SOME PROBLEMS IN TOPOLOGY

by

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Abstract

This thesis explores generalizations of category in the sense of Lusternik-Schnirelmann. The material falls naturally into two parts.

Chapter 1 is concerned with numerical invariants. Following Berstein, for a closed n-manifold M let N(M) (resp. n(M)) denote the minimum cardinality of a covering of M by open subsets, each of which embeds (resp. immerses) in $\mathbb{R}^n$. In Chapter 1 the values of N(M) and n(M) are determined when $M = \mathbb{R}P^n$ -- with the exception of $N(\mathbb{R}P^n)$ when n = 31 or 47. For a vector bundle ξ over a space X, James has defined Vecat(ξ) to be the minimum number of open subsets required to cover X with the restriction of ξ to each subset stably trivial. In Chapter 1 the categories of all vector bundles over all real projective spaces are also determined.

In Chapter 2 it is undertaken to define Lusternik-Schnirelmann category solely in terms of the category of spaces modulo weak equivalences. This is done by equipping the homotopy category with a notion of covering. The reason for doing this is to obtain a more conceptual definition of cocategory. It turns out that there are many possible definitions of cocategory. The relationship between two of them is discussed in Chapter 2.

The category of a connected space X is closely related to the Milnor filtration of X regarded as the classifying space of its loop space. This in turn is intimately related to the H-space structure on ΩX. The framework of homotopy coverings is exploited in Chapter 3 to yield a filtration of a (k-1)-connected space analogously related to its k-fold loop space. There results a theory of higher associativity for k-fold loop spaces and a dual theory for k-fold suspensions.

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Certainly the turning point in my research was brought about by the arrival of Zdzislaw Wojtkowiak. I am very grateful to him for introducing me to many areas of mathematics and for our many mathematical conversations.

During the preparation of this work I spent two years at Northwestern University. I am indebted to Professor Mark Mahowald for undertaking to supervise me while I was there. I wish to thank him for countless stimulating conversations and sailing excursions, and for introducing me to his theory of bo-resolutions at a key moment. His influence on Chapter 1 will be apparent.

While in Chicago I met Professor A.K. Bousfield. I would like to thank him for making available to me his unpublished work. The results of Chapter 3 rely heavily on this material, and the influence of his published work on Chapters 2 and 3 will be apparent.

Next I would like to thank my friends Richard Meyer and Pam Geggus for countless stimulations. Without their diversions I am sure that very little of this thesis would have got off the ground. Nancy Meyer, Ian Jarman and Rudy Buller have also made contributions to this area.

Finally, I wish to thank Alice Wagner for typing Chapter 1 and Brenda Willoughby for typing the rest.

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Introduction

In this thesis various generalizations of category in the sense of Lusternik-Schnirelmann are explored. The material falls naturally into two parts.

Chapter 1 is concerned with numerical invariants. Following Berstein [3], for a closed n-manifold M let N(M) (resp. n(M)) denote the minimum cardinality of a covering of M by open subsets, each of which embeds (resp. immerses) in $\mathbb{R}^n$. Berstein posed the problem of computing these numbers when $M = \mathbb{RP}^n$. He introduced upper and lower bounds which agreed in some cases (when $n + 1 = 2^k m$, m odd, with $k \leq 3$ or $k \equiv 3$ (4)), but in general could be arbitrarily far apart. In Chapter 1 it is shown that the values of $N(\mathbb{RP}^n)$ and $n(\mathbb{RP}^n)$ are in fact given by Berstein's lower bound - with the possible exception of $N(\mathbb{RP}^n)$ when n = 31 or 47.

Closely related is the (stable) category of a vector bundle. For a vector bundle ξ over a space X, James ([21]) has defined Vecat(ξ) to be the minimum number of open subsets required to cover X, with the restriction of ξ to each subset stably trivial. James introduced bounds for this invariant when $X = \mathbb{RP}^n$ which agreed in certain cases. In Chapter 1, the categories of all vector bundles over all real projective spaces are computed. The values of the Berstein covering numbers then follow easily. A more detailed description of these topics appears in section 2.1.

The Lusternik-Schnirelmann category of a space is a homotopy invariant. One might expect that it could be defined solely in terms of the category of spaces modulo weak homotopy equivalence. This is indeed the case and is done in Chapter 2 by equipping the homotopy category with a notion of covering. The original reason for doing this was to describe category as closely in spirit to the original definition as possible, but in such a way as to be able to apply Eckmann-Hilton duality. The outcome, it was hoped, would yield a more conceptual definition of cocategory than is usually given ([16]). It turns out that there are many possible definitions of cocategory. The relationships between them are discussed in section 2.6.

The framework of homotopy coverings also produced an unexpected byproduct. The category of a connected space X is closely related to the Milnor filtration of X regarded as the classifying space of its loop space. This is generalized in Chapter 2 to a filtration of a (k-1)-connected space X analogously related to the k-fold loop space structure on $\Omega^k X$. There results a theory of higher associativity for k-fold loop spaces and a dual theory for k-fold suspensions. A more extensive introduction to Chapters 2 and 3 can be found in section 2.1.

The results of Chapters 2 and 3 have been announced in [20]. A.K. Bousfield has recently informed us that D.W. Anderson obtained some of the results of Chapter 3 some time ago but did not publish them.

Chapter 1

Trivializing Covers of Vector Bundles

Introduction

Manifolds are usually described as the result of gluing together open subsets of a fixed Euclidean space. It is natural at the outset to ask how efficiently a given manifold can be constructed. In [6] Berstein formalized this notion by introducing the following invariants. All manifolds, embeddings, and immersions are assumed to be differentiable of class C^{∞}.

Definition. Let M be a closed n-manifold.

i) The embedding covering number $N(M)$ is the least integer k such that M can be covered by k open subsets, each of which embeds in $\mathbb{R}^n$.

ii) The immersion covering number $n(M)$ is the least integer k such that M can be covered by k open sets, each of which immerses in $\mathbb{R}^n$.

Berstein introduced upper and lower bounds for $n(M)$ and $N(M)$ when $M = \mathbb{R}P^n$. Unfortunately they agree only in certain cases. As an application of the results of this chapter all but two of these gaps can be closed. The proof of the following result will be given in section 10.

Theorem 1. Set $n + 1 = 2^k m$ where m is odd.

i) $$n(\mathbb{R}P^n) = \begin{cases} \max\{2,m\} & \text{if } k \le 3 \\ \text{the least integer} \ge \dfrac{n+1}{2(k+1)} & \text{if } k > 3 . \end{cases}$$

ii) $N(\mathbb{R}P^n) = n(\mathbb{R}P^n)$ with the possible exception of the values $n = 31$ and $n = 47$. In that case there are inequalities

$$3 \le N(\mathbb{R}P^{31}) \leqslant 4$$

$$5 \le N(\mathbb{R}P^{47}) \leqslant 6 .$$

Remark. i) A consequence of the proof of Theorem 1 is that $n(\mathbb{R}P^n)$ depends only on the homotopy type of $\mathbb{R}P^n$ and that $N(\mathbb{R}P^n)$ depends only on the topological type of $\mathbb{R}P^n$ -- at least when $n \neq 31$ or 47. It would be interesting to know whether or not this holds more generally.

ii) The values of $n(M)$ and $N(M)$ when M is either $\mathbb{C}P^n$ or $\mathbb{H}P^n$ follow easily from Berstein's techniques. The result is stated in section 10.

Using the work of Hirsch and Poenaru ([12], [20]), the problem of determining $n(M)$ reduces to deciding the minimum number of parallelizable open subsets it takes to cover M. One can ask a similar question about any vector bundle, and in [15] James introduced the following invariant.

Definition. Let ξ be a vector bundle over a space X. Then _Vecat(ξ) $\leq$ n_ provided X can be covered by n open subsets over which ξ is stably trivial.

Such a covering of X will be called a _trivializing cover of ξ_. One says that Vecat(ξ) = n if Vecat(ξ) $\leq$ n but Vecat(ξ) $\not\leq$ n-1. A trivializing cover of ξ is said to be _minimal_ if it consists of Vecat(ξ) elements.

James produced upper and lower bounds for this invariant which were close enough to determine Vecat of all complex vector bundles over real and complex projective spaces. Unfortunately there are arbitrarily large gaps between these bounds in the case of real vector bundles over real projective spaces. The purpose of this first chapter is to introduce a technique for closing these gaps. A relationship between Vecat and geometric dimension will also be discussed. The results are stated in sections 1.1 and 1.2.

1.1 The main theorems.

Prior to stating our main results we review some notions related to category in the sense of Lusternik - Schnirelmann. For further discussion the reader is referred to [15].

Definition. ([7], [8]) Let $f : X \to Y$ be a map to a path connected space Y. The category of f, $\operatorname{cat}(f)$, is less than or equal to n if X can be covered by n open subsets U_i with the restriction $f : U_i \to Y$, $i = 1,\ldots,n$, null homotopic.

One says that $\operatorname{cat}(f) = n$ if $\operatorname{cat}(f) \leq n$ but $\operatorname{cat}(f) \not\leq n - 1$.

When $Y = X$ and f is the identity map then $\operatorname{cat}(f)$ is just the Lusternik - Schnirelmann category of X. When $Y = BO$, BU or BSp and f classifies a vector bundle ξ, then $\operatorname{cat}(f)$ is just $\operatorname{Vecat}(\xi)$.

If $f : X \to Y$ and $g : Y \to Z$ then clearly $\operatorname{cat}(g \circ f) \leq \min\{\operatorname{cat}(f), \operatorname{cat}(g)\}$.

If $Y_1,\ldots,Y_n$ have base points $*$, let $T^n(Y_1,\ldots,Y_n) = \{(y_1,\ldots,y_n) \in Y_1 \times \ldots \times Y_n : \text{some } y_i = *\}$ denote the fat wedge. When all $Y_i = Y$ we shall abbreviate this $T^n(Y)$. There is the following analogue of Whitehead's definition of category [7].

<u>Proposition 1.1.1</u>. Let $f : X \to Y$ be a map to a path connected space Y. If X and Y are paracompact and locally contractible then $\mathrm{cat}(f) \leq n$ if and only if $\Delta \circ f$ factors (up to homotopy) through $T^n(Y)$, where $\Delta : Y \to Y^n$ is the iterated diagonal.

$$\begin{tikzcd} & & T^n(Y) \arrow[d] \\ X \arrow[r, "f"] \arrow[urr, dashed] & Y \arrow[r, "\Delta"] & Y^n \end{tikzcd}$$

We shall henceforth assume all spaces to be CW complexes in order to guarantee the hypotheses of Proposition 1.1.1.

The cofibre of $T^n(Y) \to Y^n$ is the iterated smash product $\overset{n}{\wedge} Y$.

<u>Definition</u>. weak $\mathrm{cat}(f) \leqslant n$ provided the composite

$$X \xrightarrow{f} Y \longrightarrow Y^n \longrightarrow \overset{n}{\wedge} Y$$

is nullhomotopic.

<u>Proposition 1.1.2</u>. i) weak $\mathrm{cat}(f) \leqslant \mathrm{cat}(f)$.

ii) Suppose X has dimension d and Y is $(m-1)$-connected. Then weak $\mathrm{cat}(f) = \mathrm{cat}(f)$ in the "stable" range $\mathrm{cat}(f) + 1 \geqslant (d+2)/m$.

proof: Part i) is immediate from the definition. Part ii) is a consequence of the following lemma of Ganea together with the observation that $T^n(Y)$ is (m-1)-connected and $\overset{n}{\bigwedge} Y$ is (mn - 1)-connected.

Lemma 1.1.3 (Ganea [10]). Let $A \to X \to X/A$ be a cofibration and $F \to X$ the fibre of the collapse. There is a canonical map $A \to F$ and $\Sigma F/A \simeq A * \Omega(X/A)$.

Note: There are other ways to prove Proposition 1.1.2 (for example using the Serre spectral sequence). We have chosen to appeal to Lemma 1.1.3 since we will be needing the identification $\Sigma F/A \simeq A*\Omega(X/A)$ later.

Now let ξ be a vector bundle over X and let E be a multiplicative cohomology theory.

Definition. An element $c \in \tilde{E}^*(X)$ is a characteristic class of ξ if $i^*(c) = 0$ for any $i:U \to X$ with $i^*(\xi)$ stably trivial.

Examples. i) $0 \in \tilde{E}^*(X)$

ii) The Stiefel-Whitney classes $w_k(\xi) \in H^k(X;Z/2)$.

iii) The reduced class $[\xi]$ of ξ itself in the appropriate K-theory.

Given an element c in a ring R let $\mathrm{nil}(c) = \min\{k : c^k = 0\}$. The following proposition and its proof are due to James [15] (but see also [6]).

<u>Proposition 1.1.4</u>. For any characteristic class c of ξ, there is an inequality $\mathrm{nil}(c) \le \mathrm{Vecat}(\xi)$.

<u>proof</u>: Suppose $X = U_1 \cup \cdots \cup U_n$ with the restriction of ξ to each U_i stably trivial. Then c is in the image of $E^*(X, U_1)$,

$$c^2 \in \mathrm{im}\ E^*(X, U_1 \cup U_2), \ldots, c^n \in \mathrm{im}\ E^*(X, U_1 \cup \cdots \cup U_n) = 0 .$$

Proposition 1.1.4 provides an effective lower bound for $\mathrm{Vecat}(\xi)$. Upper bounds come from open coverings of the base.

If V is a real or complex vector space let $P(V)$ denote the projective space of V. One can write

$$P(V \oplus W) = [P(V \oplus W) - P(V)] \cup [P(V \oplus W) - P(W)] ,$$

and observe that the inclusions

$$P(W) \longrightarrow [P(V \oplus W) - P(V)] \text{ and } P(V) \longrightarrow [P(V \oplus W) - P(W)]$$

are homotopy equivalences. We will often abbreviate this by writing $P^{n-1} \cup P^{m-1} = P^{n+m-1}$. These are the coverings used by James.

The following theorem can be extracted from [15].

Theorem 1.1.5 (James): Let ξ be a complex vector bundle over either real or complex projective space. Vecat $(\xi) = $ nil $[\xi]$ = nil (lowest non-vanishing Chern class of ξ).

proof: The easiest way to see that the upper and lower bounds agree is to choose a representative for $[\xi]$ in the Atiyah - Hirzebruch spectral sequence and observe what happens to its powers.

As we have remarked, there are arbitrarily large gaps between these bounds in the case of real vector bundles over real projective space. The following theorem closes these gaps.

Theorem 1.1.6. Let $\xi = 2^k(\mathrm{odd})h$ be a (stable) vector bundle over RP^n where h is the canonical line bundle. Then

$$\mathrm{Vecat}\,(\xi) = \begin{cases} \mathrm{nil}\ [\xi] & \text{if } k \geqslant 3 \\ \mathrm{nil}\ w_{2^k}(\xi) & \text{if } k \leqslant 3\ . \end{cases}$$

Let τ and ν denote respectively the tangent and normal bundles of RP^n and write $n+1=2^k m$ with m odd.

Corollary 1.1.7. $\operatorname{Vecat}(\tau)=\operatorname{Vecat}(\nu)$ is the least integer greater than or equal to

$$\begin{cases} m & \text{if } k \leqslant 3 \\ \dfrac{n+1}{2(k+1)} & \text{if } k \geqslant 3 \end{cases} .$$

For example, the (complex) tangent bundle of $\mathbb{C}P^{31}$ has category 32 whereas the (real) tangent bundle of RP^{31} has category 3 .

Remark: The case $k \leqslant 3$ of Theorem 1.1.6 follows from the methods of [15]. See the remark after Lemma 1.4.3.

1.2 A Relation with Geometric Dimension

On aesthetic grounds it seems as if Vecat (tangent bundle) should have some bearing on immersions of the base -- or more generally that Vecat(ξ) should relate to the geometric dimension of ξ (to be denoted g.d. (ξ)). The calculation at the end of section 1.1 shows, however, that $\mathbb{R}P^{31}$ can be covered by 3 open sets, each of which immerses in $\mathbb{R}^{31}$. The fact that $\mathbb{R}P^{31}$ doesn't immerse until $\mathbb{R}^{53}$ seems to discourage such a notion.

There is a variant of Vecat which does relate to geometric dimension. For a real vector bundle over a finite dimensional base X let $P(\xi)$ denote the projective bundle of ξ and let h_ξ be the canonical line bundle over $P(\xi)$.

<u>Definition</u>. $\bar{\nu}\text{-cat}(\xi) = \text{Vecat}(h_\xi) - \dim(\xi)$.

<u>Lemma 1.2.1</u>. If $w_j(-\xi) \neq 0$ then $\bar{\nu}\text{-cat}(\xi) \geq j$. In particular $\bar{\nu}\text{-cat}(\xi) \geq 0$.

<u>Proof</u>: Let $\alpha \in H^1(P(\xi);\, \mathbb{Z}/_2)$ be the Stiefel-Whitney class of h_ξ. We calculate nil(α). Suppose $\dim(\xi) = n$ and let $1 + \bar{w}_1 + \ldots + \bar{w}_k$, $\bar{w}_k \neq 0$, be the total Stiefel-Whitney class of $-\xi$. By definition $\alpha^n + w_1\alpha^{n-1} + \ldots + w_n = 0$, hence

$$\alpha^{n+j} = (\alpha^n + w_1\alpha^{n-1} + \ldots + w_n)(\alpha^j + \alpha^{j-1}\bar{w}_1 + \ldots + \bar{w}_j) + \alpha^{n+j}$$

which is zero if and only if $j \geq k$. It follows that nil(α) = n+k. This completes the proof.

Example. If n denotes the n-dimensional trivial bundle over X then $\overline{\nu}\text{-cat}(n) = 0$.

Lemma 1.2.2. Let ξ and η be vector bundles over X. There is an inequality

$$\overline{\nu}\text{-cat}(\xi \oplus \eta) \leq \overline{\nu}\text{-cat}(\xi) + \overline{\nu}\text{-cat}(\eta) \ .$$

In particular, $\overline{\nu}\text{-cat}(\xi \oplus 1) \leq \overline{\nu}\text{-cat}(\xi)$.

Proof: The decomposition of $P(V \oplus W)$ given after Proposition 1.1.4 can be applied fibrewise to obtain a decomposition

$$P(\xi \oplus \eta) = P(\xi) \cup P(\eta) \ .$$

The lemma follows easily from this.

Lemma 1.2.2 allows for the following definition.

Definition. $\nu\text{-cat}(\xi) = \lim\limits_{n\to\infty} \overline{\nu}\text{-cat}(\xi \oplus n)$.

Lemma 1.2.1 provides the inequality

$$j \leq \nu\text{-cat}(\xi) \leq \overline{\nu}\text{-cat}(\xi)$$

whenever $w_j(-\xi) \neq 0$. This suggests a relationship between $\nu\text{-cat}(\xi)$ and $\text{g.d.}(-\xi)$.

Theorem 1.2.3. Let ξ be a vector bundle over X. There is an inequality

$$\nu\text{-cat}(\xi) \leq \text{g.d.}(-\xi) \ .$$

Equality holds provided $\nu\text{-cat}(\xi) > \frac{1}{2}\dim(X)$.

Equality is guaranteed in Theorem 1.2.3 if, for example, $w_j(-\xi) \neq 0$ with $j > \frac{1}{2}\dim X$. In this sense one can say that ν-cat(ξ) and g.d.$(-\xi)$ agree in the metastable range.

Before proving Theorem 1.2.3 we need to recall another characterization of the category of a map. The proof of the following lemma can be found in [22] or [15].

Lemma 1.2.4. Let $f : X \to P^\infty$ classify a line bundle h. Then Vecat$(h) \leq n$ if and only if f factors (up to homotopy) through P^{n-1}.

Proof of Theorem 1.2.3: Suppose ξ has dimension n. Fix a Riemannian metric on ξ and let $E \to X$ be the associated principal $O(n)$-bundle. Let $V_{n+k,n}$ denote the Stiefel manifold of orthonormal n-frames in $\mathbb{R}^{n+k}$ and let $E_{n+k,n}$ be the space of $\mathbb{Z}/2$-equivariant maps from S^{n-1} to S^{n+k-1}. It is well-known ([11], [14]) that the inclusion $V_{n+k,n} \subset E_{n+k,n}$ is a $(2k-1)$-equivalence. Consider the following diagram of fibrations

$$\begin{tikzcd}
V_{n+k,n} \arrow[r] \arrow[d] & E_{n+k,n} \arrow[d] \\
E \times_{O(n)} V_{n+k,n} \arrow[r] \arrow[d, "p"'] & E \times_{O(n)} E_{n+k,n} \arrow[d, "q"] \\
X \arrow[r, "="] & X
\end{tikzcd}$$

It follows easily from the 5-lemma that the map between total spaces is also a $(2k-1)$-equivalence. By adding trivial bundles to ξ, n can be increased. For sufficiently large values of n, $\nu\text{-cat}(\xi) \leq k$ if and only if q admits a section (Lemma 1.2.4). Also for sufficiently large values of n, $\text{g.d.}(-\xi) \leq k$ if and only if p admits a section. The theorem follows.

The characterization of $\bar{\nu}\text{-cat}$ used in the above proof also provides a condition on $\dim(\xi)$ which guarantees the equality $\bar{\nu}\text{-cat}(\xi) = \nu\text{-cat}(\xi)$. We first require a lemma which, though a standard fact, seems not to appear explicitly in the literature.

Lemma 1.2.5. The suspension map $E_{m,n} \to E_{m+1,n+1}$ is a $\min\{2n-m-2, n-1\}$-equivalence.

Proof: The proof is by induction on n using the fibrations $\Omega^{n-1}S^{m-1} \to E_{m,n} \to E_{m,n-1}$ [14].

Corollary 1.2.6. There is an equality $\bar{\nu}\text{-cat}(\xi) = \nu\text{-cat}(\xi)$ provided either

i) $\nu\text{-cat}(\xi) \neq 0$ and $\dim(\xi) > \dim(X) - \nu\text{-cat}(\xi)$

or

ii) $\nu\text{-cat}(\xi) = 0$ and $\dim(\xi) > \dim(X) + 1$.

Example. Consider $\xi = (m+1)h$ on P^n. For most values of m and n, $\bar{\nu}\text{-cat}(\xi) = \nu\text{-cat}(\xi)$ by Corollary 1.2.6. The space $P(\xi)$ is just $P^m \times P^n$ and the line bundle h_ξ is the tensor product of the line bundles classified by the two projections. The problem of calculating $\nu\text{-cat}$

is equivalent to deciding the smallest projective space to which the map classifying

$$P^m \times P^n \to P^\infty$$

can be deformed. This is the well-known "axial-map" problem [5]. In this sense, the problem of calculating ν-cat can be thought of as an axial map problem for arbitrary vector bundles.

1.3 A specific case of Theorem 1.1.6

The main idea behind the proof of Theorem 1.1.6 is quite simple, but the bookkeeping and a few exceptional cases make the general argument somewhat involved. It seems best then to present a specific example of the argument before proceeding to the complete proof.

Consider $\xi = 2^9 h$ over RP^{39} . This bundle is trivial on RP^{17} but not on RP^{18} . The decomposition $RP^{39} = RP^{17} \cup RP^{17} \cup RP^3$ shows that $\mathrm{Vecat}(\xi) \leqslant 3$. On the other hand, $\widetilde{KO}^0(RP^{39}) = Z/2^{19}$ and $[\xi]^2 = -[2^{19}h] = 0$ so that $\mathrm{Vecat}(\xi) \geqslant 2$. We will show that the latter inequality is an equality. First some conventions.

For a space or a spectrum X let $X\langle n\rangle$ denote its (n-1)-connected cover. The spectrum for connective K-theory will be denoted bo . Since we will be dealing only with real projective spaces we can safely abbreviate RP^n to P^n . The symbol P_k^n will denote the stunted projective space P^n/P^{k-1} , and when it will cause no confusion we will use a single symbol to denote both a vector bundle and its classifying map.

Finally, the reader may wish to look ahead to Lemmas 1.5.1 - 1.5.3 since they will be used in this section.

The map $\xi = 2^9 h$ lifts (uniquely) through $\bar{\xi}$ to $BO\langle 16\rangle$. Since $\mathrm{cat}(\bar{\xi}) \geqslant \mathrm{cat}(\xi)$ it suffices to show that $\mathrm{cat}(\bar{\xi}) \leqslant 2$. Further, since $(2+1)(16) - 2 = 46 \geqslant 39$ it is enough by Proposition 1.1.2 to show that weak $\operatorname{cat}(\bar{\xi}) \leq 2$.

We are therefore led to the following composite:

$$P^{39} \longrightarrow BO\langle 16\rangle \times BO\langle 16\rangle \longrightarrow BO\langle 16\rangle \wedge BO\langle 16\rangle \ .$$

The range of this map is 31-connected so it can be considered a map of suspension spectra. Further, by Lemma 1.5.2 the range can be replaced by $bo\langle 16\rangle \wedge bo\langle 16\rangle \approx bo\langle 32\rangle \vee \Sigma^{32}Y$ where Y is 3-connected. We must consider the two components $P^{39} \to bo\langle 32\rangle$ and $P^{39} \to \Sigma^{32}Y$.

The map $P^{39} \to bo\langle 16\rangle \wedge bo\langle 16\rangle$ factors as

$$P^{39} \longrightarrow P_{17}^{22} \wedge P_{17}^{22} \longrightarrow P_{17}^{39} \wedge P_{17}^{39} \xrightarrow{\bar{\xi}\wedge\bar{\xi}} bo\langle 16\rangle \wedge bo\langle 16\rangle \ .$$

By Lemma 1.5.1 the map $P_{17}^{22} \wedge P_{17}^{22} \approx \Sigma^{16}P^5 \wedge \Sigma^{16}P^5 \to bo\langle 16\rangle \wedge bo\langle 16\rangle$ factors through $S^{16} \wedge S^{16}$. It follows that the component $P^{39} \to \Sigma^{32}Y$ is null homotopic. The component $P^{39} \to bo\langle 32\rangle$ is detected by the composite $P^{39} \to bo\langle 32\rangle \to bo$ (Lemmas 1.5.2, 1.5.3) which classifies the "O" bundle $[\xi]^2$. This completes the proof.

