

ASYMPTOTIC ANALYSIS OF A SECONDARY BIFURCATION OF THE ONE-DIMENSIONAL GINZBURG–LANDAU EQUATIONS OF SUPERCONDUCTIVITY*

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Abstract. The bifurcation of asymmetric superconducting solutions from the normal solution is considered for the one-dimensional Ginzburg–Landau equations by the methods of formal asymptotics. The behavior of the bifurcating branch depends on the parameters d , the size of the superconducting slab, and κ , the Ginzburg–Landau parameter. The secondary bifurcation in which the asymmetric solution branches reconnect with the symmetric solution branch is studied for values of (κ, d) for which it is close to the primary bifurcation from the normal state. These values of (κ, d) form a curve in the κd -plane, which is determined. At one point on this curve, called the quintuple point, the primary bifurcations switch from being subcritical to supercritical, requiring a separate analysis. The results answer some of the conjectures of [A. Aftalion and W. C. Troy, *Phys. D*, 132 (1999), pp. 214–232].

Key words. superconducting, bifurcation, symmetric, asymmetric

AMS subject classification. 82D55

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1. Introduction. When a superconducting slab of constant thickness, between the planes $x = -d$ and $x = d$, is submitted to an exterior magnetic field $(0, 0, H)$, the state of the slab can be described by the real functions $(\Psi(x), A(x))$ satisfying the Ginzburg–Landau system:

$$(GL_d) \quad \begin{cases} \frac{1}{\kappa^2} \Psi'' = \Psi(\Psi^2 + A^2 - 1) & \text{in } (-d, d), \\ \Psi'(\pm d) = 0, \\ A'' = \Psi^2 A & \text{in } (-d, d), \\ A'(\pm d) = H. \end{cases}$$

Here, Ψ is the superconducting order parameter, which can be thought of as an averaged wave function of the superconducting electrons, and $(0, A, 0)$ is the magnetic vector potential, so that $(0, 0, A')$ is the magnetic field. This model was first introduced by Ginzburg and Landau [16]. For a more detailed description of the model, one may refer to [3], [12], or [14] and the references therein.

Notice that $\Psi \equiv 0$ and $A = H(x + e)$ is always a solution for any real e . From now on we will call this a *normal solution*. It is well known that when H is too large, superconductivity is destroyed and the only solution of (GL_d) is the normal solution.

Let us first recall some basic properties of solutions of (GL_d) .

PROPOSITION 1.1. *If (Ψ, A) is a solution of (GL_d) and if Ψ is positive, then*

- (i) $|\Psi| \leq 1$ in $(-d, d)$;
- (ii) A has a unique zero d_0 in $(-d, d)$, A is increasing on $(-d, d)$, and A' is decreasing on $(-d, d_0)$ and increasing on (d_0, d) ; and

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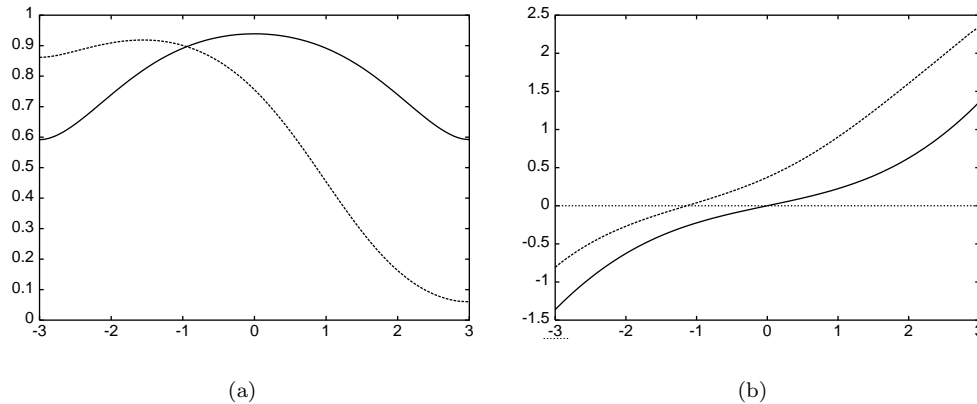


FIG. 1. Symmetric (solid) and asymmetric (dashed) solutions of (GL_d) showing (a) Ψ and (b) A , for $d = 3$, $\kappa = 0.9$.

- (iii) there exist x_1 and x_2 with $-d \leq x_1 \leq d_0 \leq x_2 \leq d$ and $x_0 \in [x_1, x_2]$ such that Ψ' is increasing on $[-d, x_1] \cup [x_2, d]$ and decreasing on $[x_1, x_2]$, and Ψ is increasing on $[-d, x_0]$ and decreasing on $[x_0, d]$.

The proof of (i) and (ii) can be found, for instance, in [7] and that of (iii) in [1].

There are two types of physically important solutions of (GL_d) : symmetric solutions and asymmetric solutions. We define a *symmetric solution* to be a solution of (GL_d) such that $\Psi > 0$, Ψ is even, and A is odd on $[-d, d]$. We define an *asymmetric solution* to be a solution of (GL_d) which satisfies $\Psi > 0$ on $[-d, d]$, yet which is not symmetric; that is, $\Psi'(0) \neq 0$ or $A(0) \neq 0$. Typical symmetric and asymmetric solutions for Ψ and A are shown in Figures 1(a) and 1(b), respectively.

In sufficiently large slabs it has been discovered experimentally (see [21] or [23]) that as H is decreased superconductivity nucleates first in surface layers of thickness $1/\kappa$. These surface layers can be explained using the asymmetric solution (GL_d) ; in a decreasing field, the first solution to bifurcate from the normal state is the asymmetric solution, which is concentrated near the boundary [20].

It is interesting to study the bifurcation diagrams of solutions of (GL_d) , which show the norm of Ψ as a function of H (see, for instance, [15], [18], or [22]). In particular, the number of intersections of the bifurcation curve with the line $H = H_0$ provides the number of solutions of (GL_d) for each H_0 . (Note in this respect that the asymmetric solutions come in pairs: if $\Psi(x)$ is a solution of (GL_d) , then so is $\Psi(-x)$.) These bifurcation diagrams for both symmetric and asymmetric solutions have been computed in [3] for various values of the parameters d and κ . For symmetric solutions we refer the reader to [3] and the asymptotic analysis in [2]. For the branch of asymmetric solutions it is found in [3] that there are two possible behaviors according to the type of bifurcation, as illustrated in Figures 2 and 3. In both figures, the branch of symmetric solutions is that bifurcating from the normal solution at the smaller value of H , and the bifurcation of the asymmetric solutions from the symmetric solution is illustrated by a square. (We call this the secondary bifurcation—the bifurcation from the normal solution being the primary bifurcation.) In the case of Figure 2 the bifurcation of asymmetric solutions is subcritical¹ and there is at most one pair of

¹Since we are not investigating stability here, we need to assign a “direction” to the bifurcation to use the labels sub- and supercritical; we choose the direction of decreasing H . This definition coincides with the usual definition based on the stability properties of the asymmetric bifurcation investigated in [10], [11].

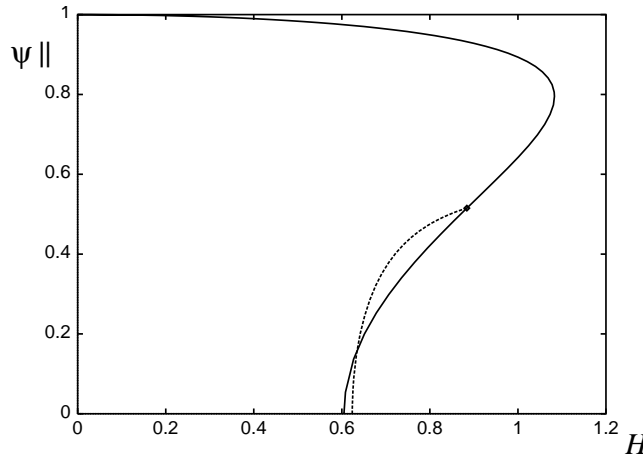


FIG. 2. Bifurcation diagrams for $a = 3$ and $\kappa = 0.35$ showing symmetric (solid) and asymmetric (dashed) solution branches.

