Applications of classification of C*-algebras

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Abstract

In this thesis, we will use classification results for C*-algebras and *-homomorphisms between them to characterise nuclear dimension equal to zero for a large class of *-homomorphisms. In particular, for certain *-homomorphisms where the codomain is a sequence algebra, having nuclear dimension equal to zero is equivalent to factoring through a simple AF-algebra. As a byproduct of characterising nuclear dimension equal to zero for *-homomorphisms between commutative C*-algebras, we develop a notion of real rank zero for inclusions of C*-algebras. Among others, we provide interesting examples from dynamics that have this property and show that full $\mathcal{O}_\infty$-stable inclusions have real rank zero.

We further use classification results for automorphisms of A$\mathbb{T}$-algebras of real rank zero to build flows with specified KMS behaviour on all unital UCT Kirchberg algebras. In the tracial case, we obtain similar results for all finite classifiable C*-algebras with real rank zero.

In the last chapter, we use classification techniques involving the trace-kernel extension to show that the property that all amenable traces are quasidiagonal is invariant under homotopy for separable exact C*-algebras that have a faithful amenable trace.

Contents

Chapter 1

Introduction

Over the last decade, there has been enormous progress in the classification programme for simple C*-algebras, culminating with abstract classification results for large classes of C*-algebras and *-homomorphisms between them. In particular, classification facilitated deep structural results, such as the realisation of certain simple C*-algebras as groupoid C*-algebras ([91]). The goal of this thesis is to apply these classification results to characterise various regularity properties for *-homomorphisms between C*-algebras as well as to build certain continuous actions of the real numbers. Among others, we employ the full force of the classification of *-homomorphisms to study the topological dimension of morphisms, and construct interesting flows for C*-algebras from prescribed KMS data.

Since the foundational paper of Murray and von Neumann ([95]), classification is a theme that has received particular interest in the study of operator algebras. In the case of von Neumann algebras, this quest culminated with the seminal work of Connes and Haagerup on the classification of injective factors ([37, 71]). In the C*-algebraic setting, the first C*-algebras to be classified were built as inductive limits of finite dimensional C*-algebras ([65, 47]). The first abstract class of simple, separable, unital, nuclear C*-algebras to be classified were the purely infinite C*-

algebras. Archetypal examples of purely infinite C*-algebras are given by the Cuntz algebras $\mathcal{O}_n$ for $n \geq 2$ introduced by Cuntz in [38]. It is now common practice to refer to simple, separable, nuclear, purely infinite C*-algebras as Kirchberg algebras. Assuming a technical condition called the universal coefficient theorem (UCT) which will be discussed later in this thesis, Kirchberg algebras were famously classified by Kirchberg and Phillips ([82, 105]) by their K_0 and K_1-groups.[1]

Following this spectacular result, the quest for classifying simple, separable, unital, nuclear C*-algebras moved into the direction of stably finite C*-algebras. Combining classification results of purely infinite C*-algebras in [82, 105] with results in the stably finite case ([67, 68, 135, 30, 27]), the following theorem has been achieved, thus completing what has been commonly called the Elliott classification programme for simple nuclear C*-algebras. The C*-algebras in the theorem below will often be called *classifiable.*

Theorem (Unital Classification)**.** The class of simple, separable, unital, nuclear C*-algebras A such that $A \cong A \otimes \mathcal{Z}$ and which satisfy the universal coefficient theorem is classified up to isomorphism by K-theory and traces.

The Jiang-Su algebra $\mathcal{Z}$, appearing in the theorem above, is a simple C*-algebra introduced in [77] by Jiang and Su that turned out to play a special role in Elliott's classification programme. We will give a more detailed description of the Jiang-Su algebra in Section 2.4.

Moreover, the unital classification theorem above is complemented by a range-of-the-invariant result that will be discussed in Section 2.6. This was a key ingredient in many fruitful connections between classification techniques and dynamic constructions. In particular, Li used the invariant to build a twisted groupoid model for any classifiable C*-algebra ([91]), which in turn is equivalent to the existence of a Cartan subalgebra ([109]). We will further use the range of the invariant in Chapter 5 to

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[1] The UCT was introduced by Rosenberg and Schochet in [120].

build actions of $\mathbb{R}$ with prescribed KMS data on classifiable C*-algebras.

Together with the classification theorem stated above, the key technical tool used in this thesis is a classification of *-homomorphisms. For the purpose of this introduction, the morphisms in the theorem below will be called *classifiable* and the invariant will be called *the total invariant*. We will review this invariant in detail in Section 2.7.

Theorem (Classification of morphisms)**.** Let A and B be unital, separable, nuclear C*-algebras such that A satisfies the UCT and B is simple and $\mathcal{Z}$-stable. Then, full unital *-homomorphisms $\theta : A \to B$ are classified up to approximate unitary equivalence by an invariant consisting of maps at the level of K-theory, K-theory with coefficients, Hausdorffised algebraic K_1-groups, and traces.

Having the classification of *-homomorphisms stated above at our disposal, we will apply it in various situations both to construct interesting *-homomorphisms with prescribed K-theoretic and tracial data and to identify regularity properties of the morphisms at the level of the invariant. In this thesis, we will build *-homomorphisms at the level of the total invariant and use the uniqueness part of the classification theorem to obtain results regarding the structure of C*-algebras and their dynamics. The main achievements of this thesis are the following:

- characterising classifiable *-homomorphisms with nuclear dimension equal to zero;
- introducing and studying a new property of inclusions called real rank zero;
- building flows on classifiable C*-algebras with specified KMS behaviour.

Furthermore, the classification results in [27] provide a new abstract approach to classification, without the need of going through internal approximation properties. In Chapter 6, we will use the techniques that originated from [121] to study the problem of when all amenable traces are quasidiagonal. We will now present the main results of this thesis.

1.1 Nuclear dimension of *-homomorphisms

With the unital classification of morphisms at our disposal ([27]), it is natural to try and understand this class of morphisms by studying various regularity properties. One such property is having finite nuclear dimension. Nuclear dimension, introduced by Winter and Zacharias in [147], is a noncommutative generalisation of covering dimension for topological spaces to C*-algebras. On the class of simple, separable, nuclear C*-algebras, finite nuclear dimension is equivalent to tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$ ([145, 131, 30, 29]). In fact, the nuclear dimension of a simple C*-algebra can only be zero, one or infinite ([30, 29]).

Since nuclear dimension of a C*-algebra is defined by means of an approximation property for the identity map, it is immediate how to generalise this notion to *-homomorphisms. If $\theta : A \to B$ is a *-homomorphism, where the domain or codomain is a simple, nuclear, $\mathcal{Z}$-stable C*-algebra, then the nuclear dimension of θ is at most one. In particular, all unital classifiable *-homomorphisms (in the sense of [27]) have nuclear dimension at most one. Hence, we will investigate criteria that will allow us to determine when the nuclear dimension of classifiable *-homomorphisms is zero. In [13], the authors pose the following problem.

Problem 1 ([13, Problem 7.3])**.** Characterise *-homomorphisms with nuclear dimension equal to zero.

Early in the theory, Winter established that a C*-algebra has nuclear dimension equal to zero if and only if it is *approximately finite dimensional* (AF) ([144, Theorem 3.4]). For *-homomorphisms, it is readily seen that factoring through an approximately finite dimensional C*-algebra yields nuclear dimension equal to zero.

Therefore, one natural question which arises is if this condition characterises all zero-dimensional morphisms up to some notion of equivalence.

In Chapter 3, we will investigate Problem 1 for *-homomorphisms covered by the classification theorem in [27]. This chapter is based on joint work with Jorge Castillejos and its content is essentially contained in [31]. Moreover, part of this section is contained in the introduction of [31]. The key idea is to describe the total invariant of classifiable *-homomorphisms with nuclear dimension equal to zero. In fact, the property of having nuclear dimension equal to zero imposes several restrictions on the total invariant which resemble the property of factoring through a simple AF-algebra. In particular, a unital *-homomorphism with nuclear dimension equal to zero vanishes on K_1, K_1 with coefficients, and Hausdorffised algebraic K_1. These facts are instrumental for the slogan that *if a *-homomorphism looks like it is factoring through an AF-algebra, then it does.* We will flesh out this slogan by using the uniqueness part of the classification of *-homomorphisms in [27] to prove the following result.

Theorem 2 (Theorem 3.5.1)**.** *Let A and B be unital, classifiable, stably finite* C*-*algebras. If $\theta : A \to B$ is a unital *-homomorphism and $\iota_B : B \to B_\infty$ is the canonical diagonal inclusion, then the following are equivalent:*

(i) $\iota_B \circ \theta$ factors through a simple, unital AF-algebra;

(ii) θ has nuclear dimension equal to zero.

The strategy for proving this theorem goes as follows. Let $\theta : A \to B$ be a unital *-homomorphism with nuclear dimension equal to zero. One can build a suitable K-theoretic invariant and use Elliott's classification of AF-algebras in [47] to obtain a simple, unital, infinite dimensional AF-algebra C. Then one builds maps from the invariant of A to the constructed invariant of the AF-algebra C. The existence part of the classification of morphisms from [27] yields a unital *-homomorphism $A \to C$. A similar strategy allows us to obtain a unital *-homomorphism $C \to B_\infty$. To show that this composition coincides with $\iota_B \circ \theta$ up to unitary equivalence, we use the uniqueness part of the classification of morphisms from [27].

In the absence of torsion in K-theory and assuming *real rank zero* on the codomain of the morphisms, one can characterise *-homomorphisms with nuclear dimension equal to zero at the level of the invariant. Moreover, not only that the restrictions imposed on the total invariant resemble the presence of an AF-algebra, but they are actually sufficient to conclude that the given *-homomorphism factors through an AF-algebra up to approximate unitary equivalence. The assumption of real rank zero on the codomain ensures a rich supply of projections which facilitates the construction of an intermediate AF-algebra.

Theorem 3 (Theorem 3.5.3)**.** *Let A and B be unital, classifiable, stably finite* C*-*algebras such that B has real rank zero and $K_0(B)$ is torsion free. Let $\theta : A \to B$ be a unital *-homomorphism. Then the following are equivalent:*

(i) θ is approximately unitarily equivalent to a *-homomorphism which factors through a simple, unital AF-algebra;

(ii) θ has nuclear dimension equal to zero;

(iii) θ vanishes at the level of K_1 and K_1 with coefficients i.e. $K_1(\theta) = 0$ and $K_1(\theta; \mathbb{Z}/n) = 0$ for all $n \in \mathbb{N}$.

A similar result can be obtained if we instead assume that the domain has real rank zero.

Theorem 4 (Theorem 3.5.4)**.** *Let A and B be unital, classifiable, stably finite* C*-*algebras such that A has real rank zero and $K_*(A)$ is torsion free. Let $\theta : A \to B$ be a unital *-homomorphism. Then the following are equivalent:*

(i) θ is approximately unitarily equivalent to a *-homomorphism which factors through a simple, unital AF-algebra;

(ii) θ *has nuclear dimension equal to zero;*

(iii) θ *vanishes at the level of* K_1 *with coefficients and Hausdorffised algebraic* K_1 *i.e.* $K_1(\theta; \mathbb{Z}/n) = 0$ *for all* $n \in \mathbb{N}$ *and* $\overline{K_1}^{\mathrm{alg}}(\theta) = 0$.

Furthermore, we will completely characterise *-homomorphisms between commutative C*-algebras, as well as unital embeddings of $\mathcal{Z}$ that have nuclear dimension equal to zero.

Theorem 5 (Theorem 3.2.5)**.** *Let* X, Y *be compact Hausdorff spaces and* $\phi : Y \to X$ *be a continuous map which induces a* **-homomorphism* $\phi^* : C(X) \to C(Y)$. *Then* ϕ^* *has nuclear dimension equal to zero if and only if* ϕ^* *factors through a commutative* C**-algebra over a* 0*-dimensional compact Hausdorff space. In this case* ϕ *is constant on connected components.*

Whereas the proof of the commutative case amounts to a careful analysis at the level of the finite dimensional approximations to show that ϕ must be constant on connected components, to study unital embeddings of $\mathcal{Z}$, one heavily uses classification. Essentially, one can construct the required finite dimensional approximations for a unital *-homomorphism $\theta : \mathcal{Z} \to B$ by using quasidiagonality of $\mathcal{Z}$ and a certain divisibility property in $K_0(B)$. Then, employing classification results, one can argue that the constructed embedding is the unique one up to approximate unitary equivalence.

Theorem 6 (Theorem 3.6.2)**.** *Let* B *be a unital, classifiable* C**-algebra and* $\theta : \mathcal{Z} \to B$ *be a unital* **-homomorphism. Then* θ *has nuclear dimension equal to zero if and only if for any* $n \in \mathbb{N}$, *there exist* $g^{(n)}, h^{(n)}$ *positive in* $K_0(B)$ *such that*

$$[1_B]_0 = n g^{(n)} + (n+1) h^{(n)}.$$

The results in Theorem 3, Theorem 4, and Theorem 5 suggest the following question.

Question 7. *Let $\theta : A \to B$ be a unital *-homomorphism with nuclear dimension equal to zero. Is θ approximately unitarily equivalent to a *-homomorphism which factors through an AF-algebra?*

1.2 Real rank zero for inclusions

In Chapter 4 we introduce the notion of real rank zero for inclusions of C*-algebras. This chapter is based on joint work with my co-supervisor James Gabe and its content is essentially contained in [62]. Moreover, this section is based on the introduction of [62].

The real rank of C*-algebras was introduced by Brown and Pedersen in [21] as a noncommutative analogue of topological dimension. Particularly, the notion of real rank zero distinguished itself due to the fact that many interesting classes of C*-algebras were shown to have this property. Unlike in the case of nuclear dimension, where only AF-algebras have nuclear dimension zero, simple purely infinite C*-algebras ([148]), irrational rotation algebras ([33, 10, 51]), AF-algebras ([21]), and von Neumann algebras ([21]) all have real rank zero. Subsequently, real rank zero started to appear in the context of the classification programme of simple nuclear C*-algebras. Most notably in [49], real rank zero facilitated access to classification by K-theory for large classes of C*-algebras.

Recall that a C*-algebra has real rank zero precisely when all of its hereditary C*-subalgebras have an approximate unit consisting of projections. Due to the abundance of projections, the ordered K_0-group of a classifiable C*-algebra with real rank zero determines both the trace space of the algebra and the pairing between K-theory and traces. Therefore, the classification invariant is highly simplified in this case.

On the other hand, having such a powerful classification of morphisms at our disposal ([27]), a natural goal is to obtain interesting examples of classifiable morphisms between potentially non-classifiable C*-algebras. Having the behaviour of classifiable C*-algebras of real rank zero in hindsight, it is natural to impose certain conditions on the *-homomorphisms classified in [27], with the aim of simplifying the total invariant. Motivated by the results in Theorem 3 and Theorem 4, where real rank zero was an instrumental tool in characterising *-homomorphisms with nuclear dimension equal to zero at the level of the total invariant, we introduce the notion of real rank zero for inclusions of C*-algebras.

Definition 8 (Definition 4.1.3)**.** We say that an inclusion of C*-algebras $A \subseteq B$ has *real rank zero* if for any nonzero positive element $a \in A$, the hereditary C*-subalgebra $\overline{aBa}$ of B has an approximate unit of projections. Moreover, we say that a *-homomorphism $\theta : A \to B$ has real rank zero if the inclusion $\theta(A) \subseteq B$ has real rank zero.

In fact, as in the case of real rank zero for C*-algebras, the counterpart notion for inclusions enjoys a series of equivalent characterisations. We will state the result in the unital case, but similar characterisations hold in the nonunital setting with appropriate modifications.

Theorem 9 (Theorem 4.1.5)**.** *Let $A \subseteq B$ be an inclusion of unital* C**-algebras such that $1_B \in A$. Then the following are equivalent:*

(i) the inclusion $A \subseteq B$ has real rank zero;

(ii) any self-adjoint in A can be approximated arbitrarily well by self-adjoints in B with finite spectrum;

(iii) any self-adjoint in A can be approximated arbitrarily well by invertible self-adjoints in B.

There are natural inclusions arising from dynamics which have real rank zero. In the following two theorems below, the strategy is to find a space X and an action α of G on X such that the crossed product $C(X) \rtimes_\alpha G$ has real rank zero. Then, the inclusions in the statements of the theorems will factor through the obtained crossed product of real rank zero. The first such result considers the canonical inclusion of the reduced group C*-algebra into its uniform Roe algebra.

Theorem 10 (Theorem 4.5.5)**.** *Let G be a countable, discrete, exact group. If we consider the action of G on $\ell^\infty(G)$ by left translation, then the canonical inclusion $\mathrm{C}^*_r(G) \subseteq \ell^\infty(G) \rtimes_r G$ has real rank zero. In fact, it factors through a* C**-algebra of real rank zero.*

Another natural inclusion is realised by embedding the reduced group C*-algebra into the reduced crossed product of $C(\partial_F G)$ by G. Here $\partial_F G$ denotes the Furstenberg boundary of G and we refer the reader to [79] for a detailed description.

Theorem 11 (Theorem 4.5.7)**.** *Let G be a countable, discrete, non-elementary hyperbolic group with no nontrivial finite normal subgroups. Then the canonical inclusion $\mathrm{C}^*_r(G) \subseteq C(\partial_F G) \rtimes_r G$ has real rank zero. In fact, it factors through a* C**-algebra of real rank zero.*

We then consider the class of inclusions classified by their total invariant in [27]. In the tracial setting, real rank zero of an inclusion between stabilised unital C*-algebras imposes restrictions on the total invariant and it can be characterised by the canonical pairing between K-theory and traces. For a unital C*-algebra C, let $T(C)$ denote the space of tracial states on C and $\mathrm{Aff}(T(C))$ the space of continuous affine functions $T(C) \to \mathbb{R}$. The natural pairing between K-theory and traces is a positive group homomorphism $\rho_C : K_0(C) \to \mathrm{Aff}(T(C))$ given by trace evaluations. Suppose that $A \subseteq B$ is a unital inclusion of C*-algebras. Then there is a canonical map $\gamma : T(B) \to T(A)$ induced by restriction and a canonical affine map $\gamma^* : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$, which is dual to γ (see [78]). Then, having real rank zero can be characterised by considering the image of the map γ^*.

Theorem 12 (see Theorems 4.4.1 and 4.4.2)**.** *Let $A \subseteq B$ be an inclusion of unital and separable* C^**-algebras such that $1_B \in A$. Suppose that A and B are simple, exact, stably finite, and $\mathcal{Z}$-stable. Then the inclusion $A \otimes \mathcal{K} \subseteq B \otimes \mathcal{K}$ has real rank zero if and only if the image of the induced map $\gamma^* : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ is contained in $\overline{\rho_B(K_0(B))}$.*

Note that the results obtained in Theorem 4.4.1 and Theorem 4.4.2 hold in greater generality, but require the introduction of various regularity properties for C*-algebras that will be defined later in this thesis.

In the absence of traces, we consider inclusions which are simple and purely infinite in the sense that all nonzero positive elements of the subalgebra are full and properly infinite in the larger algebra. Hence, we obtain a relative version of a classical result of Zhang ([148]) which says that simple purely infinite C*-algebras have real rank zero. In particular, this shows that the full inclusions classified in [61] have real rank zero.

Theorem 13 (Theorem 4.5.1)**.** *Let $A \subseteq B$ be an inclusion of* C^**-algebras such that every nonzero positive element in A is full and properly infinite in B. Then $A \subseteq B$ has real rank zero.*

One of the fundamental questions around this notion is to understand whether any inclusion with real rank zero is equivalent to an inclusion which factors through a C*-algebra of real rank zero. Since any unital inclusion with nuclear dimension equal to zero has real rank zero (Lemma 3.2.2), the question below is related to Question 7.

Question 14. *Let $\iota : A \subseteq B$ be a unital inclusion of* C^**-algebras such that $1_B \in A$ and which has real rank zero. Is ι approximately unitarily equivalent to a *-homomorphism* *which factors through a* C**-algebra of real rank zero?*

We give a positive answer for the full inclusions classified in [61]. The key ingredient in the proof of the theorem below is a classification of full, nuclear, $\mathcal{O}_\infty$-stable maps from [61] by the induced class in KK-theory.

Theorem 15 (see Theorem 4.5.4)**.** *Let $A \subseteq B$ be an inclusion of separable, exact* C**-algebras such that $1_B \in A$ and suppose that the inclusion $\iota\colon A \hookrightarrow B$ is full, nuclear, and $\mathcal{O}_\infty$-stable. Then ι is approximately unitarily equivalent to a *-homomorphism which factors through a* C**-algebra with real rank zero.*

1.3 On the bundle of KMS states

In Chapter 5, we will use the existence part of the classification of *-homomorphisms in [27] to construct continuous actions of $\mathbb{R}$ with prescribed KMS behaviour. In particular, these constructions will facilitate the study of possible bundles of KMS states that can be realised on a unital classifiable C*-algebra. This chapter is essentially contained in [97] of which I am the sole author. Moreover, the content of this section is part of the introduction of [97].

Named after mathematical physicists Kubo, Martin, and Schwinger, KMS states are a special class of states on any C*-algebra admitting a continuous action of $\mathbb{R}$. The collection of KMS states for a given flow on a C*-algebra can be quite intricate. In fact, given a flow on a unital separable C*-algebra, the collection of KMS states corresponding to that flow, also called the KMS bundle of the flow, has the structure of a *proper simplex bundle* (see for example [56, Section 2.1]).

The quest of constructing flows inducing a given KMS simplex has its roots in work of Bratteli, Elliott, Herman, and Kishimoto in [15, 16, 17], where flows were constructed on certain classes of simple C*-algebras. More recently, extensive work has been done to examine possible KMS bundles on various special classes of C*-

algebras (see for example [3, 130, 1, 36, 35, 34, 14]). Moreover, in [132, 56] Elliott and Thomsen built flows on simple AF-algebras with prescribed KMS behaviour. These results were then extended by Elliott, Sato, and Thomsen in [55, 54] to any unital classifiable C*-algebra with a unique trace, as well as any unital UCT Kirchberg algebra with no torsion in the K_1-group.

Using the new abstract classification of morphisms from [27], we will prove a similar result for any unital tracial classifiable C*-algebra with real rank zero. However, as in [55, Theorem 3.14], the assumption that the KMS bundle is compact is still needed in the case of tracial C*-algebras. Furthermore, without any restrictions on the given KMS bundle, we will use Elliott's classification of stable (nonsimple) A$\mathbb{T}$-algebras of real rank zero and of their automorphisms ([49]) to remove the assumption of no torsion in the K_1-group from [55, Theorem 5.1] and obtain the definitive result for unital UCT Kirchberg algebras. The main results we will prove in Chapter 5 are the following.

Theorem 16 (Theorem 5.4.10)**.** *Let A be a unital* UCT *Kirchberg algebra and let (S, π) be a proper simplex bundle such that $\pi^{-1}(0) = \emptyset$. Then there exists a 2π-periodic flow θ on A such that its induced* KMS *bundle (S^θ, π^θ) is isomorphic to (S, π).*

Theorem 17 (Theorem 5.5.8)**.** *Let A be a unital, stably finite, classifiable* C**-algebra of real rank zero and let (S, π) be a compact simplex bundle such that $\pi^{-1}(0)$ is affinely homeomorphic to $T(A)$. Then there exists a 2π-periodic flow θ on A such that its induced* KMS *bundle (S^θ, π^θ) is isomorphic to (S, π).*

The proof of Theorem 16 follows the strategy in [55, Theorem 5.1]. Essentially, given a unital UCT Kirchberg algebra A and a suitable simplex bundle (S, π), one can build a pair of abelian groups containing both information from the K-theory of A as well as the bundle (S, π). Then, using the classification results in [49], we can obtain a stable $A\mathbb{T}$-algebra B of real rank zero realising the constructed pair of abelian groups and such that A can be identified with a full corner of a crossed product of B and $\mathbb{Z}$. It is precisely the fact that B is an $A\mathbb{T}$-algebra rather than an AF-algebra (see [55, Theorem 5.1]) that allows us to remove the assumption that A has no torsion in K_1.

The proof of Theorem 17 uses a similar strategy. Given a unital, classifiable, stably finite C*-algebra A of real rank zero, we realise it as a full corner of a crossed product of a stabilised unital classifiable C*-algebra $B \otimes \mathcal{K}$ by $\mathbb{Z}$. A key ingredient in obtaining an appropriate automorphism of $B \otimes \mathcal{K}$ is the classification of unital *-homomorphisms from [27]. Furthermore, compared to [55, Theorem 3.14], where the corresponding result was proved for the Jiang-Su algebra $\mathcal{Z}$, extra care has to be taken in checking that A and the corner of the crossed product built above have the same pairing between K-theory and traces. The assumption that A has real rank zero is instrumental in constructing an Elliott invariant of a classifiable C*-algebra B. Precisely, one can build an ordered abelian group G and then realise the trace space of B as the space of states on G. However, to make sure that the state space on G is a metrisable Choquet simplex, the fact that A has real rank zero is heavily used. The assumption that the space S is compact in Theorem 17 seems to be more difficult to remove. In particular, it ensures that the constructed ordered abelian group is simple, and hence the C*-algebra B is simple. Then, one can use a classification of automorphisms of $B \otimes \mathcal{K}$ to obtain a suitable automorphism γ. Note that a definitive classification theorem for automorphisms of general nonsimple $\mathcal{Z}$-stable C*-algebras is currently out of reach, so the compactness assumption appears necessary at this point.

1.4 When are amenable traces quasidiagonal

In Chapter 6, we will show that not only the results obtained in [27] are important, but also the techniques used in the proofs. In particular, [27] provides a new abstract approach to classification, without the need of going through internal approximations of the C*-algebra. This abstract approach is inspired by Schafhauser's proof in [121] that any faithful amenable trace on a separable, exact C*-algebra satisfying the UCT is quasidiagonal. Essentially, using the theory of extensions, he reformulated this problem concerning approximation properties for traces into a lifting problem. We will use these techniques to study when all amenable traces on a separable, exact C*-algebra are quasidiagonal. The content of this chapter is essentially contained in [96] of which I am the sole author. Moreover, part of this section is contained in the introduction of [96].

Developing approximation properties to traces proved to be a powerful tool in the classification of operator algebras, often revealing deep structural properties. Amenability of traces dates back to the work of Connes ([37]), where he essentially showed that a II$_1$ factor is injective if and only if its unique normalised trace is amenable. Later, quasidiagonal traces were introduced by Brown in [22] and they proved to be a fundamental tool in obtaining the classification theorem of simple nuclear C*-algebras, via the quasidiagonality theorem of Tikuisis, White, and Winter ([135]). Both amenability and quasidiagonality of a trace can be witnessed by approximations through matrix algebras, the only difference being in the norm with respect to which these approximations are performed. Let us briefly recall the two notions below.

Definition (Definition 6.2.1)**.** Let A be a separable C*-algebra.

(i) A trace $\tau : A \to \mathbb{C}$ is called *amenable* if for all $n \in \mathbb{N}$, there is an integer $k(n) \geq 1$ and a completely positive and contractive (cpc) map $\phi_n : A \to M_{k(n)}(\mathbb{C})$ such that

$$\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_2 \to 0$$

and $\mathrm{tr}_{k(n)}(\phi_n(a)) \to \tau(a)$ for all $a, b \in A$, where $\|x\|_2 = \mathrm{tr}_{k(n)}(x^*x)^{1/2}$ and $\mathrm{tr}_{k(n)}$ is the unique normalised trace on $M_{k(n)}(\mathbb{C})$.

(ii) A trace $\tau : A \to \mathbb{C}$ is *quasidiagonal* if for all $n \in \mathbb{N}$ there is an integer $k(n) \geq 1$ and a completely positive and contractive (cpc) map $\phi_n : A \to M_{k(n)}(\mathbb{C})$ such that

$$\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$$

and $\mathrm{tr}_{k(n)}(\phi_n(a)) \to \tau(a)$ for all $a, b \in A$.

Since convergence in norm implies convergence in the $\|.\|_2$-norm defined above, any quasidiagonal trace is amenable. In [22], Brown asked if the converse also holds. Until the quasidiagonality theorem ([135]), not so many results were known in this direction. One such result shows that if A is any C*-algebra, then any amenable trace on its cone $C_0((0,1], A)$ is, in fact, quasidiagonal ([23, Proposition 3.2]).

Question 18. *Are all amenable traces quasidiagonal?*

Motivated by the result in [23, Proposition 3.2], we will provide an affirmative answer to Question 18 in the case of contractible C*-algebras.

Proposition 19 (Proposition 6.3.1)**.** *Let A be a separable contractible* C**-algebra. Then all amenable traces on A are quasidiagonal.*

Proposition 6.3.1 suggests that the property that all amenable traces on a separable C*-algebra are quasidiagonal has a rather topological nature. Indeed, having Voiculescu's result ([141]) saying that quasidiagonality of a separable C*-algebra is invariant under homotopy serving as conceptual evidence, we will prove that, under mild conditions, the property that all amenable traces are quasidiagonal is invariant under homotopy.

Theorem 20 (Theorem 6.4.6)**.** *Let A be a separable exact* C**-algebra with a faithful amenable trace and suppose A is homotopy dominated by some separable* C**-algebra B. If all amenable traces on B are quasidiagonal, then all amenable traces on A are quasidiagonal.*

The key technical step in proving Theorem 20 is the following result.

Theorem 21 (Theorem 6.4.1)**.** *Let A be a separable exact* C**-algebra with a faithful amenable trace τ. If $\sigma : A \to A$ is a *-homomorphism which is homotopic to the identity map on A and $\tau \circ \sigma$ is a quasidiagonal trace on A, then τ is quasidiagonal on A.*

Thesis structure

This thesis is structured as follows. In Chapter 2 we will collect preliminary facts about the the Elliott invariant (Section 2.4) and the total invariant for classifying morphisms (Section 2.7), together with regularity properties such as stability with respect to tensorial absorption of the Jiang-Su algebra and finite nuclear dimension (see Section 2.5). The main results of this thesis are in Chapters 3, 4, 5, and 6. In Chapter 3, we will use classification methods to prove the results stated in Section 1.1. The content of this chapter is essentially contained in [31], which is joint work with Jorge Castillejos. In Chapter 4, we will study inclusions of real rank zero between C*-algebras and prove the results in Section 1.2. The content of this chapter is essentially contained in [62], which is joint work with my co-supervisor James Gabe. Chapter 5 contains the results in Section 1.3. The content of this chapter is essentially contained in [97]. The last chapter proves the results stated in Section 1.4 and its content is essentially contained in [96].

Chapter 2

Preliminaries

2.1 K-theory

The K-theory of a C*-algebra A consists of a pair of abelian groups $K_0(A)$ and $K_1(A)$. In this section, we record the construction of the K_0 and K_1-groups for a C*-algebra. We refer the reader to [118] or [142] for a thorough description.

We first define the K_0-group of a unital C*-algebra. Let A be a unital C*-algebra and denote by $\mathcal{P}(M_n(A))$ the set of projections in $M_n(A)$ for any $n \in \mathbb{N}$. Then consider

$$\mathcal{P}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathcal{P}(M_n(A))$$

and equip it with a binary operation $\oplus$ given by $p \oplus q = \operatorname{diag}(p, q)$ for any $p, q \in \mathcal{P}_\infty(A)$. If $m > n$, we view $\mathcal{P}(M_n(A))$ inside $\mathcal{P}(M_m(A))$ by the diagonal embedding $p \mapsto \operatorname{diag}(p, 0)$. Moreover, we say that $p, q \in \mathcal{P}(M_n(A))$ are *Murray-von Neumann equivalent* if there exists a partial isometry $v \in M_n(A)$ such that $p = vv^*$ and $q = v^*v$. We write $p \sim q$ to mean that p and q are Murray-von Neumann equivalent. Then, one can define the *Murray-von Neumann semigroup* of A by $V(A) = \mathcal{P}_\infty(A)/\sim$. Denote the class of $p \in \mathcal{P}_\infty(A)$ in $V(A)$ by $[p]_0$.

The abelian group $K_0(A)$ is the Grothendieck completion of $V(A)$. This comes equipped with a natural order. Precisely,

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\}$$

with the positive cone given by $K_0(A)_+ = \{[p]_0 : p \in \mathcal{P}_\infty(A)\}$. This construction gives an abelian group with respect to the operation $\oplus$ and it is functorial. If A, B are unital C*-algebras and $\theta : A \to B$ is a *-homomorphism, there exists an associated group homomorphism $K_0(\theta) : K_0(A) \to K_0(B)$ given by $K_0(\theta)([p]_0 - [q]_0) = [\theta^{(n)}(p)]_0 - [\theta^{(n)}(q)]_0$ for any $p, q \in \mathcal{P}(M_n(A))$, where $\theta^{(n)}$ is the map induced by θ on $M_n(A)$.

If A is a nonunital C*-algebra, then there exists an associated split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A^\dagger \arrow[r, "\pi"] & \mathbb{C} \arrow[r] & 0, \end{tikzcd}$$

where $A^\dagger$ is the minimal unitisation of A. Then, define $K_0(A)$ to be the kernel of the homomorphism $K_0(\pi) : K_0(A^\dagger) \to K_0(\mathbb{C})$ and note that K_0 is a functor in the nonunital case too. Furthermore, it generalises the topological K^0-group, in the sense that if X is a compact Hausdorff space, then $K_0(C(X)) = K^0(X)$. A C*-algebra A is said to satisfy *cancellation*, if for any $p, q \in \mathcal{P}_\infty(A)$ such that $[p]_0 = [q]_0$, one has that $p \sim q$.

Let us introduce some further notions. An element $x \in K_0(A)_+$ is called an *order unit* in $(K_0(A), K_0(A)_+)$ if for every $y \in K_0(A)$, there exists $k \in \mathbb{N}$ such that $-kx \leq y \leq kx$.[2]If every nonzero element in $K_0(A)_+$ is an order unit, then $(K_0(A), K_0(A)_+)$ is said to be simple.

The *state space* of $K_0(A)$, written $S((K_0(A), K_0(A)_+, [1_A]_0))$ or just $S(K_0(A))$ is the set of homomorphisms $\varphi : K_0(A) \to \mathbb{R}$ such that $\varphi(K_0(A)_+) \subseteq \mathbb{R}_+$ and $\varphi([1_A]_0) = 1$.

We will now define the K_1-group of a unital C*-algebra. This is built from uni-

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[2] If A is unital and $K_0(A) \neq 0$, then $[1_A]_0$ is an order unit.

taries. Let A be a unital C*-algebra and denote by $\mathcal{U}(M_n(A))$ the set of unitaries in $M_n(A)$ for any $n \in \mathbb{N}$. Then consider

$$\mathcal{U}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathcal{U}(M_n(A))$$

and equip it with a binary operation $\oplus$ given by $u \oplus v = \operatorname{diag}(u, v)$ for any $u, v \in \mathcal{U}_\infty(A)$. If $m > n$, we view $\mathcal{U}(M_n(A))$ inside $\mathcal{U}(M_m(A))$ by the diagonal embedding $u \mapsto \operatorname{diag}(u, 1)$. Moreover, we say that $u \in \mathcal{U}(M_n(A))$ and $v \in \mathcal{U}(M_m(A))$ are *homotopic* if there exist $k \geq \max\{m, n\}$ and a continuous function $f : [0, 1] \to \mathcal{U}(M_k(A))$ such that $f(0) = u \oplus 1_{k-n}$ and $f(1) = v \oplus 1_{k-m}$. We write $u \sim_h v$ to mean that u and v are homotopic in $\mathcal{U}_\infty(A)$. We then define

$$K_1(A) := \mathcal{U}_\infty(A)/\sim_h .$$

For a nonunital C*-algebra A, one says that $K_1(A) := \mathcal{U}_\infty(A^\dagger)/\sim_h$, where $A^\dagger$ denotes the minimal unitisation of A. This defines an abelian group with respect to the operation $\oplus$. We will denote the pair $(K_0(A), K_1(A))$ by $K_*(A)$.

As in the case of K_0, K_1 is also a functor from the category of C*-algebras to the category of abelian groups. When A, B are unital C*-algebras and $\theta : A \to B$ is a *-homomorphism, $K_1(\theta) : K_1(A) \to K_1(B)$ is given by $K_1(\theta)([u]_1) = [\theta^{(n)}(u)]_1$ for any $u \in \mathcal{U}(M_n(A))$. Moreover, the functors K_0 and K_1 are continuous. If A is the inductive limit of a C*-inductive system $\{A_n, \phi_n\}_{n\in\mathbb{N}}$, then $K_i(A) = \varinjlim K_i(A_n)$ for any $i = 0, 1$ (see [118, Theorem 6.3.2] and [118, Proposition 8.2.7]).

2.2 Ordered abelian groups

This section is based on [97, Section 1.3]. We will record the notion of an ordered abelian group, together with some properties that will appear in the later chapters of this thesis. We refer the reader to [70] for a detailed account on the content of this section.

Definition 2.2.1 (see for example [115, Definition 1.1.8])**.** An ordered abelian group is a pair (G, G_+) such that G is an abelian group and G_+ is a subset of G containing 0 such that

- $G_+ + G_+ \subseteq G_+$;
- $G_+ \cap (-G_+) = \{0\}$;
- $G_+ - G_+ = G$.

We then write $x \leq y$ in G if $y - x \in G_+$. Two ordered abelian groups (G, G_+) and (H, H_+) are isomorphic if there exists a group isomorphism $\phi \colon G \to H$ such that $\phi(G_+) = H_+$.

Remark 2.2.2. If A is a stably finite C*-algebra (see Section 2.6), then the pair $(K_0(A), K_0(A))_+$ is an ordered abelian group ([6, Proposition 6.3.3]). For the purpose of this thesis, this constitutes the prototypical example of an ordered abelian group.

An ordered abelian group (G, G_+) is called *unperforated* if for any $n \in \mathbb{N}$ and any $g \in G$ such that $ng \in G_+$, then $g \in G_+$. Moreover, an ordered abelian group (G, G_+) is called *weakly unperforated* if for any $n \in \mathbb{N}$ and any $g \in G$ such that $ng \in G_+ \setminus \{0\}$, then $g \in G_+ \setminus \{0\}$. Also, (G, G_+) is said to have the *Riesz interpolation property* if for any $g_1, g_2, h_1, h_2 \in G$ such that $g_i \leq h_j$ for any $i, j = 1, 2$ there exists $z \in G$ such that $g_i \leq z \leq h_j$ for any $i, j = 1, 2$. If all inequalities are replaced by strict inequalities, then (G, G_+) is said to satisfy the *strong Riesz interpolation property.* An ordered abelian group (G, G_+) is said to have the *Riesz decomposition property* if for any $g, h_1, h_2 \in G_+$ such that $g \leq h_1 + h_2$, there exist $g_1, g_2 \in G_+$ such that $g = g_1 + g_2$ and $g_j \leq h_j$ for each $j = 1, 2$.

Recall also that a countable ordered abelian group (G, G_+) is called a *dimension group* if it is isomorphic to the inductive limit of a sequence

$$\mathbb{Z}^{r_1} \xrightarrow{\alpha_1} \mathbb{Z}^{r_2} \xrightarrow{\alpha_2} \mathbb{Z}^{r_3} \xrightarrow{\alpha_3} \dots,$$

for some natural numbers r_n, some positive group homomorphisms α_n, where $\mathbb{Z}^r$ is equipped with its standard ordering given by

$$(\mathbb{Z}^r)_+ = \{(x_1, x_2, \dots, x_r) : x_j \geq 0\}.$$

Using a classical theorem of Effros, Handelman, and Shen ([45, Theorem 2.2]), a countable ordered abelian group is a dimension group if and only if it is *unperforated* and has the *Riesz interpolation property.*

Recall that I is an *order ideal* of (G, G_+) if I is a subgroup of G such that

$$I = (I \cap G_+) - (I \cap G_+)$$

and

$$\text{if } 0 \leq y \leq x \text{ in } G_+ \text{ and } x \in I, \text{ then } y \in I.$$

An ordered abelian group (G, G_+) is called *simple* if there are no nontrivial order ideals (or any nonzero positive element is an order unit).

In fact, by Elliott's seminal classification of AF-algebras ([47]), all dimension groups are realised as the K_0-group of some AF-algebra. Furthermore, simple dimension groups correspond to simple AF-algebras.

Proposition 2.2.3 ([115, Proposition 1.4.2, Corollary 1.5.4])**.** *The ordered K_0-group of an AF-algebra is a dimension group and for every dimension group G, there is an AF-algebra A such that $K_0(A)$ and G are isomorphic as ordered abelian groups. Moreover, the dimension group G is simple if and only if the corresponding AF-algebra* *is simple.*

In Chapter 5, we will often use some of the properties of ordered groups defined above in the case when the group G is a priori graded. To avoid any confusion, we remark that the meaning of these properties is unchanged. In particular, $G = G_0 \oplus G_1$ is a *graded ordered group* if it is graded and there exists a distinguished subset G_+ such that (G, G_+) is an ordered group in the usual sense.

2.3 KK-theory and the UCT

We will make use of Kasparov's bivariant KK-theory groups in Section 4.5. We will offer a brief description, but refer the reader to [6] for a more detailed exposition.

Kasparov's KK-theory is a bifunctor from the category of separable C*-algebras to the category of abelian groups. More generally, for any separable C*-algebra A and any σ-unital C*-algebra B, there is an abelian group $\mathrm{KK}(A, B)$ which is often viewed as a group of generalised morphisms between A and B. Consider the set of pairs of *-homomorphisms (ϕ, ψ), where $\phi, \psi : A \to \mathcal{M}(B \otimes \mathcal{K})$ such that $\phi(a) - \psi(a) \in B \otimes \mathcal{K}$ for any $a \in A$.[3] Then, $\mathrm{KK}(A, B)$ is the set of such pairs of *-homomorphisms quotiented by the equivalence relation induced by homotopy. In particular, any *-homomorphism $\phi : A \to B$ induces an element in $\mathrm{KK}(A, B)$ via the pair $(\phi, 0)$. A variation of this construction which will be briefly used in Chapter 4 is given by the $\mathrm{KK}_{\mathrm{nuc}}$-groups introduced by Skandalis in [124]. A *-homomorphism $\phi : A \to \mathcal{M}(B \otimes \mathcal{K})$ is called *weakly nuclear* if the map $A \to B \otimes \mathcal{K}$ given by $a \mapsto b^*\phi(a)b$ is nuclear for all $b \in B \otimes \mathcal{K}$. Then, $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ can be defined by considering pairs of weakly nuclear *-homomorphisms as in the construction of $\mathrm{KK}(A, B)$.

The *universal coefficient theorem* ([120, Theorem 1.17]) states that for any C*-algebra that is isomorphic in the KK-category to an abelian C*-algebra, there is a

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[3] Here $\mathcal{M}(B \otimes \mathcal{K})$ stands for the multiplier algebra of the stabilisation of B by the compact operators.

natural short exact sequence that relates the KK-bifunctor, to the K_0 and K_1 functors (see for example [6, Theorem 23.1.1]). The class of C*-algebras that are isomorphic in the KK-category to an abelian C*-algebra is called the UCT class. It is a major open problem whether every nuclear C*-algebra satisfies the UCT.

We will finish this section by defining a quotient of the KK-group which will be used later in this thesis. Let A and B be separable C*-algebras. Then, one can equip $\mathrm{KK}(A,B)$ with a potentially non-Hausdorff topology ([42]). One then defines

$$\mathrm{KL}(A,B) = \mathrm{KK}(A,B)/\overline{\{0\}}.$$

The group $\mathrm{KL}(A,B)$ was first defined by Rørdam in [114] whenever A satisfies the UCT. Dadarlat's definition above generalises this group to the case when A and B do not necessarily satisfy the UCT. As in the case of the KK-group, where one can define a variant by considering only weakly nuclear representations, we can do the same for the KL-group. Let A be a separable C*-algebra, B be a σ-unital C*-algebra and define

$$\mathrm{KL}_{\mathrm{nuc}}(A,B) = \mathrm{KK}_{\mathrm{nuc}}(A,B)/\overline{\{0\}}. \tag{2.3.1}$$

2.4 Classification invariants

The invariant that was used to classify large classes of simple C*-algebras is commonly called the *Elliott invariant*. We will start with a description of the Elliott invariant. Until reaching its modern form, the Elliott invariant went through a series of iterations. If Glimm's classification of UHF-algebras in [65] only needs the K_0-group, together with the class of the unit, the order structure on K_0 made its first appearence in Elliott's classification of AF-algebras ([47]).

Any unitary in a finite dimensional C*-algebra is connected to the identity in some matrix amplification. Therefore, by continuity of the K_1-functor, any inductive limit of finite dimensional C*-algebras (or any AF-algebra) has a trivial K_1-group. However, to classify more complex inductive limits, the need for introducing the K_1-group arose. In particular, further classification results of various classes of AI and A$\mathbb{T}$-algebras ([49]) required the K_1-group as part of the invariant. In fact, the invariant consisting of the K_0-group, together with the class of the unit, and the K_1-group was used by Kirchberg and Phillips to classify simple, separable, unital, nuclear, purely infinite C*-algebras satisfying the UCT ([105, 82]). These algebras are commonly called unital UCT Kirchberg algebras.

However, all of the classes mentioned above which were successfully classified by K-theory have an abundance of projections. Precisely, both AF-algebras and Kirchberg algebras have *real rank zero.* In these cases, knowing the K_0-group recovers the space of traces of A.

We will write $T(A)$ for the space of tracial states of A and $QT(A)$ for the space of lower semicontinuous 2-quasitraces. The space of all affine functions on $T(A)$ will be denoted by $\mathrm{Aff}(T(A))$. A key observation is that, if A is a unital C*-algebra, there exists a canonical pairing map between the K_0-group and traces denoted

$$\rho_A : K_0(A) \to \mathrm{Aff}(T(A)).$$

For $n \in \mathbb{N}$ and $\tau \in T(A)$, write τ_n for the canonical non-normalised extension of τ to a tracial functional on $M_n(A)$ and define

$$\rho_A([p]_0 - [q]_0)(\tau) := \tau_n(p - q)$$

for all $\tau \in T(A)$ and all projections $p, q \in M_n(A)$.

While Kirchberg algebras have no tracial states, AF-algebras sit on the other side of the spectrum. If A is a simple, separable, unital, non-elementary AF-algebra, then $\rho_A(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$ (see for example [117, Theorem 7.2]).

Thus, in both cases, the space of traces is recovered from the K-theoretic information. The reason that underpins this phenomenon is the presence of real rank zero. Hence, as demonstrated by Thomsen in [128], to aim for classification results outside the real rank zero setting, one needs to add the space of traces and its natural pairing with the K_0-group to the invariant. In fact, this is the modern formulation of the Elliott invariant.

Definition 2.4.1. The *Elliott invariant* of a unital separable C*-algebra A, denoted $\mathrm{Ell}(A)$, is given by

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), \rho_A)\,. \tag{2.4.1}$$

A further simplification of this invariant can be obtained by considering a special class of C*-algebras which tensorially absorbs a C*-algebra called the Jiang-Su algebra. This C*-algebra proved to be quintessential for the classification programme of simple nuclear C*-algebras. The Jiang-Su algebra $\mathcal{Z}$ was introduced in [77] by Jiang and Su as an infinite dimensional, simple, separable, unital, nuclear C*-algebra that has the same K-theory and traces as the complex numbers. The Jiang-Su algebra is built in [77] as an inductive limit with building blocks of the form

$$\mathcal{Z}_{p,q} = \{f \in C\left([0,1], M_p \otimes M_q\right) : f(0) \in M_p \otimes 1_{M_q}, f(1) \in 1_{M_p} \otimes M_q\}, \tag{2.4.2}$$

where p and q are coprime. The maps of this intricate construction, as well as the building blocks of this inductive limit were chosen to ensure that $\mathcal{Z}$ is simple, nuclear, has the same K-theory as the complex numbers, and it has a unique trace. A more efficient construction of $\mathcal{Z}$, where p and q in (2.4.2) are assumed to be supernatural numbers, can be found in [123].

An important observation is that in the case of simple, separable, unital, nuclear C*-algebras which absorb the Jiang-Su algebra tensorially, the order on the K_0-group can be recovered from the traces.[4] Precisely, nonzero positive elements in K_0 are those elements that are strictly positive when evaluated on traces. Therefore, if A is a simple, separable, unital, nuclear C*-algebra which tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$, the invariant $(K_0(A),[1_A]_0,K_1(A),\mathrm{Aff}(T(A)),\rho_A)$ carries the same information as $\mathrm{Ell}(A)$. In [27], this invariant was used to successfully classify a large class of simple, separable, unital, nuclear C*-algebras.