1.4 Some arithmetic

In order to do the bookkeeping in the proof of Theorem 1.1.6 we need to establish some arithmetic conventions.

Let $I \subset KO^0(X)$ be an ideal.

<u>Definition</u>. i) $\mathrm{Vecat}(I) \leqslant n$ if X can be covered by n open sets U_i with the restriction of I to each $KO(U_i)$ equal to 0.

ii) $\mathrm{nil}(I) = \min\{k : I^k = 0\}$.

Note that when $I = ([\xi])$ is a principal ideal $\mathrm{Vecat}(I) = \mathrm{Vecat}(\xi)$ and $\mathrm{nil}(I) = \mathrm{nil}(\xi)$. It could happen that $\mathrm{nil}(I) = +\infty$.

We will need to consider the p-adic valuation $\nu_p(I)$ and the "Atiyah Hirzebruch valuation" $\nu(I)$.

<u>Definition</u>. Let $I \subset KO(X)$ be an ideal and let $\widetilde{KO}(X) \subset KO(X)$ be the augmentation ideal. Set $\underline{\nu_p(I) = \min\{k : p^k\widetilde{KO}(X) \subset I\}}$ and $\underline{\nu(I) = \min\{k : f^*I \neq 0 \text{ for some } f : Y \to X \text{ with } \dim Y = k\}}$.

The following is clear.

Lemma 1.4.1. $\nu(I \cdot J) \geq \nu(I) + \nu(J)$.

We will also need to refer to the largest "skeleton" on which $I^n = 0$.

Definition. $Q_I(n) = \nu(I^n) - 1$.

In the special case of $KO(P^d)$ certain principal ideals will be of importance.

Definition. $I_k = (2^k h)$ where h is the reduced class of the canonical line bundle.

In this case $Q_{I_k}(n)$ will be abbreviated to $Q_k(n)$.

The following lemma simply restates some standard facts about $KO(P^d)$. To avoid degenerate cases we assume $d \gg 0$.

Lemma 1.4.2. i) $I_{n-1} \cdot I_{m-1} = I_{n+m-1}$.

ii) $\nu_2(I_n) = n$; $\nu_p(I_n) = +\infty$ p odd.

iii) If $n = 4k + r - 1$ with $1 \leqslant r \leqslant 4$, then $\nu(I_n) = 8k + 2^{r-1}$.

iv) If $m > n$ then $\nu(I_m) > \nu(I_n)$.

With these conventions, the part of Theorem 1.1.6 which doesn't follow from [15] can be restated. Set $m = Q_{4k+r-1}(n)$ and suppose $k \neq 0$.

Theorem 1.1.6'. On P^m one has
$\mathrm{Vecat}(I_{4k+r-1}) = \mathrm{nil}(I_{4k+r-1}) = n$.

proof that (1.1.6') $\Longrightarrow$ (1.1.6): This uses the fact that if $\mathrm{Vecat}(\xi) \leqslant n$ on P^m then $\mathrm{Vecat}(\xi) \leqslant n$ on P^j with $j \leqslant m$. Details are left to the reader.

The following exercise uses the notation of this section and the decomposition $P^{i-1} \cup P^{j-1} = P^{i+j-1}$ to obtain an upper bound on $\mathrm{Vecat}(I)$.

Lemma 1.4.3. $\mathrm{Vecat}(I) \leqslant$ the least integer greater than $m/\nu(I)$.

proof: Write $m = d \cdot \nu(I) + e$ with $e < \nu(I)$. We must show that $\mathrm{Vecat}(I) \leqslant d+1$. One can write $P^m = \bigcup^{d} P^{\nu(I)-1} \cup P^e$. There are $d+1$ terms in this union. and by the definition of ν , the restriction of I to each term is zero.

Remark: Lemma 1.4.3 is also a consequence of a standard inequality relating the Lusternik - Schnirelmann category of a space X to the quotient of its dimension by its connectivity. The role of X is played by the stunted projective space $P^m_{\nu(I)}$.

Lemma 1.4.3 provides an upper bound for $\mathrm{Vecat}(I)$ agreeing with the lower bound given by $\mathrm{nil}(w_{2^k}(\xi))$ in case $I = (2^k h)$ with $k \leqslant 3$.

The assertion of Theorem 1.1.6' falls naturally into cases depending on the values of $4k+r-1$ and n. Using the notation of this section, Theorem 1.1.6' can be reduced to a "minimal" set of cases.

<u>Lemma 1.4.4</u>. i) $\nu_2(I_{p-1} \cdot I_{q-1}) = \nu_2(I_{p-1}) + \nu_2(I_{q-1}) + 1$.

ii) $\nu(I_p \cdot I_q) = \nu(I_p) + \nu(I_q)$ provided $q \equiv -1\ (4)$.

iii) $Q_\ell(n+4) = Q_\ell(n) + Q_\ell(4) + 1$.

<u>proof</u>: Parts i) and ii) follow from Lemma 1.4.2. Part iii) follows from ii) and the definition.

<u>Corollary 1.4.5</u>. i) Theorem 1.1.6' is true in case $r = 4$.

ii) It suffices to prove Theorem 1.1.6' in case $n \leqslant 4$ and $r \neq 4$.

<u>proof</u>: i) $Q_{4k+3}(n) = \nu((I_{4k+3})^n) - 1 = n \cdot \nu(I_{4k+3}) - 1$. Now use Lemma 1.4.3.

ii) This follows from part i), Lemma 1.4.4 iii), the decomposition $P^{i-1} \cup P^{j-1} = P^{i+j-1}$ and induction.

Finally, we turn to some numerical estimates which will be needed later.

Lemma 1.4.6: i) $Q_{4k+r-1}(n) \leqslant 8k(n+1) - 2$ provided $k > 3$ and $rn \leqslant 12$.

ii) $Q_{4k+r-1}(n) \leqslant 2 \cdot 8kn - 2$ if $nk > 3$.

iii) $Q_{4k+r-1}(n) - (8k+1)(n+1) \leqslant 7$.

proof: i)
$$\begin{aligned} Q_{4k+r-1}(n) &= \nu((I_{4k+r-1})^n) - 1 \\ &= \nu((I_{4kn+rn-1}) - 1 \\ &\leqslant \nu(I_{4kn+12-1}) - 1 \quad (\text{since } rn \leqslant 12) \\ &= 8(kn+2) + 2^3 - 1 \\ &= 8(kn+3) - 1 \\ &< 8(kn+k) - 1 = 8k(n+1) - 1 . \end{aligned}$$

Parts ii) and iii) are proved similarly.

Remark: i) Lemma 1.4.6 i) remains true when $k = 3$, $rn < 12$ and when $k = 2$, $r = 1$. This fact will be used later.

ii) Lemma 1.4.6 iii) is used to show that certain stunted projective spaces which arise are suspensions of ordinary projective spaces.

1.5 Some facts from connective K-theory

This section recalls several results from connective K-theory. Let $p:bo\langle n\rangle \to bo$ be projection from the (n-1)-connective cover and let $i:S^0 \to bo$ be the unit. Square brackets will denote pointed homotopy classes of maps.

Lemma 1.5.1. i) $i_*:[P^m,S^0] \to [P^m,bo]$ is an epimorphism.

ii) $p_*:[P^m,bo\langle n\rangle] \to [P^m,bo]$ is a monomorphism.

proof: i) $[P^m,bo]$ is a cyclic group generated by the reduced class of the canonical line bundle. This generator factors through S^0 via, say, the map of the Kahn-Priddy theorem.

ii) This follows from the Atiyah-Hirzebruch spectral sequence.

Lemma 1.5.2. In the category of spectra, the "evaluation" map $BO\langle n\rangle \to bo\langle n\rangle$ is a 2n-equivalence.

proof: This is a standard consequence of the fibration $\Omega X * \Omega X \to \Sigma\Omega X \to X$.

Lemma 1.5.3. i) $bo \wedge bo \simeq bo \vee Y$ where Y is 3-connected.

ii) $bo\langle 8k\rangle \simeq \Sigma^{8k} bo$ and the following diagram commutes:

$$
\begin{tikzcd}
bo\langle 8k\rangle \wedge bo\langle 8k\rangle \approx \Sigma^{8k}bo \wedge \Sigma^{8k}bo \approx \Sigma^{16k}bo \vee \Sigma^{16k}Y \approx bo\langle 16k\rangle \vee \Sigma^{16k}Y \arrow[d, "p \wedge p"'] \arrow[r, phantom] & \\
bo \wedge bo \arrow[r, "\mu"] & bo
\end{tikzcd}
$$

$$bo\langle 16k\rangle \vee \Sigma^{16k}Y \xrightarrow{\text{pinch}} bo\langle 16k\rangle \xrightarrow{p} bo$$

proof: i) Since bo is a ring spectrum, $bo \wedge bo \approx bo \vee Y$. The connectivity of Y is a consequence of the fact that the unit $i: S^0 \to bo$ is a 3-equivalence.

ii) This will be proved in greater generality in section 1.7.

1.6 Proof of Theorem 1.1.6' in case $k > 3$.

We can now proceed with the proof of Theorem 1.1.6' in case $k > 3$. For ease of typography set $m = Q_{4k+r-1}(n)$. We need to show that $\mathrm{Vecat}(I_{4k+r-1}) \leqslant n$ on P^m. We have reduced to the case $1 \leqslant r \leqslant 3$, $2 \leqslant n \leqslant 4$.

Let ξ generate I_{4k+r-1}. Since $\nu(I_{4k+r-1}) > 8k$ $(r \neq 4)$, ξ lifts through $\bar{\xi}$ to $BO\langle 8k\rangle$. Clearly, $\mathrm{Vecat}(\xi) \leqslant \mathrm{cat}(\bar{\xi})$. By Lemma 1.4.6, $m \leqslant 8k(n+1) - 2$ and so $\mathrm{cat}(\bar{\xi}) = \text{weak cat}(\bar{\xi})$. We are therefore led to the following composite :

$$*)\quad P^m \longrightarrow \overbrace{BO\langle 8k\rangle \times\cdots\times BO\langle 8k\rangle}^{n} \longrightarrow \overbrace{BO\langle 8k\rangle \wedge\cdots\wedge BO\langle 8k\rangle}^{n} .$$

By Lemma 1.4.6, $m \leqslant 2\cdot 8kn - 2$ so we can consider this as a map of suspension spectra. By Lemmas 1.5.2 and 1.4.6 the range can be replaced by $\overbrace{bo\langle 8k\rangle \wedge\cdots\wedge bo\langle 8k\rangle}^{n}$. The composite *) can then be rewritten

$$**)\quad P^m \longrightarrow \overbrace{P_{8k+1}^{m-(8k+1)n} \wedge\cdots\wedge P_{8k+1}^{m-(8k+1)n}}^{n} \longrightarrow \overbrace{bo\langle 8k\rangle \wedge\cdots\wedge bo\langle 8k\rangle}^{n} .$$

Since $P_{8k+1}^{m-(8k+1)n} \approx \Sigma^{8k} P^{m-(8k+1)(n+1)+1}$ (Lemma 1.4.6 iii)), Lemma 1.5.1 factors it through S^{8kn}.

An appeal to (iteration of) Lemma 1.5.3 splits $\overbrace{bo\langle 8k\rangle \wedge\cdots\wedge bo\langle 8k\rangle}^{n}$ in the form $bo\langle 8kn\rangle \vee Y'$ where Y' is

$8kn+3$ connected. The above paragraph then shows that the component of **) in Y' is null. The component in $bo\langle 8kn\rangle$ is detected by the composite $P^m \to bo\langle 8kn\rangle \to bo$ (Lemma 1.5.1), which by Lemma 1.5.3 classifies the "0" bundle $[\xi]^n$.

1.7 More facts from connective K-theory

In order to complete the proof of Theorem 1.1.6' we need a deeper study of the spectrum for connective K-theory. The basic references for this section are [2], [4] and [17].

Until the statement of Lemma 1.7.10, all spectra will be localized at 2 and have $Z_{(2)}$-cohomology of finite type. This ensures that mod 2 homology equivalences are stable homotopy equivalences.

Let $A_1 \subset A$ be the subalgebra of the mod 2 Steenrod algebra generated by S_q^1 and S_q^2. For $i = 0,1,2,4$ let $M_i = A_1/J_i$ where J_i is the left ideal with generators described as follows:

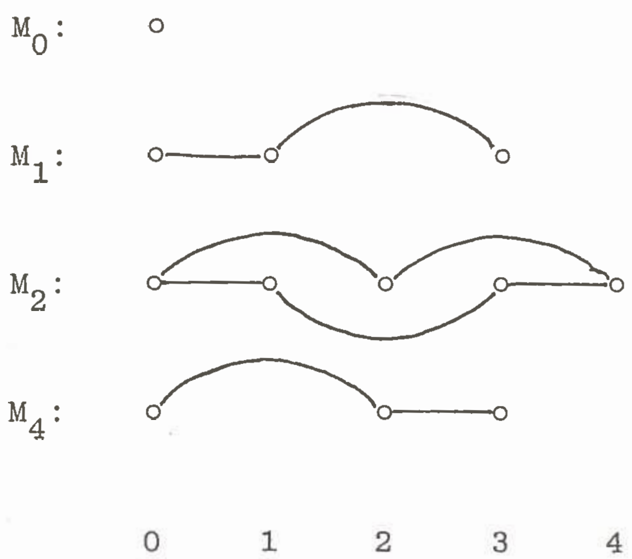
$$J_0 = (S_q^1, S_q^2)$$

$$J_1 = (S_q^2)$$

$$J_2 = (S_q^3)$$

$$J_4 = (S_q^1, S_q^2\ S_q^3) .$$

Following Mahowald we can represent M_i by a graph



in which the nodes represent a homogeneous basis for M_i, the straight lines represent multiplication by Sq^1, and the curved lines represent multiplication by Sq^2.

<u>Lemma 1.7.1</u>. For $n = 0,1,2,4$ $H^*(bo\langle i\rangle;Z/2) \approx \Sigma^i A \otimes_{A_1} M_i$ as graded A-modules.

<u>proof</u>: Proofs of this can be found in [4] and [21].

<u>Lemma 1.7.2</u>. There exist stable complexes N_i with $H^*(N_i;Z/2) \approx M_i$.

<u>proof</u>: For N_0 use the sphere S^0 . The existence of N_1 and N_4 follows from the fact that $2\eta = 0$. The existence of N_2 follows from the relation $\eta^2 \in \langle 2,\eta,2\rangle$.

Lemma 1.7.3. For $i = 0,1,2,4$ there exists a map $\Sigma^i N_i \to bo$ extending the generator of $\pi_i bo$ on the bottom cell.

proof: This follows easily from the Atiyah-Hirzebruch spectral sequence.

Since bo is a ring spectrum these maps extend to maps $\Sigma^i N_i \wedge bo \to bo$.

Corollary 1.7.4. $\Sigma^i N_i \wedge bo \approx bo\langle i\rangle$.

proof: The maps $\Sigma^i N_i \wedge bo \to bo$ lift for reasons of connectivity to $bo\langle i\rangle$. By Lemma 1.7.3 these lifts induce isomorphisms in π_i and hence in $H^i(\ ;Z/2)$. The corollary follows since the mod 2 cohomology of the spectra involved are isomorphic cyclic A modules.

Let $\bar{N}_2$ denote the 4-fold suspension of the Spanier-Whitehead dual of N_2 . This arranges matters so that both N_2 and $\bar{N}_2$ have their bottom cells in dimension zero. Observe also that N_1 and N_4 are Spanier-Whitehead 3-duals.

Corollary 1.7.5. $\Sigma^2 \bar{N}_2 \wedge bo \approx bo\langle 2\rangle$.

<u>proof</u>: The proof of Corollary 1.7.4 depended only on the cohomology of N_2 as an A_1-module.

<u>Lemma 1.7.6</u>. The duality pairings $N_2 \wedge \bar{N}_2 \to S^4$ and $N_1 \wedge N_4 \to S^3$ are projections onto wedge summands.

<u>proof</u>: There are two types of duality pairings. Their composites $S^4 \to N_2 \wedge \bar{N}_2 \to S^4$ and $S^3 \to N_1 \wedge N_4 \to S^3$ are multiplication by the Euler characteristics which are 1 .

<u>Corollary 1.7.7</u>. $N_2 \wedge \bar{N}_2 \simeq S^4 \vee Z$ where $H^*(Z;Z/2)$ is free as an A_1-module. $N_1 \wedge N_4 \approx S^3 \vee Z'$ where $H^*(Z';Z/2) \approx A_1$ as A_1-modules.

<u>proof</u>: Let $Q_0 = S_q^1$ and $Q_1 = [S_q^1, S_q^2]$ be the Milnor elements. Since $Q_0^2 = Q_1^2 = 0$ one may speak of the Q_0 and Q_1 homologies of an A_1-module. It is well known ([3], [19]) that an A_1-module is free if and only if both its Q_0 and Q_1 homologies vanish. Since $H^*(N_2 \wedge \bar{N}_2;Z/2) \simeq M_2 \otimes M_2$ the Q_0 and Q_1 homologies of $H^*(N_2 \wedge \bar{N}_2;Z/2)$ both have rank 1 . It then follows from Lemma 1.7.6 that $H^*(Z;Z/2)$ has neither Q_0 nor Q_1 homology. A similar argument works for $N_1 \wedge N_4$. A dimension count verifies the last assertion.

<u>Corollary 1.7.8</u>. The products $\Sigma N_1 \wedge \Sigma^4 N_4 \to bo$ and $\Sigma^2 N_2 \wedge \Sigma^2 \bar{N}_2 \to bo$ of the maps of Lemma 1.7.3 factor through the duality pairings composed with a generator of $\pi_8 bo$.

<u>proof</u>: An elementary Adams spectral sequence argument shows that any map $\Sigma^4 Z \to bo$ or $\Sigma^5 Z' \to bo$ must be null homotopic. The maps therefore factor through the duality pairings. Since $\pi_8 bo$ is torsion free, a calculation using the rational Hurewicz homomorphism will suffice to verify the statement about generation of $\pi_8 bo$. This is easily done, for example, by calculating the Chern characters of the maps of Lemma 1.7.3. Details are left to the reader.

The purpose of introducing the complexes N_i is to provide a calculus for the smash products of connective covers of bo. For example $bo\langle 8k+2\rangle \wedge bo\langle 8\ell+2\rangle \simeq \Sigma^{8k} bo\langle 2\rangle \wedge \Sigma^{8\ell} bo\langle 2\rangle \simeq \Sigma^{8(k+\ell)+4} N_2 \wedge \bar{N}_2 \wedge bo \wedge bo \simeq \Sigma^{8(k+\ell)+4}(S^4 \vee Z) \wedge (bo \vee Y) \simeq bo\langle 8(k+\ell+1)\rangle \vee Y'$ for some Y' . As another example $bo\langle 8k+1\rangle \wedge bo\langle 8\ell+4\rangle \simeq bo\langle 8(k+\ell+1)\rangle \vee Y'$ for some other Y' . Using the ideas of this section, the following Lemma is easily verified.

<u>Lemma 1.7.9</u>. There is an equivalence $bo\langle 8k+2\rangle \wedge bo\langle 8\ell+2\rangle \simeq bo\langle 8(k+\ell+1)\rangle \vee Y'$ and the following diagram commutes up to homotopy:

$$\begin{tikzcd}
bo\langle 8k+2\rangle \wedge bo\langle 8\ell+2\rangle \approx bo\langle 8(k+\ell+1)\rangle \vee Y' \arrow[r, "\text{pinch}"] \arrow[d, "p \wedge p"'] & bo\langle 8(k+\ell+1)\rangle \arrow[d, "p"] \\
bo \wedge bo \arrow[r, "\mu"] & bo\ .
\end{tikzcd}$$

A similar statement holds for $bo\langle 8k+1\rangle \wedge bo\langle 8\ell+4\rangle$.

Lemma 1.7.9 breaks the problem of determining a map into $bo\langle n_1\rangle \wedge bo\langle n_2\rangle$ into two parts. One part consists of a map to $bo\langle n_3\rangle$ and the other part involves a map to some other spectrum Y' . The notion of <u>simplicity</u> of a spectrum Y is useful for obtaining information about this other map. For the rest of this section all spectra will be localized at some prime p and have $Z_{(p)}$ cohomology of finite type.

If A is an abelian group then HA will denote the Eilenberg MacLane spectrum with $\pi_0 HA \approx A$.

<u>Lemma 1.7.10</u>. For a spectrum Y , the following are equivalent:

i) If $\dim X \leq n$ then any non-trivial map $X \to Y$ induces a non-zero homomorphism $H^*(Y;M) \to H^*(X;M)$ for some $Z_{(p)}$ module M.

ii) The Hurewicz homomorphism $\pi_j(Y) \to H_j(Y;Z_{(p)})$ is a split monomorphism for $j \leq n$.

iii) There is a map $Y \to \bigvee_{j\leq n} \Sigma^j H\pi_j(Y)$ which is an $(n+1)$-equivalence.

<u>proof</u>: i) $\Longrightarrow$ ii) Let F denote the fibre of $X \to X \wedge HZ_{(p)}$ and let $F^{(n)}$ denote the n-skeleton of F. Since $F^{(n)} \to X \to X \wedge HZ_{(p)}$ is null homotopic, we have by part i) $F^{(n)} \to X$ null. This results in the following diagram:

$$\begin{tikzcd} X \arrow[r] & X \wedge HZ_{(p)} \arrow[r] & \Sigma F \\ & & \Sigma F^{(n)} \arrow[u, hook] \arrow[ul, dashed] \end{tikzcd}.$$

Applying π_* yields ii).

ii) $\Longrightarrow$ iii) The splittings are elements of $\mathrm{Hom}[H_j(X;Z_{(p)}),\Pi_j(X)]$ for $j \leq n$. Thinking of these as cohomology classes gives the required map.

iii) $\Longrightarrow$ i) This is clear.

<u>Definition</u>. A spectrum Y is <u>n-simple</u> if it satisfies the equivalent conditions of Lemma 1.7.10.

<u>Example</u>. If Y is $(n-1)$-connected then Y is n-simple.

<u>Lemma 1.7.11</u>. Let Y be n-simple and suppose Z is $(k-1)$-connected. Then $Y \wedge Z$ is $(n+k)$-simple.

<u>proof</u>: This follows from condition iii).

The following is a useful criteria for n-simplicity.

<u>Lemma 1.7.12</u>. Y is n-simple if the following are satisfied for $j \leqslant n$;

i) The p-torsion in $\pi_j(Y)$ is of order P ;

ii) After tensoring with Z/p , the Hurewicz homomorphism $\pi_j(Y) \otimes Z/p \to H_j(Y) \otimes Z/p$ is a monomorphism.

<u>proof</u>: This is immediate from the next lemma.

<u>Lemma 1.7.13</u>. Let M and N be finitely generated $Z_{(p)}$-modules with torsion of order at most p . Then $f: M \to N$ is a split monomorphism if and only if $f \otimes Z/p : M \otimes Z/p \to N \otimes Z/p$ is a monomorphism.

<u>proof</u>: The forward implication is trivial. For the reverse implication, write $N = T \oplus F$ where T is the torsion subgroup and F is a free $Z_{(p)}$-module. There is clearly a

map $T \to M$ splitting the restriction of f to the torsion subgroup of M. Extend this to a map $g : N \to M$ by requiring that $g \otimes Z/p : N \otimes Z/p \to M \otimes Z/p$ split $f \otimes Z/p$. Then $g \circ f : M \to M$ is surjective by Nakayama's lemma, hence an isomorphism since M is finitely generated.

1.8 Completion of the proof of Theorem 1.1.6 (nearly)

The remaining cases confront us with greater difficulty since the estimates of Lemma 1.4.6 fail. The argument for handling these is essentially the same, however, and can be outlined as follows.

We are given a map

$$P^n \xrightarrow{\xi} BO \xrightarrow{\Delta} BO \times \cdots \times BO$$

and must show that it factors through the fat wedge. We know that ξ can be lifted through some connective cover of BO so that $\Delta \circ \xi$ can be lifted (uniquely) through

$$BO\langle n_1\rangle \times \cdots \times BO\langle n_k\rangle$$

with $n_1 \leqslant \cdots \leqslant n_k$. Equality need not hold. It will suffice to factor this map through the fat wedge.

If the n_i can be chosen sufficiently large an application of Lemma 1.1.3 will show that it is only necessary to demonstrate that the composite

$$P^m \longrightarrow BO\langle n_1\rangle \times \cdots \times BO\langle n_k\rangle \longrightarrow BO\langle n_1\rangle \wedge \cdots \wedge BO\langle n_k\rangle$$

is null homotopic. Again, if the n_i can be chosen sufficiently large, this can be considered a problem in stable homotopy theory and the range can be replaced by $bo\langle n_1\rangle \wedge \cdots \wedge bo\langle n_k\rangle$.

The condition on the n_i which we require is:

A) $m < n_1 + (n_1 + \cdots + n_k) - 1$.

Remark: In case $m = n_1 + (n_1 + \cdots + n_k) - 1$, the map $P^m \to BO\langle n_1\rangle \wedge \cdots \wedge BO\langle n_k\rangle$ can still be replaced by $P^m \to bo\langle n_1\rangle \wedge \cdots \wedge bo\langle n_k\rangle$. The problem is that the map from the fat wedge to the fibre of the collapse is only (m-1)-connected (Lemmas 1.1.3 and 1.5.2). One is confronted with one more obstruction.