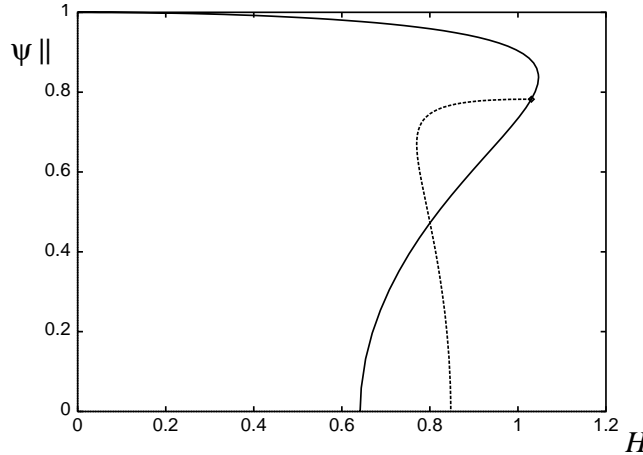


FIG. 3. Bifurcation diagrams for $a = 3$ and $\kappa = 0.5$ showing symmetric (solid) and asymmetric (dashed) solution branches.

asymmetric solutions for each given H , while in Figure 3 the bifurcation is supercritical and there are at most two pairs of asymmetric solutions.

In [3] the regimes of the parameters d and κ where each behavior of the bifurcation diagram for asymmetric solutions holds have been classified. The results of these numerical investigations are shown graphically in Figure 4. They indicate that the (d, κ) plane is the union of three connected sets A_0 , A_1 , and A_2 : In A_0 there are no asymmetric solutions, in A_1 the behavior of the asymmetric branch of Figure 2 holds, and in A_2 the behavior of the asymmetric branch of Figure 3.

In [3] the authors make the following conjecture.

CONJECTURE 1.2 (see [3]). *There exist two continuous functions $\kappa_4(d)$ and $\kappa_5(d)$ separating the (d, κ) plane into three connected regions A_0 , A_1 , and A_2 . There exists exactly one point d^* called the quintuple point (with approximate value 1.23), such that $\kappa_4(d^*) = \kappa_5(d^*) = \kappa^*$. Moreover, $\kappa_4(d) = C/d$ with C approximately equal to*

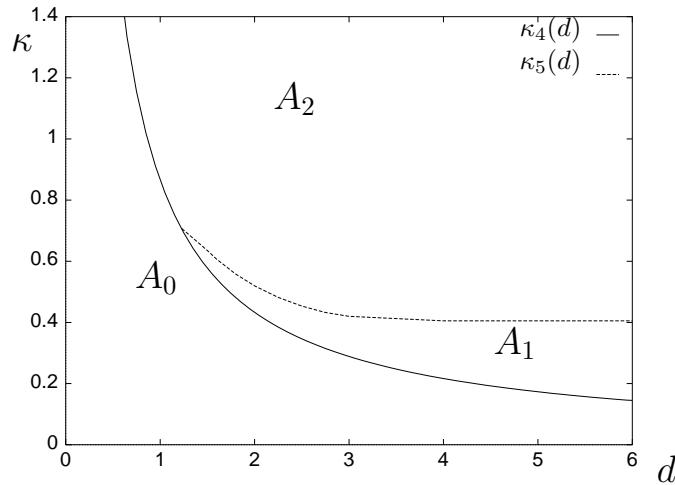


FIG. 4. Curves $\kappa_4(d)$ and $\kappa_5(d)$, which divide the κd -plane into the regions A_0 , A_1 , and A_2 , where there are zero, at most one, and at most two asymmetric solutions, respectively. The curve $\kappa_4(d)$ is the curve across which the primary asymmetric bifurcation from the normal solution disappears, while $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical.

0.90, and $\kappa_5(d)$ is defined and monotone decreasing on $[d^*, \infty)$ with $\lim_{d \rightarrow \infty} \kappa_5(d) = \kappa_{as} \simeq 0.4$.

The curve $\kappa_4(d)$ is the curve across which the asymmetric bifurcation disappears, while $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical. The point (d^*, κ^*) is known as the quintuple point since, as shown numerically in [3], a third curve $\kappa_1(d)$ also passes through this point, dividing the plane into five regions locally (see Figure 5). This curve is the curve across which the bifurcation of the symmetric solution switches from being subcritical to supercritical, and it has been analyzed asymptotically in [2]. Indeed, once we know that the curves κ_4 and κ_5 meet, it is clear that κ_1 must also pass through this point.

The secondary bifurcation (shown as a square in Figures 2 and 3) in which the two asymmetric solutions and the symmetric solution reconnect to become a single symmetric solution will be our primary interest in this paper. We will be concerned with the situation in which the secondary bifurcation is close to the primary bifurcations so that the amplitude of the solutions is still small when it takes place. Thus we will be concerned with the bifurcation diagram in the vicinity of the curve $\kappa_4(d)$.

The bifurcation of the asymmetric solution from the normal solution has been widely studied. Saint-James and De Gennes [20] examined the bifurcation in a half-space and discovered that the bifurcation of asymmetric solutions occurs at a higher field than that of symmetric solutions in an infinite slab. They also established that the bifurcation switches from being supercritical to subcritical for $\kappa \simeq 0.4$. Since then, there has been a lot of rigorous work on the topic, including that of Bolley and Helffer [5], [6], [7], [8], [9] (computation of the field where the bifurcation happens and nature of the bifurcation in various limiting cases, namely κd small and large, and d small and large) and Hastings and Troy [17] (existence of only asymmetric solutions for a certain range of magnetic field). While this work was in progress, Dancer and Hastings [13] studied the global bifurcation diagrams and proved in particular that when κd is large, there exists a branch of asymmetric solutions connecting the symmetric curve to the normal solution.

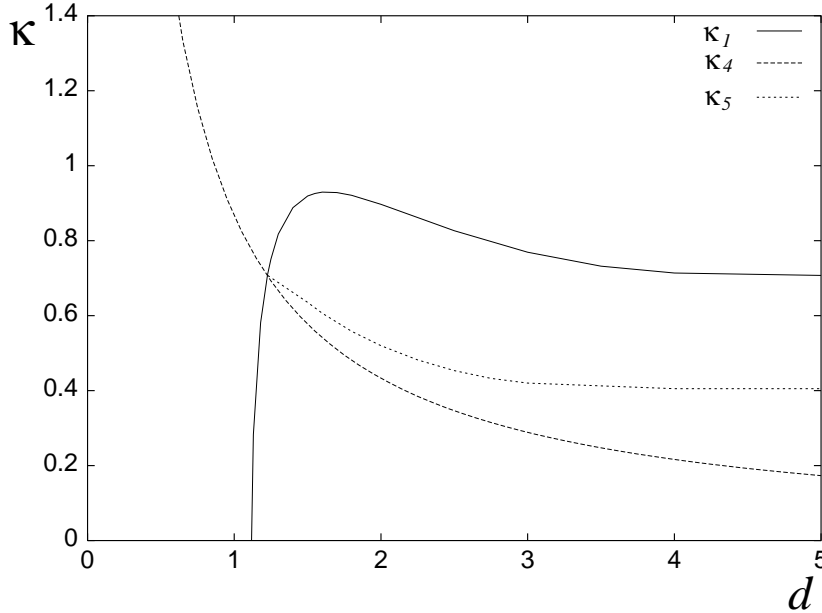


FIG. 5. The curve $\kappa_1(d)$, which is the curve across which the bifurcation of the symmetric solution switches from being subcritical to supercritical, is shown in conjunction with the curves $\kappa_4(d)$ and $\kappa_5(d)$.

In section 2, we use asymptotic analysis to study the behavior of the symmetric and asymmetric branches when their amplitudes are small and identify the curve $\kappa_4(d)$ and the quintuple point. Then in section 3, we study the response diagram in the vicinity of the quintuple point. Finally, in section 4, we present our conclusions.