Definition 2.4.2. The KT_u-invariant of a unital C*-algebra A is given by

$$\mathrm{KT}_u(A) = (K_0(A),[1_A]_0,K_1(A),\mathrm{Aff}(T(A)),\rho_A)\,. \tag{2.4.3}$$

Let A and B be unital C*-algebras. A morphism from $\mathrm{KT}_u(A)$ to $\mathrm{KT}_u(B)$ is defined to be a triple $(\alpha_0,\alpha_1,\gamma)$ with $\alpha_0 : K_0(A)\to K_0(B)$ and $\alpha_1 : K_1(A)\to K_1(B)$ be homomorphisms of abelian groups such that α_0 preserves the order and $\alpha_0([1_A]_0) = [1_B]_0$. Furthermore, $\gamma : \mathrm{Aff}(T(A))\to\mathrm{Aff}(T(B))$ is a continuous and positive linear map such that the diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, "\rho_A"] \arrow[d, "\alpha_0"] & \mathrm{Aff}(T(A)) \arrow[d, "\gamma"] \\
K_0(B) \arrow[r, "\rho_B"] & \mathrm{Aff}(T(B))
\end{tikzcd} \tag{2.4.4}$$

commutes. Such a triple $\Phi := (\alpha_0,\alpha_1,\gamma)\in\mathrm{Hom}(\mathrm{KT}_u(A),\mathrm{KT}_u(B))$ is an isomorphism if there exists $\Psi\in\mathrm{Hom}(\mathrm{KT}_u(B),\mathrm{KT}_u(A))$ such that $\Phi\circ\Psi = \mathrm{id}_{\mathrm{KT}_u(B)}$ and $\Psi\circ\Phi = \mathrm{id}_{\mathrm{KT}_u(A)}$.

Another classification invariant that will be briefly used in this thesis is the *Cuntz semigroup*. This is an invariant for C*-algebras that can be traced back to the work of Cuntz on the existence of quasitraces on simple C*-algebras ([39]). Even if we will not use the Cuntz semigroup as a classification invariant per se, we introduce it for

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[4] This is a consequence of [27, Proposition 4.13(ii)] which follows from [117] (one can also see [66, Theorem 1]).

the purpose of employing comparison properties of C*-algebras with respect to traces.

Definition 2.4.3. Let A be a C*-algebra. If $a, b \in A \otimes \mathcal{K}$ are positive, we will write $a \precsim b$ to mean that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset A \otimes \mathcal{K}$ such that $x_n b x_n^* \to a$ as $n \to \infty$. We write $a \sim b$ if $a \precsim b$ and $b \precsim a$ and the Cuntz semigroup of A, denoted by $\mathrm{Cu}(A)$, is $(A \otimes \mathcal{K})_+/\sim$. A nonzero positive element $a \in A$ is said to be *properly infinite* if $a \oplus a \precsim a$.

The Cuntz semigroup of A is *unperforated* if for any $n \in \mathbb{N}$ and $a, b \in (A \otimes \mathcal{K})_+$ such that $a \otimes 1_n \precsim b \otimes 1_n$, we have $a \precsim b$. We refer the reader to [63] for a survey on the properties and applications of the Cuntz semigroup.

2.5 Regularity properties for C*-algebras

After the pathological examples of simple nuclear C*-algebras pioneered by Villadsen in [139, 140], further more complex refinements were built by Rørdam ([116]) and Toms ([137]).[5]It then became apparent that further regularity properties ought to be imposed on the class of simple, separable, unital, nuclear C*-algebras which would allow classification by the Elliott invariant introduced in Definition 2.4.1. In the early $2000's$, there were two apparently distinct regularity properties that gained special interest. The first such condition is tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$.

Definition. A C*-algebra A is called $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$.

The second regularity property that has been considered has more of a topological flavour. In the pursuit of a suitable noncommutative counterpart of the covering dimension of topological spaces, Winter introduced covering dimension in [144]. After several iterations, among which we mention decomposition rank ([84]), the final notion appeared in [147] under the name of nuclear dimension. Nuclear dimension,

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[5] These have perforation in the K_0-group, meaning that there exists a non-positive element such that a positive multiple of itself is positive.

introduced by Winter and Zacharias in [147], is a noncommutative generalisation of covering dimension for topological spaces to C*-algebras. Finite nuclear dimension started to appear in the context of the classification programme. Before recording the definition, we define the notion of an order zero map.

A map $\phi : A \to B$ is said to be *completely positive* (or cp) if the canonical extension to matrix amplifications $\phi^{(n)} : M_n(A) \to M_n(B)$ is positive for all $n \in \mathbb{N}$. A cp map $\phi : A \to B$ between C*-algebras is said to be *order zero* if, for every $a, b \in A_+$ with $ab = 0$, one has $\phi(a)\phi(b) = 0$. The general theory for these maps was developed in [146]. In particular, there is a *-homomorphism $\pi_\phi : A \to \mathcal{M}(\mathrm{C}^*(\phi(A)))$, known as the *supporting* **-homomorphism* of ϕ, taking values in the multiplier algebra $\mathcal{M}(\mathrm{C}^*(\phi(A)))$ of $\mathrm{C}^*(\phi(A))$, and a positive contraction $h \in \mathcal{M}(\mathrm{C}^*(\phi(A))) \cap \pi(A)'$ such that

$$\phi(a) = h\pi_\phi(a), \qquad a \in A.$$

Definition 2.5.1 ([147])**.** Let A be a C*-algebra. It is said that A has *nuclear dimension at most* n, for some $n \in \mathbb{N}$ and denoted by $\dim_{\mathrm{nuc}} A \leq n$, if there exist nets $F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(n)}$ of finite dimensional C*-algebras and nets of maps $\psi_\lambda : A \to F_\lambda$ and $\eta_\lambda : F_\lambda \to A$ such that

(i) $\|\eta_\lambda \circ \psi_\lambda(a) - a\| \to 0$ for all $a \in A$,

(ii) ψ_λ is cpc,[6]

(iii) if we denote by $\eta_\lambda^{(i)}$ the restriction of η_λ to $F_\lambda^{(i)}$, then $\eta_\lambda^{(i)}$ is a cpc order zero map for all $i = 0, \ldots, n$.

If additionally each η_λ is contractive, then it is said that A has *decomposition rank* at most n. It is said that A has nuclear dimension equal to n if n is the minimum number for which we can find $(F_\lambda, \psi_\lambda, \eta_\lambda)_\lambda$ as above.

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[6] This stands for completely positive and contractive.

It is now known that, for simple, separable, unital, nuclear, non-elementary C*-algebras, finite nuclear dimension is equivalent to tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$ ([145, 30]). In fact, the nuclear dimension of a simple C*-algebra can only be zero, one or infinite ([30, 29]). By combining the work of many researchers, assuming $\mathcal{Z}$-stability or finite nuclear dimension, the following classification of simple nuclear C*-algebras has been achieved.

Theorem 2.5.2 (Unital Classification Theorem, see [27]). *Let A and B be simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C**-algebras satisfying the* UCT*. Then*

$$A \cong B \iff \mathrm{KT}_u(A) \cong \mathrm{KT}_u(B).$$

Moreover, any isomorphism from $\mathrm{KT}_u(A)$ to $\mathrm{KT}_u(B)$ lifts to an isomorphism from A to B.

We collect below a number of well-known permanence properties for the hypotheses in the statement of Theorem 2.5.2.

Proposition 2.5.3. *(see for example [97, Proposition 1.10]) Let A, B be separable* C**-algebras such that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. If A is nuclear, $\mathcal{Z}$-stable, and satisfies the UCT, then B is nuclear, $\mathcal{Z}$-stable, and satisfies the UCT.*

Proof. We first note that $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$ is nuclear and $\mathcal{Z}$-stable. Since nuclearity passes to hereditary subalgebras, B is nuclear. Moreover, B is $\mathcal{Z}$-stable by [138, Corollary 3.1]. Finally, the UCT is closed under stable isomorphisms ([6, 22.3.5 (a)]), so B satisfies the UCT. □

Proposition 2.5.4. *(see for example [97, Proposition 1.11]) Let B be a separable* C**-algebra, γ be an automorphism of B, and $e \in B$ be a full projection in $B \rtimes_\gamma \mathbb{Z}$. Then the following hold.*

(i) The corner $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is separable and unital.

(ii) If $B \rtimes_\gamma \mathbb{Z}$ is simple, then $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is simple.

(iii) If B is nuclear, then $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is nuclear.

(iv) If B satisfies the UCT, then so does $e\,(B \rtimes_\gamma \mathbb{Z})\,e$.

(v) If $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable, then so is $e\,(B \rtimes_\gamma \mathbb{Z})\,e$.

Proof. Note that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is unital as e is a projection, and it is separable since B is separable and $\mathbb{Z}$ is a countable discrete group. If $B \rtimes_\gamma \mathbb{Z}$ is simple and e is full, then $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is simple. If B is nuclear, then $B \rtimes_\gamma \mathbb{Z}$ is nuclear by [24, Theorem 4.2.4]. Since nuclearity passes to hereditary subalgebras, $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is nuclear. Suppose now that B satisfies the UCT. Since the UCT is closed under crossed products by $\mathbb{Z}$ ([6, 22.3.5 (g)]), $B \rtimes_\gamma \mathbb{Z}$ satisfies the UCT. Furthermore, $B \rtimes_\gamma \mathbb{Z}$ and $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ are stably isomorphic (see [18]) and the UCT is closed under stable isomorphisms ([6, 22.3.5 (a)]). Hence, $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ satisfies the UCT. Since $\mathcal{Z}$-stability passes to hereditary subalgebras ([138, Corollary 3.1]) and $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable, it follows that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is $\mathcal{Z}$-stable. □

In fact, both $\mathcal{Z}$-stability and finite nuclear dimension are related to a third regularity property, more algebraic in flavour, for simple C*-algebras.

Let A be a C*-algebra and $\tau : A \to \mathbb{C}$ be a trace. Recall that τ has a canonical extension to $M_k(A)$ for every $k \in \mathbb{N}$ by tensoring with the non-normalised tracial functional on M_k. Then, one can define $d_\tau : M_k(A)_+ \to [0, \infty)$ by

$$d_\tau(a) := \lim_{n\to\infty} \tau(a^{1/n}). \tag{2.5.1}$$

It is worth pointing out that if $p \in M_k(A)$ is a projection, then $d_\tau(p) = \tau(p)$ for any bounded trace τ on A.

Definition 2.5.5. A unital C*-algebra A has *strict comparison* of positive elements with respect to tracial states if

$$(\forall \tau \in T(A),\ d_\tau(a) < d_\tau(b)) \Longrightarrow a \precsim b$$

for $k \in \mathbb{N}$ and $a, b \in M_k(A)_+$.

A fact that will be used in the proof of Theorem 4.4.1 is that evaluations of the functionals d_τ at full positive elements are strictly positive.

Lemma 2.5.6. *Let B be a unital and separable* C*-*algebra and $b \in B$ be a nonzero, positive, and full contraction in B. Then $\tau(b) > 0$ and hence $d_\tau(b) > 0$ for any tracial state τ on B.*

Proof. Let $\tau \in T(B)$ such that $\tau(b) = 0$. Since $\ker(\tau) = \{x \in B : \tau(x^*x) = 0\}$ is an ideal in B, and $b \in \ker(\tau)$ is full in B, τ must be zero on B, which is a contradiction. Hence $\tau(b) > 0$ for any tracial state τ on B. Since $\tau(b) \leq d_\tau(b)$, it also follows that $d_\tau(b) > 0$ for any tracial state τ on B. ☐

The famous Toms-Winter conjecture predicts that the three regularity properties discussed in this section ($\mathscr{Z}$-stability, finite nuclear dimension, and strict comparison) are equivalent for the class of simple, separable, unital, nuclear C*-algebras (see [57] or [147, Conjecture 9.3] for the precise formulation below).

Conjecture (Toms-Winter)**.** For a separable, simple, unital, infinite dimensional and nuclear C*-algebra A, the following are equivalent:

(i) A has finite nuclear dimension;

(ii) A is $\mathscr{Z}$-stable;

(iii) A has strict comparison of positive elements.

2.6 Dichotomy and the range of the invariant

In fact, due to the work of Kirchberg, unital classifiable C*-algebras sit on two opposite ends of the spectrum: they are either stably finite or purely infinite (see for example [147, Theorem 5.4]). Precisely, the dichotomy is detected by the presence or absence of traces.

Recall that a unital C*-algebra is called *stably finite* if all isometries are unitaries. A simple unital C*-algebra A is called *purely infinite* if it is not the complex numbers and for every nonzero $a \in A_+$, there exist $x, y \in A$ such that $xay = 1_A$. Standard examples of stably finite C*-algebras are AF-algebras or the Jiang-Su algebra $\mathcal{Z}$, while famous examples of purely infinite C*-algebras are the Cuntz algebras. For $2 \leq n < \infty$, the Cuntz algebra $\mathcal{O}_n$ is the canonical C*-algebra generated by n isometries $S_1, S_2, \dots S_n$ such that $S_1S_1^* + S_2S_2^* + \dots S_nS_n^* = 1$ ([38]). The C*-algebra $\mathcal{O}_\infty$ is the C*-algebra generated by infinitely many isometries S_k for $k \in \mathbb{N}$ with orthogonal range projections.

Theorem 2.5.2 is complemented by range-of-the-invariant results both for classifiable purely infinite C*-algebras as well as the stably finite ones. Note that the purely infinite case was considered in [114], where the range of the invariant was realised by crossed product constructions, but we will quote the general result from [50]. The theorem below is folklore (see also [27, Remark 2.5] stated in terms of the KT_u-invariant) and it is implicitly contained in [50] as all of the inductive limit constructions provided in [50] are simple and have finite nuclear dimension, and hence are $\mathcal{Z}$-stable. The formulation below is from [97, Theorem 1.12]. We will use this result in the proof of Theorem 17.

Theorem 2.6.1 ([50])**.** *Let (G, G_+, v) be a simple, countable, ordered abelian group which is weakly unperforated and has distinguished order unit v, H be a countable abelian group, and X be a compact, metrisable Choquet simplex together with a weakly* *unperforated pairing* $\rho : G \times X \to \mathbb{R}$ *which determines* G_+*. Then there exists a simple, separable, unital, nuclear,* $\mathcal{Z}$*-stable* C**-algebra* A *satisfying the UCT such that*

$$\mathrm{Ell}(A) \cong (G, G_+, v, H, X, \rho).$$

Proof. Let (G, G_+, v, H, X, ρ) be as in the statement of the theorem. Then there exists a simple, separable, stable, nuclear C*-algebra B such that $\mathrm{Ell}(B) \cong (G, G_+, H, \tilde{X}, \tilde{\rho})$, where $\tilde{X}$ is the cone with base X and $\tilde{\rho}$ is the unique extension of the pairing ρ ([50, Theorem 5.2.3.2]). Since the C*-algebra B is constructed as an inductive limit, one can simply tensor with $\mathcal{Z}$ at each stage to assume that B is $\mathcal{Z}$-stable, or notice that B is simple and has finite nuclear dimension as each building block has finite nuclear dimension ([147, Proposition 2.3]). Thus, B is $\mathcal{Z}$-stable by [134, Corollary 8.7]. Moreover, the UCT is preserved by inductive limits (see [6, 22.3.5 (e)]), so B satisfies the UCT.

If $p \in B$ is a projection such that $[p]_0 = v$ in $K_0(B)$, then we take $A = pBp$. Since G is simple, p is a full projection, so A is simple, separable, and unital (see [18]). Furthermore, A is nuclear, $\mathcal{Z}$-stable, and it satisfies the UCT by Proposition 2.5.3. Finally, it is immediate to see that $\mathrm{Ell}(A) \cong (G, G_+, v, H, X, \rho)$. ☐

2.7 The total invariant

To classify *-homomorphisms, one needs a larger invariant than the aforementioned Elliott invariant. In this section, we will describe this invariant and collect some facts for later use. We refer the reader to Sections 2 and 3 in [27] for a more detailed description. This section is a slightly expanded version of [31, Section 1.4].

Let A be a unital C*-algebra. The *total K-theory* of A, denoted by $\underline{K}(A)$, is the combination of $K_0(A) \oplus K_1(A)$ together with $\bigoplus_{n\geq 2} K_*(A;\mathbb{Z}/n)$ and natural maps between them. There are multiple equivalent ways of defining the groups $K_i(A;\mathbb{Z}/n)$, but in this thesis we choose to say that $K_i(A;\mathbb{Z}/n) := K_i(A\otimes\mathcal{O}_{n+1})$ for $i = 0, 1$. These all fit into natural commuting diagrams with the Bockstein maps (see for example [27, Section 2.3]). We will use repeatedly the existence of short exact sequences

$$0 \longrightarrow K_i(A)\otimes\mathbb{Z}/n \longrightarrow K_i(A;\mathbb{Z}/n) \longrightarrow \mathrm{Tor}(K_{1-i}(A),\mathbb{Z}/n) \longrightarrow 0, \tag{2.7.1}$$

for $i = 0, 1$ and $n \geq 2$.

For unital C*-algebras A and B, a Λ-*morphism* $\underline{\alpha} : \underline{K}(A) \to \underline{K}(B)$ consists of *-homomorphisms

$$\alpha_i : K_i(A) \to K_i(B) \qquad \text{and} \qquad \alpha_i^{(n)} : K_i(A;\mathbb{Z}/n) \to K_i(B;\mathbb{Z}/n)$$

for all $i = 0, 1$ and $n \geq 2$ intertwining all the Bockstein operations. In particular, we will use that the collection of diagrams

$$\begin{tikzcd} K_0(A;\mathbb{Z}/n) \arrow[r, "\nu_{0,A}^{(n)}"] \arrow[d, "\alpha_0^{(n)}"] & K_1(A) \arrow[d, "\alpha_1"] \\ K_0(B;\mathbb{Z}/n) \arrow[r, "\nu_{0,B}^{(n)}"] & K_1(B), \end{tikzcd} \tag{2.7.2}$$

commute.[7]The collection of these Λ-morphisms is denoted $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B))$. For a unital *-homomorphism $\theta : A \to B$, we denote the induced Λ-morphism by $\underline{K}(\theta) : \underline{K}(A) \to \underline{K}(B)$.

Recall that in any unital C*-algebra A, the traces are related to the K-theory via the canonical pairing map $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$. For a *-homomorphism $\theta : A \to B$, we will denote by $\mathrm{Aff}(T(\theta)) : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ the map $\mathrm{Aff}(T(\theta))(f)(\tau) = f(\tau\circ\theta)$ for any $f \in \mathrm{Aff}(T(A))$ and any $\tau \in T(B)$.

Given a unital C*-algebra A, we can associate an abelian group $\overline{K_1}^{\mathrm{alg}}(A)$ in a

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[7] Note that $\nu_{0,A}^{(n)}$ and $\nu_{0,B}^{(n)}$ denote canonical Bockstein maps.

functorial way. We write $\mathcal{U}_n(A)$ for the group of unitaries in $M_n(A)$, and $D\mathcal{U}_n(A)$ for the subgroup of $\mathcal{U}_n(A)$ generated by commutators uvu^*v^* for $u, v \in \mathcal{U}_n(A)$. Each $\mathcal{U}_n(A)$ is included into $\mathcal{U}_{n+1}(A)$ via the canonical diagonal embedding, and we denote the direct limit by $\mathcal{U}_\infty(A)$. Then, the *Hausdorffised unitary algebraic K_1-group* of A is denoted by

$$\overline{K_1}^{\mathrm{alg}}(A) := \mathcal{U}_\infty(A)/\overline{D\mathcal{U}_\infty(A)},$$

where the closure is taken with respect to the inductive limit topology. We write $[u]_{\mathrm{alg}}$ for the class of $u \in \mathcal{U}_\infty(A)$ in $\overline{K_1}^{\mathrm{alg}}(A)$. Then, a unital *-homomorphism $\theta : A \to B$ induces a canonical group homomorphism $\overline{K_1}^{\mathrm{alg}}(\theta) : \overline{K_1}^{\mathrm{alg}}(A) \to \overline{K_1}^{\mathrm{alg}}(B)$ by $\overline{K_1}^{\mathrm{alg}}([u]_{\mathrm{alg}}) := [\theta^{(n)}(u)]_{\mathrm{alg}}$ for $u \in \mathcal{U}_n(A)$.

Importantly, we can relate $\overline{K_1}^{\mathrm{alg}}(A)$ with $\mathrm{Aff}(T(A))$ and $K_1(A)$ via canonical maps. There is a map $\mathscr{a}_A : \overline{K_1}^{\mathrm{alg}}(A) \to K_1(A)$ given by $\mathscr{a}_A([u]_{\mathrm{alg}}) = [u]_1$ for any $n \in \mathbb{N}$ and $u \in \mathcal{U}_n(A)$. Moreover, there exists a map $\mathrm{Th}_A : \mathrm{Aff}(T(A)) \to \overline{K_1}^{\mathrm{alg}}(A)$ called the Thomsen map. We refer to [129] for a construction of the Thomsen map. One fact we are going to use repeatedly is that there is a non-canonical decomposition

$$\overline{K_1}^{\mathrm{alg}}(A) \cong K_1(A) \oplus \frac{\mathrm{Aff}(T(A))}{\overline{\rho_A(K_0(A))}}, \tag{2.7.3}$$

[129, Corollary 3.3].[8]

As it was worked out in [27, Section 3], the last ingredient needed in the total invariant to obtain the classification of morphisms is certain natural maps

$$\zeta_A^{(n)} : K_0(A; \mathbb{Z}/n) \to \overline{K_1}^{\mathrm{alg}}(A), \quad n \geq 2.$$

Let $\beta : \overline{K_1}^{\mathrm{alg}}(A) \to \overline{K_1}^{\mathrm{alg}}(B)$ be a group homomorphism which is compatible with the Thomsen map (see (2.7.6)). To get a compatible morphism between the total

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[8] To see that this decomposition is non-canonical, the reader could consult [99] or [27, Example 9.11].

invariants of two unital C*-algebras A and B, we will further require that the diagram

$$\begin{tikzcd}
K_0(A;\mathbb{Z}/n) \arrow[r, "\zeta_A^{(n)}"] \arrow[d, "\alpha_0^{(n)}"] & \overline{K_1}^{\mathrm{alg}}(A) \arrow[d, "\beta"] \\
K_0(B;\mathbb{Z}/n) \arrow[r, "\zeta_B^{(n)}"] & \overline{K_1}^{\mathrm{alg}}(B)
\end{tikzcd} \tag{2.7.4}$$

commutes for any $n \geq 2$.

It was shown in [27, Proposition 3.12] that the diagram in (2.7.4) always commutes when A and B are unital C*-algebras with $K_1(A)$ torsion free. We can now define the total invariant of a unital C*-algebra, as well as morphisms between total invariants.

Definition 2.7.1 ([27, Definition 3.5]). Let A, B be unital C*-algebras. The *total invariant* of A, denoted by $\underline{K}T_u(A)$, is

$$\underline{K}T_u(A) := (\underline{K}(A), \mathrm{Aff}(T(A)), \overline{K_1}^{\mathrm{alg}}(A), [1_A]_0, \rho_A, \mathrm{Th}_A, \cancel{a}_A, (\zeta_A^{(n)})_{n\geq 2}). \tag{2.7.5}$$

Then, a morphism $(\underline{\alpha}, \beta, \gamma) : \underline{K}T_u(A) \to \underline{K}T_u(B)$ consists of a Λ-morphism $\underline{\alpha} \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ with $\alpha_0([1_A]_0) = [1_B]_0$, a positive, unital, linear map $\gamma : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$, and a group homomorphism $\beta : \overline{K_1}^{\mathrm{alg}}(A) \to \overline{K_1}^{\mathrm{alg}}(B)$ such that the diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, "\rho_A"] \arrow[d, "\alpha_0"] & \mathrm{Aff}(T(A)) \arrow[r, "\mathrm{Th}_A"] \arrow[d, "\gamma"] & \overline{K_1}^{\mathrm{alg}}(A) \arrow[r, "\cancel{a}_A"] \arrow[d, "\beta"] & K_1(A) \arrow[d, "\alpha_1"] \\
K_0(B) \arrow[r, "\rho_B"] & \mathrm{Aff}(T(B)) \arrow[r, "\mathrm{Th}_B"] & \overline{K_1}^{\mathrm{alg}}(B) \arrow[r, "\cancel{a}_B"] & K_1(B)
\end{tikzcd} \tag{2.7.6}$$

and the diagrams in (2.7.4) commute. In particular, by [27, Proposition 3.6], any unital *-homomorphism $\theta : A \to B$ induces a natural $\underline{K}T_u$-morphism given by

$$\underline{K}T_u(\theta) := (\underline{K}(\theta), \overline{K_1}^{\mathrm{alg}}(\theta), \mathrm{Aff}(T(\theta))) : \underline{K}T_u(A) \to \underline{K}T_u(B).$$

It is said that $(\underline{\alpha}, \beta, \gamma) : \underline{K}T_u(A) \to \underline{K}T_u(B)$ is *faithful* if the induced map γ^* :

$T(B) \to T(A)$ (which exists by Kadison's duality [78]) satisfies that $\gamma^*(\tau)$ is faithful for all $\tau \in T(B)$. Note that this is automatic if A is simple, which is the main case of interest of this thesis.

We can now state the main classification theorems we will use in this thesis. Note that a *-homomorphism $\phi : A \to B$ is called full if $\phi(a)$ generates B as a closed two-sided ideal for any nonzero $a \in A$.

Theorem 2.7.2 ([27, Theorem 9.3])**.** *Let A and B be unital, separable, nuclear* C*-*algebras such that A satisfies the UCT and B is simple and $\mathcal{Z}$-stable. If $(\underline{\alpha}, \beta, \gamma) : \underline{K}T_u(A) \to \underline{K}T_u(B)$ is a faithful morphism, then there exists a unital full *-homomorphism $\phi : A \to B$ such that $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$, and this ϕ is unique up to approximate unitary equivalence.*

In fact, Theorem 2.7.2 is proved by first classifying maps into a larger C*-algebra in which the codomain B sits canonically. Precisely, if B is a C*-algebra, B_∞ stands for the sequence algebra of B given by

$$B_\infty = \ell^\infty(\mathbb{N}, B) / \Big\{ (b_n)_{n \geq 1} : \lim_{n \to \infty} \|b_n\| = 0 \Big\}. \tag{2.7.7}$$

There exists a canonical embedding $\iota_B : B \to B_\infty$ given by viewing elements in B as constant sequences in B_∞.

Theorem 2.7.3 ([27, Theorem 9.1])**.** *Let A and B be unital, separable, nuclear* C*-*algebras such that A satisfies the UCT and B is simple and $\mathcal{Z}$-stable. If*

$$(\underline{\alpha}, \beta, \gamma) \colon \underline{K}T_u(A) \to \underline{K}T_u(B_\infty)$$

is a faithful morphism, then there exists a unital full *-homomorphism $\phi \colon A \to B_\infty$ such that $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$, and this ϕ is unique up to unitary equivalence.

We will often use the following special case of Theorem 2.7.2.

Lemma 2.7.4. *Suppose A and B are* C**-algebras as in Theorem 2.7.2 and suppose further that B has real rank zero. Then, for any triple $(\underline{\alpha}, \beta, \gamma)$ satisfying the diagram in* (2.7.6)*, β is determined by α_1, and the new pairing map diagrams in* (2.7.4) *are automatically satisfied. Therefore, unital full *-homomorphisms $\phi : A \to B$ are classified by total K-theory and traces.*

Proof. If B is purely infinite, then (2.7.3) shows that $\overline{K_1}^{\mathrm{alg}}(B) \cong K_1(B)$. If B is stably finite, since B is simple, nuclear, $\mathcal{Z}$-stable, and has real rank zero, it follows that the image of $K_0(B)$ under the pairing map is uniformly dense in $\mathrm{Aff}(T(B))$ ([117, Theorem 7.2]). Therefore, $\overline{K_1}^{\mathrm{alg}}(B) \cong K_1(B)$. In both cases, the isomorphism is given by $\not{a}_B$, so the diagram in (2.7.6) shows that $\beta = \not{a}_B^{-1} \circ \alpha_1 \circ \not{a}_A$. Therefore, β is determined by the K-theory maps and traces.

It remains to check that the diagrams in (2.7.4) commutes. Fix a natural number $n \geq 2$. By [27, Proposition 3.2], the diagram

$$\begin{tikzcd}
K_0(A;\mathbb{Z}/n) \arrow[r, "\nu_{0,A}^{(n)}"] \arrow[d, "\zeta_A^{(n)}"] & K_1(A) \arrow[d, "\mathrm{id}"] \\
\overline{K_1}^{\mathrm{alg}}(A) \arrow[r, "\not{a}_A"] & K_1(A),
\end{tikzcd} \tag{2.7.8}$$

commutes. Since B has real rank zero,

$$\beta \circ \zeta_A^{(n)} = \not{a}_B^{-1} \circ \alpha_1 \circ \not{a}_A \circ \zeta_A^{(n)}.$$

Using diagram (2.7.8), it follows that $\beta \circ \zeta_A^{(n)} = \not{a}_B^{-1} \circ \alpha_1 \circ \nu_{0,A}^{(n)}$. Then, $\alpha_1 \circ \nu_{0,A}^{(n)} = \nu_{0,B}^{(n)} \circ \alpha_0^{(n)}$ by (2.7.2). Finally, using diagram (2.7.8) for B, it follows that $\zeta_B^{(n)} = \not{a}_B^{-1} \circ \nu_{0,B}^{(n)}$. Thus

$$\beta \circ \zeta_A^{(n)} = \not{a}_B^{-1} \circ \nu_{0,B}^{(n)} \circ \alpha_0^{(n)} = \zeta_B^{(n)} \circ \alpha_0^{(n)}. \tag{2.7.9}$$

Therefore, the diagram in (2.7.4) commutes for any $n \geq 2$, and the conclusion follows by Theorem 2.7.2. $\square$

In the absence of tracial data, one can relax the hypotheses of Theorem 2.7.2 and obtain a classification of morphisms by KK-theory. First, let us record some definitions. Let A and B be C*-algebras with A separable and let $\theta : A \to B$ be a *-homomorphism. Then, following [60, Definition 3.16], θ is said to be *$\mathcal{O}_\infty$-stable* if $\mathcal{O}_\infty$ embeds unitally in $B_\infty \cap \theta(A)' / \operatorname{Ann}(\theta(A))$.[9] Two *-homomorphisms $\phi, \psi : A \to B$, with A separable, are said to be *approximately Murray-von Neumann equivalent* if there exists $v \in B_\infty$ such that $v^*\phi(a)v = \psi(a)$ and $v\psi(a)v^* = \phi(a)$ for all $a \in A$. We will only record the uniqueness part of the classification of morphisms in [61].

Theorem 2.7.5 ([61, Theorem 8.10])**.** *Let A be a separable* C**-algebra, and let B be a σ-unital* C**-algebra. Suppose that $\phi, \psi : A \to B$ are nuclear, $\mathcal{O}_\infty$-stable, full *-homomorphisms. The following are equivalent:*

(i) $\mathrm{KL}_{\mathrm{nuc}}(\phi) = \mathrm{KL}_{\mathrm{nuc}}(\psi)$*;*

(ii) ϕ and ψ are approximately Murray–von Neumann equivalent.

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[9] Note that $\operatorname{Ann}(\theta(A)) = \{b \in B_\infty : b\theta(A) + \theta(A)b = \{0\}\}$.

Chapter 3

Topologically zero-dimensional *-homomorphisms

This chapter is based on joint work with Jorge Castillejos and its content is essentially contained in [31]. Section 3.2 is based on joint work with my co-supervisor James Gabe in [62].

This chapter is organised as follows. Section 3.1 reviews the definition of nuclear dimension together with some standard simplifications in the case when $\theta : A \to B$ is a unital *-homomorphism with nuclear dimension equal to zero. In Section 3.2 we completely characterise maps between commutative C*-algebras having nuclear dimension equal to zero. Then, Section 3.3 focuses on properties of sequence algebras of finite dimensional C*-algebras. The main goal is Proposition 3.3.5, which shows that if the dimension of the matrix blocks grows fast enough, we can obtain a non-elementary dimension group that can be mapped into the K_0-group of the sequence algebra. This is going to provide the key ingredient in the proof of Theorem 3.5.1. We then change gears in Section 3.4 and collect a number of properties of morphisms with dimension equal to zero at the level of their total invariant. We prove the main results of this chapter in Section 3.5. We finish this chapter in Section 3.6 by providing a characterisation of unital embeddings of $\mathcal{Z}$ with nuclear dimension equal to zero.

3.1 Preliminaries

Let us first recall and extend the notion of a sequence algebra from (2.7.7). Given a sequence $(B_n)_{n\in\mathbb{N}}$ of C*-algebras, the induced *sequence algebra* is the quotient of the bounded sequences by the null sequences. This will be denoted by $\prod_{n=1}^\infty B_n / \bigoplus_{n=1}^\infty B_n$ or $\prod\limits_{\infty} B_n$ for simplicity of the notation.

For a sequence algebra $\prod\limits_{\infty} B_n$ we denote by $T_\infty\Big(\prod\limits_{\infty} B_n\Big)$ the set of *limit traces* on $\prod\limits_{\infty} B_n$, i.e. those $\tau \in T\Big(\prod\limits_{\infty} B_n\Big)$ of the form $\tau((b_n)_{n\in\mathbb{N}}) = \lim_{n\to\omega} \tau_n(b_n)$ for some sequence of traces $\tau_n \in T(B_n)$ and a free ultrafilter ω on $\mathbb{N}$.

We will often want to transfer properties of the algebras $(B_n)_{n\in\mathbb{N}}$ to their sequence algebra. In particular, one fact we will need is that real rank zero passes to sequence algebras.

Lemma 3.1.1. *Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of* C**-algebras with real rank zero. Then $\prod\limits_{\infty} B_n$ has real rank zero.*[10]

Proof. Note that by definition, a nonunital C*-algebra has real rank zero if and only if its minimal unitisation does. Therefore, by [29, Lemma 1.11] we can assume that each B_n is unital. Let $x, y \in \prod\limits_{\infty} B_n$ be positive orthogonal contractions and $\epsilon > 0$. Say the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $\prod\limits_{\infty} B_n$ represent x and y, respectively. Thus there is $N \in \mathbb{N}$ such that $\|a_n b_n\| < \epsilon^2$ for all $n > N$. Since each B_n has real rank zero, [21, Theorem 2.6] yields the existence of a projection $p_n \in B_n$ such that

$$\|(1-p_n)a_n\| < \epsilon \qquad \text{and} \qquad \|p_n b_n\| < \epsilon \qquad \text{with} \quad n > N.$$

Set $p_n := 0$ if $n \leq N$. It then follows that the sequence $(p_n)_{n\in\mathbb{N}}$ induces a projection

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[10] This result actually holds for any filter on $\mathbb{N}$.

$p \in \prod_\infty B_n$ such that

$$\|(1-p)x\| \leq \epsilon \qquad \text{and} \qquad \|py\| \leq \epsilon.$$

Again, by [21, Theorem 2.6], $\prod_\infty B_n$ has real rank zero. □

In Section 2.5, we defined finite nuclear dimension for C*-algebras. This definition can be immediately extended to *-homomorphisms.

Definition 3.1.2 ([147, 136])**.** Let A, B be C*-algebras and $\theta : A \to B$ be a *-homomorphism. It is said that θ has *nuclear dimension at most* n, for some $n \in \mathbb{N}$ and denoted by $\dim_{\text{nuc}} \theta \leq n$, if there exist nets $F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(n)}$ of finite dimensional C*-algebras and nets of maps $\psi_\lambda : A \to F_\lambda$ and $\eta_\lambda : F_\lambda \to B$ such that

(i) $\|\eta_\lambda \circ \psi_\lambda(a) - \theta(a)\| \to 0$ for all $a \in A$,

(ii) ψ_λ is cpc,

(iii) if we denote by $\eta_\lambda^{(i)}$ the restriction of η_λ to $F_\lambda^{(i)}$, then $\eta_\lambda^{(i)}$ is a cpc order zero map for all $i = 0, \ldots, n$.

If additionally each η_λ is contractive, then it is said that θ has *decomposition rank* at most n.[11]

The *-homomorphism θ has nuclear dimension equal to n if n is the minimum number for which we can find $(F_\lambda, \psi_\lambda, \eta_\lambda)_\lambda$ as above. In this case, we call $(F_\lambda, \psi_\lambda, \eta_\lambda)_\lambda$ an *n-decomposable approximating system* for θ. In fact, in the case when a *-homomorphism θ is unital and has nuclear dimension equal to zero, we can say more about the approximating system given by Definition 3.1.2. By definition, in this situation θ also has decomposition rank equal to zero. As in the case of finite decomposition rank for C*-algebras ([84, Proposition 5.1]), we can assume that the downward approximating maps are approximately multiplicative. We will record this result for later use.

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[11] If the C*-algebra A is further assumed to be separable, then we can replace nets by sequences.

Lemma 3.1.3. *[13, Proposition 1.7] Let* $\theta : A \to B$ *be a* **-homomorphism between* C^**-algebras with* A *separable and* $\dim_{\mathrm{nuc}} \theta = 0$*. Then there exists an* n*-decomposable approximating system* $(F_k, \psi_k, \eta_k)_{k\in\mathbb{N}}$ *for* θ *such that the maps* ψ_k *are approximately multiplicative, i.e. the induced map* $\psi\colon A \to \prod_{\infty} F_k$ *is a* **-homomorphism.*

Remark 3.1.4. *(i)* By the definition in [147], $\dim_{\mathrm{nuc}} A = \dim_{\mathrm{nuc}} \mathrm{id}_A$ for any separable C^*-algebra A.

(ii) If we take $\theta : A \to B$ to be a unital *-homomorphism with finite nuclear dimension, a standard simplification in the spirit of [84, Remark 5.2 (ii)] allows us to assume that the downward approximating maps ψ_k can be taken to be unital and completely positive. In particular, if $\dim_{\mathrm{nuc}} \theta = 0$, we can take the downward approximating maps to be ucp and approximately multiplicative.

(iii) Essentially the same proof as in [147, Proposition 2.3 (ii)] shows that if $\theta : A \to B$ is a *-homomorphism with $\dim_{\mathrm{nuc}} \theta = 0$, then any matrix amplification of θ, denoted by $\theta \otimes \mathrm{id}_{M_n(\mathbb{C})}$, has nuclear dimension equal to zero. This will be used in the proof of Proposition 3.4.2.

(iv) If $\phi, \psi : A \to B$ are approximately unitarily equivalent *-homomorphisms, then it is readily seen that $\dim_{\mathrm{nuc}} \phi = \dim_{\mathrm{nuc}} \psi$. This will be freely used throughout this thesis.

Lemma 3.1.5. *Let* A *and* B *be separable* C^**-algebras and* $\theta : A \to B$ *a* **-homomorphism. Then* $\dim_{\mathrm{nuc}} \theta \leq \min(\dim_{\mathrm{nuc}} A, \dim_{\mathrm{nuc}} B)$*.*

Proof. If neither A nor B have finite nuclear dimension, the conclusion follows. If $\dim_{\mathrm{nuc}} A < \infty$, since $\theta = \theta \circ \mathrm{id}_A$, we can use an approximating system of id_A to get one for θ. Note that composing with a *-homomorphism preserves the fact that the upward maps are order zero. Hence, $\dim_{\mathrm{nuc}} \theta \leq \dim_{\mathrm{nuc}} A$. Similarly, $\dim_{\mathrm{nuc}} \theta \leq \dim_{\mathrm{nuc}} B$. $\square$

Remark 3.1.6. In particular, combining the lemma above and [30, Theorem B, Corollary C], if A or B are simple, unital, nuclear, and $\mathcal{Z}$-stable, then $\dim_{\mathrm{nuc}} \theta \leq 1$.

The next proposition follows in the spirit of [147, Proposition 3.4] and it says that the matrix blocks in the approximating finite dimensional C*-algebras can be taken to be large, as long as the ranks of the irreducible representations of our domain are large. In particular, if the domain has no finite dimensional irreducible representations, the sizes of the matrix blocks go to infinity.

Proposition 3.1.7 (cf. [147, Proposition 3.4]). *Let A and B be separable* C*-*algebras and $\theta : A \to B$ be a *-homomorphism with* $\dim_{\mathrm{nuc}} \theta \leq n < \infty$. *Let $r \in \mathbb{N}$ and suppose A has no irreducible representations of rank strictly less than r. Then there exists an n-decomposable approximating system $(F_k, \psi_k, \eta_k)_{k\in\mathbb{N}}$ such that the irreducible representations of each F_k have rank at least r.*

Proof. Let $(\bar{F}_k, \bar{\psi}_k, \bar{\eta}_k)_{k\in\mathbb{N}}$ be an approximating system witnessing nuclear dimension at most n as in Definition 3.1.2. By [13, Proposition 1.7], we can further assume that the maps ψ_k are approximately order zero i.e.

$$\|\psi_k(a)\psi_k(b)\| \to 0$$

for all $a, b \in A_+$ that satisfy $ab = 0$. For each k, we write $\bar{F}_k = F_k \oplus \tilde{F}_k$, where $\tilde{F}_k$ consists of those matrix blocks of $\bar{F}_k$ with rank at most $r - 1$. Likewise, we denote by $\psi, \tilde{\psi}, \eta$, and $\tilde{\eta}$ the respective components of $\bar{\psi}$ and $\bar{\eta}$.

Consider the cpc order zero map induced by the sequence $(\tilde{\psi}_k)_{k\in\mathbb{N}}$:

$$\tilde{\psi} : A \to \prod_{\infty} \tilde{F}_k.$$

Let $h \in A$ be a strictly positive element of norm one and set

$$\mu := \limsup_{k\to\infty} \|\tilde{\psi}_k(h)\| = \|\tilde{\psi}(h)\|.^{12}$$

Since the matrix blocks of $\tilde{F}_k$ have rank at most $r-1$ for any $k \geq 1$, there exists an irreducible representation

$$\pi : \prod_{\infty} \tilde{F}_k \to M_l$$

for some $l \leq r-1$ such that $\|\pi \circ \tilde{\psi}(h)\| = \mu$.

Since π is a *-homomorphism, $\pi \circ \tilde{\psi}$ is a cpc order zero map. By the structure theorem for order zero maps ([146, Theorem 3.3]), there exist a *-homomorphism $\sigma : A \to M_l$ and $0 \leq d \leq 1_{M_l} \in M_l$ such that

$$\pi \circ \tilde{\psi}(a) = d\sigma(a) = \sigma(a)d$$

for any $a \in A$. However, by hypothesis A has no irreducible representations of rank strictly less than r, so $\sigma = 0$. Thus

$$\|\tilde{\psi}(h)\| = \mu = \|\pi\tilde{\psi}(h)\| = 0.$$

Since $\tilde{\psi}$ is a cpc order zero map and h is a strictly positive element, we can appeal again to the structure theorem of order zero maps ([146, Theorem 3.3]) to conclude that $\tilde{\psi} = 0$. Hence, $(F_k, \psi_k, \eta_k)_{k\in\mathbb{N}}$ is a system of cp approximations for θ with the required properties. $\square$

Corollary 3.1.8. *Let A and B be* C**-algebras with A simple, separable, and non-elementary.*[13] *Suppose $\theta : A \to B$ is a *-homomorphism with finite nuclear dimension. Then there is a finite dimensional approximating system $(F_k, \psi_k, \phi_k)_{k\in\mathbb{N}}$ witnessing the*

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[12] A positive element $a \in A$ is strictly positive if $\overline{aAa} = A$.

[13] A C*-algebra A is non-elementary if it is not the C*-algebra of compact operators on a Hilbert space.

finite nuclear dimension of θ where each F_k has no irreducible representations of rank less than k; i.e. all matrix blocks in the decomposition of F_k have size at least k.

Before proving the main results in this chapter, let us explain why Winter's proof that a unital C*-algebra with nuclear dimension equal to zero is an AF-algebra does not transfer to *-homomorphisms.

One natural starting point is trying to mimic Winter's proof in [144] showing that a unital C*-algebra A with nuclear dimension equal to zero is an AF-algebra. Loosely speaking, it goes as follows: using an approximating system witnessing nuclear dimension equal to zero, we would like to produce an increasing family $(F_n)_{n\in\mathbb{N}}$ of finite dimensional C*-subalgebras of A with dense union. Let us suppose we have already found F_n and want to obtain a larger finite dimensional C*-algebra F_{n+1} approximately containing a finite subset $\mathcal{F}$ of A up to a given $\epsilon > 0$. Nuclear dimension equal to zero and a perturbation argument will give a finite dimensional C*-algebra F and a unital *-homomorphism $\phi : F \to A$ such that $\phi(F)$ approximates $\mathcal{F}$ up to ϵ. To make sure F_{n+1} contains F_n, we can assume that the matrix units of F_n are contained in the finite set $\mathcal{F}$.

However, when dealing with a general *-homomorphism $\theta : A \to B$ with nuclear dimension equal to zero, a "double approximation issue" arises if we try to adopt the same strategy. Precisely, the image of the upward approximating maps $\phi(F)$ will possibly lie outside the image of θ in the bigger C*-algebra B. Therefore, the strategy employed by Winter will produce a finite dimensional C*-algebra F_{n+1} which generally will not be contained in the image of θ. Then, we cannot ensure in the next step that we can include the necessary matrix units in the finite set to be approximated. Summarising, we can obtain that for any finite subset $\mathcal{F}$ of A and any $\epsilon > 0$, there exists a finite dimensional C*-algebra F in B approximating $\theta(\mathcal{F})$ up to ϵ, but we cannot a priori produce any sort of increasing sequence of finite dimensional C*-algebras; and hence, no AF-algebra within B.

Instead, we propose a different approach to this problem. By a careful analysis at the level of K-theory of the approximations witnessing nuclear dimension equal to zero, we will produce a simple, unital AF-algebra. Then, after analysing zero-dimensional *-homomorphisms at the level of the total invariant, we will make use of [27, Theorem 9.3] to produce a factorisation of such morphisms via the AF-algebra constructed in the previous step.

3.2 The commutative case

We will first discuss Problem 1 in the commutative case. The content of this section appears in [62, Section 2], which is joint work with my co-supervisor James Gabe.

The proof is intimately related to the notion of real rank zero for inclusions which will be introduced in Chapter 4. We will begin with the following lemma that states that for unital *-homomorphisms of nuclear dimension equal to zero, the second map in the approximation can be assumed to be a *-homomorphism rather than an order zero map. This follows in the spirit of [144, Theorem 3.4]

Lemma 3.2.1. *Let* $\theta : A \to B$ *be a unital* **-homomorphism between unital* C^**-algebras with* $\dim_{\mathrm{nuc}} \theta = 0$*. Let* $\mathcal{F} \subset A$ *be a finite subset and* $\epsilon > 0$*. Then there exist a finite dimensional* C^**-algebra* F*, a ucp map* $\psi : A \to F$*, and a unital* **-homomorphism* $\eta : F \to B$ *such that* $\|\theta(x) - \eta \circ \psi(x)\| < \epsilon$ *for all* $x \in \mathcal{F}$*.*

Proof. We can assume that $1 \in \mathcal{F}$ and all the elements in $\mathcal{F}$ are contractions. Since $\dim_{\mathrm{nuc}} \theta = 0$, there exist a finite dimensional C^*-algebra F, a ucp map $\psi : A \to F$ and a cpc order zero map $\phi : F \to B$ such that

$$\|\theta(x) - \phi \circ \psi(x)\| \leq \epsilon/2$$

for all $x \in \mathcal{F}$.

By the structure theorem for order zero maps, there exists a unital *-homomorphism $\eta : F \to \mathcal{M}(C^*(\phi(F))) \cap \{\phi(1)\}'$ such that

$$\phi(a) = \phi(1)\eta(a)$$

for all $a \in F$. Note that since θ and ψ are unital, $\|\phi(1) - 1\| \leq \epsilon/2$, which yields that $\phi(1)$ is invertible whenever ϵ is sufficiently small. Hence, η can be considered as a unital *-homomorphism into B by noticing that

$$\eta(a) = \phi(1)^{-1/2}\phi(a)\phi(1)^{-1/2}.$$

Moreover, for all $x \in \mathcal{F}$,

$$\begin{aligned} \|\theta(x) - \eta \circ \psi(x)\| &\leq \|\theta(x) - \phi \circ \psi(x)\| + \|\phi(1)\eta \circ \psi(x) - \eta \circ \psi(x)\| \\ &\leq \epsilon/2 + \epsilon\|x\|/2 \leq \epsilon. \end{aligned}$$

This completes the proof. □

Lemma 3.2.2. *Let A, B be unital* C**-algebras and $\theta : A \to B$ be a unital *-homomorphism with nuclear dimension equal to zero. Then, for any $a \in A$ self-adjoint, and any $\epsilon > 0$, there exists an invertible self-adjoint $b \in B$ such that $\|\theta(a) - b\| \leq \epsilon$.*

Proof. Let $a \in A$ be a self-adjoint contraction and $\epsilon > 0$. By Lemma 3.2.1, there exist a finite dimensional C*-algebra F, a ucp map $\psi : A \to F$, and a unital *-homomorphism $\eta : F \to B$ such that $\|\theta(a) - \eta \circ \psi(a)\| \leq \epsilon/2$. Moreover, ψ is ucp and $a \in A$ is self-adjoint, so $\psi(a) = \psi(a_+) - \psi(a_-)$ is self-adjoint. Since $\psi(a)$ is self-adjoint and F is finite dimensional, there exists an invertible self-adjoint $b' \in F$ such that $\|\psi(a) - b'\| \leq \epsilon/2$. Therefore, $\eta(b')$ is self-adjoint invertible and

$$\|\theta(a) - \eta(b')\| \leq \|\theta(a) - \eta \circ \psi(a)\| + \|\psi(a) - b'\| \leq \epsilon,$$

which finishes the proof. $\square$

Remark 3.2.3. In the language of Chapter 4, we will say that unital *-homomorphisms with nuclear dimension equal to zero have real rank zero.

We now have all the necessary ingredients to characterise *-homomorphisms with nuclear dimension equal to zero between commutative C*-algebras.

