Global regularity results for vectorial elliptic problems with non-quadratic growth



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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2021

Acknowledgements

Firstly, I would like to thank my advisor Jan Kristensen for his constant support in the last four years. Thank you for always being willing to meet to discuss ideas and problems. I cannot count how often he pointed me in the direction of a reference that turned out to be applicable to the problem at hand.

I am grateful to Gui-Qiang Chen, Luc Nguyen and Gregory Seregin for helpful discussions of preliminary versions of this thesis at transfer and confirmation of status.

I would like to thank Yves Capdeboscq for setting me on the path towards regularity questions in a summer research project during my undergraduate and teaching me the basics of elliptic regularity theory in my first year. I am grateful for the opportunity to continue to study regularity questions I find interesting in the group of Felix Otto, in collaboration with Michael Goldman.

Thanks is due to André Guerra and Sauli Lindberg for the very enjoyable discussions and collaboration on the prescribed Jacobian equation in the past year. I look forward to many more discussions and joint papers in the future.

Without the PDE CDT group and countless coffee discussions, mathematical and otherwise, my experience would have been considerably poorer. Particular thanks go to Asad, André and James for lunches, punting adventures and board games as well as Vicki and Jeremy for helping to make the Covid-times bearable.

I am also grateful for the friends I've made at MCBC and OUL – without your distraction from mathematics I could not have written this thesis.

Finally, I thank EPSRC [EP/L015811/1] for funding my DPhil studies.

Abstract

In this thesis we are concerned with proving regularity properties of solutions to the following problem: Given a domain Ω in $\mathbb{R}^n$, find a map $u\colon \Omega \to \mathbb{R}^N$ minimizing the functional

$$\mathscr{F}(v) = \int_\Omega F(x, v(x), \mathrm{D}v(x)) - f(x) \cdot v(x)\,\mathrm{d}x \tag{P}$$

amongst all maps v in a suitable Sobolev space. Here, we assume that $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$. $f\colon \Omega \to \mathbb{R}^N$ is a given Sobolev map. We will usually impose Dirichlet boundary conditions

$$u = g \qquad \text{on } \partial\Omega,$$

where $g\colon \Omega \to \mathbb{R}^N$ is a given Sobolev map.

We are interested in the vectorial case and hence assume throughout that $n, N > 1$.

Given sufficiently regular data f and g, the existence of solutions to (P) can be derived by the direct method under growth and ellipticity assumptions on $F(x, y, z)$ such as

$$\lambda(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\xi|^2 \leq \partial_{zz}F(x, y, z)\xi \cdot \xi \leq \Lambda(\mu^2 + |z|^2)^{\frac{q-2}{2}}|\xi|^2 \tag{1}$$

for some $\lambda, \Lambda > 0$, $\mu \geq 0$, $1 < p \leq q < \infty$, almost all $x \in \Omega$ and all $y \in \mathbb{R}^N$, $z, \xi \in \mathbb{R}^{N\times n}$.

We also study the closely related question of regularity of weak solutions $u\colon \Omega \to \mathbb{R}^N$ in an appropriate Sobolev space to systems of the form

$$\operatorname{div} a(x, u, \mathrm{D}u) = f. \tag{2}$$

Here $a(x, y, z)\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$.

We begin by studying (2) when $a(x,y,z)$ is an elliptic structure field satisfying (1) with $p=q$ (and where $\partial_{zz}F(x,y,z)$ is replaced by $\partial_z a(x,y,z)$). We view $a(x,y,z)$ as a perturbation of a reference elliptic structure field $b(x,y,z)$, also satisfying (1) with $p=q$, and for which solutions to (2) exist, we introduce the iterative process

$$\operatorname{div} b(x,u_{n+1},Du_{n+1}) = \operatorname{div} b(x,u_n,Du_n) - \gamma \operatorname{div} a(x,u_n,\mathrm{D}u_n) + \gamma f,$$

where $\gamma \in (0,1)$. We derive conditions that ensure convergence of the process to a solution of $\operatorname{div} a(x,u,\mathrm{D}u) = 0$. Furthermore, we investigate numerical applications of the process and use the process to transfer higher regularity properties of solutions to $\operatorname{div} b(x,u,\mathrm{D}u) = \operatorname{div} f$ to solutions of $\operatorname{div} a(x,u,\mathrm{D}u) = \operatorname{div} f$.

We next consider (P) where $F(x,y,z)$ satisfies (1) with $p<q$. In this case, in addition to the standard pointwise minimisation of (P) amongst maps $v \in W^{1,p}(\Omega)$, we also consider a suitable relaxed formulation. If it holds that $F(x,y,z) = F(x,z)$ is α-Hölder-continuous in x, we prove the $W^{1,q}$-regularity of minimisers to (P) in both the pointwise and relaxed formulation under the assumption $q < \frac{(n+\alpha)p}{n}$. This is the global equivalent to known sharp $W^{1,q}_{\mathrm{loc}}$-regularity results [79, 78]. In the autonomous case $F(x,y,z) = F(z)$, we can improve our arguments to conclude the $W^{1,q}(\Omega)$-regularity of minimisers under the assumption $q < \frac{np}{n-1}$. This is again the global equivalent to a known $W^{1,q}_{\mathrm{loc}}$-regularity result [186]. Our proofs rely on a difference quotient argument on the level of the functional $\mathscr{F}(\cdot)$, that is, without relying on the Euler–Lagrange equation, and a careful construction of appropriate approximations to $W^{1,p}$-functions in order to overcome the Lavrentiev phenomenon.

Considerations coming from convex duality theory lead us to consider the growth conditions of the form

$$\lambda\Big(\mu^2+|z|^2\Big)^{\frac{p-2}{2}}|w|^2 \le \partial_{zz}F(z)w\cdot w \le \Lambda\Big(1+|\partial_z F(z)|^2\Big)^{\frac{q-2}{2(q-1)}}|w|^2. \tag{3}$$

We prove the $W^{1,q}_{\mathrm{loc}}$-regularity for minimisers of (P) when $F(x,y,z) = F(z)$ satisfies (3) with $q < \frac{np}{n-2}$ if $n>2$ or any $p \le q < \infty$ if $n=2$. We further

show that polynomials of degree $d \geq 2$ automatically satisfy the upper bound in (3). As a consequence, we are able to show the local real analyticity of minimisers of (P) in the case where $F(x, y, z) = F(z)$ is a convex polynomial satisfying a quadratic lower bound and $n = 2$.

In the final chapter, we consider the prescribed Jacobian equation: For $n > 1$ let us take a bounded domain $\Omega \subset \mathbb{R}^n$ which is both smooth and uniformly convex. Then consider

$$\begin{cases} \det \mathrm{D}u = f & \text{in } \Omega \\ u = \mathrm{id} & \text{on } \partial\Omega. \end{cases} \tag{JD}$$

Here $f \colon \Omega \to \mathbb{R}$ is integrable and satisfies the following compatibility and non-degeneracy conditions

$$\frac{1}{|\Omega|} \int_\Omega f(x) \, \mathrm{d}x = 1 \qquad \text{and} \qquad \inf_\Omega f \geq c > 0. \tag{4}$$

We study Sobolev solutions of (JD) when $f \in L^p(\Omega)$ and we focus, in particular, on the case where the solutions are in a critical or supercritical Sobolev space. Our main result is that, for $p > 1$, there is in general no solution $u \in W^{1,q}(\Omega, \mathbb{R}^n)$, $q > \max(n, p)$, if $f \in L^p(\Omega)$. If $f \in L \log L(\Omega)$, there is in general no solution $u \in W^{1,q}(\Omega, \mathbb{R}^n)$, $q > n$. We also prove some further existence and uniqueness properties of Sobolev solutions to the Jacobian equation $\det \mathrm{D}u = f$ posed in $\mathbb{R}^n$ with data in the Hardy space $\mathrm{H}^p(\mathbb{R}^n)$.

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1. Introduction

1 Overview of main results

Let Ω be a domain in $\mathbb{R}^n$. In this thesis we are concerned with regularity properties of solutions to the following problem: Find a map $u\colon \Omega \to \mathbb{R}^N$ minimizing the functional

$$\mathscr{F}(v) = \int_\Omega \Big(F(x, v(x), \mathrm{D}v(x)) - f(x) \cdot v(x) \Big) \,\mathrm{d}x \tag{P}$$

amongst all maps v in a suitable Sobolev space. Here $F(x, y, z)$ is an integrand, that is $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$. Moreover, $f\colon \Omega \to \mathbb{R}^N$ is a given sufficiently regular map. We will usually impose Dirichlet boundary conditions

$$u = g \qquad \text{on } \partial\Omega,$$

where $g\colon \Omega \to \mathbb{R}^N$ is a sufficiently regular boundary datum. This type of problem is the fundamental problem studied in the calculus of variations. To be precise, we say

Definition 1.1. Given $p > 1$, $g \in W^{1,p}(\Omega)$ and $f \in W^{-1,p'}(\Omega) = W^{-1,p'}(\Omega, \mathbb{R}^N)$, we call a Sobolev map $u \in W^{1,p}_g(\Omega) = W^{1,p}_g(\Omega, \mathbb{R}^N)$ a **(pointwise) minimiser** of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$ if it holds that $F(x, u, \mathrm{D}u) \in L^1(\Omega)$ and

$$\int_\Omega F(x, u, \mathrm{D}u) - f \cdot u \,\mathrm{d}x \le \int_\Omega F(x, u, \mathrm{D}u + \mathrm{D}\phi) - f \cdot (u + \phi) \,\mathrm{d}x$$

for all $\phi \in W^{1,p}_0(\Omega)$.

More generally, let $\mathcal{G}\colon W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be (any) extended real-valued functional on $W^{1,p}(\Omega)$. Then $u \in W^{1,p}(\Omega)$ is a **minimiser of $\mathcal{G}(\cdot)$** provided $\mathcal{G}(u) < +\infty$ and

$$\mathcal{G}(u) \le \mathcal{G}(u + \phi)$$

for all $\phi \in W^{1,p}_0(\Omega)$.

Through the Euler–Lagrange equations, (P) is closely related to systems of the form

$$\operatorname{div} a(x,u,\mathrm{D}u) = f. \tag{1.1}$$

Here $a(x,y,z)\colon \Omega\times\mathbb{R}^n\times\mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$ and we understand that the divergence operator is applied row-wise. In fact, assuming that $F(x,y,z)$ is jointly in C^2, the Euler–Lagrange equation for (P) is

$$\operatorname{div}\Big(\partial_z F(x,u(x),\mathrm{D}u(x))\Big) - \partial_y F(x,u(x),\mathrm{D}u(x)) = f. \tag{EL}$$

Here $\partial_y F$ and $\partial_z F$ denote the derivative of $F(x,y,z)$ with respect to the second and third argument respectively. Note that (EL) is of the form (1.1) with a nonlinear source term depending on $(u,\mathrm{D}u)$. If $F(x,y,z)=F(x,z)$, so that $\partial_y F(x,u\mathrm{D}u)=0$, then (EL) is of the form (1.1) with right-hand side f.

We are interested in the vectorial case, and hence assume throughout that $n,N>1$. In the scalar case, stronger results than the ones discussed in this thesis can be obtained, but we do not comment on these here, referring instead to [100] for an overview of the theory.

The study of (P) has a long history. An overview of the historical development can be found in [31] and the introduction to [100]. Initially the problem – with regards to both existence and regularity questions – was mostly approached through the Euler–Lagrange equation. A direct approach to the existence of solutions of (P) was developed later. This approach is now standard and known as the direct method of the calculus of variations. Similarly, direct approaches to regularity are known as the direct method for regularity in the calculus of variations. In order to deduce the existence of minimisers to (P) using the direct method, $F(x,y,z)$ needs to satisfy ellipticity and growth conditions. Prototype bounds that cover a large range of the operators of interest can be given through so-called controlled growth conditions:

$$\lambda(\mu^2+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \le \partial_{zz}F(x,y,z)\xi\cdot\xi \le \Lambda(\mu^2+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \tag{1.2}$$

for some $\lambda,\Lambda>0$, $\mu\ge 0$, $1<p\le q<\infty$, almost all $x\in\Omega$, and all $y\in\mathbb{R}^N$, $z,\xi\in\mathbb{R}^{N\times n}$. We remark that (1.2) implies that $F(x,y,z)$ is strictly convex in z. The regularity theory in the end-point case $p=1$ is harder and requires a different functional set-up, explaining our restriction to $p>1$.

A weaker alternative to (1.2) is

$$\lambda(\mu^2 + |z|^2)^{\frac{p}{2}} \leq F(x, y, z) \leq \Lambda(\mu^2 + |z|^2)^{\frac{q}{2}}. \tag{1.3}$$

The condition (1.3) is referred to as natural growth. (1.3) is usually supplemented with a convexity assumption, e.g., $F(x, y, z)$ is convex with respect to z.

We give a very rough sketch of the idea behind the direct method, illustrating why the given growth and ellipticity bounds are necessary. Due to the growth bounds there exists a map $u \in W^{1,p}_g(\Omega)$ for which $\mathscr{F}(u) < \infty$. In fact, assuming that $g \in W^{1,q}(\Omega)$, g provides such an example. Thus we may consider a sequence $(u_j)_{j\in\mathbb{N}} \subset W^{1,p}_g(\Omega)$ such that $(\mathscr{F}(u_j))_{j\in\mathbb{N}}$ converges to the minimum value of the problem (P). Using the ellipticity bounds we may extract a weakly convergent subsequence (u_{j_k}). The convexity of $F(x, y, z)$ in z ensures weak sequential lower semicontinuity of the functional $\mathscr{F}(\cdot)$ in $W^{1,p}(\Omega)$, and hence the weakly convergent subsequence (u_{j_k}) converges in $W^{1,p}(\Omega)$ to a minimiser of (P). The strict convexity ensures the uniqueness of the minimiser. We also see that, if $p = q$, it suffices to assume quasi-convexity in order to obtain the existence of minimisers – quasi-convexity is precisely the condition needed in order to ensure weak sequential lower semicontinuity of the functional. If quasi-convexity is assumed and $p \neq q$, then it becomes necessary to work with a relaxed version of the functional in order to deduce the existence of minimisers through the direct method. If one wishes to work with the pointwise definition, it is necessary to assume stronger quasi-convexity assumptions, see [17, 187, 143].

In the case of systems of the form (1.1) similar growth and ellipticity conditions with $\partial_{zz}F(x, y, z)$ replaced by $\partial_z a(x, y, z)$ need to be assumed in order to obtain the existence of solutions. Under growth and ellipticity conditions, the existence and uniqueness can be established using the theory of monotone operators. The procedure is classical and can be found in many textbooks, e.g [192].

The existence of solutions in Sobolev spaces is the natural starting point for regularity theory for (P). In fact, functions in Sobolev spaces are in general only weakly differentiable and hence it is natural to ask whether it is possible to prove that the minimisers are continuous, differentiable, etc. Of course, in the context of Sobolev functions

(1.1) needs to be interpreted in the weak sense, that is

$$\int_\Omega a_i(x,u,\mathrm{D}u)\mathrm{D}_i\phi\,\mathrm{d}x = \int_\Omega f D_i\phi\,\mathrm{d}x$$

for all $\phi \in C_0^\infty(\Omega,\mathbb{R}^N)$, where we implicitly sum over repeated indices.

The simplest functional to consider is the Dirichlet functional,

$$\int_\Omega |\mathrm{D}u|^2\,\mathrm{d}x$$

whose study leads to the theory of harmonic functions. Here there is a very strong and well-established regularity theory. Solutions are in general as smooth as the data allow, that is as smooth as the forcing term f, the boundary $\partial\Omega$ and boundary data g allow, see, e.g., [99].

In terms of growth conditions, the Laplacian obviously satisfies controlled growth (1.2) with $p = q = 2$ and $\lambda = \Lambda = 1$. The main question of this thesis can be roughly phrased as:

> *Which regularity properties of harmonic functions are retained when the conditions $p = q$ and/or $\lambda = \Lambda = 1$ are relaxed?*

In this introduction, we give a very rough overview of how the problems we consider in this thesis relate to this question. We refrain from giving an overview of the theory relating to each of the problems we consider, but refer to the start of each chapter for a discussion of the context and literature pertaining to each particular problem.

Already in the case where $p = q = 2$ is enforced, but $\lambda < \Lambda$, the situation changes compared to the Laplacian and irregularity occurs. The first counter-example to regularity was given in [69] where a quadratic functional is exhibited for which minimisers for $n \geq 3$ are not Hölder-continuous of any order. A number of structural conditions are known that allow to recover Hölder-regularity of solutions. In [141] for systems of the type (1.1) where $a(x,y,z)$ satisfies controlled growth conditions with $p = q = 2$ the iterative process

$$-\Delta u_{i+1} = -\Delta u_i - \gamma \mathrm{div}\ a(x,u_i,\mathrm{D}u_i)$$

is introduced. Remarkably, it is possible to show, see [141], that if for some $\alpha \in (0,1)$,

$$1 - \frac{\alpha(\alpha+n-2)}{2(n-1)} > 0 \quad \text{and} \quad \frac{(\Lambda-\lambda)^2}{(\Lambda+\lambda)^2}\Big(1-\frac{\alpha(n-2)}{n-1}\Big)\Big(1-\frac{\alpha(\alpha+n-2)}{2(n-1)}\Big)^{-1} < 1$$

then solutions to (1.1) are $C^{0,\alpha}_{\mathrm{loc}}(\Omega)$-continuous, and these conditions are sharp. We remark that boundary regularity results in the same style, as well as higher regularity results, are also proven in [141]. In Chapter 2 we extend some of these results to systems with p-growth, that is, to systems of the form (1.1) with $a(x, y, z)$ satisfying controlled growth conditions with $p = q \neq 2$.

In light of the counter-examples arising already when $q = p$, it is not surprising that when $p < q$ the situation does not improve. Already in the scalar $N = 1$ autonomous case $F(x, y, z) = F(z)$, counterexamples show that, in order to prove regularity of minimisers, p and q may not be too far apart [95], [153], [121]. Hence the main question in this context becomes to give conditions on $q - p$ that ensure regularity properties of minimisers. A crucial first step to proving interior regularity results is to obtain $W^{1,q}_{\mathrm{loc}}$-regularity of minimisers. Once $W^{1,q}_{\mathrm{loc}}$-regularity is obtained, further regularity results such as partial regularity may be obtained, see for example [2], [175]. Suppose $F(x, z)$ is a non-autonomous integrand, convex and with (p, q)-growth in z, while satisfying a uniform α-Hölder condition in x. For $n \geq 2$ counterexamples to $W^{1,q}$-regularity with $1 < p < n < n + \alpha < q$ are due to [79], see also [89]. Recent work suggests that the condition $p < n < q$ may be removed [12]. If $q < \frac{(n+\alpha)p}{n}$, it was proven in [79] for many standard examples that minimisers enjoy $W^{1,q}_{\mathrm{loc}}$-regularity. In Chapter 3 we extend this local result to a global regularity statement, showing $W^{1,q}(\Omega)$-regularity of minimisers. In the case of autonomous integrands $F(x, y, z) = F(z)$, we improve our result and show $W^{1,q}(\Omega)$-regularity of minimisers if $q < \frac{np}{n-1}$. This is the global equivalent of local regularity results in [44].

The following relaxed functional naturally appears in our approach:

$$\overline{\mathscr{F}}(u) = \inf \Big\{ \liminf_{j\to\infty} \mathscr{F}(u_j) \colon (u_j)_{j\in\mathbb{N}} \subset W^{1,q}_g(\Omega), u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega) \Big\}.$$

In fact, our regularity results apply to minimisers of the relaxed functional, too. Note that we have incorporated the boundary condition $u = g$ on $\partial\Omega$ in the definition of the relaxed functional. This is natural in the given context, but not done often in the literature.

In our analysis of minimisers for functionals with (p, q)-growth, a part is played by the question of whether minimisers u of the relaxed functional satisfy the Euler-Lagrange equations in some sense. It turns out that in the previously described set-up relaxed

minimisers are indeed very weak solutions of the Euler-Lagrange equations and moreover that, if $1 < p \leq q \leq \frac{np}{n-p}$, we have the stress-regularity $\partial_z F(\mathrm{D}u) \in L^{q'}(\Omega)$. Convex duality theory plays a major role in establishing this fact, and motivates considering integrands satisfying the following condition:

$$\lambda\Big(\mu^2 + |z|^2\Big)^{\frac{p-2}{2}} |w|^2 \leq \partial_{zz} F(z) w \cdot w \leq \Lambda\Big(1 + |\partial_z F(z)|^2\Big)^{\frac{q-2}{2(q-1)}} |w|^2 \tag{1.4}$$

which we refer to as controlled-duality (p, q)-growth. This condition is a quantitative form of Legendreness for real-valued C^2 integrands. In [182, Chapt. V] a special class of functions – called functions of Legendre type – is highlighted for their useful features in optimisation. In our set-up, these correspond precisely to integrands which are – together with their Fenchel conjugates – real-valued and strictly convex. For such integrands, the lower bound in (1.4) quantifies the strict convexity of $F(\cdot)$, while the upper bound is a quantification of the strict convexity of the Fenchel conjugate $F^*(\cdot)$. Our main result in Chapter 4 is that, in the planar case $n = 2$, if $F(z)$ is a convex polynomial satisfying a quadratic lower bound, then minimisers of (P) are locally real analytic. This will follow – adapting standard techniques – from a $W^{1,q}_{\mathrm{loc}}$-regularity result for minimisers of (P) when integrands satisfy (1.4) with $2 \leq p \leq q < \frac{np}{n-2}$ if $n > 2$, and with $2 \leq p \leq q < \infty$ if $n = 2$ combined with the observation that convex polynomials of degree $d \geq 2$ automatically satisfy (1.4).

A primary motivation for the study of functionals with (p, q)-growth comes from the following consideration in finite elasticity. If $u \colon \Omega \to \mathbb{R}^n$ describes the position of points after applying a given deformation, then $\det(\mathrm{D}u)$ gives the ratio between the volume elements of the reference configuration and the deformed configuration. We expect that the deformation energy blows up as $\det(\mathrm{D}u)$ tends to zero or infinity. Hence a reasonable model elastic energy could be

$$\int_\Omega |\mathrm{D}u|^2 + g(\det(\mathrm{D}u))\,\mathrm{d}x \tag{1.5}$$

where $g(t)$ is a convex function with $g(t) \to \infty$ as $t \to 0$ or $t \to \infty$. Energies of the form (1.5) are known as Neo-Hookean energies. The connection of the Jacobian to nonlinear elasticity has been explored in [15, 57, 84, 120]. Note that this energy will, in general, not satisfy the same growth bounds from above and below, leading naturally – as a first step – to the consideration of (p, q)-growth functionals. Nevertheless, functionals involving

the Jacobian clearly have properties not exhibited by functionals with (p, q)-growth. This inspires us to study the Jacobian and in particular the Jacobian equation, which we define below, more closely. For $n > 1$ let us take a bounded domain $\Omega \subset \mathbb{R}^n$ which is both smooth and uniformly convex. We consider the prescribed Jacobian equation, that is,

$$\begin{cases} \det \mathrm{D}u = f & \text{in } \Omega \\ u = \mathrm{id} & \text{on } \partial\Omega. \end{cases} \tag{JD}$$

which is a first-order fully non-linear underdetermined PDE. Here $f \colon \Omega \to \mathbb{R}$ is integrable and satisfies the compatibility and non-degeneracy conditions

$$\frac{1}{|\Omega|} \int_\Omega f(x) \, \mathrm{d}x = 1 \qquad \text{and} \qquad \inf_\Omega f \geq c > 0. \tag{1.6}$$

The equation (JD) has a rich geometric flavour: Faormally, it can be written in the form $u^*(f \, \mathrm{d}y) = \mathrm{d}x$, where u^* denotes the pull-back of the volume form $f(y) \, \mathrm{d}y$ under u. Hence (JD) can be read as asserting equivalence of volume forms over Ω with the same mass. Moreover, formally, equations of the form (JD) arise as the Euler–Lagrange equations of (P) when $F(x, u, \mathrm{D}u) = F(x, \mathrm{D}u)$ and the dependence on the gradient is only through the Jacobian.

Whenever f is smooth or, more generally, Hölder continuous, it is natural to look for solutions which are at least C^1 and thus (JD) can be understood classically. Under these assumptions, there is a very complete well-posedness theory going back to the seminal works of Moser [168] and Dacorogna–Moser [58], see also the book [56] and the references therein. An alternative approach to this problem was taken in [42], where the well-posedness theory for (JD) is derived from Caffarelli's regularity theory [39, 40] for the Monge–Ampère equation. We are interested in studying Sobolev solutions of (JD) when $f \in L^p(\Omega)$ and we will focus, in particular, on the case where the solutions are in a critical or supercritical Sobolev space. In contrast with the rich classical theory, almost nothing is known in this setting. Indeed, and since the Jacobian is an n-homogeneous polynomial in $\mathrm{D}u$, we have the following natural question:

Question 1.2. Let $1 \leq p < \infty$ and assume f satisfies (1.6).

(i) If $p > 1$, $f \in L^p(\Omega)$, does (JD) have a solution $u \in W^{1,np}(\Omega, \mathbb{R}^n)$? See [208, Question 3].

(ii) If $p = 1$, $f \in L \log L(\Omega)$, does (JD) have a solution $u \in W^{1,n}(\Omega, \mathbb{R}^n)$? See [120, page 206].

In Chapter 5 we prove that the answer to both parts of Question 1.2 is negative and go on to study some further existence and uniqueness properties of solutions to (JD).

1.1 Thesis outline

In the remainder of this chapter, we introduce our notation and recall a number of known results regarding in particular the theory of Besov spaces on Lipschitz domains. Chapter 2 considers the iterative process, and is based on [140]. Chapter 3 deals with functionals with (p, q)-growth, and is based on [139] as well as additional unpublished work. Chapter 4 considers functionals with controlled-duality (p, q)-growth and, in particular, polynomials in the case $n = 2$. It is based on unpublished joint work with C. de Filippis and J. Kristensen and continues joint work started in the DPhil thesis of the first autor [64]. Finally, Chapter 5 considers the Jacobian equation, is based on [107, 106, 108] and is joint work with A. Guerra and S. Lindberg.

2 Notation

In this section we introduce our notation.

2.1 Elementary notation

If $p \in [1, \infty]$ denote by $p' = \frac{p}{p-1}$ its Hölder conjugate. The symbols $a \sim b$ and $a \lesssim b$ mean that there is some constant $C > 0$, depending only on $n, N, p, q, \Omega, \mu, \lambda$ and Λ, and independent of a and b, such that $C^{-1}a \leq b \leq Ca$ and $a \leq Cb$, respectively.

c denotes a constant that may change from line to line. Important dependencies on other parameters, such as, e.g. s, will be denoted as $c(s)$. Distinguished constants will be donated by C_1, c_1, and similar quantities.

We will often write $\mathrm{J}u \equiv \det \mathrm{D}u$.

Ω will always denote an open, bounded domain in $\mathbb{R}^n$. Given a set $\omega \subset \mathbb{R}^n$, $\overline{\omega}$ will denote its closure. We use $\omega \Subset \Omega$ as shorthand for $\overline{\omega} \subset \Omega$. We write $B_r(x)$ for the usual open Euclidean balls in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$ and denote by $\mathbb{S}^{n-1}$ the

unit sphere in $\mathbb{R}^n$. For $r < R$, we write

$$\mathbb{A}(r,R) \equiv \{x \in \mathbb{R}^n : r < |x| < R\}, \qquad \mathbb{A}[r,R] \equiv \{x \in \mathbb{R}^n : r \le |x| \le R\}.$$

Here, $|\cdot|$ denotes the Euclidean norm of a vector in $\mathbb{R}^n$; for a matrix $A \in \mathbb{R}^{n\times n}$, we also write $|A|$ for its Hilbert-Schmidt norm. We denote the operator norm of a matrix by $\|A\|$. If a matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, we denote its smallest eigenvalue by λ_A and its largest by Λ_A. In particular, when $a(x,y,z)$ is a map $a\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$, differentiable in z, we view $\partial_z a(x,y,z)$ both as a matrix and as a linear form and denote $\Lambda_a = \Lambda_A$, $\Lambda_b = \Lambda_B$. We will at times write $A > 0$ to denote that A is positive definite. Id_n denotes the identity matrix in $\mathbb{R}^{n\times n}$. If there is no risk of confusion, we write $\mathrm{Id} = \mathrm{Id}_n$.

We represent by $\mathscr{L}^n$ the n-dimensional Lebesgue measure, by $\mathscr{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure, and we write $\omega_n \equiv \mathscr{L}^n(B_1(0))$. At times we write $|\Omega| = \mathscr{L}^n(\Omega)$.

We recall the definition of Muckenhoupt weights. For a fixed $1 < p < \infty$, we say that a weight $\omega\colon \mathbb{R}^n \to [0,\infty)$ belongs to A_p if ω is locally integrable and there is a constant C such that, for all balls B in $\mathbb{R}^n$, we have

$$[w]_p \coloneqq \fint_B w(x)\,\mathrm{d}x \left(\fint_B w^{-\frac{q}{p}}\,\mathrm{d}x\right)^{\frac{p}{q}} \le C < \infty. \tag{1.7}$$

Here, and in general, $\fint_B f\,\mathrm{d}x = |B|^{-1}\int_B f\,\mathrm{d}x$.

We frequently impose regularity assumptions on $\partial\Omega$ and recall the definitions here. We say $\Omega \subset \mathbb{R}^n$ is a Lipschitz ($C^{1,\alpha}$) domain if Ω is an open subset of $\mathbb{R}^n$ and for every $x \in \partial\Omega$, there exists a neighbourhood V of x in $\mathbb{R}^n$ and orthogonal coordinates $\{y_i\}_{1\le i\le n}$ such that

(i) V is a hypercube in the new coordinates:

$$V = \{(y_1,\dots,y_n\} : -a_i < y_i < a_i, 1 \le i \le n-1\}.$$

(ii) there exists a Lipschitz ($C^{1,\alpha}$) function ϕ defined in

$$V' = \{(y_1,\dots y_{n-1}) : -a_i < y_i < a_i, 1 \le i \le n-1\}$$

and such that

$$|\phi(y')| \leq a_n/2 \text{ for every } y' = (y_1, \dots, y_{n-1}) \in V',$$

$$\Omega \cap V = \{\, y = (y', y_n) \in V : y_n < \phi(y') \,\},$$

$$\partial\Omega \cap V = \{\, y = (y', y_n) \in V : y_n = \phi(y') \,\}.$$

We sometimes assume that a smooth domain Ω is uniformly convex, in the sense that the second fundamental form $-\langle \mathrm{d}\nu(\cdot), \cdot\rangle\nu$ of $\partial\Omega$ is uniformly positive. Here ν denotes the Gauss map and, viewing ν as a vector-valued differential form, $\mathrm{d}\nu$ is its differential.

We denote the cone of height ρ, aperture θ and axis in direction $\boldsymbol{n}$ by $C_\rho(\theta, \boldsymbol{n})$. That is

$$C_\rho(\theta, \boldsymbol{n}) = \{\, h \in \mathbb{R}^n : |h| \leq \rho, h \cdot \boldsymbol{n} \geq |h| \cos(\theta) \,\}. \tag{1.8}$$

Given an open set Ω we denote $\Omega_\lambda \coloneqq \{\, x \in \Omega : d(x, \partial\Omega) > \lambda \,\}$ and $\lambda\Omega \coloneqq \{\, \lambda x : x \in \Omega \,\}$. Here $d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$ denotes the distance of x from the boundary of Ω.

We will often find it useful to write for a function v defined on $\mathbb{R}^n$ and a vector $h \in \mathbb{R}^n$, $v_h(x) \coloneqq v(x + h)$. We further denote $\tau_h v \coloneqq v(x + h) - v(x)$ and $\Delta_h v \coloneqq \tau_h v / h$.

We pick a family $\{\, \phi_\varepsilon \,\}$ of radially symmetric, non-negative mollifiers of unitary mass. We denote convolution with ϕ_ε as

$$u \star \phi_\varepsilon(x) = \int_{\mathbb{R}^n} u(y)\phi_\varepsilon(x - y)\, \mathrm{d}y.$$

2.2 Tools for *p*-growth

Write $V_{\mu,s}(z) \coloneqq (\mu^2 + |z|^2)^{\frac{s-2}{4}} z$ for $\mu > 0$ and $z \in \mathbb{R}^n$. When the value of s is clear from the context we omit it and write $V_\mu(z) \coloneqq V_{\mu,s}(z)$. We recall the useful well-known inequality below:

Lemma 1.3. *For $s > -1, \mu \in [0, 1]$, $z_1, z_2 \in \mathbb{R}^N$, with $\mu + |z_1| + |z_2| > 0$,*

$$\int_0^1 (\mu^2 + |z_1 + \lambda(z_2 - z_1)^2)^{\frac{s}{2}} \lambda \, \mathrm{d}\lambda \sim (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{s}{2}}$$

with the implicit constants only depending on s. Further,

$$|V_{\mu,t}(z_1) - V_{\mu,t}(z_2)| \sim (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{t-2}{2}} |z_1 - z_2|^2.$$

The following lemma is standard, but the author has been unable to find a version in the literature with explicit bounds that tend to 1 as $p \to 2$. Hence, a proof giving the explicit constants shown in the version below can be found in the appendix.

Lemma 1.4. *For $s \geq 0$ and $\mu \geq 0$ we have*

$$\frac{1}{6^s(2s+1)}(\mu^2+|\eta|^2+|\eta-\xi|^2)^s \leq \int_0^1 (\mu^2+|t\xi+(1-t)\eta|^2)^s \,\mathrm{d}t \leq 2^s(\mu^2+|\eta|^2+|\eta-\xi|^2)^s.$$

If $s \in (-1/2, 0]$ then

$$\begin{aligned} 2^s(\mu^2+|\eta|^2+|\eta-\xi|^2)^s \leq& \int_0^1 (\mu^2+|t\xi+(1-t)\eta|^2)^s \,\mathrm{d}t \\ \leq& \frac{1}{4^s(s+1)}(\mu^2+|\eta|^2+|\xi-\eta|^2)^s. \end{aligned}$$

We also recall that the V_μ-functional enjoys a Young-type inequality.

Lemma 1.5 (cf.[3], Lemma 2.3)**.** *Let $x, y \in \mathbb{R}^{N\times n}$, $\mu \geq 0$ and $1 \leq s$. Let $\varepsilon > 0$. Then with*

$$C_\varepsilon \coloneqq \max\left(\frac{1}{4\varepsilon}, \frac{(s-1)^{s-1}}{s^s\varepsilon^{s-1}}\right),$$

it holds that

$$(\mu^2+|x|^2)^{\frac{s-2}{2}}|x||y| \leq \varepsilon|V_\mu(x)|^2 + C_\varepsilon|V_\mu(y)|^2.$$

Further, the following elementary estimate holds:

Lemma 1.6. *For $a, b \in \mathbb{R}^n$ we have*

$$\frac{3-\sqrt{5}}{2}(|a|^2+|b|^2) \leq |a|^2+|a-b|^2 \leq \frac{3+\sqrt{5}}{2}(|a|^2+|b|^2).$$

We conclude this section by noting a well-known iteration lemma, see [96] for a proof.

Lemma 1.7. *Let $h\colon [\rho_0, \rho_1] \to \mathbb{R}$ be a non-negative and bounded function, and let $\theta \in (0,1)$. Take $A, B, \gamma_1, \gamma_2 \geq 0$. Assume that*

$$h(t) \leq \theta h(s) + \frac{A}{(s-t)^{\gamma_1}} + \frac{B}{(s-t)^{\gamma_2}}$$

holds for all $\rho_0 \leq t < s \leq \rho_1$. Then

$$h(\rho_0) \leq c(\theta, \gamma_1, \gamma_2)\left\{\frac{A}{(\rho_1-\rho_0)^{\gamma_1}} + \frac{B}{(\rho_1-\rho_0)^{\gamma_2}}\right\}.$$

2.3 Function spaces

We recall some basic properties of the families of function spaces we use. The exposition follows mainly [185], alternatively the theory can be found in [200].

2.3.1 Global statements

For $0 \leq \alpha \leq 1$ and $k \in \mathbb{N}$, $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ denote the spaces of functions k-times continuously differentiable in Ω and k-times α-Hölder differentiable in Ω, respectively.

For $1 \leq p \leq \infty, k \in \mathbb{N}$, $L^p(\Omega) \coloneqq L^p(\Omega, \mathbb{R}^N)$ and $W^{k,p}(\Omega) \coloneqq W^{k,p}(\Omega, \mathbb{R}^N)$ denote the usual Lebesgue and Sobolev spaces respectively. We will also consider the homogeneous Sobolev spaces

$$\dot{W}^{k,p}(\Omega, \mathbb{R}^N) = \{u \in L^p(\Omega) \colon \}.$$

We write $W_0^{k,p}(\Omega)$ for the closure of $C_0^\infty(\Omega)$-functions with respect to the $W^{k,p}$-norm. For $g \in W^{1,1}(\Omega)$, write $W_g^{k,p}(\Omega) = g + W_0^{k,p}(\Omega)$. We freely identify $W^{k,p}$-functions with their precise representatives.

We use the following interpolation inequality: Given $p_1, p_2 \in [1, \infty]$ and for $0 < \theta < 1$ define p_θ through the identity $\frac{1}{p_\theta} \coloneqq \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then for $u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$ it holds that

$$\|u\|_{L^{p_\theta}(\Omega)} \leq \|u\|^{\theta}_{L^{p_1}(\Omega)} \|u\|^{(1-\theta)}_{L^{p_2}(\Omega)}. \tag{1.9}$$

A further family of spaces we use are the real-variable Hardy spaces. Denote the space of Schwartz functions by $\mathscr{S}(\mathbb{R}^n)$, and the space of tempered distributions by $\mathscr{S}'(\mathbb{R}^n)$. The real-variable Hardy space $\mathscr{H}^p(\mathbb{R}^n)$ is defined by fixing any $\Phi \in \mathscr{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x)\,\mathrm{d}x \neq 0$, denoting $\Phi_t(x) \coloneqq \Phi(x/t)/t^n$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and setting

$$\mathscr{H}^p(\mathbb{R}^n) \equiv \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{\mathscr{H}^p} \equiv \| \sup_{t>0} |f * \Phi_t(\cdot)| \|_{L^p} < \infty \Big\}.$$

We refer the reader to the monograph [197] for the theory of $\mathscr{H}^p(\mathbb{R}^n)$. Here we just note that $\mathscr{H}^p(\mathbb{R}^n) equiv L^p(\mathbb{R}^n)$, with equivalent norms, whenever $p \in (1,\infty)$ and that, moreover, $\mathscr{H}^1(\mathbb{R}^n) \subsetneq \{f \in L^1(\mathbb{R}^n) \colon \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0\}$. Indeed, loosely speaking, elements of $\mathscr{H}^1(\mathbb{R}^n)$ locally have an extra logarithm of integrability by a theorem of Stein [196].

For $s \in [1,\infty)$ we denote the fractional Sobolev space by $W^{s,p}(\Omega)$. We recall, in particular, the following fact from [105]: if $p \in (1,\infty)$, $\alpha > 0$ and Ω is a polytope, then

for $u \in W^{1+\alpha,p}(\Omega)$ the best approximation v to u in the space of continuous piecewise linear functions subordinate to a shape-regular triangulation of Ω of mesh-size h satisfies

$$\|u - v\|_{W^{1,p}(\Omega)} \lesssim h^{\alpha} \|u\|_{W^{1+\alpha,p}(\Omega)}. \tag{1.10}$$

The real interpolation theory we use is standard and so we refer to [21] for it, merely remarking that $[,]_{s,q}$ will denote the real interpolation functor. Let $s \in (0,1)$ and $p, q \in [1,\infty]$. We define

$$B^{s,p}_q(\Omega) \coloneqq B^{s,p}_q(\Omega, \mathbb{R}^N) = [W^{1,p}(\Omega, \mathbb{R}^N), L^p(\Omega, \mathbb{R}^N)]_{s,q}$$

and note that we may identify

$$B^{1+s,p}_q(\Omega) = [W^{2,p}(\Omega), W^{1,p}(\Omega)]_{s,q} = \{\, v \in W^{1,p}(\Omega) : Dv \in B^{s,p}_q(\Omega) \,\}.$$

We define the spaces $B^{s,p}_q(\partial B_r)$ for $r > 0$ similarly. Further, we recall that it holds that $W^{1+s,p}(\Omega) = B^{1+s,p}_p(\Omega)$ and that for $1 \leq q < \infty$, $B^{s,p}_q(\Omega)$ embeds continuously in $B^{s,p}_\infty(\Omega)$. The same properties hold for the spaces $B^{s,p}_q(\partial B_r)$. We will use a characterisation of these spaces in terms of difference quotients. Let D be a set generating $\mathbb{R}^n$, star-shaped with respect to 0. For $s \in (0,1)$, $p \in [1,\infty]$, consider

$$[v]^p_{s,p,\Omega} \coloneqq \sup_{h \in D \setminus \{\,0\,\}} \int_{\Omega_h} \left| \frac{v_h(x) - v(x)}{h} \right|^p \mathrm{d}x.$$

This characterises $B^{s,p}_\infty(\Omega)$ in the sense that

$$v \in B^{s,p}_\infty(\Omega) \Leftrightarrow v \in L^p(\Omega) \text{ and } [v]^p_{s,p,\Omega} < \infty.$$

Moreover, there are positive constants $C_1, C_2 > 0$ depending only on s, p, D, Ω, such that

$$C_1 \|v\|_{B^{s,p}_\infty(\Omega)} \leq \|v\|_{L^p(\Omega)} + [v]_{s,p,\Omega} \leq C_2 \|v\|_{B^{s,p}_\infty(\Omega)}. \tag{1.11}$$

If $\Omega = B_r(x_0)$, then C_1, C_2 are unchanged by replacing D with QD, where Q is an orthonormal matrix. In particular, when $D = C_\rho(\theta, \boldsymbol{n})$ is a cone, they are independent of the choice of axis direction $\boldsymbol{n}$.

We warn the reader that no characterisation in terms of difference quotients is available for $B^{s,p}_q(\partial B_r)$.

Let $D \coloneqq C_\rho(\theta, \boldsymbol{n})$ and $D' \coloneqq C_{t\rho}(\theta, \boldsymbol{n}))$ with $t \in (0,1]$ and $\rho_0 \leq 1$. Denote the corresponding difference-quotient quantities by $[\cdot]_{s,p,\Omega}$ and $[\cdot]'_{s,p,\Omega}$, respectively. It is straightforward to check that for $v \in B^{s,p}_\infty(\Omega)$,

$$t^s(\|v\|_{L^p(\Omega)} + [v]_{s,p,\Omega}) \lesssim \|v\|'_{L^p(\Omega)} + [v]'_{s,p,\Omega} \leq \|v\|_{L^p(\Omega)} + [v]_{s,p,\Omega} \tag{1.12}$$

where the implicit constant only depends on p.

Further, recall that $B^{s,p}_\infty(\Omega)$ may be localised: If $\{ B(x_i, r_i) \}_{i \leq M}$ is a finite collection of balls covering Ω, then $v \in B^{s,p}_\infty(\Omega)$ if and only if $v_{|\Omega \cap 2B(x_i,r_i)} \in B^{s,p}_\infty(\Omega \cap 2B(x_i, r_i))$ for $i = 1, ..., M$. Moreover, with $r \coloneqq \min(r_i)$ we have

$$\min(r, \rho_0)^s [v]^p_{s,p,\Omega} \lesssim \sum_{i=1}^M [v]_{s,p,\Omega \cap 2B(x_i,r_i)} \leq M [v]^p_{s,p,\Omega}. \tag{1.13}$$

In fact, it is immediate that

$$[v]_{s,p,\Omega \cap 2B(x_i,r_i)} \leq [v]_{s,p,\Omega}$$

from which the upper bound follows immediately. For the lower bound note that if $h < \min(r, \rho_0)$,

$$\Omega_h \subset \cup \left(\Omega \cap 2B(x_i, r_i)\right)_h .$$

Using (1.12) we obtain the lower bound.

We recall a well-known embedding theorem, see e.g. [202]:

Theorem 1. *Let $0 < s \leq 1$ and $p, p_1 \in [1, \infty]$. Assume that $s - \frac{n}{p} = -\frac{n}{p_1}$, and suppose that $v \in B^{s,p}_\infty(\Omega)$. Then for any $\varepsilon \in (0, 1 - p_1]$,*

$$\|v\|_{L^{p_1 - \varepsilon}(\Omega)} \lesssim \|v\|_{B^{s,p}_\infty(\Omega)}.$$

If $v \in B^{s,p}_\infty(\partial B_r)$ for some $r > 0$, $\varepsilon \in (0, 1 - p_1)$ and $s - \frac{n-1}{p} = -\frac{n-1}{p_1}$, we have that

$$\|v\|_{L^{p_1 - \varepsilon}(\partial B_r)} \lesssim \|v\|_{B^{s,p}_\infty(\partial B_r)}.$$

We also require a Fubini-type theorem on spheres, that may be of independent interest. The statement is probably known to the expert, but we have been unable to find a reference in the literature. In the context of fractional Sobolev spaces such a result has been obtained in [102].

Lemma 1.8. *Let $\sigma, \rho \geq 0$ and write $B_\sigma \coloneqq B_\sigma(0)$, $B_\rho \coloneqq B_\rho(0)$. Let $s \in (0,1)$ and $1 \leq \tau \leq q \leq p$. If $v \in B^{s,p}_q(B_\sigma \setminus B_\rho)$ then*

$$\int_\sigma^\rho \|v\|^\tau_{B^{s,p}_q(\partial B_r)} \,\mathrm{d}r \lesssim \|v\|^\tau_{B^{s,p}_q(B_\sigma \setminus B_\rho)}.$$

Here the implicit constant only depends on s, τ, q, p, σ and ρ.

Proof. Let $v \in B^{s,p}_q(B_\sigma \setminus B_\rho)$. Since $B^{s,p}_q(\partial B_r) = [L^p(\partial B_r), W^{1,p}(\partial B_r)]_{s,q}$,

$$\|v\|_{B^{s,p}_q(\partial B_r)} = \left(\int_0^\infty \left(t^{-s} K(v,t,r) \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}},$$

where

$$K(v,t,r) \coloneqq \inf \left\{ \|v_1\|_{L^p(\partial B_r)} + t\|v_2\|_{W^{1,p}(\partial B_r)} \colon v = v_1 + v_2 \in X \right\},$$

with $X = L^p(\partial B_r) \times \in W^{1,p}(\partial B_r)$. Using Jensen's inequality and Fubini's theorem, we find

$$\begin{aligned}
\int_\rho^\sigma \|v\|^\tau_{B^{s,p}_q(\partial B_r)} \,\mathrm{d}r &= \int_\rho^\sigma \left(\int_0^\infty (t^{-s} K(v,t,r))^q \,\mathrm{d}r \frac{\mathrm{d}t}{t} \right)^{\frac{\tau}{q}} \mathrm{d}r \\
&\lesssim \left(\int_0^\infty \int_\rho^\sigma (t^{-s} K(v,t,r))^q \,\mathrm{d}r \frac{\mathrm{d}t}{t} \right)^{\frac{\tau}{q}} \\
&\lesssim \left(\int_0^\infty t^{-sq} \int_\rho^\sigma K'(v,t,r)^q \,\mathrm{d}r \frac{\mathrm{d}t}{t} \right)^{\frac{\tau}{q}} = II,
\end{aligned}$$

where

$$\begin{aligned}
&K'(v,t,r) \coloneqq \inf \left\{ \|v_1\|_{L^p(\partial B_r)} + t\|v_2\|_{W^{1,p}(\partial B_r)} \colon v = v_1 + v_2 \text{ in } B_\sigma \setminus B_\rho, (v_1, v_2) \in Y \right\} \\
&\quad \text{with } Y \coloneqq L^p(B_\sigma \setminus B_\rho) \times W^{1,p}(B_\sigma \setminus B_\rho).
\end{aligned}$$

Note that, as $q \leq p$,

$$\begin{aligned}
\int_\rho^\sigma (\|v_1\|_{L^p(\partial B_r)} + t\|v_2\|_{W^{1,p}(\partial B_r)})^q \,\mathrm{d}r &\lesssim \int_\rho^\sigma \|v_1\|^q_{L^p(\partial B_r)} + t^q \|v_2\|^q_{W^{1,p}(\partial B_r)} \\
&\lesssim \|v_1\|^q_{L^p(B_\sigma \setminus B_r)} + t^q \|v_2\|^q_{W^{1,p}(B_\sigma \setminus B_\rho)} \\
&\lesssim \left(\|v_1\|_{L^p(B_\sigma \setminus B_r)} + t\|v_2\|_{W^{1,p}(B_\sigma \setminus B_\rho)} \right)^q.
\end{aligned}$$

Thus, using Fatou's lemma,

$$II \lesssim \left(\int_0^\infty t^{-sq} \inf \left(\|v_1\|_{L^p(B_\sigma \setminus B_r)} + t\|v_2\|_{W^{1,p}(B_\sigma \setminus B_\rho)} \right)^q \,\mathrm{d}r \frac{\mathrm{d}t}{t} \right)^{\frac{\tau}{q}} = \|v\|^\tau_{B^{s,p}_q(B_\sigma \setminus B_\rho)},$$

where the infinimum is taken over

$$\{(v_1, v_2)\colon v = v_1 + v_2 \text{ in } B_\sigma \setminus B_\rho, v_1 \in L^p(B_\sigma \setminus B_\rho), v_2 \in W^{1,p}(B_\sigma \setminus B_\rho)\}.$$

□

We next recall some basic properties of mollifications. Proofs can be found in [80].

Lemma 1.9. *Let $1 < p < \infty$. Let Ω be an open domain in $\mathbb{R}^n$ and $\omega \Subset \Omega$. Suppose $u \in W^{1,p}(\Omega)$ and consider $u_\varepsilon = u \star \phi_\varepsilon$ for $\varepsilon < d(\omega, \partial\Omega)$. Then*

(i) $u_\varepsilon \in C^\infty(\omega)$

(ii) $\|u_\varepsilon\|_{W^{1,p}(\omega)} \leq \|u\|_{W^{1,p}(\Omega)}$

(iii) *For $p \leq q \leq \frac{np}{n-p}$, $\|u_\varepsilon - u\|_{L^q(\omega)} \lesssim \varepsilon^\Theta \|u\|_{W^{1,p}(\Omega)}$ where $\Theta := 1 + n(1/q - 1/p)$ if $p < n$, $\Theta = n/q$ if $p \geq n$.*

(iv) $\|Du_\varepsilon\|_{L^\infty(\omega)} \lesssim \varepsilon^{-\frac{n}{p}} \|Du\|_{W^{1,p}(\Omega)}$.

We also recall the trace theorem in the following form, see e.g. [81].

Lemma 1.10. *Let Ω be a Lipschitz domain. Let $1 < p < \infty$. Then there is a bounded linear operator $\operatorname{Tr}\colon W^{1+\frac{1}{p},p}(\Omega) \to W^{1,p}(\partial\Omega)$. In fact, given $u \in W^{1+\frac{1}{p},p}(\Omega)$, it is possible to define $\operatorname{Tr}(u)$ to be the values of the precise representative of u on $\partial\Omega$.*

Finally we recall the following well-known result (see [81] for the ingredients of the proof) which will justify extending u by extensions of g.

Lemma 1.11. *Let $p \in [1, \infty]$. Let $V \ni \Omega$ be an open, bounded set. Suppose $u \in W^{1,p}(\Omega)$ and $v \in (u + W_0^{1,p}(\Omega)) \cap W^{1,p}(V)$. Then the map*

$$w = \begin{cases} u \text{ in } \Omega, \\ v \text{ in } V \setminus \Omega, \end{cases}$$

belongs to $W^{1,p}(V)$.

2.3.2 Local statements

We require some localised versions of results mentioned in the previous section which we collect here. We denote the local Lebesgue, Sobolev and Besov-spaces by $L^p_{\mathrm{loc}}(\Omega)$,

$W^{s,p}_{\mathrm{loc}}(\Omega)$ and $B^{s,p}_{q,\mathrm{loc}}(\Omega)$, respectively. In all cases we say $u \in X_{\mathrm{loc}}(\Omega)$ if $u \in X(\omega)$ for any $\omega \Subset \Omega$.

An important fact about translation operators is their continuity in Lebesgue spaces.

Lemma 1.12. *Let $\tilde{\Omega} \Subset \Omega$ be any open set, $h \in \mathbb{R}^n$ satisfy $|h| \in \left(0, \frac{1}{2}\operatorname{dist}(\tilde{\Omega}, \partial\Omega)\right)$ and $w \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^k)$ with $p \in [1, \infty)$ and $k \in \mathbb{N}$. Then*

$$\|\tau_h w - w\|_{L^p(\tilde{\Omega})} \to 0$$

as $|h| \to 0$. Furthermore, whenever $\Omega' \Subset \tilde{\Omega}$ is another open subset and we have that $|h| \in \left(0, \frac{1}{2}\operatorname{dist}(\Omega', \partial\tilde{\Omega})\right)$, there exists an absolute constant $c > 0$ such that

$$\int_{\Omega'} \left[|w(x)|^2 + |w(x+h)|^2\right]^{\frac{p}{2}} \, \mathrm{d}x \leq c \int_{\tilde{\Omega}} |w|^p \, \mathrm{d}x.$$

We recall the following analogue of (1.11) in the case of Sobolev spaces.

Lemma 1.13. *Let $w \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^k)$, $k \geq 1$. There holds that:*

- *If $p > 1$ and $\Omega' \Subset \Omega$ is any open set so that*

$$\sup_{|h|>0} |h|^{-p} \int_{\Omega'} |\tau_h w|^p \, \mathrm{d}x < \infty,$$

  then $Dw \in L^p(\Omega', \mathbb{R}^{k\times n})$ and $\tau_h w \to Dw$ in $L^p(\Omega')$.

- *If, in addition, $w \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^k)$, $\Omega'' \Subset \Omega$ is another open subset and we have $|h| < \frac{1}{2}\operatorname{dist}(\Omega'', \partial\Omega')$, then*

$$\int_{\Omega''} |\tau_h w|^p \, \mathrm{d}x \leq c(n,s)|h|^p \int_{\Omega'} |Dw|^p \, \mathrm{d}x.$$

We require the following local, quantified embedding theorem of fractional Sobolev spaces, see for instance [9] for a proof.

Lemma 1.14. *Let $B_r \subset \mathbb{R}^n$ be a ball with $r \leq 1$, $w \in L^p(B_r, \mathbb{R}^k)$, $p > 1$, and assume that, for $\alpha \in (0, 1]$, $S \geq 1$ and concentric balls $B_\rho \Subset B_r$, there holds*

$$\|\tau_h w\|_{L^p(B_\rho, \mathbb{R}^k)} \leq S|h|^\alpha \quad \text{for every } h \in \mathbb{R}^n \text{ with } 0 < |h| \leq \tfrac{r-\rho}{K}, \text{ where } K \geq 1\,.$$

Then $w \in W^{\beta,p}(B_\rho, \mathbb{R}^k)$ whenever $\beta \in (0, \alpha)$, and

$$\|f\|_{W^{\beta,p}(B_\rho, \mathbb{R}^k)} \leq \frac{c}{(\alpha-\beta)^{1/p}} \left(\frac{r-\rho}{K}\right)^{\alpha-\beta} S + c\left(\frac{K}{r-\rho}\right)^{n/p+\beta} \|f\|_{L^p(B_r, \mathbb{R}^k)}\,,$$

where $c \equiv c(n,p)$.

We also require a localised version of the Sobolev-embedding in the scale of fractional Sobolev spaces. The result is well-known to experts but we have chosen to give a short proof for completeness.

Lemma 1.15. *Let $B_\rho \Subset B_r \Subset \mathbb{R}^n$ be concentric balls with $r \leq 1$, $1 < p < q < \infty$ and $f \in W^{1,p}(B_r, \mathbb{R}^N)$ with $N \geq 1$. Set $\alpha := 1 - n\left(\frac{1}{p} - \frac{1}{q}\right)$. Then there exists a positive constant $c \equiv c(n, N, p, q)$ such that*

$$\|f\|_{B^{\alpha,q}_\infty(B_\rho)} \leq c\left[\|Df\|_{L^p(B_r)} + \frac{1}{r-\rho}\|f\|_{L^p(B_r)}\right]. \tag{1.14}$$

Proof. Let $\eta \in C^1_c(B_r)$ be a cut-off function satisfying

$$\mathbb{1}_{B_{\frac{\rho+r}{2}}} \leq \eta \leq \mathbb{1}_{B_{\frac{3(\rho+r)}{4}}} \qquad \text{and} \qquad |D\eta| \leq \frac{4}{r-\rho}. \tag{1.15}$$

Consider f as in the statement. Note that then $\tilde{f} := \eta f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$. By the global version of Theorem 1, see [200, Theorem 6.2.4],

$$\|\tilde{f}\|_{B^{\alpha,q}_\infty(\mathbb{R}^n)} \lesssim \|\tilde{f}\|_{W^{1,p}(\mathbb{R}^n)}.$$

By direct computation we find that

$$\|\tilde{f}\|_{W^{1,p}(\mathbb{R}^n)} \leq \|\mathrm{D}f\|_{L^p(B_r)} + \frac{8}{r-\rho}\|f\|_{L^p(B_r)}. \tag{1.16}$$

Recalling that $\|f\|_{B^{\alpha,q}_\infty(B_\rho)} = \inf \|g\|_{B^{\alpha,q}_\infty(\mathbb{R}^n)}$, where the infimum is taken over all $g \in B^{\alpha,q}_\infty(\mathbb{R}^n)$ that extend f (see [201, Chapter 3]), this concludes the proof.

□

2.4 Convexity and growth bounds

In this subsection we recall and adapt some known bounds and facts for convex functions. The main idea is that growth bounds for integrands F imply dual growth bounds for the Fenchel conjugate F^*. We will exploit these ideas to show that integrands that satisfy controlled-duality (p, q)-growth conditions satisfy an improved monotonicity estimate involving the strain $\partial_z F$. This improved monotonicity is the crucial ingredient to improved local $W^{1,q}$-regularity results for integrands satisfying controlled-duality (p, q)-growth as opposed to natural or controlled (p, q)-growth.

We begin by recalling the definition of the Fenchel conjugate of $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$. We denote

$$F^*(\xi) \coloneqq \sup_{z\in\mathbb{R}^{N\times n}} \Big(z\cdot\xi - F(z) \Big), \quad \xi \in \mathbb{R}^{N\times n}. \tag{1.17}$$

Recall that the Fenchel conjugate F^* is real-valued precisely when F is super-linear at infinity, that is when $F(z)/|z| \to \infty$ as $|z| \to \infty$. In fact, we can say the following: when F is convex, F is real-valued and super-linear precisely when F^* is. To see this recall that the convex and lower semicontinuous envelope of F coincides with the double Fenchel conjugate $F^{**} = (F^*)^*$. For later reference, we also recall the Fenchel-Young inequality stating that

$$F(z) + F^*(\xi) \geq z\cdot\xi$$

holds for all z, $\xi \in \mathbb{R}^{N\times n}$. It applies to any proper integrand F as a direct consequence of the definition of Fenchel conjugation. The equality case holds particular interest to us and occurs when $\xi \in \partial F(z)$, the subdifferential of $F(\cdot)$ at z. In particular, when F is C^1 – as in all instances considered in this thesis – the equality holds if and only if F is convex at z and $\xi = \partial_z F(z)$.

We next present our first result, which is well-known, showing how natural (p,q)-growth conditions for F are transformed to natural (q',p')-growth conditions for F^* under Fenchel conjugation.

Lemma 1.16. *Let $1 < p \leq q < \infty$. Assume $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is an integrand satisfying for constants $A_i > 0$, $i = 1,\dots,4$, the (p,q)-growth condition*

$$A_1|z|^p - A_2 \leq F(z) \leq A_3|z|^q + A_4 \tag{1.18}$$

for all $z \in \mathbb{R}^{N\times n}$. Then

$$B_3|\xi|^{q'} - B_4 \leq F^*(\xi) \leq B_1|\xi|^{p'} + B_2 \tag{1.19}$$

for all $\xi \in \mathbb{R}^{N\times n}$, where $B_2 \coloneqq A_2$, $B_4 \coloneqq A_4$,

$$B_1 \coloneqq \Big(p^{-\frac{1}{p-1}} - p^{-p'}\Big) A_1^{-\frac{1}{p-1}} \text{ and } B_3 \coloneqq \Big(q^{-\frac{1}{q-1}} - q^{-q'}\Big) A_3^{-\frac{1}{q-1}}.$$

The proof is a straightforward calculation using the definitions which can be found in the appendix.