2. Asymptotic analysis. Since the size of the domain d is a parameter which we will want to vary, we begin by rescaling distance with d so that the domain will remain fixed as $-1 < x < 1$. The Ginzburg–Landau system is then

$$(2.1) \quad \frac{1}{\kappa^2 d^2} \Psi'' = \Psi^3 - \Psi + \Psi A^2,$$

$$(2.2) \quad A'' = d^2 \Psi^2 A,$$

$$(2.3) \quad \Psi'(\pm 1) = 0,$$

$$(2.4) \quad A'(\pm 1) = Hd.$$

We wish to examine the bifurcation from the normal state to a superconducting state. Close to the bifurcation point Ψ will be small. We quantify this smallness by introducing a small parameter ε and setting

$$(2.5) \quad \Psi = \varepsilon^{1/2} f,$$

$$(2.6) \quad A = Hd(x + e) + \varepsilon q,$$

as in [4], [10], [11], [19]. Here the relative scaling of f and q is motivated by the fact that we want the nonlinear term Ψ^3 to balance with the correction to ΨA^2 in (2.1). For ease of notation we set

$$p = \frac{1}{\kappa^2 d^2}, \quad h = Hd.$$

Then f and q satisfy

$$(2.7) \quad pf'' = \varepsilon f^3 - f + f(h^2(x+e))^2 + 2\varepsilon h(x+e)q + \varepsilon^2 q^2,$$

$$(2.8) \quad q'' = d^2 f^2 (h(x+e) + \varepsilon q),$$

$$(2.9) \quad f'(\pm 1) = 0,$$

$$(2.10) \quad q'(\pm 1) = 0.$$

We formally expand all quantities in powers of ε :

$$(2.11) \quad f = f_0 + \varepsilon f_1 + \dots,$$

$$(2.12) \quad q = q_0 + \varepsilon q_1 + \dots,$$

$$(2.13) \quad e = e_0 + \varepsilon e_1 + \dots,$$

$$(2.14) \quad h = h_0 + \varepsilon h_1.$$

Leading order. Substituting the expansions (2.11)–(2.14) into (2.7)–(2.8) and equating coefficients of powers of ε we find that at leading order

$$(2.15) \quad pf_0'' + f_0 - f_0 h_0^2 (x + e_0)^2 = 0,$$

$$(2.16) \quad q_0'' = d^2 f_0^2 h_0 (x + e_0),$$

with homogeneous Neumann boundary data. Now, for each e_0 (2.15) is an eigenvalue problem for h_0 . Let θ_0 be the corresponding normalized eigenfunction (with the L^∞ norm, say). Then $f_0 = C\theta_0$, where C is undetermined at present. Now, the homogeneous version of (2.16) with homogeneous Neumann boundary conditions is satisfied by a constant, so that by the Fredholm alternative there is a solvability condition that the integral of the right-hand side is zero. Thus

$$(2.17) \quad \int_{-1}^1 \theta_0^2 (x + e_0) dx = 0.$$

This is the equation which determines e_0 (remember that θ_0 is a function of e_0). It is easy to see that $e_0 = 0$ is always a solution since, in this case, θ_0 will be even. This is the symmetric superconducting solution. However, for some values of p and d (which we will determine later) there is also a nonzero solution for e_0 . Note that if $(\theta_0(x), e_0)$ is a solution of (2.15) and (2.17), then $(\theta_0(-x), -e_0)$ is also a solution. These two solutions are the asymmetric superconducting solutions.

We are interested in the secondary bifurcation in which the two asymmetric solutions and the symmetric solution reconnect to become a single symmetric solution. In particular, we will be concerned with the situation in which the secondary bifurcation is close to the primary bifurcations, so that the amplitude of the solutions is still small when it takes place (the square in Figures 2 and 3 is close to $\Psi = 0$). In this case we expect that e will be zero for all three solutions to leading order, but will be nonzero for the asymmetric solutions at a higher order. Therefore, we expand in powers of ε as

$$(2.18) \quad f = f_0 + \varepsilon^{1/2} f_1 + \varepsilon f_2 + \dots,$$

$$(2.19) \quad q = q_0 + \varepsilon^{1/2} q_1 + \varepsilon q_2 + \dots,$$

$$(2.20) \quad e = \varepsilon^{1/2} e_1 + \varepsilon e_2 + \dots,$$

$$(2.21) \quad h = h_0 + \varepsilon h_2 + \dots,$$

$$(2.22) \quad p = p_0 + \varepsilon p_2 + \dots,$$

$$(2.23) \quad d = d_0 + \varepsilon d_2 + \dots.$$

The motivation for the $\varepsilon^{1/2}$ scaling on e here is difficult to give. It occurs because the solvability conditions on q at first and second order are satisfied automatically, leaving e_1 to be determined by the third-order solvability condition as we will see. Once e has assumed to be of order $\varepsilon^{1/2}$ the expansion for f and q must be in powers of $\varepsilon^{1/2}$. However, we still expand h in powers of ε since we know that this is the correct scaling for the symmetric solution. Finally note that we also expand p and d to allow us to consider what happens for (d, p) close to a point (d_0, p_0) on the curve κ_4 . Again, the $O(\varepsilon^{1/2})$ term in these expansions is zero, so that the corrections to p_0 and d_0 come into the solvability condition at the right order. The final justification for the form of the expansions (2.18)–(2.23) is the ability to develop them self-consistently.

Leading order. Substituting the expansions (2.18)–(2.23) into (2.7)–(2.8) and equating coefficients of powers of ε we find that at leading order

$$(2.24) \quad Lf_0 \equiv p_0 f_0'' + f_0 - f_0 h_0^2 x^2 = 0,$$

$$(2.25) \quad q_0'' = d_0^2 f_0^2 h_0 x,$$

with homogeneous Neumann boundary data. Let h_0 be the leading eigenvalue of (2.24)², and θ_0 the corresponding normalized eigenfunction (with the L^∞ norm, say). Then $f_0 = C\theta_0$, where C is undetermined at present. We see that $q_0 = C^2 a_0$, where

$$(2.26) \quad a_0'' = d_0^2 \theta_0^2 h_0 x,$$

with homogeneous Neumann boundary data. Thus q_0 is determined once C is known. Note that θ_0 is even, while a_0 is odd, so that the solvability condition on (2.26) is automatically satisfied.

First order. At first order in (2.7) and (2.8) we find

$$(2.27) \quad Lf_1 = 2f_0 h_0^2 x e_1,$$

$$(2.28) \quad q_1'' = 2d_0^2 f_0 f_1 h_0 x + d_0^2 f_0^2 h_0 e_1,$$

with homogeneous Neumann boundary data. Since the operator L is self-adjoint and f_0 satisfies the homogeneous version of (2.27), there is a solvability condition for f_1 by the Fredholm alternative, which is that the right-hand side of (2.27) must be orthogonal to f_0 . However, this condition is automatically satisfied since f_0 is even and the right-hand side of (2.27) is odd. Hence

$$(2.29) \quad f_1 = C e_1 \theta_1 + D \theta_0,$$

where

$$(2.30) \quad L\theta_1 = 2h_0^2 x \theta_0.$$

Since $L\theta_0 = 0$ we may add any multiple of θ_0 to θ_1 . However, it is convenient to fix $\theta_1(0) = 0$ so that θ_1 is odd. We see that

$$(2.31) \quad q_1 = C^2 e_1 a_1 + 2CD a_0,$$

where

$$(2.32) \quad a_1'' = 2d_0^2 \theta_0 \theta_1 h_0 x + d_0^2 h_0 \theta_0^2.$$

²We choose to examine the leading eigenfunction, since it is shown in [10], [11] that this is the one which has the possibility of being stable. Of course, a similar analysis can be performed using any eigenfunction.

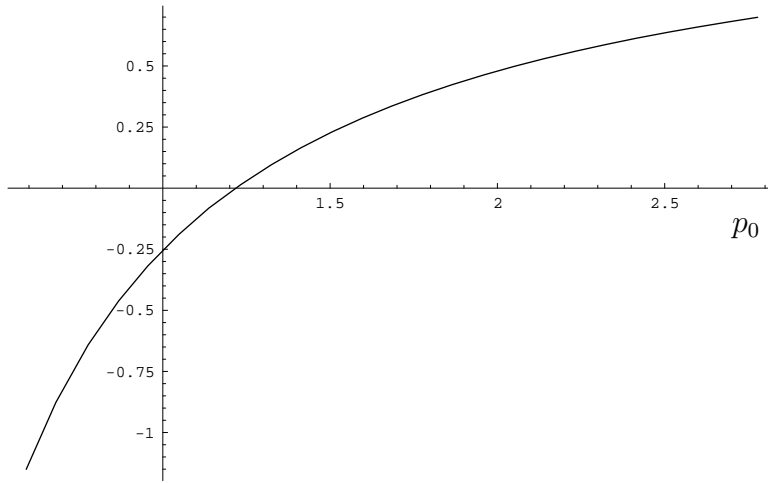


FIG. 6. The left-hand side of (2.33) as a function of p_0 .