Lemma 3.2.4. *Let X, Y be compact Hausdorff spaces and $\phi : Y \to X$ be a continuous map which induces a *-homomorphism $\phi^* : C(X) \to C(Y)$. Suppose that for any self-adjoint element $f \in C(X)$ and any $\epsilon > 0$, there exists an invertible self-adjoint element $g \in C(Y)$ such that $\|\phi^*(f) - g\| \leq \epsilon$. Then, ϕ is constant on connected components and ϕ^* factors through a commutative* C**-algebra over a 0-dimensional compact Hausdorff space which is the space of connected components of Y.*

Proof. We first prove that ϕ is constant on each connected component of Y. To prove it, let Y_0 be a connected component of Y and $y_1, y_2 \in Y_0$. If Y_0 is a singleton, we are done, so let us assume that y_1 and y_2 are distinct. Suppose for a contradiction that $\phi(y_1) \neq \phi(y_2)$. By Urysohn's lemma, we then pick a self-adjoint $f \in C(X)$ such that $f(\phi(y_1)) < 0$ and $f(\phi(y_2)) > 0$. Let $\epsilon > 0$ such that $2\epsilon < \min\{f(\phi(y_2)), |f(\phi(y_1))|\}$.

By assumption, there exists an invertible self-adjoint $g \in C(Y)$ such that $|g(y) - \phi^*(f)(y)| \leq \epsilon$ for all $y \in Y$. In particular,

$$g(y_1) \leq f(\phi(y_1)) + \epsilon < 0 \text{ and } g(y_2) \geq f(\phi(y_2)) - \epsilon > 0.$$

Since g is continuous and Y_0 is connected, $g(Y_0) \subseteq \mathbb{R}$ is connected, so there is $y_0 \in Y_0$ such that $g(y_0) = 0$. This contradicts the assumption that g is invertible. Thus, $\phi(y_1) = \phi(y_2)$, which implies that ϕ is constant on Y_0.

Let Y_c be the set of connected components of Y equipped with the quotient topology. Since ϕ is constant on each connected component, there exists a continuous map

$\psi : Y_c \to X$ given by evaluating ϕ on each connected component. Then $\phi = \psi \circ \pi$, where $\pi : Y \to Y_c$ is the canonical quotient map. Therefore, by duality, the diagram

$$\begin{tikzcd}
C(X) \arrow[rr, "\phi^*"] \arrow[dr, "\psi^*"'] & & C(Y) \\
& C(Y_c) \arrow[ur, "\pi^*"'] &
\end{tikzcd}$$

commutes, where ψ^* and π^* are the dual maps of ψ and π respectively. Since Y is compact and Hausdorff, it follows that Y_c is compact and totally disconnected. It is well-known that Y_c is Hausdorff. To see this, note that in any compact Hausdorff space, connected components are quasicomponents and hence closed (see for example [43, Proposition 1.4]). As Y is compact and Hausdorff, and in particular normal, any two connected components are contained in disjoint open sets. Consequently, the quotient space Y_c is Hausdorff and the conclusion follows. □

Theorem 3.2.5. *Let X, Y be compact Hausdorff spaces and $\phi : Y \to X$ be a continuous map which induces a *-homomorphism $\phi^* : C(X) \to C(Y)$. Then $\dim_{\mathrm{nuc}} \phi^* = 0$ if and only if ϕ^* factors through a commutative* C***-algebra over a* 0*-dimensional compact Hausdorff space.*

Proof. The if direction follows from [147, Proposition 2.4] (see also [28, Theorem 2.6] where the result is proved under no cardinality assumptions). Conversely, let us suppose that ϕ^* has nuclear dimension equal to zero. Then the result follows from Lemma 3.2.2 and Lemma 3.2.4. □

3.3 Uncountable dimension groups

One goal of this section is to show that the K_0-group of a sequence algebra of finite dimensional C*-algebras can be thought of as a dimension group which is not necessarily countable. We invite the reader to recall some standard properties of ordered groups from Section 2.2.

Since we will be mostly interested in dimension groups coming from $\mathcal{Z}$-stable AF-algebras, we need to characterise when an AF-algebra has no finite dimensional representations. For this, it is said that an ordered abelian group G has *property* (D) if for each order unit x in G, there exists an order unit y in G such that $2y \leq x$. As shown in [53, Lemma 9], the K_0-group of a unital AF-algebra A has *property* (D) if and only if A has no nonzero finite dimensional representations. Let $(F_k)_{k\in\mathbb{N}}$ be a sequence of finite dimensional C*-algebras and denote by $E := \prod_{\infty} F_k$ the sequence algebra induced by them. Our key observation is that $K_0(E)$ is almost the K_0-group of an AF-algebra, with the mention that it might not be countable.

Let us recall from Section 2.4 that the Cuntz semigroup of E is *unperforated* if for any $n \in \mathbb{N}$ and $a, b \in (E \otimes \mathcal{K})_+$ such that $a \otimes 1_n \precsim b \otimes 1_n$, we have $a \precsim b$. Recall also that a unital C*-algebra is said to have *stable rank one* if the invertible elements are dense ([110]).

Proposition 3.3.1. *Let E be as defined above. Then $K_0(E)$ is an unperforated, ordered abelian group which satisfies the Riesz interpolation property. Moreover, if for all $k \geq 1$, F_k has no irreducible representations of rank less than k, then $K_0(E)$ satisfies property (D).*

Proof. By Lemma 3.1.1, E has real rank zero. Then, [149, Corollary 1.3] shows that projections in the K_0-group satisfy the Riesz decomposition property with respect to the Murray-von Neumann equivalence. Since E is unital and has stable rank one (see for example [12, Lemma 1.21])[14], it has cancellation ([6, Proposition 6.5.1]) and hence its K_0-group satisfies the Riesz decomposition property. This condition is equivalent to the Riesz interpolation property by [70, Proposition 2.1].

Now, let $n \in \mathbb{N}$ and $x = [p]_0 - [q]_0 \in K_0(E)$ such that $nx \in K_0(E)_+$. This implies that $n[q]_0 \leq n[p]_0$ and since we have cancellation, this is equivalent to saying

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[14] Technically this was proved for $\prod_{\omega} F_k$, but the same proof works for the sequence algebra $\prod_{\infty} F_k$.

that $q \otimes 1_n \precsim p \otimes 1_n$ in the Cuntz semigroup of E. Since the Cuntz semigroup of E is unperforated ([111, Lemma 2.3(ii)]), we get that $q \precsim p$, which implies that $x \in K_0(E)_+$. Hence, $K_0(E)$ is unperforated.

For the last part, since we have strict comparison with respect to limit traces (essentially by the same proof of [12, Lemma 1.23]), order units in $K_0(E)$ correspond precisely to those elements which are strictly positive on all limit traces. Let us choose an order unit $[p]_0 \in K_0(E)$, which is induced by a sequence of projections $p_k \in M_N(F_k)$ for some natural number N. Then, if F_k has a decomposition into r matrix blocks, we can write p_k as a sum of orthogonal projections $p_k^{(1)} \oplus \ldots \oplus p_k^{(r)}$.[15]

Since p is strictly positive on all limit traces and the trace space is compact, there exists some $n \in \mathbb{N}$ such that $\tau(p) \geq 1/n$ for all limit traces τ. By hypothesis, each of the blocks of F_k has size at least k, and denote the size of the block i by $n_i^{(k)}$. Then, we can consider $q = (q_k)_{k\in\mathbb{N}}$, where $q_k = q_k^{(1)} \oplus \ldots \oplus q_k^{(r)}$, and each $q_k^{(i)}$ is a diagonal matrix with $\lfloor \frac{n_i^{(k)}}{3n} \rfloor$ entries equal to 1 on the main diagonal and the rest equal to 0. Then, for any trace τ on F_k, one has that $2\tau(q_k) < \frac{1}{n} \leq \tau(p)$, where we still denote by τ the non-normalised extension to $M_N(F_k)$. Since each F_k has strict comparison, it follows that $2q_k \precsim p_k$, so $2[q]_0 \leq [p]_0$. Note that $[q]_0$ constructed in this way is an order unit. Since $n_i^{(k)} \geq k$ for all i and all $k \in \mathbb{N}$, q_k is nonzero for k large enough. Furthermore, the trace of $q_k^{(i)}$ is $\lfloor \frac{n_i^{(k)}}{3n} \rfloor / n_i^{(k)}$, which is at least $\frac{1}{4n}$ for k large enough. Thus, q is strictly positive on all limit traces, so $[q]_0$ is an order unit. Hence the conclusion follows. $\square$

Therefore, the lack of countability is the only obstruction that prevents us to say that $K_0(E)$ is a dimension group. However, all of the abstract properties defining a dimension group together with property (D) are *separably inheritable* in the sense of Blackadar ([7, Definition II.8.5.1]). Thus, we can proceed to find a separable C*-subalgebra of the sequence algebra such that its K_0-group is a dimension group. Let

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[15] Note that r depends on k, but we suppress this for ease of notation.

us first recall the definition of a separably inheritable property.

Definition 3.3.2. [7, Definition II.8.5.1] A property P of C*-algebras is *separably inheritable* if the following hold:

(i) whenever A is a C*-algebra with property P and B is a separable C*- subalgebra of A, then there is a separable C*-subalgebra C of A which contains B and which has property P;

(ii) whenever (A_n, ϕ_n) is an inductive system of separable C*-algebras with injective connecting maps, and each A_n has property P, so does $\varinjlim A_n$.

Proposition 3.3.3 ([7, II.8.5.4, II.8.5.5])**.** *The following properties are separably inheritable:*

(i) A has real rank zero;

(ii) A has stable rank one;

(iii) $K_0(A)$ is unperforated;

(iv) $K_0(A)$ has the Riesz interpolation property;

(v) $K_1(A) = 0$.

Note that property (D) can be defined in the general context of ordered abelian groups and we are going to show that the fact that $K_0(A)$ has property (D) is a separably inheritable property. Note that by replacing the unit with any order unit, it is enough to consider the unital case.

Lemma 3.3.4. *Let A be a unital* C**-algebra such that $K_0(A)$ has property (D) and let $S \subset A$ be a countable subset. Then there exists a separable* C**-subalgebra B of A, containing S, such that $K_0(B)$ has property (D). Moreover, having a K_0-group with property (D) is a separably inheritable property.*

Proof. We can assume that $1_A \in S$. We can write $S = \{s_0, s_1, \ldots\}$ with $s_0 = 1_A$. Set $S_0 := \{1_A\}$ and $B_0 := \mathrm{C}^*(S_0)$. Since $K_0(A)$ has property (D), there exists a projection y_0 in some matrix amplification of A such that $2[y_0] \leq [1_A]$, $[y_0]$ is an order unit, and the subequivalence is realised by some partial isometry v_0. Then let S_1 be the set containing S_0, s_1, all entries of y_0, and all entries of v_0, and let $B_1 = \mathrm{C}^*(S_1)$. Then B_1 is a separable C^*-algebra, so $K_0(B_1)$ is a countable ordered abelian group. Since $1_A \in B_1$ induces an order unit in $K_0(B_1)$, any order unit in $K_0(B_1)$ induces a corresponding order unit in $K_0(A)$. Then, for each order unit $[x]_0$ in $K_0(B_1)$, there is an order unit $[y_1]_0$ in $K_0(A)$ such that $2[y_1]_0 \leq [x_1]_0$ in $K_0(A)$ via some partial isometry v_1. Performing this operation for each of the countably many order units in $K_0(B_1)$, we let S_2 be the set containing S_1, s_2, all the entries of the chosen order units and partial isometries imported from A. Let $B_2 = \mathrm{C}^*(S_2)$.

Inductively, we can choose increasing countable sets S_n and $B_n := \mathrm{C}^*(S_n)$. Now take B to be the closure of $\bigcup_{n\geq 1} B_n$. Then B is a separable C^*-subalgebra of A containing S. Furthermore, it has property (D) by construction and as any projection in B is Murray-von Neumann equivalent to a projection in some B_n.

Finally, condition (ii) in Definition 3.3.2 is seen to be satisfied by continuity of the K_0-functor. Thus, having a K_0-group with property (D) is a separably inheritable property. $\square$

Then, provided that the sizes of irreducible representations of each F_k are large enough, we can show that there is a separable C^*-subalgebra of $\prod_{\infty} F_k$ preserving a series of K-theoretic properties.

Proposition 3.3.5. *Let A be a simple, separable, unital, non-elementary C^*-algebra and let $(F_k)_{k\in\mathbb{N}}$ be a sequence of finite dimensional C^*-algebras. Suppose $\psi : A \to \prod_{\infty} F_k$ is a unital *-homomorphism. Then there exists a unital, separable C^*-algebra E of real rank zero and stable rank one such that* *(i)* $\psi(A) \subset E \subset \prod_{\infty} F_k$,

(ii) $K_0(E)$ *is unperforated and satisfies the Riesz interpolation property,*

(iii) $K_0(E)$ *has property (D),*

(iv) $K_1(E) = 0$.

Proof. Note first that $\prod_{\infty} F_k$ has real rank zero by Lemma 3.1.1, stable rank one by [12, Lemma 1.21]; and its K_0-group is unperforated and satisfies the Riesz interpolation property by Proposition 3.3.1. Moreover, since A is simple and non-elementary, it has no finite dimensional representations. Note that any finite dimensional irreducible representation of $\prod_{\infty} F_k$ induces a finite dimensional representation of A. Therefore, the same proof as in Proposition 3.1.7 shows that, up to reindexing, F_k has no irreducible representations of rank less than k. Hence $K_0\Big(\prod_{\infty} F_k\Big)$ has property (D) by Proposition 3.3.1. Furthermore, since any unitary in $\prod_{\infty} F_k$ can be represented by a sequence of unitaries and $K_1(F_k) = 0$ for any $k \geq 1$, it follows that $K_1\Big(\prod_{\infty} F_k\Big) = 0$.

Since a countable intersection of separably inheritable properties is separably inheritable ([7, Proposition II.8.5.3]), combining Proposition 3.3.3 and Lemma 3.3.4, we obtain a unital separable C*-algebra E of real rank zero and stable rank one which satisfies the required conditions. $\square$

3.4 The total invariant of zero-dimensional morphisms

In this section we will study how *-homomorphisms with nuclear dimension equal to zero act on the total invariant introduced in Section 2.7. We start with a sequence algebra version of Lemma 3.2.1.

Lemma 3.4.1. *Let* $\theta : A \to B$ *be a unital* **-homomorphism between separable* C**-algebras with* $\dim_{\mathrm{nuc}} \theta = 0$*. Then there exist finite dimensional* C**-algebras* $(F_k)_{k\in\mathbb{N}}$ *and unital* **-homomorphisms* $\psi : A \to \prod_{\infty} F_k$*,* $\eta : \prod_{\infty} F_k \to B_\infty$ *such that* $\iota_B \circ \theta = \eta \circ \psi$*.*

Proof. This follows by combining Lemma 3.1.3, *(ii)* of Remark 3.1.4, and Lemma 3.2.1. □

We start by showing that morphisms with nuclear dimension equal to zero vanish on K_1. The key observation is that the unitary group of a finite dimensional C*-algebra is connected. As any map that factors through a finite dimensional algebra vanishes on K_1, the next proposition is a necessary condition for our main results.

Proposition 3.4.2. *Let* $\theta : A \to B$ *be a unital* **-homomorphism between separable* C**-algebras. If* $\dim_{\mathrm{nuc}} \theta = 0$*, then* $K_1(\theta) = 0$*.*

Proof. Let $u \in A$ be a unitary, $\mathcal{F} = \{u, u^*, 1_A\} \subset A$, and $\epsilon \geq \delta > 0$. By hypothesis, there exist a finite dimensional C*-algebra F and cpc maps $\psi : A \to F$, $\eta : F \to B$ such that

$$\|\eta(\psi(x)) - \theta(x)\| < \delta/2, \qquad x \in \mathcal{F}.$$

By *(ii)* of Remark 3.1.4 and Lemma 3.2.1, we can further assume that η is a unital *-homomorphism and ψ is ucp and *approximately multiplicative on* $\mathcal{F}$ *up to* δ; i.e. $\|\psi(x)\psi(y) - \psi(xy)\| < \delta$ for all $x, y \in \mathcal{F}$.

We now proceed to prove that $[\theta(u)]_1 = 0$ in $K_1(B)$. Since ψ is approximately multiplicative on $\mathcal{F}$ up to δ, we obtain that

$$\|\psi(u)\psi(u)^* - 1_F\| < \delta \quad \text{and} \quad \|\psi(u)^*\psi(u) - 1_F\| < \delta.$$

By assuming that δ is sufficiently small, there exists a unitary $v \in F$ such that $\|\psi(u) - v\| < \epsilon/2$. Since the unitary group of a finite dimensional C*-algebra is connected, it follows that v is homotopic to the unit 1_F. This yields that $\eta(v)$ is homotopic to $\eta(1_F) = 1_B$ in B. Moreover,

$$\|\theta(u) - \eta(v)\| \leq \|\theta(u) - \eta \circ \psi(u)\| + \|\eta \circ \psi(u) - \eta(v)\| < \epsilon.$$

Since we can assume $\epsilon < 1$, this implies that $\theta(u)$ and $\eta(v)$ are homotopic unitaries in B. Thus, $\theta(u)$ is homotopic to 1_B and $[\theta(u)]_1 = 0$ in $K_1(B)$. The same argument works for any unitary in any matrix amplification of A since any matrix amplification of θ also has nuclear dimension equal to zero by *(iii)* of Remark 3.1.4. Hence $K_1(\theta) = 0$. □

Similarly to the result in Proposition 3.4.2, morphisms with nuclear dimension equal to zero vanish at the level of K_1-groups with coefficients. The proof uses the same strategy as in Proposition 3.4.2, the key ingredient being that matrix amplifications of Cuntz algebras have trivial K_1-groups and their unitary groups are connected. As any map that factors through an AF-algebra vanishes at the level of K_1-groups with coefficients, the next proposition is a necessary condition for our main results.

Proposition 3.4.3. *Let $\theta : A \to B$ be a unital *-homomorphism between separable* C^**-algebras with* $\dim_{\mathrm{nuc}} \theta = 0$*. Then,* $K_1(\theta; \mathbb{Z}/n) = K_1(\theta \otimes \mathrm{id}_{\mathcal{O}_{n+1}}) = 0$ *for all* $n \in \mathbb{N}$.

Proof. For all $n \geq 2$, let $\mathrm{id}_{\mathcal{O}_n}$ be the identity map on the Cuntz algebra $\mathcal{O}_n$. Observe that this proof reduces to showing that $K_1(\theta \otimes \mathrm{id}_{\mathcal{O}_n}) = 0$. This will be proved using a similar strategy as the one employed in Proposition 3.4.2. Let $u \in A \otimes \mathcal{O}_n$ be a unitary and $1 > \epsilon > \delta > 0$. Then there exist $(a_i)_{i=1}^k \subset A$ and $(x_i)_{i=1}^k \subset \mathcal{O}_n$ such that

$$\left\| u - \sum_{i=1}^{k} a_i \otimes x_i \right\| < \frac{\delta}{6}. \tag{3.4.1}$$

Set $\mathcal{F} = \{a_1, \ldots, a_k\}$, $C = \max_{i,j}(\|x_i x_j^*\|, 1)$, and let (F, ψ, η) be an approximation witnessing nuclear dimension equal to zero for the map θ. Using that $\delta < \epsilon$ and

(3.4.1), we can ensure that

$$\| (\theta \otimes \mathrm{id}_{\mathcal{O}_n}) (u) - (\eta \otimes \mathrm{id}_{\mathcal{O}_n}) \circ (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) (u)\| < \epsilon/2. \tag{3.4.2}$$

By (*(ii)*) of Remark 3.1.4 and Lemma 3.2.1, we can further assume that η is a unital *-homomorphism and ψ is unital and approximately multiplicative on $\mathcal{F}$ up to $\frac{\delta}{3Ck^2}$. It follows that

$$\begin{aligned}(\psi \otimes \mathrm{id}_{\mathcal{O}_n}) (u) (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) (u)^* &\approx_{\delta/3} (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) \left(\sum_{i=1}^{k} a_i \otimes x_i\right) (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) \left(\sum_{i=1}^{k} a_i \otimes x_i\right)^* \\ &= \sum_{i,j=1}^{k} \psi(a_i)\psi(a_j^*) \otimes x_i x_j^* \\ &\approx_{\delta/3} \sum_{i,j=1}^{k} \psi(a_i a_j^*) \otimes x_i x_j^* \\ &= (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) \left(\sum_{i,j=1}^{k} a_i a_j^* \otimes x_i x_j^*\right) \\ &= (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) \left(\left(\sum_{i=1}^{k} a_i \otimes x_i\right)\left(\sum_{i=1}^{k} a_i \otimes x_i\right)^*\right) \\ &\approx_{\delta/3} (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) (uu^*) \\ &= 1_{F\otimes\mathcal{O}_n}. \end{aligned} \tag{3.4.3}$$

Similarly, one can show $(\psi \otimes \mathrm{id}_{\mathcal{O}_n}) (u)^* (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) (u) \approx_{\delta} 1_{F\otimes\mathcal{O}_n}$. Choosing δ to be sufficiently small, there is a unitary $v \in F \otimes \mathcal{O}_n$ such that $\| (\psi \otimes \mathrm{id}_{\mathcal{O}_n}) (u) - v\| < \epsilon/2$. Combining with (3.4.2), it follows that

$$\| (\theta \otimes \mathrm{id}_{\mathcal{O}_n}) (u) - (\eta \otimes \mathrm{id}_{\mathcal{O}_n}) (v)\| < \epsilon. \tag{3.4.4}$$

This yields that $(\theta \otimes \mathrm{id}_{\mathcal{O}_n}) (u)$ is homotopic to $(\eta \otimes \mathrm{id}_{\mathcal{O}_n}) (v)$.

On the other hand, the unitary group of any matrix amplification of $\mathcal{O}_n$ is connected ([40, Theorem 1.9]), so the unitary group of $F \otimes \mathcal{O}_n$ is also connected. Furthermore, $K_1(F \otimes \mathcal{O}_n) = 0$, so $(\eta \otimes \mathrm{id}_{\mathcal{O}_n})(v)$ is homotopic to $(\eta \otimes \mathrm{id}_{\mathcal{O}_n})(1_{F\otimes\mathcal{O}_n}) = 1_{B\otimes\mathcal{O}_n}$. By transitivity, we obtain that $(\theta \otimes \mathrm{id}_{\mathcal{O}_n})(u)$ is homotopic to $1_{B\otimes\mathcal{O}_n}$ and this implies $[(\theta \otimes \mathrm{id}_{\mathcal{O}_n})(u)]_1 = 0$. As in the proof of Proposition 3.4.2, a similar argument holds for matrix amplifications, which in turn implies $K_1(\theta; \mathbb{Z}/n) = 0$ for any $n \geq 2$. $\square$

Using (2.7.1), we can restrict the codomain of the K_0-maps with coefficients.

Proposition 3.4.4. *Let A and B be separable* C**-algebras and $\theta : A \to B$ be a *-homomorphism with $K_1(\theta) = 0$. Then the image of $K_0(\theta; \mathbb{Z}/n)$ is contained in $K_0(B) \otimes \mathbb{Z}/n$ for all $n \in \mathbb{N}$. In particular, this holds if θ is unital and* $\dim_{\mathrm{nuc}} \theta = 0$.

Proof. At the level of K_0-groups with coefficients, by (2.7.1), we obtain the following short exact sequences with the maps induced by the morphism θ making each square commute

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \otimes \mathbb{Z}/n \arrow[r] \arrow[d, "K_0(\theta)\otimes \mathrm{id}"] & K_0(A; \mathbb{Z}/n) \arrow[r] \arrow[d, "K_0(\theta; \mathbb{Z}/n)"] & \mathrm{Tor}(K_1(A), \mathbb{Z}/n) \arrow[r] \arrow[d, "K_1(\theta)"] & 0 \\
0 \arrow[r] & K_0(B) \otimes \mathbb{Z}/n \arrow[r] & K_0(B; \mathbb{Z}/n) \arrow[r, "q_B"] & \mathrm{Tor}(K_1(B), \mathbb{Z}/n) \arrow[r] & 0.
\end{tikzcd}$$

Since $K_1(\theta) = 0$, we obtain $q_B \circ K_0(\theta; \mathbb{Z}/n) = 0$. This means precisely that the image of $K_0(\theta; \mathbb{Z}/n)$ is contained into $K_0(B) \otimes \mathbb{Z}/n$. The last part follows from Proposition 3.4.2. $\square$

By combining (2.7.3) and [117, Theorem 7.2], we see that a simple, infinite dimensional AF-algebra has trivial Hausdorffised algebraic K_1-group. Therefore, any map which approximately factors through such a simple AF-algebra is trivial at the level of the Hausdorffised algebraic K_1. Hence, to show that certain *-homomorphisms with nuclear dimension equal to zero approximately factor through a simple AF-algebra, the following proposition is necessary.

Proposition 3.4.5. *Let A be a simple, separable, unital, non-elementary* C^**-algebra and let $\theta : A \to B$ be a unital* **-homomorphism with* $\dim_{\mathrm{nuc}} \theta = 0$*. Then* $\overline{K_1}^{\mathrm{alg}}(\theta) = 0$.

Proof. Since $\dim_{\mathrm{nuc}} \theta = 0$ and θ is unital, Lemma 3.4.1 produces finite dimensional C^*-algebras $(F_k)_{k\in\mathbb{N}}$ and unital *-homomorphisms $\psi : A \to \prod_\infty F_k$, $\eta : \prod_\infty F_k \to B_\infty$ such that $\iota_B \circ \theta = \eta \circ \psi$. We will prove this proposition by showing that $\overline{K_1}^{\mathrm{alg}}\Big(\prod_\infty F_k\Big) = 0$. Crucially, by simplicity of A, Corollary 3.1.8 implies that, by passing to a subsequence, we can assume that the matrix blocks of F_k have size at least k. This is the reason that underpins the fact that the Hausdorffised algebraic K_1-group of $\prod_\infty F_k$ is trivial.

It will then follow that $\overline{K_1}^{\mathrm{alg}}(\iota_B) \circ \overline{K_1}^{\mathrm{alg}}(\theta) = \overline{K_1}^{\mathrm{alg}}(\iota_B \circ \theta) = 0$. Using that $\overline{K_1}^{\mathrm{alg}}(\iota_B)$ is injective, since ι_B induces an injective morphism at the level of the invariants ([27, Lemma 3.16]), we obtain $\overline{K_1}^{\mathrm{alg}}(\theta) = 0$.

Now, let us prove the claim. Recall from the proof of Proposition 3.3.5 that $K_1\Big(\prod_\infty F_k\Big) = 0$; and notice that by equation (2.7.3), the claim will follow after checking that the image of the pairing map is uniformly dense in $\mathrm{Aff}\Big(T\Big(\prod_\infty F_k\Big)\Big)$.

In order to do this, we will follow the strategy employed in the proof of [117, Theorem 7.2]. For convenience, let us denote $\prod_\infty F_k$ by E. Observe that $\overline{\rho_E(K_0(E))}$ is closed and separates points since E has real rank zero (Lemma 3.1.1). Moreover, it contains constant functions as E is unital.

After checking that $\overline{\rho_E(K_0(E))}$ is a subspace of $\mathrm{Aff}(T(E))$, we will reach the conclusion by applying a version of Kadison's Representation Theorem given in [10, Proposition 3.12 (b)]. Let us pick a nonzero positive element $[x]_0$ in $K_0(E)$. Since a matrix amplification of a sequence algebra is the sequence algebra of matrix amplifications, x is induced by a sequence of projections $p_k \in M_N(F_k)$ for some $N \in \mathbb{N}$. Then, if F_k has a decomposition into r matrix blocks with sizes $n_1^{(k)}, \dots, n_r^{(k)}$, we can write p_k as a sum of orthogonal projections $p_k^{(1)} \oplus \ldots \oplus p_k^{(r)}$.[16]

Let $n \in \mathbb{N}$ and let us denote by $\mathrm{tr}_{Nn_i^{(k)}}$ the unique normalised trace on $M_{Nn_i^{(k)}}$.

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[16] Note that r depends on k, but we suppress this for ease of notation.

Thus

$$\mathrm{tr}_{Nn_i^{(k)}}(p_k^{(i)}) = \frac{t_i}{Nn_i^{(k)}}$$

for some $t_i \in \mathbb{N}$. If $\lfloor t_i/n \rfloor < n$, we set $q_k^{(i)} := p_k^{(i)}$; and if $\lfloor t_i/n \rfloor \geq n$, we set $q_k^{(i)} \in M_{Nn_i^{(k)}}$ as the diagonal matrix with the first $\lfloor t_i/n \rfloor$ entries equal to 1 and the rest equal to 0. Then, consider $y_n := (q_k)_{k\in\mathbb{N}}$, where $q_k := q_k^{(1)} \oplus \ldots \oplus q_k^{(r)}$. Since the sizes of the blocks go to infinity and n is fixed, note that the projections $p_k^{(i)}$ will be zero on any limit trace if $\lfloor t_i/n \rfloor < n$. Thus, by construction, we obtain that

$$n\tau(y_n) \leq \tau(x) \leq (n+1)\tau(y_n) \tag{3.4.5}$$

for all limit traces τ on E.

Finally, the convex hull of limit traces on E is weak*-dense in $T(E)$ (by essentially the same proof of [98, Theorem 1.2]), which yields that for all $n \in \mathbb{N}$ there exists some nonzero positive $[y_n]_0 \in K_0(E)$ such that

$$n\tau(y_n) \leq \tau(x) \leq (n+1)\tau(y_n) \tag{3.4.6}$$

for all traces τ on E.

We claim that $\overline{\rho_E(K_0(E))}$ is a subspace of $\mathrm{Aff}(T(E))$. Note that by linearity and density of rationals in the reals, it suffices to show that for any $[x]_0 \in K_0(E)_+$ and any $k \in \mathbb{N}$, $\frac{1}{k}\rho_E([x]_0) \in \overline{\rho_E(K_0(E))}$. Let $\epsilon > 0, k \in \mathbb{N}$, and $n \in \mathbb{N}$ be such that

$$\frac{1}{nk}\tau(x) < \epsilon \tag{3.4.7}$$

for any $\tau \in T(E)$. Then, by (3.4.6), there exists $[y]_0 \in K_0(E)_+$ such that

$$nk\tau(y) \leq \tau(x) \leq (nk+1)\tau(y) \tag{3.4.8}$$

for all $\tau \in T(E)$. By rearranging the inequality in (3.4.8) and dividing by k, we obtain

$$0 \leq \frac{1}{k}\tau(x) - n\tau(y) \leq \frac{1}{k}\tau(y) \tag{3.4.9}$$

for any $\tau \in T(E)$. By the left-hand side of (3.4.8), we have

$$\tau(y) \leq \frac{1}{nk}\tau(x). \tag{3.4.10}$$

Thus, we obtain

$$0 \leq \frac{1}{k}\tau(x) - n\tau(y) \leq \frac{1}{k}\tau(y) \overset{(3.4.10)}{<} \frac{1}{nk^2}\tau(x) \overset{(3.4.7)}{<} \epsilon \tag{3.4.11}$$

for all $\tau \in T(E)$. Hence, we have shown that for all $[x]_0 \in K_0(E)_+$ and $\epsilon > 0$ there exist $n \in \mathbb{N}$ and $[y]_0 \in K_0(E)_+$ such that

$$0 \leq \frac{1}{k}\rho_E([x]_0) - n\rho_E([y]_0) < \epsilon.$$

Since ϵ was arbitrary, this yields that $\frac{1}{k}\rho_E([x]_0) \in \overline{\rho_E(K_0(E))}$. Therefore $\overline{\rho_E(K_0(E))}$ is a subspace of $\mathrm{Aff}(T(E))$ and we reach the conclusion by [10, Proposition 3.12 (b)]. □

Next, we show that the induced map in K_0 of a *-homomorphism with nuclear dimension equal to zero satisfies a certain weak divisibility condition.

Definition 3.4.6 ([104, Definition 5.1])**.** Let A be a C*-algebra and p be a nonzero projection in A. It is said that A is *weakly divisible of degree* n at p if there is a unital *-homomorphism

$$\phi : M_{n_1} \oplus M_{n_2} \oplus \ldots \oplus M_{n_r} \to pAp,$$

for some natural numbers r, and $n_1, n_2, \ldots, n_r$ where $n_j \geq n$ for all j.

This condition is equivalent to a notion of divisibility in the Murray-von Neumann semigroup $V(A)$ of A. Precisely, A is weakly divisible of degree n at p if and only if there are natural numbers $r, n_1, \dots, n_r$ and $x_1, \dots, x_r \in V(A)$ such that $n_j \geq n$ for all j and $[p]_0 = n_1 x_1 + \ldots + n_r x_r$.

Corollary 3.4.7. *Let A and B be unital, separable* C**-algebras with A simple and non-elementary. If $\theta : A \to B$ is a unital *-homomorphism with $\dim_{\mathrm{nuc}} \theta = 0$, then B is weakly divisible of degree n at 1_B for all $n \in \mathbb{N}$.*

Proof. This follows from Lemma 3.2.1 and Corollary 3.1.8. □

Remark 3.4.8. In fact, one can obtain a more explicit condition. Let $n \in \mathbb{N}$ and suppose that B is weakly divisible of degree m at 1_B for all $m \in \mathbb{N}$. Note that for all $m \geq n(n+2)$, there exist $r, s \in \mathbb{N}$ such that $m = rn + s(n+1) = n(r+s) + s$. Indeed, if m is 0 modulo n, let $s = n$ and since $m \geq n(n+2)$, we can take $r \geq 1$ such that $m = n(r+n+1)$. Else, let s be the nonzero remainder when we divide m by n and this automatically determines r. If B has cancellation of projections (for example if B has stable rank one), then for all $n \in \mathbb{N}$, there exist $g^{(n)}, h^{(n)} \in K_0(B)_+$ such that $[1_B]_0 = ng^{(n)} + (n+1)h^{(n)}$. Then, there exists a unital *-homomorphism from $M_n \oplus M_{n+1}$ to B by [115, Lemma 1.3.1].

If B is classifiable, the condition described in Remark 3.4.8 is connected with the property of having real rank zero. Any simple, separable, unital, finite, $\mathcal{Z}$-stable C*-algebra with real rank zero is weakly divisible of degree n at 1 for all $n \in \mathbb{N}$ ([117, Theorem 7.2]). The converse holds in the unique trace case ([117, Corollary 7.3]), but it is not true in general. For instance, one can consider a classifiable C*-algebra A with $K_0(A) = \mathbb{Q}$, $K_1(A) = 0$, $[1_A]_0 = 1$, and 2 extreme traces. Since there is only one state on $K_0(A)$, projections do not separate traces, so A cannot have real rank zero.

The next proposition is the key technical step in the proof of Theorem 3.5.1. Under certain hypotheses, a *-homomorphism which factors through a sequence algebra of finite dimensional C*-algebras, resembles a map which factors through a simple AF-algebra at the level of total K-theory and traces. This proposition shows precisely how to obtain such an AF-algebra by finding a suitable dimension group inside the K_0-group of the given sequence algebra.

Proposition 3.4.9. *Let A be a simple, separable, unital, exact, non-elementary, and stably finite* C**-algebra and let* $(F_k)_{k\in\mathbb{N}}$ *be a sequence of finite dimensional* C**-algebras. Suppose* $\psi : A \to \prod_\infty F_k$ *is a unital* **-homomorphism. Then there exist a simple, unital, AF-algebra C with no finite dimensional representations, together with Λ-morphisms*

$$\underline{\alpha} : \underline{K}(A) \to \underline{K}(C), \qquad \underline{\beta} : \underline{K}(C) \to \underline{K}\left(\prod_\infty F_k\right)$$

and affine maps

$$\gamma_0 : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(C)), \qquad \gamma_1 : \mathrm{Aff}(T(C)) \to \mathrm{Aff}\left(T\Big(\prod_\infty F_k\Big)\right)$$

such that

$$\underline{K}(\psi) = \underline{\beta} \circ \underline{\alpha}, \qquad \text{and} \qquad \mathrm{Aff}(T(\psi)) = \gamma_1 \circ \gamma_0, \tag{3.4.12}$$

and the following diagram commutes

$$\begin{tikzcd}
K_0(A) \arrow[r, "\alpha_0"] \arrow[d, "\rho_A"] & K_0(C) \arrow[r, "\beta_0"] \arrow[d, "\rho_C"] & K_0\Big(\prod_\infty F_k\Big) \arrow[d, "\rho_{\prod_\infty F_k}"] \\
\mathrm{Aff}(T(A)) \arrow[r, "\gamma_0"] & \mathrm{Aff}(T(C)) \arrow[r, "\gamma_1"] & \mathrm{Aff}\Big(T\Big(\prod_\infty F_k\Big)\Big).
\end{tikzcd} \tag{3.4.13}$$

Proof. By Proposition 3.3.5, there exists a unital, separable C*-algebra E such that $\psi(A) \subset E \subset \prod_\infty F_k$ and $K_0(E)$ is a dimension group with Property (D). Let $K_0(E)_{++}$ denote the set of order units of $K_0(E)$. Since E contains the unit, it follows that

$K_0(E)_{++}$ is nonempty. Then $(K_0(E), K_0(E)_{++} \cup \{0\})$ is a simple dimension group by [53, Proposition 10]. By Proposition 2.2.3, there is a simple, unital AF-algebra C with $(K_0(C), K_0(C)_+) = (K_0(E), K_0(E)_{++} \cup \{0\})$. We will freely identify these groups. Moreover, since $K_0(C)$ has property (D), C has no finite dimensional representations by [53, Lemma 9]. To define suitable K-theory maps we will essentially use that, apart from the order in K_0, the C*-algebras E and C have the same K-groups and K-groups with coefficients.

If $\psi_0 : A \to E$ is the corestriction of ψ, define $\alpha_0 : K_0(A) \to K_0(C)$ by $\alpha_0 := K_0(\psi_0)$. As $K_0(A)$ is a simple ordered group ([6, Corollary 6.3.6]), all nonzero positive elements in $K_0(A)$ are order units, so α_0 is a well-defined ordered group homomorphism. Since all the K_1-groups with coefficients of C vanish, we define $\alpha_1 : K_1(A) \to K_1(C)$ and $\alpha_1^{(n)} : K_1(A; \mathbb{Z}/n) \to K_1(C; \mathbb{Z}/n)$ to be the corresponding zero map. It remains to define $\alpha_0^{(n)} : K_0(A; \mathbb{Z}/n) \to K_0(C; \mathbb{Z}/n)$. For this, note that $K_0(C; \mathbb{Z}/n) = K_0(C) \otimes \mathbb{Z}/n = K_0(E) \otimes \mathbb{Z}/n$. Furthermore, $K_1(E) = 0$ by Proposition 3.3.5, so $K_0(E; \mathbb{Z}/n) = K_0(E) \otimes \mathbb{Z}/n$ by (2.7.1), which in turn implies that $K_0(C; \mathbb{Z}/n) = K_0(E; \mathbb{Z}/n)$. Thus, we can set $\alpha_0^{(n)} := K_0(\psi_0; \mathbb{Z}/n)$. Note that the absence of torsion in the K_0-groups yields that $K_1(C; \mathbb{Z}/n) = K_1(E; \mathbb{Z}/n)$ for all $n \geq 2$. Finally, $\underline{\alpha}$ is a well-defined Λ-morphism since $\underline{K}(\psi_0)$ is a Λ-morphism.

There is an inclusion $\iota : E \to \prod_{\infty} F_k$ such that $\iota \circ \psi_0 = \psi$. Define $\beta_1 := 0$ and $\beta_1^{(n)} := 0$. Similarly, identifying $K_0(C)$ with $K_0(E)$ and $K_0(C; \mathbb{Z}/n)$ with $K_0(E; \mathbb{Z}/n)$, let us define $\beta_0 := K_0(\iota)$ and $\beta_0^{(n)} := K_0(\iota; \mathbb{Z}/n)$. Likewise, $\underline{\beta}$ is a Λ-morphism because $\underline{K}(\iota)$ is a Λ-morphism. Furthermore, by construction, it follows that $\underline{\beta} \circ \underline{\alpha} = \underline{K}(\psi)$.

For the second part, by Proposition 2.2.3 and [53, Proposition 11], it follows that the space of states on $(K_0(E), K_0(E)_+)$ is affinely homeomorphic to the space of states on $(K_0(C), K_0(C)_+)$. Since E is stably finite, unital, and with real rank zero, the space of states on $K_0(E)$ is affinely homeomorphic to $QT(E)$, the space of quasitraces on E by [6, Theorem 6.9.1]. As A is exact, quasitraces on A are traces

[72]. Moreover, since C is an AF-algebra, states on $K_0(C)$ are traces on C. With these identifications, consider $\tilde{\gamma}_0$ to be given by the following sequence of maps

$$\tilde{\gamma}_0 : T(C) \xrightarrow{\cong} S(K_0(E)) \xrightarrow{\cong} QT(E) \xrightarrow{(\psi_0)^*} QT(A) = T(A).$$

Let γ_0 be the dual map of $\tilde{\gamma}_0$.

Set $\tilde{\gamma}_1$ to be the following sequence of maps

$$\tilde{\gamma}_1 : T\Big(\prod_\infty F_k\Big) \xrightarrow{i} QT\Big(\prod_\infty F_k\Big) \xrightarrow{\sigma_0} QT(E) \xrightarrow{\cong} T(C),$$

where i is the canonical inclusion and σ_0 is the affine map given by restriction. Likewise, let γ_1 be the dual map of $\tilde{\gamma}_1$. By construction, it follows that $\mathrm{Aff}(T(\psi)) = \gamma_1 \circ \gamma_0$.

Finally, note that the diagram in (3.4.13) commutes by construction. Precisely, $\alpha_0 = K_0(\psi_0)$, $T(C) \cong QT(E)$, and $QT(A) = T(A)$ give that the left square commute. Similarly, by using the canonical map $\mathrm{Aff}\Big(QT\Big(\prod_\infty F_k\Big)\Big) \to \mathrm{Aff}\Big(T\Big(\prod_\infty F_k\Big)\Big)$, the right square commutes as $\beta_0 = K_0(\iota)$ and $T(C) \cong QT(E)$. □

3.5 Approximate factorings through simple AF-algebras

We now proceed to prove the first main theorem of this chapter which states that a classifiable, zero-dimensional *-homomorphism factors through a simple AF-algebra when viewed as a map into the sequence algebra of its codomain. Let us recall the statement of the theorem.

Theorem 3.5.1. *Let A and B be simple, separable, unital, and nuclear* C**-algebras such that A is stably finite and satisfies the UCT and B is $\mathcal{Z}$-stable. If $\theta : A \to B$ is a unital *-homomorphism and $\iota_B : B \to B_\infty$ is the canonical diagonal inclusion, then* *the following are equivalent:*

(i) there exist a simple, unital AF-algebra C and unital *-homomorphisms $\psi : A \to C$ and $\varphi : C \to B_\infty$ such that $\iota_B \circ \theta = \varphi \circ \psi$;

(ii) θ has nuclear dimension equal to zero.

Proof. If A is finite dimensional then the statement is vacuously true, so let us suppose A is infinite dimensional. Observe then that simplicity and unitality imply that A has no finite dimensional representations. First, note that if we assume that $\iota_B \circ \theta$ factors through a simple, separable AF-algebra, then $\iota_B \circ \theta$ has nuclear dimension equal to 0 and hence θ also has nuclear dimension equal to 0 by [136, Proposition 2.5].

Conversely, let us assume θ has nuclear dimension equal to zero. By Lemma 3.4.1, there are finite dimensional C*-algebras $(F_k)_{k\in\mathbb{N}}$ and unital *-homomorphisms $\sigma : A \to \prod_\infty F_k$, $\eta : \prod_\infty F_k \to B_\infty$ such that $\iota_B \circ \theta = \eta \circ \sigma$. By Proposition 3.4.9, there exist a simple, unital AF-algebra C with no finite dimensional representations, Λ-morphisms

$$\underline{\alpha} : \underline{K}(A) \to \underline{K}(C), \qquad \underline{\beta} : \underline{K}(C) \to \underline{K}\left(\prod_\infty F_k\right)$$

and affine maps

$$\gamma_0 : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(C)), \qquad \gamma_1 : \mathrm{Aff}(T(C)) \to \mathrm{Aff}\left(T\Big(\prod_\infty F_k\Big)\right)$$

such that

$$\underline{K}(\sigma) = \underline{\beta} \circ \underline{\alpha}, \qquad \text{and} \qquad \mathrm{Aff}(T(\sigma)) = \gamma_1 \circ \gamma_0, \tag{3.5.1}$$

and the diagram (3.4.13) commutes. Additionally, since C is a simple, unital, infinite dimensional AF-algebra then $\overline{K_1}^{\mathrm{alg}}(C) = 0$ by combining [117, Theorem 7.2] and (2.7.3).

Since the diagram (3.4.13) commutes and $\overline{K_1}^{\mathrm{alg}}(C) = 0$, then the triple $(\underline{\alpha}, 0, \gamma_0)$ makes the diagram (2.7.6) commutative. By Lemma 2.7.4, it follows that $(\underline{\alpha}, 0, \gamma_0) : \underline{K}T_u(A) \to \underline{K}T_u(C)$ is a $\underline{K}T_u$-morphism, which is automatically faithful by the simplicity of A. Then, by Theorem 2.7.2 there exists a full, unital *-homomorphism $\psi : A \to C$ realising $(\underline{\alpha}, 0, \gamma_0)$. On the other hand, $(\underline{\beta}, 0, \gamma_1) : \underline{K}T_u(C) \to \underline{K}T_u\Big(\prod_{\infty} F_k\Big)$ is a $\underline{K}T_u$-morphism. Then, $\underline{K}T_u(\eta) \circ (\underline{\beta}, 0, \gamma_1) : \underline{K}T_u(C) \to \underline{K}T_u(B_\infty)$ is a $\underline{K}T_u$-morphism, which is faithful by the simplicity of C. Therefore, it induces a full, unital *-homomorphism $\varphi : C \to B_\infty$ by Theorem 2.7.3.

By construction and (3.5.1), we obtain

$$\begin{aligned}\underline{K}(\varphi \circ \psi) &= \underline{K}(\varphi) \circ \underline{K}(\psi) = \underline{K}(\eta) \circ \underline{\beta} \circ \underline{K}(\psi) \\ &= \underline{K}(\eta) \circ \underline{\beta} \circ \underline{\alpha} = \underline{K}(\eta) \circ \underline{K}(\sigma) = \underline{K}(\eta \circ \sigma) = \underline{K}(\iota_B \circ \theta). \qquad (3.5.2)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Aff}(T(\varphi \circ \psi)) &= \mathrm{Aff}(T(\varphi)) \circ \mathrm{Aff}(T(\psi)) = \mathrm{Aff}(T(\eta)) \circ \gamma_1 \circ \gamma_0 \\ &= \mathrm{Aff}(T(\eta)) \circ \mathrm{Aff}(T(\sigma)) = \mathrm{Aff}(T(\iota_B \circ \theta)). \qquad (3.5.3)\end{aligned}$$

Using again that $\overline{K_1}^{\mathrm{alg}}(C) = 0$, we obtain that $\overline{K_1}^{\mathrm{alg}}(\varphi \circ \psi) = 0$. On the other hand, Proposition 3.4.2 yields that $\overline{K_1}^{\mathrm{alg}}(\iota_B \circ \theta) = 0$. In summary,

$$\underline{K}T_u(\varphi \circ \psi) = \underline{K}T_u(\iota_B \circ \theta).$$

Therefore, $\iota_B \circ \theta$ and $\varphi \circ \psi$ are unitarily equivalent by Theorem 2.7.3. Conjugating φ by a unitary if necessary, we can assume that $\iota_B \circ \theta = \varphi \circ \psi$ and the conclusion follows. □

In general, the AF-algebra C identified above can be embedded into B_∞, but it is unclear if one can obtain such an embedding into B. It is not clear how to find a dimension group which is factoring the K_0-map in general. However, this is not an issue if we assume that the domain or codomain has real rank zero. We will prove Theorem 3, where the codomain has real rank zero, and then we will show that a similar result holds if the domain has real rank zero instead. Let us first record a result which is well-known to experts. Essentially, if B is a C*-algebra as in the statement of Theorem 3, then its K_0-group and trace space look like those of a simple, infinite dimensional AF-algebra.

Lemma 3.5.2. *Let B be a simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C**-algebra which has real rank zero, $T(B) \neq \emptyset$ and $K_0(B)$ is torsion free. Then there exists a simple, unital, infinite dimensional AF-algebra C such that $K_0(C) \cong K_0(B)$ as ordered abelian groups and $T(C) \cong T(B)$.*

Proof. We first claim that $K_0(B)$ is a simple dimension group. Note that $K_0(B)$ is a simple ordered abelian group ([6, Corollary 6.3.6]). Suppose $n[p]_0 \geq 0$ for some $n \in \mathbb{N}$ and $[p]_0 \in K_0(B)_+$. Since $K_0(B)$ is a simple ordered abelian group, either $n[p]_0 = 0$ or $n[p]_0$ is an order unit. In the former case, $[p]_0 = 0$ since $K_0(B)$ is torsion free, and in the latter, by compactness of $T(B)$ we see that there is some $\alpha > 0$ such that $\tau(p^{\oplus n}) > \alpha$ for all $\tau \in T(B)$. Therefore, $\tau(p) > 0$ for all $\tau \in T(B)$, which gives that $p \geq 0$ since B has strict comparison of projections with respect to tracial states ([117, Corollary 4.6]). Thus, $K_0(B)$ is unperforated.

