Pricing Corporate Securities and Stochastic Differential Games

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A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2001

I would like to dedicate this thesis to my dear wife, Jasmin, for her
loving support while I have been working on this, and her
encouragement when I most needed it.

Acknowledgements

First and foremost I wish to thank my supervisors, Professor William Perraudin and Dr. Sam Howison, without whose support and guidance this thesis would not have been possible. In particular, I am very grateful to Professor Perraudin for introducing me to the field of strategic modelling in continuous time finance and stochastic differential games. I would like to thank Dr. Hilary Ockendon and other members of the Oxford Centre for Industrial and Applied Mathematics (OCIAM) for their support of my work.

I would like to also thank the EPSRC and ESRC for financial support in the form of a studentship from October 1997 to September 2000 and a research grant from October 2000 till March 2001, respectively.

All the chapters of this thesis consist of original work, with the exception of the first chapter, which provides a review of the literature along with motivation for the thesis itself, and the conclusion. Chapters 3 and 4 consist of joint work with Professor William Perraudin.

Abstract

This thesis studies the effect of strategic behaviour on the pricing and optimal exercise of corporate securities. The first chapter provides a review of strategic modelling in continuous-time finance along with important results from the stochastic differential games literature.

In the second chapter we show how incentives may arise to convert bonds strategically when convertible debt is held by perfectly competitive independent creditors. We analyse a symmetric Nash equilibrium involving the sequential exercise of bonds in which creditors are indifferent between immediate and delayed conversion at an endogenously varying threshold. The indifference condition dynamically incorporates pre-emptive incentives, arising from the gradual dilution of share value, as well as the incentive to delay, stemming from increasing firm credit worthiness. Perfect competition among convertible debt holders causes fresh equity to arrive earlier than in models of simultaneous exercise ("block" models), thereby suggesting an intrinsic delayed-equity characteristic of convertible offerings. The sequential equilibria are more efficient than the block strategies for more highly leveraged firms and yield larger convertible bond values. We analyse these interactions in a stylised continuous-time structural model and illustrate how the same methods may be used to price finite-maturity convertible bonds.

The third chapter relates to real option valuation and exercise with strategic behaviour and incomplete information. Firms that possess real options to cease production may act strategically if they acquire increased market power when a competitor quits. We analyse the equilibria that arise in a simple real options model when firms are engaged in such a war of attrition game. We show that firms may adopt randomised strategies in which exits are generated by conditionally Poisson jump processes. We

generalise the model with incomplete information regarding flow costs and show that as the number of cost types increases the scope for randomised exit is less. Surprisingly, in the limit of a continuum of types, firms revert to their monopoly exit triggers.

In the fourth chapter we show how default hazards similar to those suggested by the literature on reduced form credit risk models may arise purely from the strategic behaviour of indebted firms operating in a duopoly. In so doing, our research advances attempts to reconcile structural and reduced form approaches to modelling credit risk. In equilibrium, firm defaults are generated endogenously by a randomly evolving intensity and short credit spreads are strictly positive. We generalise the model to allow for incomplete information concerning firm types and show how this leads to default intensities that evolve in a path-dependent manner through Bayesian learning.

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Chapter 1

Strategic Modelling in Continuous-Time Finance

1.1 Introduction

The seminal methods of contingent claims analysis developed by Black and Scholes (1973) and Merton (1973b) have revolutionised the pricing of securities in continuous time.[1] An important assumption of much of this literature is the absence of strategic interactions between security holders. This supposition is made even though there exists a substantial theoretical and empirical finance literature that seeks to understand the influence of strategic behaviour on claim values. There are at least a few areas in which such strategic interactions are significant:

1. The Pricing and Exercise of Convertible Securities.
2. Real Option Valuation and Exercise.
3. Corporate Debt Valuation.
4. Sovereign Debt Valuation.

The earliest implementation of strategic models in continuous-time finance was in the context of warrant pricing and exercise. Since American warrants endow holders with an option to acquire shares within a given firm, as opposed to American call options, which are mere marketable side-bets, the dilution of new shares creates incentives for strategic behaviour. Emanuel (1983) studies the optimality of sequential exercise when the whole issue is hoarded by a monpolist, while Constantinides (1984) and Constantinides and Rosenthal (1984) study the case of perfect competition under differing reinvestment policies regarding the proceeds of warrant exercise (See also Spatt and Sterbenz (1988)). Although these articles do consider the broad area of convertible securities, there is little consideration of convertible bonds. In this regard Lewis (1991) examines strategic incentives among claim holders of a firm's convertible debt where the seniority class varies. Since the bonds are non-coupon bearing, the strategic incentives in his analysis only arise at the debt's maturity and as a result of the complexity of debt claims.

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[1] For a recent review of methods in continuous-time finance see Sundaresan (2000).

The real options literature has long recognised the fact that oligopolistic competitive interactions can have a substantial impact on the timing of entry and exit into certain markets.[2] Smets (1993) develop a leader-follower equilibrium which has been applied by Grenadier (1996) in the context of real estate development (see also Grenadier (1999)). In a well known empirical case study by Ghemawat (1986), he shows that the discount retail oulet Wal-Mart devised a strategy of opening stores in the South Western U.S. in areas that were clearly too small to contain more than one such outlet. The type of first-mover advantages that Wal-Mart ceased were recognised by Lambrecht and Perraudin (2002) in their equilibrium model of pre-emptive entry with incomplete information.

Several authors have also applied the real options approach to situations where there is perfect competition in an industry. Leahy (1993) shows under typical conditions of profit uncertainty that a perfectly competitive firm should enter a market myopically, as it would with no strategic interactions. Bartolini (1993) also shows that there may be situations under which 'competitive runs' can take place, where firms suddenly enter or exit a market simultaneously.

In a series of important empirical papers, Franks and Torous (1989) and Franks and Torous (1994) study the Chapter 11 bankruptcy process in the US. Their key findings illustrate that the absolute priority rule in bankruptcy is frequently violated, legal proceedings are delayed and protracted and equity holders often extract subtantial concessions from debt holders in the run-up to bankruptcy. Using this important intuition Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) model the strategic interactions between debt holders and equity holders in a financially distressed firm. The former contribution considers a binomial setting and pays special attention to the design of debt contracts, while the latter employs a continuous-time model to model the strategic debt service from equity holders to creditors. By modelling such strategic interactions these structural credit risk models are able to replicate empirically observed default premia under typical parameter settings.[3]

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[2] See amongst others Spencer and Brander (1992), Kulatilaka and Perotti (1992), Trigeorgis (1991) and Smit and Trigeorgis (1997).

The approach initiated by Mella-Barral and Perraudin (1997) has since been extended by Mella-Barral (1999) to incorporate renegotiation of debt and by Fan and Sundaresan (2000) to include optimal dividend policy and other forms of bargaining. More recently, David (2001) has incorporated bargaining into a continuous-time model of corporate debt, where Poison putable bond holders may trigger liquidity crises in a firm by collectively exercising their put options.

The literature on strategic modelling of corporate debt is not confined, however, solely to intra-firm strategic behaviour. A number of studies have examined strategic behaviour between firms within an industry (i.e. an oligopoly) as well as the pricing of debt in a full-blown industry equilibrium. In a real options set-up similar to that employed by Leahy (1993), Fries, Miller, and Perraudin (1997) show how entry and exit by competitive firms in an industry affects the pricing of debt and equity.

As to the former oligopoly models, Lambrecht and Perraudin (1996) price perpetual corporate bonds in a duopoly, when the lenders may force the liquidation of financially distressed firms under the Chapter 7 bankruptcy code. In this set-up there are first-mover advantages in that the first to liquidate the firm obtains a larger value. Lambrecht (2001) also considers a pair of firms in financial distress, where the equity holders of each firm decide bankruptcy through their inability to service debt payments. Since the firm to wait longest enjoys a temporary monoply, there are second-mover advantages.

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[3] Jones, Mason, and Rosenfeld (1984) were among the first to recognise that *non-strategic* structural models consistenly under-estimate credit spreads with reasonable parameters.

Of the four afore-mentioned areas of strategic behaviour in continuous-time finance, sovereign debt is the only one which has no direct relation to the pricing of *corporate securities.* Most approaches to price bonds issued by sovereigns have followed the reduced-form pricing methodology.[4] A strand of literature, however, has sought to explain the relatively larger default premia on sovereign bonds compared with corporate bonds. Gibson and Sundaresan (2001) devise a strategic model of sovereign debt, where lenders may impose trade sanctions and cease a small fraction of the country's exports, while sovereigns optimally default on their debt service. This approach has been extended by Chang and Sundaresan (2001) to include the effect of sovereign credit history and reputation. A crucial aspect of such analyses is that, unlike corporate debt, sovereigns are not bound by any bankrutpcy code. Other studies in sovereign debt, although not strategic in the game-theoretic sense, price sovereign bonds under the assumption that missed payments are rolled over (see Miller and Zhang (1999) and Bartolini and Dixit (1991)).

1.2 Differential Games

1.2.1 Non-Cooperative Game Theory

The literature on strategic modelling, which was reviewed in the previous section, uses methods from non-cooperative game theory to determine the outcome of the strategic incentives. Our purpose in this sub-section is to briefly review some of the key concepts involved in a somewhat abstract manner. In these matters Fudenberg and Tirole (1993) is an excellent reference.

The cornerstone of non-cooperative game theory is the concept of a Nash equilibrium (see Nash (1950)). Suppose we have N players in a game, indexed by $i \in \{1, N\}$, with strategies $\sigma_i \in S_i$, where S_i is player i's action set. Player i's arbitrary payoff is $V_i(\sigma_{-i}, \sigma_i)$, where σ_{-i} represents the collective strategies of the other $N-1$ agents with $j \neq i$.

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[4] See Litterman and Iben (1991), Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997) and Duffie and Singleton (1999) among others.

Definition 1 *The strategies $\sigma_i^* \in S_i$, $i \in \{1, N\}$, form a pure strategy Nash equilibrium if*

$$V_i\left(\sigma_{-i}^*, \sigma_i^*\right) \geq V_i\left(\sigma_{-i}^*, \sigma_i\right), \qquad i \in \{1, N\} \tag{1.1}$$

for all $\sigma_i \in S_i$.

Players take the others' actions as given (i.e. σ_{-i}^*) and adopt the strategy that maximises their own value (i.e. σ_i^*). In a Nash equilibrium players have no incentives to deviate from their strategy in (1.1), as this cannot possibly raise their value.

The equilibrium in definition 1 is one in *pure strategies*; agents will certainly act according to σ_i. This may be contrasted with *mixed-strategy* (or randomised) Nash equilibria, in which players act randomly over their possible actions. Often this takes the form of a probability distribution over the players' action set. Once again, agents have no incentives to deviate from their randomised strategies given the other's randomised strategies.

1.2.2 Stochastic Differential Games

Many applications of non-cooperative game theory in financial economics are restricted to discrete time models where the action set is finite. Often these models employ a binomial tree for future realisations. By applying the game theoretic methods of the last sub-section to such simple models, Nash equilibria can often be derived with relative ease. In applying classical game theory to continuous-time finance models, however, a number of extensions are required. First, future realisations are given by stochastic diffential equations as opposed to the simple binomial model. Second, agents can act at any point in time, as we are working in continuous time. Third, because of the possible dynamics of the stochastic differential equations, we must look at solution concepts that allows for infinite dynamic games. Conversely, finite dynamic games allow only a finite action set for agents. Since the diffusion process allows for a continuum of possible realisations, we have *de facto*, infinite actions sets.

As a branch of differential games, stochastic differential games provide the framework within which strategic behaviour can be examined for continuous-time models.

The theory of stochastic differential games has evolved from the marriage of (i) methods of optimal stochastic control, and (ii) the afore-mentioned (static) game theory. The former is quite familiar to mathematicians and probabilists while the latter is familiar to financial economists. Since the audience of these two fields are quite diverse, the stochastic differential games literature has been couched either in more mathematical terms or as a more sophisticated type of extended form games.

Stochastic differential games fall into the two categories of (i) zero-sum, and (ii) nonzero-sum; the former entails payoffs to the players that add up to zero, while the latter has payoffs that do not add up to zero. This difference means that the former are easier to compute, especially in the 2-person case. The reason for this is that players then have incentives to maximise and minimise the same value function, leading to simplified min-max conditions for zero-sum stochastic differential games. In the rest of this chapter we shall suppose that all the games are nonzero-sum for generality.

1.3 Stochastic Differential Games with Continuous Controls

The earliest exposition of N-person nonzero-sum stochastic differential games by Friedman (1972) examines a system of m stochastic differential equations:

$$d\xi_t = f\left(t, \xi_t, u_{1t}, ..., u_{Nt}\right) dt + \sigma\left(t, \xi_t\right) dB_t \tag{1.2}$$

where $B_t = (B_{1t}, ..., B_{mt})$, the individual B_{it} are independent Brownian motions and $u_{it} \in U_i$ are the continuous controls of each of the i players in the strategy spaces, U_i. $f(t, \xi_t, u_{1t}, ..., u_{Nt})$ is an n-vector defined on $[0, \infty) \times R^n \times U_1 \times ... \times U_N$ and $\sigma(t, \xi_t)$ is an $n \times n$ matrix defined on $[0, \infty) \times R^n$. The initial condition is

$$\xi_{t=0} = \xi_0. \tag{1.3}$$

An important feature of stochastic differential games is the information pattern accessible by the agents, and these may be summarised as follows.

Definition 2 *In an N-person stochastic differential game of duration $[0, T]$, the information structure for agent i, η_{it}, is said to be:*

1. *open-loop (OL), if* $\eta_{it} = \{\xi_0\} \;\; \forall \;\; t \in [0, T]$.

2. *closed-loop perfect state (CLPS), if* $\eta_{it} = \{\xi_s : s \in [0, t]\} \;\; \forall \;\; t \in [0, T]$.

3. *memoryless perfect state (MPS) if* $\eta_{it} = \{\xi_0, \xi_t\} \;\; \forall \;\; t \in [0, T]$.

4. *feedback (perfect state) (FB) if* $\eta_{it} = \{\xi_t\} \;\; \forall \;\; t \in [0, T]$.

The information pattern represents the scope for observation by the agents. In the case of an open-loop (OL) information pattern, for instance, agents only have access to the state variable ξ_t at $t = 0$ and must condition their actions on this for all time. Conversely, under a feedack (FB) information pattern, agents have access to the realisations of ξ_t for all time.

The objective of each of the N players is to minimise[5] their cost functionals, J_i, under a feedback (perfect state) information pattern:

$$J_i\left(u_{1t}, ..., u_{Nt}\right) = \mathrm{E}_t\left\{\int_t^\tau h_i\left(s, \xi_s, u_{1s}, ..., u_{Ns}\right) ds + g_i\left(\tau, \xi_\tau\right)\right\} \tag{1.4}$$

where $h_i(t, \xi_t, u_{1t}, ..., u_{Nt})$ and $g_i(t, \xi_t)$, $i \in \{1, N\}$, are n-vectors defined on $[0, \infty) \times R^n \times U_1 \times ... \times U_N$ and $[0, \infty) \times R^n$ respectively. The term h_i in (1.4) is a cost per unit time incurred up until τ while g_i can be regarded as a terminal cost incurred at time $t = \tau$.

By setting

$$a_{ij}\left(t, \xi\right) = \frac{1}{2}\sigma\left(t, \xi\right)\sigma\left(t, \xi\right)^T \tag{1.5}$$

where $\sigma\sigma^T$ is positive definite, Fleming and Nisio (1966) have showed that the cost functional is

$$J_i = V_i(t, \xi_t) \tag{1.6}$$

where V_i is the solution of

$$\frac{\partial V_i}{\partial t} + \sum_{i,j=1}^{m} a_{ij}\left(t, \xi_t\right)\frac{\partial^2 V_i}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{m} f\left(t, \xi_t, u_1\left(t, \xi\right), ..., u_N\left(t, \xi\right)\right)\frac{\partial V_i}{\partial \xi_i} + h_i\left(t, \xi_t, u_1\left(t, \xi\right), ..., u_N\left(t, \xi\right)\right) = 0 \tag{1.7}$$

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[5] In the context of financial models it is more natural to think of choosing controls to maximise claim values. Such a modification can easily be incorporated into this setting.

with

$$V_i(\tau, \xi) = g_i\left(\tau, \xi_\tau\right). \tag{1.8}$$

One may note that the same partial differential equation should result by simply applying the Feynman-Kac representation formula to (1.4). Friedman describes the system of equations (1.6), (1.7) and (1.8) as an N-person stochastic differential game with perfect observation. Since the information pattern he selected is feedback, he also designated this a "Markovian" game. Alternatively, equations (1.2), (1.3), and (1.4) represent the same differential game. The i-th Hamiltonian associated with the stochastic differential game, (1.6), (1.7), (1.8) is given by

$$\begin{aligned} H_i\left(t, \xi_t, u_1(t, \xi), ..., u_N(t, \xi), p_i\right) &= f\left(t, \xi_t, u_1(t, \xi), ..., u_N(t, \xi)\right) \cdot p_i \\ &\quad + h_i\left(t, \xi_t, u_1(t, \xi), ..., u_N(t, \xi)\right). \quad (1.9) \end{aligned}$$

Friedman shows that if agents choose their controls, u_i, in such a manner that their Hamiltonians are minimised and the a_{ij}, and functions f, g_i and h_i satisfy certain conditions then there exists a solution to the semilinear parabolic system

$$\begin{aligned} \frac{\partial \phi_i}{\partial t} + \sum_{i,j=1}^{m} a_{ij}\left(t, \xi_t\right) \frac{\partial^2 \phi_i}{\partial \xi_i \partial \xi_j} &+ f\left(t, \xi_t, u^*\left(t, \xi, \nabla_\xi \phi\right)\right) . \nabla_\xi \phi_i \\ &+ h_i\left(t, \xi_t, u^*\left(t, \xi, \nabla_\xi \phi\right)\right) = 0 \quad (1.10) \end{aligned}$$

where

$$\phi_i(\tau, \xi) = g_i\left(\tau, \xi_\tau\right), \tag{1.11}$$

which is continuous. In equation (1.10)

$$\nabla_\xi \phi = \left(\nabla_\xi \phi_1, ..., \nabla_\xi \phi_N\right).$$

Furthermore, under these same conditions, the strategies $u^*(t, \xi_t, \nabla_\xi \phi)$ constitute a Nash equilibrium.

One may note that most expositions of stochastic differential games pose the partial differential equation for the cost functional, (1.10), directly as a Hamilton-Jacobi-Bellman (HJB) equation with minimum operators.[6] The minimisation of the Hamiltonians is synonymous with finding first-order conditions directly from the N-coupled

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[6] Once again in the case of financial problems with value maximising agents the minimum operators are replaced by maximum operators.

HJB partial differential equations.[7] Effectively, the PDE (1.10) already incorporates the minimising controls u^* that yield the first-order condition.

The existence of value in games has been examined in a different way by Uchida (1978) for nonzero-sum games[8] using martingale methods of Davis and Varaiya (1973) and Davis (1973). A crucial aspect of the analysis of Davis and Varaiya is that analogues of the time derivative and gradient of the value function can be constructed using a martingale method. Thus, the optimal value can be obtained by minimising the Hamiltonian at each point in time, without resorting to the study of the corresponding partial differential equation. It is shown under suitable technical assumptions that a Nash equilibrium holds if the *Nash condition* is satisfied. This latter condition requires that the controls be chosen in such a manner that the i-th Hamiltonian is minimised. Uchida (1979) latter found a sufficient condition for the Nash condition to hold.

For readers familiar with optimal continuous-parameter stochastic control[9] the connection with stochastic differential games with continuous controls will be clear. The former can be regarded as stochastic games when there is only player and the latter as control problems extended to more than one agent. In fact, the study by Friedman (1972) acknowledges the use of methods of optimal stochastic control to obtain the above results. Examples in the continuous-time finance literature of optimal continuous-parameter stochastic control include the work of Merton (1971) and Merton (1973a) in the context of inter-temporal portfolio decisions, the real option model of Brennan and Schwartz (1985), corporate debt models (see Brennan and Schwartz (1984)), transaction cost option pricing models (see Davis, Panas, and Zariphopoulou (1993) and Davis and Norman (1990)) and the pricing of options on a traded account (See Henderson and Hobson (2000) and Shreve and Vecer (2000)).

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[7] See, for instance, Basar and Olsder (1998).

[8] The analogous result for zero-sum games was presented by Elliott (1976) for 2 person games while sufficient conditions for the N-person case are studied by Varaiya (1976).

[9] For a review see Fleming (1969) and the book by Fleming and Soner (1993).

1.4 Stochastic Differential Stopping Games

The analysis of the previous section relates to stochastic differential games where there is a control being controlled continuously by each of the N players. Many finance problems involve discrete decisions, as opposed to such continuous decisions. In many senses the continuous control stochastic differential games can be regarded as an approximation of actual behaviour, as payments are usually made discretely. Moreover, it is quite difficult to motivate the continuous monitoring and altering of controls by claim holders, especially in the presence of standard market imperfections.

Examples of *single agent* optimal fixed decisions abound in continuous-time finance. Common examples include the optimal exercise of American options, equity holders' decision to optimally default on debt service payments, and irreversible entry and exit of firms in an industry. In many of these, the decision is made at the first hitting time of a boundary when the dynamics of the underlying are driven by a diffusion process. At the boundary the claim values satisfy some free-boundary condition when the problem is transformed to that of solving a partial differential equation.

In the game-theoretic, *multi-agent* context, N players seek to "stop" in such a way that their claim value is maximised. The order of stopping with respect to the other $N-1$ agents plays an important role in the value assigned to the players ex post. Bensoussan and Friedman (1974) and Bensoussan and Friedman (1977) study stochastic stopping games and determine the conditions under which a Nash equilibrum may exist. As discussed above, there is a strong link between such decision problems and free-boundary problems, and so Bensoussan and Friedman (1977) show that the claim values satisfy a system of variational inequalities when there is a Nash equilibrium.

1.4.1 Pure Strategy Equilibria

In this sub-section we briefly present the determination of a Nash equilibrium in pure strategies in the manner of Bensoussan and Friedman (1977) as stochastic stopping equilibria play an important role in all of the chapters of this thesis.

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and $\{\mathcal{F}_t \colon t \geq 0\}$ be a family of increasing σ-algebras. Let B_t be an n-dimensional Brownian motion with respect to $\mathcal{F}_t$, $g(x,t)$ be an n-vector defined on $R^n \times [0,\infty)$ and $\sigma(x,t)$ be an $n \times n$ matrix defined on $R^n \times [0,\infty)$, such that

$$g \text{ is continuous and bounded,} \tag{1.12}$$

$$|g(x,t) - g(x',t)| \leq C|x - x'| \text{ for all } x, x' \quad (C \text{ constant}), \tag{1.13}$$

$$\sigma \text{ is continuous and bounded,} \tag{1.14}$$

$$\partial\sigma(x,t)/\partial x \text{ is bounded and measurable,} \tag{1.15}$$

$$\sigma^{-1} \text{ is continuous and bounded.} \tag{1.16}$$

We denote by x_s, for $s \geq t$ the solution of the system of n diffusion processes

$$dx_s = g\left(x_s, s\right) ds + \sigma\left(x_s, s\right) dB_s, \qquad x_{s=t} = x_t \tag{1.17}$$

Let T be a positive number and $f_i(x,t)$, $\phi_i(x,t)$, $\psi_i(x,t)$, $h_i(x)$ $(i = 1,2)$ be functions such that

$$f_i, \phi_i, \text{ and } \psi_i \text{ are continuous and bounded in } R^n \times [0,T], \tag{1.18}$$

$$h_i \text{ is continuous and bounded,} \tag{1.19}$$

$$f_i \in L^2(R^n \times [0,T]), \tag{1.20}$$

$$\psi_i \leq \phi_i \tag{1.21}$$

for all x, t in $R^n \times [0,T]$ $(i = 1,2)$.

Under these assumptions the payoff functions, $J_t^i(\tau_1, \tau_2)$, are

$$\begin{aligned} J_t^i\left(\tau_1, \tau_2\right) &= \mathrm{E}_t \Bigg\{ \int_t^{\tau_1 \wedge \tau_2} f_i\left(x_s, s\right) ds + \phi_i\left(x_{\tau_i}, \tau_i\right) \mathcal{I}_{\tau_i < \tau_j} \\ &\qquad + \psi_i\left(x_{\tau_j}, \tau_j\right) \mathcal{I}_{\tau_i \geq \tau_j, T > \tau_j} + h_i\left(x_T\right) \mathcal{I}_{\tau_1 = \tau_2 = T} \Big\} \end{aligned} \tag{1.22}$$

where τ_i are stopping times, with $t \leq \tau_i \leq T$ and $\mathcal{I}$ are indicator functions. As in the previous section, the payoffs can be regarded as cost functionals, where f_i are running costs per unit time until either of the two players "stop", at which time sunk costs of ϕ_i or ψ_i are incurred. If the game terminates before either of the twin agents stop the sunk costs are h_i, $i = \{1,2\}$. Unlike the games of continuous parameter control, the only strategic variables here are the stopping times, τ_i. The first player to stop incurs the cost ϕ_i while the second-mover incurs ψ_i. Thus, we have a war of attrition game as $\phi_i \geq \psi_i$.[10] There are second-mover advantages because the sunk cost of moving second is less than that of moving first.

Following definition 1, we can define a Nash equilibrium for such nonzero-sum stopping games:

Definition 3 *The stopping times, $\hat{\tau}_i$, $i = \{1, 2\}$ are pure stopping times and form a pure strategy Nash equilibrium if*

$$J_t^1 (\hat{\tau}_1, \hat{\tau}_2) \leq J_t^1 (\tau_1, \hat{\tau}_2), \qquad J_t^2 (\hat{\tau}_1, \hat{\tau}_2) \leq J_t^2 (\hat{\tau}_1, \tau_2), \tag{1.23}$$

for any τ_1, τ_2.

Now consider two functions, $V_1(x,t)$, $V_2(x,t)$, where $Q = R^n \times (0,T)$, such that

$$V_i \text{ is continuous and bounded in } \overline{Q}, V_i \in L^2 \left(0, T; H^1 \left(R^n\right)\right); \tag{1.24}$$

$$V_i (x, T) = h_i (x) \qquad (x \in R^n); \tag{1.25}$$

$$V_i (x, t) \leq \phi_i (x, t) \quad \text{in } Q; \tag{1.26}$$

$$\begin{aligned} &\text{if } V_j (x, t) = \phi_j (x, t) \text{ for some } (x, t) \text{ in } Q, \\ &\qquad \text{then } V_i (x, t) = \psi_i (x, t); \end{aligned} \tag{1.27}$$

$$\begin{aligned} &\text{if } D_i = \{(x, t) \in Q; V_j (x, t) < \phi_j (x, t) \text{ for } j \neq i\}, \text{ then} \\ &\frac{\partial V_i}{\partial t} + \sum_{j,k=1}^{n} \frac{a_{jk} (x, t)}{2} \frac{\partial^2 V_i}{\partial x_j \partial x_k} + \sum_{j=1}^{n} g_j (x, t) \frac{\partial V_i}{\partial x_j} \in L^2 (D_i), \\ &\frac{\partial V_i}{\partial t} + \sum_{j,k=1}^{n} \frac{a_{jk} (x, t)}{2} \frac{\partial^2 V_i}{\partial x_j \partial x_k} + \sum_{j=1}^{n} g_j (x, t) \frac{\partial V_i}{\partial x_j} \leq f_i \quad \text{a.e. in } D_i, \\ &[V_i - \phi_i] \left[\frac{\partial V_i}{\partial t} + \sum_{j,k=1}^{n} \frac{a_{jk} (x, t)}{2} \frac{\partial^2 V_i}{\partial x_j \partial x_k} + \sum_{j=1}^{n} g_j (x, t) \frac{\partial V_i}{\partial x_j} - f_i\right] = 0 \quad \text{a.e. in } D_i. \end{aligned} \tag{1.28}$$

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[10] If $\phi_i < \psi_i$ we would have a pre-emption game.

In the above system of differential inequalities $a_{jk}(x,t)$ are the components of the matrix $a(x,t) = \sigma\sigma^T(x,t)$, and $g_j(x,t)$ are the components of $g(x,t)$.

To the value V_i, one can assign a set C_i

$$C_i = \{(x,t) \in Q; V_i\,(x,t) < \phi_i\,(x,t)\} \tag{1.29}$$

Clearly, $C_i = D_j$ for $i(\neq j) \in \{1,2\}$ and the exit time of C_i is defined by

$$\hat{\tau}_i = \inf\{s; t < \leq T, x_s \notin C_i\} \tag{1.30}$$

We can now reproduce the Theorem due to Bensoussan and Friedman (1977).

Theorem 1 *Let the assumptions of this sub-section hold. If there exists functions,* V_1, V_2 *satisfying (1.24)-(1.28), then for any* x, t, *the* $\hat{\tau}_i$ *given by (1.30) form a pure strategy Nash equilibrium. Furthermore,*

$$V_i\,(x,t) = J_t^i\,(\hat{\tau}_1, \hat{\tau}_2)\,. \tag{1.31}$$

The proof of this involves the use of an extension to Ito's Lemma (see pages 279-280 of Bensoussan and Friedman (1977)). In the same article, a Theorem is presented in the time-independent case where the function f_i takes the form $\exp[-r(s-t)]f_i^*$ and $T \to +\infty$.

The similarity between free-boundary problems, such as pricing American put options, and nonzero-sum stochastic stopping games is evident from their formulation as variational inequalities above. In particular, note the linear complementarity form in (1.28). The Theorem due to Bensoussan and Friedman above states that a pure-strategy Nash equilibrium exists in a stochastic differential stopping game if there is solution to the system of variational inequalities. Moreover, the stopping times are synonymous with the pure strategies of players in the Nash equilibrium. In general, these stopping times must be determined as part of the solution of the payoffs.

This approach to determine pure-strategy equilibria can be contrasted with a martingale based approach, which examines the stopping times themselves. In their unpublished papers, Huang and Li (1986) develop such an approach for general stochastic stopping games in continuous-time, and apply it to problems of entry and exit in Huang and Li (1992). More recently, Dutta and Rustichini (1993) have studied a special class of Markov Perfect equilibria[11] in such stopping games.

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[11] We consider this class of equilibria in the next section.

1.4.2 Mixed Strategy Equilibria

What happens to the equilibrium when the players may adopt randomised strategies? Appealing to the construction of mixed-strategies as a probability distribution over the agents' action space, one can see that the stopping times are now *randomised.* At a given point in time, agents randomly decide to "stop" given the other $N-1$ agents' probability of doing so.

The construction of such mixed-strategy stopping equilibria in *discrete-time* has received some attention by Yasuda (1985) and more recently by Chalasani and Jha (2001). The former considers mixed-strategies in a variant of the classic game due to Dynkin (1969), while the latter uses randomised strategies to determine bounds on American options with transaction costs. An important contribution of this thesis is to show how mixed-strategy Nash equilibria among a finite number of agents can be determined in *continuous time.* We are unaware of any other studies that have examined this.

In chapters 3 and 4 we show that mixed-strategy equilibria can be constructed where agents "stop" according to a conditionally Poisson point process. In continuous time, the actions of agents can be regarded as an intensity of stopping, λ, where the probability of stopping over an interval $[t, t+\delta t)$ is $\lambda \delta t$. Since the intensities of stopping are continuous in time, they can be regarded as continuous controls and the methods of the last section can be used for such games.

1.4.3 Competitive Equilibria

Competitive equilibria are an important class of equilibria often considered in the finance literature. In such equilibria there is a continuum of infinitesimally small players (i.e. an infinite number of them). Since the players are small they have no individual impact on the allocation of value in the game. On an aggregate level, however, the actions of agents does affect the value. Notable examples in the continuous-time literature of such equilibria include the real options model of Leahy (1993), as well as the model of debt valuation in an industry equilibrium by Fries, Miller, and Perraudin (1997).

The equilibria themselves, are usually of the *symmetric* mixed-strategy class. Unlike asymmetric equilibria, symmetric equilibria prescribe similar actions for all the agents in the game. Since agents are infinitesimally small, the action set of agents does not consist of intensities of stopping as discussed above. Instead, the claim values are structured, through the application of relevant boundary conditions, in such a way that the relevant indifference and optimality conditions implicit in the equilibrium hold for all agents.

1.4.4 Bayesian Equilibria

The study of Nash equilibria in games of incomplete information was first undertaken by Harsanyi (1967-68). This paved the way for the concept of *Perfect Bayesian equilibria*,[12] which require that agents' decisions be optimal given their belief and the beliefs of others and these are updated using Bayes rule. An important extension to stochastic differential games is the incorporation of incomplete information. In many circumstances, players may have incentives to hide as much private information as possible. In such cases the action set is not only contingent on the underlying diffusion processes but also on agents' beliefs. Through the action/inaction of players, individual agents revise their beliefs of their rival, and this is effected through the application of Bayes rule.

In a series of interesting papers by Lambrecht and Perraudin (1994) and Lambrecht and Perraudin (2002), real options are valued when there is incomplete information over rival cost types. In the latter contribution a threat of pre-emption is modelled and more efficient firms are able to enter the market earlier. The threat of pre-emption and agents' beliefs of their rival's cost type, cause firms to anticipate their market entry. A similar approach has also been used in the valuation of corporate debt by Lambrecht and Perraudin (1996). In this set-up the bankruptcy costs of two sets of creditors to a financially distressed firm are private information and the first firm to foreclose extracts more value in the bankruptcy settlement. More recently, Déscamps and Mariotti (2000) have studied a real option model of second-mover advantages when there is imperfect asymmetric information about the technology itself.

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[12] For a detailed discussion see Fudenberg and Tirole (1993).

An important contribution of this thesis is to build on this strand of incomplete information game-theoretic models. In chapters 3 and 4 we incorporate incomplete information regarding cost types, in a spirit akin to that of Lambrecht and Perraudin (2002). In our set-up the incomplete information takes the form of a discrete number of types, rather than a continuous distribution of types. We analyse the limit of a continuum of types and determine its impact on real option valuation in chapter 3.

1.5 Equilibria Refinements

1.5.1 Introduction

Up until now we have ignored the form of Nash equilibria in stochastic games. Many dynamic games, have a plethora of equilibria. Game theory has placed much emphasis on equilibria refinements, or means by which certain equilibria can be ruled out. These refinements often look at the robustness of the equilibria with small perturbations. In this section we briefly review some important refinements concepts.

1.5.2 Sub-Game Perfection

Introduced by Selten (1975), sub-game perfection requires that the Nash equilibrium be an equilibrium for all sub-games including those off the equilibrium path. Following the presentation of 2-person nonzero-sum stopping games of the last section, we have the following definition:

Definition 4 *The pure stopping times $\hat{\tau}_1$, $\hat{\tau}_2$ form a sub-game perfect Nash equilibrium of a 2 person stochastic stopping game if*

$$J_s^1\left(\hat{\tau}_1, \hat{\tau}_2\right) \leq J_s^1\left(\tau_1, \hat{\tau}_2\right), \qquad J_s^2\left(\hat{\tau}_1, \hat{\tau}_2\right) \leq J_s^2\left(\hat{\tau}_1, \tau_2\right) \tag{1.32}$$

for all τ_1 and τ_2 and for all starting times $s \in [t, \hat{\tau}_1 \wedge \hat{\tau}_2]$, and initial values of the state variable $x_s \in R^n$.

By varying the start time of the game, s, we consider the equilibrium in all future sub-games, with all possible future starting points for the diffusion process, x_t. Sub-game perfection is a strong refinement as it requires the solution to be a Nash equilibrium even if off the equilibrium path. Such situations may arise, for instance, if agents "forget" to act as they are required in equilibrium.

Sub-game perfection is synonymous with *strongly time consistent* equilibrium solutions (see Basar (1989)). This type of time consistency may be contrasted with *weak time consistency* which entails equilibria which hold for all sub-games, but which do not necessarily apply off the equilibrium path.

1.5.3 Markov Perfect Equilibria

Agents' strategies are *stationary Markovian* if they only depend on the current state. Alternatively, such strategies are known as "feedback" strategies.[13] For games which involve such strategies and are sub-game perfect, the corresponding equilibrium is a *Markov Perfect Equilibrium.* This equilibrium concept is particularly prevalent in finance problems, as they often have a Markovian information structure.

1.6 Outline of the Thesis

This thesis develops models of stochastic differential stopping games in corporation finance. Pure and mixed strategy equilibria are analysed in games with 2-players as well as a continuum of agents (i.e. a competitive equilibrium). The thesis makes an important contribution to the stochastic differential games literature by characterising mixed-strategy equilibria stopping strategies as conditionally Poisson point processes in continuous-time. The thesis also illustrates how a simple form of incomplete information can be incorporated into such games.

The next three chapters present models of stochastic stopping games applied to quite diverse pricing issues. These models make contributions to the three strands of literature listed in the introduction of this chapter that relate to corporate securities, namely (i) convertible securities, (ii) real options, (iii) and corporate debt.

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[13] This should not be confused with a feedback information pattern for games, which set out the information accessible by agents. Conversely, "feedback strategies" denote strategies that are only conditional on the current level of the state variable.

Chapter 2 develops models of convertible bond valuation and exercise when the security issue is held by perfectly competitive bond holders. Since the firm's management must issue new shares on conversion, the existing shares are diluted in value and this leads to incentives for creditors to exercise their option in a strategic manner. We study competitive equilibria involving both the (i) sequential and (ii) simultaneous exercise of convertible bonds.

Chapter 3 examines a duopoly model of exit when there is incomplete information over rivals' costs. We present a class of mixed-strategy Nash equilibria consisting of agents' randomising their exit decision through conditionally Poisson point processes. We then introduce incomplete information over rival cost types and show how this leads to option values that evolve in a path-dependent manner through Bayesian learning.

Chapter 4 provides further reconciliation between the two strands of the credit risk pricing literature, by endogenising default hazards in a structural model of corporate debt. The endogenised default intensities arise from the strategic behaviour between two sets of equity holders in a duopoly, and thereby provides a further rationale for surprise credit events. Finally, in chapter 5 we conclude, briefly considering extensions to the models.

Chapter 2

Competitive Bond Conversion

2.1 Introduction

The issuance of convertible securities in 2001 of 62.1 billion dollars rose significantly over the corresponding figure for the previous year (60.4 billion dollars). The increasing demand for equity-style call-protected securities, such as mandatory convertibles, is even more pronounced with 2.5 billion dollars issued in the full year 2000 and 5.5 billion dollars issued in the year 2001 up until July.[1] Given the unwillingness of lenders to supply further credit and the general economic slowdown it is all the more likely that the market for hybrid securities such as convertibles will become larger in the near future.

Using methods of contingent claims analysis, Ingersoll (1977a) and Brennan and Schwartz (1977) developed the first convertible bond pricing models. This approach has since been extended by Brennan and Schwartz (1980) to include stochastic interest rates and has been enhanced through the reduced-form modelling of bankruptcy.[2] Given the inherent complexity of convertible bonds, with dual call and conversion options, this literature has implicitly assumed that convertible bonds are converted and called *simultaneously*, as a "block". With such inter-dependence between convertible bonds *within* the issue, arbitrage-free prices and conversion and call policies can easily be determined.

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[1] See *The Financial Times*, July 31 2001, pp. 23.

[2] See Ho and Pfeffer (1996) and Davis and Lischka (1999), among others. In such reduced-form credit risk models default is a surprise event, and is given by the first jump time of a Poisson point process (see for example Duffie and Singleton (1999)).

An important limitation of the "block constraint" is that convertible bond holders are assumed to exercise their securities at the same time as others, even though the American-style feature of their claim allows them to convert independently of others. Since the exercise strategy of securities affects their value, this means that the imposition of the "block constraint" also constrains the bonds' value.

On the face of it, one might expect the relaxation of the "block constraint" to make little impact to the optimal conversion strategy. However, the firm's management must redistribute its existing equity base to all the shares following conversion. In a "block" pricing model this causes the share value to be diluted *ex-post*, and to be marginally greater *ex-ante*. Since this drop in value is predictable, independent bond investors would have incentives to take advantage of the implied arbitrage opportunity. By converting earlier than the block, bond holders could acquire shares that are worth more than the convertible bonds. If creditors were truly bound by the block constraint, such arbitrage possibilities would not be possible because all the bond holders would be constrained to act simultaneously, thereby receiving the same diluted post-conversion share value. Figure 2.1 provides a dramatic illustration of arbitrage opportunities as described above, for non-callable perpetual convertible bonds (convertible preferreds). The (block) convertible security's value clearly undercuts the pre-conversion value[3] far below the block conversion trigger. The rectification of such anomalous arbitrage opportunities in convertible bond pricing models is among the main objectives of this chapter.

From an empirical stand-point it appears that convertible bonds are converted *sequentially*. The empirical study of Mehta (1976) shows yearly decreasing issue sizes of convertible offerings made by large US industrials in the late 1960's.[4] There may be several reasons for sequential bond conversion. Convertible bond investors may have asymmetric preferences for the type of security they wish to hold. Further, because of standard market imperfections, bond investors may not be able to react as quickly to movements in share and bond prices as others. The diversity of response means

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[3] This is the value of traded shares before the block conversion.

[4] More up to date data and empirical research will be needed to rigorously validate this assertion.

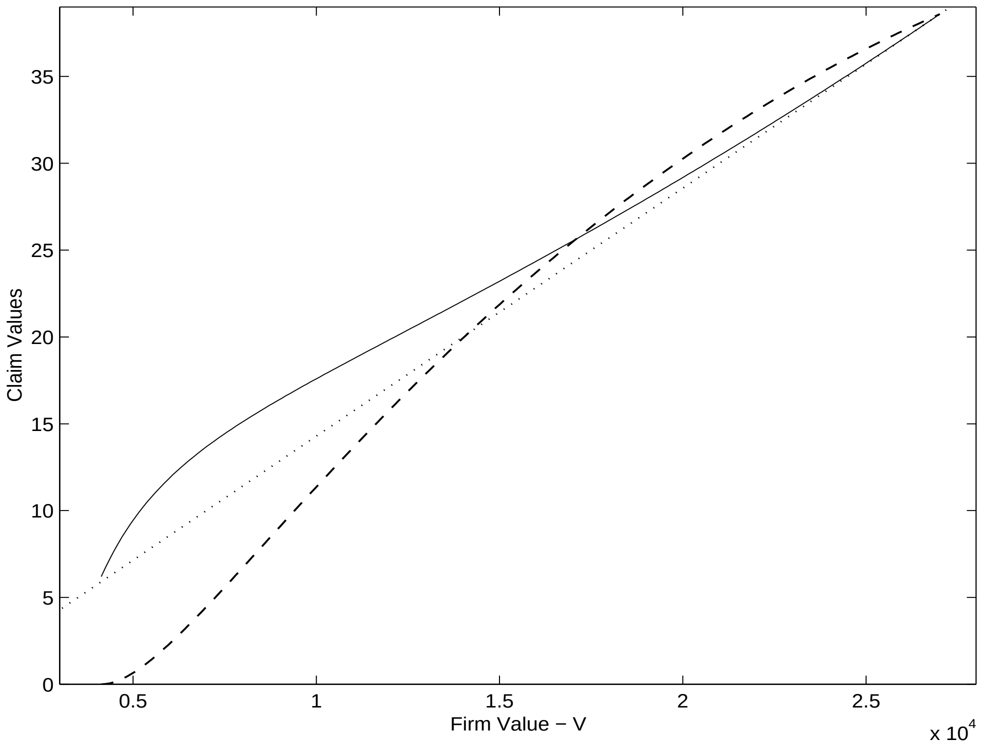

Figure 2.1: An illustration of arbitrage opportunities when the "block constraint" is enforced in convertible bond models. The perpetual non-callable convertible bond value (solid line) clearly undercuts the pre-conversion share value (dashed) well before the block trigger (near a firm value of 28,000). The post-conversion share value is dotted. The claim values are ploted using the expressions for "block" bond and share values in proposition 1. See sections 2.2.2 and 2.2.3.

that only some convertible bonds may be converted at the money. Finally, some bond investors may be "sleepy" resulting in the delayed conversion of a number of bonds.[5]

These theoretical inconsistencies and observations stress the need for a pricing methodology that allows for small portions of a convertible bond issue to be exercised independently of each other. Such a methodology must acknowledge the non-cooperative behaviour of individual bond holders. This naturally suggests the use of strategic modelling methods. In this chapter we incorporate strategic behaviour into a convertible bond pricing model. We assume there is a continuum of perfectly competitive bond holders acting non-cooperatively. We adopt a structural approach[6] for the model, thereby capturing some of the key features of debt financing: equity holders' bankruptcy decision, costs of financial distress, corporation taxes, dividend policy, business risk and leverage. Using continuous time methods, we obtain closed-form solutions for the various claim values when coupon payments are perpetual, as in convertible preferreds.[7]

Since creditors and the firm's management act non-cooperatively, we pose our model as a stochastic stopping game. Small creditors take the actions of other creditors and management as given and convert their holding into shares in a manner that their convertible bond value is maximised. Management, in its turn, acting in the best interests of equity holders, defaults on its debt service payments in such a way that share value is maximised.[8] By considering these non-cooperative actions, we analyse a symmetric Nash equilibrium which we examine in two key stages. We first consider the more tractable case of non-callable convertible debt and, later, we treat the more challenging callable case. Our analysis of competitive equilibria is similar to the approach taken by Bartolini (1993) in the context of competitive runs.

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[5] The "sleepiness" of management has also been advanced by Ingersoll (1977b) to explain delayed calls for callable convertible bonds.

[6] See amongst others Merton (1974), Black and Cox (1976), Leland (1994), Longstaff and Schwartz (1995) and Mella-Barral and Perraudin (1997). This approach to price corporate bonds may be contrasted with reduced-form models.

[7] Several studies have also developed models of sequential exercise for warrants. Emanuel (1983) considers the case of a monopoly holder of a warrant issue, while Constantinides (1984) studies the case of perfect competition. While this strand provides much insight into warrant exercise under different reinvestment policies (see also Spatt and Sterbenz (1988)), convertible bonds receive little attention.

[8] With no protective net worth covenants, bankruptcy is triggered by equity holders' inability to service its debt payments (see Leland (1994) and Mella-Barral and Perraudin (1997)).