At odd primes and rationally this map is null. At the prime 2 the range will split. Again, with carefully chosen n_i this splitting will be in the form $bo\langle n\rangle \vee Y$. The component of our map in $bo\langle n\rangle$ will be detected by the composite $P^m \to bo\langle n\rangle \to bo$ which vanishes since it classifies $[\xi]^k = 0$. The map into Y will vanish since we will be able to show that Y is at least m-simple and that our map is not detected in cohomology with coefficients in any $Z_{(2)}$ module.

We have imposed two more conditions on the sequence $\{n_i\}$.

B) After excluding at most one n_i , the number of $n_j \equiv 2(8)$ must be even and the number of $n_j \equiv 1\ (8)$ must equal the number of $n_j \equiv 4\ (8)$.

C) $\nu(\xi) > n_1$.

Remark: Condition B) guarantees the appropriate splitting and condition C) guarantees that the map $P^m \to bo\langle n_1\rangle \wedge \cdots \wedge bo\langle n_k\rangle$ will induce the zero map in cohomology with coefficients in any $Z_{(2)}$-module.

proof of remark: Condition B) places one in the situation of Lemma 1.7.9. Given condition C), one observes that the map in question factors as

$$P^m \subset P \xrightarrow{\Delta} P \wedge \cdots \wedge P \xrightarrow{\xi \wedge \cdots \wedge \xi} bo\langle n_1\rangle \wedge \cdots \wedge bo\langle n_k\rangle$$

where P denotes infinite dimensional real projective space. The map $\xi : P \to bo\langle n_1\rangle$ induces the zero map in $H^{n_1}(\ ;Z/2)$ by condition C) and hence in $H^*(\ ;Z/2)$ since $H^*(bo\langle n_1\rangle;Z/2)$ is a cyclic A-module. It follows that $P \to bo\langle n_1\rangle \wedge \cdots \wedge bo\langle n_k\rangle$ induces the zero map in mod 2 cohomology which is enough since reduction mod 2 of (untwisted) coefficients is a monomorphism in the cohomology of P.

We can now continue our computations of Vecat. The following table summarizes the remaining cases:

k	r	$4k+r-1$	n	$Q_{4k+r-1}(n)$
	1	4		8 17 27 39
1	2	5	1 2 3 4	9 23 33 47
	3	6		11 25 40 55
	1	8		16 33 51 71
2	2	9	1 2 3 4	17 39 57 79
	3	10		19 41 64 87
3	3	14	1 4	27 119

.

In each instance we will describe the appropriate sequence $\{n_i\}$ and state a lower bound for the simplicity of the resulting summand Y. We leave to the reader the task of verifying conditions A), B) and C). The statements about simplicity are proved in Appendix A.

<u>$k=3$</u>. The only case is $r=3$, $n=4$. Use the sequence (26, 26, 26, 26). The summand Y has simplicity 119.

<u>$k=2$</u>. The argument in section 1.5 works for $k=2$, $r=1$ and in fact for $k=2$, $r=2$, $n=2$. Since $Q_9(3)=Q_9(2)+Q_9(1)+1$ and $Q_q(4)=Q_q(2)+Q_q(2)+1$ this settles $k=2$, $r=2$, $n=3,4$. When $r=3$ we must deal

with $(2^{10}h)$ on P^{41}, P^{64} and P^{87}. The appropriate sequences to choose are (18, 18), (18, 18, 20) and (18, 18, 18, 18). The resulting summands Y have simplicities 47, 67 and 87.

<u>$k = 1$</u>. When $r = 1$ the argument of section 1.5 dispenses with $n = 1,2,3$. When $n = 4$ use the sequence (8, 8, 8, 9). The simplicity of Y will be 40.

When $r = 2$ it suffices to consider $n = 2$. In this case use (8, 10). The summand Y has simplicity 25.

For $r = 2$, $n = 2,3$ use the sequences (10, 10) and (10, 10, 12). The resulting summands have simplicities 31 and 43.

The case $k = 1$, $r = 3$, $n = 4$ is dealt with in the next section.

1.9 $(2^6 h)$ on P^{55}

It remains to consider the case of $2^6 h$ on P^{55}. Using the sequence (10, 12, 12, 12) the argument of the preceding section nearly goes through since $bo\langle 10\rangle \wedge bo\langle 12\rangle \wedge bo\langle 12\rangle \wedge bo\langle 12\rangle$ is already 55-simple by Example 2 in Appendix A. What fails is that the map from the fat wedge to the fibre of the collapse

$$BO\langle 10\rangle \times BO\langle 12\rangle^3 \longrightarrow BO\langle 10\rangle \wedge \bigwedge^3 BO\langle 12\rangle$$

is only 54 - connected (see the remark after condition A). We must deal with one more obstruction.

Let $T_1 \subset BO\langle 10\rangle \times BO\langle 12\rangle^3$ and $T_2 \subset BO\langle 10\rangle^4$ denote the fat wedges. Consider the following diagram in which the vertical sequences are fibrations:

$$\begin{tikzcd}
 & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & E_2/T_2 \\
P^{55} \arrow[r] \arrow[ur, dashed] & BO\langle 10\rangle \times BO\langle 12\rangle^3 \arrow[r] \arrow[d] & BO\langle 10\rangle^4 \arrow[d] & \\
 & BO\langle 10\rangle \wedge \bigwedge^3 BO\langle 12\rangle \arrow[r] & \bigwedge^4 BO\langle 10\rangle &
\end{tikzcd}$$

An application of Lemma 1.1.3 with $A = T_2$, $X = E_2$ reduces the problem to showing that the composite $P^{55} \to E_2/T_2$ is nullhomotopic.

An elementary calculation using Lemma 1.1.3 again shows that E_1/T_1 and E_2/T_2 can be replaced by $\Sigma^{-1} \overset{2}{\Lambda} bo\langle 10\rangle \wedge \overset{3}{\Lambda} bo\langle 12\rangle$ and a wedge of copies of $\Sigma^{-1} \overset{5}{\Lambda} bo\langle 10\rangle$ through dimension 59 , and hence that the map $E_1/T_1 \to E_2/T_2$ induces the zero map in cohomology through dimension 59 . We are therefore left with a stable map

$$P^{55} \longrightarrow \Sigma^{-1} \overset{5}{\Lambda} bo\langle 10\rangle$$

which is not detected in mod two cohomology.

At odd primes and rationally $bo\langle 10\rangle \simeq bo\langle 12\rangle$ so that the range is 58-connected. At the prime 2 , the methods of Appendix A show that the range is 58-equivalent to a wedge of suspensions of mod 2 Eilenberg MacLane spectra. It follows that this last obstruction vanishes.

This completes the proof of Theorem 1.1.6.

1.10 Berstein Covering Numbers

Using Corollary 1.1.7 we can now prove Theorem 1 of the introduction. The following lemma is a slight modification of a result of Berstein ([6], Proposition 2.3).

Lemma 1.10.1. Let M be a closed n-manifold with tangent bundle τ. The immersion covering number $n(M)$ is equal to $\max\{2, \mathrm{Vecat}(\tau)\}$.

Proof: The lemma is immediate from the theorem of Hirsch-Poenaru ([12], [20]) which states that a non-closed n-manifold can be immersed in $\mathbb{R}^n$ if and only if it is parallelizable, together with the observation that a non-closed manifold is parallelizable if and only if it is stably parallelizable.

Part i) of Theorem 1 is an immediate consequence of Lemma 1.10.1 and Corollary 1.1.7.

Following Berstein and James ([6], [16]), some information about the embedding numbers $N(P^n)$ can be obtained from the coverings of projective space by projective spaces of smaller dimension. Suppose $P^n = P^{n_1} \cup \ldots \cup P^{n_k}$ is a trivializing cover of the tangent bundle of P^n with $2n_i < n$, $i = 1,\ldots,k$. For dimensional reasons P^{n_i} embeds in $\mathbb{R}^n$. Let ν_1 be the normal bundle of an embedding. The total space of ν_1 also embeds in $\mathbb{R}^n$. The open subset $P^n - P^{n-n_i-1} \subset P^n$ of which P^{n_i} is a deformation retract is diffeomorphic to the normal bundle ν_2 of the embedding $P^{n_i} \subset P^n$. Since the tangent bundle of P^n

restricts to a stably trivial bundle on P^{n_i}, ν_1 is stably equivalent to ν_2. However, the dimensions of ν_1 and ν_2 are $n - n_i > n_i$, so they are already stable. It follows that the total space of ν_2 embeds in $\mathbb{R}^n$ and that $N(P^n) \leq k$.

The decompositions of P^n as a union of other projective spaces therefore place an upper bound on $N(P^n)$. A lower bound is given by the inequality $n(P^n) \leq N(P^n)$. Write n+1 in the form $2^k m$ with m odd. According to Corollary 1.4.5 and the remark after Corollary 1.1.7 these bounds agree when either $k \leq 3$ or $k \equiv 3$ (4). They also supply the bounds for $N(P^{31})$ and $N(P^{47})$.

Under certain circumstances this upper bound can be improved. Consider for example the case of P^{79} where $\text{Vecat}(\tau) = 8$. There is the decomposition $P^{79} = P^{39} \cup P^{39}$. Let ν_1 be the normal bundle of an embedding $P^{39} \subset \mathbb{R}^{79}$ and let ν_2 be the normal bundle of the inclusion $P^{39} \subset P^{79}$. The tangent bundle of P^{79} does not restrict trivially to P^{39} so ν_1 and ν_2 are not isomorphic. However, $\text{Vecat}(80h) = 4$ on P^{39}. One can therefore decompose P^{39} as a union of 4 open sets over which the tangent bundle of P^{79} does restrict trivially. Over each of these open subsets ν_1 and ν_2 are therefore isomorphic. This results in a decomposition of P^{79} into $4 + 4 = 8$ open subsets, each of which embeds in $\mathbb{R}^{79}$. It follows that $N(P^{79}) = 8$.

In general, this situation will arise whenever a minimal trivializing cover of the tangent bundle of P^n can be obtained by refining a decomposition $P^n = P^{n_1} \cup \ldots \cup P^{n_k}$ with $2n_i < n$, $i = 1,\ldots,k$. It turns out that this is nearly always the case. Let ξ be a vector bundle over P^n. According to the proof of Corollary 1.4.5 one obtains a minimal trivializing cover of ξ by refining a decomposition $P^n = P^{n_1} \cup \ldots \cup P^{n_k}$ with $\mathrm{Vecat}(\xi) \le 4$ on each P^{n_i}. The condition $2n_i < n$ is automatically satisfied if $\mathrm{Vecat}(\xi) \ge 8$. Combined with the previous estimate this shows that $n(P^n) = N(P^n)$ provided $k \le 3$, $k \equiv 3\ (4)$ or $\frac{(n+1)}{2(k+1)} \ge 8$. The only values of n which do not satisfy these inequalities are 15, 31, 47 and 63. The decomposition $P^{15} = P^7 \cup P^7$ is a minimal trivializing cover of the tangent bundle of P^{15}. The decomposition $P^{63} = P^{25} \cup P^{25} \cup P^{11}$ can be refined to a minimal trivializing cover of the tangent bundle. This completes the proof of Theorem 1.

Remark: The values of $n(M)$ and $N(M)$ when M is complex or quaternionic projective space follow easily from the methods of [6] and [16]. The lower bound nil $[\tau]$ agrees with the upper bound coming from the decomposition of projective space as a union of other projective spaces. The values are as follows:

$$n(\mathbb{C}P^n) = N(\mathbb{C}P^n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$n(\mathbb{H}P^n) = N(\mathbb{H}P^n) = n+1 .$$

1.11 A semi-explicit construction

Theorem 1.1.6 abstractly provides unusual covers of real projective spaces. In this section one of these covers is described somewhat more explicitly. This is done largely for amusement though the construction does give some insight into the cohomological properties of these "exotic" covers. It also serves to motivate the more abstract notions of the next chapter.

Consider 2^5h on P^{20}. This bundle is trivial on P^9 but not on P^{10}. The decomposition $P^{20} = P^9 \cup P^9 \cup P^0$ shows that $\mathrm{Vecat}(2^5h) \leq 3$. According to Theorem 1.1.6 however, Vecat of this bundle is actually 2. As a first step toward finding a better covering we must look for a 10-dimensional complex over which 2^5h is trivial.

Let X be the complex $P^9 \vee S^7 \cup e^{10}$, where the attaching map of the 10 cell is the sum of the double cover $S^9 \to P^9$ and the generator η^2 of $\pi_9 S^7$. The inclusion $P^9 \to P^{20}$ clearly extends over X. What's more, the map $P^9 \to X$ induces an isomorphism $\widetilde{KO}(X) \approx \widetilde{KO}(P^9)$. It follows that 2^5h is trivial on X.

The complex X is to play the role of one of the open subsets of a cover. The easiest guess is that X could also play the role of the other. We are left looking for a space to use for their intersection.

Phrased another way, we seek a space U and two maps $P_1, P_2 : U \to X$ so that the double mapping cylinder of

$$X \xleftarrow{P_1} U \xrightarrow{P_2} X$$

is P^{20}. It is however, not necessary that this double mapping cylinder actually <u>be</u> P^{20} -- only that it dominate P^{20}. With this relaxation of our requirements in mind we can ask whether a universal example of an "intersection" exists. One does.

Let U be the homotopy inverse limit of the diagram

$$\begin{tikzcd} X \arrow[dr, "i_1"'] & & X \arrow[dl, "i_2"] \\ & P^{20} & \end{tikzcd}.$$

This is the subspace of $X \times (P^{20})^I \times X$ consisting of triples (x,w,y) satisfying $w(0) = x, w(1) = y$. Restriction of the two projections gives rise to maps $p_1, p_2 : U \to X$. The composites $i_1 \circ p_1$ and $i_2 \circ p_2$ are canonically homotopic. The double mapping cylinder of

*) $$\begin{tikzcd} & U \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

therefore maps to P^{20} (the evaluation map) in such a way as to extend the maps i_1 and i_2. The diagram *) also has the following universal property: given any diagram

**) $$\begin{tikzcd} & V \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X & & X \end{tikzcd}$$

and a homotopy of $i_1 \circ q_1$ with $i_2 \circ q_2$, there is a unique map of diagrams from **) to *) which respects the fixed homotopies $i_1 \circ q_1 \sim i_2 \circ q_2$ and $i_1 \circ p_1 \sim i_2 \circ p_2$. This means that the map from the double mapping cylinder of **) to P^{20} factors through the map from the double mapping cylinder of *) to P^{20}. In particular, if the diagram **) can be used to "cover" P^{20} then so can the diagram *).

For our purposes it is no loss of generality to compose our maps to P^{20} with the inclusion $P^{20} \subset P^{\infty}$. This has certain technical advantages as will be apparent later.

We are now left with the following problem. Let X be as above and let $i_1, i_2 : X \to P^{\infty}$ be two non null-homotopic maps. Form the diagram

$$\begin{tikzcd} & U \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

where U is the appropriate homotopy inverse limit, and let Y be the double mapping cylinder. There is a canonical map $Y \to P^{\infty}$ and we must show that the inclusion $P^{20} \to P^{\infty}$ factors (up to homotopy) through Y. In this sense we will have "covered" P^{20} by two copies of X. This will give another proof that Vecat $(2^5 h) = 2$ on P^{20} since Y actually is covered by two open sets, each having the homotopy type of X, and since $2^5 h$ on P^{20} will be induced by a map to Y.

To describe the space Y we must digress and discuss fibre products over P^∞. Let $A \to P^\infty$ and $B \to P^\infty$ be two maps, and let $\tilde{A} \to A$, $\tilde{B} \to B$ be the associated double covers. The homotopy inverse limit of

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & P^\infty & \end{tikzcd}$$

can be replaced by $\tilde{A} \times_{\mathbb{Z}/2} \tilde{B}$, and the double mapping cylinder of

$$\begin{tikzcd} & \tilde{A} \times_{\mathbb{Z}/2} \tilde{B} \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

is also known as the projective join $\tilde{A} *_{\mathbb{Z}/2} \tilde{B}$. It is the quotient of $\tilde{A} * \tilde{B}$ by the obvious $\mathbb{Z}/2$ action. For example, if $A \to P^\infty$ and $B \to P^\infty$ are the inclusions of P^{n-1} and P^{m-1} respectively, the double mapping cylinder resulting from the above construction is P^{m+n-1}.

The double cover of the map $X \to P^\infty$ is the double mapping cylinder of

$$\begin{tikzcd} & S^9 \arrow[dl, "\eta^2"'] \arrow[dr, "\eta^2"] & \\ S^7 & & S^7 \end{tikzcd}$$

The group $\mathbb{Z}/2$ acts antipodally on S^9 and interchanges the two copies of S^7. Representatives of the maps η^2 can be chosen so as to admit such an action. The space Y is obtained from the join of this double mapping cylinder with itself by passing to the orbit space of the $\mathbb{Z}/2$ action. This join can be described as the homotopy limit of the diagram

$$\begin{tikzcd}
S^{15} & S^{17} \arrow[l] \arrow[r] & S^{15} \\
S^{17} \arrow[u] \arrow[d] & S^{19} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & S^{17} \arrow[u] \arrow[d] \\
S^{15} & S^{17} \arrow[l] \arrow[r] & S^{15}
\end{tikzcd}$$

(all maps are η^2) .

It is the quotient of the disjoint union of $S^{19} \times I^2$, four copies of $S^{17} \times I$, and four copies of S^{15} by the identifications suggested by the diagram. The group $\mathbb{Z}/2$ acts antipodally on S^{19}, sends each S^{17} to the diametrically opposed S^{17} via the antipodal map, and sends each S^{15} to the diametrically opposed S^{15} via the identity map. The 12 representatives of η^2 can be chosen so as to admit such an action.

The inclusion $P^9 \to X$ results in an inclusion $P^{19} \to Y$. It is represented by the $\mathbb{Z}/2$-equivariant inclusion of S^{19} into the centre of the diagram. The obstruction to extending this over P^{20} is this same inclusion considered as a free map. But this is evidently homotopic to the map $\eta^2 \circ \eta^2 : S^{19} \to S^{15}$, with range any one of the corners. Since η^4 is null-homotopic this obstruction vanishes. This proves that the inclusion $P^{20} \to P^{\infty}$ factors (up to homotopy) through Y.

Appendix A. Smashing the Connective Covers of bo .

In this appendix we calculate the simplicities of the summands which result when one smashes together connective covers of bo . This is done using the Adams spectral sequence.

The mod-2 Adams spectral sequence for a spectrum of the form $X \wedge bo$ has for its E_2-term $\mathrm{Ext}_{A_1}^{**}(H^*(X);Z/2)$, where $A_1 \subset A$ is generated by S_q^1 and S_q^2 and $H^*(X)$ is cohomology with $Z/2$ coefficients. We need to define a notion of <u>simplicity</u> for A_1-modules of the form $H^*(X)$ which will imply simplicity for $X \wedge bo$ under favorable circumstances. Although a perfectly good homotopy theory exists for the category of A_1-modules, and the definitions of section 1.7 do carry directly over, it is simpler to base a notion of simplicity on Lemma 1.7.12.

Let $A_0 \subset A$ be the subalgebra generated by S_q^1 , and let h_0 be the non-zero element of $\mathrm{Ext}_{A_1}^{1,1}(Z/2,Z/2)$ $\approx \mathrm{Ext}_{A_0}^{1,1}(Z/2,Z/2)$.

<u>Definition A.1</u>. An A_1-module M is said to be <u>n-simple</u> if the following two conditions are satisfied:

i) For $t-s \leqslant n$, the h_0-torsion of $\mathrm{Ext}_{A_1}^{s,t}(M,Z/2)$ is annihilated by h_0 ;

ii) For $t-s \leqslant n$ the "reduced Hurewicz homomorphism" $\mathrm{Ext}_{A_1}^{s,t}(M,Z/2) \otimes_{Z/2[h_0]} Z/2 \longrightarrow \mathrm{Ext}_{A_0}^{s,t}(M,Z/2) \otimes_{Z/2[h_0]} Z/2$ a monomorphism.

<u>Proposition A.2</u>: Suppose that the Adams spectral sequence for $X \wedge bo$ satisfies $E_2^{s,t} \approx E_\infty^{s,t}$ for $t-s \leqslant n$. If the A_1-module $H^*(X)$ is n-simple then $X \wedge bo$ is n-simple.

<u>proof</u>: The conditions ensure satisfaction of the conditions of Lemma 1.7.12.

<u>Remark</u>: In all of the cases under consideration the appropriate Adams spectral sequence collapses for elementary reasons.

One advantage of working with the simplicity of a module M is that it depends only on the stable class of M.

<u>Definition</u>. ([3], [4] and [19]) Two A_1-modules M and N are said to be stably equivalent (written $M \approx_s N$) if there exist projective modules P_1 and P_2 with $M \oplus P_1 \approx N \oplus P_2$.

Direct sums and tensor products preserve stable equivalence.

<u>Proposition A.3</u>. If $M \approx_s N$ and M is n-simple then N is n-simple.

<u>proof</u>: This is clear.

We need two standard lemmas from homological algebra.

Lemma A.4. Let $R_1 \rightarrowtail P_n \to \cdots \to P_1 \twoheadrightarrow M$ and $R_2 \rightarrowtail P'_n \to \cdots \to P'_1 \twoheadrightarrow M$ be two exact sequences with the P_i and P'_i projective. Then $R_1 \approx_s R_2$.

Lemma A.5. Let $\alpha : R \rightarrowtail P_n \to \cdots \to P_1 \twoheadrightarrow M$ be exact with the P_i projective. Then multiplication by the Yoneda class of α induces an isomorphism $\mathrm{Ext}^s_{A_1}(R,Z/2) \approx \mathrm{Ext}^{s+n}_{A_1}(M,Z/2)$ for $s > 0$ and an epimorphism when $s = 0$.

The simplicities of the iterated tensor products $M_1^i \otimes M_2^j \otimes M_4^k$ with $i \leq k$ can now be determined. First we will determine their stable classes. The resulting Ext "charts" can then be extracted from [4] -- though the cases we consider are easily enough calculated.

Let I denote the desuspension of the augmentation ideal of A_1. I is (-1)-connected but not 0-connected.

Lemma A.6. There are stable equivalences $\Sigma^2 M_4 \simeq_s M_2 \otimes I$, $M_2 \otimes M_2 \simeq_s \Sigma^4 Z/2$ and $M_1 \otimes M_4 \simeq_s \Sigma^2 Z/2$.

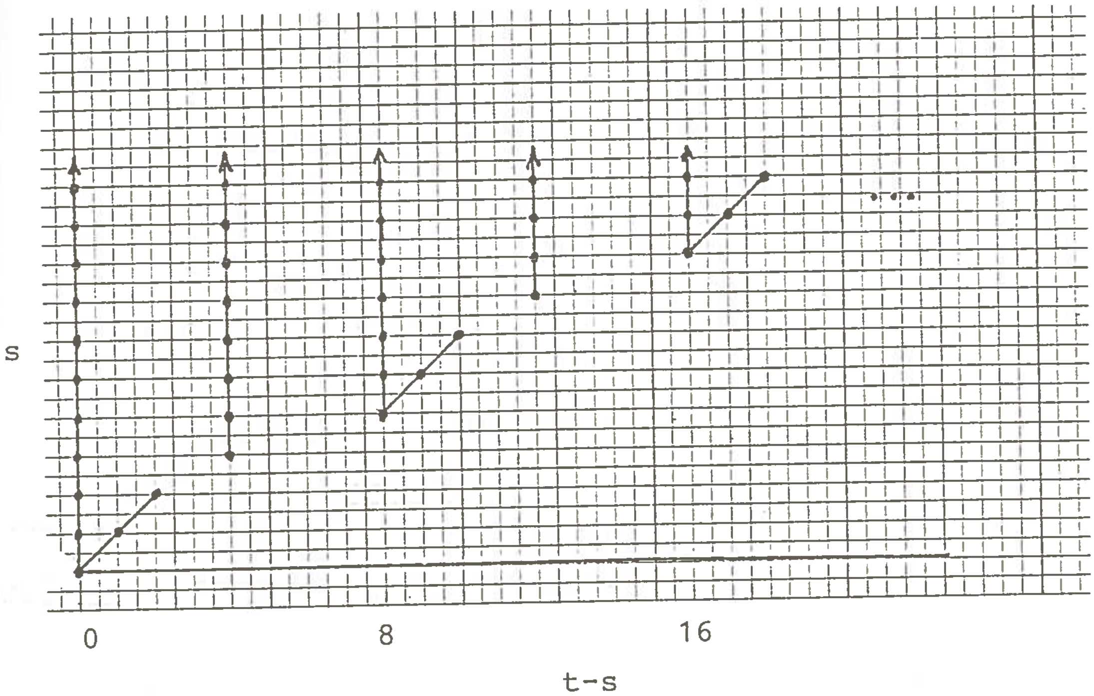
proof: The first follows from the exact sequences $\Sigma^3 M_4 \rightarrowtail A_1 \twoheadrightarrow M_2$ and $\Sigma I \otimes M_2 \rightarrowtail A_1 \otimes M_2 \twoheadrightarrow M_2$ by Lemma A.4. The remaining equivalences are contained in Corollary 1.7.7.

It follows that the A_1-modules $M_1^i \otimes M_2^j \otimes M_4^k$ with $i \leq k$ are stably isomorphic to modules of the form $\Sigma^a I^b$ or $\Sigma^a I^b \otimes M_2$. We therefore need only determine the simplicities of I^n and $I^n \otimes M_2$.