Finally, note that since B has real rank zero, the projections in B satisfy the Riesz decomposition with respect to the Murray-von Neumann equivalence ([149, Corollary 1.3]). As B has stable rank one, B has cancellation ([6, Corollary 6.5.1]), and hence $K_0(B)$ satisfies the Riesz decomposition property. Since the Riesz decomposition property is equivalent to the Riesz interpolation property by [70, Proposition 2.1], it follows that $K_0(B)$ is a dimension group ([115, Theorem 1.4.4]).

Then, by Proposition 2.2.3, there is a simple, unital, infinite dimensional AF-algebra C with

$$(K_0(C), K_0(C)_+, [1_C]_0) \cong (K_0(B), K_0(B)_+, [1_B]_0).$$

As B is unital, nuclear with real rank zero and stable rank one, traces on B are homeomorphic to states on $K_0(B)$ ([6, Theorem 6.9.1]). Likewise, traces on C are homeomorphic to states on $K_0(C)$, so $T(B) \cong T(C)$. □

We recall the statement of Theorem 3.

Theorem 3.5.3. *Let A and B be simple, separable, unital, and nuclear* C**-algebras such that A is stably finite and satisfies the UCT and B is $\mathcal{Z}$-stable, of real rank zero, with $T(B) \neq \emptyset$, and $K_0(B)$ is torsion free. Let $\theta : A \to B$ be a unital *-homomorphism. Then the following are equivalent:*

(i) there exist a simple, unital AF-algebra C and unital *-homomorphisms $\psi : A \to C$ and $\varphi : C \to B$ such that θ is approximately unitarily equivalent to $\varphi \circ \psi$;

(ii) θ has nuclear dimension equal to zero;

(iii) θ vanishes at the level of K_1 and K_1 with coefficients i.e. $K_1(\theta) = 0$ and $K_1(\theta; \mathbb{Z}/n) = 0$ for all $n \in \mathbb{N}$.

Proof. If θ is approximately unitarily equivalent to a *-homomorphism which factors through a simple AF-algebra, then θ has nuclear dimension equal to zero. On the other hand, if θ has nuclear dimension equal to zero, then $K_1(\theta) = 0$ and $K_1(\theta; \mathbb{Z}/n) = 0$ for all $n \in \mathbb{N}$ by Proposition 3.4.2 and Proposition 3.4.3, respectively. This shows that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

We will prove that (iii) $\Longrightarrow$ (i). Let us suppose that the K_1-maps with coefficients induced by θ are all trivial. We claim that θ is approximately unitarily equivalent to a *-homomorphism which factors through a simple AF-algebra. By Lemma 3.5.2, there is a simple, unital, infinite dimensional AF-algebra C with

$$(K_0(C), K_0(C)_+, [1_C]_0) \cong (K_0(B), K_0(B)_+, [1_B]_0)$$

and $T(C) \cong T(B)$. In order to simplify the proof, we will freely identify these two ordered dimension groups. We claim that, up to approximate unitary equivalence, θ factors through C.

In order to show this, we are going to use Theorem 2.7.2 to construct unital *-homomorphisms from A to C and from C to B, respectively. We start by obtaining a $\underline{K}T_u$-morphism between the $\underline{K}T_u$-invariants of A and C. Let $\alpha_0 : K_0(A) \to K_0(C)$ be given by $\alpha_0 := K_0(\theta)$. Since $K_1(C) = 0$, we will let $\alpha_1 := 0$. At the level of the K-groups with coefficients, we recall the short exact sequences in (2.7.1)

$$0 \longrightarrow K_i(C) \otimes \mathbb{Z}/n \longrightarrow K_i(C; \mathbb{Z}/n) \longrightarrow \operatorname{Tor}(K_{1-i}(C), \mathbb{Z}/n) \longrightarrow 0.$$

Since $K_0(C)$ is torsion free and $K_1(C) = 0$, it follows that $K_1(C; \mathbb{Z}/n) = 0$, which in turn implies that we must take $\alpha_1^{(n)} := 0$ for all $n \in \mathbb{N}$.

Using the short exact sequence from above for $i = 0$ and that $K_1(C) = 0$, it follows that

$$K_0(C; \mathbb{Z}/n) \cong K_0(C) \otimes \mathbb{Z}/n \cong K_0(B) \otimes \mathbb{Z}/n.$$

By Proposition 3.4.4, the image of $K_0(\theta; \mathbb{Z}/n)$ is contained in $K_0(B) \otimes \mathbb{Z}/n$, and after identifying $K_0(C, \mathbb{Z}/n)$ with $K_0(B) \otimes \mathbb{Z}/n$, we can define $\alpha_0^{(n)} : K_0(A; \mathbb{Z}/n) \to K_0(C; \mathbb{Z}/n)$ as $\alpha_0^{(n)} := K_0(\theta; \mathbb{Z}/n)$. We denote by $\underline{\alpha}$ the collection of all the morphisms $\alpha_i, \alpha_i^{(n)}$ defined above. Since $\underline{\alpha}$ coincides with $\underline{K}(\theta)$, it follows that $\underline{\alpha}$ is a compatible Λ-morphism at the level of total K-theory.

Recall from (2.7.3) that

$$\overline{K_1}^{\mathrm{alg}}(C) \cong \operatorname{Aff}(T(C))/\overline{\operatorname{im} \rho_C} \oplus K_1(C).$$

Since C is a simple, unital, infinite dimensional AF-algebra, it follows that $\overline{K_1}^{\mathrm{alg}}(C) = 0$ (again by combining (2.7.3) and [117, Theorem 7.2]). Thus, the only group homomorphism from $\overline{K_1}^{\mathrm{alg}}(A)$ to $\overline{K_1}^{\mathrm{alg}}(C)$ is the zero morphism.

Recall that $T(C) \cong S(K_0(C)) \cong S(K_0(B)) \cong T(B)$ (again, we will freely identify these simplices). Hence, we can define $\gamma : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(C))$ as $\gamma := \mathrm{Aff}(T(\theta))$. It follows that the diagram (2.7.6) commutes, and by Lemma 2.7.4, $(\underline{\alpha}, 0, \gamma)$ defines a $\underline{K}T_u$-morphism from $\underline{K}T_u(A)$ to $\underline{K}T_u(C)$. Thus, there is a full, unital *-homomorphism $\psi : A \to C$ with associated $\underline{K}T_u$-invariant equal to $(\underline{\alpha}, 0, \gamma)$ by Theorem 2.7.2. Set $\beta_0 := \mathrm{id}_{K_0(B)}$, $\beta_1 := 0$, $\beta_1^{(n)} := 0$, and $\beta_0^{(n)} := \mathrm{id}_{K_0(B)\otimes\mathbb{Z}/n}$ for all $n \geq 2$. By Lemma 2.7.4, $(\underline{\beta}, 0, \mathrm{id}_{\mathrm{Aff}(T(B))}) : \underline{K}T_u(C) \to \underline{K}T_u(B)$ gives rise to a full, unital *-homomorphism $\varphi : C \to B$. We will check now that θ and $\varphi \circ \psi$ are indeed approximately unitarily equivalent. It is clear by construction that

$$\begin{aligned}\mathrm{Aff}(T(\varphi\circ\psi)) &= \mathrm{Aff}(T(\varphi)) \circ \mathrm{Aff}(T(\theta))\\ &= \mathrm{id}_{\mathrm{Aff}(T(B))} \circ \mathrm{Aff}(T(\theta))\\ &= \mathrm{Aff}(T(\theta)).\end{aligned}$$

For the K_0-groups (with coefficients), we note that

$$K_0(\varphi\circ\psi) = K_0(\varphi)\circ K_0(\psi) = \mathrm{id}_{K_0(B)} \circ K_0(\theta) = K_0(\theta)$$

and $K_0(\varphi\circ\psi; \mathbb{Z}/n) = K_0(\theta; \mathbb{Z}/n)$ for any $n \in \mathbb{N}$. At the level of K_1-groups (with coefficients), since $K_1(C; \mathbb{Z}/n) = 0$ and $K_1(C) = 0$ it follows that $K_1(\varphi\circ\psi; \mathbb{Z}/n) = 0$ and $K_1(\varphi\circ\psi) = 0$, respectively. Moreover, by assumption $K_1(\theta; \mathbb{Z}/n) = 0$ and $K_1(\theta) = 0$.

We have then shown that $\underline{K}(\theta) = \underline{K}(\varphi\circ\psi)$ and $\mathrm{Aff}(T(\theta)) = \mathrm{Aff}(T(\varphi\circ\psi))$. As B has real rank zero, Lemma 2.7.4 yields that $\theta \sim_{\mathrm{au}} \varphi\circ\psi$, so the conclusion follows. □

A similar characterisation can be obtained when the domain is classifiable, stably finite, and with real rank zero. In the absence of torsion in the K_0-group, we can find a simple, infinite dimensional AF-algebra with the same K_0-group and trace space as the domain of the given morphism.

Theorem 3.5.4. *Let A and B be simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C*-*algebras such that A is stably finite, has real rank zero, satisfies the UCT, and $K_i(A)$ is torsion free for $i = 0, 1$. Let $\theta : A \to B$ be a unital *-homomorphism. Then the following are equivalent:*

- *(i) there exist a simple, unital AF-algebra C and *-homomorphisms $\psi : A \to C$ and $\varphi : C \to B$ such that θ is approximately unitarily equivalent to $\varphi \circ \psi$;*
- *(ii) θ has nuclear dimension equal to zero;*
- *(iii) $K_1(\theta; \mathbb{Z}/n) = 0$ for all $n \in \mathbb{N}$ and $\overline{K_1}^{\mathrm{alg}}(\theta) = 0$.*

Proof. The fact that (i) implies (ii) is clear and by combining Propositions 3.4.3 and 3.4.5, we obtain that (ii) implies (iii). It only remains to prove that (iii) implies (i). Here is where the extra real rank zero hypothesis comes into play.

Suppose that $K_1(\theta; \mathbb{Z}/n) = 0$ for all $n \in \mathbb{N}$ and $\overline{K_1}^{\mathrm{alg}}(\theta) = 0$. We claim that θ is approximately unitarily equivalent to a *-homomorphism which factors through a simple, unital AF-algebra. By Lemma 3.5.2, there exists a simple, unital, infinite dimensional AF-algebra C such that $K_0(C) \cong K_0(A)$ as ordered abelian groups and $T(A) \cong T(C)$ as Choquet simplices. We will freely identify these objects.

First we will define a Λ-morphism from $\underline{K}(A)$ to $\underline{K}(C)$. We can take $\alpha_0 : K_0(A) \to K_0(C)$ as $\alpha_0 := \mathrm{id}_{K_0(C)}$ and $\alpha_1 := 0$. Since $K_1(A)$ and $K_1(C)$ are torsion free, (2.7.1) yields that

$$K_0(A; \mathbb{Z}/n) \cong K_0(A) \otimes \mathbb{Z}/n \text{ and } K_0(C; \mathbb{Z}/n) \cong K_0(C) \otimes \mathbb{Z}/n. \tag{3.5.4}$$

Similarly, since $K_1(C) = 0$ and $K_0(C)$ is torsion free, (2.7.1) yields that $K_1(C;\mathbb{Z}/n) = 0$. We can then take $\alpha_0^{(n)} : K_0(A;\mathbb{Z}/n) \to K_0(C;\mathbb{Z}/n)$ to be the identity map and $\alpha_1^{(n)} : K_1(A;\mathbb{Z}/n) \to K_1(C;\mathbb{Z}/n)$ as the zero map.

Essentially by definition, these maps intertwine the Bockstein maps and hence the collection of these maps defines a Λ-morphism $\underline{\alpha} : \underline{K}(A) \to \underline{K}(C)$. By taking $\gamma_0 := \mathrm{id}_{\mathrm{Aff}(T(C))}$, it is immediate that the corresponding diagram (2.7.6) commutes. By Lemma 2.7.4, we obtain a full, unital *-homomorphism $\psi : A \to C$ such that $\underline{K}(\psi) = \underline{\alpha}$ and $\mathrm{Aff}(T(\psi)) = \mathrm{id}_{\mathrm{Aff}(T(C))}$.

We now proceed to define a faithful morphism from the total invariant of C to the total invariant of B. Since $K_0(C) \cong K_0(A)$ as ordered abelian groups and $K_1(C) = 0$, we can set $\beta_i : K_i(C) \to K_i(B)$ as $\beta_0 := K_0(\theta)$ and $\beta_1 := 0$. Recall we have already observed that $K_0(A;\mathbb{Z}/n) \cong K_0(C;\mathbb{Z}/n)$ and $K_1(C;\mathbb{Z}/n) = 0$. Then, we can define $\beta_i^{(n)} : K_i(C;\mathbb{Z}/n) \to K_i(B;\mathbb{Z})$ by

$$\beta_0^{(n)} = K_0(\theta, \mathbb{Z}/n) \qquad \text{and} \qquad \beta_1^{(n)} = 0. \tag{3.5.5}$$

Note that $K_1(\theta) = 0$ and $K_1(\theta;\mathbb{Z}/n) = 0$ for all $n \in \mathbb{N}$ by assumption. Therefore, the collection of maps $\underline{\beta}$ can be identified with the collection of maps $\underline{K}(\theta)$. Hence, these maps define a Λ-morphism $\underline{\beta} : \underline{K}(C) \to \underline{K}(B)$.

On the other hand, since C is a simple, unital, infinite dimensional AF-algebra, it follows by (2.7.3) and [117, Theorem 7.2] that $\overline{K_1}^{\mathrm{alg}}(C) = 0$. Hence, we can take the map $\overline{K_1}^{\mathrm{alg}}(C) \to \overline{K_1}^{\mathrm{alg}}(B)$ to be the zero map. Likewise, as $T(C) \cong T(A)$, we can set $\gamma_1 := \mathrm{Aff}(T(\theta)) : \mathrm{Aff}(T(C)) \to \mathrm{Aff}(T(B))$. It follows that the corresponding diagram (2.7.6) commutes and the diagram (2.7.4) commutes automatically since $K_1(C) = 0$ ([27, Proposition 3.12]). Then, Theorem 2.7.2 yields a full, unital *-homomorphism $\varphi : C \to B$ such that $\underline{K}(\varphi) = \underline{\beta}$, $\overline{K_1}^{\mathrm{alg}}(\varphi) = 0$ and $\mathrm{Aff}(T(\varphi)) = \mathrm{Aff}(T(\theta))$.

We will complete the proof by showing that θ and $\varphi \circ \psi$ are approximately unitarily equivalent. By hypothesis, $\overline{K_1}^{\mathrm{alg}}(\theta) = 0$. Also, $\overline{K_1}^{\mathrm{alg}}(\varphi \circ \psi) = \overline{K_1}^{\mathrm{alg}}(\varphi) \circ \overline{K_1}^{\mathrm{alg}}(\psi) = 0$. Straightforward calculations yield that

$$\underline{K}(\varphi \circ \psi) = \underline{\beta} \circ \underline{\alpha} = \underline{K}(\theta) \tag{3.5.6}$$

and that

$$\mathrm{Aff}(T(\varphi \circ \psi)) = \mathrm{Aff}(T(\theta)) \circ \mathrm{id}_{\mathrm{Aff}(T(A))} = \mathrm{Aff}(T(\theta)). \tag{3.5.7}$$

Therefore, by Theorem 2.7.2, we conclude that $\theta \sim_{\mathrm{au}} \varphi \circ \psi$. $\square$

Remark 3.5.5. Unlike in Theorem 3.5.3, the proof above depends on the assumption that $K_1(A)$ is torsion free. The reason is that unless $K_1(A)$ is torsion free, $K_0(\theta; \mathbb{Z}/n)$ can potentially hit elements outside the image of $K_0(\theta) \otimes \mathrm{id}_{\mathbb{Z}/n}$. However, composing any map $K_0(A; \mathbb{Z}/n) \to K_0(C; \mathbb{Z}/n)$ with $K_0(C; \mathbb{Z}/n) = K_0(C) \otimes \mathbb{Z}/n \to K_0(B) \otimes \mathbb{Z}/n$ will only hit elements in the image of $K_0(\theta) \otimes \mathrm{id}_{\mathbb{Z}/n}$.

3.6 Unital embeddings of $\mathcal{Z}$

In this last section we will characterise when unital embeddings of $\mathcal{Z}$ have nuclear dimension equal to zero. Unlike in Section 3.5, where finer K-theoretic invariants are required to detect the property of having nuclear dimension equal to zero, all the information for unital embeddings of $\mathcal{Z}$ lies in the K_0-map. Precisely, the weak divisibility condition provided by Corollary 3.4.7 will characterise unital embeddings of $\mathcal{Z}$ with nuclear dimension equal to zero.

The lemma below is the main technical result we need in order to build unital embeddings of $\mathcal{Z}$ which have nuclear dimension equal to zero. Essentially, one can witness quasidiagonality of $\mathcal{Z}$ via any sequence of matrix algebras with arbitrarily large sizes.

Lemma 3.6.1. *Let $\mathcal{F} \subset \mathcal{Z}$ be a finite set and $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $m \geq N$ there is a ucp map $\phi_m : \mathcal{Z} \to M_m$ which is approximately multiplicative on $\mathcal{F}$ up to ϵ.*

Proof. Let $\mathcal{F} \subset \mathcal{Z}$ be finite and let $\epsilon > 0$. We can further assume that all elements in $\mathcal{F}$ are contractions. Since $\mathcal{Z}$ is an inductive limit of *dimension drop algebras*[17] $\mathcal{Z}_{n,n+1}$ ([77, Proposition 2.5, Theorem 2.9]), there exists $n \in \mathbb{N}$ satisfying that for all $a \in \mathcal{F}$, there is a contraction $x \in \mathcal{Z}_{n,n+1}$ such that $\|a - x\| < \frac{\epsilon}{4}$. Denote by ev_0 and ev_1 the unital *-homomorphisms $\mathrm{ev}_0 : \mathcal{Z}_{n,n+1} \to M_n$ and $\mathrm{ev}_1 : \mathcal{Z}_{n,n+1} \to M_{n+1}$, given by evaluation at 0 and 1 respectively.

By Arveson's extension theorem (see for example [24, Corollary 1.5.16]), we obtain ucp maps $\pi_n : \mathcal{Z} \to M_n$ and $\pi_{n+1} : \mathcal{Z} \to M_{n+1}$ extending ev_0 and ev_1, respectively. Then, for any $a, b \in \mathcal{F}$, let $x, y \in \mathcal{Z}_{n,n+1}$ be contractions such that $\|x - a\| < \frac{\epsilon}{4}$ and $\|y - b\| < \frac{\epsilon}{4}$. Then, since π_n restricted to $\mathcal{Z}_{n,n+1}$ is a *-homomorphism, a direct estimation gives that

$$\pi_n(a)\pi_n(b) \approx_{\frac{\epsilon}{2}} \pi_n(x)\pi_n(y) = \pi_n(xy) \approx_{\frac{\epsilon}{2}} \pi_n(ab). \tag{3.6.1}$$

Hence, π_n is approximately multiplicative on $\mathcal{F}$ up to ϵ. A similar argument shows that π_{n+1} is also approximately multiplicative on $\mathcal{F}$ up to ϵ.

Take $N = n(n+2)$. As noted in Remark 3.4.8, for all $m \geq N$, there exist $r, s \in \mathbb{N}$ such that $m = rn + s(n+1)$. Therefore, by considering $\varphi_m : \mathcal{Z} \to M_m$ as $\varphi_m(a) = \pi_n(a)^{\oplus r} \oplus \pi_{n+1}(a)^{\oplus s}$, we obtain a ucp map which is approximately multiplicative on $\mathcal{F}$ up to ϵ. $\square$

We can now characterise when unital embeddings of $\mathcal{Z}$ into simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebras have nuclear dimension equal to zero.

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[17] $\mathcal{Z}_{n,n+1} = \{f \in C([0,1], M_n \otimes M_{n+1}) : f(0) \in M_n \otimes 1_{M_{n+1}}, f(1) \in 1_{M_n} \otimes M_{n+1}\}$.

Theorem 3.6.2. *Let B be a simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C*-*algebra and $\theta : \mathcal{Z} \to B$ be a unital* *-*homomorphism. Then* $\dim_{\mathrm{nuc}} \theta = 0$ *if and only if B is weakly divisible of degree n at 1_B for all $n \in \mathbb{N}$. In particular, if B is not weakly divisible of degree n at 1_B for all $n \in \mathbb{N}$, then any unital embedding of $\mathcal{Z}$ into B has nuclear dimension equal to one.*

Proof. Suppose that $\dim_{\mathrm{nuc}} \theta = 0$. By Corollary 3.4.7 we obtain that B is weakly divisible of degree n at 1 for all $n \in \mathbb{N}$.

Let us prove the converse. Let $(\mathcal{F}_k)_{k\in\mathbb{N}}$ be an increasing family of finite sets of $\mathcal{Z}$ with dense union. For each $k \in \mathbb{N}$, Lemma 3.6.1 yields a natural number N_k such that for all $m \geq N_k$ there is a ucp map $\pi_m^{(k)} : \mathcal{Z} \to M_m$ which is approximately multiplicative on $\mathcal{F}_k$ up to $\frac{1}{k}$. By assumption, B is weakly divisible of degree N_k at 1_B for any $k \in \mathbb{N}$. Then, there exist $m_1, \ldots, m_r \geq N_k$ and a unital *-homomorphism $\eta_k : M_{m_1} \oplus \ldots \oplus M_{m_r} \to B$. To simplify the notation, set $F_k := \bigoplus_{i=1}^{r} M_{m_i}$. Then consider $\psi_k := \pi_{m_1}^{(k)} \oplus \ldots \oplus \pi_{m_r}^{(k)} : \mathcal{Z} \to F_k$. Denote by $\psi : \mathcal{Z} \to \prod_\infty F_k$ and $\eta : \prod_\infty F_k \to B_\infty$ the maps induced by $(\psi_k)_{k\in\mathbb{N}}$ and $(\eta_k)_{k\in\mathbb{N}}$, respectively. Observe that both maps are unital *-homomorphisms.

Let us consider now the unital *-homomorphisms $\iota_B \circ \theta$ and $\eta \circ \psi$. We claim that these two maps are unitarily equivalent. Since $\mathcal{Z}$ has a unique trace, $\mathrm{Aff}(T(\iota_B \circ \theta)) = \mathrm{Aff}(T(\eta \circ \psi))$. Both maps induce the same map on K_0 since they are unital and $K_0(\mathcal{Z}) = \mathbb{Z}$. Similarly, they are equal to zero on K_1 since $K_1(\mathcal{Z}) = 0$.

At the level of total K-theory, note that since $\mathcal{Z}$ is KK-equivalent to $\mathbb{C}$, with the equivalence being induced by the canonical unital inclusion ([77, Lemma 2.11]), any unital embedding of $\mathcal{Z}$ induces the same class in $\mathrm{KK}(\mathcal{Z}, B_\infty)$. Since $\mathrm{KL}(\mathcal{Z}, B_\infty)$ is a quotient of $\mathrm{KK}(\mathcal{Z}, B_\infty)$, it follows that any unital embedding of $\mathcal{Z}$ induces the same class in $\mathrm{KL}(\mathcal{Z}, B_\infty)$. Then, by the universal multicoefficient theorem (see for example [27, Theorem 8.5]), we conclude that $\iota_B \circ \theta$ and $\eta \circ \psi$ induce the same Λ-morphism at the level of total K-theory.

Finally, since $K_1(\mathcal{Z}) = 0$, the decomposition formula in (2.7.3) shows that the Hausdorffised algebraic K_1-maps are completely determined by the trace maps. Therefore, $\overline{K_1}^{\mathrm{alg}}(\iota_B \circ \theta) = \overline{K_1}^{\mathrm{alg}}(\eta \circ \psi)$. Hence, $\iota_B \circ \theta$ and $\eta \circ \psi$ are unitarily equivalent by [27, Theorem 9.1].

By construction, $\dim_{\mathrm{nuc}}(\eta \circ \psi) = 0$. Thus $\iota_B \circ \theta$ has nuclear dimension equal to zero as well since nuclear dimension is preserved by unitary equivalence. Therefore, $\dim_{\mathrm{nuc}} \theta = 0$ by [136, Proposition 2.5]. The last part of the statement holds from the fact that nuclear dimension of any embedding is bounded by the nuclear dimension of $\mathcal{Z}$ (see Lemma 3.1.5). □

A natural question to follow Theorem 3.6.2 is if nuclear dimension equal to zero for unital embeddings of $\mathcal{Z}$ implies factoring through a simple AF-algebra. Before examining this question, we first recall the following classification theorem for unital maps with domain a strongly self-absorbing C*-algebra $\mathcal{D}$ that appears in [138].

Proposition 3.6.3 ([138, Corollary 1.12])**.** *Let A be a $\mathcal{Z}$-stable, unital* C**-algebra. Then all unital *-homomorphisms from $\mathcal{Z}$ to A are mutually approximately unitarily equivalent.*

With this statement in hand, we can compute the nuclear dimension of unital embeddings of $\mathcal{Z}$ into $\mathcal{Z}$-stable C*-algebras of real rank zero.

Proposition 3.6.4. *Let A be a separable, unital, $\mathcal{Z}$-stable* C**-algebra of real rank zero. Then any unital *-homomorphism from $\mathcal{Z}$ to A is approximately unitarily equivalent to a map which factors through a simple, unital AF-algebra. In particular, any unital *-homomorphism from $\mathcal{Z}$ to A has nuclear dimension equal to zero.*

Proof. By [53, Corollary 12], there is a simple, unital, infinite dimensional AF-algebra C and a unital embedding $\varphi : C \to A$. Since C is $\mathcal{Z}$-stable, there is a unital *-homomorphism $\psi : \mathcal{Z} \to C$. Hence, $\varphi \circ \psi : \mathcal{Z} \to A$ is a unital *-homomorphism with nuclear dimension equal to zero. By Proposition 3.6.3, any other unital *-homomorphism $\mathcal{Z} \to A$ is approximately unitarily equivalent to $\varphi \circ \psi$; and hence, it also has nuclear dimension equal to zero. $\square$

We finish this chapter by showing that if the trace space of B has at most two extreme points, any unital embedding of $\mathcal{Z}$ into B with nuclear dimension equal to zero is approximately unitarily equivalent to a unital *-homomorphism which factors through a simple AF-algebra. The proof boils down to constructing a dimension group within $K_0(B)$ using that the unit is weakly divisible.

Theorem 3.6.5. *Let $\theta : \mathcal{Z} \to B$ be a unital *-homomorphism into a simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C**-algebra B which has at most two extreme traces. Then the following are equivalent:*

(i) θ is approximately unitarily equivalent to a *-homomorphism $\mathcal{Z} \to B$ which factors through a simple, unital AF-algebra;

(ii) θ has nuclear dimension equal to zero;

(iii) for all $n \in \mathbb{N}$ there exist $g^{(n)}, h^{(n)} \in K_0(B)_+$ such that

$$[1_B]_0 = ng^{(n)} + (n+1)h^{(n)}. \tag{3.6.2}$$

Proof. Combining Remark 3.4.8 and Theorem 3.6.2, it follows that (ii) and (iii) are equivalent. Since (i) implies (ii), it suffices to check that (iii) implies (i).

If B has no bounded traces, then B is purely infinite ([117, Corollary 5.1]). Since B is also simple, it has real rank zero by [148]. Then, by Proposition 3.6.4, θ is approximately unitarily equivalent to a map which factors through a simple, unital AF-algebra.

Suppose now that B has a unique trace. Then the weak divisibility at the unit implies that $\rho_B(K_0(B))$ is a subgroup of $\mathbb{R}$ with arbitrarily small positive elements.

This means that $\rho_B(K_0(B))$ is uniformly dense in $\mathrm{Aff}(T(B))$ and hence B has real rank zero by [117, Theorem 7.2]. Then, Proposition 3.6.4 yields that θ is approximately unitarily equivalent to a map which factors through a simple, unital AF-algebra.

Suppose that B has two extreme traces τ_1 and τ_2. For simplicity, let us write τ_i instead of $K_0(\tau_i)$. Let us consider first the case when $\tau_1(g^{(n)}) = \tau_2(g^{(n)})$ for all but finitely many $n \geq 2$. Then, by equation (3.6.2), the corresponding $h^{(n)}$'s must be constant in traces as well (i.e. $\tau_1(h^{(n)}) = \tau_2(h^{(n)})$). We will consider the subgroup $H \subset K_0(B)$ generated by those $g^{(n)}$ and $h^{(n)}$ which are constant in traces. Let us equip H with the order induced by τ_1 and note that H has arbitrarily small positive elements in trace. Since the order on H is inherited from the order on $K_0(B)$ and H contains the unit, H is simple and weakly unperforated, so there exists some classifiable C*-algebra C with unique trace such that $(K_0(C), K_0(C)_+, [1_C]_0) \cong (H, H_+, [1_B]_0)$ and $K_1(C) = 0$ (see Theorem 2.6.1). Then, there exists a unital embedding $\psi : \mathcal{Z} \to C$ and by [27, Corollary 9.5], there exists a unital embedding $\phi : C \to B$. Using [27, Theorem 9.3], one can check that θ is approximately unitarily equivalent to $\phi \circ \psi$. Since $\rho_C(K_0(C))$ is a subgroup of $\mathbb{R}$ with arbitrarily small positive elements, C has real rank zero ([117, Theorem 7.2]). Therefore, Proposition 3.6.4 shows that any unital embedding of $\mathcal{Z}$ into C is approximately unitarily equivalent to a map which factors through a simple, separable, unital AF-algebra. Hence, the conclusion follows.

We are left with the case when there are infinitely many $n \in \mathbb{N}$ such that $\tau_1(g^{(n)}) \neq \tau_2(g^{(n)})$. By passing to a subsequence, we can assume that $\tau_1(g^{(n)}) \neq \tau_2(g^{(n)})$ for all $n \in \mathbb{N}$. Observe that this immediately implies $\tau_1(h^{(n)}) \neq \tau_2(h^{(n)})$ for all $n \in \mathbb{N}$.

We claim that $\rho_B(K_0(B))$ is dense in $\mathbb{R}^2$ and hence B has real rank zero by [117, Theorem 7.2]. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any nonzero linear functional. We will show that $f(\rho_B(K_0(B)))$ is dense in $\mathbb{R}$. Consider $\epsilon > 0$ and say $f((1,0)) = \alpha_1$ and $f((0,1)) = \alpha_2$. Then $f(\rho_B(x)) = \alpha_1\tau_1(x) + \alpha_2\tau_2(x)$ for any $x \in K_0(B)$. On the other hand, by equation (3.6.2) we have

$$\alpha_1 + \alpha_2 = f(\rho_B([1_B]_0)) = nf(\rho_B(g^{(n)})) + (n+1)f(\rho_B(h^{(n)})).$$

If $\alpha_1 + \alpha_2 \neq 0$, then either $f(\rho_B(g^{(n)})) \neq 0$ or $f(\rho_B(h^{(n)})) \neq 0$. In both cases, if we consider the element with nonzero image and choose a large enough n, we obtain an element $x \in K_0(B)$ such that

$$0 < |f(\rho_B(x))| < \epsilon. \tag{3.6.3}$$

On the other hand, if $\alpha_1 + \alpha_2 = 0$, then

$$|f(\rho_B(g^{(n)}))| = |\alpha_1(\tau_1(g^{(n)}) - \tau_2(g^{(n)}))| < \epsilon$$

for n large enough and it is strictly positive since $\tau_1(g^{(n)}) \neq \tau_2(g^{(n)})$.

By considering $-g^{(n)}$ or $-h^{(n)}$ if needed, we conclude that the image of $\rho_B(K_0(B))$ through f is a subgroup of $\mathbb{R}$ which contains arbitrarily small positive elements and hence it is dense. Then, it follows that $\rho_B(K_0(B))$ is dense in $\mathbb{R}^2$ by [73, Theorem 1.1]. Thus, B has real rank zero and the result follows by Proposition 3.6.4. □

Chapter 4

Inclusions of real rank zero

This chapter is based on joint work with my co-supervisor James Gabe and its content is essentially contained in [62].

In Section 4.1 we introduce real rank zero for inclusions. After showing a series of equivalent characterisations for this property, we offer a complete description of the commutative case. Then, Section 4.3 collects a number of permanence properties. In Section 4.4 we examine inclusions of real rank zero at the level of their total invariant. In particular, we prove that under some regularity assumptions, real rank zero can be characterised by the relative pairing of K-theory and traces. Then, Section 4.5 focuses on infinite inclusions and provides some non-trivial examples of inclusions of real rank zero.

4.1 Equivalence of definitions

Before defining real rank zero for inclusions, let us first recall the corresponding notion for C*-algebras. In [21], Brown and Pedersen showed that the property of having real rank zero can be characterised in various surprising ways.

Theorem 4.1.1 ([21, Theorem 2.6])**.** *For a* C**-algebra A, the following conditions* *are equivalent:*

- *(i) self-adjoint elements in the minimal unitisation of A can be approximated arbitrarily well by invertible self-adjoint elements in the minimal unitisation of A;*
- *(ii) the self-adjoint elements in A with finite spectrum are dense in the set of self-adjoint elements in A;*
- *(iii) every hereditary* C*-*subalgebra of A has an approximate unit (not necessarily increasing) consisting of projections;*
- *(iv) for each pair of positive orthogonal elements x, y in the minimal unitisation of A and $\epsilon > 0$, there is a projection p in the minimal unitisation of A, such that $\|(1-p)x\| \leq \epsilon$ and $\|py\| \leq \epsilon$.*

Remark 4.1.2. In [21], condition (i) of Theorem 4.1.1 was taken as the definition of real rank zero for a C*-algebra. However, it is now common practice to say that a C*-algebra has real rank zero if it satisfies any of the equivalent conditions in Theorem 4.1.1.

We define the notion of real rank zero for inclusions of C*-algebras. However, we will show that, similar to the result in Theorem 4.1.1, there are a number of equivalent characterisations.

Definition 4.1.3. We say that an inclusion of C*-algebras $A \subseteq B$ has *real rank zero* if for any nonzero positive element $a \in A$, the hereditary C*-subalgebra $\overline{aBa}$ of B has an approximate unit of projections.[18]

Since we will sometimes work with *-homomorphisms, we also define real rank zero for *-homomorphisms.

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[18] The approximate unit of projections is not assumed to be increasing.

Definition 4.1.4. Let $\theta : A \to B$ be a *-homomorphism between C*-algebras. Then we say that θ has *real rank zero* if the inclusion $\theta(A) \subseteq B$ has real rank zero.

In the spirit of [21, Theorem 2.6], let us show that there are many equivalent characterisations of when an inclusion has real rank zero. Since some of the properties require the existence of a unit, let us first deal with the unital case. We also point out that even if some of the proofs transfer from Theorem 4.1.1, a different argument is needed for the implication (iv) $\Longrightarrow$ (ii) in the theorem below.

Theorem 4.1.5. *Let $A \subseteq B$ be an inclusion of unital* C**-algebras such that $1_B \in A$. Then the following conditions are equivalent:*

(i) the inclusion $A \subseteq B$ has real rank zero;

(ii) for any self-adjoint $a \in A$ and any $\epsilon > 0$, there exists a self-adjoint $b \in B$ with finite spectrum such that $\|a - b\| \leq \epsilon$;

(iii) for any self-adjoint $a \in A$ and any $\epsilon > 0$, there exists an invertible self-adjoint $b \in B$ such that $\|a - b\| \leq \epsilon$;

(iv) for any positive $x, y \in A$ such that $xy = 0$, and any $\epsilon > 0$, there exists a projection $p \in B$ such that $\|px - x\| \leq \epsilon$ and $\|(1 - p)y - y\| \leq \epsilon$.

Proof. We are going to show that (iii) and (iv) are equivalent, and then that (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Assume condition (iii). We will show that condition (iv) holds. Let $x, y \in A$ be positive and mutually orthogonal, $\epsilon > 0$, and $\epsilon_0 > 0$ such that $\epsilon_0 + \sqrt{(4 + \epsilon_0)\epsilon_0} < \epsilon$. If $x = 0$, we take $p = 0$ and if $y = 0$, we take $p = 1$. Therefore, we can assume that x and y are nonzero. Rescaling if necessary, suppose that $\|x\| = \|y\| = 1$. If we let $a = x - y$ which is self-adjoint, then by (iii), there exists $b \in B$ invertible and self-adjoint such that $\|a - b\| \leq \epsilon_0$. Let p be the spectral projection of b supported on $(0, \infty]$. Then, we note that $pb_+ = b_+$ and $(1 - p)b_- = b_-$.[19] We claim that p satisfies

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[19] Note that $b_+ = \max(b, 0)$ and $b_- = \max(-b, 0)$. In particular, $b = b_+ - b_-$.

the required conditions in (iv). Since $xy = 0$, it follows that $a_+ = (x-y)_+ = x$. As $pb_+ = b_+$, by an application of the triangle inequality, we have that

$$\begin{aligned} \|px - x\| &\leq \|pa_+ - pb_+\| + \|a_+ - b_+\| \\ &\leq 2\|a_+ - b_+\|. \end{aligned}$$

By [21, Lemma 2.2], we get that $2\|a_+ - b_+\| \leq \delta + \epsilon_0$, where $\delta^2 = (\|a\| + \|b\|)\epsilon_0$. Since $\|a\| \leq 2$, we have that $\|b\| \leq 2 + \epsilon_0$ and hence $\delta^2 \leq (4+\epsilon_0)\epsilon_0$. Therefore,

$$\|px - x\| \leq \epsilon_0 + \sqrt{(4+\epsilon_0)\epsilon_0} < \epsilon.$$

Similarly, as $a_- = y$ and $(1-p)b_- = b_-$, we get that

$$\|(1-p)y - y\| \leq \|(1-p)a_- - (1-p)b_-\| + \|a_- - b_-\|.$$

The same argument as above gives that $\|(1-p)y - y\| \leq \epsilon$, which proves the claim.

Suppose that (iv) holds. To show (iii), let $a \in A$ be self-adjoint and $\epsilon > 0$. Applying (iv) to a_+ and a_-, there is a projection $p \in B$ such that $\|(1-p)a_+\| \leq \epsilon$ and $\|pa_-\| \leq \epsilon$.

Note that

$$\begin{aligned} \|pa_+(1-p) + (1-p)a_+p\| &\leq \|a_+(1-p)\| + \|(1-p)a_+\| \\ &\leq 2\epsilon. \end{aligned}$$

Likewise, we get that $\|pa_-(1-p) + (1-p)a_-p\| \leq 2\epsilon$, so using that $a = a_+ - a_-$, we obtain

$$\|a - (pap + (1-p)a(1-p))\| \leq 4\epsilon. \tag{4.1.1}$$

Define a self-adjoint element in B by

$$b = pap + 2\epsilon p + (1-p)a(1-p) - 2\epsilon(1-p).$$

Since $\|pa_-p\| \leq \epsilon$, we have $pa_-p \leq \epsilon p$, so $pap \geq p(a_+ - \epsilon)p \geq -\epsilon p$. Therefore, we get that $pbp \geq \epsilon p$, which shows that pbp is invertible in pBp. Similarly, $(1-p)a(1-p) \leq \epsilon(1-p)$, so $(1-p)b(1-p) \leq -\epsilon(1-p)$, which gives that $(1-p)b(1-p)$ is invertible in $(1-p)B(1-p)$. Thus, b is invertible in B, by an application of [21, Lemma 2.3] with $x = b$ and $c = d = 0$. Finally, using (4.1.1), a standard application of the triangle inequality shows that $\|a - b\| \leq 8\epsilon$.

We then move to the implication (i) $\Longrightarrow$ (iv). Let $x, y \in A$ be positive orthogonal elements and $\epsilon > 0$. Since $A \subseteq B$ has real rank zero, there exists a projection $p \in \overline{xBx}$ such that $\|px - x\| \leq \epsilon$. Since $xy = 0$, we also get that $py = 0$, so (iv) holds.

Suppose that (iv) holds. To show (ii), let $a \in A$ be self-adjoint and $\epsilon > 0$. If $a = 0$, the conclusion follows by taking $b = 0$. Suppose that a is nonzero. By replacing a with $(\|a_+\| + \|a_-\|)^{-1}(a + \|a_-\|1_B)$, we can assume that $a \in A$ is a positive contraction. Indeed, if $(x_n)_{n\geq 1}$ is a sequence of self-adjoint elements in B with finite spectrum converging to $(\|a_+\| + \|a_-\|)^{-1}(a + \|a_-\|1_B)$, then $(\|a_+\| + \|a_-\|)x_n - \|a_-\|1_B$ is a sequence of self-adjoint elements with finite spectrum converging to a.

Let $n \in \mathbb{N}$ such that $3/n < \epsilon$, so by applying condition (iv) to $(a - j/n)_+$ and $(a - j/n)_-$, for any $j \in \{1, 2, \ldots, n-1\}$ there exists a projection $p_j \in B_\infty$ such that

$$p_j(a - j/n)_+ = (a - j/n)_+ \text{ and } (1 - p_j)(a - j/n)_- = (a - j/n)_-.$$

Then, it follows that

$$p_j a = p_j((a - j/n)_+ - (a - j/n)_- + (j/n)1_B) = (a - j/n)_+ + (j/n)p_j. \qquad (4.1.2)$$

Moreover, $p_j a = a p_j$.

Let $f_j \in C_0((0,1])_+$ given by

$$f_j(t) = \begin{cases} 0 & 0 \leq t \leq \frac{j-1}{n} \\ \text{linear} & \frac{j-1}{n} \leq t \leq \frac{j}{n} \\ 1 & \frac{j}{n} \leq t \leq 1. \end{cases}$$

As the spectrum of $p_j a$ in $D_j := C^*((a - j/n)_+, p_j)$ is contained in $[j/n, 1]$ and p_j commutes with a, we get that $p_j = f_j(p_j a)$. Since $p_j F(a) = F(p_j a)$ for any polynomial F, we also get that $p_j f_j(a) = f_j(p_j a) = f_j(a p_j) = f_j(a) p_j$.

As p_{j-1} acts as an identity on $(a - (j-1)/n)_+$, it also fixes $f_j(a)$, so

$$p_{j-1} p_j = p_{j-1} f_j(a) p_j = f_j(a) p_j = p_j,$$

which gives that $p_1 \geq p_2 \geq \ldots \geq p_{n-1}$ is a decreasing sequence of projections.

Let $q_0 = 1 - p_1$, $q_j = p_j - p_{j+1}$ for $j = 1, 2, \ldots, n-2$ and $q_{n-1} = p_{n-1}$. Then, for $j = 1, 2, \ldots, n-2$ equation (4.1.2) yields that

$$q_j a = (a - j/n)_+ - (a - (j+1)/n)_+ + (j/n) q_j - (1/n) p_{j+1},$$

so

$$\|q_j a - (j/n) q_j\| \leq 2/n. \tag{4.1.3}$$

Also,

$$\|q_0 a\| = \|a - (a - 1/n)_+ - (1/n) p_1\| \leq 2/n \tag{4.1.4}$$

and

$$\left\| q_{n-1} a - \frac{n-1}{n} q_{n-1} \right\| = \left\| \left(a - \frac{n-1}{n} \right)_+ \right\| \leq 1/n. \tag{4.1.5}$$

Since all of the approximations above are in mutually orthogonal C*-algebras, putting together (4.1.3), (4.1.4), (4.1.5), and using that $\sum_{j=0}^{n-1} q_j = 1_{B_\infty}$, we get that

$$\left\| a - \sum_{j=0}^{n-1} (j/n) q_j \right\| = \left\| \sum_{j=0}^{n-1} (q_j a - (j/n) q_j) \right\| \leq 2/n. \tag{4.1.6}$$

Since $(q_j)_{j=0}^{n-1}$ is a sequence of mutually orthogonal projections that sum up to 1_{B_∞}, the map $\phi : \mathbb{C}^n \to B_\infty$ given by $(\lambda_0, \ldots, \lambda_{n-1}) \mapsto \sum_{j=0}^{n-1} \lambda_j q_j$ is a unital *-homomorphism. Let $c_j(B) = \{(b_i)_{i\geq 1} \in \ell^\infty(B) : b_i = 0 \ \forall \ i \geq j\}$ and note that $(c_j(B))_{j\geq 1}$ is an increasing sequence of ideals of $\ell^\infty(B)$ generating $c_0(B)$. Then, since $\mathbb{C}^n$ is semiprojective ([93, Lemma 14.1.5] and [93, Theorem 14.2.1]), it follows that ϕ lifts to a unital *-homomorphism $\tilde{\phi} : \mathbb{C}^n \to \ell^\infty(B)/c_j(B) \subseteq \ell^\infty(B)$ for some $j \geq 1$. Say that $\tilde{\phi}$ is given by a sequence of *-homomorphisms $(\phi_k)_{k\geq 1}$. Then $x_k = \phi_k(0, 1/n, \ldots, (n-1)/n)$ is a self-adjoint element with finite spectrum and (4.1.6) gives that $\limsup_{k\to\infty} \|a - x_k\| \leq 2/n$. Thus, for k large, we get that $\|a - x_k\| \leq 3/n < \epsilon$, so condition (ii) holds as claimed.

Suppose now that (ii) holds and we want to obtain (i). We remark for the reader's convenience that this part of the proof works exactly the same in the nonunital case. Let $a \in A_+$ be nonzero, and E be the hereditary C*-subalgebra of B generated by a. Rescaling if necessary, we can assume that $\|a\| = 1$. Given $0 < \epsilon < 1$, it suffices to find a projection $p \in E$ such that $\|a - pa\| \leq 2\epsilon$. Choose $\delta > 0$ such that $9\delta < \epsilon - \epsilon^2$ and $n \in \mathbb{N}$ such that $1 - \delta^{2/n} \leq \delta$.

By assumption, there exists $b \in B$ self-adjoint and with finite spectrum such that $\|a - b\| \leq \delta$. Since a is positive of norm 1, we can assume that $0 \leq b \leq 1$, where 1 is in the minimal unitisation of B if necessary. Since the function $t \to t^{1/n}$ is continuous, so we may also assume that

$$\|a^{1/n} - b^{1/n}\| \leq \delta. \tag{4.1.7}$$

If we denote by q the spectral projection of b corresponding to the interval $[\delta, 1]$, then $\|b - qb\| \leq \delta$. Moreover, $q \in B$ as b has finite spectrum. Recall that $1 - \delta^{2/n} \leq \delta$.

Since the spectrum of bq in qBq is contained in $[\delta, 1]$, it follows that

$$\|q - b^{2/n}q\| \leq \sup_{\delta \leq t \leq 1} |1 - t^{2/n}| \leq 1 - \delta^{2/n} \leq \delta.$$

Using the triangle inequality, we now get that

$$\|a - qa\| \leq \|b - qb\| + \|a - b\| + \|q(a - b)\| \leq 3\delta. \tag{4.1.8}$$

Similarly, (4.1.7) and the estimate $\|q - b^{2/n}q\| \leq \delta$ give that

$$\|a^{1/n}qa^{1/n} - q\| \leq 3\delta. \tag{4.1.9}$$

With the notation $z = a^{1/n}qa^{1/n}$, the last inequality implies that

$$\|z - z^2\| \leq \|z - q\| + \|z - q\|\|z + q\| \leq 9\delta < \epsilon - \epsilon^2,$$

so the spectrum of z is contained in $[0, \epsilon] \cup [1 - \epsilon, 1]$. If we consider the spectral projection p of z corresponding to the interval $[1/2, 1]$, then $\|z - p\| \leq \epsilon$ and $p \in E$ as $z \in E$. Hence, $\|p - q\| \leq \epsilon + 3\delta$ and thus

$$\|a - pa\| \leq \epsilon + 3\delta + \|a - qa\| \leq \epsilon + 6\delta < 2\epsilon,$$

which finishes the proof. □

As in [21], a similar result holds for nonunital inclusions. For any C*-algebra C, we denote its minimal unitisation by $C^\dagger$.

Theorem 4.1.6. *Let $A \subseteq B$ be an inclusion of* C**-algebras. Then the following are equivalent:*

(i) the inclusion $A \subseteq B$ has real rank zero;

(ii) *for any self-adjoint* $a \in A$ *and any* $\epsilon > 0$*, there exists a self-adjoint* $b \in B$ *with finite spectrum such that* $\|a - b\| \leq \epsilon$*;*

(iii) *for any self-adjoint* $a \in A^\dagger$ *and any* $\epsilon > 0$*, there exists an invertible self-adjoint* $b \in B^\dagger$ *such that* $\|a - b\| \leq \epsilon$*;*

(iv) *for any positive* $x, y \in A^\dagger$ *such that* $xy = 0$*, and any* $\epsilon > 0$*, there exists a projection* $p \in B^\dagger$ *such that* $\|px - x\| \leq \epsilon$ *and* $\|(1 - p)y - y\| \leq \epsilon$*.*

Proof. Conditions (iii) and (iv) are equivalent by applying Theorem 4.1.5 to the unital inclusion $A^\dagger \subseteq B^\dagger$. Suppose condition (i) holds and we claim it implies (iv). If $x, y \in A^\dagger$ are orthogonal, then at least one of x and y is in A. Suppose that $x \in A$ and let $p \in \overline{xBx}$ be a projection such that $\|px - x\| \leq \epsilon$. Since $xy = 0$ and $p \in \overline{xBx}$, it also follows that $py = 0$.