We next explore the relations between smoothness and strict convexity of integrands and their Fenchel conjugates. Recall that F is strictly convex when the inequality $F(tz + (1-t)w) < tF(z) + (1-t)F(w)$ holds for $t \in (0,1)$ and $z \neq w$. Note, in particular, that the lower bound in all the growth conditions we consider in this thesis can be interpreted as a quantified version of strict convexity.

Lemma 1.17. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{\infty\}$ be a convex and proper integrand. Then F is real-valued, super-linear and strictly convex if and only if its Fenchel conjugate F^* is real-valued, super-linear and C^1.*

In particular, we emphasize that the integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is super-linear, strictly convex and C^1 precisely when its Fenchel conjugate $F^*\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is so. In this case, additionally $\partial_z F(\cdot)$ and $\partial_\xi F^*(\cdot)$ are homeomorphisms of $\mathbb{R}^{N\times n}$, and $\partial_\xi F^*(\cdot) = \partial_z F^{-1}(\cdot)$. An implication of this fact we wish to highlight is that when F is real-valued, super-linear, strictly convex and C^1, then so is F^*. Furthermore, the following identities hold:

$$\partial_\xi F^*\Big(\partial_z F(z)\Big) = z \ \text{ and } \ \partial_z F\Big(\partial_\xi F^*(\xi)\Big) = \xi \ \text{ for all } \ z,\, \xi \in \mathbb{R}^{N\times n}. \tag{1.20}$$

The following well-known result is a consequence of these relations and is fundamental in obtaining the improved monotonicity formula of this section.

Lemma 1.18. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be super-linear, strictly convex and $\mathrm{C}^2(B_r(z_0)$ for some $z_0 \in \mathbb{R}^{N\times n}$ and $r > 0$. If*

$$\det F''(z) \neq 0$$

for all $z \in B_r(z_0)$, then the Fenchel conjugate $F^*\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is super-linear, strictly convex and C^1. In addition, F^* is C^2 on $\partial_z F\Big(B_r(z_0)\Big)$, with

$$\partial_{\xi\xi} F^*\big(\partial_z F(z)\big) = \partial_{zz} F(z)^{-1} \tag{1.21}$$

for $z \in B_r(z_0)$.

Lemma 1.18 follows from (1.20) and the inverse function theorem. Details can be found in [118, Corollary 4.2.10].

Our next task is to establish an analogue of Lemma 1.16 in the case of controlled-duality (p,q)-growth conditions.

Corollary 1.1. *Let $\leq p \leq q < \infty$, Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 integrand satisfying for constants λ, $\Lambda > 0$ and $\mu \in [0,1]$ the controlled-duality growth condition (4.1). Then $F^*\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^2, and*

$$\frac{1}{\Lambda}\Big(1+|\partial_z F(z_0)|^2\Big)^{q'-2}|\xi|^2 \leq \partial_{\xi\xi}F^*(\partial_z F(z_0))\xi\cdot\xi \leq \frac{1}{\lambda}\Big(\mu^2+|z_0|^2\Big)^{\frac{2-p}{2}}|\xi|^2 \tag{1.22}$$

for all z_0, $\xi \in \mathbb{R}^{N\times n}$.

Proof. We focus on the case $\mu > 0$. The case $\mu = 0$ is established by the same argument. Since the changes are mostly notational, we omit them here.

Due to the lower bound in (1.22), F is strictly convex and super-linear, and since $\mu > 0$ the Hessian matrix $\partial_{zz}F(z_0)$ is invertible for all $z_0 \in \mathbb{R}^{N\times n}$. Lemma 1.18 shows that F^* is C^2 and that $\partial_{\xi\xi}F^*(\partial_z F(z_0)) = \partial_{zz}F(z_0)^{-1}$ for all $z_0 \in \mathbb{R}^{N\times n}$. In order to deduce the desired estimate, note that (4.1) implies that all eigenvalues of $\partial_{zz}F(z_0)$ lie in the interval

$$\left[\lambda\Big(\mu^2+|z_0|^2\Big)^{\frac{p-2}{2}}, \Lambda\Big(1+|\partial_z F(z_0)|^2\Big)^{\frac{q-2}{2(q-1)}}\right].$$

In particular, since the eigenvalues of $\partial_{\xi\xi}F^*(\partial_z F(z_0)) = \partial_{zz}F(z_0)^{-1}$ are their reciprocals, they all lie in the interval

$$\left[\frac{1}{\Lambda}\Big(1+|\partial_z F(z_0)|^2\Big)^{\frac{2-q}{2(q-1)}}, \frac{1}{\lambda}\Big(\mu^2+|z_0|^2\Big)^{\frac{2-p}{2}}\right].$$

Since $\frac{2-q}{q-1} = q'-2$, this concludes the proof. □

We are now in a position to prove the desired monotonicity property. While this type of result probably is not surprising to the experts, we are not aware of any record of it in the literature. It is instrumental for the approach of Chapter 4, and we remark that it allows us to use the power of convex duality without having to go through the dual variational problem [211].

Corollary 1.2. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 integrand that satisfies (4.1). Then for some constant $c > 0$ the monotonicity inequality*

$$\Big\langle \partial_z F(z) - \partial_z F(w), z-w\Big\rangle \geq c\Big|V_{\mu,p}(z) - V_{\mu,p}(w)\Big|^2 + c\Big|V_{1,q'}(\partial_z F(z)) - V_{1,q'}(\partial_z F(w))\Big|^2 \tag{1.23}$$

holds for all z, $w \in \mathbb{R}^{N\times n}$.

Proof. When $\mu \in (0,1]$ the monotonicity inequality follows from Corollary 1.1, Lemma 1.3 and Lemma 1.4. In order to obtain the case $\mu = 0$ we use Corollary 1.1. Together with Lemma 1.3 and Lemma 1.4 we find that (1.23) holds for all z, $w \in \mathbb{R}^{N\times n}$ with the property that 0 does not belong to the closed segment $[z,w]$ with endpoints z, w. The constant c only depends on p, q, λ, Λ, N and n and not on z, w. Finally note that the expressions on the two sides of (1.23) are continuous in (z,w), and so we obtain the general case by approximation. □

2. Regularity and numerical approximations via an interative process

In this chapter, we focus on equations of the form (1.1) with p-growth. To be precise, we make the following assumptions on $a \equiv a(x,z)\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$:

(A1) $a(x,z)$ is measurable in Ω with respect to x and continuously differentiable with respect to z almost everywhere in $\mathbb{R}^{N\times n}$.

(A2) There are $\lambda_a, \Lambda_a > 0$, $\mu \geq 0$ such that for all $z, \xi \in \mathbb{R}^{N\times n}$,

$$\lambda_a(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\xi|^2 \leq \partial_z a(x,z)\xi \cdot \xi \leq \Lambda_a(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\xi|^2.$$

(A3) For all $x \in \Omega$, $z \in \mathbb{R}^{N\times n}$, $|a(x,z)| \lesssim 1 + |z|^{p-1}$.

For simplicity of presentation we make the further assumption that

(A4) $\partial_z a(x,z)$ is symmetric.

Remark 2.1. We have chosen to focus on fields of the form $a(x,z)$, however with minor changes to (A1) – (A3), all the theory we develop applies equally to fields of the form $a \equiv a(x,y,z)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$.

(A4) is only used to prove Lemma 2.4, which can be modified to apply to the case when $\partial_z a(x,z)$ is non-symmetric, see Remark 2.5.

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $f \in L^{p'}(\Omega, \mathbb{R}^{N\times n})$, and $g \in W^{1,p}(\Omega, \mathbb{R}^N)$. We seek to find $u \in W^{1,p}_g(\Omega, \mathbb{R}^N)$ such that

$$\operatorname{div} a(x, \mathrm{D}u) = \operatorname{div} f \text{ in } \Omega, \tag{2.1}$$

where $a(x,z)$ satisfies (A1) – (A3) and (A4), and (2.1) is to be understood in the sense of distributions.

The observation of [141] is that if $a(x,z)$ is a matrix-valued elliptic structure field satisfying controlled quadratic growth, that is (A1) – (A3) with $p=2$ (c.f. (1.3)), then considering – for sufficiently small $\gamma>0$ – the scheme

$$\begin{cases} -\Delta u_{n+1} = -\Delta u_n - \gamma \operatorname{div} a(x, \mathrm{D}u_n) + \gamma \operatorname{div} f & \text{in } \Omega, \\ u_{n+1} = g & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

with any choice of $u_0 \in W^{1,2}_g(\Omega)$, gives a sequence $\{u_n\}$ that converges in $W^{1,2}(\Omega)$ to a solution of (2.1). Our first aim is to extend this result to structure fields satisfying the p-growth assumptions stated above. Given a suitable reference field $b(x,z)$, the precise meaning of which we will specify at a later point, satisfying (A1) – (A3) and (A4), we consider the process

$$\operatorname{div} b(x, \mathrm{D}u_{n+1}) = \operatorname{div} b(x, \mathrm{D}u_n) - \gamma \operatorname{div} a(x, \mathrm{D}u_n) + \gamma \operatorname{div} f. \tag{2.3}$$

We will find that we need to restrict our attention to fields $a(x,z)$ that are sufficiently close to $b(x,z)$, in the sense that

$$K \coloneqq \frac{\Lambda_{B^{-1}A} - \lambda_{B^{-1}A}}{\lambda_{B^{-1}A} + \Lambda_{B^{-1}A}} \tag{2.4}$$

is sufficiently small. Here we assume that $A \coloneqq \partial_z a(x,z)$ and $B \coloneqq \partial_z b(x,z)$ are symmetric. $\lambda_{B^{-1}A}, \Lambda_{B^{-1}A}$ are the smallest and largest eigenvalue of $B^{-1}A$ respectively.

In fact, we cannot quite use (2.4), but rather need to be more precise in the following sense: Suppose $a(x,z)$ and $b(x,z)$ satisfy (A1) and (A2), and let $A \coloneqq \partial_z a(x,z), B \coloneqq \partial_z b(x,z)$. Note that then $\lambda_{B^{-1}A}, \Lambda_{B^{-1}A}$ have lower and upper bounds, respectively, that are independent of x,z. Denote these by $\lambda_{a,b}, \Lambda_{a,b}$, and set

$$K_{a,b} \coloneqq \frac{\Lambda_{a,b} - \lambda_{a,b}}{\lambda_{a,b} + \Lambda_{a,b}}. \tag{2.5}$$

The iterative scheme (2.3) naturally lends itself as a numerical scheme to solve (2.1) using finite element approximations. In order to avoid additional technical difficulties related to numerical approximations, in our numerical considerations we take $g=0$ and suppose Ω is a polytope. Consider a family of shape-regular triangulations $\{T_h\}_{h\in(0,1]}$ of Ω of mesh-size h. Denote by X_h the space of continuous piecewise linear (subordinate

to T_h) functions. Take $u_0 \in X_h$. Define $u^h_{n+1} \in X_h$ inductively as a solution of the problem:

$$\begin{cases} \displaystyle\int_\Omega b(x, \mathrm{D}u^h_{n+1}) \cdot \mathrm{D}\phi \,\mathrm{d}x = \int_\Omega (b(x, \mathrm{D}u^h_n) - \gamma a(x, \mathrm{D}u^h_n) + \gamma f) \cdot \mathrm{D}\phi \,\mathrm{d}x & \text{for all } \phi \in X_h, \\ u^h_{n+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

We prove in Section 3 that, under the assumptions we require to ensure convergence of the iterative process, this scheme converges in $W^{1,p}(\Omega)$ to a solution of the problem

$$\begin{cases} \displaystyle\int_\Omega a(x, \mathrm{D}u_h) \cdot \mathrm{D}\phi = \int_\Omega f \cdot \mathrm{D}\phi \,\mathrm{d}x & \text{for all } \phi \in X_h, \\ u_h = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, under a reasonable additional regularity assumption on solutions of (2.1) posed for $b(x, z)$, $u_h \to u$ in $W^{1,p}(\Omega)$ as $h \to 0$ where $u \in W^{1,p}_0(\Omega)$ is a solution of (2.1). We then present a number of numerical experiments confirming these results.

If solutions to (2.1) for a given structure field $b(x, z)$ have good regularity properties, one can hope that, if this $b(x, z)$ is used in (2.3), then these regularity properties transfer to the solution of (2.1) posed for $a(x, z)$. In this direction, [141] shows the convergence of the scheme (if $p = 2$ and with $b(x, z) = z$) in appropriate Morrey-spaces, obtaining $C^{0,\alpha}$- and $C^{1,\alpha}$-regularity of solutions to (2.1) under (sharp in the case of $C^{0,\alpha}$) smallness assumptions on the ellipticity ratio Λ_a/λ_a. An alternative proof was suggested in [146]. This motivates us to explore applications of (2.3) to the regularity theory of (2.1) in Section 4, where we also comment more closely on the available regularity theory for (2.1).

1 A basic linear algebra result

A crucial tool in our analysis of the iterative process will be the following linear algebra result from [141]: Let $A \in \mathbb{R}^{n\times n}$. Write $A \coloneqq A^+ + A^-$ where A^+, A^- are the symmetric and skew-symmetric part of A, respectively. In a slight abuse of notation, we denote the smallest and largest eigenvalue of A by λ_A and Λ_A, respectively. Set $C \coloneqq A^+A^- - A^-A^+ - (A^-)^2$, and denote by σ the largest eigenvalue of C. Introduce K^*_γ by setting

$$K^*_\gamma \coloneqq \min_{\gamma>0} \|I - \gamma A\|. \tag{2.7}$$

Lemma 2.2 (c.f. Lemma 1.1.2 in [141]). *Define K_γ and γ as*

$$\begin{cases} K_\gamma^2 := \dfrac{\sigma}{\sigma + \lambda_A^2}, \qquad \gamma := \dfrac{\lambda_A}{\sigma + \lambda_A^2} & \text{if } \sigma \geq \dfrac{\lambda_A(\Lambda_A - \lambda_A)}{2} \\ K_\gamma^2 := \dfrac{(\Lambda_A - \lambda_A)^2 + 4\sigma}{(\Lambda_A + \lambda_A)^2}, \quad \gamma := \dfrac{2}{\Lambda_A + \lambda_A} & \text{if } \sigma \leq \dfrac{\lambda_A(\Lambda_A - \lambda_A)}{2}. \end{cases}$$

Then $K_\gamma^* \leq K_\gamma$, where K_γ^* is defined as in (2.7).

Proof. For the readers convenience, we provide a sketch of the proof. Let $\{\lambda_i\}$ denote the eigenvalues of A. Expanding the square, it is easy to check that

$$\begin{aligned} \|I - \gamma A\|^2 \leq \|I - \gamma A^+\|^2 + \gamma^2\sigma &\leq \sup_i (1 - \gamma\lambda_i)^2 + \gamma^2\sigma \\ &= \max\left((1 - \gamma\lambda_A)^2, (1 - \gamma\Lambda_A)^2\right) + \gamma^2\sigma = K^2. \end{aligned}$$

By direct calculation, one verifies that

$$K_\gamma^2 = \begin{cases} (1 - \gamma\lambda_A)^2 + \gamma^2\sigma & \text{if } 0 \leq \gamma \leq \frac{2}{\lambda + \Lambda}, \\ (1 - \gamma\Lambda_A)^2 + \gamma^2\sigma & \text{if } \gamma \geq \frac{2}{\lambda + \Lambda}. \end{cases}$$

Analysing cases the result follows. □

Remark 2.3. The proof shows that the inequality $\|I - \gamma A\| \leq K_\gamma$ still holds when λ_A, Λ_A and σ are replaced by any upper bound for the smallest and largest eigenvalue of A, and the largest eigenvalue of C respectively in the definition of K_γ and γ.

We require the following version of the above lemma.

Lemma 2.4. *Suppose $A, B \in \mathbb{R}^{n\times n}$ are positive definite and symmetric. Then with the choice $\gamma := \frac{2}{\Lambda_{B^{-1}A} + \lambda_{B^{-1}A}}$ and assuming that*

$$K := \frac{\Lambda_{B^{-1}A} - \lambda_{B^{-1}A}}{\Lambda_{B^{-1}A} + \lambda_{B^{-1}A}} < 1$$

the estimate

$$\|B - \gamma A\| \leq K\|B\|$$

holds.

Proof. By Lemma 2.2 there holds

$$\|B - \gamma A\| \leq \|B\|\|I - \gamma B^{-1}A\| \leq \|B\|\frac{\Lambda_{B^{-1}A} - \lambda_{B^{-1}A}}{\Lambda_{B^{-1}A} + \lambda_{B^{-1}A}} = \|B\|K$$

□

Remark 2.5. Suppose $a(x,z)$ and $b(x,z)$ are elliptic structure fields satisfying (A1) $-$ (A3) and (A4). Set $A(x,z) = \partial_z a(x,z)$ and $B(x,z) = \partial_z b(x,z)$. By Remark 2.3 we can use Lemma 2.4 to ensure the existence of a $\gamma > 0$ so that $\|B(x,z) - \gamma A(x,z)\| \leq K_{a,b}\|B\|$ holds.

In the case where A, B are non-symmetric the choice of γ and K needs to be modified to agree with the expressions given for this case in Lemma 2.2. With these modifications the symmetry assumption (A4) may be dropped in all results of this paper.

In general, given symmetric, positive definite $A, B \in \mathbb{R}^{(N\times n)\times(N\times n)}$ and $\gamma > 0$ we define K_γ through the identity

$$\|B - \gamma A\| \coloneqq K_\gamma \|B\| \tag{2.8}$$

2 The iterative process

We call $b\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ a reference field if the following assumptions hold: $b(x,z)$ *satisfies* (A1) $-$ (A3) *and (A4). Moreover, whenever* $F \in L^{p'}(\Omega, \mathbb{R}^{N\times n})$ *and* $g \in W^{1,p}(\Omega, \mathbb{R}^N)$*, there exists a unique solution* $u \in W^{1,p}_g(\Omega)$ *to the problem*

$$\operatorname{div} b(x, \mathrm{D}u) = \operatorname{div} F. \tag{2.9}$$

(2.9) is to be understood in the sense of distributions.

Remark 2.6. The p-Laplacian $|z|^{p-2}z$ is an example of such a field. We encourage the reader to think of this case on a first reading.

We recall the definition of the iterative process. Let $u_0 \in W^{1,p}_g(\Omega)$ and $f \in L^{p'}(\Omega)$. Take $\gamma > 0$ to be chosen later. Define $u_{n+1} \in W^{1,p}_g(\Omega)$ to be the weak solution of the problem (2.9) with the choice

$$F \coloneqq b(x, \mathrm{D}u_n) - \gamma a(x, \mathrm{D}u_n) + \gamma f. \tag{2.10}$$

Note that (A3) ensures that $F \in L^{p'}(\Omega, \mathbb{R}^{N\times n})$, so that the sequence $\{u_n\}$ is well-defined.

We want to show that $u_n \to u$ in $W^{1,p}(\Omega)$, where u is a solution to (2.1). The following theorem gives conditions which guarantee convergence:

Theorem 2. *Suppose $b\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ is a reference field. Consider a second field $a\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ satisfying* (A1) – (A3) *and (A4). If*

$$\begin{cases} \dfrac{\Lambda_b}{\lambda_b} K_{a,b} \left(\dfrac{3+\sqrt{5}}{2} \right)^{\frac{p-2}{2p}} 6^{p-2}(p-1) < 1 & \text{for } p \geq 2, \\ \dfrac{\Lambda_b}{\lambda_b} K_{a,b} \left(\dfrac{3-\sqrt{5}}{2} \right)^{\frac{(p-2)}{4}} \dfrac{2^{2-p}}{p-1} < 1 & \text{for } p \leq 2, \end{cases} \tag{2.11}$$

then there is $\gamma > 0$ such that for the sequence $u_n \in W^{1,p}_g(\Omega)$ generated via (2.10) *it holds that $u_n \to u$ in $W^{1,p}(\Omega)$. Here $u \in W^{1,p}_g(\Omega)$ is the unique solution to* (2.1).

Remark 2.7. We note that if $p = 2$ and $b(x,z) = z$ we recover precisely the result in [141]. From the proof it will be clear that in fact, $\{u_n\}$ generated via (2.10) converges to u in $W^{1,p}(\Omega)$ whenever $\gamma > 0$ is such that

$$\begin{cases} R^{p-1} = \dfrac{\Lambda_b}{\lambda_b} K^* \left(\dfrac{3+\sqrt{5}}{2} \right)^{\frac{p-2}{2p}} 6^{p-2}(p-1) < 1 & \text{if } p \geq 2 \\ R^{p/2} = \dfrac{\Lambda_b}{\lambda_b} K^* \left(\dfrac{3-\sqrt{5}}{2} \right)^{\frac{(p-2)}{4}} \dfrac{2^{2-p}}{p-1} < 1 & \text{if } p \leq 2. \end{cases}$$

Here K^* is an upper bound independent of z on $\max_{x\in\Omega} K_\gamma(x,z)$ where $K_\gamma(x,z)$ is the constant from (2.8) with the choice $A(x,z) \coloneqq \partial_z a(x,z)$, $B(x,z) \coloneqq \partial_z b(x,z)$.

Further, the convergence occurs at linear rate R. To be precise, the following estimates hold:

$$\begin{aligned} \|u_{n+1} - u_n\|_{W^{1,p}(\Omega)} &\leq R^n \|u_1 - u_0\|_{W^{1,p}(\Omega)}, \\ \|u_{n+1} - u\|_{W^{1,p}(\Omega)} &\leq R^n \|u_1 - u\|_{W^{1,p}(\Omega)}, \end{aligned}$$

where u is the weak solution of (2.1).

Proof. Throughout the proof we write $A(x,z) \coloneqq \partial_z a(x,z)$ and $B(x,z) \coloneqq \partial_z b(x,z)$. Using Remark 2.5, we fix a $\gamma > 0$ such that $\|B(x,z) - \gamma A(x,z)\| \leq K_{a,b}\|B(x,z)\|$ for all $x \in \Omega$ and almost every $z \in \mathbb{R}^{N\times n}$. We write $K = K_{a,b}$.

Introduce, for $\theta \in (0,1)$, $z_n(\theta) \coloneqq (1-\theta)\mathrm{D}u_{n-1} + \theta \mathrm{D}u_n$ for $n \geq 1$.

Let $n \geq 0$. Test the equations defining u_{n+1} and u_n against $u_{n+1} - u_n$ to find

$$I \coloneqq \int_\Omega \left(b(x, \mathrm{D}u_{n+1}) - b(x, \mathrm{D}u_n) \right) \mathrm{D}(u_{n+1} - u_n) \, \mathrm{d}x$$

$$= \int_\Omega \Big(b(x, \mathrm{D}u_n) - b(x, \mathrm{D}u_{n-1}) - \gamma \left(a(x, \mathrm{D}u_n) - a(x, \mathrm{D}u_{n-1}) \right) \Big) \mathrm{D}(u_{n+1} - u_n) \, \mathrm{d}x$$
$$= \int_\Omega \int_0^1 \left(B(x, z_n(\theta)) - \gamma A(x, z_n(\theta)) \right) \mathrm{D}(u_n - u_{n-1}) \cdot \mathrm{D}(u_{n+1} - u_n) \; d\theta \, \mathrm{d}x =: II.$$

The last line follows from an application of the mean value theorem. Now proceed to estimate both sides of this equality.

We focus first on the case $p \geq 2$. Using again the mean-value theorem, by assumption (A2), the left-hand side gives:

$$\begin{aligned} I =& \int_\Omega \int_0^1 B(x, z_{n+1}(\theta) \mathrm{D}(u_{n+1} - u_n) \cdot \mathrm{D}(u_{n+1} - u_n) \, \mathrm{d}\theta \, \mathrm{d}x \\ \geq& \lambda_b \int_\Omega \int_0^1 (\mu^2 + |z_{n+1}(\theta)|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1} - u_n)|^2 \, \mathrm{d}\theta \, \mathrm{d}x \\ \geq& \frac{\lambda_b}{6^{\frac{p-2}{2}}(p-1)} \int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}u_{n+1} - \mathrm{D}u_n|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1} - u_n)|^2 \, \mathrm{d}x, \end{aligned}$$

where the last line uses Lemma 1.4.

Note that $\|B(x, w)\| \leq \Lambda_b (\mu^2 + |w|^2)^{\frac{p-2}{2}}$ by (A2). Hence, on the right-hand side, by Lemma 2.4 and Lemma 1.6,

$$\begin{aligned} II \leq& \Lambda_b K \int_\Omega \int_0^1 (\mu^2 + |z_n(\theta)|^2)^{\frac{p-2}{2}} |\mathrm{D}u_n - \mathrm{D}u_{n-1}| |\mathrm{D}u_{n+1} - \mathrm{D}u_n| \, \mathrm{d}\theta \, \mathrm{d}x \\ \leq& \Lambda_b K \int_\Omega (\mu^2 + |\mathrm{D}u_n|^2 + |\mathrm{D}u_n - \mathrm{D}u_{n-1}|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_n - u_{n-1})| |\mathrm{D}(u_{n+1} - u_n)| \, \mathrm{d}x =: A. \end{aligned}$$

We now apply Lemma 1.5 and Lemma 1.4 to find for any $\varepsilon > 0$,

$$\begin{aligned} A \leq& \Lambda_b K \Bigg(C_\varepsilon \int_\Omega V_{(\mu^2 + |\mathrm{D}u_n|^2)^{1/2}} (\mathrm{D}(u_{n+1} - u_n))^2 \, \mathrm{d}x \\ &\qquad + \varepsilon \int_\Omega V_{(\mu^2 + |\mathrm{D}u_n|^2)^{1/2}} (\mathrm{D}(u_n - u_{n-1}))^2 \, \mathrm{d}x \Bigg) \\ \leq& \Lambda_b K \Big(\varepsilon \int_\Omega V_{(\mu^2 + |\mathrm{D}u_n|^2)^{1/2}} (\mathrm{D}(u_n - u_{n-1}))^2 \, \mathrm{d}x \\ &+ C_\varepsilon \left(\frac{3 + \sqrt{5}}{2} \right)^{\frac{p-2}{2}} \int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}u_{n+1} - \mathrm{D}u_n|^2)^{\frac{p-2}{2}} |\mathrm{D}u_{n+1} - \mathrm{D}u_n|^2 \, \mathrm{d}x \Big). \end{aligned}$$

Combining the estimates for I and II and re-arranging gives

$$\begin{aligned} &\int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}u_{n+1} - \mathrm{D}u_n|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1} - u_n)|^2 \, \mathrm{d}x \\ \leq& \frac{\varepsilon C_1}{C_0 - C_\varepsilon C_2} \int_\Omega (\mu^2 + |\mathrm{D}u_n|^2 + |\mathrm{D}(u_n - u_{n-1})|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_n - u_{n-1})|^2 \, \mathrm{d}x, \end{aligned}$$

where

$$C_0 \coloneqq \frac{\lambda_b}{6^{\frac{p-2}{2}}(p-1)}, \qquad C_1 \coloneqq \Lambda_b K, \qquad C_2 \coloneqq \left(\frac{3+\sqrt{5}}{2}\right)^{\frac{p-2}{2}} \Lambda_b K.$$

Optimising in ε and using (2.11) this shows by induction that

$$\begin{aligned}&\int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}(u_{n+1}-u_n)|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1}-u_n)|^2 \,\mathrm{d}x \\ \leq & R \int_\Omega (\mu^2 + |\mathrm{D}u_n|^2 + |\mathrm{D}(u_n-u_{n-1})|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_n-u_{n-1})|^2 \,\mathrm{d}x,\end{aligned}$$

where

$$R \coloneqq \left(\frac{\Lambda_b}{\lambda_b} K \left(\frac{3+\sqrt{5}}{2}\right)^{\frac{p-2}{2p}} 6^{p-2}(p-1)\right)^{\frac{1}{p-1}} < 1.$$

As $p \geq 2$, it immediately follows that $\{u_n\}$ converges in $W^{1,p}(\Omega)$ at linear rate R. Necessarily, the limit u lies in $W^{1,p}_g(\Omega)$ and it is a solution to problem (2.1). Moreover, considering any solution $v \in W^{1,p}_g(\Omega)$ of (2.1) and repeating the above estimates with the starting point

$$\begin{aligned}&\int_\Omega (b(x,\mathrm{D}u_{n+1}) - b(x,\mathrm{D}v))\mathrm{D}(u_{n+1}-v)\,\mathrm{d}x \\ = & \int_\Omega \Big(b(x,\mathrm{D}u_n) - b(x,\mathrm{D}v) - \gamma(a(x,\mathrm{D}u_n) - a(x,\mathrm{D}v))\Big)\cdot \mathrm{D}(u_{n+1}-v)\,\mathrm{d}x,\end{aligned}$$

we find that for any $u_0 \in W^{1,p}_g(\Omega)$ the corresponding iterative process converges to v in $W^{1,p}_g(\Omega)$. In particular, the solution to (2.1) is unique.

We now consider $p < 2$. Proceed as before to obtain

$$\begin{aligned}&\int_\Omega \left(b(x,\mathrm{D}u_{n+1}) - b(x,\mathrm{D}u_n)\right)\mathrm{D}(u_{n+1}-u_n)\,\mathrm{d}x \\ \geq & \lambda_b \int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}u_{n+1}-\mathrm{D}u_n|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1}-u_n)|^2 \,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \int_0^1 \left(B(x,z(\theta)) - \gamma A(x,z(\theta))\right)\mathrm{D}(u_n-u_{n-1})\cdot \mathrm{D}(u_{n+1}-u_n)\,\mathrm{d}\theta\,\mathrm{d}x \\ \leq & \frac{\Lambda_b K}{2^{p-2}(p-1)} \Bigg(\varepsilon \int_\Omega (\mu^2 + |\mathrm{D}u_n|^2 + |\mathrm{D}u_n - \mathrm{D}u_{n-1}|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_n-u_{n-1})|^2 \,\mathrm{d}x \\ & + \left(\frac{3-\sqrt{5}}{2}\right)^{\frac{p-2}{2}} C_\varepsilon \int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}u_{n+1}-\mathrm{D}u_n|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1}-u_n)|^2 \,\mathrm{d}x\Bigg).\end{aligned}$$

Combining these estimates and our assumptions, we again find – after optimising in ε – that

$$\int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}(u_{n+1} - u_n)|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1} - u_n)|^2 \,\mathrm{d}x \to 0$$

at linear rate.

Note that by Hölder and the triangle inequality there is $c > 0$ such that

$$\begin{aligned}&\int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2 + |\mathrm{D}(u_{n+1} - u_n)|^2)^{\frac{p-2}{2}} |\mathrm{D}(u_{n+1} - u_n)|^2 \,\mathrm{d}x \\ \geq & c \frac{\|\mathrm{D}(u_{n+1} - u_n)\|^2_{L^p(\Omega)}}{\|\mathrm{D}u_n\|^{2-p}_{L^p(\Omega)} + \|\mathrm{D}u_{n+1}\|^{2-p}_{L^p(\Omega)}}.\end{aligned}$$

If $\{\mathrm{D}u_n\}$ is a bounded sequence in $W^{1,p}$, convergence of $\{u_n\}$ in $W^{1,p}(\Omega)$ and existence and uniqueness of solutions to (2.1) follow by repeating the arguments of the case $p \geq 2$. For clarity of presentation, and as it essentially follows by repeating arguments similar to those of this proof, we postpone the proof of boundedness to Lemma 2.8. □

We provide the proof of boundedness of $\{u_n\}$ under the assumptions of Theorem 2 that is still lacking to complete its proof. Note that we only require this result for $p \leq 2$, but since the general case requires little additional effort we prove the result for all $p \in (1, \infty)$.

Lemma 2.8. *Suppose the assumptions of Theorem 2 hold. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$.*

Proof. Throughout the proof, we write $A(x, z) \coloneqq \partial_z a(x, z)$ and $B(x, z) \coloneqq \partial_z b(x, z)$. Using Remark 2.5, we fix a $\gamma > 0$ such that $\|B(x, z) - \gamma A(x, z)\| \leq K_{a,b}\|B(x, z)\|$ for all $x \in \Omega$ and almost every $z \in \mathbb{R}^{N\times n}$. We write $K = K_{a,b}$. Further, write for $\theta \in (0, 1)$, $z_n(\theta) \coloneqq \theta \mathrm{D}u_n$.

Let $n \geq 0$. Test the equation defining u_{n+1} against u_{n+1} to find:

$$\begin{aligned}I &\coloneqq \int_\Omega \left(b(x, \mathrm{D}u_{n+1}) - b(x, 0)\right) \mathrm{D}u_{n+1} \,\mathrm{d}x \\ &= \int_\Omega \left(b(x, \mathrm{D}u_n) - b(x, 0)\right) \mathrm{D}u_{n+1} \\ &\quad - \gamma \left(a(x, \mathrm{D}u_n) - a(x, 0)\right) \mathrm{D}u_{n+1} + \gamma(a(x, 0) + f)\mathrm{D}u_{n+1} \,\mathrm{d}x \\ &= \int_\Omega \int_0^1 \left(B(x, z_n(\theta)) - \gamma A(x, z_n(\theta))\right) \mathrm{D}u_n \cdot \mathrm{D}u_{n+1} + \gamma a(x, 0)\mathrm{D}u_{n+1} \,\mathrm{d}\theta \,\mathrm{d}x =: II.\end{aligned}$$

The last line follows from an application of the mean value theorem. Now proceed to estimate both sides of this equality exactly as in Theorem 2.

We focus first on the case $p \geq 2$. Using again the mean-value theorem and writing $z_{n+1}(\theta) = \theta u_{n+1}$, by assumption (A2), the left-hand side gives

$$\begin{aligned} I =& \int_\Omega \int_0^1 (B(x, z_{n+1}(\theta))\mathrm{D}u_{n+1} \cdot \mathrm{D}u_{n+1} \,\mathrm{d}\theta \,\mathrm{d}x \\ \geq& \lambda_b \int_\Omega \int_0^1 (\mu^2 + |z_{n+1}(\theta)|^2)^{\frac{p-2}{2}} |\mathrm{D}u_{n+1}|^2 \,\mathrm{d}\theta \,\mathrm{d}x \\ \geq& \frac{\lambda_b}{(p-1)6^{\frac{p-2}{2}}} \int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2)^{\frac{p-2}{2}} |\mathrm{D}u_{n+1}|^2 \,\mathrm{d}x, \end{aligned}$$

where the last line uses Lemma 1.4.

Note that $\|B(x,z)\| \leq \Lambda_b(\mu^2 + |z|^2)^{\frac{p-2}{2}}$ by (A2). Hence on the right-hand side by Lemma 2.4 and Lemma 1.4, as well as (A3),

$$\begin{aligned} II \leq& \Lambda_b K \int_\Omega \int_0^1 (\mu^2 + |z_n(\theta)|^2)^{\frac{p-2}{2}} |\mathrm{D}u_n||\mathrm{D}u_{n+1}| + C\gamma(1+|f|)|\mathrm{D}u_{n+1}| \mathrm{d}\theta \,\mathrm{d}x \\ \leq& \Lambda_b K \int_\Omega (\mu^2 + |\mathrm{D}u_n|^2)^{\frac{p-2}{2}} |\mathrm{D}u_n||\mathrm{D}(u_{n+1})| \,\mathrm{d}x + C\gamma \int_\Omega (1+|f|)|\mathrm{D}u_{n+1}| \,\mathrm{d}x = I + II. \end{aligned}$$

We now apply Lemma 1.5, to find

$$I \leq \Lambda_b K \left(C_\varepsilon \int_\Omega V_\mu(\mathrm{D}u_{n+1})^2 \,\mathrm{d}x + \varepsilon \int_\Omega V_\mu(\mathrm{D}u_n)^2 \,\mathrm{d}x \right).$$

Young's inequality also gives

$$II \lesssim C\gamma(1 + \|f\|_{L^{p'}(\Omega)})\mathrm{D}u_{n+1}\|_{L^p(\Omega)} \leq \varepsilon_1 \|\mathrm{D}u_{n+1}\|^p_{L^p(\Omega)} + C(\varepsilon_1)(1 + \|f\|_{L^{p'}(\Omega)}).$$

Combining the estimates for I and II and re-arranging gives:

$$\begin{aligned} &\int_\Omega (\mu^2 + |\mathrm{D}u_{n+1}|^2)^{\frac{p-2}{2}} |\mathrm{D}u_{n+1}|^2 \,\mathrm{d}x \\ \leq& \frac{\varepsilon C_1}{C_0 - C_\varepsilon C_2} \int_\Omega (\mu^2 + |\mathrm{D}u_n|^2)^{\frac{p-2}{2}} |\mathrm{D}u_n|^2 \,\mathrm{d}x \\ &\quad + \frac{1}{C_0 - \varepsilon C_2} \left(C(\varepsilon_1)(1 + \|f\|_{L^{p'}(\Omega)}) + \varepsilon_1 \|\mathrm{D}u_{n+1}\|^p_{L^p(\Omega)} \right). \end{aligned}$$

where $C_0 \coloneqq \frac{\lambda_b}{6^{\frac{p-2}{2}}(p-1)}$ and $C_1 \coloneqq C_2 = \Lambda_b K$.

Optimising in ε, we can ensure that $\frac{\varepsilon C_1}{C_0 - C_\varepsilon C_1} < 1$. Now choosing ε_1 sufficiently small, we conclude that

$$\|V_\mu(\mathrm{D}u_{n+1})\|^2_{L^2(\Omega)} \leq \eta \|V_\mu(\mathrm{D}u_n)\|^2_{L^2(\Omega)} + C(1 + \|f\|_{L^{p'}(\Omega)}) \tag{2.12}$$

for some constant $\eta < 1$. The conclusion follows using induction.

For $p \leq 2$ we argue similarly to obtain (2.12) with the choices $C_0 \coloneqq \lambda_b$, $C_1 \coloneqq \Lambda_b K$, $C_2 \coloneqq \Lambda_b K(p-1)^{-1}4^{(2-p)2}$. Optimising in ε we can again ensure that

$$\frac{\varepsilon C_1}{C_0 - C_\varepsilon C_1} < 1,$$

and use this to conclude as before. □

3 Numerical analysis and experiments

Throughout this section we assume that $b(x,z)$ is a reference field and that $a(x,z)$ satisfies (A1) – (A3) and (A4). In order to avoid additional technical complications we assume $g = 0$ throughout this section. We further assume that the assumptions of Theorem 2 hold. In particular this will imply that with an appropriate choice of $\gamma > 0$ the iterative process (2.10) converges in $W^{1,p}(\Omega)$ to a solution of (2.1) and so in particular the iterates $\{u_n\}$ are uniformly bounded in $W^{1,p}(\Omega)$. We fix such a choice of γ from now on.

Recall that we intend to study the following numerical scheme: Suppose Ω is a polytope. Consider a sequence of shape-regular triangulations $\{T_h\}_{h\in(0,1]}$ of Ω of mesh-size h. Denote by X_h the space of continuous piecewise linear (subordinate to T_h) functions vanishing on $\partial\Omega$. Choose $u_0 \in X_h$. Define $u^h_{n+1} \in X_h$ inductively as a solution of the problem:

$$\begin{cases} \displaystyle\int_\Omega b(x, \mathrm{D}u^h_{n+1}) \cdot \mathrm{D}\phi \,\mathrm{d}x = \int_\Omega (b(x, \mathrm{D}u^h_n) - \gamma a(x, \mathrm{D}u^h_n) + \gamma f) \cdot \mathrm{D}\phi \,\mathrm{d}x & \forall \phi \in X_h. \\ u_{n+1} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.13}$$

Note that (A1) – (A3) ensure that (2.13) makes sense and moreover by the theory of monotone operators (2.13) solutions exist and are unique. Hence (2.13) is well-defined.

If $p = 2$ and $b(x,z) = z$ the proposed numerical scheme falls into the class of iterative linearised Galerkin schemes studied in [113]. We refer to this reference for an overview and further references regarding such schemes. An advantage of the proposed scheme, and of unified iterative schemes of the form studied in [113] in general, is that it is possible to show global convergence results at linear rate for a wide class of problems. This should be compared with non-linear Newton schemes for which convergence can

usually only be expected locally, albeit at quadratic rate. An example of a scheme, fitting into the framework of unified iterative schemes, that has been used to study some problems of p-Laplace type where $a(x,z) = a(x,|z|)z$ and $a(x,|z|)$ has $p-2$-growth is the so-called Kačanov iteration, originally introduced in [132], and defined by solving

$$\operatorname{div} a(x,|\mathrm{D}u_n|)\mathrm{D}u_{n+1} = \operatorname{div} f.$$

Versions of this scheme have been studied in [209, 111, 94, 72, 114], but in general have been restricted to the case $p \in [1,2]$. Note that the scheme we study cannot be used to replace these schemes as it requires the ability to solve (2.1) numerically for a reference operator. However, given a problem to which a known numerical scheme, such as the Kačanov iteration can be applied, it enables us to solve perturbations of the problem to which the known scheme cannot be applied directly. For example in the case of the Kačanov scheme, perturbations need not satisfy any structural assumption of the form $a(x,z) = a(x,|z|)z$.

3.1 Analysis of the numerical scheme

In this section we present our main results regarding convergence, a-priori and a-posteriori estimates for (2.13).

We begin with the following convergence result:

Theorem 3. *Let $u_0 \in X_h$ and consider the sequence $\{u_n^h\}$ generated by (2.13) starting from u_0. Then $u_n^h \to u^h$ in $W^{1,p}(\Omega)$ where $u^h \in X_h$ solves*

$$\begin{cases} \displaystyle\int_\Omega a(x,\mathrm{D}u^h)\cdot \mathrm{D}\phi \,\mathrm{d}x = \int_\Omega f\cdot \mathrm{D}\phi\,\mathrm{d}x & \forall \phi \in X_h \\ u^h = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Moreover there is $0 < C_1 < 1$ such that

$$\|u_{n+1}^h - u_n^h\|_{W^{1,p}(\Omega)} \le C_1^n \|u_1 - u_0\|_{W^{1,p}(\Omega)}$$

Proof. This follows from carrying out line by line the proof of Theorem 2, replacing $\{u_n\}$ with $\{u_n^h\}$ and u with u^h respectively. □

We now wish to show that $u^h \to u$ in $W^{1,p}(\Omega)$ as $h \to 0$ where $u \in W_0^{1,p}(\Omega)$ solves (2.1). The proof is standard but for completeness we include it here. We make the

following regularity assumption on $b(x,z)$: *If $F \in L^{p'}(\Omega)$, then there is $\alpha > 0$ such that* (2.9) *has a solution u in $W^{1+\alpha,p}(\Omega)$ and moreover we have the following estimate for some $c > 0$:*

$$\|u\|_{W^{1+\alpha,p}(\Omega)} \leq c\left(1 + \|u\|_{W^{1,p}(\Omega)} + \|F\|^{1/(p-1)}_{L^{p'}(\Omega)}\right) \tag{2.15}$$

Remark 2.9. (2.15) is satisfied for example when $b(x,z) = |z|^{p-2}z$ with any choice of $\alpha > 0$ such that $\alpha < \min\left(1/(p-1)^2, (p-1)^2\right)$, see [193].

We proceed to study the effect of decreasing the mesh-size h. We use the notation of Theorem 3.

Theorem 4. *Suppose the assumptions of Theorem 3 hold. Suppose moreover that* (2.15) *is satisfied. Choose $u_0 \in X_h$ and let $\{u_n^h\}$ be the sequence generated by* (2.13)*. Suppose $u \in W_0^{1,p}(\Omega)$ solves* (2.1)*. Then $u \in W^{1+\alpha,p}(\Omega)$ and $u^h \to u$ in $W^{1,p}(\Omega)$ as $h \to 0$. Moreover we have the estimate*

$$\|u_{n+1}^h - u\|_{W^{1,p}(\Omega)} \leq ch^{\frac{2\alpha}{\max(2,p)}} c\left(\|f\|_{L^{p'}(\Omega),\|u_0\|_{L^p(\Omega)}}\right) + C_1^n \|u_1 - u_0\|_{W^{1,p}(\Omega)}.$$

Proof. Consider the iterative process $\{u_n\}$ started from u_0. By Theorem 2, $u_n \to u$ in $W^{1,p}(\Omega)$. Using (2.15), we find

$$\|u_n\|_{W^{1+\alpha,p}(\Omega)} \leq c\left(1 + \|u_n\|_{W^{1,p}(\Omega)} + \|b(x,\mathrm{D}u_n) - \gamma a(x,\mathrm{D}u_n) + \gamma f\|^{1/(p-1)}_{L^{p'}(\Omega)}\right)$$

Arguing similarly to Lemma 2.8 and relying in particular on (A2), we bound the last term by $c\|u_n\|_{W^{1,p}(\Omega)}$. In particular, we deduce that there is $c(\|f\|_{L^{p'}(\Omega)}, \|u_0\|_{L^p(\Omega)})$ such that for all $n \geq 0$,

$$\|u_n\|_{W^{1+\alpha,p}(\Omega)} \leq c(\|f\|_{L^{p'}(\Omega)}, \|u_0\|_{L^p(\Omega)}).$$

Extracting a weakly convergent subsequence we conclude the same estimate holds for u.

Let v be the best approximation to u in X_h. Using (A2) and the fact that u, u_h solve (2.1) and (2.14) respectively we find

$$\begin{aligned}
&\lambda_a \int_\Omega (\mu^2 + |\mathrm{D}u|^2 + |\mathrm{D}u^h|^2)^{\frac{p-2}{2}} |\mathrm{D}u - \mathrm{D}u^h|^2 \,\mathrm{d}x \\
&\leq \int_\Omega (a(x,\mathrm{D}u) - a(x,\mathrm{D}u^h))\mathrm{D}(u - u^h)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega (a(x, \mathrm{D}u) - a(x, \mathrm{D}u^h))\mathrm{D}(u-v)\,\mathrm{d}x \\
&\lesssim \Lambda_a \int_\Omega (\mu^2 + |\mathrm{D}u|^2 + |\mathrm{D}(u-u^h)|^2)^{\frac{p-2}{2}} |\mathrm{D}(u-u^h)||\mathrm{D}(u-v)|\,\mathrm{d}x \\
&\lesssim C_\varepsilon \int_\Omega (\mu^2 + |\mathrm{D}u|^2 + |\mathrm{D}u_h|^2)^{\frac{p-2}{2}} |\mathrm{D}(u-u_h)|^2\,\mathrm{d}x \\
&\qquad + \varepsilon \int_\Omega (\mu^2 + |\mathrm{D}u|^2 + |\mathrm{D}v|^2)^{\frac{p-2}{2}} |\mathrm{D}(u-v)|^2\,\mathrm{d}x
\end{aligned}
$$

where to obtain the last line we have used Lemma 1.5. Choosing ε sufficiently large we conclude using (1.10)

$$
\begin{aligned}
I &= \int_\Omega (\mu^2 + |\mathrm{D}u|^2 + |\mathrm{D}u^h|^2)^{\frac{p-2}{2}} |\mathrm{D}(u-u^h)|^2 \\
&\lesssim \int_\Omega (\mu^2 + |\mathrm{D}u|^2 + |\mathrm{D}v|^2)^{\frac{p-2}{2}} |\mathrm{D}(u-v)|^2\,\mathrm{d}x \\
&\lesssim \|\mathrm{D}(u-v)\|^2_{L^p(\Omega)} \left(1 + \|\mathrm{D}u\|^{p-1}_{W^{1,p}(\Omega)} + \|\mathrm{D}(u-v)\|^{p-1}_{W^{1,p}(\Omega)}\right) \\
&\lesssim h^{2\alpha} \|u\|^2_{W^{1+\alpha,p}(\Omega)} \left(1 + \|u\|^{p-1}_{W^{1,p}(\Omega)} + h^{\alpha(p-1)} \|u\|^{p-1}_{W^{1+\alpha,p}(\Omega)}\right) \qquad (2.16)
\end{aligned}
$$

If $p \geq 2$, $I \geq \|\mathrm{D}(u-u_h)\|^p_{L^p(\Omega)}$ whereas if $p \leq 2$, we have by applying Hölder's inequality $I \gtrsim \|\mathrm{D}(u-u^h)\|^2_{L^p(\Omega)}(1 + \|\mathrm{D}u\|_{L^p(\Omega)} + \|\mathrm{D}u^h\|_{L^p(\Omega)})^{p-2}$. Recall, that since u, u^h solve (2.1) and (2.14) respectively, the basic estimates $\|\mathrm{D}u\|_{L^p(\Omega)} \leq \|f\|^{1/(p-1)}_{L^{p'}(\Omega)}$ and $\|\mathrm{D}u^h\|_{L^p(\Omega)} \leq \|f\|^{1/(p-1)}_{L^{p'}(\Omega)}$ apply. Hence we conclude by combining (2.16), Theorem 3 and the inequality

$$
\|u^h_{n+1} - u\|_{W^{1,p}(\Omega)} \leq \|u^h_{n+1} - u^h\|_{W^{1,p}(\Omega)} + \|u^h - u\|_{W^{1,p}(\Omega)}.
$$

□

We also have an a-posteriori error bound. The proof follows [113].

Proposition 2.10. *Assume the conditions of Theorem 3 hold. Then we have*

$$
\begin{aligned}
\|\mathrm{D}(u^h_n - u^h)\|_{L^p(\Omega)} \leq &\|\mathrm{D}(u^h_n - u^h_{n-1})\|_{L^p(\Omega)} \\
&+ C(\|f\|_{L^{p'}(\Omega)}, \|\mathrm{D}u_0\|_{L^p(\Omega)}) \|\mathrm{D}(u^h_n - u^h_{n-1})\|^{\frac{2}{\max(2,p)}}_{L^p(\Omega)}.
\end{aligned}
$$

Proof. We compute using (A2) and Lemma 1.5,

$$
\begin{aligned}
&\lambda_a \gamma \int_\Omega (\mu^2 + |\mathrm{D}(u^h - u^h_{n-1})|^2 + |\mathrm{D}u^h_{n-1}|^2)^{\frac{p-2}{2}} |\mathrm{D}u^h - \mathrm{D}u^h_{n-1}|^2\,\mathrm{d}x \\
\lesssim &\gamma \int_\Omega (a(x, \mathrm{D}u^h) - a(x, \mathrm{D}u^h_{n-1})) \cdot \mathrm{D}(u^h - u^h_{n-1})\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega \gamma f \cdot \mathrm{D}(u^h - u^h_{n-1}) + (b(x, \mathrm{D}(u^h_n)) - b(x, \mathrm{D}u^h_{n-1})) - \gamma f) \cdot \mathrm{D}(u^h - u^h_{n-1}) \,\mathrm{d}x \\
&\lesssim \Lambda_b \int_\Omega \left(\mu^2 + |\mathrm{D}u^h_{n-1}|^2 + |\mathrm{D}(u^h_n - u^h_{n-1})|^2\right)^{\frac{p-2}{2}} |\mathrm{D}(u^h_n - u^h_{n-1})||\mathrm{D}(u^h - u^h_{n-1})| \,\mathrm{d}x \\
&\lesssim \varepsilon \int_\Omega \left(\mu^2 + |\mathrm{D}u^h_{n-1}|^2 + |\mathrm{D}(u^h_n - u^h_{n-1})|^2\right)^{\frac{p-2}{2}} |\mathrm{D}(u^h_n - u^h_{n-1})|^2 \,\mathrm{d}x \\
&\qquad + C_\varepsilon \int_\Omega (\mu^2 + |\mathrm{D}(u^h - u^h_{n-1})|^2 + |\mathrm{D}u^h_{n-1}|^2)^{\frac{p-2}{2}} |\mathrm{D}u^h - \mathrm{D}u^h_{n-1}|^2 \,\mathrm{d}x
\end{aligned}
$$

Thus choosing ε sufficiently large, re-arranging, employing by now standard arguments and recalling that $\{u^h_n\}$ is bounded uniformly in $W^{1,p}(\Omega)$ we obtain

$$
\|\mathrm{D}(u^h - u^h_{n-1})\|_{L^p(\Omega)} \lesssim c\left(\|\mathrm{D}u^h\|_{L^p(\Omega)}, \|\mathrm{D}u_0\|_{L^p(\Omega)}\right) \|\mathrm{D}(u^h_n - u^h_{n-1})\|^{\frac{2}{\max(2,p)}}_{L^p(\Omega)}.
$$

By the triangle inequality and using $\|u^h\|_{W^{1,p}(\Omega)} \lesssim \|f\|^{1/(p-1)}_{L^{p'}(\Omega)}$ we conclude the desired estimate:

$$
\begin{aligned}
&\|\mathrm{D}(u^h_n - u^h)\|_{L^p(\Omega)} \\
\leq &\|\mathrm{D}(u^h_{n-1} - u^h)\|_{L^p(\Omega)} + \|\mathrm{D}(u^h_n - u^h_{n-1})\|_{L^p(\Omega)} \\
\leq &\|\mathrm{D}(u^h_n - u^h_{n-1})\|_{L^p(\Omega)} + C\left(\|f\|_{L^{p'}(\Omega)}, \|\mathrm{D}u_0\|_{L^p(\Omega)}\right) \|\mathrm{D}(u^h_n - u^h_{n-1})\|^{\frac{2}{\max(2,p)}}_{L^2(\Omega)}.
\end{aligned}
$$

□

We close this section by detailing a modification of the numerical scheme, following the algorithm outlined in [113]. We strengthen our assumptions on $\{T_h\}$ and assume $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of shape-regular triangulations with meshsize $h_k \to 0$ as $k \to \infty$. Moreover, we assume that $\{T_k\}$ is obtained from $\{T_{k-1}\}$ by refinement. Denote by X^k the space of continuous piecewise linear functions subordinate to $\{T_k\}$. We then consider the following algorithm:

Algorithm 1 A variant of the iteration scheme

Set $u^0_0 = 0$. Set $k = 0$. Choose a maximal mesh k_{max} and a sequence of tolerances $\mathrm{tol}_k \to 0$ as $k \to \infty$.
repeat
  while $\|u^k_n - u^k_{n-1}\| > \mathrm{tol}_k$ **do**
    solve $\mathrm{div}\, b(x, \mathrm{D}u^k_{n+1}) = \mathrm{div}\,(b(x, \mathrm{D}u^k_n) - \gamma(a(x, \mathrm{D}u^k_n) - f)$ in X^k
    $n \leftarrow n + 1$
  end while
  $u^{k+1}_0 \leftarrow u^k_n$
  $k \leftarrow k + 1$
until $k = k_{max}$.

From the results of this section it is clear that under the assumptions of Theorem 4 Algorithm 1 converges to a solution of (2.1) as $k_{max} \to \infty$.

3.2 Numerical experiments

The results presented in this section are obtained using Firedrake [10, 11, 60, 178, 5, 6]. Throughout this section $\Omega = [0,1]^3$. We consider shape-regular triangulations of Ω with uniformly spaced nodes at distance $h = 2^{-i}$. Note that in this set-up the mesh-size is $\sqrt{3}h$.

A linear example: We first consider a linear example where the exact solution is known. We choose,

$$f = \begin{pmatrix} 8\pi & 8\pi & 8\pi \\ 10\pi & 10\pi & 10\pi \\ 2\pi & 2\pi & 2\pi \end{pmatrix} (v,v,v)^T \quad \text{where } v = \begin{pmatrix} cos(2\pi x)sin(2\pi y)sin(2\pi z) \\ sin(2\pi x)cos(2\pi y)sin(2\pi z) \\ sin(2\pi x)sin(2\pi y)cos(2\pi z) \end{pmatrix}$$

and consider the problem

$$\begin{cases} -\operatorname{div} A\,\mathrm{D}u = \operatorname{div} f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{where } A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that with $\tilde{u}(x,y,z) = sin(2\pi x)sin(2\pi y)sin(2\pi z)$,

$$u(x,y,z) = (\tilde{u}(x,y,z), \tilde{u}(x,y,z), \tilde{u}(x,y,z))^T$$

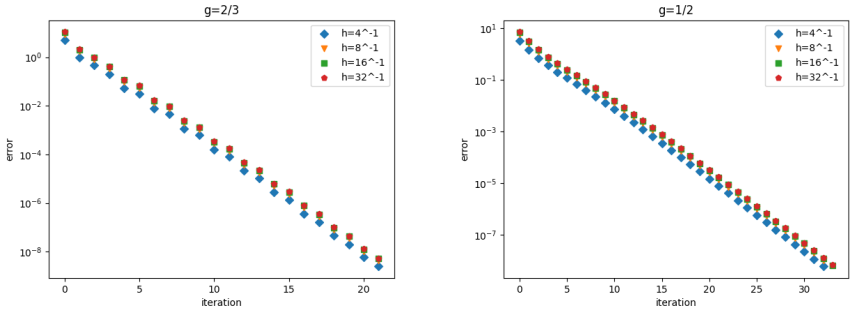
is the exact solution of this problem.

For the iteration scheme we choose $u_0 = 0$, $b(x,z) = z$ and $\gamma = 2/3$ or $\gamma = 1/2$. We solve each iteration step using GMRES with an incomplete LU factorisation to precondition the problem. Our convergence condition is $\|u^h_{n+1} - u^h_n\|_{W^{1,p}(\Omega)} \leq 10e-9$.

We record the H^1-error of the numerical solution v computed using the iterative scheme (2.13) in Table 2.1. We also record the number of iterations needed.

h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
$\gamma = 2/3$, error	6.6924	5.3767	3.2456	1.6786	0.8436
$\gamma = 2/3$, iterations	1	22	22	22	22
$\gamma = 1/2$, error	6.6924	5.3767	3.2456	1.6786	0.8436
$\gamma = 1/2$, iterations	1	33	33	34	34

Table 2.1: $\|u - v\|_{H^1(\Omega)}$



(a) $\gamma = 2/3$ (b) $\gamma = 1/2$

Figure 2.1: $\|u_n^h - u_h\|_{H^1(\Omega)}$

We also compute the numerical solution u^h directly using a LU-factorisation. We record the H^1-distance between u_n^h and u^h for two different choices of γ in Fig.2.1.

A non-linear example: We also consider the non-linear problem

$$\begin{cases} -\text{div}\, A\text{D}u + |u|^4 u = (y, x^2, z^2 + x^2)^T & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{with } A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the iteration scheme we chose $b(x, z) = z$, $\gamma = 2/3$ and $u_0 = 0$. We employ the iteration scheme (2.13), solving each iteration step using GMRES with an incomplete LU factorisation to precondition the problem. Our convergence condition is given by $\|u_{n+1}^h - u_n^h\|_{W^{1,p}(\Omega)} \leq 10e - 9$.

In order to verify our method we compare the solution we obtain through our iteration scheme with the solution computed by the standard 'solve'-method in Firedrake. This utilises a non-linear Newton linesearch scheme where the linear step is computed using GMREs. Denote the solution obtained in this way with u_{dir}. We record the H^1-distance $\|u_{dir} - u_n^h\|_{H^1(\Omega)}$ in Fig.2.2.

We moreover compare computing times for (2.13) and Algorithm (1) in Table 2.2 (for Algorithm 1 we choose $\text{tol}_i = 10^{-(i+4)}$ when $h = 2^{-i}$). For the reader's convenience we have additionally included computation times for the standard 'solve'-method in Firedrake both when using GMRES at the linear step and when enforcing a LU-factorisation.

Remark 2.11. We note that in the iteration schemes we assemble each required stiffness matrix only once. However we do not apply a scheme optimised for the Laplacian. Improving our computation in this way a further reduction in runtime should be achieved.