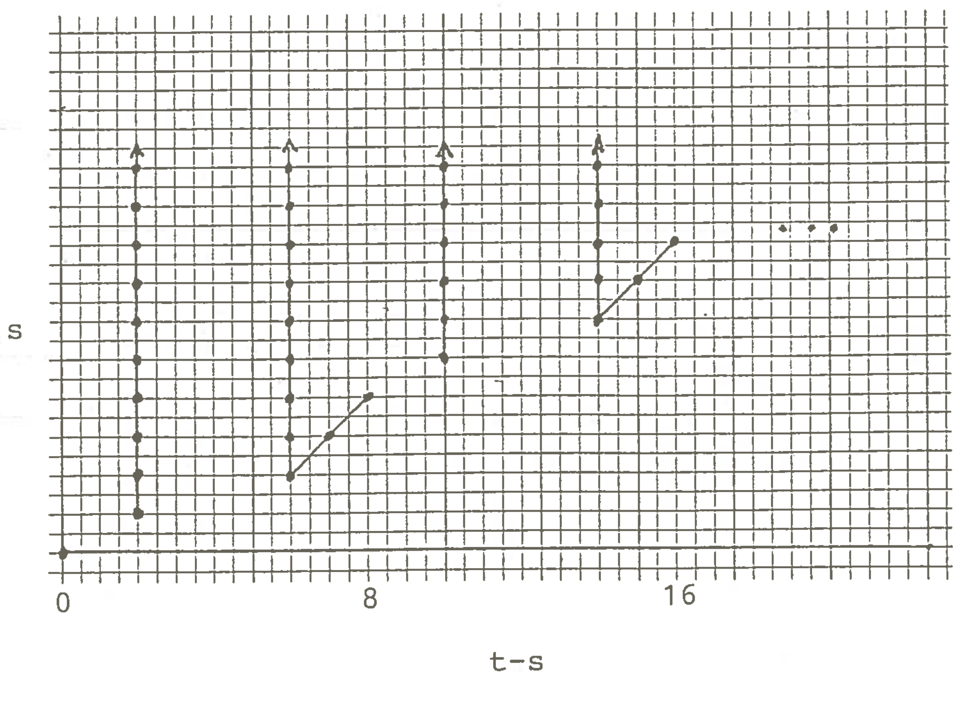
<u>Proposition A.7</u>. Write $n = 4k + r$ and $n + 2 = 4\ell + s$ with $0 \leq r,s \leq 3$. Then I^n has simplicity $8k + 2^r - 1$ and $I^n \otimes M_2$ has simplicity $8\ell + 2^s - 3$.

<u>proof</u>: From Lemma A.5 one knows that $\mathrm{Ext}_{A_1}^{s,t}(I^n, Z/2) \approx \mathrm{Ext}_{A_1}^{s+n,t+n}(Z/2, Z/2)$, and that $\mathrm{Ext}_{A_1}^{s,t}(I^n \otimes M_2, Z/2) \approx \mathrm{Ext}_{A_1}^{s+n,t+n}(M_2, Z/2)$ for $s > 0$. In terms of the usual $(t-s,s)$ display one shifts ones gaze up n units from $\mathrm{Ext}_{A_1}^{s,t}(Z/2,Z/2)$ and $\mathrm{Ext}_{A_1}^{s,t}(M_2,Z/2)$. These charts are displayed on the following page. The result then follows from Proposition A.3 and the definition. The fact that the h_0-"towers" originate at $s = 0$ is a consequence of the "epimorphism" part of Lemma A.5.

<u>Corollary A.8</u>. For j = 0,1,2,4 $N_j \wedge N_4 \wedge bo$ is 3-simple.



$\mathrm{Ext}^{st}_{A_1}(\mathbb{Z}/2,\ \mathbb{Z}/2)$



$\mathrm{Ext}^{st}_{A_1}(M_2, \mathbb{Z}/2)$

We need one more fact from connective K-theory [17], [18].

Theorem A.9. Localized at 2 there is an equivalence

$$bo \wedge bo \approx (S^0 \vee \Sigma^4 N_4 \vee Z) \wedge bo$$

where Z is 7-connected.

Lemma A.10. For $n \equiv 0,1,2$ or 4 (8) there is a splitting

$$bo\langle n\rangle \wedge \overset{j}{\Lambda}\, bo \approx bo\langle n\rangle \vee Y$$

where Y is $(n+7)$-simple.

Proof: Use Corollary 1.7.4, Corollary A.8, Theorem A.9 and induction.

We can now proceed with the calculations needed for section 1.8. Rather than do each case used in 1.8 we will work two illustrative examples. Verification of the remaining simplicity estimates will be left to the reader.

Example 1. Consider $bo\langle 26\rangle \wedge bo\langle 26\rangle \wedge bo\langle 26\rangle \wedge bo\langle 26\rangle$. We have $\overset{4}{\Lambda}\, bo\langle 26\rangle \simeq \Sigma^{104} N_2 \wedge \overline{N}_2 \wedge N_2 \wedge \overline{N}_2\ \overset{9}{\Lambda}\, bo$. By Corollary 1.7.7 $N_2 \wedge \overline{N}_2 \wedge N_2 \wedge \overline{N}_2 \approx S^8 \vee V$ where $H^*(V;\mathbb{Z}/2)$ is A_1 free. It follows that $\Sigma^{104} N_2 \wedge \overline{N}_2 \wedge N_2 \wedge \overline{N}_2\ \overset{4}{\Lambda}\, bo \approx \Sigma^{104} S^8\ \overset{4}{\Lambda}\, bo \vee HU$ where the latter summand is a wedge of mod-2 Eilenberg-MacLane spectra. It cannot contribute to simplicity. Appealing to Lemma A.10 therefore yields a splitting $\overset{4}{\Lambda}\, bo\langle 26\rangle \approx bo\langle 112\rangle \vee Y$ where Y is $112 + 7 = 119$ - simple.

<u>Example 2</u>. Consider $bo\langle 10\rangle \wedge bo\langle 12\rangle \wedge bo\langle 12\rangle \wedge bo\langle 12\rangle$ $\simeq \Sigma^{46} N_2 \wedge N_4 \wedge N_4 \wedge N_4 \overset{4}{\wedge} bo$. We approximate this with $\Sigma^{46} N_2 \wedge N_4 \wedge N_4 \wedge N_4 \wedge bo$. The module $M_2 \otimes M_4 \otimes M_4 \otimes M_4$ is stably isomorphic to $\Sigma^{-6} I^3 \otimes M_2^4 \simeq_s \Sigma^2 I^3$ which is 9-simple. It then follows from Proposition A.2 and Lemma 1.7.11 that $\Sigma^{46} N_2 \wedge N_4 \wedge N_4 \wedge N_4 \wedge bo$ is $46 + 9 = 55$ - simple. Smashing with further copies of bo doesn't change the simplicity by Lemma 1.7.11. It follows that $bo\langle 10\rangle \wedge bo\langle 12\rangle \wedge bo\langle 12\rangle \wedge bo\langle 12\rangle$ is 55-simple.

Chapter 2

Homotopy Coverings and Co-coverings

2.1 Introduction

The Lusternik-Schnirelmann category of a space X is usually defined to be one less than the minimum number of open subsets U_i it takes to cover X with the inclusions $U_i \to X$ null homotopic (this is a different normalization from that used in Chapter 1 where, for example, contractible spaces had category 1). As a homotopy invariant it is associated with cohomological nilpotence. It was Ganea ([16]) who first formulated category in such a way as to be able to apply Eckmann-Hilton duality. He was then able to define a notion of "cocategory" which bore a similar relation to homotopical nilpotence.

To define category Ganea constructed (for a path-connected pointed topological space X) a sequence of spaces B_iX inductively, starting with $B_0X = PX \to X$ and obtaining $B_{i+1}X$ as the mapping cone $B_iX \cup CF_iX$, where $F_iX \to B_iX$ is defined by the fibre sequence $F_iX \to B_iX \to X$. These spaces can be pieced together to form a sequence $B_0X \to B_1X \to \ldots$ filtering the homotopy type of X. Ganea then defined cat $X \leq n$ if $B_nX \to X$ admits a right homotopy inverse. He was able to show that this definition agrees with the original one in suitable cases (when X is path connected, paracompact, and say locally contractible).

Ganea's formulation of cocategory was the dual of this. Beginning with $X \to CX = B^0X$, and having defined $X \to B^iX$, one obtains $B^{i+1}X$ as the homotopy fibre of $B^iX \to B^iX/X$.

$$\begin{tikzcd}
 & B^{i+1}X \arrow[d] & \\
X \arrow[ur, dashed] \arrow[r] & B^iX \arrow[r] & B^iX/X
\end{tikzcd}$$

The B^iX fit together to form a tower. Ganea defined cocat $X \leq n$ if $X \to B^nX$ admits a left homotopy inverse.

One unsatisfactory feature of this formulation is its lack of resemblance to the original definition of category. Consequently, whatever it is that cocategory measures about a space is somewhat obscure.

The Lusternik-Schnirelmann category of a space is a homotopy invariant. One might expect that it could be defined using only properties of the category of spaces modulo weak homotopy equivalences. This is indeed the case and will be done in this chapter by equipping the homotopy category with a notion of covering. It can in fact be shown that these "homotopy coverings" satisfy the axioms of a Grothendieck topology, though some care must be taken since the homotopy category does not in general admit limits. We will not pursue this matter here (though see Corollary 2.3.3). In the language of homotopy coverings the cocategory of a space X becomes the minimum number of spaces required to write X as a "homotopy intersection" of contractible spaces. Hopefully this will shed some light on the meaning of cocategory.

Ganea's filtration of X by the B_iX is the Milnor filtration of X regarded as the classifying space of its loop space. There is a close relationship between the construction of this filtration and the loop space structure on ΩX. The ideas of this chapter allow one to generalize this to a filtration of an (n-1)-connected space constructed from spaces weakly equivalent to products of its n-fold loop space, using only properties of the multiplication. There results a characterisation of iterated loop spaces which can be dualized to a characterisation of iterated suspensions. Along with these characterisations come spectral sequences whose edge homomorphisms are the iterated homotopy and homology suspensions. These latter topics will be treated in Chapter 3, though in section 2.6 various descriptions of these new filtrations are given.

2.2 Homotopy limits and colimits

Formulation of the notion of homotopy covering requires the theory of homotopy limits ([6], [35]) and the framework of homotopical algebra ([25]). In this section we recall the material needed from these areas. For further information the reader is referred to [6], [13] and [25].

Let $\underline{S}$ be the category of simplicial sets and let S^A and $[A,\underline{S}]$ both denote the category of functors from a small category A to $\underline{S}$. An object of $\underline{S}^A$ is called an A-diagram of simplicial sets. When it will cause no confusion a single symbol will be used to denote both a category and its nerve.

We will have occasion to consider categories of contravariant functors from A to $\underline{S}$. Rather than reserve a special notation, the symbols $[A,\underline{S}]$ and $\underline{S}^A$ will be used to denote these categories as well. It will be clear from context which category is meant. The results of this section are true with only trivial modifications in the case of categories of contravariant functors (replace the domain category by its opposite). These remarks apply especially to sections 2.3-2.6.

For the remainder of this work the term "space" will mean simplicial set. Eventually our definitions will depend on functors between the categories obtained from certain diagram categories by inverting the weak equivalences. Since the categories of A-diagrams of spaces and A-diagrams of simplicial sets become isomorphic after inverting the weak equivalences ([25]), these definitions carry over to the category of topological spaces. A further discussion appears in section 2.4.

Finally, it will be convenient to adopt some notation from [10] and use the symbol

$$A_1,\ldots,A_n{}^F B_1,\ldots,B_m$$

to indicate that F is a functor from

$$A_1 \times \ldots \times A_n \times B_1^{op} \times \ldots \times B_m^{op}$$

to $\underline{S}$.

<u>Definition</u>. Let X and Y be functors from A to $\underline{S}$ (i.e. the situation $({}_A X, {}_A Y)$). The function complex $\hom_A[X,Y]$ is the simplicial set whose n-simplices are the natural transformations from $\Delta(n) \times X$ to Y. It is the equalizer of the diagram

$$\prod_a (Y_a)^{X_a} \rightrightarrows \prod_{a \leftarrow a'} (Y_a)^{X_{a'}} .$$

<u>Definition</u>. Let X and Y be contravariant and covariant functors from A to $\underline{S}$ respectively (i.e. the situation $(X_A, {}_A Y)$). The <u>balanced product</u> $X \times_A Y$ is the co-equalizer of the diagram

$$\coprod_a X_a \times Y_a \rightleftarrows \coprod_{a \leftarrow a'} X_a \times Y_{a'} .$$

The function complex and balanced product functors satisfy a strong form of adjointness. The proof of the following proposition is straightforward.

Proposition 2.2.1. In the situation $({}_A X, {}_B Y_A, {}_B Z)$ there is a natural (in X,Y and Z) isomorphism of spaces

$$\hom_A[X, \hom_B[Y,Z]] \approx \hom_B[Y \times_A X, Z] .$$

Examples. i) Suppose A is a category with a single object a, whose morphisms form a group G. A covariant functor from A to $\underline{S}$ is a left G space and a contravariant functor from A to S is a right G-space. The balanced product $X \times_A Y$ is the orbit space $Xa \times_G Ya$ and the function complex $\hom_A[X,Y]$ is the space of equivariant maps.

ii) Let $* : A \to \underline{S}$ denote the functor which sends all maps to the identity map of a point. It can be viewed as either contravariant or covariant. In this case one can make the identifications

$$* \times_A X = \lim_{\to A} X$$

$$\hom_A[*, X] = \lim_{\leftarrow A} X .$$

We wish to introduce "homotopy" versions of the balanced products and function complexes. Given the situation $(X_A, {}_A Y)$ the obvious way to obtain a "homotopy" version of $X \times_A Y$ would be to replace the colimit in the definition with the diagonal of the simplicial space

$$\coprod_{a_0} Xa_0 \times Ya_0 \leftleftarrows \coprod_{a_0 \leftarrow a_1} Xa_0 \times Ya_1 \; \substack{\leftarrow\\\leftarrow\\\leftarrow} \; \coprod_{a_0 \leftarrow a_1 \leftarrow a_2} Xa_0 \times Ya_2 \cdots .$$

Similarly, given $({}_A X, {}_A Y)$ the obvious homotopy function complex would be obtained as the total complex ([6]) of the co-simplicial space

$$\prod_{a_0} \hom[Xa_0,Ya_0] \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} \prod_{a_0\leftarrow a_1} \hom[Xa_1,Ya_0] \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} \prod_{a_0\leftarrow a_1\leftarrow a_2} \hom[Xa_2,Ya_0] \cdots .$$

These are indeed the correct notions, though it is more convenient to formulate them a bit differently.

Given an object a of A, let A/a denote the category of objects over a. This is the category whose objects are the maps $a \leftarrow a_0$ and whose morphisms are the sequences $a \leftarrow a_0 \leftarrow a_1$. By taking nerves, one obtains a covariant functor $A/- : A \to \underline{S}$.

Dually, there is a contravariant functor $A\backslash - : A \to \underline{S}$ obtained by forming the category of objects under a given object.

Definition ([6]). Let X be a functor from A to $\underline{S}$.

i) The homotopy direct limit of X, $\underset{\longrightarrow A}{\text{holim}}\, X$ (or just $\underset{\longrightarrow}{\text{holim}}\, X$), is the space $A\backslash - \times_A X$.

ii) The homotopy inverse limit of X, $\underset{\longleftarrow A}{\text{holim}}\, X$ (or just $\underset{\longleftarrow}{\text{holim}}\, X$), is the space $\hom_A[A/-,X]$.

Examples. i) Consider again the case in which A has one object and the morphisms from a group G. In this case the functors A/- and A\- correspond to free acyclic left and right G-spaces respectively. Call them both EG. If X is a left G space regarded as an object of $\underline{S}^A$, then $\underset{\longrightarrow}{\text{holim}}\, X$ is the homotopy theoretic orbit space $EG \times_G X$, and $\underset{\longleftarrow}{\text{holim}}\, X$ is the homotopy theoretic fixed point space $\hom_G[EG,X]$.

ii) If A is the category $\cdot \to \cdot$ and a functor X is given by the map $f : U_0 \to U_1$ then $\underrightarrow{\mathrm{holim}}\, X$ is the mapping cylinder of f.

iii) If A is the category

$$\begin{tikzcd} \cdot \arrow[dr] & & \cdot \arrow[dl] \\ & \cdot & \end{tikzcd}$$

and a functor $X : A \to \underline{S}$ is given by the diagram

$$\begin{tikzcd} U \arrow[dr] & & V \arrow[dl] \\ & W & \end{tikzcd},$$

then $\underleftarrow{\mathrm{holim}}\, X$ is the usual homotopy theoretic fibre product $U \times_W W^I \times_W V$, where I is the nerve of A.

iv) If A has a terminal object a then the natural map $Xa \to \underrightarrow{\mathrm{holim}}\, X$ is a weak homotopy equivalence. Similarly, if A has an initial object a, then the natural map $\underleftarrow{\mathrm{holim}}\, X \to Xa$ is a weak homotopy equivalence.

Just as ordinary direct and inverse limits are adjoints to certain "constant" functors, so are homotopy direct and inverse limits.

<u>Proposition 2.2.2</u> ([6]). i) The functor $\underrightarrow{\mathrm{holim}} : \underline{S}^A \to \underline{S}$ is left adjoint to the functor $\hom[A\backslash -, -]$.

ii) The functor $\underleftarrow{\mathrm{holim}} : \underline{S}^A \to S$ is right adjoint to the functor $A/- \times -$.

Proposition 2.2.2 is a special case of a more general, stronger adjunction (Proposition 2.2.5). Before stating it we need to introduce more "homotopy" notions.

Let $X : \underline{A} \to \underline{S}$ be a functor. For each object a of $\underline{A}$, X determines functors (also called X) from $\underline{A}/a$ and $\underline{A}\backslash a$ to $\underline{S}$. Define new functors $\tilde{X}$ and $\underset{\sim}{X}$ by the formulae

$$\tilde{X}a = \underset{\longrightarrow \underline{A}/a}{\mathrm{holim}}\ X$$

$$\underset{\sim}{X}a = \underset{\longleftarrow \underline{A}\backslash a}{\mathrm{holim}}\ X \ .$$

If X is contravariant set $\tilde{X}a = \underset{\longrightarrow \underline{A}\backslash a}{\mathrm{holim}}\ X$. There are natural transformations $X \to \tilde{X}$ and $\underset{\sim}{X} \to X$ which induce weak homotopy equivalences $Xa \to \tilde{X}a$ and $\underset{\sim}{X}a \to Xa$ for all objects a of $\underline{A}$.

The functors $\tilde{\ }$ and $\underset{\sim}{\ }$ satisfy a strong form of adjointness.

Lemma 2.2.3. i) In the situation $({}_{\underline{A}}X, {}_{\underline{A}}Y)$ there is a natural (in X and Y) isomorphism of function complexes

$$\mathrm{hom}[X,\underset{\sim}{Y}] \approx \mathrm{hom}[\tilde{X},Y] \ .$$

ii) In the situation $(X_{\underline{A}}, {}_{\underline{A}}Y)$ there is a natural (in X and Y) isomorphism of balanced products

$$\tilde{X} \times_{\underline{A}} Y \approx X \times_{\underline{A}} \tilde{Y} \ .$$

Proof: The proof is a straightforward calculation. We sketch the details since it involves a useful technique. Given a simplicial space X construct another simplicial space PX by setting $(PX)_n = X_{n+1}$ and by omitting the last face and degeneracy maps. The omitted face maps give rise to a diagram

$$PPX \rightrightarrows PX \to X$$

which is easily seen to represent X as a co-equalizer. Taking diagonals results in a representation of the diagonal of X as a co-equalizer. The choice of the functor P(-) is not unique and another representation of the diagonal of X as a co-equalizer can be obtained by using all but the zeroth face and degeneracy maps of X to form PX. There is a dual construction CX for a cosimplicial space X which results in a representation of the total complex Tot X as an equalizer

$$\mathrm{Tot}\, X \to \mathrm{Tot}\, CX \rightrightarrows \mathrm{Tot}\, CCX$$

in two ways. Applying these constructions to the simplicial and cosimplicial spaces

$$\coprod_{a_0} Xa_0 \times Ya_0 \;\overset{\leftarrow}{\underset{\leftarrow}{\to}}\; \coprod_{a_0 \leftarrow a_1} Xa_0 \times Ya_1 \; \cdots$$

and

$$\prod_{a_0} \hom[Xa_0, Ya_0] \;\overset{\to}{\underset{\to}{\leftarrow}}\; \prod_{a_0 \leftarrow a_1} \hom[Xa_1, Ya_0] \; \cdots$$

gives the desired result.

Definition. i) Given $({}_AX, {}_AY)$, the homotopy function complex $\underset{\sim}{\hom}_A[X,Y]$ is the space $\hom_A[X,\underset{\sim}{Y}] = \hom_A[\tilde{X},Y]$.

ii) Given $(X_A, {}_AY)$ the homotopy balanced product $X \tilde{\times}_A Y$ is the space $\tilde{X} \times_A Y = X \times_A \tilde{Y}$.

Example. Let $* : A \to \underline{S}$ be the functor which sends all maps to the identity map of a point. The functor $\tilde{*}$ is the functor A/- if $*$ is regarded as being covariant, and it is the functor $A\backslash-$ if $*$ is regarded as being contravariant. It follows that there are isomorphisms

$$\underrightarrow{\mathrm{holim}}\; X = \tilde{*} \times_A X = * \mathbin{\tilde{\times}_A} X = * \times_A \tilde{X}$$

$$\underleftarrow{\mathrm{holim}}\; X = \mathrm{hom}_A[\tilde{*},X] = \underset{\sim}{\mathrm{hom}}_A[*,X] = \mathrm{hom}_A[*,\underset{\sim}{X}] \; .$$

In particular the homotopy inverse and direct limits of X are the ordinary inverse and direct limits of $\tilde{X}$ and $\underset{\sim}{X}$ respectively.

The homotopy function complexes and balanced products satisfy a strong form of "homotopy" adjointness.

<u>Proposition 2.2.4</u>. Consider the situation $({}_AX, {}_BY_A, {}_BZ)$. There is a natural (in X,Y and Z) isomorphism of homotopy function complexes

$$\underset{\sim}{\mathrm{hom}}_A[X,\underset{\sim}{\mathrm{hom}}_B[Y,Z]] = \underset{\sim}{\mathrm{hom}}_B[Y \mathbin{\tilde{\times}_A} X,Z] \; .$$

<u>Proof</u>:

$$\begin{aligned}\underset{\sim}{\mathrm{hom}}_A[X,\underset{\sim}{\mathrm{hom}}_B[Y,Z]] &= \mathrm{hom}_A[\tilde{X},\mathrm{hom}_B[Y,\underset{\sim}{Z}]] \\ &= \mathrm{hom}_B[Y \times_A \tilde{X},\underset{\sim}{Z}] \\ &= \underset{\sim}{\mathrm{hom}}_B[Y \mathbin{\tilde{\times}_A} X,Z] \; .\end{aligned}$$

Next we need to discuss Kan extensions. Let $f : A \to B$ be a functor between small categories. Composition with f gives a functor $\mathrm{res}_f : S^B \to S^A$. This functor has both a left and a right adjoint as we shall see.

Let $\Delta[A,B] : A^{op} \times B \to \underline{S}$ be the functor $(a,b) \to \hom[fa,b]$ and let $\Delta[B,A] : B^{op} \times A \to \underline{S}$ be the functor $(b,a) \to \hom[b,fa]$. It is easy to see that

$$\Delta[B,A] \times_B X = \mathrm{res}_f X : A \to \underline{S}$$

and

$$\hom_B[\Delta[A,B],X] = \mathrm{res}_f X : A \to \underline{S} .$$

Indeed, the coequalizer diagram defining $\Delta[B,A] \times_B X$ is the same as that defining $\underrightarrow{\lim}_{B/f-} X = X(f-)$, and the equalizer diagram defining $\hom_B[\Delta[A,B],X]$ is the same as that defining $\underleftarrow{\lim}_{B\backslash f-} X = X(f-)$.

<u>Definition</u>. Let $f : A \to B$ be a functor between small categories.

i) The <u>left Kan extension</u> of f is the functor $\Delta[A,B] \times_A - : \underline{S}^A \to \underline{S}^B$.

ii) The <u>right Kan extension</u> of f is the functor $\hom_A[\Delta[B,A],-] : \underline{S}^A \to \underline{S}^B$.

By Proposition 2.2.1 the right and left Kan extensions of f are right and left adjoints of res_f respectively.

<u>Definition</u>. Let $f : A \to B$ be a functor between small categories.

i) The <u>left homotopy Kan extension</u> $\underrightarrow{\mathrm{ind}}_f : \underline{S}^A \to \underline{S}^B$ is given by

$$\underrightarrow{\mathrm{ind}}_f X = \Delta[A,B] \tilde{\times}_A X .$$

The <u>right homotopy restriction</u> $\underset{\leftarrow}{\mathrm{res}}_f : \underline{S}^B \to \underline{S}^A$ is given by

$$\underset{\leftarrow}{\mathrm{res}}_f X = \utilde{\mathrm{res}_f} X \ .$$

ii) The <u>right homotopy Kan extension</u> $\underset{\leftarrow}{\mathrm{ind}}_f : \underline{S}^B \to \underline{S}^A$ is given by

$$\underset{\leftarrow}{\mathrm{ind}}_f X = \utilde{\mathrm{hom}}_A[\Delta[B,A],X] \ .$$

The <u>left homotopy restriction</u> $\underset{\rightarrow}{\mathrm{res}}_f : \underline{S}^B \to \underline{S}^A$ is given by

$$\underset{\rightarrow}{\mathrm{res}}_f X = \widetilde{\mathrm{res}_f} X \ .$$

<u>Proposition 2.2.5</u>. Let $f : A \to B$ be a functor between small categories. There are natural isomorphisms

$$\mathrm{hom}_B[X, \underset{\leftarrow}{\mathrm{res}}_f Y] = \mathrm{hom}_A[\underset{\rightarrow}{\mathrm{ind}}_f X, Y]$$

$$\mathrm{hom}_B[X, \underset{\leftarrow}{\mathrm{ind}}_f Y] = \mathrm{hom}_A[\underset{\rightarrow}{\mathrm{res}}_f X, Y]$$

of function complexes.