Now the homogeneous version of (2.32) is satisfied by a constant. Thus there is a solvability condition on a_1 , which is

$$(2.33) \quad \int_{-1}^1 (2\theta_0\theta_1x + \theta_0^2) dx = 0.$$

This condition defines the curve κ_4 . Note that since θ_0 and θ_1 are functions of p_0 only, that is, they are independent of d_0 , condition (2.33) gives the curve κ_4 as $p = \text{constant} \approx 1.218$. The left-hand side of (2.33) is shown as a function of p_0 in Figure 6, from which it appears that there is a unique p_0 satisfying (2.33). We believe this to be true, but note that it is not necessary for the following analysis, for which p_0 may be any solution of (2.33).

With this solvability condition satisfied we may add any constant to a_1 . For definiteness we fix $a_1(0) = 0$. Note that a_1 is even.

Second order. Proceeding to the next order in the expansion in (2.7) we find

$$(2.34) \quad Lf_2 = -p_2f_0'' + f_0^3 + 2h_0h_2x^2f_0 + 2h_0xq_0f_0 + f_0h_0^2(e_1^2 + 2xe_2) + 2xe_1f_1h_0^2,$$

$$(2.35) \quad q_2'' = 2d_0d_2f_0^2h_0x + 2d_0^2f_0f_2h_0x + d_0^2f_1^2h_0x + d_0^2f_0^2h_2x \\ + d_0^2f_0^2q_0 + d_0^2f_0^2h_0e_2 + 2d_0^2f_0f_1h_0e_1.$$

As before, since f_0 satisfies the homogeneous version of (2.34) there is a solvability condition which is derived by multiplying by f_0 and integrating. We find, after some manipulation, that the terms involving D and e_1 cancel, leaving

$$(2.36) \quad h_2 = \alpha_0C^2 + H_2p_2,$$

where

$$(2.37) \quad \alpha_0 = -\frac{\int_{-1}^1 \left(\theta_0^4 - \frac{2(a_0')^2}{d_0^2} \right) dx}{2h_0 \int_{-1}^1 x^2\theta_0^2 dx},$$

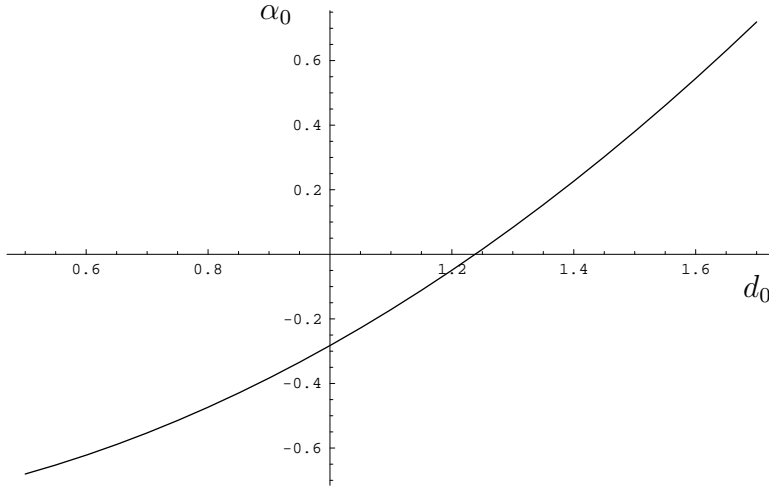


FIG. 7. α_0 as a function of d_0 on the curve κ_4 .

$$(2.38) \quad H_2 = -\frac{\int_{-1}^1 (\theta'_0)^2 dx}{2h_0 \int_{-1}^1 x^2 \theta_0^2 dx}.$$

Thus, when $\alpha_0 \neq 0$, C is determined in terms of h_2 . Figure 7 shows α_0 as a function of d_0 on the curve κ_4 .

The leading order solution is now determined. However, we need to proceed further down the expansions to determine the coefficients e_1 and D of the first-order solution, in order to distinguish between the symmetric and asymmetric branches. We have

$$(2.39) \quad f_2 = C^3 \theta_2 + Cp_2 \phi_2 + Ce_1^2 \Omega_2 + (Ce_2 + De_1) \theta_1 + E \theta_0,$$

where

$$(2.40) \quad L\theta_2 = \theta_0^3 + 2h_0 \alpha_0 x^2 \theta_0 + 2h_0 x a_0 \theta_0,$$

$$(2.41) \quad L\phi_2 = -\theta_0'' + 2h_0 H_2 x^2 \theta_0,$$

$$(2.42) \quad L\Omega_2 = h_0^2 \theta_0 + 2x h_0^2 \theta_1.$$

Again, we can add any multiple of θ_0 to θ_2 , ϕ_2 , and Ω_2 . For definiteness we fix these functions by requiring that they are equal to zero at the origin. The solvability condition on q_2 from (2.36) turns out to be automatically satisfied due to (2.33). Then we find

$$(2.43) \quad q_2 = C^4 a_2 + \frac{2C^2 d_2 a_0}{d_0} + C^2 p_2 b_2 + C^2 e_1^2 g_2 + (C^2 e_2 + 2CDe_1) a_1 + (D^2 + 2CE) a_0,$$

where

$$(2.44) \quad a_2'' = 2d_0^2 \theta_0 h_0 x \theta_2 + d_0^2 \theta_0^2 \alpha_0 x + d_0^2 \theta_0^2 a_0,$$

$$(2.45) \quad b_2'' = 2d_0^2 \theta_0 h_0 x \phi_2 + d_0^2 \theta_0^2 H_2 x,$$

$$(2.46) \quad g_2'' = 2d_0^2 \theta_0 h_0 x \Omega_2 + d_0^2 h_0 x \theta_1^2 + 2d_0^2 \theta_0 \theta_1 h_0.$$

Again, we may add a constant to a_2 , b_2 , and c_2 . We fix them by requiring that they are zero at the origin. Then θ_2 , ϕ_2 , and Ω_2 are even, while a_2 , b_2 , and g_2 are odd.

Third order. Proceeding to third order in (2.7) we find

$$(2.47) \quad Lf_3 = -p_2 f_1'' + 3f_0^2 f_1 + 2xe_3 h_0^2 f_0 + 2e_1 e_2 h_0^2 f_0 + 2h_0 e_1 q_0 f_0 \\ + 4h_0 h_2 x e_1 f_0 + 2f_0 h_0 x q_1 + 2h_0 h_2 x^2 f_1 + (2xe_2 + e_1^2) h_0^2 f_1 \\ + 2h_0 x q_0 f_1 + 2xe_1 h_0^2 f_2,$$

$$(2.48) \quad q_3'' = 2d_0 d_2 (2f_0 f_1 h_0 x + f_0^2 h_0 e_1) + d_0^2 f_0^2 (h_0 e_3 + h_2 e_1 + q_1) \\ + 2f_0 f_1 d_0^2 (h_0 e_2 + q_0 + h_2 x) + d_0^2 (2f_0 f_2 + f_1^2) h_0 e_1 \\ + d_0^2 (2f_0 f_3 + 2f_1 f_2) h_0 x.$$

As usual there are solvability conditions. Multiplying (2.48) by f_0 and integrating, using the expressions for f_1 , etc., we find after some manipulation that the only term remaining is that which is multiplied by $C^3 D$. Hence we must have $D = 0$, so that f_1 is odd. Then

$$(2.49) \quad f_3 = C^3 e_1 \theta_3 + C e_1 p_2 \phi_3 + C e_1^3 \Omega_3 + C e_1 e_2 \Omega_2 + (E e_1 + C e_3) \theta_1 + F \theta_0,$$

where

$$(2.50) \quad L\theta_3 = 3\theta_0^2 \theta_1 + 2\theta_0 h_0 a_0 + 4\theta_0 h_0 \alpha_0 x + 2\theta_1 h_0 \alpha_0 x^2 + 2h_0 x a_0 \theta_1 \\ + 2\theta_2 x h_0^2 + 2\theta_0 h_0 x a_1,$$

$$(2.51) \quad L\phi_3 = -\theta_1'' + 4h_0 H_2 \theta_0 x + 2\theta_1 h_0 x^2 H_2 + 2x h_0^2 \phi_2,$$

$$(2.52) \quad L\Omega_3 = \theta_1 h_0^2 + 2x h_0^2 \Omega_2.$$

Again, we choose θ_3 , ϕ_3 , and Ω_3 to be zero at the origin, so that all are odd.