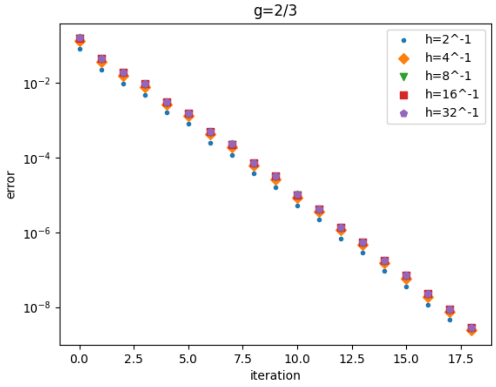


Figure 2.2: $\|u_{dir} - u_n^h\|_{L^2(\Omega)}$

h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
iteration scheme	0.2635	0.3098	0.6894	7.852	162.8
Algorithm 1	0.4748	1.108	1.951	10.12	165.2
Newton using LU	0.0270	0.0304	0.0772	4.133	308.8
Newton using GMRES	0.0044	0.0070	0.0380	0.4842	8.888

Table 2.2: runtime in seconds

We further remark that from our experiments it seems that for this problem $\gamma = 2/3$ is the optimal choice. However note that a direct calculation of the constants λ_a and Λ_a in (A2) using Sobolev embedding gives $\lambda_a = 1$, $\Lambda_a = 2 + c$ for some $c > 0$ and hence $\gamma < 2/3$ for the optimal choice. It would be interesting to understand this effect.

4 Improved regularity results

The study of the regularity of elliptic systems of the form (2.1) is well established with a very long list of important results. For an introduction we refer to [97] and [100] as well as the classical [144]. It is well known that, in general, without stronger structural assumptions than ellipticity and controlled p-growth, solutions to (2.1) will not be $C^{1,\alpha}$- or even $C^{0,\alpha}$- regular. In fact, problems already arise in the simplest case when $p = 2$. An initial counterexample for systems with $a(x, y, z) = a(x, z)$ (with $n \geq 3$) was developed in [69], where a quadratic system was given for which solutions are not Hölder-continuous and in fact fail to be bounded. While the x-dependence of the system in [69] is discontinuous, almost at the same time an example was found of a

higher-order elliptic operator with real analytic coefficients but discontinuous solutions [158]. We remark at this point that, if $n = 2$, solutions to (2.1) are Hölder continuous [165]. When $a(x, y, z)$ does indeed depend on y, it is possible to give examples where $a(x, y, z)$ is real analytic but solutions to (2.1) are nowhere continuous; see [194, 128], building on examples with real analytic fields $a(x, y, z)$ but discontinuous solutions in [101]. In this set-up an example of a system in optimal dimensions $n > 2$, $N = 2$ with discontinuous solutions is given in [93]. However the lack of regularity occurs already without x-dependence of the coefficients. Even when $a(x, y, z) = a(z)$ is quadratic and real analytic, $C^{1,\alpha}$-regularity of solutions need not hold if $n > 2$ as [172] shows.

Due to this lack of regularity, the general regularity theory of systems of type (2.1) proceeds through notions of partial regularity, that is regularity outside of a context-dependent small relatively closed set. Our focus here lies on results holding everywhere in the domain and so we refer the reader to [97, 100, 162] for results and references in this direction.

Regularity results holding in the full domain are known only in a few special cases. The radial Uhlenbeck structure $a(x, y, z) = a(|z|)z$, originally developed in [204], is well known to guarantee full $C^{1,\alpha}$-regularity. In [199] the result is generalised to fields of the form $g(z, z)z$ where $g(\cdot, \cdot)$ is a symmetric, positive definite bilinear form. In particular, this theory covers the model case of the p-Laplace operator $a(x, y, z) = |z|^{p-2}z$. There now exists a vast literature concerning the regularity theory of (2.1) if $a(x, y, z) = a(|z|)z$ and we refer the reader to [98, 155, 156, 147, 75, 74] for further results and references.

A second direction of everywhere regularity results when $a(x, y, z) = a(z)$ concerns the case where the modulus of continuity of $\partial_z a(z)$ is not too large compared to the ellipticity constant of $a(z)$. $C^{1,\alpha}_{\mathrm{loc}}$-regularity of solutions to (2.1) is shown in this set-up in [61, 62, 63].

A further direction, closely related to the results we obtain in this section, are results of Cordes–Nirenberg type. For linear elliptic PDE, the classical Cordes–Nirenberg results [173, 55] state that solutions of

$$a_{ij}(x)\partial_{ij}u(x) = f(x) \tag{2.17}$$

have interior $C^{1,\alpha}$-regularity if $f \in L^\infty$ and $|a_{ij} - \delta_{ij}| < \varepsilon$ for sufficiently small ε. That is, $C^{1,\alpha}$-regularity is obtained when the opeator $a \equiv (a_{ij})$ is sufficiently close to the

Laplacian in an appropriate sense. In [38] the result is extended to fully non-linear elliptic equations with solutions in the sense of viscosity solutions. A paper of a similar spirit is [76] where local BMO-regularity for extremals of (a suitable relaxed version of) $\operatorname{dist}_K^2(\mathrm{D}u)$ is shown. Here $\operatorname{dist}_K(z)$ denotes the distance of $z \in \mathbb{R}^{N\times n}$ to the compact set $K \subset \mathbb{R}^{N\times n}$.

We also comment on the case where $a(x,z) = a(x)z + b(x,z)z$ and $\|b(x,z)\| \leq c(n)\lambda_a$. Here $a(x)$ is a positive definite matrix, with smallest eigenvalue uniformly bounded below in Ω by $\lambda_a > 0$ and $c(n)$ is a sufficiently small constant depending only on n. Then bounded solutions of (2.1) are α-Hölder-continuous for some $\alpha > 0$ [207].

We intend to obtain regularity statements under similar conditions on $a(x,z)$ and $b(x,z)$ as those imposed in Theorem 2. Our focus lies on perturbations of operators satisfying the following structure of p-Laplace type:

$$b(x,z) = (B(x)z \cdot z)^{\frac{p-2}{2}} B(x)z \tag{2.18}$$

where $B(x)$ is measurable in x and satisfies for $x \in \Omega$, $\xi \in \mathbb{R}^{N\times n}$,

$$\lambda|\xi|^2 \leq B(x)\xi \cdot \xi \leq \Lambda|\xi|^2. \tag{2.19}$$

We note that $b(x,z)$ satisfies (A1) $-$ (A3) with $\mu = 0$. We will further assume that $B(x)$ is symmetric, so that (A4) holds.

4.1 A first example: Calderón–Zygmund type estimates

Recall the following global Calderón–Zygmund type result from [133]:

Theorem 5. *Let $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,\alpha}$-boundary for some $\alpha \in (0,1]$. Suppose that the coefficients of $B(x)$ belong to $VMO(\Omega)$ and that $F \in L^{\frac{q}{p-1}}(\Omega)$ for some $q > p$. Then there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ of (2.9) with $g = 0$. Moreover, there is a constant $C_0 > 0$ so that*

$$\int_\Omega |\mathrm{D}u|^q \,\mathrm{d}x \leq C_0 \int_\Omega |F|^{\frac{q}{p-1}} \,\mathrm{d}x.$$

Here C_0 depends only on $n, p, q, \lambda, \Lambda$, the VMO data of B and $\partial\Omega$. In particular, this implies that $u \in W_0^{1,q}(\Omega)$.

We will use fields satisfying the assumptions of Theorem 5 as reference fields and apply the iterative process to obtain the following result.

Theorem 6. *Let $1 < p < q < \infty$. Suppose $f \in W^{1,q/(p-1)}(\Omega)$ and let $g = 0$. Suppose a satisfies* (A1) – (A3) *and (A4) with $\mu = 0$. Suppose there is B such that* (2.19) *and the assumptions of Theorem 5 are satisfied. Suppose, moreover, that with this choice of $b(x, z)$ the assumptions of Theorem 2 are satisfied. Finally suppose that*

$$C_1 \coloneqq \frac{2^{\frac{q-p-1}{q}} C_0^{(p-1)/q} \Lambda_b K_{a,b}}{(p-1)} < 1$$

where $K_{a,b}$ is given by (2.5)*. Then the solution u of* (2.1) *satisfies*

$$\|\mathrm{D}u\|_{L^q(\Omega)} \lesssim 1 + \|f\|_{L^{q/(p-1)}}^{\frac{1}{p-1}}.$$

Proof. Let $\{u_n\}$ be the sequence generated by the iterative process with $u_0 = 0$ and with the choice of γ as in in Theorem 2. By Theorem 5, Lemma 2.4, (A2) and (A3), we have for some $c > 0$ the estimate

$$\begin{aligned}
\int_\Omega |\mathrm{D}u_{n+1}|^q \,\mathrm{d}x \leq & C_0 \int_\Omega |b(x, \mathrm{D}u_n) - \gamma a(x, \mathrm{D}u_n) + \gamma f|^{\frac{q}{p-1}} \,\mathrm{d}x \\
\leq & C_0 \int_\Omega |b(x, \mathrm{D}u_n) - b(x, 0) - \gamma(a(x, \mathrm{D}u_n) + a(x, 0)) + \gamma f|^{\frac{q}{p-1}} \,\mathrm{d}x + c \\
\leq & 2^{\frac{q-p-1}{p-1}} C_0 \Lambda_b^{\frac{q}{p-1}} K^{\frac{q}{p-1}} \int_\Omega \left(\int_0^1 \theta^{p-2} \,\mathrm{d}\theta \right)^{\frac{q}{p-1}} |\mathrm{D}u_n|^q \\
& + 2^{\frac{q-p-1}{p-1}} C_0 \int_\Omega |f|^{\frac{q}{p-1}} \,\mathrm{d}x + c \\
= & \frac{2^{\frac{q-p-1}{p-1}} C_0 \Lambda_b^{\frac{q}{p-1}} K^{\frac{q}{p-1}}}{(p-1)^{q/(p-1)}} \int_\Omega |\mathrm{D}u_n|^q + 2^{\frac{q-p-1}{p-1}} C_0 \int_\Omega |f|^{\frac{q}{p-1}} \,\mathrm{d}x + c.
\end{aligned}$$

Thus, as by assumption $C_1 < 1$, by induction we find,

$$\|\mathrm{D}u_{n+1}\|_{L^q(\Omega)} \lesssim \frac{C_1^{\frac{q}{p-1}}}{1 - C_1^{\frac{q}{p-1}}} (1 + \|f\|_{L^{q/(p-1)}}^{\frac{1}{p-1}}).$$

Extracting a weakly convergent subsequence, and noting that $u_n \to u$ in $W^{1,p}(\Omega)$, by Theorem 2 where u is the solution to (2.1) we find the desired estimate holds. □

4.2 A second example: Weighted estimates and Hölder continuity

Global weighted estimates for fields of the form (2.18) are considered in [176, 161]. We remind the reader that we introduced A_p-weights in (1.7). A special case of the main result of [176] is the following weighted regularity estimate:

Theorem 7 (c.f. Theorem 2.1 in [176]). *Let $1 < p < q < \infty$ and let w be an $A_{q/p}$ weight. Suppose Ω is a C^1-domain. Suppose $B \in VMO(\Omega)$ and B satisfies (2.19). Then there exists a positive constant $C_2 > 0$ such that the following holds: For a given vector field $F \in L_w^{q/(p-1)}(\mathbb{R}^n)$, the boundary value problem (2.9) with $g = 0$ has a unique weak solution $u \in W_0^{1,p}(\Omega)$ satisfying $\mathrm{D}u \in L_w^q(\Omega)$ with the estimate*

$$\|\mathrm{D}u\|_{L_w^q(\Omega)} \leq C_2 \|F\|^{\frac{1}{p-1}}_{L_w^{\frac{q}{p-1}}(\Omega)}.$$

Here the constant C_2 depends only on $n, p, q, R, \lambda, \Lambda$ and $[w]_{q/p}$.

We can use our iteration scheme to perturb this regularity result as follows:

Theorem 8. *Suppose a is a field satisfying (A1) − (A3) and (A4) with $\mu = 0$ and, moreover, there is symmetric B satisfying the assumptions of Theorem 7, and such that the assumptions of Theorem 2 apply with this choice of b. Finally suppose that*

$$C_3 := \frac{2^{\frac{q-p-1}{q}} C_2^{\frac{p-1}{q}} K_{a,b} \Lambda_b}{(p-1)} < 1$$

where $K_{a,b}$ is given by (2.5). Then the solution v to the boundary value problem (2.1) satisfies the estimate

$$\|\mathrm{D}v\|_{L_w^q(\Omega)} \lesssim 1 + \|f\|^{\frac{1}{p-1}}_{L_w^{\frac{q}{p-1}}(\Omega)}.$$

Proof. The proof is similar to the proof of Theorem 6. Let $\{u_n\}$ be given by the iterative process with $u_0 = 0$ and γ chosen as in Theorem 2.

Note that $b(x,0) - \gamma a(x,0) + \gamma f \in L_w^{\frac{q}{p-1}}(\Omega)$. By Theorem 7, Lemma 2.4, (A2) and (A3), we have the estimate

$$\begin{aligned}
\|\mathrm{D}u_{n+1}\|^q_{L_w^q(\Omega)} \leq & C_2 \int_\Omega |b(x, \mathrm{D}u_n) - \gamma a(x, \mathrm{D}u_n) + \gamma f|^{\frac{q}{p-1}} w(x)\,\mathrm{d}x \\
\leq & 2^{\frac{q-p-1}{p-1}} C_2 \int_\Omega |b(x, \mathrm{D}u_n) - b(x,0) - \gamma(a(x, \mathrm{D}u_n) - a(x,0))|^{\frac{q}{p-1}} w(x)\,\mathrm{d}x \\
& + C_2 |b(x,0) - \gamma a(x,0) + \gamma f|^{\frac{q}{p-1}} w(x)\,\mathrm{d}x \\
\leq & 2^{\frac{q-p-1}{p-1}} C_2 K_{a,b}^{\frac{q}{p-1}} \Lambda_b^{\frac{q}{p-1}} 2^{-\frac{q}{p-1}} \|\mathrm{D}u_n\|^q_{L_w^q(\Omega)} + c(C_2, p, q) \left(1 + \|f\|^{\frac{q}{p-1}}_{L_w^{\frac{q}{p-1}}(\Omega)}\right).
\end{aligned}$$

Hence by induction, as by assumption $C_3 < 1$,

$$\|\mathrm{D}u_{n+1}\|_{L_w^q(\Omega)} \lesssim \frac{C_3^{\frac{q}{p-1}}}{1 - C_3^{\frac{q}{p-1}}} \left(1 + \|f\|^{\frac{q}{p-1}}_{L_w^{\frac{q}{p-1}}(\Omega)}\right).$$

Extracting a weakly convergent subsequence, and noting that $u_n \to u$ in $W^{1,p}(\Omega)$ by Theorem 2 where u solves (2.1), we conclude the desired estimate, after passing to the limit in the above estimate. □

We note that of particular interest is the choice $w(x) = |x|^\alpha$ for $\alpha > 0$, which can be used to obtain estimates in the familiar Morrey spaces and hence through the Morrey–Sobolev embedding allows to obtain continuity statements. Recall the definition of the $L^{q,\theta}$-Morrey-norm:

$$\|u\|_{L^{q,\theta}(\Omega)} \coloneqq \sup_{0<r<\operatorname{diam}(\Omega), z\in\Omega} r^{\frac{\theta-n}{q}} \|u\|_{L^q(B_r(z)\cap\Omega)},$$

where $\theta \in (0, n)$. We assume that all the assumptions and the notation of Theorem 8 hold and show how to deduce estimates in Morrey spaces.

Fix $z \in \Omega$, $r \in (0, \operatorname{diam}(\Omega))$ and choose for $\rho \in (0, \theta)$,

$$w(x) \coloneqq \min\left(|x-z|^{-n+\theta-\rho}, r^{-n+\theta-\rho}\right).$$

Then by Theorem 8,

$$\|\mathrm{D}u\|^q_{L^{q,\theta}(B_r(z)\cap\Omega)} \le r^{n-\theta+\rho}\|\mathrm{D}u\|^q_{L^q_w(B_r(z)\cap\Omega)} \le r^{n-\theta+\rho} c(C,p,q)\left(1 + \|f\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}_w(\Omega)}\right).$$

It remains to estimate $\|f\|_{L^{\frac{q}{p-1}}_w(\Omega)}$. For this we proceed exactly as [161] but provide the argument here for the sake of completeness. We will show that

$$\|f\|_{L^{\frac{q}{p-1}}_w(\Omega)} \le c\|f\|^q_{L^{q,\theta}(\Omega)} r^{-\rho},$$

which will conclude the proof.

It is convenient to introduce f' where $|f'|^{p-2}f' \coloneqq f$. For $\alpha > 0$ we denote the set $E_\alpha \coloneqq \{x \in \Omega \colon |f'| > \alpha\}$. Then we can write

$$\begin{aligned}\|f\|^q_{L^{\frac{q}{p-1}}_w(\Omega)} = \|f'\|^q_{L^q_w(\Omega)} =& q\int_0^\infty \alpha^q \int_{E_\alpha} w(x)\,\mathrm{d}x \frac{d\alpha}{\alpha} \\ \le& q\int_0^\infty \alpha^q \int_0^{r^{-n+\theta-\rho}} \left|E_\alpha \cap B_{\beta^{\frac{1}{-n+\theta-\rho}}}(z)\right| \,\mathrm{d}\beta \frac{d\alpha}{\alpha}.\end{aligned}$$

We now estimate the inner integral as follows:

$$\int_0^{r^{n+\theta-\rho}} \left|E_\alpha \cap B_{\beta^{\frac{1}{-n+\theta-\rho}}}(z)\right| d\beta \le \sum_{i=1}^\infty 2^{-i} r^{-n+\theta-\beta} \left|E_\alpha \cap B_{r2^{\frac{-i}{-n+\theta-\rho}}}(z)\right|$$

$$\leq 2\int_0^{\frac{1}{2}r^{-n+\theta-\rho}} \beta \left| E_\alpha \cap B_{\beta^{\frac{1}{-n+\beta-\rho}}}(z) \right| \frac{d\beta}{\beta}.$$

Now, returning to the original estimate, and applying Fubini's theorem, we conclude

$$\begin{aligned}\|f\|^q_{L_w^{\frac{q}{p-1}}(\Omega)} \leq & 2q\int_0^\infty \alpha^q \int_0^{\frac{1}{2}r^{-n+\theta-\rho}} \beta \left| E_\alpha \cap B_{\beta^{\frac{1}{-n+\theta-\rho}}}(z) \right| \frac{\mathrm{d}\beta}{\beta}\frac{\mathrm{d}\alpha}{\alpha} \\ \leq & 2q\|f\|^q_{L^{q,\theta}(\Omega)} \int_0^{\frac{1}{2}r^{-n+\theta-\rho}} \beta^{1+\frac{n-\theta}{-n+\theta-\rho}} \leq c\|f\|^q_{L^{q,\theta}(\Omega)} r^{-\rho}.\end{aligned}$$

This gives the desired result.

3. Higher integrability for non-autonomous convex functionals with (p,q)-growth

In this chapter our aim is to prove a global $W^{1,q}$-regularity result for general non-autonomous functionals with (p,q)-growth. We begin by recalling some properties of Lipschitz domains in Section 2 before introducing one of our main technical tools—Whitney-Besicovitch coverings—in Section 3. Before proving a regularity result for minimisers of the relaxed functional in Section 5, and for minimisers in the pointwise sense in Section 7, we gather some properties of the relaxed functional in Section 4. In Section 6 we improve our result for minimisers of the relaxed functional in the case of autonomous functionals. In Section 8 we adapt our assumptions in order to deal with anisotropic functionals. At the end of this chapter, in Section 9, we present a list of examples to which the results we derive in this chapter apply.

1 Regularity for functionals with (p, q)-growth

We intend to study functionals of the form (P) satisfying

$$|z|^p - 1 \lesssim F(x,z) \lesssim 1 + |z|^q$$

where $p \leq q$. In addition, $F \equiv F(x,z)$ will satisfy a further convexity assumption in z.

The study of regularity theory for minimisers in this set-up started with the seminal papers [153], [154]. We remark that [154] was inspired by the following consideration in finite elasticity. If $u\colon \Omega \to \mathbb{R}^n$ describes the position of points after applying a given deformation, then $\det \mathrm{D}u$ gives the ratio between the volume elements of the reference configuration and the deformed configuration. We expect that the deformation energy

blows up as $\det \mathrm{D}u$ tends to zero or infinity. Hence a reasonable model of an elastic energy, known in the literature as Neo-Hookean, could be

$$\int_\Omega |\mathrm{D}u|^2 + g(\det(\mathrm{D}u))\,\mathrm{d}x$$

where $g(t)$ is a convex function with $g(t) \to \infty$ as $t \to 0$ or $t \to \infty$. Note that this energy will, in general, not satisfy the same growth bounds from above and below, leading naturally to the consideration of (p, q)-growth functionals. We will return to the study of similar problems in the next chapter.

Recall that already when $p = q$ counter-examples to regularity arise. Hence it is not surprising that when $p < q$ the situation does not improve. Already in the scalar ($N = 1$) autonomous case ($F(x, z) = F(z)$) counterexamples show that in order to prove regularity of minimisers p and q may not be too far apart [95], [153], [121]. A crucial first step to proving interior regularity results is to obtain $W^{1,q}_{\mathrm{loc}}$-regularity of minimisers. Once $W^{1,q}_{\mathrm{loc}}$-regularity is obtained, further regularity results such as partial regularity may be obtained, see for example [2], [175]. Hence the goal in this chapter is to obtain $W^{1,q}(\Omega)$-regularity results.

The counter-examples show that it is necessary to require $q - p = O(1/n)$ as $n \to \infty$. Hence the task at hand is to make the $O(1/n)$-term precise. In order to give the reader an idea of the available results we list, to our knowledge, the best known $W^{1,q}_{\mathrm{loc}}$-regularity results for general autonomous convex functionals with (p, q)-growth (when $n \geq 2$). Under natural growth conditions, it suffices to assume $q < \frac{np}{n-1}$ [44] in order to obtain $W^{1,q}_{\mathrm{loc}}$-regularity of minimisers. To obtain the same conclusion under controlled growth conditions the gap may be widened to $q < p\left(1 + \frac{2}{n-1}\right)$ [186], and under controlled duality growth conditions it suffices to take $q < \frac{np}{n-2}$ [66]. In the last instance, if $n = 2$ any exponent $q < \infty$ will do. We note that in all three cases higher integrability goes hand in hand with a higher differentiability result. Partial $C^{1,\alpha}$-regularity of pointwise minimisers (and relaxed minimisers with $X = W^{1,p}(\Omega)$, $Y = W^{1,q}_{\mathrm{loc}}(\Omega)$, see below for a definition) for autonomous quasiconvex functionals has been obtained in [187, 188, 189] under natural growth conditions with $q < 1 + \frac{\min(2,p)}{2n}$. The borderline-case $p = n - 1$, $q = n$ was studied in [46]. Finally, we refer to [24] for results and references in the case of parabolic systems with (p, q)-growth.

While our intent is to not impose further structure conditions on the functional be-

yond (p,q)-growth or assume a-priori bounds on minimisers, doing so enables to improve the results listed so far. Without going into further detail, we refer to [29], [43] for results and further references in these directions. Local boundedness of minimisers for convex non-autonomous (p,q)-growth functionals under natural growth conditions and the additional assumption $F(x,2z) \lesssim 1+F(x,z)$ is studied in [119]. A particular focus of research has been the special cases of the double-phase functional $F(x,z) = |z|^p + a(x)|z|^q$ and functionals with $p(x)$-growth. For an introduction and further references with regards to these special cases, we refer to the introductions of [18], [73] and [183], respectively.

In general, a summary of the available regularity theory for functionals with (p,q)-growth can be found in [162] and the updated [163].

In order to comment further on the results of this chapter, we introduce the precise assumptions we make: *Let $0 < \alpha \leq 1$. We assume that $F(x,z)$ is measurable in x, continuously differentiable in z, and moreover satisfies for some $\lambda, \Lambda > 0$, $\mu \geq 0$, $1 < p \leq q$, $\alpha \in (0,1)$, almost every $x, y \in \Omega$ and every $z \in \mathbb{R}^{N\times n}$,*

$$z \mapsto F(x,z) - \lambda(\mu^2 + |z|^2)^{\frac{p}{2}} \text{ is convex,} \tag{H1}$$

$$F(x,z) \leq \Lambda(1 + |z|^q), \tag{H2}$$

$$|F(x,z) - F(y,z)| \leq \Lambda |x-y|^\alpha \left(1 + |z|^2\right)^{\frac{q}{2}}. \tag{H3}$$

If $p < n$, we further suppose $q \leq \frac{np}{n-p}$.

We remark that our conditions clearly imply the following bounds for $z_1, z_2 \in \mathbb{R}^{N\times n}$ and almost every $x \in \Omega$,

$$\nu(\mu^2 + |z|^2 + |w|^2)^{\frac{p-2}{2}} \leq \frac{F(x,z) - F(x,w) - \langle \partial_z F(x,w), z-w\rangle}{|z-w|^2}, \tag{H4}$$

$$|z|^p - 1 \lesssim F(x,z), \tag{H5}$$

$$|F(x,z_1) - F(x,z_2)| \lesssim \Lambda |z_1 - z_2| \left(1 + |z_1|^2 + |z_2|^2\right)^{\frac{q-1}{2}}. \tag{H6}$$

Further, for a pointwise minimiser u of $\mathscr{F}$, $\mathscr{F}(u) < \infty$, coupled with (H5), Sobolev embedding and Young's inequality gives $\mathrm{D}u \in L^p(\Omega)$.

Note that our assumptions allow the integrand to depend on x, that is we consider non-autonomous integrands. In this set-up, for $n \geq 2$, counterexamples to $W^{1,q}$-regularity with $1 < p < n < n + \alpha < q$ are due to [79], see also [89]. Recent work suggests that the condition $p < n < q$ may be removed [12]. If $q < \frac{(n+\alpha)p}{n}$, it was proven

in [79] for many standard examples that minimisers enjoy $W^{1,q}_{\mathrm{loc}}$-regularity. Using [78] the result may be extended to integrands satisfying in addition the following condition:

there is $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $x \in \Omega$ *there is* $\hat{y} \in \overline{B_\varepsilon(x) \cap \Omega}$ *such that*

$$F(\hat{y}, z) \leq F(y, z) \qquad \forall\, y \in \overline{B_\varepsilon(x) \cap \Omega},\ z \in \mathbb{R}^{N \times n}. \tag{H7}$$

Our aim in this chapter is to extend this result up to the boundary. We will return to a discussion of the need for this condition shortly, but remark for now that (H7) is very similar to Assumption 2.3. in [213]. Further, similar conditions have been considered in the case of radial integrands $F(x, z) = F(x, |z|)$ in [112]. (H7) holds for many of the commonly considered examples, see Chapter 3.9 as well as [79], [78]. $W^{1,q}_{\mathrm{loc}}$-regularity is in general not known if $q = \frac{(n+\alpha)p}{n}$. An exception are functionals modeled on the double-phase functional where $F(x, y, z) = |z|^p + a(x)|z|^q$, [18], see also [68].

It is worth pointing out that not many results up to the boundary are known in the setting of (p, q)-growth. In fact, the results of this chapter seem to be the first general results in this setting.

One of the few results on regularity up to the boundary in the setting of (p, q)-growth functionals is [33], where Lipschitz regularity up to the boundary is obtained for minimisers of scalar autonomous functionals satisfying nonstandard growth conditions and the structure condition $F(x, z) = b(|z|)$. The growth conditions considered include (p, q)-growth. For functionals satisfying a structural assumption inspired by the double-phase functional, Caldéron–Zygmund estimates valid up to the boundary are obtained in [37]. The author of this thesis, together with I. Chlebicka and C. de Filippis, also studied regularity up to the boundary for the $p(x)$-Laplacian $F(x, y, z) = k(x)|z|^{p(x)}$ with $p \leq p(x) \leq q$ under the additional difficulty that solution maps are constrained to lie on a manifold in [52].

Proofs of $W^{1,q}_{\mathrm{loc}}$-regularity in the (p, q)-growth framework often involve two major steps. First, an a-priori estimate for minimisers of a suitably regularised q-growth functional is obtained using difference quotient methods. Second, taking limits the estimate is transferred to minimisers of the original functional. This two-step approach is taken due to the obstructions caused by the Lavrentiev phenomenon, which describes the pos-

sibility that

$$\inf_{u\in W^{1,p}_g(\Omega)} \mathscr{F}(u) < \inf_{u\in W^{1,q}_g(\Omega)} \mathscr{F}(u). \tag{3.1}$$

The phenomenon is also of interest in nonlinear elasticity [92], where it is related to the formation defects, such as fractures. A first example of the Lavrentiev phenomenon was given in [145]. In the context of (p,q)- growth functionals the theory was further developed in [210], [212],[213]. The Lavrentiev phenomenon is closely related to properties of the relaxed functional. We adopt the viewpoint and terminology of [36] and consider a topological space X of weakly differentiable functions and a dense subspace $Y \subset X$. We introduce the following sequentially lower semi-continuous (slsc) envelopes

$$\overline{\mathscr{F}}_X \coloneqq \sup\{\, \mathscr{G}\colon X \to [0,\infty] : \mathscr{G} \text{ slsc }, \mathscr{G} \leq \mathscr{F} \text{ on } X \,\}$$
$$\overline{\mathscr{F}}_Y \coloneqq \sup\{\, \mathscr{G}\colon X \to [0,\infty] : \mathscr{G} \text{ slsc }, \mathscr{G} \leq \mathscr{F} \text{ on } Y \,\}$$

and define the Lavrentiev gap functional for $u \in X$ as

$$\mathscr{L}(u,X,Y) \coloneqq \begin{cases} \overline{\mathscr{F}}_Y(u) - \overline{\mathscr{F}}_X(u) & \text{if } \overline{\mathscr{F}}_X(u) < \infty \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

Note that the gap functional is non-negative.

In this chapter we always consider the choice $X \coloneqq W^{1,p}_g(\Omega)$ endowed with the weak topology and $Y \coloneqq W^{1,q}_g(\Omega)$. Since $F(x,\cdot)$ is convex, standard methods then show that $\overline{\mathscr{F}}_X(\cdot) = \mathscr{F}(\cdot)$, see [100, Chapter 4]. In this set-up, an alternative, equivalent definition of the relaxed functional is useful:

Definition 3.1. $u \in W^{1,p}_g(\Omega)$ is a $\boldsymbol{W^{1,q}}$**-relaxed minimiser** (which we usually refer to as a relaxed minimiser) of $\mathscr{F}$ in the class $W^{1,p}_g(\Omega)$ if u minimises the relaxed functional

$$\overline{\mathscr{F}}(v) = \inf\left\{ \liminf_{j\to\infty} \int_\Omega \Big(F(\mathrm{D}v_j) - f\cdot v_j\Big)\,\mathrm{d}x : (v_j) \subset Y, v_j \rightharpoonup v \text{ weakly in } X \right\} \tag{3.3}$$

amongst all $v \in X \coloneqq W^{1,p}_g(\Omega)$ where $Y \coloneqq W^{1,q}_g(\Omega)$.

We use this opportunity to point out that the counterexamples to regularity only apply to pointwise minimisers as opposed to relaxed minimisers.

It is at this second step of excluding the Lavrentiev phenomenon that the condition (H7) comes into play. In order to pass to the limit in the a-priori estimate obtained

for the regularised functional, it is necessary to show that the relaxed functional agrees with the pointwise functional at the minimum u of the pointwise functional. In practice, considering (3.3) we need to find a sequence $u_n \in W^{1,q}_g(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\mathscr{F}(u_n) \to \mathscr{F}(u)$. In the local case for convex autonomous functionals such a sequence is easily found by considering mollifications $u \star \phi_\varepsilon$. Applying Jensen's inequality and monotone convergence, we calculate for $B_r \Subset \Omega$,

$$\mathscr{F}_{B_r}(u_\varepsilon) \coloneqq \int_{B_r} F(\mathrm{D}u \star \phi_\varepsilon)\,\mathrm{d}x \leq \int_{B_r} F(\mathrm{D}u) \star \phi_\varepsilon\,\mathrm{d}x \to \int_{B_r} F(\mathrm{D}u)\,\mathrm{d}x =: \mathscr{F}_{B_r}(u).$$

as $\varepsilon \to 0$. Using the minimality of u, we conclude $\overline{\mathscr{F}}(u) = \mathscr{F}(u)$ as desired. Note that the inequality fails when $F(\cdot)$ is not autonomous. In order for the argument to carry over to the non-autonomous case, we need to be able to estimate $F(x, \mathrm{D}u(y))$ by $F(y, \mathrm{D}u(y))$ where $y \in B_\varepsilon(x)$. Assumption (H7) enables us to do exactly this. While (H7) is often easy to check in practice, it does not seem to the author that it is a natural assumption and investigating whether it can be replaced is an interesting problem.

The relaxed functional is an interesting object of study in its own right. It originates in Lebesgue's definition of generalised area and has been suggested as a way to give meaning to (P) for more general functionals than those covered by (1.2) in [190, 191]. It has also been used to study (P) with solutions in BV, see [103] and the references therein. In the context of (p, q)-growth, it was first introduced in [152] and by now there is an extensive literature both on the relaxed functional in this context as well as on the Lavrientiev phenomenon and gap functional. An overview can be found in [35], [91]. Considering the common choice $X \coloneqq W^{1,p}(\Omega)$ endowed with the weak toplogy and $Y \coloneqq W^{1,q}_{\mathrm{loc}}(\Omega)$ a question related to the Lavrentiev phenomenon is to study measure representations of $\overline{\mathscr{F}}(\cdot)$. We refer to [87, 1] for results and further references in this direction, but point out that, in this context, an argument using a Whitney cover of Ω was given in [88] that is similar to arguments in this thesis.

The main result of Chapter 3 is a global regularity result for pointwise and relaxed minimisers of non-autonomous functionals with (p,q)-growth:

Theorem 9. *Suppose Ω is a Lipschitz domain. Let $g \in W^{1+\alpha,q}(\Omega)$. Assume $F(x,\cdot)$ satisfies* (H1)–(H3) *with $1 < p \leq q < \frac{(n+\alpha)p}{n}$. If u is a relaxed minimiser of $\mathscr{F}(\cdot)$ in the*

class $W^{1,p}_g(\Omega)$*, then* $u \in W^{1,q}(\Omega)$*. Moreover, for any* $0 \leq \beta < \alpha$ *there is* $\gamma > 0$ *such that*

$$\|u\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \overline{\mathscr{F}}(u) + \|g\|_{W^{1+\alpha,q}(\Omega)} + \|f\|_{L^{q'}(\Omega)}\right)^{\gamma}. \tag{3.4}$$

Suppose that Ω *is a* $C^{1,\alpha}$*-domain and that* $g \in W^{1+\max\left(\alpha,\frac{1}{q}\right),q}(\Omega)$*. Assume that* $F(x,\cdot)$ *satisfies, in addition,* (H7) *and let* u *be a pointwise minimiser of* $\mathscr{F}(\cdot)$ *in the class* $W^{1,p}_g(\Omega)$*. Then* $u \in W^{1,q}(\Omega)$ *and the estimate* (3.4) *holds.*

2 Some properties of Lipschitz and $C^{1,\alpha}$domains

In this section we recall some properties of Lipschitz and $C^{1,\alpha}$-domains. For further details we refer to [104].

Let Ω be a Lipschitz domain. Then Ω satisfies a uniform exterior cone condition [104, Section 1.2.2]: there are $\rho_0, \theta_0 > 0$ and a map $\boldsymbol{n}\colon \mathbb{R}^n \to \mathbb{S}^{n-1}$ such that for every $x \in \mathbb{R}^n$

$$C_{\rho_0}(\theta_0, \boldsymbol{n}(x)) \subset \left\{ h \in \mathbb{R}^n : |h| \leq \rho_0, (B_{3\rho_0}(x) \setminus \Omega) + h \subset \mathbb{R}^n \setminus \Omega \right\}. \tag{3.5}$$

Recall that we defined our notation for cones in (1.8). Moreover, there is a smooth vector field transversal to $\partial\Omega$, i.e., there exist $\kappa > 0$ and $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$X \cdot \nu \geq \kappa \tag{3.6}$$

a.e. on $\partial\Omega$, where ν is the exterior unit normal to $\partial\Omega$ [104, Lemma 1.5.1.9.].

We remark that every Lipschitz domain can be written as a finite union of strongly star-shaped Lipschitz domains:

Lemma 3.2. *Suppose* Ω *is a Lipschitz domain. Then there are* $M > 0$ *and strongly star-shaped Lipschitz domains* Ω_i*,* ω_i*,* $i = 1, \dots M$*, with* $\omega_i \Subset \Omega_i$ *relative to* Ω*, such that* $\partial\omega_i \cap \partial\Omega \Subset \partial\Omega_i \cap \partial\Omega$ *if* $\partial\Omega_i \cap \partial\Omega \neq \emptyset$*, and*

$$\Omega = \bigcup_{i=1}^{M} \Omega_i = \bigcup_{i=1}^{M} \omega_i.$$

Moreover, given $g \in W^{1+\frac{1}{q},q}(\Omega)$ *and* $u \in W^{1,p}_g(\Omega)$*, we can ensure that* $u \in W^{1,p}(\partial\Omega_i)$ *for* $i = 1, \dots, N$*, with* $\|u\|_{W^{1,p}(\partial\Omega_i)} \lesssim \|u\|_{W^{1,p}(\Omega_i)} + \|g\|_{W^{1+1/q,q}(\Omega)}$.

Proof. We recall the main points of the proof and refer to [41, Proposition 2.5.3] for details. Let $x \in \partial\Omega$. Then, for ε and ρ sufficiently small, we may assume that I_x is a neighbourhood of x such that, writing $\hat{y} = (y_1, \dots, y_{n-1})$,

$$I_x \cap \overline{\Omega} = \{\, y \in \mathbb{R}^n : -\varepsilon < y_n \leq \tau(\hat{y}), \qquad \hat{y} \in B_\rho(0) \,\}$$

where τ is a Lipschitz function. We may further assume that $\rho \leq \frac{\tau(0)}{2c}$, where c denotes the Lipschitz constant of τ. We now consider

$$\Omega_x^{\varepsilon',\rho'} \coloneqq \{\, y \in \mathbb{R}^n : -\varepsilon' < y_n < \tau(\hat{y}), \qquad \hat{y} \in B_{\rho'}(0) \,\}$$

with $\varepsilon' \in (0, \varepsilon)$, $\rho' \in (0, \rho)$. A straightforward argument shows that $\Omega^{\varepsilon',\rho'}$ is strongly star-shaped with respect to 0 as well as a Lipschitz domain. Since

$$\int_0^\varepsilon \int_0^\rho \int_{\partial\Omega_x^{\varepsilon',\rho'}} (|u|^p + |\mathrm{D}u|^p) \lesssim \int_{I_x \cap \Omega} |\mathrm{D}u|^p \,\mathrm{d}x,$$

we can choose $\tilde{\varepsilon} \in (0, \varepsilon)$ and $\tilde{\rho} \in (0, \rho)$ such that

$$\int_{\partial\Omega_x^{\tilde{\varepsilon},\tilde{\rho}} \setminus \partial\Omega} |u|^p + |\mathrm{D}u|^p \lesssim c(\varepsilon, \rho) \|u\|^p_{W^{1,p}(\Omega)}.$$

We now set $\Omega_x \coloneqq \Omega_x^{\tilde{\varepsilon},\tilde{\rho}}$ and $\omega_x \coloneqq \Omega_x^{\tilde{\varepsilon}/2,\tilde{\rho}/2}$. If $x \in \Omega$, by a similar argument we find a ball $B_{\tilde{\varepsilon}}(x)$ with $\tilde{\varepsilon} \in (0, d(x, \partial\Omega)/2)$ such that $\|u\|_{W^{1,p}(\partial B_{\tilde{\varepsilon}}(x))} \lesssim c(d(x, \partial\Omega)) \|u\|_{W^{1,p}(B_{\tilde{\varepsilon}}(x))}$ and set $\Omega_x \coloneqq B_{\tilde{\varepsilon}}(x)$, $\omega_x \coloneqq B_{\tilde{\varepsilon}/2}(x)$. A standard compactness argument shows that the cover $\{\, \omega_x \,\}_{x \in \Omega}$ of Ω contains a finite subcover with the desired properties. □

The next lemma enables us to stretch a small neighbourhood of the boundary in a controlled manner. This will be crucial in constructing sequences with improved integrability but unchanged boundary behaviour.

Lemma 3.3. *Suppose Ω is a $C^{1,\alpha}$-domain. Then there is a family of domains $\Omega^s \ni \Omega$ and a family of $C^{1,\alpha}$-diffeomorphisms $\Psi_s \colon \Omega^s \to \Omega$ such that*

(i) *$J\Psi_s \to 1$ and $|\mathrm{D}\Psi_s - \mathrm{Id}| \to 0$ uniformly in Ω^s as $s \nearrow 1$. Equivalently, $J\Psi_s^{-1} \to 1$ and $|\mathrm{D}\Psi_s^{-1} - \mathrm{Id}| \to 0$ uniformly in Ω as $s \nearrow 1$.*

(ii) *If $g \in W^{1+\frac{1}{q},q}(\Omega)$, then there is an extension $\hat{g}$ of g to Ω^s such that $\hat{g} \in W^{1,q}(\Omega^s)$ and $\hat{g} \circ \Psi_s^{-1} \in g + W^{1,q}(\Omega)$.*

Proof. Let $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be a smooth vector field transversal to $\partial\Omega$. Such a field exists by (3.6). Fix $t_0 > 0$. Given $z \in \partial\Omega$ consider the flow

$$\begin{cases} \frac{dh_z}{dt} & = X(h(t)) \\ h_z(s) & = z \end{cases}$$

for $|t| \leq 2t_0$ and set $\Psi(z,t) \coloneqq h_z(t)$. After possibly reducing the value of t_0, the maps Ψ, Ψ^{-1} are $C^{1,\alpha}$-regular diffeomorphisms on a neighbourhood of $\partial\Omega$ which we denote V. Moreover, the Jacobians of Ψ, Ψ^{-1} are bounded.

Let $\frac{1}{2} \leq s < 1$. Consider $\tau_s \colon [-t_0, t_0] \to [-t_0, t_0]$, a sequence of strictly monotonically increasing smooth maps with

$$\tau_s(-t_0) = -t_0 \qquad \tau_s(t_0) = s^{-1}t_0 \qquad \tau_s'(-t_0) = -1$$

and such that $\tau_s \to \mathrm{Id}$ in $C^2([-t_0, t_0])$ as $s \to 1$. Define

$$\Psi_s^{-1}(x) = \begin{cases} \Psi\left(x_0, \tau_s(t)\right) & \text{for } x = \Psi(x_0, t) \in V \\ x & \text{else.} \end{cases}$$

Set $\Omega^s \coloneqq \Psi_s^{-1}(\Omega) \subset V \cup \Omega$. Using the chain rule, we note that $\Psi_s^{-1} \colon \Omega \to \Omega^s$ is a $C^{1,\alpha}$-regular diffeomorphism. Denote its inverse by $\Psi_s \colon \Omega^s \to \Omega$ and use the inverse function theorem and the chain rule to see that $\Psi_s \to \mathrm{Id}$ in $C^1(\Omega^s)$ as $s \nearrow 1$. In particular, also $J\Psi_s \to 1$ uniformly in Ω^s as $s \nearrow 1$. To obtain $\hat{g}$, simply set

$$\hat{g}(x) = \begin{cases} g(\Psi(x_0, 0)) & \text{for } x = \Psi(x_0, t_0) \in \Omega^s \setminus \Omega, \\ g(x) & \text{for } x \in \Omega. \end{cases}$$

□

We conclude this section by noting a number of extensions we may carry out if Ω is a Lipschitz domain: Let $\Omega \Subset B(0,R)$ and $s \in [1, \infty)$. Due to [184], if $g \in W^{s,p}(\Omega)$, then there is an extension $\tilde{g} \in W^{s,p}(\mathbb{R}^n, \mathbb{R}^N)$ with

$$\|\tilde{g}\|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|g\|_{W^{s,p}(\Omega)}. \tag{3.7}$$

Further, we extend F to a functional on $B(0,R) \times \mathbb{R}^{n \times m}$, denoted $\tilde{F}$, that satisfies

$$|\tilde{F}(x,z) - \tilde{F}(y,z)| \lesssim \Lambda |x-y|^\alpha (1 + |z|^2)^{\frac{q}{2}},$$
$$|\tilde{F}(x,z)| \lesssim \Lambda (1 + |z|^2)^{\frac{q}{2}}$$

by setting for $x \in B(0,R)$,

$$\tilde{F}(x,z) \coloneqq \inf_{y\in\Omega}\left(F(y,z) + \Lambda\left(1+|z|^2\right)^{\frac{q}{2}}|x-y|\right).$$

We will often not distinguish between $\tilde{F}$ and F and write F, assuming it to be defined on $B(0,R)$.

3 Whitney–Besicovitch coverings

In this section we define Whitney–Besicovitch coverings, which will be a key tool in our proof of regularity for pointwise minimisers. Recall that a family of dyadic cubes $\{\,Q_i\,\}_{i\in I}$ with mutually disjoint interiors is called a Whitney-covering of Ω if

$$\bigcup_{i\in I} Q_i = \bigcup_{i\in I} 2Q_i = \Omega \qquad 5Q_i \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset.$$

Whitney–Besicovitch coverings are a refinement of Whitney covers that incorporate properties of Besicovitch coverings, in particular they allow us to control the multiplicity of the cover uniformly, and were introduced in [136]. A nice presentation of the theory is given in [137]. To be precise, we define:

Definition 3.4. A family of cubes $\{\,K_i\,\}_{i\in I}$ is called a Whitney–Besicovitch-covering (WB-covering) of Ω if there is a triple (δ, M, ε) of positive numbers such that

$$\bigcup_{i\in I} \frac{1}{1+\delta} K_i = \bigcup_{i\in I} K_i = \Omega, \tag{3.8}$$

$$\sum_{i\in I} \chi_{K_i} \leq M, \tag{3.9}$$

$$K_i \cap K_j \neq \emptyset \Rightarrow |K_i \cap K_j| \geq \varepsilon \max(|K_i|, |K_j|). \tag{3.10}$$

The existence of a Whitney-covering for Ω is classical. The refinement to a WB-covering can be found in [137]:

Theorem 10 (cf. Theorem 3.15, [137])**.** *Let Ω be an open subset of $\mathbb{R}^n$ with non-empty complement. Let $\{\,Q_i\,\}$ be a family of cubes which are a Whitney covering of Ω. Then the cubes $K_i \coloneqq \left(1+\frac{1}{6}\right)Q_i$ are a WB-covering of Ω with $\delta \coloneqq \frac{1}{6}$, $\varepsilon \coloneqq \frac{1}{14^n}$ and $M \leq 6^n - 4^n + 1$. Moreover, for this covering $|K_i|^{\frac{1}{n}} \sim \operatorname{dist}(K_i, \partial\Omega)$ with the implicit constants only depending on δ, ε and n.*

It will be of crucial importance to us that there exists a partition of unity associated to a WB-covering.

Theorem 11 (cf. Theorem 3.19, [137]). *Suppose the cubes $\{ K_i \}_{i\in I}$ form a WB-covering of Ω with constants (δ, M, ε). Then there is a family $\{ \psi_i \}_{i\in I}$ of infinitely differentiable functions that form a partition of unity on Ω with the following properties:*

$$\begin{aligned}
&\operatorname{supp}(\psi_i) \subset \frac{1+\frac{\delta}{2}}{1+\delta} K_i, \\
&\psi_i(x) \geq \frac{1}{M} \quad \text{for } x \in \frac{1}{1+\delta} K_i, \\
&|\mathrm{D}\psi_i| \leq c \frac{1}{|K_i|^{\frac{1}{n}}},
\end{aligned}$$

for all $i \in I$.

4 Properties of the relaxed and the regularised functional

In this section, we collect some results regarding the relaxed functional $\overline{\mathscr{F}}$ and its relation to a regularised version of $\mathscr{F}(\cdot)$. For local versions of $\overline{\mathscr{F}}(\cdot)$ these results are well-known, c.f. Section 6 in [153] and [79]. We remark that we will re-visit many of the results in this section in a more general set-up in Chapter 4. Define for $u \in W^{1,p}(\Omega)$

$$\mathscr{F}_\varepsilon(u) \coloneqq \begin{cases} \int_\Omega F(x, \mathrm{D}u)\,\mathrm{d}x + \varepsilon \int_\Omega |\mathrm{D}u|^q\,\mathrm{d}x & \text{if } u \in W^{1,q}(\Omega) \\ \infty & \text{else.} \end{cases}$$

Write $F_\varepsilon(x,z) \coloneqq F(x,z) + \varepsilon|z|^q$. If $F(\cdot, z)$ is measurable for every $z \in \mathbb{R}^{N\times n}$ and $F(x,\cdot)$ is continuously differentiable for almost every $x \in \Omega$, it is easy to see that the same properties apply to $F_\varepsilon(x,z)$. Moreover, if F satisfies (H1)–(H3), then F_ε satisfies the bounds

$$\nu(\mu^2 + |z|^2 + |w|^2)^{\frac{p-2}{2}} \leq \frac{F_\varepsilon(x,z) - F_\varepsilon(x,w) - \partial_z F_\varepsilon(x,w)\cdot(z-w)}{|z-w|^2} \tag{3.11}$$

$$|F_\varepsilon(x,z)| \leq (\Lambda+\varepsilon)(1+|z|^2)^{\frac{q}{2}} \tag{3.12}$$

$$|F_\varepsilon(x,z_1) - F_\varepsilon(x,z_2)| \lesssim |z_1 - z_2|(\Lambda+\varepsilon)(1+|z_1|^2+|z_2|^2)^{\frac{q-1}{2}} \tag{3.13}$$

$$|F_\varepsilon(x,z) - F_\varepsilon(y,z)| \leq \Lambda |x-y|^\alpha (1^2+|z|^2)^{\frac{q}{2}}, \tag{3.14}$$

for any $z, z_1, z_2 \in \mathbb{R}^{N\times n}$. Further, note that if F satisfies (H7), then so does F_ε.

Minimisers of $\mathscr{F}_\varepsilon(\cdot)$ and $\overline{\mathscr{F}}(\cdot)$ are related as follows:

Lemma 3.5 (c.f. Lemma 6.4. in [153]). *Let $g \in W^{1,q}(\Omega)$. Suppose F satisfies* (H1) *and* (H2). *Assume u is a relaxed minimiser of $\mathscr{F}$ in the class $W^{1,p}_g(\Omega)$ and u_ε is the pointwise minimiser of $\mathscr{F}_\varepsilon$ in the class $W^{1,q}_g(\Omega)$. Then $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$ as $\varepsilon \to 0$. Moreover, up to passing to a subsequence, $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$.*

Proof. Note that by our assumptions $f \cdot u \in L^1(\Omega)$. Hence existence and uniqueness of u_ε follows from the direct method and strict convexity, respectively. We further note that

$$\overline{\mathscr{F}}(u) \leq \liminf_{\varepsilon\to 0} \mathscr{F}(u_\varepsilon) \leq \liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon).$$

To prove the reverse implication note that for any $v \in W^{1,q}_g(\Omega)$,

$$\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(v) = \mathscr{F}(v) = \overline{\mathscr{F}}(v).$$

The last equality applies since $\mathscr{F}$ is continuous on $W^{1,q}_g(\Omega)$ and consequently, $\overline{\mathscr{F}} = \mathscr{F}$ on $W^{1,q}_g(\Omega)$. By definition of $\overline{\mathscr{F}}(\cdot)$ the inequality above extends to all $v \in W^{1,p}_g(\Omega)$. In particular, it holds with the choice $v = u$. Thus $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$.

Using (H4) we may extract a (non-relabelled) subsequence of u_ε so that $u_\varepsilon \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ for some $v \in W^{1,p}(\Omega)$. Note that,

$$\overline{\mathscr{F}}(v) \leq \liminf \mathscr{F}(u_\varepsilon) \leq \liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) = \overline{\mathscr{F}}(u).$$

Thus v is a relaxed minimiser of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$.

Using (H1), we find for $w_1, w_2 \in W^{1,q}_g(\Omega)$

$$\begin{aligned}
&\overline{\mathscr{F}}\left(\frac{w_1+w_2}{2}\right) - \lambda \int_\Omega \left(\mu^2 + \left|\frac{\mathrm{D}w_1 + \mathrm{D}w_2}{2}\right|^2\right)^{\frac{p}{2}} \,\mathrm{d}x \\
\leq &\overline{\mathscr{F}}(w_1) + \overline{\mathscr{F}}(w_2) - \lambda \int_\Omega \left(\mu^2 + |\mathrm{D}w_1|^2\right)^{\frac{p}{2}} + \left(\mu^2 + |\mathrm{D}w_1|^2\right)^{\frac{p}{2}} \,\mathrm{d}x
\end{aligned}$$

Re-arranging and using the fact that there is $c > 0$ such that for $a, b \in \mathbb{R}^{N\times n}$,

$$\left(\mu^2 + \left|\frac{a+b}{2}\right|^2\right)^{\frac{p}{2}} + c|a-b|^p \leq \left(\mu^2 + |a|^2\right)^{\frac{p}{2}} + \left(\mu^2 + |b|^2\right)^{\frac{p}{2}},$$

we deduce for some $c > 0$,

$$\overline{\mathscr{F}}\left(\frac{w_1+w_2}{2}\right) + c\|\mathrm{D}w_1 - \mathrm{D}w_2\|^p_{L^p(\Omega)} \leq \frac{1}{2}\left(\overline{\mathscr{F}}(w_1) + \overline{\mathscr{F}}(w_2)\right). \tag{3.15}$$

Using the definition of $\overline{\mathscr{F}}(\cdot)$ and weak lower semicontinuity of norms, we see that this estimate extends to $w_1, w_2 \in W^{1,p}_g(\Omega)$. In particular, $\overline{\mathscr{F}}(\cdot)$ is convex and so $u = v$. Moreover, making the choice $w_1 \coloneqq u, w_2 \coloneqq u_\varepsilon$ in the estimate and using weak lower semicontinuity of $\overline{\mathscr{F}}(\cdot)$, as well as that $\overline{\mathscr{F}}(u_\varepsilon) \to \overline{\mathscr{F}}(u)$, shows that $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$. □

A useful criterion for establishing the equality $\overline{\mathscr{F}}(u) = \mathscr{F}(u)$ is the following:

Lemma 3.6 (cf. [35])**.** *Let $1 < p \leq q < \infty$. For $u \in W^{1,p}_g(\Omega)$ with $\mathscr{F}(u) < \infty$, we have $\overline{\mathscr{F}}(u) = \mathscr{F}(u)$ if and only if there is a sequence $\{u_k\} \in W^{1,q}_g(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\mathscr{F}(u_\varepsilon) \to \mathscr{F}(u)$ as $\varepsilon \to 0$.*

We close this section by showing that relaxed minimisers are very weak solutions of the Euler–Lagrange system. We first recall a Lemma from [66], see also [45].

Lemma 3.7. *Suppose F satisfies* (H1) *and* (H2)*. Then*

$$\partial_z F(z) \cdot z \gtrsim |z|^q + |\partial_z F(z)|^{q'} - 1.$$

Proof. Consider the Fenchel-conjugate of the partial integrand $F(x, \cdot)$,

$$F^*(x, \xi) \coloneqq \sup_{\zeta \in \mathbb{R}^{n\times m}} \left(\xi \cdot \zeta - F(x, \zeta)\right) \qquad (x, \xi) \in \Omega \times \mathbb{R}^{n\times m}.$$

The Fenchel–Young inequality in the extremal case yields that

$$\partial_z F(z) \cdot z = F(x, z) + F^*(z, \partial_z F(x, z)).$$

To conclude, note by direct calculation that $F^*(x, \partial_z F(x, z)) \gtrsim |\partial_z F(x, \xi)|^{q'} - 1$. □

We adapt an argument from [66].

Lemma 3.8. *Let $1 < p \leq q \leq \frac{np}{n-p}$ and $\Omega \subset \mathbb{R}^n$. Let $g \in W^{1,q}(\Omega)$ and suppose F satisfies* (H1) *and* (H2)*. Let u be a relaxed minimiser of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$. Then $\partial_z F(x, \mathrm{D}u) \in L^{q'}(\Omega)$, and*

$$\int_\Omega \partial_z F(x, \mathrm{D}u) \cdot \mathrm{D}\phi - f\phi \,\mathrm{d}x = 0 \qquad \forall \phi \in W^{1,q}_0(\Omega).$$

Proof. By the direct method we obtain $u_\varepsilon \in W^{1,q}_g(\Omega)$ minimising $\mathscr{F}_\varepsilon(\cdot)$ in the class $W^{1,q}_g(\Omega)$. Denote $\sigma_\varepsilon \coloneqq \partial_z F_\varepsilon(\cdot, \mathrm{D}u_\varepsilon)$ and $\mu_\varepsilon \coloneqq |\mathrm{D}u_\varepsilon|^{q-2}\mathrm{D}u_\varepsilon$. Then u_ε satisfies the Euler–Lagrange system

$$\int_\Omega (\sigma_\varepsilon + \varepsilon\mu_\varepsilon) \cdot \mathrm{D}\phi - f\phi \,\mathrm{d}x = 0 \qquad \forall \phi \in W^{1,q}_0(\Omega).$$

Choose $\phi = u_\varepsilon - g$ and use Lemma 3.7, Hölder and Young's inequality to find that for any $\delta > 0$,

$$\begin{aligned}\int_\Omega |\mathrm{D}u_\varepsilon|^p + |\sigma_\varepsilon|^{q'} + \varepsilon|\mathrm{D}u_\varepsilon|^q \,\mathrm{d}x &\lesssim \int_\Omega (\sigma_\varepsilon + \varepsilon\mu_\varepsilon) \cdot \mathrm{D}u_\varepsilon \,\mathrm{d}x \\ &= \int_\Omega (\sigma_\varepsilon + \varepsilon\mu_\varepsilon) \cdot \mathrm{D}g + f \cdot (u_\varepsilon - g) \,\mathrm{d}x \\ &\leq \delta \left(\|\sigma_\varepsilon\|_{L^{q'}(\Omega)}^{q'} + \|u_\varepsilon\|_{L^q(\Omega)}^q + \varepsilon \|\mathrm{D}u_\varepsilon\|_{L^q(\Omega)}^q \right) \\ &\qquad + C(\delta) \left(1 + \|g\|_{L^q(\Omega)}^q + \|\mathrm{D}g\|_{L^q(\Omega)}^q + \|f\|_{L^{q'}(\Omega)}^{q'} \right).\end{aligned}$$

Noting that $\|u_\varepsilon\|_{L^q(\Omega)} \lesssim \|\mathrm{D}u_\varepsilon\|_{L^p(\Omega)}$ by our choice of q, after ensuring δ is sufficiently small we may rearrange to conclude

$$\limsup_{\varepsilon \searrow 0} \int_\Omega |\sigma_\varepsilon|^{q'} \,\mathrm{d}x \lesssim 1 + \int_\Omega \left(|f|^{q'} + |g|^q + |\mathrm{D}g|^q \right) \mathrm{d}x.$$

Since $\partial_z F(x,z)$ is Carathéodory and $\mathrm{D}u_\varepsilon \to \mathrm{D}u$ in measure by Lemma 3.5, we also have $\sigma_\varepsilon \to F_z(\cdot, \mathrm{D}u)$ in measure. Finally, note that $\sigma_\varepsilon + \varepsilon\mu_\varepsilon \rightharpoonup F_z(\cdot, \mathrm{D}u)$ in $L^1(\Omega)$, so the latter is row-wise divergence free and since it is also in $L^{q'}(\Omega)$, the result is established. □

5 Regularity of minimisers of the relaxed functional

The aim of this section is to prove the first part of Theorem 9 concerning relaxed minimisers. For the convenience of the reader we recall the statement:

Theorem 12. *Suppose F satisfies* (H1)–(H3) *with $1 < p \leq q < \frac{(n+\alpha)p}{n}$. If u is a relaxed minimiser of $\mathscr{F}(\cdot)$ in the class $W_g^{1,p}(\Omega)$, then $u \in W^{1,q}(\Omega)$. Moreover, for any $0 \leq \beta < \alpha$ there is $\gamma > 0$ such that we have the estimate*

$$\|u\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \overline{\mathscr{F}}(u) + \|g\|_{W^{1+\alpha,q}(\Omega)} + \|f\|_{L^{q'}(\Omega)} \right)^\gamma.$$

The key to proving the theorem is an a-priori estimate for minimisers of the regularised functional:

Lemma 3.9. *Suppose the assumptions of Theorem 9 hold. Let $v_\varepsilon + g$ be the minimiser of $\mathscr{F}_\varepsilon(\cdot)$ in the class $W_g^{1,q}(\Omega)$. Then for any $0 \leq \beta < \alpha$ there is $\gamma > 0$ such that the estimate*

$$\|v_\varepsilon + g\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \mathscr{F}(v_\varepsilon) + \|g\|_{W^{1+\alpha,q}(\Omega)} + \|f\|_{L^{q'}(\Omega)} \right)^\gamma$$

holds, with the implicit constant independent of ε and γ.

