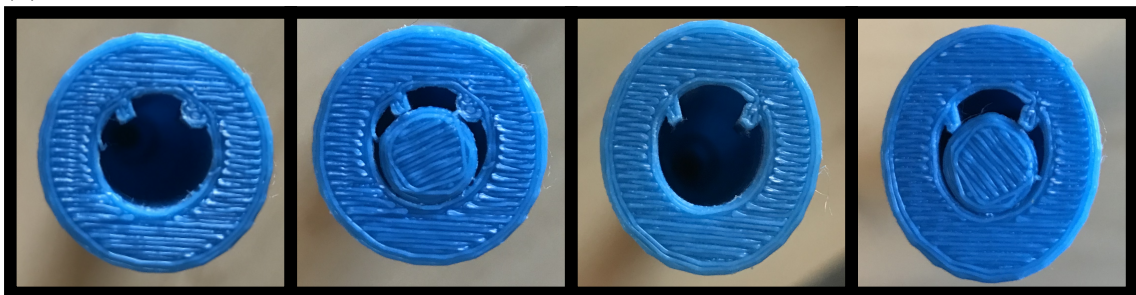
We finally settled on the designs in Figure 4.2b,c. These were 3D printed at Boston Scientific Corporation, Minnetonka.



(a) Plastic tubes printed at Boston Scientific Corporation, Marlborough, MA. March, 2017.



(b) Metal tubes machined at Boston Scientific Corporation, Minnetonka, MN. August, 2017.



(c) Plastic tubes 3D printed at the Mathematical Institute, Oxford, UK. December, 2017.

Figure 4.16: Various iterations on the experimental design to validate flow through elliptical pipes.

Appendix 4.B Data analysis

The raw data from the experiments, obtained as mass measurements at one second intervals is displayed in Figure 4.17.

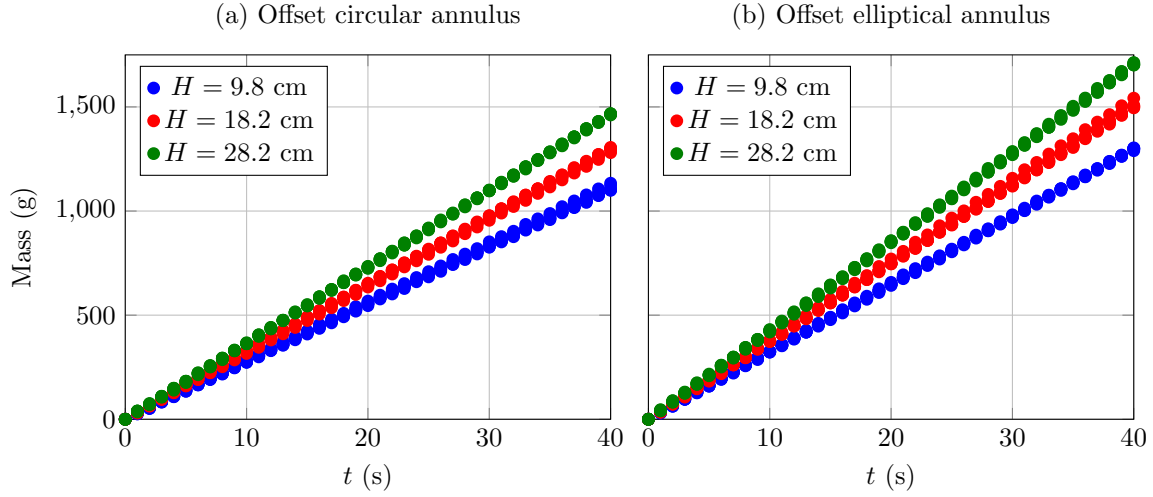


Figure 4.17: The raw data from the experiments described in Section 4.2.1.

The flow rate was approximated from the data by taking the final mass measurement for each run and dividing it by the total time that the run was performed for (sixty seconds). Six runs were performed for each value of H for each configuration (circular or elliptical). The mean flow rates, Q , and standard deviations, ς , were extracted for each configuration.

The mass balance used was a Mettler Toledo™ which has a readability of 0.01 g. Therefore, the error in the final mass measurement was taken to be ± 0.005 g, and the error in the estimated flow rate for each run to be $\sqrt{2}(0.005 \text{ g})/60 \text{ s}$. Therefore, the systematic error in the mean flow rate was $\Delta = (\sqrt{12})(0.005/60)/6 \text{ g/s}$. We assume that the standard deviation in the measurements and the systematic errors are independent and apply formula (2.43) for the total error. This is taken as the vertical error bar on the experimental data in Figure 4.4.

The fluid height in the bucket was maintained between marked lines, a distance of 1.5 cm apart. Therefore, the values for H are taken as the centre of these marked intervals, with standard deviation 0.75 cm. This is shown as the horizontal error bar on the experimental data in Figure 4.4.

Appendix 4.C Numerical details

We constructed an unstructured mesh of triangular elements via the use of the Triangle library within an `oomph-lib` driver code (Shewchuk, 1996). We used Taylor-Hood finite elements (second-order for velocity and first-order for pressure). The equations were solved using arclength continuation in the Reynolds number, implemented using `oomph-lib`'s `arc_length_step_solve` function. A single, automatic mesh adaptation was enabled at each Reynolds number. The maximum element area was set to 0.1 and the minimum to 10^{-6} . A typical number of elements was $\mathcal{O}(10^6)$.

The value of U to calculate Re was taken to be the purely viscous approximation, equation (4.30). This estimation ensured the resulting dimensionless velocities through the working channel were $\mathcal{O}(1)$. The corresponding Reynolds numbers for the physical properties in Table 4.1 and each value of H – with the range representing the interval of ϕ_{∞} values – were:

$$Re(H = 30 \text{ cm}) \approx 598 - 662, \quad Re(H = 60 \text{ cm}) \approx 1011 - 1115, \quad (4.51a,b)$$

$$Re(H = 90 \text{ cm}) \approx 1397 - 1536, \quad Re(H = 120 \text{ cm}) \approx 1758 - 1927. \quad (4.51c,d)$$

Appendix 4.D Estimated contraction loss

An analytical formula for the pressure loss due to abrupt contraction is provided by Kays (1950) to be

$$K = \frac{1 - K_{cb}\sigma^2 C_c^2 - 2C_c + C_c^2 2K_d}{C_c^2} - (1 - \sigma^2), \quad (4.52)$$

where σ is the contraction ratio, $K_{cb} = 1.543$ and $K_d = 1.2$ for Poiseuille flow through parallel plates. The value of C_c (the *coefficient of contraction*) as a function of σ can be estimated by interpolating experimental measurements provided by Rouse (1946).

Appendix 4.E Effect of l

Figures 4.18a,b plot the re-dimensionalised flux as a function of ρ for $l = 0.25$ cm and $l = 1$ cm, respectively, as opposed to $l = 0.5$ cm. The symbols correspond to the legend in Figure 4.8. We see that the general trend of the data is preserved; H has the most significant effect on Q , although we also see a small increase in Q with ϕ_{∞} . The red circle in Figure 4.18b highlights where solutions were observed to become asymmetric for $\phi_{\infty} = 0$, $H = 60$ cm. We were unable to obtain numerical solutions up to the desired Reynolds number

for $\phi_\infty = 0$ and $l = 1$ cm with $H = 90$ or $H = 120$, observable by the green and yellow circles terminating around $\rho = 0.7$ g/cm³ in Figure 4.18b. We encountered difficulties with numerical convergence in this parameter regime, even using arc-length continuation in the Reynolds number. This is likely due to the presence of multiple solutions and limit points in the bifurcation structure. This will be explored in Chapter 5.

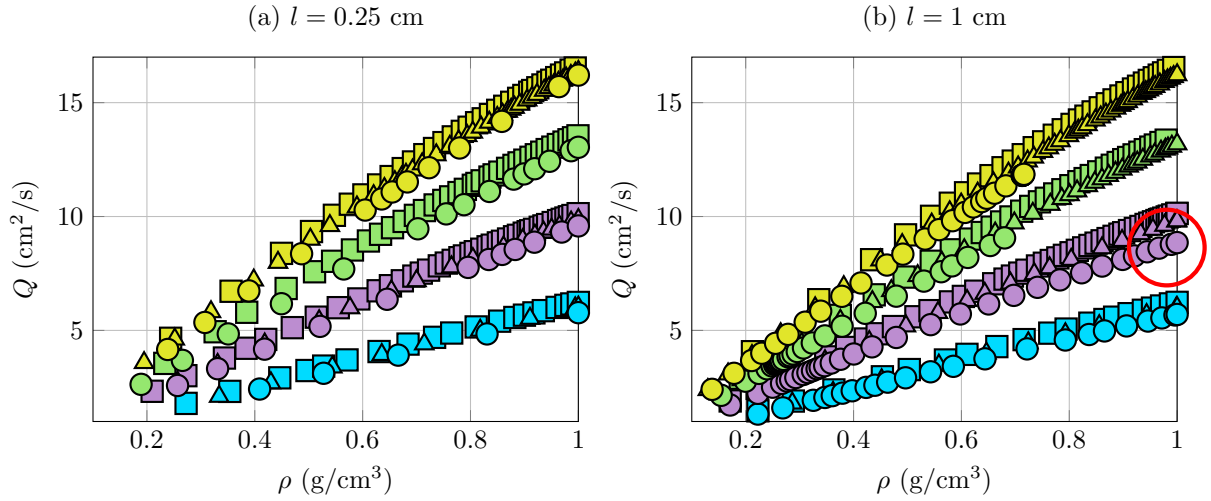


Figure 4.18: The results of the numerical continuation for (a) $l = 0.25$ cm and (b) $l = 1$ cm. Colours differentiate H (blue: $H = 30$ cm, purple: $H = 60$ cm, green: $H = 90$ cm, yellow: $H = 120$ cm) and shapes differentiate ϕ_∞ (circles: $\phi_\infty = 0$, triangles: $\phi_\infty = 0.5$, and squares: $\phi_\infty = 1$).

CHAPTER 5

Modelling flow in the renal pelvis

The complexity of things - the things
within things - just seems to be endless.
I mean nothing is easy, nothing is simple.

(Alice Munro)

Synopsis

We consider the fluid dynamics of irrigation fluid within the renal pelvis, resulting from the emerging jet through the working channel and return flow through the access sheath. Representing the renal pelvis as a two-dimensional rectangular cavity, we investigate the effects of flow rate and cavity size on flow structure and subsequent clearance time of debris. We model fluid flow with the steady Navier-Stokes equations and incompressibility condition, imposing a Poiseuille profile at the inlet boundary for the jet of saline, and zero-stress on the outlets. Resulting flow patterns in the cavity contain competing vortical structures. We demonstrate the existence of multiple solutions dependent on the Reynolds number of the flow and the aspect ratio of the cavity. We complement numerical predictions with PIV experiments. We model the clearance of an initial debris cloud via an advection-diffusion equation. We determine how the initial position of the debris cloud within the flow, and the Péclet number – characterising the relationship between advection and diffusion – affect clearance time, which is prolonged by entrapment of debris within closed streamlines. We discuss a flow manipulation strategy to extract debris from vortices and decrease washout time. We return to the systems model from Chapter 2, and discuss the cavity pressure variation as a potential explanation for the observed deviation between predicted kidney pressure and experiment.

5.1 Introduction

In this chapter, we analyse the fluid dynamics within a rectangular cavity representing the renal pelvis as introduced in Chapter 4. We determined in Section 4.4 that an imposed parabolic flow profile with specified flux at the inlet to the representative renal pelvis could be equated to a corresponding ureteroscopic set-up (saline bag height and scope position), through analytical approximations with inertial adjustments to the linear pressure drops through the irrigation tubing and working channel. In contrast, replacing the channels for outflow through the access sheath with zero-stress boundary conditions was found to require more careful consideration. For symmetric flow patterns, or for domains with a single outlet (corresponding to a scope fully-offset within the access sheath), the two formulations produced qualitatively identical flow patterns within the renal pelvis. However, for a partially offset scope, the flow structure was dependent on which boundary condition was imposed. In this chapter, although we will consider only symmetric domains and those corresponding to a fully-offset scope, we will also investigate symmetry-breaking within the symmetric domain. Therefore, we anticipate that flow solutions may deviate between numerical formulations with outflow channels and those without. Due to this, it may be desirable to consider domains with outflow channels. However, as we will compute bifurcation diagrams which require hundreds of simulations on each domain we do replace the outflow channels with zero-stress conditions on a truncated domain for computational tractability. We are driven in this chapter, by comparison of the numerical simulations with particle image velocimetry experiments, and with our reduced formulation of the problem we obtain excellent agreement. We also demonstrate, in Appendix 5.A, that the bifurcation structure is relatively unchanged between the two formulations for a particular scenario.

In Chapter 2 we determined the effects of operating conditions (saline bag height, tip deflection, and the presence of working tools) on irrigation flow rate in ureteroscopy. As good surgical visibility is often equated with high irrigation flux (Bach *et al.*, 2011), a significant result is the quantification of the large decrease in flow rate caused by working tools in the working channel (for fixed driving pressure drop). To mitigate this, we discussed optimising the cross-sectional shape of a working channel containing a working tool to maximise flow in Chapter 3. However, thus far, we have overlooked the effect of the behaviour of the emerging jet on efficient clearance of debris from the kidney, and to address this we now turn to modelling flow within the renal pelvis. Key questions are to determine the effects of kidney geometry and flow rate on flow structure, and subsequently, the influence of flow structure on the time required to clear debris. Precisely predicting flows in the kidney during ureteroscopy presents a difficult task as renal structure is complex and patient-specific; additionally, the placement of surgical instruments in relation

to stones and the kidney walls varies during any given procedure. However, a distinctive feature of irrigation flow is an emerging jet into an enclosed renal cavity with a return flow through the access sheath. Thus, we will investigate flow in a highly simplified geometry through numerical simulations combined with flow visualisation experiments.