The solvability condition on q_3 is obtained by integrating (2.49) to give, after some manipulation,

$$(2.53) \quad e_1 (\alpha_1 C^2 + p_2 \beta_1 + \gamma_1 e_1^2) = 0,$$

where

$$(2.54) \quad \alpha_1 = \int_{-1}^1 [\theta_0^2 a_1 + 2\theta_0 \theta_1 a_0 + 2h_0 \theta_0 \theta_2 + 2\theta_0 h_0 x \theta_3 + 2h_0 x \theta_1 \theta_2] dx,$$

$$(2.55) \quad \beta_1 = \int_{-1}^1 [\theta_0^2 H_2 + 2\theta_0 \theta_1 H_2 x + 2h_0 \theta_0 \phi_2 + 2h_0 x \theta_0 \phi_3 + 2h_0 x \theta_1 \phi_2] dx,$$

$$(2.56) \quad \gamma_1 = \int_{-1}^1 [2h_0 \theta_0 \Omega_2 + h_0 \theta_1^2 + 2\theta_0 h_0 x \Omega_3 + 2h_0 x \theta_1 \Omega_2] dx.$$

Hence for $\alpha_1 C^2 / \gamma_1 < -p_2 \beta_1 / \gamma_1$ there are three solutions for e_1 , namely

$$(2.57) \quad e_1 = 0, \quad e_1 = \pm \left(\frac{-p_2 \beta_1 - \alpha_1 C^2}{\gamma_1} \right)^{1/2},$$

while for $\alpha_1 C^2 / \gamma_1 > -p_2 \beta_1 / \gamma_1$ there is only the zero solution. Thus for the secondary bifurcation point to exist we require that $p_2 \beta_1 / \gamma_1 < 0$ (so that the primary asymmetric

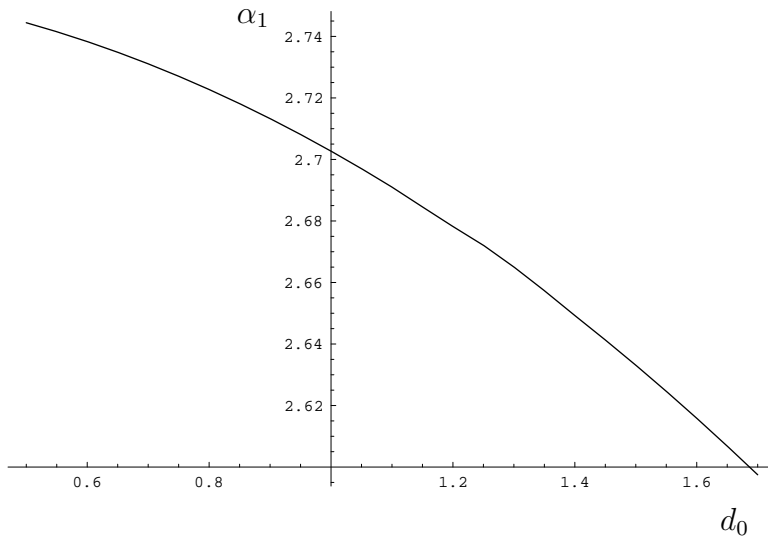


FIG. 8. α_1 as a function of d_0 on the curve $\kappa_4(d)$.

bifurcation exists) and $\alpha_1/\gamma_1 > 0$ (so that the asymmetric branch then reconnects with the symmetric branch), in which case it occurs when $C^2 = -p_2\beta_1/\alpha_1$.

Figure 8 shows α_1 as a function of d_0 on the curve κ_4 . We find that β_1 and γ_1 are independent of d_0 , and are given approximately by $\beta_1 = 1.734$, $\gamma_1 = 3.889$. Thus we see that the asymmetric solution exists for $p_2 < 0$.

We have now determined the solution at leading and first order. We see that the norms of the three solutions are the same to leading order. Note that e_1 is monotone decreasing to zero as C increases from zero to its value at the secondary bifurcation, so that the asymmetric solution becomes symmetric as the secondary bifurcation is approached.

3. The quintuple point. Suppose now that p_0 and d_0 are such that $\alpha_0 = 0$, which corresponds to the quintuple point since the sign of α_0 determines whether the primary bifurcations are sub- or supercritical. Then C is not determined by (2.36). In this case (2.53) gives e_1 in terms of C for each solution, but we need to proceed to fourth order to determine C .

Fourth order. At fourth order in (2.7) we find

(3.58)

$$\begin{aligned}
 Lf_4 = & -p_4f_0'' - p_2f_2'' + 3f_0^2f_2 + 3f_0f_1^2 + 2h_0^2xe_4f_0 + (2e_1e_3 + e_2^2)h_0^2f_0 \\
 & + 2h_0h_2(2xe_1 + e_1^2)f_0 + (2h_0h_4 + h_2^2)x^2f_0 + 2h_0xq_2f_0 + 2h_0e_1q_1f_0 + 2h_0e_2q_0f_0 \\
 & + 2h_2xq_0f_0 + q_0^2f_0 + 2xh_0^2e_3f_1 + 2h_0^2e_1e_2f_1 + 2h_0q_0e_1f_1 + 4h_0h_2xe_1f_1 \\
 & + 2h_0xq_1f_1 + 2h_0h_2x^2f_2 + (2xe_2 + e_1^2)h_0^2f_2 + 2h_0xq_0f_2 + 2xh_0^2e_1f_3.
 \end{aligned}$$

As usual, the solvability condition is derived by multiplying by f_0 and integrating. After some manipulation, we find that the terms involving E cancel, to leave

(3.59)
$$h_4 = \alpha_2C^4 + \beta_2C^2 + \gamma_2 + p_2\delta_2e_1^2 + \eta_2C^2e_1^2 + \nu_2e_1^4,$$

where

$$(3.60) \quad \alpha_2 = -\frac{1}{2h_0 \int_{-1}^1 x^2 \theta_0^2 dx} \int_{-1}^1 [3\theta_0^3 \theta_2 + 2h_0 x a_2 \theta_0^2 + \theta_0^2 a_0^2 + 2h_0 x a_0 \theta_0 \theta_2] dx,$$

$$(3.61) \quad \beta_2 = \lambda_2 d_2 + \mu_2 p_2,$$

$$(3.62) \quad \lambda_2 = -\frac{2}{d_0 \int_{-1}^1 x^2 \theta_0^2 dx} \int_{-1}^1 x \theta_0^2 a_0 dx,$$

$$(3.63) \quad \mu_2 = -\frac{1}{2h_0 \int_{-1}^1 x^2 \theta_0^2 dx} \int_{-1}^1 [\theta_0' \theta_2' + 3\theta_0^3 \phi_2 + 2h_0 x \theta_0^2 b_2 + 2x a_0 \theta_0^2 H_2 + 2h_0 H_2 x^2 \theta_0 \theta_2 + 2h_0 x a_0 \theta_0 \phi_2] dx,$$

$$(3.64) \quad \gamma_2 = p_4 H_2 + p_2^2 H_4,$$

$$(3.65) \quad H_4 = -\frac{1}{2h_0 \int_{-1}^1 x^2 \theta_0^2 dx} \int_{-1}^1 [\theta_0^2 x^2 H_2^2 + \theta_0' \phi_2' + 2h_0 H_2 x^2 \theta_0 \phi_2] dx,$$

$$(3.66) \quad \delta_2 = -\frac{1}{2h_0 \int_{-1}^1 x^2 \theta_0^2 dx} \int_{-1}^1 [\theta_0' \Omega_2' + 2h_0 H_2 \theta_0^2 + 4h_0 H_2 x \theta_0 \theta_1 + 2h_0 H_2 x^2 \theta_0 \Omega_2 + h_0^2 \theta_0 \phi_2 + 2h_0^2 x \theta_0 \phi_3] dx,$$

$$(3.67) \quad \eta_2 = -\frac{1}{2h_0 \int_{-1}^1 x^2 \theta_0^2 dx} \int_{-1}^1 [3\theta_0^3 \Omega_2 + 3\theta_0^2 \theta_1^2 + 2h_0 x \theta_0^2 g_2 + 2h_0 \theta_0^2 a_1 + 2h_0 \theta_1 \theta_0 a_0 + 2h_0 x a_1 \theta_0 \theta_1 + h_0^2 \theta_0 \theta_2 + 2h_0 x a_0 \theta_0 \Omega_2 + 2h_0^2 x \theta_0 \theta_3] dx,$$

$$(3.68) \quad \nu_2 = -\frac{1}{2h_0 \int_{-1}^1 x^2 \theta_0^2 dx} \int_{-1}^1 [h_0^2 \theta_0 \Omega_2 + 2h_0^2 x \theta_0 \Omega_3] dx.$$