<u>Proof</u>:

$$\begin{aligned} \mathrm{hom}_B[X, \underset{\leftarrow}{\mathrm{res}}_f Y] &= \mathrm{hom}_B[\tilde{X}, \mathrm{res}_f Y] \\ &= \mathrm{hom}_B[\tilde{X}, \mathrm{hom}_A[\Delta[B,A],Y]] \\ &= \mathrm{hom}_A[\Delta[B,A] \times_B \tilde{X}, Y] \\ &= \mathrm{hom}_A[\underset{\rightarrow}{\mathrm{ind}}_f X, Y] \ . \end{aligned}$$

The other isomorphism follows similarly.

Proposition 2.2.2 is the special case $B = *$ of the above proposition.

As with homotopy limits, the right homotopy Kan extension of X is the ordinary right Kan extension of $\underset{\approx}{X}$ and the left homotopy Kan extension of X is the ordinary left Kan extension of $\tilde{X}$.

<u>Examples</u>. i) Let A and B be categories with one object whose morphisms form groups H and G respectively. Suppose $f : A \to B$ corresponds to an inclusion $H \subset G$. The functor $\Delta[A,B]$ is the set G regarded as a left G-space and as a right H-space. Given an H-space X (regarded as a functor $A \to \underline{S}$), the space $\tilde{X}$ is easily seen to be $EH \times X$ where EH is a free acyclic left H space. The space $\underset{\rightarrow}{\mathrm{ind}}_f X$ is the left G-space $G \times_H (EH \times X)$.

ii) Let $f : A \to B$ be the inclusion

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \to \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad .$$

A functor $X : A \to \underline{S}$ is represented by a diagram

$$\begin{tikzcd} & V \arrow[d, "j"] \\ U \arrow[r, "i"'] & W \end{tikzcd} \quad .$$

Given such an X, the functor $\underset{\rightarrow}{\mathrm{ind}}_f X$ is represented by the diagram

$$\begin{tikzcd}
\phi \arrow[r] \arrow[d] & V \arrow[d] \\
U \to (U \times I) \underset{i}{\cup} W \underset{j}{\cup} (I \times V) &
\end{tikzcd} .$$

The functor $\underset{\leftarrow}{\mathrm{ind}}_f X$ is represented by the diagram

$$\begin{tikzcd}
U \times_W W^I \times_W V \arrow[r] \arrow[d] & W^{I_2} \times_W V \arrow[d] \\
U \times_W W^{I_1} \arrow[r] & W
\end{tikzcd}$$

where I_1, I_2 and I are the nerves of the categories $\cdot\!\to\!\cdot$, $\cdot\!\leftarrow\!\cdot$ and $\cdot\!\to\!\cdot\!\leftarrow\!\cdot$ respectively.

iii) If $f : A \to B$ is the inclusion of a full subcategory, the adjunction maps

$$F \to \underset{\leftarrow}{\mathrm{res}}_f \circ \underset{\to}{\mathrm{ind}}_f F$$

and

$$\underset{\to}{\mathrm{res}}_f \circ \underset{\leftarrow}{\mathrm{ind}}_f F \to F$$

are weak equivalences of functors.

Finally we need to recall part of Quillen's theory of homotopical algebra ([25], [26]).

<u>Definition</u> ([5], [25]). A closed model category consists of a category $\underline{C}$ together with three classes of maps in $\underline{C}$ called fibrations, cofibrations and weak equivalences, satisfying CM1 - CM5 below. A map of $\underline{C}$ is called a trivial fibration (resp. cofibration) if it is a fibration (resp. cofibration) and a weak equivalence.

CM1. $\underline{C}$ is closed under finite limits and colimits.

CM2. For $W \xrightarrow{f} X \xrightarrow{g} Y$ in $\underline{C}$ if any two of f, g and gf are weak equivalences then so is the third.

CM3. If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

CM4. Given a solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration, the dotted arrow exists if either i or j is a weak equivalence.

CM5. Any map f can be factored as $f = pi$ and $f = qj$ with i a trivial cofibration, p a fibration, j a cofibration and q a trivial fibration.

By CM1 $\underline{C}$ has both an initial object ϕ and a terminal object e. An object X of $\underline{C}$ is said to be fibrant if $X \to e$ is a fibration, and cofibrant if $\phi \to X$ is a cofibration. The subcategories $\underline{C}_c, \underline{C}_f$ and $\underline{C}_{cf}$ are the full subcategories of $\underline{C}$ consisting of the cofibrant objects, the fibrant objects, and the objects which are both fibrant and cofibrant, respectively.

Given a closed model category $\underline{C}$ one forms the homotopy category $ho\underline{C}$ by formally inverting the weak equivalences. The usual set theoretic difficulties are overcome since, by CM5 the inclusion $ho\underline{C}_{cf} \to ho\underline{C}$ is an equivalence of categories, and since one can introduce a notion of "homotopy" between maps, with the property that weak equivalences in $\underline{C}_{cf}$ have "homotopy" inverses, so that $ho\underline{C}_{cf}$ is obtained from $\underline{C}_{cf}$ by passage to homotopy classes of maps.

We will refer to the morphisms of $ho(\underline{C})$ as morphisms and to those of $\underline{C}$ as maps. A morphism ϕ is said to be <u>represented</u> by a map f if $\gamma f = \phi$, where $\gamma : \underline{C} \to ho(\underline{C})$ is the localization functor. An arbitrary morphism need <u>not</u> be represented by a map, though one easily checks that morphisms from cofibrant objects to fibrant objects are.

<u>Examples</u>. i) Consider the category $\underline{S}$ of simplicial sets. Define fibrations to be Kan fibrations, cofibrations to be inclusions, and weak equivalences to be maps which are weak homotopy equivalences after passing to the geometric realization. With these definitions the category $\underline{S}$ becomes a closed simplicial model category ([25]).

ii) Let $\underline{T}$ be the category of topological spaces. Define fibrations to be Serre fibrations, weak equivalences to be weak homotopy equivalences, and cofibrations to be the maps which have the left lifting property (CM4) with respect to the class of trivial fibrations. A map $X \to Y$ between topological spaces is said to be <u>free</u> if it is an inclusion and if Y can be obtained from X by the process of attaching cells (not necessarily in increasing order of dimension). It can be shown that the cofibrations are precisely the retracts of the free maps. One can show that cofibrant spaces are paracompact by adapting the usual proof for CW complexes ([24]). With these definitions the category T is a closed model category ([25]).

iii) Let A be a small category. A map $X \to Y$ in $\underline{S}^A$ is said to be a fibration (resp. weak equivalence) if for each object a of A the induced map $Xa \to Ya$ is a fibration (resp. weak equivalence). A map is a cofibration if it has the left lifting property with respect to the class of trivial fibrations. With these definitions $\underline{S}^A$ becomes a closed model category ([13]).

We will also be needing the notion of derived functor which appears in homotopical algebra. Let A,B and C be categories and suppose given a diagram

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\gamma"'] & C \\ B & \end{tikzcd}$$

of functors. The <u>left derived functor</u> of F with respect to γ is a functor $L^\gamma F : B \to C$ together with a natural transformation $L^\gamma F \circ \gamma \to F$ satisfying the following universal property: Given any functor $G : B \to C$ and a natural transformation $G \circ \gamma \to F$, there is a unique natural transformation $G \to L^\gamma F$ so that the diagram

$$\begin{tikzcd} G \circ \gamma \arrow[r] \arrow[d] & F \\ L^\gamma F \circ \gamma \arrow[ur] & \end{tikzcd}$$

commutes. In other words $L^{\gamma}F$ is the functor from B to C such that $L^{\gamma}F \circ \gamma$ is the closest to F from the left. There is a similar definition of the <u>right</u> derived functor of F with respect to γ.

In practice one considers the case where $\gamma : \underline{C} \to ho\underline{C}$ is the localization of a closed model category with respect to its weak equivalences. In this case the left and right derived functors of F with respect to γ will be denoted LF and RF respectively. The following propositions and definition can be found in [25] p.I, 4.2-4.4.

<u>Proposition 2.2.5</u> ([25]). Let $\underline{C}$ be a closed model category and let $F : \underline{C} \to \underline{B}$ be a functor. Suppose that F carries weak equivalences in $\underline{C}_c$ into isomorphisms in $\underline{B}$. Then $LF : ho\underline{C} \to \underline{B}$ exists. Furthermore, the map $LF(X) \to F(X)$ is an isomorphism if X is cofibrant.

<u>Definition</u> ([25]). Let $F : \underline{C} \to \underline{C}'$ be a functor between model categories. The <u>total left derived functor</u> of F is the functor $\underline{\underline{L}}F : ho\underline{C} \to ho\underline{C}'$ given by $\underline{\underline{L}}F = L^{\gamma}(\gamma' \circ F)$ where $\gamma : \underline{C} \to ho\underline{C}$ and $\gamma' : \underline{C}' \to ho\underline{C}'$ are the localization functors.

Of course there is a dual to Proposition 2.2.5 involving the existence of right derived functors and the isomorphism of $F(X) \to RF(X)$ when X is fibrant. There is also a notion of the <u>total right derived functor</u> $\underline{\underline{R}}F$ of a functor $F : C \to C'$ between closed model categories.

Proposition 2.2.6 ([25]). Let $\underline{C}$ and $\underline{C}'$ be closed model categories and let $L : \underline{C} \to \underline{C}'$, $R : \underline{C}' \to \underline{C}$ be a pair of adjoint functors, L being the left and R being the right adjoint. Suppose L preserves cofibrations and carries weak equivalences in $\underline{C}_c$ to weak equivalences in $\underline{C}'$. Suppose also that R preserves fibrations and that R carries weak equivalences in $\underline{C}'_f$ to weak equivalences in $\underline{C}$. Then the total derived functors $\underline{\underline{L}}(L)$ and $\underline{\underline{R}}(R)$ exist and are adjoint.

Let Δ be the category of (isomorphism classes of) finite ordered sets and let $D \subset A$ be the subcategory of surjective maps (i.e. the degeneracies). Given a map $f : X \to Y$ of simplicial sets, the set of simplices of Y not in the image of f does not form a simplicial set but it does form a D-set.

Let A be a small category and let $\mathring{A} \subset A$ be the subcategory consisting only of identity maps. For a category $\underline{C}$ admitting limits, the restriction functor $\underline{C}^A \to \underline{C}^{\mathring{A}}$ has a left adjoint, the left Kan extension. Call and A-diagram of D-sets free if it is the left Kan extension of an $\mathring{A}$-diagram of D-sets. A map $f : X \to Y$ of $\underline{S}^A$ will be called free if the A-diagram of D-sets $Y - f(X)$ is free. A free map in $\underline{S}^A$ is easily seen to be a cofibration. In fact one can show that the cofibrations of $\underline{S}^A$ are precisely the retracts of the free maps ([13]).

Proposition 2.2.7. Let $X \to Y$ be a morphism of $\underline{S}^A$ with property that $Xa \to Ya$ is a cofibration for all objects a of A. The map $\tilde{X} \to \tilde{Y}$ is a free map, hence a cofibration.

Proof: Let $F : \underline{S}^A \to \underline{S}^A$ be the composition of the restriction $\underline{S}^A \to \underline{S}^{\dot{A}}$ with its left Kan extension. The functor F is then a cotriple. For an object X of $\underline{S}^A$ let F_*X be the usual simplicial resolution of X with respect to F. Given an inclusion $X \to Y$, the complement $F_*Y - F_*X$ is a free A-diagram of $D \times D$ sets. Restricting to the diagonal shows that the map $\mathrm{diag}(F_*X) \to \mathrm{diag}(F_*Y)$ is free. However, it follows easily from the definition that the diagonal of F_*X is precisely $\tilde{X}$. This completes the proof.

Corollary 2.2.8. Let $f : A \to B$ be a functor between small categories. The adjoint pairs $(\underset{\rightarrow}{\mathrm{ind}}_f, \underset{\leftarrow}{\mathrm{res}}_f)$ and $(\underset{\rightarrow}{\mathrm{res}}_f, \underset{\leftarrow}{\mathrm{ind}}_f)$ both satisfy the conditions of Proposition 2.2.6.

Proof: The result is immediate from Proposition 2.2.7 and the following results of Bousfield-Kan ([6], Ch.XI, 5.5, 5.6 and Ch.XII, 4.2).

Proposition 2.2.9. Let $f : X \to Y \in \underline{S}^A$ be such that the maps $Xa \to Ya$ are fibrations for every $a \in A$. Then f induces a fibration $\underleftarrow{\mathrm{holim}}\, f : \underleftarrow{\mathrm{holim}}\, X \to \underleftarrow{\mathrm{holim}}\, Y$.

<u>Proposition 2.2.10</u>. Let $f : X \to Y \in \underline{S}^A$ be such that for every $a \in A$ the map $Xa \to Ya$ is a weak equivalence between fibrant objects. Then f induces a weak equivalence $\underleftarrow{\mathrm{holim}}\, f : \underleftarrow{\mathrm{holim}}\, X \to \underleftarrow{\mathrm{holim}}\, Y$.

<u>Proposition 2.2.11</u>. Let $f : X \to Y \in \underline{S}^A$ be such that $fa : Xa \to Ya$ is a weak equivalence for every $a \in A$. Then f induces a weak equivalence

$$\underrightarrow{\mathrm{holim}}_{\,f} : \underrightarrow{\mathrm{holim}}\, X \to \underrightarrow{\mathrm{holim}}\, Y.$$

<u>Remark</u>, It folows from Proposition 2.2.11 that the morphisms $\underline{\underline{L}}\, \underrightarrow{\mathrm{res}}_f X \to \underrightarrow{\mathrm{res}}_{\underline{f}} X$ and $\underline{\underline{L}}\, \underrightarrow{\mathrm{ind}}_f X \to \underrightarrow{\mathrm{ind}}_f X$ are isomorphisms for <u>all</u> X, not just for X cofibrant.

2.3 Homotopy covers

In this section we formulate the notion of a covering up to homotopy. For motivation, suppose one has written $X = U_0 \cup U_1 \cup U_2$. One can represent this by the diagram

$$\begin{tikzcd}
 & & U_0 & & \\
 & U_{01} \arrow[ur] \arrow[ddl] & & U_{02} \arrow[ul] \arrow[ddr] & \\
 & & U_{012} \arrow[ul] \arrow[ur] \arrow[d] & & \\
U_1 & & U_{12} \arrow[ll] \arrow[rr] & & U_2
\end{tikzcd}$$

in which $U_{ij} = U_i \cap U_j$ etc, and the maps are the inclusions. Such a diagram is a contravariant functor from the category of non-empty subsets of $\{0,1,2\}$ to the category of spaces. The condition that X be the union of the U_i is equivalent to the condition that X be the direct limit of this functor.

For a set S let $\underline{C}(S)$ be the category of non-empty subsets of S and inclusions. For any category C let C_+ denote the category obtained from C by adjoining an initial object $+$.

<u>Definition</u>. A <u>homotopy covering</u> of a space X is a contravariant functor $F : \underline{C}(S) \to \underline{S}$ for some set S, together with morphisms $p : \underrightarrow{\mathrm{holim}}\, F \to X$, $s : X \to \underrightarrow{\mathrm{holim}}\, F$ of $\mathrm{ho}(\underline{S})$ such that $p \circ s$ is equal to the identity morphism of X.

Given a homotopy covering F of X we give priority to the spaces $F\{\alpha\}$ $\alpha \in S$ over their "intersections" by saying that the $F\{\alpha\}$ <u>homotopy cover</u> X via the morphisms $F\{\alpha\} \to \underrightarrow{\text{holim}}\, F \xrightarrow{p} X$. We also say that X <u>can be homotopy covered</u> by a family of maps $\{p_\alpha : U_\alpha \to X \mid \alpha \in S\}$ if there exist a homotopy covering $F : \underline{C}(S) \to \underline{S}$ of X, and for each $\alpha \in S$ a commutative diagram

$$\begin{tikzcd}
F\{\alpha\} \arrow[r] & \underrightarrow{\text{holim}}\, F \arrow[d, "p"] \\
U_\alpha \arrow[u, "w"] \arrow[r, "p_\alpha"'] & X
\end{tikzcd}$$

in $ho(\underline{S})$, with $w : U_\alpha \to F\{\alpha\}$ an isomorphism.

<u>Example</u>. A connected space X is a co h-space if and only if it can be homotopy covered by two points (A simplicial set is said to be a co h-space if it can be given the structure of a co-group object in $ho(\underline{S})$). Indeed, the homotopy direct limit of any diagram of the form $* \leftarrow A \to *$ is ΣA, and it is well-known that a space is a co h-space if and only if it is a retract (in ho(S)) of a suspension.

Ordinarily one starts with a collection of maps $\{p_\alpha : U_\alpha \to X\}$ and wishes to know whether or not X can be homotopy covered by it. In practice it is impossible to go through the entire category of spaces looking for intersections to make up a cover. One's natural instinct is to ask if there are universal intersections. There are. Before describing them however, we need to discuss homotopy coverings from a slightly different point of view.

Let $*$ denote the set with one element. The unique map $S \to *$ gives rise to a functor $e : \underline{C}(S)_+ \to \underline{C}(*)_+$. A contravariant functor $F : \underline{C}(S)_+ \to \underline{S}$ with $F(+) = X$ can be thought of as a contravariant functor $F' : \underline{C}(S) \to \underline{S}$ together with a natural transformation to the constant functor at X. The functor $\underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e F$ is isomorphic in $\mathrm{ho}[\underline{C}(*)_+, \underline{S}]$ to $p : \underrightarrow{\mathrm{holim}}\ F' \to X$ for some map p. If in addition p has a right inverse s when regarded as a morphism of $\mathrm{ho}(\underline{S})$, then we have constructed a homotopy covering of X. Accordingly, we shall call a contravariant functor $F : \underline{C}(S)_+ \to \underline{S}$ with $F(+) = X$ a <u>homotopy covering</u> of X if it comes equipped with a right inverse of $\underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e F$ regarded as a morphism of $\mathrm{ho}(\underline{S})$.

$$
\begin{tikzcd}[column sep=small]
U_0 \arrow[dr] & U_{01} \arrow[l] \arrow[r] \arrow[d] & U_1 \arrow[dl] \\
 & X &
\end{tikzcd}
\xrightarrow{\ \underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e\ }
\begin{tikzcd}
U_0 \cup U_{01} \times I \cup U_1 \arrow[d] \arrow[r, dashed, "\equiv"] & U_0 \cup U_{01} \times I \cup U_1 \arrow[d, shift right, "p"'] \\
(U_0 \cup U_{01} \times I \cup U_1) \times I \cup X \arrow[r, dashed, "\approx"'] & X \arrow[u, shift right, "s"']
\end{tikzcd}
$$

Actually, all homotopy coverings arise in this way -- up to isomorphism. Suppose (F', p, s), $F' : \underline{C}(S) \to \underline{S}$, is a homotopy covering of X. By composing p and s with isomorphisms if necessary we may assume X is fibrant. We may also assume that F' is cofibrant so that $\underline{\underline{L}}\ \underrightarrow{\mathrm{holim}}\ F' = \underrightarrow{\mathrm{holim}}\ F'$. In this case there is an actual map $p : \underrightarrow{\mathrm{holim}}\ F' \to X$ representing the morphism p. Since

$\underrightarrow{\text{holim}}\, F' = \varinjlim \tilde{F}'$, this gives rise by adjunction to a map $\tilde{F}' \to X$, where we have used X to denote the constant functor at X. This latter map can be reinterpreted as a contravariant functor $F : \underline{C}(S)_+ \to S$ satisfying $F(+) = X$. This is the desired functor.

For a set S let $\underline{C}(S)^k$ denote the full subcategory of $\underline{C}(S)$ consisting of subsets of cardinality less than or equal to $k+1$. A collection of maps $\{p_\alpha : U_\alpha \to X | \alpha \in S\}$ corresponds to a contravariant functor $G : \underline{C}(S)^0_+ \to \underline{S}$. It is straightforward to check that X can be homotopy covered by $\{p_\alpha : U_\alpha \to X | \alpha \in S\}$ if and only if G is isomorphic in $\text{ho}[\underline{C}(S)^0_+, \underline{S}]$ to $\underline{\underline{L}}\ \underrightarrow{\text{res}}_i\ F$, for some homotopy covering $F : \underline{C}(S)_+ \to \underline{S}$ of X, where $i : \underline{C}(S)^0 \to \underline{C}(S)$ is the inclusion.

More generally, let $I \subset \underline{C}(S)$ be any subcategory and let $i : I_+ \to \underline{C}(S)_+$ be the inclusion. We will say that a contravariant functor $G : I_+ \to \underline{S}$ with $G(+) = X$ <u>can be extended to a homotopy covering</u> of X if it is isomorphic in $\text{ho}[I_+, S]$ to $\underline{\underline{L}}\ \underrightarrow{\text{res}}_i F$ for some homotopy covering F of X.

<u>Proposition 2.3.1</u>. Let $I \subset \underline{C}(S)$ be a full subcategory and let $i : I_+ \to \underline{C}(S)_+$ be the inclusion. A contravariant functor $G : I_+ \to \underline{S}$ with $G(+) = X$ can be extended to a homotopy covering of X if and only if $\underline{\underline{L}}\ \underrightarrow{\text{ind}}_e \circ \underline{\underline{R}}\ \underleftarrow{\text{ind}}_i G \in \text{ho}[\underline{C}(*)_+, \underline{S}]$ admits a right inverse when regarded as a morphism of $\text{ho}(\underline{S})$.

Proof: If $\underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_i G$ admits a right inverse then $\underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_i G$ can be considered a homotopy covering of X. Since $I \subset \underline{C}(S)$ is a full subcategory, the adjunction co-unit $\underline{\underline{L}}\ \underrightarrow{\mathrm{res}}_i \circ \underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_i G \to G$ is an isomorphism. Conversely, suppose that there is an isomorphism $\underline{\underline{L}}\ \underrightarrow{\mathrm{res}}_i F \approx G$ where $F : \underline{C}(S)_+ \to \underline{S}$ is a homotopy covering of X. Taking adjoints and applying $\underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e$ results in a morphism $\underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e F \to \underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_i G$ whose component at the object $+$ is isomorphic to the identity morphism of X. Any right inverse of $\underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e F$ therefore results in a right inverse of $\underline{\underline{L}}\ \underrightarrow{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_i G$. This completes the proof.

In the above proposition one should think of the functor $\underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_i G$ as being obtained from G by using homotopy inverse limits to fill in the missing "intersections".

The special case of Proposition 2.3.1 with I the subcategory $\underline{C}(S)^0$ assumes a particularly manageable form.

Proposition 2.3.2. Suppose $\{p_\alpha : U_\alpha \to X \mid \alpha \in S\}$ is a collection of maps to a fibrant space X. Let $p : E \to B$ be the map obtained by first converting all of the p_α into fibrations, forming their fibre join, and then converting it into a fibration. The space X can be homotopy covered by $\{p_\alpha : U_\alpha \to X \mid \alpha \in S\}$ if and only if p admits a section.

Proof: Let $G : \underline{C}(S)^0_+ \to \underline{S}$ be the functor associated with $\{p_\alpha\}$ and let $i : \underline{C}(S)^0 \to \underline{C}(S)$ be the inclusion. If $\{q_\alpha : E_\alpha \to X \mid \alpha \in S\}$ denotes the result of converting the p_α into fibrations then the functor $\underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_i G$ is isomorphic (in $\mathrm{ho}[\underline{C}(S)_+, \underline{S}]$) to the functor which assigns to a subset $A \subset S$ the fibre product of $\{q_\alpha \mid \alpha \in A\}$ and has the obvious effect on maps. It follows that $\underline{\underline{L}}\ \underset{\rightarrow}{\text{ind}}_e \circ \underline{\underline{R}}\ \underset{\leftarrow}{\text{ind}}_i G$ is isomorphic to $p : E \to X$. But the map p admits a right inverse in $ho(\underline{S})$ if and only if it does so in $\underline{S}$ since it is a fibration between cofibrant-fibrant objects. This completes the proof.

Proposition 2.3.2 can be used to show that homotopy coverings satisfy a form of associativity.

Corollary 2.3.3. Suppose X can be homotopy covered by $\{p_\alpha : U_\alpha \to X | \alpha \in S\}$, and for each $\alpha \in S$, U_α can be homotopy covered by $\{q_\beta : V_\beta \to U_\alpha | \beta \in S_\alpha\}$. Then X can be homotopy covered by $\{p_\alpha \circ q_\beta : V_\beta \to X | \beta \in S_\alpha, \alpha \in S\}$.

Proof: We may assume that all maps have been converted into fibrations. Let $q'_\alpha : E_\alpha \to X$ be the result of forming the fibre join of the $\{p_\alpha \circ q_\beta | \beta \in S\}$ and converting it into a fibration. Since U_α can be homotopy covered by $\{q_\beta : V_\beta \to U_\alpha | \beta \in S_\alpha\}$ the maps $p_\alpha : U_\alpha \to X$ lift through q'_α. It follows that the fibre join of the $\{p_\alpha | \alpha \in S\}$ lifts through the fibre join of $\{q'_\alpha | \alpha \in S\}$. Since the former admits a section after being converted into a fibration so does the latter. Since the formation of fibre joins is associative, the fibre join of $\{q'_\alpha | \alpha \in S\}$ is the same as the fibre join of $\{p_\alpha \circ q_\beta : V_\beta \to X | \beta \in S_\alpha, \alpha \in S\}$. This completes the proof.