Following the remark in the proof of Theorem 4.1.5, the fact that (ii) implies (i) is obtained in the proof of (ii) $\implies$ (i) in Theorem 4.1.5. Then, it only remains to prove that (iv) implies (ii). Let $a \in A$ be self-adjoint, $\epsilon > 0$, and suppose that (iv) holds. Then, applying Theorem 4.1.5 to the inclusion $A^\dagger \subseteq B^\dagger$, there exists $b \in B^\dagger$ self-adjoint with finite spectrum such that $\|a - b\| \leq \epsilon/2$. Then $b = b_0 + \lambda 1$, so $b_0 \in B$ is self-adjoint, has finite spectrum, and $\|a - b_0\| \leq \epsilon$. Hence the conclusion follows. □

Remark 4.1.7. In light of Theorem 4.1.6, we could have chosen any of the conditions above to characterise when an inclusion $A \subseteq B$ has real rank zero. However, we opted for the one appearing in Definition 4.1.3 because it can be stated both in the unital and nonunital case. However, we now say that an inclusion $A \subseteq B$ has real rank zero if it satisfies any of the conditions in Theorem 4.1.6, or in Theorem 4.1.5 if the inclusion is unital.

Naturally, if at least one of the C*-algebras has real rank zero, then the inclusion has real rank zero.

Lemma 4.1.8. *Let $A \subseteq B$ be an inclusion of* C**-algebras. If there exists an intermediate* C**-algebra with real rank zero, then the inclusion $A \subseteq B$ has real rank zero. In particular, if either A or B has real rank zero, then the inclusion $A \subseteq B$ has real rank zero. Moreover, a* C**-algebra C has real rank zero if and only if the inclusion $C \subseteq C$ has real rank zero.*

Proof. Suppose first that there exists a C*-algebra C with real rank zero such that $A \subseteq C \subseteq B$. We claim that the inclusion $A \subseteq B$ has real rank zero. For this, let $a \in A$ be self-adjoint and $\epsilon > 0$. By Theorem 4.1.1, there exists a self-adjoint $b \in C$ with finite spectrum such that $\|a - b\| \leq \epsilon$. Since $b \in B$, we conclude that $A \subseteq B$ has real rank zero by Theorem 4.1.6. In particular, it follows that if A or B has real rank zero, then the inclusion $A \subseteq B$ has real rank zero. Therefore, if C is a C*-algebra with real rank zero, the inclusion $C \subseteq C$ has real rank zero. Suppose now that the inclusion $C \subseteq C$ has real rank zero. To show that C has real rank zero, let $c \in C$ be self-adjoint and $\epsilon > 0$. Then, by Theorem 4.1.6, there exists a self-adjoint $d \in C$ with finite spectrum such that $\|c - d\| \leq \epsilon$. Hence, C has real rank zero by Theorem 4.1.1. □

Lemma 4.1.9. *Let A, B, and C be* C**-algebras. Then the set of* **-homomorphisms from A to B with real rank zero is point-norm closed. Moreover, if $\phi : C \to B$ and $\psi : A \to C$ are* **-homomorphisms, at least one with real rank zero, then $\phi \circ \psi$ has real rank zero.*

Proof. Let $\theta_n : A \to B$ be a sequence of *-homomorphisms with real rank zero converging pointwise to $\theta : A \to B$. Then θ has real rank zero by using condition (ii) in Theorem 4.1.6 for the sequence of inclusions $\theta_n(A) \subseteq B$. Suppose now $\phi : C \to B$ and $\psi : A \to C$ are *-homomorphisms and ψ has real rank zero. Let $a \in A$ be self-adjoint and $\epsilon > 0$. Then there exists $c \in C$ self-adjoint with finite spectrum such that $\|\psi(a) - c\| \leq \epsilon$. Therefore, $\phi(c)$ is a self-adjoint element with finite spectrum and $\|\phi(\psi(a)) - \phi(c)\| \leq \epsilon$. Hence $\phi \circ \psi$ has real rank zero. The case when ϕ has real rank zero follows similarly. □

Lemma 4.1.10. *Let* $A \subseteq B$ *be an inclusion of* C**-algebras. If the inclusion* $A \subseteq B$ *is approximately unitarily equivalent to a map factoring through a* C**-algebra of real rank zero, then* $A \subseteq B$ *has real rank zero.*

Proof. Let C be a C*-algebra of real rank zero, $\psi : A \to C$ and $\phi : C \to B$ be *-homomorphisms, and $(u_\lambda)_\lambda$ be a net of unitaries in the multiplier algebra of B such that $\mathrm{Ad}(u_\lambda) \circ \phi \circ \psi$ converges pointwise to the inclusion $A \subseteq B$.

Since C has real rank zero, combining Lemma 4.1.8 and Lemma 4.1.9 shows that the *-homomorphisms $\mathrm{Ad}(u_\lambda) \circ \phi \circ \psi$ have real rank zero. Then, by the first part of the Lemma 4.1.9, we get that the inclusion $A \subseteq B$ has real rank zero. □

4.2 Inclusions of commutative C*-algebras

In this section we will give a complete description of when an inclusion of commutative C*-algebras has real rank zero. The results essentially follow from Section 3.2.

Theorem 4.2.1. *Let* X, Y *be compact Hausdorff spaces and* $\phi : Y \to X$ *be a continuous map which induces an inclusion* $C(X) \subseteq C(Y)$ *denoted by* ϕ^**. Then the inclusion* $C(X) \subseteq C(Y)$ *has real rank zero if and only if it factors through a commutative* C**-algebra over a* 0*-dimensional compact Hausdorff space. In particular,* ϕ *is constant on connected components.*

Proof. The if direction follows from Lemma 4.1.10 and [21, Proposition 1.1]. Conversely, let us suppose that the inclusion has real rank zero. The result follows from Lemma 3.2.4. □

Having Theorem 4.2.1 at our disposal, we can show that the property of having real rank zero coincides with nuclear dimension zero for *-homomorphisms between commutative C*-algebras. Recall the following result from Lemma 3.2.2.

Proposition 4.2.2 (Lemma 3.2.2). *Let A, B be unital* C**-algebras and $\theta : A \to B$ be a unital* **-homomorphism with nuclear dimension equal to zero. Then θ has real rank zero.*

Theorem 4.2.3. *Let X, Y be compact Hausdorff spaces and $\phi : Y \to X$ be a continuous map which induces a* **-homomorphism $\phi^* : C(X) \to C(Y)$. The following are equivalent:*

(i) ϕ^* has nuclear dimension equal to zero;

(ii) ϕ^* has real rank zero;

(iii) ϕ^* factors through a commutative C**-algebra with nuclear dimension equal to zero.*

Proof. The fact that (i) implies (ii) follows from Proposition 4.2.2. If ϕ^* has real rank zero, then it follows by Theorem 4.2.1 that ϕ^* factors through a commutative C*-algebra of real rank zero. This algebra has nuclear dimension equal to zero by [21, Proposition 1.1] and [147, Proposition 2.4] (see also [28, Theorem 2.6]). The fact that (iii) implies (i) is immediate. □

We end this section by applying Theorem 4.2.1 to obtain a class of *-homomorphisms with real rank zero.

Theorem 4.2.4. *Let G be a compact, connected group, X, Y be compact Hausdorff G-spaces and $\phi : Y \to X$ be an equivariant continuous map. If $\phi^* : C(X) \to C(Y)$ has real rank zero, then the induced morphism $\tilde{\phi} : C(X) \rtimes_r G \to C(Y) \rtimes_r G$ factors through a* C**-algebra of real rank zero, and in particular $\tilde{\phi}$ has real rank zero.*

Proof. Let Y_c be the set of connected components of Y equipped with the quotient topology. Since ϕ^* has real rank zero, then ϕ is constant on connected components by

Theorem 4.2.1. Then, if Y_c is the space of connected components of Y, there exists a continuous map $\psi : Y_c \to X$ such that $\phi = \psi \circ \pi$, where $\pi : Y \to Y_c$ is the canonical quotient map. We equip the space Y_c with the trivial action of G and claim that the maps π and ψ are G-equivariant.

First note that since G is connected, the orbit Gy is connected for any $y \in Y$. Let $g \in G$ and $y \in Y$. Then, since $g \cdot y$ and y are in the same connected component of Y, we have that

$$\pi(g \cdot y) = \pi(y) = g \cdot \pi(y).$$

Hence, the map π is G-equivariant.

Similarly, since $g \cdot y$ and y are in the same connected component of Y, the proof of Theorem 4.2.1(which is contained in Lemma 3.2.4) shows that $\phi(g \cdot y) = \phi(y)$. Therefore, using the equivariance of ϕ, one gets that

$$\psi(g \cdot \pi(y)) = \psi(\pi(y)) = \phi(y) = \phi(g \cdot y) = g \cdot \phi(y) = g \cdot \psi(\pi(y)).$$

Hence, the *-homomorphism ϕ^* factors equivariantly as

$$\begin{tikzcd}
C(X) \arrow[rr, "\phi^*"] \arrow[dr, "\psi^*"'] & & C(Y) \\
& C(Y_c). \arrow[ur, "\pi^*"'] &
\end{tikzcd}$$

Therefore, we obtain the commuting diagram

$$\begin{tikzcd}
C(X) \rtimes_r G \arrow[rr, "\tilde{\phi}"] \arrow[dr] & & C(Y) \rtimes_r G \\
& C(Y_c) \rtimes_r G. \arrow[ur] &
\end{tikzcd}$$

Since G acts trivially on Y_c, we have that $C(Y_c) \rtimes_r G \cong C(Y_c) \otimes \mathrm{C}^*_r(G)$ ([143, Lemma 2.73]). Then, G is compact and connected, so $\mathrm{C}^*_r(G)$ has real rank zero by [80, Theorem 1]. Furthermore, recall from the proof of Lemma 3.2.4 that Y_c is compact, Hausdorff, and totally disconnected. In particular, $C(Y_c)$ has nuclear dimension equal to zero. Thus, $C(Y_c)$ is an AF-algebra by [144, Theorem 3.4]. Then, $C(Y_c) \otimes \mathrm{C}^*_r(G)$ has real rank zero by [21, Theorem 3.2] and we reach the conclusion. $\square$

4.3 Permanence properties

Like in [21], real rank zero is preserved under inductive limits of inclusions.

Proposition 4.3.1. *Suppose that* $A = \varinjlim(A_n, \phi_n)$ *and* $B = \varinjlim(B_n, \psi_n)$ *are two* C*-*inductive limits and there exist inclusions* $A_n \subseteq B_n$ *for all* $n \in \mathbb{N}$*, such that* $\psi_n|_{A_n} = \phi_n$*. If there exists* $N \in \mathbb{N}$ *such that* $A_n \subseteq B_n$ *has real rank zero for all* $n > N$*, then* $A \subseteq B$ *has real rank zero.*

Proof. Since $\psi_n|_{A_n} = \phi_n$ for any $n \in \mathbb{N}$, the sequence of inclusions $A_n \subseteq B_n$ induce an inclusion $A \subseteq B$. If the inclusion $A \subseteq B$ is nonunital, then consider the unital inclusion $A^\dagger \subseteq B^\dagger$ with unital connecting maps. If this inclusion has real rank zero, then by Theorem 4.1.6, the inclusion $A \subseteq B$ has real rank zero. Therefore, we can assume that $A \subseteq B$ is a unital inclusion, with A_n, B_n unital for all $n \geq 1$ and unital connecting maps.

Let $a \in A$ be a self-adjoint element and $\epsilon > 0$. Then there exist $n > N$ and a self-adjoint element $a_n \in A_n$ such that $\|a - \phi_{n,\infty}(a_n)\| \leq \epsilon/2$, where $\phi_{n,\infty} : A_n \to A$ is an inductive limit connecting map. Since $\psi_k|_{A_k} = \phi_k$ for any $k \in \mathbb{N}$, we get that $\phi_{n,\infty}(a_n) = \psi_{n,\infty}(a_n)$, so

$$\|a - \psi_{n,\infty}(a_n)\| \leq \epsilon/2.$$

By assumption, $A_n \subseteq B_n$ has real rank zero, so there exists an invertible self-adjoint $b_n \in B_n$ such that $\|a_n - b_n\| \leq \epsilon/2$. Since $\psi_{n,\infty}$ is a unital *-homomorphism, $\psi_{n,\infty}(b_n)$ is a self-adjoint invertible element. Then, triangle inequality gives that $\|a - \psi_{n,\infty}(b_n)\| \leq \epsilon$. Hence $A \subseteq B$ has real rank zero by Theorem 4.1.5. $\square$

Then, we can show that real rank zero of an inclusion is preserved if we consider an intermediate inclusion into a hereditary subalgebra.

Proposition 4.3.2. *Let $A \subseteq B$ be an inclusion of* C^**-algebras and D be a hereditary subalgebra of B such that $A \subseteq D$. If $A \subseteq B$ has real rank zero, then the inclusion $A \subseteq D$ has real rank zero.*

Proof. Let $a \in A_+$ be nonzero. Then $\overline{aBa}$ has an approximate unit of projections. Since $\overline{aDa} = \overline{aBa}$, the inclusion $A \subseteq D$ has real rank zero. $\square$

Corollary 4.3.3. *Let $A \subseteq B$ be an inclusion of* C^**-algebras and $a \in A$ be a positive element. If $A \subseteq B$ has real rank zero, then the induced inclusion $\overline{aAa} \subseteq \overline{aBa}$ has real rank zero.*

Proof. Since $A \subseteq B$ has real rank zero, the inclusion $\overline{aAa} \subseteq B$ also has real rank zero. Hence, $\overline{aAa} \subseteq \overline{aBa}$ has real rank zero by Proposition 4.3.2. $\square$

The behaviour of inclusions of real rank zero under extensions is more intricate to study. The next three propositions follow the proof of [21, Theorem 3.14].

Proposition 4.3.4. *Let A, B be* C^**-algebras and I,J be closed 2-sided ideals of A and B respectively. Suppose that $A \subseteq B$, $I \subseteq J$, and the inclusion $A \subseteq B$ has real rank zero. Denote the induced map between the quotients by $\theta : A/I \to B/J$. Then $I \subseteq J$ and $\theta(A/I) \subseteq B/J$ have real rank zero. Moreover, for any nonzero positive element $x \in A$, any projection in $\overline{xAx}/\overline{xIx}$ lifts to a projection in $\overline{xBx}$.*

Proof. Let $a \in I_+$ and $\epsilon > 0$. Since $\overline{aBa}$ has an approximate unit of projections, there exists a projection $p \in \overline{aBa}$ such that $\|pa - a\| \leq \epsilon$. However, $a \in J$, so $p \in \overline{aJa}$. Thus, $I \subseteq J$ has real rank zero.

Let us now consider the inclusion $\theta(A/I) \subseteq B/J$. Let $a \in A$ be a self-adjoint element and $\epsilon > 0$. Then, by Theorem 4.1.6, there exists a self-adjoint element with finite spectrum $b \in B$ such that $\|a - b\| \leq \epsilon$. But $b + J$ is still a self-adjoint element with finite spectrum and

$$\|\theta(a + I) - (b + J)\| = \|(a - b) + J\| \leq \epsilon.$$

Hence, $\theta(A/I) \subseteq B/J$ has real rank zero.

Let $x \in A_+$ be nonzero. By Corollary 4.3.3, the inclusion $\overline{xAx} \subseteq \overline{xBx}$ has real rank zero. Therefore, replacing the inclusion $A \subseteq B$ with $\overline{xAx} \subseteq \overline{xBx}$, it suffices to check that any nonzero projection in $\theta(A/I)$ lifts to a projection in B. Let $a \in A$ be a self-adjoint element of norm 1 such that $a + J$ is a nonzero projection and let $0 < \epsilon < 1$. Then there exists $b \in B$ self-adjoint with finite spectrum and norm 1 such that $\|a - b\| \leq \epsilon^2$. Since b is a finite linear combination of its spectral projections, we can write $b = \sum\limits_{k=1}^{n} \lambda_k p_k$, for some $n \in \mathbb{N}$, $\lambda_k \in \mathbb{R}$, and $p_k \in B$ pairwise orthogonal projections.

Let us denote by $\pi : B \to B/J$ the canonical quotient map. Using that $\pi(a)$ is a projection, a direct estimation yields that

$$\left\| \pi \left(\sum_{k=1}^{n} (\lambda_k - \lambda_k^2) p_k \right) \right\| = \|\pi(b - b^2 - a + a^2)\| \leq 3\epsilon^2. \tag{4.3.1}$$

Therefore, $p_k \in J$ whenever $|\lambda_k - \lambda_k^2| \geq 4\epsilon^2$.

Let S and T be the subsets of $\{1, 2, \ldots, n\}$ consisting of those k such that $|1 - \lambda_k| < 2\epsilon$ and $|\lambda_k| < 2\epsilon$, respectively. Setting $p = \sum\limits_{k \in S} p_k$ and observing that $p_k \in J$ if k is not contained in the union of S and T, it follows that

$$\|\pi(p - b)\| \leq \left\| \sum_{k \in S} p_k - \sum_{k \in S \cup T} \lambda_k p_k \right\| \leq 2\epsilon.$$

Then, $\|\pi(p - a)\| \leq 2\epsilon + \epsilon^2$, so there is a positive contraction $e \in J$ such that $\|(1 - e)(p - a)\| \leq 3\epsilon + \epsilon^2$. Hence, $z = (1 - e)a(1 - e) + ep + pe - epe \in B$ is self-adjoint such that $z - a \in J$ and $\|p - z\| \leq 3\epsilon + \epsilon^2$. We choose $0 < \epsilon < 1$ small enough such that $\|p - z\| < \frac{1}{16}$. Since $\|z + p\| \leq \|z - p\| + 2\|p\| \leq 2 + \frac{1}{16}$, it follows that

$$\|z - z^2\| = \|(z - p) - (z^2 - p^2)\| < \frac{1}{4},$$

so the spectrum of z has a gap around $\frac{1}{2}$. Let

$$f(t) = \begin{cases} 0 & t < \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases}$$

be a continuous function on the spectrum of z. Then $f(z)$ is a projection and it is a lift of the projection $a + J$. Hence, for any positive nonzero $x \in A$, any projection in $\overline{xAx}/\overline{xIx}$ lifts to a projection in $\overline{xBx}$. $\square$

Remark 4.3.5. Let us reflect on the converse of the above proposition. Even if the inclusions $I \subseteq J$ and $\theta(A/I) \subseteq B/J$ have real rank zero, and any projection in $\theta(A/I)$ lifts to a projection in B, we cannot conclude that $A \subseteq B$ has real rank zero. For example, if $I = \{0\}$ and $J = B$, then $I \subseteq J$ and $\theta(A/I) \subseteq B/J$ have real rank zero. Moreover, any projection in $\theta(A/I)$ lifts to a projection in B. However, the fact that $A \subseteq B$ has real rank zero does not hold in general.

The next two propositions provide partial converses to Proposition 4.3.4.

Proposition 4.3.6. *Let A, B be C^*-algebras and I,J be closed 2-sided ideals of A and B respectively such that $A \subseteq B$ and $I \subseteq J$. Suppose that A/I has real rank zero and the inclusion $I \subseteq J$ has real rank zero. Moreover, assume that for any hereditary subalgebra D of A, any projection in $(D+I)/I$ lifts to a projection in D. Then $A \subseteq B$ has real rank zero.*

Proof. Let $x, y \in A^\dagger$ be positive orthogonal elements and $\epsilon > 0$. Since $xy = 0$, at least one of them is in A, so let us assume that $x \in A$. Let $C = \overline{xAx}$ and $\pi : A \to A/I$ be the canonical quotient map. Then $\pi(C)$ is the hereditary subalgebra of A/I generated by $\pi(x)$. By applying Corollary 4.3.3 to the inclusion $A/I \subseteq A/I$ and the element $\pi(x)$, it follows that $\pi(C)$ has real rank zero. Therefore, there exists a projection $q \in \pi(C)$ such that

$$\|q\pi(x) - \pi(x)\| \le \epsilon. \tag{4.3.2}$$

By assumption, we can lift q to a projection $p \in C$.

Consider $(1-p)\overline{xIx}(1-p)$ and $(1-p)\overline{xJx}(1-p)$. Then, since the inclusion $I \subseteq J$ has real rank zero, a similar proof to that of Proposition 4.3.2 shows that the inclusion $(1-p)\overline{xIx}(1-p) \subseteq (1-p)\overline{xJx}(1-p)$ has real rank zero. Furthermore, (4.3.2) gives $\|\pi(x-px)\| \le \epsilon$, so we can find a projection $r \in (1-p)\overline{xJx}(1-p)$ such that

$$\|(x-px) - r(x-px)\| \le 2\epsilon. \tag{4.3.3}$$

Note that p and r are orthogonal, so let $p_1 = p + r$, which is a projection in $\overline{xB^\dagger x}$. Since $xy = 0$, it follows that $p_1 y = 0$. Then, (4.3.3) yields that $\|x - p_1 x\| \le 2\epsilon$, so $A \subseteq B$ has real rank zero by (iv) of Theorem 4.1.6. □

Proposition 4.3.7. *Let A, B be* C*-*algebras and I,J be closed 2-sided ideals of A and B respectively such that $A \subseteq B$ and $I \subseteq J$. Denote the induced map between the quotients by $\theta : A/I \to B/J$. Suppose that J has real rank zero, any projection in B/J lifts to a projection in B, and $\theta(A/I) \subseteq B/J$ has real rank zero. Then $A \subseteq B$ has real rank zero.*

Proof. Let $x, y \in A^\dagger$ be positive orthogonal elements and $\epsilon > 0$. Since $xy = 0$, at least one of them is in A, so let us assume that $x \in A$. Let $C = \overline{xAx}$ and $\pi : A \to A/I$ the canonical quotient map. Then $\theta(\pi(C))$ is the hereditary subalgebra of B/J generated by $x + J$. By Corollary 4.3.3, the inclusion $\theta(\pi(C)) \subset \overline{(x+J)B/J(x+J)}$ has real rank zero, so there exists a projection $q \in \overline{(x+J)B/J(x+J)}$ such that

$$\|(x+J) - q(x+J)\| \leq \epsilon. \tag{4.3.4}$$

By assumption, q lifts to a projection p in B, which can be assumed to be in $\overline{xBx}$ by [21, Lemma 3.13] since J has real rank zero.

Since J has real rank zero, we get that $(1-p)\overline{xJx}(1-p)$ has real rank zero. Moreover, (4.3.4) gives that $\|(x - px) + J\| \leq \epsilon$, so there exists a projection $r \in (1-p)\overline{xJx}(1-p)$ such that

$$\|(x - px) - r(x - px)\| \leq 2\epsilon. \tag{4.3.5}$$

Note that p and r are orthogonal, so let $p_1 = p + r$ be a projection in $\overline{xB^\dagger x}$. Since $xy = 0$, it follows that $p_1 y = 0$. Then, by (4.3.5), we also have that $\|x - p_1 x\| \leq 2\epsilon$, so $A \subseteq B$ has real rank zero by (iv) of Theorem 4.1.6. $\square$

4.4 K-theoretic properties

In this section we characterise inclusions of real rank zero in certain cases by appealing to the pairing between K-theory and traces. The motivation for this approach comes from [117], where Rørdam proved that for a certain class of C*-algebras, real rank zero is equivalent to the image of the pairing map in K-theory to be uniformly dense in the set of affine functions on the trace space ([117, Theorem 7.2]). The proof of the next result essentially follows the strategy in [117, Proposition 7.1] and [113, Theorem 7.2].

Recall from Section 2.4 that if $A \subseteq B$ is a unital inclusion of C*-algebras, then there is a canonical map $\gamma : T(B) \to T(A)$ induced by restriction and a canonical affine map $\gamma^* : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$, which is dual to γ (see [78]). Moreover, for a unital C*-algebra C, there exists a natural pairing between K-theory and traces denoted by $\rho_C : K_0(C) \to \mathrm{Aff}(T(C))$.

Let $A \subseteq B$ be a unital inclusion of C*-algebras as in the theorem below. The idea of the proof is to use functional calculus on a given positive element a to get an approximate unit for a and then construct projections between the elements forming the approximate unit of the positive element a. Using functional calculus, one can find a positive element b such that its trace evaluations sit between evaluations of the approximate unit. Then, the map $\tau \mapsto \tau(b)$ is in the image of γ^*, so the hypotheses allow us to replace b with a projection q in some matrix amplification of B. One crucially uses strict comparison to pass from inequalities in trace to Cuntz subequivalence. To finish the proof, the assumption of stable rank one on B is used to construct a projection in B starting from q, which lives in some matrix amplification of B. We invite the reader to recall the notion of strict comparison from Definition 2.5.5. Note that the case when $A = B$ in the theorem below recovers the if direction in [117, Proposition 7.1]. We do not assume exactness since the comparison property we assume is with respect to tracial states unlike in [117] where the comparison is with respect to quasitraces.

Theorem 4.4.1. *Let $A \subseteq B$ be a unital inclusion of separable* C**-algebras such that every nonzero positive element in A is full in B. Suppose further that B has stable rank one and strict comparison of positive elements with respect to tracial states. If $\gamma^*\left(\mathrm{Aff}(T(A))\right) \subseteq \overline{\rho_B(K_0(B))}$, then the inclusion $A \otimes \mathcal{K} \subseteq B \otimes \mathcal{K}$ has real rank zero.*

Proof. We will first show that the inclusion $A \subseteq B$ has real rank zero. Let $a \in A$ be a nonzero, positive contraction in A and $E = \overline{aBa}$. Then, for any $\epsilon > 0$ define

$f_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_\epsilon(t) = \begin{cases} 0 & t \leq \epsilon \\ \frac{t-\epsilon}{\epsilon} & \epsilon \leq t \leq 2\epsilon \\ 1 & 2\epsilon \leq t. \end{cases}$$

We claim that E has an approximate unit of projections. For this, it suffices to find projections $p_j \in B$ such that

$$f_{\delta_1}(a) \leq p_1 \leq f_{\delta_2}(a) \leq p_2 \leq \ldots,$$

where $\delta_j = 16^{-j}$. Then $\{p_j\}_{j\geq 1}$ will be an approximate unit for E. For each $\delta > 0$, it suffices to find a projection $p \in B$ such that

$$f_\delta(a) \leq p \leq f_{\delta/16}(a).$$

Denote by $\sigma(a)$ the spectrum of a in B. If $\sigma(a) \cap (\delta/8, \delta/2) = \emptyset$, then we can take the projection $p = f(a)$, where

$$f(t) = \begin{cases} 0 & t \leq \delta/8 \\ \text{linear} & \delta/8 \leq t \leq \delta/2 \\ 1 & \delta/2 \leq t. \end{cases} \tag{4.4.1}$$

If there exists $\lambda \in \sigma(a) \cap (\delta/8, \delta/2)$, then take g to be a continuous function, pointwise smaller than $f_{\delta/8}$, with open support $(\delta/8, \delta/2)$. One can note that $f_{\delta/8}(a) \geq g(a) + f_{\delta/2}(a)$ with g and $f_{\delta/2}$ orthogonal, and hence

$$d_\tau(f_{\delta/2}(a)) + d_\tau(g(a)) \leq d_\tau(f_{\delta/8}(a)), \tag{4.4.2}$$

for all $\tau \in T(B)$. Since $\sigma(a) \cap (\delta/8, \delta/2) \neq \emptyset$, $g(a)$ is nonzero. By assumption, $g(a)$ is full in B, so $d_\tau(g(a)) > 0$ for any $\tau \in T(B)$ by Lemma 2.5.6. Therefore,

$$d_\tau(f_{\delta/2}(a)) < d_\tau(f_{\delta/8}(a)), \tag{4.4.3}$$

for all $\tau \in T(B)$.

Consider the function f defined in (4.4.1). Then,

$$\tau(f(a)) < \tau(f_{\delta/8}(a)) \leq d_\tau(f_{\delta/8}(a)), \tag{4.4.4}$$

for all $\tau \in T(B)$. If we consider a continuous function h, pointwise smaller than f, and with open support $(\delta/8, \delta/2)$, we see that

$$\tau(f_{\delta/2}(a)^{1/n}) + \tau(h(a)) < \tau(f(a))$$

for any $\tau \in T(B)$ and any $n \in \mathbb{N}$. Since $\sigma(a) \cap (\delta/8, \delta/2) \neq \emptyset$, $h(a)$ is nonzero and hence full by assumption, so $\tau(h(a)) > 0$ by Lemma 2.5.6. Combining this with (4.4.4) yields that

$$d_\tau(f_{\delta/2}(a)) < \tau(f(a)) < d_\tau(f_{\delta/8}(a))$$

for any $\tau \in T(B)$.

The map $\tau \mapsto \tau(f(a))$ is in the image of γ^*, which is contained in $\overline{\rho_B(K_0(B))}$ by assumption. Therefore, we can find $g \in K_0(B)$ such that

$$d_\tau(f_{\delta/2}(a)) < \tau(g) < d_\tau(f_{\delta/8}(a)), \tag{4.4.5}$$

for all $\tau \in T(B)$. Since B has strict comparison of positive elements with respect to tracial states, we get that $g = [q]_0 \in K_0(B)_+$, for some projection $q \in M_n(B)$.

Furthermore, $d_\tau(q) = \tau(q)$, so strict comparison of B yields that

$$f_{\delta/2}(a) \precsim q \precsim f_{\delta/8}(a).$$

Then, one can follow the last part of the proof of [113, Theorem 7.2] to obtain a projection $p \in B$ such that $f_\delta(a) \leq p \leq f_{\delta/16}(a)$. Precisely, since $q = f_\epsilon(q)$ for $0 < \epsilon < 1/2$ and $q \precsim f_{\delta/8}(a)$, it follows that $q = rf_{\delta/8}(a)r^*$ for some $r \in M_{n,1}(B)$ (see [83, Proposition 2.6] or [113, Proposition 2.4]). If we let $v = rf_{\delta/8}(a)^{1/2}$, then $q = vv^*$ and $q' = v^*v$ is a projection in $\overline{f_{\delta/8}(a)Bf_{\delta/8}(a)}$ which is equivalent to q.

Moreover, $f_{\delta/2}(a) \precsim q'$ and the subequivalence also holds relative to $\overline{f_{\delta/8}(a)Bf_{\delta/8}(a)}$, as well as its minimal unitisation. Furthermore, $f_{\delta/8}(a)$ is full in B, so the hereditary subalgebra $\overline{f_{\delta/8}(a)Bf_{\delta/8}(a)}$ is stably isomorphic to B by [18, Theorem 2.8]. Since stable rank one is preserved by stable isomorphisms ([110, Theorem 3.6]), it follows that $\overline{f_{\delta/8}(a)Bf_{\delta/8}(a)}$ has stable rank one. Then, by definition, the minimal unitisation of $\overline{f_{\delta/8}(a)Bf_{\delta/8}(a)}$ has stable rank one. Hence, by [113, Proposition 2.4], there exists a unitary u in the minimal unitisation of $\overline{f_{\delta/8}(a)Bf_{\delta/8}(a)}$ such that

$$uf_\delta(a)u^* \in q'Bq'.$$

Put $p = u^*q'u$ and note that it is a projection such that $f_\delta(a) \leq p$, since $f_\delta(a) \leq q'$. Since $p \in \overline{f_{\delta/8}(a)Bf_{\delta/8}(a)}$, it also follows that $p \leq f_{\delta/16}(a)$, which finishes the argument. Thus, the inclusion $A \subseteq B$ has real rank zero.

Let $n \in \mathbb{N}$ and consider the unital inclusion $M_n(A) \subseteq M_n(B)$. Since $A \subseteq B$ is full, so is the inclusion $M_n(A) \subseteq M_n(B)$ ([61, Lemma 3.16]). Moreover, one can see from Definition 2.5.5 that strict comparison passes to matrix amplifications, and so does stable rank one ([110, Theorem 3.3]). Lastly, one also has that $\gamma^*\left(\mathrm{Aff}(T(M_n(A)))\right) \subseteq \overline{\rho_{M_n(B)}(K_0(M_n(B)))}$. Hence the proof above shows that the inclusion $M_n(A) \subseteq M_n(B)$ has real rank zero. Thus, $A \otimes \mathcal{K} \subseteq B \otimes \mathcal{K}$ has real rank zero by Proposition 4.3.1. $\square$

The next theorem provides a converse for Theorem 4.4.1.

Theorem 4.4.2. *Let $A \subseteq B$ be a unital inclusion of separable* C*-*algebras such that A is exact, has stable rank one, and no nonzero finite dimensional representations. If $A \otimes \mathcal{K} \subseteq B \otimes \mathcal{K}$ has real rank zero, then $\gamma^*\left(\mathrm{Aff}(T(A))\right) \subseteq \overline{\rho_B(K_0(B))}$.*

Proof. If $T(B)$ is the empty set, then the conclusion is vacuously true, so let us assume that $T(B) \neq \emptyset$. Let f be a strictly positive continuous affine function on $T(A)$. Since A is unital and exact, all quasitraces are traces by [72]. Then, there exists a positive element $a \in A \otimes \mathcal{K}$ such that $d_\tau(a) = f(\tau)$ for all $\tau \in T(A)$ ([4, Theorem 7.14] see also the second theorem in [4, pp.38]). Since the inclusion $A \otimes \mathcal{K} \subseteq B \otimes \mathcal{K}$ has real rank zero, there exists an approximate unit of projections $(p_n)_{n \geq 1}$ for the hereditary subalgebra generated by $a \in A \otimes \mathcal{K}$ in $B \otimes \mathcal{K}$. By [150, Proposition 1.2], this approximate unit can be assumed to be increasing.

In particular, for any $\tau \in T(B)$, $\tau(p_n) = d_\tau(p_n)$ converges pointwise to $d_\tau(a) = f(\tau)$, where $f(\tau)$ denotes the evaluation of f at the restriction of τ to A. But since f is continuous, Dini's theorem gives that $\tau(p_n)$ converges uniformly to $f(\tau)$. Thus, $\gamma^*(f)$ lies in $\overline{\rho_B(K_0(B))}$.

Now, if f is any continuous affine function on $T(A)$, there exists some $n \in \mathbb{N}$ such that $f + n$ is strictly positive. Since $\gamma^*(n)$ is the constant function n and the inclusion is unital, $\gamma^*(n) \in \overline{\rho_B(K_0(B))}$. Hence, $\gamma^*(f)$ lies in $\overline{\rho_B(K_0(B))}$. $\square$

Remark 4.4.3. I do not know if real rank zero passes to matrix amplifications, hence the assumption that the stabilised inclusion has real rank zero in the theorem above.

Question 4.4.4. *Let $A \subseteq B$ be an inclusion of* C*-*algebras and suppose that $A \subseteq B$ has real rank zero. Does the induced inclusion $M_2(A) \subseteq M_2(B)$ have real rank zero?*

In fact, further assuming strict comparison on B, similarly to [117, Theorem 7.2], we can show that projections coming from $A \otimes \mathcal{K}$ can be weakly divided in $K_0(B)_+$.

Theorem 4.4.5. *Let $A \subseteq B$ be a unital inclusion of separable* C^**-algebras such that A is exact, has stable rank one, and no nonzero finite dimensional representations. Suppose further that B is simple, stably finite, and has strict comparison of projections with respect to tracial states. If $A \otimes \mathcal{K} \subseteq B \otimes \mathcal{K}$ has real rank zero, then for all $x \in K_0(A)_+$ and all $n \geq 2$, there exist $g_x^{(n)}, h_x^{(n)} \in K_0(B)_+$ such that*

$$K_0(\iota)(x) = n g_x^{(n)} + (n+1) h_x^{(n)},$$

where ι denotes the inclusion map of A into B.

Proof. Let $x \in K_0(A)_+$ and $n \geq 2$. If $K_0(\iota)(x) = 0$, then we are done, so let us suppose that $K_0(\iota)(x) \in K_0(B)_+$ is nonzero. Since B is simple, stably finite and $K_0(\iota)(x) \in K_0(B)$ is positive and nonzero, it follows that $K_0(\iota)(x)$ is an order unit (see [6, Corollary 6.3.6]). As $T(B)$ is compact by unitality of B, we get that $\inf\limits_{\tau \in T(B)} \tau(K_0(\iota)(x)) > 0$. Let $\epsilon < \frac{1}{2n(n+1)} \inf\limits_{\tau \in T(B)} \tau(K_0(\iota)(x))$.

Recall that $\rho_A(x) : T(A) \to \mathbb{R}$ is the affine continuous function given by $\rho_A(x)(\tau) = \tau(x)$ for any $\tau \in T(A)$. Then $\rho_A(x) \in \mathrm{Aff}(T(A))$ and since this is a vector space, $\alpha \rho_A(x) \in \mathrm{Aff}(T(A))$ for any $\alpha \in \mathbb{R}$. Take $\alpha_n = \frac{\frac{1}{n} + \frac{1}{n+1}}{2}$ and note that $\gamma^*(\alpha_n \rho_A(x))(\tau) = \alpha_n \tau(K_0(\iota)(x))$ for any $\tau \in T(B)$. Then, by Theorem 4.4.2, the image of the canonical map $\gamma^* : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ induced by ι is contained in $\overline{\rho_B(K_0(B))}$. Hence, there exists $y_n \in K_0(B)$ such that

$$|\tau(y_n) - \alpha_n \tau(K_0(\iota)(x))| < \epsilon$$

for any $\tau \in T(B)$. Therefore,

$$\begin{aligned} n\tau(y_n) &< n\alpha_n \tau(K_0(\iota)(x)) + n\epsilon \\ &\leq n\Big(\alpha_n + \frac{1}{2n(n+1)}\Big)\tau(K_0(\iota)(x)) \end{aligned}$$

$$= \tau(K_0(\iota)(x))$$

for any $\tau \in T(B)$. By the choice of ϵ and α_n, we also get that $\tau(y_n) > 0$ for any $\tau \in T(B)$. But B has strict comparison, so $0 \leq ny_n \leq K_0(\iota)(x)$. A similar calculation using that $(n+1)\tau(y_n) > (n+1)\alpha_n\tau(K_0(\iota)(x)) - (n+1)\epsilon$ and strict comparison of B yields that $(n+1)y_n \geq K_0(\iota)(x)$.

Finally, set $g_x^{(n)} = (n+1)y_n - K_0(\iota)(x)$ and $h_x^{(n)} = K_0(\iota)(x) - ny_n$. It is immediate to check that $K_0(\iota)(x) = ng_x^{(n)} + (n+1)h_x^{(n)}$. □

4.5 Purely infinite inclusions

Following Kirchberg and Rørdam in [83], a (not necessarily simple) C*-algebra is *purely infinite* if every nonzero positive element $a \in A$ is properly infinite, in the sense that $a \oplus a$ is Cuntz subequivalent to a. In this section we consider inclusions $A \subseteq B$ where each nonzero positive element in A is properly infinite in B.

A classical result of Zhang ([148]) shows that any simple purely infinite C*-algebra has real rank zero. The next theorem can be thought of as the corresponding result for inclusions.

Theorem 4.5.1. *Let $A \subseteq B$ be an inclusion of* C**-algebras such that every nonzero positive element in A is full and properly infinite in B. Then $A \subseteq B$ has real rank zero. Moreover, for any nonzero positive element $a \in A$, $\overline{aBa}$ has an increasing approximate unit of full, properly infinite projections.*

Proof. Let $a \in A_+$ be nonzero and $\epsilon > 0$. To show that the inclusion $A \subseteq B$ has real rank zero, it suffices to show that there exists a projection $p \in \overline{aBa}$ such that $p(a-\epsilon)_+ = (a-\epsilon)_+$. Consider the spectrum of a in B denoted by $\sigma(a)$. If $\sigma(a) \cap (\epsilon/2, \epsilon) = \emptyset$, then we can take p to be the orthogonal complement of the spectral projection of a supported on $(\epsilon/2, \epsilon)$.

Else, we will construct such a projection p by obtaining a suitable scaling element in the sense of [8, Definition 1.1] (see also [102, Remark 2.4]).[20] Following [102, Remark 2.4], to obtain such a projection p, it suffices to find a contraction $x \in \overline{aBa}$ such that

$$x^*xxx^* = xx^*, \quad x^*x(a-\epsilon)_+ = (a-\epsilon)_+, \quad xx^*(a-\epsilon)_+ = 0.$$

Let $f : [0, \infty) \to [0, 1]$ be a continuous function with open support $(\epsilon/2, \epsilon)$. Then, $f(a)$ is full and properly infinite in B, so $a \precsim f(a)$ ([83, Proposition 3.5]), with the Cuntz subequivalence taking place in the hereditary subalgebra $\overline{aBa}$.

Applying [102, Remark 2.5] to the subequivalence $a \precsim f(a)$, we can find a contraction $x \in \overline{aBa}$ such that

$$x^*x(a-\epsilon/2)_+ = (a-\epsilon/2)_+ \tag{4.5.1}$$

and

$$xx^* \in \overline{f(a)Bf(a)} \subseteq \overline{(a-\epsilon/2)_+B(a-\epsilon/2)_+}.$$

It follows that $(x^*x)(xx^*) = xx^*$, so x is a scaling element in the sense of [8, Definition 1.1]. Moreover, as $(a-\epsilon)_+ \in \overline{(a-\epsilon/2)_+B(a-\epsilon/2)_+}$, $x^*x(a-\epsilon)_+ = (a-\epsilon)_+$. Furthermore, $xx^*(a-\epsilon)_+ = 0$, as $f(a)$ and $(a-\epsilon)_+$ are orthogonal. Following [102, Remark 2.4], $v = x + (1-x^*x)^{1/2}$ is an isometry in the unitisation of $\overline{aBa}$ such that $p = 1 - vv^*$ is a projection in $\overline{aBa}$ with $p(a-\epsilon)_+ = (a-\epsilon)_+$. Hence $A \subseteq B$ has real rank zero.

Note that the projection p found above is full and properly infinite in B since $(a-\epsilon)_+ \precsim p$. Applying the proof above for every $n \in \mathbb{N}$, one can find a sequence of properly infinite projections $p_n \in \overline{\left(a-\frac{1}{n+1}\right)_+ B \left(a-\frac{1}{n+1}\right)_+}$ such that $p_n\left(a-\frac{1}{n}\right)_+ =$

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[20] A contraction x is called a scaling element if $x^*xxx^* = xx^*$.

$\left(a - \frac{1}{n}\right)_+$. Hence, $(p_n)_{n \geq k}$, for any k such that $p_k \neq 0$, is an increasing approximate unit for $\overline{aBa}$ consisting of full, properly infinite projections. $\square$

Corollary 4.5.2. *Let $A \subseteq B$ be an inclusion of* C*-*algebras. Then the following are equivalent:*

(i) every nonzero positive element in A is full and properly infinite in B;

(ii) for any nonzero positive element $a \in A$, $\overline{aBa}$ has a full properly infinite projection.

Proof. Theorem 4.5.1 shows that (i) implies (ii), so it remains to show that (ii) implies (i). Let $a \in A$ be a nonzero positive element and let $p \in \overline{aBa}$ be a full properly infinite projection. Then, a is in the ideal generated by p, so $a \precsim p$ ([83, Proposition 3.5]). Since $p \in \overline{aBa}$, it also follows that $p \precsim a$ ([83, Proposition 2.7 (i)]), so a is full. Then

$$a \oplus a \precsim p \oplus p \precsim p \precsim a,$$

which implies that a is properly infinite. $\square$

Remark 4.5.3. Inclusions with infinite behaviour of this type have been essential in proving that various natural constructions give rise to purely infinite C*-algebras. For example, in [119, Theorem 3.3], Rørdam and Sierakowski showed that, under some assumptions, the crossed product $A \rtimes_r G$ is purely infinite if and only if any nonzero positive element in A is properly infinite in $A \rtimes_r G$. Similarly, in [11], Bönicke and Li studied when a groupoid C*-algebra is purely infinite. If $\mathcal{G}$ is a groupoid with unit space $\mathcal{G}^{(0)}$, they used the canonical inclusion $C_0(\mathcal{G}^{(0)}) \subseteq \mathrm{C}^*_r(\mathcal{G})$ to describe when $\mathrm{C}^*_r(\mathcal{G})$ is purely infinite (see [11, Theorem B]).

Assuming nuclearity and $\mathcal{O}_\infty$-stability, we can strenghten the result in Theorem 4.5.1 and show that *-homomorphisms classified in [61] are approximately factoring through a C*-algebra of real rank zero. In many cases of interest, this algebra can be assumed to be a Kirchberg algebra. The idea of the proof below is to use a result of Kumjian ([87]) to find a simple purely infinite C*-algebra C which is KK-equivalent to the domain of our given morphism. Then, using the classification results in [61], we can show that the given morphism is equivalent to a morphism which factors through C.

First, let us recall the following definitions from Theorem 2.7.5. Let A and B be C*-algebras with A separable and let $\theta : A \to B$ be a *-homomorphism. Then, following [60, Definition 3.16], θ is said to be *$\mathcal{O}_\infty$-stable* if $\mathcal{O}_\infty$ embeds unitally in $B_\infty \cap \theta(A)'/\operatorname{Ann}(\theta(A))$. Two *-homomorphisms $\phi, \psi : A \to B$, with A separable, are *approximately Murray-von Neumann equivalent* if there exists $v \in B_\infty$ such that $v^*\phi(a)v = \psi(a)$ and $v\psi(a)v^* = \phi(a)$ for all $a \in A$.

If $\theta : A \to B$ is a full, $\mathcal{O}_\infty$-stable *-homomorphism, then any nonzero and positive element $\theta(a)$ is properly infinite in B ([13, Lemma 4.4]). Thus, the inclusion $\theta(A) \subseteq B$ has real rank zero by Theorem 4.5.1. In fact, under some mild additional assumptions, we can show that θ is equivalent to a *-homomorphism which factors through a C*-algebra of real rank zero.

Theorem 4.5.4. *Let A be a separable exact* C*-*algebra, B be a σ-unital* C*-*algebra and $\theta : A \to B$ be a full, nuclear, $\mathcal{O}_\infty$-stable *-homomorphism. Then θ is approximately Murray-von Neumann equivalent to a *-homomorphism which factors through a separable exact* C*-*algebra C with real rank zero. Moreover, if A is* KK-*equivalent to a nuclear* C*-*algebra,*[21] *then C can be taken to be a Kirchberg algebra.*[22]

Proof. Building on the work of Pimsner in [106], Kumjian showed in [87] that there exists a simple, separable, purely infinite C*-algebra C and an inclusion $\iota : A \to C$ such that ι is a KK-equivalence. The existence of such C is proved in [87, Theorem

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[21] For example if A is nuclear or if it satisfies the UCT.

[22] Recall that a simple, separable, nuclear, purely infinite C*-algebra is called a Kirchberg algebra.

3.1] (note that nuclearity is only needed to obtain nuclearity of C), while the fact that ι induces a KK-equivalence follows from [106, Corollary 4.5]. Since A is exact, C can also be chosen to be exact by [44, Corollary 3.6].

Since the product of a KK-element with a $\mathrm{KK}_{\mathrm{nuc}}$-element is in $\mathrm{KK}_{\mathrm{nuc}}$ ([124, Proposition 2.2(b)]), the product $\mathrm{KK}_{\mathrm{nuc}}(\theta) \circ \mathrm{KK}(\iota)^{-1}$ is an element of $\mathrm{KK}_{\mathrm{nuc}}(C, B)$. Then, there exists a full, nuclear, $\mathcal{O}_\infty$-stable *-homomorphism $\psi : C \to B$ such that $\mathrm{KK}_{\mathrm{nuc}}(\psi) = \mathrm{KK}_{\mathrm{nuc}}(\theta) \circ \mathrm{KK}(\iota)^{-1}$([61, Theorem A]). Now we apply the uniqueness part of the classification of full, nuclear, $\mathcal{O}_\infty$-stable *-homomorphisms. Precisely, since $\mathrm{KK}_{\mathrm{nuc}}(\psi \circ \iota) = \mathrm{KK}_{\mathrm{nuc}}(\theta)$, θ and $\psi \circ \iota$ are approximately Murray-von Neumann equivalent ([61, Theorem 8.10]). The fact that C has real rank zero follows from [148] (alternatively apply Theorem 4.5.1 to the inclusion $C \subseteq C$).

Suppose now that A is KK-equivalent to a separable, nuclear C*-algebra D. Let $x \in \mathrm{KK}(A, D)$ be an element inducing the equivalence. Note that since D is nuclear, $\mathrm{KK}_{\mathrm{nuc}}(A, D) = \mathrm{KK}(A, D)$. Applying [87, Theorem 3.1] to the C*-algebra D, there exists a simple, separable, purely infinite C*-algebra C and an inclusion $\iota : D \to C$ such that ι is a KK-equivalence. Since D is nuclear, it follows from [87, Theorem 3.1] that we can further assume that C is nuclear, so C is a Kirchberg algebra.