Thus we have a coupled system (2.53), (3.59) for C and e_1 . Note that $\alpha_1, \beta_1, \gamma_1, \alpha_2, \lambda_2, \mu_2, \eta_2, H_2, H_4, \delta_2,$ and ν_2 depend only on p_0 and d_0 , evaluated at the quintuple point, and are therefore simply fixed numbers. Numerically we find that at the quintuple point

$$\begin{aligned} d_0 &= 1.24, & \kappa_0 &= 0.73, & \alpha_1 &= 2.67, & \beta_1 &= 1.734, & \gamma_1 &= 3.889, \\ \alpha_2 &= -0.39, & \lambda_2 &= 1.32, & \mu_2 &= 0.16, & \eta_2 &= -5.57, \\ H_2 &= -0.053, & H_4 &= 0.048, & \delta_2 &= -1.81, & \nu_2 &= -2.03. \end{aligned}$$

Let us consider the behavior of the solution of (2.53), (3.59) as p_2 and d_2 vary. The first important curve is that given by $p_2 = 0$, which determines whether the asymmetric solution exists or not, and therefore corresponds to the curve κ_4 . We find that the asymmetric solution exists only for $p_2 < 0$. The second important curve is $\beta_2 = 0$, since this decides whether the symmetric bifurcation is sub or supercritical and corresponds to the curve κ_1 . Hence κ_1 corresponds to

$$(3.69) \quad \lambda_2 d_2 + \mu_2 p_2 = 0.$$

The final curve κ_5 determines whether the asymmetric bifurcation is sub or supercritical. If the asymmetric solution exists we may replace e_1^2 in (3.59) with

$$(3.70) \quad -\frac{\alpha_1 C^2 + p_2 \beta_1}{\gamma_1},$$

giving

$$(3.71) \quad h_4 = \left(\alpha_2 - \frac{\alpha_1 \eta_2}{\gamma_1} + \frac{\alpha_1^2 \nu_2}{\gamma_1^2} \right) C^4 + \left(\lambda_2 d_2 + p_2 \left(\mu_2 - \frac{\alpha_1 \delta_2}{\gamma_1} - \frac{\beta_1 \eta_2}{\gamma_1} + \frac{2\alpha_1 \beta_1 \nu_2}{\gamma_1^2} \right) \right) C^2 + \gamma_2 + p_2^2 \left(\frac{\beta_1 \nu_2}{\gamma_1^2} - \frac{\beta_1 \delta_2}{\gamma_1} \right).$$

Whether the bifurcation is sub or supercritical depends on the coefficient of C^2 . Thus we see that κ_5 is given by

$$(3.72) \quad \lambda_2 d_2 + p_2 \left(\mu_2 - \frac{\alpha_1 \delta_2}{\gamma_1} - \frac{\beta_1 \eta_2}{\gamma_1} + \frac{2\alpha_1 \beta_1 \nu_2}{\gamma_1^2} \right) = 0,$$

with the constraint that $p_2 < 0$. Thus we have been able to determine the lines in the $p_2 d_2$ -plane (two infinite, one semi-infinite) across which the behavior of the bifurcation diagram changes.

These may be translated to the κd -plane by noting that

$$p_2 = -\frac{2(d_2 \kappa_0 + d_0 \kappa_2)}{d_0^3 \kappa_0^3}.$$

In Figures 9 and 10 we show the curves κ_1 , κ_4 , and κ_5 in the pd - and κd -planes, respectively. In Figures 11–16 we show a selection of bifurcation diagrams. Figures 11 and 12 correspond to regions where no asymmetric solutions exist and the curve is a curve of symmetric solutions. In Figures 13 to 16 the asymmetric branch is the one that bifurcates from the normal solution at the higher field.

Note that the bifurcation point for the symmetric solution is $h_4 = \gamma_2$, while that for the asymmetric solution is

$$h_4 = \gamma_2 + p_2^2 \left(\frac{\beta_1 \nu_2}{\gamma_1^2} - \frac{\beta_1 \delta_2}{\gamma_1} \right) \approx \gamma_2 + 0.577 p_2^2.$$

The coefficient of p_2^2 here is positive, so that the asymmetric solution always bifurcates at a higher value of h in the vicinity of the quintuple point (in fact numerical simulations indicate that this is always true).

Note also that we may calculate the angle between the two solutions at the secondary bifurcation point. We find that

$$(3.73) \quad \frac{dh_4}{dC^2} = \beta_2 - \frac{2\alpha_2 \beta_1 p_2}{\alpha_1}$$

for the symmetric branch and

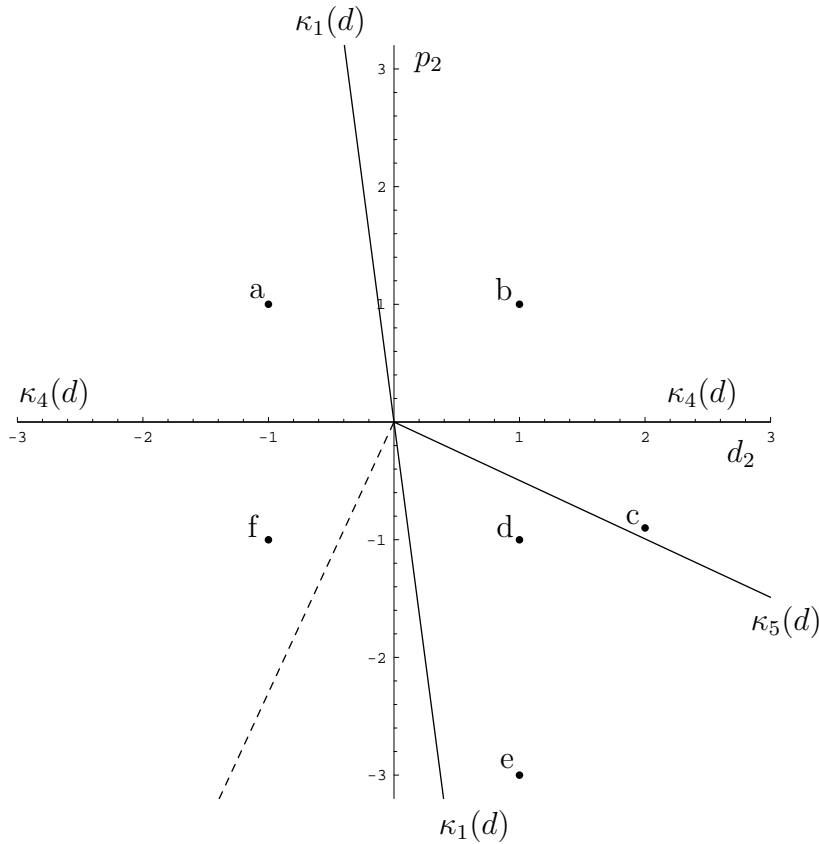


FIG. 9. The curves κ_1 , κ_4 , and κ_5 in the pd -plane. The bifurcation diagrams for the points a–f are shown in Figures 11–16. The curve κ_4 is the curve across which the primary bifurcation to an asymmetric solution from the normal solution disappears, $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical and κ_1 is the curve across which the primary bifurcation to a symmetric solution switches from being subcritical to supercritical. Also shown (dashed) is the curve across which the nose in the asymmetric solution branch disappears by meeting the secondary bifurcation. The nose is present between this curve and the curve κ_5 .

$$(3.74) \quad \frac{dh_4}{dC^2} = \beta_2 - \frac{2\alpha_2\beta_1p_2}{\alpha_1} - \frac{\alpha_1\delta_2p_2}{\gamma_1} + \frac{\beta_1\eta_2p_2}{\gamma_1}$$

for the asymmetric branch. Since $p_2 < 0$ for the asymmetric branch to exist we find that dh_4/dC^2 is larger for the asymmetric branch than for the symmetric branch at the secondary bifurcation point. This means that the asymmetric branch lies to the left of the symmetric branch close to the secondary bifurcation point, but to the right of it near the primary bifurcation points. Hence the two branches must always cross at some point.