Remark. In fact it is possible to prove a stronger result in which one requires that the homotopy covering of the U_α extend to a homotopy covering X. Since we have no need for this stronger result we omit the details.

2.4 The relationship with ordinary coverings

Let $\underline{T}$ denote the category of topological spaces. The singular complex and geometric realization functors provide an equivalence between the categories $ho(\underline{S}^A)$ and $ho(\underline{T}^A)$ whenever A is a small category. Since homotopy coverings can be defined solely in terms of the categories $ho(\underline{S}^A)$, the definition carries over to topological spaces.

Proposition 2.4.1. A cofibrant topological space X can be homotopy covered by a collection of maps $\{p_\alpha : U_\alpha \to X | \alpha \in S\}$ if and only if it can be covered by open subsets V_α, $\alpha \in S$, with the property that for each α the inclusion map $V_\alpha \to X$ factors through p_α up to homotopy.

Proof: We may assume without loss of generality that the p_α are fibrations. It follows from Proposition 2.3.1 (as in the proof of Proposition 2.3.2) that X can be homotopy covered by $\{p_\alpha : U_\alpha \to X | \alpha \in S\}$ if and only if the fibre join of the p_α, $p : E \to X$ admits an inverse in $ho(\underline{T})$. Since p is a fibration and X is cofibrant this is true if and only if p admits a section. The result now follows from a theorem of A.S. Svarc ([34] Prop.2 p.67) which states that a paracompact space X can be covered by open subsets V_α, $\alpha \in S$, over which fibrations $E_\alpha \to X$, $\alpha \in S$, admit sections respectively, if and only if the fibre join of the $\{E_\alpha \to X\}$ admits a section.

2.5 Formulations of category

We can now define Lusternik-Schnirelmann category.

Definition. The category of a space X, cat X, is the least integer n for which X can be homotopy covered by n + 1 points.

Of course this definition carries over to the category of topological spaces, where according to Proposition 2.4.1, it agrees with the usual definition when X is cofibrant.

It is interesting to compare the above definition with that of Ganea ([17]). To define category Ganea first constructed (for a path connected topological space X) a sequence of spaces

$$\begin{tikzcd} B_0X \arrow[r] \arrow[dr] & B_1X \arrow[r] \arrow[d] & B_2X \arrow[r] \arrow[dl] & \cdots \\ & X & & \end{tikzcd}$$

inductively, starting with $B_0X = PX \to X$ and obtaining $B_{n+1}X$ as the cofibre of the inclusion of the homotopy fibre of $B_nX \to X$. He then defined cat$X \leq n$ if and only if $B_nX \to X$ admits a right homotopy inverse.

To compare his definition with ours, consider the way in which $B_{n+1}X$ is obtained from B_nX. One begins with the diagram

$$(*) \qquad \begin{tikzcd} B_nX \arrow[dr] & & * \arrow[dl] \\ & X & \end{tikzcd} ,$$

forms the homotopy theoretic fibre product

$$\begin{tikzcd} & F_nX \arrow[dl] \arrow[dr] & \\ B_nX \arrow[dr] & & PX \arrow[dl] \\ & X & \end{tikzcd} ,$$

and then forms the double mapping cylinder of the upper two arrows

$$\begin{tikzcd} & F_nX \arrow[dl] \arrow[dr] & & & \\ B_nX \arrow[dr] & & PX \arrow[dl] \arrow[r] & & B_{n+1}X \arrow[d] \\ & X & & & X \end{tikzcd} .$$

Visibly this is the same as applying (with the notation $\underline{C}(\{0,1\})^0_+ \overset{i}{\to} \underline{C}(\{0,1\})_+ \overset{e}{\to} \underline{C}(*)_+$) $\underline{\underline{L}}\, \underset{\to}{res}_e \circ \underline{\underline{R}}\, \underset{\leftarrow}{ind}_i$ to the diagram (*). It follows that $B_{n+1}X$ admits a (weak) right homotopy inverse if and only if X can be homotopy covered by B_nX and a point. In the language of homotopy coverings Ganea's definition of cat X reads as follows:

<u>Definition</u>. Set cat $X = 0$ if and only if X is weakly contractible. Having defined cat $X \leq n$ set cat $X \leq n + 1$ if and only if X can be homotopy covered by a point and a space Y with cat $Y \leq n$.

Since the formation of homotopy coverings is associative (Corollary 2.3.3) this definition is equivalent to ours.

2.6 Filtrations of X

Proposition 2.3.1 converts questions like "How many copies of $p : U \to X$ does it take to homotopy cover X?" into questions about which of a collection of maps admits a weak right homotopy inverse. In fact these maps fit together to form a sequence. To see this let C denote the category of non-empty subsets of $\{0,1,2,...\}$ and let C^k be the full subcategory of subsets of cardinality less than or equal to $k + 1$. Given a map $p : U \to X$ let $F_n : C^0_+ \to \underline{X}$ be the contravariant functor defined by

$$F_n(a) = \begin{cases} U & \text{if } a \leq n \\ X & \text{if } a = + \\ \phi & \text{otherwise} \end{cases}$$

with the obvious effect on maps. According to Proposition 2.3.1 the space X can be homotopy covered by $n + 1$ copies of the map $p : U \to X$ if and only if $\underline{\underline{L}}\ \underset{\to}{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_i F_n$ $(C^0_+ \xrightarrow{i} C_+ \xrightarrow{e} C(*)_+)$ admits a right inverse when regarded as a morphism of $\mathrm{ho}(\underline{S})$. The obvious natural transformations $F_n \to F_{n+1}$ result in a sequence.

$$\cdots \to \underline{\underline{L}}\ \underset{\to}{\mathrm{ind}}_e \circ \mathrm{R}\ \underset{\leftarrow}{\mathrm{ind}}_i F_n \to \underline{\underline{L}}\ \underset{\to}{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_i F_{n+1} \to \cdots$$

of spaces over X. This sequence is isomorphic to the one obtained by converting $p : U \to X$ into a fibration and then forming the usual sequence of iterated fibre joins.

When U is a point and X is connected the above sequence can be constructed from the weak homotopy types of products of copies of ΩX using only projections and loop multiplication. In fact the restriction of $\underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_i\ F_n$ to C is isomorphic to the functor which sends a subset $A \subset \{0,1,2,...\}$ to the function complex $\mathrm{hom}[(\underline{C}(A),\underline{C}(A)^0),(X,*)]$ if $A \subset \{0,...,n\}$, and to ϕ otherwise. This sequence can be identified with the Milnor filtration of X regarded as the classifying space of its loop space. There is a close relationship between the construction of this filtration and the loop space structure of ΩX. Indeed, an h-space is an A_n-space in the sense of Stasheff ([31]) if and only if the first n stages of the Milnor construction of its classifying space exist ([33]).

Using the ideas of this chapter this can be generalised to a filtration of an (n-1)-connected space built out of spaces weakly equivalent to products of its n-fold loop spaces, using only properties of the multiplication.

For a fixed n define functors $F_m : C_+^{n-1} \to \underline{S}$ by setting

$$F_m(+) = X$$

$$F_m(A) = \begin{cases} \phi \text{ if } & A \not\subset \{0,...,m\} \\ \\ * \text{ if } & A \subset \{0,...,m\}. \end{cases}$$

Asking whether or not F_m can be extended to a homotopy covering of X leads one to a sequence

$$\cdots \to \underline{\underline{L}}\ \underset{\rightarrow}{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_i F_m \to \underline{\underline{L}}\ \underset{\rightarrow}{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_i F_{m+1} \to \cdots$$

of spaces over X. It is easy to see that $\underline{\underline{L}}\ \underset{\rightarrow}{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_i F_n$ is isomorphic to the "evaluation map" $\Sigma^n\Omega^n X \to X$ and that the functor $\underline{\underline{R}}\ \underset{\to}{\text{ind}}_i\ F_m$ is constructed out of spaces weakly equivalent to products of $\Omega^n X$ using only properties of the multiplication. In fact, the restriction of $\underline{\underline{R}}\ \underset{\leftarrow}{\text{ind}}_i\ F_m$ to C is isomorphic to the functor

$$A \to \begin{cases} \hom[(\underline{C}(A),\ \underline{C}(A)^{n-1}),(X,*)] \\ \qquad \text{if } A \subset \{0,\ldots,m\} \\ \phi \text{ otherwise} \end{cases},$$

and the map

$$\hom[(\underline{C}(A),\underline{C}(A)^{n-1}),(X,*)] \to \prod_{\sigma} \hom[(\underline{C}(\sigma),\underline{C}(\sigma)^{n-1}),(X,*)]$$

is a weak equivalence, where σ runs through the $\binom{m}{n}$ different $(n+1)$-element subsets of $\{0,\ldots,m\}$ which contain 0.

The topological situation for which one would construct such a functor would arise when trying to cover a space with open subsets in such a way that the inclusion maps of the k-fold unions of them, $k \leq n$, were compatibly null homotopic.

As with single loop spaces, the sequence of functors $\underline{\underline{R}}\ \underset{\leftarrow}{\text{ind}}_i\ F_m$ stores the algebraic information needed to reconstruct an $(n-1)$-connected space from its n-fold loop space. The resulting recognition principle will appear in the next chapter, along with a more detailed study of the filtration. The remainder of this section will consist of two more descriptions of this filtration.

In the category of topological spaces the domain of the map $\underline{\underline{R}}\ \underset{\leftarrow}{\text{ind}}_i\ F_m$ can be described as the space of maps from the m-simplex to X which send at least one $(n-1)$-face to the base point. This can be seen by decomposing this function complex into a union of (contractible) subspaces of maps which send a given (n-1)-face to the base point.

In the single loop space case, the m^{th} stage of the Milnor filtration can also be described as the homotopy theoretic fibre product of the iterated diagonal and the inclusion of the fat wedge.

$$\begin{tikzcd} B_m X \arrow[r] \arrow[d] & T^{m+1} X \arrow[d] \\ X \arrow[r] & \prod^{m+1} X \end{tikzcd}$$

This can be viewed as follows: $\prod^{m+1} X$ is the space of maps from the zero skeleton of the m-simplex to X; $T^{m+1}X$ is the space of maps from the zero skeleton of the m-simplex to X which send some vertex to the base point; X can be replaced with the space of maps from the m-simplex to X; $B_m X$ is the space of maps from the m-simplex to X which send some vertex to the base point. The above square is then easily generalized to the case of n-fold loop spaces. The result is:

$$\begin{tikzcd} \{\text{maps } \Delta_m \to X \mid \text{some } (n-1)\text{-face is sent to } *\} \arrow[r] \arrow[d] & \{\text{maps } \Delta_m^{(n-1)} \mid \text{some } (n-1)\text{-face is sent to } *\} \arrow[d] \\ \hom[\Delta_m, X] \arrow[r] & \hom[\Delta_m^{(n-1)}, X]. \end{tikzcd}$$

This should be compared with Ginsburg's description of the Milnor filtration of X ([18]).

2.7 Co-coverings and cocategory

Much of the preceding material can be dualized. To begin with, suppose one has written $X = U^0 \cap U^1 \cap U^2$. This can be represented by the diagram

$$\begin{tikzcd}
 & & U^0 \arrow[dl] \arrow[dr] & & \\
 & U^{01} \arrow[dr] & & U^{02} \arrow[dl] & \\
 & & U^{012} & & \\
U^1 \arrow[uur] \arrow[rr] & & U^{12} \arrow[u] & & U^2 \arrow[ll] \arrow[uul]
\end{tikzcd}$$

in which $U^{ij} = U^i \cup U^j$ etc. Such a diagram is a contravariant functor $F : \underline{C}(\{0,1,2\}) \to \underline{S}$ and the condition that X be the intersection of the $\{U^i\}$ is equivalent to the condition that X be the inverse limit of this functor. The dual of a homotopy covering is therefore some sort of homotopy theoretic intersection.

Throughout this section we will be dealing with covariant functors, so that $\underline{S}^A$ and $[A,\underline{S}]$ will both denote the category of covariant functors from a small category A to $\underline{S}$.

<u>Definition</u>. A <u>homotopy co-covering</u> of a space X is a covariant functor $F : \underline{C}(S) \to \underline{S}$ for some set S, together with morphisms $X \overset{i}{\to} \underleftarrow{\mathrm{holim}}\, F \overset{r}{\to} X$ of $\mathrm{ho}(\underline{S})$ such that $r \circ i$ is the identity morphism of X.

Given a homotopy co-covering F of X we give priority to the spaces $F\{\alpha\}$ $\alpha \in S$ over their "unions" by saying that the $F\{\alpha\}$ <u>homotopy co-cover</u> X via the morphisms

$X \xrightarrow{i} \underleftarrow{\operatorname{holim}}\, F \to F\{\alpha\}$. We also say that X <u>can be homotopy co-covered</u> by a family of maps $\{i_\alpha : X \to U_\alpha | \alpha \in S\}$ if there exists a homotopy co-covering $F : \underline{C}(S) \to \underline{S}$ of X, and for each $\alpha \in S$ a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_\alpha"] \arrow[d] & U_\alpha \\
\underleftarrow{\operatorname{holim}}\, F \arrow[r] & F\{\alpha\} \arrow[u, "w"']
\end{tikzcd}$$

in $\mathrm{ho}(\underline{S})$ with $w : F\{\alpha\} \to U_\alpha$ an isomorphism.

Again, let i and e be the functors $\underline{C}(S)^k_+ \xrightarrow{i} \underline{C}(S)_+ \xrightarrow{e} \underline{C}(*)_+$. As with homotopy coverings, a homotopy co-covering can be regarded as a covariant functor $F : \underline{C}(S)_+ \to \underline{S}$ with $F(+) = X$, together with a left inverse of $\underline{\underline{R}}\, \underleftarrow{\mathrm{ind}}_e F$ when regarded as a morphism of $\mathrm{ho}(\underline{S})$. A collection of maps $\{X \to U_\alpha | \alpha \in S\}$ determines a covariant functor $G : \underline{C}(S)^0_+ \to \underline{S}$ and it is easily seen that X can be homotopy co-covered by $\{X \to U_\alpha | \alpha \in S\}$ if and only if G is isomorphic in $\mathrm{ho}[\underline{C}(S)^0, \underline{S}]$ to $\underline{\underline{R}}\, \underleftarrow{\mathrm{res}}_i F$, for some homotopy co-covering $F : \underline{C}(S)_+ \to \underline{S}$ of X.

More generally, let $I \subset \underline{C}(S)$ be any subcategory and let $i : I_+ \to \underline{C}(S)_+$ be the inclusion. We say that a functor $G : I_+ \to \underline{S}$ with $G(+) = X$ <u>can be extended to a homotopy co-covering</u> of X if it is isomorphic in $\mathrm{ho}[I_+, \underline{S}]$ to $\underline{\underline{R}}\, \underleftarrow{\mathrm{res}}_i F$ for some homotopy co-covering F of X.

Proposition 2.7.1. Let $I \subset \underline{C}(S)$ be a full subcategory and let $i : I_+ \to \underline{C}(S)_+$ be the inclusion. A covariant functor $G : I_+ \to \underline{S}$ with $G(+) = X$ can be extended to a homotopy co-covering of X if and only if $\underline{\underline{R}}\ \underset{\leftarrow}{\text{ind}}_e \circ \underline{\underline{L}}\ \underset{\rightarrow}{\text{ind}}_i G \in \text{ho}[\underline{C}(*)_+, \underline{S}]$ admits a left inverse when regarded as a morphism of $\text{ho}(\underline{S})$.

The proof of Proposition 2.7.1 is the dual of the proof of Proposition 2.3.1.

Unfortunately there seems to be no dual for Proposition 2.3.2. Consequently the dual of Corollary 2.3.3 may not be true. This is largely due to the fact that the Hilton-Eckmann dual of the join operation is not associative. There result many possible definitions of cocategory. We single out the two most important.

Definition. i) The symmetric cocategory of a space X, sym-cocat X, is the least integer k for which X can be homotopy co-covered by k+1 points.

ii) The inductive cocategory of X, ind-cocat X, is the least integer k for which X can be homotopy co-covered by a point and a space Y with ind-cocat Y = k-1, starting with ind-cocat X = 0 if and only if X is weakly contractible.

Ganea's definition of cocategory ([16]) is the Hilton-Eckmann dual of his definition of category. In view of the reading given in section 2.5 it is equivalent to our definition of inductive cocategory.

Symmetric and inductive cocategory certainly appear to be distinct notions though at present it is not known whether they coincide or differ. The following theorem sums up the present state of affairs.

Theorem 2.7.2. i) ind-cocat $X \leq$ sym-cocat X.

ii) The following are equivalent:

a) ind-cocat $X = 0$

b) sym-cocat $X = 0$

c) X is contractible.

iii) The following are equivalent:

a) ind-cocat $X = 1$

b) sym-cocat $X = 1$

c) X is an h-space .

iv) If X is connected and cat $X \leq n$, then for any fibrant space Y, the symmetric and inductive cocategories of the pointed function complex hom[$(X,*)$,$(Y,*)$] are both $\leq n$.

v) If there is a non-trivial n-fold Whitehead product in $\pi_* X$ then sym-cocat $X \geq$ ind-cocat $X \geq n$.

vi) If π is a group of nilpotence class n (i.e. $\Gamma_n \pi \neq 1$, $\Gamma_{n+1}\pi = 1$) then ind-cocat$(K(\pi,1))$ = sym-cocat$(K(\pi,1)) = n$.

vii) If $F \to E \to B$ is a fibration then ind-cocat $F \leq$ ind-cocat $E + 1$.

Proof: For part i) suppose that $F : \underline{C}(S)_+ \to \underline{S}$ is a homotopy co-covering of X with $F\{\alpha\}$ weakly contractible for $\alpha \in S$. Fix $\alpha \in S$, let $S' = S - \{\alpha\}$, and define a map $f : S \to \{0,1\}$ by $f^{-1}\{1\} = S'$, $f^{-1}\{0\} = \{\alpha\}$. This determines a functor (also to be called f) $f : \underline{C}(S)_+ \to \underline{C}(\{0,1\})_+$. The functor $\underline{C}(S)_+ \to \underline{C}(*)_+$ admits a factorization $\underline{C}(S)_+ \xrightarrow{f} \underline{C}(\{0,1\})_+ \xrightarrow{e} \underline{C}(*)_+$. Since $\underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_{e\circ f}F \approx \underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_f F$ admits a left inverse when regarded as a morphism of ho($\underline{S}$), $\underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_f F$ inherits the structure of a homotopy covering of X. But $\underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_f F(\{0\})$ is weakly contractible and $\underline{\underline{R}}\ \underleftarrow{\mathrm{ind}}_f F(\{1\})$ has symmetric cocategory $\leq |S| - 1$. The result now follows by induction.

Part ii) is immediate from the definition.

Part iii) follows from Proposition 2.7.1 and the fact that a space X is an h-space if and only if it is a retract of $\Omega\Sigma X$ in ho($\underline{S}$).

For part iv) let $F : \underline{C}(S)_+ \to \underline{S}$ be a homotopy covering of X by n+1 contractible spaces. We may, without loss of generality, replace $\underline{S}$ by the category $\underline{S}_*$ of pointed simplicial sets. The result now follows from the observation that $\mathrm{hom}[F,(Y,*)] : \underline{C}(S)_+ \to \underline{S}_*$ is a homotopy co-covering of the pointed function complex $\mathrm{hom}[(X,*),(Y,*)]$. This in turn follows from the result of Bousfield-Kan ([6], Ch.XII, Proposition 4.1) which states that for $F : I \to \underline{S}_*$ and $Y \in \underline{S}_*$ there is an isomorphism

$$\mathrm{hom}_*(\underrightarrow{\mathrm{holim}}\ F,Y) \approx \underleftarrow{\mathrm{holim}}(F,Y)$$

where hom_* is the pointed function complex functor.

For part v), the inequality ind-cocat $X \geq n$ is proved in [16]. The result then follows from part i). We will give another proof of the inequality sym-cocat $X \geq n$ in Chapter 3 (Theorem 3.2.2).

For part vi), the inequality ind-cocat$(K(\pi,1)) \geq n$ is supplied b part iv). Let S be the set $\{0,\ldots,n\}$ and let $F : \underline{C}(S)_+^0 \to \underline{S}$ be the covariant functor $F(+) = K(\pi,1)$, $F(a) = *$ for $a = \{0\},\ldots,\{n\}$. It will be shown in Chapter 3 (Theorem 3.3.1) that the map $\underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_e \circ \underline{\underline{L}}\ \underset{\rightarrow}{\mathrm{ind}}_i F$ induces an isomorphism of fundamental groups. Coupled with Proposition 2.7.1 this will provide the inequality sym-cocat$(K(\pi,1)) \leq n$.

Part vii) is proved in [16]. It is also an immediate consequence of the definition. This completes the proof of Theorem 2.7.2.

By dualizing the argument at the beginning of section 2.6 one obtains from the notion of sym-cocat a tower of fibrations under a space X. We will see in the next chapter that the homotopy inverse limit of this tower is the $\mathbb{Z}$-nilpotent completion of X when X is connected. Applying π_* results in a spectral sequence. The E^2 term of this spectral sequence can be computed by applying the cobar construction to ΣX (in ho$(\underline{S})$), passing to homotopy groups, and computing the cohomology of the resulting complex. An easy consequence of the Hilton-Milnor theorem ([19], [23]) is that in a range of about three times the connectivity of X the spectral sequence collapses to an exact sequence

$$\pi_n(\Sigma X) \to \pi_n(\Sigma X \wedge X) \to \pi_{n-2}(X) \to \pi_{n-1}(\Sigma X)$$

which is the EHP sequence.

It follows from the construction that this spectral sequence is the dual of the Rothenberg-Steenrod, Milnor-Moore spectral sequence ([22], [28]). It was first discovered by M.G. Barratt in a slightly different setting ([2]). One advantage of our formulation is that it is made clear that the differentials are the obstructions to maps $S^n \to X$ being co h-maps compatible with higher associativity. We will also obtain an improved convergence result. Finally, using the ideas of 2.6 the spectral sequence can be generalized to one involving the n-fold suspension of X. This and the related recognition principle are the subject of the next chapter.

Chapter 3

Recognition Principles

3.1 Simplicial and cosimplicial spaces

The filtrations of Chapter 2 are best understood in the language of simplicial and cosimplicial spaces. In this section we recall what is needed from these areas. For further information the reader is referred to [5], [6] and [29].

Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \dots < n\}$, $n \geq 0$, and whose morphisms are the non-decreasing maps. A simplicial space is a contravariant functor from Δ to $\underline{S}$ and a cosimplicial space is a covariant functor from Δ to $\underline{S}$. The cosimplicial space given by $[n] \to \Delta(n)$ is denoted $\underset{\sim}{\Delta}$, where $\Delta(n)$ is the standard n-simplex. There is a natural filtration of $\underset{\sim}{\Delta}$ by the cosimplicial spaces $\underset{\sim}{\Delta}^k$, $k \geq 0$, where the value of $\underset{\sim}{\Delta}^k$ on the object $[n]$ is the k-skeleton of the standard n-simplex.

The total complex or geometric realization $|X|$ of a simplicial space $[n] \to X_n$ is the space $X \times_\Delta \underset{\sim}{\Delta}$. It is naturally isomorphic to the diagonal of X. The total complex of a cosimplicial space $[n] \to Y^n$, TotY, is the function complex $\hom_\Delta[\underset{\sim}{\Delta}, Y]$. The filtration of $\underset{\sim}{\Delta}$ by $\underset{\sim}{\Delta}^k$ results in a filtration of $|X|$ by $|X|^k = X \times_\Delta \underset{\sim}{\Delta}^k$ and a representation of TotY as the inverse limit of a tower $\{\mathrm{Tot}_k Y\} = \{\hom_\Delta[\underset{\sim}{\Delta}^k, Y]\}$.

Applying homology (or cohomology) to the filtration of $|X|$ results in a spectral sequence whose E^2-term can be computed as the homology of the chain complex associated with the graded simplicial abelian group $[n] \to H_*X_n$. One does not get a good spectral sequence for the homotopy groups of a cosimplicial space without additional assumptions.

The categories of simplicial and cosimplicial spaces form closed model categories ([5], [6]). A map $X \to Y$ between simplicial spaces is a <u>cofibration</u> (resp. <u>weak equivalence</u>) if for each $[n]$ the map $X_n \to Y_n$ is a cofibration (resp. weak equivalence). The <u>weak equivalences</u> in the category of cosimplicial spaces are the termwise weak equivalences. The cofibrations are the termwise cofibrations which induce isomorphisms of the maximal augmentations, where the <u>maximal augmentation</u> of a cosimplicial space X is the equalizer of $X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1$. The fibrations in the categories of simplicial and cosimplicial spaces can be defined to be the classes of morphisms which have the right lifting property with respect to the trivial cofibrations. More direct descriptions are given in [6] (p.II, Ch.X, §4) and [5] (Appendix B).

If Y is a cosimplicial pointed space, the <u>normalized</u> <u>homotopy groups</u> of Y, π'_*Y are given by

$$\pi'_t Y^s = \begin{cases} \pi_t Y^s \cap \ker s_*^0 \cap \dots \cap \ker s_*^{s-1} \\ \qquad \text{if } s \geq t \\ 0 \text{ otherwise .} \end{cases}$$

There is another process which we will also refer to as normalization. If A is a simplicial (resp. cosimplicial) abelian group, let N_*A (resp. N^*A) be the chain complex given in (co-)dimension n by $N_sA = A_s/(s_0A_{s-1} + \dots + s_{s-1}A_{s-1})$ (resp. $N^sA = A^s \cap \ker s^0 \cap \dots \cap \ker s^{s-1}$) with differential $\Sigma(-1)^i d_i$ $(\Sigma(-1)^i d^i)$.