Then, there exists a full, nuclear, $\mathcal{O}_\infty$-stable *-homomorphism $\phi : A \to C$ such that

$$\mathrm{KK}_{\mathrm{nuc}}(\phi) = \mathrm{KK}_{\mathrm{nuc}}(\iota) \circ x$$

([61, Theorem A]). Applying existence again, there exists a full, nuclear, $\mathcal{O}_\infty$-stable *-homomorphism $\psi : C \to B$ such that

$$\mathrm{KK}_{\mathrm{nuc}}(\psi) = \mathrm{KK}_{\mathrm{nuc}}(\theta) \circ x^{-1} \circ \mathrm{KK}_{\mathrm{nuc}}(\iota)^{-1}.$$

Since $\mathrm{KK}_{\mathrm{nuc}}(\psi \circ \phi) = \mathrm{KK}_{\mathrm{nuc}}(\theta)$, θ and $\psi \circ \phi$ are approximately Murray-von Neumann equivalent by [61, Theorem 8.10]. Hence, θ is approximately Murray-von Neumann equivalent to a *-homomorphism which factors through a Kirchberg algebra. $\square$

There are natural inclusions arising from dynamics that can be shown to have real rank zero. In particular, this is the case for the canonical inclusion of the reduced group C*-algebra into its uniform Roe algebra.

Theorem 4.5.5. *Let G be a countable, discrete, exact group. If we consider the action of G on $\ell^\infty(G)$ by left translation, then the canonical inclusion $\mathrm{C}^*_r(G) \subseteq \ell^\infty(G) \rtimes_r G$ has real rank zero. In fact, it factors through a* C**-algebra of real rank zero.*

Proof. Let us denote this inclusion by ι. If the group is non-amenable, there is an action α of G on the Cantor space X such that $C(X) \rtimes_{\alpha,r} G$ is a Kirchberg algebra ([119, Theorem 6.11]). Since the C*-algebra $C(X) \rtimes_{\alpha,r} G$ is simple and purely infinite, it has real rank zero by [148].

Suppose now that the group G is amenable. Then, combining [125, Theorem 5.4] and [125, Lemma 5.2], there exists a totally disconnected, compact Hausdorff space X and a free minimal action α of G on X such that the transformation groupoid $X \rtimes_\alpha G$ is almost finite.[23] Since the groupoid is also minimal and has totally disconnected unit space, the crossed product $C(X) \rtimes_\alpha G$ has real rank zero by [5, Theorem 3.2].[24]

We will now combine the two cases above. Let X be a compact Hausdorff space and an action α of G on X such that the crossed product $C(X) \rtimes_\alpha G$ has real rank zero. Fix $x \in X$ and let $C(X) \to \ell^\infty(G)$ be the equivariant *-homomorphism given by $f \mapsto (f(\alpha_g(x)))_{g \in G}$. This induces a unital *-homomorphism $C(X) \rtimes_{\alpha,r} G \to \ell^\infty(G) \rtimes_r G$.

It is readily seen that in both cases, the composition

$$\mathrm{C}^*_r(G) \to C(X) \rtimes_{\alpha,r} G \to \ell^\infty(G) \rtimes_r G \tag{4.5.2}$$

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[23] All groupoids in [125] are assumed to be Hausdorff.

[24] Note that in [5], minimal almost finite groupoids are assumed to be ample with compact unit space. In this case, ampleness is ensured by the fact that X is totally disconnected and compact. Moreover, the definitions of almost finiteness in [5] and [125] coincide, as observed in the comments following [5, Remark 2.1].

is the canonical inclusion ι. Hence, the inclusion $\mathrm{C}^*_r(G) \subseteq \ell^\infty(G) \rtimes_r G$ factors through a C^*-algebra of real rank zero. $\square$

Remark 4.5.6. Note that the inclusion $\mathrm{C}^*_r(G) \subseteq \ell^\infty(G) \rtimes_r G$ has real rank zero even if neither $\mathrm{C}^*_r(G)$ nor $\ell^\infty(G) \rtimes_r G$ have real rank zero. For example, take $G = \mathbb{F}_2 \times \mathbb{Z}^2$ and note that its uniform Roe algebra $\ell^\infty(G) \rtimes G$ does not have real rank zero by [90, Corollary 3.5]. Furthermore, if $\mathrm{C}^*_r(G)$ had real rank zero, then $\mathrm{C}^*_r(G) \otimes \mathcal{O}_2$ would have real rank zero by [102, Corollary 4.7], and hence the primitive ideal space of $\mathrm{C}^*_r(G)$ would have a basis of compact open subsets by [102, Corollary 4.3(ii)]. However, the primitive ideal space of $\mathrm{C}^*_r(G) \cong \mathrm{C}^*_r(\mathbb{F}_2) \otimes C(\mathbb{T}^2)$ is $\mathbb{T}^2$ (by simplicity of $\mathrm{C}^*_r(\mathbb{F}_2)$ shown in [108]), which is not totally disconnected, and hence it does not have a basis of compact open subsets. Thus, $\mathrm{C}^*_r(G)$ does not have real rank zero either.

Another natural inclusion from dynamics is realised by embedding the reduced group C^*-algebra into the reduced crossed product of $C(\partial_F G)$ by G. Here $\partial_F G$ denotes the Furstenberg boundary of G and we refer the reader to [79] for a detailed description.

Theorem 4.5.7. *Let G be a countable, discrete, non-elementary hyperbolic group with no nontrivial finite normal subgroups. Then the canonical inclusion $\mathrm{C}^*_r(G) \subseteq C(\partial_F G) \rtimes_r G$ has real rank zero. In fact, it factors through a* C^**-algebra of real rank zero.*

Proof. If we denote the Gromov boundary of G by ∂G, then there is an equivariant inclusion $C(\partial G) \subseteq C(\partial_F G)$ ([79, Remark 5.6]). Therefore, there is an induced unital *-homomorphism $C(\partial G) \rtimes_r G \to C(\partial_F G) \rtimes_r G$. It is readily seen that the composition

$$\mathrm{C}^*_r(G) \to C(\partial G) \rtimes_r G \to C(\partial_F G) \rtimes_r G \tag{4.5.3}$$

is the canonical inclusion $\mathrm{C}^*_r(G) \subseteq C(\partial_F G) \rtimes_r G$.

We will now show that the crossed product $C(\partial G) \rtimes_r G$ is simple and purely infinite. Since the group G is hyperbolic, the elements of G are elliptic, parabolic, or hyperbolic ([64, Section 8.3]). Then, as G is non-elementary and has no nontrivial finite normal subgroups, all elements of G are hyperbolic ([64, Proposition 8.28, Theorem 8.29]).[25] Therefore, any element of G fixes exactly two points of ∂G ([64, Corollary 8.20]). Hence, the interior of fixed points is empty, so the action of G on ∂G is topologically free ([89, Definition 4]). Moreover, by [89, Example 2.1], the action of G on ∂G is a strong boundary action (in the sense of [89, Definition 1]). Hence, $C(\partial G) \rtimes_r G$ is simple and purely infinite by [89, Theorem 5]. In particular, $C(\partial G) \rtimes_r G$ has real rank zero by [148]. Thus, the conclusion follows. $\square$

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[25] An element $g \in G$ is called hyperbolic if for any $x \in \partial G$, the orbit $n \in \mathbb{Z} \mapsto g^n x$ is a quasi-geodesic (see [64, Proposition 8.21]).

Chapter 5

Bundles of KMS states on classifiable C*-algebras

The content of this chapter is based on results I obtained in a single-author paper and it is essentially contained in [97]. We will start with some preliminaries on KMS states and Elliott's classification of (nonsimple) A$\mathbb{T}$-algebras of real rank zero from [49]. In Section 5.4 we will prove Theorem 16, while in Section 5.5 we will prove Theorem 17.

5.1 KMS states

In this section, we record some facts about KMS states. We refer the reader to [133] for a detailed account of the topic. A *flow* θ on a C*-algebra A is a continuous one-parameter group of automorphisms $(\theta_t)_{t\in\mathbb{R}}$.

Definition 5.1.1. Let A be a C*-algebra, θ be a flow on A, and $\beta \in \mathbb{R}$. Then, ω is said to be a β-KMS *weight* for θ on A if ω is a weight on A such that

$$\omega \circ \theta_t = \omega \text{ for all } t \in \mathbb{R}$$

and

$$\omega(a^*a) = \omega\left(\theta_{-\frac{i\beta}{2}}(a)\theta_{-\frac{i\beta}{2}}(a)^*\right),\ a \in A \text{ analytic.}^{26}$$

If, in addition, ω has norm 1, then it is called a β-KMS state for θ on A.

Remark 5.1.2. Note that a 0-KMS state for θ is a θ-invariant tracial state.

If A is a separable, unital C*-algebra and θ is a flow on A, we consider the collection of KMS states for θ on A. Following the notation in [55, Section 2], for each $\beta \in \mathbb{R}$, let S^θ_β denote the set of β-KMS states for θ. If we denote the set of states on A by $S(A)$, let

$$S^\theta = \left\{(\omega, \beta) \in S(A) \times \mathbb{R} : \omega \in S^\theta_\beta\right\},$$

and equip it with the relative topology from $S(A) \times \mathbb{R}$. In particular, since A is separable, the topology on S^θ is metrisable. Let $\pi^\theta : S^\theta \to \mathbb{R}$ be the projection onto the second coordinate. Then, the pair (S^θ, π^θ) will be called the KMS *bundle* of θ. An abstract characterisation of this bundle can be found in [55] as we review below.

Fix a second countable locally compact Hausdorff space S and let π be a continuous map from S to $\mathbb{R}$. The pair (S, π) is said to be a *simplex bundle* if the inverse image $\pi^{-1}(t)$ is a compact metrisable Choquet simplex in the relative topology from S for any $t \in \mathbb{R}$. For such pairs (S, π), we denote by $\mathcal{A}(S, \pi)$ the set of continuous functions f from S to $\mathbb{R}$ such that the restriction $f|_{\pi^{-1}(t)}$ of f to $\pi^{-1}(t)$ is affine for any $t \in \mathbb{R}$.

Definition 5.1.3. [55, Definition 2.1] A simplex bundle (S, π) is *proper*, if

- $\pi^{-1}(K)$ is compact in S for any compact subset K of $\mathbb{R}$,
- for any $x \neq y$ in S, there exists $f \in \mathcal{A}(S, \pi)$ such that $f(x) \neq f(y)$.

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[26] An element $a \in A$ is analytic for θ if the map $t \in \mathbb{R} \mapsto \theta_t(a) \in A$ extends to an entire analytic map $\mathbb{C} \to A$.

If S is also compact, then (S, π) is called a *compact simplex bundle.* Two proper simplex bundles (S, π) and (S', π') are *isomorphic* if there exists a homeomorphism $\phi : S \to S'$ such that

$$\pi' \circ \phi = \pi \text{ and } \phi : \pi^{-1}(\beta) \to {\pi'}^{-1}(\beta) \text{ is affine for all } \beta \in \mathbb{R}.$$

Remark 5.1.4. By [133, Theorem 9.2.2], the collection of KMS states for a flow on a unital, separable C*-algebra is a proper simplex bundle. Moreover, a proper simplex bundle (S, π) is second countable, locally compact, and Hausdorff, so the topology on S is metrisable.

Lemma 5.1.5. *Let $\eta : (S', \pi') \to (S, \pi)$ be a bijection between proper simplex bundles such that $\pi \circ \eta = \pi'$. If either η or η^{-1} is continuous, then η is a homeomorphism.*

Proof. Suppose that η is continuous. It remains to show that η^{-1} is continuous. We show that η maps closed sets to closed sets. By [101, Corollary], it suffices to check that η is a proper map i.e. the preimage of any compact set is compact. Let $K \subseteq S$ be compact and note that $\pi(K)$ is compact since π is continuous. Since $\pi \circ \eta = \pi'$, it follows that

$$\eta^{-1}(K) = \eta^{-1}(\pi^{-1}(\pi(K))) = (\pi')^{-1}(\pi(K)).$$

But $(\pi')^{-1}(\pi(K))$ is compact since (S', π') is a proper simplex bundle (see Definition 5.1.3), so $\eta^{-1}(K)$ is compact, which finishes the argument. The case when η^{-1} is assumed to be continuous is completely symmetric. $\square$

5.2 Elliott's classification of A$\mathbb{T}$-algebras of real rank zero

This section contains a summary of Elliott's classification of (possibly nonsimple) A$\mathbb{T}$-algebras of real rank zero from [49]. Recall that an A$\mathbb{T}$-algebra is an inductive limit of algebras of the form $\bigoplus_{i=1}^{k} M_{n_i}(C(\mathbb{T}))$.

Let A be a unital C*-algebra, $K_*(A)$ be the graded group $K_0(A) \oplus K_1(A)$, equipped with the order

$$K_*(A)_+ = \left\{ \begin{array}{c} ([p]_0, [u \oplus (1_n - p)]_1) : p \in \mathcal{P}(M_n(A)) \text{ for some } n \in \mathbb{N}, \\ u \in \mathcal{U}(pM_n(A)p) \end{array} \right\},$$

and

$$D_*(A) = \{([p]_0, [u \oplus (1 - p)]_1) \in K_*(A)_+ \colon p \in A\}.$$

If A is nonunital, then define

$$K_*(A)_+ = K_*(A) \cap K_*(A^\dagger)_+,$$

and

$$D_*(A) = K_*(A) \cap D_*(A^\dagger),$$

where $A^\dagger$ is the minimal unitisation of A. If A is an A$\mathbb{T}$-algebra of real rank zero, consider the pair $(K_*(A), K_*(A)_+)$. Since A also has stable rank one, the pair $(K_*(A), K_*(A)_+)$ is a graded ordered group ([49, Theorem 3.2]). Then, the graded group $K_*(A)$ together with the *graded dimension range* $D_*(A)$ determine the pair $(K_*(A), K_*(A)_+)$ and constitutes the invariant used by Elliott in his classification of A$\mathbb{T}$-algebras of real rank zero. If A is simple, the invariant $(K_*(A), K_*(A)_+)$ reduces to the usual K-theory triple $(K_0(A), K_0(A)_+, K_1(A))$ appearing in the Elliott invariant of a nonunital C*-algebra.

We record the following classification of isomorphisms between A$\mathbb{T}$-algebras of real rank zero from [49]. Note that the statement does not appear in [49], but, as observed in [49, Remark 7.3], the proof of [49, Theorem 7.1] not only produces a classification of A$\mathbb{T}$-algebras of real rank zero, but of isomorphisms between them.

Theorem 5.2.1 ([49]). *Let A and B be A$\mathbb{T}$-algebras of real rank zero. Then A is isomorphic to B if and only if there is a graded group isomorphism $\alpha : K_*(A) \to K_*(B)$ such that $\alpha(D_*(A)) = D_*(B)$. Moreover, for each such α, there exists an isomorphism $\phi : A \to B$ such that $K_*(\phi) = \alpha$.*

We also recall the range-of-the-invariant result in [49] for A$\mathbb{T}$-algebras of real rank zero.

Theorem 5.2.2 ([49, Theorem 8.1]). *Let $G_* = G_0 \oplus G_1$ be a graded ordered group which is the inductive limit of a sequence $G_{*1} \to G_{*2} \to \ldots$ of graded ordered groups such that each $G_{*i} = G_{0i} \oplus G_{1i}$ is the direct sum of finitely many basic building blocks of the form $\mathbb{Z} \oplus \mathbb{Z}$ with the order determined by strict positivity in the first component. If G_* has the Riesz interpolation property, then there exists a stable A$\mathbb{T}$-algebra of real rank zero A such that $K_*(A) \cong G_*$ and $K_*(A)_+ \cong (G_*)_+$.*

We record the following folklore result relating traces and positive homomorphisms on the K_0-group of A$\mathbb{T}$-algebras of real rank zero.

Proposition 5.2.3. *Let B be a separable, stable A$\mathbb{T}$-algebra of real rank zero. Then the canonical map from the densely defined lower semicontinous traces on B to positive group homomorphisms $K_0(B) \to \mathbb{R}$ is an affine homeomorphism.*

Proof. This follows from [69, Theorem 12.3] (or [9, Theorem III.1.3]) since any lower semicontinuous extended quasitrace on an A$\mathbb{T}$-algebra is an extended trace ([25, Theorem 6]). □

5.3 Finite Rokhlin dimension

The Rokhlin dimension of a group action on a unital C*-algebra was introduced in [75] as a tool to obtain finite nuclear dimension and hence $\mathcal{Z}$-stability of the corresponding crossed product. In [74], Hirshberg and Phillips extended finite Rokhlin dimension to actions on possibly nonunital C*-algebras. We will use the simplified definition from [76].

Definition 5.3.1 ([76, Definition 4.6])**.** An automorphism γ of a C*-algebra A is said to have *Rokhlin dimension d* if d is the least nonnegative integer with the following property. For any finite set $F \subseteq A$, integer $p \geq 1$ and $\epsilon > 0$, there are positive contractions $f_0^{(l)}, \dots, f_{p-1}^{(l)} \in A$, $l \in \{0, 1, \dots, d\}$, such that:

(i) $\|f_k^{(l)} f_j^{(l)} a\| < \epsilon$ for every $a \in F$, $l \in \{0, 1, \dots, d\}$, $j \neq k \in \{0, 1, \dots, p-1\}$;

(ii) $\left\| \left(\sum_{l=0}^{d} \sum_{k=0}^{p-1} f_k^{(l)} \right) a - a \right\| < \epsilon$ for every $a \in F$;

(iii) $\|[f_j^{(l)}, a]\| < \epsilon$ for every $a \in F$, $l \in \{0, 1, \dots, d\}$, $j \in \{0, 1, \dots, p-1\}$;

(iv) $\left\| \left(\gamma(f_j^{(l)}) - f_{j+1}^{(l)} \right) a \right\| < \epsilon$ for every $a \in F$, $l \in \{0, 1, \dots, d\}$, $j \in \{0, 1, \dots, p-1\}$, where $f_p^{(l)} := f_0^{(l)}$.

We will use finite Rokhlin dimension as a tool to detect when densely defined lower semicontinuous traces on a crossed product by $\mathbb{Z}$ can be identified with densely defined lower semicontinuous invariant traces on the algebra. The next lemma essentially follows from [92, Proposition 2.3], as the proof also works in the nonunital and nonsimple setting. That the proof works in the nonunital setting was previously observed in [76, Lemma 4.8]. However, the proof in [92, Proposition 2.3], only deals with bounded traces. To generalise the result for densely defined lower semicontinuous traces, we will further assume the existence of an approximate unit of projections.

Lemma 5.3.2 (cf.[92, Proposition 2.3])**.** *Let A be a separable* C*-*algebra with an approximate unit of projections $(q_r)_{r \in \mathbb{N}}$ and γ be an automorphism of A which has finite*

Rokhlin dimension. Then the restriction map from densely defined lower semicontinuous traces on $A \rtimes_\gamma \mathbb{Z}$ to γ-invariant densely defined lower semicontinuous traces on A is bijective.

Proof. Using the strategy in [132, Lemma 3.4], we will adapt Liao's proof of [92, Proposition 2.3]. Let $P : A \rtimes_\gamma \mathbb{Z} \to A$ be the canonical conditional expectation. Given a densely defined lower semicontinuous trace τ on $A \rtimes_\gamma \mathbb{Z}$, we will show that $\tau = \tau|_A \circ P$. Since q_r is a projection and hence contained in the Pedersen ideal of A (see [103, Remark 5.6.3]), we have that $\tau(q_r) < \infty$ (see for example [103, Proposition 5.6.7]).[27]Moreover,

$$\tau(x) = \lim_{r\to\infty} \tau(q_r x q_r)$$

and

$$\tau(P(x)) = \lim_{r\to\infty} \tau(q_r P(x) q_r)$$

for any positive element x in $A \rtimes_\gamma \mathbb{Z}$. Therefore, to show that $\tau = \tau|_A \circ P$, it suffices to show that

$$\tau(q_r x q_r) = \tau(q_r P(x) q_r)$$

for any positive element x in $A \rtimes_\gamma \mathbb{Z}$ and any $r \in \mathbb{N}$. Since τ is nonzero, we can assume that $\tau(q_r) > 0$ for any $r \in \mathbb{R}$.

Let u be the unitary in $\mathcal{M}(A \rtimes_\gamma \mathbb{Z})$ implementing the action. Since elements of the form au^n generate the crossed product $A \rtimes_\gamma \mathbb{Z}$, it suffices to show that $\tau(q_r a u^n q_r) = 0$ for any $a \in A$, $r \in \mathbb{N}$, and $n \in \mathbb{Z} \setminus \{0\}$. We will further assume that $n \geq 1$ as the case $n < 0$ follows in a similar fashion.

Recall that the automorphism γ has finite Rokhlin dimension, say $d \in \mathbb{N}$. Let $r, n \in \mathbb{N}$, $a \in A$ be a contraction, and $\epsilon > 0$. Let $p = 2n$ and $\epsilon' = \frac{\epsilon}{(6(d+1)n+1)\tau(q_r)}$. Using *(ii)* of Definition 5.3.1, it follows that there exist $\{f_k^{(\ell)} : 0 \leq k \leq p-1,\ 0 \leq \ell \leq d\}$ in

---

[27] The Pedersen ideal of A is the smallest two-sided dense ideal in A.

A such that

$$\left\| \left(\sum_{\ell=0}^{d} \sum_{k=0}^{p-1} f_k^{(\ell)} \right) a - a \right\| < \epsilon'. \tag{5.3.1}$$

Combining *(i)*, *(iii)*, and *(iv)* of Definition 5.3.1, we can further assume that

$$\| {f_k^{(\ell)}}^{\frac{1}{2}} a \gamma^n ({f_k^{(\ell)}}^{\frac{1}{2}}) \| < \epsilon' \quad (\ell = 0, 1, ..., d;\ k = 0, ..., p-1). \tag{5.3.2}$$

By *(iii)* of Definition 5.3.1, we can ensure that

$$\| {f_k^{(l)}}^{\frac{1}{2}} q_r - q_r {f_k^{(l)}}^{\frac{1}{2}} \| < \epsilon' \tag{5.3.3}$$

for any $0 \leq k \leq p-1$ and any $0 \leq l \leq d$.

Then

$$|\tau(q_r a u^n q_r)| \overset{(5.3.1)}{\leq} \left| \sum_{\ell=0}^{d} \sum_{k=0}^{p-1} \tau(q_r f_k^{(\ell)} a u^n q_r) \right| + \tau(q_r)\epsilon'. \tag{5.3.4}$$

Fix k, l and let $x := {f_k^{(l)}}^{\frac{1}{2}} a u^n$. Using that τ is a trace and that x is a contraction, we get that

$$\begin{aligned} |\tau(q_r x (q_r {f_k^{(l)}}^{\frac{1}{2}} - {f_k^{(l)}}^{\frac{1}{2}} q_r))| &= |\tau(q_r x (q_r {f_k^{(l)}}^{\frac{1}{2}} - {f_k^{(l)}}^{\frac{1}{2}} q_r) q_r)| \\ &\leq \tau(q_r) \| x (q_r {f_k^{(l)}}^{\frac{1}{2}} - {f_k^{(l)}}^{\frac{1}{2}} q_r) \| \\ &\overset{(5.3.3)}{\leq} \tau(q_r)\epsilon'. \end{aligned} \tag{5.3.5}$$

Similarly, one has that

$$\begin{aligned} |\tau({f_k^{(\ell)}}^{\frac{1}{2}} q_r {f_k^{(\ell)}}^{\frac{1}{2}} a u^n q_r) - \tau(q_r f_k^{(\ell)} a u^n q_r)| &= |\tau(q_r ({f_k^{(\ell)}}^{\frac{1}{2}} q_r - q_r {f_k^{(\ell)}}^{\frac{1}{2}}) x q_r)| \\ &\leq \tau(q_r)\epsilon'. \end{aligned}$$

Therefore, using the last two estimations above, we get that

$$|\tau(q_r {f_k^{(\ell)}}^{\frac{1}{2}} a u^n {f_k^{(\ell)}}^{\frac{1}{2}} q_r) - \tau(q_r f_k^{(\ell)} a u^n q_r)|$$

$$
\begin{aligned}
&\leq |\tau(q_r f_k^{(\ell)^{\frac{1}{2}}} a u^n q_r f_k^{(\ell)^{\frac{1}{2}}}) - \tau(q_r f_k^{(\ell)} a u^n q_r)| + \tau(q_r)\epsilon' \\
&= |\tau(f_k^{(\ell)^{\frac{1}{2}}} q_r f_k^{(\ell)^{\frac{1}{2}}} a u^n q_r) - \tau(q_r f_k^{(\ell)} a u^n q_r)| + \tau(q_r)\epsilon' \\
&\leq 2\tau(q_r)\epsilon'.
\end{aligned}
$$

Thus, from (5.3.4) we now get that

$$
|\tau(q_r a u^n q_r)| \leq \sum_{\ell=0}^{d} \sum_{k=0}^{p-1} \left| \tau(q_r f_k^{(\ell)^{\frac{1}{2}}} a u^n f_k^{(\ell)^{\frac{1}{2}}} q_r) \right| + (2(d+1)p+1)\tau(q_r)\epsilon'
$$

$$
\begin{aligned}
&= \quad \textstyle\sum_{\ell=0}^{d} \sum_{k=0}^{p-1} \left| \tau(q_r f_k^{(\ell)^{\frac{1}{2}}} a \gamma^n (f_k^{(\ell)^{\frac{1}{2}}}) u^n q_r) \right| + (2(d+1)p+1)\tau(q_r)\epsilon' \\
&\overset{(5.3.2)}{\leq} \quad ((d+1)p + 2(d+1)p + 1)\tau(q_r)\epsilon' \\
&= \quad (6(d+1)n+1)\tau(q_r)\epsilon' = \epsilon.
\end{aligned}
$$

Hence, $\tau(q_r a u^n q_r) = 0$ for any $r, n \in \mathbb{N}$, which yields that $\tau(q_r x q_r) = \tau(q_r P(x) q_r)$ for any $r \in \mathbb{N}$ and any $x \in A \rtimes_\gamma \mathbb{Z}$. Therefore, the canonical restriction induces an injection from the space of densely defined lower semicontinuous traces on $A \rtimes_\gamma \mathbb{Z}$ into the space of γ-invariant densely defined lower semicontinuous traces on A. Furthermore, given a γ-invariant trace τ on A, and the canonical conditional expectation $P : A \rtimes_\gamma \mathbb{Z} \to A$, $\tau \circ P$ is a trace on $A \rtimes_\gamma \mathbb{Z}$ which restricts to τ. To see the tracial property, let $a, b \in A$ and $m, n \in \mathbb{Z}$. Then, by the equivariance of the conditional expectation and its A-bimodule property, one gets that

$$
\tau \circ P(a u^n b u^m) = \tau \circ P(u^{-m} u^m a u^n b u^m) = \tau \circ \gamma^{-m} \circ P(u^m a u^n b) = \tau \circ \gamma^{-m} \circ P(u^m a u^n) b.
$$

Since τ is a γ-invariant trace and P is an A-bimodule map, we get that

$$
\tau \circ P(a u^n b u^m) = \tau(b P(u^m a u^n)) = \tau(P(b u^m a u^n)),
$$

which shows that $\tau \circ P$ is a trace on $A \rtimes_\gamma \mathbb{Z}$. Thus, this restriction is also surjective. $\square$

5.4 KMS bundles on Kirchberg algebras

In this section, we will prove Theorem 16. The construction of the required flow follows the strategy in [55, Section 5], together with some modifications that allow us to remove the assumption of no torsion in the K_1-group. We briefly describe the construction for the convenience of the reader, but we will often refer to [55, Section 5].

Throughout this section, let A be a unital UCT Kirchberg algebra and (S, π) be a proper simplex bundle such that $\pi^{-1}(0) = \emptyset$. The main step in proving Theorem 16 is realising A as a full corner of a crossed product $B \rtimes_\gamma \mathbb{Z}$, with B being a stable, possibly nonsimple, A$\mathbb{T}$-algebra with real rank zero. Comparing with the construction in [55, Theorem 5.1], it is precisely constructing B to be an A$\mathbb{T}$-algebra rather than an AF-algebra that allows for torsion in $K_1(A)$. We collect the differences to the proof of [55, Theorem 5.1] in Remark 5.4.11.

We first need to construct suitable K-theory groups, together with an isomorphism for each group. We will then obtain B and γ by using Elliott's classification of morphisms between stable, possibly nonsimple A$\mathbb{T}$-algebras with real rank zero from [49]. We proceed to build the K-theory of the desired A$\mathbb{T}$-algebra B. Since A will be identified with a full corner of a crossed product of B with $\mathbb{Z}$, the construction of $K_0(B)$ needs to recover $K_0(A)$ via the Pimsner-Voiculescu exact sequence. To recover the simplex bundle (S, π), we will ensure that $K_0(B)$ contains a dense subset of $\mathcal{A}(S, \pi)$. We will now sketch the relevant parts of the construction in [55, Theorem 5.1]. If $S = \emptyset$, then we can take θ to be the trivial flow on A, so assume that $S \neq \emptyset$ and $\pi^{-1}(0) = \emptyset$.

Lemma 5.4.1 (cf.[55, Theorem 5.1]). *Let (S, π) be a nonempty proper simplex bundle such that $\pi^{-1}(0) = \emptyset$. Then there exist a countable torsion free ordered group (G, G_+) and an automorphism α of (G, G_+) such that $G = \mathbb{Q}G_0$, where G_0 is a countable* *subgroup of* $\mathcal{A}(S,\pi)$*,* $\alpha(g)(x) = e^{-\pi(x)}g(x)$ *for any* $g \in G$ *and* $x \in S$*, and the following conditions hold:*

(i) *for any* $n \in \mathbb{N}$*,* $\epsilon > 0$*, and* $f \in \mathcal{A}(S,\pi)$ *such that* $\mathrm{supp}(f) \subseteq \pi^{-1}([-n,n])$*, there exists* $g \in G_0$ *with* $\mathrm{supp}(g) \subseteq \pi^{-1}([-n,n])$ *and*

$$\sup_{x\in S} |f(x) - g(x)| < \epsilon;$$

(ii) (G, G_+) *is a dimension group satisfying the strong Riesz interpolation property;*

(iii) *the maps* α *and* $\mathrm{id} - \alpha$ *are automorphisms of* G*, and* $\alpha(G_+) = G_+$*;*

(iv) *if* I *is an order ideal in* G *such that* $\alpha(I) = I$*, then* $I = \{0\}$ *or* $I = G$*;*

(v) *if* $\psi\colon G \to \mathbb{R}$ *is a positive group homomorphism such that* $\psi(1) = 1$*, where* 1 *denotes the constant function* 1 *on* S *and* $\beta \in \mathbb{R}$ *satisfies* $\psi \circ \alpha = e^{-\beta}\psi$*, then there exists a unique* $\omega \in \pi^{-1}(\beta)$ *such that* $\psi(g) = g(\omega)$ *for all* $g \in G$*.*

Proof. The construction of the pair (G, G_+) is the one provided in [55, Theorem 5.1]. Since the topology on S is second countable, we can choose a countable subgroup G_0 of $\mathcal{A}(S,\pi)$ satisfying *(i)* in the statement of the lemma. We can also ensure that

- the function

  $$x \mapsto e^{n\pi(x)}(1 - e^{-\pi(x)})^m f(x)$$

  is in G_0 for any $f \in G_0$ and any $n, m \in \mathbb{Z}$;

- the functions

  $$x \mapsto (\chi_{(-\infty,0]} \circ \pi)(x)e^{n\pi(x)}(1 - e^{-\pi(x)})^m$$

  and

  $$x \mapsto (\chi_{[0,\infty)} \circ \pi)(x)e^{n\pi(x)}(1 - e^{-\pi(x)})^m$$

are in G_0 for any $n, m \in \mathbb{Z}$. Note that the functions $\chi_{(-\infty,0]} \circ \pi$ and $\chi_{[0,\infty)} \circ \pi$ are continuous on S because $\pi^{-1}(0) = \emptyset$.

Let $G = \mathbb{Q}G_0$ and $G_+ = \{f \in G : f(x) > 0, x \in S\} \cup \{0\}$.

That (G, G_+) is a dimension group satisfying the strong Riesz interpolation property follows from [55, Lemma 5.3]. Since the function

$$x \mapsto e^{n\pi(x)}(1 - e^{-\pi(x)})^m f(x)$$

is in G_0 for any $f \in G_0$ and any $n, m \in \mathbb{Z}$, we get that α is an automorphism of G and $\alpha(G_+) = G_+$. Furthermore, note that $\pi^{-1}(0) = \emptyset$, so $\mathrm{id} - \alpha$ is also an automorphism of G. This proves *(iii)*. Condition *(iv)* follows from the proof of [55, Lemma 5.5], while the existence of an element $\omega \in \pi^{-1}(\beta)$ as in *(v)* is shown in [55, Lemma 5.6] (note that both the group G and the map ψ appearing in the proof of [55, Lemma 5.6] coincide with the 'G' and 'ψ' in the statement of the lemma). We will give the details for the convenience of the reader.

Let $\psi\colon G \to \mathbb{R}$ be a positive group homomorphism such that $\psi(1) = 1$, where 1 denotes the constant function 1 on S and let $\beta \in \mathbb{R}$ such that $\psi \circ \alpha = e^{-\beta}\psi$. Let $f \in G$ and we will prove that $|\psi(f)| \leq \sup\limits_{x \in S} |f(x)|$. If not, there exist $m, n \in \mathbb{N}$ such that $|f(x)| < \frac{n}{m} < |\psi(f)|$ for any $x \in S$. Since $-n < mf(x) < n$ for any $x \in S$, we get that $-n < mf < n$ in G. Moreover, $\psi(1) = 1$, so $|\psi(f)| \leq \frac{n}{m}$, which is a contradiction. Hence, ψ must be norm-contractive.

Let $\mathcal{A}_{\mathbb{R}}(S, \pi)$ be the subspace of $\mathcal{A}(S, \pi)$ consisting of functions that have a limit at infinity and let $\mathcal{A}_{\mathbb{R},0}(S, \pi)$ be the subspace of $\mathcal{A}_{\mathbb{R}}(S, \pi)$ consisting of functions vanishing at infinity. Note that every element of $\mathcal{A}_{\mathbb{R}}(S, \pi)$ can be approximated in norm by elements of the form

$$q\chi_{(-\infty,0]} \circ \pi + q\chi_{[0,\infty)} \circ \pi + f$$

for $q \in \mathbb{Q}$ and $f \in \mathcal{A}_{\mathbb{R},0}(S,\pi)$. Therefore, by *(i)* above and since the set of elements with compact support is dense in $\mathcal{A}_{\mathbb{R},0}(S,\pi)$, any element in $\mathcal{A}_{\mathbb{R},0}(S,\pi)$ can be approximated in norm by elements in G. Using the decomposition above, any element in $\mathcal{A}_{\mathbb{R}}(S,\pi)$ can be approximated in norm by elements in G, so we can extend ψ to a linear contraction on $\mathcal{A}_{\mathbb{R}}(S,\pi)$. Then, by the Hahn-Banach theorem, there exists a contractive extension of ψ to all continuous real-valued functions on S with a limit at infinity. For ease of notation, we will still denote this extension by ψ. Since $\psi(1) = 1$, the extension is positive and so there exists a bounded Borel measure m on S such that

$$\psi(f) = \int_S f(s) \, \mathrm{d}m(x)$$

for all $f \in \mathcal{A}_{\mathbb{R},0}(S,\pi)$. The proof now follows the last part of the proof of [56, Lemma 4.12]. Let $C_c(\mathbb{R})$ denote the set of continuous real-valued compactly supported functions on $\mathbb{R}$ and note that there exists a well-defined map from $C_c(\mathbb{R})$ into $\mathcal{A}_{\mathbb{R},0}(S,\pi)$ given by $F \mapsto F \circ \pi$. Since $\psi \circ \alpha = e^{-\beta}\psi$, it follows that the measure $m \circ \pi^{-1}$ on $\mathbb{R}$ satisfies

$$\int_{\mathbb{R}} e^{-t} F(t) \, \mathrm{d}m \circ \pi^{-1}(t) = e^{-\beta} \int_{\mathbb{R}} F(t) \, \mathrm{d}m \circ \pi^{-1}(t) \ \ \forall F \in C_c(\mathbb{R}).$$

Therefore, $m \circ \pi^{-1}$ is concentrated at the point β and hence m is concentrated at the point $\pi^{-1}(\beta)$. We can therefore define a linear functional $\psi' : \mathrm{Aff}\pi^{-1}(\beta) \to \mathbb{R}$ by

$$\psi'(f) = \psi(\hat{f}) = \int_S \hat{f}(x) \, \mathrm{d}m(x) \ ,$$

where $\hat{f} \in \mathcal{A}_{\mathbb{R},0}(S,\pi)$ is any element with $\hat{f}|_{\pi^{-1}(\beta)} = f$. The existence of such an element $\hat{f}$ is ensured by [56, Lemma 4.4(1)]. If $f \geq 0$, we can ensure that $\hat{f} \geq -\epsilon$ for any $\epsilon > 0$ ([56, Lemma 4.4(2)]), so ψ' is a positive linear functional. Since every state of $\mathrm{Aff}\pi^{-1}(\beta)$ is given by evaluation at a point in $\pi^{-1}(\beta)$, there exist $\omega \in \pi^{-1}(\beta)$ and a number $\lambda \geq 0$ such that

$$\psi(g) = \lambda g(\omega) \tag{5.4.1}$$

for all $g \in \mathcal{A}_{\mathbb{R},0}(S,\pi)$. In particular, this holds for any $g \in G \cap \mathcal{A}_{\mathbb{R},0}(S,\pi)$.

An element $f \in G$ can be writen as a sum $f = f_- + f_0 + f_+$, where $f_\pm, f_0 \in G$, f_0 has compact support and there are natural numbers $n_\pm \in \mathbb{N}$ such that $e^{n_- \pi} f_- \in \mathcal{A}_{\mathbb{R},0}(S,\pi)$ and $e^{-n_+ \pi} f_+ \in \mathcal{A}_{\mathbb{R},0}(S,\pi)$. Then $\psi(f_0) = \lambda f_0(\omega)$, and we also have that

$$\psi(f_-) = \psi(\alpha^{n_-}(e^{n_- \pi} f_-)) = e^{-\beta n_-} \psi(e^{n_- \pi} f_-) = e^{-\beta n_-} \lambda e^{n_- \pi(\omega)} f_-(\omega) = \lambda f_-(\omega).$$

Similarly, $\psi(f_+) = \lambda f_+(\omega)$, so $\psi(f) = \lambda f(\omega)$. Since $\psi(1) = 1$, it follows that $\lambda = 1$, which finishes the argument.

We now prove the uniqueness of the element ω found above. Suppose that there exists $\omega' \neq \omega$ in $\pi^{-1}(\beta)$ such that $g(\omega) = g(\omega')$ for all $g \in G$. Then, by *(i)* above, $h(\omega) = h(\omega')$ for any $h \in \mathcal{A}(S,\pi)$ with compact support. As $\omega, \omega' \in \pi^{-1}(\beta)$, there exists $f \in \mathrm{Aff}(\pi^{-1}(\beta))$ such that $f(\omega) \neq f(\omega')$. Therefore, there exist $n \in \mathbb{N}$ and $h \in \mathcal{A}(S,\pi)$ such that the support of h is contained in $\pi^{-1}([-n,n])$ and $h(\omega) \neq h(\omega')$, which is a contradiction. Hence, ω is unique. $\square$

The pair (G, G_+) only contains information about the simplex S. To complete the construction, the key ingredient is the following proposition which follows as a direct application of [114, Proposition 3.5].

Proposition 5.4.2. *There exist countable, abelian, torsion free groups* $H_0 \neq \{0\}$ *and* H_1*, automorphisms* κ_0 *and* κ_1 *of* H_0 *and* H_1 *respectively and homomorphisms* $q_i : H_i \to K_i(A)$ *for* $i = 0, 1$ *such that*

$$0 \longrightarrow H_i \xrightarrow{\mathrm{id}-\kappa_i} H_i \xrightarrow{q_i} K_i(A) \longrightarrow 0$$

is exact for $i = 0, 1$.

Proof. We apply [114, Proposition 3.5] to the pairs $(K_0(A), 0)$ and $(K_1(A), 0)$. If $H_0 = \{0\}$, which happens only when $K_0(A) = 0$, we take $H_0 = \mathbb{Q}$ and $\kappa_0(x) = 2x$. $\square$

Lemma 5.4.3. *Set* $\Gamma = H_0 \oplus G$ *and if* $p : \Gamma \to G$ *is the canonical projection, let*

$$\Gamma_+ = \{x \in \Gamma : p(x) \in G_+ \setminus \{0\}\} \cup \{0\}.$$

Then the following conditions hold:

(i) *the pair* (Γ, Γ_+) *is a dimension group;*

(ii) *for any* $g \in \Gamma_+$*, there exist* $h_1, h_2 \in \Gamma_+$ *such that* $g = 2h_1 + 3h_2$ *i.e.* Γ_+ *is weakly divisible;*

(iii) *If* I *is a nonzero order ideal of* Γ*, then* $I = H_0 \oplus p(I)$*;*

(iv) *the only order ideals* I *in* Γ *such that* $(\kappa_0 \oplus \alpha)(I) = I$ *are* $I = \{0\}$ *and* $I = \Gamma$*;*

(v) *the ordered group* (Γ, Γ_+) *has no nonzero liminary subquotients i.e. no subquotient has all its prime quotients isomorphic to* $\mathbb{Z}$.

Proof. We first check *(i)*. As (G, G_+) is a dimension group satisfying the strong Riesz interpolation property (Lemma 5.4.1*(ii)*) and H_0 is nonzero and torsion free, (Γ, Γ_+) is a dimension group by [46, Lemma 3.2].

To check *(ii)*, it suffices to find $h \in \Gamma_+$ such that $2h \leq g \leq 3h$. We then let $h_1 = 3h - g$ and $h_2 = g - 2h$. Let $g \in \Gamma_+$ be nonzero. Since $G = \mathbb{Q}G$, there exists $h' \in G_+$ nonzero such that $\frac{7}{3}h' \leq p(g) \leq \frac{8}{3}h'$. Then, $p(g - 2(0, h')) = p(g) - 2h' > 0$ and $p(3(0, h') - g) = 3h' - p(g) > 0$, so we can take $h = (0, h')$ to conclude that $2h \leq g \leq 3h$ in Γ.

We now check *(iii)*. Let I be a nonzero order ideal of Γ. Since $I = (I \cap \Gamma_+) - (I \cap \Gamma_+)$, there exist $h' \in H_0$ and $g \in G_+ \setminus \{0\}$ such that $(h', g) \in I$. Then, for any

$h \in H_0$, one has that

$$0 < (h + h', g) < 2(h', g)$$

in Γ. Since I is an order ideal, it follows that $(h + h', g) \in I$. Combining this with the fact that $(h', g) \in I$, yields that $(h, 0) \in I$. Hence $H_0 \oplus 0 \subseteq I$. If $g \in p(I)$, there exists $h' \in H_0$ such that $(h', g) \in I$. If $h \in H_0$, then $(h - h', 0) \in I$, so $(h, g) \in I$. Hence, we get that $I = H_0 \oplus p(I)$.

The proof of *(iv)* is contained in [55, Lemma 5.5], but we will include the details for the convenience of the reader. Let I be a nonzero order ideal of Γ such that $(\kappa_0 \oplus \alpha)(I) = I$. By condition *(iii)*, it follows that $I = H_0 \oplus p(I)$. Furthermore, $p(I)$ is a nonzero order ideal in G such that $\alpha(p(I)) = p(I)$. Condition *(iv)* of Lemma 5.4.1 yields that $G = p(I)$. Thus, $I = \Gamma$.

We prove *(v)* by contradiction. Let $J \subseteq I$ be order ideals of Γ such that I/J is nonzero and liminary in the category of ordered groups i.e. any prime quotient of I/J is isomorphic to $\mathbb{Z}$. Then, by *(iii)* above, $I = H_0 \oplus p(I)$. Suppose that there exists a surjection $q : (H_0 \oplus p(I))/J \to \mathbb{Z}$ and let (h, i) be positive in $H_0 \oplus p(I)$ such that $q(h, i) = 1$. As $p(I) \subseteq G$ is an order ideal, it follows that $\frac{1}{2}i \in p(I)$. If $(h, \frac{1}{2}i) \in J$, then $(0, \frac{1}{2}i) \in J$ by *(iii)*. Therefore, $(h, i) \in J$, which is a contradiction, so $(h, \frac{1}{2}i) \notin J$. Hence, as $0 \leq (h, \frac{1}{2}i) \leq (h, i)$, one gets that

$$0 \leq q(h, \frac{1}{2}i) \leq 1.$$

If $q(h, \frac{1}{2}i) = 0$, then $q(4h, 2i) = 0$. But $(h, i) \leq (4h, 2i)$, so $1 = q(h, i) \leq 0$, which is a contradiction. If $q(h, \frac{1}{2}i) = 1$, then $q(0, \frac{1}{2}i) = 0$ as $q(h, i) = 1$, which gives that $q(0, 2i) = 0$. But $(h, i) \leq (0, 2i)$, so $1 = q(h, i) \leq 0$, which is a contradiction. Thus, the subquotient I/J is not liminary. □

Lemma 5.4.4. *Choose* $w \in H_0$ *such that* $q_0(w) = [1_A]_0$. *Let* $v = (w, 1) \in \Gamma_+$, *where* 1 *stands for the constant function* 1 *on* S. *Let* $\varphi : \Gamma \to \mathbb{R}$ *be a positive homomorphism* *and* $\beta \in \mathbb{R}$ *such that* $\varphi(v) = 1$ *and* $\varphi \circ (\kappa_0 \oplus \alpha) = e^{-\beta}\varphi$. *Then, there is a unique* $\omega \in \pi^{-1}(\beta)$ *such that* $\varphi(h, g) = g(\omega)$ *for all* $(h, g) \in \Gamma$. *In particular, there are no positive homomorphisms on* Γ *sending* v *to* 1 *that are invariant under the automorphism* $\kappa_0 \oplus \alpha$.

Proof. This follows as in [55, Lemma 5.6]. Let $h \in H_0$ and $n \geq 1$. Then $n(h, 0) + v \in \Gamma_+$ and $-n(h, 0) + v \in \Gamma_+$, which implies that $n\varphi(h, 0) + 1 \geq 0$ and $-n\varphi(h, 0) + 1 \geq 0$ for all $n \geq 1$. Hence, $\varphi(h, 0) = 0$, which yields a positive homomorphism $\psi : G \to \mathbb{R}$ such that $\psi \circ p = \varphi$. Since $\varphi(v) = 1$ and $p(v) = 1$, it follows that $\psi(1) = 1$. Moreover, $\varphi \circ (\kappa_0 \oplus \alpha) = e^{-\beta}\varphi$ and $p \circ (\kappa_0 \oplus \alpha)(g) = \alpha(g)$ for any $g \in G$, so $\psi \circ \alpha = e^{-\beta}\psi$. Thus, *(v)* of Lemma 5.4.1 shows that there is a unique $\omega \in \pi^{-1}(\beta)$ such that $\psi(g) = g(\omega)$ for all $g \in G$. Hence, $\varphi(h, g) = g(\omega)$ for all $(h, g) \in \Gamma$. The last statement follows by taking $\beta = 0$, as $\pi^{-1}(0) = \emptyset$. ☐

We consider the triple $(\Gamma, \Gamma_+, H_1 \oplus G)$ and we claim that it realises the K-theory of some stable A$\mathbb{T}$-algebra with real rank zero. This will follow from Elliott's classification of A$\mathbb{T}$-algebras with real rank zero from [49]. We first need to impose a suitable order on the graded group $\Gamma \oplus (H_1 \oplus G)$. We will do so using [52, Theorem 4.28].

Lemma 5.4.5. *Consider the graded group* $\Lambda_* = \Gamma \oplus (H_1 \oplus G)$ *and define*

$$(\Lambda_*)_+ = \{((h_0, g_0), (h_1, g_1)) \in \Lambda_* \colon (h_0, g_0) \in \Gamma_+ \setminus \{0\}, g_1 \in I(g_0)\} \cup \{0\},$$

where $I(g_0)$ *is the order ideal of* G *generated by* g_0. *Then the following conditions hold:*

- *(i)* $(\Lambda_*, (\Lambda_*)_+)$ *is a countable graded ordered group;*
- *(ii)* $(\Lambda_*, (\Lambda_*)_+)$ *is unperforated;*
- *(iii)* $(\Lambda_*, (\Lambda_*)_+)$ *has the Riesz decomposition property and the Riesz interpolation property;*

(iv) $(\Lambda_*, (\Lambda_*)_+)$ *is the inductive limit of a sequence* $\Lambda_{*1} \to \Lambda_{*2} \to \dots$ *of graded ordered groups such that each* $\Lambda_{*i} = \Lambda_{0i} \oplus \Lambda_{1i}$ *is the direct sum of finitely many basic building blocks of the form* $\mathbb{Z} \oplus \mathbb{Z}$ *with the order determined by strict positivity in the first component.*

Proof. Conditions *(i)* and *(iii)* follow as a direct application of [52, Theorem 4.28]. It is worth noting that in [52], the term ideal stands for what is called order ideal in this chapter (see [52, Remark 4.27]).

We first need to define a map from order ideals of Γ to subgroups of $H_1 \oplus G$. If I is a nonzero order ideal of Γ, then $I = H_0 \oplus p(I)$ by *(iii)* of Lemma 5.4.3. We then consider the map from order ideals of Γ to subgroups of $H_1 \oplus G$ given by $H_0 \oplus p(I) \mapsto H_1 \oplus p(I)$ which maps 0 to 0. This map also preserves inclusions, upward directed unions, and maps Γ into $H_1 \oplus G$. Condition *(i)* now follows from [52, Theorem 4.28].