Theorem 9 is now a consequence of combining Lemma 3.9 and Lemma 3.5.

Proof of Theorem 9. Let $v_\varepsilon \in W^{1,q}_g(\Omega)$ be the minimiser of $\mathscr{F}_\varepsilon(\cdot)$ in the class $W^{1,q}_g(\Omega)$. For $0 \le \beta < \alpha$, by Lemma 3.9, $\{ v_\varepsilon \}$ is uniformly bounded in $W^{1,\frac{np}{n-\beta}}(\Omega)$. From Lemma 3.5 we know that $\mathscr{F}_\varepsilon(v_\varepsilon) \to \overline{\mathscr{F}}(u)$, and, up to passing to a subsequence, $v_\varepsilon \to u$ in $W^{1,p}(\Omega)$. Thus we may extract a non-relabelled subsequence such that $v_\varepsilon \rightharpoonup u$ weakly in $W^{1,\frac{np}{n-\beta}}(\Omega)$. This allows us to pass to the limit in Lemma 3.9 to conclude. □

We proceed to prove Lemma 3.9:

Proof of Lemma 3.9. By direct methods and strict convexity deriving from (3.11), we obtain a sequence of minimisers $\{v_\varepsilon\} \subset W^{1,q}_0(\Omega)$ that solve

$$\min_{v \in W^{1,q}_0(\Omega)} \mathscr{F}_\varepsilon(v+g)$$

Moreover, v_ε satisfies the Euler–Lagrange equation

$$\int_\Omega (\partial_z F_\varepsilon(x, \mathrm{D}v_\varepsilon + \mathrm{D}g)) \cdot \mathrm{D}\phi - f\phi \,\mathrm{d}x = 0 \qquad \forall \phi \in W^{1,q}_0(\Omega).$$

In particular, for any $v \in W^{1,q}_0(\Omega)$,

$$\int_\Omega (\partial_z F_\varepsilon(x, \mathrm{D}v_\varepsilon + \mathrm{D}g)) \cdot \mathrm{D}(v - v_\varepsilon) - f \cdot (v - v_\varepsilon) = 0.$$

Using this identity, we see for $v \in W^{1,q}_0(\Omega)$,

$$\begin{aligned}\mathscr{F}_\varepsilon(v+g) - \mathscr{F}_\varepsilon(v_\varepsilon + g) = &\int_\Omega F_\varepsilon(x, \mathrm{D}v + \mathrm{D}g) - F_\varepsilon(x, \mathrm{D}v_\varepsilon + \mathrm{D}g) - f \cdot (v - v_\varepsilon)\,\mathrm{d}x \\ &- \int_\Omega (\partial_z F_\varepsilon(x, \mathrm{D}v_\varepsilon + \mathrm{D}g)) \cdot \mathrm{D}(v - v_\varepsilon) - f \cdot (v - v_\varepsilon)\,\mathrm{d}x \\ &\gtrsim \|V_{\mu,p}(\mathrm{D}v + \mathrm{D}g) - V_{\mu,p}(\mathrm{D}v_\varepsilon + \mathrm{D}g)\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.16}$$

where to obtain the last line we have used (3.11) and Lemma 1.3.

Let $\rho_0 > 0$ and $\boldsymbol{n} \colon \mathbb{R}^n \to \mathbb{S}^{n-1}$ be so that the uniform cone property (3.5) holds. Possibly reducing ρ_0, assume, without loss of generality, that $\Omega + B_{3\rho_0}(x) \subset B(0,R)$ for all $x \in \Omega$. Here $B(0,R) \ni \Omega$ is the ball defined in Section 2. Given $x_0 \in \Omega$, let $0 \le \phi = \phi_{x_0,\rho_0} \le 1$ be a smooth cut-off supported in $B_{2\rho_0}(x_0)$ with $\phi(x) = 1$ in $B_{\rho_0}(x_0)$ and $|\mathrm{D}^k\phi(x)| \le C_k {\rho_0}^{-k}$ for some $C_k > 0$ and $k \in N$. For functions v_1, v_2 defined on $\mathbb{R}^k$ for some $k \in \mathbb{N}$ and $h \in \mathbb{R}^k$, introduce

$$T_h v_1 \coloneqq \phi v_{1,h} + (1-\phi) v_1$$

If $v_1, v_2 \colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$ then write in order to simplify notation

$$G_h(x, v_1, v_2) \coloneqq F_\varepsilon(x, v_{1,h} + v_2) - F_\varepsilon(x, v_1 + v_2).$$

We claim that for every $x_0 \in \mathbb{R}^n$ there is a constant $C = C(n, \rho_0, \Lambda, \Omega)$ such that for all $v \in W^{1,q}_0(\Omega)$

$$\begin{aligned}\sup_{h\in C_{\rho_0}(\theta_0,\boldsymbol{n}(x_0))} &\frac{\mathscr{F}_\varepsilon(T_h\tilde{v}+g) - \mathscr{F}_\varepsilon(v+g)}{|h|^\alpha}\\ \leq& C\left(1 + \|\mathrm{D}v\|^q_{L^q(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}\right).\end{aligned} \tag{3.17}$$

Here $\tilde{v}$ is the extension by 0 to $\mathbb{R}^n$ of v.

Let $v \in W^{1,q}_0(\Omega)$ and $h \in C_{\rho_0}(\theta_0, \boldsymbol{n}(x_0))$. We estimate:

$$\begin{aligned}&\mathscr{F}_\varepsilon(T_h\tilde{v}+g) - \mathscr{F}_\varepsilon(v+g)\\ =&\int_\Omega F_\varepsilon(x, T_h\mathrm{D}\tilde{v} + \mathrm{D}\phi(\tilde{v}_h - v) + \mathrm{D}g) - F_\varepsilon(x, T_h\mathrm{D}\tilde{v} + \mathrm{D}g)\,\mathrm{d}x\\ &\quad+\int_\Omega F_\varepsilon(x, T_h\mathrm{D}\tilde{v} + \mathrm{D}g) - F_\varepsilon(x, \mathrm{D}\tilde{v} + \mathrm{D}g)\,\mathrm{d}x - \int_\Omega f\cdot(T_h\tilde{v} - v)\,\mathrm{d}x\\ =:&A_1 + A_2 + A_3.\end{aligned}$$

Using (3.12), Hölder's inequality and recalling that $\tilde{g}$ denotes a $W^{1+\alpha,q}$-extension of g to $\mathbb{R}^n$, we find

$$\begin{aligned}|A_1| \leq&(\Lambda+\varepsilon)\int_\Omega |\mathrm{D}\phi\tau_h\tilde{v}|\Big(1 + |T_h\mathrm{D}\tilde{v} + \mathrm{D}g|^2 + |\mathrm{D}\phi\tau_h\tilde{v}|^2\Big)^{\frac{q-1}{2}}\,\mathrm{d}x\\ \leq&\|\mathrm{D}\phi\tau_h\tilde{v}\|_{L^q(B_{2\rho_0}(x_0))}\Big(1 + \|T_h\mathrm{D}\tilde{v}\|_{L^q(B_{2\rho_0}(x_0))}\\ &+ \|\mathrm{D}\phi\tau_h\tilde{v}\|_{L^q(B_{2\rho_0}(x_0))} + \|\mathrm{D}\tilde{g}\|^q_{L^q(B_{2\rho_0}(x_0))}\Big)^{q-1}.\end{aligned}$$

Recalling that as $\tilde{v} \in W^{1,q}_0(\Omega, \mathbb{R}^N)$,

$$\|\mathrm{D}\phi\tau_h\tilde{v}\|_{L^q(B_{2\rho_0}(x_0))} \lesssim |h|\|\mathrm{D}\tilde{v}\|_{L^q(B_{2\rho_0}(x_0))},$$

we conclude using (3.7) and the definition of T_h,

$$|A_1| \lesssim |h|\Big(1 + \|\mathrm{D}v\|^q_{L^q(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)}\Big).$$

We now turn to A_2. By convexity of $F_\varepsilon(\cdot)$,

$$F_\varepsilon(x, T_h\mathrm{D}\tilde{v} + \mathrm{D}g) - F_\varepsilon(x, \mathrm{D}\tilde{v} + \mathrm{D}g)$$

$$\begin{aligned}
&\leq (1-\phi)F_\varepsilon(x, \mathrm{D}v + \mathrm{D}g) + \phi F_\varepsilon(x, \mathrm{D}\tilde{v}_h + \mathrm{D}g) - F_\varepsilon(\mathrm{D}v + \mathrm{D}g)\\
&= \phi G_h(x, \mathrm{D}\tilde{v}, \mathrm{D}g).
\end{aligned}$$

In particular, using a change of coordinates and since in $B_{3\rho_0} \setminus \Omega$, $\mathrm{D}\tilde{v}_h = \mathrm{D}\tilde{v} = 0$,

$$\begin{aligned}
|A_2| &\leq \int_{B_{2\rho_0}(x_0)} \phi G_h(x, \mathrm{D}\tilde{v}, \mathrm{D}g)\,\mathrm{d}x\\
&= \int_{B_{2\rho_0}(x_0)+h} \phi(x-h) F_\varepsilon(x-h, \mathrm{D}\tilde{v} + \mathrm{D}\tilde{g}_{-h})\,\mathrm{d}x - \int_{B_{2\rho_0}(x_0)} \phi(x) F_\varepsilon(x, \mathrm{D}\tilde{v} + \mathrm{D}\tilde{g})\,\mathrm{d}x\\
&= \int_{B_{3\rho_0}(x_0)} \tau_{-h}\phi F_\varepsilon(x-h, \mathrm{D}\tilde{v} + \mathrm{D}\tilde{g}_{-h}) + \phi(x) G_{-h}(x-h, \mathrm{D}\tilde{g}, \mathrm{D}\tilde{v})\,\mathrm{d}x\\
&\qquad + \int_{B_{2\rho_0}(x_0)} \phi(x)\Big(F_\varepsilon(x-h, \mathrm{D}\tilde{v} + \mathrm{D}\tilde{g}) - F_\varepsilon(x, \mathrm{D}\tilde{v} + \mathrm{D}\tilde{g})\Big)\,\mathrm{d}x.
\end{aligned}$$

Using (3.12), (3.14), the regularity of ϕ and g as well as (3.7) to estimate each term in turn, we conclude

$$A_2 \lesssim |h|^\alpha \left(1 + \|\mathrm{D}v\|^q_{L^q(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)}\right).$$

Finally, using Young's inequality and Sobolev embedding,

$$|A_3| \leq \int_\Omega |f \cdot (\phi \tau_h v)| \lesssim \|f\|_{L^{q'}(\Omega)} \|v - v_h\|_{L^q(\Omega)} \lesssim h \left(\|f\|^{q'}_{L^{q'}(\Omega)} + \|\mathrm{D}v\|^q_{L^q(\Omega)}\right)$$

This proves the claim.

We are now ready to finish the proof. Note that $T_h\tilde{v}_\varepsilon \in W^{1,q}_0(\Omega)$ if $h \in C_{\rho_0}(\theta_0, \boldsymbol{n}(x_0))$. Thus combining (3.16) and (3.17) with the choice $v = T_h\tilde{v}_\varepsilon$ we find for $x_0 \in \mathbb{R}^n$,

$$\begin{aligned}
&\sup_{h \in C_{\rho_0}(\theta_0, \boldsymbol{n}(x_0))} |h|^{-\alpha} \|V_{\mu,p}(\mathrm{D}v_\varepsilon + \mathrm{D}g) - V_{\mu,p}(\mathrm{D}T_h\tilde{v}_{\varepsilon,h} + \mathrm{D}g)\|^2_{L^2(B_{\rho_0}(x_0))}\\
&\lesssim 1 + \|\mathrm{D}w\|^q_{L^q(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}.
\end{aligned}$$

Assume first $p > 2$. Then we deduce after recalling the definition of T_h, using the triangle inequality and regularity of $\tilde{g}$,

$$\begin{aligned}
&\sup_{h \in C_{\rho_0}(\theta_0, \boldsymbol{n}(x_0))} |h|^{-\alpha} \|\mathrm{D}(\tilde{v}_{\varepsilon,h} + \tilde{g}_h) - \mathrm{D}(v_\varepsilon + g)\|^p_{L^p(B_{\rho_0}(x_0))} \qquad (3.18)\\
&\lesssim 1 + \|\mathrm{D}w\|^q_{L^q(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}.
\end{aligned}$$

Using the characterisation of Besov spaces (1.11), we conclude for all $x_0 \in \mathbb{R}^n$,

$$[\mathrm{D}v_\varepsilon]^p_{\frac{\alpha}{p},p,B_{\rho_0}(x_0)} \lesssim \left(1 + \|\mathrm{D}v_\varepsilon\|^q_{L^q(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}\right). \qquad (3.19)$$

Covering Ω by a finite number of balls of radius ρ_0 and using (1.13), we deduce

$$\|v_\varepsilon\|^p_{B_\infty^{1+\frac{\alpha}{p},p}(\Omega)} \lesssim \left(1 + \|v_\varepsilon\|^q_{W^{1,q}(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}\right). \tag{3.20}$$

Recall now that $B_\infty^{1+\frac{\alpha}{p},p}(\Omega)$ embeds continuously into $W^{1,\frac{np}{n-\beta}}(\Omega)$ for any $\beta < \alpha$ by Theorem 1. Choose β such that $q < \frac{p(\beta+n)}{n}$ and use (1.9) with $\theta = \frac{np}{\beta}\left(\frac{1}{p} - \frac{1}{q}\right)$ to see

$$\|Dv_\varepsilon\|_{L^q(\Omega)} \leq \|Dv_\varepsilon\|^{1-\theta}_{L^p(\Omega)} \|Dv_\varepsilon\|^\theta_{L^{\frac{np}{n-\beta}}(\Omega)}. \tag{3.21}$$

As $q < \frac{(n+\beta)p}{n}$, it follows that $q\theta < p$. Thus using (3.21) in (3.20), we find after using Young's inequality,

$$\|v_\varepsilon\|^p_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim 1 + \frac{1}{2}\|v_\varepsilon\|^p_{W^{1,\frac{np}{n-\beta}}(\Omega)} + C(\theta)\|v_\varepsilon\|^{\frac{\theta q}{(\theta q - p)}}_{W^{1,p}(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}.$$

Re-arranging and recalling (H5) we obtain the desired conclusion.

Assume now $p < 2$. In this case (3.18) does not hold in this form any more. Instead arguing as before up to this point and using Hölder, we find

$$\begin{aligned}
&\|\mathrm{D}v_\varepsilon + \mathrm{D}g - \mathrm{D}\tilde{v}_{\varepsilon,h} - \mathrm{D}_h\tilde{g}\|^2_{L^p(\Omega)} \left(\int_\Omega |\mathrm{D}v_\varepsilon|^p + |\mathrm{D}\tilde{v}_{\varepsilon,h}|^p + |\mathrm{D}g|^p + |\mathrm{D}\tilde{g}_h|^p \,\mathrm{d}x\right)^{\frac{p-2}{p}} \\
\lesssim &1 + \|\mathrm{D}w\|^q_{L^q(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}.
\end{aligned}$$

In particular, proceeding as before we obtain

$$\begin{aligned}
\|v_\varepsilon\|^2_{B_\infty^{1+\frac{\alpha}{2}}(\Omega)} \lesssim &\left(1 + \|v_\varepsilon\|^q_{W^{1,q}(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}\right) \\
&\times \left(\|v_\varepsilon\|_{W^{1,p}(\Omega)} + \|g\|_{W^{1,p}(\Omega)}\right)^{\frac{2-p}{p}}.
\end{aligned}$$

It is now straightforward to check that applying the interpolation argument from the case $p \geq 2$ gives the desired conclusion. □

It is useful to note that in fact our argument can be used to obtain a local version of the a-priori estimate. For this we proceed as follows, giving details only for the case $p \geq 2$ as the case $p < 2$ can be dealt with similarly: Let $0 < r < s < t < 2r$ with $|s - t| < \rho_0 < 1$. Fix $y \in \Omega$. Given $x_0 \in \Omega \cap B_s(y)$, let $0 \leq \phi = \phi_{x_0,\rho_0} \leq 1$ be a smooth cut-off supported in $B_{|s-t|}(x_0)$ with $\phi(x) = 1$ in $B_{|s-t|/3}(x_0)$ and $|\mathrm{D}^k\phi(x)| \leq C_k|s-t|^{-k}$

for some $C_k > 0$, independent of x_0, and $k \in N$. Proceeding as before, we obtain for $h \in C_{|s-t|/3}(\theta_0, \boldsymbol{n}(x_0))$:

$$\begin{aligned}&\sup_{h\in C_{|s-t|/3}(\theta_0,\boldsymbol{n}(x_0))} |h|^{-\alpha}\|\mathrm{D}(\tilde{v}_{\varepsilon,h}+\tilde{g}_h)-\mathrm{D}(v_\varepsilon+g)\|^p_{L^p(B_{|s-t|/3}(x_0))}\\ \lesssim&|s-t|^{-1}\left(1+\|\mathrm{D}w\|^q_{L^q(\Omega\cap B_{|s-t|}(x_0))}+\|g\|^q_{W^{1+\alpha,q}(\Omega\cap B_{|s-t|}(x_0))}+\|f\|^{q'}_{L^{q'}(\Omega\cap B_{|s-t|}(x_0))}\right).\end{aligned}$$

Using (1.12) we deduce

$$\begin{aligned}\|D(\tilde{v}_{\varepsilon,h}+\tilde{g}_h)\|^p_{B^{\alpha/p,p}_\infty(B_{|s-t|/3}(x_0))} \lesssim&|s-t|^{-1-\alpha/p}\Big(1+\|\mathrm{D}w\|^q_{L^q(\Omega\cap B_{|s-t|}(x_0))}\\ &+\|g\|^q_{W^{1+\alpha,q}(\Omega\cap B_{|s-t|}(x_0))}+\|f\|^{q'}_{L^{q'}(\Omega\cap B_{|s-t|}(x_0))}\Big).\end{aligned}$$

Now cover $B_s(y)$ with balls of radius $|s-t|/6$. Note that it is possible to do so with $c(n)|s-t|^{-n}$ balls and that moreover the union of these balls is contained in $B_t(y)$. Thus using (1.13) we find

$$\begin{aligned}&\|D\tilde{v}_{\varepsilon,h}\|^p_{B^{\alpha/p,p}_\infty(B_s(y))}\\ \lesssim&|s-t|^{-1-2\alpha/p}\left(1+\|\mathrm{D}w\|^q_{L^q(\Omega\cap B_t(y))}+\|g\|^q_{W^{1+\alpha,q}(\Omega\cap B_t(y))}+\|f\|^{q'}_{L^{q'}(\Omega\cap B_t(y))}\right).\end{aligned}$$

Arguing as before we hence arrive at

$$\begin{aligned}\|v_\varepsilon\|^p_{W^{1,\frac{np}{n-\beta}}(\Omega\cap B_s(y))} \lesssim&\frac{1}{2}\|v_\varepsilon\|^p_{W^{1,\frac{np}{n-\beta}}(\Omega\cap B_t(y))}+|t-s|^{-\tilde{\tau}}\Bigg(1+C(\theta)\|v_\varepsilon\|^{\frac{\theta q}{(\theta q-p)}}_{W^{1,p}(\Omega\cap B_t(y))}\\ &+\|g\|^q_{W^{1+\alpha,q}(\Omega\cap B_t(y))}+\|f\|^{q'}_{L^{q'}(\Omega\cap B_t(y))}\Bigg)\end{aligned}$$

with $\tilde{\tau}=(1+2\alpha/p)\frac{\theta}{\theta q-p}$. Employing Lemma 1.7 we conclude the desired estimate:

$$\begin{aligned}\|v_\varepsilon\|_{W^{1,\frac{np}{n-\beta}}(\Omega\cap B_r(t))} \lesssim&r^{-\tilde{\tau}}\Bigg(1+\int_{B_{2r}(y)\cap\Omega}F_\varepsilon(x,D(g+v_\varepsilon))\,\mathrm{d}x\\ &+\|g\|_{W^{1+\alpha,q}(\Omega\cap B_{2r}(y))}+\|f\|_{L^{q'}(\Omega\cap B_{2r}(y))}\Bigg)^\gamma.\end{aligned}$$

6 An improved estimate for autonomous functionals

Restricting to autonomous integrands $F(x,y,z)=F(z)$, the previous section shows that if $q<\frac{(n+1)p}{n}$ then relaxed minimisers are $W^{1,q}(\Omega)$-regular. The reader will notice

that this leaves a gap to the conditions we cited for autonomous integrands. We recall, in particular, the condition $q < \frac{np}{n-1}$ obtained in [44] for convex autonomous integrands under natural (p,q)-growth conditions with $2 \leq p$. The aim of this section is to close this gap. While the argument in [44] relies on convex duality theory, we enhance our previous argument with a trick first employed in [20] and used in the context of vector-valued (p,q)-growth in [186]. To be precise, optimising the choice of cut-off function in the argument in dependence on the minimiser we can employ the embedding from Theorem 1 on spheres instead of balls. Due to the dimensional dependence of the constants in the embedding this will give an improvement.

Noting (H5) and adding a constant to F if necessary, we may assume that $F(z) \geq 0$ for all $z \in \mathbb{R}^{N\times n}$. We also assume that $f = 0$. The assumption that $f = 0$ can be easily removed, but in order to clearly show the changes as compared to the previous section, we focus on the case $f = 0$.

Our goal is to prove the following theorem, which is a global analogue of the main result in [44]:

Theorem 13. *Suppose Ω is a Lipschitz domain. Let $g \in W^{2,q}(\Omega)$, $f = 0$, and $2 \leq p < n-1$. Assume $F \equiv F(z)$ satisfies* (H1) *and* (H2) *with $1 < p \leq q < \frac{np}{n-1}$. If u is a relaxed minimiser of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$, then $u \in W^{1,q}(\Omega)$. Moreover, for any $0 \leq \beta < 1$ there is $\gamma > 0$ such that we have the estimate*

$$\|u\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \overline{\mathscr{F}}(u) + \|g\|_{W^{2,q}(\Omega)}\right)^{\gamma}.$$

Remark 3.10. It would be desirable to remove the constraint $p < n-1$, which does not occur in the interior case. However it is not clear to the author how to do so.

The key to proving the theorem is again an appropriate a-priori estimate for minimisers of the regularised integrand:

Lemma 3.11. *Suppose the assumptions of Theorem 13 hold. Let $v_\varepsilon + g$ be the minimiser of $\mathscr{F}_\varepsilon(\cdot)$ in the class $W^{1,q}_g(\Omega)$. Then for any $0 \leq \beta < 1$ there is $\gamma > 0$ such that the estimate*

$$\|v_\varepsilon + g\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \mathscr{F}(v_\varepsilon) + \|g\|_{W^{2,q}(\Omega)}\right)^{\gamma}$$

holds, with the implicit constant independent of ε and γ.

As before, Theorem 13 is now a consequence of combining Lemma 3.11 and Lemma 3.5.

Proof of Theorem 13. Let $v_\varepsilon + g \in W^{1,q}_g(\Omega)$ be the minimiser of $\mathscr{F}_\varepsilon(\cdot)$ in the class $W^{1,q}_g(\Omega)$. From Lemma 3.5 we know that $\mathscr{F}_\varepsilon(v_\varepsilon + g) \to \overline{\mathscr{F}}(u)$ andm up to passing to a subsequencem $v_\varepsilon + g \to u$ in $W^{1,p}(\Omega)$. For $0 \leq \beta < 1$ by Lemma 3.11, $\{ v_\varepsilon \}$ is thus uniformly bounded in $W^{1,\frac{np}{n-\beta}}(\Omega)$. In particular we may extract a non-relabelled subsequence such that $v_\varepsilon \rightharpoonup u$ weakly in $W^{1,\frac{np}{n-\beta}}(\Omega)$. This allows to pass to the limit in Lemma 3.11 to conclude. □

We next aim to prove Lemma 3.9 by combining the arguments from the previous section with the idea of applying embedding theorems on spheres instead of balls from [186]. We begin by recording a version of the key lemma, first found in [20], adapted to our purposes.

Lemma 3.12. *Fix $n \geq 2$ and let $t > 1$. For given $0 < \rho < \sigma < \infty$ with $\sigma - \rho < 1$ and $w \in L^1(B_\sigma)$ consider*

$$J(\rho, \sigma, w) \coloneqq \inf \left\{ \int_{B_\sigma} |w|(|\mathrm{D}\phi| + |\mathrm{D}\phi|^t)\,\mathrm{d}x \colon \phi \in C^1_0(B_\sigma), \phi \geq 0, \phi = 1 \text{ in } B_\rho \right\}.$$

Then for every $\delta \in (0,1)$:

$$J(\rho, \sigma, w) \leq (\sigma - \rho)^{-t-1/\delta} \left(\int_\rho^\sigma \left(\int_{\partial B_r} |w| \right)^\delta \right)^{\frac{1}{\delta}}.$$

Moreover, given $\varepsilon > 0$, if $|w| \geq 1$ in B_σ, we can choose a radial symmetric $\tilde{\phi} \in C^1_0(B_\sigma)$ with $\tilde{\phi} \geq 0$, $\tilde{\phi} = 1$ in B_σ, such that

$$\int_{B_\sigma} |w|(|\mathrm{D}\tilde{\phi}| + |\mathrm{D}\tilde{\phi}|^t)\,\mathrm{d}x \leq (\sigma - \rho)^{-t-1/\delta} \left(\int_\rho^\sigma \left(\int_{\partial B_r} |w| \right)^\delta \right)^{\frac{1}{\delta}} + \varepsilon,$$

and, in addition, for every $\rho \leq r_1 < r_2 \leq \sigma$,

$$\frac{|\tilde{\phi}(r_2) - \tilde{\phi}(r_1)|}{(r_2 - r_1)} \leq (\sigma - \rho)^{-t-1/\delta} \left(\int_\rho^\sigma \left(\int_{\partial B_r} |w| \right)^\delta \right)^{\frac{1}{\delta}} + \varepsilon.$$

Proof. The estimate follows by considering appropriate radially symmetric cut-off functions. Indeed, for any $\varepsilon \geq 0$,

$$J(\rho, \sigma, w) \leq \inf \left\{ \int_\rho^\sigma (|\phi'| + |\phi'|^t) \left(\int_{\partial B_r} |w| + \varepsilon \right) \mathrm{d}r \colon \phi \in C^1([\rho, \sigma]), \phi(\rho) = 1, \phi(\sigma) = 0 \right\}$$

$=: J_{1d,\varepsilon}$.

Note that since $w \in L^1(B_\sigma)$, and using a standard approximation argument, we can replace $\phi \in C^1([\rho,\sigma])$ with $\phi \in W^{1,\infty}(\rho,\sigma)$ in the definition of $J_{1d,\varepsilon}$. Set

$$\tilde{\phi}(r) \coloneqq 1 - \left(\int_\rho^\sigma b(r)^{-1}\,\mathrm{d}r\right)^{-1} \int_\rho^r b(r)^{-1}\,\mathrm{d}r \quad \text{where } b(r) \coloneqq \int_{\partial B_r} |w|\,\mathrm{d}\mathscr{H}^{n-1} + \varepsilon.$$

Clearly $\tilde{\phi} \in W^{1,\infty}(\rho,\sigma)$ and $\tilde{\phi}(\rho) = 1$, $\tilde{\phi}(\sigma) = 0$. Hence,

$$J_{1d,\varepsilon} \leq \int_\rho^\sigma (|\tilde{\phi}'| + |\tilde{\phi}'|^t) b(r)\,\mathrm{d}r = \frac{\sigma-\rho}{\int_\rho^\sigma b(r)^{-1}\,\mathrm{d}r} + \frac{\int_\rho^\sigma b(r)^{1-t}}{\left(\int_\rho^\sigma b(r)^{-1}\right)^t} =: I + II.$$

Note that by Hölder's inequality for any $s > 1$,

$$(\sigma-\rho) = \int_\rho^\sigma \left(\frac{b}{b}\right)^{\frac{s-1}{s}} \leq \left(\int_\rho^\sigma b^{s-1}\right)^{\frac{1}{s}} \left(\int_\rho^\sigma b^{-1}\right)^{\frac{s-1}{s}},$$

Thus for any $\delta > 0$,

$$I \leq (\sigma-\rho)^{-(1+\delta)/\delta} \left(\int_\rho^\sigma \left(\int_{\partial B_r} |w|\,\mathrm{d}\mathscr{H}^{n-1} + \varepsilon\right)^\delta\right)^{\frac{1}{\delta}}.$$

We now focus on the second term. We find using Jensen's inequality

$$\int_\sigma^\rho b(r)^{1-t} \leq (\sigma-\rho)^{2-t} \left(\int_\sigma^\rho b(r)^{-1}\right)^{t-1}.$$

Hence we have for any $\delta > 0$, upon also re-using the estimate for the term I,

$$\begin{aligned} II \leq& (\sigma-\rho)^{2-t} \left(\int_\sigma^\rho b(r)^{-1}\right)^{-1} \\ \leq& (\sigma-\rho)^{1-t-(1+\delta)/\delta} \left(\int_\rho^\sigma b(r)^\delta\right)^{1/\delta} \end{aligned}$$

Collecting estimates and letting $\varepsilon \to 0$ the result follows.

For the "moreover" part, note that since $|w| \geq 1$,

$$\frac{|\tilde{\phi}(r_1) - \tilde{\phi}(r_2)|}{r_2 - r_1} = \frac{\int_{r_1}^{r_2} b(r)^{-1}\,\mathrm{d}r}{(r_2-r_1)\int_\rho^\sigma b(r)^{-1}\,\mathrm{d}r} \leq \frac{1}{\int_\rho^\sigma b(r)^{-1}\,\mathrm{d}r}.$$

The claim now follows estimating as for the term I. □

We note the following consequence of the previous lemma. Set $\xi \coloneqq \frac{n-1}{n-2}$ if $n > 2$, $\xi \coloneqq \infty$ if $n = 2$. Suppose $w \in L^q(B_\sigma)$, $t > 1$ and $q < \frac{np}{n-1}$. Then for any $\delta \in (0, 1]$,

$$J(\sigma, \rho, |w|^q) \leq (\sigma - \rho)^{-t-1/\delta} \left(\int_\rho^\sigma \|w\|^{q\delta}_{L^q(\partial B_r)} \right)^{\frac{1}{\delta}}.$$

By Hölder's inequality we find

$$\begin{aligned}\left(\int_\rho^\sigma \|w\|^{q\delta}_{L^q(\partial B_r)} \,\mathrm{d}r \right)^{\frac{1}{\delta}} &\leq \left(\int_\rho^\sigma \|w\|^{\theta q\delta}_{L^p(\partial B_r)} \|w\|^{(1-\theta)q\delta}_{L^{\xi p}(\partial B_r)} \,\mathrm{d}r \right)^{\frac{1}{\delta}} \\ &\leq \left(\int_\rho^\sigma \|w\|^{\theta q\delta s/(s-1)}_{L^p(\partial B_r)} \,\mathrm{d}r \right)^{\frac{s-1}{s\delta}} \left(\int_\rho^\sigma \|w\|^{(1-\theta)q\delta s)}_{L^{\xi p}(\partial B_r)} \,\mathrm{d}r \right)^{\frac{1}{\delta s}},\end{aligned}$$

with

$$\frac{\theta}{p} + \frac{1-\theta}{\xi p} = \frac{1}{q}, \qquad s > 1.$$

We now make the admissible choice

$$\delta \coloneqq \frac{p}{q}, \qquad s \coloneqq \frac{1}{1-\theta},$$

to find

$$J(\sigma, \rho, |w|^q) \leq (\sigma - \rho)^{-t-q/p} \left(\int_{B_\sigma \setminus B_\rho} |w|^p \right)^{\frac{\xi}{\xi-1}(1-\frac{q}{\xi p})} \left(\int_\sigma^\rho \|w\|^p_{L^{\xi p}(\partial B_r)} \right)^{\frac{\xi}{\xi-1}\left(\frac{q}{p}-1\right)}. \tag{3.22}$$

Here we understand that $\frac{\infty}{\infty-1} = 1$.

By the same argument if $|w| \geq 1$ in B_σ and $\varepsilon > 0$, we can pick a radial test function $\tilde{\phi}$ such that for almost every $\rho \leq r_1 < r_2 \leq \sigma$,

$$\begin{aligned}&\int_{B_\sigma} |w|^q (|\mathrm{D}\tilde{\phi}| + |\mathrm{D}\tilde{\phi}|^p) \,\mathrm{d}x + \frac{|\tilde{\phi}(r_2) - \tilde{\phi}(r_1)|}{|r_2 - r_1|} \\ \leq &(\sigma - \rho)^{-p-q/p} \left(\int_{B_\sigma \setminus B_\rho} |w|^p \right)^{\frac{\xi}{\xi-1}(1-\frac{q}{\xi p})} \left(\int_\sigma^\rho \|w\|^p_{L^{\xi p}(\partial B_r)} \right)^{\frac{\xi}{\xi-1}\left(\frac{q}{p}-1\right)} + \varepsilon.\end{aligned} \tag{3.23}$$

Having at hand (3.23) we can proceed to prove Lemma 3.11.

Proof of Lemma 3.11. By direct methods and strict convexity deriving from (H1) we obtain a sequence of minimisers $\{v_\varepsilon\} \in W^{1,q}_0(\Omega)$ that solve

$$\min_{v \in W^{1,q}_0(\Omega)} \mathscr{F}_\varepsilon(v + g).$$

Moreover, v_ε satisfies the Euler–Lagrange equation

$$\int_\Omega \partial_z F_\varepsilon(\mathrm{D}v_\varepsilon + \mathrm{D}g) \cdot \mathrm{D}\phi \,\mathrm{d}x = 0 \qquad \forall \phi \in W_0^{1,q}(\Omega). \tag{3.24}$$

For notational simplicity, we suppress the dependence on ε for the time being and write $v = v_\varepsilon$.

Let $\rho_0 > 0$ and $\boldsymbol{n}\colon \mathbb{R}^n \to \mathbb{S}^{n-1}$ be so that the uniform cone property (3.5) holds. Without loss of generality we may assume that $\rho_0 < 1$. Let $\rho_0 \geq \sigma > \rho \geq \rho_0/2$. Fix $x_0 \in \Omega$ and let $\phi \in C_c^1(\mathbb{R}^n)$ be radially symmetric, monotonic decreasing with $\phi = 1$ in $B_\rho(x_0)$, $\operatorname{supp}\phi \subset B_\sigma(x_0)$. Denote the extension of v by 0 to $\mathbb{R}^n$ by $\tilde{v}$ and recall that we write $\tilde{v}_h(x) \coloneqq \tilde{v}(x+h)$. For $h \in \mathbb{R}^n$ define $T_h\mathrm{D}\tilde{v} \coloneqq \phi \mathrm{D}\tilde{v}_h + (1-\phi)\mathrm{D}\tilde{v}$ as well as $T_h v \coloneqq \phi\tilde{v}_h + (1-\phi)\tilde{v}$.

Then by (H1) and (3.24) for $h \in C_{\rho_0/4}(\theta_0, \boldsymbol{n}(x_0))$,

$$\begin{aligned}
&\int_\Omega F_\varepsilon(\mathrm{D}T_h\tilde{v} + \mathrm{D}g) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g)\,\mathrm{d}x \\
\geq &\int_\Omega \lambda(\mu^2 + |\mathrm{D}T_h\tilde{v}|^2 + |\mathrm{D}\tilde{v}|^2 + |\mathrm{D}g|^2)^{\frac{p-2}{2}}\phi^2 |\mathrm{D}\tilde{v}_h - \mathrm{D}\tilde{v}|^2 && (3.25)\\
&\quad + \partial_z F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g) \cdot \mathrm{D}(\phi(v_h - v))\,\mathrm{d}x && (3.26)\\
\gtrsim &\lambda \int_{B_\rho(x_0)} |V_{\mu,p}(\mathrm{D}\tilde{v}_h) - V_{\mu,p}(\mathrm{D}\tilde{v})|^2\,\mathrm{d}x. && (3.27)
\end{aligned}$$

To obtain the last line we also used Lemma 1.3 and noted that by our choice of h, $\phi\tilde{v}_h \in W_0^{1,q}(\Omega)$.

We further estimate for $h \in C_{\rho_0/4}(\theta_0, \boldsymbol{n}(x_0))$,

$$\begin{aligned}
&|h|^{-1}\int_\Omega F_\varepsilon(\mathrm{D}T_h\tilde{v} + \mathrm{D}g) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g)\,\mathrm{d}x \\
=&|h|^{-1}\int_\Omega F_\varepsilon(T_h\mathrm{D}\tilde{v} + \mathrm{D}\phi(\tilde{v}_h - v) + \mathrm{D}g) - F_\varepsilon(T_h\mathrm{D}\tilde{v} + \mathrm{D}g)\,\mathrm{d}x \\
&\quad + |h|^{-1}\int_\Omega F_\varepsilon(T_h\mathrm{D}\tilde{v} + \mathrm{D}g) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g)\,\mathrm{d}x \\
=:&A_1 + A_2.
\end{aligned}$$

We first find using (3.12) and Young's inequality,

$$\begin{aligned}
|A_1| \lesssim &|h|^{-1}\int_\Omega |\mathrm{D}\phi||\tilde{v}_h - \tilde{v}|\Big(1 + (|T_h\mathrm{D}\tilde{v}| + |\mathrm{D}\phi\tau_h\tilde{v}| + |\mathrm{D}g|)^{q-1}\Big)\,\mathrm{d}x \\
\leq &\varepsilon_1 \int_{B_\sigma(x_0)} \frac{|\tilde{v}_h - \tilde{v}|^q}{|h|^q}\,\mathrm{d}x + C(\varepsilon_1)\int_\Omega (|\mathrm{D}\phi| + |\mathrm{D}\phi|^q)^{\frac{q}{q-1}} w_1\,\mathrm{d}x
\end{aligned}$$

$$\lesssim \varepsilon_1 \int_{B_{\sigma+\rho_0/4}(x_0)} |\mathrm{D}\tilde{v}|^q \,\mathrm{d}x + C(\varepsilon_1) \int_\Omega \left(|\mathrm{D}\phi| + |\mathrm{D}\phi|^{\frac{q^2}{q-1}} \right) w_1 \,\mathrm{d}x$$

Here $w_1 \coloneqq 1 + |\mathrm{D}g|^q + |\mathrm{D}\tilde{v}|^q + |\mathrm{D}\tilde{v}_h|^q + |\tilde{v}|^q + |\tilde{v}_h|^q$.

We now turn to A_2. By convexity of $F_\varepsilon(\cdot)$,

$$\begin{aligned} F_\varepsilon(T_h \mathrm{D}\tilde{v} + \mathrm{D}g) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g) \leq& (1-\phi) F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g) + \phi F_\varepsilon(\mathrm{D}\tilde{v}_h + \mathrm{D}g) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g) \\ =& \phi(F_\varepsilon(\mathrm{D}\tilde{v}_h + \mathrm{D}g) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g)). \end{aligned}$$

We thus note

$$\begin{aligned} |A_2| \leq& |h|^{-1} \int_{B_\sigma(x_0)} \phi(F_\varepsilon(\mathrm{D}\tilde{v}_h + \mathrm{D}\tilde{g}_h) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}g)) \,\mathrm{d}x \\ &+ |h|^{-1} \int_{B_\sigma(x_0)} \phi(F_\varepsilon(\mathrm{D}\tilde{v}_h + \mathrm{D}g) - F_\varepsilon(\mathrm{D}\tilde{v}_h + \mathrm{D}\tilde{g}_h)) \,\mathrm{d}x \\ =& B_1 + B_2. \end{aligned}$$

Here $\tilde{g}_h$ is a $W^{2,q}(\mathbb{R}^n)$ extension of g to $\mathbb{R}^n$ with $\|\tilde{g}\|_{W^{2,q}(\mathbb{R}^n)} \lesssim \|g\|_{W^{2,q}(\Omega)}$ whose existence is guaranteed by (3.7).

Now, using a change of coordinates and since in $B_{3\rho_0} \setminus \Omega$, $\mathrm{D}\tilde{v}_h = \mathrm{D}\tilde{v} = 0$,

$$\begin{aligned} |B_1| \leq& |h|^{-1} \int_{B_\sigma(x_0)} \phi(F_\varepsilon(\mathrm{D}\tilde{v}_h + \mathrm{D}\tilde{g}_h) - F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}\tilde{g})) \,\mathrm{d}x \\ =& |h|^{-1} \int_{B_\sigma(x_0)+h} \phi(x-h) F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}\tilde{g}) \,\mathrm{d}x - |h|^{-1} \int_{B_\sigma(x_0)} \phi(x) F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}\tilde{g}) \,\mathrm{d}x \\ \leq& |h|^{-1} \int_{B_{\sigma+\rho_0/4}(x_0) \setminus B_{3\rho/4}(x_0)} |\phi(x-h) - \phi(x)| F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}\tilde{g}) \,\mathrm{d}x \\ \leq& |h|^{-1} \int_{B_{\sigma+\rho_0/4}(x_0) \setminus B_{3\rho/4}(x_0)} |\phi(|x-h|) - \phi(|x|)| F_\varepsilon(\mathrm{D}\tilde{v} + \mathrm{D}\tilde{g}) \,\mathrm{d}x. \end{aligned}$$

Finally, we estimate B_2 using (3.12) and Hölder,

$$\begin{aligned} |B_2| \leq& |h|^{-1} \int_{B_\sigma(x_0))} |\mathrm{D}\tilde{g} - \mathrm{D}\tilde{g}_h| (1 + |\mathrm{D}\tilde{v}_h| + |\mathrm{D}\tilde{v}| + |\mathrm{D}g| + |\mathrm{D}\tilde{g}_h|)^{q-1} \,\mathrm{d}x \\ \lesssim& |h|^{-1} \|\mathrm{D}g - \mathrm{D}\tilde{g}_h\|_{L^q(B_{\sigma+\rho_0}(x_0))} \Big(1 + \|\mathrm{D}\tilde{v}\|^{q-1}_{L^q(B_\sigma(x_0))} + \|\mathrm{D}\tilde{v}_h\|^{q-1}_{L^q(B_\sigma(x_0))} \\ &+ \|\mathrm{D}\tilde{g}\|^{q-1}_{L^q(B_\sigma(x_0))} + \|\mathrm{D}\tilde{g}_h\|^{q-1}_{L^q(B_\sigma(x_0))} \Big) \\ \lesssim& \|g\|_{W^{2,q}(B_\sigma(x_0))} \Big(1 + \|\mathrm{D}\tilde{v}\|^{q-1}_{L^q(B_\sigma(x_0))} + \|\mathrm{D}\tilde{v}_h\|^{q-1}_{L^q(B_\sigma(x_0))} \\ &+ \|\mathrm{D}\tilde{g}\|^{q-1}_{L^q(B_\sigma(x_0))} + \|\mathrm{D}\tilde{g}_h\|^{q-1}_{L^q(B_\sigma(x_0))} \Big) = D_1. \end{aligned}$$

To obtain the last line we used Lemma 1.13.

Collecting terms we conclude the estimate

$$\begin{aligned}
&[V_{\mu,p}(\mathrm{D}\tilde{v})]^2_{B^{1/2,2}_\infty(B_\rho)}\\
\lesssim&\varepsilon_1\int_{B_{\sigma+\rho_0/4}(x_0)}|\mathrm{D}\tilde{v}|^q\,\mathrm{d}x + C(\varepsilon_1)\int_{B_{\sigma+\rho_0/4}(x_0)}(|\mathrm{D}\phi|+|\mathrm{D}\phi|^{\frac{q^2}{q-1}})w_1\,\mathrm{d}x\\
&+|h|^{-1}\int_{B_{\sigma+\rho_0/4}(x_0)\setminus B_{\rho-\rho_0}(x_0)}|\phi(x-h)-\phi(x)|F_\varepsilon(\mathrm{D}\tilde{v})\,\mathrm{d}x\\
\lesssim&\varepsilon_1\int_{B_{\sigma+\rho_0/4}(x_0)}|\mathrm{D}\tilde{v}|^q\,\mathrm{d}x + C(\varepsilon_1)\int_{B_{\sigma+\rho_0/4}(x_0)}(|\mathrm{D}\phi|+|\mathrm{D}\phi|^{\frac{q^2}{q-1}})w_2\,\mathrm{d}x\\
&+|h|^{-1}\int_{B_{\sigma+\rho_0/4}(x_0)\setminus B_{3\rho/4}(x_0)}|\phi(x-h)-\phi(x)|F_\varepsilon(\mathrm{D}\tilde{v})\,\mathrm{d}x + D_1 \qquad (3.28)
\end{aligned}$$

where $w_2 \coloneqq 1+|V_{\mu,p}(\mathrm{D}\tilde{v}_\varepsilon)|^{\frac{2q}{p}}+|V_{\mu,p}(\mathrm{D}\tilde{v}_{\varepsilon,h})|^{\frac{2q}{p}}+|V_{\mu,p}(\tilde{v}_\varepsilon)|^{\frac{2q}{p}}+|V_{\mu,p}(\tilde{v}_{\varepsilon,h})|^{\frac{2q}{p}}$.

We immediately note that with a choice of $\phi\in C^1_c(B_\sigma)$ such that $|\mathrm{D}\phi|\le 2/(\sigma-\rho)$ we conclude

$$[V_{\mu,p}(\mathrm{D}\tilde{v})]^2_{B^{1/2,2}_\infty(B_\rho)} \lesssim \frac{1}{\varepsilon(\sigma-\rho)^{\frac{q^2}{q-1}}}\left(1+\int_\Omega F_\varepsilon(\mathrm{D}\tilde{v})\,\mathrm{d}x+\|g\|_{W^{2,q}(\Omega)}\right). \qquad (3.29)$$

Choosing $\rho=\rho_0/2$, $\sigma=\rho$ and applying a standard covering argument we deduce that $V_{\mu,p}(\mathrm{D}\tilde{v})\in B^{1/2,2}_\infty(\mathbb{R}^n)$. This observation ensures that our calculations below are valid.

Combining (3.28) and (3.23) with $w\coloneqq w_2^{\frac{1}{q}}$ we find for any $\varepsilon_2>0$,

$$\begin{aligned}
&\int_\Omega 1+|V_{\mu,p}(\mathrm{D}\tilde{v})|^2\,\mathrm{d}x + [V_{\mu,p}(\mathrm{D}\tilde{v})]^2_{B^{1/2,2}_\infty(B_\rho(x_0))}\\
\lesssim&\frac{C(\varepsilon_1)}{(\sigma+\rho_0/2-\rho)^{q^2/(q-1)+q/p}}\\
&\times\left(\int_{B_{\sigma+\rho_0/2}(x_0)\setminus B_{\rho-\rho_0/4}(x_0)}1+|V_{\mu,p}(\mathrm{D}\tilde{v})|^2+|V_{\mu,p}(\mathrm{D}\tilde{v}_h)|^2\,\mathrm{d}x\right)^{\frac{\xi}{\xi-1}(1-\frac{q}{\xi p})}\\
&\times\left(\int_{\rho-\rho_0/4}^{\sigma+\rho_0/2}1+\|V_{\mu,p}(\mathrm{D}\tilde{v})\|^2_{L^{2\xi}(\partial B_r(x_0))}+\|V_{\mu,p}(\mathrm{D}\tilde{v}_h)\|^2_{L^{2\xi}(\partial B_r(x_0))}\,\mathrm{d}r\right)^{\frac{\xi}{\xi-1}\left(\frac{q}{p}-1\right)}\\
&\times\left(1+\int_\Omega F_\varepsilon(\mathrm{D}\tilde{v})\,\mathrm{d}x\right)+\varepsilon_1\int_\Omega|\mathrm{D}\tilde{v}|^q\,\mathrm{d}x+\varepsilon_2+D_1.
\end{aligned}$$

We have also used Poincaré inequality, $\int_\Omega 1+|V_{\mu,p}(\mathrm{D}\tilde{v}_\varepsilon)|^2\,\mathrm{d}x\ge|\Omega|$ and that

$$\frac{\xi}{\xi-1}(1-q/(\xi p)+q/p-1)=q/p\ge 1.$$

Let $0 < \alpha < 1/2$. Choose $\rho = \rho_0/2$, $\sigma = \rho_0$, and use first Theorem 1, then Lemma 1.8 to conclude

$$\begin{aligned}
&\int_\Omega 1 + |V_{\mu,p}(\mathrm{D}\tilde{v})|^2 \,\mathrm{d}x + [V_{\mu,p}(\mathrm{D}\tilde{v})]^2_{B^{1/2,2}_\infty(B_\rho(x_0))\cap\Omega} \\
\lesssim & C(\varepsilon_1)\rho_0^{-q^2/(q-1)-q/p} \left(\int_\Omega 1 + |V_{\mu,p}(\mathrm{D}\tilde{v})|^2\right)^{\frac{\xi}{\xi-1}(1-\frac{q}{\xi p})} \\
&\quad \times \left(1 + \|V_{\mu,p}(\mathrm{D}\tilde{v})\|^2_{L^2(\Omega)} + [V_{\mu,p}(\mathrm{D}\tilde{v})]^2_{B^{\alpha,2}_\infty(\Omega)}\right)^{\frac{\xi}{\xi-1}\left(\frac{q}{p}-1\right)} \\
&\quad \times \left(1 + \int_\Omega F_\varepsilon(\mathrm{D}\tilde{v})\,\mathrm{d}x\right) + \varepsilon_1 \int_\Omega |\mathrm{D}\tilde{v}|^q \,\mathrm{d}x + \varepsilon_2 + D_1.
\end{aligned}$$

By a standard covering argument and (1.13), we deduce

$$\begin{aligned}
&\int_\Omega 1 + |V_{\mu,p}(\mathrm{D}\tilde{v})|^2 \,\mathrm{d}x + [V_{\mu,p}(\mathrm{D}\tilde{v})]^2_{B^{\alpha,2}_\infty(\Omega)} \\
\lesssim & C(\varepsilon-1)\rho_0^{-q^2/(q-1)-q/p} \left(\int_\Omega 1 + |V_{\mu,p}(\mathrm{D}\tilde{v})|^2 \,\mathrm{d}x\right)^{\frac{\xi}{\xi-1}(1-\frac{q}{\xi p})} \\
&\quad \times \left(1 + \|V_{\mu,p}(\mathrm{D}\tilde{v}_\varepsilon)\|^2_{L^2(\Omega)} + [V_{\mu,p}(\mathrm{D}\tilde{v})]^2_{B^{\alpha,2}_\infty(\Omega)}\right)^{\frac{\xi}{\xi-1}\left(\frac{q}{p}-1\right)} \\
&\quad \times \left(1 + \int_\Omega F_\varepsilon(\mathrm{D}\tilde{v})\,\mathrm{d}x\right) + \varepsilon_1 \int_\Omega |\mathrm{D}\tilde{v}|^q \,\mathrm{d}x + \varepsilon_2 \\
&\quad + \|g\|_{W^{2,q}(\Omega)} \left(1 + \|\mathrm{D}v\|^{q-1}_{L^q(\Omega)} + \|\mathrm{D}g\|^{q-1}_{L^q(\Omega)}\right).
\end{aligned}$$

Fix $0 < \beta < 1$. We now note that by interpolation between $L^p(\Omega)$ and $L^{\frac{np}{n-\beta}}(\Omega)$,

$$\begin{aligned}
\|\mathrm{D}v\|^{q-1}_{L^q(\Omega)} \leq & \|\mathrm{D}v\|^{(1-\theta)(q-1)}_{L^p(\Omega)} \|\mathrm{D}v\|^{\theta(q-1)}_{L^{\frac{np}{n-\beta}}(\Omega)} \\
\leq & \|V_{\mu,p}(\mathrm{D}v)\|^{2(1-\theta)(q-1)/p}_{L^2(\Omega)} \|V_{\mu,p}(\mathrm{D}v)\|^{2\theta(q-1)/p}_{L^{\frac{2n}{(n-\beta)}}(\Omega)}
\end{aligned}$$

with $\theta = \frac{n}{\beta}\left(1 - \frac{p}{q}\right)$. Since $p < n-1$ it is straightforward to check that $\frac{2\theta(q-1)}{p} < 2$ if β is chosen sufficiently close to 1.

Since $\frac{q}{p} < 1 + \frac{1}{n-1}$ and by the definition of ξ we find

$$\frac{\xi}{\xi-1}\frac{q-p}{p} < 1.$$

Choose ε_1 sufficiently small and $\varepsilon_2 = 1$. After re-arranging and using the embedding of $B^{\alpha,2}_\infty(\Omega)$ into $W^{1,2n/(n-\beta)}(\Omega)$ (which holds for a sufficiently large choice of α), we hence deduce the estimate

$$[V_{\mu,p}(\mathrm{D}\tilde{v}_\varepsilon)]^2_{B^{\alpha,2}_\infty(\Omega)} \lesssim \left(1 + \|V_{\mu,p}(\mathrm{D}\tilde{v}_\varepsilon)\|^2_{L^2(\Omega)} + \|g\|^q_{W^{2,q}(\Omega)}\right)^{\frac{\tilde{\beta}}{1-\tilde{\beta}}} \left(1 + \int_\Omega F_\varepsilon(\mathrm{D}\tilde{v}_\varepsilon)\,\mathrm{d}x\right)^{\frac{1}{1-\tilde{\beta}}},$$

where $\tilde{\beta} = \max\left(\frac{\xi}{\xi-1}\frac{q-p}{p}, \frac{2n(q-1)(q-p)}{\beta pq}\right)$.

As $\|V_{\mu,p}(\mathrm{D}\tilde{v}_\varepsilon)\|^2_{L^2(\Omega)} \lesssim 1 + \mathscr{F}_\varepsilon(\mathrm{D}\tilde{v}_\varepsilon) + \|g\|_{W^{2,q}(\Omega)}$, using Lemma 3.6 we conclude that the bound we have obtained is independent of ε.

Finally, note that by Sobolev embedding for any $\beta < 1$, we can find $\alpha \in (0, 1/2)$ such that $\|\mathrm{D}v_\varepsilon\|_{L^{\frac{np}{n-\beta}}(\Omega)} \lesssim \|\mathrm{D}v_\varepsilon\|_{B^{2\alpha/p,p}_\infty(\Omega)} \lesssim \|V_{\mu,p}(\mathrm{D}v_\varepsilon)\|_{B^{\alpha,2}_\infty(\Omega)}$. Considering the regularity of g, the result now follows. □

Remark 3.13. We note that starting from (3.28), and employing an appropriate iteration lemma, it is not difficult to obtain a local version of the a-priori estimate. This proceeds very similarly to the discussion at the end of the last section and we refrain from giving details here.

7 Regularisation and pointwise minimisers

In this section we prove the second part of Theorem 9 concerning pointwise minimisers. The proof employs regular approximations to $u \in W^{1,p}_g(\Omega)$ using a partition of unity adapted to a Whitney–Besicovitch covering of Ω. This construction is motivated by the following considerations: For autonomous convex integrands it is easy to see that, considering the mollified functions $u \star \phi_\varepsilon$ shows that $\mathscr{L}(\cdot, X, Y) = 0$ when $X = W^{1,p}(\Omega)$ and $Y = W^{1,q}(\omega)$ for some fixed $\omega \Subset \Omega$. Recall that the Lavrentiev gap functional $\mathscr{L}(\cdot, X, Y)$ was defined in (3.2). We remind the reader that we already discussed this argument at the start of the chapter. Our situation, where we intend to employ a similar argument, differs in two aspects: We consider non-autonomous integrands and require results that are valid up to the boundary. We use assumption (H7) to deal with the x-dependence. In order to obtain results valid up to the boundary, our main idea is to use a two-parameter mollification $u \star \phi_{\varepsilon\delta(x)}$ where $\delta(x) \sim d(x, \partial\Omega)$. We will implement this using a Whitney–Besicovitch covering.

We begin by collecting a number of technical lemmas in the next subsection. After proving Theorem 9 in Section 7.2, we discuss implications for the Lavrentiev gap integrand in Section 7.3.

Throughout this section we set $\Theta \coloneqq 1 + n\left(\frac{1}{q} - \frac{1}{p}\right)$ if $p < n$, $\Theta \coloneqq \frac{n}{q}$ if $p \geq n$.

7.1 Technical lemmas

We begin with an observation already noted in [78, Theorem 3.1]. For completeness we give the proof.

Lemma 3.14. *Assume $1 < p \le q \le \frac{(n+\alpha)p}{n}$. Let Ω be a domain and $u \in W^{1,p}(\Omega)$. Suppose $F(x,\cdot)$ satisfies* (H1)-(H3) *and* (H7)*. Then for $x \in \Omega$ and $\varepsilon \le \min(\varepsilon_0, d(x,\partial\Omega))$,*

$$F(x, \mathrm{D}u(\cdot) \star \phi_\varepsilon(x)) \lesssim 1 + \Big(F(\cdot, \mathrm{D}u(\cdot)) \star \phi_\varepsilon\Big)(x).$$

Proof. Let $C > 0$ be an arbitrary fixed constant. We claim that if $|z| \le C\varepsilon^{-\frac{n}{p}}$ and $x \in \overline{B_\varepsilon(x_0) \cap \Omega}$, then

$$F(x,z) \lesssim 1 + \min_{y \in \overline{B_\varepsilon(x) \cap \Omega}} F(y,z). \tag{3.30}$$

Set $G_\varepsilon(x,z) \coloneqq \min_{y \in \overline{B_\varepsilon(x)\cap\Omega}} F(x,z)$. By (H3), $G_\varepsilon(x,z) \ge F(x,z) - \Lambda\varepsilon^\alpha(1+|z|^2)^{\frac{q}{2}}$. Hence for $\delta \in (0,1)$, $\varepsilon \in (0,\varepsilon_0)$ and $|z| \le C\varepsilon^{-\frac{n}{p}}$, using (H5),

$$\begin{aligned}
G_\varepsilon(x,z) \ge& \delta F(x,z) - \delta\varepsilon^\alpha \Lambda(1+|z|^2)^{\frac{q}{2}} + (1-\delta)(|z|^p - 1) \\
\gtrsim& \delta F(x,z) - \delta\Lambda C^{q-p}\varepsilon^{\alpha + \frac{n(p-q)}{p}}|z|^p + (1-\delta)|z|^p - \delta(\Lambda\varepsilon^\alpha + 1) - 1 \\
\ge& \delta F(x,z) + \Big(1 - \delta - \Lambda C^{q-p}\delta\varepsilon_0^{\alpha + \frac{n(p-q)}{p}}\Big)|z|^p - \Lambda(\varepsilon_0^\alpha + 1) - 1.
\end{aligned}$$

The last line relies on $q \le \frac{(n+\alpha)p}{n}$. Choosing δ sufficiently small and re-arranging, (3.30) follows.

Note that by Lemma 1.9

$$|\mathrm{D}u \star \phi_\varepsilon| \le \|\mathrm{D}u\|_{L^p(\Omega)}\varepsilon^{-\frac{n}{p}}. \tag{3.31}$$

Thus using Jensen's inequality and (H7) we may estimate for $\varepsilon \in (0,\varepsilon_0)$ with some $\hat{y} \in B_\varepsilon(x)$,

$$\begin{aligned}
G_\varepsilon(x, \mathrm{D}u \star \phi_\varepsilon) =& F(\hat{y}, \mathrm{D}u \star \phi_\varepsilon) \le \int_{B_1} F(\hat{y}, \mathrm{D}u(y))\phi_\varepsilon(x-y)\,\mathrm{d}y \\
\le& (F(\cdot, \mathrm{D}u(\cdot)) \star \phi_\varepsilon)(x).
\end{aligned}$$

In particular, using (3.30) we conclude that

$$F(x, \mathrm{D}u \star \phi_\varepsilon) \lesssim 1 + \Big(F(\cdot, \mathrm{D}u(\cdot)) \star \phi_\varepsilon\Big)(x).$$

□

Partitions of unity feature heavily in our approximation approach. The next lemma studies the behaviour of $F(x,z)$ when z is a sum built using a partition of unity.