For a fibrant cosimplicial space Y, the $E^1_{s,t}$-term of the homotopy spectral sequence associated with the tower of fibrations $\{\mathrm{Tot}_n Y\}$ is the normalized homotopy group $\pi_t^{\prime} Y^s$. The spectral sequence converges to $\varprojlim \pi_*(\mathrm{Tot}_n Y)$.

The cohomology spectral sequence of a simplicial space admits products. If $x \in H^i(X_p)$ and $y \in H^j(X_q)$ then the cup product $x \cup y$ can be computed on the E^1-term of the spectral sequence using the (graded) Alexander-Whitney formula ([15]).

$$x \cup y = (-1)^{iq} d_*^{\prime} x \cup d_*^{\prime\prime} y$$

where $d' : [p] \to [p+q]$ is given by $d'(s) = s$ and $d'' : [q] \to [p+q]$ is given by $d''(s) = s+p$. This converges to the cup product in $H^*(|X|)$

Dually, the ideas of [7] can be used to endow the homotopy spectral sequence of a fibrant cosimplicial pointed space Y with a Whitehead product structure. If $x \in \pi_i^{\prime} Y^p$ and $y \in \pi_j^{\prime} Y^q$ then the Whitehead product $[x,y]$ can be computed on the E^1 term of the homotopy spectral sequence by the formula

$$[x,y] = (-1)^{iq}[d'x, d''y] \in \pi_{i+j-1}(Y^{p+q})$$

where d' and d'' are as above. These products are bilinear and satisfy the usual graded Jacobi law (from E^2 onward) where the grading of $\pi_t^{\prime}(X^s)$ is t-s. The differentials in the homotopy spectral sequence act as graded Lie derivations and the Whitehead products converge to the Whitehead products in $\pi_*(\mathrm{Tot}\, Y)$ ([7], [8]).

For a complicial pointed space Y let $\mathbb{Z}Y$ denote the prolongation of the functor which assigns to a pointed set the free abelian group on that set, modulo the subgroup generated by the base point. In the unpointed case let $\mathbb{Z} \otimes Y$ be the prolongation of the free abelian group functor. The homotopy spectral sequence for $\mathbb{Z}Y$ has for its E^1_{st} term the normalized homology of Y.

$$E^1_{st} = H'_t Y^s = \begin{cases} H_t Y^s \cap \ker s^0 \cap \dots \cap \ker s^{s-1} & \text{if } t \geq s \\ 0 & \text{otherwise} \end{cases}$$

It is called the (integral) homology spectral sequence of Y and converges to $\varprojlim \pi_{t-s}(\mathrm{Tot}_n \mathbb{Z}Y)$. We would like to know when this can be identified with $H_{t-s}(\mathrm{Tot}\, Y)$. The following theorem is a special case of a result of Bousfield ([9]). A sketch of the proof will be given in Appendix B. We assume familiarity on the part of the reader with the usual "pro-theoretic" notions for towers ([1], [6], [12]).

<u>Theorem 3.1.1</u> (Bousfield). Let Y be a fibrant cosimplicial pointed space such that

i) $\pi'_{n+k} Y^n = *$ if $k \leq 0$;

ii) Y^n is simply connected, $n \geq 0$;

iii) For each k there are only finitely many n such that $\pi'_{n+k} Y^n \neq *$.

Then the map $\{H_*\mathrm{Tot}\ Y\} \to \{\pi_*\mathrm{Tot}_n \mathbb{Z}Y\}$ is a pro-isomorphism of towers of graded abelian groups. It follows that the (integral) homology spectral sequence of Y converges strongly in the sense that:

i) For each pair (s,t) there is an $r < \infty$ with the property that $E^r_{st} = E^\infty_{st}$;

ii) For all $n \geq 0$, $\{E^\infty_{s,t} \mid t-s = n\}$ is the set of filtration quotients of a finite filtration of $H_n(\mathrm{Tot}\ Y)$.

For later convenience we require another description of this spectral sequence. Associated with a cosimplicial abelian group A is a bicomplex $\{A^q_p, \partial, \delta\}$ with $\partial = \sum_{i=0}^{p} (-1)^i d_i : A^q_p \to A^q_{p-1}$ and $\delta = \sum_{i=0}^{q+1} (-1)^i d^i : A^q_p \to A^{q+1}_p$. The total complex $T(A)$ is given by

$$T(A)_n = \begin{cases} \prod_j A^n_{n+j} & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

with differential $d = \partial + (-1)^{n+1}\delta : T(A)_n \to T(A)_{n-1}$. A decreasing filtration $\{F^k T(A)\}_{k \geq 0}$ is given by $F^k T(A)_n = \prod_{j \geq k} A^n_{n+j}$. There results a spectral sequence converging to $H_* T(A)$.

Associated to any simplicial abelian group X is its non-normalized chain complex, also denoted X. There is a map $\phi : \mathrm{Tot}\ A \to T(A)$ given as follows. An n-simplex u of Tot A is a map $\underset{\sim}{\Delta} \times \Delta(n) \to A$. The component of this in A^k extends to a homomorphism

$f : \mathbb{Z} \otimes (\Delta(k) \times \Delta(n)) \to A$. The component of $\phi(u)$ in A^n_{n+k} is $f(i_k \underline{\otimes} i_n)$ where $\underline{\otimes}$ denotes the Eilenberg-Zilber product ([14]). One easily checks that this induces a map of towers $\{\mathrm{Tot}_n A\} \to \{T(A)/F_{n+1}T(A)\}$ and that the resulting map of spectral sequences is an isomorphism at E^2. It follows that the homology spectral sequence of a cosimplicial pointed space Y coincides with the spectral sequence associated with the above filtration of $T(\mathbb{Z}Y)$. It also follows that the map $\{\pi_*\mathrm{Tot}_n \mathbb{Z}Y\} \to \{H_*(T(\mathbb{Z}Y)/F_{n+1}T(\mathbb{Z}Y))\}$ is a pro-isomorphism.

This description enables us to make the relationship between $H_*\mathrm{Tot}\, Y$ and H_*Y^n more explicit. Let $\underline{Y}^n = Y^n/d^0Y^{n-1} \cup \ldots \cup d^nY^{n-1}$. The functor Tot is right adjoint to the functor $\underset{\sim}{\Delta}\times -$. The co-unit of this adjunction provides for each n a map

$$(\Delta(n),\dot{\Delta}(n)) \times (\mathrm{Tot}\, Y,*) \to (Y^n, d^0Y^{n-1} \cup \ldots \cup d^nY^{n-1})$$

and hence a map

$$\Sigma^n\mathrm{Tot}\, Y \to \underline{Y}^n .$$

The E^∞_{st} term of the homology spectral sequence of Y is the intersection of the image of $H_t(\Sigma^s\mathrm{Tot}\, Y)$ with the image of $H_t(Y^s)$ in $H_t(\underline{Y}^s)$. This is useful when, for example, $Y^0,\ldots,Y^{k-1}$ are contractible. In that case $\mathrm{Tot}_k Y$ is weakly equivalent to $\Omega^k Y^k$ and the above remark implies that the "edge homomorphism" $H_n\mathrm{Tot}\, Y \to H_{n+k}Y^k$ is induced by the map $\Sigma^k\mathrm{Tot}\, Y \to Y^k$ weakly adjoint to the projection $\mathrm{Tot}\, Y \to \mathrm{Tot}_k Y$.

Under certain circumstances one can obtain a homotopy spectral sequence for a simplicial pointed space X ([5]). This depends on knowing when the total complex of a termwise homotopy cartesian square of simplicial spaces is homotopy cartesian. The following condition is due to Bousfield-Friedlander ([5], Appendix B).

A simplicial space X is said to satisfy the <u>π_*-Kan condition</u> at a zero simplex $a \in X_m$ if for every collection of elements $\{x_i \in \pi_t(X_{m-1}, d_i a) \mid i \in \{0,\ldots,m\},\ i \neq k\}$ which satisfy the compatibility condition $d_i x_j = d_{j-1} x_i$ for $i < j$, $i,j \neq k$, there exists an element $x \in \pi_t(X_m, a)$ such that $d_i x = x_i$ for all $i \neq k$. A simplicial space is said to satisfy the <u>π_*-Kan condition</u> if for each $m,t \geq 1$ it satisfies the π_t-Kan condition at each $a \in X_m$. This condition depends only on the isomorphism class of X considered as a simplicial object over the category $ho(\underline{S})$.

If X is a simplicial space then $\pi_t X$ is a simplicial set, so it makes sense to talk about $\pi_s \pi_t X$.

<u>Theorem 3.1.2</u>. ([5], Theorem B.4). Let

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

be a termwise homotopy cartesian square of simplicial spaces. If X and Y satisfy the π_*-Kan condition and if $\pi_0 X \to \pi_0 Y$ is a Kan fibration then the square

$$\begin{tikzcd} {|V|} \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ {|W|} \arrow[r] & {|Y|} \end{tikzcd}$$

is homotopy cartesian.

Theorem 3.1.4. Let X be a simplicial pointed space satisfying the π_*-Kan condition. There is a spectral sequence $\{E^r_{st}\}_{r\geq 2}$ converging to $\pi_{s+t}(|X|)$ with $E^2_{st} = \pi_s\pi_t X$.

As with the homology spectral sequence of a cosimplicial space, the relationship between $\pi_t X_s$ and $\pi_{s+t}|X|$ can be made explicit. Let $M_s X \subset X_{s-1} \times \dots \times X_{s-1}$ be the space of (s+1)-tuples $(x_0,\dots,x_s)$ satisfying $d_i x_j = d_{j-1} x_i$ for $i<j$. There is a map $d : X_s \to M_s X$ given by $d(x) = (d_0 x,\dots,d_s x)$. Let $F_s \to X_s$ be its fibre. The functor $\underset{\sim}{\Delta} \times_\Delta -$ is left adjoint to the functor $\hom_\Delta[\underset{\sim}{\Delta},-]$. The adjunction unit provides for each s a map $X_s \to \hom[\Delta(s),|X|]$ and by restriction a map $F_s \to \Omega^s|X|$. The E^∞_{st} term of the homotopy spectral sequence is the quotient of $\pi_t F_s$ obtained by identifying elements whose images coincide in either $\pi_t\Omega^s|X| = \pi_{t+s}X$ or $\pi_t X^s$. In case $X_0,\dots,X_{k-1}$ are contractible this implies that the "edge homomorphism" of the homotopy spectral sequence is induced by the map $X_{k-1} \to \Omega^k|X|$ weakly adjoint to the inclusion $\Sigma^k X_{k-1} \approx |X|^k \to |X|$.

Finally, we will need to relate the filtrations and towers of fibrations which arise from simplicial and cosimplicial spaces to the kinds of filtrations and towers of fibrations which appeared in Chapter 2. In order to do this the realizations $|X|^k$ and $\mathrm{Tot}_k Y$ must be modified.

Let C_m denote the category of non-empty subsets of $\{0,\ldots,m\}$ and let $C_m^k \subset C_m$ be the full subcategory consisting of subsets of cardinality $\leq k+1$. Let $\underset{\sim}{\Delta}'$ (resp. $\underset{\sim}{\Delta}^{k'}$) be the covariant functor $[m] \to C_m$ (resp. $[m] \to C_m^k$). The cosimplicial spaces $\underset{\sim}{\Delta}'$ and $\underset{\sim}{\Delta}^{k'}$ are cofibrant and weakly equivalent to $\underset{\sim}{\Delta}$ and $\underset{\sim}{\Delta}^k$ respectively.

If X is a simplicial space let $|X|'$ (resp. $|X|^{k'}$) be the space $\underset{\sim}{\Delta}' \times_\Delta X$ ($\underset{\sim}{\Delta}^{k'} \times_\Delta X$). For a cosimplicial space Y let $\mathrm{Tot}'Y$ (resp. $\mathrm{Tot}_k'Y$) be the space $\hom_\Delta(\underset{\sim}{\Delta}',Y)$ (resp. $\hom_\Delta(\underset{\sim}{\Delta}^{k'},Y)$). The spaces $|X|^k$ and $|X|^{k'}$ are weakly equivalent - as are $\mathrm{Tot}_k Y$ and $\mathrm{Tot}_k' Y$ if Y is fibrant.

Assigning to a (k+1)-element subset of $\{0,1,\ldots\}$ the object $[k] \in \Delta$ results in a functor $j_k : C_k \to \Delta$. For a simplicial (cosimplicial) space X let $U_k X$ ($U^k X$) be the composition of X with j_k. The following proposition is well-known but seems not to have appeared in print.

<u>Proposition 3.1.4</u>. i) Let Y be a fibrant cosimplicial space. There are natural weak equivalences $\mathrm{Tot}_k' Y \to \underleftarrow{\mathrm{holim}}\, U^k Y$ which are compatible in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Tot}'_k Y \arrow[r] \arrow[d] & \underleftarrow{\mathrm{holim}}\ U^k Y \arrow[d] \\
\mathrm{Tot}'_{k-1} Y \arrow[r] & \underleftarrow{\mathrm{holim}}\ U^{k-1} Y .
\end{tikzcd}$$

ii) Let X ba a simplicial space. There are natural weak equivalences $\underrightarrow{\mathrm{holim}}\ U_k X \to |X|^{k'}$ which are compatible in the sense that the following diagram commutes:

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}\ U_{k-1} X \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}\ U_k X \arrow[d] \\
|X|^{k-1'} \arrow[r] & |X|^{k'} .
\end{tikzcd}$$

<u>Proof</u>: The functor $Y \to \underleftarrow{\mathrm{holim}}\ U^k Y$ is a co-represented by the cosimplicial space $\Delta[C_k,\Delta] \times_{C_k} C_k/-$. In general, given a functor $F : A \to B$ and an object $b \in B$, the <u>category of A objects over b</u>, A/b, is the category whose objects are maps $g : Fa \to b$ and whose morphisms from $g_1 : Fa_1 \to b$ to $g_2 : Fa_2 \to b$ are the morphisms $h : a_1 \to a_2 \in A$ satisfying $g_2 \circ Fh = g_1$. The cosimplicial space $\Delta[C_k,\Delta] \times_{C_k} C_k/-$ is isomorphic to $[m] \to C_k/[m]$. The category $C_k/[m]$ is the category associated to the partially ordered set of pairs (σ,f) where $\sigma \subset \{0,\dots,k\}$ and $f : \sigma \to \{0,\dots,m\}$ is a non-decreasing map, with $(\sigma,f) \leq (\sigma',f')$ if $\sigma \subset \sigma'$ and f' extends f. Assigning to (σ,f) the image $f(\sigma) \subset \{0,\dots,m\}$ yields a functor

$C_k/[m] \to C^k_m$. These functors are compatible as [m] and k vary, so there is a map from the cosimplicial space $[m] \to C_k/[m]$ to the cosimplicial space $[m] \to C^k_m$. This gives the compatible family of maps in part i). Both of the cosimplicial spaces $[m] \to C_k/[m]$ and $[m] \to C^k_m$ are cofibrant so it remains to check (for part i)) that the map $C_k/[m] \to C^k_m$ is a weak equivalence. For this it suffices to show that for each $\sigma \in C^k_m$ the category of $C_k/[m]$- objects over σ is weakly contractible (by either Quillen's Theorem A [27], or the observation that the functor $\sigma \to (C_k/[m])/\sigma$ is the left homotopy Kan extension of the functor $C_k/[m] \to \underline{S}$ sending all morphisms to the identity map of a point).

For a given $\sigma \in C^k_m$, the category of $C_k/[m]$ objects over σ is isomorphic to $C_k/[n]$ where n+1 is the order of σ. We are therefore reduced to showing that the nerve of the category $C_k/[n]$ is weakly contractible if $n \leq k$. Suppose by induction we have shown this for $k' < k$ (the case $k = 0$ is trivial). For $j = 0,\ldots,n$ let $U_j \subset C_k/[n]$ be the full subcategory consisting of pairs (σ,f) with the property that $(\sigma,f) \leq (\sigma \cup \{k\},f')$ where $f'(k) = j$. Every simplex of $C_k/[n]$ is in some U_j so the U_j cover $C_k/[n]$. Let $U'_j \subset U_j$ be the full subcategory of pairs (σ,f) with $k \in \sigma$. The category U'_j is contractible since it has an initial object. On the other hand, there is a functor $U_j \to U'_j$ sending (σ,f) to $(\sigma \cup \{k\},f')$ where $f'(k) = j$, and, since $(\sigma,f) \leq (\sigma \cup \{k\},f')$, a natural transformation from the identity functor of U_j to this functor. It follows that U_j is contractible for all j. Finally, observe that for $j_1 < \ldots < j_\ell$, $\ell \geq 2$, the category $U_{j_1} \cap \ldots \cap U_{j_\ell}$ is non-empty and isomorphic to $C_{k-1}/[j_1]$, hence contractible by the induction hypothesis. It follows that $C_k/[n]$ is weakly equivalent to the nerve of the covering $\{U_j | j = 0,\ldots,n\}$ which in turn is isomorphic to C_n, hence contractible. This proves part i). Part ii) follows in a similar manner since $\underrightarrow{\mathrm{holim}}\, U_k X$ is the same as $X \times_\Delta (\Delta[C_k,\Delta] \times_{C_k} C_k/-)$. This completes the proof.

3.2 Certain simplicial and cosimplicial spaces

For a pointed fibrant space X let $\underset{\sim}{\Omega}^k X$ be the simplicial space $[m] \to \hom[(C_m, C_m^{k-1}), (X,*)]$ and let $\underset{\sim}{\Sigma}^k X$ be a fibrant cosimplicial space weakly equivalent to $[m] \to (C_m/C_m^{k-1}) \wedge X$. The total complex of the simplicial space $\hom[\underset{\sim}{\Delta}', X]$ is weakly equivalent to X and so there is a map $|\underset{\sim}{\Omega}^k X| \to X$. Dually, there is a map $X \to \mathrm{Tot}\underset{\sim}{\Sigma}^k X$.

When $k = 1$ the above constructions are closely related to the notions of category and symmetric cocategory. Indeed, if $F : C^1_{n+} \to \underline{S}$ is the contravariant functor which sends all non-identity morphisms to the inclusion map $* \to X$, then it follows from section 2.6 and Proposition 3.1.4 that $\underline{\underline{L}}\ \underset{\to}{\mathrm{ind}}_e \circ \underline{\underline{R}}\ \underset{\leftarrow}{\mathrm{ind}}_i F$ is isomorphic in $\mathrm{ho}[\underline{C}(*)_+, \underline{S}]$ to $|\underset{\sim}{\Omega}^1 X|^n \to X$, where e and i are the functors $C^1_{n+} \overset{i}{\to} C_{n+} \overset{e}{\to} \underset{\sim}{C}(*)_+$. It follows that X can be homotopy covered by $n + 1$ points if and only if the map $|\underset{\sim}{\Omega}^1 X|^n \to X$ admits a weak right homotopy inverse. Dually X can be homotopy co-covered by $n + 1$ points if and only if the map $X \to \mathrm{Tot}_n(\underset{\sim}{\Sigma}^1 X)$ admits a weak left homotopy inverse. We can therefore study the notions of category and symmetric cocategory, and the related filtrations and towers, by studying $\underset{\sim}{\Omega}^1 X$ and $\underset{\sim}{\Sigma}^1 X$. The generalizations mentioned in section 2.6 and 2.7 correspond to $\underset{\sim}{\Omega}^k X$ and $\underset{\sim}{\Sigma}^k X$.

The first thing one would like to know is the relationship between X, $|\underset{\sim}{\Omega}^k X|$ and $|\mathrm{Tot}(\underset{\sim}{\Sigma}^k X)|$.

Theorem 3.2.1. If X is (k-1)-connected then the map $\underset{\sim}{\Omega}^k X \to \hom(\underset{\sim}{\Delta}', X)$ induces a weak equivalence $|\underset{\sim}{\Omega}^k X| \to |\hom(\underset{\sim}{\Delta}', X)| \approx X$.

Recall from [6] (p.I, ch. III, §6) that a $\mathbb{Z}$-tower for a space X consists of a tower of fibrations $\{Y_n\}$ and a map $\{X\} \to \{Y_n\}$ which induces a pro-isomorphism $\{H_i(X;\mathbb{Z})\} \to \{H_i(Y_n;\mathbb{Z})\}$ for each i. A $\mathbb{Z}$-nilpotent tower for X is a $\mathbb{Z}$-tower $\{X\} \to \{Y_n\}$ in which each Y_n is nilpotent, Any two $\mathbb{Z}$-nilpotent towers for X have the same weak pro-homotopy type. In particular if X is nilpotent and $\{Y_n\}$ is any $\mathbb{Z}$-nilpotent tower for X then the map $\{X\} \to \{Y_n\}$ is a weak pro-homotopy equivalence. In any case the map $X \to \underleftarrow{\text{holim}}\, Y_n$ is a $\mathbb{Z}$-nilpotent completion for X.

Theorem 3.2.2. The tower $\{\text{Tot}_n \underset{\sim}{\Sigma}^k X\}$ is a $\mathbb{Z}$-nilpotent tower for X.

Corollary 3.2.3. If X is a (k-1)-connected, pointed, fibrant space, then the homology spectral sequence of the simplicial space $\underset{\sim}{\Omega}^k X$ converges strongly to $H_* X$. Its E^1 term depends only on the homology of products of $\Omega^k X$, and the differential d^1 uses only properties of the multiplication. The "edge" homomorphism is the iterated homology suspension.

Corollary 3.2.4. The homotopy spectral sequence of the cosimplicial space $\underset{\sim}{\Sigma}^k X$ converges to the homotopy groups of the $\mathbb{Z}$-nilpotent completion of X. It converges strongly if and only if X is nilpotent. The E^1-term of this spectral sequence depends only on the homotopy groups of wedges of $\Sigma^k X$ and the differential d^1 uses only properties of the co-multiplication. The "edge" homomorphism is the iterated homotopy suspension.

If A is an object of an abelian category C, there is for each $n \geq 0$ a unique simplicial object $K(A,n)$ over C satisfying

$$N_t K(A,n) = \begin{cases} 0 & \text{if} \quad t \neq n \\ A & \text{if} \quad t = n \end{cases} .$$

Such an object is characterised by the following properties:

i) $K(A,n)_t = 0$ if $t < n$;

ii) $K(A,n)_n = A$;

iii) The map $P_m = \prod_{j=1}^{\binom{m}{n}} p_j^* : K(A,n)_m \to \prod^{\binom{m}{n}} K(A,n)_n$ is an isomorphism, where p_j runs through the $\binom{m}{n}$ injections $[n] \to [m]$ satisfying $p_j(0) = 0$.

This fact is not quite standard but follows easily from the observations:

i) For any object A one can construct a simplicial object X satisfying i) - iii) above;

ii) If X satisfies i) - iii) above then the kernel of $PX \to X$ satisfies i) - iii) with n replaced by $n-1$, where PX is the simplicial path space obtained as in the proof of Lemma 2.2.3 by omission of the last face and degeneracy map.

If G is a group object in a category admitting products one can still form a simplicial object $K(G,1)$ characterized equivalently by properties i) - iii) above, or the fact that its Moore complex consists only of G in dimension 1.

The simplicial space $\underset{\sim}{\Omega}^n X$ satisfies conditions i) - iii) when regarded as a simplicial object over $ho(\underline{S})$. It follows that the normalized homotopy groups satisfy

$$N_s \pi_*(\underset{\sim}{\Omega}^n X) = \begin{cases} \pi_* \Omega^n X & \text{if } s = n \\ 0 & \text{otherwise} . \end{cases}$$

The dual remark applies to the cosimplicial space $\underset{\sim}{\Sigma}^n X$ and one obtains the description

$$N_s H_*(\underset{\sim}{\Sigma}^n X) = \begin{cases} \bar{H}_*(\Sigma^n X) & \text{if } s = n \\ 0 & \text{otherwise} . \end{cases}$$

Proof of Theorem 3.2.1: The simplicial space $\underset{\sim}{\Omega}^k X$ satisfies the π_*-Kan condition since for each $t \geq 0$ the simplicial object $\pi_t \underset{\sim}{\Omega}^k X$ is isomorphic to $K(\pi_{t+n} X, n)$. In view of the above remark, the homotopy spectral sequence collapses to its edge homomorphism. The theorem follows.

The proof of Theorem 3.2.2 requires the following vanishing theorem of Bousfield ([9]).

Theorem 3.2.5. Let X be a cosimplicial pointed space with each X^s simply connected. If λ is a positive real number such that $\pi_t' X^s = *$ for $s > \lambda t$ and A is an abelian group, then $H_t'(X^s; A) = 0$ for $s > \lambda t$.

Proof (Bousfield): For a cosimplicial space X let P^nX be the prolongation of the n^{th} Postnikov section functor. If M is a cosimplicial abelian group let $H_*(X;M)$ be the graded cosimplicial abelian group $[s] \to H_*(X^s;M_s)$. We will prove by induction on n the following statement: Suppose M is a cosimplicial abelian group with $NM^s = 0$ for $s > k$. Then $N^sH_t(P^nX;M) = 0$ for $s > \lambda t + k$. The result is trivial for $n = 0$.