Note that the secondary bifurcation point may be above or below the nose of the symmetric branch, and that there may or may not be a nose in the asymmetric branch. If there is a nose in the asymmetric branch, we find it occurs at

$$(3.75) \quad C^2 = \frac{(\alpha_1\delta_2\gamma_1 + \beta_1\eta_2\gamma_1 - 2\alpha_1\beta_1\nu_2 - \gamma_1^2\mu_2)p_2 - \gamma_1^2\lambda_2d_2}{2(\alpha_2\gamma_1^2 + \alpha_1^2\nu_2 - \alpha_1\eta_2\gamma_1)} \approx -0.266d_2 - 0.536p_2.$$

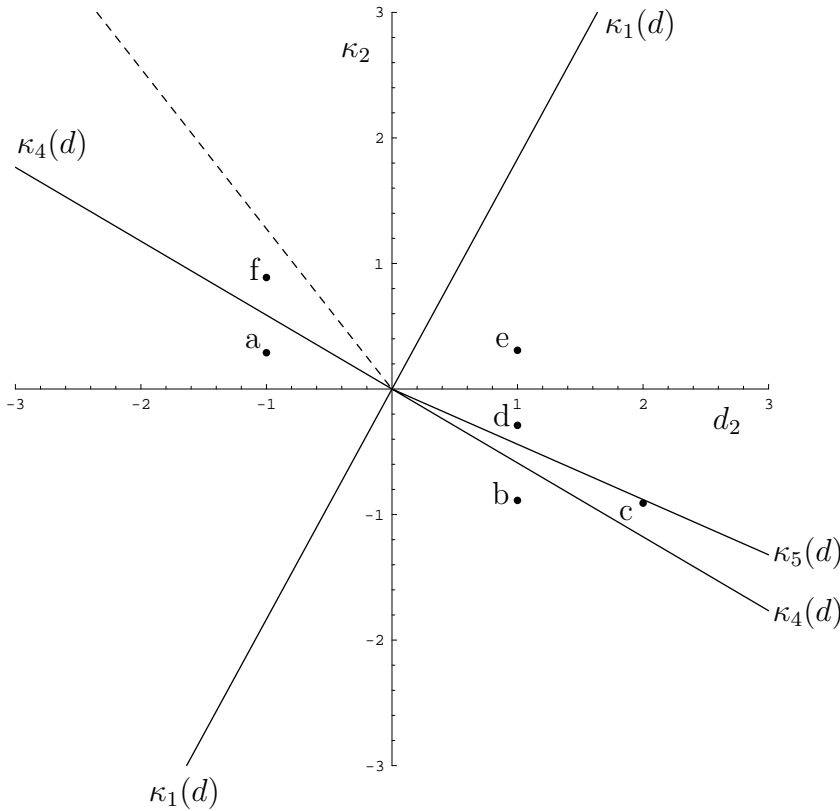


FIG. 10. The curves κ_1 , κ_4 and κ_5 in the kd -plane. The bifurcation diagrams for the points a–f are shown in Figures 11–16. The curve κ_4 is the curve across which the primary bifurcation to an asymmetric solution from the normal solution disappears, $\kappa_5(d)$ is the curve across which it switches from being subcritical to supercritical and κ_1 is the curve across which the primary bifurcation to a symmetric solution switches from being subcritical to supercritical. Also shown (dashed) is the curve across which the nose in the asymmetric solution branch disappears by meeting the secondary bifurcation. The nose is present between this curve and the curve κ_5 .

Hence, the condition for there to be a nose in the asymmetric branch is that this lies below the value of C at which the secondary bifurcation occurs, which is

$$(3.76) \quad C^2 = -p_2\beta_1/\alpha_1 \approx -0.652164p_2.$$

Hence there is a nose in the asymmetric branch if and only if

$$(3.77) \quad \alpha_1\gamma_1^2\lambda_2d_2 + (\alpha_1\beta_1\eta_2\gamma_1 - \alpha_1^2\delta_2\gamma_1 - 2\alpha_2\beta_1\gamma_1^2 + \alpha_1\gamma_1^2\mu_2)p_2 > 0,$$

which approximates to

$$(3.78) \quad d_2 - 0.435p_2 > 0.$$

Since $p_2 < 0$ for the asymmetric branch to exist, we see that there will be a nose in this branch unless d_2 is sufficiently negative. Since $d_2 = p_2 = -1$ in Figure 16 there is no nose in the asymmetric branch there.

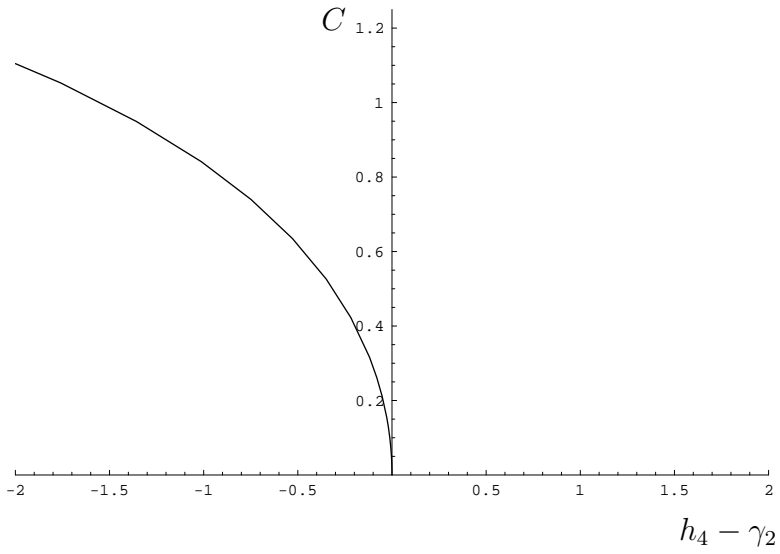


FIG. 11. The bifurcation diagram for $d_2 = -1$, $p_2 = 1$, corresponding to point (a) in Figures 9 and 10. The symmetric solution is shown; there is no asymmetric solution.

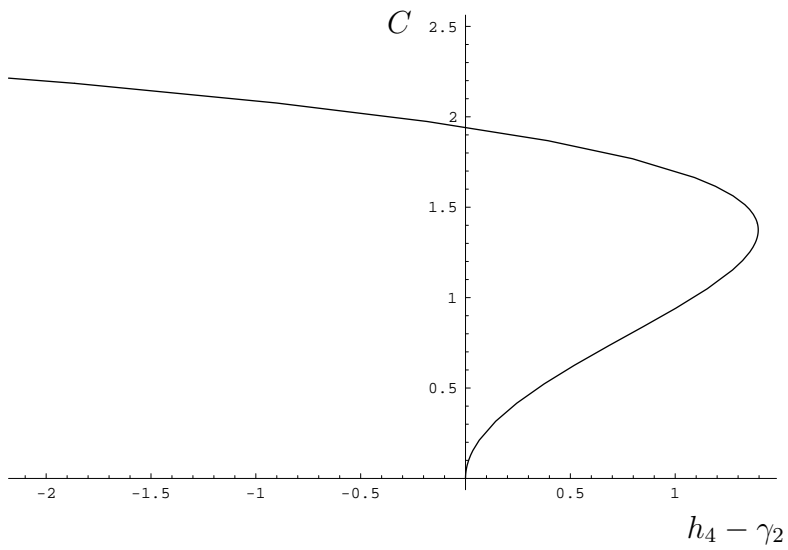


FIG. 12. The bifurcation diagram for $d_2 = 1$, $p_2 = 1$, corresponding to point (b) in Figures 9 and 10. The symmetric solution is shown; there is no asymmetric solution.