As in that study, the equilibria are symmetric and consist of the indifference of competitive creditors between immediate and delayed conversion at an endogenous upper threshold. Below the threshold, creditors have no exercise incentives as their value exceeds the conversion value of the bond. Claim values are structured so that these symmetric preferences hold for all bond holders.

The intuition for agents' indifference at the conversion threshold is as follows. As bonds are converted the share value becomes progressively more diluted. Thus, creditors have incentives to convert earlier, thereby acquiring *temporarily* more valuable shares. Counter-balancing this incentive, remaining convertible bonds become progressively more valuable as they are converted. The reason for this is that the firm has less debt to service and so the default trigger is lower. Thus, bonds become progressively less likely to default. Creditors' indifference at the upper conversion threshold hinges on the trade-off between these two effects. Since the indifference trigger is increasing in time, conversion is *sequential* and resembles models of "barrier control" such as the one by Pindyck (1988). Whenever the firm value reaches new 'highs' and the barrier is reached bonds are converted, and the option value associated with conversion is taken out of the money. This process repeats itself until all the debt is converted.

In the *callable* case conversion takes place somewhat earlier than in the *non-callable* case and always precedes the firm's call policy. Thus, the firm's call policy places an upper bound on creditors' actions and induces them to convert earlier to extract the relatively undiluted share value.

Our competitive model has significant implications for the purpose of convertible debt offerings even though we suppose that management has no particular bias in its rationale for making the offering.[9] Since sequential competitive conversion precedes block conversion, convertible debt is intrinsically closer to delayed-equity. Moreover, there are no associated negative signalling effects with the arrival of new equity,[10] as conversion is voluntary. The delayed-equity property of the offering is thus preserved.

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[9] The most cited arguments for convertible debt financing are as a debt "sweetener" allowing management to make smaller coupon payments to creditors (See Brennan and Kraus (1987) and Brennan and Schwartz (1988)) and that it gives management access to delayed equity financing (See Constantinides and Grundy (1989) and Stein (1992)). Our basic assumptions suggest no natural bias for either of these two purposes.

[10] Mikkelson (1981) reports an average abnormal return (AAR) of -2.08% around the time of call announcements.

The strategic behaviour of creditors in equilibrium has a substantial impact on the value of convertible bonds and shares. In general, these values exceed the corresponding ones in the block. The reason for this is due to the fact that bankruptcy takes place later when conversion is sequential. Further, the competitive equilibria Pareto-dominate the block strategies when there is no tax shield and do so *with* the tax shield for early sub-games. Moreover, our sequential model suggests that callable convertible debt is more efficient than non-callable convertible debt, thereby illustrating the importance of call provisions from efficiency considerations.[11] Collectively, these results emphasise the importance of pricing methods that utilise sequential conversion equilibria and they demonstrate the bearing this has on the design of convertible debt contracts.

In a wider context, this chapter adds to a growing literature of corporate debt models that include strategic behaviour and which was the subject of review in the previous chapter. The majority of these models consider either a duopoly of firms or a pair of non-cooperative claim holders within the firm. The notable exception is Fries, Miller, and Perraudin (1997) which examines debt and equity values in a full-blown competitive industry equilibrium. Our study bears most similarity, in terms of its broad approach, to this last article, although the perfect competition in our setting is among the holders of the convertible debt *within* a *single firm.*

Section 2.2 of this chapter briefly analyses the convertible bond and share values with the block constraint. For clarity we determine the block call and conversion policies. Section 2.3 motivates the presence of perfectly competitive convertible bond holders. This section also examines deviation incentives for such creditors from the block strategy. These incentives are similar to arbitrage opportunities. By studying their comparative statics, we determine when the block strategies yield competitive Nash equilibria.

Section 2.4 presents the main results on the *sequential* symmetric Nash equilibria. We first develop this when the issue is non-callable and generalise this to the callable case. Section 2.5 discusses the implications of these equilibria including efficiency considerations and issues related to the design of convertible contracts. Section 2.6 then considers finite-maturity convertible bonds with a stochastic interest rate. Since dividend and coupon payments are discrete, the equilibrium is somewhat different in that the action set applies only at certain times. Section 2.7 considers extensions to the model and, finally, section 2.8 concludes. Proofs of lemmas and propositions are consolidated in section 2.9 at the end of the chapter.

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[11] It is usually argued that call provisions primarily endow the firm's capital structure with greater flexibility.

2.2 Block Convertible Bonds

2.2.1 Basic Assumptions

We assume that the owner-manager of a firm issues n_0 convertible bonds and ψ_0 shares at time $t = 0$ to finance its activities.[12] Alternatively, the firm is a pure-equity operation prior to this with ψ_0 shares outstanding, when the convertible offering is made.

For simplicity we suppose the risk-free interest rate, r, is initially constant. Payments are fully contractible[13] and management's fixed payout policy is to distribute a proportion of firm value $\delta < r$ per unit time to current share holders. Each convertible bond receives a continuous coupon yield of c per unit time and this debt service is paid by the equity holders to the convertible security holders. Equity holders take advantage of the tax deductibility of debt, and so the initial debt service per unit time is $(1 - \tau)n_0 c$, where τ is the tax rate. The constant conversion ratio for each convertible is γ. When the convertible offering is callable, we assume that there is no call notice period[14] and bonds may be redeemed by creditors for the fixed call price,

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[12] As we stated in the introduction, the two main rationales for convertible debt financing are as (1) a cheaper means of raising debt, and (2) as a way of getting delayed-equity into the firm's financial structure. We suppose that management has no bias towards either of these two particular rationales in financing its activities.

[13] Even if the ψ_0 shares are held by inside equity holders, who combine ownership with control rights, this assumption is required for the future shares arising from bond conversion. Other studies overcome the question of differing control rights by assuming that only senior debt is issued and no further equity (See Aghion and Bolton (1992) and Anderson and Sundaresan (1996)).

[14] Allowing for such periods (typically 30 days) only makes the analysis more complicated without adding substantial new insights.

K.

Markets are perfect and there are no transaction costs or constraints on buying and selling[15] the firm's assets. Specifically, the asset value of the firm, V_t, evolves according to the lognormal diffusion process

$$dV_t = (r - \delta) V_t dt + \sigma V_t dB_t \tag{2.1}$$

under the risk-neutral probability measure,[16] where σ^2 is the instantaneous variance of the return of the firm, and B_t is a standard Brownian motion. Management is proscribed by security holders from selling further equity.[17]

Since the coupon payment, payout rate, conversion ratio and call price are all constant, the model is time-independent, as are the various claim values. This has the advantage that closed-form solutions are obtained. Although the time-independence means that the convertible bonds are, in fact, convertible preferred shares, we designate them as "convertible bonds" for generality.

We now present an important definition for the block case.

Definition 5 *A block convertible bond is an indivisible bond issue which can be converted and called only as a block.*

Under this definition the specific ownership of the issue is irrelevant as all the bonds are converted or called simultaneously.

2.2.2 The Convertible Bond Value and Share Value

Since each convertible bond receives a continuous coupon, c, per unit time, the block bond value, $\hat{W}(V)$, satisfies the following ordinary differential equation *before* conversion:

$$\frac{\sigma^2 V^2}{2} \frac{d^2 \hat{W}}{dV^2} + (r - \delta) V \frac{d\hat{W}}{dV} + c = r\hat{W}. \tag{2.2}$$

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[15] This is consistent with Merton (1974), although it differs with the structural models of corporate debt of Brennan and Schwartz (1978) and Leland (1994).

[16] Note that it is not necessary for the firm's assets to be traded in the market. If the firm's equity is freely traded, the risk-neutral measure can be used (See Ericsson and Reneby (1999)).

[17] Alternatively, standard anti-dilution clauses must be introduced into the convertible offering.

At the first hitting time of the lower default trigger, $\hat{V}_b$, bankruptcy takes place and the residual firm value *net* of bankruptcy costs is distributed among the n_0 bond holders. Thus

$$\hat{W}(\hat{V}_b) = \frac{1-\alpha}{n_0}\hat{V}_b$$

where $\alpha \in [0, \psi_0/(\psi_0 + \gamma n_0))$ is the fixed proportion of firm value lost in legal and administrative costs of bankruptcy.[18] At an upper threshold, $\hat{V}$, all the bonds are either (1) converted into equity, or (2) redeemed for the call price, K. In both cases the company turns into a pure-equity operation after conversion/redemption. In the former case, the post-conversion share value is given by $V/(\psi_0 + \gamma n_0)$, because of the γn_0 new shares. Since the bonds are convertible into γ shares, the upper boundary condition is $\hat{W}(\hat{V}) = \gamma \hat{V}/(\psi_0 + \gamma n_0)$. In the latter case, the share holders bear the cost of redeeming n_0 bonds, and so, the post-conversion share value is $(V - n_0 K)/\psi_0$. When the bonds are redeemed, individual bonds equal the call price, so $\hat{W}(\hat{V}) = K$.

The shares receive a net dividend payment per unit time of $\delta V - (1-\tau)n_0 c$. There are ψ_0 shares before conversion, and so their individual value, $\hat{S}(V)$, satisfies the following ordinary differential equation

$$\frac{\sigma^2 V^2}{2}\frac{d^2\hat{S}}{dV^2} + (r-\delta)\,V\frac{d\hat{S}}{dV} + \left[\frac{\delta V - (1-\tau)\,n_0 c}{\psi_0}\right] = r\hat{S}. \tag{2.3}$$

By strict priority of claims, shares are left with no value: $\hat{S}(\hat{V}_b) = 0$, in bankruptcy. As in the models of Leland (1994) and Mella-Barral and Perraudin (1997) we assume that there are no protective covenants on the bonds, and the equity holders decide the timing of bankruptcy at $\hat{V}_b$ by ceasing to make coupon payments to the creditors. Since share holders can always inject further cash, they decide this in an optimal way, implying the smooth-pasting condition $d\hat{S}(\hat{V}_b)/dV = 0$.

In the interests of tractability we make the following important assumption about the default trigger, $\hat{V}_b$, in this chapter.

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[18] The upper bound on α ensures that the residual value of the bond in bankruptcy exceeds the post-conversion value of the shares (i.e. $[(1-\alpha)\hat{V}_b]/n_0 > \gamma\hat{V}_b/(\psi_0 + \gamma n_0)$). If this were not the case bond holders would have incentives to convert at $\hat{V}_b$. In the presence of other debt, however, this boundary condition would change as convertible bonds are usually junior to other non-equity claims.

Assumption 1 *There is an option value of defaulting such that the default trigger is strictly less than the discounted value of the firm's debt service:*

$$\hat{V}_b < \frac{(1-\tau)\, n_0 c}{r} \tag{2.4}$$

This assumption will hold for most cases. The reason for this is as follows. The zero-NPV rule consists of defaulting at the first time the firm's assets equals the discounted value of debt service payments.[19] With uncertainty, however, there is an option value associated with defaulting and so equity holders delay this decision. Note, however, that assumption 2.4 is stronger than merely asserting that the debt is risky.[20]

The upper boundary condition on the share value, depends on whether the bonds are converted or redeemed. In the former case, the shares equal their diluted value $\hat{S}(\hat{V}) = \hat{V}/(\psi_0 + \gamma n_0)$. In the latter case, equity holders bear the payment of the call price to the n_0 convertible bonds $\hat{S}(\hat{V}) = (\hat{V} - n_0 K)/\psi_0$.

We summarise the results of this sub-section in the following proposition:

Proposition 1 *The value of a block convertible bond, $\hat{W}(V) = \hat{W}(V;i)$, is given by*

$$\begin{aligned}
\hat{W}(V;i) \;=\; & \left\{ \left(\frac{\gamma \hat{V}}{\psi_0 + \gamma n_0}\right) \mathcal{I}_{A,B} + K\,\mathcal{I}_C - \frac{c}{r} \right. \\
& \left. - \left[\frac{(1-\alpha)}{n_0}\hat{V}_b - \frac{c}{r}\right] \left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-} \right\} g\left(V; \hat{V}_b, \hat{V}\right) \\
& + \frac{c}{r} + \left[\frac{(1-\alpha)}{n_0}\hat{V}_b - \frac{c}{r}\right] \left(\frac{V}{\hat{V}_b}\right)^{\xi_-}
\end{aligned} \tag{2.5}$$

for $i \in \{A, B, C\}$, $V \in (\hat{V}_b, \hat{V})$. The block share value, $\hat{S}(V) = \hat{S}(V;i)$, is given by

$$\begin{aligned}
\hat{S}(V;i) = \frac{1}{\psi_0} & \left\{ V - \frac{(1-\tau)\, n_0 c}{r} - \left[\hat{V}_b - \frac{(1-\tau)\, n_0 c}{r}\right] \left(\frac{V}{\hat{V}_b}\right)^{\xi_-} \right\} \\
& + \frac{n_0}{\psi_0} \left\{ \frac{(1-\tau)\, c}{r} - \left[\frac{\gamma \hat{V}}{\psi_0 + \gamma n_0} \mathcal{I}_{A,B} + K\,\mathcal{I}_C\right] \right. \\
& \left. + \left[\frac{\hat{V}_b}{n_0} - \frac{(1-\tau)\, c}{r}\right] \left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-} \right\} g\left(V; \hat{V}_b, \hat{V}\right)
\end{aligned} \tag{2.6}$$

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[19] This is when the firm value, $\hat{V}_b$, equals: $E_t[\int_t^\infty (1-\tau) n_0 c \exp[-r(s-t)] ds] = [(1-\tau) n_0 c]/r$.

[20] This would be the case if $\hat{V}_b < (n_0 c)/(r[1-\alpha])$, which always holds under the assumption.

for $i \in \{A, B, C\}$*, and* $V \in (\hat{V}_b, \hat{V})$*, where*

$$g\left(V; \hat{V}_b, \hat{V}\right) = \left(\left(\frac{V}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V}{\hat{V}_b}\right)^{\xi_-}\right) / \left(\left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-}\right),$$

$\mathcal{I}$ *is the indicator function and* ξ_+ *and* ξ_- *are the positive and negative roots of the quadratic equation* $\sigma^2\xi(\xi - 1) + 2(r - \delta)\xi = 2r$*. The variable* $i \in \{A, B, C\}$ *indicates three possible cases for the block call/conversion of bonds at* $\hat{V}$*; A: the bonds are voluntarily converted by creditors; B: the bonds are converted following the receipt of a call (forced conversion), or C: the bonds are redeemed. When the firm value equals* $\hat{V}_b$*, the equity holders default on their debt service payments, and this is given by the root of the non linear equation*[21] *resulting from imposing the boundary condition* $d\hat{S}(\hat{V}_b; i)/dV = 0$*.*

The proof of this proposition may be found in section 2.9.1.

The *net firm value* (*net* of bankruptcy costs and *gross* of the tax shield) under the block strategy, $\hat{v}(V)$, is given by the sum of the bonds and shares determined in proposition 1: $\hat{v}(V) = \psi_0\hat{S}(V) + n_0\hat{W}(V)$.

2.2.3 Block Call and Conversion Strategies

Consider *non-callable* convertible bonds. Since there is no call policy to consider, creditors are free to convert and will do so in a manner that their value is maximised. This implies the smooth-pasting condition onto the post-conversion share value: $d\hat{W}(V_{co}; A)/dV = \gamma/(\psi_0 + \gamma n_0)$, where $V_{co} = \hat{V} = \hat{V}(A)$ is the block conversion strategy. Applying this to equation (2.5) we obtain

$$\begin{aligned}\left(\frac{c}{r}\right)\frac{\partial g\left(V_{co}; \hat{V}_b, V_{co}\right)}{\partial V} &= \left[\frac{(1-\alpha)\hat{V}_b}{n_0} - \frac{c}{r}\right]\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}\left\{\frac{\xi_-}{V_{co}} - \frac{\partial g\left(V_{co}; \hat{V}_b, V_{co}\right)}{\partial V}\right\} \\ &\quad + \left(\frac{\gamma}{\psi_0 + \gamma n_0}\right)\left[V_{co}\frac{\partial g\left(V_{co}; \hat{V}_b, V_{co}\right)}{\partial V} - 1\right] \qquad (2.7)\end{aligned}$$

and the conversion strategy, V_{co}, is the root of (2.7).

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[21] The equation is (2.48) and is in the proof of proposition 1 in section 2.9.1.

Now, consider the case of *callable* convertible bonds. We review a couple of important results from the early convertible debt pricing literature.

Theorem 2 *(Ingersoll (1977a) Theorem VII, pp. 319). Whenever it is optimal to make a voluntary block conversion of a non-callable bond, it will also be optimal to make a block conversion of a callable bond.*

Theorem 3 *(Ingersoll (1977a) Theorem VIII, pp. 319). The possibility of a block voluntary conversion does not affect the optimal call policy.*

The thrust of these results is that the absence of call protection makes no difference to V_{co} and the determination of V_{co} has no effect on the call policy. When the bonds are callable, the first of the call and conversion policies to come into the money will dominate the outcome.

Unfortunately, we cannot use other Theorems due to Brennan and Schwartz (1977) and Ingersoll (1977a) on call policies with such liberty as the ones above. The reason for this is that the setting they employ is devoid of corporation taxes and bankruptcy costs. By the Modigliani-Miller Theorem, this means that minimising the convertible bond value is not necessarily synonymous with maximizing the share value. This subtlety means that in seeking the optimal call policy we must look for call strategies that *maximise* the share value.

From the equity holders' view point the bonds will be redeemed by the bond holders on the receipt of a call if $V/(\psi_0 + \gamma n_0) < (V - n_0 K)/\psi_0$. Alternatively, if $V/(\psi_0 + \gamma n_0) \geq (V - n_0 K)/\psi_0$, the bond holders prefer to convert their security into shares. The limited liability of the firm in conjunction with these results then implies the following no-arbitrage condition on the share value

$$\hat{S}(V) \geq \max\left[0, \min\left(\frac{V}{\psi_0 + \gamma n_0}, \frac{(V - n_0 K)}{\psi_0}\right)\right]. \tag{2.8}$$

Superficially one would expect the share value to satisfy the following smooth-pasting condition $d\hat{S}(V^*)/dV = 1/(\psi_0 + \gamma n_0)$, if the call policy, V^*, exceeds the firm value $V_{ca} = K(n_0 + \psi_0/\gamma)$.[22] Conversely, if the call policy is less than V_{ca}, one would expect the share value to be maximised by selecting the trigger implicit in the following smooth-pasting condition: $d\hat{S}(V^*)/dV = 1/\psi_0$.

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[22] The reason this is superficial is that the smooth-pasting condition may yield either a (i) maximum or (ii) a minimum.

Consider the first case, where $V^* \geq V_{ca}$. It actually turns out, by considering the concavity of the share value, that the policy V^* always yields a *minimised* share value if V^* exceeds a certain constant (see the proof of lemma 1). The share value is then maximised by calling earlier than V^*. In the limit, the optimal call policy becomes $V^* = V_{ca}$, which is identical to the call policy with no taxes and bankruptcy costs and $K \geq c/r$ (see Ingersoll (1977a)'s Theorems III and IV, pages 299-301).

Lemma 1 *Suppose* $K \geq [(1-\tau)c]/r$ *and* $V^* \geq V_{ca}$ *where* V^* *is the root of* $d\hat{S}(V^*)/dV = 1/(\psi_0 + \gamma n_0)$. *If*

$$V^* > \left(n_0 + \frac{\psi_0}{\gamma}\right)\left(\frac{(1-\tau)\,c}{r}\right)\left(\frac{\xi_+}{\xi_+ - 1}\right)\left(\frac{\xi_-}{\xi_- - 1}\right), \tag{2.9}$$

$$\frac{d\hat{S}\,(y)}{dV} > \frac{1}{\psi_0 + \gamma n_0} \qquad \forall \qquad y \in (V_{ca}, V^*) \tag{2.10}$$

and

$$\hat{S}\,(V_{ca}) < \frac{K}{\gamma} \tag{2.11}$$

for all call policies over the interval (V^*, V_{co}), *then the optimal block call policy is* $\hat{V} = \hat{V}(B) = V_{ca}$.

The proof of this lemma may be found in section 2.9.2.

Unfortunately, the endogeneity of $\hat{V}_b$ (through the smooth-pasting condition) make the presentation of stronger results more difficult. Nonetheless, satisfaction of inequality (2.9) above for many parametrisations is often sufficient to guarantee that V_{ca} is the optimal call policy. Somewhat surprisingly the influence of corporation taxes has little impact on the call policies when compared with the results of Ingersoll (1977a) and we obtain the same call policy, under conditions that hold for many contracts. The main effect of the taxes, however, is to influence the type of convertible bonds for which this policy applies. Inequality (2.9) implies that call prices $K > [(1-\tau)c/r][\xi_-/(\xi_- - 1)][\xi_+/(\xi_+ - 1)]$ satisfy lemma 1.

Consider now the second case where $V^* < V_{ca}$. It turns out that the aforementioned smooth-pasting condition can only hold under certain conditions as we show in the next lemma.

Lemma 2 *If* $K < [(1-\tau)c]/r$ *and* $V^* < V_{ca}$ *where* V^* *is the root of* $d\hat{S}(V^*)/dV = 1/\psi_0$, *the optimal block call policy is* $\hat{V}(C) = V^*$.

The proof of this lemma may be found in section 2.9.3.

For contracts with call prices, $K > [(1-\tau)c]/r$, but which do not satisfy (2.9) it is not clear what the outcome is.[23] The next proposition sums up the results on conversion and call policies from this sub-section.

Proposition 2 *Given proposition 1 and the assumptions of this section*

1. *The convertible bonds are converted voluntarily as a block* ($i = A$) *and* $\hat{V} = \hat{V}(A) = V_{co}$ *if:*
   (a) *The offering is non-callable, or*
   (b) *the offering is callable and* $V_{co} < \min(\hat{V}(B), \hat{V}(C))$.
2. *The convertible bonds are converted as a block as a result of a call (i.e. a forced conversion)* ($i = B$) *and* $\hat{V} = \hat{V}(B) = V_{ca}$ *if the offering is callable,* $V_{ca} < V_{co}$ *and lemma 1 holds.*
3. *The convertible bonds are redeemed as a block for* K ($i = C$) *and* $\hat{V} = \hat{V}(C) = V^*$, *where* V^* *is the root of* $d\hat{S}(V^*)/dV = 1/\psi_0$ *if lemma 2 holds.*

The proof of this proposition may be found in section 2.9.4.

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[23] Since the emphasis of this chapter is on studying competitive effects on claim values, we refrain from an involved analysis of block call strategies.

2.3 Deviation Incentives from the Block Strategy with Perfect Competition

2.3.1 Perfectly Competitive Creditors

The previous section determined convertible bond values by constraining them to be converted and called as a block. The purpose of this section is two-fold. First, we characterise the ownership of the convertible bonds under perfect competition. Under the block constraint, this issue is irrelevant for pricing as the bonds must be called or converted together. Without the block constraint, the ownership of the bond issue affects the scope for strategic behaviour. The number of agents as well as their respective market share and modes of interaction will crucially determine individual creditors' incentives and actions. We also drop the block constraint and determine circumstances under which bond holders have incentives to abide by the block strategies. In particular, we wish to determine whether agents should hasten or postpone their conversion with respect to the block. In the process we will determine when the block strategies are synonymous with Nash equilibria.

We assume there is a continuum (i.e. an infinite number) of identical creditors. Formally, the continuum of bond holders is distributed on the interval $[0, n_0]$, with each creditor holding convertible bonds of measure zero. Being infinitesimally small, each creditor's conversion decision does not alter the aggregate number of remaining bonds.

Assumption 2 *With perfect competition there is a continuum of identical bond holders with infinitesimally small holdings of convertible debt.*

In addition, we assume that creditors are unable to collude or communicate with each other regarding their conversion policies. This assumption makes the analysis of game-theoretic behaviour more tractable.[24] This can be rationalised given our assumption of perfect competition. Since agents are infinitesimally small, collusion with others makes no impact to agents' value. Moreover, many convertible debt offerings are made to outside investors on secondary bond markets or through other intermediaries. Even if communication were possible it is unlikely that agents could identify other bond holders, let alone collude with them.

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[24] Without this assumption we would have to allow for cooperative behaviour among agents as well as bargaining. Since there are an infinite number of bond holders, inclusion of this would make the analysis very much more complicated, if not impossible.

Assumption 3 *With perfect competition individual creditors are unable to communicate their decisions to one another and collusion is impossible.*

We formalise the concept of a competitive convertible bond issue:

Definition 6 *A competitive convertible bond is an infinitely divisible bond issue that is held by a continuum of independent, infinitesimally small creditors. The bond issue may be converted and traded in independent, infinitely divisible portions.*

Most importantly, definition 6 allows perfectly competitive[25] agents to act independently of each other.

2.3.2 Pre-emptive Incentives among Competitive Creditors

With the afore-mentioned distribution of the convertible offering, consider the incentives of individual creditors given that all the other bond holders stick to the block strategy.

A small creditor will have incentives to convert their holding prior to the block strategy if $\hat{W}(V) < \gamma \hat{S}(V)$ for $V \leq \hat{V}$, as the conversion value is always available. If this inequality holds, small creditors can make a risk-free profit, $\gamma \hat{S}(V) - \hat{W}(V)$, by converting their bonds into shares earlier, selling them and buying the convertible bonds back. In order to rule out this arbitrage opportunity when $i = A$ or B (i.e. voluntarily or forcible conversion as a block), the first derivative of the shares at the block strategy, $\hat{V}$, must be greater than or equal to the first derivative of the bond value:

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[25] Throughout this chapter we will use the term "competitive" to designate bond issues that are subject to definition 6. We also use the terms "competitive", "small" and "infinitesimal" in relation to convertible bond holders and security values, when definition 6 applies.

$$\gamma \frac{d\hat{S}(\hat{V})}{dV} \geq \frac{d\hat{W}(\hat{V})}{dV}. \tag{2.12}$$

If this necessary condition does not hold, the bond value will undercut the share value before $\hat{V}$ and there will be pre-emptive incentives. Figure 2.1 provides an example of a situation in which this inequality does *not* hold. Small creditors are able to take advantage of the larger non-diluted share value by preempting others. Unlike many strategic real options models, however, the preemption incentive here is temporary.[26] Once all the bonds have been converted, share value will settle to its diluted post-conversion level, $V/(\psi_0 + \gamma n_0)$.

Now, consider the special case of convertible debt which is voluntarily convertible as a block (i.e. $i = A$ from propositions 1 and 2). Differentiating (2.6) and (2.5) and substituting in the inequality (2.12), we obtain

$$\begin{aligned} 0 \leq\; & \frac{\gamma n_0}{\psi_0 + \gamma n_0}\left(1 - \hat{V}\frac{\partial g\left(\hat{V};\hat{V}_b,\hat{V}\right)}{\partial V}\right) - \hat{V}_b\left(\frac{\xi_-}{\hat{V}}\left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-} - \frac{\partial g\left(\hat{V};\hat{V}_b,\hat{V}\right)}{\partial V}\left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-}\right) \\ & + \frac{(1-\tau)\,n_0 c}{r}\left(\frac{\partial g\left(\hat{V};\hat{V}_b,\hat{V}\right)}{\partial V}\left[1 - \left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-}\right] + \frac{\xi_-}{\hat{V}}\left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-}\right) \end{aligned}$$

Substituting for the term $\gamma n_0/(\psi_0 + \gamma n_0)$ from equation (2.7), we then obtain after some rearrangement

$$\left(\frac{\tau n_0 c}{r}\right)\hat{V}\frac{\partial g\left(\hat{V};\hat{V}_b,\hat{V}\right)}{\partial V} \leq \left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-}\left(\hat{V}\frac{\partial g\left(\hat{V};\hat{V}_b,\hat{V}\right)}{\partial V} - \xi_-\right)\left(\alpha\hat{V}_b + \frac{\tau n_0 c}{r}\right) \tag{2.13}$$

Note that $(\hat{V}/\hat{V}_b)^{\xi_-}$ is positive and $\hat{V}\partial g(\hat{V};\hat{V}_b,\hat{V})/\partial V > 1$. Thus, both sides of (2.13) are positive. This inequality must hold if there are to be no incentives for preemptive conversion in the block. An interesting implication of this is that if there are no corporation taxes (i.e. $\tau = 0$) the inequality holds.

Lemma 3 *Consider convertible bonds that are voluntarily convertible as a block ($i = A$). If there are no corporation taxes then the necessary condition (2.12) holds.*

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[26] For example, in the model of Lambrecht and Perraudin (2002) the first-mover acquires the whole surplus for the entire time horizon of the game.

The proof of this lemma may be found in section 2.9.6.

The presence of corporation taxes is crucial to induce early conversion as it confers greater ex-ante value to the shares. Since (block) conversion removes the tax shield ex-post, by turning the firm into a pure-equity operation, agents in the block get none of the tax deductibility of debt. If there were other non-convertible debt present, however, the influence of corporation taxes would not be as extreme.[27]

Now, consider the case of convertible bonds that are redeemed as a block ($i = C$). Since the conversion value is increasing in V_t, small creditors would have incentives to convert preemptively as close as possible to $\hat{V}$. In the limit, the conversion value is given by $\gamma \hat{S}(\hat{V}; C) = [\gamma(\hat{V} - n_0 K)]/\psi_0$. Since, $\hat{V} < V_{ca} = K(n_0 + \psi_0/\gamma)$, from lemma 2, $\gamma \hat{S}(\hat{V}; C) < K$. So bond holders have no incentives to convert before the block as the conversion value is less than the redemption value, K, which is guaranteed:

Lemma 4 *Consider convertible bonds that are redeemed as a block ($i = C$). There are no preemptive conversion incentives for competitive bond holders in the block.*

2.3.3 Delay Incentives among Competitive Creditors

So far we have only examined preemptive incentives. In the case of convertible debt that is voluntarily block convertible (i.e. $i = A$), creditors are free to convert after the block[28] and so this kind of deviation must also be considered.

Suppose an infinitesimally small bond holder delays conversion after the block. By assumption 2, the number of remaining bonds tends to zero with the company becoming a pure-equity operation (i.e. $\hat{V}_b \downarrow 0$), and so the *delayed* convertible bond value, $W_{de}(V)$, will be given by

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[27] If management relevered after the block conversion, share holders would presumably get some tax deductibility on other debt payments.

[28] For the other two cases, where (1) the bonds are forcibly converted, $i = B$, and (2) the bonds are redeemed, $i = C$, creditors have no option to voluntarily delay conversion because the bonds are called at $\hat{V}$.

$$\begin{aligned} W_{de}(V) &= \lim_{\epsilon \downarrow 0} \left[\frac{c}{r} + \left[\frac{(1-\alpha)}{\epsilon}\epsilon - \frac{c}{r} \right] \left(\frac{V}{\epsilon} \right)^{\xi_-} \right. \\ &\quad \left. + \left\{ \left(\frac{\gamma V_{de}}{\psi_0 + \gamma [n_0 - \epsilon]} \right) - \frac{c}{r} - \left[\frac{(1-\alpha)}{\epsilon}\epsilon - \frac{c}{r} \right] \left(\frac{V_{de}}{\epsilon} \right)^{\xi_-} \right\} g(V; \epsilon, V_{de}) \right] \\ &= \frac{c}{r} + \left[\frac{\gamma V_{de}}{\psi_0 + \gamma n_0} - \frac{c}{r} \right] \left(\frac{V}{V_{de}} \right)^{\xi_+} \end{aligned}$$

where V_{de} is the delayed conversion trigger. Since the deviating creditor is the "last" remaining bond holder, they will convert in a manner that maximises their convertible bond value. The trigger, V_{de}, must therefore satisfy a smooth-pasting condition onto the *post-conversion* share value (i.e. $dW_{de}(V_{de})/dV = \gamma/(\psi_0 + \gamma n_0)$)

$$\frac{\gamma}{\psi_0 + \gamma n_0} = \frac{\xi_+}{V_{de}} \left[\frac{\gamma V_{de}}{\psi_0 + \gamma n_0} - \frac{c}{r} \right] \quad \Rightarrow \quad V_{de} = \frac{c}{r} \left(\frac{\xi_+}{\xi_+ - 1} \right) \left[n_0 + \frac{\psi_0}{\gamma} \right]. \quad (2.14)$$

This ensures that the "last" bond holder's value exceeds the conversion value for $V \leq V_{de}$, with equality taking place at V_{de}. So, if the delayed trigger exceeds the block strategy (i.e. if $V_{de} > V_{co}$), then the value of the "last" bond will exceed that of the block bond at V_{co} (i.e. $W_{de}(V_{co}) > \gamma V_{co}/(\psi_0 + \gamma n_0) = \hat{W}(V_{co})$). Agents will then have incentives to postpone their conversion after the block. It can be shown that there are always incentives to delay conversion if the convertible bonds are credit-risky (See the proof of lemma 5):

Lemma 5 *If the bonds are voluntarily convertible as a block (i.e. $i = A$), there will always be delaying incentives in the block, and the conversion strategy $\hat{V}$ is not a Nash equilibrium.*

The proof of this lemma may be found in section 2.9.7.

The intuition for lemma 5 is as follows. Since the post-conversion value is the same in both delayed and block cases, the only difference between the two is the default trigger. Since the latter has a higher trigger, $\hat{V}_b > 0$, the likelihood of default is greater in the block case, and so the delayed bond value is greater.

We summarise the results of the past two sub-sections in the following proposition:

Proposition 3 *Given propositions 1 and 2 and lemmas 4 and 5,*

1. *If the bonds are voluntarily convertible as a block ($i = A$), the block strategy is not a Nash equilibrium. There will always be incentives to delay conversion and if inequality (2.12) fails to hold there will also be preemptive incentives.*
2. *If the bonds are forcibly-convertible ($i = B$) and inequality (2.12) fails to hold, small creditors will have preemptive incentives and the block strategy is not a Nash equilibrium.*
3. *If bonds are redeemed as a block ($i = C$), there are no preemptive conversion incentives and $\hat{V}(C)$ is a Nash equilibrium.*

The proof of this proposition may be found in section 2.9.8. Note that case C is rather uninteresting as the convertible bonds will never be converted even though they are nominally convertible securities. Because of this we refrain from here on to consider such convertibles.

Proposition 3 is essentially about instances in which the block strategies are synonymous with (pure-strategy) competitive Nash equilibria. Since the block constraint forces simultaneous action, these equilibria are symmetric and the corresponding strategies are pure.[29] When the block strategies fail to yield Nash equilibria small creditors have incentives either to anticipate or postpone conversion with respect to the block. Under such circumstances, the block model would yield inaccurate prices for the convertible bonds and shares. The reason for this is that bond investors would take advantage of the implied arbitrage opportunities, thereby eliminating the arbitrage profit and altering the prices of securities in the market.

2.3.4 Comparative Statics of the Pre-emptive Incentives

Among the incentives examined in the previous sub-section, there were no clear-cut results on preemptive incentives in cases $i = A$ and B. We conduct a series of comparative statics results, by measuring the degree to which there are preemptive incentives. We determine the way in which these incentives vary by altering firm and convertible debt contract primitives. An important consequence of this is that we can identify the types of contracts and firms for which the block strategies are Nash equilibria.

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[29] Pure-strategy Nash equilibria here involve agents adopting (pure) stopping times. In our stochastic setting these are synonymous with agents converting at the first hitting time of a threshold (here $\hat{V}$).

The basic quantity we measure is the difference between the first derivatives of the conversion value and the bond value in the *block* case (i.e. $\gamma d\hat{S}(\hat{V})/dV - d\hat{W}(\hat{V})/dV$). When this quantity is positive, there are no *local* preemptive incentives, as shown in (2.12). Conversely, negative quantities reflect the existence of such incentives. The magnitude of these differences then reflect how great these incentives may be.

For these comparative statics we choose base case parameters. The short rate is set at 6%, with volatility of 20% (i.e. $\sigma = 0.2$), while the payout rate is 3%. We assumed that the convertible debt offering consists of 500 perpetual bonds (i.e. $n_0 = 500$), with conversion ratio, 5, ($\gamma = 5$). There are 1000 shares initially. The debt to equity ratio is, therefore, initially $\gamma n_0/(\psi_0 + \gamma n_0) = 2500/3500 = 71.4\%$.[30] We set the coupon rate at 0.8 which is equivalent to a 4% annual rate on a 20 Dollar principal payment. We take the proportion of costs due to bankruptcy to amount to a quarter of firm value ($\alpha = 0.25$) and a corporation tax rate of 30%. Finally, we set a call price of 25 US Dollars, and so $K > c/r$. Using these base case parameters, the following block conversion strategy was obtained: $V_{co} = 27,524$, as well as $V_{de} = 28,000$. Among the comparative statics listed in Table 1, the results concerning firm leverage merit some discussion. In order to avoid the selection of firm leverage affecting other parameters, we set the total number of shares after conversion fixed at 3,500 (i.e. $\psi_0 + \gamma n_0 = 3,500$). So when we vary ψ_0 we also vary n_0 such that the total number of post-conversion shares is fixed at $3,500$.

Broadly speaking, the preemptive incentives are usually present in the non-callable case, while this is rarely the case for callable convertible debt. The exceptions to this pattern in the non-callable case are when (1) firm leverage is very low, or (2) taxes are very low. The latter result is merely an illustration of lemma 3. The former

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[30] This percentage seems to be approximately consistent with the leverage level opimizing net firm value in Leland (1994)'s perpetual model of corporate debt (see his Figure 7, pp. 1231). Nonetheless, our motivation in setting a relatively high leverage level is to highlight the effect of competition.

Comparative Static	Parameter Value	Non-callable Case ($i = A$) $\gamma\hat{S}'(\hat{V}; A) - \hat{W}'(\hat{V}; A)$ ($\times 10^{-4}$)	Callable Case ($i = B$) $\gamma\hat{S}'(\hat{V}; B) - \hat{W}'(\hat{V}; B)$ ($\times 10^{-4}$)
Firm Volatility (σ)	15%	−6.75	+2.45
	20%	−5.11	+4.65
	30%	−3.12	+5.88
	40%	−2.07	+5.67
Call Price (K)	15 USD	−5.11	+20.3
	25 USD	−5.11	+4.65
	45 USD	−5.11	−7.83
	65 USD	−5.11	−1.30
Firm Leverage (ψ_0) N.B: $\gamma = 5$ and $n_0 = \frac{3{,}500-\psi_0}{\gamma}$	100 $\Rightarrow$ 97.1%	−43.7	−8.38
	500 $\Rightarrow$ 85.7%	−11.8	+3.91
	1,000 $\Rightarrow$ 71.4%	−5.11	+4.65
	1,500 $\Rightarrow$ 57.1%	−2.74	+4.69
	2,000 $\Rightarrow$ 42.9%	−1.44	+4.62
	2,500 $\Rightarrow$ 28.6%	−0.357	+4.52
	3,000 $\Rightarrow$ 14.3%	+0.556	+4.40
	3,300 $\Rightarrow$ 5.7%	+1.15	+4.33
Bankruptcy Costs (α)	0%	−5.05	+4.89
	10%	−5.08	+4.80
	25%	−5.11	+4.65
	50%	−5.18	+4.41
	80%	−4.90	+4.12
	100%	−4.95	+3.92
Taxes (τ)	0%	+1.71	+11.2
	1%	+1.79	+11.0
	15%	−2.00	+8.03
	30%	−5.11	+4.65
	40%	−6.82	+2.30
Dividend Policy (δ)	1%	−1.47	+15.4
	3%	−5.47	+4.65
	5%	−9.43	−10.8

Table 2.1: Pre-emptive Incentives when $i = A$ and B. The base case parameters used for these comparative statics are outlined in the text.

result shows that when the relative issue size of convertible debt is small preemptive incentives are less significant.[31]

In the callable case, preemptive incentives arise only when (1) the call price is very high or (2) the dividend policy is very high.[32] As the call price increases, callable convertible debt bears mimics non-callable convertible debt (see proposition 2). Since convertible bond holders stand to obtain larger dividend payments with increases in the payout rate, δ, we obtain the second result. If firm's issue convertible bonds with a relatively low call price and set a modest dividend policy, the block call strategy is more likely to yield a Nash equilibrium

So far we have only taken a broad look at the results. What about trends within each set of comparative statics? An interesting feature of Table 2.1 is the variation of the preemptive effects with firm leverage, $\gamma n_0/(\psi_0 + \gamma n_0)$. As the leverage decreases, the preemptive incentive in the non-callable case becomes smaller. This is not always the case for callable convertible debt. The effect tails off for lower leverage ratios, where the preemptive incentives gradually become a little stronger again. Pre-emptive incentives decrease with the firm's business risk, σ, for both callable and non-callable issues of convertible debt.

2.4 Competitive Bond Conversion and Optimal Call Strategies

In the previous sections we considered block strategies which involve the *simultaneous* exercise of bonds. As we have shown, the block exercise of bonds does not always yield a Nash equilibrium under perfect competition. In this section our objective is to determine symmetric competitive equilibria involving the *sequential* conversion of bonds.

We analyse the equilibria in three stages. First, we consider the case where the bonds are non-callable. Since management has no recourse to call the bonds, the determination of an equilibrium is more straight-forward. In the second stage, we tackle the more complicated case, where the bonds are callable. Creditors must take into account management's call decision as well as the actions of other creditors. Acting in the best interests of the current share holders, management holds a monopoly on the call decision of the firm, and it may also call bonds *sequentially.* In the final stage, which we treat in section 1.6, we consider finite-maturity convertible bonds with discrete dividend and coupon payments and stochastic interest rates.

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[31] These incentives dissappear completely when the leverage is between 14.3% and 28.6%.

[32] There may also be preemptive incentives if the firm's leverage is unrealistically high.

2.4.1 A Competitive Equilibrium for Non-Callable Convertible Bonds

2.4.1.1 Competitive Nash Equilibria

Consider the strategic interactions between a continuum of perfectly competitive bond holders. Since agents are unable to collude or communicate with each other, the only information which individual creditors receive regarding the actions of others is the aggregate number of convertible bonds remaining, $n_t \in [0, n_0]$. This affects the net dividend payment to share holders, and thereby influences the default decision of management, as well as bond holders' decision to convert their holdings into shares.

Broadly speaking, there are two sets of incentives for creditors when deciding their conversion and these are closely related to the analysis of the previous section. First, bond holders have incentives to convert sooner than others. The total equity value is divided among the share holders, and so, the share value becomes more diluted as bonds are converted. By converting earlier, creditors can acquire shares that are *temporarily* less diluted in value.[33] Second, by delaying their conversion, bond holders retain marginally safer debt. The intuition for this is as follows. Since the effect of converting bonds is to reduce the total debt service to bond holders, the firm's default trigger is lower. This means that the risk of default is marginally less for outstanding bonds, and hence, the debt is more valuable.

Given the inherent symmetry of type of the creditors, we look for a symmetric Nash equilibrium, which hinges crucially on the afore-mentioned trade-off between preemptive and delayed conversion.[34] Specifically, the equilibrium consists of the indifference of creditors between (1) immediate and (2) delayed conversion at an upper threshold, where each convertible bond equals γ shares. For firm values below the threshold, agents have no incentive to convert as the convertible bond value exceeds the value of the shares. By symmetry of types, these preferences hold for all agents.

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[33] As n_t declines, the share holders also bear less debt service. However, the dilutive effect is more dominant for higher firm values, where the conversion option comes into the money.

Whenever the indifference threshold is reached, some infinitesimal bond holders surrender their conversion option, as required by the equilibrium. Since the threshold is decreasing in n_t, the conversion of bonds brings the option associated with conversion out of the money. This process repeats itself whenever the trigger is reached until all the bonds are converted. The sequential conversion here resembles Pindyck (1988)'s model of "barrier control" for a firm expanding its operations. In that study, whenever an upper trigger is reached the firm increases its capacity by exercising some infinitesimal expansion options. An important difference between that study and our model, is that the firm in the former case operates in a monopoly and makes its expansion decisions as a monopoly holder. In our setting, the "barrier control" arises from non-cooperative behaviour among the continuum of creditors.

Formally, our model of sequential conversion is a symmetric mixed-strategy Nash equilibrium with an infinite number of players. In static mixed-strategy models, agents "mix" their pure strategies, by assigning probabilities to them, in such a way that the indifference of other agents is maintained. In the context of stochastic continuous-time stopping games, such as this one, mixed-strategy equilibria prescribe randomised stopping times[35] for the players. The randomisations describe the probability that agents "stop", thereby converting their bonds, and they are required to fulfill the indifference of other creditors. Since the bond holders in our model are infinitesimally small and have no individual influence on n_t, however, there are no randomised stopping times prescribed for them. In the model agents' indifference is maintained through the boundary conditions on the claim values.

Our model also fits into a strand of literature that studies stochastic models of competitive equilibria. Examples include the real options model of Leahy (1993), competitive runs (see Bartolini (1993)) and debt in industry equilibrium (see Fries, Miller, and Perraudin (1997)). Unlike the first and last of these contributions, however, the actions of creditors at the upper boundary in our study have no feedback-like effect on the value of the firm, V_t. The reason for this is that no money changes hands on exercising the conversion option, and so, by the Modigliani-Miller Theorem, firm value remains unchanged. If we specified a lognormal diffusion process for the share value, as opposed to the asset value of the firm, we would have such a feedback effect with a regulated geometric Brownian motion (see Harrison (1985)) for the share value. At the conversion threshold share value would be reflected downwards.

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[34] One could make an argument for asymmetries stemming from differing attitudes to risk. Even if agents had differing degrees of risk-aversion, however, this would make no difference as the risk-neutral measure is still the appropriate one for pricing.

[35] In continuous time, Poisson point processes describe such randomised strategies (see chapter 3, while in discrete time, probabilities of stopping apply at each decision node (see Chalasani and Jha (2001)).

Of the three afore-mentioned articles, Bartolini's study of competitive entry and exit decisions bears most resemblance to our model. In his set-up, the firms' profit stream is decreasing in the number of active firms in the industry and there are linear costs of entry and exit. As in our model, the equilibrium entry and exit triggers are specified in such a way that agents are indifferent to immediate and delayed entry/exit at the respective trigger. An important difference with our model, however, is that his triggers are not always increasing in time. For some cases they are fixed, resulting in the simultaneous (block) entry/exit of remaining firms ('a competitive run').