Lemma 3.15. *Let Ω be a domain with $\Omega = \bigcup_{i\in I}\Omega_i$, where $\Omega_i \subset \Omega$ is open relative to Ω. Suppose $1 < p \le q < \frac{(n+1)p}{n}$ and suppose $\{\psi_i\}_{i\in I}$ is a partition of unity subordinate to $\{\Omega_i\}$, and such that*

$$\|\mathrm{D}\psi_i\|_{L^\infty(\Omega_i)} \sim \|\mathrm{D}\psi_j\|_{L^\infty(\Omega_j)} \qquad \text{if } \operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset. \tag{3.32}$$

Assume that there is $c > 0$ such that $\|\mathrm{D}\psi_i\|_{L^\infty(\Omega_i)} \ge c > 0$ for all $i \in I$ and that there is $M > 0$ such that for every $x \in \Omega$,

$$\#\{i \in I \colon x \in \operatorname{supp}\psi_i\} \le M. \tag{3.33}$$

Take $m' \ge 1$ such that $m'\Big(\Theta - \frac{n(q-1)}{p}\Big(1-\frac{p}{q}\Big)\Big) \ge 1$.

Suppose F satisfies (H4) *and* (H2). *Let $u \in W^{1,p}(\Omega)$. Assume that there are functions $u_\varepsilon^i \in W^{1,p}(\Omega_i)$, $\varepsilon_1 > 0$ and $C = C(\|u\|_{W^{1,p}(\Omega)})$ such that for each $i \in I$, $\varepsilon \in (0,\varepsilon_1)$, $u_\varepsilon^i \to u$ in $W^{1,p}(\Omega_i)$ as $\varepsilon \to 0$ and*

$$\begin{aligned} \|u_\varepsilon^i - u\|_{L^q(\Omega_i)} &\lesssim C\|\mathrm{D}\psi_i\|_{L^\infty(\Omega)}^{-m\Theta}\varepsilon^\Theta, \\ \|\mathrm{D}u_\varepsilon^i\|_{L^q(\Omega_i)} &\lesssim C\big(\varepsilon\|\mathrm{D}\psi_i\|_{L^\infty(\Omega)}^{-m}\big)^{-\frac{n}{p}\left(1-\frac{p}{q}\right)}. \end{aligned} \tag{3.34}$$

for some $m \ge m'$. Then $u_\varepsilon \coloneqq \sum_{i\in I} u_\varepsilon^i\psi_i \to u$ in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$ and

$$\int_\Omega F(x,\mathrm{D}u_\varepsilon)\,\mathrm{d}x \lesssim c\Big(\|u\|_{W^{1,p}(\Omega)}\Big) + \int_\Omega F\Big(x, \sum_{i\in I}\mathrm{D}u_\varepsilon^i\psi_i\Big)\,\mathrm{d}x.$$

Proof. We remark that by the restriction on q, m' is well-defined. Let $\varepsilon \in (0,\varepsilon_1)$. We first show that $u_\varepsilon \in W^{1,p}(\Omega)$. By (3.33) and strong convergence of u_ε^i, we see that $u_\varepsilon \to u$ in $L^p(\Omega)$. Thus we focus on

$$\mathrm{D}u_\varepsilon = \sum_{i\in I}\mathrm{D}u_\varepsilon^i\psi_i + \sum_{i\in I} u_\varepsilon^i \otimes \mathrm{D}\psi_i =: A_1 + A_2.$$

Now due to (3.33), $A_1 \in L^p(\Omega)$. For A_2 note using that $\|\mathrm{D}\psi_i\|_{L^\infty(\Omega_i)} \ge c > 0$ and again (3.33),

$$\|A_2\|_{L^q(\Omega)} \lesssim C\sum_{i\in I}\varepsilon^\Theta\|\mathrm{D}\psi_i\|_{L^\infty(\Omega)}^{-m\Theta+1} \lesssim C\varepsilon^\Theta.$$

To obtain the last inequality we also noted the choice of m'. Hence $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$. Using (H6), we find using also the bound on A_2,

$$\begin{aligned}\int_\Omega F(x, \mathrm{D}u_\varepsilon)\,\mathrm{d}x &\lesssim \int_\Omega F(x, A_1) + |A_2|(1 + |A_1|^{q-1} + |A_2|^{q-1})\,\mathrm{d}x \\ &\lesssim c(\|u\|_{W^{1,p}(\Omega)}) + \int_\Omega F(x, A_1)\,\mathrm{d}x + \int_\Omega |A_1|^{q-1}|A_2|\,\mathrm{d}x.\end{aligned}$$

Set for $j \in I$, $I_j = \{i \in I \colon \operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset\}$. Then using Hölder's inequality and Lemma 1.9

$$\begin{aligned}\int_\Omega |A_1|^{q-1}|A_2|\,\mathrm{d}x \leq& \sum_{j\in I}\int_{\Omega_j} |A_2||\mathrm{D}u_\varepsilon^j|^{q-1}\,\mathrm{d}x \\ \lesssim& c(\|u\|_{W^{1,p}(\Omega)}) \sum_{j\in I} (\varepsilon \|\mathrm{D}\psi_j\|_{L^q(\Omega_j)}^{-m})^{-\frac{n}{p}\left(1-\frac{p}{q}\right)} \sum_{I_j} \varepsilon^\Theta \|\mathrm{D}\psi_i\|_{L^\infty(\Omega_i)}^{-m\Theta+1} \\ \lesssim& c(\|u\|_{W^{1,p}(\Omega)})\varepsilon^\tau \sum_{i\in I} \|\mathrm{D}\psi_i\|_{L^\infty(\Omega_i)}^{-m\tau+1}\end{aligned}$$

with $\tau \coloneqq \Theta - \frac{n(q-1)}{p}\left(1-\frac{p}{q}\right) > 0$ by our restriction on q. By our choice of m', the fact that $\|\mathrm{D}\psi_i\|_{L^\infty(\Omega_i)} \geq c > 0$ and (3.33), we conclude that the right-hand side is bounded uniformly in ε. This concludes the proof. □

We next specialise to a specific choice of u_ε^i and deal with the convergence of the term $\int_\Omega F\left(x, \sum_{i\in I} \mathrm{D}u_\varepsilon^i \psi_i\right)\mathrm{d}x$.

Lemma 3.16. *Let Ω be a domain. Suppose $1 < p \leq q < \frac{(n+\alpha)p}{n}$. Choose $m' \geq 1$ such that $m'\left(\Theta - \frac{n(q-1)}{p}\left(1-\frac{p}{q}\right)\right) \geq 1$. Given $u \in W^{1,p}(\Omega)$, $\varepsilon \in \left(0, 1/(1+1/6)^n\right)$ and $m \geq 1$, write*

$$u_\varepsilon \coloneqq \sum_{i\in I} u \star \phi_{\varepsilon\delta_i}\psi_i \qquad \text{with } \delta_i = |K_i|^{\frac{m}{n}}. \tag{3.35}$$

Here $\{K_i\}_{i\in I}$ is a WB-covering of Ω and $\{\psi_i\}_{i\in I}$ the partition of unity associated to this covering by Theorem 11. Assume that F satisfies (H4)–(H3) *and* (H7)*. Then if $m > m'$, up to passing to a subsequence if necessary, $u_\varepsilon \in W_u^{1,p}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$, $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$, and*

$$\int_\Omega F(x, \mathrm{D}u_\varepsilon)\,\mathrm{d}x \to \int_\Omega F(x, \mathrm{D}u)\,\mathrm{d}x \qquad \text{as } \varepsilon \searrow 0.$$

Proof. The well-definedness of u_ε is ensured by Theorem 10 and Theorem 11.

By Theorem 11 and Lemma 1.9, as well as our choice of m', we may apply Lemma 3.15 to find that $u_\varepsilon \in W^{1,p}(\Omega)$ and $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$ as $\varepsilon \searrow 0$. By (3.9) and Lemma

1.9, it is clear that $u_\varepsilon \in W^{1,q}_{\mathrm{loc}}(\Omega)$. We turn to the boundary behaviour of u_ε. By a density argument we may assume that $u \in C(\overline{\Omega})$. Note that for $x \in \Omega$,

$$|u_\varepsilon(x) - u(x)| \leq \sum_{i \in I : x_n \in K_i} |u \star \rho_{\varepsilon\delta_i}(x) - u(x)|\psi_i \leq M \max_{y \in B_r(x)} |u(y) - u(x)|$$

where $r \coloneqq \max\{\, \delta_i : i \in I, x \in K_i \,\}$. But if $x \in K_i$, then $\delta_i \lesssim |K_i|^{\frac{1}{n}} \sim d(x, \partial\Omega)$. It follows that $u_\varepsilon \in W^{1,p}_u(\Omega)$.

Finally, again by Lemma 3.15,

$$\int_\Omega F(x, \mathrm{D}u_\varepsilon)\,\mathrm{d}x \lesssim c\Big(\|u\|_{W^{1,p}(\Omega)}\Big) + \int_\Omega F\Big(x, \sum_{i \in I} \mathrm{D}u \star \phi_{\varepsilon\delta_i}\psi_i\Big)\,\mathrm{d}x.$$

For $\varepsilon < \varepsilon_0 \left(\max_{i \in I} |K_i|^{\frac{1}{n}}\right)^{-1}$, we find using Jensen's inequality, (3.9), Lemma 3.14, Lemma 1.9 and the dominated convergence theorem,

$$\begin{aligned}
\int_\Omega F\Big(x, \sum_{i \in I} \mathrm{D}u \star \phi_{\varepsilon\delta_i}\psi_i\Big) &\leq \sum_{i \in I} \int_\Omega \int_{B_1} F(x, \mathrm{D}u(x - \varepsilon\delta_i y))\phi(y)\,\mathrm{d}y\psi_i\,\mathrm{d}x \qquad (3.36)\\
&\lesssim 1 + \sum_{i \in I} \int_\Omega (F(\cdot, \mathrm{D}u(\cdot)) \star \phi_{\varepsilon\delta_i})(x)\psi_i\,\mathrm{d}x\\
&\xrightarrow{\varepsilon \to 0} 1 + \sum_{i \in I} \int_\Omega F(x, \mathrm{D}u)\psi_i\,\mathrm{d}x = 1 + \mathscr{F}(\mathrm{D}u) < \infty.
\end{aligned}$$

An application of a version of the dominated convergence theorem concludes the proof.

□

7.2 Proof of Theorem 9

We are now able to prove the second part of Theorem 9.

Proof of Theorem 9. Let $u \in W^{1,p}_g(\Omega)$. Let $\Omega^s, \Psi_s, \hat{g}$ be as constructed in Lemma 3.3. Extend u by $\hat{g}$ to Ω^s, still denoting the extension by u. Define

$$F^s(x, z) \coloneqq F\left(\Psi_s^{-1}(x), z\mathrm{D}\Psi_s(\Psi_s^{-1}(x))\right) \qquad \mathscr{F}^s(u) \coloneqq \int_\Omega F^s(x, \mathrm{D}u) - f \cdot u\,\mathrm{d}x \qquad (3.37)$$
$$u^s(x) \coloneqq u(\Psi_s^{-1}(x)).$$

By Lemma 1.11, $u \in W^{1,p}(\Omega^s)$. Since $|\mathrm{D}\Psi_s - \mathrm{Id}| \to 0$ uniformly in Ω, we find $u^s \to u$ in $W^{1,p}(\Omega)$. With the change of coordinates $\tilde{x} = \Psi_s^{-1}(x)$, we obtain

$$\int_\Omega F^s(x, \mathrm{D}u^s)\,\mathrm{d}x = \int_\Omega F(\Psi_s^{-1}(x), \mathrm{D}u(\Psi_s^{-1}(x))\mathrm{D}\Psi_s^{-1}(x)\mathrm{D}\Psi_s(\Psi_s^{-1}(x)))$$

$$= \int_\Omega F(\Psi_s^{-1}(x), \mathrm{D}u(\Psi_s^{-1}(x))\,\mathrm{d}x$$
$$= \int_\Omega F(\tilde{x}, \mathrm{D}u(\tilde{x}))J_{\Psi_s}\,\mathrm{d}\tilde{x} + \int_{\Omega^s\setminus\Omega} F(\tilde{x}, \mathrm{D}u(\tilde{x}))J_{\Psi_s}\,\mathrm{d}\tilde{x} \to \int_\Omega F(\mathrm{D}u)\,\mathrm{d}\tilde{x}$$

as $s \nearrow 1$, using (H2) and absolute continuity of the integral since $u \in W^{1,q}(\Omega^s \setminus \Omega)$.

We check that there is $s_0 > 0$ such that $F^s(x,z)$ satisfies (H1)-(H3) and (H7) with constants independent of s for $s \in (0, s_0)$. Using the properties of Ψ_s obtained from Lemma 3.3, straightforward arguments show that (H1)–(H3) hold if $s \in (0, s_0)$ for some $s_0 > 0$. Supoose (H7) holds for F. Let $\varepsilon \in (0, \varepsilon_0)$. Choosing τ sufficiently small, we can ensure that $|x - y| \le \tau$ implies $|\psi_s^{-1}(x) - \psi_s^{-1}(y)| < \varepsilon$ for all $s \in (0, s_0)$. Thus,

$$\min_{x\in\overline{B_\tau(x_0)\cap\Omega}} F^s(x,z) = \min_{x\in\overline{B_\varepsilon(\Psi_s^{-1}(x_0))\cap\Omega}} F\left(x, z\mathrm{D}\Psi_s(\Psi_s^{-1}(x))\right),$$

and so $F^s(x,z)$ satisfies (H7).

Let now u be a pointwise minimiser of $\mathscr{F}(\cdot)$ in the class $W_g^{1,p}(\Omega)$. Using our observations on u^s, Lemma 3.16 and a diagonal subsequence argument, we find a sequence $u_k \in W_g^{1,q}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and, in addition,

$$\int_\Omega F^{1/k}(x, \mathrm{D}u_k)\,\mathrm{d}x \to \int_\Omega F(x, \mathrm{D}u)\,\mathrm{d}x.$$

Fix $0 \le \beta < \alpha$. Since F^s satisfies (H1)–(H3) we may apply Lemma 3.9 to find that the pointwise minimisers v_k of

$$\mathscr{F}_{\varepsilon_k}^{1/k}(v) \coloneqq \int_\Omega F^{1/k}(x, \mathrm{D}v) + \varepsilon_k|\mathrm{D}v|^q - f\cdot v\,\mathrm{d}x$$

satisfy for some $\gamma > 0$ the apriori estimate

$$\|v_k\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \mathscr{F}_{\varepsilon_k}^{1/k}(v_k) + \|g\|_{W^{1+\max\left(\alpha,\frac{1}{q}\right),q}(\Omega)} + \|f\|_{L^{q'}(\Omega)}\right)^\gamma.$$

Here $\varepsilon_k \coloneqq \left(1 + \varepsilon^{-1} + \|\mathrm{D}u_k\|_{L^q(\Omega)}\right)^{-1}$. Using the minimality of v_k and then the choice of u_k and ε_k, we deduce

$$\|v_k\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \mathscr{F}_{\varepsilon_k}^{1/k}(u_k) + \|g\|_{W^{1+\max\left(\alpha,\frac{1}{q}\right),q}(\Omega)} + \|f\|_{L^{q'}(\Omega)}\right)^\gamma$$
$$\lesssim \left(1 + \mathscr{F}(u) + \|g\|_{W^{1+\max\left(\alpha,\frac{1}{q}\right),q}(\Omega)} + \|f\|_{L^{q'}(\Omega)}\right)^\gamma.$$

In particular we note that the estimate is independent of s and ε. Choosing now β sufficiently large that $q \le \frac{np}{n-\beta}$, we may extract a subsequence of $\{v_k\}$ converging

weakly in $W^{1,q}(\Omega)$ to some v and, by weak lower semicontinuity of norms, $v \in W^{1,q}(\Omega)$. Moreover, by weak semicontinuity and minimality of v_k in the class $W^{1,q}_g(\Omega)$,

$$\mathscr{F}(v) = \lim_{k\to\infty} \int_\Omega F^{1/k}(x, \mathrm{D}v) - f \cdot v \,\mathrm{d}x \leq \lim_{k\to\infty} \mathscr{F}^{1/k}_{\varepsilon_k}(x, v_k) \leq \lim_{k\to\infty} \mathscr{F}^{1/k}_{\varepsilon_k}(u_k) = \mathscr{F}(u).$$

By minimality of u in the class $W^{1,p}_g(\Omega)$, we conclude $\mathscr{F}(v) = \mathscr{F}(u)$. But then by convexity of $\mathscr{F}(\cdot)$ it follows that $v = u$, concluding the proof. □

We note that our proof in the case of autonomous integrands can be improved in the following way, following arguments from [109]:

Corollary 3.1. *Suppose Ω is a $C^{1,\alpha}$-domain for some $\alpha > 0$ and $g \in W^{1+\frac{1}{q},q}(\Omega)$. Suppose $F(z)$ satisfies* (H1) *and* (H2) *with $1 < p \leq q < \infty$. Then*

$$\mathscr{L}(\cdot, X, Y) = 0$$

where $X \coloneqq W^{1,p}_g(\Omega)$ and $Y \coloneqq W^{1,q}(\Omega)$.

Moreover, if $g \in W^{2,q}(\Omega)$, $f = 0$, $2 \leq p < n-1$ and $1 < p \leq q < \frac{np}{n-1}$, then if u is a minimiser of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$, $u \in W^{1,q}(\Omega)$. Further for any $0 \leq \beta < 1$ there is $\gamma > 0$ such that we have the estimate

$$\|u\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \overline{\mathscr{F}}(u) + \|g\|_{W^{2,q}(\Omega)}\right)^\gamma.$$

Proof. In order to prove the first part of the theorem, we use the ideas of the proof of Theorem 9 with minor changes which enables us to avoid the use of Lemma 3.16. Given $u \in W^{1,p}_g(\Omega)$, it suffices to exhibit a sequence $\{u_n\} \subset W^{1,q}_g(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\mathscr{F}(u_n) \to \mathscr{F}(u)$. Our first goal is to improve the regularity of u near the boundary.

Let $\Omega^s, \Psi_s, \hat{g}$ be as constructed in Lemma 3.3. Extend u by $\hat{g}$ to Ω^s, still denoting the extension by u. Define

$$u^s(x) \coloneqq u(\Psi_s^{-1}(x)).$$

By Lemma 1.11, $u \in W^{1,p}(\Omega^s)$. Since $|\mathrm{D}\Psi_s - \mathrm{Id}| \to 0$ uniformly in Ω, we find $u^s \to u$ in $W^{1,p}(\Omega)$. With the change of coordinates $\tilde{x} = \Psi_s^{-1}(x)$ we may compute

$$\int_\Omega F(\mathrm{D}u^s)\,\mathrm{d}x = \int_\Omega F(\mathrm{D}u(\tilde{x}))J_{\Psi_s}\,\mathrm{d}\tilde{x} + \int_{\Omega^s\setminus\Omega} F(\mathrm{D}u(\tilde{x}))J_{\Psi_s}\,\mathrm{d}\tilde{x} \to \int_\Omega F(\mathrm{D}u)\,\mathrm{d}\tilde{x}$$

as $s \nearrow 1$, using (H2) and absolute continuity of the integral since $u \in W^{1,q}(\Omega^s \setminus \Omega)$.

In particular, using a diagonal argument and our work so far we may assume without loss of generality that there is a neighbourhood U of $\partial\Omega$ such that $u \in W^{1,q}(U)$. Let $\eta \in C_C^\infty(\Omega)$ be such that $\eta = 1$ in $\Omega \setminus U$ and $0 \leq \eta \leq 1$. We intend to apply a mollification argument in the interior and consider

$$u_\varepsilon \coloneqq \eta u \star \phi_e + (1-\eta)u.$$

We note that $u_\varepsilon \in W_g^{1,q}(\Omega)$ and that $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$. Finally we consider

$$\begin{aligned}\mathscr{F}(u_\varepsilon) &= \int_{\{\eta=1\}} F(\mathrm{D}u \star \phi_e) + \int_{\{\eta=0\}} F(\mathrm{D}u)\,\mathrm{d}x + \int_{\{0<\eta<1\}} F(\mathrm{D}u_\varepsilon)\,\mathrm{d}x \\ &\leq \int_{\{\eta=1\}} F(\mathrm{D}u) \star \phi_e\,\mathrm{d}x + \int_{\{\eta=0\}} F(\mathrm{D}u)\,\mathrm{d}x + \int_{0<\eta<1} F(\mathrm{D}u_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

To obtain the last line we used Jensen's inequality. Thus we can conclude that we have $\mathscr{F}(u_\varepsilon) \to \mathscr{F}(u)$ using a variant of dominated convergence as long as we can bound the third term uniformly in ε. However here we note that

$$\mathrm{D}u_\varepsilon = \eta \mathrm{D}u \star \phi_e + (1-\eta)\mathrm{D}u + \mathrm{D}\eta \otimes (u \star \phi_e - u).$$

Since $u \in W^{1,q}(U)$, we see that $\mathrm{D}u_\varepsilon \to \mathrm{D}u$ in $W^{1,q}(U)$ as $\varepsilon \to 0$. In particular, using (H2) and noting that $\{0 < \eta < 1\} \subset U$, we find that

$$\int_{\{0<\eta<1\}} F(\mathrm{D}u_\varepsilon)\,\mathrm{d}x \lesssim \|\mathrm{D}u\|_{W^{1,q}(U)}$$

where the implicit constant depends on η but is independent of ε.

Thus we have shown the first part of the claim. The second part of the claim follows directly from combining the first part with Theorem 13. □

We conclude this section by noting that our proofs also allow us to obtain a higher differentiability result:

Corollary 3.2. *Assume the assumptions of Theorem 9 hold. Let u be a relaxed (pointwise) minimiser of $\mathscr{F}(\cdot)$ in the class $W_g^{1,p}(\Omega)$. Then $u \in W^{1+\beta,p}(\Omega)$ for $\beta < \frac{\alpha}{\max(2,p)}$. Moreover, there is $\gamma > 0$ such that*

$$\|u\|_{W^{1+\beta,p}(\Omega)} \lesssim \left(1 + \mathscr{F}(u) + \|g\|_{W^{1+\max\left(\alpha,\frac{1}{q}\right),q}(\Omega)} + \|f\|_{L^{q'}(\Omega)}\right)^\gamma.$$

We briefly comment on how to prove Corollary 3.2.

Proof of Corollary 3.2. Once Theorem 9 is available, there is no problem in making sense of the Euler–Lagrange equation for pointwise minimisers u. Hence we may repeat the arguments of Lemma 3.9 with $\mathscr{F}_\varepsilon(\cdot)$ and $v_\varepsilon + g$ replaced by $\mathscr{F}(\cdot)$ and u, respectively. The equivalent estimate to (3.20) and Theorem 1 then give the result. In the case of relaxed minimisers, using Lemma 3.8 the same argument applies. □

7.3 Implications for the Lavrentiev gap integrand

Note that while we were able to show improved integrability and differentiability results for minimisers, under the assumptions of Theorem 9 we considered the perturbed integrands $\mathscr{F}^s(\cdot)$. In particular, our results show that the Lavrentiev gap integrand vanishes at minimisers but do not give information about the behaviour at other points. The aim of this subsection is to adapt our arguments in order to conclude that the Lavrentiev gap integrand vanishes on all of $W^{1,p}_g(\Omega)$. To be precise our aim is to show the following:

Proposition 1. *Suppose Ω is a Lipschitz domain and $g \in W^{1+\frac{1}{q},q}(\Omega)$. Let $X \coloneqq W^{1,p}_g(\Omega)$ endowed with the weak topology. Suppose that $1 < p \leq q < \frac{(n+\alpha)p}{n}$ and that F satisfies* (H1)–(H3) *and* (H7). *Then with the choice $Y \coloneqq W^{1,q}_{\mathrm{loc}}(\Omega)$,*

$$\mathscr{L}(\cdot, X, Y) = 0 \text{ on } X.$$

If $1 < p \leq q < min\Big(p+1, \Big(1+\frac{\alpha}{(\alpha+1)n}\Big)p\Big)$, then $\mathscr{L}(\cdot, X, Y) = 0$ on X with the choice $Y \coloneqq W^{1,q}_g(\Omega)$.

Note that we are unable to recover the full range of q for which we prove $W^{1,q}(\Omega)$-regularity in Theorem 9. The author suspects this to be an artifact of the proof. If $Y = W^{1,q}(\omega)$ where $\omega \Subset \Omega$, then the corresponding result is shown in [78].

We intend to use Lemma 3.2 to reduce to the case of a star-shaped domain. Thus we focus first on improving the regularity near the boundary for the sequence $\{u_\varepsilon\}$ from Lemma 3.16 in the case of a star-shaped domain.

Lemma 3.17. *Let $1 < p \leq q < \Big(1+\frac{\alpha}{(\alpha+1)n}\Big)p$. Suppose Ω is a strongly star-shaped Lipschitz domain. Let $\gamma' \Subset \gamma \subset \partial\Omega$ where γ' and γ are connected. Suppose $\Omega' \subset \Omega$*

with $\partial\Omega' \cap \partial\Omega = \gamma'$ and such that if $N_{\gamma'}$ is a neighbourhood of γ', then $\Omega' \setminus N_{\gamma'} \Subset \Omega$. If $u \in W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega) \cap W^{1,q}(\gamma)$, then there is $\{u_n\} \subset W^{1,p}_u(\Omega) \cap W^{1,q}(\Omega')$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ and $\int_{\Omega'} F(x, \mathrm{D}u_n)\,\mathrm{d}x \to \int_{\Omega'} F(x, \mathrm{D}u)\,\mathrm{d}x$.

Proof. Without loss of generality, assume that Ω is strongly star-shaped with respect to 0. Let $u \in W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega) \cap W^{1,q}(\gamma)$. Consider the homogeneous extension of u to $\mathbb{R}^n$ by setting

$$u(x) \coloneqq \begin{cases} u(x) & \text{if } x \in \Omega \\ \frac{1}{\lambda'} u(\lambda' x) & \text{if } x \notin \Omega \text{ where } \lambda' = \sup\{\, \lambda > 0 : \lambda x \in \overline{\Omega}\,\}. \end{cases}$$

For $\frac{1}{2} < s < 1$ we define $u^s(x) \coloneqq su(x/s)$. It is easy to check that $u_s \in W^{1,p}_u(\Omega)$ and $u_s \to u$ in $W^{1,p}(\Omega)$ as $s \nearrow 1$. Let $\{K_i\}_{i\in I}$ be a WB-covering of Ω, and let $\{\psi_i\}_{i\in I}$ be the partition of unity associated to this covering by Theorem 11. The existence of $\{K_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ is ensured by Theorem 10 and Theorem 11.

Take $m \geq 1$ such that $m\Big(\Theta - \frac{n(q-1)}{p}\Big(1 - \frac{p}{q}\Big)\Big) > 1$. Define with $\delta_i \coloneqq |K_i|^{\frac{m}{n}}$,

$$u^s_\varepsilon \coloneqq \sum_{i\in I} u^s \star \phi_{\varepsilon\delta_i}\psi_i \qquad u_\varepsilon \coloneqq \sum_{i\in I} u \star \phi_{\varepsilon\delta_i}\psi_i.$$

Note that for s, ε sufficiently small, u^s_ε is well-defined. By Theorem 11 and Lemma 1.9, we may apply Lemma 3.15 to u^s_ε in order to see that $\{u^s_\varepsilon\} \in W^{1,p}_u(\Omega)$ with $u^s_\varepsilon \to u^s$ in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$. Further by Lemma 1.9 noting that u^s_ε is $W^{1,q}$-regular in a neighbourhood of γ', $u^s_\varepsilon \in W^{1,q}(\Omega')$. Using Lemma 3.15,

$$\begin{aligned}\int_{\Omega'} F(x, \mathrm{D}u^s_\varepsilon)\,\mathrm{d}x \lesssim& c(\|u^s\|_{W^{1,p}(\Omega)}) + \int_{\Omega'} F\Big(x, \sum_{i\in I} \mathrm{D}u^s_\varepsilon\psi_i\Big)\,\mathrm{d}x \\ \lesssim& c\Big(\|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\partial\Omega)}\Big) + \int_{\Omega'} F\Big(x, \sum_{i\in I} \mathrm{D}u^s \star \phi_{\varepsilon\delta_i}\psi_i\Big)\,\mathrm{d}x.\end{aligned}$$

Let $1 - s$ be sufficiently small that $\partial\Big(s^{-1}\Omega' \setminus \Omega\Big) \subset \gamma$ and, in addition, for any neighbourhood N_γ of γ, $(s^{-1}\Omega' \cap \Omega) \setminus N_\gamma \Subset \Omega$. Then using the change of coordinates $x \to sx$,

$$\begin{aligned}\int_{\Omega'} F(x, \sum_{i\in I} \mathrm{D}u^s_\varepsilon\psi_i)\,\mathrm{d}x =& s^{-n} \int_{s^{-1}\Omega'} F(sx, A)\,\mathrm{d}x \\ \leq& s^{-n} \int_{s^{-1}\Omega'\setminus\Omega} F(sx, A)\,\mathrm{d}x + s^{-n} \int_{s^{-1}\Omega'\cap\Omega} F(sx, A)\,\mathrm{d}x =: I + II\end{aligned}$$

with $A \coloneqq \sum_{i\in I}\Big(\mathrm{D}u(\cdot)\star\phi_{\varepsilon\delta_i}\Big)(x)\psi_i(sx)$. Now note that $\|u\|_{W^{1,q}(s^{-1}\Omega'\setminus\Omega)} \lesssim \|u\|_{W^{1,q}(\gamma)}$, so using (H2), Lemma 1.9 and (3.9), we see that

$$I = s^{-n}\int_{s^{-1}\Omega'\setminus\Omega} F(sx,A)\,\mathrm{d}x \lesssim s^{-n}\int_{s^{-1}\Omega'\setminus\Omega} 1+|A|^q\,\mathrm{d}x \lesssim 1+\|u\|_{W^{1,q}(\gamma)},$$

independently of s and ε. Using (H3) we find,

$$\begin{aligned} II &\leq \int_{s^{-1}\Omega'\cap\Omega} F(sx,A)-F(x,A)\,\mathrm{d}x + \int_{s^{-1}\Omega'\cap\Omega} F(x,A)\,\mathrm{d}x \\ &\lesssim (1-s)^\alpha \int_{s^{-1}\Omega'\cap\Omega} |A|^q\,\mathrm{d}x + \int_\Omega F(x,A)\,\mathrm{d}x. \end{aligned}$$

We know from Lemma 3.16 that the second term is bounded. The first term we estimate as follows: Note that if $x\in\operatorname{supp}\psi_i(s\cdot)\cap\Omega$, then $|K_i|^{\frac{1}{n}} \sim d(x,\partial\Omega) \gtrsim (1-s)$ and so $\delta_i \gtrsim (1-s)^m$. Thus using Lemma 1.9 we find for some sufficiently small $c>0$

$$(1-s)^\alpha\int_{s^{-1}\Omega'\cap\Omega}|A|^q\,\mathrm{d}x \lesssim (1-s)^\alpha\int_{s^{-1}\Omega'\cap\Omega}\sum_{i\in I:|K_i|^{\frac{1}{n}}\geq c(1-s)} (\varepsilon\delta_i)^{-\frac{n}{p}(q-p)}\|u\|^p_{W^{1,p}(K_i)}.$$

Recalling the definition of δ_i, if $\alpha - m\frac{n}{p}(q-p)\geq 0$, then the right-hand side is bounded independently of s. Recalling the choice of m we find that we need to have that $q<(1+\frac{\alpha}{(\alpha+1)n})p$.

By a version of the dominated convergence theorem we hence deduce, since it holds that $\mathrm{D}u^s_\varepsilon \to \mathrm{D}u^s$ almost everywhere in Ω',

$$\int_{\Omega'} F(x,\mathrm{D}u^s_\varepsilon)\,\mathrm{d}x \to \int_{\Omega'} F(x,\mathrm{D}u_\varepsilon)\,\mathrm{d}x \qquad \text{as } s\nearrow 1.$$

By Lemma 3.16

$$\int_{\Omega'} F(x,\mathrm{D}u_\varepsilon)\,\mathrm{d}x \to \int_{\Omega'} F(x,\mathrm{D}u)\,\mathrm{d}x \qquad \text{as } \varepsilon\searrow 0.$$

Thus by a diagonal subsequence argument, we can extract a subsequence of (u^s_ε) with all the desired properties. □

We can now finally prove Proposition 1.

Proof of Proposition 1. Note that the first part of Proposition 1 is a consequence of Lemma 3.16 and Lemma 3.6.

For the second part by Lemma 3.6 it suffices to exhibit a sequence $\{u_n\} \subset W^{1,q}_g(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\mathscr{F}(u_n) \to \mathscr{F}(u)$ as $n \to \infty$. By Lemma 3.2 we may write

$$\Omega = \bigcup_{i=1}^{M} \Omega_i = \bigcup_{i=1}^{N} \omega_i$$

where Ω_i is a strongly star-shaped Lipschitz domain, $\omega_i \Subset \Omega_i$ relative to Ω and further $\partial\omega_i \cap \partial\Omega \Subset \partial\Omega_i \cap \partial\Omega$ if $\partial\Omega_i \cap \Omega \neq \emptyset$. We may also assume that $u \in W^{1,p}(\partial\Omega_i)$ for $i = 1, \dots, M$, with $\|u\|_{W^{1,p}(\partial\Omega_i)} \lesssim \|u\|_{W^{1,p}(\Omega_i)} + \|g\|_{W^{1+1/q,q}(\Omega)}$. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{\omega_i\}$. By Lemma 3.17, we find sequences $\{u^i_n\} \subset W^{1,p}(\Omega_i) \cap W^{1,q}(\omega_i)$ such that $u^i_n \to u$ in $W^{1,p}(\Omega)$ and $\int_{\omega_i} F(x, \mathrm{D}u^i_n)\,\mathrm{d}x \to \int_{\omega_i} F(x, \mathrm{D}u)\,\mathrm{d}x$ where $i = 1, \dots, M$. Consider

$$u_n \coloneqq \sum_{i=1}^{N} u^i_n \psi_i.$$

Then $u_n \in W^{1,q}_g(\Omega)$ and $u_n \to u$ in $W^{1,p}(\Omega)$. Moreover, using (H6), Hölder's inequality, Sobolev's embedding and Jensen's inequality,

$$\begin{aligned}
\int_\Omega F(x, \mathrm{D}u_n)\,\mathrm{d}x &\lesssim \sum_{i,j=1}^{N} \int_\Omega |u^i_n \otimes \mathrm{D}\psi_i|(1 + |u^j_n \otimes \mathrm{D}\psi_j|^{q-1} + |\mathrm{D}u^j_n \otimes \mathrm{D}\psi_j|^{q-1})\,\mathrm{d}x \\
&\qquad + \int_\Omega F\Big(x, \sum_{i=1}^{N} \mathrm{D}u^i_n \psi_i\Big)\,\mathrm{d}x \\
&\lesssim \sum_{i,j=1}^{n} \|u^i_n\|_{W^{1,p}(\Omega_i)} \|u^j_n\|^{q-1}_{W^{1,p}(\Omega_j)} + \sum_{i=1}^{N} \int_{\omega_i} F(x, \mathrm{D}u^i_n)\psi_i\,\mathrm{d}x \\
&\to \sum_{i=1}^{n} \|\mathrm{D}u\|^q_{L^p(\Omega_i)} + \int_\Omega F(x, \mathrm{D}u)\,\mathrm{d}x < \infty
\end{aligned}$$

as $n \to \infty$. To obtain the second line we used the assumption that $q \leq \dfrac{np}{n-p}$. Thus we conclude by a variant of the dominated convergence theorem that $\mathscr{F}(u_n) \to \mathscr{F}(u)$ as $n \to \infty$. □

8 Regularity under alternatives to assumption (H4)

In this section we explore some alternatives to (H4) which only require minor modifications of the proofs. To be precise, we consider the following assumptions: Assume one of the following holds for some $\mu \geq 0, \lambda > 0$, for all $z, w \in \mathbb{R}^{N\times n}$, and almost every

$x \in \Omega$:

$$\begin{cases} \lambda \left(\mu^2 + |z|^2 + |w|^2\right)^{\frac{p(x)-2}{2}} |z-w|^2 \leq F(x,z) - F(z,w) - \partial_z F(x,w) \cdot (z-w) \\ \text{where } p(x) \in C^{0,\alpha}(\Omega) \text{ and there is } \varepsilon > 0 \text{ such that } 1 < p \leq p(x) \leq q \end{cases} \tag{H1.1}$$

$$\begin{cases} \sum_{i=1}^n \left(\mu_i^2 + |z_i|^2 + |w_i|^2\right)^{\frac{p_i-2}{2}} |z_i - w_i|^2 \leq F(x,z) - F(z,w) - \partial_z F(x,w) \cdot (z-w) \\ \text{where } z_i = (z_i)_{1\leq j\leq n}^j, w_i = (w_i)_{1\leq j\leq n}^j \text{ and } 1 < p = p_1 \leq p_2 \leq ... \leq p_n = q \end{cases} \tag{H1.2}$$

Before starting the proofs of versions of Theorem 9 adapted to this set-up, we remark that replacing (H4) with (H1.1) or (H1.2) also causes a change in the proof of Lemma 3.5: Using the notation of Lemma 3.5, (3.15) needs to be replaced by

$$\overline{\mathscr{F}}\left(\frac{w_1+w_2}{2}\right) + \frac{\nu}{p}\left(\int_\Omega |\mathrm{D}w_1 - \mathrm{D}w_2|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{1}{2}\left(\overline{\mathscr{F}}(w_1) + \overline{\mathscr{F}}(w_2)\right).$$

or

$$\overline{\mathscr{F}}\left(\frac{w_1+w_2}{2}\right) + \frac{\nu}{p}\left(\int_\Omega \sum_{i=1}^n |\mathrm{D}_i w_1 - \mathrm{D}_i w_2|^{p_i}\,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{1}{2}\left(\overline{\mathscr{F}}(w_1) + \overline{\mathscr{F}}(w_2)\right),$$

respectively. Arguing as before, either estimate suffices to conclude that $\mathrm{D}u_\varepsilon \to \mathrm{D}u$ almost everywhere in Ω.

We then have:

Theorem 14. *Suppose $g \in W^{1+\alpha,q}(\Omega)$ and F satisfies assumptions* (H2), (H3) *and either* (H1.1) *or* (H1.2). *Suppose $1 \leq p < q < \frac{(n+\alpha)p}{n}$. Then the conclusion regarding the regularity of relaxed minimisers in Theorem 9 still holds. If Ω is a $C^{1,\alpha}$-domain, $g \in W^{1+\max(\alpha,1/q),q}(\Omega)$, and F satisfies in addition* (H7), *then the conclusion of Theorem 9 regarding pointwise minimisers holds.*

Proof. For simplicity we only focus on the case $p \geq 2$. The case $p < 2$ follows from similar considerations. Assume first that (H1.1) holds. Let $\Omega \Subset B(0,R)$. Extend $p(x)$ to a $C^{0,\alpha}$-function on $B(0,R)$ with $p(x) \in [p,q]$ for $x \in B(0,R)$ by setting

$$p(x) \coloneqq \inf_{y\in\Omega}\left(p(y) + \|p\|_{C^{0,\alpha}(\Omega)}|x-y|\right).$$

for $x \in B(0,R) \setminus \Omega$. The first change to the argument occurs in the proof of the a-priori estimate in Lemma (3.9). Instead of (3.16), we obtain

$$\mathscr{F}_\varepsilon(v+g) - \mathscr{F}_\varepsilon(v_\varepsilon+g) \gtrsim \int_\Omega \left(\mu^2 + |\mathrm{D}v + \mathrm{D}g|^2 + |\mathrm{D}v_\varepsilon + \mathrm{D}g|^2\right)^{\frac{p(x)-2}{2}} |\mathrm{D}v - \mathrm{D}v_\varepsilon|^2 \,\mathrm{d}x.$$

We now argue as before to conclude that

$$\int_\Omega |\mathrm{D}v_\varepsilon|^{\frac{np(x)}{n-\beta}} \,\mathrm{d}x \lesssim 1 + \frac{1}{2}\|Dv_\varepsilon\|^p_{L^{\frac{np}{n-\beta}}(\Omega)} + C(\theta)\|v_\varepsilon\|^{\frac{\theta q}{(\theta q - p)}}_{W^{1,p}(\Omega)} + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}.$$

Noting that

$$\int_\Omega |\mathrm{D}v_\varepsilon|^{\frac{np}{n-\beta}} \,\mathrm{d}x \lesssim \int_\Omega |\mathrm{D}v_\varepsilon|^{\frac{np(x)}{n-\beta}} \,\mathrm{d}x + 1,$$

we finish the proof without further change.

For the second part of the proof of Theorem 9 we only need to verify that (H1.1) holds for F^s. The computation is straightforward.

We now assume that (H1.2) holds. Instead of (3.16), we have

$$\mathscr{F}_\varepsilon(v+g) - \mathscr{F}_\varepsilon(v_\varepsilon+g) \gtrsim \int_\Omega \sum_{i=1}^n V_{(\mu_i^2+|Dv_i+Dg_i|^2)^{\frac{1}{2}},p_i)}(Dv_i - Dv_{\varepsilon,i})^2.$$

Repeating the arguments given in Lemma 3.9, we use this to obtain

$$\begin{aligned}\int_\Omega \sum_{i=1}^n |\mathrm{D}v_{\varepsilon,i}|^{\frac{np_i}{n-\beta}} \,\mathrm{d}x \lesssim & 1 + \frac{1}{2}\|Dv_\varepsilon\|^p_{L^{\frac{np}{n-\beta}}(\Omega)} + C(\theta)\|v_\varepsilon\|^{\frac{\theta q}{(\theta q - p)}}_{W^{1,p}(\Omega)} \\ & + \|g\|^q_{W^{1+\alpha,q}(\Omega)} + \|f\|^{q'}_{L^{q'}(\Omega)}.\end{aligned}$$

The proof now concludes by the same arguments as before by noting that

$$\int_\Omega |\mathrm{D}v_\varepsilon|^{\frac{np}{n-\beta}} \,\mathrm{d}x \lesssim 1 + \int_\Omega \sum_{i=1}^n |\mathrm{D}v_{\varepsilon,i}|^{\frac{np_i(x)}{n-\beta}} \,\mathrm{d}x$$

In the second part of the proof of Theorem 9 we need to verify that (H1.2) holds for $F^s(x,\cdot)$. The argument is straightforward · □

Remark 3.18. It is straightforward to adapt the arguments of Theorem 14 to growth conditions that combine (H1.1) and (H1.2), e.g.,

$$\begin{cases} \sum_{i=1}^n \left(\mu_i^2 + |z_i|^2 + |w_i|^2\right)^{\frac{p_i(x)-2}{2}} |z_i - w_i|^2 \le F(x,z) - F(z,w) - \partial_z F(x,w)\cdot(z-w) \\ \text{where } z_i = (z_i)^j_{1\le j\le n}, w_i = (w_i)^j_{1\le j\le n} \text{ and } 1 < p \le p_i(x) \le q. \end{cases} \tag{H1.3}$$

9 Examples

In this section we list a number of examples to which the results of this chapter apply. We highlight in particular that we can treat the double-phase integrand (iii), the anisotropic $p(x)$-Laplacian (v) as well as more general anisotropic integrands, e.g. (iv). The theory developed in this paper applies to all the integrands listed below:

(i) $\mathscr{F}_1(u) \coloneqq \int_\Omega a(x)F(\mathrm{D}u)\,\mathrm{d}x$, where $1 \le a(\cdot) \le L$,

(ii) $\mathscr{F}_2(u) \coloneqq \int_\Omega \sum_{i=1}^n a_i(x)F_i(\mathrm{D}_i u)\,\mathrm{d}x$, where $1 \le a_i(\cdot) \le L$,

(iii) $\mathscr{F}_3(u) \coloneqq \int_\Omega \Big(|\mathrm{D}u|^p + a(x)|\mathrm{D}u|^q\Big)\,\mathrm{d}x$ where $0 \le a(x) \in C^{0,\alpha}(\Omega)$,

(iv) $\mathscr{F}_4(u) \coloneqq \int_\Omega \Big(|\mathrm{D}u|^p + |a^{i,j}_{\alpha,\beta}(x)\mathrm{D}_i u^\alpha \mathrm{D}_j u^\beta|^{\frac{q}{2}}\Big)\,\mathrm{d}x$ where $a^{i,j}_{\alpha,\beta}(\cdot) \in C^{0,\alpha}(\Omega)$ and for all $x \in \Omega$ and $\xi \in \mathbb{R}^{n\times m}$,

$$\lambda(x)|\xi|^2 \le a^{i,j}_{\alpha,\beta}\xi^\alpha_i \xi^\beta_j \le L\lambda(x)|\xi|^2$$

with $L \ge 1$ and $0 \le \lambda(\cdot) \in C^{0,\alpha}(\Omega)$. where $0 \le a_i(\cdot) \in C^{0,\alpha}(\Omega)$, and further $1 < p \le p_i \le q$,

(v) $\mathscr{F}_5(u) \coloneqq \int_\Omega \sum_{i=1}^n |\mathrm{D}_i u|^{p_i(x)}\,\mathrm{d}x$, where $p \le p_i(x) \le q$ and $p_i(x) \in C^{0,\alpha}(\Omega)$,

(vi) $\mathscr{F}_6(u) \coloneqq \int_\Omega |\mathrm{D}u|^{p(x)} \log(1+|\mathrm{D}u|)\,\mathrm{d}x$, where $1 < p \le p(x) \le q$,

(vii) $\mathscr{F}_7(u) \coloneqq \int_\Omega \Big(|\mathrm{D}u|^q + a(x)\max(|\mathrm{D}_n u|, 0)\Big)\,\mathrm{d}x$ where $q > 2$, $0 \le a(\cdot) \in C^{0,\alpha}(\Omega)$,

(viii) $\mathscr{F}_8(u) \coloneqq \int_\Omega F(x, \mathrm{D}u)\,\mathrm{d}x$ where $F(x,z) = h(a(x), z)$ where

- (i) $t \to h(t,z)$ is increasing,
- (ii) $h(x,z)$ is convex in the second argument,
- (iii) $a(x) \in C(\overline{\Omega})$,
- (iv) $F(x,z)$ satisfies (H4)–(H3) (or one of (H1.1), (H1.2) instead of (H4)).

The first five examples are standard. Example (vi) has been studied as a model case (with $p(x) = \text{const} > 1$) in [154]. Example (viii) is inspired by [177], whereas the last example is taken from [78].

Before making two observations that are helpful to verify that $W^{1,q}$-regularity holds for minimisers of these examples, let us extract the precise statement for the proof of which we use (H7).

Corollary 3.3. *Suppose Ω is a $C^{1,\alpha}$-domain. Suppose $g \in W^{1+\max\left(\alpha,\frac{1}{q}\right),q}(\Omega)$. Assume $\mathscr{F}(\cdot)$ satisfies* (H4)–(H3) *with $1 < p \leq q < \frac{(n+\alpha)p}{n}$. Suppose u is a pointwise minimiser of $\mathscr{F}(\cdot)$ in the class $\in W^{1,p}_g(\Omega)$ and one of the following holds:*

(i) *There is a sequence $\{u_k\} \subset W^{1,p}_g(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\mathscr{F}(u_k) \to \mathscr{F}(u)$ as $k \to \infty$.*

(ii) *There is $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and almost every $x \in \Omega$,*

$$F(x, \mathrm{D}u \star \phi_\varepsilon) \lesssim 1 + (F(\cdot, \mathrm{D}u(\cdot)) \star \phi_\varepsilon)(x).$$

Then $u \in W^{1,q}(\Omega)$.

Proof. Item (i) is precisely the conclusion from Lemma 3.6 we need in order to run the proof of Theorem 9. Item (ii) is the conclusion of Lemma 3.14 which is the only place (H7) is used. Hence, up to making these changes, the proof of Theorem 9 can be carried out as before. □

We highlight two ways of combining integrands for which the assumptions of Corollary 3.3 hold. First, if each $F_i(x,z)$, $i \in \{1, \dots, M\}$ for some $M \in \mathbb{N}$ satisfies assumption (ii) of Corollary 3.3, then $\sum_i F_i(x,z)$ also satisfies the assumption.

Second, assume G satisfies(H1)–(H3) and (i) in Corollary 3.3. Consider F and suppose that

$$G(x,z) - 1 \lesssim F(x,z) \lesssim G(x,z) + 1 \tag{3.38}$$

for almost every $x \in \Omega$ and every $z \in \mathbb{R}^{N\times n}$. Then from Lemma 3.6 applied to G there is $u_\varepsilon \in W^{1,p}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\int_\Omega G(x, \mathrm{D}u_\varepsilon)\,\mathrm{d}x \to \int_\Omega G(x, \mathrm{D}u)\,\mathrm{d}x$. Thus, by a version of the dominated convergence theorem and (3.38), $\int_\Omega F(x, \mathrm{D}u_\varepsilon)\,\mathrm{d}x \to \int_\Omega F(x, \mathrm{D}u)\,\mathrm{d}x$. Hence $F(x,z)$ also satisfies assumption (i) of Corollary 3.3.

Using these observations it is straightforward to see that the regularity results of this chapter apply to all the given examples.

4. Regularity under controlled-duality (p, q)-growth

In this chapter we consider autonomous integrands $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying the following growth condition:

$$\lambda\Big(\mu^2 + |z|^2\Big)^{\frac{p-2}{2}}|\xi|^2 \leq \partial_{zz}F(z)\xi\cdot\xi \leq \Lambda\Big(1 + |\partial_z F(z)|^2\Big)^{\frac{q-2}{2(q-1)}}|\xi|^2 \tag{4.1}$$

for all z, $\xi \in \mathbb{R}^{N\times n}$, where $2 \leq p \leq q < \infty$, λ, $\Lambda > 0$ and $\mu \in [0, 1]$. We refer to this growth condition as controlled-duality (p, q)-growth. It's not difficult to see that (4.1) implies, for some $c_1, c_2, c_3 > 0$ and all $z \in \mathbb{R}^{N\times n}$, the bound

$$c_1|z|^p - c_2 \lesssim F(z) \lesssim c_3(1 + |z|^q). \tag{4.2}$$

We recall that we used (4.1) in Chapter 1.2.4 to derive an improved monotonicity formula (1.23). This improved monotonicity formula lies at the heart of many of the results of this chapter. We further remark that from the point of convex duality theory (4.1) is a natural assumption as the results of Chapter 1.2.4 illustrate.

We note immediately that (4.1) implies that such an integrand F satisfies controlled (p, q)-growth, that is

$$\lambda\Big(\mu^2 + |z|^2\Big)^{\frac{p-2}{2}}|w|^2 \leq \partial_{zz}F(z)w\cdot w \leq c\Lambda\Big(1 + |z|^2\Big)^{\frac{q-2}{2}}|w|^2, \tag{4.3}$$

where z, $w \in \mathbb{R}^{N\times n}$ and $c > 0$ is a constant only depending on q. It is easy to see that the converse is false, that is, (4.1) is strictly stronger than (4.3) when $q > 2$. Nevertheless this means that our comments at the start of Chapter 3 apply equally here. This leaves us free to focus on motivating (4.1).

We saw in the proof of Lemma 3.8 that when considering the question of whether the relaxed functional satisfies the Euler-Lagrange system in some sense, the regularity

of the stress $\partial_z F(\mathrm{D}u)$ and duality considerations play a crucial role. We have already commented that these dual convexity considerations play a key role in the study of $W^{1,q}_{\mathrm{loc}}$-regularity of minimisers of (P) satisfying natural (p,q)-growth conditions with $q < \frac{np}{n-1}$ [44]. The considerations of this chapter are motivated by the approach of that paper. Recall in particular from Chapter 1.2.4 that growth bounds of the form (4.1) are preserved under Fenchel conjugation. To be precise we saw in Lemma 1.1 that for C^2-integrands (4.1) implies that the Fenchel conjugate $F^*(\cdot)$ satisfies

$$\frac{1}{\Lambda}\Big(1+|\partial_z F(z_0)|^2\Big)^{q'-2}|w|^2 \leq \partial_{\xi\xi}F^*(\partial_z F(z_0))w\cdot w \leq \frac{1}{\lambda}\Big(\mu^2+|z_0|^2\Big)^{\frac{2-p}{2}}|w|^2 \tag{4.4}$$

for all z_0, $w \in \mathbb{R}^{N\times n}$. In other words, the upper bound in (4.1) corresponds to a lower-bound of controlled p-growth type on $F^*(\cdot)$. Integrands for which both $F(\cdot)$ and $F^*(\cdot)$ are real-valued and strictly convex are referred to as functions of Legendre type in [182] and appear in optimisation problems.

The aim of this chapter is to show the following theorem:

Theorem 15. *Let $n = 2$ and $\Omega \subset \mathbb{R}^2$. Consider $F\colon \mathbb{R}^{N\times 2} \to \mathbb{R}$, satisfying (4.1) and suppose that F is a convex polynomial of degree $d \geq 2$ satisfying the lower bound $\partial_{zz}F(z)w\cdot w \geq \lambda|w|^2$ for some $\lambda > 0$ and all $z, w \in \mathbb{R}^{N\times 2}$. Suppose $u \in W^{1,p}(\Omega)$ is a minimiser of the corresponding functional $\mathscr{F}(\cdot)$. Then u is real analytic in the interior of Ω.*

Remark 4.1. The author believes that assuming a non-degenerate quadratic lower bound is not needed in order for the result to hold. The statement should hold whenever $F(\cdot)$ is a convex polynomial of degree $d \geq 2$ satisfying a lower bound of the form $\partial_z F(z)\xi\cdot\xi \geq \lambda(\mu+|z|^2)^{\frac{p-2}{2}}|\xi|^2$ with $p \geq 2$ for some $\lambda > 0$, $\mu \geq 0$. The non-degenerate quadratic lower bound is only applied in the argument of Section 3.

We remark that, even in simple cases, there are open questions concerning convex polynomials that are related to questions arising in the calculus of variations. An example is [180] where already in a simple set-up the problem of finding the extremals of cones of convex polynomials is not fully settled.

The proof splits into two main parts. We first prove an a-priori estimate in the style of the results of Chapter 3 for functionals satisfying controlled-duality (p,q)-growth. The

key observation here is the improved monotonicity formula of Lemma 1.2. Adapting well-known results in the case $n = 2$ this allows us to prove $C^{1,\beta}_{\mathrm{loc}}(\Omega)$-regularity of minimisers for any $\beta \in (0, 1)$. We then go on to investigate controlled-duality growth more closely. In particular, we show that for convex polynomials of degree $d \geq 2$ the upper bound in the natural growth condition implies the corresponding controlled-duality growth condition. Combining these two observations with standard results showing that $C^{1,\beta}_{\mathrm{loc}}$-regularity of minimisers of real analytic functionals implies local real analyticity of minimisers, will prove the theorem.

However, we intend to start this chapter with a detour. We will revisit the relaxed functional in a set-up that is slightly different from that explored in Chapter 3.2 and that will allow us to formulate the results in the framework of Γ-convergence. This abstract approach is useful as it prepares the path to obtaining similar results to those obtained in this chapter in more abstract settings, e.g. for Banach-space valued maps.

1 The relaxed functional and the Euler-Lagrange system

Throughout this section we fix two exponents p, q satisfying $1 < p < q < \infty$, and assume that Ω is a bounded $W^{1,p}$-extension domain in $\mathbb{R}^n$. We will impose further restrictions on p and q as necessary.

Given a Carathéodory integrand $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying natural (p, q)-growth conditions of the form

$$c_1|z|^p - c_2 \leq F(x, z) \leq c_3\Big(|z|^q + 1\Big) \quad \forall\, (x, z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{4.5}$$

where $c_1 > 0$, c_2, $c_3 \in \mathbb{R}$ are constants, we define the relaxed functional for each $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ by

$$\overline{\mathscr{F}}(u) \coloneqq \inf\left\{ \liminf_{j\to\infty} \int_\Omega F(\cdot, \mathrm{D}u_j)\ \mathrm{d}x : \begin{array}{l} u_j \in W^{1,q}(\Omega, \mathbb{R}^N) \\ u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^N) \end{array} \right\}. \tag{4.6}$$

The reader should compare this definition to (3.3). In particular, we emphasize that here we do not impose a Dirichlet boundary condition. The main reason is the potentially large gap $q - p$ allowed in this chapter.

Now, noting that the $W^{1,p}$ weak topology is metrizable on norm bounded sets we see that $\overline{\mathscr{F}}(\cdot)$ is nothing but the weak lower semicontiuous envelope of the functional I_F

defined on $W^{k,p}$ as

$$I_F(u) \coloneqq \begin{cases} \displaystyle\int_\Omega F(\cdot, \mathrm{D}u)\ \mathrm{d}x & \text{if } u \in W^{1,q}(\Omega, \mathbb{R}^N) \\ +\infty & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^N) \setminus W^{1,q}(\Omega, \mathbb{R}^N). \end{cases}$$

due to the lower bound in (4.5). In particular, this further implies that

$$\int_\Omega \Big(c_1 |\mathrm{D}u|^p - c_2 \Big)\ \mathrm{d}x \leq \overline{\mathscr{F}}(u) \tag{4.7}$$

for all $u \in W^{1,p}(\Omega)$.

Remark 4.2. We note that there is some flexibility in the choice of space when defining the relaxed functional $\mathcal{F}$. Indeed, the assumptions on $\overline{\mathscr{F}}$ – in particular the upper bound in (4.5) – imply that the functional $u \mapsto \int_\Omega F(\cdot, \mathrm{D}u)\ \mathrm{d}x$ is strongly continuous on $W^{1,q}(\Omega)$. Thus we could replace $W^{1,q}(\Omega)$ by $W^{1,s}(\Omega)$ for any $s \in [q, \infty]$ or, in fact, with any other subset of $W^{1,q}(\Omega)$ that is strongly dense in $W^{1,q}(\Omega)$.

As we saw in Chapter 3 it is useful to consider regularised functionals $\mathscr{F}_\varepsilon(\cdot)$ with integrands $F_\varepsilon(\cdot) = F() + \varepsilon|\cdot|^s$ where $s \geq q$. We also saw that in the set-up of Chapter 3 these functionals were related to the relaxed functional in the limit. The situation is no different now, but we choose to view the results through the lense of Γ-convergence.

Proposition 2. *Assume $F \colon \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is a Carathéodory integrand satisfying* (4.5)*, and define for $s \geq q$ and $\varepsilon > 0$,*

$$\mathscr{F}_\varepsilon(u) \coloneqq \int_\Omega \left(F(\cdot, \mathrm{D}u) + \varepsilon |\mathrm{D}u|^s \right) \mathrm{d}x, \tag{4.8}$$

where we understand this as $+\infty$ when $u \in W^{1,p}(\Omega) \setminus W^{1,s}(\Omega)$. Then

$$\mathscr{F}_\varepsilon \to \overline{\mathscr{F}} \ \text{ as } \ \varepsilon \searrow 0$$

in the sense of Γ-convergence on $W^{1,p}$ with its weak topology.

For the convenience of the reader, we recall that Proposition 2 requires us to prove the following two inequalities:

$$\liminf_{\varepsilon \searrow 0} \mathscr{F}_\varepsilon(u_\varepsilon) \geq \overline{\mathscr{F}}(u)$$

whenever $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}$, and for each $u \in W^{1,p}$ there exists $(u_\varepsilon) \subset W^{1,p}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}$ and

$$\limsup_{\varepsilon \searrow 0} \mathscr{F}_\varepsilon(u_\varepsilon) \leq \overline{\mathscr{F}}(u).$$

This definition, and much more information about Γ-convergence, can be found in [59].

Proof. It is clear that the first inequality holds. In order to prove the second inequality, fix $u \in W^{1,p}(\Omega)$. By density of $W^{1,s}(\Omega)$ in $W^{1,p}(\Omega)$ and the definition of $\overline{\mathscr{F}}(\cdot)$ there is $\{v_j\} \subset W^{1,s}(\Omega)$ so that $v_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $I_F(v_j) \to \overline{\mathscr{F}}(u)$. The claim holds trivially if $v_j = 0$ for all j. Thus without loss of generality we may assume that $v_1 \neq 0$. Define

$$\varepsilon_j \coloneqq \frac{1}{j \max\Big\{\int_\Omega |\mathrm{D}v_i|^s \, \mathrm{d}x : 1 \leq i \leq j\Big\}}. \tag{4.9}$$

Note that $\{\varepsilon_j\}$ is a strictly decreasing sequence converging to 0, and set

$$u_\varepsilon \coloneqq \begin{cases} v_j & \text{if } \varepsilon \in [\varepsilon_j, \varepsilon_{j-1}),\ j \geq 2, \\ v_1 & \text{if } \varepsilon \geq \varepsilon_1. \end{cases} \tag{4.10}$$

Then $\{u_\varepsilon\} \subset W^{1,s}(\Omega)$, $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$ as $\varepsilon \searrow 0$, as required. □

The next result is a standard consequence of the Γ-convergence established in Proposition 2, though the general context makes it difficult to find an easily accessible reference in the literature, and we therefore give the short proof below.