We now prove *(ii)*. Let $((h_0, g_0), (h_1, g_1)) \in \Lambda_*$ and $n \in \mathbb{N}$ such that $n((h_0, g_0), (h_1, g_1))$ is positive in Λ_*. Recall from Proposition 5.4.2 that H_0 and H_1 are torsion free. Then Λ_* is torsion free, so we can assume that $(h_0, g_0) \in \Gamma \setminus \{0\}$. By *(i)* of Lemma 5.4.3, Γ is a dimension group and hence unperforated, so $(h_0, g_0) \in \Gamma_+$. Furthermore, since $n((h_0, g_0), (h_1, g_1)) \in (\Lambda_*)_+$, we have that ng_1 is in the order ideal of G generated by ng_0, say I_0. As g_0 is positive, it follows that $g_0 \in I_0$, so I_0 is generated by g_0. Since $ng_1 \in I_0$, there exists $k \in \mathbb{N}$ such that

$$-kng_0 \leq ng_1 \leq kng_0$$

in I_0. Recall from Lemma 5.4.1 that $G = \mathbb{Q}G$. In particular, G is divisible, so it follows that $-kg_0 \leq g_1 \leq kg_0$, which implies that $0 \leq g_1 + kg_0 \leq 2kg_0$. This yields that $g_1 + kg_0 \in I_0$ and so $g_1 \in I_0$. Thus $((h_0, g_0), (h_1, g_1)) \in (\Lambda_*)_+$, which shows *(ii)*.

By *(i)* of Lemma 5.4.3, (Γ, Γ_+) is a dimension group, so it is weakly unperforated and satisfies the Riesz interpolation property, which coincides with the Riesz decomposition property (see [70, Proposition 2.1]). The correspondence $H_0 \oplus p(I) \mapsto H_1 \oplus p(I)$ preserves finite sums and intersections. Moreover, no subquotient of Γ is liminary by *(v)* of Lemma 5.4.3, so Λ_* satisfies the Riesz decomposition property by [52, Theorem 4.28], thus proving *(iii)*.

Condition *(iv)* follows from [48, Theorem 5.2] and the fact that $H_1 \oplus G$ is torsion free. □

To apply classification results, we will need to construct an A$\mathbb{T}$-algebra which is $\mathcal{Z}$-stable. Building on Winter's techniques from [145], in [112], Robert and Tikuisis show that under some extra assumptions, finite nuclear dimension implies $\mathcal{Z}$-stability even in the nonsimple setting. We will use this result to prove that the A$\mathbb{T}$-algebra we obtain is $\mathcal{Z}$-stable.

Lemma 5.4.6. *There exists a separable, stable A$\mathbb{T}$-algebra B with real rank zero such that the following conditions hold:*

(i) $(K_0(B), K_0(B)_+, K_1(B)) \cong (\Gamma, \Gamma_+, H_1 \oplus G)$;

(ii) $D_*(B) \cong K_*(B)_+ \cong (\Lambda_*)_+$;

(iii) *B is $\mathcal{Z}$-stable;*

Proof. Consider the graded group $\Lambda_* = \Gamma \oplus (H_1 \oplus G)$ as in Lemma 5.4.5. Combining *(iii)* and *(iv)* of Lemma 5.4.5 with Theorem 5.2.2, it follows that there exists a separable, stable A$\mathbb{T}$-algebra B with real rank zero such that *(i)* holds and $K_*(B)_+ \cong (\Lambda_*)_+$. Since B is stable, we can identify $K_*(B)_+$ with the graded dimension range $D_*(B)$ (see Section 5.2), so *(ii)* is also satisfied.

We will prove that B is $\mathcal{Z}$-stable. Since B is an A$\mathbb{T}$-algebra, it follows that B has finite decomposition rank (see [84, Example 4.2]). We are going to use [112, Corollary 7.11]. Since B has finite decomposition rank and real rank zero, no quotient of B has a simple purely infinite ideal and the space of primitive ideals of B has a basis of compact open sets (see the remark (a) after [112, Corollary 7.11]). Furthermore, the Murray-von Neumann semigroup of B is weakly divisible by *(ii)* of Lemma 5.4.3, which yields that B has no nonzero elementary ideal quotients (see [127, Theorem 9.1] together with the implication (1) $\implies$ (3) of [127, Theorem B]). Therefore, B is $\mathcal{Z}$-stable by [112, Corollary 7.11]. □

In the next lemma, we will realise the automorphisms $\kappa_0 \oplus \alpha$ and $\kappa_1 \oplus \alpha$ as the K-theory of an automorphism γ of B. We can further choose γ such that the crossed product $B \rtimes_\gamma \mathbb{Z}$ is simple using a variation of a result by Kishimoto from [85]. Recall that an automorphism σ on a C*-algebra E is called *properly outer* if for every nonzero σ-invariant closed two-sided ideal I of E and for every unitary u in $\mathcal{M}(I)$ one has $\|\sigma|_I - \mathrm{Ad}(u)\| = 2$.

Lemma 5.4.7. *Let B be the $A\mathbb{T}$-algebra obtained in Lemma 5.4.6. Then there exists an automorphism γ of B such that the following conditions hold:*

- *(i)* $K_0(\gamma) = \kappa_0 \oplus \alpha$ *and* $K_1(\gamma) = \kappa_1 \oplus \alpha$*;*
- *(ii)* $B \rtimes_\gamma \mathbb{Z}$ *is simple;*
- *(iii)* $B \rtimes_\gamma \mathbb{Z}$ *is* $\mathcal{Z}$*-stable;*
- *(iv) the canonical restriction induces a bijection between the cone of densely defined lower semicontinuous traces on $B \rtimes_\gamma \mathbb{Z}$ and the cone of γ-invariant densely defined lower semicontinuous traces on B.*

Proof. Consider the automorphism of $\Gamma \oplus (H_1 \oplus G)$ given by $(\kappa_0 \oplus \alpha) \oplus (\kappa_1 \oplus \alpha)$. Since $\alpha(G_+) = G_+$ by *(iii)* of Lemma 5.4.1, $(\kappa_0 \oplus \alpha) \oplus (\kappa_1 \oplus \alpha)$ preserves $(\Lambda_*)_+$, so we can apply Elliott's classification of automorphisms of $A\mathbb{T}$-algebras with real rank zero. Thus, by Theorem 5.2.1, there exists $\gamma' \in \mathrm{Aut}(B)$ such that $K_0(\gamma') = \kappa_0 \oplus \alpha$ and $K_1(\gamma') = \kappa_1 \oplus \alpha$.

We claim that we can replace γ' by another automorphism γ which induces the same maps in K-theory and the crossed product has finite nuclear dimension. Building on the work in [126], we can take an automorphism γ of B with finite Rokhlin dimension such that $K_i(\gamma) = K_i(\gamma')$ for $i = 0, 1$ ([76, Lemma 4.7]). Since B has finite nuclear dimension and γ has finite Rokhlin dimension, it follows that the crossed product $B \rtimes_\gamma \mathbb{Z}$ has finite nuclear dimension ([126, Theorem 6.2] or [74, Theorem 3.1]).

We now show that $B \rtimes_\gamma \mathbb{Z}$ is simple. Since α^k is non-trivial for all $k \neq 0$, then $K_0(\gamma)^k$ is non-trivial, which implies that no non-trivial power of γ is inner. Since B is stable and has stable rank one, the fact that $K_0(\gamma)^k$ is non-trivial, implies that there exists a projection $p_k \in B$ such that $\gamma^k(p_k)$ is not equivalent to p_k. Then, if u is a unitary in $\mathcal{M}(B)$, $\gamma^k(p_k)$ is not equivalent to up_ku^*, so

$$\|\gamma^k(p_k) - up_ku^*\| = 1.$$

By *(iv)* of Lemma 5.4.3, the only order ideals I in Γ such that $(\kappa_0 \oplus \alpha)(I) = I$ are $I = \{0\}$ and $I = \Gamma$. Thus, γ^k is properly outer by the implication (iii) $\implies$ (ii) in [100, Theorem 6.6]. Hence, the crossed product $B \rtimes_\gamma \mathbb{Z}$ is simple by [100, Theorem 7.2]. Furthermore, $B \rtimes_\gamma \mathbb{Z}$ has finite nuclear dimension, so it is $\mathcal{Z}$-stable by [134, Corollary 8.7].

Note that B has real rank zero, so it has an approximate unit of projections. As γ also has finite Rokhlin dimension, *(iv)* follows from Lemma 5.3.2. □

We now have all the necessary ingredients to show that A can be realised as a corner of a crossed product by the integers.

Lemma 5.4.8. *Let B the the stable $A\mathbb{T}$-algebra with real rank zero from Lemma 5.4.6, and let γ be the automorphism of B from Lemma 5.4.7. Then there exists a projection $e \in B$ such that $e\,(B \rtimes_\gamma \mathbb{Z})\,e \cong A$.*

Proof. Recall that $v = (w, 1) \in \Gamma_+$, where $w \in H_0$ is such that $q_0(w) = [1_A]_0$. Let e be a projection in B such that $[e]_0 = v$ in $K_0(B)$. We will prove that $A \cong e\,(B \rtimes_\gamma \mathbb{Z})\,e$ using the Kirchberg-Phillips classification theorem ([81, 105]). First, we claim that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is a simple, separable, unital, nuclear, purely infinite C*-algebra satisfying the UCT.

Note that $B \rtimes_\gamma \mathbb{Z}$ is simple by *(ii)* of Lemma 5.4.7, so e is a full projection. Moreover, B is nuclear, satisfies the UCT, and $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable by *(iii)* of Lemma 5.4.7. Therefore, by Proposition 2.5.4, $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra which satisfies the UCT. By [117, Corollary 5.1], to show that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is purely infinite, it now suffices to prove that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ has no quasitraces. Since $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is unital and nuclear, any quasitrace is a trace by [72]. Therefore, it suffices to show that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ has no tracial states. If τ is a tracial state on $e\,(B \rtimes_\gamma \mathbb{Z})\,e$, then it extends to a lower semicontinuous (possibly unbounded) trace τ' on $B \rtimes_\gamma \mathbb{Z}$ by [147, Corollary 5.2]. By *(iv)* of Lemma 5.4.7, the restriction of τ' to B induces a positive group homomorphism $\tau'_* : K_0(B) \to \mathbb{R}$ such that $\tau'_*(v) = 1$ and $\tau'_* \circ K_0(\gamma) = \tau'_*$. Recall from *(i)* of Lemma 5.4.6 and *(i)* of Lemma 5.4.7 that $K_0(B) = \Gamma$ and $K_0(\gamma) = \kappa_0 \oplus \alpha$. Thus, we obtain a contradiction with Lemma 5.4.4. Hence, $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ has no tracial states, so it is purely infinite by [117, Corollary 5.1].

Therefore, by the Kirchberg-Phillips' classification theorem, it suffices to check that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ and A have the same Elliott invariant. First note that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ and $B \rtimes_\gamma \mathbb{Z}$ have the same K-groups since they are stably isomorphic ([18]). Applying the Pimsner-Voiculescu exact sequence ([107, Theorem 2.4]) to B and the automorphism γ, we get the six-term exact sequence

$$\begin{tikzcd}
H_0 \oplus G \arrow[r, "\mathrm{id}-K_0(\gamma)"] & H_0 \oplus G \arrow[r] & K_0(B \rtimes_\gamma \mathbb{Z}) \arrow[d] \\
K_1(B \rtimes_\gamma \mathbb{Z}) \arrow[u] & H_1 \oplus G \arrow[l] & H_1 \oplus G. \arrow[l, "\mathrm{id}-K_1(\gamma)"]
\end{tikzcd}$$

By *(i)* of Lemma 5.4.7, $\mathrm{id}_{H_1\oplus G} - K_1(\gamma) = (\mathrm{id}_{H_1} - \kappa_1) \oplus (\mathrm{id}_G - \alpha)$, which is injective by Proposition 5.4.2 and since $\mathrm{id}_G - \alpha$ is an automorphism of G (*(iii)* of Lemma 5.4.1). Then, the downward map $K_0(B \rtimes_\gamma \mathbb{Z}) \to H_1 \oplus G$ is zero. This further implies that the map $H_0 \oplus G \to K_0(B \rtimes_\gamma \mathbb{Z})$ is surjective, which yields that

$$K_0(B \rtimes_\gamma \mathbb{Z}) \cong (H_0 \oplus G)/(\mathrm{id}_{H_0\oplus G} - K_0(\gamma))(H_0 \oplus G).$$

By Lemma *(i)* of 5.4.7, $\mathrm{id}_{H_0\oplus G} - K_0(\gamma) = (\mathrm{id}_{H_0} - \kappa_0) \oplus (\mathrm{id}_G - \alpha)$. By *(iii)* of Lemma 5.4.1, $\mathrm{id}_G - \alpha$ is an automorphism of G, so

$$K_0(B \rtimes_\gamma \mathbb{Z}) \cong H_0/(\mathrm{id}_{H_0} - \kappa_0)(H_0) \cong K_0(A),$$

where the last isomorphism is given by Proposition 5.4.2. Note that $[e]_0 = (w, 1)$ in $K_0(B)$, where w is mapped to $[1_A]_0$ by the isomorphism $H_0/(\mathrm{id}_{H_0} - \kappa_0)(H_0) \cong K_0(A)$. Therefore, the resulting isomorphism $K_0(e\,(B \rtimes_\gamma \mathbb{Z})\,e) \to K_0(A)$ sends $[e]_0$ to $[1_A]_0$.

Further examining the Pimsner-Voiculescu exact sequence and using that $\mathrm{id}_{H_0\oplus G} - K_0(\gamma)$ is injective, it follows that the map $K_1(B \rtimes_\gamma \mathbb{Z}) \to H_0 \oplus G$ is zero. Therefore, we obtain the exact sequence

$$0 \longrightarrow H_1 \oplus G \xrightarrow{\mathrm{id} - K_1(\gamma)} H_1 \oplus G \longrightarrow K_1(B \rtimes_\gamma \mathbb{Z}) \longrightarrow 0,$$

which yields that

$$K_1(B \rtimes_\gamma \mathbb{Z}) \cong (H_1 \oplus G)/(\mathrm{id}_{H_1\oplus G} - (\kappa_1 \oplus \alpha))(H_1 \oplus G) \cong H_1/(\mathrm{id}_{H_1\oplus G} - \kappa_1)(H_1),$$

where the last identification follows as $\mathrm{id}_G - \alpha$ is an automorphism on G (*(iii)* of Lemma 5.4.1). Hence $K_1(B \rtimes_\gamma \mathbb{Z}) \cong K_1(A)$ by Proposition 5.4.2. Thus, $e\,(B \rtimes_\gamma \mathbb{Z})\,e \cong A$ by the Kirchberg-Phillips classification theorem (see for example [115, Theorem

8.4.1(iv)]). □

We will now finish the proof of Theorem 16. Recall from Lemmas 5.4.6 and 5.4.7 that B is a stable $A\mathbb{T}$-algebra with real rank zero and γ is an automorphism on B with specified behaviour on K-theory such that $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable.

Consider the dual action $\hat{\gamma}$ on $B \rtimes_\gamma \mathbb{Z}$ as a 2π-periodic flow. Recall that $\hat{\gamma}_t(f)(x) = e^{-ixt} f(x)$ for any $t \in \mathbb{R}, f \in C_c(\mathbb{Z}, B)$ and $x \in \mathbb{Z}$, so that $\hat{\gamma}$ acts trivially on B. Since $e \in B$, $\hat{\gamma}$ restricts to an action on $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ which we denote by θ. First, we need a result which will relate KMS states on $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ to positive homomorphisms on $K_0(B)$. The last statement of the following proposition is not explicitly contained in [56, Lemma 4.1]. However, it is implied by the remark leading to [56, Corollary 4.2].

Proposition 5.4.9. *[56, Lemma 4.1] Let D be a* C*-*algebra. Let ρ be an automorphism of D and let $q \in D$ a projection in D which is full in $D \rtimes_\rho \mathbb{Z}$. Let $\hat{\rho}$ denote the restriction of the dual action on $D \rtimes_\rho \mathbb{Z}$ to $q(D \rtimes_\rho \mathbb{Z})q$ and consider it as a 2π-periodic flow. If $P : D \rtimes_\rho \mathbb{Z} \to D$ is the canonical conditional expectation, then for each $\beta \in \mathbb{R}$, the map $\tau \mapsto \tau \circ P|_{q(D\rtimes_\rho\mathbb{Z})q}$ is an affine homeomorphism from the densely defined lower semicontinuous traces on D satisfying*

$$\tau \circ \rho = e^{-\beta}\tau \quad \textit{and} \quad \tau(q) = 1,$$

*onto the set of β-*KMS *states for the dual action $\hat{\rho}$ on $q(D \rtimes_\rho \mathbb{Z})q$. Moreover, the inverse is given by the map $\omega \mapsto \hat{\omega}|_D$, where $\hat{\omega}$ is a β-*KMS *weight for the dual action on $D \rtimes_\rho \mathbb{Z}$, which extends ω.*

Proof. We will only comment on the proof of the last statement. Let ω be a β-KMS state for the dual action $\hat{\rho}$ on $q(D \rtimes_\rho \mathbb{Z})q$. Then, by [88, Remark 3.3], there exists a unique β-KMS weight $\hat{\omega}$ for the dual action on $D \rtimes_\rho \mathbb{Z}$ which extends ω. The result now follows by [132, Lemma 3.1]. □

Theorem 5.4.10. *Let A be a unital* UCT *Kirchberg algebra and let (S, π) be a proper simplex bundle such that $\pi^{-1}(0) = \emptyset$. Then there exists a 2π-periodic flow θ on A such that its induced* KMS *bundle (S^θ, π^θ) is isomorphic to (S, π).*

Proof. The proof follows the strategy in [55, Lemma 5.8]. Recall that if $S = \emptyset$, then we take θ to be the trivial flow on A. Therefore, we assume that S is nonempty and we claim that the KMS bundle (S^θ, π^θ) of θ is isomorphic to (S, π). This will finish the proof.

Let $(\omega, \beta) \in S^\theta$. By [88, Remark 3.3], there exists a unique β-KMS weight $\hat{\omega}$ for the dual action $\hat{\gamma}$ on $B \rtimes_\gamma \mathbb{Z}$ which extends ω. Furthermore, $B \rtimes_\gamma \mathbb{Z}$ is simple by *(ii)* of Lemma 5.4.7, so e is full in $B \rtimes_\gamma \mathbb{Z}$. Then, by Proposition 5.4.9, $\hat{\omega}|_B$ is a densely defined lower semicontinuous trace on B such that $\hat{\omega}|_B \circ \gamma = e^{-\beta}\hat{\omega}|_B$ and $\hat{\omega}|_B(e) = 1$. Therefore, $(\hat{\omega}|_B)_*$ is a positive homomorphism on $K_0(B)$ such that

$$(\hat{\omega}|_B)_* \circ K_0(\gamma) = e^{-\beta}(\hat{\omega}|_B)_* \text{ and } (\hat{\omega}|_B)_*([e]_0) = 1.$$

Then, by Lemma 5.4.4, there exists a unique $\omega' \in \pi^{-1}(\beta)$ such that

$$(\hat{\omega}|_B)_*(h, g) = g(\omega') \tag{5.4.2}$$

for all $(h, g) \in \Gamma \cong K_0(B)$. Define $\varphi : S^\theta \to S$ by $\varphi(\omega, \beta) = \omega'$ and note that $\pi \circ \varphi = \pi^\theta$ as $\omega' \in \pi^{-1}(\beta)$. The restriction $\varphi : (\pi^\theta)^{-1}(\beta) \to \pi^{-1}(\beta)$ is affine by construction for any $\beta \in \mathbb{R}$.

We claim that φ is surjective. Let $\mu \in S$ and define $\mathrm{ev}_\mu : K_0(B) \to \mathbb{R}$ by $\mathrm{ev}_\mu(h, g) = g(\mu)$. Recall that $K_0(\gamma) = \kappa_0 \oplus \alpha$ by *(i)* of Lemma 5.4.7 and $\alpha(g)(x) = e^{-\pi(x)}g(x)$ for any $g \in G$. Then, ev_μ is a positive group homomorphism such that $\mathrm{ev}_\mu \circ K_0(\gamma) = e^{-\pi(\mu)} \mathrm{ev}_\mu$ and $\mathrm{ev}_\mu(v) = 1$ as $v = (w, 1)$. Then, by Proposition 5.2.3, there exists a unique densely defined lower semicontinuous trace τ_μ on B such that $(\tau_\mu)_* = \mathrm{ev}_\mu$. By Proposition 5.4.9, there exists a $\pi(\mu)$-KMS state ω for θ such that

$(\hat{\omega}|_B)_* = \mathrm{ev}_\mu$. Then $\varphi(\omega, \pi(\mu)) = \mu$, so φ is surjective.

We further check that φ is injective. Let $(\omega_1, \beta_1), (\omega_2, \beta_2) \in S^\theta$ such that $\varphi(\omega_1, \beta_1) = \varphi(\omega_2, \beta_2)$. Then,

$$\beta_1 = \pi(\varphi(\omega_1, \beta_1)) = \pi(\varphi(\omega_2, \beta_2)) = \beta_2.$$

Moreover, by the definition of the map φ,

$$(\hat{\omega}_1|_B)_* = (\hat{\omega}_2|_B)_*.$$

By Proposition 5.4.9, $\hat{\omega}_1|_B$ and $\hat{\omega}_2|_B$ are densely defined lower semicontinuous traces on B such that

$$(\hat{\omega}_1|_B)(e) = (\hat{\omega}_2|_B)(e) = 1.$$

Since $(\hat{\omega}_1|_B)_* = (\hat{\omega}_2|_B)_*$, it follows that $\hat{\omega}_1|_B = \hat{\omega}_2|_B$ (Proposition 5.2.3). Then, Proposition 5.4.9 yields that $\omega_1 = \omega_2$, which shows that φ is injective.

If we show that $\varphi^{-1} : S \to S^\theta$ is continuous, then φ is a homeomorphism by Lemma 5.1.5. Recall from Remark 5.1.4 that both S^θ and S are metrisable and let ω'_n be a sequence in S which converges to ω'. If $\varphi^{-1}(\omega'_n) = (\omega_n, \beta_n)$ and $\varphi^{-1}(\omega') = (\omega, \beta)$, then (5.4.2) yields that $(\hat{\omega}_n|_B)_*(h, g)$ converges to $(\hat{\omega}|_B)_*(h, g)$ for any $(h, g) \in \Gamma \cong K_0(B)$. Then, $\hat{\omega}_n|_B$ converges to $\hat{\omega}|_B$ by Proposition 5.2.3, so ω_n converges to ω by Proposition 5.4.9. Moreover, $\beta_n = \pi(\omega'_n)$, which converges to $\pi(\omega') = \beta$ by continuity of π. Thus, (ω_n, β_n) converges to (ω, β), so φ^{-1} is continuous. Together with Lemma 5.4.8, this yields the conclusion of Theorem 16. $\square$

Remark 5.4.11. Note that in [55, Section 5], to realise the K-theory of a stable AF-algebra, the authors apply Proposition 5.4.2 to the pair $(K_0(A), K_1(A))$. Applying Proposition 5.4.2 to the pairs $(K_0(A), 0)$ and $(K_1(A), 0)$ instead of the pair $(K_0(A), K_1(A))$ is precisely what allows us to avoid assuming that $K_1(A)$ is torsion free. This idea has its roots in [114, Theorem 3.6], where Rørdam showed that any pair of countable discrete abelian groups can be realised as the K-theory of an endomorphism crossed product of an $A\mathbb{T}$-algebra. Then, since we are working with an A$\mathbb{T}$-algebra, rather than an AF-algebra, different arguments are needed in Lemmas 5.4.6 and 5.4.7 to show that B is $\mathcal{Z}$-stable and the crossed product is simple and $\mathcal{Z}$-stable. Furthermore, the invariant for classifying possibly nonsimple A$\mathbb{T}$-algebras of real rank zero is more intricate than in the case of AF-algebras (see Section 5.2), so more detailed K-theory groups have to be constructed.

5.5 KMS bundles on unital tracial classifiable C*-algebras

In this section, we will prove Theorem 17. The construction of the required flow is in the spirit of [56]. Adapting the construction in [56] and using classification of unital *-homomorphisms between classifiable C*-algebras from [27], we will extend [56, Theorem 3.1] to any simple, separable, unital, nuclear, stably finite, $\mathcal{Z}$-stable C*-algebra, with real rank zero, and which satisfies the UCT.

Throughout this section, A will be a simple, separable, unital, nuclear, stably finite, $\mathcal{Z}$-stable C*-algebra, with real rank zero, and which satisfies the UCT. We also let (S, π) be a compact simplex bundle such that $\pi^{-1}(0) \cong T(A)$. We will freely identify these simplices. The main step in proving Theorem 17 is realising A as a unital corner of a crossed product $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$, with B being a simple, separable, unital, nuclear, stably finite, $\mathcal{Z}$-stable C*-algebra satisfying the UCT. We first need to build the K-theory of the desired C*-algebra B. Let $\mathcal{A}_0(S, \pi)$ denote the set of elements $f \in \mathcal{A}(S, \pi)$ for which $f(T(A)) = 0$.

Lemma 5.5.1. *Let (S, π) be a compact simplex bundle such that $\pi^{-1}(0) \cong T(A)$. Then there exists a countable subgroup G_0 of $\mathcal{A}_0(S, \pi)$ such that the following conditions hold:*

(i) for any $f \in \mathcal{A}_0(S,\pi)$ and any $\epsilon > 0$, there is an element $g \in G_0$ such that

$$\sup_{x\in S} |f(x) - g(x)| < \epsilon;$$

(ii) the map $\eta : g \mapsto e^{-\pi} g$ is an automorphism on G_0 and $\mathrm{id} - \eta$ is also an automorphism of G_0;

(iii) For any $g \in G_0$, there exist $g_1, g_2 \in G_0 \cap \mathcal{A}_0(S,\pi)_+$ such that $g = g_1 - g_2$.

Proof. Since S is compact and the topology on S is second countable, we can choose a countable subgroup G_0 of $\mathcal{A}_0(S,\pi)$ such that for all $\epsilon > 0$ and all $f \in \mathcal{A}_0(S,\pi)$, there is an element $g \in G_0$ such that

$$\sup_{x\in S} |f(x) - g(x)| < \epsilon.$$

Moreover, we can ensure that if $f \in G_0$, then

$$x \mapsto e^{n\pi(x)}(1 - e^{-\pi(x)})^m f(x)$$

is in G_0 for all $m, n \in \mathbb{Z}$. As in [56, Property 4.5], we can ensure that η and $\mathrm{id} - \eta$ are surjective. Since elements in G_0 are supported away from $\pi^{-1}(0)$, it also follows that $\mathrm{id} - \eta$ is injective, so we obtain *(ii)*. To check *(iii)*, let $g \in G_0$ and $g_1, g_2 \in \mathcal{A}_0(S,\pi)_+$ such that $g = g_1 - g_2$. Using compactness of S and *(i)*, we can then ensure that $g_1, g_2 \in G_0$. □

Lemma 5.5.2. *There exist a countable ordered abelian group (G, G_+, v) with distinguished order unit v, a homomorphism $\hat{L} : G \to \mathcal{A}(S,\pi)$, and an automorphism α of (G, G_+, v) such that the following conditions hold:*

(i) (G, G_+, v) is simple and weakly unperforated;

(ii) $\hat{L}(G)$ *is uniformly dense in* $\mathcal{A}(S,\pi)$*;*

(iii) the space of states on G*, denoted by* $S(G)$*, is a metrisable Choquet simplex;*

(iv) the homomorphism $\mathrm{id}-\alpha : G \to G$ *is injective;*

(v) there is an isomorphism $\Sigma : G/(\mathrm{id}-\alpha)(G) \to K_0(A)$*;*

(vi) if $\beta \in \mathbb{R}$ *and* φ *is a state on* (G, G_+, v) *with the property that*

$$\varphi \circ \alpha = e^{-\beta}\varphi,$$

then there is a unique $s \in \pi^{-1}(\beta)$ *such that* $\varphi = \mathrm{ev}_s \circ \hat{L}$*.*[28] *In particular, the space of* α*-invariant states on* G *is affinely homeomorphic to* $T(A)$*.*

Proof. Set

$$G = \left(\bigoplus_{\mathbb{Z}} K_0(A)\right) \oplus G_0.$$

We will define an order on G adapting the construction in [56, Section 4.2]. The map

$$r : \mathcal{A}(S,\pi) \to \mathrm{Aff}(T(A))$$

given by restriction is surjective ([56, Lemma 4.4(1)]), so we can choose a linear map

$$L : \mathrm{Aff}(T(A)) \to \mathcal{A}(S,\pi) \text{ such that } r \circ L = \mathrm{id} \text{ and } L(1) = 1. \tag{5.5.1}$$

If $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ is the canonical pairing map of A, let us consider the homomorphism $\hat{L} : G \to \mathcal{A}(S,\pi)$ given by

$$\hat{L}(\xi, g)(x) = g(x) + \sum_{n\in\mathbb{Z}} L(\rho_A(\xi_n))(x)e^{n\pi(x)} \tag{5.5.2}$$

---

[28] The map ev_s denotes evaluation at $s \in \pi^{-1}(\beta)$.

for all $\xi = (\xi_n)_{n\in\mathbb{Z}}$, $g \in G_0$, and $x \in S$. Note that the map $\hat{L}$ is well-defined since ξ has only finitely many nonzero entries.

Define

$$G_+ = \{(\xi, g) \in G : \hat{L}(\xi, g)(x) > 0,\ x \in S\} \cup \{0\} \tag{5.5.3}$$

and set $v = (1^{(0)}, 0)$, where $1^{(0)}$ is the sequence with $[1_A]_0$ in the zero entry and zero elsewhere.

To show *(i)*, we will first show that the triple (G, G_+, v) defines an ordered abelian group. The construction of G_+ in (5.5.3) yields that $G_+ \cap (-G_+) = \{0\}$. We will check that $G = G_+ - G_+$. Let $(\xi, g) \in G$. As A is stably finite, $(K_0(A), K_0(A)_+)$ is an ordered abelian group ([6, Proposition 6.3.3]), so there exist sequences $\xi^{(1)}, \xi^{(2)}$ such that $\xi^{(i)}_n \in K_0(A)_+$ for any $i = 1, 2$ and $n \in \mathbb{N}$ and $\xi = \xi^{(1)} - \xi^{(2)}$. By *(iii)* of Lemma 5.5.1, there exist $g_1, g_2 \in G_0 \cap \mathcal{A}_0(S, \pi)_+$ such that $g = g_1 - g_2$. Thus, $(\xi, g) = (\xi^{(1)}, g_1) - (\xi^{(2)}, g_2)$, which shows that $G = G_+ - G_+$.

To show that (G, G_+, v) is simple, we use the assumption that S is compact. Let $(\xi, g) \in G_+ \setminus \{0\}$. By definition, this yields that $\hat{L}(\xi, g)(x) > 0$ for any $x \in S$. Since S is compact, there exists $n \in \mathbb{N}$ such that $\hat{L}(\xi, g)(x) \geq \frac{1}{n}$ for any $x \in S$. This implies that (ξ, g) is an order unit, so (G, G_+, v) is a simple ordered abelian group.

To see that (G, G_+) is weakly unperforated, let $n \in \mathbb{N}$ and $(\xi, g) \in G$ such that $n(\xi, g) \in G_+ \setminus \{0\}$. Therefore, by linearity of L, it follows that $\hat{L}(\xi, g)(x) > 0,\ x \in S$. Hence, $(\xi, g) \in G_+ \setminus \{0\}$, so (G, G_+) is weakly unperforated.

We claim that *(ii)* follows by combining *(i)* of Lemma 5.5.1 and the fact that A has real rank zero. Let $f \in \mathcal{A}(S, \pi)$ and $\epsilon > 0$. Note $f|_{T(A)} \in \mathrm{Aff}(T(A))$. Since A is unital, simple, exact, finite, $\mathcal{Z}$-stable, and has real rank zero, it follows that $\rho_A(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$ ([117, Theorem 7.2]). Therefore, there exists $y \in K_0(A)$ such that

$$f(\tau) - \epsilon/2 < \rho_A(y)(\tau) < f(\tau) + \epsilon/2$$

for any $\tau \in T(A)$. As $T(A)$ is closed in S and S is compact, [17, Lemma 2.3] shows that there exists $h \in \mathcal{A}(S, \pi)$ extending $\rho_A(y)$ such that

$$f(x) - \epsilon/2 < h(x) < f(x) + \epsilon/2 \tag{5.5.4}$$

for any $x \in S$. Then, $h - L(\rho_A(y)) \in \mathcal{A}_0(S, \pi)$, so by *(i)* of Lemma 5.5.1, there exists $g \in G_0$ such that

$$|h(x) - L(\rho_A(y))(x) - g(x)| < \epsilon/2 \tag{5.5.5}$$

for any $x \in S$. Now consider the element $(y^{(0)}, g) \in G$, where $y^{(0)}$ is the sequence which is constant 0 apart from the zero entry which is equal to y. Then,

$$\begin{aligned} \sup_{x\in S} |\hat{L}(y^{(0)}, g)(x) - f(x)| &= \sup_{x\in S} |g(x) + L(\rho_A(y))(x) - f(x)| \\ &\overset{(5.5.5)}{<} \sup_{x\in S} |f(x) - h(x)| + \epsilon/2 \\ &\overset{(5.5.4)}{<} \epsilon. \end{aligned}$$

Thus, $\hat{L}(G)$ is uniformly dense in $\mathcal{A}(S, \pi)$.

We will now show *(iii)*. This follows the strategy in *(ii)*. Using [2, Corollary II.3.11] and Kadison's duality ([78]), it suffices to check that G satisfies the strong Riesz interpolation property. Suppose that $(\xi_i, g_i) < (\eta_j, h_j)$ in G for any $i, j = 1, 2$. Therefore, $\hat{L}(\xi_i, g_i)(x) < \hat{L}(\eta_j, h_j)(x)$ for any $x \in S$. Note that $\hat{L}(\xi_i, g_i)$ and $\hat{L}(\eta_j, h_j)$ restrict to affine functions on $T(A)$.

Since A is unital, simple, exact, finite, $\mathcal{Z}$-stable, and has real rank zero, it follows that $\rho_A(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$ ([117, Theorem 7.2]). Thus, $\rho_A(K_0(A))$ satisfies the strong Riesz interpolation property with respect to the strict ordering in $\mathrm{Aff}(T(A))$ ([46, Lemma 3.1]), so there exists $y \in K_0(A)$ such that

$$\hat{L}(\xi_i, g_i)(\tau) < \rho_A(y)(\tau) < \hat{L}(\eta_j, h_j)(\tau) \tag{5.5.6}$$

for any $\tau \in T(A)$ and any $i, j = 1, 2$. As $T(A)$ is closed in S and S is compact, [17, Lemma 2.3] shows that there exists $f \in \mathcal{A}(S, \pi)$ extending $\rho_A(y)$ such that

$$\hat{L}(\xi_i, g_i)(x) < f(x) < \hat{L}(\eta_j, h_j)(x) \tag{5.5.7}$$

for any $x \in S$ and any $i, j = 1, 2$.

Since S is compact, we can let $\delta > 0$ be smaller than $f(x) - \hat{L}(\xi_i, g_i)(x)$ and $\hat{L}(\eta_j, h_j)(x) - f(x)$ for any $x \in S$ and any $i, j = 1, 2$. By *(i)* of Lemma 5.5.1, there exists $g \in G_0$ such that

$$|f(x) - L(\rho_A(y))(x) - g(x)| < \delta \tag{5.5.8}$$

for any $x \in S$. Now consider the element $(y^{(0)}, g) \in G$, where $y^{(0)}$ is the sequence which is constant 0 apart from the zero entry which is equal to y. We claim that

$$\hat{L}(\xi_i, g_i)(x) < \hat{L}(y^{(0)}, g)(x) < \hat{L}(\eta_j, h_j)(x)$$

for any $x \in S$. First note that for any $x \in S$ and $i = 1, 2$ we have that

$$\begin{aligned}
\hat{L}(y^{(0)}, g)(x) - \hat{L}(\xi_i, g_i)(x) &= g(x) + L(\rho_A(y))(x) - \hat{L}(\xi_i, g_i)(x) \\
&\overset{(5.5.8)}{>} f(x) - \delta - \hat{L}(\xi_i, g_i)(x) \\
&> 0,
\end{aligned}$$

by the choice of δ. A similar calculation shows that $\hat{L}(\eta_j, h_j)(x) > \hat{L}(y^{(0)}, g)(x)$ for any $x \in S$ and $j = 1, 2$. Hence, by the definition of the order on G, we obtain that

$$(\xi_i, g_i) < (y^{(0)}, g) < (\eta_j, h_j)$$

for any $i, j = 1, 2$, which shows that G satisfies the strong Riesz interpolation property.

Thus, $S(G)$ is a Choquet simplex. Moreover, $S(G)$ is metrisable since G is countable.

We will now define the automorphism α of G. Let α be the automorphism of (G, G_+, v) given by

$$\alpha(\xi, f) = (\sigma(\xi), e^{-\pi} f),$$

where $\sigma(\xi) = (\xi_{n+1})_{n\in\mathbb{Z}}$ is the left shift on $\bigoplus_{\mathbb{Z}} K_0(A)$ and $e^{-\pi}$ denotes the function $x \mapsto e^{-\pi(x)}$. To check *(iv)*, let $(\xi, g) \in G$ such that $(\mathrm{id} - \alpha)(\xi, g) = 0$. Then $\sigma(\xi) = \xi$ and $e^{-\pi} g = g$. Since σ is the shift on $\bigoplus_{\mathbb{Z}} K_0(A)$, it follows that $\xi = 0$. As g is supported away from $\pi^{-1}(0)$ by Lemma 5.5.1, it follows that $g = 0$. Thus, $\mathrm{id} - \alpha$ is injective.

We will now check *(v)*. Consider the map $\Sigma_0 : G \to K_0(A)$ given by

$$\Sigma_0((\xi_n), f) = \sum_{n\in\mathbb{Z}} \xi_n,$$

for any $(\xi_n) \in \bigoplus_{\mathbb{Z}} K_0(A)$ and $f \in G_0$. We claim that the kernel of Σ_0 is $(\mathrm{id} - \alpha)(G)$. If $(\xi, g) \in G$, then $\Sigma_0((\mathrm{id} - \alpha)(\xi, g)) = 0$. Conversely, let $(\xi, g) \in G$ such that $\Sigma_0(\xi, g) = 0$ and choose $N \in \mathbb{N}$ such that $\xi_n = 0$ for all $|n| \geq N$. We follow the proof in [132, Lemma 4.6]. Set $z_n = 0$ for $n \geq N$ and

$$z_n = \xi_n + z_{n+1}, \; n < N.$$

Then

$$z_{-N} = \xi_{-N} + \sum_{i=1}^{2N} \xi_{-N+i} = 0$$

since $\Sigma_0(\xi, g) = 0$. This yields that $z_n = 0$ for any $n \leq -N$. If we set $z = (z_n)_{n\in\mathbb{Z}}$, we get that $(\mathrm{id} - \sigma)(z) = \xi$. By condition *(ii)* in Lemma 5.5.1 and since g is supported away from $\pi^{-1}(0)$, it also follows that there exists $f \in G_0$ such that $g = (1 - e^{-\pi})f$, so $(\mathrm{id} - \alpha)(z, f) = (\xi, g)$. Thus, the kernel of Σ_0 is $(\mathrm{id} - \alpha)(G)$, so Σ_0 induces an isomorphism

$$\Sigma : G/(\mathrm{id} - \alpha)(G) \to K_0(A).$$

The proof of *(vi)* follows as in [55, Lemma 3.8]. Let $\beta \in \mathbb{R}$ and φ be a state on (G, G_+, v) such that $\varphi \circ \alpha = e^{-\beta}\varphi$.

The map $\hat{L}$ can be extended uniquely to a $\mathbb{Q}$-linear map on $\mathbb{Q} \otimes_{\mathbb{Z}} G$. For ease of notation, we will identify $\hat{L}$ with its extension $\hat{L} : \mathbb{Q} \otimes_{\mathbb{Z}} G \to \mathcal{A}(S, \pi)$. Then, we can also extend φ uniquely to a $\mathbb{Q}$-linear state $\hat{\varphi}$ of $\mathbb{Q} \otimes_{\mathbb{Z}} G$. Suppose that $\hat{L}(\xi, g) = 0$ for some $(\xi, g) \in \mathbb{Q} \otimes_{\mathbb{Z}} G$. Then, $\frac{1}{n}v + (\xi, g)$, $\frac{1}{n}v - (\xi, g) \in (\mathbb{Q} \otimes_{\mathbb{Z}} G)_+$, so

$$-\frac{1}{n} \leq \hat{\varphi}(\xi, g) \leq \frac{1}{n}$$

for any $n \in \mathbb{N}$. Thus, $\hat{\varphi}(\xi, g) = 0$, which implies that $\hat{\varphi}$ factors through $\hat{L}$. In other words, there is a $\mathbb{Q}$-linear map

$$\psi : \hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G) \to \mathbb{R}, \text{ such that } \psi \circ \hat{L} = \hat{\varphi}.$$

We claim that ψ is contractive with respect to the supremum norm. Let $f \in \hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$ and suppose by compactness of S that $|f(x)| < \frac{n}{m}$ for any $x \in S$, for some $n, m \in \mathbb{N}$. Then, there exists $w \in \mathbb{Q} \otimes_{\mathbb{Z}} G$ such that $\hat{L}(w) = f$. Since

$$-n = -n\hat{L}(v)(x) < m\hat{L}(w)(x) < n\hat{L}(v)(x) = n$$

for any $x \in S$, we get that $-nv < mw < nv$ in $\mathbb{Q} \otimes_{\mathbb{Z}} G$. Since $\psi \circ \hat{L} = \hat{\varphi}$ and $\hat{\varphi}(v) = 1$, it follows that $|\psi(f)| = |\hat{\varphi}(w)| \leq \frac{n}{m}$. If there exists $g \in \hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$ such that $|\psi(g)| > \sup_{x \in S} |g(x)|$, then there exist $n, m \in \mathbb{N}$ such that

$$|\psi(g)| > \frac{n}{m} > \sup_{x \in S} |g(x)|.$$

But by the argument above, if $|g(x)| < \frac{n}{m}$ for any $x \in S$, then $|\psi(g)| \leq \frac{n}{m}$, so ψ must be norm-contractive.

It follows from *(ii)* above that $\hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$ is uniformly dense in $\mathcal{A}(S, \pi)$. By continuity of ψ, we get that ψ extends uniquely to a linear norm-contractive map $\psi : \mathcal{A}(S, \pi) \to \mathbb{R}$. The fact that there exists a unique $s \in \pi^{-1}(\beta)$ such that $\psi = \mathrm{ev}_s$ follows as in [55, Lemma 3.8]. Hence, $\varphi = \mathrm{ev}_s \circ \hat{L}$ as required. $\square$

Remark 5.5.3. Note that the assumption that A has real rank zero was instrumental in showing that $S(G)$ is a Choquet simplex.

We now build a suitable K_1-group.

Proposition 5.5.4. *There exist a countable, abelian, torsion free group H, an automorphism κ of H, and a homomorphism $q : H \to K_1(A)$ such that*

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\mathrm{id}-\kappa"] & H \arrow[r, "q"] & K_1(A) \arrow[r] & 0 \end{tikzcd}$$

is exact.

Proof. We apply [114, Proposition 3.5] to the pair $(K_1(A), 0)$. $\square$

We can now construct a classifiable C*-algebra B. To show that we can identify A with a corner of a crossed product of $B \otimes \mathcal{K}$, we will produce an automorphism of $B \otimes \mathcal{K}$ such that the crossed product is $\mathcal{Z}$-stable. We first show that we can choose an automorphism of $B \otimes \mathcal{K}$ which has finite Rokhlin dimension. Combining this with the fact that $B \otimes \mathcal{K}$ has finite nuclear dimension will yield that the obtained crossed product has finite nuclear dimension.

Lemma 5.5.5. *Let $\rho_G : G \to \mathrm{Aff}(S(G))$ be the canonical map given by $\rho_G(g)(\phi) = \phi(g)$ for any $g \in G$ and $\phi \in S(G)$. There exist a simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C**-algebra B, satisfying the* UCT *and an automorphism γ of $B \otimes \mathcal{K}$ such that the following conditions hold:*

(i) $\mathrm{Ell}(B) \cong (G, G_+, v, H, S(G), \rho_G)$*;*

(ii) $K_0(\gamma) = \alpha$ *and* $K_1(\gamma) = \kappa$*;*

(iii) $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ *is simple and* $\mathcal{Z}$*-stable;*

(iv) the restriction map is a bijection from the densely defined lower semicontinuous traces on $(B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}$ *onto the* γ*-invariant densely defined lower semicontinuous traces on* $B \otimes \mathcal{K}$*.*

Proof. To construct a C*-algebra B as in the statement of the lemma, we apply Elliott's range-of-the-invariant result (see Theorem 2.6.1). Lemma 5.5.2 shows that (G, G_+, v) is a simple, weakly unperforated, countable ordered abelian group and $S(G)$ is a metrisable Choquet simplex. Thus, by Theorem 2.6.1, there exists a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra B, satisfying the UCT such that $\mathrm{Ell}(B) \cong (G, G_+, v, H, S(G), \rho_G)$.

We construct an automorphism γ' of $B \otimes \mathcal{K}$ which satisfies *(ii)*. Set $p_1 = 1_B \otimes e_{11}$, where e_{11} is a minimal projection in $\mathcal{K}$, and say that $\alpha([p_1]_0) = [p_2]_0$ for some projection $p_2 \in B \otimes \mathcal{K}$. Then, the inclusion $\phi : B \otimes \mathcal{K} \to B \otimes \mathcal{K} \otimes \mathcal{K}$ given by $\phi(x) = x \otimes e_{11}$ induces group isomorphisms $K_i(\phi) : K_i(B) \to K_i(B \otimes \mathcal{K})$ for $i = 0, 1$. By [18], we get isometries $v_i \in \mathcal{M}(B\otimes\mathcal{K}\otimes\mathcal{K})$ such that $v_i v_i^* = p_i \otimes 1_{\mathcal{M}(\mathcal{K})}$ for $i = 1, 2$. Thus, the maps

$$\eta_i := \mathrm{Ad}(v_i) \circ \phi : B \otimes \mathcal{K} \to p_i(B \otimes \mathcal{K})p_i \otimes \mathcal{K}$$

induce isomorphisms at the level of K_0 and K_1 for any $i = 1, 2$ (see [55, Lemma 3.5]).

We then set

$$\alpha' := K_0(\eta_2) \circ \alpha \circ K_0(\eta_1)^{-1}$$

to be a unital ordered group isomorphism from $K_0(p_1(B\otimes\mathcal{K})p_1)$ to $K_0(p_2(B\otimes\mathcal{K})p_2)$.

Similarly, set

$$\kappa' := K_1(\eta_2) \circ \kappa \circ K_1(\eta_1)^{-1}$$

to be a group isomorphism from $K_1(p_1(B \otimes \mathcal{K})p_1)$ to $K_1(p_2(B \otimes \mathcal{K})p_2)$.

By construction of B, the canonical map from tracial states on B to states on $K_0(B)$ is an affine homeomorphism. Therefore, the tracial states on $p_i(B \otimes \mathcal{K})p_i$ are determined by states on $K_0(p_i(B \otimes \mathcal{K})p_i)$ for any $i = 1, 2$. Hence, it is immediate to see that there exists a unique linear isomorphism $\sigma : \mathrm{Aff}(T(p_1(B \otimes \mathcal{K})p_1)) \to \mathrm{Aff}(T(p_2(B \otimes \mathcal{K})p_2))$, which is compatible with α' via the canonical pairing maps. Moreover, since B is nuclear, $\mathcal{Z}$-stable, and satisfies the UCT, and all these properties are preserved by stable isomorphisms (see Proposition 2.5.3), so are $p_i(B \otimes \mathcal{K})p_i$ for $i = 1, 2$. Then, p_1, p_2 are full projections, so $p_i(B \otimes \mathcal{K})p_i$ is simple, separable, unital for $i = 1, 2$. Thus, by [27, Corollary 9.5], there exists a unital *-isomorphism $\psi : p_1(B \otimes \mathcal{K})p_1 \to p_2(B \otimes \mathcal{K})p_2$ such that $(K_0(\psi), K_1(\psi), \mathrm{Aff}(T(\psi))) = (\alpha', \kappa', \sigma)$.

Set γ' to be the following sequence of maps

$$\gamma' : B \otimes \mathcal{K} \xrightarrow{\eta_1} p_1(B \otimes \mathcal{K})p_1 \otimes \mathcal{K} \xrightarrow{\psi \otimes \mathrm{id}_{\mathcal{K}}} p_2(B \otimes \mathcal{K})p_2 \otimes \mathcal{K} \xrightarrow{\eta_2^{-1}} B \otimes \mathcal{K}.$$

By construction, we have $K_0(\gamma') = \alpha$ and $K_1(\gamma') = \kappa$.