2.4.1.2 Partial Differential Equations for the Claim Values and Equilibrium Boundary Conditions

Since conversion is now sequential, the net dividend payment to individual shares is an explicit function of the number of bonds remaining: $(\delta V_t - [1-\tau]n_t c)/(\psi_0 + \gamma[n_0 - n_t])$. The competitive non-callable (with non-callability denoted by the index N) share value, $S(V,n) = S(V,n;N)$,[36] is thus given by the solution to the partial differential equation

$$\frac{\sigma^2 V^2}{2}\frac{\partial^2 S}{\partial V^2} + (r-\delta)\,V\frac{\partial S}{\partial V} + \left[\frac{\delta V - (1-\tau)\,nc}{\psi_0 + \gamma\,[n_0 - n]}\right] = rS. \qquad (2.15)$$

By strict priority, share holders have no value in bankruptcy: $S(V_b(n), n) = 0$, where $V_b(n) = V_b(n;N)$ is the varying competitive default trigger. As before, equity holders choose this point in an optimal fashion: $\partial S(V_b(n), n)/\partial V = 0$. Applying these familiar value-matching and smooth-pasting conditions to a general solution of the share implies that its value is given by

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[36] For convenience, we supress the dependence on $i = N$ in the notation.

$$\begin{aligned} S\left(V,n\right) &= \frac{1}{\psi_0+\gamma\left[n_0-n\right]}\Bigg\{V-\frac{\left(1-\tau\right)nc}{r} \\ &\quad +\left[\frac{\xi_- -1}{\xi_+-\xi_-}V_b-\frac{\xi_-}{\xi_+-\xi_-}\frac{\left(1-\tau\right)nc}{r}\right]\left(\frac{V}{V_b}\right)^{\xi_+} \\ &\quad -\left[\frac{\xi_+ -1}{\xi_+-\xi_-}V_b-\frac{\xi_+}{\xi_+-\xi_-}\frac{\left(1-\tau\right)nc}{r}\right]\left(\frac{V}{V_b}\right)^{\xi_-}\Bigg\}, \end{aligned} \tag{2.16}$$

for $V \in (V_b, \bar{V})$, where $\bar{V}$ is an upper threshold to be defined. Applying assumption 1 to the competitive case, the default trigger now satisfies the following inequality

$$V_b < \frac{\left(1-\tau\right)nc}{r}.$$

The competitive bond value, $W(V,n) = W(V,n;N)$, satisfies the following partial differential equation[37]

$$\frac{\sigma^2V^2}{2}\frac{\partial^2 W}{\partial V^2}+\left(r-\delta\right)V\frac{\partial W}{\partial V}+c=rW. \tag{2.17}$$

In bankruptcy, the bonds still recover the residual firm value, net of bankruptcy costs

$$W\left(V_b(n),n\right)=\frac{1-\alpha}{n}V_b\left(n\right). \tag{2.18}$$

Unlike the block case, the bond value at this boundary condition varies as n is decreasing and the default trigger itself is a function of n_t. At the upper conversion trigger, $\bar{V}(n) = \bar{V}(n;N)$, individual bonds equal their conversion value

$$W\left(\bar{V}(n),n\right)=\gamma S\left(\bar{V}(n),n\right). \tag{2.19}$$

In fact, by the absence of arbitrage, condition (2.19) holds as an inequality and lower bound for the convertible bond value. By the absence of arbitrage, the following inequality must also hold

$$\frac{\partial W\left(\bar{V}\left(n\right),n\right)}{\partial V}\leq\gamma\frac{\partial S\left(\bar{V}\left(n\right),n\right)}{\partial V}. \tag{2.20}$$

---

[37] Although equation (2.17) is almost identical to (2.2), we include partial derivatives here to emphasise that the competitive claim values are contingent on both the firm value, V_t, as well as the state variable, n_t.

If this were not the case, the bond value would undercut the share value and the arbitrage bound, (2.19), would be violated for firm values $V < \bar{V}$. Agents would have incentives to convert before $\bar{V}(n)$ and the Nash equilibrium would no longer hold.

Applying boundary conditions (2.18) and (2.19) to a standard solution of (2.17) yields the competitive *non-callable* convertible bond value

$$\begin{aligned} W\left(V,n\right) = \quad & \left\{ \gamma S\left(\bar{V},n\right) - \frac{c}{r} - \left[\frac{(1-\alpha)}{n}V_b - \frac{c}{r}\right]\left(\frac{\bar{V}}{V_b}\right)^{\xi-} \right\} g\left(V;V_b,\bar{V}\right) \\ & + \frac{c}{r} + \left[\frac{(1-\alpha)}{n}V_b - \frac{c}{r}\right]\left(\frac{V}{V_b}\right)^{\xi-}, \end{aligned} \tag{2.21}$$

for $V \in (V_b, \bar{V})$ where the default and conversion triggers, V_b and $\bar{V}$,[38] are functions of n to be determined. In order to determine these two more boundary conditions are needed.

The Nash equilibrium requires that creditors be indifferent between (1) immediate and (2) delayed conversion of their holdings at the upper conversion threshold, $\bar{V}(n)$. The former part of this indifference is fulfilled by boundary condition (2.19). Since the bond value equals the conversion value at $\bar{V}(n)$, agents are indifferent to convert their holdings into shares or to retain them. The latter part of the indifference requires that the bond value be invariant with changes in n:

$$\frac{\partial W\left(\bar{V}(n),n\right)}{\partial n} = 0. \tag{2.22}$$

Changes in the number of outstanding securities have no effect on the convertible bond value and, therefore bond holders are indifferent to allow others to convert first. Condition (2.22) also acknowledges the fact that creditors behave rationally and anticipate the actions of other agents. Together, boundary conditions (2.19) and (2.22) imply that the *immediate* conversion value, $\gamma S(\bar{V}(n), n)$, is equal to the competitive bond value, $W(\bar{V}(n), n)$, and this is locally invariant with changes in n. It is important to realise that (2.22) is *not* a smooth-pasting condition nor is it an optimality condition. It is required in partial fulfilment of the indifference in the Nash equilibrium.

As for the share value, the relevant boundary condition at $\bar{V}(n)$, is determined by considering the influence of converting Δn securities on the *total* equity value. Since no cash changes hands, the *total* equity value must equal the new equity value that results when there is a decline Δn in the number of securities:

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[38] For convenience we sometimes drop the dependence on n in the notation of the two triggers.

$$(\psi_0 + \gamma\left[n_0 - n\right]) S(\bar{V}(n), n) = (\psi_0 + \gamma\left[n_0 - n + \Delta n\right]) S(\bar{V}(n), n - \Delta n)$$

Taking the limit as $\Delta n \downarrow 0$, to give

$$(\psi_0 + \gamma\left[n_0 - n\right]) \lim_{\Delta n \downarrow 0} \left[\frac{S(\bar{V}(n), n) - S(\bar{V}(n), n - \Delta n)}{\Delta n}\right] = \gamma \lim_{\Delta n \downarrow 0} \left[S(\bar{V}(n), n - \Delta n),\right]$$

we obtain the boundary condition

$$\frac{\partial S\left(\bar{V}(n), n\right)}{\partial n} = \frac{\gamma S\left(\bar{V}(n), n\right)}{\psi_0 + \gamma\left[n_0 - n\right]}. \tag{2.23}$$

Boundary conditions (2.19), (2.22) and (2.23) can be usefully combined by *totally differentiating* boundary condition (2.19) with respect to n

$$dn\left\{\frac{\partial W\left(\bar{V}(n), n\right)}{\partial n} + \frac{\partial W\left(\bar{V}(n), n\right)}{\partial V}\frac{d\bar{V}\left(n\right)}{dn}\right\} = $$
$$dn\left\{\gamma\frac{\partial S\left(\bar{V}(n), n\right)}{\partial n} + \gamma\frac{\partial S\left(\bar{V}(n), n\right)}{\partial V}\frac{d\bar{V}\left(n\right)}{dn}\right\}.$$

Applying boundary conditions (2.22) and (2.23) and rearranging, then yields the following differential equation for the evolution of the conversion threshold

$$\frac{d\bar{V}\left(n\right)}{dn} = \frac{\left(\gamma^2/\left[\psi_0 + \gamma\left[n_0 - n\right]\right]\right) S\left(\bar{V}\left(n\right), n\right)}{W_V\left(\bar{V}\left(n\right), n\right) - \gamma S_V\left(\bar{V}\left(n\right), n\right)} \tag{2.24}$$

where subscripts refer to partial derivatives.

Differential equation (2.24) is crucial to the analysis of competitive equilibrium. It can be recast to show the relevant smooth-pasting condition for the bond value at $\bar{V}(n)$

$$\frac{\partial W\left(\bar{V}(n), n\right)}{\partial V} = \gamma\frac{\partial S\left(\bar{V}(n), n\right)}{\partial V} + \left(\frac{\gamma^2 S\left(\bar{V}(n), n\right)}{\left(\psi_0 + \gamma\left[n_0 - n\right]\right)}\right) / \left(\frac{d\bar{V}(n)}{dn}\right). \tag{2.25}$$

The first term on the right-hand side of (2.25) is familiar to us as the term arising from the smooth-pasting condition with fixed n. The second term, however, takes into account the dilutive effect on bond holdings of variations in n, along with the corresponding changes in $\bar{V}(n)$. Note, that since the arbitrage condition (2.20) holds as a strict inequality, $d\bar{V}/dn$ is decreasing in n. This is because the numerator of (2.24) is always positive (as the share value is always greater than or equal to zero) and the denominator of (2.24) is negative by (2.20). The decreasing nature of $d\bar{V}(n)/dn$ is required for the Nash equilibrium. Were this not the case, conversion of bonds at $\bar{V}(n)$ would take the subsequent conversion options deeper in the money (as there is a drop in n) and, future bonds would be exercised sub-optimally.

The initial condition, $\bar{V}(0)$, of the differential equation (2.24) is determined by considering the optimal strategy of the "last" infinitesimal creditor, or the strategy as $n \downarrow 0$. This is identical to the delayed strategy considered in the last section: $\bar{V}(0) = V_{de}$.

Equity holders anticipate the arrival of new shares at the conversion boundary, $\bar{V}(n)$, and so application of the boundary condition (2.23) to the share value, (2.16), determines the optimal default point, $V_b(n)$, as a function of the amount of debt remaining and the conversion trigger, $\bar{V}(n)$:

$$\frac{dV_b}{dn} = \frac{(1-\tau)\,c}{r}\left[\frac{\xi_+ - \xi_- + \xi_-\left(\frac{\bar{V}}{V_b}\right)^{\xi_+} - \xi_+\left(\frac{\bar{V}}{V_b}\right)^{\xi_-}}{\left[\left(\frac{\bar{V}}{V_b}\right)^{\xi_-} - \left(\frac{\bar{V}}{V_b}\right)^{\xi_+}\right]\left[(\xi_+ - 1)(\xi_- - 1) - \xi_+\xi_-\frac{(1-\tau)nc}{rV_b}\right]}\right]. \qquad (2.26)$$

When all the creditors convert their holdings, the firm turns into a pure-equity operation, and so liquidation is determined by the positive net-worth requirement: $V_b(0) = 0$.

2.4.1.3 Convertible Bond and Equity Values

Using the afore-mentioned boundary conditions we obtain the following proposition:

Proposition 4 *Consider competitive non-callable convertible debt ($i = N$). A symmetric Nash equilibrium consists of:*

1. *the indifference of creditors between immediate and delayed conversion at the first hitting time of the trigger, $\bar{V} = \bar{V}(n; N)$;*

2. *creditors' preference to hold their bonds when* $V \in (V_b(n;N), \bar{V}(n;N))$*;*

3. *the firm's optimal default decision at the first hitting time of the firm value,* $V_b = V_b\,(n;N)$.

The twin triggers, $\bar{V}(n;N)$ *and* $V_b(n;N)$*, are determined from the coupled ordinary differential equations, (2.24) and (2.26), with initial conditions*

$$\bar{V}(0;N) = \frac{c}{r}\left(\frac{\xi_+}{\xi_+ - 1}\right)\left[n_0 + \frac{\psi_0}{\gamma}\right]$$

and

$$V_b(0;N) = 0$$

respectively. The value of the competitive convertible bond value, $W\,(V, n;N)$*, and share value,* $S\,(V, n;N)$*, are given by (2.21) and (2.16), respectively.*

The necessary conditions for the equilibrium include:

1. *that the arbitrage condition (2.20) holds as a strict inequality for all* $n \in [0, n_0]$*;*

2. *that* $\bar{V}(n;N) > V_b(n;N)$ *for all* $n \in [0, n_0]$*;*

3. *and that* $W(V, n;N) > \gamma S(V, n;N)$ *for all* $V \in (V_b, \bar{V})$ *and* $n \in [0, n_0]$.

The proof of this proposition may be found in section 2.9.9.

In order to see heuristically that the conversion trigger, $\bar{V}(n)$, is synonymous with a symmetric Nash equilibrium, consider deviations by small agents of two kinds. First, there are deviations consisting of different conversion triggers, $V^*(n) \neq \bar{V}(n)$, for a given n, and second, there are deviations concerning the order of conversion with others: i.e. $V^*(n^*)(\neq \bar{V}(n))$ for $n^* \neq n$.

Consider deviations of the first kind. Suppose $V^*(n) = \bar{V}(n) - \epsilon$, for arbitrarily small ϵ. Here we examine a deviation involving a small creditor converting earlier than the conversion strategy. Small creditors will have incentives for early conversion if $W(V^*(n), n) < \gamma S(V^*(n), n)$. But since the arbitrage bound (2.20) holds as well as (2.19), the bond value is always greater than this share value. Further, note that deviations of the type: $V^*(n) > \bar{V}(n)$, are impossible. The reason for this is that creditors will convert continuously at $\bar{V}(n)$, thereby lowering the value of n and preventing the firm value from ever exceeding the trigger. Now, consider deviations of the second kind. By the indifference condition on the bond value, $\partial W(\bar{V}(n), n)/\partial n = 0$, creditors have no incentives to act marginally earlier or later as they cannot raise their value.

Although the equilibrium in proposition 4 illustrates *weak* time-consistency[39], as agents have no incentives to deviate from the equilibrium for all n_t, it should be noted that it is *not* Markov perfect. The reason is that Markov perfect equilibrium require that the Nash equilibrium hold for each proper sub-game, *starting for all possible realisations of the state variable.* Clearly, the equilibrium above cannot then apply if one considers sub-games at time s when the firm value exceeds the conversion trigger. The reason for this is that agents actions are only specified for $V \leq \bar{V}$.

2.4.2 A Competitive Equilibrium for Callable Convertible Bonds

2.4.2.1 Management's Optimal Call Policy

Since most convertible bonds are callable, the analysis of the previous sub-section would be far from complete in the absence of this important feature. In this sub-section we examine perpetually *callable* convertible debt, when $K \geq [(1-\tau)c]/r$. The determination of an equilibrium is now complicated by the fact that creditors must also take into account the actions of management, which acts as a monopoly holder of the firm's equity. Management, in its own turn, must evaluate the effect of bond holders' actions when devising its call policy.

Since bonds are converted sequentially, management may call the bonds as a block at any stage of the game, for all possible $n \in [0, n_0]$. By the absence of arbitrage, the share value must satisfy the following inequality

$$\hat{S}(V,n) \geq \max\left[0, \min\left(\frac{V}{\psi_0 + \gamma n_0}, \frac{(V - nK)}{\psi_0 + \gamma\left[n_0 - n\right]}\right)\right]. \qquad (2.27)$$

---

[39] See Basar (1989).

where the block share value for varying n_t, $\hat{S}(V, n)$, is given by

$$\begin{aligned} \hat{S}\left(V, n\right) &= \frac{1}{\psi_0 + \gamma\left[n_0 - n\right]} \left\{ V - \frac{\left(1-\tau\right)nc}{r} - \left[\hat{V}_b - \frac{\left(1-\tau\right)nc}{r}\right]\left(\frac{V}{\hat{V}_b}\right)^{\xi_-} \right. \\ &+ \left. \left\{ \frac{\left(1-\tau\right)nc}{r} - \frac{\gamma n \hat{V}}{\psi_0 + \gamma n_0} + \left[\hat{V}_b - \frac{\left(1-\tau\right)nc}{r}\right]\left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-} \right\} g\left(V; \hat{V}_b, \hat{V}\right) \right\} \end{aligned} \tag{2.28}$$

It turns out that the point separating the interval of firm value over which bonds are converted and redeemed is still V_{ca}. One may note that in equation (2.28) bonds are converted and never redeemed. In fact, it may be shown that whenever $K \geq [(1-\tau)c]/r$ the call strategy cannot be less than V_{ca} for all n_t (see the proof of lemma 6). With varying n_t we obtain the analogous result to lemma 1

Lemma 6 *Suppose $K \geq [(1-\tau)c]/r$ and $V^*(n) \geq V_{ca}$ where $V^*(n)$ is the root of $d\hat{S}(V^*(n), n)/dV = 1/(\psi_0 + \gamma n_0)$. If*

$$V^*(n) > \left(n_0 + \frac{\psi_0}{\gamma}\right)\left(\frac{\left(1-\tau\right)c}{r}\right)\left(\frac{\xi_+}{\xi_+ - 1}\right)\left(\frac{\xi_-}{\xi_- - 1}\right), \quad \forall \quad n \in [0, n_0], \tag{2.29}$$

$$\frac{d\hat{S}\left(y, n\right)}{dV} > \frac{1}{\psi_0 + \gamma n_0}, \qquad \forall \qquad y, n \in (V_{ca}, V^*), [0, n_0] \tag{2.30}$$

and

$$\hat{S}\left(V_{ca}, n\right) < \frac{K}{\gamma} \tag{2.31}$$

for all call policies over the interval (V^*, V_{co}) and $n \in [0, n_0]$, then the optimal block call policy is $\hat{V}(n) = V_{ca}$ for all $n \in [0, n_0]$.

The proof of this lemma may be found in section 2.9.10.

Although the conditions for this call policy are numerous, many parametrisations of the model do satisfy them. Management's optimal call policy is the same for all sub-games, with each such game starting along the continuum of $n_t \in [0, n_0]$. Even though the analysis of the firm's call policy in lemma 6 is conducted in isolation from creditors' actions, the call policy is time-consistent in the sense that management always has incentives to stick to the same policy regardless of the unfoldment of future information.

2.4.2.2 The Competitive Equilibrium

Since the conversion, call and default strategies are all Markov[40] with no other historical dependence, we look, once again, for a competitive equilibrium in feedback strategies. For each portion of the game, there should be no incentives for agents to deviate from their strategies. Thus, proceeding recursively from the last convertible bond holder (in the limit that $n \downarrow 0$), we look for a symmetric equilibrium where management and creditors have no incentives to deviate.

In the limit $n \downarrow 0$, with the assumptions of lemma 6 holding, management will call the "last" bonds if possible at V_{ca}. From the analysis of the previous section, the "last" bond holders would voluntarily convert at V_{de}. Thus, the first of these two strategies to come into the money will dominate the voluntary/forced conversion of bonds in the limit. The callable competitive conversion trigger, $\bar{V}(n;C)$, where C denotes that the bonds are callable, is therefore given by

$$\bar{V}\left(0;C\right) = \min\left[V_{de}, V_{ca}\right] \tag{2.32}$$

in the limit $n \downarrow 0$. Thus, $\bar{V}(0;C) \leq V_{ca}$.

How does the competitive conversion trigger vary with $n > 0$? Using methods from the equilibrium in the non-callable case it can be shown that the conversion strategy, $\bar{V}(n;C)$, precedes V_{ca} for all n_t.

Lemma 7 *If the necessary condition for the competitive equilibrium*

$$\gamma\frac{\partial S(\bar{V}(n;C),n;C)}{\partial V} \geq \frac{\partial W(\bar{V}(n;C),n;C)}{\partial V} \tag{2.33}$$

holds for all $n \in [0, n_0]$*, the conversion policy* $\bar{V}(n;C) < V_{ca}$ *for all* $n \in (0, n_0]$.

The proof of this lemma may be found in section 2.9.11.

An important implication of this lemma is that management cannot access its afore-mentioned optimal call policy, V_{ca}, as the competitive conversion trigger comes into the money first. The only action that could be taken in defiance of $\bar{V}(n;C)$ by creditors would be to call the bonds earlier. The reason for this is that bonds are converted continuously at $\bar{V}(n;C)$ and so the firm value is never allowed to rise above the trigger level. We show in the proof of proposition 5 that management has no incentives to deviate from allowing creditors to convert voluntarily at $\bar{V}(n)$. Since management has incentives to adopt a policy of inaction, the conversion trigger is synonymous once again with a Nash equilibrium, which is presented formally in the next proposition.

---

[40] These are also known as "feedback" strategies.

Proposition 5 *Consider callable convertible debt with $K \geq [(1-\tau)c]/r$ when the conditions of lemmas 6 and 7 hold. A symmetric competitive Nash equilibrium consists of:*

1. *the indifference of creditors between immediate and delayed conversion at the first hitting time of the trigger, $\bar{V} = \bar{V}\,(n;C)$;*
2. *creditors' preference to hold their bonds when $V \in (V_b(n;C), \bar{V}(n;C))$;*
3. *the firm's optimal default decision at the first hitting time of the firm value, $V_b = V_b\,(n;C)$.*

The conversion and default triggers are given by the coupled ordinary differential equations (2.24) and (2.26) with initial conditions

$$\bar{V}\,(0;C) = \min\left[K\left(n_0 + \frac{\psi_0}{\gamma}\right), \bar{V}\,(0;N)\right]$$

and

$$V_b(0;C) = 0$$

respectively, and where $S(V,n) = S(V,n;C)$ and $W(V,n) = W(V,n;C)$ are the competitive share and convertible bond values given by (2.16) and (2.21).

In addition to the necessary conditions listed in proposition 4, a further necessary condition for the competitive Nash equilibrium, is that

$$V_b\,(n;C) \leq \left(\frac{\xi_-}{\xi_- - 1}\right)\frac{(1-\tau)\,nc}{r}, \qquad \forall\ n \in [0, n_0] \tag{2.34}$$

The proof of this proposition may be found in section 2.9.12. As in proposition 4, the equilibrium is not Markov perfect.

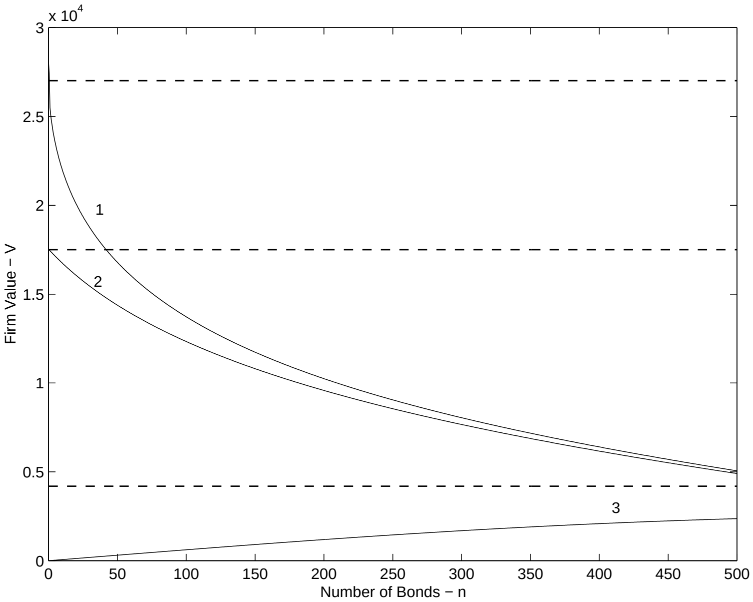

Figure 2.2: Competitive conversion triggers, $\bar{V}(n;N)$ (labelled 1), $\bar{V}(n;C)$ (labelled 2) and the default trigger, $V_b(n;N)$. The corresponding block conversion, $\hat{V}(A)$, call, $\hat{V}(B)$, and default strategy, $\hat{V}_b(B)$, are given by the dashed horizontal lines starting from the top one and coming down.

2.4.3 A Numerical Example

We consider a numerical example to get insights into the effect of the sequential equilibria on the claim values. For Figures 2.1-2.14, we use the same base case parameters as those in the previous section. Exceptions are indicated in the captions. With the base case parameters, Figure 2.1 shows the conversion values of the bonds both after the block conversion (this is the dotted line), and if conversion were to take place before the block (this line is dashed). The former share value is merely $\gamma V/(\psi_0 + \gamma n_0)$, while the latter is $\gamma \hat{S}(V;A)$. The solid line shows the value of the convertible bond, $\hat{W}(V;A)$. The presence of preemptive incentives is confirmed also from Table 2.1 (the difference in first derivatives between the bond and conversion value is -5.11×10^{-4}).

The sequential nature of the conversion and call policies can be seen clearly in Figures 2.2 and 2.3. These sequential strategies prescribe the conversion of bonds at a much earlier stage than the block case (see plots labelled "1" and "2" as well as the upper horizontal lines). In the case of *callable* convertible debt, competitive conversion strictly precedes both the block call and conversion strategies. This is

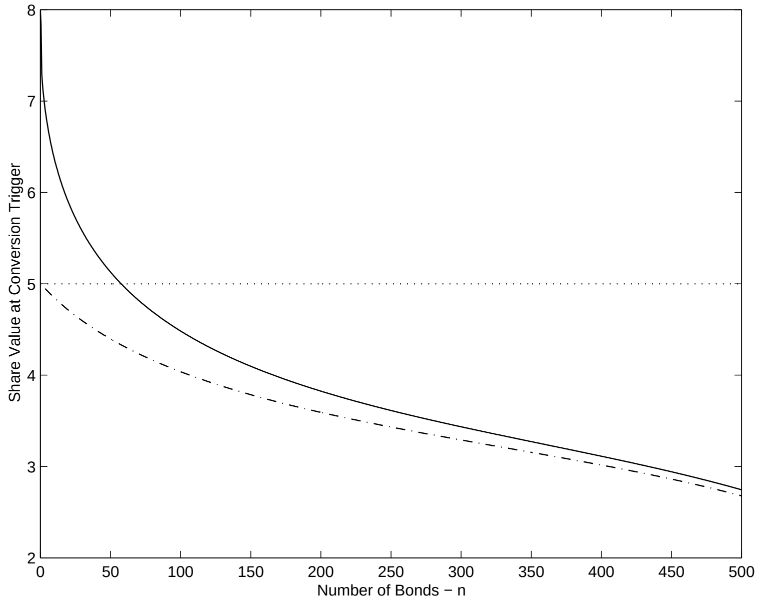

Figure 2.3: Evolution of the share value at the conversion boundary in both the non-callable case (solid line), $S(\bar{V}(n;N),n;N)$, and the callable case, $S(\bar{V}(n;C),n;C)$ (dot-dashed line). The share value at which management would call, in the absence of creditors' actions is also shown as the horizontal dotted line.

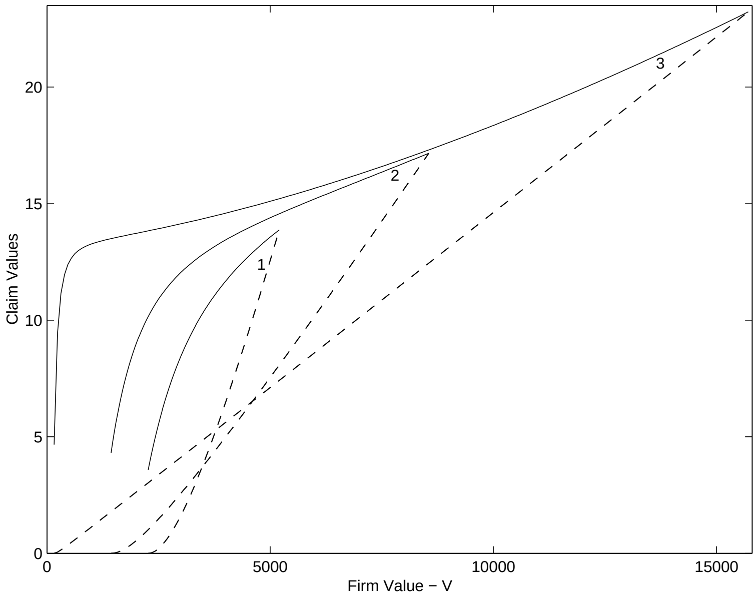

Figure 2.4: Competitive callable convertible bond values, $W(V,n;C)$, (solid lines) and conversion values, $\gamma S(V,n;C)$. The numbers 1, 2, 3 correspond to the following number of bonds remaining, n, respectively 475, 250 and 25, for each pair of plots.

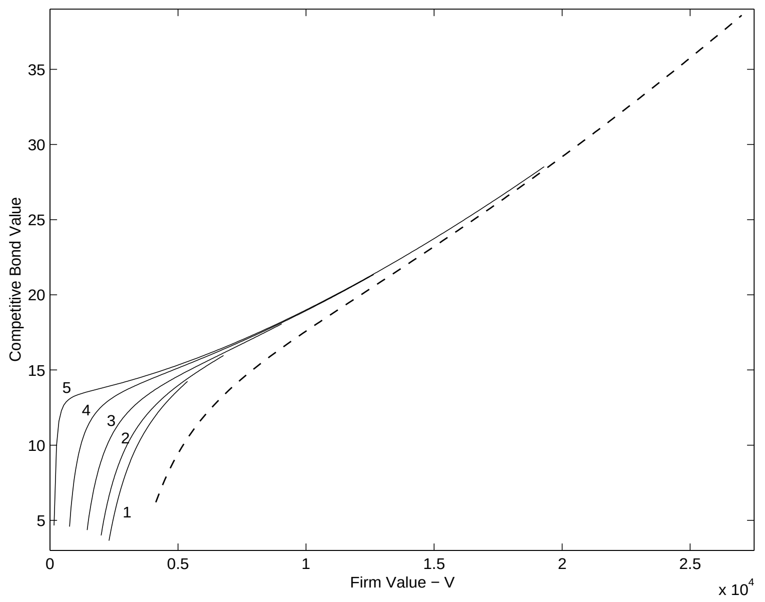

Figure 2.5: Competitive Non-Callable Bond value, $W(V, n; N)$, compared with the corresponding block value (this line is dashed). The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

not the case for non-callable convertible debt as $\bar{V}(0; N) = V_{de} > V_{co}$ from lemma 5. Thus, the last bonds will be delayed with respect to the block. Moreover, the share value at which conversion take place is decreasing in n.

Figure 2.4 illustrates the sequential equilibrium and its effect on claim values. At the conversion trigger, the bond value is invariant to infinitesimal changes in n and is also equal to the conversion value. Below the indifference trigger the bond values exceed the conversion value, and so creditors have no incentive to convert. Clearly, the sufficient and necessary condition of the equilibrium holds ($\gamma S_V(\bar{V}, n; C) > W_V(\bar{V}, n; C)$) and this ensures that agents have no incentives to deviate from the equilibrium. This may be contrasted with the block strategy where the conversion value is undercut (see Figure 2.1). At the lower default trigger, $V_b(n; C)$, management optimally defaults on its debt service, with the share value smooth-pasting onto zero value. The Figures illustrate the key trade-offs at the indifference trigger. Convertible bond values are increasing in time (see Figures 2.5 and 2.6), while the share values are decreasing in time (for high V_t only, see Figures 2.7 and 2.8).

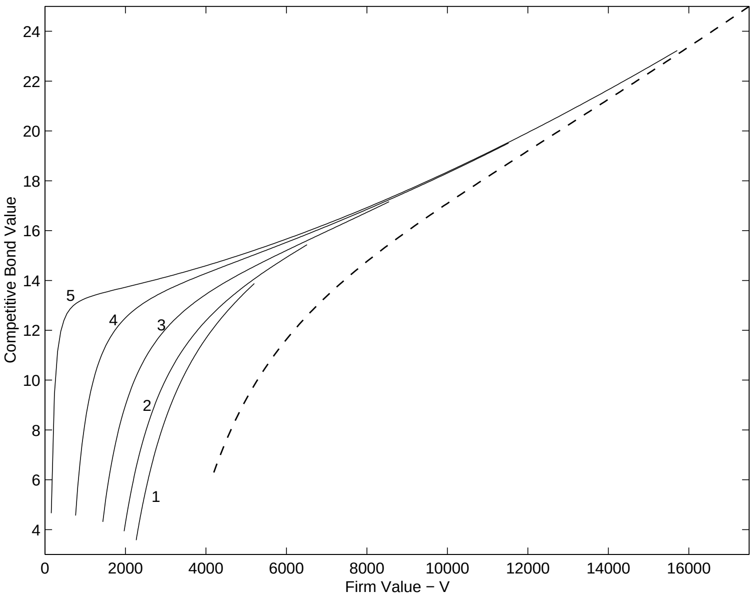

Figure 2.6: Competitive Callable Bond value, $W(V, n; C)$, compared with the corresponding block value (this line is dashed). The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

An interesting feature of the competitive bond values is that the convertible bond values exceed the block values, when the issue is both *callable* and *non-callable* (see Figures 2.5 and 2.6). This is not necessarily the case for the share value. When the issue is non-callable, the share value for low n_t and high V_t undercuts the block value (See plot number 5 as well as the dashed line in Figure 2.7). This is not the case with callable debt. Recall that Table 2.1 shows there are preemptive incentives in the non-callable case while such incentives do not exist in the callable case, when base case parameters are used (the corresponding quantities are -5.11×10^{-4} and $+4.65 \times 10^{-4}$ respectively). The intuition here is that the block strategy results in preemptive incentives when the bonds are *non-callable*, causing them to supersede the competitive share values.

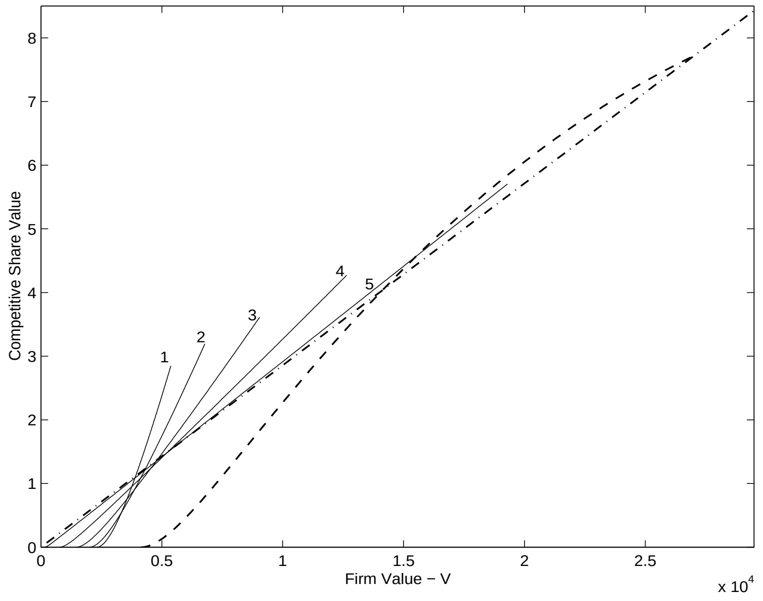

Figure 2.7: Competitive Share value when the bonds are non-callable, $S(V, n; N)$, compared with the block value, $\hat{S}(V; A)$, (this line is dashed) and the ex-post share value (this line is dot-dashed). The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

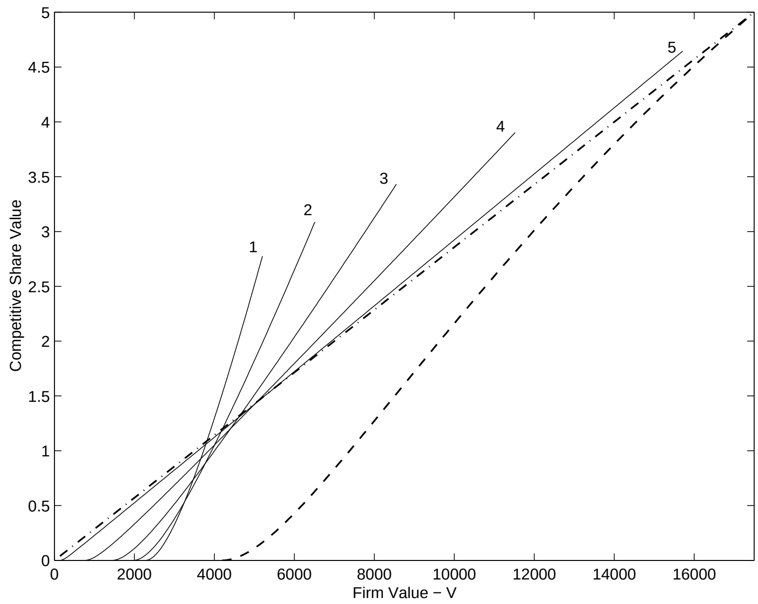

Figure 2.8: Competitive Share value when the bonds are callable, $S(V, n; C)$, compared with the block value, $\hat{S}(V; B)$, (this line is dashed) and the ex-post share value (this line is dot-dashed). The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

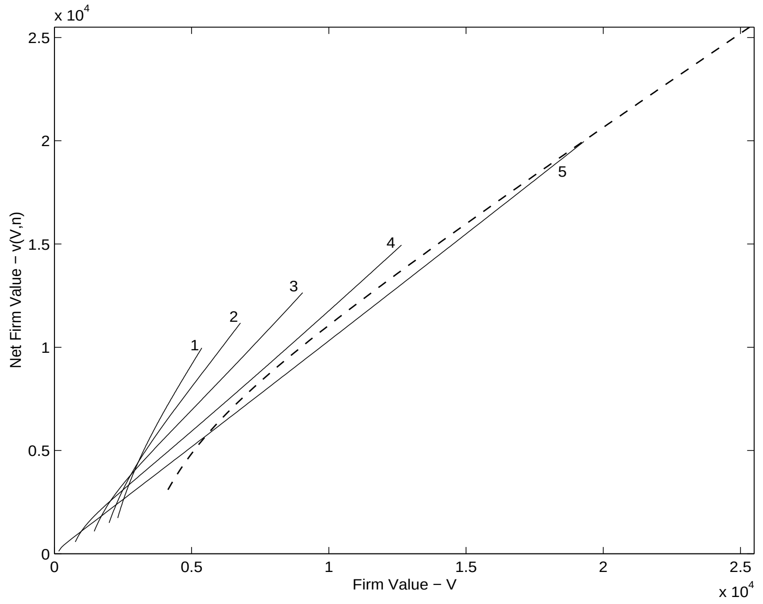

Figure 2.9: Net firm value in the competitive equilibrium with non-callable convertible debt, $v(V, n; N) = (\psi_0 + \gamma[n_0 - n])S(V, n; N) + nW(V, n; N)$, compared with the block value (this line is dashed). The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

2.5 Implications of the Equilibrium

2.5.1 Efficiency and Design of Convertible Debt Contracts

The efficiency of the equilibria can be studied by determining the *net firm value*, $v(V, n; i) = (\psi_0 + \gamma[n_0 - n])S(V, n; i) + nW(V, n; i)$ for $i = \{N, C\}$ and comparing it with the block case. Alternatively, using the Modigliani and Miller (1958) Theorem, this is the firm value *gross* of the tax benefit of debt and *net* of bankruptcy costs. Both the tax benefit of debt and bankruptcy costs increase with n. The reason for this is that more debt is outstanding and the default trigger is higher (see Figure 2.2) as n increases. The *net* firm value, therefore, involves a dynamic trade-off between the tax advantage of debt and inefficiencies associated with bankruptcy. Without corporation taxes the sequential equilibria Pareto-dominate the block strategies (see Figures 2.11 and 2.12). The effect is stronger for earlier stages of the game. Moreover, the percentage increase of the *net* firm value over the firm value (i.e. $[v(V, n; C) -$

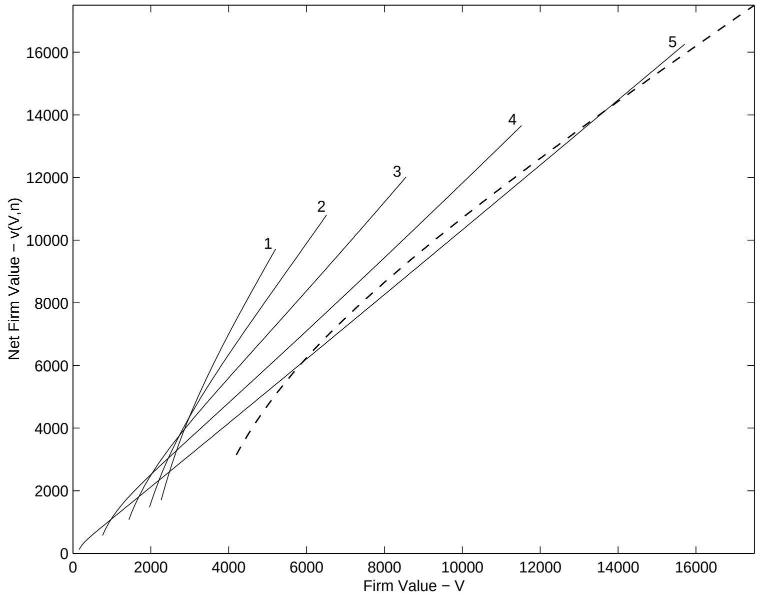

Figure 2.10: Net firm value in the competitive equilibrium with callable convertible debt, $v(V, n; C) = (\psi_0 + \gamma[n_0 - n])S(V, n; C) + nW(V, n; C)$, compared with the block value (this line is dashed). The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

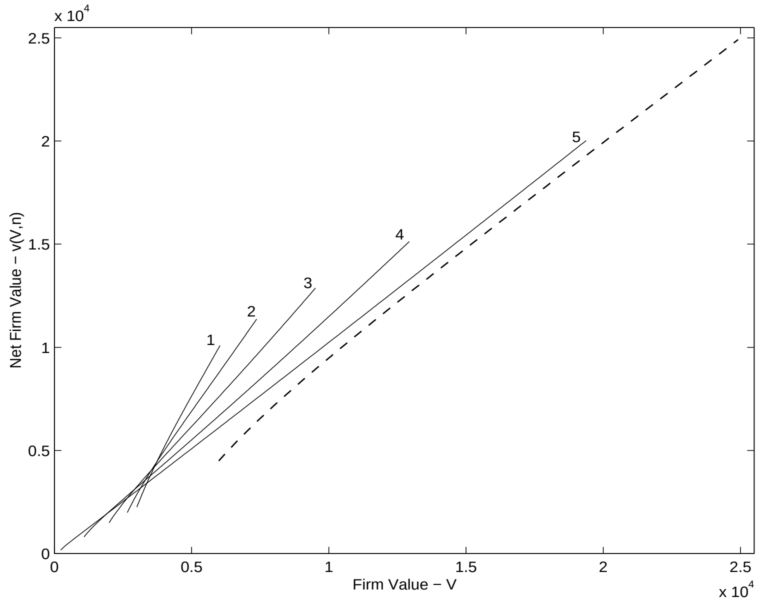

Figure 2.11: **No Taxes:** Net firm value in the competitive equilibrium with non-callable convertible debt, $v(V, n; N) = (\psi_0 + \gamma[n_0 - n])S(V, n; N) + nW(V, n; N)$, compared with the block value (this line is dashed). In this particular plot there are no taxes. The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

$V]/V$) can be substantial (see Figures 2.9-2.12). These results obtain even though for many of the plots the leverage level is far lower than the block leverage level, which is fixed. One reason for this result is that sequential conversion raises the credit quality of the company, thereby reducing the costs of financial distress.

With taxes present and for later stages of the game the sequential equilibria no longer necessarily Pareto-dominate the block case, unless firm value is very low or high. With such extremes, the likelihood of losing the tax shield through bankruptcy or conversion is imminent. The reason for the main result, however, is that block conversion maintains the tax shield at its highest level for all sub-games. As the remaining bonds are converted, the tax shield becomes smaller in the sequential equilibria and results in a relative loss of value, compared with the block case.

The cut-off point, where the sequential equilibria cease to Pareto-dominate the block case, appear to lie between the plots for 125 and 25 bonds remaining (See the plots labelled 4 and 5 respectively in Figures 2.9 and 2.10). These correspond to leverages of 17.9% and 3.6% respectively. In the case of straight debt financing, firm value-maximising managers have incentives to raise the ratio to at least about 20%.[41]. Since our setting does not accommodate other non-convertible debt it is difficult to evaluate these Pareto effects in an isolated fashion from leverage effects. Nonetheless, given the low cut-off point for leverages between 3.6 and 17.9%, it seems likely that the Pareto dominance of the sequential equilibria will be strict over the block strategies for firms with other tranches of debt outstanding. In our setting a natural way around this would be to assume that the firm makes further convertible debt offerings, whenever the leverage level reaches a certain low value.

Figure 2.13 illustrates the relative efficiency of the callable case over the non-callable case. Once again, the effect becomes less pronounced as the number of bonds decreases. Moreover, the call feature causes the corresponding share values to exceed the non-callable ones (see Figure 2.14). From the standpoint of efficiency call features have a positive effect on the efficiency of the sequential equilibria.

An interesting consequence of our choice of base case parameters is that the block call strategy, $\hat{V}(B)$, yields a pure-strategy Nash equilibrium for callable convertible debt. The reason for this is that condition (2.12) is satisfied (see the quantities in Table 2.1). Since *sequential* share and bond values $S(V, n; C)$ and $W(V, n; C)$ exceed the block ones, however, security holders do better under the sequential Nash equilibrium. Moreover, the efficiency results for callable convertible debt in the preceding paragraphs show that the sequential equilibria Pareto-dominate the "block" Nash equilibrium (i.e. the equilibrium with simultaneous actions).

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[41] This figure relates to short term debt of about 6 months maturity (see Figure 1, pp. 997 of Leland and Toft (1996)). As the maturity of standard corporate debt contracts tends to infinity, as in our model, the optimal ratios increase.

Collectively, the results of this sub-section demonstrate the importance of the sequential competitive equilibria from the perspective of contract design. Given perfect competition, management will have incentives to tailor convertible debt contracts in such a way that claim values are maximised. Unlike the literature on security design, however, the implications of our study relate to inefficiencies arising from the block constraint and the use of non-Pareto-optimal equilibria. Conversely, this literature has focused primarily on inefficiencies associated with information acquisition (see Boot and Thakor (1993)), the allocation of control rights,[42] and the management of financial distress (see Anderson and Sundaresan (1996)). By designing convertible debt offerings in a suitable way and accounting for competition, first-best outcomes may be realised.