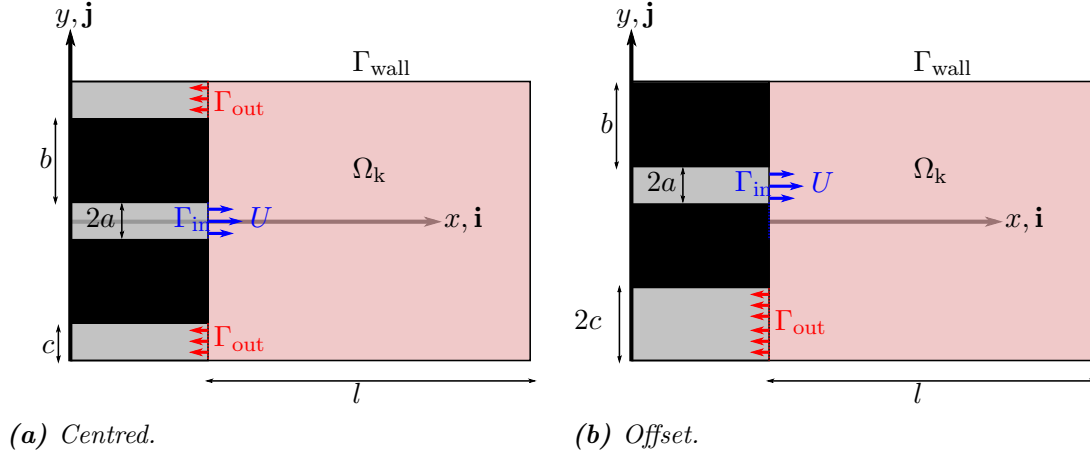


Figure 5.1: The idealised geometry. The light pink rectangle denotes the domain, Ω_k . The black rectangles represent the solid scope shaft, with grey regions indicating inflow through the working channel and outflow through the sheath alongside the scope shaft. The labelled lengths indicate: cavity length l , working channel half-width a , scope shaft half-width $a+b$, and cavity width $a+b+c$. The two considered scope positions are (a) centred and (b) fully offset.

We consider a two-dimensional Cartesian coordinate system. The idealised geometry is depicted in Figure 5.1, with Figures 5.1a,b representing the scope as centred and offset within the access sheath, respectively. The black rectangles symbolise the solid scope shaft, with grey areas denoting fluid-filled inflow and outflow channels. The light-pink region in Figures 5.1a,b, denotes the renal pelvis domain Ω_k with inflow and outflow boundaries Γ_{in} and Γ_{out} . This domain is characterised by three dimensionless aspect ratios, e.g., a/b , a/c , and a/l . As introduced in Chapter 4, inertial effects of the fluid in the renal pelvis are significant. Thus, we must consider the full steady Navier-Stokes equations, combined with incompressibility condition, equations (1.1). We assume fixed values for a , b , and c motivated by typical working channel, scope shaft, and access sheath dimensions in ureteroscopy. Therefore, we focus on quantifying the flow structure as a function of three important parameters: the flow rate, captured mathematically by the dimensionless Reynolds number, the scope position (centred or offset), and the length l , of the cavity (see Figure 5.1). In a ureteroscopy context, these parameters could be varied by the operating staff by changing the height of the irrigation bag, or by adjusting the distance between the scope tip and proximal kidney walls, respectively.

The existence of multiple solutions to the nonlinear two-dimensional Navier-Stokes equations in domains such as Figure 5.1 are expected, as flow in a channel with a sudden expansion is one of the classical examples of nonlinear bifurcation phenomena in fluid mechanics

(Drikakis, 1997). The tendency for a stream of fluid entering an expanded section to curve towards a proximal boundary is an example of the Coanda effect; an increase in velocity near the wall leads to a decrease in pressure, establishing a pressure gradient and resulting cross-stream velocity across the channel (Wille & Fernholz, 1965). It is well-known for expanding channels that in a low-Reynolds number regime, the flow is symmetric (and the solution to the steady Navier-Stokes equations unique). As the Reynolds number increases, however, steady asymmetric solutions emerge, and the critical Reynolds number at which these develop depends on the ratio between the channel widths downstream and upstream of the sudden expansion (Sobey & Drazin, 1986). Several modifications on the classic expansion channel problem have also been studied, such as the addition of a downstream contraction, shown to restabilise the symmetric state above certain Reynolds numbers (Mizushima *et al.*, 1996; Mullin *et al.*, 2002). Bifurcation phenomena of flow in a cavity with one inlet and two outlets was considered by Mizushima & Takahashi (1999) – distinct from the geometry in Figure 5.1a in that the outlets are on the opposite wall of the cavity from the inlet. Two different numerical boundary conditions were implemented at the two outlets (equal pressure and equal flux), and although the flow patterns were qualitatively similar, the structure of the bifurcation diagram was dependent on the choice of boundary condition. Laminar jet flow in a rectangular cavity with outlets similar to the considered geometry in Figure 5.1a has been solved numerically by Jelić *et al.* (2007) and Kolšek *et al.* (2007). Several flow distributions, including asymmetric patterns, were observed for different cavity dimensions, and it was postulated that cavity length is a key parameter in controlling the nature of the flow. However, a full characterisation of the numerical solutions, and the parameter regimes under which they exist, was not presented. Solutions in a geometry similar to 5.1a with $b = 0$ (i.e., with a simple boundary between inlet and outlets) have also been determined by Battaglia *et al.* (1998). Here, symmetry-breaking bifurcations were found, and solutions presented for different ratios of a/c .

In this chapter, we combine physical experiments with numerical simulations to investigate cavity flow properties. Our experimental technique to visualise and quantitatively measure instantaneous velocity fields is particle image velocimetry (PIV) in a pseudo two-dimensional set-up. Tracer particles are added to the flow, and are illuminated in a plane with images captured at least twice within a short time interval. The displacement of the particles between the two images, measured through post-processing techniques, provides a measure of the velocity field (Raffel *et al.*, 1998). We solve the steady Navier-Stokes equations in the cavity domain, Ω_k , pictured in Figure 5.1 using a finite element numerical scheme detailed in Section 5.4.

The resulting flow patterns, which depend on the Reynolds number of the flow and the aspect ratio of the cavity, are characterised by multiple vortices. For the symmetric domain

(Figure 5.1a) we demonstrate the existence of multiple solutions and compute bifurcation diagrams using deflated numerical continuation (Farrell *et al.*, 2015). By scaling the aspect ratio of the cavity into the governing equations, we are able to compute bifurcation diagrams in this parameter, as well as the Reynolds number of the flow. It has been shown that small physical imperfections in an experimental apparatus can lead to significant disconnections in the bifurcation diagram (Fearn *et al.*, 1990; Mullin *et al.*, 2002; Hawa & Rusak, 2000), and indeed, we observe a disconnected pitchfork in our experimental results. For both the centred and offset-scope domains (Figures 5.1a,b, respectively), we compare the flow patterns obtained through numerical simulation to those extracted from the PIV experiments, with excellent qualitative agreement.

Finally, we explore how different flow structures may affect the time taken to clear the field-of-view in ureteroscopy by solving an advection-diffusion equation, for an imposed initial cloud of stone dust, assuming that the concentration of dust has no effect on the background advecting velocity. Due to the vortical flow structure, debris can become trapped in regions of closed streamlines, resulting in increased washout times. A passive tracer advected by a non-uniform two-dimensional flow field and subject to weak diffusion undergoes enhanced mixing through shear dispersion, first discovered by Taylor in 1953 through his study of shear-flow dispersion in a pipe (Taylor, 1953). The interaction of velocity gradients with weak diffusivity produces an enhanced spatial flux of the tracer. This governing theory can be extended beyond dispersion by laminar Poiseuille flow to the mixing of a passive scalar within closed streamlines. Rhines & Young (1983) determined the time for expulsion of a tracer from closed streamlines which is split into two stages. The first stage is rapid due to principles of shear dispersion, with duration scaling with $Pe^{1/3}$. During this period, the tracer concentration averages throughout the vortex. A second slow stage allows the tracer to escape the closed streamlines which requires the full diffusion time. Motivated by this work, we propose, and validate through numerical simulations, that the time required for a debris cloud to escape a vortex and exit the cavity is split into three phases: the first two as described by Rhines & Young (1983) enable the concentration to escape the vortex, and the third, the time required for the tracer to advect out of the cavity, which depends on the flow speed and distance between the vortex and the outlet. We quantify how the initial position of the debris cloud and the flow structure, in terms of vortex size and location, affect washout time. This relates to work in earlier sections of this chapter which quantify the link between the geometry of the domain and the Reynolds number of the flow on streamline patterns.

5.1.1 Chapter outline

We begin by describing the PIV experiments: the experimental set-up, followed by a description of observed mirror-pair asymmetric flow patterns. We then present the theo-

retical formulation and numerical methods used for the finite-element simulations. Next, we compute and discuss bifurcation diagrams for the symmetric configuration (Figure 5.1a). We compare to good agreement, the theorised bifurcation diagram with one extracted from the PIV experiments, and present qualitative comparisons between velocity fields from the numerical simulations and PIV experiments. We next turn to discussing the advection and diffusion of a passive scalar within the computed velocity fields focussing, in particular, on the effects of flow structure, tracer position and the Péclet number on cavity clearance times.

5.2 Particle image velocimetry experiments

5.2.1 Experimental set-up

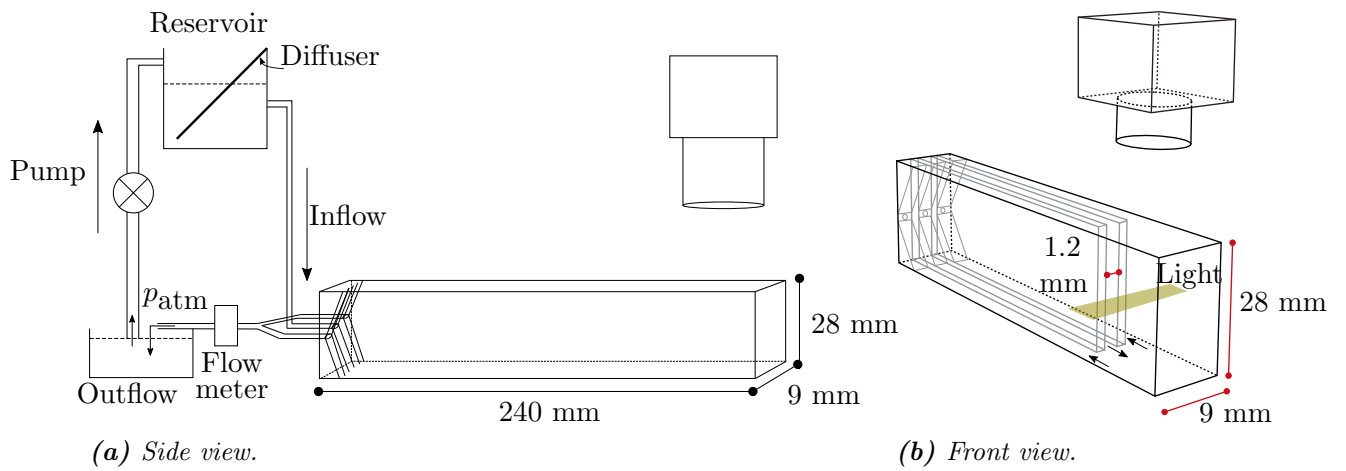


Figure 5.2: Schematics of the PIV set-up. Measurements provided in mm units; diagrams not to scale. Figure (a) provides a side-view of the full set-up with main components (reservoir, pump, flow meter, and camera) labelled. Figure (b) indicates a more detailed view of the experimental rig for the inflow, outflows, and cavity. The inner walls demonstrate the division between the inflow and outflow channels. The LED light sheet, which illuminates a plane of the cavity, is shown in yellow.

A schematic of the experimental set-up is shown in Figure 5.2. A rectangular acrylic chamber, containing parallel inflow and outflow channels, lies horizontally. There were two positions for the inflow channel: centred, in which case there were two outflow channels of equal width; and offset, where there was a single outflow channel (see the two positions in Figure 5.1).

The centre channel had width 1.2 mm (the diameter of the working channel of Boston Scientific’s LithoVue™ ureteroscope). The depth of the chamber was approximately 25 times the inflow channel width to approximate a domain of infinite parallel plates. Water

flowed from a suspended reservoir to the inflow channel through a piece of vertical tubing. The height of the reservoir could be adjusted to control the driving pressure, and subsequent flux into the chamber. The parallel channels terminated a finite distance from the end of the chamber (see Figure 5.2b), creating a cavity. Two different cavity lengths, 13 mm and 5 mm, were tested experimentally. The water left the chamber through the two outlets and the flow rate was measured using a flowmeter (FLR1000 Series Omega™), before collecting in an open air container. The fluid was pumped from the collecting container to the reservoir at the required rate to maintain the height of the reservoir. A diagonal acrylic sheet acted as a diffuser to minimise fluid recirculation in the reservoir.

To visualise the flow, the water was seeded with a dilute suspension of particles, designed to be advected with the flow. Two particle types were used: (A) silver-coated glass particles of 12 μm diameter and density 1.22 g/cm^3 (10089-SLVR, TSI Inc.), and (B) polyamide seeding particles of 5 μm diameter and density 1.03 g/cm^3 (PSP-5 Dantec Dynamics). The experiment images in Figures 5.3 and 5.6 contain particle type (B). An LED light illuminated a two-dimensional sheet at the centre of the chamber (Figure 5.2b), and a Phantom™ Miro LAB 320 high-speed camera recorded images at a speed between 2000 - 5600 fps. PIV experiments were performed for a range of reservoir heights (1.8-72 cm above the inflow channel). As the flow was gravitationally-driven, once the rig was set-up, the flow was allowed to continuously circulate. For each reservoir height, the flow rate was recorded, and particles introduced for flow visualisation. The wait-time between introducing particles and recording the image depended on the type of data sought. For purely qualitative images, we introduced particles to a clear cavity, and immediately captured the resulting patterns (Figures 5.3 and 5.6a,c). To enable the extraction of quantitative velocity fields by comparing subsequent image pairs, we waited sufficient time for the particles to diffuse uniformly throughout the cavity before recording the images. Images were analysed using PIVlab (Thielicke & Stamhuis, 2014), an open source tool in MATLAB.

5.2.2 The existence of asymmetric flow patterns

In the longer (13 mm) cavity, when the height of the suspended reservoir was above 27 cm, we observed asymmetric flow patterns, as seen in Figure 5.3a.