Assuming the above statement for $n' < n$ let $\{E^r_{ij}, d^r\}$ be the cosimplicial spectral sequence given in codimension s by the Serre spectral sequence of the fibration $P^nX^s \to P^{n-1}X^s$ with coefficients in M^s. Since N^* is an exact functor it suffices to show that $N^sE^2_{ij} = 0$ for $s > \lambda(i + j) + k$. Since E^2_{ij} $= H_i(P^{n-1}X\ ;\ H_j(K(\pi_nX,n);M))$ it suffices by the induction hypothesis to show that $N^sH_j(K(\pi_{n+1}X,n+1);M) = 0$ for $s > \lambda j + k$. This requires two results of Dold-Puppe [11].

For a functor T from abelian groups to abelian groups recall that the cross-effects of T, $T^r(M_1,\ldots,M_r)$ are defined by the equation ([11], [15])

$$T(M_1 \oplus \ldots \oplus M_r) = \sum_{\sigma\subset\{1,\ldots,r\}} T^{|\sigma|}(M_\sigma)$$

where $T^{|\sigma|}(M_\sigma)$ denotes $T^j(M_{i_1},\ldots,M_{i_j})$ if $\sigma = \{i_1,\ldots,i_j\}$. The degree of T is the least integer d for which the $(d + 1)$-fold cross-effects vanish. For example if T is the functor $T(M) = H_j(K(M,n);\mathbb{Z})$ then T has degree given by the greatest integer in j/n.

The two results of Dold-Puppe which we need are:

i) ([11], Hilfsatz 6.10). Let $V(-,-)$ be a functor from pairs of abelian groups to abelian groups satisfying $V(0,M) = V(M,0) = 0$. If A_i, $i = 1,2$, are cosimplicial abelian groups with $N^s A_i = 0$ for $s > p_i$ then $N^s V(A_1,A_2) = 0$ for $s > p_1 + p_2$.

ii) ([11], Hilfsatz 4.23). Let T be a covariant functor from abelian groups to abelian groups of degree $\leq d$ with $T(0) = 0$. If A is a cosimplicial abelian group with $N^s A = 0$ for $s > p$, then $N^s T(A) = 0$ for $s > dp$.

Combining i) with the universal coefficient theorem reduces us to showing that $N^s H_j(K(\pi_n X,n),\mathbb{Z}) = 0$ for $s > j$, which follows from ii). This completes the proof of Theorem 3.2.5.

<u>Proof of Theorem 3.2.2</u>: For a cosimplicial space X let $X(n)$ be the right Kan extension of the restriction of X to the full subcategory of Δ consisting of the objects $[0],\ldots,[n]$. If X is fibrant then so is $X(n)$. There are natural maps $X_{(n+1)} \to X_{(n)}$ and the tower $\{\mathrm{Tot}\, X(n)\}$ is the same as $\{\mathrm{Tot}_n X\}$. The normalized homotopy of $X(n)$ satisfies

$$\pi_t' X(n)^s = \begin{cases} \pi_t' X^s & \text{if } s \leq n \\ 0 & \text{otherwise} . \end{cases}$$

Let X be as in the statement of Theorem 3.2.2. Since $(\underset{\sim}{\Sigma}^k X)^j = *$ for $j \le k-1$, it follows from the above and the Whitehead product structure in the homotopy spectral sequence that all $([\frac{n}{k}] + 1)$-fold Whitehead products in $\pi_*(\mathrm{Tot}_n \underset{\sim}{\Sigma}^k X) = \pi_*(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n))$ vanish. The space $\mathrm{Tot}_n(\underset{\sim}{\Sigma}^k X)$ is therefore nilpotent. It remains to show that $\{H_* X\} \to \{H_* \mathrm{Tot}\underset{\sim}{\Sigma}^k X(n))\}$ is a pro-isomorphism. Since $\mathrm{Tot}\underset{\sim}{\Sigma}^k X(k) \approx \Omega^k \Sigma^k X$, the map is a split monomorphism. We must therefore show that the tower $\{H_*(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n))/H_*(X)\}$ is pro-trivial - that is, for each n,j there exists an r with the property that the map $H_j(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n+r)) \to H_j(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n))/H_j(X)$ is zero.

Using the Hilton-Milnor theorem one easily shows that $\pi_t^{\prime}(\underset{\sim}{\Sigma}^k X^s) = 0$ for $s \ge t - 1$. It follows that $\pi_t^{\prime}(\underset{\sim}{\Sigma}^k X(n)^s) = 0$ for $s > \frac{n}{n+1}\, t$. By Theorems 3.1.1 and 3.2.5 the homology spectral sequence of $\underset{\sim}{\Sigma}^k X(n)$ converges strongly and satisfies $N^s H_t(\underset{\sim}{\Sigma}^k X(n)) = 0$ for $s > \frac{n}{n+1}\, t$. But the groups $N^s H_t(\underset{\sim}{\Sigma}^k X(n))$ coincide with $N^s H_t \underset{\sim}{\Sigma}^k X$ if $s \le n$, and hence vanish in the range $0 \le s \le n$, $s \ne k$. Finally, the map $H_* X \to H_*(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n))$ corresponds to the "edge" homomorphism $H_* X \to H_{*+k}(\Sigma^k X)$. It follows that the groups $H_{t-s}(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n))/H_{t-s} X$ can be obtained from a spectral sequence whose E^1_{st}-terms vanish in the range $0 \le s \le n$, $s > \frac{n}{n+1}\, t$. This implies that the map

$$H_j(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n+r))/H_j X \to H_j(\mathrm{Tot}\underset{\sim}{\Sigma}^k X(n))/H_j X$$

is the zero homomorphism if $r \ge n(j-1)$. This completes the proof of Theorem 3.2.2.

<u>Remark</u>. It was shown in the above proof that all $(n+1)$-fold Whitehead products vanish in $\pi_* \mathrm{Tot}_n(\underset{\sim}{\Sigma}^1 X)$. This gives another proof of Theorem 2.7.2, part v).

3.3 The tower of fundamental groups

If G is a group let $\Gamma_n G \subset G$ be the n-fold commutator subgroup. The case $k = 1$ of the following result was used in the proof of Theorem 2.7.2.

Theorem 3.3.1. Let X be a connected space. There are isomorphisms

i) $\pi_1 \mathrm{Tot}_{nk+r}(\underset{\sim}{\Sigma}^k X) \approx \pi_1 \mathrm{Tot}_{nk}(\underset{\sim}{\Sigma}^k X)$ if $0 \leq r < k$;

ii) $\pi_1 \mathrm{Tot}_{nk}(\underset{\sim}{\Sigma}^k X) \approx \pi_1 X / \Gamma_{n+1} \pi_1 X$.

Proof: Using the Hilton-Milnor theorem, one easily checks that the groups $\pi'_{s+1} \underset{\sim}{\Sigma}^k X^s$ are zero for $s \not\equiv 0(k)$, and that under the Whitehead product structure, every element of $\pi'_{nk+1} \underset{\sim}{\Sigma}^k X^{nk}$ can be expressed as an n-fold Whitehead product of elements of $\pi'_{k+1} \underset{\sim}{\Sigma}^k X^k = \pi_{k+1} \Sigma^k X \approx (\pi_1 X)_{ab}$. Part i) follows easily from this. For part ii) let us abbreviate $\pi_1 \mathrm{Tot}_{nk}(\underset{\sim}{\Sigma}^k X)$ as $\pi(n)$. We first claim that $\ker(\pi(n+r) \to \pi(n))$ is isomorphic to $\Gamma_{n+1}\pi(n+r)$. That it contains Γ_{n+1} follows from the Whitehead product structure of the spectral sequence. On the other hand, any representative of $x \in \ker(\pi(n+r) \to \pi(n))$ on the E^1 term of the homotopy spectral sequence can be written as an (n+1)-fold commutator of elements of $\pi'_{k+1} \underset{\sim}{\Sigma}^k X^k \approx (\pi_1 X)_{ab}$ -- all of which survive the spectral sequence. It follows that x is an element of Γ_{n+1} modulo $\ker(\pi(n+r) \to \pi(n+1))$. The claim now follows by induction on r. From this we can conclude that the map $\pi(n) \to \pi(1) \approx (\pi_1 X)_{ab}$, $n \geq 1$, is the abelianization and hence, since $\pi(n)$ is nilpotent, that $\pi_1 X \to \pi(n)$ is surjective.

Let $F_{n+1} = \ker(\pi_1 X \to \pi(n))$ and $\Gamma_{n+1} = \Gamma_{n+1}\pi_1 X$. It follows from the above that $\Gamma_{n+1} \subset F_{n+1}$ and that the composite $\Gamma_{n+1} \to F_{n+1} \to F_{n+1}/F_{n+r}$ is surjective for all $r \geq 0$. We now appeal to [6] for pro-isomorphisms

$$\{\pi(n)\} \approx \{\pi_1 \mathrm{Tot}_s \underset{\sim}{\Sigma}^k X\} \approx \{\pi_1 \mathbb{Z}_s X\} \quad \text{([6] p.I, ch.III 6.4 and Theorem 3.2.2)}$$

$$\approx \{\pi_1 \mathbb{Z}_s K(\pi_1 X, 1)\} \quad \text{([6] p.1, ch.V, 5.1)}$$

$$\approx \{\pi_1 X/\Gamma_{s+1}\} \quad \text{([6] p.1, ch.IV, 2.4).}$$

It follows that for each $n > 0$ there is an r such that $F_{n+r} \subset \Gamma_{n+2}$. This implies that $\Gamma_{n+1} \to F_{n+1}/\Gamma_{n+2}$ is surjective and so $\Gamma_{n+1} \to F_{n+1}$ must also be. This completes the proof.

3.4 Recognition principles

The following are the recognition principles associated with Theorems 3.2.1 and 3.2.2.

Theorem 3.4.1. Let X be a simplicial space such that

i) $X_0, \ldots, X_{k-1}$ are contractible;

ii) The map $P_n = \prod_{j=1}^{\binom{n}{k}} p_j^* : X_n \to \prod^{\binom{n}{k}} X_k$ is a weak equivalence where p_j runs through the $\binom{n}{k}$ injections $[k] \to [n]$ satisfying $p_j(0) = 0$. Then the map $X_k \to \Omega^k|X|$ weakly adjoint to the inclusion $\Sigma^k X_k \approx |X|^k \to |X|$ is a weak equivalence.

Proof: That X satisfies the π_*-Kan condition and that the homotopy spectral sequence converges and collapses to its edge homomorphism follow from the discussion preceding the proof of Theorem 3.2.1. The theorem follows easily.

Theorem 3.4.2. Let X be a fibrant cosimplicial space such that

i) $X^0, \ldots, X^{k-1}$ are contractible;

ii) X^k is k-connected;

iii) For each n the map $i_n = \bigvee_{j=1}^{\binom{n}{k}} p_{j*} : \bigvee^{\binom{n}{k}} X^k \to X^n$ is a weak equivalence where p_j runs through the $\binom{n}{k}$ injections $[k] \to [n]$ satisfying $p_j(0) = 0$. If in addition the homotopy spectral sequence of X converges strongly then the map $\Sigma^k \mathrm{Tot} X \to X^k$ weakly adjoint to the projection $\mathrm{Tot}\ X \to \mathrm{Tot}_k X \approx \Omega^k X^k$ is a weak equivalence.

Remark. Strong convergence is guaranteed if for example X^k is k+1-connected, or more generally, if for each j, the set $\{\pi'_{n+j} X^n | n \geq 0\}$ contains only finitely many non-zero groups. Some condition like this is necessary since, for example, the map $\Sigma^k \mathrm{Tot}(\underset{\sim}{\Sigma}^k \mathbb{R}P^2) \to \Sigma^k \mathbb{R}P^2$ is not a weak equivalence since $\mathbb{R}P^2$ is $\mathbb{Z}$ bad ([6], p.1, ch.VII, 5.2).

Proof: It follows as in the proof of Theorem 3.2.2 that the map $\{\Sigma^k \mathrm{Tot}_n X\} \to \{X^k\}$ is a pro-homology equivalence. Since all spaces involved are simply connected it is a weak pro-homotopy equivalence. On the other hand the map $\{\mathrm{Tot}\ X\} \to \{\mathrm{Tot}_n X\}$ is a weak pro-homotopy equivalence by assumption. This implies that $\{\Sigma^k \mathrm{Tot}\ X\} \to \{\mathrm{Tot}_n X\}$ is also, and hence so is $\{\Sigma^k \mathrm{Tot}\ X\} \to \{X^k\}$. This completes the proof.

Theorems 3.4.1 and 3.4.2 should be compared to [30], Proposition 1.5.

It is interesting to replace the category of spaces with the category of sets, and the weak equivalences with isomorphisms in the statement of Theorem 3.4.1. When k = 1 one is demanding a set X with a distinct element e, and a binary operation $\circ : X \times X \to X$ satisfying

i) $x \circ x = e$;

ii) $e \circ x = x$;

iii) $(x \circ y) \circ (z \circ y) = x \circ z$.

These axioms are equivalent to the group axioms with the operation $\circ$ corresponding to $(x,y) \to y^{-1}x$. Axiom iii) corresponds to associativity. When $k \geq 2$ one finds a description of abelian groups based on the $(k+1)$-ary operation corresponding to

$$(x_0,\ldots,x_k) \to \Sigma(-1)^i x_i \ .$$

Let us call a space Y an $A_m(k)$ space if there exists a simplicial space X satisfying the conditions of Theorem 3.4.1 for $n \leq m$, with X_k weakly equivalent to Y. By the above discussion, an $A_2(1)$ space is an h-space with an inverse, and an $A_3(1)$ space is an associative h-space with inverse. The notion of an $A_m(k)$-space therefore provides a theory of higher associativity for k-fold loop spaces - and a dual theory for k-fold suspensions. The following theorem makes precise the sense in which the differentials in the spectral sequences of Corollaries 3.2.3 and 3.2.4 are the obstructions to maps $S^n \to \Sigma^k X$ or $\Omega^k X \to K(A,n)$ being compatible with this higher structure.

<u>Theorem 3.4.3</u>. i) Let $\{E^r_{st}, d_r\}_{r\geq 1}$ be the homotopy spectral sequence for $\underset{\sim}{\Sigma}^k X$. An element $\alpha \in E^1_{k,n} \approx \pi_n \Sigma^k X$ satisfies $d_j(\alpha) = 0$ for $1 \leq j \leq r$ if and only if there is a cosimplicial space S weakly equivalent to $\underset{\sim}{\Sigma}^k S^{n-k}$ in codimensions $\leq r + k - 1$, and a map $S \to X$ which represents α in codimension k.

ii) Let $\{E_r^{st}, dr\}_{r\geq 1}$ be the cohomology spectral sequence of the cosimplicial space $\underset{\sim}{\Omega}^k X$ with coefficients in an abelian group A. An element $\alpha \in E_1^{kn} \approx H^n(\Omega^k X;A)$ satisfies $d_j(\alpha) = 0$ for $1 \leq j \leq r$ if and only if there is a cosimplicial space K weakly equivalent to $\underset{\sim}{\Omega}^k K(A,n+k)$ in dimensions $\leq r + k - 1$, and a map $\underset{\sim}{\Omega}^k X \to K$ which represents α in dimension k.

Proof: We will prove part i). The proof of ii) is dual. For a fibrant cosimplicial pointed space Y let $F_n^m Y$ be the fibre of $\mathrm{Tot}_m Y \to \mathrm{Tot}_{n-1} Y$. An element $\alpha \in E_{st}^1 = \pi_t' Y^s$ is an element of $\pi_{t-s} F_s^s$. It survives to E^r if and only if it is in the image of $\pi_{t-s}(F_s^{s+r-1} Y) \to \pi_{t-s}(F_s^s Y)$. The functor F_n^m is right adjoint to $\underset{\sim}{\Delta}^m / \underset{\sim}{\Delta}^{n-1} \wedge -$. It follows that $\alpha \in \pi_n \Sigma^k X = E_{k,n}^1$ survives to E^r if and only if there is a map $\underset{\sim}{\Delta}^{k+r-1} / \underset{\sim}{\Delta}^{k-1} \wedge S^{n-k} \to \underset{\sim}{\Sigma}^k X$ representing α in codimension k. This completes the proof.

Appendix B

Bousfield's work on the homology spectral sequence of a cosimplicial space

The results of Chapter 3 depended on Bousfield's Theorem 3.1.1. Since this result has not yet been published we will give a sketch of its proof in this appendix. This material is taken from a private communication and the author is very grateful to Professor Bousfield for making it available, and for his permission to reproduce it here. A homology spectral sequence for cosimplicial spaces has also been obtained by Z. Wojtkowiak (unpublished) and it was through him that this author first learned of its existence.

The proof of Theorem 3.1.1 requires several preparatory lemmas. A filtration $X \to Y$ between cosimplicial spaces is said to be minimal if each $X^s \to Y^s$ is a minimal fibration. The following lemma can be proved using the techniques of [6] ch.X.

Lemma B.1. A fibration $X \to Y$ of termwise connected cosimplicial spaces, is minimal if and only if the associated fibrations $X^s \to Y^s \times_{M^sX} M^sY$ are minimal where M^s is the matching space functor ([6] ch.X, 4.5).

If X is a cosimplicial space, the left (resp. right) Kan extension of the restriction of X to the full subcategory of Δ consisting of $[0],\ldots,[n]$ will be denoted sk^nX (resp. csk^nX). Recall that the maximal augmentation $a(X)$ of X is the equalizer of $X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1$.

<u>Lemma B.2</u>. Let $u : W \to Y$ be a map of termwise connected cosimplicial spaces inducing a weak equivalences $a(W) \to a(Y)$ and $W^0 \to Y^0$. Then u can be factored as a composition $W \xrightarrow{e} X \xrightarrow{f} Y$ where e is a weak equivalence and f is a minimal fibration.

<u>Proof</u>: Let $X^0 = Y^0$ with $e = u : W^0 \to X^0$ and $f = \mathrm{Id} : X^0 \to Y^0$. To construct X^1, form the pushout

$$\begin{tikzcd}
W^0 \underset{a(W)}{\cup} W^0 \arrow[r, "{(d^0,d^1)}"] \arrow[d, "{(u,u)}"'] & W^1 \arrow[d, "\beta"] \\
Y^0 \underset{a(W)}{\cup} Y^0 \arrow[r] & B
\end{tikzcd}$$

and observe that β is a weak equivalence since (u,u) is a weak equivalence and (d^0,d^1) is a cofibration. Factor the obvious map $B \to Y^1$ as $B \xrightarrow{\gamma} X^1 \xrightarrow{f} Y^1$ where γ is a weak equivalence and f is a minimal fibration. Let $e = \gamma\beta : W^1 \to X^1$ be the composite weak equivalence, and let $d^0, d^1 : X^0 \to X^1$ and $s^0 : X^1 \to X^0$ be the obvious maps. Note that γf is a cofibration since $f\gamma f$ is, and thus $a(X) = a(Y)$. Now suppose inductively that $\{W^j \xrightarrow{e} X^j \xrightarrow{f} Y^j\}_{0 \le j \le s}$ has been constructed for some $s \geq 2$ and consider the following diagram

$$\begin{tikzcd}
(sk^{s-1}W)^s \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & W^s \arrow[rr] \arrow[d, "\beta"] & & (csk^{s-1}W)^s \arrow[d] \\
(sk^{s-1}X)^s \arrow[r] \arrow[d] & B \arrow[r] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & (csk^{s-1}X)^s \arrow[d] \\
(sk^{s-1}Y)^s \arrow[rr] & & Y^s \arrow[r] & (csk^{s-1}Y)^s
\end{tikzcd}$$

.

Since $a(W) \to a(X)$ and $W^j \to X^j$ are weak equivalences for all $j < s$, $(sk^{s-1}W)^s \to (sk^{s-1}X)^s$ is also. Thus, since α is a cofibration, β is a weak equivalence. Factor the map $B \to C$ as a composite $B \xrightarrow{\gamma} X \xrightarrow{\eta} C$ where γ is a weak equivalence and η is a minimal fibration. Then let e and f be the maps $\gamma\beta : W^s \to X^s$ and $\theta\eta : X^s \to Y^s$ respectively, and define the obvious coface and codegeneracy maps between X^{s-1} and X^s. This completes the inductive step. The resulting map $f : X \to Y$ is a minimal fibration by Lemma B.1. This completes the proof.

If X is a cosimplicial space, G a cosimplicial group and $X \times G \to X$ a principal action, the orbit map $p : X \to X/G = B$ is a fibration, but <u>not</u> necessarily induced from a map to $\overline{W}G$. It is easy to see that p is an induced fibration precisely when it admits a pseudo-cross section. A <u>pseudo-cross section</u> is a cross section compatible with all of the simplicial and cosimplicial operators, except possibly the simplicial d_0. It can also be shown that the principal fibration p admits a pseudo cross section if and only if the induced map $a(X) \to a(B)$ does. This holds in particular if B is cofibrant.

We can now introduce a Moore-Postnikov tower for a cosimplicial space. Let X be a fibrant cosimplicial space with each X^s connected. There is a fibration $q : X \to csk^0X$. Factor the map $\phi \to csk^0X$ into a cofibration followed by a trivial fibration $B \to X$, and form the fibre square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "j"'] & X \arrow[d] \\ B \arrow[r] & \mathrm{csk}^0 X \end{tikzcd}$$

Finally, factor j into a weak equivalence followed by a minimal fibration $f : \underline{X} \to B$. The map f is isomorphic in $h_0[\cdot \to \cdot, \underline{S}]$ to q. One can now form the Moore-Postnikov factorization of f

$$\underline{X} \to \cdots \to \underline{X}^{(n)} \to \underline{X}^{(n-1)} \to \cdots \to \underline{X}^{(0)} = B .$$

For $n \geq 1$ and $s \geq 0$, let $\pi_n X^s \backslash \pi_n X^0$ denote the kernel of the codegeneracy map $\pi_n X^s \to \pi_n X^0$. The cosimplicial simplicial abelian group $K(\pi_n \underline{X} \backslash \pi_n \underline{X}^0, n)$ acts principally on $\underline{X}^{(n)}$, making $\underline{X}^{(n)} \to \underline{X}^{(n-1)}$ a principal fibration. Since $\underline{X}^{(n)}$ and $\underline{X}^{(n-1)}$ are cofibrant, a pseudo-cross section exists, and there is a fibre square

$$\begin{tikzcd} \underline{X}^{(n)} \arrow[r] \arrow[d] & WK(\pi_n \underline{X} \backslash \pi_n \underline{X}^0, n) \arrow[d] \\ X^{(n-1)} \arrow[r] & \overline{W}K(\pi_n X \backslash \pi_n X^0, n) \end{tikzcd}$$

for each $n \geq 1$.

A cosimplicial (pointed) space is said to be <u>convergent</u> if the tower map

$$\{\tilde{H}_i(\mathrm{Tot}_s X)\} \to \{\pi_i \mathrm{Tot}_s \mathbb{Z}X\} \approx \{H_i(T(\mathbb{Z}X)/F^{s+1}T(\mathbb{Z}X))\}$$

is a pro-isomorphism for each i.

<u>Lemma B.3</u>. Let $X \xrightarrow{f} Z \xleftarrow{s} Y$ be a diagram of convergent fibrant cosimplicial spaces with g a fibration. For each $s \geq 0$ suppose that Y^s is contractible and that X^s and Z^s are connected and nilpotent with $\pi_1 X^s \to \pi_1 Z^s$ onto. For each sufficiently large s suppose that $\mathrm{Tot}_s X$ and $\mathrm{Tot}_s Z$ are connected and that $\pi_1 \mathrm{Tot}_s X \to \pi_1 \mathrm{Tot}_s Z$ is surjective. Then $X \times_Z Y$ is convergent.

<u>Proof</u>: The proof is similar to that of [4], Lemma 7.1, and uses [12] along with the fact that each $\mathrm{Tot}_s X$ is nilpotent when X is fibrant with each X^s nilpotent.

<u>Lemma B.4</u>. For $n \geq 1$ let M be a cosimplicial abelian group with $\pi^s M = 0$ for $s \geq n$. Then the cosimplicial space $K(M,n)$ is convergent.

<u>Proof</u>: For an abelian group A let $K(A,{}^s_n)$ be the cosimplicial simplicial abelian group with

$$N^j N_k K(A,{}^s_n) = \begin{cases} A & \text{if} \quad j = s \quad \text{and } k = n \\ 0 & \text{otherwise} \end{cases},$$

and let $E(A,{}^s_n)$ be the cosimplicial abelian group with

$$N^j N_k E(A,{}^s_n) = \begin{cases} A & \text{if} \quad j = s \ \text{ or } s+1 \quad \text{and} \quad k = n \\ 0 & \text{otherwise} \end{cases}$$

and with $\delta : N^s N_n E(A^s_n) \to N^{s+1} N_n E(A^s_n)$ the identity map. Since $\mathrm{Tot}_k E(A,{}^s_n)$ is contractible if $k > s$ and $n > s+1$ it is convergent. There is a principal fibration $K(A,{}^{s+1}_{n}) \to E(A,{}^s_n) \to K(A,{}^s_n)$. Since $K(A,{}^0_n)$ is clearly convergent, it follows by induction and Lemma B.3 that

$K(A,{}^s_n)$ is convergent for $s < n$. Now form a short exact sequence $M \to I \to J$ with I and J cosimplicial divisible abelian groups with divisible cohomotopy, and with $\pi^s I = \pi^s J = 0$ for $s \geq n$. $K(I,n)$ and $K(J,n)$ are convergent since each is a product of a cosimplicially contractible space with a finite product of $K(A,{}^s_n)$'s with $s < n$. The result now follows from Lemma B.3.

Proof of Theorem 3.1.1. Using the Moore-Postnikov tower together with Lemmas B.3 and B.4 one inductively shows that each $X^{(n)}$ is convergent. This implies (under the assumptions of the theorem) that X is convergent.

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for Chapter 1

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