2.5.2 Convertible Debt as Delayed Equity Financing

An important finding of the sequential competitive equilibria relates to the purpose of convertible debt financing. Since the competitive conversion option comes into the money a lot sooner than the corresponding block option, convertible debt contracts are *intrinsically* more equity-like.[43] Under perfect competition, the delayed-equity motive for issuing convertible debt is stronger. It is interesting that this result holds even though our model has avoided any ex-ante bias in the convertible offering. Since the sequential equilibrium involves a "barrier control"-like conversion policy, "backdoor equity" will be acquired in spurts, rather like separate issues of equity. Since only one issue of convertible debt is made at time $t = 0$, however, there are probably no associated adverse pricing effects with the "new" equity issues.

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[42] See Aghion and Bolton (1992) and Hart and Moore (1994).

[43] By "intrinsic" we mean the natural characteristics of convertible debt, when (i) the issue is held by perfectly competitive creditors and (ii) there is no block restriction on conversion.

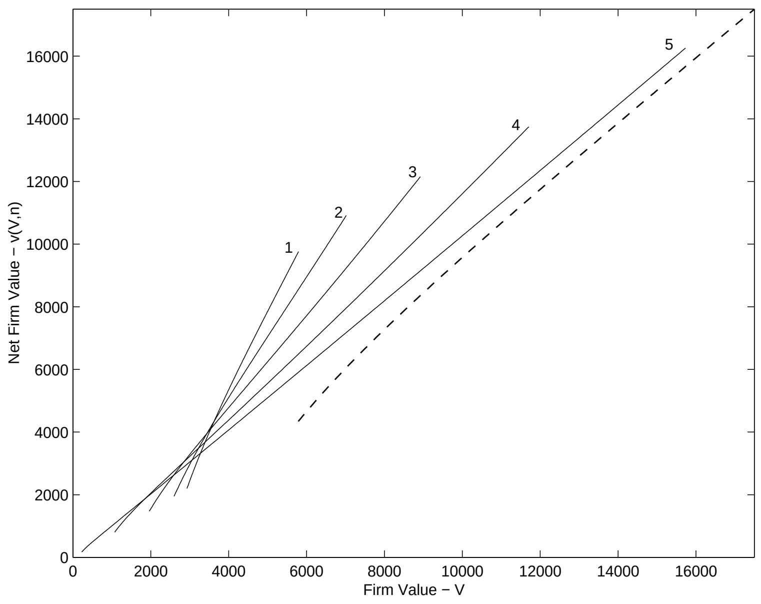

Figure 2.12: **No Taxes:** Net firm value in the competitive equilibrium with callable convertible debt, $v(V, n; C) = (\psi_0 + \gamma[n_0 - n])S(V, n; C) + nW(V, n; C)$, compared with the block value (this line is dashed). In this particular plot, there are no taxes. The numbered plots (the solid lines) 1, 2, 3, 4, 5 correspond to the following number of bonds remaining, n, respectively 475, 375, 250, 125 and 25.

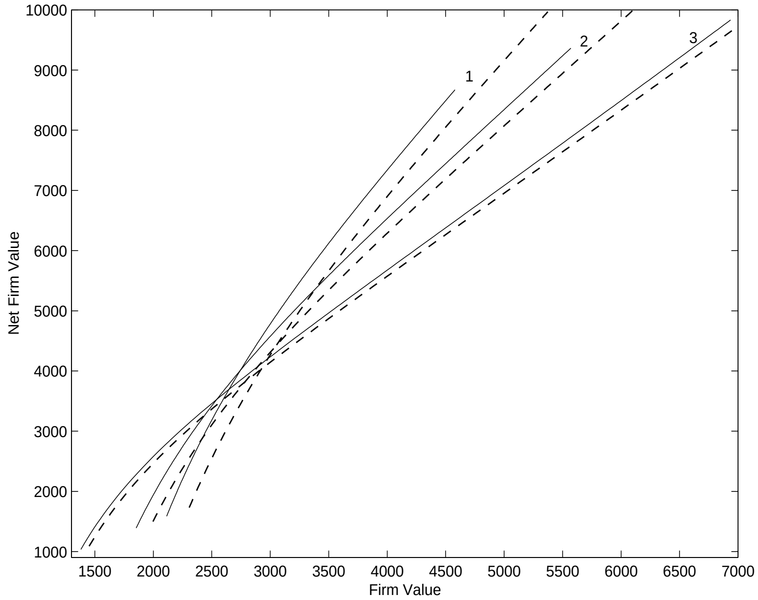

Figure 2.13: Comparison of competitive net firm value in the callable (solid lines) and non-callable (dashed lines) cases. In this particular plot, $K = 15$ U.S. Dollars. The numbered plots (the solid lines) 1, 2, 3 correspond to the following number of bonds remaining, n, respectively 475, 375 and 250.

Several studies have documented the fact that conversion-forcing calls are perceived by the market as a signal of future declines in profit. Mikkelson (1981) finds an average abnormal return (AAR) of -2.08% near the time of call announcements. By calling the debt, management avoids potential costs of financial distress and can ensure that there are no unconverted convertibles remaining.[44] As shown in the empirical study by Smith (1986), the effect of issuing convertibles results in an AAR of -2.07%. The combined effect of issuing convertible debt and subsequently forcing conversion results in a greater AAR of -4.15% than issuing equity outright with an AAR of -3.14% (See Nyborg (1996)). Thus, if firms issue convertible debt with the intention of getting delayed equity into their capital structures, they will be reluctant to force conversion. From management's perspective this difficulty can be overcome if voluntary conversion can be synthesised.

If investors and management use the competitive conversion strategy, management will never have a need to force conversion. This consequence suggests an even stronger

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[44] The models by Harris and Raviv (1985) and Nyborg (1995) hinge crucially on the assumption that conversion-forcing calls signal negative information to investors.

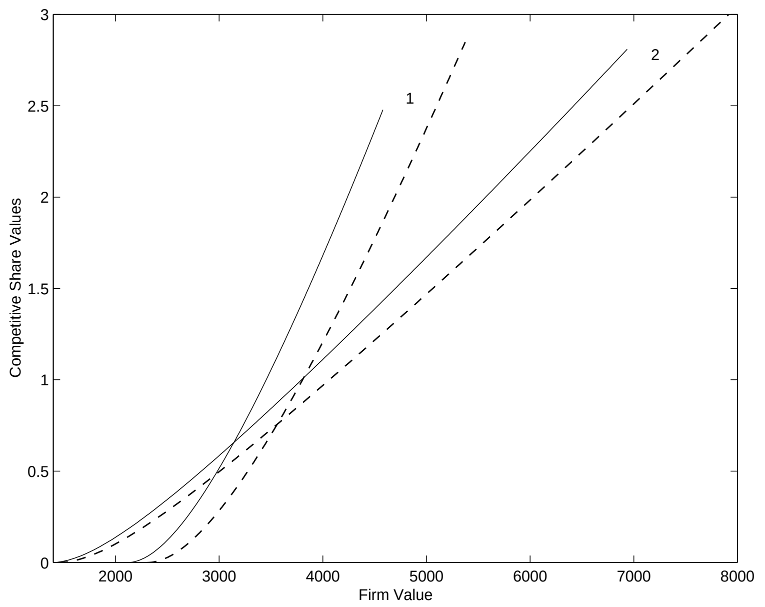

Figure 2.14: Comparison of competitive share values in the callable (solid lines) and non-callable (dashed lines) cases. In this particular plot, $K = 15$ U.S. Dollars. The numbered plots (the solid lines) 1, 2 correspond to the following number of bonds remaining, n, respectively 475 and 250.

intrinsic delayed-equity characteristic of convertible debt. Our implications relating to the delayed equity rationale for convertible debt may be contrasted with Stein (1992), where forced-conversions are required to bring about “backdoor equity financing”, and Nyborg (1995), where the link between voluntary conversion and delayed equity financing arises as a theoretical prediction.

2.5.3 Default Probabilities of High-Yield Convertible Bonds

The empirical study by Rosengren (1993) shows that convertible high-yield bonds have lower probabilities of default than their non-convertible counterparts. In our model the default probabilities of convertible bonds under the sequential equilibria are generally less than the corresponding figures in the block case. The reason for this is that the default trigger is generally lower (see Figure 2.2) and so, the probability of hitting the barrier $V_b(n;i)$ is generally lower than that of hitting $\hat{V}_b(i)$. High-yield convertible bonds are generally synonymous with low current firm values. The consequence of this is that the default probabilities will be even more sensitive to the the lower default trigger, V_b.

A full analysis of the default probabilities of convertible bonds is beyond the scope of this chapter. Future empirical research in this direction, however, must ascertain the degree to which bonds are converted gradually.

2.6 Pricing Convertible Bonds under Perfect Competition

2.6.1 Basic Assumptions

The analyses of propositions 4 and 5 examined extreme cases where the bonds are either *perpetually* non-callable or callable. The convertible debt resembled convertible preferreds with continuous dividend and coupon payments. In this section we present a methodology for pricing finite-maturity convertible bonds under perfect competition, with discrete payments.

We assume that management makes an aggregate dividend payment of D to all share holders on a quarterly basis (on dividend dates t_D) and a semi-annual coupon of $\bar{c}$ (on coupon dates t_c) to each convertible bond. Moreover, the convertible debt contract has a maturity of T with a principal payment of P per bond. With these alterations, the firm value evolves according to the diffusion process

$$dV_t = r_t V_t dt + \sigma V_t dB_t \tag{2.35}$$

under the risk neutral measure. At any dividend date the firm value satisfies the jump condition: $V_{t_{D-}} = V_{t_{D+}} - D$, where the signs denote infinitesimal times before and after the payment.

Following Brennan and Schwartz (1980), we assume that there is also a diffusion process for the short rate, r_t. We use the one-factor model of Cox, Ingersoll, and Ross (1985)

$$dr_t = \beta\left(\bar{r} - r_t\right) + \sigma_r r_t^{1/2} dZ_t \tag{2.36}$$

under the objective probability measure, where σ_r^2 is the variance of the instantaneous rate of return, $\bar{r}$ is the mean to which the short rate reverts, β is a positive constant and Z_t is another Brownian motion. Under the risk-neutral measure, the value of the competitive convertible bonds is given by the partial differential equation

$$\begin{aligned}\frac{\partial W}{\partial t}+\frac{\sigma^2V^2}{2}\frac{\partial^2 W}{\partial V^2}+\rho\sigma\sigma_r r^{\frac{1}{2}}V\frac{\partial^2 W}{\partial V\partial r} &+ \frac{\sigma_r^2 r}{2}\frac{\partial^2 W}{\partial r^2}+rV\frac{\partial W}{\partial V}\\ &+ \left[\beta\left(\bar{r}-r\right)-\lambda\sigma_r r^{\frac{1}{2}}\right]\frac{\partial W}{\partial r}=rW \qquad (2.37)\end{aligned}$$

where $\lambda(t,V,r)$ is the market price of interest rate risk and ρ is the correlation between the two Brownian motions. The share value is given by the partial differential equation

$$\begin{aligned}\frac{\partial S}{\partial t}+\frac{\sigma^2V^2}{2}\frac{\partial^2 S}{\partial V^2}+\rho\sigma\sigma_r r^{\frac{1}{2}}V\frac{\partial^2 S}{\partial V\partial r} &+ \frac{\sigma_r^2 r}{2}\frac{\partial^2 S}{\partial r^2}+rV\frac{\partial S}{\partial V}\\ &+ \left[\beta\left(\bar{r}-r\right)-\lambda\sigma_r r^{\frac{1}{2}}\right]\frac{\partial S}{\partial r}=rS. \qquad (2.38)\end{aligned}$$

Since coupon and dividend payments are now discrete, there are no such terms in the partial differential equations. Instead, these payments introduce various jump conditions in the solutions.

2.6.2 Equilibrium Boundary Conditions

At the maturity of the convertible debt contract, either (1) bond holders convert their holdings into shares, or (2) they prefer to receive the principal payment on the bond, or (3) the firm defaults with creditors receiving the residual value net of bankruptcy costs. These outcomes depend on the realisation of the firm value at maturity,

$$W(T,V,r,n)=\begin{cases}\gamma V/\left(\psi_0+\gamma n_0\right), & V>P\left(n_0+\psi_0/\gamma\right),\\ P, & Pn/\left(1-\alpha\right)\le V\le P\left(n_0+\psi_0/\gamma\right),\\ \left[\left(1-\alpha\right)V\right]/n, & V<Pn/\left(1-\alpha\right).\end{cases} \qquad (2.39)$$

In the three cases outlined above, either (1) the shares become diluted in value from the conversion of the remaining bonds, or (2) share holders must bear the cost of the principal payment, or (3) the firm defaults, and by absolute priority equity holders get nothing. This results in the following final condition on the share value:

$$S(T,V,r,n) = \begin{cases} V/\left(\psi_0 + \gamma n_0\right), & V > P\left(n_0 + \psi_0/\gamma\right), \\ \frac{V-n(1-\tau)P}{\psi_0+\gamma[n_0-n]}, & Pn/\left(1-\alpha\right) \le V \le P\left(n_0 + \psi_0/\gamma\right), \\ 0, & V < Pn/\left(1-\alpha\right). \end{cases} \qquad (2.40)$$

Note that both final conditions (2.39) and (2.40) allow for the fact that bonds may be converted gradually as $n \le n_0$.

As in the perpetual cases, there will be similar lower boundary conditions on the bond and share values. Specifically, condition (2.18) applies as well as smooth-pasting and value-matching conditions on the share value.

An important consequence of the presence of discrete payments is that the bonds are now convertible only at specific times:

Theorem 4 *(Brennan and Schwartz (1977) Lemma 1, pp. 1702). It will never be optimal to convert an uncalled convertible bond except immediately prior to a dividend date or to an adverse change in the conversion terms, or at maturity.*

Since we do not consider changes in conversion terms, competitive *sequential* bond conversion can only take place immediately prior to a dividend date and at maturity.

As in the perpetual case, we look for a symmetric Nash (competitive) equilibrium, consisting of agents' indifference between immediate and delayed conversion. By Theorem 4, bond holders only have incentives to convert voluntarily immediately before a dividend date, t_D (i.e. they will only convert at t_{D-}). Since creditors must now act simultaneously at a specific time, the indifference condition is different from that in the previous equilibria. Indifference of creditors now requires that the conversion value equal the convertible bond value immediately after the dividend date, t_{D+}. In order to satisfy this equality there must be a specific number of bonds, $n_+ \le n_-$, remaining after the date. For different realisations of the state variables, V_t and r_t, the number of remaining bonds for the indifference condition will vary as a function of these variables (i.e. $n_+ = n_+(V,r)$). Thus, unlike the perpetual case, we obtain an interval of indifference over $V_t \in (\underline{V}, \overline{V})$[45]

$$\begin{aligned}
W\left(t_{D-}, V, r, n_-\right) &= \gamma S\left(t_{D+}, V - D, r, n_+\left(V, r\right)\right) + \frac{\gamma\left[D - \left(1 - \tau\right) n_+\left(V, r\right) \bar{c}\, \mathcal{I}_{t_D = t_c}\right]}{\psi_0 + \gamma\left[n_0 - n_+\left(V, r\right)\right]} \\
&= W\left(t_{D+}, V - D, r, n_+\left(V, r\right)\right) + \bar{c}\, \mathcal{I}_{t_D = t_c}.
\end{aligned} \tag{2.41}$$

The indicator functions in (2.41) allows for the twin annual dividend dates that coincide with coupon dates. On such dates, share holders must bear the cost of the coupons to the n_+ remaining share holders. The first line of (2.41) expresses the fact that the convertible bond satisfies a no-arbitrage jump condition if the bond is converted, while the second line expresses the same condition if it the bond is retained. Together, these conditions make creditors indifferent between converting and retaining their bonds. V_t and r_t are free parameters, and so $n_+(V, r)$ must be adjusted to ensure that indifference condition (2.41) holds. Whenever the firm value hits the upper interval, $(\underline{V}, \overline{V})$, at the dividend date t_{D-}, $[n_- - n_+(V, r)]$ convertible bonds must be exercised to fulfill the indifference condition (2.41). As in previous sections, the share value is generally increasing in n while the convertible bond value is decreasing in n, for high firm values. As the realisation of the firm value at the date rises (i.e. $V_{t_{D-}}$), more bonds must be converted to satisfy the indifference. This is because bond values must receive further compensation for the delay in conversion, and this also lowers the share value.

At an upper threshold, $\tilde{V}$, the indifference requires all remaining bonds to be exercised (i.e. $n_+(\tilde{V}, r) = 0$). The upper support of the interval $(\underline{V}, \overline{V})$ is, thus, equal to the minimum of the firm's call strategy and $\tilde{V}$

$$\overline{V} = \min\left[V_c\left(t_{D+}\right), \tilde{V}\right] \tag{2.42}$$

where $V_c(t)$ is the firm's time-dependent call strategy.[46] The lower support of the interval, $\underline{V}$, satisfies $n_- = n_+(\underline{V}, r)$. This separates the (1) interval over which creditors have no incentive to convert (i.e. $(V_b(t_D, n_-, r), \underline{V}])$ and (2) the interval over which agents are first indifferent according to (2.41). For firm values below $\underline{V}$, the convertible bond value exceeds the conversion value and so creditors have no incentive to convert, yielding the jump condition

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[45] Formally, this indifference will take place over a domain including an interval of short rate values. Since the effect of changing firm value has a more natural heuristic interpretation, however, we focus on this in the presentation.

[46] Often the call price evolves at an accretion rate, thereby making it time-dependent.

$$W\left(t_{D-}, V, r, n_{-}\right) = W\left(t_{D+}, V - D, r, n_{-}\right) + \bar{c}\,\mathcal{I}_{t_D = t_c}, \qquad V \in (V_b(t_D, n_{-}, r), \underline{V}] \tag{2.43}$$

where $V_b(t_D, n_-, r)$ is the default trigger. It is important to realise that boundary conditions (2.41) and (2.43) are crucial to ensure a competitive Nash equilibrium.

As for the share value, the *total* share value before the dividend date must equal *total* share value after the jump condition with appropriate cash flows

$$\begin{aligned}&\left(\psi_0 + \gamma\left[n_0 - n_{-}\right]\right) S\left(t_{D-}, V, r, n_{-}\right) = \\ &\quad \left(\psi_0 + \gamma\left[n_0 - n_{+}\left(V, r\right)\right]\right) S\left(t_{D+}, V, r, n_{+}\left(V, r\right)\right) + D - n_{+}\left(V, r\right)\left(1 - \tau\right)\bar{c}\,\mathcal{I}_{t_D = t_c}.\end{aligned} \tag{2.44}$$

Moreover, between dividend dates the convertible bond value must be less than the call price, $K(t)$ if the call protection has expired[47]

$$W\left(t, V, r, n\right) \leq K(t). \tag{2.45}$$

When the bonds are non-callable the following upper boundary condition applies between dividend dates on the convertible bond value

$$\lim_{V \to \infty}\left[\frac{\partial W\left(t, V, r, n\right)}{\partial V}\right] = \gamma \lim_{V \to \infty}\left[\frac{\partial S\left(t, V, r, n\right)}{\partial V}\right] \tag{2.46}$$

and likewise for the share value

$$\lim_{V \to \infty}\left[S\left(t, V, r, n\right)\right] = \frac{V}{\psi_0 + \gamma\left[n_0 - n\right]}. \tag{2.47}$$

Proceeding recursively from $t = T$ and using an appropriate finite-difference scheme (see for example Brennan and Schwartz (1977)), jump conditions (2.41) and (2.43) must be applied at each dividend date along with the other boundary conditions.

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[47] Most convertible bonds have an initial period of 4-5 years during which they are call-protected.

2.7 Extensions

2.7.1 Protective Covenants in Bankruptcy

The analysis throughout this chapter has assumed that equity holders decide the default trigger in an optimal fashion by maximising the value of their claim. However, in many debt contracts (see Black and Cox (1976) and Longstaff and Schwartz (1995)) there are protective clauses allowing creditors to force liquidation if the firm value reaches a certain threshold. In our setting where the amount of debt is progressively decreasing, it would be natural to assume that such protective covenants force bankruptcy at a trigger which is declining with the amount of debt present.[48] The creditors could exogenously fix the default trigger in the following way

$$V_b\left(n\right) = \eta\left(\frac{nc}{r}\right)$$

in the convertible debt contract, where η prescribes the amount of protection given to bond holders. When $\eta = 1$, the convertible preferreds recover, gross of bankruptcy costs their riskless investment value, c/r.

Since the default trigger in this protective case is still increasing in the amount of debt, the competitive equilibria will retain similar characteristics to those studied in the main sections of this chapter. With other tranches of debt, however, senior creditors may agree to renegotiate the debt,[49] in which case, the default trigger may change from a varying threshold to a fixed one. In the absence of other debt, however, this is unlikely as competitive bond holders are unable to collude.

2.7.2 Non-Convertible Debt and Senior Debt

An obvious limitation of our study is that there is no other debt outstanding and the firm becomes increasingly more like a pure-equity operation with conversion. It seems natural that other tranches of debt will be present in most capital structures given the tax advantage. By extending the model to include a tranche of non-convertible debt, the relative recovery rates of the different debt issues could be fixed exogenously. This would entail allocating a fraction of firm value to each class of creditor. Clearly, such extensions would be useful in examining the efficiency implications of the equilibria under more typical capital structures.

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[48] A disadvantage of a fixed trigger is that it may give rise to unrealistic situations where the recovery value of the bond in bankruptcy is greater than the riskless investment value. This will be the case for the last bonds to be converted and will create an artificial delay incentive.

[49] See Mella-Barral and Perraudin (1997) and Mella-Barral (1999).

2.8 Conclusion

This chapter has considered issues relating to strategic behaviour in pricing and exercising convertible bonds. In the process two types of symmetric competitive equilibrium have been studied; one involving the *simultaneous* actions of agents (the block case) and the other involving the *sequential* exercise of bonds. The former solution concept is by far more common in the pricing literature, where the simultaneity of actions is implicitly assumed. We have determined the circumstances under which these so called-"block strategies" yield Nash equilibria. As we have shown, there are many instances, especially when the issue is non-callable, where there are incentives to deviate from the block strategies.

Although the sequential equilibria are initially developed in a simplified perpetual setting, the resulting strategies are quite complicated, given by a system of first order differential equations. We have shown that the claim values and net firm value arising from the equilibria generally exceed the corresponding ones in the block. Thus, from efficiency considerations as well as the perspective of individual security holders, agents are better off sticking to the sequential equilibrium. Moreover, the equilibrium plays an important role in influencing the characteristics of convertible debt financing. Since sequential bond conversion takes place voluntarily and before the block conversion strategy, the issue has more of a delayed-equity nature.

The general approach taken in this chapter is to determine convertible bond values contingent on the value of the firm's assets, V_t. Our motivation in using V_t as a diffusion process was to facilitate easier study of the efficiency effects of the various equilibria. The more recent convertible debt pricing literature uses a diffusion process for the share value (see for example Davis and Lischka (1999)). The methodology in this chapter, and particularly section 2.6, could naturally be applied to this approach.

Future empirical research should be able to ascertain the degree to which gradual conversion is observed in practice. The results of such an analysis will be important in informing the theoretical predictions of this chapter. From a theoretical standpoint, however, the effect of competition is significant for the arbitrage-free pricing of convertible securities. The methodology presented in section 2.6 can be used to price convertible bonds in an arbitrage-free fashion. Managers will also have incentives to price the offering in this way so that the value of the firm's securities can be raised.

2.9 Proofs of Propositions

2.9.1 Proof of Proposition 1

The general solution of the differential equation for the share value, (2.3) is

$$\hat{S}\left(V\right)=A_0+A_1V+A_2V^{\xi_+}+A_3V^{\xi_-}.$$

Take derivatives and substitute in equation (2.3). Equating coefficients on similar terms yields $A_0=-[(1-\tau)n_0c]/(r\psi_0)$, $A_1=1/\psi_0$, and $\xi_{+/-}$ are the positive and negative roots of the equation $\sigma^2\xi(\xi-1)+2(r-\delta)\xi=2r$. Applying the boundary condition $\hat{S}(\hat{V}_b)=0$ and the upper boundary condition in the text, then yields the expression for the share value after a little algebraic manipulation. Applying the smooth-pasting condition $(d\hat{S}(\hat{V}_b)/dV=0)$ to (2.6) yields the default trigger as the root to the following equation

$$\begin{aligned}0 &= \frac{\partial g\left(\hat{V}_b;\hat{V}_b,\hat{V}\right)}{\partial V}\left\{\frac{\left(1-\tau\right)n_0c}{r}-n_0\left[\frac{\gamma\hat{V}}{\psi_0+\gamma n_0}\mathcal{I}_{A,B}+K\mathcal{I}_C\right]\right.\\ &\left.+\left[\hat{V}_b-\frac{\left(1-\tau\right)n_0c}{r}\right]\left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-}\right\}+\frac{\xi_-\left(1-\tau\right)n_0c}{r\hat{V}_b}+1-\xi_-. \qquad (2.48)\end{aligned}$$

The general solution of the differential equation for the convertible bond value, (2.2) is

$$\hat{W}\left(V\right)=B_0+B_1V^{\xi_+}+B_2V^{\xi_-}.$$

Take derivatives and substitute in equation (2.2). Equating coefficients on similar terms yields $B_0=c/r$, and $\xi_{+/-}$ are the same positive and negative roots as before. Applying the boundary condition $\hat{W}(\hat{V}_b)=(1-\alpha)\hat{V}_b/n_0$ and the upper conversion condition described in the text yields (2.5) after some algebraic manipulation. □

2.9.2 Proof of Lemma 1

Proof that the share value is a minimum at V^* if (2.9) applies.

Consider the second derivative of the share value at $V = V^*$

$$\begin{aligned}
\psi_0 \frac{d^2\hat{S}\left(V^*\right)}{dV^2} &= \left\{ \frac{\partial^2 g\left(V^*;\hat{V}_b,V^*\right)}{\partial V^2} - \frac{\xi_-\left(\xi_- - 1\right)}{\left(V^*\right)^2} \right\} \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-} \left[\hat{V}_b - \frac{\left(1-\tau\right)n_0 c}{r}\right] \\
&\quad + \frac{\partial^2 g\left(V^*;\hat{V}_b,V^*\right)}{\partial V^2} \left\{ \frac{\left(1-\tau\right)n_0 c}{r} - \frac{\gamma n_0 \hat{V}}{\psi_0 + \gamma n_0} \right\}. \qquad (2.49)
\end{aligned}$$

Substituting for the term $[\hat{V}_b - ((1-\tau)n_0c)/r](V^*/\hat{V}_b)^{\xi_-}$ from the equation arising from the smooth-pasting condition $d\hat{S}(V^*)/dV = 1/(\psi_0 + \gamma n_0)$:

$$\begin{aligned}
&\left[\hat{V}_b - \frac{\left(1-\tau\right)n_0 c}{r}\right] \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-} \left\{ V^* \frac{\partial g\left(V^*;\hat{V}_b,V^*\right)}{\partial V} - \xi_- \right\} = \\
&\frac{\gamma n_0 V^*}{\psi_0 + \gamma n_0} \left\{ V^* \frac{\partial g\left(V^*;\hat{V}_b,V^*\right)}{\partial V} - 1 \right\} - V^* \frac{\partial g\left(V^*;\hat{V}_b,V^*\right)}{\partial V} \frac{\left(1-\tau\right)n_0 c}{r} \qquad (2.50)
\end{aligned}$$

we then obtain

$$\begin{aligned}
\psi_0 \left(V^*\right)^2 \frac{d^2\hat{S}\left(V^*\right)}{dV^2} &= \left\{ \frac{\left(V^*\right)^2 g_{VV} - \xi_-\left(\xi_- - 1\right)}{V^* g_V - \xi_-} \right\} \left\{ \frac{\gamma n_0 V^*}{\psi_0 + \gamma n_0} \left(V^* g_V - 1\right) - V^* g_V \frac{\left(1-\tau\right)n_0 c}{r} \right\} \\
&+ \left(V^*\right)^2 g_{VV} n_0 \left\{ \frac{\left(1-\tau\right)c}{r} - \frac{\gamma V^*}{\psi_0 + \gamma n_0} \right\} \qquad (2.51)
\end{aligned}$$

where g_{VV} is shorthand for $\partial^2 g(V^*;\hat{V}_b,V^*)/\partial V^2$ and g_V for $\partial g(V^*;\hat{V}_b,V^*)/\partial V$. Denoting $A_1 = \gamma V^*/(\psi_0 + \gamma n_0)$and $A_2 = (1-\tau)c/r$ we then obtain

$$\begin{aligned}
\frac{d^2\hat{S}\left(V^*\right)}{dV^2} &= \left\{ \frac{n_0}{\psi_0 \left(V^*\right)^2 \left(V^* g_V - \xi_-\right)} \right\} \left\{ \left[\left(V^*\right)^2 g_{VV} - \xi_-\left(\xi_- - 1\right)\right] \left[A_1\left(V^* g_V - 1\right) - V^* g_V A_2\right] \right. \\
&\quad \left. + \left(V^* g_V - \xi_-\right) \left(V^*\right)^2 g_{VV} \left(A_2 - A_1\right) \right\} \qquad (2.52) \\
&= \left\{ \frac{n_0}{\psi_0 \left(V^*\right)^2 \left(V^* g_V - \xi_-\right)} \right\} \left\{ A_1 \left[\left(\left(V^*\right)^2 g_{VV} - \xi_-\left(\xi_- - 1\right)\right) \left(V^* g_V - 1\right) \right. \right. \\
&\qquad \left. - \left(V^*\right)^2 g_{VV} \left(V^* g_V - \xi_-\right) \right] \\
&\quad \left. + A_2 \left[\left(V^*\right)^2 g_{VV} \left(V^* g_V - \xi_-\right) - V^* g_V \left(\left(V^*\right)^2 g_{VV} - \xi_-\left(\xi_- - 1\right)\right) \right] \right\} \qquad (2.53) \\
&= \left\{ \frac{n_0}{\psi_0 \left(V^*\right)^2 \left(V^* g_V - \xi_-\right)} \right\} \left\{ \left(\xi_- - 1\right) A_1 \left[\left(V^*\right)^2 g_{VV} - \xi_- V^* g_V + \xi_-\right] \right. \\
&\qquad \left. - \xi_- A_2 \left[\left(V^*\right)^2 g_{VV} - \left(\xi_- - 1\right) V^* g_V\right] \right\} \qquad (2.54)
\end{aligned}$$

In equation (2.54)

$$\begin{aligned}
(V^*)^2 g_{VV} - \xi_- V^* g_V + \xi_- &= \left[\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}\right]^{-1} \left[\xi_+ (\xi_+ - 1) \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+}\right. \\
&\quad - \xi_- (\xi_- - 1) \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-} - \xi_- \xi_+ \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} \\
&\quad \left. + \xi_- \xi_+ \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-} + \xi_- \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} - \xi_- \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}\right] \\
&= \frac{(\xi_+ - 1)(\xi_+ - \xi_-)\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+}}{\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}} \qquad (2.55)
\end{aligned}$$

and

$$\begin{aligned}
(V^*)^2 g_{VV} - (\xi_- - 1) V^* g_V &= \left[\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}\right]^{-1} \left[\xi_+ (\xi_+ - 1) \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+}\right. \\
&\quad - \xi_- (\xi_- - 1) \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-} - \xi_+ (\xi_- - 1) \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} \\
&\quad \left. + \xi_- (\xi_- - 1) \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}\right] \\
&= \frac{\xi_+ (\xi_+ - \xi_-)\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+}}{\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}}. \qquad (2.56)
\end{aligned}$$

Making these substitutions back into (2.54) we then obtain

$$\begin{aligned}
\frac{d^2 \hat{S}(V^*)}{dV^2} &= \underbrace{\left\{\frac{n_0 (\xi_+ - \xi_-)\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+}}{\psi_0 (V^*)^2 (V^* g_V - \xi_-)\left(\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}\right)}\right\}}_{>0} \\
&\quad \times \left\{(\xi_- - 1)(\xi_+ - 1) A_1 - \xi_- \xi_+ A_2\right\} \qquad (2.57)
\end{aligned}$$

The first term is clearly strictly positive as all the individual terms therein are positive. Thus, in order for the share to be a minimum at V^* the share value must be strictly concave. This means that

$$(\xi_- - 1)(\xi_+ - 1) A_1 < \xi_- \xi_+ A_2 \qquad (2.58)$$

Rearranging (2.58) one then obtains inequality (2.9) in the lemma. □

Proof that the share value is maximised at V_{ca}.

Since the share value is concave at V^* management can do better calling slightly earlier at $V^* - \epsilon$ say. The reason for this is that $\hat{S}(V^* - \epsilon) < (V^* - \epsilon)/(\psi_0 + \gamma n_0)$ by the concavity of the claim. If $V^* > V_{ca}$, we must then consider alternative call policies of three categories: call policies over the interval (i) $[V_{ca}, V^*)$, (ii) $[0, V_{ca})$ and, (iii) (V^*, V_{co}). By Theorems 2 and 3 the conversion policy, V_{co}, limits how much management can delay calling.

Consider the first case and a given policy $y \in [V_{ca}, V^*)$. If the first derivative of the share value always exceeds $1/(\psi_0 + \gamma n_0)$, then management will always have incentives to act slightly earlier. The reason for this is that the share value will undercut the post-conversion share value as its gradient at y is greater. Since this inequality holds for the whole interval $[V_{ca}, V^*)$ in the proposition, management will have incentives to anticipate calling the bonds at V_{ca}.

In the second case, calling results in redemptions and so the difference in value between the shares and their post-conversion value is

$$\begin{aligned}
\psi_0 \left[\hat{S}\left(V\right) - \frac{V - n_0 K}{\psi_0} \right] &= \underbrace{\left[1 - g\left(V; \hat{V}_b, y\right) \right]}_{>0} \left\{ \underbrace{n_0 K - \frac{\left(1-\tau\right) n_0 c}{r}}_{\geq 0} \right. \\
&\qquad \left. - \left(\frac{y}{\hat{V}_b} \right)^{\xi_-} \underbrace{\left[\hat{V}_b - \frac{\left(1-\tau\right) n_0 c}{r} \right]}_{<0} \right\} \\
&> 0.
\end{aligned} \tag{2.59}$$

This result holds for all $y \in [0, V_{ca})$ and so there are no incentives to call before V_{ca}.

In the final case, we can examine the effect of call policies on the value of the share value at V_{ca} over the upper interval. If the share value at $V = V_{ca}$ ever exceeds K/γ, which is guaranteed under the V_{ca} policy, this policy ceases to be the optimal strategy. □

2.9.3 Proof of Lemma 2

Consider the smooth-pasting condition, $d\hat{S}(V^*)/dV = 1/\psi_0$, that applies when $V^* < V_{ca}$

$$\underbrace{\left[\hat{V}_b - \frac{(1-\tau)\,n_0 c}{r}\right]\left(\frac{V^*}{\hat{V}_b}\right)^{\xi_-}}_{<0}\;\underbrace{\left\{V^*\frac{\partial g\left(V^*;\hat{V}_b,V^*\right)}{\partial V} - \xi_-\right\}}_{>0} = n_0 V^*\frac{\partial g\left(V^*;\hat{V}_b,V^*\right)}{\partial V}\left\{K - \frac{(1-\tau)\,c}{r}\right\}. \tag{2.60}$$

Note that the curly-bracketed term on the first line of (2.60) is always positive as both $V^* g_V(V^*;\hat{V}_b,V^*)$ and ξ_- are always positive. Thus, the only way that the smooth-pasting can be satisfied is if the second line is negative: $K < (1-\tau)c/r$. □

2.9.4 Proof of Proposition 2

Part 1.

The first part of the proposition follows on from the results of the early part of the sub-section. Also note that Theorems 2 and 3 imply that V_{co} must be less than the firm's call policy when the offering is callable. □

Part 2.

Again the results follow from the application of the twin Theorems 2 and 3. The results of lemma 1 are important to ensure that V_{ca} is the optimal call policy. □

Part 3.

The results of lemma 2 are necessary for this part of the proposition. Note, however, that the call strategy always precedes V_{co} here. The reason for this is sketched below.

Lemma: $V^* < V_{co}$ **if** $K < (1-\tau)c/r$.

Since bonds are redeemed, the block redemption value exceeds the conversion value: $K > \gamma V^*/(\psi_0 + \gamma n_0)$. When the convertible bonds are non-callable, the conversion option gives creditors an option value, which is greater than the investment value of the bond $\gamma V_{co}/(\psi_0 + \gamma n_0) > c/r$. This option is always available as the conversion value is increasing in V_t.

Since, $(1-\tau)c/r > K$, this in turn implies that

$$\frac{\gamma V_{co}}{\psi_0 + \gamma n_0} > \frac{c}{r} > \frac{(1-\tau)\,c}{r} > K > \frac{\gamma V^*}{\psi_0 + \gamma n_0} \quad \Rightarrow \quad V_{co} > V^*. \qquad \square$$

2.9.5 Proof of Lemma 3

Whenever inequality (2.13) is satisfied there are no preemptive incentives in the block conversion strategy. Setting $\tau = 0$, inequality (2.13) becomes

$$0 \leq \underbrace{\left(\frac{\hat{V}}{\hat{V}_b}\right)^{\xi_-}}_{\geq 0} \underbrace{\left(\hat{V}\frac{\partial g\left(\hat{V};\hat{V}_b,\hat{V}\right)}{\partial V} - \xi_-\right)}_{>0} \alpha \hat{V}_b. \tag{2.61}$$

But the right hand side of (2.61) is always positive as $(\hat{V}/\hat{V}_b)^{\xi_-}$ is positive as is $\hat{V}\partial g(\hat{V};\hat{V}_b,\hat{V})/\partial V$. Further, α is positive and the default trigger is non-negative $\hat{V}_b \geq 0$. Thus inequality (2.61) is satisfied and so there are no preemptive incentives in the block with $\tau = 0$. □

2.9.6 Proof of Lemma 4

The proof of this is sketched out in the text preceding the lemma. □

2.9.7 Proof of Lemma 5

Proof that $V_{de} > V_{co}$

Suppose this did not hold

$$\frac{\gamma V_{de}}{\psi_0 + \gamma n_0} \leq \frac{\gamma V_{co}}{\psi_0 + \gamma n_0}. \tag{2.62}$$

Substituting for V_{de} and the term $\gamma V_{co}/(\psi_0 + \gamma n_0)$ from (2.7), into (2.62) we obtain

$$\frac{c}{r}\left(\frac{\xi_+}{\xi_+ - 1}\right) \le \frac{\frac{c}{r}V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} + \left[\frac{1-\alpha}{n_0}\hat{V}_b - \frac{c}{r}\right]\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}\left[V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} - \xi_-\right]}{V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} - 1}. \tag{2.63}$$

Rearranging this we, then have

$$\begin{aligned}\left[\frac{1-\alpha}{n_0}\hat{V}_b - \frac{c}{r}\right]\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}\left[V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} - \xi_-\right] \ge \\ \frac{c}{r}\left[\left(\frac{\xi_+}{\xi_+ - 1}\right)\left(V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} - 1\right) - V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V}\right]\end{aligned} \tag{2.64}$$

and so,

$$\begin{aligned}\underbrace{\left[\frac{1-\alpha}{n_0}\hat{V}_b - \frac{c}{r}\right]}_{<0}\underbrace{\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}}_{>0}\underbrace{\left[V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} - \xi_-\right]}_{>0} \ge \\ \underbrace{\frac{c}{r\left(\xi_+ - 1\right)}}_{>0}\left[V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} - \xi_+\right].\end{aligned} \tag{2.65}$$

Consider the term $V_{co}\partial g\left(V_{co};\hat{V}_b,V_{co};\right)/\partial V$ in (2.65)

$$V_{co}\frac{\partial g\left(V_{co};\hat{V}_b,V_{co};\right)}{\partial V} = \frac{\xi_+\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_+} - \xi_-\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}}{\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}} > \frac{\xi_+\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_+} - \xi_+\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}}{\left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_+} - \left(\frac{V_{co}}{\hat{V}_b}\right)^{\xi_-}} = \xi_+.$$

Thus, the right hand side of (2.65) is positive while the left hand side is negative. So the inequality cannot be true and this contradicts the original assertion. Thus, $V_{co} < V_{de}$. □

Proof that $W_{de}(V_{co}) > \hat{W}(V_{co};A)$

Now consider the second derivative of $W_{de}(V)$:

$$\begin{aligned}\frac{d^2W_{de}(V)}{dV^2} &= \underbrace{\frac{\xi_+\left(\xi_+ - 1\right)}{V^2}}_{>0}\underbrace{\left(\frac{V}{V_{de}}\right)^{\xi_+}}_{>0}\underbrace{\left[\frac{\gamma V_{de}}{\psi_0 + \gamma n_0} - \frac{c}{r}\right]}_{>0} \\ &> 0.\end{aligned}$$

Thus, the delayed bond value is strictly convex for all $V \in [0, V_{de}]$. Since W_{de} value-matches onto the post-conversion value $(\gamma V)/(\psi_0+\gamma n_0)$ at V_{de}, $W_{de}(V) > (\gamma V)/(\psi_0+\gamma n_0)$ for all $V < V_{de}$. Thus, $W_{de}(V_{co}) > \hat{W}(V_{co}; A)$. Creditors therefore retain a larger convertible bond value, $W_{de}(V_{co})$, at V_{co} if they postpone their conversion till V_{de}. □

2.9.8 Proof of Proposition 3

The proposition follows on naturally from lemmas 4 and 5 as well as the requirement that inequality (2.12) holds. When the latter requirement does not hold, small creditors have incentives to convert before $\hat{V}$. By definition, $\hat{V}$ is not, then, a Nash equilibrium as small agents can do better acting earlier. □

2.9.9 Proof of Proposition 4

The claim values are given by solutions of the form

$$W\left(V,n\right) = D_0\left(n\right) + D_1\left(n\right)V^{\xi_+} + D_2\left(n\right)V^{\xi_-}$$

and,

$$S\left(V,n\right) = E_0\left(n\right) + E_1\left(n\right)V + E_2\left(n\right)V^{\xi_+} + E_3\left(n\right)V^{\xi_-}.$$

Take derivatives with respect to V and substitute in differential equations (2.17) and (2.15). Equating coefficients, $D_0 = c/r$, $E_0 = -[(1-\tau)nc]/[r(\psi_0 + \gamma[n_0 - n])]$ and $E_1 = 1/(\psi_0 + \gamma[n_0 - n])$. Applying boundary conditions (2.18) and (2.19) to the convertible bond value then implies that it has the form shown in (2.21). Applying boundary conditions, $S(V_b, n) = 0$ and $\partial S(V_b, n)/\partial V = 0$, also implies that the equity value is given by (2.16).

Applying boundary condition (2.23) to the solution for the equity value (2.16) yields the ordinary differential equation for the default trigger, (2.26). Totally differentiating boundary condition (2.19) as shown and applying conditions (2.22) and (2.23) in the text then yields the ordinary differential equation for the the conversion trigger, (2.24). The initial conditions on these two triggers are discussed in the text.

The first necessary condition for the Nash equilibrium must hold, namely that

$$\frac{\partial W\left(\bar{V}\left(n\right),n\right)}{\partial V} < \gamma \frac{\partial S\left(\bar{V}\left(n\right),n\right)}{\partial V} \tag{2.66}$$

for all $n \in (0, n_0]$. If this is the case, competitive agents have no incentives to convert before $\bar{V}(n)$ as $W(V,n) > \gamma S(V,n)$ for all $V < \bar{V}(n)$. Furthermore, whenever the threshold $\bar{V}(n)$ is reached continuous conversion of bonds brings the conversion option out of the money. i.e. if Δn securities are converted at $\bar{V}(n)$, the conversion trigger immmediately after conversion will be $\bar{V}(n-\Delta n) > \bar{V}(n)$. The reason for this is that $d\bar{V}/dn$ is strictly negative if (2.66) holds by direct substitution into (2.24).

The second necessary condition merely ensures that there is an interval over which both the firm's default decision and creditors' conversion decisions are simultaneously out of the money. Were this not the case there would be ambiguities associated with simultaneous moves of the agents.

The third necessary condition ensures that there are no incentives for creditors to convert before $\bar{V}(n; N)$. □

2.9.10 Proof of Lemma 6

The proof for this is very similar to that for lemma 1. The main difference is that the new share value $\hat{S}(V,n)$ must be used in the equations. It may be easily verified that the relevant expressions for the lemma can be obtained by substituting $\psi_0 + \gamma[n_0 - n]$ for ψ_0 and n for n_0 in the equations in the proof of lemma 1. □

2.9.11 Proof of Lemma 7

By lemma 6 management's call policy is fixed for all n_t. Moreover, since $\bar{V}(0; C) = \min[V_{ca}, V_{de}]$, $\bar{V}(0; C) \leq V_{ca}$. If the necessary condition in the lemma holds then the denominator of (2.24) is negative while the denominator is positive. This implies that $d\bar{V}(n; C)/dn < 0$ and so $\bar{V}(n; C) \leq V_{ca}$ completing the proof. □

2.9.12 Proof of Proposition 5

The proof is very similar to the one for proposition 4. The main difference is the initial condition $\bar{V}(0;C)$. What we must show, however, is that management has no incentives to call the bonds earlier than $\bar{V}(n;C)$ for all $n \in [0, n_0]$. This may be shown if the competitive share value exceeds the share value on redemption for all n.