In these cases, it was possible to force the flow profile to adopt the ‘mirror’ orientation. To achieve this, we manually blocked the outflow that the jet was inclined towards (e.g. the lower outflow in Figure 5.3a), forcing the jet to move across the cavity (Figures 5.3a-e). We then released the closed outflow, and the new flow pattern persisted (Figure 5.3f). For the shorter cavity (5 mm), or for reservoir heights below 27 cm, closing and re-opening

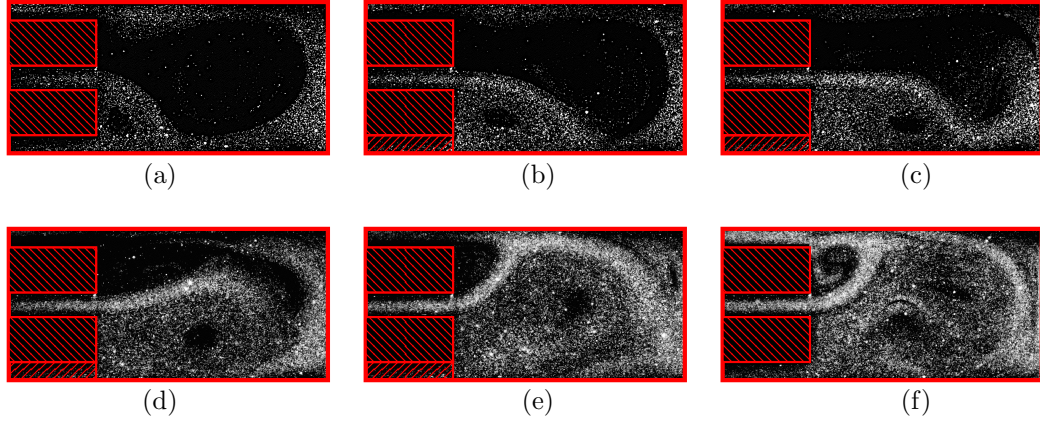


Figure 5.3: A time-series of the experiment technique to ‘switch’ the direction of the jet. The images are at 0.1 second intervals. The bottom outflow was closed between (a) and (b) and re-opened between (e) and (f). This experiment was performed at a head height of 61 cm ($1.2 \text{ cm}^3/\text{s}$), and a cavity length of 13 mm.

the outflow tube only disturbed the flow before it returned to its original pattern. For very low reservoir heights (3 cm) for the longer cavity, and for all reservoir heights for the shorter cavity, the flow patterns appeared symmetric.

5.3 Theoretical formulation

The experimental result depicted by Figure 5.3 suggests the existence of multiple stable flow solutions in our representative geometry. To interrogate this, we determine numerical solutions to the steady Navier-Stokes equations (1.1) – without the influence of gravity – considering the two-dimensional domain in Cartesian coordinates pictured in Figure 5.1. We consider fluid with density ρ and viscosity μ ; these are taken to be the properties of water (Table 2.2). The velocity field is given by $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ and pressure by p . Non-dimensionalising (x, y) with a , $\mathbf{u}$ with a velocity scaling U , and p with $\mu U/a$, the dimensionless steady Navier-Stokes equations (1.1) are

$$\text{Re}(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (5.1a,b)$$

where $\text{Re} = \rho U a / \mu$. We impose a fully-developed parabolic profile at the inlet boundary and take U to be maximum velocity. The prescribed dimensionless inflow is then

$$u = 1 - y^2, \quad v = 0, \quad \text{on } \Gamma_{\text{in}}. \quad (5.2)$$

We approximate the outflow geometry by prescribing zero normal and tangential stress on boundaries Γ_{out} in Figure 5.1. We determined, through comparing solutions with both

geometries, that these render nearly identical results with details provided in Appendix 5.A. Thus, we prescribe $\boldsymbol{\sigma}\mathbf{n} = \mathbf{0}$, which provides

$$2u_x - p = 0, \quad u_y + v_x = 0, \quad \text{on } \Gamma_{\text{out}}. \quad (5.3a,b)$$

On the walls (black solid lines in Figure 5.1), we assume no-slip

$$u = v = 0, \quad \text{on } \Gamma_{\text{wall}}. \quad (5.4)$$

We are interested in varying the dimensionless length of the cavity; thus, to allow for continuous numerical continuation in this geometric parameter, we consider the following dimensionless ratio

$$\alpha = l'/l, \quad (5.5)$$

where l' is a considered (dimensionless) cavity length and l is the length of our finite-element mesh.¹ We scale our coordinate system and vertical velocity component accordingly

$$x = \alpha x^*, \quad v = v^*/\alpha, \quad (5.6a,b)$$

where x^* and v^* are the scaled variables. The coordinate transformation (5.6a) is chosen so that $x = l'$ corresponds to $x^* = l$, and (5.6b) so that equation (5.1b) remains independent of α (Mullin *et al.*, 2002). Equations (5.1) in component form become

$$\frac{\text{Re}}{\alpha} \left(u \frac{\partial u}{\partial x^*} + v^* \frac{\partial u}{\partial y} \right) = -\frac{1}{\alpha} \frac{\partial p}{\partial x^*} + \left(\frac{1}{\alpha^2} \frac{\partial^2 u}{\partial x^{*2}} + \frac{\partial^2 u}{\partial y^2} \right), \quad (5.7a)$$

$$\frac{\text{Re}}{\alpha^2} \left(u \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y} \right) = -\frac{\partial p}{\partial y} + \left(\frac{1}{\alpha^3} \frac{\partial^2 v^*}{\partial x^{*2}} + \frac{1}{\alpha} \frac{\partial^2 v^*}{\partial y^2} \right), \quad (5.7b)$$

$$\frac{\partial u}{\partial x^*} + \frac{\partial v^*}{\partial y} = 0. \quad (5.7c)$$

Thus, we can solve equations (5.7) for (u, v^*) on a fixed domain of length $x^* = l$ for given α , and subsequently compute

$$\mathbf{u}(x, y) = u(\alpha x^*, y)\mathbf{i} + \alpha^{-1}v^*(\alpha x^*, y)\mathbf{j}, \quad (5.8)$$

to obtain the solution to equations (5.1) on a domain of length $x = l'$.

¹In our simulations $l \neq 1$, and instead, corresponds to 1 cm in dimensional units (approximately seventeen times the half-width of the inflow channel). This is a standard distance between the tip of the scope and the wall of the kidney during ureteroscopy, and allows for a fixed mesh size to be used for relevant values of α .

5.4 Numerical and experimental results

We computed numerical solutions to equations (5.7) using the open-source finite element library **Firedrake** (Balay *et al.*, 2018; 1997; Dalcin *et al.*, 2011; Rathgeber *et al.*, 2016; Hendrickson & Leland, 1995; Mitchell & Müller, 2016; Kirby & Mitchell, 2018; Amestoy *et al.*, 2001; 2006; Deuffhard, 2011). The solver we used was provided by the authors of Farrell *et al.* (2019), but extends the work of that paper to use the exactly divergence-free Scott-Vogelius element instead, as explained in Wechsung (2019); Scott & Vogelius (1985); John *et al.* (2017). We built a structured, rectangular mesh of triangular elements using **Gmsh**, demonstrated in Figure 5.4. We used a mesh that was symmetric across $y = 0$, as asymmetries in the mesh can tend the numerical solution towards a particular branch which can complicate the exploration of bifurcations from a symmetric solution. The values for b and c were chosen from the measurements of the experimental apparatus, scaled by a . The length of the mesh was taken to be equivalent to 1 cm in dimensional units; hence the two experimental cavity lengths correspond to values of approximately $\alpha = 0.5$ and $\alpha = 1.3$.

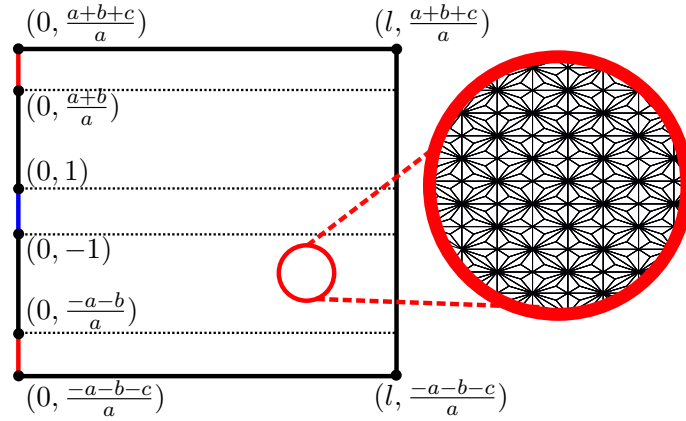


Figure 5.4: The structured mesh of rectangular elements, built in *Gmsh*.

5.4.1 Isolated solutions

To estimate the Reynolds number for a particular experiment, we take the measured volumetric flux, Q , and divide by the cross-sectional area of the inflow channel. This gives an estimate of the average velocity, and to obtain the peak velocity for a Poiseuille inflow, we multiply this by 1.54, which is the relation between the average and maximum velocity for Poiseuille flow through a rectangular channel of the relevant dimensions (Happel & Brenner, 1983). This is a slight correction from the relationship between average and maximum velocity for Poiseuille flow through parallel plates, which is 1.5 (Lamb, 1916).

The experiment in Figure 5.3 corresponds to $\alpha = 1.3$ and $Re = 34$. Solving equations (5.7) for these parameters, providing the Newton solver with a uniform initial guess, $\mathbf{u} = \mathbf{0}$, we obtain a symmetric flow pattern (streamlines shown in Figure 5.5a). Motivated by the experiment in Figure 5.3, we attempt to find asymmetric solutions by manipulating the flow pattern. We replace the boundary condition on the lower outflow with a zero-velocity condition, forcing the jet to exit via the top outflow (Figure 5.5b). This provides an asymmetric $\mathbf{u}$ which we use as the initial guess for the Newton solver with the original boundary conditions. This technique produces the asymmetric flow pattern in Figure 5.5c, and if we perform numerical continuation in the Reynolds number, the asymmetry persists. The structure of the flow in Figure 5.5c shows the existence of a large vortex near the centre of the cavity, along with other smaller vortices against the cavity walls. Regions of closed streamlines may be of particular interest in a ureteroscopy context, as we postulate these could inhibit stone dust from advecting quickly out through the access sheath. We will explore the role of vortical flow structure on the clearance of stone dust in detail in Section 5.5.

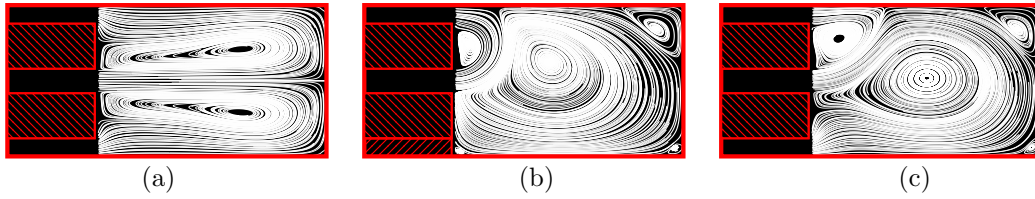


Figure 5.5: Figures (a)-(c) show a sequence of numerical streamlines for $\alpha = 1.3$ $Re = 34$. Figure (a) is initially computed by solving equations (5.7) on the computational domain with initial guess $\mathbf{u} = \mathbf{0}$ provided to the Newton solver. To produce (b), the boundary condition on the bottom outflow is replaced by a zero-velocity condition (demonstrated graphically by the blocked outflow channel). Using the solution pictured in (b) as the initial guess for the Newton solver with the true boundary conditions, we compute the asymmetric solution (c).

We display a comparison between numerical streamlines and images of the corresponding experiments in Figure 5.6 for the longer cavity length at two Reynolds numbers, $Re = 7$ and $Re = 34$. When flow rate is low, the experiment shows the jet emerging symmetrically into the cavity. This pattern is shown in Figure 5.6a, where tracer particles indicate streamlines originating within the inflow channel. A symmetric pattern at low flow rates is also predicted by theory, and the numerical trick demonstrated in Figure 5.5 does not produce an asymmetric solution. We note that Figures 5.6a,c, were produced purely for qualitative comparison and thus images were captured shortly after particles were introduced to the flow-field. Thus, the concentration of particles is low in Figures 5.6a,c in the areas corresponding to closed-streamlines in Figures 5.6b,d as more time would be required for the particles to diffuse into the vortices. A uniform particle distribution is necessary for the quantitative comparisons in the following sections.

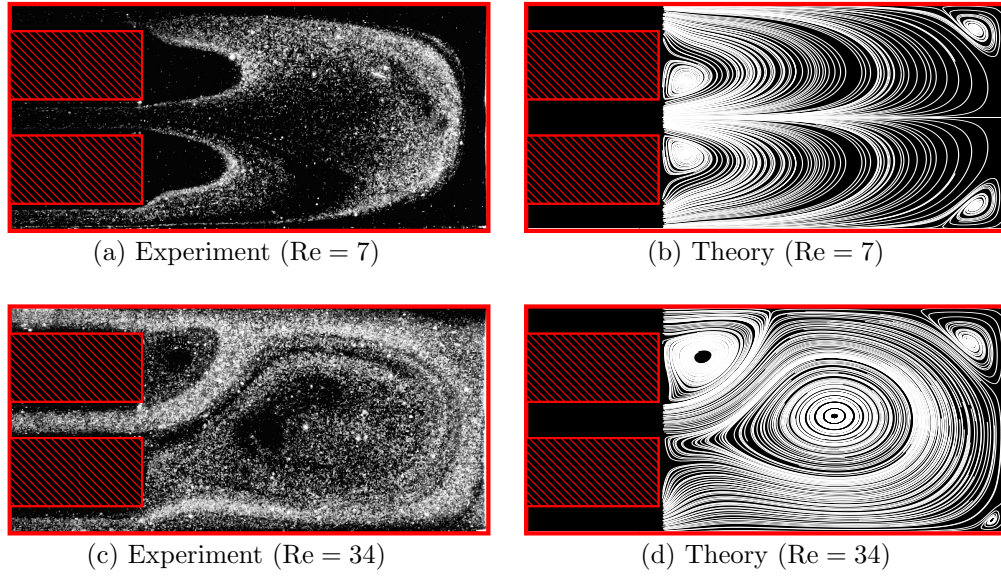


Figure 5.6: A comparison of qualitative ‘for visualisation’ experiments with theoretical streamlines.