Proposition 3. *We retain the assumptions of Proposition 2 and let $g \in W^{1,p}(\Omega)$. Then there exists $\{g_\varepsilon\} \subset W^{1,s}(\Omega)$ with*

$$g_\varepsilon \rightharpoonup g \ \text{ weakly in } \ W^{1,p}(\Omega), \quad \varepsilon \int_\Omega |\mathrm{D}g_\varepsilon|^s \, \mathrm{d}x \to 0 \tag{4.11}$$

and

$$\inf \mathscr{F}_\varepsilon\Big(W^{k,s}_{g_\varepsilon}\Big) \to \inf \overline{\mathscr{F}}\Big(W^{1,p}_g\Big) \ \text{ as } \ \varepsilon \searrow 0,$$

Conversely, if $\{g_\varepsilon\}$ is any family in $W^{1,s}(\Omega)$ satisfying (4.11) and $u_\varepsilon \in W^{1,s}_{g_\varepsilon}(\Omega)$ minimises $\mathscr{F}_\varepsilon(\cdot)$ for each $\varepsilon > 0$, then, if $\sup_{\varepsilon>0} \mathscr{F}_\varepsilon(u_\varepsilon) < +\infty$, any $W^{1,p}(\Omega)$-weak cluster point for u_ε as $\varepsilon \searrow 0$ is a minimiser for $\overline{\mathscr{F}}(\cdot)$ on $W^{1,p}_g(\Omega)$.

Proof. First note that the liminf inequality from Proposition 2 and the lower bound in (4.5) guarantee that for any family $\{g_\varepsilon\} \subset W^{1,s}(\Omega)$ satisfying (4.11) we have

$$\liminf_{\varepsilon \searrow 0} \inf \mathscr{F}_\varepsilon\Big(W^{1,s}_{g_\varepsilon}\Big) \geq \inf \overline{\mathscr{F}}\Big(W^{1,p}_g\Big).$$

The last part of the proposition is also an easy consequence of the coercivity and the liminf inequality. We leave the details to the interested reader, c.f. also Lemma 3.5. We

turn to the first part of the proposition. By the direct method we find $u \in W^{1,p}_g(\Omega)$ so that $\overline{\mathscr{F}}(u) = \inf \mathscr{F}\left(W^{1,p}_g\right)$. The limsup inequality of Proposition 2 gives us a family $\{u_\varepsilon\} \subset W^{1,s}(\Omega)$ so that $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and

$$\limsup_{\varepsilon \searrow 0} \mathscr{F}_\varepsilon(u_\varepsilon) \leq \overline{\mathscr{F}}(u).$$

We conclude the proof by noting that $g_\varepsilon \coloneqq u_\varepsilon$ have all the required properties. □

We summarise our findings regarding the relaxed functional in the following proposition.

Proposition 4. *We retain the assumptions of Proposition 2 and assume, in addition, $F(x,\cdot)$ is convex for each fixed $x \in \Omega$. Then $\overline{\mathscr{F}}\colon W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is the lower semicontinuous envelope of I_F in the strong $W^{1,p}(\Omega)$ topology. Furthermore, it is a convex, proper, weakly lower semicontinuous functional on $W^{1,p}(\Omega)$ that is coercive in the sense of* (4.7)*, and extends I_F to $W^{1,p}(\Omega)$. More precisely we have*

$$\overline{\mathscr{F}}(u) \geq \int_\Omega F(\cdot, \mathrm{D}u)\ \mathrm{d}x \tag{4.12}$$

for all $u \in W^{1,p}(\Omega)$, with equality when $u \in W^{1,q}(\Omega)$. Finally, if we define the functionals $\mathscr{F}_\varepsilon$ as in Proposition 2, then

$$\mathscr{F}_\varepsilon \to \overline{\mathscr{F}} \ \text{ as } \ \varepsilon \searrow 0$$

in the sense of Γ-convergence on $W^{1,p}(\Omega)$ with its strong topology.

Proof. The weak lower semicontinuity, coercivity and properness (note that $\overline{\mathscr{F}}(\cdot)$ is finite on $W^{1,q}(\Omega)$) have already been established and do not require the convexity assumption. The remaining statements follow from the convexity of the integrand with respect to the z-variable. Due to this assumption the epigraph of I_F is a convex subset of $W^{1,p}(\Omega)$, and its weak closure in $W^{1,p}(\Omega)$ remains convex. However this closure is the epigraph of $\overline{\mathscr{F}}(\cdot)$ that therefore also is convex. By Mazur's lemma, the weak and strong closures of a convex subset of a Banach space coincide, hence $\overline{\mathscr{F}}(\cdot)$ is also the strong lower semicontinuous envelope of I_F in $W^{1,p}(\Omega)$. The inequality (4.12) for $u \in W^{1,p}(\Omega)$ now follows from Fatou's lemma, whereas the equality for $u \in W^{1,q}(\Omega)$ follows directly from the definition of relaxed functional. We turn to the proof of Γ-convergence. The liminf inequality is a direct consequence of the definition of $\overline{\mathscr{F}}(\cdot)$. For the limsup inequality

we fix $u \in W^{1,p}(\Omega)$. Since $W^{1,s}(\Omega)$ is strongly dense in $W^{1,p}(\Omega)$ and $\overline{\mathscr{F}}(\cdot)$ is the strong lower semicontinuous envelope of I_F on $W^{1,p}(\Omega)$, we find a sequence $\{v_j\} \subset W^{1,s}(\Omega)$ so $v_j \to u$ strongly in $W^{1,p}(\Omega)$ and $I_F(v_j) \to \overline{\mathscr{F}}(u)$. Define ε_j as in (4.9) and u_ε as in (4.10). Then $\{u_\varepsilon\} \subset W^{1,s}(\Omega)$, $u_\varepsilon \to u$ strongly in $W^{1,p}(\Omega)$ and $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$ as $\varepsilon \searrow 0$. This concludes the proof. □

We finally apply the above results to minimisers of $\overline{\mathscr{F}}(\cdot)$, employing the convexity condition that

$$z \mapsto F(x,z) - \lambda|z|^p \tag{4.13}$$

is convex for each fixed $x \in \Omega$, where $\lambda \in (0, c_1)$ is a constant.

Proposition 5. *Assume $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand satisfying* (4.5) *and* (4.13). *If $v_\varepsilon \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and $\mathscr{F}_\varepsilon(v_\varepsilon) \to \overline{\mathscr{F}}(v) < +\infty$ as $\varepsilon \searrow 0$, then $v_\varepsilon \to v$ strongly in $W^{1,p}(\Omega)$ and $\varepsilon \int_\Omega |\mathrm{D}v_\varepsilon|^s \,\mathrm{d}x \to 0$ as $\varepsilon \searrow 0$. In particular, if $u \in W^{1,p}(\Omega)$ is a minimiser for $\overline{\mathscr{F}}(\cdot)$, then there exist minimisers $u_\varepsilon \in W^{1,s}(\Omega)$ for $\mathscr{F}_\varepsilon$ so that $u_\varepsilon \to u$ strongly in $W^{1,p}(\Omega)$ and $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$ as $\varepsilon \searrow 0$.*

Proof. Let v_ε be as in the statement of the proposition. Then $v_\varepsilon \in W^{1,s}(\Omega)$ for all $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. Since $\overline{\mathscr{F}}(\cdot)$ is the weakly lower semicontinuous envelope of I_F on $W^{1,p}(\Omega)$ and $s \geq q$ we deduce that $I_F(v_\varepsilon) \to \overline{\mathscr{F}}(v)$ and $\varepsilon \int_\Omega |\mathrm{D}v_\varepsilon|^s \,\mathrm{d}x \to 0$ as $\varepsilon \searrow 0$. As $F(x,\cdot) - \lambda|\cdot|^p$ is convex for each $x \in \Omega$ we infer from Proposition 4 that

$$\liminf_{\varepsilon \searrow 0} I_{F-\lambda|\cdot|^p}(v_\varepsilon) \geq \overline{\mathscr{F}}(v) - \lambda \int_\Omega |\mathrm{D}v|^p \,\mathrm{d}x.$$

The left-hand side can be rewritten as

$$\liminf_{\varepsilon \searrow 0} \left(I_F(v_\varepsilon) - \lambda \int_\Omega |\mathrm{D}v_\varepsilon|^p \,\mathrm{d}x \right) = \overline{\mathscr{F}}(v) - \lambda \limsup_{\varepsilon \searrow 0} \int_\Omega |\mathrm{D}v_\varepsilon|^p \,\mathrm{d}x.$$

In particular we have

$$\limsup_{\varepsilon \searrow 0} \int_\Omega |\mathrm{D}v_\varepsilon|^p \,\mathrm{d}x \leq \int_\Omega |\mathrm{D}v|^p \,\mathrm{d}x.$$

It is straightforward to see that this shows that $\mathrm{D}v_\varepsilon \to \mathrm{D}v$ strongly in $L^p(\Omega)$, and hence $v_\varepsilon \to v$ strongly in $W^{1,p}(\Omega)$ as $\varepsilon \searrow 0$.

Turning to the last part of the Proposition, let $u \in W^{1,p}(\Omega)$ be a minimiser for $\overline{\mathscr{F}}(\cdot)$. Due to Proposition 4, $v \mapsto \mathcal{F}(v) - \lambda \int_\Omega |\mathrm{D}v|^p \,\mathrm{d}x$ is convex in $W^{1,p}(\Omega)$, so that u

is the unique minimiser for $\overline{\mathscr{F}}(\cdot)$ on $W^{1,p}_u(\Omega)$. Using Proposition 3, we find minimisers $u_\varepsilon \in W^{1,s}(\Omega)$ for $\mathscr{F}_\varepsilon$ so that $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$ and any $W^{1,p}(\Omega)$-weak cluster point for (u_ε) as $\varepsilon \searrow 0$ is a minimiser for $\overline{\mathscr{F}}$ on $W^{1,p}_u(\Omega)$. By uniqueness, and considering a subsequence argument we conclude that $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. Using the first part of the proposition, the convergence is strong in $W^{1,p}(\Omega)$. This concludes the proof. □

We now restrict our attention to the case

$$1 < p \leq q \leq p^*, \tag{4.14}$$

where p^* is the Sobolev conjugate exponent to p in n dimensions:

$$p^* \coloneqq \begin{cases} \dfrac{np}{n-p} & \text{if } 1 < p < n, \\ \infty & \text{if } p \geq n. \end{cases}$$

Our goal is to show a local result concerning the Euler-Lagrange equation in analogy to Lemma 3.8. We note that similar results were proven in [45], but that our results here are localized in Ω. As part of the proof we obtain regularity of the stress $\partial_z F(\mathrm{D}u)$ of relaxed minimisers.

Proposition 6. *Retain the assumptions and notation of Proposition 5. Then we have* $\partial_z F(\mathrm{D}u) \in L^{q'}_{\mathrm{loc}}(\Omega)$ *and, further,*

$$F_z(\cdot, \mathrm{D}u_\varepsilon) \rightharpoonup F_z(\cdot, \mathrm{D}u) \ \text{ weakly in } \ L^{q'}_{\mathrm{loc}} \ \text{ and in measure on } \Omega, \ \text{ as } \ \varepsilon \searrow 0.$$

Furthermore,

$$\int_\Omega \partial_z F(\cdot, \mathrm{D}u) \cdot \mathrm{D}\phi \, \mathrm{d}x = 0 \tag{4.15}$$

holds for all compactly supported $W^{1,q}$ maps ϕ on Ω.

Proof. Assume that $u \in W^{1,p}(\Omega)$ minimises $\overline{\mathscr{F}}(\cdot)$. The existence of the family $\{u_\varepsilon\}$ of minimisers for $\mathscr{F}_\varepsilon(\cdot)$ so that $u_\varepsilon \to u$ strongly in $W^{1,p}(\Omega)$ and $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$ as $\varepsilon \searrow 0$ follows from Proposition 5. For this family, we first note the obvious bound

$$\int_\Omega \Big(c_1 |\mathrm{D}u_\varepsilon|^p - c_2 + \varepsilon |\mathrm{D}u_\varepsilon|^s \Big) \, \mathrm{d}x \leq \mathscr{F}_\varepsilon(u_\varepsilon),$$

and its consequence

$$\int_\Omega \Big(|\mathrm{D}u_\varepsilon|^p + \varepsilon |\mathrm{D}u_\varepsilon|^s \Big) \, \mathrm{d}x \leq \frac{c_2}{c_1} \mathcal{L}^n(\Omega) + 1 + \frac{1}{c_1} \overline{\mathscr{F}}(u) \tag{4.16}$$

for sufficiently small ε, say for all $\varepsilon \in (0, \delta)$. Next we employ elementary convexity bounds in connection with the Euler-Lagrange system for $\mathscr{F}_\varepsilon$. The Fenchel-Young inequality in the extremal case yields for all (x, z) the identity

$$F(x,z) + F^*(x, \partial_z F(x,z)) = \partial_z F(x,z) \cdot z,$$

where F^* is defined as in (1.17). Using Lemma 1.16 we find

$$\partial_z F(x,z) \cdot z \geq A\Big(|z|^p + |\partial_z F(x,z)|^{q'}\Big) - B \tag{4.17}$$

for all (x, z), where A, $B > 0$ are constants whose precise values are not important. If we denote

$$\sigma_\varepsilon \coloneqq \partial_z F(\cdot, \mathrm{D}u_\varepsilon) \text{ and } \mu_\varepsilon \coloneqq |\mathrm{D}u_\varepsilon|^{s-2}\mathrm{D}u_\varepsilon,$$

then the Euler-Lagrange system for $\mathscr{F}_\varepsilon$ in its weak formulation is

$$\int_\Omega (\sigma_\varepsilon + \varepsilon\mu_\varepsilon) \cdot \mathrm{D}\phi \, \mathrm{d}x = 0 \quad \forall\, \phi \in W^{1,s}_0.$$

Fix a ball $B_{2R} = B_{2R}(x_0) \subset \Omega$ and a smooth cut-off function η satisfying

$$\mathbb{1}_{B_R} \leq \eta \leq \mathbb{1}_{B_{2R}} \qquad \text{and} \qquad |\mathrm{D}\eta| \leq cR^{-1}.$$

We test the Euler-Lagrange system with $\phi \coloneqq \eta^s(u_\varepsilon - a_\varepsilon)$, where a_ε are constants that will be chosen shortly. A routine calculation using (4.17) and Young's inequality yields

$$\begin{aligned}\int_\Omega \eta^s \Big(|\mathrm{D}u_\varepsilon|^p + |\sigma_\varepsilon|^{q'} + \varepsilon|\mathrm{D}u_\varepsilon|^s\Big) \, \mathrm{d}x \leq & c\int_\Omega \Big(\eta^s + |u_\varepsilon - a_\varepsilon|^q |\mathrm{D}\eta|^q\Big) \, \mathrm{d}x \\ & + c\varepsilon \int_\Omega |u_\varepsilon - a_\varepsilon|^s |\mathrm{D}\eta|^s \, \mathrm{d}x \end{aligned} \tag{4.18}$$

for a constant c whose precise value is not important. We now set

$$a_\varepsilon(x_0) \coloneqq (u_\varepsilon)_{B_{2R}}.$$

Using the properties of η and Poincaré's inequality (here we use $q \leq p^*$), we estimate the right-hand side of (4.18) from above by

$$c\int_{B_{2R}} \Big(1 + \varepsilon|\mathrm{D}u_\varepsilon|^s\Big) \, \mathrm{d}x + c\left(\fint_{B_{2R}} |\mathrm{D}u_\varepsilon|^p \, \mathrm{d}x\right)^{\frac{q}{p}} \mathscr{L}^n(B_{2R}).$$

Using (4.16) we find

$$\int_\Omega \eta^s \Big(|\mathrm{D}u_\varepsilon|^p + |\sigma_\varepsilon|^{q'} + \varepsilon\langle \mathrm{D}u_\varepsilon\rangle^s\Big) \, \mathrm{d}x \leq c\Big(1 + \overline{\mathscr{F}}(u) + \overline{\mathscr{F}}(u)^{\frac{q}{p}}\Big)$$

for $\varepsilon \in (0,\delta]$. Since the ball $B_{2R} \subset \Omega$ was arbitrary, it follows in particular that the family $\left\{\sigma_\varepsilon\right\}$ is bounded in $L^{q'}$ locally on Ω as $\varepsilon \searrow 0$. Now since $\partial_z F(x,z)$ is Carathéodory and $\mathrm{D}u_\varepsilon \to \mathrm{D}u$ in measure in Ω also $\sigma_\varepsilon \to \partial_z F(\cdot,\mathrm{D}u)$ in measure on Ω. Using (4.16) again we estimate

$$\int_\Omega \left|\varepsilon\mu_\varepsilon\right|^{s'} \,\mathrm{d}x \leq \varepsilon^{\frac{1}{s-1}} \mathscr{F}_\varepsilon(u_\varepsilon) \leq \varepsilon^{\frac{1}{s-1}} \left(1 + \overline{\mathscr{F}}(u)\right)$$

for $\varepsilon \in (0,\delta]$. In particular, we deduce that $\varepsilon\mu_\varepsilon \to 0$ in $L^{s'}$ on Ω, and thus we find $\sigma_\varepsilon + \varepsilon\mu_\varepsilon \rightharpoonup \partial_z F(\cdot,\mathrm{D}u)$ weakly in $L^{s'}$ locally on Ω. Therefore, we may pass to the limit $\varepsilon \searrow 0$ in the Euler-Lagrange system for $\mathscr{F}_\varepsilon(\cdot)$ to deduce that

$$\int_\Omega \partial_z F(\cdot,\mathrm{D}u) \cdot \mathrm{D}\phi \,\mathrm{d}x = 0$$

holds for all $\phi \in \mathrm{C}_c^\infty(\Omega,\mathbb{R}^N)$. By Fatou's lemma we have for each ball $B \Subset \Omega$,

$$\int_B \left|\partial_z F(\cdot,\mathrm{D}u)\right|^{q'} \,\mathrm{d}x \leq \liminf_{\varepsilon \searrow 0} \int_B \left|\sigma_\varepsilon\right|^{q'} \,\mathrm{d}x < +\infty,$$

hence $\partial_z F(\cdot,\mathrm{D}u)$ is locally in $L^{q'}$ on Ω and so (4.15) follows too. □

Remark 4.3. The result in Proposition 6 implies that $F(\cdot,\mathrm{D}u) \in L^1$ and

$$\int_\Omega F(\cdot,\mathrm{D}u) \,\mathrm{d}x < \int_\Omega F(\cdot,\mathrm{D}u + \mathrm{D}\phi) \,\mathrm{d}x \tag{4.19}$$

for maps $\phi \in W_0^{1,q}(\Omega)$ that do not vanish identically. The map u is, therefore, in particular a very weak minimizer of the variational integral $v \mapsto \int_\Omega F(\cdot,\mathrm{D}v) \,\mathrm{d}x$ and a very weak solution to its Euler-Lagrange system. Well-known examples show that this is strictly weaker than being a minimizer for the relaxed functional $\mathcal{F}(\cdot)$.

2 Improved integrability under controlled-duality (p,q)-growth

In this section we establish $W^{1,q}_{\mathrm{loc}}$-regularity of minimisers of (P) for integrands F satisfying (4.1) if $2 \leq p \leq q < \frac{np}{n-2}$ for $n > 2$, $2 \leq p \leq q < \infty$ if $n = 2$. We stress that we assume no further structure conditions on $F(z)$. In particular, neither a radial structure nor a doubling condition is assumed to hold.

The outline of the proof is similar to the proof of the $W^{1,q}$-regularity results in Chapter 3. We need to prove an a-priori estimate for relaxed minimisers and to overcome the Lavrentiev phenomenon. The second step is considerably easier in the local autonomous

case and hence we incorporate it as a step of the proof rather than stating a separate result. Our main idea for the first step is to exploit the monotonicity formula of Lemma 1.2. This will allow us to improve the differentiability of $\partial_z F(\mathrm{D}u)$ which, in turn, will allow a further improvement in differentiability of u. Given an integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying (4.1), it will be useful to introduce the local quantity

$$\mathscr{F}(v, B_r) = \int_{B_r} F(\mathrm{D}v)\,\mathrm{d}x$$

for $B_r \Subset \Omega$ and $v \in W^{1,1}(\Omega)$. We then prove the following a-priori estimate:

Proposition 7. *Let $2 \le p \le q < \frac{np}{n-2}$ if $n > 2$, $2 \le p < q$ if $n = 2$. Suppose $F(\cdot) \in C^2(\mathbb{R}^{N\times n})$ satisfies (4.1). Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a minimiser of $\mathscr{F}(\cdot)$. Then*

$$\mathrm{D}u \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}) \quad \textit{for all} \ \ \gamma \in \left[1, \frac{np}{n-2}\right],$$
$$\partial_z F(\mathrm{D}u) \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}) \quad \textit{for all} \ \ \gamma \in \left[1, \frac{nq'}{n-2}\right].$$

Further

$$V_{\mu,p}(\mathrm{D}u) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}) \quad \textit{and} \quad V_{\mu,q'}(\partial_z F(\mathrm{D}u)) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}).$$

Moreover, if $B_\rho \Subset B_r \Subset \Omega$ are concentric balls we have the estimate that

$$\|\partial_z F Du\|_{L^{\frac{nq'}{n-2}}(B_\rho)} + \|Du\|_{L^{\frac{np}{n-2}}(B_\rho)} \le \frac{c}{(r-\rho)^{\gamma}}\left[1 + \mathscr{F}(u, B_r)\right]^{\gamma'},$$

with $\gamma, \gamma' \equiv \gamma(n, p, q)$ and $c \equiv c(n, N, \lambda, \Lambda, p, q, \|u\|_{W^{1,p}(B_r)}, \mathscr{F}(u, B_r))$.

Remark 4.4. We comment that many of the primitive examples of autonomous integrands with (p, q)-growth satisfy controlled-duality (p, q)-growth. These include

$$\mathscr{F}_1(u) = \int_\Omega \sum_{i=1}^{n} |D_i u|^{p_i}$$

with $p = \min_i p_i \le p(x) \le q = \max_i q_i$ or

$$\mathscr{F}_2(u) = \int_\Omega |\mathrm{D}u|^p + \sum_{i=1}^{n} |D_i u|^{p_i}$$

with $p \le p_i \le q = \max_i p_i$. In fact, the class of Legendre functions enables the construction of a wide range of examples of integrands satisfying controlled-duality (p, q)-growth. As an example, the Legendre functions given in [26] show that the following integrand enjoys controlled-duality (p, q)-growth:

$$F(z) = F(z_1, z_2) = |z|^2 + \frac{1}{2}(z_1^2 - 4z_1 + 3)\mathbb{1}_{z_1 \le 1} - \log(z_1)\mathbb{1}_{z_1 \ge 1}.$$

Proof. For notational simplicity we focus on the case where $r = 1$, that is we consider concentric balls $B_\rho \Subset B_1 = B \Subset \Omega$. The general case follows by the same argument.

We begin by setting up some notation and making a number of basic comments. As suggested at numerous points in this thesis already, we regularize u via mollification $u_\varepsilon \coloneqq u \star \phi_\varepsilon$. Then

$$u_\varepsilon \to u \quad \text{strongly in} \quad W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N). \tag{4.20}$$

Moreover, using Jensen's inequality,

$$F(\mathrm{D}u_\varepsilon) \leq F(\mathrm{D}u) \star \phi_\varepsilon \to F(\mathrm{D}u) \quad \text{strongly in} \quad L^1_{\mathrm{loc}}(\Omega). \tag{4.21}$$

Define the auxiliary integrands

$$W^{1,p}(B_\rho, \mathbb{R}^N) \ni w \mapsto \mathscr{F}_\varepsilon(w, B_\rho) := \int_{B_\rho} F_\varepsilon(\mathrm{D}w) \ \mathrm{d}x,$$

where, as usual,

$$F_\varepsilon(z) := F(z) + \kappa_\varepsilon(\mu^2 + |z|^2)^{\frac{q}{2}}, \tag{4.22}$$

and $\kappa_\varepsilon \in (0,1)$ is defined as

$$\kappa_\varepsilon := \left(\varepsilon^{-1} + 1 + \|\mathrm{D}u_\varepsilon\|^q_{L^q(B_\rho)}\right) \to 0. \tag{4.23}$$

as $\varepsilon \to 0$. By direct methods and strict convexity deriving from (4.1), we have that there exists a unique solution $v_\varepsilon \in W^{1,q}(B_\rho, \mathbb{R}^N)$ to the Dirichlet problem

$$u_\varepsilon + W^{1,q}_0(B, \mathbb{R}^N) \ni w \mapsto \min \mathscr{F}_\varepsilon(w, B).$$

Moreover, v_ε satisfies the Euler-Lagrange equation

$$0 = \int_B \partial_z F_\varepsilon(\mathrm{D}v_\varepsilon) \cdot \mathrm{D}\phi \ \mathrm{d}x \quad \text{for all} \quad \phi \in W^{1,q}_0(B, \mathbb{R}^N). \tag{4.24}$$

Notice that, by the minimality of v_ε in the class $u_\varepsilon + W^{1,q}_0(B, \mathbb{R}^N)$ and (4.2)

$$\begin{aligned}
\nu \int_B |\mathrm{D}v_\varepsilon|^p \ \mathrm{d}x + \kappa_\varepsilon \int_B (\mu^2 + |\mathrm{D}v_\varepsilon|^2)^{\frac{q}{2}} \ \mathrm{d}x &\leq \int_B \left[F_\varepsilon(\mathrm{D}v_\varepsilon) + L\right] \ \mathrm{d}x \\
&\leq \int_B \left[F(\mathrm{D}u_\varepsilon) + L\right] \ \mathrm{d}x + \kappa_\varepsilon \int_B (\mu^2 + |\mathrm{D}v_\varepsilon|^2)^{\frac{q}{2}} \ \mathrm{d}x \\
&\leq \mathscr{F}(u, B_\rho) + L|B_\rho| + \mathsf{o}(\varepsilon) =: \mathcal{M}.
\end{aligned} \tag{4.25}$$

To obtain the last line we have used (4.21) and (4.23). Moreover, using (4.25) and Poincaré inequality we deduce

$$\|v_\varepsilon\|_{W^{1,p}(B_\rho)} \le \|u\|_{W^{1,p}(B_\rho)} + c\mathcal{M} + \mathsf{o}(\varepsilon), \tag{4.26}$$

for $c \equiv c(n, N, \nu, L, p)$.

We are now prepared to proceed with the main part of the proof. We first derive an a-priori estimate, uniform in ε, for v_ε using the difference quotient method: Select parameters $\rho/2 \le \tau_1 < \tau_2 \le \rho$, a cut-off function $\eta \in C^1_c(B)$ so that

$$\mathbb{1}_{B_{\tau_1}} \le \eta \le \mathbb{1}_{B_{(\tau_2+\tau_1)/2}} \quad \text{and} \quad |\mathrm{D}\eta| \le \frac{8}{\tau_2 - \tau_2},$$

a vector $h \in \mathbb{R}^n$ with $|h| \le h_0 = \frac{1}{2}\min\{1, \operatorname{dist}(\operatorname{supp}(\eta), \partial B)\}$ and test (4.24) against $\phi := \tau_{-h}(\eta^2 \tau_h v_\varepsilon)$. Using integration by parts, we then obtain

$$0 = \int_B \tau_h \partial F_\varepsilon(\mathrm{D}v_\varepsilon) \cdot \left[\eta^2 \tau_h \mathrm{D}v_\varepsilon + 2\eta \mathrm{D}\eta \otimes \tau_h v_\varepsilon\right] \,\mathrm{d}x =: I + II.$$

Given $w \in W^{1,q}(B_\rho, \mathbb{R}^N)$ introduce

$$\mathcal{D}(h, \mathrm{D}w) := \left(\mu^2 + |\mathrm{D}w(x+h)|^2 + |\mathrm{D}w(x)|^2\right).$$

We first estimate I. Using Lemma 1.3, Lemma 1.4 and (4.1)

$$\begin{aligned}
|I| =& \int_{B_\rho} \eta^2 \tau_h \partial_z F(\mathrm{D}v_\varepsilon) \cdot \tau_h \mathrm{D}v_\varepsilon \,\mathrm{d}x + q\kappa_\varepsilon \int_{B_\rho} \eta^2 \tau_h \left((\mu^2 + |\mathrm{D}v_\varepsilon|^2)^{\frac{q-2}{2}} \mathrm{D}v_\varepsilon\right) \cdot \tau_h \mathrm{D}v_\varepsilon \,\mathrm{d}x \\
\ge& c \int_{B_\rho} \eta^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{p-2}{2}} |\tau_h \mathrm{D}v_\varepsilon|^2 \,\mathrm{d}x + c \int_{B_\rho} \eta^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'-2}{2}} |\tau_h \partial_z F(\mathrm{D}v_\varepsilon)|^2 \,\mathrm{d}x \\
&+ c\kappa_\varepsilon \int_{B_\rho} \eta^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\tau_h \mathrm{D}v_\varepsilon|^2 \,\mathrm{d}x \\
\ge& c \int_{B_\rho} \eta^2 |\tau_h V_{\mu,p}(\mathrm{D}v_\varepsilon)|^2 \,\mathrm{d}x + c \int_{B_\rho} \eta^2 |\tau_h V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon)|^2 \,\mathrm{d}x \\
&+ c\kappa_\varepsilon \int_{B_\rho} \eta^2 |\tau_h V_{\mu,q}(\mathrm{D}v_\varepsilon)|^2 \,\mathrm{d}x,
\end{aligned}$$

with $c \equiv c(n, N, \lambda, \Lambda, p, q)$. We next estimate II. By Lemma 1.3, Lemma 1.4, the mean value theorem, Hölder and Young inequalities, we obtain

$$\begin{aligned}
|II| \le& c \int_{B_\rho} \eta |\mathrm{D}\eta| \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\pm\frac{q'-2}{4}} |\partial_z F(Dv_\varepsilon)||\tau_h v_\varepsilon| \,\mathrm{d}x \\
&+ c\kappa_\varepsilon \int_{B_\rho} \eta^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\tau_h \mathrm{D}v_\varepsilon||\mathrm{D}\eta||\tau_h v_\varepsilon| \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
\leq & \iota \int_{B_\rho} \eta^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'-2}{2}} |\tau_h \partial_z F(\mathrm{D}v_\varepsilon)|^2 \ \mathrm{d}x \\
& + \frac{c}{\iota} \int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{2-q'}{2}} |\tau_h v_\varepsilon|^2 \ \mathrm{d}x \\
& + \iota \kappa_\varepsilon \int_{B_\rho} \eta^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\tau_h (\mathrm{D}v_\varepsilon)|^2 \ \mathrm{d}x \\
& + \frac{c\kappa_\varepsilon}{\iota} \int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\tau_h v_\varepsilon|^2 \ \mathrm{d}x \\
\leq & \iota c \int_{B_\rho} \eta^2 |V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon)|^2 \ \mathrm{d}x \\
& + \frac{c}{\iota} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 |\tau_h v_\varepsilon|^2 \ \mathrm{d}x \right)^{\frac{2}{q}} \\
& + \iota \kappa_\varepsilon c \int_{B_\rho} \eta^2 |V_{\mu,q}(\mathrm{D}v_\varepsilon)|^2 \ \mathrm{d}x + \frac{c\kappa_\varepsilon |h|^2}{\iota} \int |\mathrm{D}\eta|^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \ \mathrm{d}x,
\end{aligned}
$$

for $c \equiv c(n, N, \Lambda, p, q)$ and arbitrary $\iota \in (0, 1)$. Combining estimates, choosing $\iota > 0$ sufficiently small and re-arranging, we conclude

$$
\begin{aligned}
& \int_{B_\rho} \eta^2 |\tau_h V_{\mu,p}(\mathrm{D}v_\varepsilon)|^2 \ \mathrm{d}x + \int_{B_\rho} \eta^2 |\tau_h V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon))|^2 \ \mathrm{d}x \\
& \qquad + \kappa_\varepsilon \int_{B_\rho} \eta^2 |\tau_h V_{\mu,q}(\mathrm{D}v_\varepsilon)|^2 \ \mathrm{d}x \\
\leq & c \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 |\tau_h v_\varepsilon|^q \ \mathrm{d}x \right)^{\frac{2}{q}} \\
& \qquad + c|h|^2 \kappa_\varepsilon \int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \ \mathrm{d}x, \qquad (4.27)
\end{aligned}
$$

where $c \equiv c(n, N, \lambda, \Lambda, p, q)$.

We now split the rest of the proof into three steps. We first improve the integrability of $\partial_z F(Dv_\varepsilon)$. Next, we use this improvement to improve the integrability of v_ε. In the third step, we pass to the limit as $\varepsilon \to 0$.

Step 1: Higher integrability of $\boldsymbol{\partial_z F(\mathrm{D}v_\varepsilon)}$ Set $\alpha := 1 - n\left(\frac{1}{p} - \frac{1}{q}\right)$. Dividing both sides of the inequality (4.27) by $|h|^{2\alpha}$ we obtain

$$
\begin{aligned}
\mathcal{I}(h) = & \int_{B_\rho} \eta^2 \left| \frac{\tau_h V_{\mu,p}(\mathrm{D}v_\varepsilon)}{|h|^\alpha} \right|^2 \ \mathrm{d}x + \int_{B_\rho} \eta^2 \left| \frac{\tau_h V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon))}{|h|^\alpha} \right|^2 \ \mathrm{d}x \\
& + \kappa_\varepsilon \int_{B_\rho} \eta^2 \left| \frac{\tau_h V_{\mu,q}(\mathrm{D}v_\varepsilon)}{|h|^\alpha} \right|^2 \ \mathrm{d}x \\
\leq & c \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|^\alpha} \right|^q \ \mathrm{d}x \right)^{\frac{2}{q}}
\end{aligned}
$$

$$+ c|h|^{2(1-\alpha)}\kappa_\varepsilon \int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \ \mathrm{d}x = S_1(h) + S_2(h). \quad (4.28)$$

We first estimate $S_2(h)$. Using Lemma 1.12 and Lemma 1.13 as well as (4.25), we find

$$\begin{aligned} S_2(h) &= \left(c|h|^{2(1-\alpha)}\kappa_\varepsilon \int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \ \mathrm{d}x \right) \\ &\leq \frac{c\kappa_\varepsilon}{(\tau_2-\tau_1)^2} \int_{B_{(\tau_2+\tau_1)/2}} (\mu^2 + |\mathrm{D}v_\varepsilon|^2)^{\frac{q}{2}} \ \mathrm{d}x \leq \frac{c\mathcal{M}}{(\tau_2-\tau_1)^2} \end{aligned}$$

Moreover, using Lemma 1.15, Lemma 1.12 and (4.26) we find

$$\begin{aligned} S_1(h) =& c \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|^\alpha} \right|^q \ \mathrm{d}x \right)^{\frac{2}{q}} \\ \leq& \frac{c}{(\tau_2-\tau_1)^4} \left(\int_{B_{(\tau_2+\tau_1)/2}} (\mu^2 + |\partial_z F(\mathrm{D}v_\varepsilon)|^2)^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}} \|v_\varepsilon\|^2_{W^{1,p}(B_{(\tau_2+\tau_1)/2})} \\ \leq& \frac{c\left(1 + \mathscr{F}(u, B_\rho)^2 + \|u\|^2_{W^{1,p}(B_\rho)}\right)}{(\tau_2-\tau_1)^4} \left(\int_{B_{\tau_2}} (\mu^2 + |\partial_z F(\mathrm{D}v_\varepsilon)|^2)^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}}, \end{aligned}$$

for $c \equiv c(n, N, \nu, L, p)$. Returning to (4.28) we hence deduce

$$\sup_{|h|<h_0} \mathcal{I}(h) \leq \frac{c}{(\tau_2-\tau_1)^2} + \frac{c}{(\tau_2-\tau_1)^4} \left(\int_{B_{\tau_2}} (\mu^2 + |\partial_z F(\mathrm{D}v_\varepsilon)|^2)^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}},$$

with $c \equiv c(n, N, \nu, L, p, q, \mathscr{F}(u, B_\rho), \|u\|_{W^{1,p}(B_\rho)})$. Therefore, applying Lemma 1.14, we have obtained for any $\beta \in (0, \alpha)$,

$$\begin{aligned} &\|V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon))\|_{W^{\beta,2}(B_{\tau_1})} + \|V_{\mu,p}(\mathrm{D}v_\varepsilon)\|_{W^{\beta,2}(B_{\tau_1})} \\ \leq& \frac{c}{(\alpha-\beta)^{\frac{1}{2}}} (\tau_2-\tau_1)^{\alpha-\beta} \left[\frac{1}{(\tau_2-\tau_1)^2} + \frac{1}{(\tau_2-\tau_1)^2} \|V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{q-2}{q}}_{L^2(B_{\tau_2})} \right] \\ &+ \frac{c}{(\tau_2-\tau_1)^{n/2+\beta}} \|V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^2(B_{\tau_2})}, \end{aligned} \quad (4.29)$$

where $c \equiv c(n, N, \lambda, \Lambda, p, q, \mathscr{F}(u, B_\rho), \|u\|_{W^{1,p}(B_\rho)})$. Applying now Lemma 1 we conclude

$$\begin{aligned} &\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{q'}{2}}_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})} \leq c\|V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon))\|_{L^{\frac{2n}{n-2\beta}}} \\ &\leq \frac{c}{(\alpha-\beta)^{\frac{1}{2}}} (\tau_2-\tau_1)^{\alpha-\beta} \left[\frac{1}{(\tau_2-\tau_1)^2} + \frac{1}{(\tau_2-\tau_1)^2} \|V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon))\|^{\frac{q-2}{q}}_{L^2(B_{\tau_2})} \right] \\ &\quad + \frac{c}{(\tau_2-\tau_1)^{n/2+\beta}} \|V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^2(B_{\tau_2})} \\ &\leq \frac{c}{(\alpha-\beta)^{\frac{1}{2}}} (\tau_2-\tau_1)^{\alpha-\beta} \left[\frac{1}{(\tau_2-\tau_1)^2} + \frac{1}{(\tau_2-\tau_1)^2} \|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{2-q'}{2}}_{L^{q'}(B_{\tau_2})} \right] \end{aligned}$$

$$+\frac{c}{(\tau_2-\tau_1)^{n/2+\beta}}\left[1+\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{q'}{2}}_{L^{q'}(B_{\tau_2})}\right] \tag{4.30}$$

$$=\frac{c}{(\alpha-\beta)^{\frac{1}{q'}}}(\tau_2-\tau_1)^{2(\alpha-\beta)/q'}\left[\frac{1}{(\tau_2-\tau_1)^{4/q'}}+\frac{1}{(\tau_2-\tau_1)^{4/q'}}\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{2-q'}{q'}}_{L^{q'}(B_{\tau_2})}\right]$$

$$+\frac{c}{(\tau_2-\tau_1)^{n/2+\beta}}\left[1+\|\partial-zF(\mathrm{D}v_\varepsilon)\|_{L^{q'}(B_{\tau_2})}\right]. \tag{4.31}$$

Noting that $\beta>0$, we have $1<q'<\frac{nq'}{n-2\beta}$. In particular, we may apply the following interpolation inequality on the right-hand side of (4.31),

$$\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{q'}(B_{\tau_2})}\le\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\theta}_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{1-\theta}_{L^1(B_{\tau_2})}, \tag{4.32}$$

where $\theta\in(0,1)$ is determined through the identity

$$\frac{1}{q'}=\frac{\theta(n-2\beta)}{nq'}+1-\theta. \tag{4.33}$$

In particular, we obtain,

$$\begin{aligned}
&\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})}\\
&\quad\le\frac{c}{(\alpha-\beta)(\tau_2-\tau_1)^{\frac{4-2(\alpha-\beta)}{q'}}}\left[1+\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{(2-q')\theta}{q'}}_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{(2-q')(1-\theta)}{q'}}_{L^1(B_{\tau_2})}\right]\\
&\quad+\frac{c}{(\tau_2-\tau_1)^{\frac{n+2\beta}{2}}(B_{\tau_2})}\left[1+\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\theta}_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{1-\theta}_{L^1(B_{\tau_2})}\right],
\end{aligned} \tag{4.34}$$

with $c\equiv c(n,N,\lambda,\Lambda,p,q,\mathscr{F}(u,B_\rho),\|u\|_{W^{1,p}(B_)})$. Using Young's inequality with conjugate exponents

$$\left(\frac{q'}{(2-q')\theta},\frac{q'}{q'-(2-q')\theta}\right)\quad\text{and}\quad\left(\frac{1}{\theta},\frac{1}{1-\theta}\right)$$

in (4.34) we deduce

$$\begin{aligned}
\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})}\le&\frac{1}{2}\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\\
&+\frac{c}{(\alpha-\beta)^{\tilde\gamma_1}(\tau_2-\tau_1)^{\gamma_1}}\left[1+\mathscr{M}^{\frac{1}{q'}}+\|\partial_z F(\mathrm{D}v_\varepsilon)\|^{\frac{(2-q')(1-\theta)}{q'-(2-q')\theta}}_{L^1(B_{\tau_2})}\right]\\
\le&\frac{1}{2}\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\\
&+\frac{c}{(\alpha-\beta)^{\tilde\gamma_1}(\tau_2-\tau_1)^{\gamma_1}}\left[1+\mathscr{F}(u,B_\rho)^{\frac{(2-q')(1-\theta)}{q'-(2-q')\theta}}\right],
\end{aligned}$$

where we set $\gamma_1 := \max\left\{\frac{4-2(\alpha-\beta)}{q'-(2-q')\theta}, \frac{n+2\beta}{2(1-\theta)}\right\}$ and c depends on n, N, λ, Λ, p, q, $\mathscr{F}(u, B_\rho)$ and $\|u\|_{W^{1,p}(B_\rho)}$. To obtain the last line, we also recalled the definition of $\mathscr{M}$ and used (4.2). Finally, by means of Lemma 1.7, we obtain

$$\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\rho/2})} \leq \frac{c}{\rho^{\gamma_1}}, \tag{4.35}$$

for $c \equiv c(n, N, \lambda, \Lambda, p, q, \mathcal{F}(u, B_\rho), \|u\|_{W^{1,p}(B_\rho)})$. Combining (4.35) with a standard covering argument we can conclude that $\partial_z F(\mathrm{D}v_\varepsilon) \in L^{\frac{nq'}{n-2\beta}}_{\mathrm{loc}}(B_\rho, \mathbb{R}^{N\times n})$. Moreover, by the same argument, whenever $B_s \Subset B_t \Subset B_\rho$ are concentric balls, we have

$$\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{q'}(B_s)} \leq c\|\partial_z F(\mathrm{D}v_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_s)} \leq \frac{c}{(t-s)^{\gamma_1}}\left[1+\mathcal{F}(u, B_\rho)\right]^{\tilde{\gamma}_1}, \tag{4.36}$$

for all $\beta \in (0, \alpha)$, with γ_1 and c as in (4.35) and $\tilde{\gamma}_1 := 4\max(1, \frac{(2-q')(1-\theta)}{q'-(2-q')\theta} + \frac{1}{q}')$. Note that c blows up as $\beta \to \alpha$. However, we will fix the value of β in dependence only on p, q and n at a later stage and thus this does not introduce any additional dependencies.

Step 2: Higher integrability for $\mathrm{D}v_\varepsilon$

In order to obtain higher integrability for $\mathrm{D}v_\varepsilon$ we distinguish two cases: $n > 2$ and $n = 2$.

Case 1: $n > 2$ Having improved the integrability of $\partial_z F(\mathrm{D}v_\varepsilon)$, we can return to (4.27) and divide both sides of the inequality by $|h|^2$, thus obtaining

$$\begin{aligned}
\mathcal{I}(h) =& \int_{B_\rho} \eta^2 \left| \frac{\tau_h V_{\mu,p}(\mathrm{D}v_\varepsilon)}{|h|} \right|^2 \mathrm{d}x + \int_{B_\rho} \eta^2 \left| \frac{\tau_h V_{\mu,q'}(\partial_z F(\mathrm{D}v_\varepsilon)}{|h|} \right|^2 \mathrm{d}x \\
\leq& c\left(\int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'}{2}} \mathrm{d}x\right)^{\frac{q-2}{q}} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|} \right|^q \mathrm{d}x\right)^{\frac{2}{q}} \\
&+ c\kappa_\varepsilon \int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \mathrm{D}v_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \mathrm{d}x = T_1(h) + T_2(h).
\end{aligned} \tag{4.37}$$

We estimate $T_2(h)$ in exactly the same way as we estimated $S_2(h)$ in (4.28). Thus we focus on $T_1(h)$. Using Lemma 1.12 and (4.36) we have

$$\begin{aligned}
T_1(h) =& c\left(\int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}v_\varepsilon))^{\frac{q'}{2}} \mathrm{d}x\right)^{\frac{q-2}{q}} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|} \right|^q \mathrm{d}x\right)^{\frac{2}{q}} \\
\leq& \frac{c}{(\tau_2-\tau_1)^{(2-q')\gamma_1}} \|\mathrm{D}v_\varepsilon\|^2_{L^q(B_{\tau_2})}.
\end{aligned}$$

Utilising this estimate in (4.37) and applying the Sobolev embedding, we obtain

$$\|\mathrm{D}v_\varepsilon\|^{\frac{p}{2}}_{L^{\frac{np}{n-2}}(B_{\tau_1})} \leq c\|\mathrm{D}V_{\mu,p}(\mathrm{D}v_\varepsilon)\|_{L^2(B_{\tau_1})}$$

$$\leq \frac{c}{(\tau_2-\tau_1)^{\frac{(2-q')\gamma_1}{2}}}\|\mathrm{D}v_\varepsilon\|_{L^q(B_{\tau_2})} + \frac{c\mathcal{M}^{\frac{1}{2}}}{(\tau_2-\tau_1)} + \|\mathrm{D}v_\varepsilon\|^{\frac{p}{2}}_{L^p(B_{\tau_2})}, \tag{4.38}$$

with $c \equiv c(n, N, \lambda, \Lambda, p, q, \|u\|_{W^{1,p}(B_\rho)}, \mathscr{F}(u, B_\rho))$. Assuming $p < q$ (else there is nothing to prove), we can use the interpolation inequality

$$\|\mathrm{D}v_\varepsilon\|_{L^q(B_{\tau_2})} \leq \|\mathrm{D}v_\varepsilon\|^{\theta}_{L^{\frac{np}{n-2}}(B_{\tau_2})}\|\mathrm{D}v_\varepsilon\|^{1-\theta}_{L^p(B_{\tau_2})} \tag{4.39}$$

where $\theta := \frac{n(q-p)}{2q}$ to find

$$\begin{aligned}\|\mathrm{D}v_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\tau_1})} \leq & \frac{c}{(\tau_2-\tau_2)^{\frac{(2-q')\gamma_1}{p}}}\|Dv_\varepsilon\|^{\frac{2\theta}{p}}_{L^{\frac{np}{n-2}}(B_{\tau_2})}\|\mathrm{D}v_\varepsilon\|^{\frac{2(1-\theta)}{p}}_{L^p(B_{\tau_2})} \\ & + \frac{\mathcal{M}^{\frac{1}{p}}}{(\tau_2-\tau_1)^{2/p}} + \|\mathrm{D}v_\varepsilon\|_{L^p(B_{\tau_2})}.\end{aligned}$$

Using the definition of q, it is easy to check that $2\theta/p < 1$, so we can use Young inequality with conjugate exponents

$$\left(\frac{p}{2\theta}, \frac{p}{p-2\theta}\right)$$

to conclude

$$\|\mathrm{D}v_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\tau_1})} \leq \frac{1}{2}\|\mathrm{D}v_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\tau_1})} + \frac{c}{(\tau_2-\tau_1)^{\gamma_2}}\left[\|\mathrm{D}v_\varepsilon\|_{L^p(B_{\tau_2})}+1\right]^{\tilde{\gamma}_2},$$

where we set $\gamma_2 := \max\left\{\frac{2}{p}, \frac{(2-q')\gamma_1}{p-2\theta}\right\}$, $c \equiv c(n, N, \lambda, \Lambda, p, q, \|u\|_{W^{1,p}(B_\rho)}, \mathscr{F}(u, B_\rho))$ and $\tilde{\gamma}_2 := \frac{2p}{p-2\theta}\max((2-q')\gamma_1, \frac{2(1-\theta)}{p}, 1)$. By Lemma 1.7 and (4.25) we conclude

$$\|\mathrm{D}v_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\rho/2})} \leq \frac{c}{\rho^{\gamma_2}}\left[\|\mathrm{D}v_\varepsilon\|_{L^p(B_\rho)}+1\right]^{\tilde{\gamma}_2} \leq \frac{c}{\rho^{\gamma_2}}\left[\mathscr{F}(u, B_\rho)+1\right]^{\tilde{\gamma}_2}.$$

As before, we can apply the same argument to arbitrary concentric balls $B_s \Subset B_t \Subset B_\rho$ in order to obtain

$$\|\mathrm{D}v_\varepsilon\|_{L^{\frac{np}{n-2}}(B_s)} \leq \frac{c}{(t-s)^{\gamma_2}}\left[\mathscr{F}(u, B_\rho)+1\right]^{\tilde{\gamma}_2}, \tag{4.40}$$

for $c(n, N, \lambda, \Lambda, p, q, \|u\|_{W^{1,p}(B_\rho)}, \mathscr{F}(u, B_\rho))$.

Case 2: $n = 2$ We modify (4.38) by using the limiting case of Sobolev embedding theorem:

$$\|\mathrm{D}v_\varepsilon\|^{\frac{p}{2}}_{L^m(B_{\tau_1})} \leq c\|\mathrm{D}v_{\mu,p}(Dv_\varepsilon)\|_{L^2(B_{\tau_1})}$$

$$\leq \frac{c}{(\tau_2-\tau_1)^{\frac{(2-q')\gamma_1}{2}}}\|\mathrm{D}v_\varepsilon\|_{L^q(B_{\tau_2})} + \frac{c\mathcal{M}^{\frac{1}{2}}}{(\tau_2-\tau_1)} + c\|\mathrm{D}v_\varepsilon\|^{\frac{p}{2}}_{L^p(B_{\tau_2})}, \tag{4.41}$$

for all $m > p$. Specifically, we choose

$$m > \max\left\{\frac{p^2 q}{(pq-2q+2p)}, q\right\} > 0. \tag{4.42}$$

Note that as $p \geq 2$, $q(p-2)+2p \geq 2p$, so this is a finite positive quantity. We now use the interpolation inequality

$$\|\mathrm{D}v_\varepsilon\|_{L^q(B_{\tau_2})} \leq \|\mathrm{D}v_\varepsilon\|^{\frac{2\theta}{p}}_{L^m(B_{\tau_2})}\|\mathrm{D}v_\varepsilon\|^{\frac{2(1-\theta)}{p}}_{L^p(B_{\tau_2})} \quad \text{with} \quad \theta \coloneqq \frac{m(q-p)}{q(m-p)} \tag{4.43}$$

in order to find

$$\begin{aligned}\|\mathrm{D}v_\varepsilon\|_{L^m(B_{\tau_1})} \leq & \frac{c}{(\tau_2-\tau_1)^{\frac{(2-q')\gamma_1}{p}}}\|\mathrm{D}v_\varepsilon\|^{\theta}_{L^m(B_{\tau_2})}\|\mathrm{D}v_\varepsilon\|^{1-\theta}_{L^p(B_{\tau_2})} \\ & + \frac{c\mathcal{M}^{\frac{1}{p}}}{(\tau_2-\tau_1)^{\frac{1}{p}}} + c\|\mathrm{D}v_\varepsilon\|_{L^p(B_{\tau_2})}.\end{aligned}$$

Our choice of m ensures that $2\theta/p < 1$, so using Young's inequality with exponents

$$\left(\frac{p}{2\theta}, \frac{p}{p-2\theta}\right)$$

we find

$$\|\mathrm{D}v_\varepsilon\|_{L^m(B_{\tau_1})} \leq \frac{1}{2}\|\mathrm{D}v_\varepsilon\|_{L^m(B_{\tau_2})} + \frac{c}{(\tau_2-\tau_1)^{\gamma_2'}}\left[\mathscr{F}(u,B_\rho)+1\right]^{\tilde{\gamma}_2},$$

with $\gamma_2' \coloneqq \max\left\{\frac{1}{p}, \frac{(2-q')\gamma_1}{p-2\theta}\right\}$, $c \equiv c(n,N,\lambda,\Lambda,p,q,\|u_{W^{1,p}(B_\rho)},\mathscr{F}(u,B_\rho))$ and the choice $\tilde{\gamma}_2 = \max(1,\tilde{\gamma}_1,\frac{1}{p},\frac{2(1-\theta)}{p-2\theta})$. Using Lemma 1.7 we finally conclude

$$\|\mathrm{D}v_\varepsilon\|_{L^m(B_{\rho/2})} \leq \frac{c}{\rho^{\gamma_2'}}\left[1+\mathcal{F}(u,B_\rho)\right]^{\tilde{\gamma}_2}. \tag{4.44}$$

Step 3: Convergence

We now proceed to take limits as $\varepsilon \to 0$. The argument follows familiar lines. By (4.1), the minimality of v_ε in the class $u_\varepsilon + W^{1,q}_0(B_\rho,\mathbb{R}^N)$, Lemma 1.4, (4.23) and (4.21) we have

$$\begin{aligned}\int_{B_\rho}|\mathrm{D}v_\varepsilon - \mathrm{D}u|^p \ \mathrm{d}x \leq & c\int_{B_\rho}|V_{\mu,p}(\mathrm{D}v_\varepsilon) - V_{\mu,p}(\mathrm{D}u)|^2 \ \mathrm{d}x \\ \leq & c\int_{B_\rho}\left[F_\varepsilon(\mathrm{D}v_\varepsilon) - F_\varepsilon(\mathrm{D}u) - \partial_z F_\varepsilon(\mathrm{D}v_\varepsilon)\cdot(\mathrm{D}u-\mathrm{D}v_\varepsilon)\right] \ \mathrm{d}x\end{aligned}$$

$$\leq c \int_{B_\rho} \left[F_\varepsilon(\mathrm{D}u_\varepsilon) - F_\varepsilon(\mathrm{D}u) \right] \ \mathrm{d}x \to 0,$$

thus $\mathrm{D}v_\varepsilon \to \mathrm{D}u$ strongly in $L^p(B_\rho, \mathbb{R}^{N\times n})$ and, up to non-relabelled subsequences, $\mathrm{D}v_\varepsilon \to \mathrm{D}u$ a.e. in B_ρ. Now fix concentric balls $B_s \Subset B_t \Subset B_\rho$. We can use weak lower semicontinuity in (4.40)-(4.44) to obtain that for some $\tilde{\gamma} > 0$,

$$\|\mathrm{D}u\|_{L^q(B_s)} \leq c\|\mathrm{D}u\|_{L^m(B_s)} \leq \frac{c}{(t-s)^\gamma} \left[1 + \mathscr{F}(u, B_\rho)\right]^{\tilde{\gamma}}$$
$$\text{where } m := \begin{cases} \frac{np}{n-2} & \text{if } \ n > 2, \\ \text{any } m > q & \text{if } \ n = 2, \end{cases}$$

with $c \equiv c(n, N, \lambda, \Lambda, p, q, \|u\|_{W^{1,p}(B_\rho)}, \mathscr{F}(u, B_\rho))$ and $\gamma := \max\{\gamma_2, \gamma_2'\}$. Now we study the convergence of $\{\partial_z F(\mathrm{D}v_\varepsilon)\}$.

By (4.36), (4.21), (4.23) and a well-known variant of the dominated convergence theorem, we have that $\partial_z F(\mathrm{D}v_\varepsilon) \to \partial_z F(\mathrm{D}u)$ strongly in $L^1(B_\rho, \mathbb{R}^{N\times n})$. In particular, up to non-relabelled subsequences, we find $\partial_z F(\mathrm{D}v_\varepsilon) \to \partial_z F(\mathrm{D}u)$ a.e. in B_ρ, so by Fatou lemma and (4.36) we conclude that

$$\|\partial_z F(\mathrm{D}u)\|_{L^{q'}(B_s)} \leq c\|\partial_z F(\mathrm{D}u)\|_{L^{\frac{nq'}{n-2\beta}}(B_s)} \leq \frac{c}{(t-s)^{\gamma_1}} \left[1 + \mathcal{F}(u, B_\rho)\right]^{\tilde{\gamma}}.$$

Using a standard covering argument, we conclude

$$\mathrm{D}u \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}) \quad \text{and} \quad \partial_z F(\mathrm{D}u) \in L^{q'}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}). \tag{4.45}$$

Step 4: Weak differentiability of $V_{\mu,p}(\mathrm{D}u)$ and $\partial_z F(\mathrm{D}u)$

Having established (4.45), we can consider the Euler-Lagrange equation of $\mathscr{F}(\cdot)$ directly. Repeat precisely the procedure presented in *Step 1* up to (4.27) – only using $F(\cdot)$ and u instead of $F_\varepsilon(\cdot)$ and v_ε, as well as picking $\eta \in C^1_c(B_\rho)$ with $\mathbb{1}_{B_{\rho/2}} \leq \eta \leq \mathbb{1}_{B_\rho}$ and $|\mathrm{D}\eta| \lesssim \rho^{-1}$. Divide both sides of the inequality obtained in this way by $|h|^2$ to find

$$\begin{aligned}
&\int_{B_\rho} \frac{|\tau_h| V_{\mu,p}(\mathrm{D}u)|^2}{|h|^2} \, \mathrm{d}x \int_{B_\rho} \eta^2 \left| \frac{\tau_h V_{\mu,q'}(\partial_z F(\mathrm{D}u))}{|h|} \right|^2 \ \mathrm{d}x \\
&\quad \leq \left[c \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \mathcal{D}(h, \partial_z F(\mathrm{D}u))^{\frac{q'}{2}} \ \mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\rho} |\mathrm{D}\eta|^2 \left| \frac{\tau_h u}{|h|} \right|^q \ \mathrm{d}x \right)^{\frac{2}{q}} \right] \\
&\quad \leq \frac{c}{\rho^{-2}} \|\partial_z F(\mathrm{D}u)\|^{2-q'}_{L^{q'}(B_\rho)} \|\mathrm{D}u\|^2_{L^q(B_\rho)} < \infty,
\end{aligned}$$

with $c \equiv c(n, N, \lambda, \Lambda, p, q)$. To obtain the last line we used (4.45). Employing a standard covering argument using (1.13) we deduce that

$$V_{\mu,q'}(\partial_z F(\mathrm{D}u)), V_{\mu,p}(\mathrm{D}u) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}).$$

In particular, by Sobolev embedding $\partial_z F(\mathrm{D}u) \in L^{\frac{nq'}{n-2}}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$. Thus the proof is complete. □

3 A Morrey-estimate in the case $n = 2$

We now focus on the case $n = 2$. Note that we have already obtained the following regularity results in this case

$$\begin{aligned} u &\in W^{2,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,d}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \\ V_{1,q'}(\partial_z F(\mathrm{D}u)) &\in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^{d}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times 2}) \end{aligned} \tag{4.46}$$

for any $d \in [1, \infty)$. Adapting arguments from [23] we can use this information to deduce that $\mathrm{D}u \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times 2})$ for any $\beta \in (0, 1)$. In fact, the argument applies to any C^2-regular integrand satisfying (4.1) in the non-degenerate case $\mu = 1$.