Proof that $S(V,n;C) \geq (V-nK)/(\psi_0+\gamma[n_0-n])$ for all $n \in (0,n_0]$

Consider

$$
\begin{aligned}
S\left(V,n;C\right) - \frac{V-nK}{\psi_0+\gamma\left[n_0-n\right]} &= \frac{1}{\psi_0+\gamma\left[n_0-n\right]}\left\{ n \underbrace{\left(K - \frac{\left(1-\tau\right)c}{r}\right)}_{\geq 0}\right. \\
&+ \frac{1}{\xi_+-\xi_-}\left(\left(\frac{V}{V_b}\right)^{\xi_+}\underbrace{\left[\left(\xi_--1\right)V_b - \xi_-\frac{\left(1-\tau\right)nc}{r}\right]}_{\geq 0}\right. \\
&+ \left.\left.\left(\frac{V}{V_b}\right)^{\xi_-}\underbrace{\left[\xi_+\frac{\left(1-\tau\right)nc}{r} - \left(\xi_+-1\right)V_b\right]}_{>0}\right)\right\} \qquad (2.67) \\
&> 0
\end{aligned}
$$

The term on the third line above is strictly positive as $[(1-\tau)c]/r > V_b$ and $\xi+ > \xi + -1$. □

Thus, the share value always exceeds the corresponding share value if the bonds were converted before $\bar{V}(n;C)$. Since this is the case for all $n \in [0,n_0]$, management never has incentives to deviate from this strategy for all time and so the equilibrium holds for all $n \in [0,n_0]$. □

Chapter 3

Real Options and Myopic Investment Policies in a War of Attrition

3.1 Introduction

The importance of option values in capital budgeting decisions has attracted considerable attention in recent years among corporate planners and business strategists. Discussions of the practical relevance of real options include Kester (1984) and Leslie and Michaels (1997). However, traditional real option analysis of investment decisions (for a summary, see Dixit and Pindyck (1994)) has the drawback that it ignores the possible impact on a firm of possible investment by competitors and the scope that this creates for strategic behaviour.

Recently, several authors have examined how real option models are affected by introducing strategic behaviour. In particular, Spencer and Brander (1992) look at duopoly models in which uncertainty creates option values. Kulatilaka and Perotti (1992) study a firm which can invest in advance of competitors and thus acquire market power, but which faces uncertainty about market conditions. Trigeorgis (1991) and Smit and Trigeorgis (1997) employ binomial models to examine strategic investment decisions. Smets (1993) develops a leader-follower model of real option decisions which is applied to a real estate investment problem by Grenadier (1996).

In a further step, Lambrecht and Perraudin (1994) and Lambrecht and Perraudin (2002) develop strategic real option models *with incomplete information.* Adopting a framework similar to the classic real option studies of McDonald and Siegel (1986) and Dixit (1989), Lambrecht and Perraudin (2002) suppose that firms may preempt each other by early investment. They show that ignorance of competitors' costs leads to an equilibrium in which firm's learn about their competitor's type from the fact that the other firm has not so far acted.

In this chapter, we analyse a war of attrition, real option problem which again is a simple specialisation of the well-known models of McDonald and Siegel (1986) and Dixit (1989). The problem we examine is exit from a productive activity by firms in a duopoly. We suppose that if one firm exits, the other enjoys a discrete enhancement in market power and hence each firm has an incentive to out-wait its competitor. Incomplete information is introduced by assuming that there are a discrete number of private cost types.

The early literature on war of attrition games was developed by Maynard Smith (1974), Bishop, Cannings, and Maynard Smith (1978) and Riley (1980) in the field of mathematical biology. A non-exhaustive list of applications in economics includes Bliss and Nalebuff (1984), Bulow and Klemperer (1999), Fudenberg and Tirole (1986) and Ghemawat and Nalebuff (1985). Of these, the latter two consider exit problems of firms. In all of these papers, the basic state variable is time and there is therefore no option value associated with optimal stopping.

A duopoly exit problem similar to ours in which the state variable is a diffusion process and hence options values appear is analysed by Lambrecht (2001). Lambrecht only considers pure strategy equilibria, assumes complete information and focuses on the impact on leverage. Huang and Li (1986) present a general theory of continuous time stopping games and Huang and Li (1992) apply it to entry and exit in a duopoly when the demand function is driven by a stochastic differential equation. They determine the pure strategy Nash equilibria in the exit case, as in Lambrecht (2001), by endogenising the entry decisions. Fine and Li (1989) examine firms' exit decisions in stochastically declining industries in discrete time. Dutta and Rustichini (1993) develop a general approach to stopping problems and apply it to product innovations and asset sales.

All the above studies focus on pure strategy equilibria alone. These have drawbacks in that the equilibria are generally not unique and the payoffs obtained by similar or identical firms are often completely different. (Gilbert and Harris (1984) discuss some of the weaknesses of asymmetric equilibria in a deterministic model of competition.)

A contribution of this chapter is to analyse Nash equilibria involving randomised strategies under both complete and incomplete information. The equilibria which result are Pareto inefficient but imply symmetric outcomes for similar firms. Exits are generated randomly by jumps in conditionally Poisson point processes. In some regions of the state variable, firms may be "locally risk-averse" despite the fact that equity, when there is an exit option, resembles a call option.

The structure of the chapter is as follows. Section 3.2 describes exit models with complete information. Both pure and mixed strategy equilibria are analysed. Section 3.3 sets out our incomplete information model with two types of firm and shows that there exist Bayesian mixed strategy equilibria for certain parameter values. This is extended to include multiple types. Section 3.4 considers the limit of the incomplete information model when there is a continuum of cost types, and section 3.5 concludes. Proofs of propositions as well as the derivation of certain equations are consolidated in section 3.6 at the end of the chapter.

3.2 Exit Models with Complete Information

3.2.1 Basic Assumptions

Suppose that agents are risk-neutral and can borrow or lend at the constant safe interest rate, r. Consider two firms, labelled 1 and 2, with cash flows per unit time

$$x_t - w_i \ , \qquad \text{for } \ i = 1, 2,$$

$$\text{where} \qquad dx_t = \mu x_t dt + \sigma x_t dB_t,$$

where x_t is an income flow, μ, σ, w_1 and w_2 are constant parameters, and B_t is a standard arithmetic Brownian motion. Without loss of generality, suppose that the continuous flow costs satisfy $w_1 \leq w_2$, i.e., firm 1 is at least as efficient as firm 2.

Assume that either firm may exit the industry irreversibly, at which time it receives a constant scrapping value, γ. Suppose that, if one firm exits, the firm which remains enjoys enhanced market power. A simple way to represent this is to suppose that if one firm exits the industry, the income flow, $x_t - w_i$, that the other firm receives, jumps up by a constant amount, $\delta > 0$, after which the dynamics of $x_t - w_i$ are the same as before the exit.

3.2.2 A Non-Strategic Model

If $\delta = 0$, one can value each firm in isolation since there is then no interaction between the two firms' payoffs. Let $\hat{V}_i(x_t)$ denote the value of the i-th firm when $\delta = 0$. Under risk neutrality, standard arguments imply that the expected return from investing $\hat{V}_i$ in safe bonds, $r\hat{V}_i$, equals the income, $x_t - w_i$, and the expected capital gains on $\hat{V}_i$. Applying Ito's lemma to $\hat{V}_{it} = \hat{V}_i(x_t)$ to obtain the expected capital gains implies that $\hat{V}_i(x_t)$ must satisfy the differential equation

$$r\hat{V}_i(x) = x - w_i + \mu x \frac{d\hat{V}_i(x)}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2\hat{V}_i(x)}{dx^2} \quad . \tag{3.1}$$

The stationarity of the firm's exit problem means that its optimal exit strategy will consist of exiting the industry when x_t first hits a low level $\underline{\hat{x}}_i$. One may therefore derive $\hat{V}_i(x_t)$ by solving equation (3.1), subject to the no-bubbles conditions[1] $\lim_{x\uparrow\infty} \left[\hat{V}_i(x) - x/(r-\mu) + w_i/r\right] = 0$ and the value-matching condition $\hat{V}_i(\underline{\hat{x}}_i) = \gamma$. This yields

$$\hat{V}_i(x_t) = \frac{x_t}{r-\mu} - \frac{w_i}{r} + \left[\gamma - \frac{\underline{\hat{x}}_i}{r-\mu} + \frac{w_i}{r}\right]\left(\frac{x_t}{\underline{\hat{x}}_i}\right)^{\xi} \quad , \tag{3.2}$$

$$\text{where} \quad \xi \equiv \frac{-(\mu - \sigma^2/2) - \sqrt{(\mu - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \quad . \tag{3.3}$$

---

[1] This says that the firm value approaches its unlimited liability value as x becomes large. The unlimited liability value is $\mathrm{E}_t \int_t^{\infty} \exp[-r(\tau - t)](x_t - w)d\tau = x_t/(r-\mu) - w/r$.

Maximizing the expression for $\hat{V}_i(x_t)$ in (3.2) with respect to the closure point $\hat{\underline{x}}_i$ yields the optimal exit trigger:[2]

$$\hat{\underline{x}}_i = \frac{\xi}{\xi - 1}\left(\frac{w_i}{r} + \gamma\right)(r - \mu) \quad . \tag{3.4}$$

3.2.3 Pure Strategy Equilibria

Now, consider what happens when $\delta > 0$, so each firm has an incentive to out-wait the other. At any given time, the state consists of the current level of x_t and the indicator variables I_1 and I_2, where $I_i = 1$ if firm i has exited and $I_i = 0$ otherwise. We restrict attention to Markov strategies,[3] i.e. those which depend on the current level of the state variables. Restricting the strategy space in this way is common in the differential games literature (see Isaacs (1965) and Friedman (1971)) but has been questioned, most notably by Stinchcombe (1992).[4] Firm i will only have an exit decision if the state variable is less than or equal to the monopolistic or duopolistic triggers, for $j \neq i$. Formally, we define the strategies:[5]

Definition 7 *For $i = 1, 2$, let firm i's strategy space, S_i, be defined as the set of pairs $\{\underline{x}_i^d, \underline{x}_i^m\}$, where if $I_j = 0$ for $j \neq i$ (and the i-th firm is operating in a duopoly), firm i exits the first time x_t hits $\underline{x}_i^d$, and if $I_j = 1$ (and the i-th firm is operating in a monopoly), firm i exits the first time x_t hits $\underline{x}_i^m$.*

Note that simultaneous moves, which occur either if one agent moves immediately after the other or if the strategies prescribe identical triggers for both firms, do not generate any ambiguities here since both agents receive payoffs of γ on the exit date.

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[2] $\hat{\underline{x}}_i$ may be obtained by solving either a first order condition $\partial \hat{V}_i(x)/\partial \hat{\underline{x}}_i = 0$ or the high contact condition $\partial \hat{V}_i(\hat{\underline{x}}_i)/\partial x = 0$.

[3] Also known as "feedback strategies".

[4] Stinchcombe, however, notes that "ODE differential games, as now formulated give a logically coherent set of strategies and an often appealing subset of the set of subgame perfect equilibria," (see Stinchcombe (1992), p. 259). On definitions of strategy spaces in continuous time, see also Fudenberg and Tirole (1986), Dutta and Rustichini (1995) and Simon and Stinchcombe (1989).

[5] It turns out to be important that strategies be specified in terms of trigger strategy pairs. One may show that with the payoffs structured as they are in this chapter, if strategies consist of single exit triggers, there is no pure strategy Nash equilibrium. This is in contrast to the model in Lambrecht (2001) which has different payoffs.

By defining the strategy space in terms of exit triggers, we depart from the formulation employed by Bensoussan and Friedman (1974) and Bensoussan and Friedman (1977).[6] They study Nash equilibria in continuous-time optimal stopping games in which agents' strategies are value functions (that are solutions to free-boundary problems) and trigger strategies are defined implicitly. The "value function" approach permits one to obtain general results on existence of Nash equilibria but equilibrium refinements such as sub-game perfection are hard to analyse. This is discussed in Dutta and Rustichini (1993).

Proposition 6 *Suppose the two firms have strategy spaces* S_1 *and* S_2 *and* $w_1 \leq w_2$. *There exists a sub-game perfect Nash equilibrium in which firm 2 (the less efficient firm) exits at* $\hat{\underline{x}}_2$ *leaving firm 1 as a monopolist. Firm 1 then exits subsequently when* x_t *first hits* $\hat{\underline{x}}_1$. *The strategies followed by the firms in the duopoly are given by*

$$\left\{\underline{x}_1^d, \underline{x}_1^m\right\} = \left\{\underline{x}_1^*, \hat{\underline{x}}_1\right\}, \qquad \left\{\underline{x}_2^d, \underline{x}_2^m\right\} = \left\{\hat{\underline{x}}_2, \hat{\underline{x}}_2\right\} \tag{3.5}$$

where $\underline{x}_i^*$ *is the root of* $\hat{V}_i(x_t+\delta) - \left[\frac{x_t}{r-\mu} - \frac{w_i}{r} + \left[\gamma - \frac{\hat{\underline{x}}_i}{r-\mu} + \frac{w_i}{r}\right]\left(\frac{x_t}{\hat{\underline{x}}_i}\right)^{\xi}\right] = 0$. *If* $\hat{\underline{x}}_1 > \underline{x}_2^*$, *there is also an equilibrium in which firm 1 exits at* $\hat{\underline{x}}_1$ *and firm 2 subsequently exits when* x_t *crosses* $\hat{\underline{x}}_2$. *If* $w_1 < w_2$, *this second Nash equilibrium is not sub-game perfect. The strategies followed by the firms in this second equilibrium are given by*

$$\left\{\underline{x}_1^d, \underline{x}_1^m\right\} = \left\{\hat{\underline{x}}_1, \hat{\underline{x}}_1\right\}, \qquad \left\{\underline{x}_2^d, \underline{x}_2^m\right\} = \left\{\underline{x}_2^*, \hat{\underline{x}}_2\right\} \tag{3.6}$$

If the second firm to exit (i.e., the winner) is of type i, with real option value: $W_i\left(x_t, \hat{\underline{x}}_i, \hat{\underline{x}}_j\right)$, *and the first firm to exit (i.e., the loser) is of type j, with real option value:* $L_j\left(x_t, \hat{\underline{x}}_j\right)$, *the values in the equilibria are:*

$$\begin{aligned} W_i\left(x_t, \hat{\underline{x}}_i, \hat{\underline{x}}_j\right) &= \begin{cases} x/(r-\mu) - w_i/r + \Big[\hat{V}_i(\hat{\underline{x}}_j+\delta) - & \\ \qquad \hat{\underline{x}}_j/(r-\mu) + w_i/r\Big]\left(x/\hat{\underline{x}}_j\right)^{\xi}, & \text{for } \ I_j = I_i = 0 \\ \hat{V}_i(x) & \text{for } \ I_j = 1, \ I_i = 0 \\ \gamma & \text{for } \ I_i = I_j = 1 \end{cases}, \\ L_j\left(x_t, \hat{\underline{x}}_j\right) &= \begin{cases} \hat{V}_j(x) & \text{for } \ I_i = I_j = 0 \\ \gamma & \text{for } \ I_j = 1 \end{cases}. \end{aligned} \tag{3.7}$$

The proof of this proposition may be found in section 3.6.1. The real option values

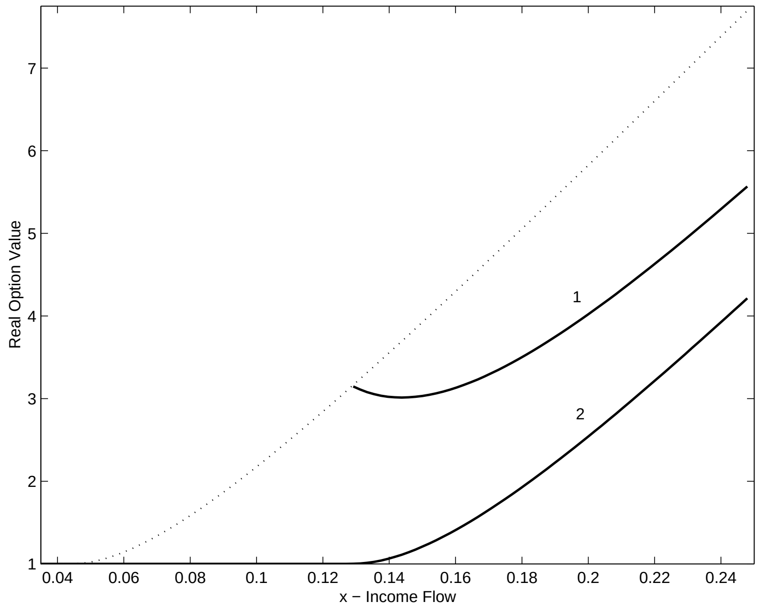

Figure 3.1: Unique pure-strategy sub-game perfect Nash equilibrium arising when $\hat{x}_1 \leq x_2^*$. The following parameters were used: $r = 0.05, \mu = 0.0, \sigma = 0.1, \gamma = 1, w_1 = 0.05, w_2 = 0.09, \delta = 0.025$.

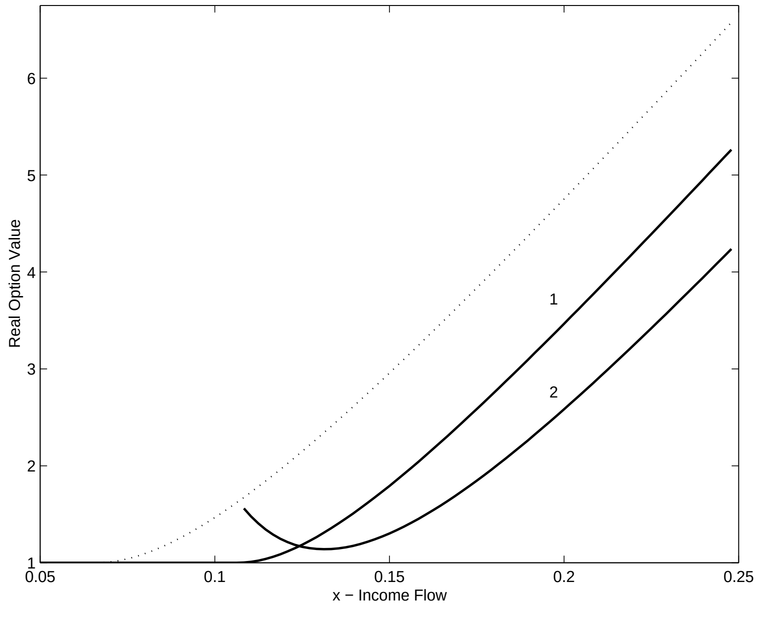

Figure 3.2: The pure-strategy sub-game perfect Nash equilibrium arising when $\hat{x}_1 > x_2^*$. The following parameters were used: $r = 0.05, \mu = 0.0, \sigma = 0.1, \gamma = 1, w_1 = 0.08, w_2 = 0.09, \delta = 0.025$.

and strategies described in proposition 6 are depicted in Figures 3.1-3.2. Figure 3.1 illustrates the single subgame-perfect equilibrium while Figure 3.2 shows the non-subgame perfect equilibrium in which the less efficient firm exits second.[7]

3.2.4 Mixed Strategies under Complete Information

When the firms are symmetric, i.e., $w_1 = w_2 = w$, pure strategy equilibria in which firms exit when the state variable x_t first hits some level are not the only type of possible equilibrium. Suppose that in any interval, $[t, t + \Delta]$, agents exit randomly according to some probability. If the probability that firm i exits in $[t, t + \Delta]$ for small Δ is $\lambda_t \Delta$, then we may think of exits being given by the first jump time of a point process with rate of jump $\{\lambda_{i\tau}\}_{\tau=t}^{\infty}$. For simplicity, we restrict agent's choices to Markov jump rates which are functions of the current level of the state variable, x_t. We define the strategy spaces to be:

Definition 8 *For $i = 1, 2$, let firm i's space of mixed strategies, denoted M_i, be defined as the set of conditionally Poisson processes with non-negative jump rates $\lambda_{it} = \lambda_i(x_t)$.*

To analyse the two firm's selection of jump rates $\lambda_i(x_t)$, note that under risk-neutrality, the required returns, $rV_i(x_t)$ must again equal the firms' income flows plus expected capital gains, as in equation (3.1). The possibility of jumps adds extra terms in the expression for capital gains. Hence,

$$\begin{aligned} rV_i &= x - w_i + \mu x \frac{dV_i}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2 V_i}{dx^2} + \max_{\lambda_i \geq 0} \{\lambda_i(\gamma - V_i)\} + \lambda_j(\hat{V}_i(x + \delta) - V_i), \quad (3.8) \\ rV_j &= x - w_j + \mu x \frac{dV_j}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2 V_j}{dx^2} + \max_{\lambda_j \geq 0} \{\lambda_j(\gamma - V_j)\} + \lambda_i(\hat{V}_j(x + \delta) - V_j). \quad (3.9) \end{aligned}$$

To interpret equation (3.8), if firm j exits, firm i's value, V_i, jumps by $\hat{V}_i(x + \delta) - V_i$. On the other hand, if firm i itself exits, V_i jumps by $\gamma - V_i$. Equation (3.9) shows the symmetric case of firm j.

---

[6] We presented their determination of Nash equilibria in section 1.4.1.

[7] For these plots the following parameters were employed: $r = 0.025$, $\mu = 0$, $\sigma = 0.1$, $\gamma = 1$, $\delta = 0.0625$, $w_1 = 0.12$ and $w_2 = 0.15$.

Each firm chooses its own random jump rate so that the firms' capital gains are maximised with respect to λ_i. This means that the two equations for firm value, (3.8) and (3.9), constitute a pair of linked Hamilton-Jacobi-Bellman equations just as in a stochastic differential game. (See Basar and Olsder (1998) for a survey of these games, and Friedman (1972), Uchida (1978) and Uchida (1979)[8])

By the absence of arbitrage, the real option values must always be greater than the scrapping value, as firms may always exit from the industry. Examination of equations (3.8) and (3.9) reveals that as long as $V_i > \gamma$, firm i will have an incentive to reduce λ_i. The reason for this is that the term $\lambda_i(\gamma - V_i)$ is then negative. Since jump rates are necessarily non-negative, this implies that if $V_i > \gamma$, then $\lambda_{it} = 0$, and if $V_i = \gamma$, the hazard of the i-th firm remains undetermined, $\lambda_i \geq 0$.[9]

When $V_i = \gamma$, the i-th firm is indifferent about the level of λ_{it}. Using this fact, we can find a pair of functions, $\lambda_i(x_t)$ and $\lambda_j(x_t)$ such that both firms are in equilibrium. Thus, for $x > \hat{\underline{x}}$, $\lambda_i(x_t) = \lambda_j(x_t) = 0$, while for $x \leq \hat{\underline{x}}$, the hazards $\lambda_i(x_t) = \lambda_j(x_t)$ are chosen so as to satisfy

$$r\gamma \quad = \quad x - w_i + \lambda_i \left[\hat{V}_i(x + \delta) - \gamma \right] \quad . \tag{3.10}$$

In the lower region, where $x \leq \hat{\underline{x}}$, each claim value equals the scrapping value, γ. Thus, firms exit randomly without regret. However, if claim values are constant, there is no expected capital gain attributable to the continuous evolution of the state variable, x_t. So the time value of the investment $r\gamma$, must be compensated for through expected capital gains stemming from jumps in value when the competitor firm exits. For firm i to be in equilibrium, the other firm's (j's) exit jump rate must be chosen so as to equate expected capital gains to the risk-neutral required return, rV_i, i.e., to satisfy equation (3.10). We summarise this discussion in the following proposition.

---

[8] The existence of an equilibrium point in a stochastic differential game is proved in Friedman (1972) by making use of auxiliary results from the study of parabolic PDEs. The existence of Nash equilibria in nonzero-sum stochastic differential games is set out in Uchida (1978) and shown to be synonymous with the satisfaction of the "Nash condition" in Uchida (1979), under certain technical conditions. In addition, an analogous nonzero-sum "Isaac's Condition" is obtained for the implementation of the Nash condition. The Nash condition is satisfied by maximising the terms containing each agent's control with respect to that control. In our setting the maximisation is particularly simple as only one term is involved with two simplifying regimes ($V_i = \gamma$ and $V_i > \gamma$).

[9] Note that if $V_i < \gamma$ then $\lambda_{it} \to \infty$ implying immediate exit, as the player would have incentives to raise the control in the term $\lambda_{it}\left[\gamma - V_i\right]$.

Proposition 7 *Suppose the two firms have strategy spaces M_1 and M_2 and $w_1 = w_2$. In a mixed-strategy, feedback, Nash equilibrium, the firm values $V_i(x_t)$ are*

$$V_i(x_t) = \begin{cases} \hat{V}_i(x_t) & x_t \in (\underline{\hat{x}}_i, \infty) \\ \gamma & x_t \in (\underline{\hat{x}}_i - \delta, \underline{\hat{x}}_i] \end{cases} \tag{3.11}$$

where $\underline{\hat{x}} \equiv \underline{\hat{x}}_i$ for $i = 1, 2$, and each firm exits at the first jump time of a conditionally Poisson process with rate of jump $\lambda_{it} = \lambda_i(x_t) \in M_i$ where

$$\lambda_i(x) = \begin{cases} 0 & x_t \in (\underline{\hat{x}}_i, \infty) \\ (r\gamma - x + w_i)/(\hat{V}_i(x + \delta) - \gamma) & x_t \in (\underline{\hat{x}}_i - \delta, \underline{\hat{x}}_i] \end{cases} . \tag{3.12}$$

The proof of this proposition may be found in section 3.6.2.

The actions in the mixed strategy equilibrium are specified entirely by the hazards and the other firm's survivorship. We can obtain some intuition as to the agents' preferences by studying the hazards. In a short time increment, δt, the probability of exit is *increasing* in the sum of (1) the lost interest gained if the firm exited immediately, $r\gamma\delta t$, and (2) the cost of staying in,[10] $(w_i - x_t)\,\delta t$. The probability of exit is *decreasing* in the monopoly power acquired in exiting, $\hat{V}_i(x + \delta) - \gamma$.

The equilibrium described in Proposition 7 is depicted in Figure 3.3, which shows the symmetric firm values, $V_i(x)$, smooth-pasting onto the scrapping value γ at $\underline{\hat{x}}_i$ just below 0.09. As x_t falls below $\underline{\hat{x}}_i$, the hazard rate for exit, $\lambda_i(x_t)$ becomes larger, rising until it explodes to infinity as x_t approaches $\underline{\hat{x}}_i - \delta$. At this particular income flow value, the monopoly power acquired is equal to zero, and the firm exits immediately with nothing to gain in waiting. Figure 3.3 also shows $\hat{V}_i(x + \delta)$ which is the level to which a firm's value jumps if its competitor exits the industry first.

Although the mixed-strategy equilibrium of proposition 7 is less efficient than the pure-strategy equilibria of proposition 6, it should be stressed that the Pareto-efficiency is not strict. The losing firm obtains the same real option value in either equilibrium.

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[10] Normally, $x_t - w$ is the profit stream, but when $x_t < w$ the firm needs to inject capital to keep the real option alive.

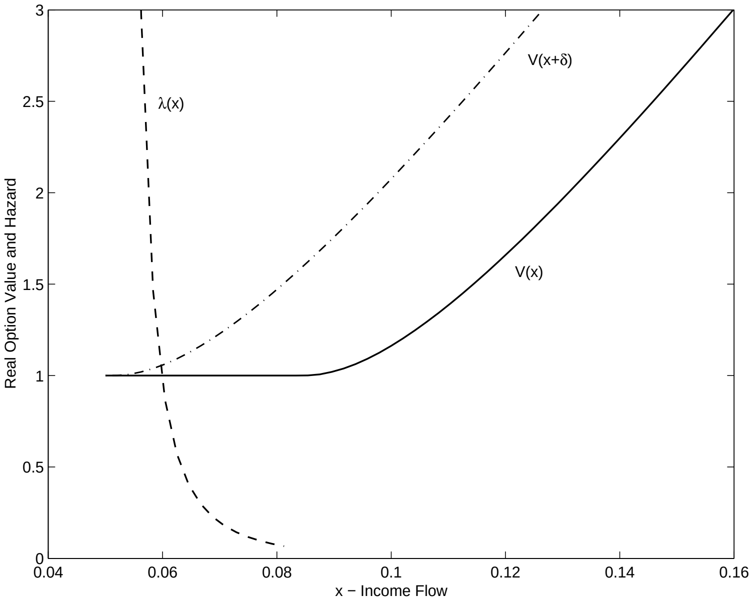

Figure 3.3: The mixed-strategy Nash equilibrium with complete information when $w_1 = w_2$. The corresponding hazard if also shown as well as the monopoly power payoff to which the winner jumps. The following parameters were used: $r = 0.025, \mu = 0.0, \sigma = 0.1, \gamma = 1, w_1 = w_2 = 0.09, \delta = 0.0021$.

3.3 Exit with Incomplete Information

3.3.1 Filtering with Two Types

So far, we have assumed that each firm's type is known with certainty. Instead, assume there are two types of firm with revenues

$$\begin{cases} x_t - w_1 & (\text{type} \quad 1 \quad \text{firm}) \\ x_t - w_2 & (\text{type} \quad 2 \quad \text{firm}) \end{cases} .$$

where $w_1 < w_2$. We suppose that the firms have prior probabilities, $\overline{P}$, at time $t = 0$ that the other is of type 1.

By analogy with the model described in the last section, we may look for a mixed strategy equilibrium in which firms of type i exit at the first jump time of conditionally Poisson processes with rates of jump, $\tilde{\lambda}_{1t}$ and $\tilde{\lambda}_{2t}$. As time goes by and neither firm has exited, the firms will up-date their estimate of the probability that the other firm is of type 1 conditional on no jumps having taken place, which we denote P_t.

We define a mixed strategy Bayesian Nash equilibrium in our model as follows:

Definition 9 *Let a mixed-strategy, Bayesian, Nash equilibrium denote a Nash equilibrium in which admissible strategies for type i firms (for* $i = 1, 2$*) in the strategy space* $\tilde{M}_i$ *consist of exiting at the first jump time of a point process which, conditional on* x_t *and* P_t*, is a Poisson process with jump rate,* $\tilde{\lambda}_{it} = \tilde{\lambda}_i(x_t, P_t)$*.*

In section 3.6.3, we analyse the filtering problem that the firms face. Given its initial prior, $\overline{P}$, and knowledge of the strategies its competitor follows if it is either type 1 ($\tilde{\lambda}_1$) or type 2 ($\tilde{\lambda}_2$), the i-th firms infer the *conditional* probability, P_t that the other firm is type 1 given that neither firm has exited and this solves:

$$\frac{dP_t}{dt} = \left[\tilde{\lambda}_2\left(x_t, P_t\right) - \tilde{\lambda}_1\left(x_t, P_t\right)\right]\left(1 - P_t\right) P_t \qquad \text{subject to} \qquad P_0 = \overline{P} \ . \tag{3.13}$$

3.3.2 The Mixed-Strategy Nash Equilibrium with 2 Types

Consider the Hamilton-Jacobi-Bellman equations satisfied by the firm values for the two types which we denote $\tilde{V}_1$ and $\tilde{V}_2$. These are:

$$\begin{aligned} r\tilde{V}_1 &= x - w_1 + \mu x \frac{\partial \tilde{V}_1}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 \tilde{V}_1}{\partial x^2} + \max_{\tilde{\lambda}_1 \geq 0}\left\{\tilde{\lambda}_1(\gamma - \tilde{V}_1)\right\} \\ &\quad + P(1-P)(\tilde{\lambda}_2 - \tilde{\lambda}_1)\frac{\partial \tilde{V}_1}{\partial P} + [P\tilde{\lambda}_1 + (1-P)\tilde{\lambda}_2](\hat{V}_1(x+\delta) - \tilde{V}_1), \qquad (3.14) \\ r\tilde{V}_2 &= x - w_2 + \frac{\partial \tilde{V}_2}{\partial x}\mu x + \frac{\partial^2 \tilde{V}_2}{\partial x^2}\frac{\sigma^2}{2}x^2 + \max_{\tilde{\lambda}_2 \geq 0}\left\{\tilde{\lambda}_2(\gamma - \tilde{V}_2)\right\} \\ &\quad + P(1-P)(\tilde{\lambda}_2 - \tilde{\lambda}_1)\frac{\partial \tilde{V}_2}{\partial P} + [P\tilde{\lambda}_1 + (1-P)\tilde{\lambda}_2](\hat{V}_2(x+\delta) - \tilde{V}_2). \qquad (3.15) \end{aligned}$$

The coefficients of the derivatives $\partial\tilde{V}_i/\partial P$, i.e., $P(1-P)(\tilde{\lambda}_2 - \tilde{\lambda}_1)$ are the time-derivatives of P and come from equation (3.13). Note also that in equations (3.14) and (3.15), the i-th firm maximises with respect to $\tilde{\lambda}_i$ only where it appears as a coefficient on $(\gamma - \tilde{V}_i)$ because the other places in which $\tilde{\lambda}_i$ appear reflect the possibility that the other firm is of type 1 and hence are not in the control of firm i.

In the symmetric, complete information case, firms randomise in such a way that the rival is compensated for the sunk cost incurred in inefficiently keeping their project alive. With incomplete information about types, the rival firms are compensated in a similar way. In this equilibrium, however, the more efficient firm only randomises when it is certain that its rival is of the more efficient type. The equilibrium then reverts to the one depicted in proposition 7 with symmetric types w_1 if both firms are of type 1. The less efficient firms randomise with a hazard that is increasing in P_t. As it becomes more likely that their opponent is of the more efficient type, the less efficient firm has a smaller chance of winning and so randomises with a larger hazard. By substituting $\tilde{V}_2(x_t, P_t) = \gamma$ into equation (3.15) the hazards must satisfy:

$$P_t\tilde{\lambda}_1 + (1 - P_t)\,\tilde{\lambda}_2 = \lambda_2(x_t),$$

where λ_2 is as defined in (3.12). Since the more efficient firms never randomise until $P_t = 1$, $\tilde{\lambda}_1 = 0$ and yields the simple expression for $\tilde{\lambda}_2 = \lambda_2/(1 - P_t)$. We summarise the formal Bayesian Nash equilibrium in the following proposition:

Proposition 8 *Suppose the two firms have strategy spaces $\tilde{M}_i$ and $\tilde{M}_j$ where the two possible types are $w_1 < w_2$. In a mixed-strategy, feedback, Nash equilibrium with incomplete information in which the firms exit at the first jump time of the conditionally Poisson processes with jumps*

$$\begin{aligned} \tilde{\lambda}_1(x_t, P_t) &= 0, && (3.16)\\ \tilde{\lambda}_2(x_t, P_t) &= \lambda_2(x_t)/(1 - P_t), && (3.17) \end{aligned}$$

where the beliefs of agents regarding the probability, $P_t < 1$, evolve according to

$$P_t = \overline{P}\exp\left[\int_0^t \lambda_2\left(x_\tau\right) d\tau\right] \qquad (3.18)$$

with symmetric priors $P_0 = \overline{P} \in (0,1)$. The equilibrium real option values for the two firms are:

$$\begin{aligned} \tilde{V}_1\left(x_t, P_t\right) &= V_1\left(x_t\right) + \mathrm{E}_t\left\{\int_t^T \left(\hat{V}_1\left(x_\tau + \delta\right) - V_1\left(x_\tau\right)\right)\right. && (3.19)\\ &\qquad \left.\times\lambda_2\left(x_\tau\right)\exp\left[-\int_t^\tau \left(r + \lambda_2\left(x_s\right)\right) ds\right] d\tau\right\}, \\ \tilde{V}_2(x_t, P_t) &= V_2(x_t)\ . && (3.20) \end{aligned}$$

At the random time, T, the game turns into one of complete information (as in proposition 7), and this satisfies $P_T = 1$ in (3.18).

The proof of this proposition may be found in section 3.6.4.

The probability conjecture is dependent on the historical path of the state variable x_t as well as the prior, $\overline{P}$. An interesting feature of this equilibrium is that the strategies are contingent on the entire history of the game, even though feedback strategies are employed.[11]

Since the value of a type 1 firm, (3.19), cannot be determined in closed-form one must resort to numerical methods. This option value satisfies the following partial differential equation:

$$r\tilde{V}_1 = \frac{\sigma^2 x^2}{2}\frac{\partial^2 \tilde{V}_1}{\partial x^2} + \mu x \frac{\partial \tilde{V}_1}{\partial x} + \lambda_2 P \frac{\partial \tilde{V}_1}{\partial P} + \lambda_2 \left[\hat{V}_1\left(x+\delta\right) - \tilde{V}_1\right] + x - w_1 \tag{3.21}$$

which may solved numerically[12] subject to the boundary conditions $\lim_{x\to\infty}(\tilde{V}_1(x,P) - x/(r-\mu) - w/r) = 0$, $\lim_{P\to 1}\tilde{V}_1(x,P) = \hat{V}_1(x)$ and $\lim_{x\downarrow \underline{\hat{x}}_2 - \delta}\tilde{V}_1(x,P) = \hat{V}_1(\underline{\hat{x}}_2 - \delta)$.

In Figure 3.4 we show the real option value $\tilde{V}_1$ as a function of P_t. The real option value was computed by solving the partial differential equation (3.21) using a fully-implict finite-difference scheme. The more efficient firm's real option, $\tilde{V}_1$, is decreasing in P_t as can be seen in Figure 3.4. The intuition is clear: as the likelihood increases that the opponent is of the more efficient type, it is less likely that the firm can win without randomizing (i.e. if the other firm is of type 2). Nonetheless, it is interesting to note that even with a very small probability that the opponent is of the less efficient type, $\tilde{V}_1 > \gamma$, and so there are benefits to wait to allow the other to reveal its type. As the probability increases, the randomisation of the less efficient firm increases dramatically, as can be seen in Figure 3.5, and in the limit $P_t \to 1$, the less efficient firm exits immediately.

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[11] This solution concept falls under a sub-set of a class of integral games. Fershtman, Kamien, and Muller (1991) study Nash equilibria of integral games with open-loop strategies, i.e. strategies that are adopted at time $t = 0$ and are retained for the remainder of the game. In our set-up, the strategies are feedback (i.e. Markovian) and the integral equation for P_t can be reduced to a non-linear differential equation as the hazard, $\tilde{\lambda}_2(x_t, P_t)$, is functionally time-independent. This makes the solution easier than more general integral games. The explicit evolution of the state variables, P_t, through an integral equation can be seen in equation (3.18).

[12] The numerical method for this may be found in the Appendix in section A.1.

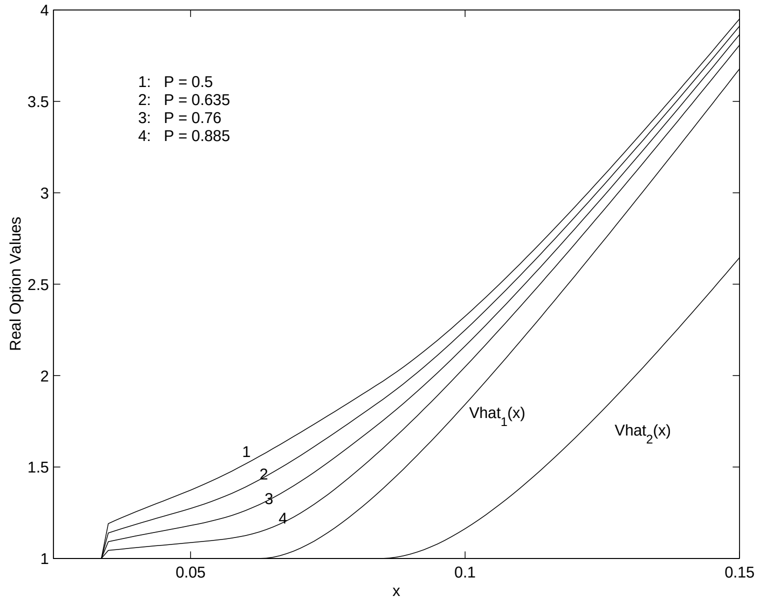

Figure 3.4: The mixed-strategy Nash equilibrium with incomplete information. Plots are shown for $\tilde{V}_1$ at $P_t = \overline{P} = 0.5, 0.635, 0.76, 0.885, 1$. The following parameters were used: $r = 0.025, \mu = 0.0, \sigma = 0.1, \gamma = 1, w_1 = 0.06, w_2 = 0.09, \delta = 0.05, \overline{P} = 0.5$.

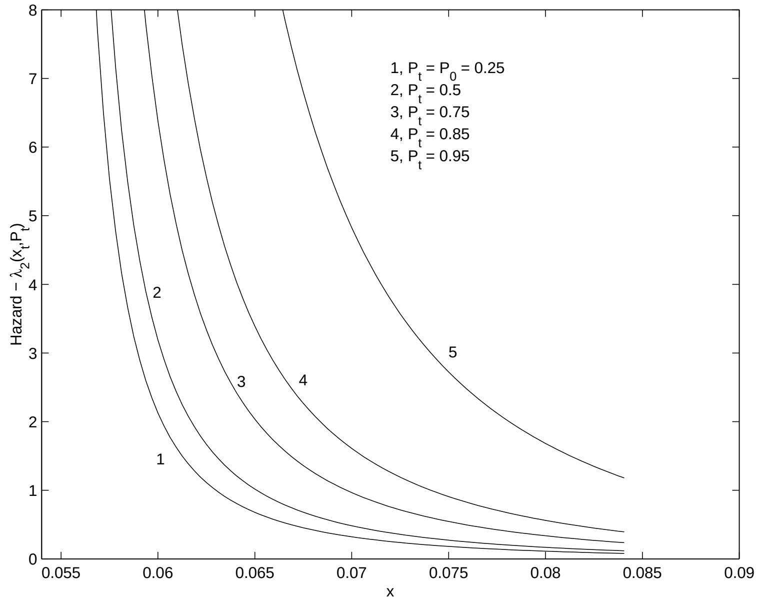

Figure 3.5: The mixed-strategy Nash equilibrium with incomplete information randomisation for type 2 firms: $\tilde{\lambda}_2(x_t, P_t)$. The following parameters were used: $r = 0.025, \mu = 0.0, \sigma = 0.1, \gamma = 1, w_1 = 0.06, w_2 = 0.09, \delta = 0.0312, \overline{P} = 0.25$.

3.3.3 Filtering with n Types

We extend the previous analysis by having n private cost types and these are increasing:

$$w_{i-1} < w_i < w_{i+1}, \qquad \forall \qquad i \in \{2, n-1\}.$$

As in the two-type game, we look for a Bayesian Nash equilibrium, which is demonstrated heuristically before presenting the formal result. The equilibrium consists of types being sequentially 'filtered out' of the war of attrition game. At the start of the game, (i.e. before any type is filtered out and all n types are 'in') all agents assign similar priors to the fact that their opponent is of the n different types. At this stage of the game only the least efficient firms randomise their exit decision while all more efficient types wait. Over time, the probabilities evolve through Bayesian updating until it is evident that neither firm is of the least efficient type (i.e. neither is with cost type w_n). As in the 2 type model, the probabilities evolve through the movement of the state variable x_t, which thus alters the exiting type's randomisation. Once it is clear that neither firm is of the least efficient type, type n is 'filtered out' of the model and the game continues, this time with $n-1$ types. This process continues until the less efficient of the two firms exit with their randomised exit decision.

As before probabilities are assigned to the conjectures of both agents. Let the probability that agents assign to their opponent being of type $j < m$, at time t, when type m is the least efficient type remaining (i.e. w_m is the largest cost type left in the game) be given by: P_{jt}^m. So the probability that agents assign to their opponent being of type m when type m is being filtered out is given by: $1 - \sum_{j=1}^{m-1} P_{jt}^m$. Clearly, given the sequential nature of filtering, the superscript index m in the probabilities is decreasing with time and the subscript j is always less than m.

Definition 10 *Let a mixed-strategy, Bayesian, Nash equilibrium denote a Nash equilibrium in which admissible strategies for type i firms (for* $i = \{1, m\}$ *where* m *is the least efficient type remaining) in the strategy space* M_i^m *consist of exiting at the first jump time of a point process which, conditional on* x_t *and* P_{it}^m*, (*$i = \{1, m-1\}$*), is a Poisson process with jump rate,* $\tilde{\lambda}_{it} = \tilde{\lambda}_i(x_t, P_{1t}^m, ..., P_{m-1,t}^m)$.

As in the two-type model, Bayesian updating implies that the conditional probabilities evolve according to the Riccatti type equations

$$\frac{dP^m_{jt}}{dt} = P^m_{jt}\left[\left(1-\sum_{k=1}^{m-1}P^m_{kt}\right)\tilde{\lambda}_{mt} + \sum_{k=1,k\neq j}^{m-1} P^m_{kt}\tilde{\lambda}_{kt} - \left(1-P^m_{jt}\right)\tilde{\lambda}_{jt}\right] \tag{3.22}$$

for $j \in \{1, m-1\}$ and with boundary conditions, $\overline{P}^m_j = P^{m+1}_{j,T^{m+1}}$, where T^{m+1} is the time at which type $m+1$ is filtered out. Clearly, type m is filtered out at time T^m, and the probability of firms being of type m is equal to zero: $1-\sum_{k=1}^{m-1}P^m_{kT^m} = 0$. We have the natural continuity condition on the probabilities that $P^{m-1}_{jT^m} = P^m_{jT^m}$ for $j \in \{1, m-2\}$. Equation (3.22) can be verified by Bayes rule as in the case with 2 types and the derivation is in section 3.6.5.

3.3.4 The Mixed-Strategy Nash Equilibrium with n Discrete Types

By financial market equilibrium, the return on safe bonds must equal the income flow to agents plus the capital gains of the real option and the weighted probabilities of winning or losing the game. Using Ito's Lemma this implies that the real option value for a type i firm when type m is randomizing, V^m_i, is given by the m-coupled Hamilton-Jacobi-Bellman partial differential equations

$$\begin{aligned} rV^m_i &= x - w_i + \frac{\sigma^2 x^2}{2}\frac{\partial^2 V^m_i}{\partial x^2} + \mu x \frac{\partial V^m_i}{\partial x} + \max_{\tilde{\lambda}_i \geq 0}\left\{\tilde{\lambda}_i\left[\gamma - V^m_i\right]\right\} \\ &+ \sum_{k=1}^{m-1} P^m_k \left\{\left(1-\sum_{l=1}^{m-1}P^m_l\right)\tilde{\lambda}_m + \sum_{l=1,l\neq k}^{m-1} P^m_l \tilde{\lambda}_l - \left(1-P^m_k\right)\tilde{\lambda}_k\right\}\frac{\partial V^m_i}{\partial P^m_k} \\ &+ \left\{\sum_{k=1}^{m-1} P^m_k \tilde{\lambda}_k + \left[1-\sum_{k=1}^{m-1}P^m_k\right]\tilde{\lambda}_m\right\}\left\{\hat{V}_i(x+\delta) - V^m_i\right\} \end{aligned} \tag{3.23}$$

where $i \in \{1, m\}$. The coefficients on the derivatives $\partial V^m_i / \partial P^m_k$ are the time derivatives of the probabilities and come from equation (3.22). As before, in PDE (3.23), the i-th firm only maximises with respect to his control, $\tilde{\lambda}_i$, when it appears as a coefficient on $(\gamma - V^m_i)$ because the other places where it appears reflect the possibility that the opponent is of type i. Note that in the notation of this sub-section, $V^2_i = \tilde{V}_i$ from the two type model (see proposition 8) and $V^1_1 = V_1$, from the complete information model of proposition 7.