Figures 5.7a,d and Figures 5.7b,e display velocity colourmaps for the numerical and experimental velocity fields for $Re = 35$ and $\alpha = 0.5$ and 1.3 , respectively. These suggest the existence of a critical cavity length at which asymmetric solutions emerge. Although less relevant in a symmetry-breaking context, we have also compared experimental and theoretical velocity fields for the ‘offset scope’ configuration (Figure 5.1b) in Figures 5.7 (c) and (f), respectively. All three geometries in Figure 5.7 demonstrate good agreement between PIV and simulation.

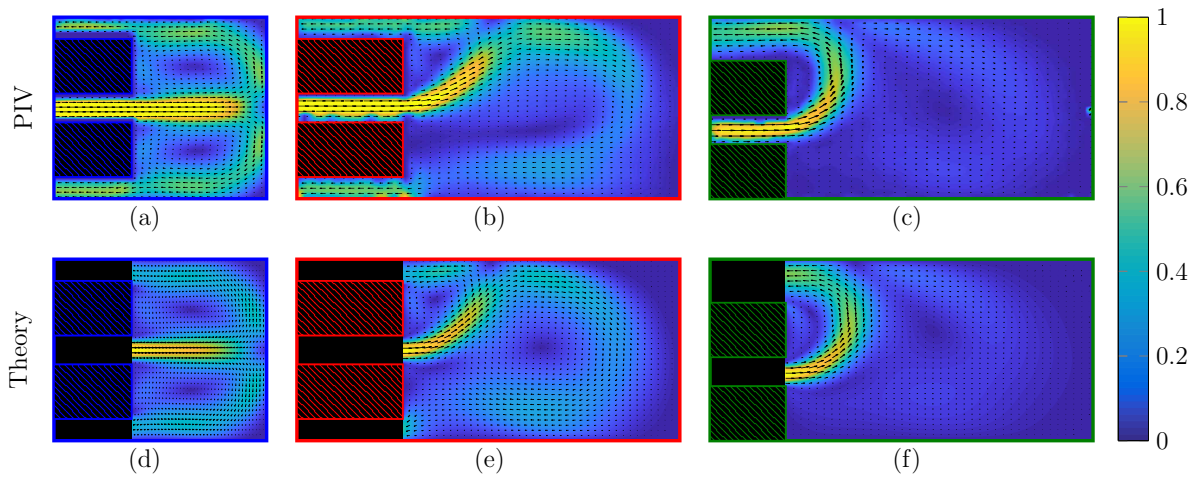


Figure 5.7: A comparison for (upper) numerical simulations with (lower) PIV experiments showing the velocity magnitude and vector directions. Left $\alpha = 0.5$, middle and right $\alpha = 1.3$ (with far right the ‘offset scope’ configuration). All are for $Re = 35$.

As we have determined the existence of multiple solutions to the steady Navier-Stokes

equation on the symmetric domain (Figure 5.1a) to be dependent on the length of the cavity and the Reynolds number of the flow, we will quantify this in the following subsection by computing bifurcation diagrams as functions of Re and α .

5.4.2 Bifurcation diagrams

To visualise the multiple solutions as functions of parameters Re and α , we now compute bifurcation diagrams. We use a recently developed algorithm, deflated continuation, for computing bifurcation diagrams that combines classical continuation with a deflation technique that eliminates known branches from consideration. This allows for the discovery of disconnected branches with Newton’s method, an advantage over other methods for computing bifurcation diagrams of numerical solutions, such as arc-length continuation and branch switching (Farrell *et al.*, 2015). To test the stability of the branches at representative points on the bifurcation diagram, we determine transient behaviour by using the computed steady-state solution (with an added random perturbation) as the initial condition to a time-dependent finite-element solver for the incompressible Navier-Stokes equations, implemented in **Firedrake**.²

For expanding channels, the maximum value of the cross-stream velocity along the centreline ($y = 0$) provides a good norm of the solution for visualising bifurcation phenomena, as this value will be zero for symmetric flows, and non-zero otherwise (Battaglia *et al.*, 1997). However, due to our confined geometry, the cross-stream velocity along the centreline of the cavity may have two peaks of opposite sign; see, for example, Figure 5.6d. Thus, we define a bifurcation metric, v_m , to be the value of the first peak; that is, the one with smaller x -value coordinate. Denoting the cross-stream velocity along the centreline $v_c = v(x, 0)$, v_m is hence defined as

$$v_m = \begin{cases} \max(v_c), & \operatorname{argmax}(v_c) < \operatorname{argmin}(v_c), \\ \min(v_c), & \text{otherwise.} \end{cases} \quad (5.9)$$

In Figure 5.8a, we plot the theoretical bifurcation for $\alpha = 1.3$ as a function of Reynolds number – branch stability is indicated by solid lines (stable) and dashed lines (unstable) – as well as the relevant experimental data. We obtain a pitchfork in the numerical solutions, where the appearance of the asymmetric branches implies loss of stability of the symmetric branch. Figure 5.8 demonstrates good quantitative agreement between theory

²We take $\mathbf{u}(\mathbf{x}) + \epsilon(\mathbf{x})$ as the initial condition, where $\mathbf{u}$ is the calculated steady-state solution and ϵ is an independent normally distributed random variable with mean zero and standard deviation 0.01, at each value of $\mathbf{x}$. If the transient solution moves away from the initial condition, the branch is marked unstable; otherwise it is stable. We terminate the solver once a steady-state has been reached – i.e., when the solver requires zero Newton iterations to update the solution from the previous time-step.

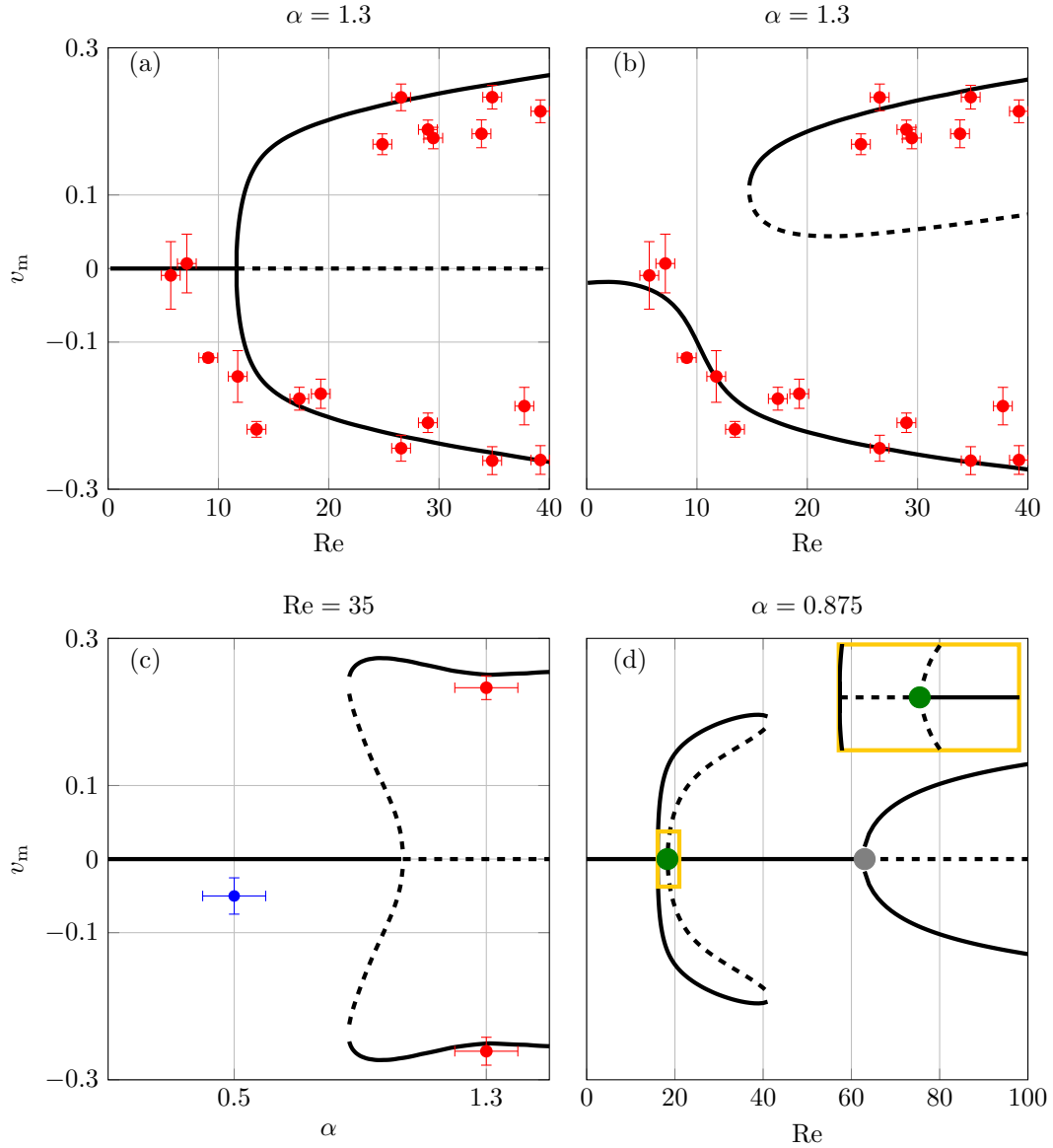


Figure 5.8: Black lines denote results of numerical simulations. Solid and dashed lines are solution branches conjectured to be stable and unstable, respectively. Figure (a) A plot of v_m as a function of Reynolds number. Data points from PIV experiments for $\alpha = 1.3$ are in red, with error bars providing the standard deviation. Figure (b) We add a small asymmetry to the domain; here the upper b separation is 10% smaller than the lower one. Figure (c) shows the bifurcation structure as a function of α for fixed $Re = 35$. Data points are from the PIV experiments with red for $\alpha = 1.3$ and blue for $\alpha = 0.5$. Figure (d) is a function of Re for fixed $\alpha = 0.875$. The inset plot shows a zoomed-in view of the area encompassed by the yellow box (from $Re = 16.14$ to $Re = 21$). The grey and green dots denote a subcritical point and a supercritical point, respectively, which will collide to form a C-coalescence point as α is increased.

and experiment; however, for $Re < 24$, only one asymmetric jet direction was obtained experimentally, and we were unable to switch the direction of the flow via the experimental technique described in Section 5.2.2. This indicates a disconnected pitchfork in the exper-

imental results, as is often found, due to small imperfections in the experimental set-up (Fearn *et al.*, 1990; Mullin *et al.*, 2002; Hawa & Rusak, 2000). To illustrate the effect of imperfections on the bifurcation diagram, in Figure 5.8b we add a small asymmetry to the domain (we take the upper outflow to be 10% smaller than the lower one), and compute the resulting bifurcation diagram, comparing this, again, with the experimental data. Although it is impossible to quantify the particular imperfections in the physical set-up, as these exist in an infinite-dimensional space, we see that an asymmetry in the computational domain can disconnect the bifurcation diagram, demonstrating better agreement with experiment. Additionally it is worth recalling when comparing bifurcation results between numerics and experiment, that flows in the physical experiments, although conducted in a geometry where the third dimension is nearly 25 times larger than the width of the inflow channel, will inherently be three-dimensional. The effect of the distance between the bounding walls in the third dimension on bifurcation phenomena in a sudden expansion channel was explored numerically by Guevel *et al.* (2018). The critical Reynolds number for the first primary steady bifurcation was shown to converge to the value for two-dimensional flow, albeit slowly, as the distance between the bounding walls in the third-dimension was increased.

As we have scaled the length of the cavity into the governing equations, we are also able to compute bifurcation diagrams as a function of α . An example is shown for fixed $Re = 35$ in Figure 5.8c, where we also plot experimental measurements for the two experimental cavity lengths at this Reynolds number. As predicted, for cavity lengths below a critical value ($\alpha \approx 0.87$), only the symmetric solution exists.

Interestingly, in the numerical solutions, we observe a hysteresis loop and a small range of α for which both the symmetric and asymmetric branches may be stable. Now fixing $\alpha = 0.875$, we vary Re , to obtain the bifurcation diagram shown in Figure 5.8d. A similar hysteresis loop at low Reynolds number has been shown for flow in a symmetric channel with an expanded section, emerging at a particular aspect ratio and Reynolds number, termed a C+ coalescence point (Mullin *et al.*, 2002). As the aspect ratio is increased, the subcritical point in the hysteresis loop (analogous to the green point in Figure 5.8) moves and eventually coalesces with the supercritical point (in grey), colliding to form a C- coalescence point. Although we have only computed slices through the bifurcation structure that exists in a three-dimensional space, and have not explicitly determined the trajectories of limit points, we postulate a similar behaviour for the bifurcation structure in our geometry as for the channel with an expanded section (Mullin *et al.*, 2002). Although not shown, we have also computed eight additional solution branches at $Re > 100$ and $\alpha = 0.875$. This is not surprising given that a rich structure to solutions of the Navier Stokes equations in a two-dimensional expansion channel at Reynolds numbers less than a few hundred is well-established (Alleborn *et al.*, 1997).

The results in this section highlight that even for the simplest symmetric representative geometry, complex asymmetric flow patterns can result. In the following section, we consider how the structure of the flow affects the time required for a passive tracer (representing debris within the kidney) to exit the cavity.

5.5 The effect of flow structure on cavity clearance

We consider a passive tracer in a steady velocity field, the latter computed as a solution to equations (5.7). As a proof-of-concept, we take the flow field for $\text{Re} = 34$ and $\alpha = 1.3$. We assume an initial concentration in the cavity (sufficiently low such that the tracer does not affect the flow) that is subsequently transported through the fluid by advection and diffusion.

5.5.1 Theoretical formulation

We consider the tracer concentration $c(\mathbf{x}, t)$ to diffuse with coefficient, $\mathcal{D}$, and to be passively advected by the flow of fluid within the cavity (Figure 5.9).

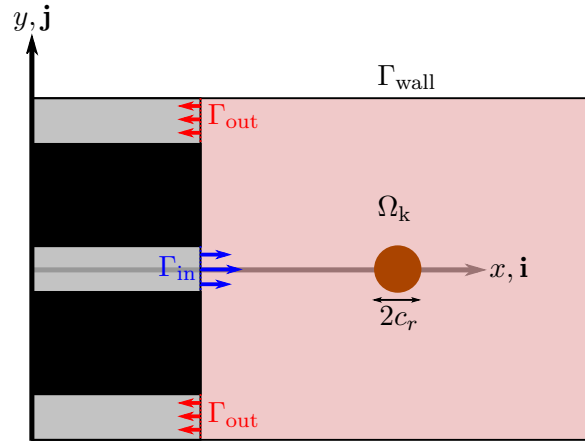


Figure 5.9: We consider the domain in Figure 5.1a, with the addition of a passive tracer of concentration $c(\mathbf{x}, t)$ with an initially circular distribution of radius c_r .