We give an overview of the proof, the key point of which is to apply a version of Gehring's lemma which gives an improvement in integrability on a logarithmic scale. Consider discs $B_\rho \Subset B_{r/2} \Subset \Omega$, not necessarily concentric. Further introduce a cut-off $\eta \in C^1_c(B_\rho)$ with $\mathbb{1}_{B_{\rho/2}} \le \eta \le \mathbb{1}_{B_\rho}$ and $|\mathrm{D}\eta| \lesssim \rho^{-1}$. Take $h \in \left(0, \min(1, \frac{\mathrm{dist}(\partial B_\rho, \mathrm{supp}\,\eta)}{10000})\right)$ and the competitor $\phi_s = \tau_{-h_s}(\eta^2(\tau_{h_s}u - (\mathrm{D}u)_{B_\rho}h_s)$ where $h_s := he_s$. Here $\{e_1, e_2\}$ is the standard orthonormal basis of $\mathbb{R}^2$. We now denote

$$\Gamma(\mathrm{D}u) := \int_0^1 \partial_{zz} F(Du + \lambda\tau_{h_s}\mathrm{D}u)\,\mathrm{d}\lambda.$$

Note that since $\partial_{zz}F(\cdot)$ is symmetric and positive definite due to (4.1), so is $\Gamma(\cdot)$. Using integration by parts, the mean value theorem, Cauchy-Schwartz and (4.1), we estimate

$$\begin{aligned} &\int_{B_\rho} \eta^2 \Gamma(\mathrm{D}u)\tau_{h_s}\mathrm{D}u \cdot \tau_{h_s}\mathrm{D}u\,\mathrm{d}x \\ =& -2\int_{B_\rho} \eta\tau_{h_s}\partial_z F(\mathrm{D}u)(D_s u - (\mathrm{D}u)_{B_\rho}h_s) \otimes D\eta\,\mathrm{d}x \\ =& -2\int_{B_\rho} \eta\left(\sqrt{\Gamma(\mathrm{D}u)}\tau_{h_s}\mathrm{D}u \cdot \sqrt{\Gamma(\mathrm{D}u)}(\tau_{h_s}u - (\mathrm{D}u)_{B_\rho}h_s)\right) \otimes \mathrm{D}\eta\,\mathrm{d}x \end{aligned}$$

$$\leq 2\Lambda \int_{B_\rho} \left(\eta^2 \Gamma(\mathrm{D}u)\tau_{h_s}\mathrm{D}u \cdot \tau_{h_s}\mathrm{D}u\right)^{\frac{1}{2}} (1+|\partial_z F(\mathrm{D}u)|^2)^{\frac{q-2}{4(q-1)}} |\mathrm{D}\eta||\tau_{h,s}u - (\mathrm{D}u)_{B_\rho} h_s| \,\mathrm{d}x$$
$$\leq c \int_{B_\rho} \left(\eta^2 \Gamma(\mathrm{D}u)\tau_{h_s}\mathrm{D}u \cdot \tau_{h_s}\mathrm{D}u\right)^{\frac{1}{2}} (1+|V_{\mu,q'}(\partial_z F(\mathrm{D}u)|)^{\frac{q-2}{q}} |\mathrm{D}\eta||\tau_{h,s}u - (\mathrm{D}u)_{B_\rho} h_s| \,\mathrm{d}x$$

where $c = c(\Lambda, q)$. We now divide both sides by h^2, sum over $s \in \{1,2\}$ and let $h \to 0$ to deduce

$$\fint_{B_\rho/2} \mathscr{Y}(u)^2 \,\mathrm{d}x \leq \frac{c(\Lambda,q)}{\rho} \fint_{B_\rho} \sigma \mathscr{Y}(u) |Du - (Du)_{B_\rho}| \,\mathrm{d}x \tag{4.47}$$

where we have denoted

$$\mathscr{Y}(u) \coloneqq \sqrt{\partial_{zz} F(\mathrm{D}u)\mathrm{D}^2 u \cdot \mathrm{D}^2 u} \qquad \sigma \coloneqq (1+|V_{\mu,q'}(\partial_z F(\mathrm{D}u))|)^{\frac{q-2}{q}}.$$

Here we have also used (4.46), Lemma 1.12, Lemma 1.13, Fatou's Lemma (on the left-hand side) as well as dominated convergence (on the right-hand side). Combining (4.47), (4.46) and a covering argument, we see that $\mathscr{Y}(u) \in L^2_{\mathrm{loc}}(\Omega)$. We consider now (4.47) and apply Hölder's inequality and Sobolev-Poincaré, denoting $\delta \coloneqq 4/3$. We find

$$\begin{aligned}\fint_{B_\rho} \sigma \mathscr{Y}(u) |Du - (Du)_{B_\rho}| \,\mathrm{d}x \; &\leq \left(\fint_{B_\rho} (\sigma \mathscr{Y}(u))^\delta \,\mathrm{d}x\right)^{\frac{1}{\delta}} \left(\fint_{B_\rho} |Du - (Du)_{B_\rho}|^{\delta'}\right)^{\frac{1}{\delta'}} \\ &\leq c\rho \left(\fint_{B_\rho} (\sigma \mathscr{Y}(u))^\delta \,\mathrm{d}x\right)^{\frac{1}{\delta}} \left(\fint_{B_\rho} |D^2 u|^\delta\right)^{\frac{1}{\delta}}.\end{aligned}$$

Note that here we have used the fact that in two dimensions δ' coincides with the Sobolev exponent δ^*. As $p \geq 2$ and using (4.1), we note that

$$|D^2 u|^2 \leq (1+|\mathrm{D}u|^2)^{\frac{p-2}{2}} |D^2 u|^2 \leq c(N,\lambda,p)\mathscr{Y}(u)^2.$$

Using this fact, we have shown that

$$\left(\fint_{B_\rho/2} \mathscr{Y}(u)^2 \,\mathrm{d}x\right)^{\frac{\delta}{2}} \leq c \left(\fint_{B_\rho} \sigma^\delta \mathscr{Y}(u)^\delta\right).$$

Our next task is to analyse the exponential integrability of σ. Utilising (4.46) and Trudinger's inequality, see [99, Theorem 7.15], we find a constant α_0, depending only on $\|V_{1,q'}(\partial_z F(\mathrm{D}u))\|_{W^{1,2}(B_r)}$ such that

$$\fint_{B_{r/2}} \exp\left(\alpha_0 \left(1+|V_{1,q'}(\partial_z F(\mathrm{D}u))|\right)^2\right) \mathrm{d}x < \infty.$$

Note that for $\kappa \in (0, 4/q)$, $(1+|V_{1,q'}(\partial_z F(\mathrm{D}u))|)^2 \geq \sigma^2$. Thus independently of $\alpha \in (0,1)$ we have

$$\fint_{B_{r/2}} \exp(\alpha\sigma^2)\,\mathrm{d}x \leq c < \infty \tag{4.48}$$

where c depends only on α, κ and $\|V_{1,q'}(\partial_z F(\mathrm{D}u))\|_{W^{1,2}(B_r)}$.

We are now in the precise situation of [23, Lemma 1.2] and hence there exists a positive constant c_0 depending on N, λ, Λ, p and q such that

$$\mathscr{Y}(u)^\delta \in L^{\frac{2}{\delta}}_{\mathrm{loc}}(\log L)^{c_0\alpha}(B_\rho) \Rightarrow \mathscr{Y}(u) \in L^2_{\mathrm{loc}}(\log L)^{c_0\alpha}(B_\rho) \tag{4.49}$$

for all $\alpha > 0$. We note that $|\mathrm{D}\partial_z F(\mathrm{D}u)| \leq c(\lambda, q)\mathscr{Y}(u)\sigma$ by (4.1) and (4.46) as well as the elementary observation that there is $c(\alpha) > 0$ so that if $a, b \geq 0$, then

$$(ab)^2 \log^\alpha(e + ab) \leq 2^\alpha a^2 \log^{\alpha+2}(e + a) + c(\alpha)\exp(6b).$$

A proof of this inequality can be found in [23, Inequality (2.12)]. Thus given any $B_\xi \Subset B_\rho$ we compute

$$\begin{aligned}
&\int_{B_\xi} |\mathrm{D}\partial_z F(\mathrm{D}u)|^2 \log^\alpha(e + |\mathrm{D}\partial_z F(\mathrm{D}u)|)\,\mathrm{d}x \\
\leq &\int_{B_\xi} |c\mathscr{Y}(u)\sigma|^2 \log^\alpha(e + |c\mathscr{Y}(u)\sigma|)\,\mathrm{d}x \\
\leq &2^\alpha \int_{B_\xi} |\mathscr{Y}(u)|^2 \log^{\alpha+2}(e + \mathscr{Y}(u))\,\mathrm{d}x + c^\alpha \int_{B_\xi} \exp(6\sigma c)\,\mathrm{d}x.
\end{aligned}$$

However recalling (4.48) and (4.49) this quantity is finite for all $\alpha > 0$. Applying a covering argument we see that $|\mathrm{D}\partial_z F(\mathrm{D}u)| \in L^2_{\mathrm{loc}}(\log L)^\alpha(\Omega)$ for any $\alpha > 0$. In particular, choosing $\alpha > 1$, we have $\partial_z F(\mathrm{D}u) \in C_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$. Applying [131, Example 5.3] we deduce that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$. In particular, (4.1) now implies standard p-growth when applied at u – in other words the degeneracy of non-standard growth becomes immaterial and we may conclude by standard arguments, see e.g. [100], that $\mathrm{D}u \in C^{0,\beta}$ for any $\beta \in (0,1)$.

4 Controlled-duality growth conditions and homogeneity

In this section we discuss the upper bound in what we have called the controlled-duality growth condition. We have already commented that while controlled-duality

growth implies controlled growth, the converse fails in general. However within restricted classes of functionals the two conditions are the same. In particular, the aim of this section is to show that whenever $F(\cdot)$ is a convex polynomial of degree $d \geq 2$ the two conditions agree and in fact the conditions are in turn equivalent to the corresponding natural growth condition. Before formally stating the result let us recall that an integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be d-homogeneous provided $F(tz) = |t|^d F(z)$ holds for all $z \in \mathbb{R}^{N\times n}$ and $t \in \mathbb{R}$.

Proposition 8. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a convex integrand. Assume that for $z \in \mathbb{R}^{N\times n}$,*

$$F(z) = \sum_{j=1}^{m} F_j(z) \tag{4.50}$$

where $F_j\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ are of class C^2 and d_j-homogeneous with $2 \leq d_1 < \ldots < d_m$. Then there exists a constant $c > 0$ so that for all z, $w \in \mathbb{R}^{N\times n}$ we have

$$\partial_{zz}F(z)w \cdot w \leq c\Big(1 + |\partial_z F(z)|^2\Big)^{\frac{d_m-2}{2(d_m-1)}}|w|^2.$$

The proof follows easily once we have established the following key lemma.

Lemma 4.5. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be convex, C^2 and d-homogeneous for some $d \geq 2$. Then there exist a subspace V of $\mathbb{R}^{N\times n}$ and constants $0 < c \leq C$ so that $F(z) = 0$ for all $z \in V^\perp$, and*

$$c|z|^{d-2}|w|^2 \leq \partial_{zz}F(z)[w,w] \leq C\Big|\partial_z F(z)\Big|^{\frac{d-2}{d-1}}|w|^2$$

holds for all z, $w \in V$.

Proof. By adding a constant if necessary, we may assume that F is non-negative. Since if $F(\cdot) \equiv 0$ there is nothing to prove we may also assume that F is strictly positive somewhere. Then $z \mapsto F(z)^{\frac{1}{d}}$ is a convex 1-homogeneous and nonnegative integrand. In particular, it must be the support function for a compact and convex set K in $\mathbb{R}^{N\times n}$ that contains 0 [182, Corollary 13.2.2], i.e.

$$F(z)^{\frac{1}{d}} = \max_{\xi\in K} \xi \cdot z$$

for all $z \in \mathbb{R}^{N\times n}$. As F is not identically zero, $K \setminus \{0\} \neq \emptyset$, and because F is even, K must be symmetric:

$$\max_{\xi\in K} \xi \cdot z = \max_{\xi\in K} \xi \cdot (-z) = \max_{\xi\in -K} \xi \cdot z$$

holds for all $z \in \mathbb{R}^{N\times n}$. Thus $K = -K$. Set $V \coloneqq \operatorname{span} K$ and note that if $z \in V^{\perp}$, then $F(z) = 0$. Define

$$m \coloneqq \min\Big\{f(z) : z \in V \text{ and } |z| = 1\Big\},$$
$$M \coloneqq \max\Big\{f(z) : z \in V \text{ and } |z| = 1\Big\}$$

and note that then $m|z|^d \leq f(z) \leq M|z|^d$ holds for all $z \in V$. Since K is convex, symmetric and $K \setminus \{0\} \neq \emptyset$ we must have $m > 0$. Introduce for $\varepsilon > 0$, $F_\varepsilon(z) \coloneqq F(z) + \varepsilon|z|^d$. Note that F_ε is strictly convex, d-homogeneous and C^2. Hence the Fenchel conjugate of $F_\varepsilon|_V$,

$$g_\varepsilon \equiv \Big(F_\varepsilon|_V\Big)^* \colon V \to \mathbb{R}$$

is also strictly convex, d'-homogeneous and C^1 on V by Lemma 1.18. In view of the bound for $F_\varepsilon|_V$ we may use Lemma 1.16 to deduce that

$$c\Big(M+\varepsilon\Big)^{-\frac{1}{d-1}}|\xi|^{d'} \leq g_\varepsilon(\xi) \leq c\Big(m+\varepsilon\Big)^{-\frac{1}{d-1}}|\xi|^{d'}$$

for all $\xi \in V$, where we may take $c \coloneqq d^{-\frac{1}{d-1}} - d^{-d'}$. Using the Euler relation, that is $d' g_\varepsilon(\xi) = \partial_z g_\varepsilon(\xi)\cdot\xi$, and the local Lipschitz bound for g_ε we deduce that

$$d'c\Big(M+\varepsilon\Big)^{-\frac{1}{d-1}}|\xi|^{d'-1} \leq |\partial_z g_\varepsilon(\xi)| \leq 2^{2d'+1}c\Big(m+\varepsilon\Big)^{-\frac{1}{d-1}}|\xi|^{d'-1}$$

holds for all $\xi \in V$. Now note that $z = \partial_z g_\varepsilon\Big(\partial_z F_\varepsilon(z)\Big)$ and combine this fact with the above to estimate

$$|z| \leq 2^{2d'+1}c\Big(m+\varepsilon\Big)^{-\frac{1}{d-1}}\Big|\partial_z F(z) + \varepsilon d|z|^{d-2}z\Big|^{d'-1}.$$

We proceed to prove the upper bound. Since F is C^2 and d-homogeneous we have with

$$C \coloneqq \max\Big\{\partial_{zz}F(z)w\cdot w : z, w \in V \text{ and } |z| = |w| = 1\Big\}$$

that

$$\partial_{zz}F_\varepsilon(z)[w,w] \leq (C+\varepsilon d)|z|^{d-2}|w|^2,$$

hence

$$\partial_{zz}F_\varepsilon(z)w\cdot w \leq (C+\varepsilon d)c_d\Big(m+\varepsilon\Big)^{-\frac{d-2}{d-1}}\Big|\partial_z F(z) + \varepsilon d|z|^{d-2}z\Big|^{\frac{d-2}{d-1}}|w|^2.$$

Letting $\varepsilon \to 0$ we obtain the desired upper bound with $C \coloneqq C(d)m^{-\frac{d-2}{d-1}}$. The lower bound is found by arguing in a similar fashion. $\square$

Having established Lemma 4.5 we can prove Proposition 8.

Proof of Proposition 8. First observe that

$$\frac{F(tz)}{t^{d_m}} \to F_m(z) \text{ as } t \to \infty$$

pointwise for any $z \in \mathbb{R}^{N\times n}$. Thus $F_m(\cdot)$ is convex. Apply Lemma 4.5 to find a subspace V_m of $\mathbb{R}^{N\times n}$ and constants c_m, C_m with $0 < c_m \leq C_m$ so that $F_m(z) = 0$ for $z \in V_m^{\perp}$ and further

$$c_m|z|^{d_m-2}|w|^2 \leq \partial_{zz}F_m(z)w \cdot w \leq C_m\Big|\partial_z F_m(z)\Big|^{\frac{d_m-2}{d_m-1}}|w|^2 \tag{4.51}$$

for z, $w \in V_m$. Now consider the restriction of f to $V_m^{\perp}$. It is clearly convex and since $F_m|_{V_m^{\perp}} = 0$ we find arguing as before that the restriction $F_{m-1}|_{V_m^{\perp}}$ must be convex. Hence Lemma 4.5 applies and yields a subspace V_{m-1} of $V_m^{\perp}$ and two constants c_{m-1}, C_{m-1} with $0 < c_{m-1} \leq C_{m-1}$ so that $F_{m-1}(z) = 0$ for $z \in V_{m-1}^{\perp} \cap V_m^{\perp}$, and also

$$c_{m-1}|z|^{d_{m-1}-2}|w|^2 \leq \partial_{zz}F_{m-1}(z)w \cdot w \leq C_{m-1}\Big|\partial_z F_{m-1}(z)\Big|^{\frac{d_{m-1}-2}{d_{m-1}-1}}|w|^2 \tag{4.52}$$

for z, $w \in V_{m-1}$. We proceed in this way until $V_{j-1}^{\perp} \cap V_j^{\perp} = \{0\}$, or until we reach f_1. Considering the lower bounds in (4.51), (4.52) and the corresponding bounds in the subsequent subspaces, it is clear that F_j will be the dominant term in the subspace V_j. Thus the proof is complete. □

As a direct consequence of Proposition 8 we obtain the desired statement about polynomial functionals.

Corollary 4.1. *If $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a convex polynomial of degree $d \geq 2$, then there exists a constant $c > 0$ so that for all z, $w \in \mathbb{R}^{N\times n}$,*

$$\partial_{zz}F(z)w \cdot w \leq c\Big(1 + |\partial_z F(z)|^2\Big)^{\frac{d-2}{2(d-1)}}|w|^2.$$

5 Proof of Theorem 15

Theorem 15 is an immediate consequence of combining the results of this chapter.

Proof of Theorem 15. Due to Corollary 4.1, under the assumptions of the theorem we may apply the results of Section 3. We deduce that minimisers are $C^{1,\beta}_{\text{loc}}$-regular for any

$\beta \in (0,1)$. Local real analyticity now follows from standard theory originally developed in [166], see also [167]. $\square$

5. The Jacobian equation with $\mathscr{H}^p$-data

We now return to a model that already served as motivation for the study of functionals with (p, q)-growth in the previous chapter. Consider the following typical problem in nonlinear elastostatics: Given a bounded domain $\Omega \subset \mathbb{R}^n$,

$$\text{minimise } \mathscr{W}[u] \coloneqq \int_\Omega W(x, \mathrm{D}u)\,\mathrm{d}x, \qquad \text{among all } u \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^n). \tag{5.1}$$

In this context, it is standard to assume that $u_0 \colon \overline{\Omega} \to \mathbb{R}^n$ is an orientation-preserving diffeomorphism and that, in addition, the stored-energy function $W \colon \overline{\Omega} \times \mathrm{GL}^+(n) \to \mathbb{R}$ is continuous [14, 13].

We remind the reader that in order for the direct method to be applicable $\mathscr{W}(\cdot)$ needs to be sequentially weakly lower semicontinuous and, more importantly for our purposes here, coercive [50]. However, one is sometimes led to consider non-coercive energy functions and this is the case, for instance, when W depends on $\mathrm{D}u$ only through $\mathrm{J}u \coloneqq \det \mathrm{D}u$, see e.g. [51, 85, 90] for examples in the study of elastic crystals and [57] for a different example. In this case, a solution of (5.1) is found, at least formally, by solving

$$\begin{cases} \mathrm{J}u = f & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \qquad \text{where } f \text{ is such that } W(x, f(x)) \equiv \min_{\xi > 0} W(x, \xi). \tag{5.2}$$

Note that this is nothing but an equation of the form (JD) as introduced in the introduction. There is a good existence theory for (JD) whenever there is some $\alpha > 0$ such that u_0 is a $C^{1,\alpha}$-diffeomorphism and $f \in C^{0,\alpha}$ [56, 58, 168, 208]. An alternative approach to the problem in this setting was taken in [42], where the well-posedness theory for (JD) is derived from Caffarelli's regularity theory [39, 40] for the Monge–Ampère equation.

In contrast, little is known when f is just an L^p function, although some results can be found in [84, 122, 142, 149, 181].

In light of the above discussion, we are led to study the prescribed Jacobian equation

$$\mathrm{J}u = f \tag{5.3}$$

posed either in Ω or in $\mathbb{R}^n$ with data $f \in L^p(\Omega)$ ($L \log L$ if $p = 1$) or $f \in \mathrm{H}^p(\mathbb{R}^n)$ respectively, where $p \geq 1$. If posed in the domain with boundary condition $u = \mathrm{id}$ on $\partial\Omega$, a simple integration by parts argument shows that we need to enforce the compatibility condition $⨍_\Omega f = 1$. The results concerning (5.3) presented in this chapter are based on the works [107, 106, 108] in collaboration with A. Guerra and S. Lindberg. Note that (5.3) is a first-order fully non-linear underdetermined PDE. The equation (5.3) also has a rich geometric flavour: formally, it can be written as $u^*(f \,\mathrm{d}y) = \mathrm{d}x$, where u^* denotes the pull-back of the volume form $f(y)\,\mathrm{d}y$ under u. Hence (5.3) can be read as asserting equivalence of volume forms over Ω with the same mass.

We begin by studying (5.3) in domains. For the convenience of the reader, we collect our assumptions and the precise formulation of the problem: Solve

$$\begin{cases} \det \mathrm{D}u = f & \text{in } \Omega, \\ u = \mathrm{id} & \text{on } \partial\Omega. \end{cases} \tag{JD}$$

Here $f \colon \Omega \to \mathbb{R}$ is integrable and satisfies the compatibility and non-degeneracy conditions

$$\frac{1}{|\Omega|} \int_\Omega f(x)\,\mathrm{d}x = 1 \qquad \text{and} \qquad \inf_\Omega f \geq c > 0. \tag{5.4}$$

Since the Jacobian is an n-homogeneous polynomial in $\mathrm{D}u$, we posed the following natural question, c.f. Question 1.2:

Question 5.1. Let $1 \leq p < \infty$ and assume f satisfies (5.4).

(i) If $p > 1$, $f \in L^p(\Omega)$, does (JD) have a solution $u \in W^{1,np}(\Omega, \mathbb{R}^n)$? See [208, Question 3].

(ii) If $p = 1$, $f \in L \log L(\Omega)$, does (JD) have a solution $u \in W^{1,n}(\Omega, \mathbb{R}^n)$? See [120, page 206].

The case $p = \infty$ is ruled out from Question 5.1 and it was addressed in [34, 160], see also [205], where it was shown that in this case the answer is negative, even when

f is continuous. Partial positive results were obtained in [181]. The case $p = 1$, where one must replace $L^1(\Omega)$ with $L \log L(\Omega)$ [120], also deserves special attention. Indeed, the Jacobian benefits from an improved integrability, first noticed by MÜLLER [170] and then generalised in [54]; an estimate up to the boundary was obtained in [120] and, in the planar case, a beautiful sharp version can be found in [8]. As noticed in [110], this improved integrability is essentially a by-product of the weak continuity of the Jacobian, see Section 1.1 for more details. We note that the special behaviour of the Jacobian at the endpoints $p = 1$, $p = \infty$ is not unexpected: for the divergence equation, which is a linearisation of the prescribed Jacobian equation, one also has non-existence of solutions when $f \in L^1$ or $f \in L^\infty$, due to the failure of estimates in these cases [164, 174], see also [83, 135] and [27, §2].

We finally comment on the restriction $1 \leq p$. In the case $n/(n+1) \leq p < 1$, Müller has posed an outstanding problem about the range of the *distributional Jacobian* $u \mapsto \operatorname{Det} \mathrm{D}u$, see [171, page 659]. In [142] it was shown that for any $1 < q < n$ and any $f \in L^{q/n}(\Omega)$ there is a *pointwise solution* $u \in W^{1,q}(\Omega, \mathbb{R}^n)$ of $\mathrm{J}u = f$ and one may prescribe any boundary datum $g \in W^{1-1/q,q}(\partial\Omega)$. The free choice of the boundary datum precises the well-known fact that the pointwise Jacobian loses its geometric content when $q < n$. Nevertheless, for $q \geq n$, as in this thesis, the distributional Jacobian and the pointwise Jacobian agree. We do not discuss distributional Jacobians further, instead referring the reader to [4, 14, 32, 169].

Before addressing Question 5.1, we recall some properties of the Jacobian in more detail and collect some useful results for mappings of finite distortion- noting that solutions of (JD) in $W^{1,p}(\Omega, \mathbb{R}^n)$ with $p \geq n$ will be mappings of finite distortion.

1 Properties of the Jacobian

1.1 Improved integrability of the Jacobian

In this section we recall standard facts concerning the improved integrability of the Jacobian. This phenomenon was first noticed by MÜLLER in [170] and then generalised by COIFMAN–LIONS–MEYER–SEMMES in [54]. The latter paper shows that the Jacobian determinant of maps in $\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ lies in the Hardy space $\mathscr{H}^1(\mathbb{R}^n)$. These results are essentially local and, as we are interested in the behaviour of Jacobians in domains,

we will need the following global version from [120]:

Theorem 5.2. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be such that $\mathrm{J}u \geq 0$ a.e. in Ω. Then, for any $p > n$,*

$$\|\mathrm{J}u\|_{L \log L(\Omega)} \leq C(\|u\|_{W^{1-1/p,p}(\partial\Omega)} + \|\mathrm{D}u\|_{L^n(\Omega)}).$$

There is a more general version of Theorem 5.2 when $\mathrm{J}u$ changes sign, but we will not need it here. The space $L \log L(\Omega)$ in the theorem is a particular case of an Orlicz–Zygmund space and it can be defined as the space of measurable functions $f \colon \Omega \to \mathbb{R}$ such that

$$\|f\|_{L \log L(\Omega)} \coloneqq \int_\Omega |f(x)| \log \left(e + \frac{|f(x)|}{\|f\|_{L^1}} \right) \mathrm{d}x < \infty. \tag{5.5}$$

The above expression defines a complete norm, although it is not immediate that it satisfies the triangle inequality, see [127, §8]. The standard Luxemburg norm in $L \log L(\Omega)$ is different but equivalent to the one in (5.5); we chose the latter since it is easier to use in calculations. We refer the reader to [196] for the relation between the space $L \log L$ and local Hardy spaces.

1.2 The Lusin (N) property and the change of variables formula

The following notions are very relevant in relation to the change of variables formula:

Definition 5.3. Let $u \colon \Omega \to \mathbb{R}^n$ be a continuous map which is differentiable a.e. in Ω. Then:

(i) u has the *Lusin (N) property* if $|u(E)| = 0$ for any $E \subset \Omega$ such that $|E| = 0$;

(ii) u has the *(SA) property* if $|u(E)| = 0$ for any open set $E \subset \Omega$ with $\mathrm{J}u = 0$ a.e. in E.

In the one-dimensional case, the Lusin (N) property is well understood: for instance, on an interval, a continuous function of bounded variation has the Lusin (N) property if and only if it is absolutely continuous. However, in higher dimensions, the situation is much more complicated, although we have the following characterisation, proved in [157]:

Proposition 5.4. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map with $\mathrm{J}u \geq 0$ in Ω. Then u has the Lusin (N) property if and only if it has the (SA) property.*

We remark that Proposition 5.4 is in general false if $\mathrm{J}u \not\geq 0$, see [179] for a counterexample. The following result, see [151], is also useful for our purposes:

Proposition 5.5. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map such that, for some $K \geq 1$,*

$$\operatorname{diam}(u(B_r(x)) \leq K \operatorname{diam}(u(\partial B_r(x)) \qquad \text{for all } B_r(x) \Subset \Omega. \tag{5.6}$$

Then u has the Lusin (N) property.

The change of variables formula is closely related to the Jacobian determinant:

Theorem 5.6. *Let $u \in C^0(\Omega, \mathbb{R}^n) \cap W^{1,n}(\Omega, \mathbb{R}^n)$ be a map with the Lusin (N) property. Then*

$$\int_E |\mathrm{J}u| \, \mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) \, \mathrm{d}y \qquad \text{for all measurable sets } E \subset \Omega, \tag{5.7}$$

where $\mathcal{N}$ is the multiplicity function*, defined as $\mathcal{N}(y, u, E) \equiv \#\{x \in E : u(x) = y\}$.*

The reader may find the proof of Theorem 5.6, together with a wealth of information on geometric properties of Sobolev maps, in [86].

2 Mappings of finite distortion

In this section we gather a few results from the theory of mappings of finite distortion, which is detailed in [115, 125], see also [7] for the planar case.

Definition 5.7. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be such that $0 \leq \mathrm{J}u \in L^1_{\mathrm{loc}}(\Omega)$. We say that u is a *map of finite distortion* if there is a function $K \colon \Omega \to [1, \infty]$ such that $K(\cdot) < \infty$ a.e. in Ω and

$$|\mathrm{D}u(x)|^n \leq K(x) \, \mathrm{J}u(x) \quad \text{ for a.e. } x \text{ in } \Omega.$$

If u has finite distortion, we can set $\mathrm{K}u(x) = \frac{|\mathrm{D}u|^n}{\mathrm{J}u(x)}$ if $\mathrm{J}u(x) \neq 0$ and $\mathrm{K}u(x) = 1$ otherwise; this function is the (optimal) *distortion* of u.

We state some basic properties of the topological degree, referring the reader to [86] for further details.

Lemma 5.8. *For $u \in C^0(\overline{\Omega}, \mathbb{R}^n)$ and $x \in \mathbb{R}^n \backslash u(\partial\Omega)$, the* topological degree *of u at x with respect to Ω, denoted by* $\deg(x, u, \Omega)$*, has the following properties:*

(i) $\deg(\cdot, u, \Omega) \colon \mathbb{R}^n \backslash u(\partial\Omega) \to \mathbb{Z}$;

(ii) *if* $u = v$ *on* $\partial\Omega$, *then* $\deg(\cdot, u, \Omega) = \deg(\cdot, v, \Omega)$ *on* $\mathbb{R}^n \setminus u(\partial\Omega)$;

(iii) *if* $x \notin u(\partial\Omega)$ *and* $\deg(x, u, \Omega) \neq 0$ *then* $x \in u(\Omega)$.

For our purposes, the following classical result [206] is particularly relevant:

Theorem 5.9. *Let* $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ *be a map of finite distortion. Then* u *has a continuous representative with the Lusin (N) property. In fact, whenever* $r < R$ *and* $B_R(x_0) \subset \Omega$,

$$\Big(\operatorname{diam}_{B_r(x_0)} u\Big)^n \leq \frac{C}{\log(R/r)} \int_{B_R(x_0)} |\mathrm{D}u|^n \,\mathrm{d}x;$$

Theorem 5.9 gives no improvement in the supercritical case, since the continuous representative of a map $u \in W^{1,p}$, $p > n$, has the Lusin (N) property. For further use we isolate the following fact that is used in the proof of Theorem 5.9.

Proposition 5.10. *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ *be a map of finite distortion; then* (5.6) *holds. In fact, if we measure the diameter in* $\mathbb{R}^2$ *with respect to the* ℓ^∞ *norm, we can take* $K = 1$.

Theorem 5.9 shows that solutions of (JD) satisfy the change of variables formula:

Corollary 5.1. *Let* $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ *be a solution of* (JD) *and* (5.4). *Then*

$$\mathscr{L}^n(u(E)) = \int_E f \,\mathrm{d}x \qquad \text{for all measurable sets } E \subseteq \Omega.$$

Proof. The non-degeneracy assumption (5.4) implies that u is a map of finite, and even integrable, distortion. Hence, by Theorem 5.9, the coarea formula applies, see for instance [86, Theorem 5.23]: if $\mathcal{N}$ is the multiplicity function, that is we define $\mathcal{N}(y, u, E) \equiv \#\{x \in E : u(x) = y\}$, then

$$\int_{u(E)} \mathcal{N}(y, u, E) \,\mathrm{d}y = \int_E f \,\mathrm{d}x, \qquad \text{for all measurable sets } E \subseteq \Omega. \tag{5.8}$$

Moreover, u is injective a.e., that is $\mathcal{N}(y, u, \Omega) \leq 1$ for a.e. $y \in \mathbb{R}^n$: it suffices to apply (5.8) with $E = \Omega$, recalling that $\int_\Omega f \,\mathrm{d}x = |\Omega|$ by (5.4). Hence the conclusion follows from (5.8). □

It is worthwhile mentioning that solutions of (JD) in a sufficiently good Sobolev space are homeomorphisms. The sharp statement follows for instance from the main result in [116]:

Theorem 5.11. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map with distortion in $L^{n-1}(\Omega)$. If there is a compact set $K \subset \Omega$ such that $u|_{\Omega \setminus K}$ is discrete then u is open and discrete.*

In fact, this can be used to prove,

Corollary 5.2. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ be a map with distortion in $L^1(\Omega)$. Then u is open and discrete.*

In general dimensions $n > 2$ and considering maps with distortion in $L^{n-1}(\Omega)$, discreteness may fail [117]. In the same set-up, openness was conjectured to hold by Iwaniec and Šverák [126].

Recall that a map is *discrete* if the preimage of every point is locally finite. A counterexample due to Ball [15] shows that the conclusion of Theorem 5.11 does not hold if the distortion is in $L^q(\Omega)$ for $q < n - 1$. We use Theorem 5.11 to deduce:

Corollary 5.3. *Let $u \in W^{1,n(n-1)}(\Omega, \mathbb{R}^n)$ be a solution of* (JD), (5.4). *Then $u \in \mathrm{Hom}(\overline{\Omega}, \overline{\Omega})$.*

Proof. It follows from our assumptions that the distortion of u is in $L^{n-1}(\Omega)$. Consider a sufficiently small neighbourhood Ω_δ of Ω and extend u to be the identity in $\Omega_\delta \setminus \Omega$. Theorem 5.11, applied in Ω_δ, shows that $u|_\Omega$ is open and discrete and it follows that $u \in \mathrm{Hom}(\overline{\Omega}, u(\overline{\Omega}))$, see e.g. [115, Theorem 3.27]. Moreover, $u(\overline{\Omega}) = \overline{\Omega}$. Indeed, it suffices to prove that $u(\Omega) \subseteq \Omega$, since

$$u(\overline{\Omega}) = u(\Omega) \sqcup u(\partial\Omega) = u(\Omega) \sqcup \partial\Omega \tag{5.9}$$

where the unions are disjoint, and $\Omega \subseteq u(\Omega)$ by Lemma 5.8. Suppose that there is $x \in \Omega$ such that $u(x) \in \mathbb{R}^n \setminus \overline{\Omega}$. Take $y \in \Omega$ such that $u(y) \in \Omega$ and consider a continuous path in Ω joining x and y. The image of such a path under u must cross $\partial\Omega$ somewhere, which contradicts (5.9). □

In fact, a simple argument using the change of variables formula shows that the inverse map u^{-1} is in $W^{1,n}(\Omega, \Omega)$. Note that the situation in the planar case is particularly pleasant, since then $n - 1 = 1$. Other results in the direction of Corollary 5.3 can be found in [15, 198].

3 Polar coordinates and radial stretchings

Our analysis of Question 5.1 will be based on understanding solutions with radial symmetry. Such solutions are most naturally expressed in polar coordinates. While the theory of polar coordinates for Sobolev mappings in the plane is in principle well understood, we have found it difficult to find the precise statements we require in the literature and hence include them here. Given a planar Sobolev map $u \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$, we consider polar coordinates both in the domain and in the target; that is, we want to write

$$u(re^{i\theta}) = \psi(r,\theta)\exp(i\gamma(r,\theta)) \tag{5.10}$$

for some functions $\psi\colon (0,\infty)\times[0,2\pi] \to [0,\infty)$ and $\gamma\colon (0,\infty)\times[0,2\pi]\to\mathbb{R}$, where furthermore we must have the compatibility conditions

$$\psi(r,0) = \psi(r,2\pi) \quad \text{and} \quad \gamma(r,0) - \gamma(r,2\pi) \in 2\pi\mathbb{Z} \qquad \text{for all } r. \tag{5.11}$$

We will freely identify $(r,\theta) \equiv re^{i\theta}$, adopting either notation whenever it is more convenient.

The existence of a representation as in (5.10) is a standard problem in lifting theory:

Proposition 5.12. *Let $0 \le R_1 < R_2$ and $p \ge 2$. Let $u \in W^{1,p}(\mathbb{A}(R_1,R_2),\mathbb{R}^2)$ and, if $p = 2$, suppose moreover that u is continuous. Assume $u^{-1}(0) \subseteq \{0\}$. Then there are continuous functions*

$$\psi \in W^{1,p}\left([R_1,R_2]\times[0,2\pi]\right), \qquad \gamma \in W^{1,p}\left((\max\{R_1,\varepsilon\},R_2)\times[0,2\pi]\right),$$

where $\varepsilon \in (0,R_2)$ is arbitrary, which satisfy (5.11) *and such that the representation* (5.10) *holds.*

Proof. Let $\varepsilon > 0$ and consider the keyhole domains

$$\begin{aligned}\mathbb{A}_{1,\varepsilon} &\coloneqq [\max\{R_1,\varepsilon\},R_2]\times[\varepsilon,2\pi-\varepsilon],\\ \mathbb{A}_{2,\varepsilon} &\coloneqq [\max(R_1,\varepsilon),R_2]\times([0,\pi-\varepsilon]\cup[\pi+\varepsilon,2\pi]).\end{aligned}$$

We freely identify $\mathbb{A}_{i,\varepsilon}$ with the respective domains in $\mathbb{R}^2$.

We first show the existence of a representation (5.10) in each $\mathbb{A}_{i,\varepsilon}$. Note that if $u \in W^{1,p}$ then $\psi \coloneqq |u|$ is also in $W^{1,p}$ and is continuous whenever u is. Thus, since

$0 \notin u(\mathbb{A}_{i,\varepsilon})$, it suffices to prove the existence of a continuous function $\gamma_i \in W^{1,p}(\mathbb{A}_{i,\varepsilon}, \mathbb{R})$ such that $u/|u| = e^{i\gamma_i}$ for $i = 1, 2$. Since u is continuous, $u/|u| \in W^{1,p}(\mathbb{A}_{i,\varepsilon}, \mathbb{S}^1)$, and so the existence of γ_i follows from the results in [22], see also [28].

Thus, for almost every $(r, \theta) \in \mathbb{A}_{1,\varepsilon} \cap \mathbb{A}_{2,\varepsilon}$,

$$\psi(r,\theta)e^{i\gamma_1(r,\theta)} = u(re^{i\theta}) = \psi(r,\theta)e^{i\gamma_2(r,\theta)} \iff \gamma_1(r,\theta) - \gamma_2(r,\theta) = 2\pi k(r,\theta),$$

where $k(r, \theta) \in \mathbb{Z}$. As γ_1, γ_2 are continuous in $\mathbb{A}_{1,\varepsilon} \cap \mathbb{A}_{2,\varepsilon}$, we must have

$$k(r,\theta) = \begin{cases} k_1 & \text{for } \varepsilon < \theta < \pi - \varepsilon, \\ k_2 & \text{for } \pi + \varepsilon < \theta < \varepsilon. \end{cases}$$

Without loss of generality, upon redefining γ_1 we may assume $k_1 = 0$. Hence we may define

$$\gamma_\varepsilon(r,\theta) \coloneqq \begin{cases} \gamma_1(r,\theta) & \text{if } (r,\theta) \in \mathbb{A}_{1,\varepsilon}, \\ \gamma_2(r,\theta) & \text{if } (r,\theta) \in \mathbb{A}_{2,\varepsilon}. \end{cases}$$

By a similar argument, we see that we may take $\gamma_\varepsilon = \gamma_\delta$ in $\mathbb{A}(R_1, R_2) \setminus (B_\delta \cup B_\varepsilon)$, so that in fact $u = \psi(r,\theta)e^{i\gamma(r,\theta)}$ with $\gamma \in W^{1,p}(\max(R_1, \varepsilon), R_2) \times [0, 2\pi])$ for all $\varepsilon > 0$. The conclusion follows. □

Remark 5.13. The conclusion of Proposition 5.12 is false if $p < 2$, see [28, §4].

We record the following formulae

Corollary 5.4. *In the setting of Proposition 5.12, we have a.e. the formulae*

$$\mathrm{J}u = \frac{\psi}{r}\left(\partial_r\psi\,\partial_\theta\gamma - \partial_\theta\psi\partial_r\gamma\right), \tag{5.12}$$

$$|\mathrm{D}u|^2 = |\partial_r\psi|^2 + |\psi\partial_r\gamma|^2 + \frac{|\partial_\theta\psi|^2}{r^2} + \frac{|\psi\partial_\theta\gamma|^2}{r^2}. \tag{5.13}$$

Proof. It is not difficult to formally derive the above formulae whenever the representation (5.10) holds. To make the argument rigorous it suffices to note that, due to the regularity of ψ and γ, the right-hand sides in (5.12)–(5.13) define locally integrable functions. Thus the corollary follows by a standard density argument. □

A function $f\colon B_R(0) \to \mathbb{R}$ is *radially symmetric* if $|x| = |y| \implies f(x) = f(y)$ and we identify any such function with a function $f\colon [0, +\infty) \to \mathbb{R}$ in the obvious way. Under natural assumptions, and when the data is radially symmetric, equation (5.3) admits a

unique spherically symmetric solution, that is, a solution of the form $u(x) = \rho(|x|)\frac{x}{|x|}$; we refer to these maps as *radial stretchings*. More precisely, c.f. [16, Lemma 4.1], we have:

Lemma 5.14. *Let $n > 1$ and $1 \leq p < \infty$. The map $u(x) \coloneqq \rho(|x|)\frac{x}{|x|}$ is in $W^{1,p}(B_R(0), \mathbb{R}^n)$ if and only if ρ is absolutely continuous in $(0, 1)$, and*

$$\|\mathrm{D}u\|^p_{L^p(B_R(0))} \approx_n \int_0^R \left(|\dot{\rho}(r)|^p + \left| \frac{\rho(r)}{r} \right|^p \right) r^{n-1} \, \mathrm{d}r < \infty;$$

a similar statement holds for $p = \infty$. In this case, for a.e. x in $B_R(0)$, and writing $r = |x|$, we have the formulae

$$\mathrm{J}u(x) = \frac{1}{r^{n-1}} \dot{\rho}(r) \rho^{n-1}(r), \tag{5.14}$$

$$\mathrm{D}u(x) = \frac{\rho(r)}{r} \mathrm{Id} + \left(\dot{\rho}(r) - \frac{\rho(r)}{r} \right) \frac{x \otimes x}{r^2}. \tag{5.15}$$

In particular, if u solves (5.3), then

$$\rho^n(r) = \int_0^r n f(s) s^{n-1} \, \mathrm{d}s. \tag{5.16}$$

We also record here the standard notation $\partial_r u(x) \equiv (\mathrm{D}u(x)) \cdot \frac{x}{r}$; in particular, if u is a radial stretching as in Lemma 5.14, then

$$\partial_r u(x) = \dot{\rho}(r) \frac{x}{r}. \tag{5.17}$$

Based on Lemma 5.14 we define:

Definition 5.15. When $p \in [1, \infty)$, $k \in \mathbb{Z}$ and $f \in \mathscr{H}^p(\mathbb{R}^2)$ is radially symmetric, a mapping of the form

$$\phi_k(z) \coloneqq \phi_k(re^{i\theta}) \equiv \frac{\rho(r)}{\sqrt{|k|}} e^{2\pi i k \theta}, \qquad \rho^2(r) \coloneqq \int_0^r 2sf(s) \, \mathrm{d}s$$

is called a *generalised radial stretching*. We simply refer to ϕ_1 as a *radial stretching*.

Generalised radial stretchings are also spherically symmetric in the sense that they map circles centred at zero to circles centred at zero. The subscript k denotes the topological degree of the map ϕ_k.

The following proposition gives information on the regularity of radial stretchings solving (5.3) and follows easily from Lemma 5.14, see also [208, Theorem 6].

Proposition 5.16. *Let $\Omega = B_R(0)$ and $p \in [1, +\infty]$. If $f \in L^p(\Omega)$ is radially symmetric and satisfies* (5.4) *then the unique radial stretching u solving* (5.3) *is in $W^{1,p}(B_R(0))$ and*

$$\|\mathrm{D}u\|_{L^p(B_R(0))} \lesssim_n \frac{\|f\|_{L^p(B_R(0))}}{(\inf f)^{\frac{n-1}{n}}} + R^{\frac{n-1}{p}} \|f\|^{\frac{1}{n}}_{L^p(B_R(0))}. \tag{5.18}$$

Moreover, this inequality is sharp: in general $u \notin W^{1,q}$ for any $q > p$.

Proof. We assume that $1 \leq p < \infty$, as the case $p = \infty$ is similar. Using (5.16) and Jensen's inequality, we deduce that

$$\left|\frac{\rho(r)}{r}\right|^p \lesssim \left(⨍_{B_r(0)} f(x)\,\mathrm{d}x\right)^{\frac{p}{n}} \leq \frac{1}{r}\left(\int_{B_r(0)} f(x)^p\,\mathrm{d}x\right)^{\frac{1}{n}} \leq \frac{1}{r}\|f\|^{\frac{p}{n}}_{L^p(B_R(0))}$$

and therefore, integrating in r,

$$\int_0^R \left|\frac{\rho(r)}{r}\right|^p r^{n-1}\,\mathrm{d}r \lesssim R^{n-1}\|f\|^{\frac{p}{n}}_{L^p(B_R(0))}.$$

Since f satisfies (5.4), we deduce that $(\inf f)^{\frac{1}{n}} \leq \rho(r)/r$. Thus, from (5.14), we have

$$\dot\rho(r) = \frac{r^{n-1}}{\rho^{n-1}(r)} f(r) \quad \Longrightarrow \quad \int_0^R |\dot\rho(r)|^p r^{n-1}\,\mathrm{d}r \lesssim \frac{\|f\|^p_{L^p(B_R(0))}}{(\inf f)^{\frac{p(n-1)}{n}}}$$

and the desired estimate follows from Lemma 5.14. To see that this inequality is sharp, it suffices to note that for $\delta > 0$ there is a constant C_δ such that $C_\delta f(r) \leq \dot\rho(r)$ for $r \in (\delta, R)$. By choosing $f \in L^p(0, R)$ such that $f \notin L^q(\delta, R)$ for $q > p$ we have $\mathrm{D}u \notin L^q(B_R(0))$. □

Clearly it is not the case that any radially symmetric $f \in \mathscr{H}^p(\mathbb{R}^2)$ admits generalised radial stretchings as solutions of (5.3): for instance, we must have

$$\text{either } \int_0^r 2sf(s)\,\mathrm{d}s \leq 0 \text{ for a.e. } r, \qquad \text{or } \int_0^r 2sf(s)\,\mathrm{d}s \geq 0 \text{ for a.e. } r.$$

Even if this condition holds, there may not be generalised radial stretchings with the required regularity:

Example 5.17. If $f = 1_{\mathbb{A}(1,2)}$ then the map ϕ_k is in $\bigcup_{1\leq p<2} W^{1,p}\backslash W^{1,2}(B_2, \mathbb{R}^2)$, for any $k \in \mathbb{Z}$.

The claim in Example 5.17 follows readily from Lemma 5.14. In general, it is not completely clear what the relation between the regularity of f and the regularity of ϕ_k is, but further to the work in this chapter, see also [148, §3] and [208, §7].

4 Non-existence of solutions in domains

We are now ready to answer Question 5.1. It will turn out that the answer is negative in a rather strong sense. Indeed, let us consider for $1 \leq p < \infty$ the complete metric spaces

$$X_p \equiv \left\{ f \in L^p(\Omega) : f \text{ satisfies (5.4)} \right\}, \qquad \text{for } p > 1$$
$$X_1 \equiv \left\{ f \in L \log L(\Omega) : f \text{ satisfies (5.4)} \right\}.$$

We then have:

Theorem 16. *Fix $0 < c < 1$ and $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a smooth, uniformly convex domain. There is $f \in X_p$ such that the Dirichlet problem* (JD) *has no solution $u \in W^{1,np}(\Omega, \mathbb{R}^n)$. In fact, for a Baire-generic $f \in X_p$,* (JD) *has no solution in $\bigcup_{n \leq q,\, p<q} W^{1,q}(\Omega, \mathbb{R}^n)$.*

It will be clear from the proof that the boundary condition $u = \mathrm{id}$ on $\partial\Omega$ in (JD) can be relaxed to the requirement that $u(x) \in \partial\Omega$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial\Omega$.

We remark that this result leaves open the following question:

Question 5.18. For $n \leq p = q < \infty$ and $f \in X_p$, does (JD) have a solution such that $u \in W^{1,q}(\Omega, \mathbb{R}^n)$?

Question 5.18 is natural from the viewpoint of the regularity theory for optimal transport maps. In Optimal Transportation, the underdetermined equation (5.3) is usually turned into a determined elliptic equation by imposing the constraint $u = \mathrm{D}\phi$ for some convex scalar function ϕ: thus ϕ satisfies the Monge–Ampère equation $\det \mathrm{D}^2\phi = f$, see [30, 70, 203]. The relation between the prescribed Jacobian equation and the Monge–Ampère equation is analogous to that of their linear counterparts, the divergence equation and the Poisson equation, see Table 5.1. Moreover, the estimates for the Monge–Ampère equation scale as in these linear problems, which is precisely the scaling in Question 5.18.

However, the failure of a certain estimate for the determined problem does not imply that the same happens for the underdetermined problem, as shown by Bourgain–Brezis [27]. Surprisingly, even in the linear setting, *solutions of the determined problem do not have optimal regularity for the underdetermined problem*! This is also the case

for the non-linear problem (5.3), at least if we only require $\inf_\Omega f \geq 0$. Indeed, at [208, page 293], Ye gives an example in $\Omega = B \subset \mathbb{R}^2$ such that (JD) has a smooth solution but the corresponding solution of the Monge–Ampère equation is only $C^{1,1}$ regular.

	Linear	Non-linear
Determined	$\triangle\phi = f$	$\det \mathrm{D}^2\phi = f$
Underdetermined	$\operatorname{div} u = f$	$\det \mathrm{D}u = f$

Table 5.1: Particular solutions to the underdetermined problem are obtained by taking $u = \mathrm{D}\phi$ for ϕ a solution of the corresponding determined problem.

Theorem 16 will follow from a sequence of lemmas. The main idea is that for data with bad integrability behaviour near the boundary, solutions that are radial stretchings will inherit the bad behaviour. In a second step, we show that in this set-up all solutions are comparable in energy to radial stretchings near the boundary. We begin by taking $\Omega = B_1(0)$.

Suppose $\inf_\Omega f(x) \geq c > 0$. For $p \in [1, +\infty)$ and $\eta \in [0, 1)$, let

$$\begin{aligned} Z_p^\eta &\coloneqq \left\{ f \in L^p(B_1) : \fint_{B_1} f \,\mathrm{d}x = 1,\ f \geq (1-\eta)c \right\}, \qquad \text{for } p > 1 \\ Z_1^\eta &\coloneqq \left\{ f \in L\log L(B_1) : \fint_{B_1} f \,\mathrm{d}x = 1,\ f \geq (1-\eta)c \right\}; \end{aligned} \tag{5.19}$$

and write $Z_p \equiv Z_p^0$. We make a few immediate remarks about the sets Z_p^η:

(i) clearly $Z_p^{\eta_1} \subset Z_p^{\eta_2}$ if $\eta_1 < \eta_2$;

(ii) each set Z_p^η is a complete metric space under the distance

$$\operatorname{dist}_p(f, g) \coloneqq \begin{cases} \|f - g\|_{L^p(B_1)}, & p > 1 \\ \|f - g\|_{L\log L(B_1)}, & p = 1 \end{cases};$$

(iii) if $c = 1$ then the only elements of Z_p are functions which are 1 a.e. in B_1. Thus we assume throughout this section, without loss of generality, that $c < 1$.

We write $B_{Z_p^\eta}(f, \varepsilon)$ for a ball of radius ε around f, under this distance. Given $f \in Z_p$ and $\max\{p, n\} \leq q$, we consider the *q-energy of the datum* f, defined by

$$\mathcal{E}_q(f) \coloneqq \inf \left\{ \int_{B_1} |\mathrm{D}v|^q \,\mathrm{d}x : v \text{ solves (JD)} \right\}. \tag{5.20}$$

Thus (JD) admits a $W^{1,q}$ solution if and only if $\mathcal{E}_q(f) < \infty$. We say that w is a *q-energy-minimal solution* for f if w solves (JD) and moreover

$$\mathcal{E}_q(f) = \int_{B_1(0)} |\mathrm{D}w|^q \,\mathrm{d}x.$$

We begin with the following lemma:

Lemma 5.19. *Fix $1 \le p < \infty$ and let $\varepsilon, \eta > 0$ be arbitrary. For any $f \in Z_p$ there are sequences $f_j \in B_{Z_p^\eta}(f, \varepsilon)$ and $R_j \nearrow 1$ such that*

(i) $\mathrm{dist}_p(f_j, f) \to 0$ *as* $j \to \infty$;

(ii) f_j *is radially symmetric in* $\mathbb{A}(R_j, 1)$;

(iii) *if u_j is the radial stretching such that $\mathrm{J}u_j = f_j$ in $\mathbb{A}(R_j, 1)$ and $u_j = \mathrm{id}$ on $\mathbb{S}^{n-1}$, then*

$$\lim_{j\to\infty} \int_{\mathbb{A}(R_j,1)} |\partial_r u_j(x)|^q \,\mathrm{d}x = +\infty$$

for any $q > p$.

Proof. For $\gamma, R \in (\frac{3}{4}, 1)$, consider the functions

$$f_{\gamma,R}(x) \coloneqq \begin{cases} \dfrac{\gamma^n}{⨍_{B_R} f \,\mathrm{d}x} f(x), & 0 < |x| < R, \\ \dfrac{M[\gamma R + M(|x| - R)]^{n-1}}{|x|^{n-1}}, & R \le |x| < 1, \end{cases} \qquad \text{where } M \coloneqq \frac{1-\gamma R}{1-R}. \tag{5.21}$$

The choice of M ensures that $⨍_{B_1} f_{\gamma,R} \,\mathrm{d}x = 1$, since

$$\int_R^1 f_{\gamma,R}(r) r^{n-1} \,\mathrm{d}r = \frac{1-(\gamma R)^n}{n}. \tag{5.22}$$

Fix $\alpha \in (-\frac{q}{p}, -1)$ and choose $\gamma = \gamma(R)$ in such a way that

$$1 - \gamma R = (1-R)^{1+\alpha/q}; \tag{5.23}$$

in particular, we have $\gamma(R) \to 1$ as $R \to 1$. Take any sequence $R_j \nearrow 1$ and consider the associated numbers $\gamma_j = \gamma_j(R_j)$. We will prove that, for j large enough, $f_j \equiv f_{\gamma_j,R_j}$ satisfies the properties above.

Concerning the lower bounds of the sequence, for $r \in (R, 1)$, straightforward algebraic manipulations show that

$$f_{\gamma,R}(r) = M \left(\frac{\gamma R + M(r-R)}{r} \right)^{n-1} \ge 1 \times \gamma^{n-1};$$

thus $f_{\gamma,R}(r) > c$ for γ sufficiently close to 1. For $r \in (0,R)$, clearly it holds that $f_{\gamma,R}(r) \geq \gamma^n c / \fint_{B_R} f \to c$ as $\gamma, R \to 1$. Hence the lower bounds are satisfied. Moreover, (ii) clearly holds.

For (iii), denote by $u_{\gamma,R}(x) = \rho_{\gamma,R}(r)\frac{x}{r}$ the unique radial stretching solving the equation $\mathrm{J}u_{\gamma,R} = f_{\gamma,R}$ in $\mathbb{A}(R,1)$ and such that $u_{\gamma,R} = \mathrm{id}$ on $\mathbb{S}^{n-1}$. By Lemma 5.14 we find that

$$\rho_{\gamma,R}(r) = \gamma R + M(r-R), \qquad r \in (R,1] \tag{5.24}$$

and so, by (5.17), since $r^{n-1}\,\mathrm{d}r \approx \mathrm{d}r$ for $r \in (\frac{1}{2},1)$,

$$\int_{\mathbb{A}(R,1)} |\partial_r u_{\gamma,R}(x)|^q \,\mathrm{d}x \approx \int_R^1 |\dot\rho(r)|^q \,\mathrm{d}r = M^q(1-R) = \frac{(1-\gamma R)^q}{(1-R)^{q-1}} = (1-R)^{1+\alpha}.$$

Thus, since $\alpha < -1$, we see that (iii) also holds.

Hence it remains to prove (i), and we split the proof into two cases.

Case $p > 1$: Whenever $\gamma, R \nearrow 1$, we have

$$\int_{B_R} |f - f_{\gamma,R}|^p \,\mathrm{d}x = \left| \frac{\gamma^n}{\fint_{B_R} f \,\mathrm{d}x} - 1 \right|^p \int_{B_R} |f|^p \,\mathrm{d}x \to 0.$$

Thus, since $\int_{\mathbb{A}(R,1)} |f(x)|^p \,\mathrm{d}x \to 0$ as $R \to 1$, in order to prove that $f_{\gamma,R} \to f$ in $L^p(B_1)$ it suffices to show that, as $\gamma, R \nearrow 1$,

$$\int_{\mathbb{A}(R,1)} |f_{\gamma,R}(x)|^p \,\mathrm{d}x \to 0. \tag{5.25}$$

For this, note that for $|x| = r \in (R,1)$, since $\gamma R < 1$ and $R > \frac{1}{2}$,

$$f_{\gamma,R}(x) \leq \frac{M}{r^{n-1}} \leq 2^{n-1}\frac{(1-\gamma R)}{1-R}.$$

Thus, as $r^{n-1}\,\mathrm{d}r \approx \mathrm{d}r$ for $r \in (\frac{1}{2},1)$,

$$\int_{\mathbb{A}(R,1)} |f_{\gamma,R}(x)|^p \,\mathrm{d}x \approx \int_R^1 |f_{\gamma,R}(r)|^p \,\mathrm{d}r \lesssim \frac{(1-\gamma R)^p}{(1-R)^{p-1}} = (1-R)^{\alpha\frac{p}{q}+1}.$$

Since $\alpha > -\frac{q}{p}$ we have that $\alpha\frac{p}{q} + 1 > 0$ and so (5.25) is proved.

Case $p = 1$: As before, we have that, as $\gamma, R \nearrow 1$,

$$\|f_{\gamma,R} - f\|_{L\log L(B_R)} = \left| 1 - \frac{\gamma^n}{\fint_{B_R} f \,\mathrm{d}x} \right| \|f\|_{L\log L(B_R)} \to 0$$

and also $\|f\|_{L\log L(\mathbb{A}(R,1))} \to 0$ as $R \to 1$. Thus it suffices to show $\|f_{\gamma,R}\|_{L\log L(\mathbb{A}(R,1)} \to 0$ as $R \to 1$. We make the simple observation that

$$1 - (\gamma R)^n = (1-\gamma R)(1 + \gamma R + \cdots + (\gamma R)^{n-1}) \approx_n (1-\gamma R).$$

Then, using the estimates on $f_{\gamma,R}$ from the previous case and (5.22), we obtain

$$\begin{aligned}
\|f_{\gamma,R}\|_{L\log L(\mathbb{A}(R,1))} &\approx \int_{\mathbb{A}(R,1)} f_{\gamma,R}(x) \log\left(e + \frac{f_{\gamma,R}(x)}{\|f_{\gamma,R}\|_{L^1(\mathbb{A}(R,1))}}\right) \mathrm{d}x \\
&\lesssim \int_R^1 \frac{1-\gamma R}{1-R} \log\left(e + \frac{1-\gamma R}{(1-R)(1-\gamma^n R^n)}\right) \mathrm{d}r \\
&\leq (1-R)^{\alpha/q+1} \log\left(e + \frac{1}{1-R}\right).
\end{aligned}$$

The right-hand side converges to zero as $R \to 1$, since $\alpha/q + 1 > 0$. □

Our goal is to show that $\mathcal{E}_q(f_{\gamma,R}) \to +\infty$ as $\gamma, R \to 1$. The idea is that energy-minimal solutions are controlled by the radial solution.