To check condition *(iii)*, we will first show that we can choose γ such that the crossed product is simple and has finite nuclear dimension. Building on work in [126], we can take an automorphism γ of $B \otimes \mathcal{K}$ with finite Rokhlin dimension such that $K_i(\gamma) = K_i(\gamma')$ for $i = 0, 1$ ([76, Lemma 4.7] or [75, Theorem 3.4]). Since $B \otimes \mathcal{K}$ is simple, nuclear, and $\mathcal{Z}$-stable, it has finite nuclear dimension ([29, Theorem A]). Moreover, γ has finite Rokhlin dimension, so the crossed product $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ has finite nuclear dimension ([126, Theorem 5.2] or [74, Theorem 3.1]). Since $K_0(\gamma)^n \neq \mathrm{id}$ for all $n \neq 0$, no non-trivial power of γ is inner. Furthermore, $B \otimes \mathcal{K}$ is simple, so $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ is simple by [85, Theorem 3.1]. Thus, we get *(iii)* by [134, Corollary

8.7].

Since B is unital and $\mathcal{K}$ has real rank zero, $B \otimes \mathcal{K}$ has an approximate unit of projections. As γ also has finite Rokhlin dimension, *(iv)* follows from Lemma 5.3.2. $\square$

As B is unital and simple, the restriction map $\tau \mapsto \tau|_{B\otimes e_{11}}$ is a linear order-preserving isomorphism from the cone of densely defined lower semicontinuous traces on $B\otimes\mathcal{K}$ to the space of tracial states on B ([41, Proposition 4.7]). But, by construction, the space of tracial states on B is affinely homeomorphic to the space of states on $K_0(B)$, so we identify states on $K_0(B)$ with densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$. We will now show that A can be identified with a corner of the crossed product $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$.

Lemma 5.5.6. *Let B be the* C*-*algebra and γ be the automorphism of $B \otimes \mathcal{K}$ both given by Lemma 5.5.5 and $p = 1 \otimes e_{11} \in B \otimes \mathcal{K}$, where e_{11} is a minimal projection in $\mathcal{K}$. Then $p\,((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z})\,p \cong A$.*

Proof. We will use the classification theorem in [27, Theorem 9.9]. First, we check that $p\,((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z})\,p$ is a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra satisfying the UCT. Note that $(B\otimes\mathcal{K}) \rtimes_\gamma \mathbb{Z}$ is simple by *(iii)* of Lemma 5.5.5, so p is a full projection. Moreover, $B\otimes\mathcal{K}$ is nuclear, satisfies the UCT, and $(B\otimes\mathcal{K}) \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable by *(iii)* of Lemma 5.5.5. Therefore, by Proposition 2.5.4, $p\,((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z})\,p$ is a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra which satisfies the UCT.

It remains to check that the Elliott invariant of $p\,((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z})\,p$ is isomorphic to the Elliott invariant of A. As discussed in Section 2.4, we do not need to check that the positive cones in the K_0-groups coincide. Note first that the space of tracial states on $p\,((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z})\,p$ is linearly isomorphic to the space of densely defined lower semicontinuous traces on $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ by [41, Proposition 4.7]. By *(iv)* of Lemma 5.5.5, the latter is in a bijective correspondence to the space of γ-invariant densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$. As observed previously, since B is simple, unital and $S(K_0(B)) \cong T(B)$, it follows that we can identify the space of γ-invariant densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ with the space of $K_0(\gamma)$-invariant states on $K_0(B)$. But the latter is homeomorphic to $\pi^{-1}(0) \cong T(A)$ by *(vi)* of Lemma 5.5.2. Hence, $T\left(p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p\right) \cong T(A)$.

To compute the K-groups of $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p$, we apply the Pimsner-Voiculescu exact sequence in [107, Theorem 2.4] to the C*-algebra $B \otimes \mathcal{K}$ and the automorphism γ. Identifying $K_i(B \otimes \mathcal{K})$ with $K_i(B)$ for $i = 0, 1$, we get that

$$\begin{tikzcd}
K_0(B) \arrow[r, "\mathrm{id}-K_0(\gamma)"] & K_0(B) \arrow[r] & K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \arrow[d] \\
K_1((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \arrow[u] & K_1(B) \arrow[l] & K_1(B). \arrow[l, "\mathrm{id}-K_1(\gamma)"']
\end{tikzcd}$$

Recall from *(i)* of Lemma 5.5.5 that $K_1(B) = H$ and from *(ii)* of Lemma 5.5.5 that $K_1(\gamma) = \kappa$. Then, Proposition 5.5.4 gives that $\mathrm{id} - K_1(\gamma)$ is injective. Therefore, the map $K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \to K_1(B)$ is zero, which yields that the map $K_0(B) \to K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z})$ is surjective. Thus,

$$K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \cong K_0(B)/(\mathrm{id} - K_0(\gamma))(K_0(B)). \tag{5.5.9}$$

Since $K_0(B) = G$ by *(i)* of Lemma 5.5.5 and $K_0(\gamma) = \alpha$ by *(ii)* of Lemma 5.5.5, we get that $K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \cong K_0(A)$ by *(v)* of Lemma 5.5.2. This gives that,

$$K_0\left(p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p\right) \cong K_0(A).$$

Since $\Sigma([p]_0) = [1_A]_0$, it follows that the K_0-isomorphism is compatible with the position of the unit.

Combining *(ii)* of Lemma 5.5.5 and *(iv)* of Lemma 5.5.2 yields that the map

$\mathrm{id} - K_0(\gamma)$ is injective. Therefore, we obtain the short exact sequence

$$0 \longrightarrow H \xrightarrow{\mathrm{id}-K_1(\gamma)} H \longrightarrow K_1((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \longrightarrow 0.$$

It follows that $K_1((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \cong K_1(A)$ by Proposition 5.5.4. Hence,

$$K_1\left(p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p\right) \cong K_1(A).$$

To apply [27, Theorem 9.9], it remains to check that $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p$ and A have the same pairing between K-theory and traces. Let τ be a tracial state on A and τ_* be the induced state on $K_0(A)$. Recall that the homeomorphism from $T(A)$ onto $T(p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p)$ is given by following sequence of mappings. One first uses *(vi)* of Lemma 5.5.2 to send τ to $\mathrm{ev}_\tau \circ \hat{L}$, which is a $K_0(\gamma)$-invariant state on $K_0(B)$. Then, *(i)* of Lemma 5.5.5 yields that $\mathrm{ev}_\tau \circ \hat{L}$ corresponds to a unique γ-invariant tracial state on B. By *(iv)* of Lemma 5.5.5, this extends uniquely to a densely defined lower semicontinuous trace on $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$. As mentioned previously, the latter corresponds to a tracial state on $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p$ ([41, Proposition 4.7]).

Therefore, it suffices to check that

$$\tau_* \circ \Sigma_0 = \mathrm{ev}_\tau \circ \hat{L},$$

where $\Sigma_0 : K_0(B) \to K_0(A)$ is the homomorphism in *(v)* of Lemma 5.5.2 which induces the isomorphism Σ. If $(\xi, g) \in K_0(B)$, then

$$\tau_* \circ \Sigma_0(\xi, g) = \tau_* \left(\sum_{n \in \mathbb{Z}} \xi_n \right) = \sum_{n \in \mathbb{Z}} \rho_A(\xi_n)(\tau).$$

On the other hand,

$$(\mathrm{ev}_\tau \circ \hat{L})(\xi, g) = g(\tau) + \sum_{n\in\mathbb{Z}} L(\rho_A(\xi_n))(\tau) e^{n\pi(\tau)}.$$

Since $\pi(\tau) = 0$ and $g \in G_0$ is supported away from $T(A)$ by Lemma 5.5.1, it follows that

$$(\mathrm{ev}_\tau \circ \hat{L})(\xi, g) = \sum_{n\in\mathbb{Z}} L(\rho_A(\xi_n))(\tau).$$

Furthermore, recall from (5.5.1) that for any $f \in \mathrm{Aff}(T(A))$, $L(f)|_{T(A)} = f$. Thus,

$$(\mathrm{ev}_\tau \circ \hat{L})(\xi, g) = \sum_{n\in\mathbb{Z}} \rho_A(\xi_n)(\tau) = \tau_* \circ \Sigma_0(\xi, g).$$

Hence, as A and $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right)p$ have the same Elliott invariant, they are isomorphic by [27, Theorem 9.9]. $\square$

We will now finish the proof of Theorem 17. Consider the dual action $\hat{\gamma}$ on $C = (B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ as a 2π-periodic flow. Recall that $\hat{\gamma}_t(f)(x) = e^{-ixt} f(x)$ for any $t \in \mathbb{R}, f \in C_c(\mathbb{Z}, B \otimes \mathcal{K})$ and $x \in \mathbb{Z}$. Since $p \in B \otimes \mathcal{K}$, $\hat{\gamma}(p) = p$, so $\hat{\gamma}$ restricts to an action on $pCp \cong A$ which we denote by θ. We claim that the KMS-bundle (S^θ, π^θ) is isomorphic to (S, π). We will crucially use [132, Lemma 3.1], so we state it for the convenience of the reader.

Lemma 5.5.7. *[132, Lemma 3.1] Let D be a* C*-*algebra and $\sigma \in \mathrm{Aut}(D)$ be an automorphism of D. Let $\hat{\sigma}$ be the dual action on $D \rtimes_\sigma \mathbb{Z}$ considered as a 2π-periodic flow. For $\beta \in \mathbb{R}$, the restriction map $\omega \mapsto \omega|_D$ is a bijection from the β-*KMS *weights for $\hat{\sigma}$ onto the densely defined lower semicontinuous traces τ on D with the property that $\tau \circ \sigma = e^{-\beta}\tau$. The inverse is the map $\tau \mapsto \tau \circ P$, where $P : D \rtimes_\sigma \mathbb{Z} \to D$ is the canonical conditional expectation.*

Theorem 5.5.8. *Let A be a unital, stably finite, classifiable* C*-*algebra with real* *rank zero and let (S, π) be a compact simplex bundle such that $\pi^{-1}(0)$ is affinely homeomorphic to $T(A)$. Then there exists a 2π-periodic flow θ on A such that its induced* KMS *bundle (S^θ, π^θ) is isomorphic to (S, π).*

Proof. This follows the strategy in [55, Lemma 3.13]. Let $(\omega, \beta) \in S^\theta$. By [88, Remark 3.3], ω extends uniquely to a β-KMS weight $\hat{\omega}$ for $\hat{\gamma}$ on C. By Lemma 5.5.7, the restriction $\hat{\omega}|_{B\otimes\mathcal{K}}$ is a densely defined lower semicontinuous trace on $B \otimes \mathcal{K}$ such that

$$\hat{\omega}|_{B\otimes\mathcal{K}} \circ \gamma = e^{-\beta}\hat{\omega}|_{B\otimes\mathcal{K}}.$$

Since $\hat{\omega}(1 \otimes e_{11}) = 1$, *(vi)* of Lemma 5.5.2 gives a unique $s \in \pi^{-1}(\beta)$ such that

$$(\hat{\omega}|_{B\otimes\mathcal{K}})_* = \mathrm{ev}_s \circ \hat{L}. \tag{5.5.10}$$

Then, we can define a map $\xi : S^\theta \to S$ by $\xi(\omega, \beta) = s$. By construction, $\pi \circ \xi = \pi^\theta$ and the restriction $\xi : (\pi^\theta)^{-1}(\beta) \to \pi^{-1}(\beta)$ is affine for any $\beta \in \mathbb{R}$.

We will first check that the map ξ is injective. Let $(\omega_i, \beta_i) \in S^\theta$ such that $\xi(\omega_1, \beta_1) = \xi(\omega_2, \beta_2) = s$ for $i = 1, 2$. Since $\pi \circ \xi = \pi^\theta$, it follows that $\beta_1 = \beta_2$. By construction of the map ξ, we have that

$$(\hat{\omega}_1|_{B\otimes\mathcal{K}})_* = (\hat{\omega}_2|_{B\otimes\mathcal{K}})_*.$$

As the space of densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ is in a bijective correspondence to the space of states on $K_0(B)$, it follows that $\hat{\omega}_1|_{B\otimes\mathcal{K}} = \hat{\omega}_2|_{B\otimes\mathcal{K}}$, so $\hat{\omega}_1 = \hat{\omega}_2$ by Lemma 5.5.7. Since $\hat{\omega}_i$ is an extension of ω_i for $i = 1, 2$, we get that $\omega_1 = \omega_2$, so the map ξ is indeed injective.

We will now check that ξ is also surjective. Let $s \in S \cap \pi^{-1}(\beta)$ for some $\beta \in \mathbb{R}$.

Then $\mathrm{ev}_s \circ \hat{L}$ is a state on $K_0(B)$ such that

$$\mathrm{ev}_s \circ \hat{L} \circ K_0(\gamma) = e^{-\beta}\, \mathrm{ev}_s \circ \hat{L}.$$

By construction, states on $K_0(B)$ are uniquely induced by tracial states on B (Lemma 5.5.5). These are in one-to-one correspondence with densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ ([41, Proposition 4.7]). Therefore, there exists a unique densely defined lower semicontinuous trace τ on $B \otimes \mathcal{K}$ such that $\tau_* = \mathrm{ev}_s \circ \hat{L}$ and $\tau_* \circ K_0(\gamma) = e^{-\beta}\tau_*$. Since the canonical map from densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ to states on $K_0(B)$ is a bijection (Lemma 5.5.5 and [41, Proposition 4.7]), we get that $\tau \circ \gamma = e^{-\beta}\tau$. If $P : C \to B \otimes \mathcal{K}$ is the canonical conditional expectation, then the restriction $\tau \circ P|_{pCp}$ is a β-KMS state for θ by Lemma 5.5.7. By construction, $\xi(\tau \circ P|_{pCp}, \beta) = s$, so ξ is surjective.

If we show that $\xi^{-1} : S \to S^\theta$ is continuous, then ξ is a homeomorphism by Lemma 5.1.5. Recall from Remark 5.1.4 that both S^θ and S are metrisable and let s_n be a sequence in S which converges to s. Then $\hat{L}(g)(s_n)$ converges to $\hat{L}(g)(s)$ for any $g \in G \cong K_0(B)$. If $\xi^{-1}(s_n) = (\omega_n, \beta_n)$ and $\xi^{-1}(s) = (\omega, \beta)$, then (5.5.10) yields that

$$\lim_{n\to\infty} (\hat{\omega}_n|_{B\otimes\mathcal{K}})_*(g) = (\hat{\omega}|_{B\otimes\mathcal{K}})_*(g)$$

for any $g \in K_0(B)$. Since the canonical map from the densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ to states on $K_0(B)$ is a bijection (follows again from Lemma 5.5.5 and [41, Proposition 4.7]), $\hat{\omega}_n|_{B\otimes\mathcal{K}}$ converges pointwise to $\hat{\omega}|_{B\otimes\mathcal{K}}$. Thus, $\hat{\omega}_n$ converges pointwise to $\hat{\omega}$ by Lemma 5.5.7, so ω_n converges pointwise to ω. Moreover, $\beta_n = \pi(s_n)$ converges to $\pi(s) = \beta$ by continuity of π, so (ω_n, β_n) converges to (ω, β). This shows that ξ^{-1} is continuous. Hence, the KMS bundle (S^θ, π^θ) is isomorphic to (S, π). Combining this with Lemma 5.5.6 finishes the proof. □

Chapter 6

Amenable and Quasidiagonal traces

The content of this chapter is based on results I obtained in a single-author paper and it is essentially contained in [96]. Before delving into the proofs, let us first recall some standard facts about extensions of C*-algebras.

6.1 Extension theory

Extension theory has its roots in the work of Busby in [26]. However, the theory started to get a lot more attention following the remarkable work of L. Brown, Douglas, and Fillmore ([19, 20]) who managed to classify essentially normal operators. This was the first instance when extension theory was shown to have deep consequences to classification in operator theory.

We will give a brief introduction to extension theory of C*-algebras. These concepts will be crucially used in proving the main results of this chapter.

Definition 6.1.1. Let A and I be C*-algebras. By an extension $E = (\alpha, E, \beta)$ of A by I, we mean a C*-algebra E together with morphisms α and β for which the sequence

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\alpha"] & E \arrow[r, "\beta"] & A \arrow[r] & 0 \end{tikzcd}$$

is exact.

The set of all such extensions will be denoted by $\mathrm{Ext}(A, I)$. Moreover, an extension is trivial if it splits i.e. there exists a *-homomorphism $\gamma : A \to E$ such that $\beta \circ \gamma = \mathrm{id}_A$. We denote by $\mathcal{M}(I)$ the multiplier algebra of I and we can use the universal property of the multiplier algebra to get a unique *-homomorphism $\sigma : E \to \mathcal{M}(I)$ such that $\sigma \circ \alpha$ is the inclusion map of I into $\mathcal{M}(I)$. Thus, we can define the *Busby invariant* of an extension of A by I as the unique *-homomorphism $\tau_E : A \to \mathcal{M}(I)/I$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] & E \arrow[r, "\beta"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, "\tau_E"] & 0 \\ 0 \arrow[r] & I \arrow[r] & \mathcal{M}(I) \arrow[r] & \mathcal{M}(I)/I \arrow[r] & 0 \end{tikzcd}$$

commutes.

Conversely, if we have a morphism $\tau_E : A \to \mathcal{M}(I)/I$ and we let $\rho : \mathcal{M}(I) \to \mathcal{M}(I)/I$ be the canonical surjection, then we can form the pullback associated with these morphisms by considering $PB = \{x \oplus a \in \mathcal{M}(I) \oplus A : \rho(x) = \tau_E(a)\}$. Then, by including I into PB and projecting onto A, we get an extension

$$\eta : \begin{tikzcd} 0 \arrow[r] & I \arrow[r] & PB \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

with Busby invariant τ_E.

Two extensions η_1 and η_2 have the same Busby invariant if and only if there exists an isomorphism ψ for which the diagram

$$\begin{tikzcd}
\eta_1 : 0 \arrow[r] & I \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{id}"] & E_1 \arrow[r, "\beta_1"] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\eta_2 : 0 \arrow[r] & I \arrow[r, "\alpha_2"] & E_2 \arrow[r, "\beta_2"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes ([142, Corollary 3.2.12]).

However, to apply this theory later, we want to obtain some extra structure on $\mathrm{Ext}(A, I)$. To do this, let us fix an arbitrary extension

$$\begin{tikzcd}
\eta : 0 \arrow[r] & I \arrow[r] & E \arrow[r, "\pi"] & A \arrow[r] & 0
\end{tikzcd}$$

with Busby invariant $\beta : A \to \mathcal{M}(I)/I$.

First, we will be interested in semisplit extensions. The extension η is called *semisplit* if there is a completely positive and contractive splitting $\phi : A \to E$. Note that by the Choi-Effros lifting theorem ([32]), this is automatic if A is nuclear.

We will mostly be interested in trivial extensions. The main strategy for producing such extensions is to obtain extensions which are absorbing in some suitable sense. We will first consider a notion of simplicity for morphisms and extensions.

Definition 6.1.2. An extension of A by I with Busby invariant β is called *full* if for all nonzero $a \in A$, $\beta(a)$ generates $\mathcal{M}(I)/I$ as a two-sided ideal. An extension η is called *unitizably full* if the unitized extension

$$\begin{tikzcd}
\eta^\dagger : 0 \arrow[r] & I \arrow[r] & E^\dagger \arrow[r, "\pi^\dagger"] & A^\dagger \arrow[r] & 0
\end{tikzcd}$$

is full, where $A^\dagger$ and $E^\dagger$ are the forced unitizations of A and E. If A or E is unital, then a new unit is adjoined. Note that the Busby invariant of the unitized extension is given by $\beta^\dagger : A^\dagger \to \mathcal{M}(I)/I$, $\beta^\dagger(a + \lambda) = \beta(a) + \lambda$ for all $a \in A$ and $\lambda \in \mathbb{C}$.

To obtain further structure, we will define an additive operation on extensions.

Given two extensions η_1 and η_2 of A by I with Busby invariants $\beta_1, \beta_2 : A \to \mathcal{M}(I)/I$, η_1 and η_2 are said to be *strongly unitarily equivalent* if there is a unitary $u \in \mathcal{M}(I)$ such that $\mathrm{Ad}(\rho(u)) \circ \beta_1 = \beta_2$ where $\rho : \mathcal{M}(I) \to \mathcal{M}(I)/I$ is the quotient map.

Suppose that A and I are separable C*-algebras and I is stable by tensoring with the compacts. Then, $\mathcal{M}(I)$ will contain a copy of the Cuntz algebra $\mathcal{O}_2$, so there are isometries s_1, $s_2 \in \mathcal{M}(I)$ such that $s_1 s_1^* + s_2 s_2^* = 1$. Given extensions η_1 and η_2 of A by I with Busby invariants $\beta_1, \beta_2 : A \to \mathcal{M}(I)/I$, define a *-homomorphism $\beta : A \to \mathcal{M}(I)/I$ by

$$\beta(a) = s_1\beta_1(a)s_1^* + s_2\beta_2(a)s_2^*.$$

We denote the extension with this Busby invariant by $\eta_1 \oplus \eta_2$. Up to strong unitary equivalence, $\eta_1 \oplus \eta_2$ is independent of the choice of isometries s_1 and s_2.

When A and I are separable and I is stable, let $\mathrm{Ext}^{-1}(A, I)$ denote the set of equivalence classes of semisplit extensions of A by I, where two semisplit extensions η_1 and η_2 are equivalent if there is a trivial extension η of A by I such that $\eta_1 \oplus \eta$ and $\eta_2 \oplus \eta$ are strongly unitarily equivalent. For a semisplit extension η of A by I, let $\langle \eta \rangle$ denote the element in $\mathrm{Ext}^{-1}(A, I)$ induced by η. The set $\mathrm{Ext}^{-1}(A, I)$ is now an abelian group with addition given by $\langle \eta_1 \rangle + \langle \eta_2 \rangle = \langle \eta_1 \oplus \eta_2 \rangle$.[29]

The zero element of $\mathrm{Ext}^{-1}(A, I)$ is the equivalence class of a split extension. Moreover, for a semisplit extension η of A by I, $\langle \eta \rangle = 0$ if and only if there is a split extension η' of A by I such that $\eta \oplus \eta'$ is a split extension. When the algebra A is not assumed to be nuclear, it is often the case that a refinement of $\mathrm{Ext}^{-1}(A, I)$ is needed. For this, we will have to consider weakly nuclear extensions.

A completely positive splitting $\sigma : A \to E$ for an extension η is called *weakly nuclear* if for all $x \in I$, the map $A \to I$ given by $a \to x\sigma(a)x^*$ is nuclear. Then an extension is called weakly nuclear if it admits a weakly nuclear splitting. By only considering the weakly nuclear extensions in $\mathrm{Ext}^{-1}(A, I)$, one obtains an abelian

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[29] See for example [6, Corollary 15.8.4] for the case when A is nuclear.

group $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$. Considering weakly nuclear extensions is important since we can obtain lifts by applying the Choi-Effros lifting theorem in [32].

From now on, we will interchangeably refer to an extension both by a short exact sequence and by its Busby invariant.

6.2 Examples of amenable traces which are quasidiagonal

Here, by a *trace* τ on a C*-algebra A we mean a positive *contractive* linear functional such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. In particular, if A is unital and $\tau(1) = 1$, then we say τ is a *tracial state.* Since we are going to consider traces composed with *-homomorphisms, we cannot expect that traces of this form will be states, so we need to consider all tracial functionals with norm less than or equal to 1.

Definition 6.2.1. Let A be a separable C*-algebra.[30]

(i) A trace $\tau : A \to \mathbb{C}$ is called *amenable* if for all $n \in \mathbb{N}$, there is an integer $k(n) \geq 1$ and a cpc map $\phi_n : A \to M_{k(n)}(\mathbb{C})$ such that

$$\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_2 \to 0$$

and $\mathrm{tr}_{k(n)}(\phi_n(a)) \to \tau(a)$ for all $a, b \in A$, where $\|x\|_2 = \mathrm{tr}_{k(n)}(x^*x)^{1/2}$ and $\mathrm{tr}_{k(n)}$ is the unique normalised trace on $M_{k(n)}(\mathbb{C})$.

(ii) A trace $\tau : A \to \mathbb{C}$ is called *quasidiagonal* if for all $n \in \mathbb{N}$, there is an integer $k(n) \geq 1$ and a cpc map $\phi_n : A \to M_{k(n)}(\mathbb{C})$ such that

$$\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$$

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[30] Note that one can define these notions for non-separable C*-algebras by replacing sequences with nets.

and $\mathrm{tr}_{k(n)}(\phi_n(a)) \to \tau(a)$ for all $a, b \in A$.

Throughout, ω will stand for a fixed free ultrafilter on $\mathbb{N}$. Then, we can define

$$\mathcal{Q}_\omega := \ell^\infty(\mathcal{Q})/\{(x_n)_{n=1}^\infty \in \ell^\infty(\mathcal{Q}) : \lim_{n\to\omega} \|x_n\| = 0\}$$

to be the uniform ultrapower of the universal UHF-algebra $\mathcal{Q}$. Similarly,

$$\mathcal{R}^\omega := \ell^\infty(\mathcal{R})/\{(x_n)_{n=1}^\infty \in \ell^\infty(\mathcal{R}) : \lim_{n\to\omega} \|a_n\|_2 = 0\}$$

represents the tracial ultrapower of the hyperfinite II_1 factor $\mathcal{R}$. Let tr_ω and tr^ω denote the traces on $\mathcal{Q}_\omega$ and $\mathcal{R}^\omega$ induced by the unique traces on $\mathcal{Q}$ and $\mathcal{R}$.

For convenience, we will record the following result which appears, for example, in [121, Proposition 1.3].

Proposition 6.2.2. *Let A be a separable C^*-algebra.*

(i) A trace τ is amenable on A if and only if there is a *-homomorphism $\phi : A \to \mathcal{R}^\omega$ with a cpc lift $A \to \ell^\infty(\mathcal{R})$ such that $\mathrm{tr}^\omega \circ \phi = \tau$.

(ii) A trace τ is quasidiagonal on A if and only if there is a *-homomorphism $\phi : A \to \mathcal{Q}_\omega$ with a cpc lift $A \to \ell^\infty(\mathcal{Q})$ such that $\mathrm{tr}_\omega \circ \phi = \tau$.

We now provide examples of when amenable traces can be shown to be quasidiagonal. One of the most general such results is given by the quasidiagonality theorem in [135]. Let us record a slightly stronger version from [59].

Theorem 6.2.3. *[59, Theorem 3.8] Let A be a separable, exact C^*-algebra satisfying the* UCT. *Then every faithful amenable trace on A is quasidiagonal.*

Further interesting examples are provided by the class of group C^*-algebras. In particular, we record some known examples regarding traces on full group C^*-algebras of some non-amenable groups.

Proposition 6.2.4. *[22, Proposition 4.1.14] Let $\mathbb{F}_n$ be the free group on n generators. Then, all amenable traces on* $\mathrm{C}^*(\mathbb{F}_n)$ *are quasidiagonal.*

A similar result can be obtained in the case of groups with Property (T). A group G has *Kazhdan's Property (T)* if every unitary representation which has almost invariant vectors actually has invariant vectors.

Theorem 6.2.5. *[24, Corollary 6.4.10] If G is a countable discrete group with Property (T), then all amenable traces on* $\mathrm{C}^*(G)$ *are quasidiagonal.*

6.3 The contractible case and Schafhauser's approach

Before proving the main results of this chapter, we take a short detour and examine the case when A is a contractible C*-algebra. Since contractible C*-algebras are homotopy equivalent to 0, this is an instance where the property that all amenable traces are quasidiagonal is homotopy invariant.

Proposition 6.3.1. *Let A be a separable contractible* C**-algebra. Then all amenable traces on A are quasidiagonal.*

Proof. Let τ be an amenable trace on A. Since A is contractible, the identity map on A is homotopic to the zero map. Therefore, there exists a *-homomorphism $\theta : A \to C_0((0,1],A)$ such that $\theta(a)(1) = a$ for all $a \in A$.

Now observe that τ factorises as

$$A \xrightarrow{\theta} C_0((0,1],A) \xrightarrow{\tau\circ \mathrm{ev}_1} \mathbb{C}.$$

Since τ is amenable on A and ev_1 is a *-homomorphism, $\tau \circ \mathrm{ev}_1$ is amenable on the cone $C_0((0,1],A)$, so quasidiagonal by [23, Proposition 3.2]. Therefore, there exists a *-homomorphism $\phi : C_0((0,1], A) \to \mathcal{Q}_\omega$ with a cpc lift $\psi : C_0((0,1], A) \to \ell^\infty(\mathcal{Q})$ such that $\mathrm{tr}_\omega \circ \phi = \tau \circ \mathrm{ev}_1$, where tr_ω is the induced trace on $\mathcal{Q}_\omega$.

Now $\phi \circ \theta : A \to \mathcal{Q}_\omega$ is a *-homomorphism with a cpc lift $\psi \circ \theta : A \to \ell^\infty(\mathcal{Q})$ such that

$$\mathrm{tr}_\omega \circ \phi \circ \theta = (\tau \circ \mathrm{ev}_1) \circ \theta = \tau.$$

Thus, τ is a quasidiagonal trace on A. □

A standard application of Kaplansky's density theorem (see for example [94, Theorem 4.3.3]) shows that the map $\pi : \mathcal{Q}_\omega \to \mathcal{R}^\omega$ induced by the canonical inclusion $\mathcal{Q} \hookrightarrow \mathcal{R}$ is a surjection. More precisely, if we take some sequence $(x_1, x_2, \ldots) \in \ell^\infty(\mathcal{R})$ and x to be its canonical image in $\mathcal{R}^\omega$, then, by Kaplansky's density theorem, we get $a_n \in \mathcal{Q}$ such that $\|a_n\| \le \|x_n\|$ and $\|a_n - x_n\|_{2,\tau} \le \frac{1}{n}$.

The kernel of the surjection $\pi : \mathcal{Q}_\omega \to \mathcal{R}^\omega$ is called the trace-kernel ideal. Concretely, this ideal will be denoted by

$$J := \{(a_n) \in \mathcal{Q}_\omega : \lim_{n\to\omega} \|a_n\|_{2,\mathrm{tr}^\omega} = 0\}.$$

Therefore, one obtains an extension

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & \mathcal{Q}_\omega \arrow[r, "\pi"] & \mathcal{R}^\omega \arrow[r] & 0, \end{tikzcd}$$

known as the trace-kernel extension. Schafhauser's breakthrough rephrased the quasidiagonality of a trace into a lifting problem.

We are now in the position to state the key idea in Schafhauser's approach in [121]. If τ is any amenable trace on a separable exact C*-algebra A, Proposition 6.2.2 gives a trace-preserving *-homomorphism $\phi : A \to \mathcal{R}^\omega$ which has a cpc lift into $\ell^\infty(\mathcal{R})$. Putting together ϕ and the canonical quotient map π, we get the following pullback extension

$$\begin{tikzcd}
\eta_0 : 0 \arrow[r] & J \arrow[r] \arrow[d, "\mathrm{id}"] & E_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
\eta : 0 \arrow[r] & J \arrow[r] & \mathcal{Q}_\omega \arrow[r, "\pi"] & \mathcal{R}^\omega \arrow[r] & 0.
\end{tikzcd} \tag{6.3.1}$$

Recall that a map $\sigma : A \to E_0$ is *weakly nuclear* if for all $x \in J$, the map $A \to J$ given by $a \mapsto x\sigma(a)x^*$ is nuclear. Since η_0 is a pullback extension, one can note that η_0 has a *-homomorphic splitting $\sigma : A \to E_0$ if and only if there exists a lift $\psi : A \to \mathcal{Q}_\omega$ of ϕ. The only if direction is clear, and if we have a lift ψ, then we can define $\sigma(a) = a \oplus \psi(a)$. Note that if ψ is nuclear, then σ is weakly nuclear.

The key point is that η_0 has a weakly nuclear *-homomorphic splitting if and only if τ is quasidiagonal. The only if direction appears in [121, Theorem 1.2], and the converse, even not spelt out explicitly, is contained in the proof of [122, Proposition 4.3]. Let us include a proof for the convenience of the reader.

Lemma 6.3.2. *Let τ be an amenable trace on a separable exact* C**-algebra A. Then the pullback extension η_0 constructed in* (6.3.1) *has a weakly nuclear *-homomorphic splitting if and only if τ is quasidiagonal.*

Proof. The only if direction is shown in [121, Theorem 1.2]. Conversely, suppose τ is quasidiagonal. Proposition 6.2.2 then implies that there exists a trace-preserving *-homomorphism $\psi : A \to \mathcal{Q}_\omega$ with a cpc lift $A \to \ell^\infty(\mathcal{Q})$. Since A is exact, [59, Proposition 3.1] gives that ψ is nuclear.

Then, we have that $\mathrm{tr}^\omega \circ \phi = \mathrm{tr}^\omega \circ (\pi \circ \psi)$, with $\pi \circ \psi$ and ϕ nuclear by exactness of A ([121, Lemma 5.1]). Since $\mathcal{R}^\omega$ is a finite factor, by a consequence of Connes' theorem ([37]), these two maps are approximately unitarily equivalent (see [122, Proposition 1.1]). Moreover, since A is separable, by a reindexing argument (see for example [60, Lemma 4.1]), ϕ and $\pi \circ \psi$ are unitarily equivalent. Let $\tilde{v}$ be a unitary in $\mathcal{R}^\omega$ such that $\phi = \mathrm{Ad}(\tilde{v})\pi \circ \psi$. As the unitary group of $\mathcal{R}^\omega$ is path-connected, there exists a unitary

$v \in \mathcal{Q}_\omega$ such that $\pi(v) = \tilde{v}$. Therefore, replacing ψ by $\mathrm{Ad}(v)\psi$, we can assume that $\pi \circ \psi = \phi$. We then let $\sigma(a) = a \oplus \psi(a)$ to show that the extension η_0 splits. But ψ is nuclear, so η_0 has a weakly nuclear *-homomorphic splitting. $\square$

Let us end this section with an observation that we are going to use in the proof of Theorem 21. This is in the spirit of deunitization tricks used in [135] and [121]. Precisely, the first part is very similar to the techniques used in [121, Theorem 1.2], and the latter part is [135, Proposition 1.4], without assuming nuclearity of A.

Lemma 6.3.3. *Let τ be an amenable trace on a separable* C**-algebra A. Then $\frac{1}{2}\tau$ is amenable. Moreover, if $\frac{1}{2}\tau$ is quasidiagonal, then τ is quasidiagonal.*

Proof. With the notation as in Definition 6.2.1, let $\phi_n : A \to M_{k(n)}(\mathbb{C})$ be cpc maps, approximately multiplicative in 2-norm approximating the trace τ. If we denote by ι_n the canonical embedding into the top left corner $M_{k(n)}(\mathbb{C}) \to M_{2k(n)}(\mathbb{C})$, let $\psi_n = \iota_n \circ \phi_n$. Then, ψ_n is a sequence of cpc maps, approximately multiplicative in 2-norm such that $\mathrm{tr}_{2k(n)}(\psi_n(a)) \to \frac{1}{2}\tau(a)$ for all $a \in A$. Thus, the trace $\frac{1}{2}\tau$ is amenable.

For the last part, if $\frac{1}{2}\tau$ is quasidiagonal, then we can follow the strategy in the implication (ii)(c) $\implies$ (ii)(b) in [135, Proposition 1.4]. Suppose there exist approximately multiplicative cpc maps $\psi_n : A \to M_{k(n)}(\mathbb{C})$ approximating the trace $\frac{1}{2}\tau$. If A is unital, then, for n large enough, $\psi_n(1)$ is approximately a projection, so let $p_n \in M_{k(n)}(\mathbb{C})$ be a projection close to $\psi_n(1)$. Then, we can consider ψ_n as a cpc map into the corner $p_n M_{k(n)}(\mathbb{C}) p_n$ and it is still approximately multiplicative since $\psi_n(1)$ approximately commutes with the image of ψ_n. Finally, exactly as in [135, Proposition 1.4], one can note that the sequence of cpc maps ψ_n approximates the trace τ. Thus, τ is quasidiagonal.

If A is nonunital, then we can pass to the unitization and the same proof follows since a trace on A is quasidiagonal if and only if the induced trace on the unitization is quasidiagonal ([22, Proposition 3.5.10]). ☐

6.4 Proof of main results

Let us first recall the statement of Theorem 21.

Theorem 6.4.1. *Let A be a separable exact* C**-algebra with a faithful amenable trace* τ. *If* $\sigma : A \to A$ *is a* **-homomorphism which is homotopic to the identity map on* A *and* $\tau \circ \sigma$ *is a quasidiagonal trace on* A*, then* τ *is quasidiagonal on* A.

The proof of Theorem 6.4.1 is heavily motivated by Theorem 1.2 in [121]. Essentially, we can build two extensions with homotopic Busby invariants and, since one splits, the other will split as well. There are *three* key steps in the proof of [121, Theorem 1.2]: obtain a separable version of the extension η_0 built in (6.3.1), show it is nuclearly absorbing, and show that it has class 0 in $\mathrm{Ext}_{\mathrm{nuc}}$. Recall that an extension has class 0 if it has a weakly nuclear splitting after taking the direct sum with an extension with a weakly nuclear splitting. Moreover, an extension η is called *nuclearly absorbing* if the extension obtained by taking the direct sum of η with any extension with a weakly nuclear *-homomorphic splitting is strongly unitarily equivalent to η.

In [121, Theorem 1.2], the UCT is needed to show that the relevant $\mathrm{Ext}_{\mathrm{nuc}}$ class vanishes. We are going to avoid assuming the UCT by using that the relevant class in $\mathrm{Ext}_{\mathrm{nuc}}$ is preserved under homotopy of the Busby invariant.

Therefore, we will break the proof into propositions illustrating these steps. Let us start by producing a separable version of (6.3.1). Crucially, we will use the fact that the property of being an admissible kernel (in the sense of [121, Definition 2.1]) is separably inheritable ([121, Proposition 4.1]).[31] The following result is essentially Proposition 4.2 of [121], with the only modification being that we can make sure that

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[31] Recall the notion of separable inheritability from Definition 3.3.2.

B_0 contains a specified separable preimage of π. Note that the case $C = \{0\}$ is exactly Proposition 4.2 in [121].

Proposition 6.4.2 (cf. [121, Proposition 4.2])**.** *Consider an extension*

$$0 \longrightarrow I \longrightarrow B \xrightarrow{\pi} D \longrightarrow 0$$

such that I is an admissible kernel, B and D are unital. Suppose $D_0 \subset D$ is any separable, unital subalgebra and C is a separable C**-subalgebra of B such that $\pi(C) \subset D_0$. Then, there exists a separable, unital subalgebra $B_0 \subset B$ such that $C \subset B_0$, $\pi(B_0) = D_0$ and $B_0 \cap I$ is an admissible kernel.*

Proof. Let S_0 be countable dense in C. Fix a countable dense subset $T \subset D_0$ such that $\pi(S_0) \subset T$ and let $S \subset B$ countable such that $S_0 \subset S$ and $\pi(S) = T$. The rest of the proof now follows verbatim as in [121, Proposition 4.2]. □

Combining this and Lemma 6.3.2, one can show that a quasidiagonal trace produces a separable extension with a weakly nuclear *-homomorphic splitting. To show this, let us move into the set-up of Theorem 6.4.1, but without assuming faithfulness. Let A be a separable exact C*-algebra, τ an amenable trace on A, and $\sigma : A \to A$ a *-homomorphism which is homotopic to the identity map on A such that $\tau \circ \sigma$ is a quasidiagonal trace on A. Using the notation in Proposition 6.2.2, let $\phi : A \to \mathcal{R}^\omega$ be a *-homomorphism witnessing amenability of τ.

Proposition 6.4.3. *There exists an extension*

$$\eta' : 0 \longrightarrow J_0 \longrightarrow Q_0 \xrightarrow{\pi_0} R_0 \longrightarrow 0$$

with J_0 a separable admissible kernel, $\phi(A) \subseteq R_0$, and such that if we denote by ϕ_0 the corestriction of ϕ to R_0, then the pullback extension induced by π_0 and $\phi_0 \circ \sigma$ has a weakly nuclear *-homomorphic splitting.

Proof. Following Proposition 4.3 in [121], we get a separable, unital C*-subalgebra $R_0 \subset \mathcal{R}^\omega$ such that R_0 is simple, $\phi(A) \subset R_0$ and ϕ is nuclear as a map $A \to R_0$. Denote by ϕ_0 the corestriction of ϕ to R_0. Applying Proposition 6.4.2 to the trace-kernel extension η, we obtain a separable, unital subalgebra $Q_0 \subset \mathcal{Q}_\omega$ such that $J_0 = Q_0 \cap J$ is an admissible kernel and $\pi(Q_0) = R_0$. We denote by π_0 the restriction of π from Q_0 into R_0. Then, we can form a pullback extension.

$$\begin{tikzcd}
\eta_1' : 0 \arrow[r] & J_0 \arrow[r] \arrow[d, "\mathrm{id}"] & E_1' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\phi_0 \circ \sigma"] & 0 \\
\eta' : 0 \arrow[r] & J_0 \arrow[r] & Q_0 \arrow[r, "\pi_0"] & R_0 \arrow[r] & 0
\end{tikzcd} \tag{6.4.1}$$

By Lemma 6.3.2, the pullback extension induced by π and $\phi \circ \sigma$ has a weakly nuclear *-homomorphic splitting induced by a *-homomorphism $\psi : A \to \mathcal{Q}_\omega$ which witnesses the quasidiagonality of the trace $\tau \circ \sigma$. Since $\pi(\psi(A)) = \phi \circ \sigma(A) \subset R_0$, Proposition 6.4.2 allows us to assume that $\psi(A) \subset Q_0$, so ψ will also give a weakly nuclear *-homomorphic splitting for the extension η_1'. □

With the notation from Proposition 6.4.3, we consider the pullback extension

$$\begin{tikzcd}
\eta_0' : 0 \arrow[r] & J_0 \arrow[r] \arrow[d, "\mathrm{id}"] & E_0' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\phi_0"] & 0 \\
\eta' : 0 \arrow[r] & J_0 \arrow[r] & Q_0 \arrow[r, "\pi_0"] & R_0 \arrow[r] & 0
\end{tikzcd} \tag{6.4.2}$$

induced by π_0 and ϕ_0, and we will prove that η_0' has a weakly nuclear *-homomorphic splitting.

If A and B are separable C*-algebras, a direct consequence of the canonical identification $\mathrm{KK}^1(A,B) \cong \mathrm{Ext}^{-1}(A,B)$ ([6, Corollary 18.5.4]), says that two semisplit extensions with homotopic Busby invariants have the same class in $\mathrm{Ext}^{-1}(A,B)$ ([6, Corollary 15.10.1]). The key technical fact we are claiming is that the same holds when all the relevant extensions are weakly nuclear. Precisely, if β_1 and β_2 are nuclear Busby invariants, homotopic via a path of nuclear maps, then the extensions they induce have the same class in $\mathrm{Ext}_{\mathrm{nuc}}$. This is a direct consequence of [86, Corollary 1.8]. In [86, Corollary 1.8], Kucerovsky and Ng assume that all extensions are weakly nuclear, and by an absorbing extension they mean an extension which absorbs a weakly nuclear split extension. With these clarifications, [86, Corollary 1.8] translates to $\mathrm{KK}^1_{\mathrm{nuc}}(A,B) \cong \mathrm{Ext}_{\mathrm{nuc}}(A,B)$. Therefore, if we take β_1 and β_2 as above, since the homotopy is induced by nuclear maps, Theorem 4.4 in [121] shows that the extensions induced by β_1, β_2, and the maps realising the homotopy are weakly nuclear. Since $\mathrm{KK}^1_{\mathrm{nuc}}(A,B)$ is invariant under homotopy via nuclear maps, β_1 and β_2 induce the same element in $\mathrm{KK}^1_{\mathrm{nuc}}(A,B)$. Thus, they induce extensions with the same class in $\mathrm{Ext}_{\mathrm{nuc}}(A,B)$ by [86, Corollary 1.8]. Let us apply this observation to the extensions η_0' and η_1' defined in (6.4.2) and (6.4.1).

Proposition 6.4.4. *With the notation from* (6.4.2)*, the class of the extension* η_0' *is* 0 *in* $\mathrm{Ext}_{\mathrm{nuc}}(A, J_0)$.

Proof. Denote by $\mathcal{M}(J_0)$ the multiplier algebra of J_0. Then, if $\beta : R_0 \to \mathcal{M}(J_0)/J_0 = Q(J_0)$ is the Busby invariant of the extension η', then $\beta \circ \phi_0$ is the Busby invariant of η_0' and $\beta \circ \phi_0 \circ \sigma$ is the Busby invariant of η_1'. Since σ is homotopic to the identity on A, the Busby invariants of the extensions η_0' and η_1' are homotopic and since ϕ_0 is nuclear, the homotopy is realised via nuclear maps. Therefore, the observation above implies that the extensions η_0' and η_1' have the same class in $\mathrm{Ext}_{\mathrm{nuc}}(A, J_0)$.

Finally, η_1' is a split extension by Proposition 6.4.3, so the class of η_0' is also 0 in $\mathrm{Ext}_{\mathrm{nuc}}(A, J_0)$. □

The last step to conclude that η_0' is a split extension is to show that η_0' is nuclearly absorbing. This is the point where the faithfulness of τ comes into picture and, as in [121], it is required to prove that η_0' is unitizably full (see [121, Section 2]). Since a unital *-homomorphism cannot be unitizably full, we need to split into cases, and we first handle the case where ϕ_0 is nonunital.

Proposition 6.4.5. *Let A be a separable exact* C**-algebra, τ be a faithful amenable trace on A, and $\sigma : A \to A$ be a *-homomorphism which is homotopic to the identity map on A such that $\tau \circ \sigma$ is a quasidiagonal trace on A. Suppose further that A is nonunital or A is unital and $\phi_0(1) \neq 1$. Then η_0' is nuclearly absorbing.*

Proof. The fact that η_0' is unitizably full follows verbatim from [121, Theorem 4.4]. Since J_0 is a separable admissible kernel, J_0 is stable and has the corona factorization property ([122, Proposition 3.3.(3)]). Combining this with the fact that η_0' is unitizably full, we get that η_0' is nuclearly absorbing by [58, Theorem 2.6]. ☐

Finally, the proof of our main result is a combination of the previous lemmas, so let us assume the same notation as in the pullback extensions built in (6.4.2) and (6.4.1). Having our previous results at hand, the nonunital case follows verbatim as in [121, Theorem 1.4]. For the case when τ is a tracial state, instead of following the proof of [121, Theorem 1.4] which is using classification of normal *-homomorphisms into II$_1$ factors, we are going to use Lemma 6.3.3.

Proof of Theorem 6.4.1. Suppose first that A is nonunital or A is unital and $\phi_0(1) \neq 1$. By combining Proposition 6.4.4 and Proposition 6.4.5, we get that η_0' has a weakly nuclear *-homomorphic splitting. Then, by making use of exactness of A and faithfulness of τ, one can use the proof of [121, Theorem 1.4] to get a nuclear *-homomorphism $\psi_0 : A \to \mathcal{Q}_\omega$ such that $\pi \circ \psi_0 = \phi$. But since ψ_0 is nuclear, the Choi-Effros lifting theorem implies that ψ_0 has a cpc lift $A \to \ell^\infty(\mathcal{Q})$. Moreover, by choice of ϕ, we have $\mathrm{tr}^\omega \circ \phi = \tau$, so

$$\tau = \mathrm{tr}^\omega \circ \pi \circ \psi_0 = \mathrm{tr}_\omega \circ \psi_0.$$

Thus, by Proposition 6.2.2, τ is quasidiagonal.

Finally, if A is unital and ϕ_0 is unital, then τ is a tracial state. By Lemma 6.3.3, $\frac{1}{2}\tau$ is amenable, so one can consider the same problem with $\frac{1}{2}\tau$ and $\frac{1}{2}\tau \circ \sigma$, and since

$\frac{1}{2}\tau$ is not a state, the proof above will give that $\frac{1}{2}\tau$ is quasidiagonal. Hence, τ is quasidiagonal by Lemma 6.3.3. $\square$

Theorem 20 now follows easily from Theorem 6.4.1 by unravelling the definitions and using a simple trick to pass quasidiagonality from a faithful trace to all amenable traces on A. We will first recall the statement of the theorem.

Theorem 6.4.6. *Let A be a separable exact* C**-algebra with a faithful amenable trace and suppose A is homotopy dominated by some separable* C**-algebra B. If all amenable traces on B are quasidiagonal, then all amenable traces on A are quasidiagonal.*

Proof. Let $\phi : A \to B$ and $\psi : B \to A$ be two *-homomorphisms such that $\psi \circ \phi$ is homotopic to the identity map on A.

Now let τ_0 be a faithful amenable trace on A. Then, $\tau_0 \circ \psi$ is amenable on B, so quasidiagonal by assumption. Thus, precomposing with ϕ gives a quasidiagonal trace on A. Therefore, by Theorem 6.4.1, τ_0 must be quasidiagonal on A. If τ is any amenable trace on A, for any $n \in \mathbb{N}$ the convex combination $\tau_n := \frac{n-1}{n}\tau + \frac{1}{n}\tau_0$ is a faithful amenable trace on A, so quasidiagonal by the same argument.

Then, the set of quasidiagonal traces is weak*-closed ([22, Proposition 3.5.1]) and τ_n converges weak* to τ, so τ must be quasidiagonal. $\square$

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