Conservation of mass provides

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = \mathbf{u}c - \mathcal{D}\nabla c, \quad (5.10a,b)$$

where $\mathbf{u}$ is determined a priori by solving equations (5.7) with boundary conditions (5.2), (5.3), and (5.4). We note that due to the incompressibility of the fluid, equation (5.10) is

equivalent to

$$\frac{\partial c}{\partial t} = \mathcal{D}\nabla^2 c - \mathbf{u} \cdot \nabla c. \quad (5.11)$$

We impose no diffusion of the tracer through the inflow, outflow, or solid walls:

$$\nabla c \cdot \hat{\mathbf{n}} = 0, \quad \text{on } \Gamma_{\text{in}}, \Gamma_{\text{wall}}, \Gamma_{\text{out}}, \quad (5.12)$$

where $\hat{\mathbf{n}}$ is the unit vector normal to each boundary. We assume an initial tracer concentration of density C that is uniform in a circle of radius c_r , centred at $\mathbf{x}_c = (x_c, y_c)$, which we approximate by

$$c(\mathbf{x}, 0) = \frac{C}{2} \left[1 + \tanh \left(\frac{d(\mathbf{x})}{\delta} \right) \right], \quad (5.13)$$

where δ controls the width of the step function approximation and

$$d(x, y) = c_r - \sqrt{(x - x_c)^2 + (y - y_c)^2} \quad (5.14)$$

ensures an initially circular distribution.

We nondimensionalise in the usual manner, where we use the advective time scale:

$$c = C\hat{c}, \quad t = a/U\hat{t}, \quad \mathbf{u} = U\hat{\mathbf{u}}, \quad \mathbf{x} = a\hat{\mathbf{x}}, \quad (5.15\text{a-d})$$

and hats denote dimensionless variables. Thus, equation (5.11) becomes

$$\frac{\partial \hat{c}}{\partial \hat{t}} = \frac{1}{\text{Pe}} \hat{\nabla}^2 \hat{c} - \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{c}, \quad (5.16)$$

where Pe is the Péclet number

$$\text{Pe} = Ua/\mathcal{D}, \quad (5.17)$$

which characterises the relative influence of advection with respect to diffusion. Henceforth we will drop the hat notation when referring to dimensionless variables.

As in Section 5.3, we scale α into the equations to enable efficient variation in this parameter without modifying the numerical mesh. Using (5.6), equation (5.16) becomes

$$\frac{\partial c}{\partial t} = \frac{1}{\text{Pe}} \left(\frac{1}{\alpha^2} \frac{\partial^2 c}{\partial (x^*)^2} + \frac{\partial^2 c}{\partial y^2} \right) - \frac{1}{\alpha} \left(u \frac{\partial c}{\partial x} + v^* \frac{\partial c}{\partial y} \right). \quad (5.18)$$

The dimensionless, scaled, initial condition (5.13), (5.14) is

$$c(x^*, y, 0) = \frac{1}{2} \left[1 + \tanh \left(\frac{d(\mathbf{x}^*)}{\delta} \right) \right], \quad (5.19\text{a})$$

$$d(x^*, y) = c_r - a\sqrt{(\alpha x^* - x_c)^2 + (y - y_c)^2}, \quad (5.19\text{b})$$

and boundary conditions (5.12) are

$$\nabla^* c \cdot \hat{\mathbf{n}} = 0, \quad \text{on } \Gamma_{\text{in}}, \Gamma_{\text{wall}}, \Gamma_{\text{out}}, \quad (5.20)$$

where $\nabla^* = (\alpha^{-1} \frac{\partial}{\partial x^*}, \frac{\partial}{\partial y})$.

In ureteroscopy, the primary purpose of irrigation is to clear the field of view. Thus, it is paramount to have fast clearance of debris, and thus we are interested in the rate at which concentration exits the cavity. We can define the percentage loss at time t as

$$\% \text{ loss} = \frac{\iint_{\Omega_k} [c(x^*, y, 0) - c(x^*, y, t)] d\Omega_k}{\iint_{\Omega_k} c(x^*, y, 0) d\Omega_k}. \quad (5.21)$$

We define a ‘washout time’ metric, T_{90} corresponding to the time required for 90% of the tracer to exit the cavity, i.e.,

$$\frac{\iint_{\Omega_k} [c(x^*, y, 0) - c(x^*, y, T_{90})] d\Omega_k}{\iint_{\Omega_k} c(x^*, y, 0) d\Omega_k} = 0.9, \quad (5.22)$$

where the initial concentration is set by equations (5.19). We note that as concentration only advects through the cavity outlets by equation (5.20), T_{90} can also be defined by

$$\frac{\int_0^{T_{90}} \left[\int_{\Gamma_{\text{out}}} c \mathbf{u} \cdot \hat{\mathbf{n}} dy \right] dt}{\iint_{\Omega_k} c(x^*, y, 0) d\Omega_k} = 0.9. \quad (5.23)$$

We have verified that both formulas agree, and use the former as it less prone to accumulation of numerical error.

5.5.2 Effect of flow pattern

In Section 5.4.1, we determined the existence of multiple flow solutions in a symmetric domain, and additionally, computed flow patterns corresponding to both centred and offset inlets. In this section, we initially place the dust cloud in the centre of the cavity, and consider the effect of different flow patterns on the washout time. In Figure 5.10, we plot equation (5.21) as a function of time, for three different advecting flow patterns: symmetric, asymmetric, and offset. All are at $\text{Re} = 34$, cavity length corresponding to $\alpha = 1.3$, and $\text{Pe} = 10^3$. The washout times T_{90} are indicated by the colourbar to the right of the streamline figures, corresponding to the colors of the circular dust clouds in Figures 5.10a,b,c, and the lines in Figure 5.10d. The symmetric flow pattern provides the shortest washout time, and the offset one the longest. Placing the tracer initially in the centre for the symmetric flow profile allows it to be immediately transported along streamlines

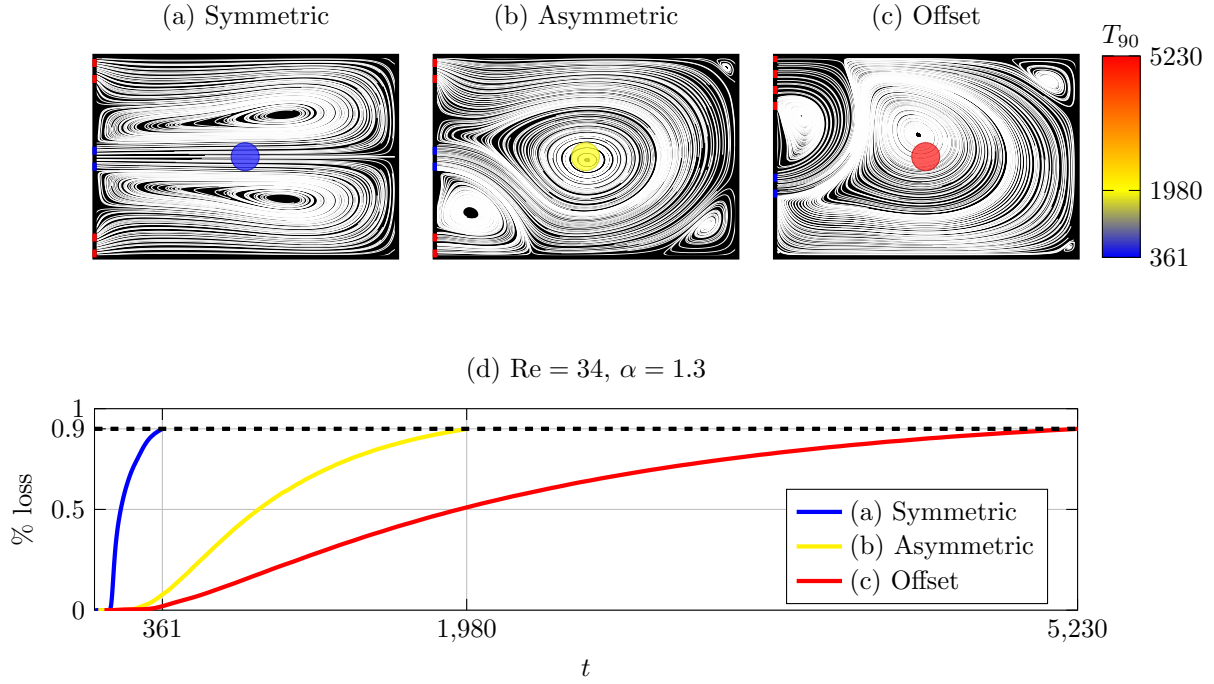


Figure 5.10: The % loss for $Pe = 10^3$ as a function of time for three different flow patterns. The corresponding patterns are shown in Figures (a), (b), and (c), with the colour of the initial concentration indicating the value of T_{90} .

that advect out of the cavity, with only trace amounts of debris entering the two vortices (Figure 5.10a). In contrast, both the asymmetric and offset flow patterns, contain vortices of closed streamlines surrounding the initial concentration, causing much longer washout times in an advection dominated regime ($Pe = 10^3$). To further explore the possibility of vortical streamlines trapping debris and causing longer washout times, we examine the effect of the initial position of a tracer within the flow pattern in Figure 5.10b, as well as the Péclet number, in the following sections.

5.5.3 Effect of tracer position

We have seen in Section 5.4.1 that velocity patterns in the characteristic domain are highly spatially heterogeneous, and contain vortices at several locations. Thus, it is of interest to see how the initial position of the concentration, $\mathbf{x}_c$, affects T_{90} . We consider 5 equally spaced positions within the cavity, which we label A , B , C , D , and E , as indicated in Figure 5.11. The colours of the tracers in Figure 5.11 demonstrate the corresponding washout times, equation (5.22). Results are shown for three different values of Pe – 10^3 , 10^2 , and 10^1 – in Figures 5.11 a, b, and c, respectively. Initial placement of the tracer in positions of vortical flow (A , B , and C) results in larger washout times than initial placement in streamlines that advect directly out of the cavity (D , E). This difference is

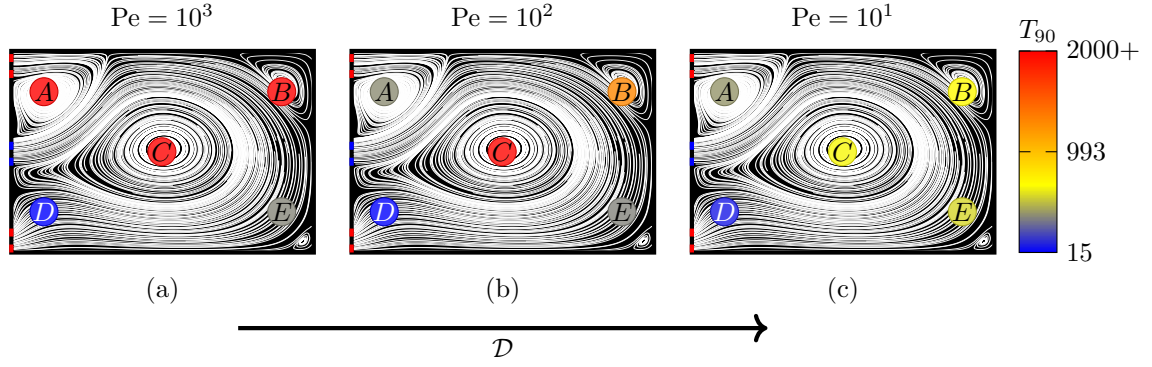


Figure 5.11: T_{90} for five values of $\mathbf{x}_c$. Figures (a), (b), and (c) are for $Pe = 10^3$, 10^2 , and $Pe = 10^1$.

more pronounced for larger values of Pe . As we decrease Pe (analogous to increasing the diffusion coefficient as we have scaled on the advective timescale), diffusion out of areas with closed-streamlines is faster, and total washout time decreases for positions A , B , and C . Interestingly, an increase in diffusion is not always advantageous in decreasing washout – E has a slower washout time for $Pe = 10^1$ than $Pe = 10^2$, as the tracer is initially on streamlines that advect directly out of the cavity, so diffusion away from these streamlines is counterproductive to washout.

5.5.4 Effect of Pe

According to Rhines & Young (1983), the expulsion of a tracer from closed streamlines occurs in two stages: a rapid concentration-averaging phase dominated by shear-augmented diffusion over a time $Pe^{1/3}(\ell/U)$, and a slow phase requiring the full diffusion time ($\ell^2/\mathcal{D}_{\text{eff}}$), where ℓ is the lengthscale of the flow (Rhines & Young, 1983). The Péclet number is based on $\mathcal{D}_{\text{eff}}$ which is proportional to $\mathcal{D}$, but exceeds it if the streamlines of the vortex are non-circular (due to principles of shear dispersion). Thus, for fluid that initiates in a vortex, we propose that clearance time is split into three phases of dimensional time scales:

$$T_a \sim (\ell_{\mathcal{D}}/U)Pe^{1/3}, \quad T_b \sim (\ell_{\mathcal{D}}^2/Ua)Pe, \quad T_c \sim \ell_U/U, \quad (5.24\text{a-c})$$

where $\ell_{\mathcal{D}}$ is the diffusive distance required (i.e., vortex size for initial positions A , B , and C), and ℓ_U the advective distance (i.e., the distance of the vortex from an outlet). T_a is the time required for the tracer to distribute uniformly throughout the vortex and reach the dividing streamline, T_b is the time to escape the closed streamlines via diffusion, and T_c is the time to advect from the vortex to an outlet. Scaling equations (5.24) by the

advective time-scale, as in equation (5.15b) we obtain the dimensionless forms

$$\hat{T}_a \sim (\ell_{\mathcal{D}}/a)\text{Pe}^{1/3}, \quad \hat{T}_b \sim (\ell_{\mathcal{D}}/a)^2\text{Pe}, \quad \hat{T}_c \sim \ell_U/a. \quad (5.25\text{a-c})$$

We note that, as we have scaled on the advective time, $\hat{T}_c$ is independent of Pe . We will henceforth drop the hat notation and refer to dimensionless timescales (5.25).