Proposition 5.20. *Fix $1 \leq p < q$ and suppose that $v \in W^{1,n}(B_1, \mathbb{R}^n)$ is a solution of $\mathrm{J}v = f_{\gamma,R}$ with $v = \mathrm{id}$ on ∂B_1, where $f_{\gamma,R}$ is as in* (5.21) *and* (5.23). *There is a constant $C = C(n,q) > 0$ such that*

$$\int_{\mathbb{A}(R,1)} |\partial_r u(x)|^q \,\mathrm{d}x \leq C \int_{\mathbb{A}(R,1)} |\partial_r v(x)|^q \,\mathrm{d}x,$$

where $u = u_{\gamma,R}$ is the radial stretching such that $\mathrm{J}u_{\gamma,R} = f_{\gamma,R}$ in $\mathbb{A}(R,1)$ and $u = \mathrm{id}$ on $\mathbb{S}^{n-1}$.

Proof. Throughout the proof θ will denote an element of $\mathbb{S}^{n-1}$ and we write $r\theta$ for the corresponding element in a sphere of radius r.

Since $R > \frac{1}{2}$, from Hölder's inequality,

$$\begin{aligned}
2^{n-1} \fint_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)|^q r^{n-1} \,\mathrm{d}r \,\mathrm{d}\theta &\geq \fint_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)|^q \,\mathrm{d}r \,\mathrm{d}\theta \\
&\geq \left(\fint_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)| \,\mathrm{d}r \,\mathrm{d}\theta\right)^q.
\end{aligned}$$

On the other hand, since $\partial_r u(r\theta) = M\theta$ for all $r \in (R,1)$, c.f. (5.24), we can estimate

$$\left(\fint_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r u(r\theta)| \,\mathrm{d}r \,\mathrm{d}\theta\right)^q = \fint_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r u(r\theta)|^q \,\mathrm{d}r \,\mathrm{d}\theta$$

$$\geq \fint_{\mathbb{S}^{n-1}} \fint_R^1 |\partial_r u(r\theta)|^q r^{n-1} \,\mathrm{d}r \,\mathrm{d}\theta.$$

Thus, the proposition will be proved once we show that

$$\int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)| \,\mathrm{d}r \,\mathrm{d}\theta \geq \frac{1}{4n^2} \int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r u(r\theta)| \,\mathrm{d}r \,\mathrm{d}\theta$$

for all $R \in (\frac{1}{2}, 1)$. Note that $\int_R^1 |\partial_r u(r\theta)| \,\mathrm{d}r = (1-R)M = 1 - \gamma R$ for all $\theta \in \mathbb{S}^{n-1}$.

Let us take the set

$$\Theta \coloneqq \{\theta \in \mathbb{S}^{n-1} : r \mapsto u(r\theta) \text{ is absolutely continuous in } [R, 1]\}.$$

For $\lambda \in (0, 1)$ to be chosen later, let

$$\Theta_1 \coloneqq \left\{\theta \in \Theta : \int_R^1 |\partial_r v(s\theta)| \,\mathrm{d}s \leq \lambda\right\}, \qquad \Theta_2 \coloneqq \left\{\theta \in \Theta : \int_R^1 |\partial_r v(s\theta)| \,\mathrm{d}s > \lambda\right\}.$$

As Sobolev functions are absolutely continuous on almost all lines [159, §1.1.3], we have $\mathscr{H}^{n-1}(\Theta) = \mathscr{H}^{n-1}(\mathbb{S}^{n-1})$ and so the set $\mathbb{S}^{n-1} \setminus \Theta$ is an $\mathscr{H}^{n-1}$-null set. Fubini's theorem and the fact that v satisfies the Lusin (N) property, c.f. Theorem 5.9, then implies

$$\mathscr{L}^n\left(v(\Theta_1 \times [R, 1])\right) + \mathscr{L}^n\left(v(\Theta_2 \times [R, 1])\right) = \mathscr{L}^n(v(\mathbb{A}(R, 1)).$$

By the fundamental theorem of calculus, for $\theta \in \Theta_1$, $r \in [R, 1]$,

$$1 - |v(r\theta)| \leq |v(\theta) - v(r\theta)| \leq \int_r^1 |\partial_r v(s\theta)| \,\mathrm{d}s \leq \lambda$$

and thus $v(\Theta_1 \times [R, 1]) \subset \mathbb{A}[1 - \lambda, 1]$. Combined with the change of variables formula from Corollary 5.1, (5.22) and Bernoulli's inequality, this estimate yields

$$\begin{aligned}
\mathscr{L}^n(v(\Theta_2 \times [R, 1])) &= \int_{\mathbb{A}(R,1)} f_{\gamma,R}(x) \,\mathrm{d}x - \mathscr{L}^n(v(\Theta_1 \times [R, 1])) \\
&\geq \omega_n \left[(1 - \gamma^n R^n) - (1 - (1 - \lambda)^n)\right] \\
&\geq \omega_n \left(1 - n\lambda - \gamma^n R^n\right),
\end{aligned}$$

since $\mathscr{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$. Moreover, by Markov's inequality and (5.22),

$$\begin{aligned}
\mathscr{L}^n(v(\Theta_2 \times [R, 1])) &= \int_{\Theta_2 \times [R,1]} f_{\gamma,R}(x) \,\mathrm{d}x \\
&= \mathscr{H}^{n-1}(\Theta_2) \frac{1 - \gamma^n R^n}{n} \leq \frac{1 - \gamma^n R^n}{n\lambda} \int_{\mathbb{A}(R,1)} |\partial_r v(r\theta)| \,\mathrm{d}r \,\mathrm{d}\theta.
\end{aligned}$$

Combining the last two estimates and choosing $\lambda \coloneqq \frac{1-\gamma^n R^n}{2n}$, we find

$$\frac{1}{4} \int_{\mathbb{A}(R,1)} |\partial_r u(r\theta)| \,\mathrm{d}r \,\mathrm{d}\theta = \frac{n\omega_n(1 - \gamma R)}{4} \leq \frac{n\omega_n(1 - (\gamma R)^n)}{4} \leq \int_{\mathbb{A}(R,1)} |\partial_r v(r\theta)| \,\mathrm{d}r \,\mathrm{d}\theta$$

since $(\gamma R)^n < \gamma R < 1$. The conclusion follows. □

Remark 5.21. It is clear from the proof that the boundary condition $v = \mathrm{id}$ on $\partial B_1(0)$ can be weakened to the requirement that $v(\theta) \in \mathbb{S}^{n-1}$ for $\mathscr{H}^{n-1}$-a.e. $\theta \in \mathbb{S}^{n-1}$. Note that this condition is independent of the choice of representative in the equivalence class of $\{v \in W^{1,q}(B_1(0), \mathbb{R}^n)\}$. The argument above carries through simply by replacing the set Θ with $\Theta \cap \{v(\theta) \in \mathbb{S}^{n-1}\}$.

Combining Lemma 5.19 with Proposition 5.20, we immediately obtain:

Corollary 5.5. *Let $1 \leq p < q < \infty$ and $n \leq q$. For any $\varepsilon, \eta > 0$ and any $f \in Z_p$, there is a sequence $\{f_j \in B_{Z_p^\eta}(f, \varepsilon)\}$ such that $\mathrm{dist}_p(f_j, f) \to 0$ and $\mathcal{E}_q(f_j) \to \infty$.*

We are ready to prove Theorem 16:

Proof. We focus on the case $\Omega = B_1(0)$ first. Fix $\delta > 0$ sufficiently small; in particular, $\delta < 1 - c$ suffices. We replace c with $c/(1-\delta) < 1$ in the definition (5.19) of Z_p^η, in order to account for the slightly worse lower bound satisfied by the perturbations of Lemma 5.19.

Let us first fix q such that $n \leq q$ and $p < q$. For $k \in \mathbb{N}$ let $Y_k \coloneqq \{f \in Z_p^\delta : \mathcal{E}_q(f) \leq k\}$. Note that each Y_k is closed in Z_p^δ. Indeed, given a sequence $\{f_j\} \subset Y_k$ such that we have $\mathrm{dist}_p(f_j, f) \to 0$ for some $f \in Z_p^\delta$, if v_j are energy-minimal solutions corresponding to f_j, then from weak compactness there is a function $v \in W^{1,q}(B_1, \mathbb{R}^n)$ such that $v_j \rightharpoonup v$ in $W^{1,q}(B_1, \mathbb{R}^n)$. By the weak continuity of the Jacobian and weak lower semicontinuity of the q-Dirichlet energy, it follows that $\mathcal{E}_q(v) \leq \liminf_j \mathcal{E}_q(v_j) \leq k$.

We show that Y_k has empty interior in Z_p^δ. Indeed, take $f_0 \in Y_k$ and consider an arbitrary $\varepsilon > 0$. We claim that $B_{Z_p^\delta}(f_0, \varepsilon) \not\subset Y_k$. For this, it suffices to show that $B_{Z_p}(f_0, \varepsilon) \not\subset Y_k$, since $B_{Z_p}(f_0, \varepsilon) \subset B_{Z_p^\delta}(f_0, \varepsilon)$. Any such ball $B_{Z_p}(f_0, \varepsilon)$ contains an element f_1 which is in Z_p; thus, by replacing f_0 with f_1 and shrinking ε if need be, we can assume without loss of generality that $f_0 \in Z_p$. By Corollary 5.5, there are $f_j \in B_{Z_p^\delta}(f_0, \varepsilon) \subset Y_k$ such that $\mathrm{dist}_p(f_j, f) \to 0$ but $j \leq \mathcal{E}_q(f_j)$. This proves the claim.

Since each set Y_k is closed and has empty interior, the Baire Category Theorem implies that $\bigcup_{k \in \mathbb{N}} Y_k$ is meagre in Z_p^δ. Equivalently, $\mathrm{J}(\mathrm{id} + W_0^{1,q}(\Omega, \mathbb{R}^n))$ is meagre in Z_p^δ. For $p \geq n$, we have that

$$\mathrm{J}\left(\mathrm{id} + \bigcup_{n \leq q,\, p < q} W_0^{1,q}(\Omega, \mathbb{R}^n)\right) = \bigcup_{j=1}^{\infty} \mathrm{J}\left(\mathrm{id} + W_0^{1,q_j}(\Omega, \mathbb{R}^n)\right), \qquad q_j \searrow p,$$

and the right-hand side is a countable union of meagre sets, hence meagre, so the conclusion follows. The case $p < n$ is identical.

We conclude the proof by considering the case where Ω is a general smooth and uniformly convex domain. There is a smooth diffeomorphism $\varphi\colon \overline{\Omega} \to \overline{B_1(0)}$ with $\mathrm{J}\varphi = \omega_n/|\Omega|$, see for instance [90, Theorem 5.4] for a more general version of this fact; the existence of such a map can also be deduced from Caffarelli's regularity theory for the Monge–Ampère equation, see e.g. [70, Theorem 3.3]. We can now identify every map $u\colon \Omega \to \mathbb{R}^n$ with a map $v\colon B_1(0) \to \mathbb{R}^n$ with the same regularity, through $v \equiv \varphi \circ u \circ \varphi^{-1}$ and moreover, when $x \in \partial B_1(0)$, we have $v(x) = x$, as $\varphi(\partial\Omega) = \partial B_1(0)$. Since $\mathrm{J}v = \mathrm{J}u \circ \varphi^{-1}$, the general statement follows from the case $\Omega = B_1(0)$. □

5 Regular solutions to the Jacobian equation in $\mathbb{R}^n$

We now consider (5.3) posed in $\mathbb{R}^n$. In addition to the already mentioned motivations for studying the problem, we further remark that in $\mathbb{R}^n$, there is an intimate connection between the Jacobian equation, commutator estimates and factorization theorems in Hardy and L^p spaces [122, 123, 150].

To be precise, we intend to study the following problem in this section:

$$\mathrm{J}u \coloneqq \det \mathrm{D}u = f \qquad \text{in } \mathbb{R}^n. \tag{JRn}$$

The already mentioned improved integrability result of [54] shows that

$$u \in \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \quad \Longrightarrow \quad \mathrm{J}u \in \mathscr{H}^1(\mathbb{R}^n).$$

The same type of statement holds for various other compensated compactness quantities. Coifman & al. proceeded to ask whether compensated compactness quantities are surjective into the Hardy space $\mathscr{H}^1(\mathbb{R}^n)$. That this question must be formulated in terms of homogeneous Sobolev spaces, if it is to have a positive answer, was shown in [149]. The following problem remains open:

Question 5.22. For $p \in [1, \infty)$ and $f \in \mathscr{H}^p(\mathbb{R}^n)$, is there $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ solving (JRn)?

We note that the case $p > 1$ was posed by Iwaniec, see also Conjecture 5.29 below for a stronger formulation of Question 5.22.

It is very difficult to predict the answer to Question 5.22, and in both directions, the currently available evidence is only tentative. A positive answer seems currently out of reach, as there is no systematic way of building solutions to (JRn) for a general discontinuous f, although see [181] for some endpoint cases.

Partially motivated by the negative result of Theorem 16 we establish some tools that are potentially useful for disproving the conjecture as well as some further evidence towards a negative answer.

Apart from the nonlinear character of the Jacobian, the main obstacle in studying existence and regularity of solutions of (JRn) is the *underdetermined* nature of the equation: it implies that there is a multitude of possible solutions to rule out. This is the main reason why Question 5.22 is much harder than regularity results for (JD), as the lack of boundary conditions makes the problem even more underdetermined. Indeed, the space $\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ is extremely large, and so there is an abundance of possible solutions to consider, and moreover it contains many poorly-behaved maps, especially when $p = 1$. We now make precise this last point.

When $p = 1$, there are continuous maps in $\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ which do not satisfy the change of variables formula (5.7), as they map a null set E into a set of positive measure, and hence

$$0 = \int_E f \, \mathrm{d}x < |u(E)| \leq \int_{\mathbb{R}^n} \#\Big(u^{-1}(y) \cap E\Big) \, \mathrm{d}y.$$

It is thus possible for (JRn) to hold a.e. in $\mathbb{R}^n$, and hence also in the sense of distributions, and yet for its geometric information to be completely lost! To be precise, in [151] the following example is given:

Example 5.23. *There is a continuous map $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ such that*

$$\mathrm{J}u = 0 \text{ a.e. in } \mathbb{R}^2 \quad \text{and} \quad u([0,1] \times \{0\}) = [0,1]^2.$$

Maps as above violate the *Lusin (N) property*; their existence is classical and goes back to the work of Cesari [48].

We are interested in considering *admissible solutions* of (JRn), i.e. solutions for which the geometric information of the equation is preserved; here our choice of terminology is inspired by the fluid dynamics literature. As a first step we show that surjectivity of the Jacobian is equivalent to the existence of solutions satisfying an a-priori estimate:

Corollary 5.6. *Fix $1 \leq p < \infty$. The following statements are equivalent:*

(i) $\mathrm{J}\colon \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ *is surjective;*

(ii) *There is a* bounded *operator* $E\colon \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ *such that* $\mathrm{J}\circ E = \mathrm{Id}$*;*

(iii) *For all* $f \in \mathscr{H}^p(\mathbb{R}^n)$ *there is* $u \in \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ *such that* $\mathrm{J}u = f$ *and*

$$\|\mathrm{D}u\|^n_{L^{np}(\mathbb{R}^n)} \lesssim \|f\|_{\mathscr{H}^p(\mathbb{R}^n)}. \tag{5.26}$$

In fact, we obtain this result as a consequence of a more general nonlinear open mapping principle.

Theorem 5.24. *Let X and Y be Banach spaces such that $\mathbb{B}_{X^*}$ is sequentially weak* compact. We make the following assumptions:*

(A1) $T\colon X^* \to Y^*$ *is a weak*-to-weak* sequentially continuous operator.*

(A2) $T(au) = a^s T(u)$ *for all $a > 0$ and $u \in X^*$, where $s > 0$.*

(A3) *For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}\colon X^* \to X^*$, $\sigma_k^{Y^*}\colon Y^* \to Y^*$ such that*

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \qquad \text{for all } k \in \mathbb{N}, \qquad \sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0 \qquad \text{for all } f \in Y^*.$$

Then the following conditions are equivalent:

(i) $T(X^*)$ *is non-meagre in Y^*.*

(ii) $T(X^*) = Y^*$.

(iii) T *is open at the origin.*

(iv) *For every $f \in Y^*$ there exists $u \in X^*$ such that*

$$Tu = f, \qquad \|u\|^s_{X^*} \leq C\|f\|_{Y^*}. \tag{5.27}$$

A sufficient condition for $\mathbb{B}_{X^*}$ to be sequentially weak* compact is that X is a *weak Asplund space* [195, Theorem 3.5]. For instance, reflexive or separable spaces are weak Asplund [71].

Proof. We have (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) and so we just prove (i) ⇒ (iv).

Assume that (i) holds. We may write $T(X^*)$ as a union $\cup_{\ell=1}^{\infty} K_\ell$, where

$$K_\ell \equiv \Big\{ f \in Y^* : \text{there exists } u \in X^* \text{ with } Tu = f \text{ and } \|u\|_{X^*}^s \leq \ell \|f\|_{Y^*} \Big\}.$$

Since balls in X^* are sequentially weak* compact, by (A1), the sets K_ℓ are norm-closed. Now, by the Baire Category Theorem, some K_ℓ contains a closed ball $\bar{B}_r(f_0)$.

Our aim is to solve (5.27) whenever $\|f\|_{Y^*} = r$; assumption (A2) then implies the claim. Suppose, therefore, that $\|f\|_{Y^*} = r$. For every $k \in \mathbb{N}$ we have that $f_0 + (\sigma_k^{Y^*})^{-1} f \in \bar{B}_r(f_0)$. Hence, we may choose $u_k \in X^*$ such that $Tu_k = f_0 + (\sigma_k^{Y^*})^{-1} f$ and

$$\|\sigma_k^{X^*} u_k\|_{X^*}^s = \|u_k\|_{X^*}^s \leq \ell \|f_0 + (\sigma_k^{Y^*})^{-1} f\|_{Y^*} \leq \ell(\|f_0\|_{Y^*} + r).$$

Since balls in X^* are sequentially weak* compact, after passing to a subsequence if need be, $\sigma_k^{X^*} u_k$ converges weakly* to some $u \in X^*$, so that $T(\sigma_k^{X^*} u_k) \overset{*}{\rightharpoonup} Tu$. By the lower semicontinuity of the norm we have

$$\|u\|_{X^*}^s \leq \liminf_{k \to \infty} \|\sigma_k^{X^*} u_k\|_{X^*}^s \leq \ell(\|f_0\|_{Y^*} + r).$$

On the other hand, (A3) gives

$$T(\sigma_k^{X^*} u_k) = \sigma_k^{Y^*}(Tu_k) = \sigma_k^{Y^*} f_0 + f \overset{*}{\rightharpoonup} f,$$

so that, by (A1), $Tu = f$. Thus u solves (5.27) and the proof is complete. □

Theorem 5.24 applies to the Jacobian equation and immediately gives Corollary 5.6.

We now focus on the case $n = 2$ for simplicity and we assume throughout that $\mathrm{J} \colon \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2) \to \mathscr{H}^p(\mathbb{R}^2)$ is surjective. We are particularly interested in the case $p = 1$. Corollary 5.6 yields the quite surprising fact that

the existence of rough solutions implies the existence of admissible solutions.

This fact is made precise in the following result:

Theorem 5.25. *Let $\Omega \subset \mathbb{R}^2$ be an open set and take $f \in \mathscr{H}^1(\mathbb{R}^2)$ such that $f \geq 0$ in Ω. Assume that $\mathrm{J} \colon \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R}^2)$ is onto. Then there is a solution $u \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ of* (JD) *such that:*

(i) *u is continuous in Ω;*

(ii) *u has the Lusin (N) property in Ω;*

(iii) *$\int_{\mathbb{R}^2} |\mathrm{D}u|^2 \,\mathrm{d}x \leq C\|f\|_{\mathscr{H}^1}$ with $C > 0$ independent of f.*

In particular, u satisfies the change of variables formula (5.7)*. Moreover, let $\Omega' \subseteq \Omega$ be an open set such that $f = 0$ a.e. in Ω'. Then:*

(iv) *For any set $E \subset \Omega'$, we have $u(\partial E) = u(\overline{E})$;*

(v) *For $y \in u(\Omega')$, if C denotes a connected component of $u^{-1}(y) \cap \Omega'$ then C intersects $\partial\Omega'$.*

In the supercritical regime $p > 1$, the first part of Theorem 5.25 holds automatically. The second part of Theorem 5.25 also holds in any dimension if p is taken to be sufficiently large. Before proceeding with the proof, we note that (iv) is a type of degenerate monotonicity which had already appeared in the study of the hyperbolic Monge–Ampère equation [53, 134].

Proof. The point of the proof is to perturb f appropriately; then the solution u is obtained as a limit of mappings of integrable distortion.

Take an exhaustion of Ω by bounded open sets Ω_j: that is, $\Omega_j \subset \Omega_{j+1}$ and further $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Fix j and let B_j^+ be a ball containing Ω_j and let B_j^- be another ball, disjoint from Ω_j, and with the same volume as B_j^+. Consider the perturbations

$$f_{\varepsilon,j} \equiv f + \varepsilon a_j, \qquad a_j \equiv \chi_{B_j^+} - \chi_{B_j^-},$$

which satisfy $f_{\varepsilon,j} \geq \varepsilon$ a.e. in Ω_j. Clearly $a_j \in \mathscr{H}^1(\mathbb{R}^2)$, being bounded, compactly supported and with zero mean. Hence, as $\varepsilon \to 0$, $f_{\varepsilon,j} \to f$ in $\mathscr{H}^1(\mathbb{R}^2)$ and, from Corollary 5.6, we see that we can choose solutions $u_{\varepsilon,j}$ of $\mathrm{J}u_{\varepsilon,j} = f_{\varepsilon,j}$ such that, for all $\varepsilon > 0$,

$$\int_{\mathbb{R}^2} |\mathrm{D}u_{\varepsilon,j}|^2 \leq C\|f_{\varepsilon,j}\|_{\mathscr{H}^1} \leq C(j).$$

Since the maps $u_{\varepsilon,j}$ have integrable distortion in Ω_j, we can apply Theorem 5.9 to conclude that the family $\{u_{\varepsilon,j}\}_\varepsilon$ is equicontinuous; we also normalise the maps so that $u_{\varepsilon,j}(x_0) = 0$ for some fixed $x_0 \in \Omega_1$. Therefore, by taking a diagonal subsequence of

$\{u_{\varepsilon,j}\}$, we get a sequence $\{u_k\}$ which converges both locally uniformly in Ω and weakly in $\dot{W}^{1,2}(\mathbb{R}^2,\mathbb{R}^2)$ to a limit u. This already proves (i) and (iii).

To prove (ii) it suffices to show that $|u(E)| = 0$ whenever $E \subset \Omega$ is a *bounded* null set; we may therefore assume that $E \subset \Omega_j$ for some large j. We note that, for each $\varepsilon > 0$, the map $u_{\varepsilon,j}$ satisfies (5.6), c.f. Proposition 5.10. In Ω_j, u is the uniform limit of a subsequence of $(u_{\varepsilon,j})_\varepsilon$, and hence u also satisfies (5.6). Thus, by Proposition 5.5, u has the Lusin (N) property in Ω_j, which implies (ii).

For (iv) we may again suppose that $E \subset \Omega_j \cap \Omega'$ for some large j. Since j is fixed we write for simplicity $u_\varepsilon \equiv u_{\varepsilon,j}$ in the rest of the proof. As u_ε has integrable distortion in Ω_j, it is open in Ω_j and hence $\partial u_\varepsilon(E) \subseteq u_\varepsilon(\partial E)$. Suppose, for the sake of contradiction, that there is $y \in u(\overline{E})\backslash u(\partial E)$. On the one hand, there is some $\delta > 0$ such that, for all ε small enough,

$$B_\delta(y) \cap \partial u_\varepsilon(\operatorname{int} E) \subset B_\delta(y) \cap u_\varepsilon(\partial E) = \emptyset;$$

on the other hand, since $y \in u(\operatorname{int} E)$, for all ε small enough,

$$B_\delta(y) \cap u_\varepsilon(\operatorname{int} E) \neq \emptyset.$$

It follows that $B_\delta(y) \subseteq u_\varepsilon(\operatorname{int} E)$. We also have that $|u_\varepsilon(\operatorname{int} E)| \to 0$ as $\varepsilon \to 0$: by the change of variables formula,

$$|u_\varepsilon(\operatorname{int} E)| \leq \int_{u_\varepsilon(\operatorname{int} E)} \mathcal{N}(y, u_\varepsilon, \operatorname{int} E)\,\mathrm{d}y = \int_E \mathrm{J}u_\varepsilon = \varepsilon|E| \to 0.$$

Thus, since $|B_\delta(y)| \leq |u_\varepsilon(E)|$, a contradiction is reached by sending $\varepsilon \to 0$.

Finally, (v) follows from (iv), as shown for instance in [134, Lemma 2.10]. □

In view of the change of variables formula, it is useful to control the multiplicity function- for instance, this seems essential if one intends to disprove the surjectivity of the Jacobian through a geometric argument involving perturbations of an appropriate datum. The following proposition establishes such control, again under the assumption that the Jacobian is surjective.

Proposition 5.26. *Let $\Omega \subset \mathbb{R}^2$ be an open set and let*

$$Y \coloneqq \{f \in \mathscr{H}^p(\mathbb{R}^2) : f \geq c \text{ a.e. in } \Omega\},$$

where $c > 0$. Suppose that $\{f_j\} \subset Y$ is a sequence converging weakly to f in $\mathscr{H}^p(\mathbb{R}^2)$. For any maps $u_j \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$ satisfying $\mathrm{J}u_j = f_j$ and the a priori estimate (5.26)*, we have that*

$$\sup_j \sup_{y \in u_j(\Omega')} \mathcal{N}(y, u_j, \Omega') < \infty, \qquad \text{whenever } \Omega' \Subset \Omega.$$

Proof. We claim that the sequence u_j is equicontinuous and converges uniformly in Ω' to $u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$ solving $\mathrm{J}u = f$. Once the claim is proved, the conclusion follows: u has integrable distortion in Ω and so by Theorem 5.9 it is at most m-to-one in Ω', for some $m \in \mathbb{N}$. Thus, for all j sufficiently large, u_j is also at most m-to-one in Ω': if not, there are arbitrarily large j and points $x_1^{(j)}, \dots, x_{m+1}^{(j)} \in \Omega'$ such that $u_j(x_i^{(j)}) = y$ for some $y \in \mathbb{R}^n$ and all $i \in \{1, \dots, m+1\}$. By compactness, we can further assume that $x_i^{(j)} \to x_i$ for $i = 1, \dots, m+1$. However, there are at least two different points $y_1 \neq y_2$ such that

$$\{y_1, y_2\} \subset u(\{x_1, \dots, x_{m+1}\});$$

for the sake of definiteness, say $u(x_1) = y_1, u(x_2) = y_2$. Let $\varepsilon < |y_1 - y_2|$ and take j sufficiently large so that, for $i = 1, 2$,

$$|u_j(x_i^{(j)}) - u_j(x_i)| < \frac{\varepsilon}{4}, \qquad |u_j(x_i) - u(x_i)| < \frac{\varepsilon}{4};$$

this is possible from equicontinuity of the sequence u_j and the fact that it converges to u uniformly. The triangle inequality gives $|y_1 - y_2| = |u(x_1) - u(x_2)| < \varepsilon$, a contradiction.

To prove the claim, we assume that the Jacobian is surjective and we use Corollary 5.6. If $p > 1$ we appeal to Morrey's inequality,

$$[u_j]_{C^{0,1-2/p}(\mathbb{R}^2)} \lesssim_p \|\mathrm{D}u_j\|_{L^{2p}(\mathbb{R}^2)} \leq C,$$

while for $p = 1$ we use Theorem 5.9 instead. Either way, after normalizing the maps so that $u_j(x_0) = 0$, where $x_0 \in \Omega$, we see that the sequence (u_j) is precompact in the local uniform topology over Ω'. Hence we may assume that u_j converges to some map $u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$ uniformly in Ω' and also weakly in $\dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$. □

6 Nonuniqueness of energy minimisers in $\mathbb{R}^2$

In this section, we investigate an alternative approach to attacking the issue of the Jacobian equation being underdetermined. A natural idea is to look for solutions which

minimise an appropriate energy function: it may be that such solutions are better behaved and unique. Since we are interested in determining the optimal Sobolev regularity of solutions to (JD), the natural energy to consider is the *np*-Dirichlet energy:

Definition 5.27. For each $f \in L^1_{\mathrm{loc}}(\Omega)$, we define the *np-energy of* f as

$$\mathcal{E}_{np}(f,\Omega) \equiv \inf\left\{ \int_\Omega |\mathrm{D}v|^{np}\,\mathrm{d}x : v \in \dot{W}^{1,np}(\Omega,\mathbb{R}^n) \text{ satisfies } \mathrm{J}v = f \text{ a.e. in } \Omega \right\}.$$

Given $f \in L^1_{\mathrm{loc}}(\Omega)$, we say that $u \in \dot{W}^{1,np}(\Omega,\mathbb{R}^n)$ is a *np-energy minimiser for* f if

$$\int_\Omega |\mathrm{D}u|^{np}\,\mathrm{d}x = \mathcal{E}_{np}(f,\Omega) \qquad \text{and} \qquad \mathrm{J}u = f \text{ a.e. in } \Omega.$$

We contrast this definition with that of the theory of Optimal Transport: there, a typical choice of cost functional to minimise is

$$\int_\Omega \frac{|u(x)-x|^2}{2}\, f(x)\,\mathrm{d}x,$$

see [30, 70]. While this energy does single out a unique solution of (JD), in general there are other solutions of (JD) with better regularity, see the discussion around Table 5.1.

Establishing regularity of energy minimisers for (JRn) is a difficult task, even in the incompressible case $f = 1$. There is an extensive literature on the topic, and we refer the reader to [19, 49, 82, 129, 130], as well as the references therein, for further information.

In this section we are particularly concerned with the following question:

Question 5.28. Is energy minimisation an effective selection criterion for (JRn)?

Iwaniec suggested in [124] that in order to answer Question 5.22 one should prove (5.26) for all *np-energy minimisers*. To achieve this goal, Iwaniec has proposed that one construct a *Lagrange multiplier* for every *np*-energy minimiser. This turns out to be a very difficult task since the standard methods fail, see [106]. Nonetheless, when $p = 1$ and $n = 2$, uniformly bounded Lagrange multipliers were constructed in [148, 150] for a large class of *np*-energy minimisers, which then automatically satisfy the estimate (5.26); we also note that the methods of [148, 150] can be partly adapted to all the cases $n \geq 2$, $p \in [1,\infty)$.

In [124], see also [25], Iwaniec went further than Question 5.22 and posed the following:

Conjecture 5.29. For $p \in [1, \infty)$, a continuous map $E\colon \mathscr{H}^p(\mathbb{R}^n) \to \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ exists such that $\mathrm{J} \circ E = \mathrm{Id}$.

Conjecture 5.29 proposes the existence of a continuous *right inverse* for the Jacobian (abbreviated in this paragraph as r.i.). We have the following trivial chain of implications:

J has a bounded, F-differentiable r.i. $\implies$ J has a bounded, continuous r.i. $\implies$ J has a bounded r.i. $\implies$ J has a r.i.

Corollary 5.6 states the equivalence of the last two statements, while the first statement is false by results in [106]. Finally, and concerning the second statement, Iwaniec pointed out in [124] that it is plausible that p-energy minimisers are unique up to rotations: if so, this would pave a road towards a solution of Conjecture 5.29. However, and as further negative evidence towards Question 5.22 and Conjecture 5.29, there are some data which admit uncountably many p-energy minimisers:

Theorem 5.30. *Fix $1 \leq p < \infty$. There is a radially symmetric function $f \in \mathscr{H}^p(\mathbb{R}^2)$ which has uncountably many $2p$-energy minimisers, modulo rotations.*

Remark 5.31. Here, and elsewhere in this section, we say two maps $u, v\colon \mathbb{R}^n \to \mathbb{R}^n$ are *equal modulo rotations* if there is $Q \in \mathrm{SO}(n)$ such that $u(x) = Qv(x)$ for a.e. x.

A more informative statement can be found in Corollary 5.7, at the end of the section. The proof of Theorem 5.30 relies mostly on elementary tools and the most sophisticated result that we use is the following:

Theorem 5.32 (Sierpiński)**.** *Let (X_n) be disjoint closed sets such that $I = \bigcup_{n\in\mathbb{N}} X_n$, where $I = [a, b] \subset \mathbb{R}$. There is one and only one $n \in \mathbb{N}$ such that X_n is non-empty.*

Theorem 5.32 is only needed to obtain uncountably many distinct minimisers, as non-uniqueness follows already from more elementary means. We also note that Theorem 5.32 holds more generally for a compact, connected Hausdorff space, see e.g. [77, Theorem 6.1.27]. In the case of an interval the proof is simple and so we give it here for the sake of completeness:

Proof. Take $Y \equiv \bigcup_n \partial X_n = I \setminus \bigcup_n \mathrm{int}(X_n)$, which is closed, thus a complete metric space.

We observe that the set Y has empty interior in I, i.e. any open interval L contains an open set U disjoint from Y. Indeed, from the Baire Category Theorem we see that there is an open set $U \subseteq L$ and some X_m which is dense in U. Since X_m is closed, we must have $U \subseteq \operatorname{int} X_m$ and thus U is disjoint from Y.

By the Baire Category Theorem there is also some open subinterval J of I and some $n \in \mathbb{N}$ such that ∂X_n is dense in $Y \cap J$. Since ∂X_n is closed we have $\partial X_n \cap J = Y \cap J$. Thus $(Y \backslash \partial X_n) \cap J = \emptyset$.

Suppose now that $X_n \neq I$. It follows that J intersects $Y \backslash \partial X_n$. Indeed, since Y has empty interior in I, J intersects $I \backslash X_n$ and so it intersects $\operatorname{int}(X_k)$ for some k. Actually, J must intersect ∂X_k: otherwise, $\operatorname{int}(X_k) \cap J$ is non-empty, open and closed in J, thus $\operatorname{int} X_k = J$, since J is connected; clearly this is impossible, since X_k is disjoint from X_n. So we proved that J intersects $Y \backslash \partial X_n$, contradicting the previous paragraph. □

We are now ready to begin the proof of Theorem 5.30, whose core idea is contained in the following lemma.

Lemma 5.33. *Let u be a $2p$-energy minimiser for a given radially symmetric function $f \in \mathscr{H}^p(\mathbb{R}^2)$. For $\alpha_0 \in [0, 2\pi]$, consider the set*

$$X_{\alpha_0} \coloneqq \{\alpha \in [0, 2\pi] : u_\alpha = u_{\alpha_0} \text{ modulo rotations}\}, \quad \text{where } u_\alpha(z) \coloneqq u(e^{i\alpha} z). \tag{5.28}$$

Assume that $f \in C^0(B_R)$ has a sign. If $X_{\alpha_0} = [0, 2\pi]$ then there is $k \in \mathbb{Z} \setminus \{0\}$ such that

$$u(z) = \phi_k(z) \quad \text{in } B_R, \text{ modulo rotations},$$

where ϕ_k is as in Definition 5.15.

Proof. If $X_{\alpha_0} = [0, 2\pi]$ then, for any $\alpha \in [0, 2\pi]$ and $z \in B_R$, we have $|u(e^{i\alpha} z)| = |u(z)|$; that is, circles in B_R, centred at zero, are mapped to circles centred at zero.

For each $r \in (0, R)$, we have $0 \notin u(\mathbb{S}_r)$. Indeed, for each ball $B \Subset B_R$, there is $c = c(B) > 0$ such that $f \geq c$ in B (or $f \leq -c$, but by reversing orientations we can always consider the first case without loss of generality). Thus, in B_r, u is a map of integrable distortion and so it is both continuous and open [126]. Therefore $\partial(u(B_r)) \subseteq u(\partial B_r) = u(\mathbb{S}_r)$ and we see that $u(\mathbb{S}_r) \neq \{0\}$. Since $u(\mathbb{S}_r)$ is a circle, we conclude that $0 \notin u(\mathbb{S}_r)$.

By Proposition 5.12 we may write

$$u(r,\theta) = \psi(r,\theta)e^{i\gamma(r,\theta)} \tag{5.29}$$

where $\psi \in W^{1,2p}([0,R]\times[0,2\pi])$ and $\gamma \in W^{1,2p}([\varepsilon,R]\times[0,2\pi])$ satisfy (5.11) and $\varepsilon > 0$ is arbitrary. For $r < R$, $u(\mathbb{S}_r) = \mathbb{S}_{r'}$, that is, $\psi(r,\theta)$ is independent of θ. Thus, by (5.12), $\mathrm{J}u = f$ reduces to

$$\partial_r(\psi^2)\partial_\theta\gamma = 2rf(r), \tag{5.30}$$

which is valid for almost every $(r,\theta) \in (0,R]\times[0,2\pi]$. Since both ψ and the right-hand side are independent of θ we must have $\gamma(r,\theta) = k\theta + \beta(r)$ and additionally there is the compatibility constraint (5.11) which yields $k \in \mathbb{Z}$. We may assume that $k \neq 0$: otherwise (5.30) shows that $f = 0$ a.e., which is impossible. Since u is a $2p$-energy minimiser, (5.13) readily implies that β is constant. We integrate both sides of (5.30), using $\psi(0) = 0$, to find

$$\psi(r)^2 = \frac{1}{k}\int_0^r 2sf(s)\,\mathrm{d}s \qquad \text{for } r < R.$$

Thus, modulo rotations, $u = \phi_k$ in B_R. □

In fact, the same argument applied in an annulus $\mathbb{A}(R_0,R)$ gives the following variant:

Lemma 5.34. *Consider the setup of Lemma 5.33, but replace B_R by $\mathbb{A}(R_0,R)$. Then there is $k \in \mathbb{Z}\setminus\{0\}$ and $c \in \mathbb{R}$ such that, in $\mathbb{A}(R_0,R)$,*

$$u(z) = \psi(r)e^{2\pi ik\theta} \text{ modulo rotations}, \qquad \text{where } \psi(r)^2 = \frac{1}{k}\int_{R_0}^r 2sf(s)\,\mathrm{d}s + c.$$

We now combine the previous two lemmas.

Lemma 5.35. *There is a radially symmetric $f \in \mathscr{H}^p(\mathbb{R}^2)$, admitting a $2p$-energy minimiser u, for which we have $X_0 \neq [0,2\pi]$, where X_0 is as in* (5.28).

Proof. We take a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned}
&f \in C^1(\mathbb{R}^2) \text{ is radially symmetric,}\\
&\int_{B_2} f\,\mathrm{d}x = \int_{\mathbb{R}^2} f\,\mathrm{d}x = 0\\
&f(r) < 0 \text{ if } r \in [0,1)\cup(3,4), \quad f(r) > 0 \text{ if } r \in (1,3)\cup(4,6),\\
&f(r) = [(6-r)^+]^2 \text{ if } r \in (5,+\infty).
\end{aligned} \tag{5.31}$$

As f decays sufficiently fast near $\mathbb{S}_6$, we can apply [138, Theorem 4] to see that there is $v \in C^1(\overline{B_6}, \mathbb{R}^2)$ such that $\mathrm{J}v = f$ and $v = 0$ on $\mathbb{S}_4$; in particular, by extending v by zero outside B_6, we have $v \in W^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$. Since the $2p$-Dirichlet energy is convex, the direct method, combined with the sequential weak continuity of the Jacobian, shows that f has at least one $2p$-energy minimiser and we call it u, using it to define the sets in (5.28).

Suppose, for the sake of contradiction, that $X_0 = [0, 2\pi]$. Using Lemmas 5.33 and 5.34, we deduce that there are angles $\alpha, \alpha' \in [0, 2\pi)$, numbers $k, k' \in \mathbb{Z}$ and $c \in \mathbb{R}$ such that

$$u = e^{i\alpha}\phi_k \text{ in } B_1, \qquad u = e^{i\alpha'}\left(\psi(r)e^{2\pi i k'\theta}\right) \text{ in } \mathbb{A}(1,3),$$

where, for $r \in (1,3)$,

$$\psi(r)^2 = \frac{1}{k'}\int_1^r 2sf(s)\,\mathrm{d}s + c.$$

In the notation of Definition 5.15, we must have

$$e^{i\alpha+ik\theta}\frac{\rho(1)}{\sqrt{|k|}} \equiv \mathrm{Tr}_{\mathbb{S}_1} u_{|B_1} = \mathrm{Tr}_{\mathbb{S}_1} u_{|\mathbb{A}(1,3)} \equiv e^{i\alpha'+ik'\theta}c$$

in $L^{2p}(\mathbb{S}_1)$. It is easy to conclude that $\alpha = \alpha'$, $k = k'$ and $c = \rho(1)/\sqrt{|k|}$, and so, modulo rotations, actually $u = \phi_k$ in B_3. Now note that for f as in (5.31), we have

$$\int_0^3 |\dot{\rho}(r)|^2 r\,\mathrm{d}r \gtrsim \int_{3/2}^2 \frac{1}{-\int_0^r 2sf(s)\,\mathrm{d}s}\,\mathrm{d}r = \int_{3/2}^2 \frac{1}{\int_r^2 2sf(s)ds}\,\mathrm{d}r \gtrsim \int_{3/2}^2 \frac{1}{4-r^2}\,\mathrm{d}r = +\infty,$$

where all implicit constants depend on f, and so by Lemma 5.14 $u \notin W^{1,2}(B_3, \mathbb{R}^2)$, which is a contradiction. Alternatively, one can infer that $u \notin W^{1,2}(B_3, \mathbb{R}^2)$ from [148, Theorem 3.4]. □

Proof of Theorem 5.30. Let f and u be as in Lemma 5.35.

For each $\alpha \in [0, 2\pi]$, we claim that the set X_α is closed. The case $p > 1$ is clear, as u is automatically continuous. Indeed, given a sequence $\alpha_j \in X_\alpha$ such that $\alpha_j \to \alpha_\infty$, we find numbers $\beta_j \in [0, 2\pi]$ such that, for all $z \in \mathbb{C}$,

$$u_\alpha(z) = e^{i\beta_j}u_{\alpha_j}(z) = e^{i\beta_j}u(e^{i\alpha_j}z). \tag{5.32}$$

By passing to subsequences we can assume that $\beta_j \to \beta_\infty$ and by continuity of u we thus see that $u_{\alpha_\infty} = u(e^{i\alpha_\infty}\cdot)$ is equal to u_α, modulo rotations. In the case $p = 1$ the argument

is similar but slightly more delicate. By the choice of f, u is necessarily continuous in the *disconnected* open set

$$\Omega \equiv B_1 \cup \mathbb{A}(1,3) \cup \mathbb{A}(3,4) \cup \mathbb{A}(4,6) \cup \mathbb{A}(6,+\infty). \tag{5.33}$$

Indeed, continuity in the first four sets in (5.33) follows from the choice of f, together with the theory of mappings of finite distortion [115], while continuity in $\mathbb{A}(6,+\infty)$ follows from [148]: 2-energy minimisers are even Lipschitz continuous in the interior of open sets where $f = 0$. We thus see that we can still pass to the limit in (5.32) for all $z \in \Omega$; as $\mathbb{R}^2 \backslash \Omega$ is a null-set, the claim follows.

We may write, for some index set A,

$$[0, 2\pi] = \bigcup_{\alpha \in A} X_\alpha, \qquad \text{where the union is disjoint.}$$

For distinct $\alpha, \alpha' \in A$, X_α and $X_{\alpha'}$ correspond to distinct equivalence classes of $2p$-energy minimisers, where two maps are in the same equivalent class if they are equal up to a rotation. Thus, by Lemma 5.35, we must have $\#A > 1$. But now Theorem 5.32 shows that A must be uncountable. □

We also note that the proof of Lemma 5.33 yields the following corollary:

Corollary 5.7. *Let $f \in \mathscr{H}^p(\mathbb{R}^2)$ be radially symmetric and suppose u is its unique $2p$-energy minimiser, modulo rotations. If u is continuous then $u = \phi_k$ for some $k \in \mathbb{Z} \backslash \{0\}$.*

Clearly the continuity assumption is not restrictive if $p > 1$.

Proof. As in the proof of Lemma 5.33 we conclude that u maps circles centred at zero to circles and that $(r, \theta) \mapsto |u(re^{i\theta})|$ is independent of θ. Thus we write simply $|u(r)|$.

We show that the set $\{r \in (0,\infty) \colon |u(r)| > 0\}$ is connected. Suppose, by way of contradiction, that there are $r_1 < r_2 < r_3$ such that $|u(r_1)|, |u(r_3)| > 0$ but $|u(r_2)| = 0$. We get another $2p$-energy minimiser for f by setting

$$v(z) \coloneqq \begin{cases} u(z), & |z| \leq r_2, \\ e^{i\pi} u(z), & |z| > r_2, \end{cases}$$

contradicting the assumption that the $2p$-energy minimiser for f is unique modulo rotations.

Thus we can write, for some $0 \leq R_1 \leq R_2 \leq \infty$,

$$\{r \in (0,\infty)\colon |u(r)| > 0\} = (R_1, R_2).$$

We can use Lemma 5.34 to conclude that $u = \phi_k$ in $\mathbb{A}(R_1, R_2)$, modulo rotations. Moreover, clearly we must have $f(r) = 0$ if $r \notin (R_1, R_2)$. Thus $\phi_k(z) = 0$ if $r \notin (R_1, R_2)$ and so $u = \phi_k$ outside $\mathbb{A}(R_1, R_2)$ as well. □

6. Summary and outlook

In this thesis, we have discussed regularity results for a number of problems with non-quadratic growth conditions, ranging from p-growth to non-polynomial growth. Here we briefly summarise the main results and indicate possible extensions.

We began by investigating an iterative scheme that allowed to solve quasi-linear systems with p-growth, including numerically, and that allowed for the transfer of regularity properties from a given reference operator to perturbations of it. This raises in particular the following questions:

1. Is it possible to apply a similar scheme to minimisers of (P)?
2. In light of the sharp $C^{0,\alpha}$-regularity results obtained in [141], are the conditions obtained for convergence of the iterative process in the p-growth case and with regards to higher regularity properties sharp?

We next investigated functionals with (p,q)-growth and obtained global regularity results. In particular we showed $W^{1,q}$-regularity of pointwise and relaxed minimisers for α-Hölder-continuous functionals with respect to the x-variable under the assumption $q < \frac{(n+\alpha)p}{n}$. For autonomous functionals, and considering relaxed minimisers, we were able to improve this assumption to $q < \frac{np}{n-1}$. We then considered functionals with controlled-duality (p,q)-growth. We have left open in particular the following questions:

1. Does $W^{1,q}(\Omega)$-regularity hold in the edge-case $q = \frac{(n+\alpha)p}{n}$ and $q = \frac{np}{n-1}$ respectively?
2. Can the full range of q be recovered in Corollary 1 – that is, considering functionals $F(x,z)$ satisfying (H1)–(H3) and (H7) with $q < \frac{(n+\alpha)p}{n}$, is it possible to show that the Lavrentiev gap functional between $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ vanishes?

3. A final question in this direction, and perhaps the most interesting one, concerns the assumption (H7). Finding (sharp) conditions which guarantee that the Lavrentiev gap functional vanishes would be very interesting.

4. It might also be of interest to investigate whether the methods of Chapter 3 can be adapted to apply to obstacle problems such as those investigated in [67, 65, 47].

5. Can the methods of Chapter 4 be adapted to Banach-space valued maps?

6. When one carries out the arguments of Chapter 4 for non-autonomous functionals satisfying a α-Hölder condition in the x-variable, the sharp bound $q < \frac{(n+\alpha)p}{n}$ is recovered. Is it possible to improve on this when considering minimisers of the relaxed functional?

In the final part of the thesis we investigated the Jacobian equation $Ju \equiv \det \mathrm{D}u = f$ with Sobolev-data $f \in L^p(\Omega)$ where $p \geq n$. We showed that generically no solutions exist in $W^{1,q}(\Omega)$ if $q > p$. The main question we left open is

1. Do solutions exist in $W^{1,p}(\Omega)$ if $f \in L^p(\Omega)$ where $p \geq n$? An answer to this question would likely also be useful for investigating the longstanding problem of surjectivity of the Jacobian operator $J\colon \mathrm{H}^p(\mathbb{R}^n) \to \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$.

In an attempt to provide tools that can facilitate to disprove the conjecture that the map $J\colon \mathrm{H}^p(\mathbb{R}^n) \to \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ is surjective, we utilised a nonlinear open mapping principle. A key assumption in this principle was that the operator under considerations was weak* to weak*-continuous.

1. Is weak*-to-weak*-continuity a fundamental requirement for an open mapping principle to exist or is this assumption an artifact of our proof?

Appendices

A Proof of Lemma 1.4

We restate and prove Lemma 1.4 here.

Lemma .1. *For $s \geq 0$ and $\mu \geq 0$ we have*

$$\frac{1}{6^s(2s+1)}(\mu^2+|\eta|^2+|\eta-\xi|^2)^s \leq \int_0^1 (\mu^2+|t\xi+(1-t)\eta|^2)^s \,\mathrm{d}t \leq 2^s(\mu^2+|\eta|^2+|\eta-\xi|^2)^s.$$

If $s \in (-1/2, 0]$ we have

$$\begin{aligned} 2^s(\mu^2+|\eta|^2+|\eta-\xi|^2)^s \leq& \int_0^1 (\mu^2+|t\xi+(1-t)\eta|^2)^s \,\mathrm{d}t \\ \leq& \frac{1}{4^s(s+1)}(\mu^2+|\eta|^2+|\xi-\eta|^2)^s. \end{aligned}$$

Proof. We first consider $s \geq 0$. Then by Young's inequality,

$$\begin{aligned} \int_0^1 (\mu^2+|t\xi+(1-t)\eta|^2)^s \,\mathrm{d}t \leq& 2^s \int_0^1 (\mu^2+|\eta|^2+t^2|\xi-\eta|^2)^s \,\mathrm{d}t \\ \leq& 2^s(\mu^2+|\eta|^2+|\xi-\eta|^2)^s. \end{aligned}$$

For the lower bound, note that by symmetry we may assume $|\eta| \geq |\xi|$. Dividing both sides by $|\eta|^s$ we see that moreover we may assume $|\eta| = 1$. Rotating coordinate axis if necessary we may even assume that $\eta = e_1$, the unit vector in the first coordinate direction. Note that

$$|\eta + t(\xi-\eta)| \geq |1-t|\xi-\eta||.$$

Write $\tau = |\xi - \eta|$. Note that $\tau \in [0, 2]$.

We first assume $\tau \leq 1$. Then as $\tau \geq 0$,

$$\begin{aligned} &\int_0^1 (\mu^2+|t\xi+(1-t)\eta|^s \,\mathrm{d}t \geq \int_0^1 (\mu^2+(1-t\tau)^2)^s \,\mathrm{d}t \geq 2^{-s}\int_0^1 (\mu+1-t\tau)^{2s} \,\mathrm{d}t \\ =& \frac{1}{2^s(2+1)\tau}\left((\mu+1)^{2s+1}-(\mu+1-\tau)^{2s+1}\right) \\ =& \frac{1}{2^s(2s+1)\tau}(\mu+1)^{2s}\left((\mu+1)\left(1-\left(\frac{\mu+1-\tau}{\mu+1}\right)^{2s}\right)+\tau\left(\frac{\mu+1-\tau}{\mu+1}\right)^{2s}\right) \\ \geq& \frac{1}{2^s(2s+1)}(\mu+1)^{2s}\left((\mu+1)\left(1-\left(\frac{\mu+1-\tau}{\mu+1}\right)^{2s}\right)+\left(\frac{\mu+1-\tau}{\mu+1}\right)^{2s}\right) \\ \geq& \frac{1}{2^s(2s+1)}(\mu+1)^{2s} \end{aligned}$$

$$\geq \frac{1}{4^s(2s+1)}(\mu^2+1+\tau^2)^s.$$

Next assume $\tau \geq 1$. Then

$$\begin{aligned}
&\int_0^1 (\mu^2+|t\xi+(1-t)\eta|^s\,\mathrm{d}t \geq \int_0^1 (\mu^2+(1-t\tau)^2)^s\,\mathrm{d}t \\
\geq& 2^{-s}\int_0^{1/\tau}(\mu+1-t\tau)^{2s}\,\mathrm{d}t + 2^{-s}\int_{1/\tau}^1 (\mu+t\tau-1)^{2s} \\
=& \frac{1}{2^s(2s+1)\tau}\left((\mu+1)^{2s+1}-\mu^{2s+1}+(\mu+\tau-1)^{2s+1}-\mu^{2s+1}\right) \\
\geq& \frac{1}{2^s(2s+1)\tau}(\mu+1+\tau)^{2s}\left(\left(\frac{\mu+1}{\mu+1+\tau}\right)^{2s}+(\tau-1)\left(\frac{\mu+\tau-1}{\mu+1+\tau}\right)^{2s}\right).
\end{aligned}$$

Define

$$f(\tau)=\frac{1}{\tau}\left(\left(\frac{\mu+1}{\mu+1+\tau}\right)^{2s}+(\tau-1)\left(\frac{\mu+\tau-1}{\mu+1+\tau}\right)^{2s}\right).$$

Note that

$$\begin{aligned}
0 \leq& \frac{1}{\tau^2}\left((1-\frac{2}{1+\mu+\tau})^{2s}+4s(\tau-1)\tau\frac{(\mu+\tau-1)^{2s-1}}{(\mu+\tau+1)^{2s+1}}-\frac{(\mu+1)^{2s}(1+\mu+\tau+2s\tau)}{(\mu+\tau+1)^{2s+1}}\right) \\
=& f'(\tau).
\end{aligned}$$

Hence we conclude

$$\begin{aligned}
\int_0^1 (\mu^2+|t\xi+(1-t)\eta|^s\,\mathrm{d}t \geq& \frac{1}{2^s(2s+1)}(\mu+\tau)^{2s}\left(\frac{\mu+1}{\mu+3}\right)^{2s} \\
\geq& \frac{1}{6^s(2s+1)}(\mu+\tau)^{2s}.
\end{aligned}$$

This completes the case $s \geq 0$.

We now turn to $s \in (-\frac{1}{2}, 0)$. We find

$$\begin{aligned}
\int_0^1 (\mu^2+|t\eta+(1-t)\xi|^2)^s\,\mathrm{d}t \geq& \int_0^1 (\mu^2+2(|\eta|^2+|\xi-\eta|^2)^2)^s\,\mathrm{d}t \\
\geq& 2^s(\mu^2+|\eta|^2+|\eta-\xi|^2)^s.
\end{aligned}$$

We turn to the upper bound. Again set $\tau=|\xi-\eta|$. First assume $\tau \leq 1$.Then

$$\begin{aligned}
\int_0^1 (\mu^2+|t\xi+(1-t)\eta|)^s\,\mathrm{d}t \leq& \int_0^1 (\mu^2+(1-t\tau)^2)^s\,\mathrm{d}t \\
\leq& 2^{-s}\int_0^1 (\mu+1-t\tau)^{2s}
\end{aligned}$$

$$=\frac{1}{2^s(2s+1)\tau}\left((\mu+1)^{2s+1}-(\mu+1-\tau)^{2s+1}\right).$$

Set

$$f(\tau)=\frac{1}{\tau}\left((\mu+1)^{2s+1}-(\mu+1-\tau)^{2s+1}\right)$$

and note

$$f'(\tau)=(1+\mu)((1+\mu-\tau)^{2s}-(1+\mu)^{2s})+2s\tau(1+\mu-\tau)^{2s}\geq 0$$

to conclude

$$\begin{aligned}\int_0^1(\mu^2+|t\eta+(1-t)\xi|^2)^s\,\mathrm{d}t\leq&\frac{1}{2^s(2s+1)}((\mu+1)^{2s+1}-\mu^{2s+1})\\\leq&\frac{(\mu+1)^{2s}}{2^s(2s+1)}\\\leq&\frac{(\mu+1+\tau^2)^{2s}}{4^s(2s+1)}.\end{aligned}$$

If $\tau\geq 1$ we estimate

$$\begin{aligned}&\int_0^1(\mu^2+|t\xi+(1-t)\eta|)^s\,\mathrm{d}t\\\leq&\int_0^1(\mu^2+(1-t\tau)^2)^s\,\mathrm{d}t\\\leq&2^{-s}\int_0^{1/\tau}(\mu+1-t\tau)^{2s}+2^{-s}\int_{1/\tau}^1(\mu+t\tau-1)^{2s}\\=&\frac{1}{2^s(2s+1)\tau}\left(\mu+1)^{2s+1}-\mu^{2s+1}+(\mu+\tau-1)^{2s+1}-\mu^{2s+1}\right)\\\leq&\frac{1}{2^s(2s+1)\tau}\left((\mu+1)^{2s}+(\tau-1)(\mu+\tau-1)^{2s}\right).\end{aligned}$$

Define

$$g(\tau)=\frac{1}{\tau}\left((\frac{\mu+1}{\mu+\tau})^{2s}+(\tau-1)\left(\frac{\mu+\tau-1}{\mu+\tau}\right)^{2s}\right)$$

and note that

$$\begin{aligned}0\leq g'(\tau)=&\frac{1}{\tau^2(\mu+\tau-1)}(m+\tau)^{-1-2s}(-(\mu+1)^{2s}(\mu+\tau-1)(\mu+\tau+2s\tau)\\&+(m+\tau-1)^{2s}(\mu^2+(1+2s)(\tau-1)\tau+\mu(2\tau-1)))\\\Leftrightarrow(\frac{\mu+1}{\mu+\tau-1})^{2s}>&\frac{\mu^2+(1+2s)\tau(\tau-1)+\mu(2\tau-1)}{(\mu+\tau-1)(\mu+\tau+2s\tau)}.\end{aligned}$$

By Bernoulli's inequality it suffices to check that

$$\begin{aligned}\mu+\tau-1+2s(\tau-2) &\geq \frac{\mu^2+(1+2s)\tau(\tau-1)+\mu(2\tau-1)}{\mu+\tau+2s\tau}\\ \Leftrightarrow 2s\mu(\tau-2)+2s\tau(\tau-2)+4s^2\tau(\tau-2) &\geq 0.\end{aligned}$$

It is straightforward to see that the last inequality holds. Thus $g(\tau)$ is increasing and we can conclude

$$\begin{aligned}\int_0^1 (\mu^2+|t\xi+(1-t)\eta|)^s \,\mathrm{d}t \leq& \frac{(\mu+\tau)^{2s}}{2^s(2s+1)}\\ \leq& \frac{(\mu+\tau+1)^{2s}}{4^s(2s+1)}.\end{aligned}$$

□

B Proof of Lemma 1.16

Lemma .2. *Assume $F\colon \mathbb{M}\to\mathbb{R}$ is an integrand satisfying for exponents $1<p\leq q<\infty$ and constants $A_i>0$ the (p,q)-growth condition*

$$A_1|z|^p - A_2 \leq F(z) \leq A_3|z|^q + A_4 \tag{1}$$

for all $z\in\mathbb{M}$, then

$$B_3|\xi|^{q'} - B_4 \leq F^*(\xi) \leq B_1|\xi|^{p'} + B_2 \tag{2}$$

for all $\xi\in\mathbb{M}$, where $B_2=A_2$, $B_4=A_4$,

$$B_1 = \left(p^{-\frac{1}{p-1}} - p^{-p'}\right)A_1^{-\frac{1}{p-1}} \text{ and } B_3 = \left(q^{-\frac{1}{q-1}} - q^{-q'}\right)A_3^{-\frac{1}{q-1}}.$$

Proof. Using the definition of $F^*(\cdot)$ as well as the lower bound in (1) we have

$$F^*(\xi) \leq \sup_{z\in\mathbb{R}^{n\times N}} (z\cdot\xi - A_1|z|^p + A_2).$$

The supremum is clearly obtained when $z=\lambda\xi$ where $\lambda\in[0,\infty)$. Optimising over λ gives $\lambda = \dfrac{|\xi|^{1/(p-1)-1}}{(A_1p)^{1/(p-1)}}$. Using this value in the expression above immediately gives the desired result.

For the lower bound, using the definition of $F^*(\cdot)$ as well as the upper bound in (1) we have

$$F^*(\xi) \geq \sup_{z\in\mathbb{R}^{n\times N}} (z\cdot\xi - A_3|z|^q - A_4).$$

Again the supremum is obtained when $z = \lambda\xi$ where $\lambda \in [0, \infty)$. Optimising over λ gives $\lambda = \dfrac{|\xi|^{1/(q-1)-1}}{(A_3 q)^{1/(q-1)}}$. Using this value in the expression above immediately gives the desired result. $\square$

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