It will be useful to define both the product and the sum of the probabilities of the $m-1$ cost types when m is randomising:

$$R_{mt} = \prod_{k=1}^{m-1} P_{kt}^{m}, \qquad S_{mt} = \sum_{k=1}^{m-1} P_{kt}^{m} \tag{3.24}$$

as the dimensionality of the PDEs can thereby be substantially reduced. Note that in the 2-type case, both of these variables are simply given by the probability, P_t.

Proposition 9 *Suppose the two firms have strategy spaces M_i^m and M_j^m, where there are m increasing cost types, $w_i < w_{i+1}, \forall, i \in \{1, n-1\}$. In a mixed-strategy, feedback, Nash equilibrium with incomplete information and m types remaining, the equilibrium consists of a recursive series of sub-games, in each of which the least efficient type is filtered out. In a given sub-game, with m types remaining, the firms exit at the first jump time of conditionally Poisson processes with jumps*

$$\tilde{\lambda}_j(x_t, R_{mt}, S_{mt}) = 0, \qquad j \in \{1, m-1\}, \tag{3.25}$$

$$\tilde{\lambda}_m(x_t, R_{mt}, S_{mt}) = \lambda_m(x_t)/(1 - S_{mt}) \tag{3.26}$$

where the beliefs of agents evolve according to[13]

$$R_{mt} = \overline{R}_m \exp\left[(m-1)\int_{T_{m+1}}^{t} \lambda_m(x_\tau)\, d\tau\right], \tag{3.27}$$

$$S_{mt} = \overline{S}_m \exp\left[\int_{T_{m+1}}^{t} \lambda_m(x_\tau)\, d\tau\right] \tag{3.28}$$

with priors $R_{m,T_{m+1}} = \overline{R}_m \in (0,1)$ and $S_{m,T_{m+1}} = \overline{S}_m \in (0,1)$ and where T_{m+1} denotes the time at which the $m+1$-th type was filtered out (the end of the previous sub-game). The equilibrium real option values, $V_i^m(x_t, R_{mt}, S_{mt}) = V_i^m(x_t, P_{1t}^m, ..., P_{m-1,t}^m)$, are:

$$\begin{aligned} V_i^m(x_t, R_{mt}, S_{mt}) &= V_i^{m-1}(x_t, R_{m-1,t}, S_{m-1,t}) \qquad (3.29)\\ &+ \mathrm{E}_t\left\{\int_t^{T_m} \left(\hat{V}_i(x_\tau + \delta) - V_i^{m-1}(x_\tau, R_{m-1,\tau}, S_{m-1,\tau})\right)\right. \\ &\qquad \left. \lambda_m(x_\tau)\exp\left[-\int_t^{\tau}(r + \lambda_m(x_s))\, ds\right] d\tau\right\}, \\ V_m^m(x_t, R_{mt}, S_{mt}) &= V_m(x_t) \qquad (3.30) \end{aligned}$$

---

[13] Note that the two state variables in the next two equations may be expressed in terms of the other $R_{mt}/\overline{R}_m = (S_{mt}/\overline{S}_m)^{m-1}$. In the interests of clarity, however, we retain both the product and sum of probabilities as state variables.

for $i \in \{1, m-1\}$. At the random time, T_m, which satisfies $S_{m,T_m} = 1$, the m-th type is filtered out and the game proceeds to the next sub-game with $m-1$ types remaining.

The proof of this proposition may be found in section 3.6.6.

Since there are no closed-form solutions for the real option values, V_i^m, we must resort to numerical methods[14] using the partial differential equations:[15]

$$rV_i^m = x - w_i + \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_i^m}{\partial x^2} + \mu x \frac{\partial V_i^m}{\partial x} + \lambda_m R_m \frac{\partial V_i^m}{\partial R_m} + \lambda_m \left[\hat{V}_i(x+\delta) - V_i^m\right] \tag{3.31}$$

with boundary conditions:

$$\lim_{x_t \to \infty} V_i^m(x_t, R_{mt}, S_{mt}) = x_t/(r-\mu) - w_i/r, \tag{3.32}$$

$$\lim_{x_t \downarrow \hat{\underline{x}}_m - \delta} V_i^m(x_t, R_{mt}, S_{mt}) = V_i^{m-1}(\hat{\underline{x}}_m - \delta, R_{m-1,t}S_{m-1,t}), \tag{3.33}$$

$$V_i^m(x_{T_m}, R_{m,T_m}, 1) = V_i^{m-1}(x_{T_m}, R_{m-1,T_m}, S_{m-1,T_m}). \tag{3.34}$$

The boundary conditions show that the real option values are solved recursively between each of the times above starting with T_1, where the final boundary condition is simply $V_1(x_t)$ (from proposition 2). The n sub-games are separated by the following random times:

$$0 < T_n < T_{n-1} < ... < T_2 < T_1 \tag{3.35}$$

where the subscripts denote the type that has been filtered out at that stage of the game. An interesting feature of the state variables, R_t and S_t,[16] is that they experience discontinuous jumps as soon as a type is filtered out. In the case of R_t, the jump is *upwards* and equal to $R_{m-1,t_+} = R_{m,t_-}(1/P^m_{m-1,t_-} - 1)$, for limiting times t_- and t_+, immediately before and after T_m respectively. In the case of S_t, the jump is *downwards* and equal to $S_{m-1,t_+} = 1 - P^m_{m-1,t_-}$. Within each sub-game, however, both state variables are increasing in time as can be confirmed from (3.27) and (3.28).

We demonstrate the real option values arising in a 3-type game in Figure 3.6. Using a fully-implicit finite-difference scheme to determine numerical solutions to the recursive PDEs.

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[14] The numerical method for this may be found in the Appendix in section A.1.

[15] In the proof of proposition 8 we show that the differential equations for the real options simplify to these expressions with the change of variable specified earlier.

[16] Here, we deliberately omit any dependence on m to show that these variables apply over the course of the entire game.

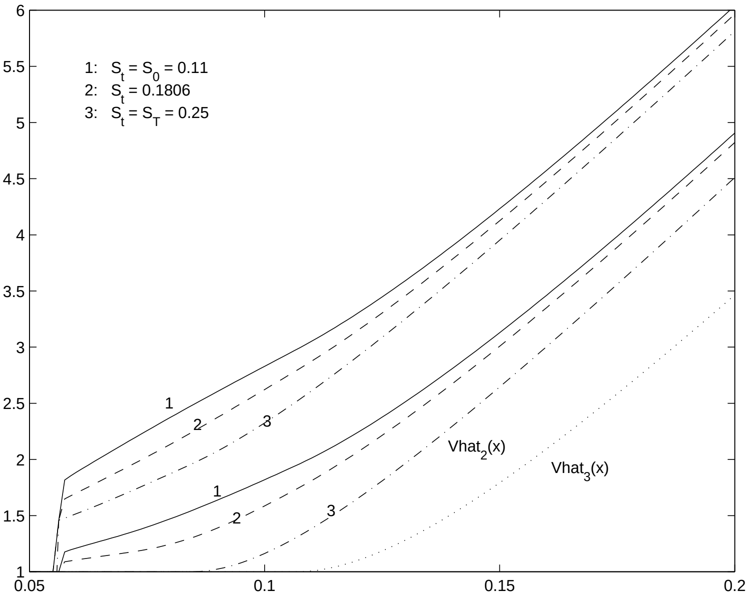

Figure 3.6: The mixed-strategy Nash equilibrium with incomplete information for 3 types. The following parameters were used: $r = 0.025, \mu = 0.0, \sigma = 0.1, \gamma = 1, w_1 = 0.06, w_2 = 0.09, w_3 = 0.12, \delta = 0.05, \overline{P}_1 = 1/3, \overline{P}_2 = 1/3$.

3.4 Exit with Incomplete Information and a Continuum of Types

3.4.1 Filtering with an Infinite Number of Types

We can extend the previous proposition to the case where there is a continuum of types. The first important point to notice is that the hazards will all become infinite over the entire interval over which the least efficient firm randomises (i.e. over $x_t \in (\underline{\hat{x}}_m - \delta, \underline{\hat{x}}_m]$ for type m). The reason for this is that $S_{mt} \to 1$ and so the hazard (3.26) tends to infinity. This means that firms revert to trigger strategies, which are, in fact, their non-strategic strategies! Thus, firms no longer randomise their exit even though we are looking at a limit of a mixed-strategy Nash equilibrium.

The strategies of agents will be given by their monopoly exit triggers, $\underline{\hat{x}}_i = \underline{\hat{x}}(w_i)$, with each firm exiting at the first hitting time of the trigger. An important consequence of this is that the beliefs of agents evolve in a different way. There will no longer be a dependence on the randomising type, m and probabilities evolving from differential equations. Rather, a new state variable $\check{x}_t = \inf\{x_s s \in [0,t]\}$ represents the lowest realisation of the income flow and thus bounds the possible remaining types in the game. This type of learning is used in other strategic real option models of entry such as those studied by Lambrecht and Perraudin (2002) and Lambrecht and Perraudin (1994).

Definition 11 *For $i \neq j$, let firm i's strategy space, S_i, be defined as the trigger $x_i(w_i)$, where firm i exits at the first time x_t hits $x_i(w_i)$.*

If there is a continuum of types over the support $[w_L, w_U]$, where $w_L < w_U$, then these will correspond to a continuum of exit triggers over the support $[\hat{\underline{x}}(w_L), \hat{\underline{x}}(w_L)]$. Instead of setting initial priors for agents' conjectures as to their opponents type, we have instead a cumulative distribution function over the support of types: $F_j(x) = G(w(x))$ for the i-th agent, where

$$w\left(x\right) = r\left[\left(\frac{\xi - 1}{\xi}\right)\left(\frac{x}{r-\mu}\right) - \gamma\right] \quad . \tag{3.36}$$

With such a specification the probability that the opponent is of a less efficient type is given by: $\Pr(w_i < w_j) = 1 - G(w_i)$. Whenever the income flow hits new lows, $\check{x}$, and neither firm has exited, the agents infer new information as to the possible type of their opponent. By Baye's rule the conditional distribution of the other agent's trigger is then given by:

$$F_j(x_t|\check{x}_t) = \frac{F_j(x_t)}{F_j(\check{x}_t)} = \frac{G(w)}{G(w(\check{x}_t))} \quad . \tag{3.37}$$

3.4.2 The Bayesian Nash Equilibrium with a Continuum of Types

In the multiple type model the real option values were given by the solution of recursive partial differential equations, (3.31). Since the probabilities become infinitesimally small in the continuous limit, the product of probabilities, R_{mt}, tends to zero. Firms may first exit when $x_i = \hat{\underline{x}}_i$, at which time their hazards would be infinite. Since the hazard is equal to zero for the upper interval (i.e. $(\hat{\underline{x}}_i, \infty)$) the partial

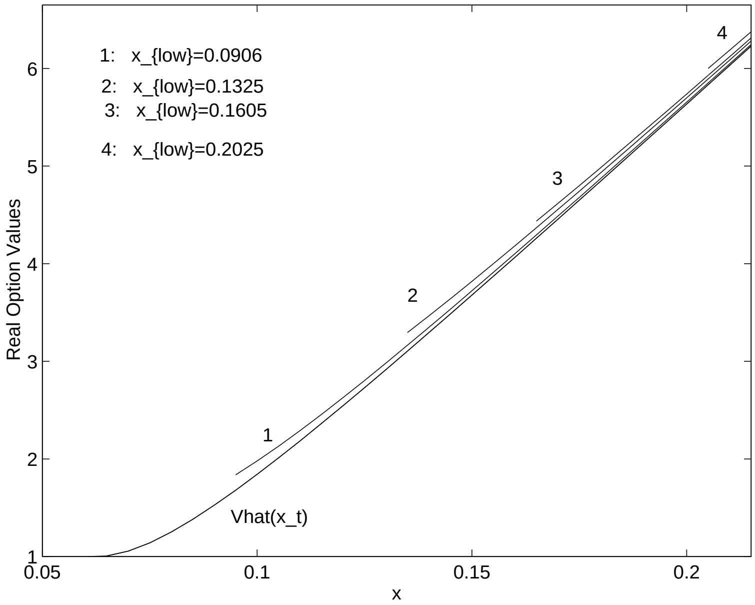

Figure 3.7: The Bayesian Perfect Nash equilibrium with incomplete information. The following parameters were used: $r = 0.025, \mu = 0.0, \sigma = 0.1, \gamma = 1, \delta = 0.025, w_i = 0.06$. The distribution for the types was $F_j(w) = \frac{w^{-\alpha} - w^{-\alpha}}{w^{-\alpha} - \overline{w}^{-\alpha}}$, with parameters: $w = 0.03, \overline{w} = 0.25, \alpha = 0.03$.

differential equation simplifies itself to a ordinary differential equation:[17]

$$rV_i\left(x_t, \check{x}_t | x_i\right) = x - w_i + \mu x \frac{\partial V_i\left(x_t, \check{x}_t | x_i\right)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 V_i\left(x_t, \check{x}_t | x_i\right)}{\partial x^2}, \qquad x_t \in \left[\check{x}_t, \infty\right) \quad . \tag{3.38}$$

As before the solution to the real option must satisfy a standard unlimited liability boundary condition as the income flow tends to infinity. As for the lower boundary condition, this is applied at $\check{x}_t$, below which there is uncertainty as to the firm's rival cost type. For infinitesimal drops in $\check{x}_t$, there is a small chance that the rival firm will exit thereby yielding the higher payoff to the i-th firm. The probability that this will take place in the limit is given by the conditional probability density function of cost types. We thus have the boundary condition

$$\frac{\partial V_i\left(\check{x}, \check{x} | x_i\right)}{\partial \check{x}} = \frac{F_j'\left(\check{x}\right)}{F_j\left(\check{x}\right)} \left[\hat{V}_i\left(\check{x} + \delta\right) - V_i\left(\check{x}, \check{x} | x_i\right)\right], \qquad x_i < \check{x} \quad . \tag{3.39}$$

The third and final boundary condition can be applied by considering what happens when $\check{x}_t = x_i$. When this is the case, the i-th firm is the next to exit and will do so when $x_t = x_i$. By the absence of arbitrage this real option value must always be greater than the scrapping value. Thus the real option value in this case is equal to the non-strategic one:

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[17] Note, however, that we retain partial derivatives to emphasise the fact that there is also a second state variable, $\check{x}_t$.

$$V_i\left(x, x_i | x_i\right) = \hat{V}_i\left(x\right) \tag{3.40}$$

By applying these boundary conditions to a standard solution of (3.38) one may then obtain the following result.

Proposition 10 *In the limit of a continuum of types in proposition 9, there is a Bayesian Nash equilibrium where the i-th firm exits at the first hitting time of the trigger* $x_i = \hat{\underline{x}}_i = \hat{\underline{x}}(w_i)$ *and its belief regarding j's type evolves according to the distribution* $F_j(x)/F_j(\check{x})$. *The real option value for a firm of type i is:*

$$\begin{aligned} V_i\left(x, \check{x} | x_i\right) = \frac{x}{r-\mu} &- \frac{w_i}{r} + \left[\frac{F_j\left(x_i\right)}{F_j\left(\check{x}\right)}\right]\left[\gamma - \frac{x_i}{r-\mu} - \frac{w_i}{r}\right]\left(\frac{x}{x_i}\right)^{\xi} \\ &+ \int_{x_i}^{\check{x}} \frac{F_j'\left(v\right)}{F_j\left(\check{x}\right)}\left[\hat{V}_i\left(v+\delta\right) - \frac{v}{r-\mu} - \frac{w_i}{r}\right]\left(\frac{x}{v}\right)^{\xi} dv, \end{aligned} \tag{3.41}$$

for $x \in (\check{x}, \infty)$.

The proof of this proposition may be found in section 3.6.7.

When $\check{x} = x_i$, the i-th firm knows it cannot win and so exits optimally as it would do in a monopoly. Contingent on this final outcome and learning up until that point, the real option value is computed above.

In Figure 3.8 we plot the real option value by computing the expression (3.41). We assume the types follow a Pareto distribution:

$$F_j(x) = G(w) = \frac{\underline{w}^{-\alpha} - w\left(x\right)^{-\alpha}}{\underline{w}^{-\alpha} - \bar{w}^{-\alpha}}, \qquad \alpha \neq 0, \qquad w \in [\underline{w}, \bar{w}]. \tag{3.42}$$

A surprising feature of the equilibrium in proposition 10 is that firms exit at their non-strategic triggers even though there are second-mover advantages. The equilibrium suggests that agents should adopt a myopic exit policy as if there were no strategic incentives. The kind of myopia here is similar to that considered by Leahy (1993) in his model of competitive investment under uncertainty. Leahy's

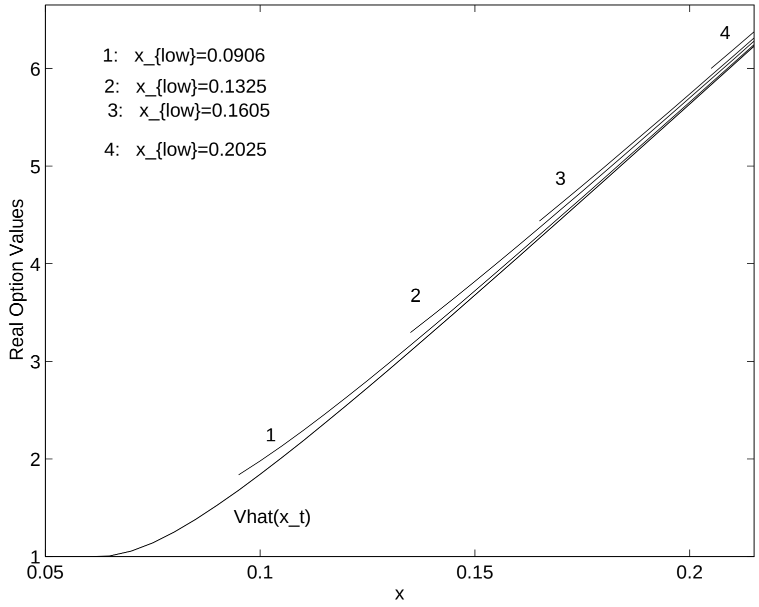

Figure 3.8: The Bayesian Nash equilibrium with incomplete information. The following parameters were used: $r = 0.025, \mu = 0.0, \sigma = 0.1, \gamma = 1, \delta = 0.025, w_i = 0.06$. The distribution for the types was $F_j(w) = \frac{w^{-\alpha} - w^{-\alpha}}{w^{-\alpha} - \overline{w}^{-\alpha}}$, with parameters: $w = 0.03, \overline{w} = 0.25, \alpha = 0.03$.

competitive equilibrium suggests that the investment policies of two firms operating in a monopoly and a perfectly competitive industry are the same. Although the equilibrium prescribes 'myopic' exit policies, the real option values are not identical to those in a monopoly, as the prospect of winning the game adds extra value to the firm's exit option. This effect can be seen clearly in Figure 3.8. As more types are filtered out the real option value becomes smaller, until $\check{x} = x_i$ and the option value is identical to the monopoly one.

3.5 Conclusion

In this chapter, we have analysed equilibria in a simple real options model of industry exit. Despite the simplicity of our basic real options model (which is a stripped down version of the classic model of Dixit (1989)), the equilibria which arise turn out to be more complicated.

We show that pure strategy equilibria Nash equilibria exist in which firms choose an exit trigger conditional on whether their competitor has left or not. The only pure strategy equilibrium which is sub-game perfect is the one in which the firm with higher costs exits first.

An interesting finding, however, is that there exist equilibria with mixed strategies in the sense that firms exit randomly at the first jump time of conditionally Poisson point processes. In the case of symmetric firms with complete information, random exits occur when the state variable falls below the level at which firms would exit in the absence of strategic interactions.

When information about the competitor firm's costs is incomplete, there may exist mixed-strategy Bayesian Nash equilibria in which firms filter the information they have (that the other firm has not so far exited), up-dating their conditional probabilities that the other firm is of one or other type.

3.6 Proofs of Propositions

3.6.1 Proof of Proposition 6

The root $\underline{x}_i^*$ of the equation $\hat{V}_i(x+\delta) - \left[\frac{x_t}{r-\mu} - \frac{w_i}{r} + \left[\gamma - \frac{\hat{\underline{x}}_i}{r-\mu} + \frac{w_i}{r}\right]\left(\frac{x_t}{\hat{\underline{x}}_i}\right)^{\xi}\right] = 0$ gives the exit trigger at which the i-th firm is indifferent between "losing the game" and exiting first at $\hat{\underline{x}}_i$ or "winning the game" when the other firm exits first at $\underline{x}_i^*$. For each firm, equilibrium triggers must lie in the interval $[\underline{x}_i^*, \hat{\underline{x}}_i]$ since they can always achieve $\hat{V}_i(x)$ or better by exiting at $\hat{\underline{x}}_i$.

Consider the Nash equilibrium

$$\left\{\underline{x}_1^d, \underline{x}_1^m\right\} = \left\{\underline{x}_1^*, \hat{\underline{x}}_1\right\} \ , \tag{3.43}$$

$$\left\{\underline{x}_2^d, \underline{x}_2^m\right\} = \left\{\hat{\underline{x}}_2, \hat{\underline{x}}_2\right\} \ . \tag{3.44}$$

Since $\underline{x}_2^* > \underline{x}_1^*$, 2's optimal response to 1 playing $\underline{x}_1^d = \underline{x}_1^*$ while both are still operating, is to exit at $\hat{\underline{x}}_2$. Since 1 obtains the maximum possible payoff, its strategy must be an optimal response to 2's.

To see that the equilibrium is sub-game perfect, suppose that 2 deviates from $\{\hat{\underline{x}}_2, \hat{\underline{x}}_2\}$ by adopting some lower duopoly exit trigger $\underline{x}_2^d \in [\underline{x}_2^*, \hat{\underline{x}}_2)$. Since $\underline{x}_2^* > \underline{x}_1^*$, firm 1's optimal response will always be $\{\underline{x}_1^*, \hat{\underline{x}}_1\}$ which will ensure it wins the game and gets the maximum possible claim value.

Now, suppose that $\hat{\underline{x}}_1 > \underline{x}_2^*$ and consider the alternative Nash equilibrium

$$\left\{\underline{x}_1^d, \underline{x}_1^m\right\} = \left\{\hat{\underline{x}}_1, \hat{\underline{x}}_1\right\} \ , \tag{3.45}$$

$$\left\{\underline{x}_2^d, \underline{x}_2^m\right\} = \left\{\underline{x}_2^*, \hat{\underline{x}}_2\right\} \ , \tag{3.46}$$

Again, the fact that this is a Nash equilibrium may be verified by checking firm 1's optimal response to firm 2's strategy is the hypothesised $\left\{\underline{x}_1^d, \underline{x}_1^m\right\}$ pair. 2's claim value is the highest possible so 2's strategy must be the optimal response.

However, this second Nash equilibrium is not sub-game perfect, since if 1 deviates by choosing some $\underline{x}_1^d \in [\underline{x}_1^*, \underline{x}_2^*)$, 2's optimal response will no longer be the hypothesised $\underline{x}_2^d$.

Given the boundary conditions:

1. $\lim_{x\to\infty}[W_k(x) - x/(r-\mu) - w_k/r] = 0$ for $k = 1, 2$,
2. if i denotes the "winning" firm which exits second, then $W_i(x_{trigger}) = \hat{V}(x_{trigger} + \delta)$,
3. if j denotes the losing firm which exits first, then $L_j(x_{trigger}) = \gamma$ and $L_j'(x_{trigger}) = 0$,

the firm values given in the proposition may be obtained using standard methods. □

3.6.2 Proof of Proposition 7

For there to be an equilibrium, the coupled HJB ordinary differential equations must be satisfied for $V_i(x_t) \geq \gamma$ for all x_t and the relevant maximisation must be achieved. Maximizing: $\lambda_{it}[\gamma - V_i]$ with respect to $\lambda_{it} \geq 0$ for $V_i \geq \gamma$ and $i \in \{1, 2\}$, yields the optimal control: $\lambda_{it} = 0$ for $V_i(x_t) > \gamma$ (as any other positive control would leave the term negative), while for $V_i(x_t) = \gamma$, the agent is indifferent to his control, as the term is always equal to zero. By substituting, the solution $V_i(x_t) = \gamma$ into the HJB equation, however, the best response control of agent j is determined: $\lambda_{jt} = (r\gamma + w_i - x_t)/(\hat{V}_i(x_t + \delta) - \gamma)$. Thus, the hazards are

$$\lambda_{it} = \begin{cases} (r\gamma + w_j - x_t)/(\hat{V}_j(x_t + \delta) - \gamma), & x \in (x_j^* - \delta, x_j^*) \\ 0, & x \in [x_i^*, \infty) \end{cases} \tag{3.47}$$

where x_i denotes the boundary between income flow values for which $V_i(x_t) = \gamma$ and $V_i(x_t) > \gamma$. As the two cost types are the same ($w_i = w_j$) the two boundaries are also the same: $x_i^* = x_j^* = x^*$. Substitution of the hazards (3.47) back into the HJB equation over the two regimes, results in the claim value being equal to γ for $x \in (\underline{\hat{x}}_i - \delta, \underline{\hat{x}}_i)$ and satisfying the ODE

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) + x - w_i = rV_i(x) \tag{3.48}$$

for $x \in [x^*, \infty)$. The solution to the above ordinary differential equation is given by the non-strategic value, $\hat{V}_i(x)$, which has a continuous first-derivative at the boundary. Thus, $x^* = \underline{\hat{x}}_i$. □

3.6.3 Derivation of Equation (3.13)

Suppose we observe the sample path of a point process and do not know whether jumps are generated by the jump rate $\tilde{\lambda}_1(x_t)$ or $\tilde{\lambda}_2(x_t)$. Let P_t be the probability that jumps are generated by $\tilde{\lambda}_{1t}$ conditional on observing the past path of x_t and hence of $\tilde{\lambda}_{1t}$ or $\tilde{\lambda}_{2t}$, and suppose that P_t is initially equal to a given prior, $\overline{P}$, i.e., $P_0 = \overline{P}$.

P_t may be up-dated using Bayes' Rule. The analysis may be performed conditional on the time path of $\{x_\tau\}_{\tau=0}^{t}$ so the jump rates, $\tilde{\lambda}_{1t}$ or $\tilde{\lambda}_{2t}$, may be treated as functions of time. By Bayes' Rule

$$\text{Prob}\left\{\tilde{\lambda} = \tilde{\lambda}_1 | \text{no jump by } t+\Delta\right\} = \frac{\text{Prob}\left\{\tilde{\lambda} = \tilde{\lambda}_1 \;\; \textit{and} \;\; \text{no jump by } t+\Delta\right\}}{\text{Prob}\left\{\text{no jump by } t+\Delta\right\}}, \tag{3.49}$$

where all the probabilities are conditional on no jump by t. Writing out the probabilities for a small increment in time, Δ, we get:

$$P_{t+\Delta} = \frac{(1-\tilde{\lambda}_{1t}\Delta)P_t}{(1-\tilde{\lambda}_{1t}\Delta)P_t + (1-\tilde{\lambda}_{2t}\Delta)(1-P_t)}, \tag{3.50}$$

$$\frac{P_{t+\Delta} - P_t}{\Delta} = \frac{(\tilde{\lambda}_{2t} - \tilde{\lambda}_{1t})P_t(1-P_t)}{[(1-\tilde{\lambda}_{1t}\Delta)P_t + (1-\tilde{\lambda}_{2t}\Delta)(1-P_t)]}. \tag{3.51}$$

Taking the limit as $\Delta \downarrow 0$ yields the Riccatti equation in (3.13). □

3.6.4 Proof of Proposition 8

Satisfying the Maximisation Condition

For an equilibrium, the coupled HJB PDE's, (3.14) and (3.15) must be satisfied for $\tilde{V}_i(x_t, P_t) \geq \gamma$ and the maximisations must be achieved. Maximizing: $\tilde{\lambda}_i[\gamma - \tilde{V}_i]$ with respect to $\tilde{\lambda}_i \geq 0$ for $\tilde{V}_i \geq \gamma$ yields the optimal control: $\tilde{\lambda}_i = 0$ for $\tilde{V}_i > \gamma$ (as any other positive control would leave the term negative), while for $\tilde{V}_i = \gamma$, the real option holder is indifferent to his hazard, as the term is always equal to zero. By substituting, the solutions $\tilde{V}_i(x_t, P_t) = \gamma$ into the HJB PDE's, however, the following coupled equations must be satisfied:

$$\tilde{V}_1(x_t, P_t) = \gamma \qquad \Rightarrow \qquad P_t\tilde{\lambda}_1 + (1-P_t)\,\tilde{\lambda}_2 = \frac{r\gamma + w_i - x_t}{\hat{V}_1\,(x_t+\delta) - \gamma} \tag{3.52}$$

$$\tilde{V}_2(x_t, P_t) = \gamma \qquad \Rightarrow \qquad P_t\tilde{\lambda}_1 + (1 - P_t)\,\tilde{\lambda}_2 = \frac{r\gamma + w_i - x_t}{\hat{V}_2\,(x_t + \delta) - \gamma} \tag{3.53}$$

Hypothesis: $\tilde{V}_1 > \gamma$ for all $P_t < 1$

If $\tilde{V}_1 > \gamma$ for all $P_t < 1$, then $\tilde{\lambda}_1 = 0$ when $P_t < 1$. The reason for this is that type 1 agents maximise the term, $\tilde{\lambda}_1[\gamma - \tilde{V}_1]$, with respect to $\tilde{\lambda}_1$. Since $\tilde{V}_1 > \gamma$ the term is always negative unless the hazard is equal to zero.

An important consequence of the fact that $\tilde{V}_1 > \gamma$, is that equation (3.52), no longer applies. Substituting $\tilde{\lambda}_1 = 0$ in (3.53) then implies that the other hazard, $\tilde{\lambda}_2$, is given by

$$\tilde{\lambda}_2(x, P) = \frac{1}{1 - P}\left[\frac{r\gamma + w_2 - x}{\hat{V}_2\,(x + \delta) - \gamma}\right] \tag{3.54}$$

when $\tilde{V}_2 = \gamma$. Substituting these two hazards into the PDE for $\tilde{V}_1$, (3.14), then implies that the PDE is

$$r\tilde{V}_1 = \frac{\sigma^2 x^2}{2}\frac{\partial^2 \tilde{V}_1}{\partial x^2} + \mu x \frac{\partial \tilde{V}_1}{\partial x} + P\lambda_2 \frac{\partial \tilde{V}_1}{\partial P} + \lambda_2\left[\hat{V}_1\,(x + \delta) - \tilde{V}_1\right] + x - w_1. \tag{3.55}$$

In the case of the less efficient firm, the real option value is either greater than or equal to γ. In the former case, $\tilde{\lambda}_2$ is equal to zero and so substituting both zero hazards into the HJB PDE, (3.15), then implies that the differential equation is

$$r\tilde{V}_2 = \frac{\sigma^2 x^2}{2}\frac{\partial^2 \tilde{V}_2}{\partial x^2} + \mu x \frac{\partial \tilde{V}_2}{\partial x} + x - w_2 \tag{3.56}$$

which is the same differential equation as in the non-strategic case. Thus, when $\tilde{V}_2 > \gamma$, for some upper interval, (x_2^*, ∞), the option value satisfies differential equation, (3.56), while for a lower interval, $\tilde{V}_2 = \gamma$. In order to maximise the real option value, it must satisfy value-matching and smooth-pasting conditions at x_2^*: $\tilde{V}_2(x_2^*, P) = \gamma$, and $\partial\tilde{V}_2(x_2^*, P)/\partial V = 0$. At $x_2^* - \delta$, the winner's option value (if of type w_2) is $\hat{V}_2(x_2^* - \delta + \delta) = \hat{V}_2(x_2^*) = \gamma$ and so there is no longer an incentive to wait. So the interval for which $\tilde{V}_2 = \gamma$, is $(x_2^* - \delta, x_2^*]$. The upper boundary condition is a standard unlimited liability one, and so the real option value is identical to the one in proposition 7

$$\tilde{V}_2(x, P) = \begin{cases} \hat{V}_2(x), & x \in (\underline{\hat{x}}_2, \infty) \\ \gamma, & x \in (\underline{\hat{x}}_2 - \delta, \underline{\hat{x}}_2) \end{cases} \tag{3.57}$$

The only difference here is the smooth-pasting trigger. The hazard, $\tilde{\lambda}_2$, in (3.54), then applies over the interval, $(\hat{\underline{x}}_2 - \delta, \hat{\underline{x}}_2]$ while elsewhere it is equal to zero.

As for the boundary conditions on the more efficient firm's real option value, $\tilde{V}_1$. First, note that as $P_t \to 1$, the game turns into a game of complete information; and both firms are of type w_1. So the option value tends towards the one in proposition 7 with $w = w_1$. Thus, $\tilde{V}_1(x_t, 1) = V_1(x_t)$, where this latter value is defined in (3.11). As $x_t \downarrow \hat{\underline{x}}_2 - \delta$, $\tilde{\lambda}_2 \to \infty$. So, once again, the game turns into one of complete information and $\lim_{x \downarrow \hat{\underline{x}}_2 - \delta} \tilde{V}_1(x, P) = V_1(\hat{\underline{x}}_2 - \delta)$. Finally, a standard unlimited liability boundary condition applies as $x_t \to \infty$.

Substituting the twin hazards into the Riccati equation, the probabilities evolve according to a much simpler differential equation, whose solution can be written out by inspection:

$$\frac{dP_t}{dt} = P_t \lambda_2(x_t) \quad \Rightarrow \quad P_t = \overline{P} \exp\left[\int_0^t \lambda_2(x_s) ds\right] \tag{3.58}$$

where $\overline{P} = P_0$. Since the integrand in (3.58) is non-negative the probability is non-decreasing in time.

Proof that the Hypothesis: $\tilde{V}_1 > \gamma$, for $P_t < 1$ holds

Since P_t is non-decreasing in time (see (3.58)) define T as the first time that the probability equals unity:

$$1 = \overline{P} \exp\left[\int_0^T \lambda_2\left(x_s\right) ds\right] \qquad \Rightarrow \qquad T = \inf\left\{s > 0 | P_s = 1\right\} \tag{3.59}$$

Also, consider the hazard $\tilde{\lambda}_2$, from (3.54):

$$\tilde{\lambda}_2\left(x_t\right) = \frac{\lambda_2\left(x_t\right)}{1 - P_t} = \frac{\lambda_2\left(x_t\right)}{1 - \overline{P} \exp\left[\int_0^t \lambda_2\left(x_s\right) ds\right]} = \frac{\lambda_2\left(x_t\right) \exp\left[-\int_0^t \lambda_2\left(x_s\right) ds\right]}{\exp\left[-\int_0^t \lambda_2\left(x_s\right) ds\right] - \overline{P}} \tag{3.60}$$

Now, consider the value of the option, $\tilde{V}_1(x_t, P_t)$. This may be written down as the sum of (1) the probability of facing a type 1 agent multiplied by the payoff if that is the case, plus (2) the weighted-probability that the opponent is of the less-efficient type. Since the stopping time for the winner's payoff is generated by a point process, $\tilde{\lambda}_2$, this latter term must be conditioned on the fact that exit has not taken place up until a given time multiplied by the probability of an exit in the next time increment. The payoff for the former term is simply $V_1(x_t)$ from proposition 7. So the equity value of the more efficient firm is given by:

$$\begin{aligned} V_1\left(x_t, P_t\right) &= P_t V_1\left(x_t\right) + \left(1 - P_t\right)\left\{V_1\left(x_t\right) + \right. \\ &\quad \left. \mathrm{E}_t\left(\int_t^T \left(\hat{V}_1\left(x_\tau + \delta\right) - V_1\left(x_\tau\right)\right)\tilde{\lambda}_{2\tau}\exp\left[-\int_t^\tau \left(r + \tilde{\lambda}_{2s}\right)ds\right]d\tau\right)\right\} \\ &= V_1\left(x_t\right) + \left(1 - P_t\right) \times \\ &\quad \mathrm{E}_t\left\{\int_t^T \left(\hat{V}_1\left(x_\tau + \delta\right) - V_1\left(x_\tau\right)\right)\tilde{\lambda}_{2\tau}\exp\left[-\int_t^\tau \left(r + \tilde{\lambda}_{2s}\right)ds\right]d\tau\right\} \end{aligned}$$

The term $\tilde{\lambda}_{2\tau}\exp[-\int_t^\tau (r + \tilde{\lambda}_{2s})ds]$ can be simplified, using (3.60):

$$\begin{aligned} \tilde{\lambda}_{2\tau}\exp\left[-\int_t^\tau \left(r + \tilde{\lambda}_{2s}\right)ds\right] &= \left(\frac{\lambda_2\left(x_\tau\right)\exp\left[-r\left(\tau - t\right)\right]}{1 - P_t\exp\left[\int_t^\tau \lambda_2\left(x_s\right)ds\right]}\right) \times \\ &\quad \exp\left[\int_t^\tau \frac{-\lambda_2\left(x_s\right)\exp\left[-\int_t^s \lambda_2\left(x_\nu\right)d\nu\right]}{\exp\left[-\int_t^s \lambda_2\left(x_\nu\right)d\nu\right] - P_t}ds\right] \\ &= \left(\frac{\lambda_2\left(x_\tau\right)\exp\left[-r\left(\tau - t\right)\right]}{1 - P_t\exp\left[\int_t^\tau \lambda_2\left(x_s\right)ds\right]}\right) \times \\ &\quad \exp\left\{\left[\log\left(\exp\left[-\int_t^s \lambda_2\left(x_\nu\right)d\nu\right] - P_t\right)\right]_{s=t}^{s=\tau}\right\} \\ &= \left(\frac{\lambda_2\left(x_\tau\right)\exp\left[-r\left(\tau - t\right)\right]}{1 - P_t\exp\left[\int_t^\tau \lambda_2\left(x_s\right)ds\right]}\right) \times \\ &\quad \left(\frac{\exp\left[-\int_t^\tau \lambda_2\left(x_\nu\right)d\nu\right] - P_t}{1 - P_t}\right) \\ &= \left(\frac{\lambda_2\left(x_\tau\right)\exp\left[-r\left(\tau - t\right)\right]}{1 - P_t}\right)\exp\left[-\int_t^\tau \lambda_2\left(x_\nu\right)d\nu\right] \end{aligned}$$

Substituting this back into the expression for $\tilde{V}_1(x_t, P_t)$ we obtain:

$$\begin{aligned} \tilde{V}_1\left(x_t, P_t\right) &= V_1\left(x_t\right) + \\ &\quad \mathrm{E}_t\left\{\int_t^T \left(\hat{V}_1\left(x_\tau + \delta\right) - V_1\left(x_\tau\right)\right)\lambda_2\left(x_\tau\right)\exp\left[-\int_t^\tau \left(r + \lambda_2\left(x_s\right)\right)ds\right]d\tau\right\} \end{aligned} \tag{3.61}$$

Since P can only go up, for any time path of x_τ starting at x_t, $T - t$ cannot be higher if P_t is larger. This follows obviously from the fact that $P_\tau = P_t\exp[\int_0^\tau \lambda_2(x_s)ds]$. Given this sample-path-by-sample-path result, the integral must be smaller when P_t

is larger. Thus, $\partial \tilde{V}_1 / \partial P < 0$, and the more efficient firm's value is strictly decreasing in P_t. Since the final condition on the option value is greater than or equal to γ (i.e. $\tilde{V}_1(x,1) \geq \gamma$), we have that $\tilde{V}_1 > \gamma$ for all $P_t \in [\overline{P}, 1)$. Thus the hypothesis required is verified and this completes the proof. □

3.6.5 Derivation of Equation (3.22)

Suppose we observe the sample path of a point process and do not know whether jumps are generated by the various jump rates $\tilde{\lambda}_i(x_t)$, $i \in \{1, m-1\}$. Let P_i^m be the probability that jumps are generated by $\tilde{\lambda}_{it}$ conditional on observing the past path of x_t and hence of $\tilde{\lambda}_{it}$, when type m is the least efficient type remaining.

The probabilities may be up-dated using Bayes' Rule. The analysis may be performed conditional on the time path of $\{x_\tau\}_{\tau=0}^{t}$ so the jump rates, $\tilde{\lambda}_{it}$, may be treated as functions of time. By Bayes' Rule

$$\text{Prob}\left\{\tilde{\lambda} = \tilde{\lambda}_i | \text{no jump by } t+\Delta\right\} = \frac{\text{Prob}\left\{\tilde{\lambda} = \tilde{\lambda}_i \ \ and \ \ \text{no jump by } t+\Delta\right\}}{\text{Prob}\left\{\text{no jump by } t+\Delta\right\}}, \tag{3.62}$$

where all the probabilities are conditional on no jump by t. Writing out the probabilities for a small increment in time, Δ, we get:

$$\begin{aligned}
P^m_{i,t+\Delta} &= \frac{(1-\tilde{\lambda}_{it}\Delta)P^m_{it}}{\sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} + (1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}, \\
P^m_{i,t+\Delta} - P^m_{it} &= \frac{(1-\tilde{\lambda}_{it}\Delta)P^m_{it} - P^m_{it}\sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} - P^m_{it}(1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}{\sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} + (1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}, \\
\frac{P^m_{i,t+\Delta} - P^m_{it}}{P^m_{it}} &= \frac{(1-\tilde{\lambda}_{it}\Delta) - \sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} - (1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}{\sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} + (1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}, \\
\frac{P^m_{i,t+\Delta} - P^m_{it}}{P^m_{it}} &= \frac{-\tilde{\lambda}_{it}\Delta + \sum_{j=1}^{m-1}\tilde{\lambda}_{jt}\Delta P^m_{jt} + \tilde{\lambda}_{mt}\Delta(1-\sum_{j=1}^{m-1}P^m_{jt})}{\sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} + (1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}, \\
\frac{P^m_{i,t+\Delta} - P^m_{it}}{P^m_{it}\Delta} &= \frac{\tilde{\lambda}_{mt}(1-\sum_{j=1}^{m-1}P^m_{jt}) + \sum_{j=1}^{m-1}\tilde{\lambda}_{jt}P^m_{jt} - \tilde{\lambda}_{it}}{\sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} + (1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}, \\
\frac{P^m_{i,t+\Delta} - P^m_{it}}{P^m_{it}\Delta} &= \frac{\tilde{\lambda}_{mt}(1-\sum_{j=1}^{m-1}P^m_{jt}) + \sum_{j=1,j\neq i}^{m-1}\tilde{\lambda}_{jt}P^m_{jt} - (1-P^m_{it})\tilde{\lambda}_{it}}{\sum_{j=1}^{m-1}(1-\tilde{\lambda}_{jt}\Delta)P^m_{jt} + (1-\tilde{\lambda}_{mt}\Delta)(1-\sum_{j=1}^{m-1}P^m_{jt})}.
\end{aligned}$$

Taking the limit as $\Delta \downarrow 0$ yields the Riccatti equation in (3.22). $\square$

3.6.6 Proof of Proposition 9

The proof of this proposition is quite similar to that of proposition 8.

Satisfying the Maximisation Conditions

For an equilibrium, the coupled HJB PDE's, must be satisfied for $V_i^m \geq \gamma$ and the maximisations must be achieved. Maximizing: $\tilde{\lambda}_i[\gamma - V_i^m]$ with respect to $\tilde{\lambda}_i \geq 0$ for $\tilde{V}_i \geq \gamma$ yields the optimal control: $\tilde{\lambda}_i = 0$ for $\tilde{V}_i > \gamma$ (as any other positive control would leave the term negative), while for $\tilde{V}_i = \gamma$, the real option holder is indifferent to his hazard, as the term is always equal to zero. By substituting, the solutions $\tilde{V}_i(x_t, P_t) = \gamma$ into the HJB PDE's, however, the following coupled equations must be satisfied:

$$V_i^m(x_t, P_{1t}^m, ..., P_{m-1,t}^m) = \gamma \quad \Rightarrow \quad \sum_{k=1}^{m-1} P_{kt}^m \tilde{\lambda}_i + \left(1 - \sum_{k=1}^{m-1} P_{kt}^m\right) \tilde{\lambda}_m = \frac{r\gamma + w_i - x_t}{\hat{V}_i\left(x_t + \delta\right) - \gamma} \tag{3.63}$$

when $i \in \{1, m\}$.

Hypothesis: $V_i^m > \gamma$ for all $i < m$ and $S_{mt} < 1$

If $V_i^m > \gamma$ for all $S_{mt} < 1$ and $i \in \{1, m-1\}$, then all the corresponding hazards, $\tilde{\lambda}_{i \in \{1, m-1\}} = 0$. The reason for this is that the more efficient agents maximise the term, $\tilde{\lambda}_i[\gamma - V_i^m]$, with respect to $\tilde{\lambda}_i$. Since $V_i^m > \gamma$ the term is always negative unless the hazard is equal to zero.

An important consequence of this, is that equation (3.63), only applies when $i = m$. Substituting the other zero hazards in when $i = m$, then implies that the hazard, $\tilde{\lambda}_m$, is given by:

$$\tilde{\lambda}_m(x_t, P_{1t}^m, ..., P_{m-1,t}^m) = \frac{1}{1 - \sum_{k=1}^{m-1} P_{kt}^m} \left[\frac{r\gamma + w_m - x}{\hat{V}_m\left(x + \delta\right) - \gamma}\right] = \frac{\lambda_m(x_t)}{1 - S_{mt}} \tag{3.64}$$

when $V_m^m = \gamma$.