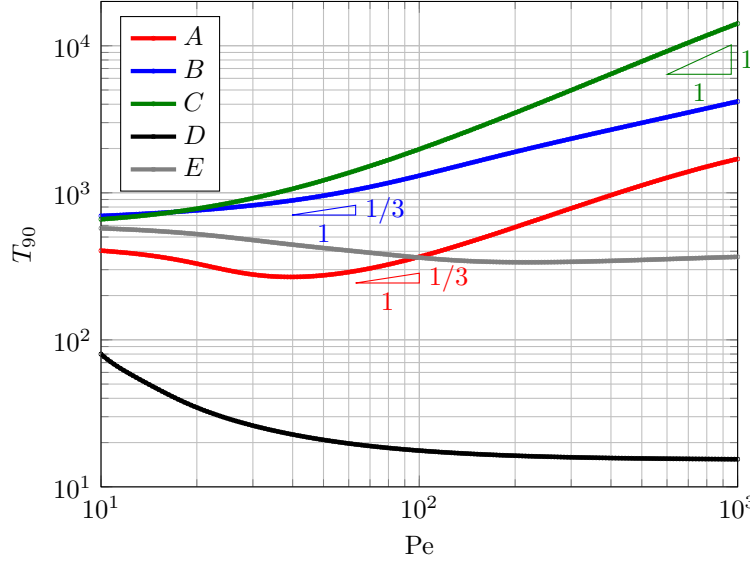


Figure 5.12: A log-log plot of T_{90} as a function of Pe for dust placed in positions A , B , C , D , and E . Points at $\text{Pe} = 10^1, 10^2$, and 10^3 , correspond to the snapshots in Figure 5.11. Péclet numbers considered: $\text{Pe} = 10^x$ for 200 evenly distributed x values $x \in [1, 3]$.

In Figure 5.12 we plot T_{90} as a function of Pe for concentration initiating at positions A , B , C , D , and E (Figure 5.11). For positions D and E which are not in vortices, $\ell_{\mathcal{D}} = 0$, and hence for large Pe , where diffusion is negligible, we will expect T_{90} to scale with T_c , which is independent of Pe , according to equations (5.25c). This is seen by the flat grey and black lines as we increase Pe in Figure 5.12. For concentration that is initially in a vortex (positions A , B , and C) we see the competing effects of T_a , T_b , and T_c . For moderate Pe and small vortices where $\ell_{\mathcal{D}}/a < 1$, T_a may dominate over T_b , and this is seen by the initial $\text{Pe}^{1/3}$ scaling for positions A and B . For $\ell_{\mathcal{D}}/a > 1$, however, T_b will quickly dominate as Pe increases; hence we see a linear slope emerging for position C . For the Pe range considered, this is not seen for positions A and B , where the vortices are small, and hence a delicate balance exists between T_a , T_b , and T_c . We also note a decrease in T_{90} with diffusion for positions A , D , and E with Pe when Pe is approximately $10^1 \leq \text{Pe} \leq 10^2$. Here, diffusion takes concentration on to the closed streamlines in the centre of the cavity, where it will require more time to escape.³

³As tracer originating at position A is already in a region of closed streamlines, diffusion allows some tracer to escape via the top of the vortex and and advect through the proximal outflow. However,

Although not explored here, it is interesting to note that Rhines & Young (1983) also predict the difference in washout times between vortices with circular and elliptical streamlines on T_b , the time required to escape closed streamlines after averaging has taken place. As expected, T_b decreases with increasing eccentricity of elliptical streamlines.

5.5.5 Effect of instantaneous flow switch

The results in Figures 5.11 and 5.12 demonstrate that remaining within a steady vortex can induce extremely large times required to exit the cavity. Thus, we hypothesise that disrupting the flow pattern may improve washout times. Motivated by the experiment in Figure 5.3, where we saw mirror asymmetric flows could be established, with the opportunity to switch between them by blocking and re-opening an outflow, we instantaneously switch $\mathbf{u}$ in equation 5.18 from one mirror branch to the other at a time s (see Figure 5.13a,b). We denote the washout time resulting from a switch at time s by T_{90}^s . Fig-

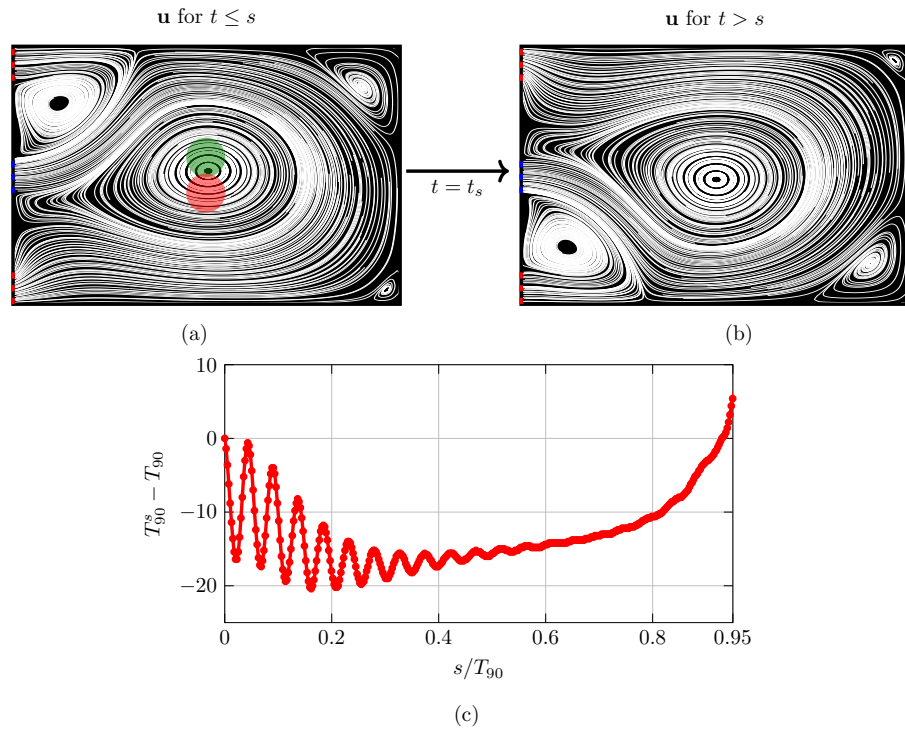


Figure 5.13: Figure (a) shows the velocity field for $t < s$, and (b) for $t \geq s$. Figure (c) is the decrease in washout time as a function of switch-time. The green and red circles in Figure (a) indicate positions for the bulk of the tracer concentration where “flow switching” corresponds to peaks and troughs in Figure (c), respectively.

ure 5.13c shows the change in T_{90}^s from T_{90} (when no “flow-switching” is imposed) as a

a significant proportion of the tracer still diffuses to the large central vortex, explaining the initial decrease in T_{90} in Figure 5.12.

function of s/T_{90} for concentration initially placed in the centre of the cavity (position C in Figure 5.11) and $Pe = 10^2$. We see that instantaneously switching from Figure 5.13a to Figure 5.13b at time $t = s$, when $s < 0.93T_{90}$ results in a decreased washout time, T_{90}^s . The oscillations in Figure 5.13c occur when switching during the advective averaging phase, T_a (equation 5.25). During this initial period, as position C doesn't align precisely with the centre of the vortex, the tracer swirls along the streamlines. Switching from Figure 5.13a to Figure 5.13b when the tracer is above the centreline of the vortex (for example at the position indicated in green in Figure 5.13a) moves the tracer further towards the outside of the vortex. This is beneficial both because the tracer is closer to streamlines that advect directly out of the cavity and because the velocity is faster, thereby increasing shear-enhanced diffusion of the tracer. Switching once the tracer has distributed uniformly throughout the vortex, i.e., $s > T_a$, still moves some of the tracer on to the advective streamlines (as the vortex is not central within the cavity), but as the position of the tracer is relatively constant, we see a dampening of the oscillations in Figure 5.13c. Although not shown, for higher values of Re the vortex is less central within the cavity, and hence switching flow branches can move the tracer onto a streamline that advects directly out of the cavity, thus inducing further reductions in washout time. We caution, however, that switching the flow once the tracer has diffused out of the vortex can actually increase washout times. This is seen in Figure 5.13c, where, when $s/T_{90} \rightarrow 1$, $T_{90}^s - T_{90} > 0$. This is similar to the effect seen in Figure 5.11, where an increase in diffusion caused an increase in washout time for tracer E , which is initially placed on streamlines that advect out of the cavity. Once the tracer has reached open streamlines that leave the cavity, switching the advecting flow pattern may move tracer back onto closed streamlines, slowing washout times. This “flow-switching” experiment draws parallels with a classical example of chaotic mixing: the blinking vortex model, where alternate activation and deactivation of a pair of separated vortices results in enhanced spreading of a tracer (Aref, 1984).

5.6 Summary and discussion

In this chapter, we have investigated the fluid mechanics of flow in an idealised representation of the renal pelvis, consisting of an emerging jet into a rectangular cavity, with a return flow through parallel channels. We then progressed to consider how flow patterns affect the time required for successful clearance of debris by coupling the velocity field to an advection-diffusion equation.

We began by studying symmetry-breaking in the considered geometry, which we determined occurs as a function both of Reynolds number and the aspect ratio of the cavity. We

adopted a complementary approach, performing both numerical simulations and particle image velocimetry (PIV) experiments, and obtained excellent qualitative and quantitative agreement. We computed bifurcation diagrams as functions both of cavity aspect ratio and the Reynolds number of the flow via numerical deflation techniques. A rich bifurcation structure was observed including hysteresis loops for a small range of cavity lengths and Reynolds numbers. The PIV experiments confirmed the presence of a disconnected pitchfork bifurcation as a function of Reynolds number for fixed aspect ratio. We determined both critical Reynolds numbers and aspect ratios for the existence of asymmetric flow patterns.

For a particular Reynolds number and cavity length, we studied the effect of initial placement of a passive tracer within the flow field on the time required for 90% of the concentration to advect out of the cavity (the washout time). We calculated results for a variety of Péclet numbers, and found that when the tracer is initially placed in a vortex, the washout time (scaled by the advective time-scale) is comprised of three contributions: first, a time for the tracer to distribute uniformly throughout the vortex via mechanisms of shear-enhanced diffusion which scales with $Pe^{1/3}$; second, a time for the tracer to escape the closed streamlines requiring the full diffusion time which scales with Pe ; third, a time for the tracer to advect from the vortex to a cavity outlet, which depends primarily on the advecting velocity and is hence nearly independent of Pe . The dominant scaling depends on the initial position of the tracer in relation to vortices within the cavity and streamlines that advect out of the cavity.

Finally, we considered the effect of switching from one asymmetric solution to its mirror pair on the advection and diffusion of the passive tracer, and found that this often results in a reduction in washout time from the “no-switching” case.

The results in this chapter provide valuable insight into the dependence of successful clearance of the kidney on the structure of the flow, and indicate that increasing flux may not be the only method through which to improve clearance. Extrapolating the results of our study in a simple two-dimensional cavity, we hypothesise that debris within the working space during ureteroscopy may get trapped in vortices, resulting in large washout times. Disturbing the flow can increase shear-dispersion and move debris on to streamlines that advect out of the cavity, resulting in shorter times required to clear the field-of-view.

An important extension to the work in this chapter will be to consider the effect of the third dimension on flow patterns and resulting washout times. Solving the steady Navier-Stokes equations in a two-dimensional rectangular domain has allowed us to explore fundamental principles, such as the existence of multiple solutions, and the effect of vortical flow on the clearance of debris that is passively advected with the addition of weak diffusion. Although the results in this chapter in relation to ureteroscopy are purely

illustrative, they serve to highlight the importance of flow rate and kidney geometry on flow structure, and consequently on the time required to clear the field-of-view. Thus, a high flux may not always correlate with good surgical visibility if the flow patterns remain relatively steady and debris is trapped in vortices. In this case, disrupting the flow to extract debris from regions of closed streamlines may be necessary to clear the intrarenal view.

5.7 Epilogue: pressure variation in the renal pelvis

Returning to the systems model discussed in Chapter 2, we recall a discrepancy between the predicted intrarenal pressure, modelled as a single point, and experimental data from *ex-vivo* experiments (Figure 2.10b, where kidney A does not fall within the estimated range). One possible explanation is the position of the pressure probe within the kidney in the experiments; although it was placed proximal to the end of the scope, its position was likely to vary. To test this, we explore the degree to which pressure is likely to vary in the renal cavity, using the dimensions of the working channel, scope shaft, and access sheath listed in Table 2.4 to determine geometric parameters a , b , and c , respectively (Figure 5.1a). As the cross-sectional area of the working channel ($A = 0.013 \text{ cm}^2$) is smaller than the cross-sectional area of the rectangular channel in the PIV experiments ($A = 0.364 \text{ cm}^2$), the mean velocity within the cross-section $\bar{U} = Q/A$ is considerably larger for the same flux. The velocity scaling for the numerical simulations is taken as the peak of the Poiseuille inlet with relevant flux; for a circular channel, this is $U = 2\bar{U}$. Thus, the relevant Reynolds numbers for simulations representing the flow rates in Figure 2.10a, and thus measured renal pressures in 2.10b, range from $\text{Re} = 0$ to $\text{Re} = 1883$. Results of the simulation for a representative flux $Q = 1.4 \text{ cm}^3/\text{s}$ ($\text{Re} = 1500$, $\alpha = 1$) are shown in Figures 5.14a,b. The dimensions of the working channel, scope, and access sheath used for the experiments are indicated along the vertical axis of Figure 5.14a in cm units. By initially placing a zero-velocity condition on one of the outflow boundaries, as outlined in Section 5.4.1, we ensure the asymmetric solution branch, as we have theorised this to be stable. The higher Reynolds number causes a larger central vortex than the flow pattern considered in Section 5.5.2, for example, for $\text{Re} = 34$ and $\alpha = 1.3$. The corresponding pressure contour plot is shown in Figure 5.14b, and the colourbar indicates pressure in units of cm H_2O . As flow is driven by pressure gradients, the same velocity solution will exist if an arbitrary constant is added to the pressure field; thus, we shift the pressure solution so the minimum value is at zero. Hence, Figure 5.14b indicates the pressure variation in the cavity, rather than absolute pressure values. We see that the pressure varies significantly on a spatial scale with a maximum variation of 90 cm H_2O . In Figure 2.10b, the deviation between P^* for kidney A and the predicted values increases with P_{in} .