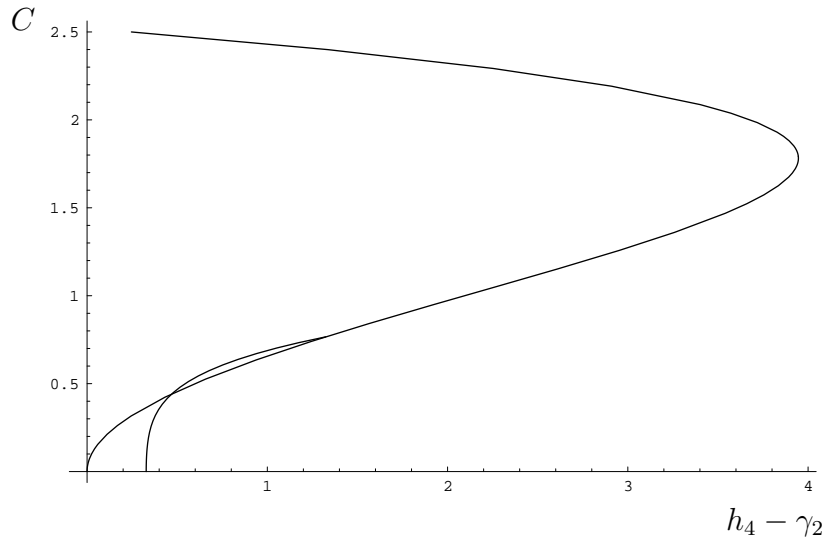


FIG. 13. The bifurcation diagram for $d_2 = 2$, $p_2 = -0.9$, corresponding to point (c) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of h_4 .

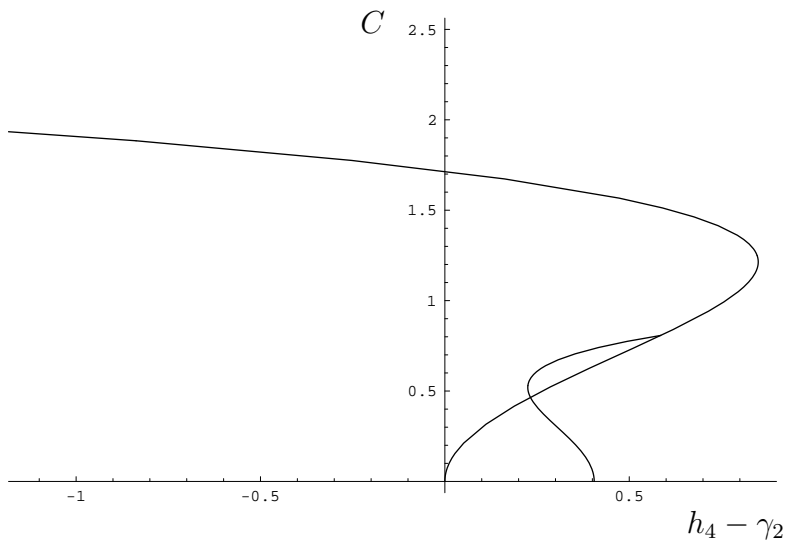


FIG. 14. The bifurcation diagram for $d_2 = 1$, $p_2 = -1$, corresponding to point (d) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of h_4 .

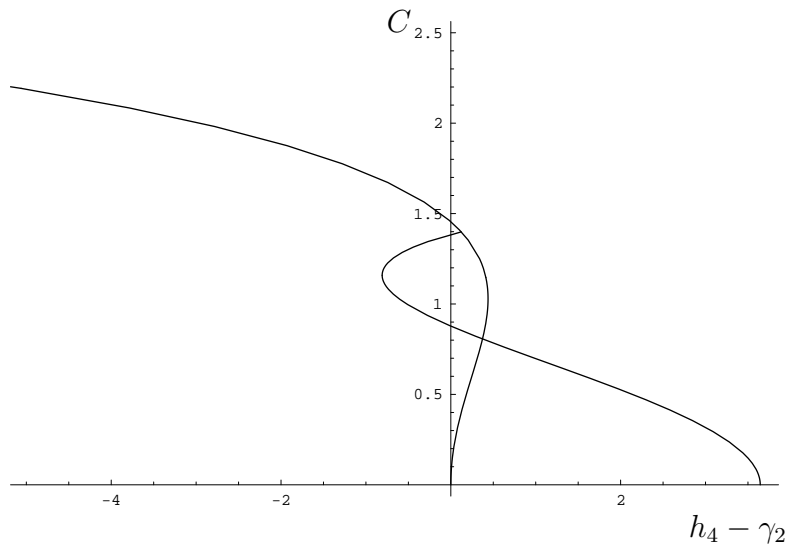


FIG. 15. The bifurcation diagram for $d_2 = 1$, $p_2 = -3$, corresponding to point (e) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of h_4 .

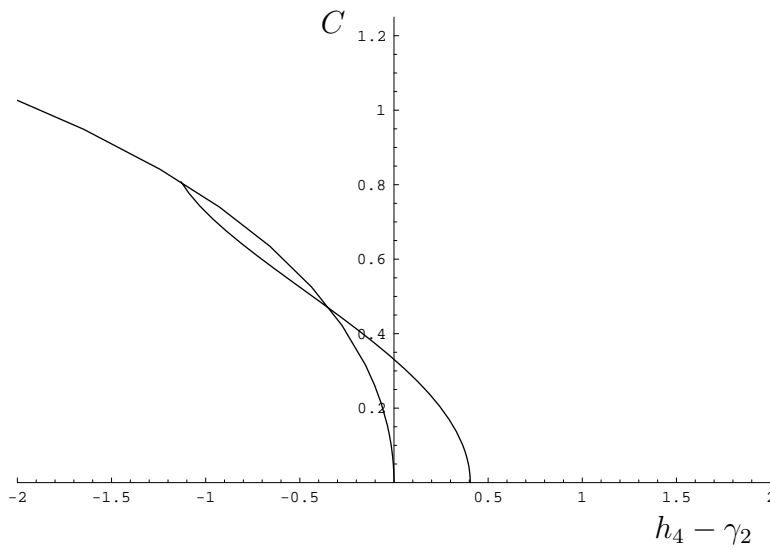


FIG. 16. The bifurcation diagram for $d_2 = -1$, $p_2 = -1$, corresponding to point (f) in Figures 9 and 10. The asymmetric solution is the one bifurcating from the higher value of h_4 .

4. Conclusion. We have examined the bifurcation from the normal solution of superconducting solutions of the Ginzburg–Landau equations on a slab (GL_d), using the applied magnetic field as the bifurcation parameter. There are bifurcation branches corresponding to both symmetric and asymmetric solutions, and their existence and behavior depends on the additional parameters d , the slab thickness, and κ , the Ginzburg–Landau parameter.

It was found numerically in [3] that there are three distinct regions of the κd -plane, A_0 , A_1 , and A_2 , in which there are, respectively, no asymmetric solutions, at most one, and at most two asymmetric solutions.

Here we have been concerned with the curve they labeled $\kappa_4(d)$, which divides A_0 from $A_1 \cup A_2$. This is the curve across which the asymmetric solution ceases to exist and is given by (2.33). We find it is of the form $\kappa d = C$, where C is constant and approximately equal to 0.90.

Close to κ_4 the secondary bifurcation in which the asymmetric solutions connect with the symmetric solution to form a single symmetric solution lies close to the normal solution, i.e., close to the primary bifurcations. Thus, close to κ_4 , we have been able to see the secondary bifurcation explicitly through our asymptotic analysis.

Since the leading-order problem corresponds to a linearization about the normal state, the amplitude of the leading-order solution is not determined at leading order but by a solvability condition at higher order. We found that along most of the curve κ_4 this solvability condition occurs at second order and leads to the perturbation of the magnetic field from the bifurcation value being quadratic in the amplitude of the leading-order solution as given by (2.36). The symmetric and asymmetric solutions are the same to leading order, and we were able to distinguish between them at first order and identify the secondary bifurcation point.

However, at one point on κ_4 , (called the quintuple point and given by $\alpha_0 = 0$, where α_0 is defined by (2.37)) the primary bifurcations switch from being subcritical to supercritical, and at this point the amplitude of the solutions is determined by a solvability condition at fourth order, with the perturbation of the magnetic field from the bifurcation value being quartic in the amplitude of the leading-order solution, as given by (3.59). In this case the symmetric and asymmetric solutions have different amplitudes at leading order. This is the point at which A_0 , A_1 , and A_2 meet, and there are five different qualitative behaviours for the bifurcation diagram in the vicinity of this point. This rich structure is captured completely by our asymptotic analysis. In particular, we are able to establish that the asymmetric branch lies to the left of the symmetric branch close to the secondary bifurcation point, but to the right of it near the primary bifurcation points (near the normal solution), so that the two branches must always cross at some point. We have also determined the condition for there to be a nose in the asymmetric branch.

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