Evolution of Beliefs

Substituting the hazards into the Riccatti equations, (3.22), also yields more simple ordinary differential equations for the evolution of the probabilities

$$\frac{dP^m_{kt}}{dt} = \lambda_m(x_t)P^m_{kt}, \qquad k \in \{1, m-1\}. \tag{3.65}$$

The solution of this can be written out by inspection:

$$P^m_{kt} = \overline{P}^m_k \exp\left[\int_{T_{m+1}}^t \lambda_m(x_s)ds\right] \tag{3.66}$$

where T_{m+1} is the time at which the $m+1$-th type is filtered out and $\overline{P}^m_k = P^m_{k,T_{m+1}}$. Using the definitions of R_{mt} and S_{mt}, their evolution can also be written out by inspection:

$$R_{mt} = \overline{R}_m \exp\left[(m-1)\int_{T_{m+1}}^t \lambda_m(x_s)ds\right] \quad , \tag{3.67}$$

$$S_{mt} = \overline{S}_m \exp\left[\int_{T_{m+1}}^t \lambda_m(x_s)ds\right] \quad . \tag{3.68}$$

Thus, both R_{mt} and S_{mt} are increasing in time.

Partial Differential Equations for the Real Option Values

Substituting the hazards into the HJB PDEs for V_i^m, then implies that the PDEs are:

$$rV_i^m = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_i^m}{\partial x^2} + \mu x \frac{\partial V_i^m}{\partial x} + \lambda_m \sum_{k=1}^{m-1} P^m_k \frac{\partial V_i^m}{\partial P^m_k} + \lambda_m \left[\hat{V}_i(x+\delta) - V_i^m\right] + x - w_i$$

when $i < m$. By substituting R_{mt} for the probabilities, we can reduce the dimensionality of the PDE:

$$rV_i^m = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_i^m}{\partial x^2} + \mu x \frac{\partial V_i^m}{\partial x} + \lambda_m R_m \frac{\partial V_i^m}{\partial R_m} + \lambda_m \left[\hat{V}_i(x+\delta) - V_i^m\right] + x - w_i$$

In the case of the least efficient firm ($i = m$), the real option value is either greater than or equal to γ. In the former case, $\tilde{\lambda}_m$ is equal to zero. Since the hazards are all zero we obtain the differential equation:

$$rV_m^m = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_m^m}{\partial x^2} + \mu x \frac{\partial V_m^m}{\partial x} + x - w_m \tag{3.69}$$

which is the same differential equation as in the non-strategic case. Thus, when $V_m^m > \gamma$, for some upper interval, (x_m^*, ∞), the option value satisfies differential equation, (3.69), while for a lower interval, $V_m^m = \gamma$. In order to maximise the real option value, it must satisfy value-matching and smooth-pasting conditions at x_m^*: $V_m^m(x_m^*, ...) = \gamma$, and $\partial V_m^m(x_m^*, ...)/\partial V = 0$. The upper boundary condition is a standard unlimited liability one, and so the real option value is identical to the one in proposition 7

$$V_m^m(x, ...) = \begin{cases} \hat{V}_m(x), & x \in (\hat{\underline{x}}_m, \infty) \\ \gamma, & x \in (\hat{\underline{x}}_m - \delta, \hat{\underline{x}}_m) \end{cases} \tag{3.70}$$

The only difference here is the smooth-pasting trigger. The hazard, $\tilde{\lambda}_m$, in (3.64), then applies over the interval, $(\hat{\underline{x}}_m - \delta, \hat{\underline{x}}_m]$ while elsewhere it is equal to zero.

As for the boundary conditions on the real option values when $i < m$. First, note that as $S_m \to 1$, and no one has exited the probability of either agent being of the m-th type is equal to zero. The reason for this is that $1 - S_{mt}$ represents this probability. Thus, in this limit, the m-th type is filtered out. Likewise, in the limit that $x \downarrow \hat{\underline{x}}_m - \delta$, $\tilde{\lambda}_m \to \infty$, the m-th type is filtered out. In both limits the real option value must equal the corresponding value when there are $m - 1$ types left. Thus, we have the boundary conditions:

$$\begin{aligned} V_i^m(x_t, R_m, S_m) &= V_i^{m-1}(x_t, R_{m-1}, S_{m-1}), & (3.71) \\ \lim_{x \downarrow \hat{\underline{x}}_m - \delta} V_i^m(x_t, R_m, S_m) &= V_i^{m-1}(\hat{\underline{x}}_m - \delta, R_{m-1}, S_{m-1}), & (3.72) \\ \lim_{x \to \infty} V_i^m(x_t, R_m, S_m) &= \frac{x_t}{r - \mu} - \frac{w_i}{r}, & (3.73) \end{aligned}$$

where the last boundary condition is a standard unlimited liability condition.

Proof that the Hypothesis: $V_i^m > \gamma$, for $i < m$ and $S_{mt} < 1$ holds

T_m is the first time that S_{mt} equals unity in the sub-game where m is the least efficient type:

$$1 = \overline{S}_m \exp\left[\int_0^{T_m} \lambda_m(x_s)\, ds\right] \qquad \Rightarrow \qquad T_m = \inf\{s > 0 | S_{ms} = 1\} \tag{3.74}$$

Also, consider the hazard $\tilde{\lambda}_m$, from (3.64):

$$\tilde{\lambda}_m(x_t) = \frac{\lambda_m(x_t)}{1 - S_{mt}} = \frac{\lambda_m(x_t)}{1 - \overline{S}_m \exp\left[\int_{T_m}^t \lambda_m(x_s)\, ds\right]} = \frac{\lambda_m(x_t) \exp\left[-\int_{T_m}^t \lambda_m(x_s)\, ds\right]}{\exp\left[-\int_{T_m}^t \lambda_m(x_s)\, ds\right] - \overline{S}_m} \tag{3.75}$$

Now, consider the value of the option, $V_i^m(x_t, R_{mt}, S_{mt})$. This may be written down as the sum of (1) the probability of facing an agent of type $j < m$ agent multiplied by the payoff if that is the case, plus (2) the weighted-probability that the opponent is of type m. Since the stopping time for the winner's payoff is generated by a point process, $\tilde{\lambda}_m$, this latter term must be conditioned on the fact that exit has not taken place up until a given time multiplied by the probability of an exit in the next time increment. The payoff for the former term is simply V_i^{m-1} from the recursive nature of the game. So the real option value is:

$$\begin{aligned}
V_i^m(x_t, R_{mt}, S_{mt}) &= S_{mt} V_i^{m-1}(x_t, R_{m-1,t}, S_{m-1,t}) + (1 - S_{mt}) \left\{ V_i^{m-1}\left(x_t, R_{m-1,t}, S_{m-1,t}\right) \right. \\
&+ \mathrm{E}_t \left(\int_t^{T_m} \left(\hat{V}_i\left(x_\tau + \delta\right) - V_i^{m-1}\left(x_t, R_{m-1,t}, S_{m-1,t}\right) \right) \right. \\
&\qquad \left. \left. \tilde{\lambda}_{m\tau} \exp\left[- \int_t^\tau \left(r + \tilde{\lambda}_{ms} \right) ds \right] d\tau \right) \right\} \\
V_i^m(x_t, R_{mt}, S_{mt}) &= V_i^{m-1}\left(x_t, R_{m-1,t}, S_{m-1,t}\right) + (1 - S_{mt}) \times \\
&\mathrm{E}_t \left\{ \int_t^{T_m} \left(\hat{V}_1\left(x_\tau + \delta\right) - V_i^{m-1}\left(x_t, R_{m-1,t}, S_{m-1,t}\right) \right) \right. \\
&\qquad \left. \tilde{\lambda}_{m\tau} \exp\left[- \int_t^\tau \left(r + \tilde{\lambda}_{ms} \right) ds \right] d\tau \right\} \qquad (3.76)
\end{aligned}$$

The term $\tilde{\lambda}_{m\tau} \exp[-\int_t^\tau (r + \tilde{\lambda}_{ms}) ds]$ can be simplified:

$$\begin{aligned}
\tilde{\lambda}_{m\tau} \exp\left[- \int_t^\tau \left(r + \tilde{\lambda}_{ms} \right) ds \right] &= \left(\frac{\lambda_m\left(x_\tau\right) \exp\left[-r\left(\tau - t\right)\right]}{1 - S_{mt} \exp\left[\int_t^\tau \lambda_m\left(x_s\right) ds\right]} \right) \times \\
&\quad \exp\left[\int_t^\tau \frac{-\lambda_m\left(x_s\right) \exp\left[-\int_t^s \lambda_m\left(x_\nu\right) d\nu\right]}{\exp\left[-\int_t^s \lambda_m\left(x_\nu\right) d\nu\right] - S_{mt}} ds \right] \\
&= \left(\frac{\lambda_m\left(x_\tau\right) \exp\left[-r\left(\tau - t\right)\right]}{1 - S_{mt} \exp\left[\int_t^\tau \lambda_m\left(x_s\right) ds\right]} \right) \times \\
&\quad \exp\left\{ \left[\log\left(\exp\left[- \int_t^s \lambda_m\left(x_\nu\right) d\nu \right] - S_{mt} \right) \right]_{s=t}^{s=\tau} \right\} \\
&= \left(\frac{\lambda_m\left(x_\tau\right) \exp\left[-r\left(\tau - t\right)\right]}{1 - S_{mt} \exp\left[\int_t^\tau \lambda_m\left(x_s\right) ds\right]} \right) \times \\
&\quad \left(\frac{\exp\left[-\int_t^\tau \lambda_m\left(x_\nu\right) d\nu\right] - S_{mt}}{1 - S_{mt}} \right) \\
&= \left(\frac{\lambda_m\left(x_\tau\right) \exp\left[-r\left(\tau - t\right)\right]}{1 - S_{mt}} \right) \exp\left[- \int_t^\tau \lambda_m\left(x_\nu\right) d\nu \right]
\end{aligned}$$

Substituting this back into the expression for $V_i^m(x_t, R_{mt}, S_{mt})$ we obtain:

$$\begin{aligned} V_i^m\left(x_t, R_{mt}, S_{mt}\right) &= V_i^{m-1}\left(x_t, R_{m-1,t} S_{m-1,t}\right) + \\ & \mathrm{E}_t\left\{\int_t^{T_m}\left(\hat{V}_i\left(x_\tau+\delta\right)-V_i^{m-1}\left(x_\tau, R_{m-1,\tau}, S_{m-1,\tau}\right)\right)\right. \\ & \left.\lambda_m\left(x_\tau\right)\exp\left[-\int_t^\tau\left(r+\lambda_m\left(x_s\right)\right)ds\right]d\tau\right\} \end{aligned} \tag{3.77}$$

First note that $\hat{V}_i(x_\tau + \delta) > V_i^{m-1}(x_\tau, R_{m-1,\tau}, S_{m-1,\tau})$. The reason for this is that the former is the winner's value in the war of attrition game. Now since R_{mt} and S_{mt} can only increase with, for any time path of x_τ starting at x_t, $T_m - t$ cannot be higher if R_{mt} and S_{mt} are larger. This follows obviously from the explicit expressions for R_{mt} and S_{mt} in equations (3.67) and (3.68). Given this sample-path-by-sample-path result, the integral in equation (3.77) must be smaller when R_{mt} and S_{mt} are larger.

Thus, $\partial V_i^m/\partial S_m < 0$, and $\partial V_i^m/\partial R_m < 0$ and the more efficient option values are strictly decreasing in S_{mt} and R_{mt} over each sub-game. By altering the index $i \in \{1, m-1\}$ the decreasing nature of the option values applies for all sub-games. Since the last recursive option value is greater than or equal to γ (i.e. $V_i^i \geq \gamma$), we have that $V_i^m > \gamma$ for all $i < m$. Thus the hypothesis required is verified and this completes the proof. □

3.6.7 Proof of Proposition 10

Proof using the Differential Equation and Boundary Conditions

The standard solution to (3.38) is:

$$V_i\left(x, \check{x} | x_i\right) = \frac{x}{r-\mu} - \frac{w_i}{r} + B\left(\check{x}\right) x^\xi \tag{3.78}$$

Here the upper boundary condition has already been applied, thereby removing a term multiplied by x to a positive power. Applying condition, (3.39), we then obtain:

$$\check{x}^\xi \frac{dB\left(\check{x}\right)}{d\check{x}} = \frac{F_j'\left(\check{x}\right)}{F_j\left(\check{x}\right)}\left[\hat{V}_i\left(\check{x}+\delta\right) - \frac{\check{x}}{r-\mu} + \frac{w_i}{r} - B\left(\check{x}\right)\check{x}^\xi\right] \quad , \tag{3.79}$$

$$\frac{dB\left(\check{x}\right)}{d\check{x}} + \frac{F_j'\left(\check{x}\right)}{F_j\left(\check{x}\right)} B\left(\check{x}\right) = \frac{F_j'\left(\check{x}\right)}{F_j\left(\check{x}\right)}\left[\hat{V}_i\left(\check{x}+\delta\right) - \frac{\check{x}}{r-\mu} + \frac{w_i}{r}\right]\left(\frac{1}{\check{x}}\right)^\xi \quad , \tag{3.80}$$

$$F_j\left(\check{x}\right)\frac{dB\left(\check{x}\right)}{d\check{x}}+F_j'\left(\check{x}\right)B\left(\check{x}\right)=F_j'\left(\check{x}\right)\left[\hat{V}_i\left(\check{x}+\delta\right)-\frac{\check{x}}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{1}{\check{x}}\right)^{\xi}\quad , \tag{3.81}$$

$$\frac{\partial}{\partial\check{x}}\left[F_j\left(\check{x}\right)B\left(\check{x}\right)\right]=F_j'\left(\check{x}\right)\left[\hat{V}_i\left(\check{x}+\delta\right)-\frac{\check{x}}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{1}{\check{x}}\right)^{\xi}\quad , \tag{3.82}$$

$$B\left(\check{x}\right)=\frac{1}{F_j\left(\check{x}\right)}\int F_j'\left(\check{v}\right)\left[\hat{V}_i\left(v+\delta\right)-\frac{v}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{1}{v}\right)^{\xi}dv\quad . \tag{3.83}$$

Applying boundary condition (3.40) then yields

$$B\left(x_i\right)=\left[\gamma-\frac{x_i}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{1}{x_i}\right)^{\xi}\quad , \tag{3.84}$$

which is applied as a boundary condition to (3.83):

$$\begin{aligned}B\left(\check{x}\right)\quad &= \quad \frac{1}{F_j\left(\check{x}\right)}\int_{x_i}^{\check{x}}F_j'\left(\check{v}\right)\left[\hat{V}_i\left(v+\delta\right)-\frac{v}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{1}{v}\right)^{\xi}dv\\ &+\quad \frac{F_j\left(x_i\right)}{F_j(\check{x})}\left[\gamma-\frac{x_i}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{1}{x_i}\right)^{\xi}\quad . \qquad \Box\end{aligned} \tag{3.85}$$

Proof using a Conditioning Argument

Since the trigger x_i is the only strategic variable, the real option value can be written down as the sum of a weighted payoff corresponding to firm i winning the game as well as a payoff corresponding to losing the game. If rival firm j has a higher trigger than i, i will win, while it loses if the opposite is the case:

$$\begin{aligned}V_i\left(x,\check{x}|x_i\right)=\frac{x}{r-\mu}\quad &-\quad \frac{w_i}{r}+\Pr\left(x_i>x_j\right)\times\left[\gamma-\frac{x_i}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{x}{x_i}\right)^{\xi}\\ &+\quad \int_{v|x_i<x_j}F_j'\left(v|\check{x}\right)\left[\hat{V}_i\left(v+\delta\right)-\frac{v}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{x}{v}\right)^{\xi}dv,\end{aligned} \tag{3.86}$$

$$\begin{aligned}V_i\left(x,\check{x}|x_i\right)=\frac{x}{r-\mu}\quad &-\quad \frac{w_i}{r}+\frac{F_j\left(x_i\right)}{F_j\left(\check{x}\right)}\left[\gamma-\frac{x_i}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{x}{x_i}\right)^{\xi}\\ &+\quad \int_{\check{x}}^{x_i}\frac{F_j'\left(v\right)}{F_j\left(\check{x}\right)}\left[\hat{V}_i\left(v+\delta\right)-\frac{v}{r-\mu}+\frac{w_i}{r}\right]\left(\frac{x}{v}\right)^{\xi}dv.\ \Box\end{aligned} \tag{3.87}$$

Chapter 4

Default Hazards and the Term Structure of Credit Spreads in a Duopoly

4.1 Introduction

Industry competition can substantially influence the values of claims on firms. Declining performance in one firm may signal the industrial ascendancy of others, while the liquidation of one company may precipitate the acquisition of temporary monopoly power by another. Interactions of this kind are common in oligopolies and give rise to different incentives among debt and equity holders. While there has been much study of how capital structure decisions affect strategic behavior in product markets,[1] few studies have examined how competition affects capital structure decisions and the pricing of securities.

An exception to this is Lambrecht (2001) which investigates the order of bankruptcy, subsequent entry and debt exchange offers within a stochastic continuous-time duopoly model. While Lambrecht's study provides insight on how business failure and renegotiation are related to macro-economic variables as well as firm and industry characteristics, he devotes no attention to the behaviour of default premia in oligopolies. In this chapter, we analyse strategic behaviour in a duopoly model and study its impact on credit spreads. Our continuous-time structural model reconciles the two strands of the credit risk pricing literature and provides further theoretical justification for the existence of surprise credit events by implicitly modelling the default intensity.

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[1] See, for example, Brander and Lewis (1986) and Maksimovic (1988).

The recent literature on pricing risky debt has two principal strands. First, several authors have refined and extended the so-called structural models of corporate default first suggested by Merton (1974) and Black and Cox (1976).[2] Second, reduced form models for pricing risky debt have been developed which may be fitted directly to risky bond prices but have no very obvious link to the borrower's financial position.[3] An important difference between structural and reduced-form models is the way in which they assume that default is triggered. In structural models, bankruptcy occurs when the firm's underlying asset value crosses a threshold.[4] By contrast, in reduced form models, borrowers may in principle jump into a default state at any time. The likelihood that they will is described by a hazard of default. An advantage of models which permit jumps into default is that they can explain the fact that credit spreads on very short-term bonds appear to be strictly positive. The liquidity spreads found in bond markets mean that it is hard to judge whether this is so for high credit quality short-term debt but it is almost certainly true for lower quality credit exposures.

In our model, we obtain endogenous default intensities, that are functions of firm and industry characteristics. The default intensities are the randomised strategies of a pair of equity holders in a non-cooperative Nash equilibrium.[5] The basic intuition for the bankruptcy game between equity holders is as follows. Default occurs when equity holders decide to cease injecting capital to meet debt service payments, and payoffs are structured so that the last to exit is relatively better off. The trade-off between winning the higher payoff and of waiting inefficiently, cause the equity holders to randomise their default decision through a conditionally Poisson process, as in reduced-form models. Equity holders' randomised default strategies in turn affect debt values and the corresponding credit spreads. The randomised strategies are equivalent to default hazards and so our model resembles other hybrid structural models.

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[2] See Brennan and Schwartz (1978), Longstaff and Schwartz (1995), Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) amongst other contributions.

[3] See, amongst others, Litterman and Iben (1991), Jarrow, Lando, and Turnbull (1997), Jarrow and Turnbull (1995), and Duffie and Singleton (1999).

[4] If the asset value follows a diffusion process, the probability of default in the next instant of time is either zero (if asset values are a discrete distance from the default threshold) or one (if the threshold is reached).

[5] The randomised strategies are very much like the exit hazards in the previous chapter, and our model, thus, results in a mixed-strategy equilibrium.

Recently, several other studies have sought to reconcile the two branches of the literature on corporate debt by showing circumstances in which structural models generate default hazards similar to those which arise in reduced-form models. The simplest approach is to include jump components in the diffusion process driving firm assets within a structural model. See Zhou (1997), El Jahel (1999) and Cathcart and El-Jahel (1998).[6] This approach also leads to hybrid models that can explain the empirical fact of positive short credit spreads but which are not entirely compatible with the reduced form literature. Madan and Unal (2000) have refined this approach by assuming that jumps are triggered by cash-shortages in non-interest-rate sensitive components of the firm's assets.

Another approach is to suppose that the firm's asset value within a structural model is imperfectly observed by the market. Again, bankruptcy is triggered when asset values cross a threshold but the bankruptcy event will be a surprise for investors and debt values will jump. This idea has been explored by Duffie and Lando (2000). They show that the first hitting time of an imperfectly-observed Brownian motion behaves like the first jump time of a Poisson process. Hence, such imperfect information generates default hazards similar to those proposed in the reduced-form valuation literature.[7] Finally, Cao and Wei (2000) have considered the behaviour of credit spreads when the indebted firm has short positions in vulnerable options. The presence of these extra corporate liabilities generates positive short term credit spreads.

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[6] In contrast, Cathcart and El-Jahel (1998) introduce a conditionally Poisson signalling process. The jumps precipitate default but are not linked to firm asset values in any direct way.

[7] Lambrecht and Perraudin (1996) also incorporated incomplete information into a structural model of risky debt, thereby generating "surprise defaults". However, in their model the quantity that is imperfectly observed by investors is the trigger level for bankruptcy (i.e., a random variable) rather than the firm's asset value (i.e., a stochastic process). The structure of Bayesian up-dating that this implies meant that surprise bankruptcies can only occur when the state variable hits new lows and thus the hazard of instantaneous default is either zero or infinite.

In most of these approaches, the default intensity generating jumps is either exogenously specified or implied directly by assumptions about effects of different kinds.[8] So the link between default hazards and the firm's capital structure and macro-economic variables is still relatively weak. Our model has the advantage of yielding endogenous default hazards that are a function of the characteristics of the firm as well as reflecting the market environment in which it operates. The endogenous hazards we derive provide intuitions about the causes of random defaults in firms and have a number of appealing features that we discuss.

The structure of the chapter is as follows. Section 4.2 models the hazard rate in a duopoly in which firms issue perpetual debt. Section 4.3 discusses the structure of these hazards. Section 4.4 studies the impact of the default hazards on credit spreads and demonstrates the main result of the chapter: positive short credit spreads. Section 4.5 generalises the hazard rate to asymmetric settings with incomplete information and considers some extensions. Section 4.6 concludes and the various proofs of propositions are consolidated in section 4.7.

4.2 Default Intensities in a Duopoly

4.2.1 Equity and Debt Values in a Monopoly

Suppose that all agents are risk-neutral and there is a constant interest rate, r. Consider a firm that enters a product market by investing a fixed start-up amount, K. Equity holders have a maximum sum $J < K$ to invest in the firm. To fund the difference, $K - J$, they issue perpetual debt that pays a continuous coupon, c. Following Mella-Barral and Perraudin (1997), we suppose that the firm's total profit flow is $x_t - w$, where w is a constant continuous flow cost and x_t is a geometric Brownian motion

$$dx_t = \mu x_t dt + \sigma x_t dB_t \tag{4.1}$$

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[8] Zhou (1997), El Jahel (1999) and Cathcart and El-Jahel (1998) directly specify jump processes that generate defaults. In Duffie and Lando (2000), the conditional distribution of the asset value is exogenous, while in Madan and Unal (2000) the distribution of cash shortages is exogenously given.

with constant volatility and drift parameters σ and $\mu < r$.[9] The net income flow to equity holders is, therefore, $x_t - w - c$, while bond holders receive c.

Application of Ito's Lemma and financial market equilibrium with risk-neutral agents imply that the *monopoly*[10] equity value, $\hat{V}(x)$, and debt value, $\hat{D}(x)$, satisfy

$$\begin{aligned} r\hat{V}(x) &= x - w - c + \frac{\sigma^2 x^2}{2}\frac{d^2\hat{V}(x)}{dx^2} + \mu x \frac{d\hat{V}(x)}{dx}\ , && (4.2) \\ r\hat{D}(x) &= c + \frac{\sigma^2 x^2}{2}\frac{d^2\hat{D}(x)}{dx^2} + \mu x \frac{d\hat{D}(x)}{dx}\ . && (4.3) \end{aligned}$$

As x_t tends to infinity, the likelihood of default diminishes and so the equity approaches the expected value of discounted net earnings: $\lim_{x_t \to \infty} \hat{V}(x_t) = x_t/(r-\mu) - (w+c)/r$,[11] while the debt value approaches its riskless value, i.e., $\lim_{x_t \to \infty} \hat{D}(x_t) = c/r$.

Boundary conditions for low levels of x_t are generated by what happens in the event of default. There is now considerable evidence of deviations from absolute priority in the allocation of firm value between stake holders during bankruptcy, especially in the case of Chapter 11 bankruptcies in the US. The extensive powers given to management in the Chapter 11 process and their ability to delay legal proceedings[12] allow equity holders to extract value in bankruptcy settlements.[13] Consistent with this evidence, we suppose that in the event of bankruptcy equity holders extract a constant value γ_E. Thus, $\hat{V}(\hat{x}_b) = \gamma_E$, at the default trigger, $\hat{x}_b$. Furthermore, we assume that bankruptcy involves dead weight administrative and legal costs equal to a fraction, ϕ, of total firm value, $\hat{W}(x)$. In bankruptcy, debt holders therefore obtain

$$\hat{D}(\hat{x}_b) = (1-\phi)\,\hat{W}(\hat{x}_b) - \gamma_E,$$

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[9] There is no difference between constructing a model in which the state variable is an earnings flow or assuming that the state variable is the unlimited liability value of the firm's underlying assets since it is straightforward to show that, given our assumptions, assets are linearly related to earnings.

[10] Throughout this chapter we use hatted functions and variables to denote values and parameters relating to the non-strategic, monopoly case.

[11] Specifically:

$$\lim_{x\to\infty} \hat{V}(x_t) = \mathrm{E}_t\left[\int_t^{\infty} (x_s - w - c)\exp\left[-r\,(s-t)\right] ds\right].$$

[12] See Franks and Torous (1989) and Brown (1989).

[13] Eberhart, Moore, and Roenfeldt (1990) find that on average equity holders receive 7 percent of firm value in Chapter 11.

where the total firm value is equal to the sum of an unlimited liability claim to the firm's entire income stream plus an exit option to liquidate the firm: $\hat{W}(x_t) = x_t/(r-\mu)-w/r+[\gamma - \hat{x}/(r-\mu)+w/r]\,(x_t/\hat{x})^{\xi}$. Here, γ is the liquidation value, ξ is a negative constant and $\hat{x}$ is the liquidation point at which the firm value satisfies value-matching and smooth-pasting conditions for optimal liquidation of the pure-equity firm.

Finally, we assume, as in Leland (1994) and Mella-Barral and Perraudin (1997) that there are no net worth covenants on the debt. This implies that bankruptcy occurs when equity holders decide to cease injecting capital. The bankruptcy trigger, $\hat{x}_b$, is therefore determined so as to maximise the equity value. This implies the smooth-pasting (optimality) condition $\hat{V}'(\hat{x}_b) = 0$ where $\hat{x}_b$ ($> \hat{x}$) is the trigger level for bankruptcy.

Standard methods imply that debt and equity values in the simple monopoly case are as follows.

Proposition 11 *The values of a monopoly firm's equity, $\hat{V}(x_t) = \hat{V}(x_t; w)$, and debt, $\hat{D}(x_t) = \hat{D}(x_t; w)$, prior to bankruptcy are*

$$\begin{aligned}
\hat{V}\left(x_t; w\right) &= \frac{x_t}{r-\mu} - \frac{w+c}{r} + \left[\gamma_E - \frac{\hat{x}_b}{r-\mu} + \frac{w+c}{r}\right]\left(\frac{x_t}{\hat{x}_b}\right)^{\xi}, && (4.4)\\
\hat{D}\left(x_t; w\right) &= \frac{c}{r} + \left[(1-\phi)\hat{W}(\hat{x}_b) - \gamma_E - \frac{c}{r}\right]\left(\frac{x_t}{\hat{x}_b}\right)^{\xi}, && (4.5)
\end{aligned}$$

for $x_t \in [\hat{x}_b, \infty)$. The trigger point for bankruptcy is

$$\hat{x}_b = \hat{x}_b\left(w\right) = \frac{\xi}{\xi-1}\left(\gamma_E + \frac{w+c}{r}\right)[r-\mu] \qquad (4.6)$$

and $\xi \equiv \left(-(\mu-\sigma^2/2) - \sqrt{(\mu-\sigma^2/2)^2 + 2\sigma^2 r}\right)/\sigma^2$.

4.2.2 Pure and Mixed Strategy Equilibria in a Duopoly

Now, consider the strategic interaction between two identical levered firms. Suppose that if one firm exits first, the other obtains some monopoly power. The term "monopoly power" should not be interpreted in the literal sense as reorganisation often involves the firm being subsequently run as an impaired pure-equity operation by creditors. In many cases, however, the reorganised firm ends up being liquidated or partially dismantled by creditors (see Franks and Torous (1989)). Specifically, we assume that when one firm exits, the earnings flow variable, x_t, obtained by the remaining firm jumps up by a fixed amount Δ and subsequently evolves according to the same geometric Brownian motion as in equation (4.1) starting from the new higher level.

The prospect of acquiring monopoly power gives each firm an incentive to outwait its competitor, delaying the decision of equity holders in a financially distressed firm to cease injecting capital. Counter-balancing this incentive, equity holders must inject capital to avoid bankruptcy. The longer the firm waits, the greater the costs incurred. The model therefore resembles a war of attrition as in chapter 3.

A study which closely resembles our own in that it focuses on levered firms in a stochastic duopoly model is Lambrecht (2001). In the duopoly he examines, Lambrecht (2001) shows that when firms are identical, there exist two subgame perfect, pure strategy Nash equilibria. These consist of the losing firm exiting first at the trigger which would be optimal for a monopolist. Lambrecht's analysis implies interesting results on the order of firms' departure from industries and relates these to the firms' "fitness" and "fatness", as discussed in the empirical study by Zingales (1998).

Although we shall not focus on them, there *are* asymmetric pure strategy equilibria in our model like those examined by Lambrecht.[14] Under this solution concept, default is triggered when the state variable, x_t, reaches a lower threshold.[15] A striking feature of the pure strategy equilibria however is their extreme asymmetry. Although firms are ex ante identical, one firm extracts the entire "surplus" on offer in the game. Experimental evidence suggests that game-playing agents are often reluctant to accept severely asymmetric allocations. Much of this evidence[16] is in the context of Nash bargaining, but the results have significance for game theory in general.

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[14] In chapter 3 the corresponding symmetric equilibria were discussed in proposition 6.

[15] The expressions implied by these equilibria will bear much similarity to those in proposition 6 for real options.

[16] For example, Weg, Rapoport, and Felsenthal (1990) experimentally test different bargaining outcomes, when players make alternating offers over an infinite horizon with discounting. They reject the hypothesis that players prefer sub-game perfect equilibria (SPE solution) and accept the hypothesis that agents prefer alternative 'focal points', such as a 'split-the-difference' (STD) rule, where the surplus is divided evenly between the two players. Ochs and Roth (1989) find similar

A second and perhaps more serious disadvantage with the pure strategy equilibria is that the debt values are greater than or equal to those one would observe in a monopoly.[17] This means that the corresponding default premia are smaller than those in the monopoly case. A common criticism of structural models is the small size of the default premia when they are parameterised in a plausible way.[18]

A significant contribution of this chapter is that it introduces a class of randomised strategies and solves for a symmetric equilibrium in which bankruptcy occurs at the first jump time of a point process with rate of jump, λ_t.[19] Since λ_t will turn out to be a function of the contemporaneous levels of the state variables, the random bankruptcy process becomes a conditionally Poisson process. This is important since it means that the pricing expressions in our model resemble reduced-form models for valuing defaultable debt of the kind developed by Duffie and Singleton (1999).

4.2.3 Claim Values and Default Intensities in a Duopoly

As in the monopoly described above, we consider two identical firms that issue infinite maturity debt. We suppose that the two sets of equity holders precipitate bankruptcy when they decide to cease payments to creditors. Since the firms randomise their default decisions, there will be extra terms in the differential equation representing the probabilities of default by one of the two firms in the duopoly. In a time increment δt, the probability of default with a conditionally Poisson process, λ_t, equals $\lambda_t \delta t$. As the two firms are identical, it is natural to look for a symmetric Nash equilibrium. By

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results in their experiment.

[17] With identical firms, one firm exits non-strategically while the other reaps the rewards of monopoly power. The debt value of the first firm is the same as the monopoly value and the second firm's debt value is clearly larger.

[18] Jones, Mason, and Rosenfeld (1984) is a standard reference for this problem. Using Merton (1974)'s structural model, they found that credit spreads were consistently underestimated.

[19] This is similar to the mixed-strategy equilibrium of the previous chapter. As in that chapter, our randomised equilibrium is Pareto-inefficient. The inefficiency arises purely from the fact that in a pure-strategy equilibrium the winner is known ex ante, resulting in a larger claim value for the winner. Conversely, in the mixed-strategy equilibrium both equity values are the same and equal to that of the loser in the pure-strategy equilibrium. An important implication of this is that the effect of the inefficiency only applies to the claim value of the winner.

financial market equilibrium with risk-neutral agents, the return on safe bonds must equal the net income to equity holders plus the capital gains and the probability-weighted payoffs that arise when one or other firm defaults. Applying the generalised form of Ito's lemma for jump-diffusions, one obtains a differential equation for each of the duopoly equity values, $V(x)$:

$$rV = x - w - c + \mu x \frac{dV}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2V}{dx^2} + \max_{\lambda \geq 0} \{\lambda [\gamma_E - V]\} + \lambda \left[\hat{V}(x + \Delta) - V\right]. \tag{4.7}$$

The main difference between the above Hamilton-Jacobi-Bellman equation and the equation we encountered in the monopoly case, (equation (4.2)), is the presence of two payoffs, representing the gain to either firm of randomly "losing" or "winning" the game. If a firm loses the game, it defaults first and its equity value jumps by an amount $\gamma_E - V$. If it wins the game, the other exits and the equity of the remaining firm jumps by $\hat{V}(x + \Delta) - V$. Since each set of equity holders only control their own default decision, maximisation operators appear only on the terms that result from this decision (i.e. $\lambda [\gamma_E - V(x)]$).

Two facts influence the equilibrium:

1. Since the equity holders can at any time decide to default receiving γ_E, the absence of arbitrage implies that $V(x) \geq \gamma_E$.
2. To maximise their value, equity holders choose their (non-negative) hazard rate $\lambda(x)$ to maximise $\lambda(x)(\gamma_E - V(x))$, taking the other firm's randomisation, $\lambda(x)$, as given.

These two facts imply that $\lambda(x) = 0$ if $V(x) > \gamma_E$ (as any other positive hazard would leave the term $[\gamma_E - V]$, which is in control of the equity holders, negative) and $\lambda(x) \geq 0$ only if $V(x) = \gamma_E$. By substituting the solution $V(x) = \gamma_E$, into the HJB equation, however, the hazard (in this case the other firm's response) must satisfy:

$$r\gamma_E + w + c - x = \lambda \left[\hat{V}(x + \Delta) - \gamma_E\right]. \tag{4.8}$$

Thus, the equity holders are compensated for their inefficient waiting, where $V(x) = \gamma_E$, by the possibility that the other firm defaults, with the randomisation rate, λ, given in (4.8).

One may distinguish between an interval over which $\lambda(x) = 0$ and an interval over which $\lambda(x) > 0$. By symmetry, the two intervals will be the same for the two firms. For x less than some level, x^*, $V(x) = \gamma_E$ and $\lambda(x)$ is given by equation (4.8). For $x \geq x^*$, $V(x) > \gamma_E$ and $\lambda(x) = 0$.

To derive the equity value, we solve equation (4.7) for $x > x^*$ imposing similar unlimited liability boundary conditions as in the monopoly case and value-matching and smooth-pasting conditions at x^*. This is simple because with a zero hazard the equation is just the monopoly differential equation (4.2). Effectively, one may think of the equity holders as deciding on the switching point x^*, at which they start randomizing.

As in the case of the equity, the debt values are influenced by the possibility that either firm may default. The value of debt must satisfy an analogous equation to (4.3), with the addition of two probability-weighted payoffs, corresponding to the impact on bond holders' claim values when their equity holders "win" or "lose" the game. By financial market equilibrium and Ito's Lemma, the value of debt must satisfy the differential equation:

$$rD = c + \mu x \frac{dD}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2 D}{dx^2} + \lambda \left[(1 - \phi) \hat{W}(x) - \gamma_E + \hat{D}(x + \Delta) - 2D \right] \qquad (4.9)$$

As x_t tends to infinity the risk of default disappears, so the debt must equal its riskless value (i.e. $\lim_{x_t \to \infty} D(x_t) = c/r$). The lower boundary condition is obtained by noting that in the limit as the earnings tend to $x^* - \Delta$ both agents exit with hazards tending to infinity. The reason for this is that the equity holders stand to gain nothing from waiting further in this limit (since $\hat{V}(x^* - \Delta + \Delta) = \gamma_E$ and so the hazard in equation (4.8) tends to infinity).

We thus arrive at the following important result concerning the duopoly default intensity[20] and claim values:

Proposition 12 *Under the assumptions of this section, the value of each firm's equity in a complete information, feedback, Nash equilibrium with randomised strategies,*

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[20] Throughout this chapter we will use the terms 'hazards', 'randomizing strategies', 'strategies' and 'default intensities' interchangeably for $\lambda(x_t)$. It is important to note, however, that $\lambda(x_t)$ is not a standard hazard, as in reduced form models. It is both a default intensity and an "intensity" of a sudden upwards discontinuous jump Δ in the earnings x_t. Both of these effects must be incorporated into other pricing expressions.

prior to bankruptcy of either firm is

$$V\left(x\right)=V\left(x;w\right)=\begin{cases}\gamma_E & \text{for } x\in\left(\hat{x}_b-\Delta,\hat{x}_b\right]\\ \hat{V}(x) & \text{for } x\in\left(\hat{x}_b,\infty\right)\end{cases}. \tag{4.10}$$

The default hazard rate is

$$\lambda\left(x\right)=\lambda\left(x;w\right)=\begin{cases}\left(r\gamma_E+w+c-x\right)/(\hat{V}\left(x+\Delta\right)-\gamma_E) & \text{for } x\in\left(\hat{x}_b-\Delta,\hat{x}_b\right],\\ 0 & \text{for } x\in\left(\hat{x}_b,\infty\right).\end{cases} \tag{4.11}$$

The corresponding duopoly debt value is given by the solution to equation (4.9) with the following boundary conditions:

$$\lim_{x\to\infty}D(x)=c/r,$$

$$\lim_{x\downarrow\hat{x}_b-\Delta}D(x)=\frac{1}{2}\left[(1-\Phi)\left(\hat{W}\left(\hat{x}_b-\Delta\right)+\hat{W}\left(\hat{x}_b\right)\right)\right]-\gamma_E.$$

Since these result bear great similarity to proposition 7, that relates to real options, we omit details of the proof.

The hazards shown in equation (4.11) have several interesting properties. First, equity holders will not default at a point higher than their non-strategic trigger $\hat{x}_b$. The reason for this is that the equity holders can always obtain the monopoly equity value, by exiting at $\hat{x}_b$. Second, neither agent will default at an income value equal to or lower than $\hat{x}_b-\Delta$. In the limit as $x\downarrow\hat{x}_b^d-\Delta$ each group of equity holders becomes indifferent between foreclosing first or second since either leads to a post-exit payoff of γ_E.

Figure 4.1 illustrates the monopoly and duopoly security values and the associated hazard. In all the numerical calculations of this section, the same baseline parameters are used. The short rate is set at 6 percent, which is the approximate rate in the U.S, while the drift is 0. The continuous flow cost is set at 0.15 and the coupon rate at 0.3. The liquidation value of the firm in the pure-equity case, γ, is set equal to 2, while the deviation from absolute priority was 0.2, which approximately equals 7 percent[21]

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[21] This is the average value calculated by Eberhart, Moore, and Roenfeldt (1990) in their empirical study of Chapter 11 reorganisations.

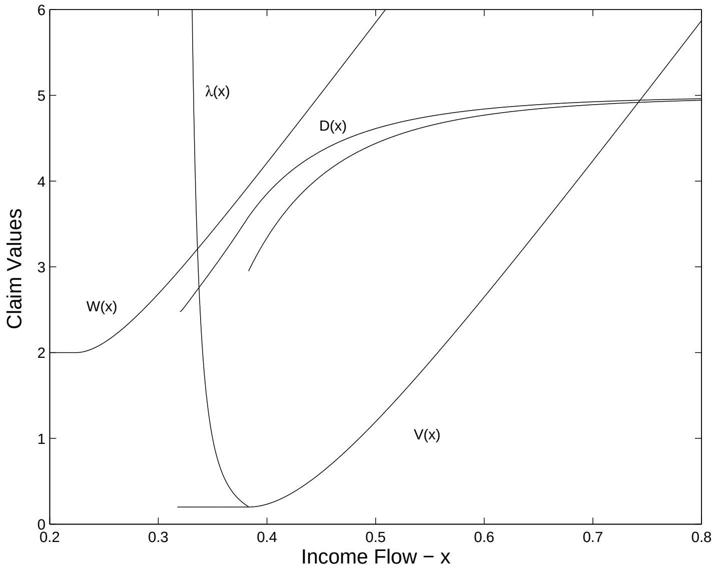

Figure 4.1: Monopoly and Duopoly Security Values and the Default Hazard.
Base case parameters as described in the text were used for all figures in this section.

of $\hat{W}(x)$ over the randomizing interval. The volatility of the earnings process, σ, is set in such a way that the firm value's volatility over the interval, $(x_b^d, x_b^d + \Delta]$, approximately equals 15 percent. We thus used an earnings volatility of 8 percent. Finally, the monopoly jump was set in the base case to a modest level of 0.066 (less than half the flow cost, w). This jump was chosen so that the predictable default point was well above the liquidation point in the pure-equity case. i.e. $x_b^d \gg \hat{x}$.[22] We also set the costs of bankruptcy, ϕ, to 0.2.

Figure 4.1 shows that the monopoly default trigger is just below an income flow of 0.4. This is the point at which the hazard first becomes non-zero as x_t decreases. The hazard tends to infinity as earnings approach the duopoly default trigger which is clearly well above the smooth-pasting firm liquidation trigger, $\hat{x}$.

---

[22] Formally, the relation $x_b^d > \hat{x}$, can be accomplished if:

$$\left(\frac{\xi}{\xi - 1}\right)(r - \mu)\left[\gamma_E + \frac{c}{r} - \gamma\right] > \Delta.$$

4.3 The Structure of the Default Hazards

4.3.1 Features of the Hazards

If one examines Figure 4.1 and equation (4.11), several features of the default intensities stand out. First, unlike the default hazards that arise in other hybrid structural models, the default intensities in our model are entirely endogenous and reflect the following features of the firm and the environment in which it operates: (i) macroeconomic variables (through the interest rate, drift parameter as well as the volatility); (ii) the capital structure of the firm (through the coupon rate); (iii) the interaction of real investment decisions with debt (through the continuous flow cost w of the financial activity); (iv) shareholder incentives in deviations from the absolute priority rule (through γ_E); (v) oligopoly effects (through the size in monopoly jump, Δ).

Second, the hazards can assume very large values for some levels of the state variable, x_t. Indeed, when earnings approach $x_b^d = \hat{x}_b - \Delta$ the default hazard explodes in that $\lambda_t \to \infty$. In contrast, many reduced-form models, drawing from methods in fixed-income pricing, specify mean-reverting processes for the default intensity. It is very unlikely in such models that the default hazard will ever assume high values.

Third, the hazard rate is non-increasing in the income flow which implies a sensible negative correlation between the firm's financial well-being and the risk of default. Fourth, the recovery rate on the firm's debt in the event of default is random since bankruptcy may occur at any point in a discrete interval of state variable values.

4.3.2 Comparative Statics of the Default Hazard

The simplicity of the default hazards enables one to derive several comparative statics.

Proposition 13 *Given the default intensity $\lambda(x_t)$ calculated in proposition 12, the following relations apply:*

$$\begin{array}{llllll} (i) & \dfrac{\partial \lambda}{\partial x} \leq 0, & (ii) & \dfrac{\partial \lambda}{\partial \sigma} \leq 0, & (iii) & \dfrac{\partial \lambda}{\partial c} \geq 0, \\ (iv) & \dfrac{\partial \lambda}{\partial w} \geq 0, & (v) & \dfrac{\partial \lambda}{\partial \gamma_E} \geq 0, & (vi) & \dfrac{\partial \lambda}{\partial \Delta} \leq 0. \end{array} \tag{4.12}$$

Result (i) shows that the intuitively reasonable relation between short-maturity credit spreads (equal to the hazard) and firm profitability holds in our model. Result (ii) shows that higher volatility reduces the default hazard and consequently short-maturity spreads. The intuition here is that lower volatility reduces the equity value of the firm in the event that it wins the game and hence a larger hazard is required to maintain the two firms in equilibrium.

Results (iii) and (iv) show that increases in total firm costs either through greater debt service costs or greater flow costs are associated with a larger hazard and consequently greater short-maturity credit spreads. The intuition is that higher costs make it less attractive to equity-holders to maintain a firm in operation and so a larger hazard by the competitor firm and greater chance of a jump rise in equity values is required to maintain equilibrium. Note, moreover, that results (iii) and (iv) confirm the empirical finding of Zingales (1998) that the 'fattest' and 'fittest' firms, respectively, are most likely to survive market shake-outs.[23]

Results (v) and (vi) show the impact on the hazard of increases in γ_E (positive) and of Δ (negative). The intuition is that equity-holders require a greater hazard of exit by their competitor to maintain value equal to γ_E for a given cash-flow $x - c - w$ and a smaller hazard if the jump in cash flows the event of exit by the competitor, Δ, is larger.

Figures 4.2 and 4.3 illustrate other important comparative statics of the default hazards. The default hazard increases with the short rate (as can be seen in Figure 4.2) while the growth parameter of the income process, μ, has a dramatic effect on the hazards (see Figure 4.3). It is interesting to note that even with strict priority (i.e. $\gamma_E = 0$), default hazards are positive and substantial (see Figure 4.4). So our results are independent of the arguably ad hoc assumption of deviations from absolute priority.