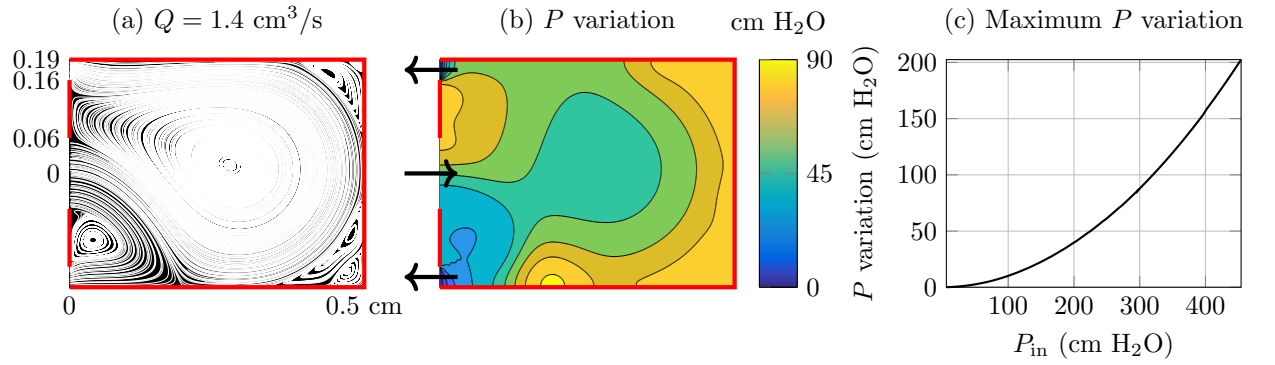


Figure 5.14: Plot of predicted (a) streamlines and (b) pressure variation, within a rectangular cavity with prescribed parabolic inflow for $Q = 1.4 \text{ cm}^3/\text{s}$ and zero-stress outflow conditions. Cavity measurements labelled in (a) in cm units, corresponding to working channel, scope, and sheath dimensions. Pressure variation in (b) in cm H₂O, shifted so the minimum pressure in the cavity is at zero. Figure (c) plots the difference between the minimum and maximum pressure values in the cavity, as a function of P_{in} . The relationship between P_{in} and Q prescribed for each simulation is determined from equation (2.41) with R_{sc} taken as the value for a scope centred in the sheath, equation (2.25).

To estimate P_{in} for a given simulation with a prescribed flow rate, we use the relationship for the steady-state flow rate, equation (2.41), assuming R_{sc} for a scope centred within the access sheath. Using this approximation, the simulation in Figures 2.10a,b corresponds to $P_{\text{in}} = 276 \text{ cm H}_2\text{O}$. We can then plot the maximum pressure variation (difference between maximum and minimum pressure in the cavity) as a function of P_{in} (Figure 5.14c), and we see that the variation increases with P_{in} . In a sense, this predicted variation corresponds to large error bars on the P^* data points in Figure 2.10b, explaining our inability to always accurately predict the pressure detected by the probe. Additionally, the pressure variation in the cavity increases with P_{in} , possibly explaining the increase in deviation between the experiments and prediction for kidney A in Figure 2.10b.

Appendix 5.A Outflows

In Chapter 5 we solve equations (5.7) in the light-pink domain in Figure 5.1a, prescribing zero-stress boundary conditions, equations (5.3), on Γ_{out} , and a Poiseuille profile on Γ_{in} , equation (5.2).

Here, we validate truncating the numerical domain to include only the cavity, and not the outflow channels present in the particle image velocimetry experiments (Section 5.2), by simulating flow on the domain pictured in Figure 5.15, with outflow channels of dimensional length 24 cm (see Figure 5.2a), denoted l_{out} (when non-dimensionalised with respect to $a = 0.06$ cm). As we scaled the horizontal coordinate by α and solve the scaled equations (5.7), we build a mesh with outflow channels of length l_{out}/α . As outflow conditions

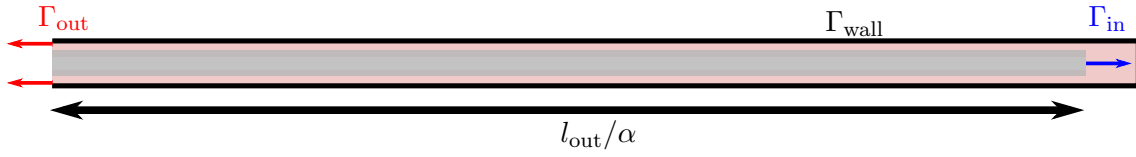


Figure 5.15: The considered numerical domain including outflow channels.

we prescribe zero normal stress and parallel outflow

$$2u_x - p = 0, \quad v = 0, \quad \text{on } \Gamma_{\text{out}}, \quad (5.26a,b)$$

and as before, a Poiseuille inlet profile, equation (5.2).

We plot the bifurcation diagram for $\alpha = 1.3$, $\text{Re} \in [0, 40]$, in Figure 5.16 (black marks) including points for the simulations at the same parameters in the domain with outflow channels with conditions (5.26) (grey marks). We see the bifurcation structure remains unchanged, with only small discrepancies between the numerical results.

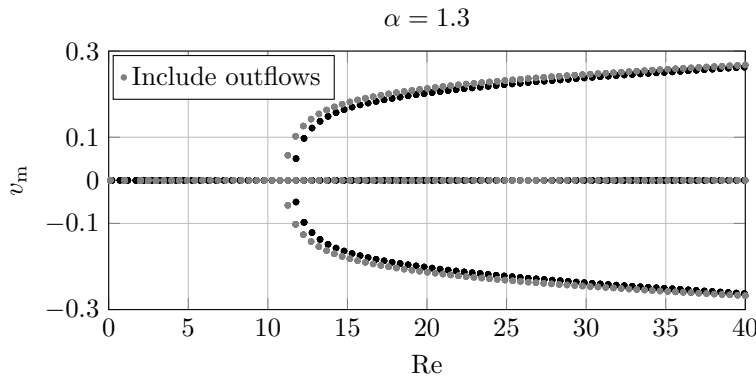


Figure 5.16: A plot of v_m as a function of Reynolds number, comparing the results in Figure 5.8a (black marks) with those including outflows (grey marks).

CHAPTER 6

Summary and future work

Mere mechanical difficulties can be overcome.

(Brunel)

6.1 Summary of the thesis

This thesis has focussed on understanding the fluid mechanics of ureteroscope irrigation through mathematical modelling, numerical simulation, and bench-top, *ex-vivo*, and particle image velocimetry (PIV) experiments.

We began in Chapter 2 by modelling the effects of saline bag height, tip deflection, and the presence of working tools on irrigation flow rate. We found that flow rate increases non-linearly with saline bag height, due to effects of viscous dissipation through the irrigation tubing. Tip deflection was incorporated by considering Dean flow through the deflected tip, and we determined only a minimal (at most 5%) decrease in flow rate with tip deflection. Working tools, however, presented a significant hindrance to irrigation flow for fixed pressure drop and flow decreased monotonically with tool size. As the position of the working tool within the working channel is unknown, and flow resistance also decreases monotonically with tool proximity to the edge of the channel, we allowed for uncertainty in working tool position within the modelling framework, resulting in predictive ‘wedges’. The modelled influences of bag height, tip deflection, and working tools agreed with the results of bench-top experiments and provided a physical underpinning for the conclusions of previous experimental studies. The considered domain for modelling flow through a working channel containing a working tool was the region between a solid

rod and surrounding hollow cylinder. The simplicity of this framework rendered it applicable to modelling flow back from the renal pelvis through the access sheath containing the scope shaft. Thus, we turned to flow through the working channel and back through the sheath as flow through distinct pipes connected by a pressure point at the exit of the working channel and entrance to the sheath, representing the intrarenal pressure. This systems model was closed with a constitutive law for kidney compliance; a relationship between fluid pressure and volume within the renal pelvis. We used an exponential relationship, motivated by previous biological studies and validated with *ex-vivo* experimental data. The model predictions contained a degree of uncertainty resulting from the variable position of the scope within the sheath; analogous to modelling flow through a working channel containing a working tool of unknown position, we obtained predictive ‘wedges’. The steady-state predictions of the systems model were compared against data from *ex-vivo* experiments, to good agreement for the flow rate predictions. The pressure data involved more variability, leading to some discrepancy between the estimated range and the experimental results. We hypothesised this to be caused by the disregard for pressure variation in the renal pelvis within the systems modelling framework. We revisited this hypothesis in Chapter 5. The key parameters affecting the steady-state solution for renal pressure from the systems model were the ratio between geometric resistance through the working channel to resistance back through the access sheath, and the inlet pressure. For a fixed inlet pressure, the addition of a working tool to an empty working channel increased inflow resistance, thereby decreasing kidney pressure and subsequent flow rate. However, if a set flow rate was required and the inlet pressure was adjusted appropriately when a working tool was inserted, the kidney pressure was unaffected by the addition of auxiliary working tools. Minimising flow resistance through the access sheath both decreased kidney pressures and increased flow rate for fixed inlet pressure.

As high flow and low intrarenal pressure are both desirable attributes of ureteroscopy irrigation, we turned, in Chapter 3, to minimising flow resistance in a domain of coaxial cylinders, and to considering the question of where a scope (tool) will naturally sit within the sheath (working channel). This latter question was explored by perturbing the position of the inner cylinder within the channel to generate cross-sectional hydrodynamic forces. Ensuring that the perturbation was small, the problem decoupled into unidirectional (Poiseuille) flow down the axis and Stokes flow in the cross-section. We considered geometries within the class of elliptical shapes; this was motivated by manufacturing simplicity and physiological compatibility. For clarity, we explored in detail three sub-cases: I) a circle in an ellipse, II) an ellipse in a circle, and III) an ellipse in an ellipse with the same eccentricity and orientation. We highlighted counter-intuitive phenomena, both in the axial and cross-sectional flows. A notable result for minimising axial flow resistance was the presence of optimal elliptical shapes. This inspired us to determine the full opti-

mal geometry within the class of elliptical shapes, fixing only the available cross-sectional area. Turning to the cross-sectional flows and forces, resistance to imposed motion increased with distance between the rod and cylinder centres, and an apparent correlation between the position of minimal axial flux and of maximum cross-sectional resistance was found in the case of III. We concluded by determining the optimal elliptical arrangement for the relevant dimensions of the access sheath and scope shaft. This design modification was theorised to increase flow rates and to decrease kidney pressure by over 50% from the optimal circular configuration of the same cross-sectional area.

Motivated by the predictions of optimal elliptical shapes for minimal axial flow resistance and the implications for ureteroscope design, we conducted validation experiments in Chapter 4. A significant challenge was to minimise potential sources of pressure loss in the experimental apparatus unaccounted for by viscosity. Even following several design iterations, we were unable to explain a large discrepancy between fully-developed theory and the experimental result. Thus, we considered additional pressure drops which occur at pipe entrances and exits caused by dissipation of kinetic energy due to fluid inertia. These adjustments are predicted to scale quadratically with flow rate. To determine the influence of inertia on relationships between flow rate and pressure in ureteroscopy, we solved the steady Navier-Stokes equations numerically in thirty-six configurations representing variations in the clinical arrangement due to saline bag height, scope shaft position within the access sheath, and renal pelvis length. We determined the influence of renal pelvis length on flow rate to be insignificant, although this parameter was found to control a symmetry breaking behaviour for the symmetric configuration which we explored in great detail in Chapter 5. As anticipated, the results of the numerical simulations demonstrated an increase in flow rate with saline bag height and a small increase with scope proximity to the edge of the sheath. Although these behaviours were also captured by analytical formulas for fully-developed flow, the magnitude of the flow rate was over-predicted by the linear, inertia-less equations. By subtracting the viscosity-induced pressure drops through the tubing and working channel from the total drops determined from the numerical simulations, we confirmed the residual pressure loss to scale quadratically with flow rate. The inertial adjustment to the pressure drop through the irrigation tubing was independent of scope position, in contrast to the inertial adjustment to the pressure drop through the working channel, which varied with scope position due to geometry-specific effects of the jet emerging into the renal pelvis. Pressure corrections within the renal pelvis and through the access sheath were impossible to account for with simple energy-loss coefficients. However, we found that the total flow rate through the system as a function of saline bag height and scope position could be accurately predicted only by adjusting for inertial effects through the tubing and working channel, avoiding additional complications within the renal pelvis and access sheath. We emphasise that although irrigation flux is

relatively unaffected by flow patterns within the renal pelvis, an understanding of flow patterns in the kidney and how these affect the clearance of stone particles and debris is paramount for good visibility during ureteroscopy.

We turned to focus on the fluid mechanics within the renal pelvis in Chapter 5. Our domain was comprised of an idealised, two-dimensional rectangular cavity of dimensions appropriate for ureteroscopy, with a single inlet (flow through the working channel) and two outlets (flow through the access sheath). We also considered a geometry representing the scope fully offset from the centre of the access sheath which contained a single outflow boundary. We modelled fluid flow with the steady, Navier-Stokes equations and incompressibility, imposing a parabolic velocity profile at the inlet and zero-stress at the outlet boundaries. We found the existence of multiple solutions, both symmetric and asymmetric, characterised by competing vortical structures. Through numerical deflated continuation, we tracked the bifurcation diagrams as functions of the Reynolds number of the flow and aspect ratio of the cavity. We determined the existence of critical Reynolds numbers and critical aspect ratios at which bifurcations emerged. To complement our numerical solutions, we also performed PIV experiments in a pseudo two-dimensional set-up. We found excellent qualitative and quantitative agreement between theory and experiment. We concluded the chapter by modelling the influence of flow structure on cavity clearance. We added a circular cloud of *debris* (a passive tracer) to be advected and diffused by the previously computed flow patterns. We determined that washout-time was prolonged if debris became entrapped in vortices, and that this effect was reduced as diffusion, with respect to advection, decreased, characterised by the Péclet number. For a debris cloud initially placed within a vortex, washout time was characterised by three phases representing: the time required for the tracer to average throughout the vortex and reach a dividing streamline, the time to escape the closed streamlines via diffusion, and the time to advect from the vortex to an outlet. We explored a strategy to decrease washout time by removing debris from vortices – instantaneously switching flow solutions from one asymmetric branch to its mirror pair. Although the results are purely illustrative, due to the geometric discrepancy between a two-dimensional rectangular cavity and the true renal pelvis, the principle of disrupting established flows to enhance the release of trapped debris may be a transferable prospect to ureteroscopy. We concluded by revisiting the discrepancy between predicted and measured kidney pressure in Chapter 2, supporting the proposed explanation of unaccounted for spatial variation in kidney pressure in the systems modelling approach.