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[23] Lambrecht (2001) finds analogous results in his model of pure strategy equilibria.

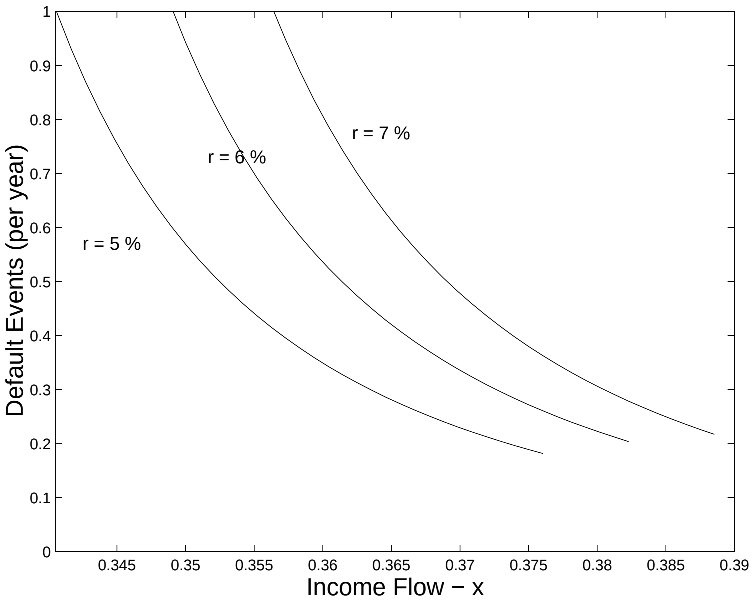

Figure 4.2: Interest Rate Effects on the Default Hazards.

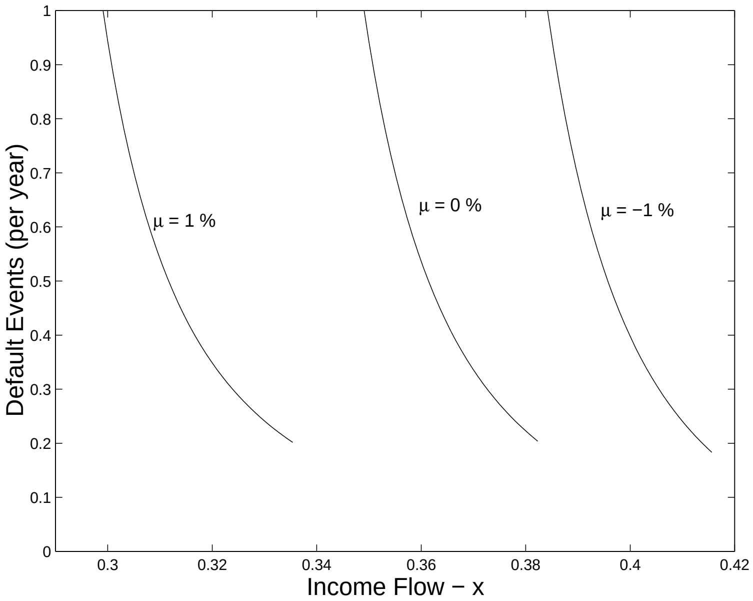

Figure 4.3: Drift Effects on the Default Hazards.

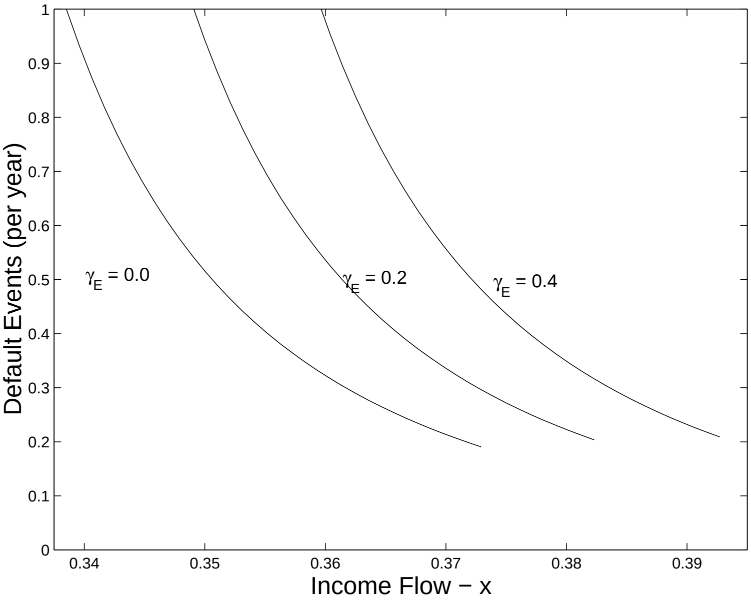

Figure 4.4: Deviations from Absolute Priority Effects on the Default Hazards.

4.4 The Term-Structure of Credit Spreads

4.4.1 Credit Spreads in a Duopoly

In this section, we study the impact of the default hazards on spreads of different maturities. For simplicity, we suppose that the firms' liabilities consist predominantly of infinite maturity debt like that described above but that they have issued a marginal amount of a pure discount bond. The discount bond issue is assumed to be so small that it does not affect the equilibrium hazard rate so we can concentrate on pricing it while avoiding the complications that arise if the firm has a complex, time-varying capital structure.

Let $D(t, x_t)$ denote a zero-coupon bond with a terminal maturity T. Suppose that in the event of bankruptcy, holders receive a fraction, $(1-\psi)$, of a riskless bond (i.e. recovery of treasury, see Jarrow and Turnbull (1995)) with the same maturity and contractual cash flow as the original zero-coupon bond. The value of the defaultable bond satisfies the following partial differential equation:

$$\frac{\partial D}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 D}{\partial x^2} + \mu x \frac{\partial D}{\partial x} + \lambda \left[(1-\psi)\, e^{-r[T-t]} + \hat{D}\,(t, x+\Delta) - 2D \right] = rD \quad (4.13)$$

with the final boundary condition, $D(T, x_T) = 1$ and the lower predictable boundary condition: $D(t, x_b^d) = (1 - \psi) \exp[-r(T - t)]$. As the earnings state variable tends to infinity, the prospect of bankruptcy diminishes and so the bond value tends to that of a riskless bond: $\lim_{x_t \to \infty} D(t, x_t) = \exp[-r(T - t)]$. Should the other firm default first, the firm's income flow will experience a jump Δ and the value of the bond will equal that of a bond in a structural model with no strategic-interaction, $\hat{D}(t, x_t)$.[24] The hazard in the differential equation (4.13) is given by equation (4.11) in proposition 12.

The credit spread of a zero-coupon bond equals the difference between its yield to maturity and that of a riskless bond:

$$CS(t, x_t) = -\frac{\log[D(t, x_t)]}{T - t} - r.$$

In Figure 4.5, we compare the credit spreads arising on the zero-coupon bond issued by a firm operating in a duopoly and in a monopoly.[25] Since the hazards are time-independent, the lower predictable bankruptcy triggers x_b^d and $\hat{x}_b$ are constant over time. We follow Longstaff and Schwartz (1995) in measuring credit spreads at ratios of the income flow with respect to these two points, $x_t/\hat{x}_b$ and $x_t/(\hat{x}_b - \Delta)$ monopoly and duopoly cases, respectively.[26] Clearly the effect of strategic behaviour is substantial with a strictly positive duopoly default premium at the short end, even for income flows well above the predictable bankruptcy trigger.

The plots in Figure 4.5 resemble the credit term structures reported by various empirical studies. For example, see Litterman and Iben (1991), Sarig and Warga

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[24] This debt value is given by the solution to the simpler partial differential equation:

$$\frac{\partial \hat{D}}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 \hat{D}}{\partial x^2} + \mu x \frac{\partial \hat{D}}{\partial x} = r\hat{D} \tag{4.14}$$

with final and upper boundary conditions similar to those of the duopoly case and the lower boundary condition: $\hat{D}(t, \hat{x}_b) = (1 - \psi) \exp[-r(T - t)]$.

[25] We used a fully explicit finite-difference scheme to discretize the PDE (4.13). In this scheme the income flow spacing, δx, was set such that $x_{max}/\delta x = 200$, where the maximum income flow in the model, $x_{max} = 0.7$. As for the time-step this was set such that $T/\delta t = 4000$, where the maturity $T = 10$. The other parameters we employed were: $r = 0.06$, $\mu = 0.0$, $\sigma = 0.08$, $\gamma = 0.5$, $\Delta = 20 \times \delta x$, $w = 0.08$, $c = 0.3$, $\phi = 0.3$, $\psi = 0.3$, $T = 10$, and $\gamma_E = 0.05$.

[26] Other measures could have been used. For example, we could have compared spreads at equal discrete earnings, δ, from the respective predictable bankruptcy points. The comparison would have been similar, however.

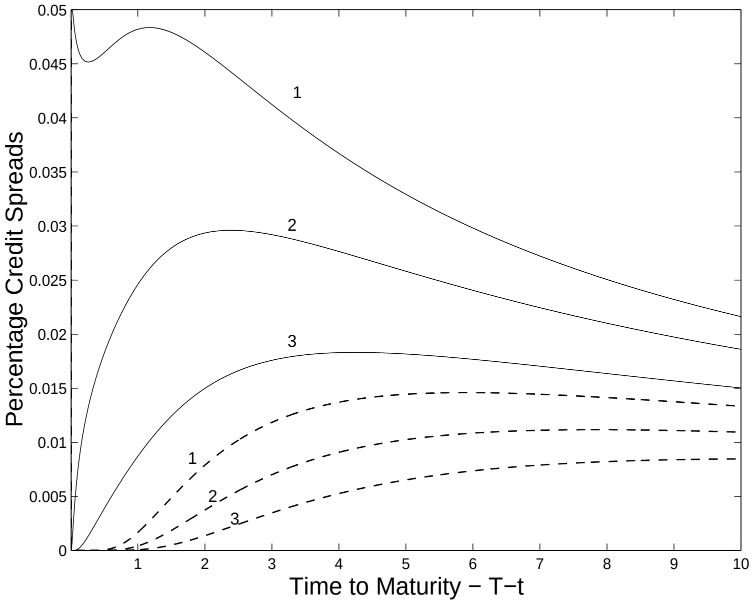

Figure 4.5: Monopoly and Duopoly Credit Spreads

Dotted lines indicate monopoly credit spreads while solid lines represent duopoly spreads. The different lines show spreads evaluated at different ratios between the state variable x_t and the predictable default points, $\hat{x}_b$ in the monopoly case and $\hat{x}_b - \Delta$ in the duopoly case. The labels '1', '2' and '3' refer to spreads evaluated at the ratios 1.25, 1.30 and 1.35, respectively.

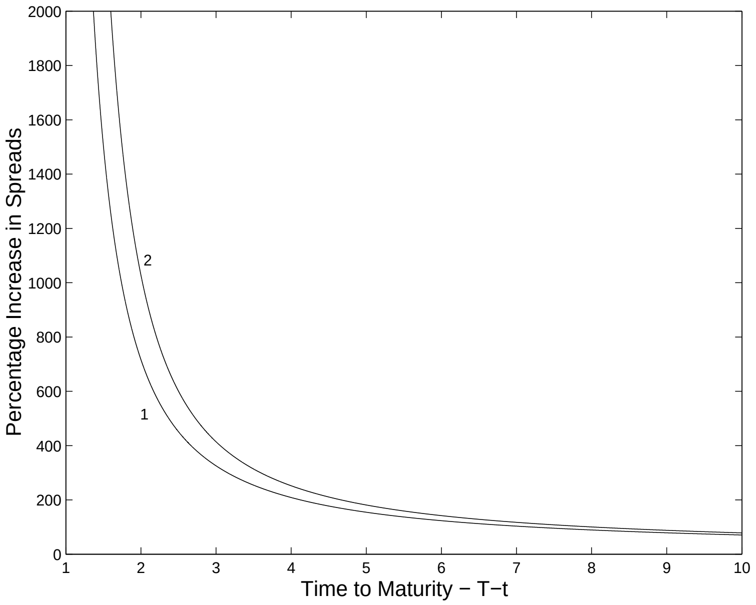

Figure 4.6: Duopoly to Monopoly Credit Spreads Ratios in Percent
These are taken at different ratios of the income flow from the predictable default point. The labels '1', '2', and '3' refer to income flow ratios 1.25, 1.30 and 1.35.

(1989) and Fons (1994). Few structural models are able to replicate the downward sloping term structures for low quality debt in their studies. Our results also help to resolve the problems with structural models documented by Jones, Mason, and Rosenfeld (1984) in that, as Figure 4.5 shows, spreads are higher for all maturities in the duopoly model. For further empirical evidence on the term structure of credit spreads see Duffee (1998) and Helwege and Turner (1999).

To measure the effect of strategic behaviour on the credit spreads, we define "a duopoly percentage increase in credit spreads over the monopoly case":

$$PIC\left(t,x_t\right)=100\times\frac{CS\left(t,x_t\right)-\hat{CS}\left(t,x_t\right)}{\hat{CS}\left(t,x_t\right)}$$

where $\hat{CS}\left(t,x_t\right)=-\log\left[\hat{D}\left(t,x_t\right)\right]/\left(T-t\right)-r$, is the analogous monopoly credit spread. In Figure 4.6, this quantity is plotted as a function of time to maturity again keeping the ratio between the state variable and the predictable default trigger constant in the monopoly and duopoly cases. Perhaps surprisingly, for maturities in excess of 1 year, the increase in spreads resulting from strategic behaviour tends to increase with credit quality. In other words, the impact of the default hazards is felt throughout a wide range of x_t values, even though the hazards are non-zero only in a limited lower interval, $(x_b^d, \hat{x}_b]$.

4.5 The Generalised Hazard Rate

4.5.1 Learning with Incomplete Information

An apparent shortcoming of the analysis of the previous sections is that the Nash equilibrium is a knife-edge case. As is true of mixed strategy equilibria in many other contexts, asymmetries in parameters across agents cause the equilibrium to break down. This would seem to limit the interest of the analysis. However, an equilibrium with multiple types may be sustained if there is incomplete information as in the previous chapter. One may also view the introduction of incomplete information as desirable as the model is then more realistic. In a duopoly, a significant risk for firms is that their conjectures about their rivals may be incorrect.

In this section, we generalise our model to include incomplete information over firms types. While this adds little to the more important implications of the previous sections, it answers the possible criticism that we are examining a knife-edge case. To be specific, we determine the default hazard when each firm in the duopoly may have flow costs equal to one of two levels, $w_1 < w_2$. Evidently, one could designate other parameters in the model as a source of incomplete information but it is natural to think that information on flow costs will be private to the firm.

We develop a Bayesian model in which equity holders act rationally and filter past events to revise their conjectures about each other's type. In our case, firms acquire new information from the fact that their competitor *has not so far defaulted.* We assume that each firm's prior at date 0 that its competitor is of the more efficient type (i.e., has costs w_1) is $\overline{P}$. For date $t > 0$, we denote the filtered probability that the other firm has costs w_1 as P_t. As in the previous chapter,[27] it may be easily shown that if the other has not so far defaulted, then in the period up to the other's default,

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[27] See the derivation of equation (3.13) in section 3.6.3.

P_t evolves over time according to the Riccati-type equation

$$\frac{dP_t}{dt} = P_t\left(1 - P_t\right)\left[\tilde{\lambda}_2\left(x_t, P_t\right) - \tilde{\lambda}_1\left(x_t, P_t\right)\right] \tag{4.15}$$

where, $\tilde{\lambda}_i$ are the default hazards of firms in this two-type environment. As a conditional expectation, P_t is, of course a martingale. One may show that the upward drift in P_t shown on the right hand side of equation (4.15) is compensated by the chance that the other firm will default, in which case P_t will jump to zero. Further note that the reason that the evolution of P_t prior to default has no diffusion term is a consequence of the fact that the new information that each firm acquires about its competitor in any instant of time $(t, t+\delta)$ comes not from the level of x_t but from the fact that it does not default in $(t, t+\delta)$ when the hazards $\tilde{\lambda}_2$ and $\tilde{\lambda}_1$ are known and $\tilde{\lambda}_2 > \tilde{\lambda}_1$.

4.5.2 Default Hazards with Incomplete Information

As in the perpetual debt duopoly model of Section 2, we look for a symmetric Nash equilibrium. The presence of incomplete information introduces a second state variable, P_t, that affects the pricing equations. Ito's lemma and financial market equilibrium imply that the value of equity in a firm, V_i, of type $i \in \{1, 2\}$ satisfies a Hamilton-Jacobi-Bellman partial differential equation[28]

$$\begin{aligned} rV_i &= \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_i}{\partial x^2} + \mu x \frac{\partial V_i}{\partial x} + P\left(1-P\right)\left[\tilde{\lambda}_2 - \tilde{\lambda}_1\right]\frac{\partial V_i}{\partial P} + x - w_i - c \\ &+ \max_{\tilde{\lambda}_i \geq 0}\left\{\tilde{\lambda}_i\left[\gamma_E - V_i\right]\right\} + \left[P\tilde{\lambda}_1 + \left(1-P\right)\tilde{\lambda}_2\right]\left[\hat{V}\left(x+\Delta; w_i\right) - V_i\right] \end{aligned} \tag{4.16}$$

As before, the i-th agent maximises terms involving their own hazard rate. The second term on the second line of (4.16) reflects the probability of winning the game for the i-th agent. Conditional on no defaults occurring up to time t, the opponent will default in the next instant with probability $P_t\tilde{\lambda}_1$ if it is of the more efficient type and with probability $(1-P_t)\tilde{\lambda}_2$, if it is of the less efficient type. The absence of arbitrage opportunities implies that $V_i \geq \gamma_E$, as the equity holders can always default. When the i-th firm's equity value strictly exceeds γ_E, equity holders have no incentive to default so $\tilde{\lambda}_i = 0$. Substituting from the no-arbitrage condition (i.e. $V_i = \gamma_E$) into the HJB partial differential equation reveals that the twin default intensities satisfy:

---

[28] Debt values satisfy similar equations but we omit the details since the focus of this section is default hazards with incomplete information.

$$\frac{r\gamma_E + w_i + c - x}{\hat{V}(x + \Delta; w_i) - \gamma_E} = P\tilde{\lambda}_1 + (1 - P)\tilde{\lambda}_2, \qquad i \in \{1, 2\}. \tag{4.17}$$

Equation (4.17) prescribes a linear system satisfied by the hazards. Since $w_1 < w_2$, and $\hat{V}(x + \Delta; w_1) > \hat{V}(x + \Delta; w_2)$, the system can only possibly yield solutions to the hazards if one of the hazards is equal to zero. This will be true whenever one of the equity values exceeds γ_E. It may be shown that the value of a firm of the efficient type is always greater than γ_E as long as there is uncertainty about its rival's type (i.e., $P_t < 1$). Hence, the equilibrium will involve randomisation by less efficient firms (if they are present) in some upper range of the earning process x_t until one exits or P_t equals unity. If $P_t = 1$ and both firms remain, then they behave as in the complete information equilibrium described before.

The incomplete information equilibrium is summarised in the following proposition.

Proposition 14 *Under the assumptions of this section, a symmetric, feedback, Bayesian Nash equilibrium consists of less efficient firms defaulting at the first jump time of conditionally Poisson processes with intensity:*

$$\tilde{\lambda}_1(x_t, P_t) = 0, \qquad P_t \in [\overline{P}, 1), \tag{4.18}$$

$$\tilde{\lambda}_2(x_t, P_t) = \lambda(x_t; w_2) / [1 - P_t], \qquad P_t \in [\overline{P}, 1). \tag{4.19}$$

When $P = 1$, the game reverts to the symmetric equilibrium of proposition 12, (with w replaced by w_1).

In equilibrium, the equity values are

$$V_2(x_t, P_t) = V(x_t; w_2) \tag{4.20}$$

and

$$\begin{aligned} V_1(x_t, P_t) \;=\; & V(x_t; w_1) + \mathrm{E}_t \left\{ \int_t^T \left(\hat{V}(x_\tau + \Delta; w_1) - V(x_\tau; w_1) \right) \right. \\ & \left. \lambda_2(x_\tau) \exp\left[-\int_t^\tau (r + \lambda_2(x_s))\, ds \right] d\tau \right\} \end{aligned} \tag{4.21}$$

where $\lambda_2(x_t) = \lambda(x_t; w_2)$ and $T = \inf\{s > 0 | P_s = 1\}$, is the first time that the conditional probability equals unity.

Once again, since this proposition bears much similarity to proposition 8, relating to real options, we omit details of the proof.

The equilibrium has a ‘type-filtering’ property, whereby more efficient types delay their randomizing until they are convinced that their opponent is also efficient. Meanwhile, the less efficient firm (if one is present), randomises its default decision with a hazard that increases as the state variable, x_t, declines and as the probability tends to unity. The dramatic effect of learning on the hazard, through P_t, can be seen in equation (4.19).

Substituting the hazards given in Proposition 14, into the Riccati equation for the probability, one obtains that the beliefs of the players evolve according to the simpler differential equation

$$\frac{dP_t}{dt} = P_t\lambda_2(x_t) \text{ which implies } P_t = \overline{P}\exp\left[\int_0^t \lambda_2(x_s)ds\right]. \tag{4.22}$$

The hazard, $\tilde{\lambda}_2$, therefore, equals

$$\tilde{\lambda}_2 = \frac{\lambda_2(x_t)}{1-\overline{P}\exp\left[\int_0^t \lambda_2(x_s)ds\right]} = -\frac{\partial}{\partial t}\left\{\log\left[\overline{P}-\exp\left(-\int_0^t \lambda_2(x_s)ds\right)\right]\right\}. \tag{4.23}$$

As one may see from equation (4.23), the incomplete information default hazards depend on the prior, $\overline{P}$, and the historical time path of the state variable, x_t. An interesting feature of the default intensity, as presented in equation (4.23), is that it is actually a time-derivative of a function of the probability,[29] and thus confirms the important relationship between learning and default events. Duffie and Lando (2000) also obtain default intensities that are first derivatives of a function. In their case the function is the conditional distribution of the firm’s assets and the derivative is with respect to the underlying firm value.

4.5.3 A Numerical Example

If both firms are of type w_1, the time τ at which they conclude that the other is of type w_1 is given implicitly by:

$$\overline{P}\exp\left[\int_0^\tau \lambda_2(x_s)\,ds\right] = 1. \tag{4.24}$$

---

[29] The argument of the logarithm in equation (4.23) can be expressed as $(P_t/\overline{P})/(P_t-1)$.

Using a Monte Carlo approach, we estimate the expected time, $\mathrm{E}_0(\tau)$, when the parameters are those of the baseline given in section 4.2, and assuming that $x_0 = 1/2(\hat{x}_b + \hat{x}_b^d)$ and $\overline{P} = 0.5$. With a time-step of 0.001 years and 80,000 simulations, $\mathrm{E}_0(\tau)$ was estimated to be 5.8 years.

Clearly, the time required to resolve incomplete information can be quite drawn out and indeed longer than the maturity of many corporate bonds. Note that the initial income flow, x_0, is at a value that is halfway over the randomizing interval. Since the firm is assumed to be solvent when it first enters the debt contract, x_0 will usually be well above the monopoly bankruptcy trigger, and the expected time $\mathrm{E}_0(\tau)$ will, therefore, be even greater.

4.5.4 Extensions

The incomplete information version of the model outlined in this section can be extended to include a discrete number $n > 2$ of types.[30] The results would be similar in that successively less efficient types would randomise on disjoint intervals of the state variables and we would obtain the same "type-filtering" property described above.

One may note that the assumption that there are no net-worth covenants and, hence, that equity holders decide the timing of bankruptcy is not a prerequisite for obtaining equilibria in which game theoretic default intensities occur. An alternative approach would be to consider a setting in which senior creditors decide when to liquidate the firm, at which point they obtain a constant value, γ_D. The prospect that the firm might become a monopolist whereupon defaultable debt values will jump up may induce the senior bond holders to delay their liquidation decision and, instead, randomise this decision through a conditionally Poisson point process. If the firms are symmetric, and the continuous coupon yield is replaced with a finite maturity principal payment, keeping all other parameters the same as in section 4.2, one may show that the symmetric default intensities in such an equilibrium are:

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[30] The main point of this section is to illustrate how asymmetries can be incorporated into the duopoly model. If there were more than 2 types, the results would be similar to those described in chapter 3.

$$\lambda\left(t,x\right)=\frac{r\gamma_D}{\hat{D}\left(t,x+\Delta\right)-\gamma_D}$$

where $\hat{D}(t,x)$ is the monopoly senior debt value. Thus, endogenous default hazards in firms, arising from strategic behaviour, can exist both in the presence and the absence of protective bond covenants.

So far we have not discussed the incentives that may result after one of the two firms becomes bankrupt. The winner will clearly reap the rewards of temporary monopoly power in the industry and if earnings recover to high levels, one might expect another firm to enter the monopoly, causing the equilibrium to revert to the one described in this section. The industry will, thus, go through a cycle alternating between duopoly and monopoly market structures. This intuition is important as it means that the default hazards will be present and influence claim values for a significant portion of the time in such an industry.

4.6 Conclusion

This chapter has examined the behaviour of credit spreads in a duopoly when the firms equity holders play a non-cooperative war of attrition game against each other. We show that there are Nash equilibria in which each firm defaults on the first jump time of a conditionally Poisson process, the jump rate of which is a function of the firm's earnings. Asymmetries in firm types may be introduced into the model by including incomplete information.

Using this framework, we demonstrate that surprise defaults may occur even in a complete information structural model of defaultable debt in which the underlying information is generated by a diffusion state variable. The fact that defaults may be a surprise in turn implies that our model can generate strictly positive short-maturity credit spreads for low credit quality bond issuers.

Our analysis advances attempts to reconcile structural models of debt valuation with the reduced form approach. The reduced form approach prices defaultable debt by specifying a hazard that the borrower will jump into default at different levels of a set of state variables. The structural model developed in this chapter yields endogenous hazard rates for default for bond issuers that depend on firm-specific parameters and variables describing the firm's profitability and its financial environment.

The hazard rates we obtain have interesting properties. For example, as the state variable for firm profitability approaches certain low levels, the hazards explode to infinity so default takes place for certain. Standard reduced-form models of defaultable debt valuation usually adopt mean-reverting hazard rate specifications similar to those employed in the default-free term structure literature. Our analysis suggests that hazard rate specifications should allow for discontinuous hazard rates that explode to infinity on some sample paths.

4.7 Proof of Propositions

4.7.1 Proof of Proposition 11

The proof of this is standard and is sketched in the text before the proposition with discussion of the boundary conditions on the ordinary differential equations. □

4.7.2 Proof of Proposition 13

We start with some important results that are required by the proofs:

Lemma: $\partial\xi/\partial\sigma > 0$.

The parameter ξ is the negative root of the fundamental equation:

$$\xi = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\sigma}{\sigma^2}}. \tag{4.25}$$

Differentiating this directly with respect to σ yields:

$$\frac{\partial\xi}{\partial\sigma} = \frac{2}{\sigma^3}\left\{\frac{\mu\sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\sigma}{\sigma^2}} + r + \mu\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)}{\sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\sigma}{\sigma^2}}}\right\}. \tag{4.26}$$

Note that in equation (4.26), the magnitude of the square root term in the numerator is always greater than $\mu/\sigma^2 - 1/2$, by Pythagoras. Thus, the both the numerator and denominator are always positive and $\partial\xi/\partial\sigma > 0$.

Lemma: $r\gamma_E + w + c - x > 0$ **for all** $x \in (x_b^d, \hat{x}_b]$.

It can be readily shown that the term: $r\gamma_E - x_t + w + c$ is always positive, by showing that its value at the maximum x_t (for which the hazard is non- zero. i.e. the smooth-pasting point, $\hat{x}_b$) is positive:

$$r\gamma_E - \hat{x}_b + w + c = [r\gamma_E + w + c]\left(1 - \left(\frac{\xi}{\xi - 1}\right)\left(\frac{r - \mu}{r}\right)\right) \tag{4.27}$$

The curved bracketed term in (4.27) is always positive for all values of σ, μ and r as $\xi/(\xi - 1) < 1$ and $(r - \mu)/r \leq 1$. □

Lemma: $(x_t + \Delta/\hat{x}_b)^{\xi} < 1$ **for all** $x \in (x_b^d, \hat{x}_b]$.

Over the interval mentioned $(x_t + \Delta/\hat{x}_b)$ is greater than one as the lower bound is $\hat{x}_b - \Delta + \Delta = \hat{x}_b$. By noticing that ξ is negative, the proof of this result is completed. □

Lemma: $\partial \hat{V}(x+\Delta)/\partial \xi > 0$.

By directly differentiating the real option value we obtain:

$$\begin{aligned} \frac{\partial \hat{V}\left(x+\Delta\right)}{\partial \xi} &= \log\left(\frac{x+\Delta}{\hat{x}_b}\right)\left[\gamma_E - \frac{\hat{x}_b}{r-\mu} + \frac{w+c}{r}\right]\left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi}, \\ &= \log\left(\frac{x+\Delta}{\hat{x}_b}\right)\left(\gamma_E + \frac{w+c}{r}\right)\left(\frac{-1}{\xi-1}\right)\left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi}, \\ &> 0. \end{aligned}$$

where we the second line was calculated using the expression for $\hat{x}_b$. □

For ease of exposition we re-write the hazard:

$$\lambda(x_t) = \frac{r\gamma_E + w + c - x_t}{\hat{V}\left(x_t+\Delta\right) - \gamma_E} \tag{4.28}$$

Part 1: $\partial \lambda / \partial x \leq 0$

Differentiating the hazard with respect to x, we have:

$$\frac{\partial \lambda(x_t)}{\partial x} = -\frac{\left[\hat{V}(x_t+\Delta) - \gamma_E\right] + \left(r\gamma_E - x_t + w + c\right)\hat{V}'\left(x+\Delta\right)}{\left[\hat{V}\left(x_t+\Delta\right) - \gamma_E\right]^2} \tag{4.29}$$

where,

$$\hat{V}'\left(x+\Delta\right) = \frac{1}{r-\mu} + \left(\frac{\xi}{x+\Delta}\right)\left[\gamma_E - \frac{\hat{x}_b}{r-\mu} + \frac{w+c}{r}\right]\left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi}$$

which after some rearrangement can be simplified to:

$$\hat{V}'(x+\Delta) = \frac{1}{r-\mu}\left[1 - \left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi-1}\right] \tag{4.30}$$

Clearly the expression in (4.30) is always positive using the second result above. The first result also implies that the other terms in the numerator of (4.29) are positive and this completes the proof. □

Part 2: $\partial\lambda/\partial\sigma \leq 0$

Differentiating the hazard with respect to σ, we have:

$$\frac{\partial\lambda(x_t)}{\partial\sigma} = -\underbrace{\frac{r\gamma_E - x + w + c}{\left[\hat{V}\left(x_t+\Delta\right) - \gamma_E\right]^2}}_{>0}\underbrace{\frac{\partial\hat{V}\left(x+\Delta\right)}{\partial\xi}}_{>0}\underbrace{\frac{\partial\xi}{\partial\sigma}}_{>0} < 0 \qquad \square \tag{4.31}$$

Parts 3 and 4: $\partial\lambda/\partial w \geq 0$ **and** $\partial\lambda/\partial c \geq 0$**:**

Differentiating with respect to w we obtain:

$$\frac{\partial\lambda(x_t)}{\partial w} = \frac{\left[\hat{V}(x_t+\Delta) - \gamma_E\right] - (r\gamma_E - x_t + w + c)\left[\frac{1}{r}\left(\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi} - 1\right)\right]}{\left[\hat{V}(x_t+\Delta) - \gamma_E\right]^2} \tag{4.32}$$

Using the second result, the last bracketed term in the numerator of (4.32) is negative and thus the product of this and $-(r\gamma_E + w + c - x)$ is positive. This completes the proof. □

The proof that $\partial\lambda/\partial c \geq 0$ is almost identical to the one presented here for the cost flow, w.

Part 5: $\partial\lambda/\partial\gamma_E \geq 0$**:**

Differentiating with respect to γ_E we obtain:

$$\frac{\partial\lambda(x_t)}{\partial\gamma_E} = \frac{r\left[\hat{V}(x_t+\Delta) - \gamma_E\right] - (r\gamma_E - x_t + w + c)\left[\frac{x_t+\Delta}{\hat{x}_b} - 1\right]}{\left[\hat{V}(x_t+\Delta) - \gamma_E\right]^2} \tag{4.33}$$

Using the second result, the last bracketed term in the numerator of (4.33) is negative and thus the product of this and $-(r\gamma_E + w + c - x)$ is positive. This completes the proof. □

Part 6: $\partial\lambda/\partial\Delta \leq 0$

Differentiating the hazard with respect to Δ:

$$\frac{\partial\lambda(x_t)}{\partial\Delta} = -\frac{(r\gamma_E - x_t + w + c)\left[\frac{1}{r-\mu} + \frac{\xi}{x_t+\Delta}\left[\gamma_E - \frac{\hat{x}_b}{r-\mu} + \frac{w+c}{r}\right]\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi}\right]}{\left[\hat{V}(x_t+\Delta) - \gamma_E\right]^2}$$

$$\frac{\partial \lambda(x_t)}{\partial \Delta} = -\frac{(r\gamma_E - x_t + w + c)\left[\frac{1}{r-\mu} + \frac{\xi}{x_t+\Delta}\left[\gamma_E + \frac{w+c}{r}\right]\left[1 - \frac{\xi}{\xi-1}\right]\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi}\right]}{\left[\hat{V}(x_t+\Delta) - \gamma_E\right]^2}$$

$$\frac{\partial \lambda(x_t)}{\partial \Delta} = -\frac{(r\gamma_E - x_t + w + c)\left[\frac{1}{r-\mu} - \frac{1}{x_t+\Delta}\left[\gamma_E + \frac{w+c}{r}\right]\left[\frac{-\xi}{\xi-1}\right]\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi}\right]}{\left[\hat{V}(x_t+\Delta) - \gamma_E\right]^2}$$

$$\frac{\partial \lambda(x_t)}{\partial \Delta} = -\frac{(r\gamma_E - x_t + w + c)\,\frac{1}{r-\mu}\left[1 - \left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi-1}\right]}{\left[\hat{V}(x_t+\Delta) - \gamma_E\right]^2} \tag{4.34}$$

Analyzing (4.34), it is evident that the first bracketed term in the numerator is always positive from the first result. The second bracketed term in the numerator of (4.34) is always also positive from the second result, and this completes the proof. □

Chapter 5

Conclusion

5.1 Summary of the Thesis

This thesis has developed models of strategic behaviour in continuous-time finance, relating to the pricing and exercise of convertible securities, corporate debt and real option valuation. The models are formally stopping games, involving discrete multilateral decisions and have largely focused on mixed-strategy equilibria.

With two players the action space consists of agents stopping according to conditionally Poisson point processes, while in the limit of an infinite number of agents, as is the case under perfect competition, the action set ceases to be defined by such point processes. Instead, the action space consists of intervals of inaction as well as an upper threshold of indifference in acting. An interesting common feature of the equilibria we study is that they are mostly symmetric. As we have discussed, the appeal of symmetric equilibria stems to a large extent from agents' unwillingness to accept extreme ex ante allocation of value.

The implications of these analyses are broad and can be classified into two main inter-linked categories. First, the techniques provide a theoretical framework within which one can price financial or real options whose value is influenced by the actions of various claim holders. In such instances the omission of strategic considerations in the contingent claims framework may result in inaccurate predictions. In this regard the last three chapters have illustrated how one can price corporate securities taking into account the influence of others' actions.

Second, the incorporation of such methods helps to reconcile theoretical and empirical results. In this regard, we summarize some of the key findings of the second and fourth chapters:

1. Perfect competition among convertible bond holders may explain the empirically observed gradual conversion of convertible bonds (see Mehta (1976)).
2. Strategic behaviour can help explain why firms default according to probabilistic intensities.
3. Following on from the strategic structural models of Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997), we find that strategic behaviour between firms provides further reconciliation between the reduced-form and structural models of credit risk.

5.2 Extensions

The chapter on convertible securities only examined the case of perfectly competitive agents. There are a number of other interesting cases that could be examined including: (i) the case of two or more convertible bond investors as well as (ii) a large bond holder holding a large proportion of the issue along with a continuum of small investors. Study of the former would help to understand incentives and strategic behaviour in instances where a small number of investors attempt to hoard the entire issue. Spatt and Sterbenz (1988) have considered this in the case of warrant financing. Study of the latter along with the monopoly case should inform the benefits of the extent of an investor's monopoly.

The second chapter has concentrated mainly on the optimality of bond conversion from the perspective of convertible bond holders and *not* management. Future research should address the optimality of firm call policy in a richer setting incorporating call announcement periods and, consequently, costs of failed forced-conversions (see Jaffee and Schleifer (1990)). Such a continuous-time setting may also incorporate reputation effects as in the separating equilibrium model of Harris and Raviv (1985).

The chapters on real options and corporate debt valuation could be extended to include other market structures than the duopoly considered. More importantly, it would be interesting to compare the shape of the credit spreads in the model of chapter 4 with those observed empirically when there is a diffusion process for the short rate. Such an exercise would then permit study of the correlation between spreads on Treasuries as well as defaultable bonds.

Appendix A

Numerical Methods for Solving the Differential Equations

A.1 Finite-Difference Scheme for Partial Differential Equations (3.21) and (3.31)

Differential equations (3.21) and (3.31) are quite similar and so we only consider the latter one, which is the more general case:

$$rV_i^m = x - w_i + \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_i^m}{\partial x^2} + \mu x \frac{\partial V_i^m}{\partial x} + \lambda_m R_m \frac{\partial V_i^m}{\partial R_m} + \lambda_m \left[\hat{V}_i(x+\delta) - V_i^m \right].$$

Since the product of probabilities, R_t is like a proxy for time (as this is strictly increasing in time), it is natural to treat this as a time-like variable and the equation as a backward-parabolic partial differential equation.

The first derivative with respect to x is approximated with a forward difference and the second derivative with respect to x with a central difference. Finally the first derivative with respect to R is a forward difference. This last specification ensures that the scheme is *fully-implicit* and not explicit. The reason for choosing this is that the derivative $\partial V_i^m / \partial R$ is only non-zero for the lower interval, $(\underline{\hat{x}}_m - \delta, \underline{\hat{x}}_m]$, because of the discontinuity in the hazard, λ_m. This means that an explicit scheme would not allow us to obtain the real option values recursively for $x > \underline{\hat{x}}_m$. By adopting a fully-implicit scheme and a successive over relaxation algorithm (SOR),[1] we make an initial approximation to the real option value and iteratively obtain converging better approximations until our specified tolerance level is reached. The finite-difference scheme which then results for an approximate option value, $V_i^m(x,R) \approx V_i^m(n\delta x + \underline{\hat{x}}_m - \delta, l\delta R + R_{m+1,T_{m+1}}) = V_i^m(n,l)$, is:

$$B^* V_i^m(n,l) = AV_i^m(n-1,l) + BV_i^m(n,l+1) + CV_i^m(n+1,l) + D, \tag{A.1}$$

where

$$\delta x = \frac{x_{max} - (\underline{\hat{x}}_m - \delta)}{N}, \qquad \delta R = \frac{R_{m,T_m} - \overline{R}_m}{L},$$

where $N+1$ and $L+1$ are the total number of x and R grid points, respectively and x_{max} is the maximum income flow under consideration. Recall that $\overline{R}_m$ is the prior product of the probabilities remaining at the start of the sub-game in which the m-th type is 'filtered out'.

In the finite-difference scheme (A.1) the coefficients are:

$$\begin{aligned} B^* &= r + \mu\left(n + \frac{x_{min}}{\delta x}\right) + \sigma^2\left(n + \frac{x_{min}}{\delta x}\right)^2 \\ &\qquad + \lambda_m\left(n\delta x + x_{min}\right)\left(1 + l + \frac{\overline{R}_m}{\delta R}\right), \qquad &(A.2) \\ A &= \frac{\sigma^2}{2}\left(n + \frac{x_{min}}{\delta x}\right)^2, &(A.3) \\ B &= \lambda_m\left(n\delta x + x_{min}\right)\left(l + \frac{\overline{R}_m}{\delta R}\right), &(A.4) \\ C &= \frac{\sigma^2}{2}\left(n + \frac{x_{min}}{\delta x}\right)^2 + \mu\left(n + \frac{x_{min}}{\delta x}\right), &(A.5) \\ D &= n\delta x + x_{min} - w_i + \lambda_m\left(n\delta x + x_{min}\right)\hat{V}_i\left(n\delta x + x_{min} + \delta\right) &(A.6) \end{aligned}$$

where $x_{min} = \underline{\hat{x}}_m - \delta$.

The relevant boundary conditions that must be applied to the numerical solution include:

$$\begin{aligned} V_i^m(N,l) &= \frac{N\delta x}{r-\mu} - \frac{w_i}{r}, &(A.7) \\ V_i^m(0,l) &= V_i^{m-1}(n^*, l^* = 0), &(A.8) \\ V_i^m(n,L) &= V_i^{m-1}(n^*, l^* = 0). &(A.9) \end{aligned}$$

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[1] For details see Smith (1985) or Richtmyer and Morton (1967).

In the above conditions, n^* and l^* refer to indices in the subsequent sub-game where,

$$\delta x^* = \frac{x_{max} - \left(\underline{\hat{x}}_{m-1} - \delta\right)}{N}, \qquad \delta R^* = \frac{R_{m-1,T_{m-1}} - \overline{R}_{m-1}}{L}.$$

The finite-difference scheme is solved recursively, starting with $m = 2$, as the $m = 1$ case is available in closed-form (see proposition 7). The overall prior beliefs in the game are determined from the priors, $\overline{R}_m$. By altering the size of this prior in each sub-game, the players place different weights on each of the types and, thus, influence the form of the discrete distribution of cost types.

A.2 Numerical Method for Solving the Ordinary Differential Equation (4.9)

Since this is a linear ordinary differential equation we solve it over two separate intervals:

$$D(x) = \begin{cases} \underline{D}(x), & x \in [\hat{x}_b - \Delta, \hat{x}_b), \\ \\ \overline{D}(x), & x \in [\hat{x}_b, \infty) \end{cases} \tag{A.10}$$

where

$$r\overline{D} = c + \frac{\sigma^2 x^2}{2}\frac{d^2\overline{D}}{dx^2} + \mu x \frac{d\overline{D}}{dx} \tag{A.11}$$

and $\underline{D}(x)$ satisfies the differential equation (4.9). The solution to the former is available in closed-form

$$\overline{D}(x) = \frac{c}{r} + \left[\underline{D}(\hat{x}_b) - \frac{c}{r}\right]\left(\frac{x}{\hat{x}_b}\right)^{\xi} \tag{A.12}$$

while the latter may be treated as a two point boundary value problem[2] with boundary conditions

$$\lim_{x \downarrow \hat{x}_b - \Delta} \underline{D}(x) = \frac{1}{2}\left[(1-\Phi)\left(\hat{W}(\hat{x}_b - \Delta) + \hat{W}(\hat{x}_b)\right)\right] - \gamma_E, \tag{A.13}$$

$$\underline{D}(\hat{x}_b) = K \tag{A.14}$$

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[2] For details on solving such problems, see for example Atkinson (1989).

where K is a constant. The continuity of the debt value implies the following smooth-pasting condition at $\hat{x}_b$

$$\frac{d\underline{D}\left(\hat{x}_b\right)}{dx} = \frac{\xi}{\hat{x}_b}\left[K - \frac{c}{r}\right]. \tag{A.15}$$

By iteration we vary the upper boundary condition, $\underline{D}(\hat{x}_b) = K$, until the smooth-pasting condition (A.15) is satisfied.

A.3 Finite-Difference Scheme for the Partial Differential Equation (4.13)

The differential equation (4.13) was discretised using a *fully-explicit* finite-difference scheme. This entails approximating the time derivative with a backward difference, the first derivative of the income flow with a forward difference and the second derivative of the income flow with a central difference. Under these circumstances the duopoly bond value, $D(t,x)$, may be approximated as $D(m\delta t, n\delta x) = D(m,n)$, with $\delta t = T/M$, and $\delta x = (x_{max} - x_b^d)/N$, where $N+1$ and $M+1$ are the number of income flow and time grid points respectively. The finite-difference scheme is then

$$D(m-1,n) = A_1 D(m,n-1) + A_2 D(m,n) + A_3 D(m,n+1) + A_4, \tag{A.16}$$

where

$$A_1 = \delta t\left\{\frac{\sigma^2}{2}\left(n + \frac{x_b^d}{\delta x}\right)^2\right\}, \tag{A.17}$$

$$A_2 = 1 - \delta t\left\{r + \frac{\sigma^2}{2}\left(n + \frac{x_b^d}{\delta x}\right)^2 + \mu\left(n + \frac{x_b^d}{\delta x}\right) + 2\lambda\left(n\delta x + x_b^d\right)\right\}, \tag{A.18}$$

$$A_3 = \delta t\left\{\frac{\sigma^2}{2}\left(n + \frac{x_b^d}{\delta x}\right)^2 + \mu\left(n + \frac{x_b^d}{\delta x}\right)\right\}, \tag{A.19}$$

$$A_4 = \delta t\lambda(n\delta x + x_b^d)\left\{(1-\psi)\exp\left[-r\left(T - m\delta t\right)\right] + \hat{D}\left(m, n + \Delta/\delta x\right)\right\} \tag{A.20}$$

where $\hat{D}(m, n+\Delta/\delta x) = \hat{D}(m\delta t, n\delta x + \Delta) \approx \hat{D}(t, x+\Delta)$ is the approximate value of the non-strategic debt value in PDE (4.14), whose finite difference scheme is given by

$$\hat{D}(m-1,n) = A_1\hat{D}(m,n-1) + A_2^*\hat{D}(m,n) + A_3\hat{D}(m,n+1). \tag{A.21}$$

In the scheme (A.21), the coefficient $A_2^* = A_2 + 2\delta t\lambda(n\delta x + x_b^d)$.

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