The work in this thesis provides the foundations of a mechanistic understanding of ureteroscopy irrigation, and some insights into how the process may be improved. It also uncovers a plethora of unanswered questions which may be addressed with further

mathematical study. Several of these avenues for further research are detailed in the following section.

6.2 Future work

6.2.1 The coaxial assumption

In Chapters 2 and 3 we modelled flow through the working channel containing a working tool, and flow through the sheath containing the scope shaft, as flow through a cylindrical channel containing a coaxial, cylindrical rod. The assumption of coaxiality between the rod and bounding cylinder was key to the analysis as it ensured a constant cross-section along the length of the cylinders. As a consequence, the geometric resistance was determined via solutions to Poisson’s equation in an annular domain of arbitrary geometry. However, in practice, tools and the scope shaft are unlikely to be perfectly aligned with the working channel and access sheath, respectively, and incorporating an axial tilt to the rod generates cross-sectional hydrodynamic forces along the length of a stationary rod. Recent work to address this has considered flow through a two-dimensional channel containing a plate with a small tilt and small offset from the channel centre. The results indicate that generated forces will cause the plate to rotate and will push it towards one of the bounding walls (Reynolds, 2019). This may be important from a ureteroscopy design perspective, indicating that, although a working tool (scope shaft) is likely to move towards the edge of the working channel (access sheath), resulting in desirably lower resistances, additional features to ‘pin’ the tool’s (scope shaft’s) position may be required to ensure that it remains coaxial with the working channel (access sheath).

6.2.2 Analysis of multiple solutions

The work in Chapter 5 highlights the complex flow structure and multiple solutions that exist in a geometry of the sort pictured in Figure 5.1. Although we presented bifurcation structures to compare with experimental data (Figure 5.8), and tested branch stability at particular points with a transient solver, a thorough consideration of stability, through the computation of eigenvalues of the transient system, was missing from the analysis. We can conjecture the stability of solutions for simple bifurcations, such as single pitchforks; however, more complex bifurcation structures exist in this geometry and computing branch stability and the trajectory of limit points is necessary for a more in-depth understanding of the bifurcation structure. For example, in Figure 6.1, we plot the bifurcation diagram for aspect ratio $\alpha = 0.875$, with Reynolds number ranging up to $\text{Re} = 200$ – this is a

continuation of the diagram in Figure 5.8d, where the domain ranged up to $Re = 100$. For this particular geometry, at $Re = 200$ there exist a minimum of 11 solutions, and though a number of these will be unstable and hence irrelevant in practice, the complexity of Figure 6.1 indicates a rich bifurcation structure.

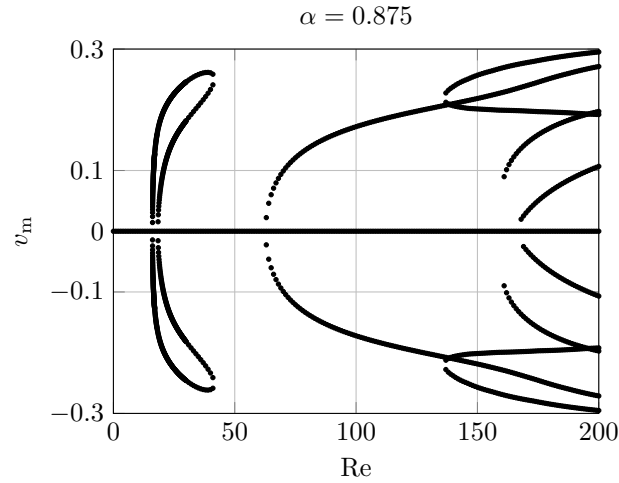


Figure 6.1: A plot of v_m , equation (5.9), as a function of Reynolds number for $\alpha = 0.875$, equation (5.5). Black dots are the results of numerical simulations. This plot continues the Reynolds number range shown in Figure 5.8d up to $Re = 200$ (although without indicating branch stability).

It is also worthwhile to note that the existence of two parameters controlling the bifurcation structure, Re and α , distorts bifurcation structures into three-dimensional space, with Figures 5.8 and 6.1 presenting only two-dimensional slices. Even the simplest of pitchfork bifurcations is unfolded to a complex three-dimensional cusp catastrophe with the existence of two symmetry-breaking parameters (Cliffe & Winters, 1984). A valuable extension to our preliminary study of multiple solutions in Chapter 5 is to compute the trajectory of bifurcation limit points and the stability of solution branches in (Re, α) space, thereby constructing a holistic understanding of how observable flow structure varies with flow rate and cavity geometry.

The effect of domain imperfections on solutions, as introduced in Figure 5.8b to gain improved agreement with experiment, is a practical concern. As perfect symmetry in any physical set-up is only achievable in theory, we expect realistic solution structures to be disconnected. The degree of detachment can be measured by the location of the detached limit point (see Figure 5.8b) and will increase with the amount of imposed asymmetry in the domain (perhaps taken as the difference between outflow widths, as considered in Figure 5.8b). The locus of the detached limit point as a function of Reynolds number and the imposed asymmetry forms a cusp singularity at the point where the domain is symmetric (Cliffe & Winters, 1984). We found, in Section 5.5.5, that the

existence of multiple solutions, and the ability to switch between them at fixed Re and α can be beneficial to ureteroscopy by decreasing the time required for debris clearance. Thus, discerning disconnect as a function of imposed imperfection could be beneficial to determine how much ‘outflow-blocking’ is required to switch solution branches.

6.2.3 3D renal flows in realistic geometries

Although the reduced two-dimensional domain in Figure 5.1 is useful for exploring the qualitative structure of intrarenal flows, the geometry is far from the complex biological structure of the kidney. This consists of a central cavity at the ureteral entrance (the renal pelvis) as well as multiple chamber-like offshoots, known as calyces (Figure 6.2).

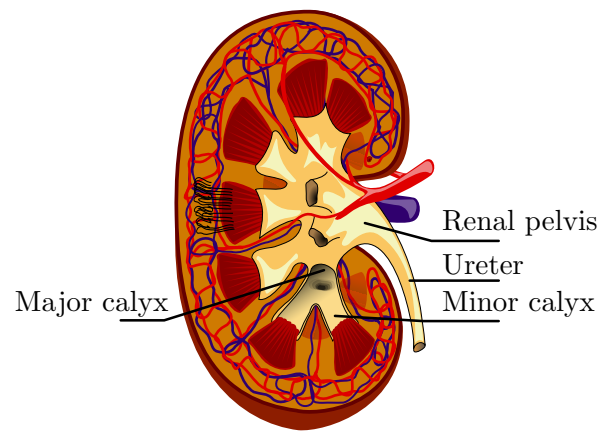


Figure 6.2: Renal structure. Adapted from drawing by Piotr Michał (PioM). File obtained from https://commons.wikimedia.org/wiki/File:KidneyStructures_PioM.svg with licensing <https://creativecommons.org/licenses/by-sa/3.0/>.

Thus, obtaining clinically relevant flow patterns and pressure variation would require complex three-dimensional experiments (physical and/or numerical) on realistic kidney domains, encompassing the wide range of patient-specific structural variations. Due to flow over the calyceal orifices, we anticipate additional regions of vortical flow and hence low pressure within the calyces (Pozrikidis, 1994), which could lead to debris migration and entrapment. Due to the confined geometric structure of the calyces, extraction of the trapped stone particles via simple flow manipulation may be impossible, necessitating manual extraction by stone basketing or irrigating directly into a calyx.

6.2.4 Deformability of physiological structures

The work in this thesis has focussed on flow within rigid structures. For the working channel and access sheath this is a realistic assumption; the working channel is embedded

within the solid scope shaft and the access sheath wall is reinforced with a stainless steel coil. For flows within the renal pelvis, however, this is a simplifying assumption. In Chapter 2 we did consider an elastic kidney characterised by an exponential curve relating changes in irrigation mass to changes in intrarenal pressure, although spatial variation in kidney pressure was neglected. In Chapter 5 we considered a rigid cavity for the renal pelvis which allowed for flow patterns and spatially heterogeneous pressures, but ignored the deformability of the kidney walls inherent to the true physiological structure. Amalgamating complex flow patterns within the renal pelvis with deformability of the tissue walls remains an important task for modelling ureteroscopy irrigation.

Deformability is additionally relevant if ureteroscopy is performed without an access sheath. In this clinical practice, irrigation fluid flows out through the gap between the scope shaft and the ureteral wall which is an elastic biological tissue. Asymptotic solutions in the aspect ratio of a slender elastic tube have rendered a nonlinear relationship between flow rate and pressure drop (Anand & Christov, 2018); this may be used to capture the effects of a deformable ureter on flow rate during ureteroscopy without an access sheath. Furthermore, even with the presence of an access sheath, the entrance to the sheath from the renal pelvis is usually placed distal to the ureteropelvic junction (UPJ), a functional valve connecting the renal pelvis to the ureter. Thus, fluid flowing from the renal pelvis flows through a short, deformable constriction before entering the access sheath. The presence of the UPJ was included in the Oratis *et al.* (2018) systems model of ureteroscopy irrigation, although only as a short, rigid tube of radius 0.02 cm and length 0.5 cm. Hence, incorporating effects of elasticity of the UPJ may be another step towards refining a physiologically accurate model of ureteroscopy irrigation.

6.2.5 The influence of kidney stones

Although the clearance of a passive tracer, as considered in Section 5.5 sheds light on the important effects of flow structure, diffusion, and the initial placement of debris, on clearance rates, it remains to incorporate the effect of larger, solid kidney stones with the ability to affect the flow field. Fluid structure interaction problems, such as the benchmark problem of flow in a two-dimensional channel past a cylinder with an attached elastic component, have been solved numerically in `oomph-lib` (Heil & Hazel, 2006), and finite element codes of this sort may provide the basis for solving the fully coupled problem. Of particular interest will be the forces on the stone by the jet of irrigation fluid, as these may lead to undesirable effects such as *retropulsion*, which moves the stone away from the view of the operating clinician. The addition of large kidney stones within a computational framework is further complicated, as their structure will be externally affected by concurrent laser lithotripsy degrading the solid structures into smaller fragments more easily advected by the surrounding flow.

6.2.6 Oscillatory inlet pressure

In this thesis, we have considered steady inlet pressures, resulting from gravitationally-driven flows from a hanging saline bag. However, increasing the bag height or applying a pressure cuff to the bag are not the only methods for increasing inlet pressure; oscillatory pumps may be employed or boluses of fluid added via hand or foot-controlled devices (Pedro, 2012). Thus, inlet pressure is not always steady, introducing a necessity for time-dependence within the mathematical model. Introducing pulsatile flow to a systems lumped-parameter model such as the one presented in Chapter 2, has demonstrated that the dampening of oscillations through the working channel can be controlled by varying the amplitude and frequency of inlet pressure oscillations (Reynolds, 2019). Ongoing research will introduce pulsatile flows to the renal pelvis, as we hypothesise that time-dependent flows may break up vortices, leading to decreased debris washout times.

6.2.7 Heat modelling and stone cavitation

Although we have focussed on the fluid mechanics of ureteroscopy in this thesis, there is another important physical process: the transfer of thermal energy from a firing laser during laser lithotripsy. Recent experimental studies have recorded high temperatures resulting from ureteroscopic lasers, which is alarming due to possible damage to surrounding tissue (Aldoukhi *et al.*, 2017; Winship *et al.*, 2019). Additionally, the thermal process may couple to fluid flow, potentially inducing fluid cavitation which enhances stone break-up. Modelling these thermomechanical processes will allow for valuable insight into the mechanisms of stone break-up which can be harnessed to improve stone lasering protocol.

6.2.8 Optimal tip geometry and clinical protocol

We solved in Chapter 3 a question of optimising the cross-sectional geometry of cylindrical components of ureteroscopy systems (the working tools, working channel, scope shaft, and access sheath) to minimise axial flow resistance. However, the work in Chapter 5 motivates a second optimisation question: *what is the optimal scope tip geometry for ureteroscopy?* To address this, careful consideration of an appropriate cost-function will be required. One candidate may be ‘washout time’ for a passive tracer, such as the metric provided by equation (5.22). However, optimising flows based on their impact on debris without considering the coupling between kidney stones and flow structures may neglect key physical phenomena, such as the effect of tip geometry on the fluid forces imposed on a stone. Thus, another candidate cost-function may be the magnitude of repulsive forces. The experimental PIV set-up, developed to investigate cavity flows in Chapter 5,

will be essential for validating identified tip designs, and further testing will be required in an *ex-vivo* set-up, such as the one in Chapter 2. Through the EPSRC Doctoral Prize scheme we have received funding to pursue the avenue of proposed research relating to optimal design of the scope tip and hence this will comprise our immediate future research.

In addition to improving tip geometry, the results in Section 5.5.5 inspire questions of optimising clinical protocol, such as: *how, and in what manner, to disturb flow to minimise clearance times?* In Section 5.5.5 we considered an instantaneous switch of flow direction, as an approximation to the experimental technique for switching the flow pattern by blocking and re-opening the appropriate outflow channel (Figure 5.3). However, in actuality, disrupting the flow structure is a time-dependent phenomenon. It is thus of interest to determine the influence of various dynamic disruptions to a steady flow on shear-enhanced diffusion and other washout mechanisms.

6.3 Concluding remarks

This thesis accentuates the capacity of mathematical modelling to guide advancements in biomedicine. We are optimistic about the introduction of quantitative techniques to many facets of ureteroscopy: to instruct design of medical equipment via optimisation techniques; to provide infrastructure to compute intrarenal pressures from measured flow rates; and to automatically adjust tuneable parameters, such as inlet pressure or scope position, to improve visualisation while maintaining intrarenal pressures below a critical threshold. Notwithstanding the multitude of extensions proposed in Section 6.2, the work in this thesis – mathematical models based on fundamental physics and complementary experiments – provides critical first steps towards improving procedural efficiency and reducing patient risk.

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