Investigating the Effects of Procedural Variation on Multiplicative Reasoning



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Abstract

Procedural variation uses varied sequences of questions to draw children's attention to the invariant critical features of a mathematical concept. This pre- and post-test intervention study investigates the effects of teaching using procedural variation on 7-8-year-old children's distributive and scalar multiplicative reasoning. The first research question examines whether Year 3 children are capable of recognising the distributive and scalar relationships between sequences of multiplication questions. The second research question considers whether teaching using procedural variation can improve Year 3 children's distributive and scalar reasoning. 184 children were recruited from six Year 3 classes and were randomly allocated to three interventions: distributive reasoning, scalar reasoning or the active control. Each intervention comprised three 30-minute sessions; the distributive and scalar reasoning groups received procedural variation teaching, while the active control focused on times tables. The pre- and post-test comprised distributive and scalar reasoning tasks and was administered immediately before and after the interventions. Many of the children were capable of using procedural variation to recognise the distributive relationships between multiplications but could not identify the scalar relationships. There were significant post-test differences between groups, with the experimental groups outperforming the control group. There was also a significant interaction between the distributive and scaling groups and their respective tasks, meaning that procedural variation teaching had a specific effect on distributive or scalar reasoning. This is the first intervention study to empirically evaluate the effects of teaching using procedural variation and it therefore makes a small yet valuable contribution to primary mathematics research.

Table of Contents

Acknowledgements ..... **i**

Abstract ..... **ii**

Table of Contents ..... **iii**

List of Tables ..... **vi**

List of Figures ..... **vii**

List of Appendices ..... **viii**

1. Introduction **1**

1.1 Procedural Variation 1

1.2 Aims of the Present Study 2

1.3 Structure of the Dissertation 2

2. Literature Review **4**

2.1 The Variation Theory of Learning 4

2.1.1 The Chinese tradition 4

2.1.2 The Variation Theory of Learning 5

2.1.3 Conceptual and procedural variation 6

2.1.4 The purpose of procedural variation 6

2.1.5 Patterns of variation and invariance 7

2.1.6 The object of learning 9

2.1.7 Variation pedagogy 10

2.2 Current Research in the Field of Variation Theory 12

2.2.1 Contrasting research paradigms 12

2.2.2 Learning studies 13

2.2.3 Learning studies in primary mathematics 14

2.2.4 Critique of the learning study design 15

2.2.5 An experimental approach to variation theory 15

2.2.6 An intervention study of variation theory 17

2.3 Variation Theory in Primary Mathematics 17

2.3.1 A 'paradigm shift'? 17

2.3.2 The use of procedural variation in primary mathematics 18

2.4 Multiplicative Reasoning 19

2.4.1 Operational invariants 19

2.4.2 Multiplicative reasoning 19

2.4.3 Distributive reasoning 20

2.4.4 Scalar reasoning 20
2.4.5 Multiplicative reasoning and procedural variation 21
2.5 The Present Study 21
3. Methodology 23
3.1 Design 23
3.2 Ethics 23
3.3 Recruitment 24
3.4 Sample and Groups 24
3.5 Measures 25
3.6 Reliability and Validity 27
3.7 Interventions 28
3.7.1 Distributive reasoning 29
3.7.2 Scalar reasoning 31
3.7.3 Active control 32
3.8 Pilot Study 32
4. Results 34
4.1 Children's Multiplicative Reasoning in the Pre-Test 34
4.1.1 Descriptive statistics 34
4.1.2 Conceptual and arithmetic errors 36
4.2 Gender and Multiplicative Reasoning 39
4.2.1 Descriptive statistics 39
4.2.2 Parametric assumptions of the ANOVA 41
4.2.3 Mixed ANOVA between genders and within tasks 41
4.3 Post-Test Differences between Intervention Groups 42
4.3.1 Descriptive statistics 42
4.3.2 Parametric assumptions of the ANCOVA 46
4.3.3 Mixed ANCOVA between groups and within tasks 47
4.4 Qualitative Analysis of the Children's Workings 50
4.4.1 Distributive addition 51
4.4.2 Distributive subtraction 52
4.4.3 Scalar doubling 53
4.4.4 Scalar tripling and quadrupling 54
4.4.5 Further inferences of children's reasoning processes 55
4.5 Summary of Results 56

5. Discussion 57
5.1 Conclusions 57
5.1.1 Children's use of procedural variation 57
5.1.2 The effects of procedural variation teaching 59
5.2 Implications for Mathematics Pedagogy 60
5.3 Strengths of the Present Study 61
5.4 Limitations of the Present Study 62
5.5 Future Research 62
References 64
Appendices 73

List of Tables

3.1 Gender Distribution Across Intervention Groups 25
4.1 Means and Standard Deviations of Correct Scores in Pre-Test Tasks 35
4.2 Types of Answers in the Distributive Addition Task 37
4.3 Types of Answers in the Distributive Subtraction Task 38
4.4 Means and Standard Deviations of Correct Scores in Pre-Test Tasks between Genders 40
4.5 Mixed ANOVA Between Genders and Within Tasks on Pre-Test Total Correct Score 42
4.6 Means and Standard Deviations of Correct Scores in Pre- and Post-Test Distributive Tasks 43
4.7 Means and Standard Deviations of Correct Scores in Pre- and Post-Test Scaling Tasks 44
4.8 Mixed ANCOVA Between Groups and Within Tasks on Post-Test Scores 47

List of Figures

1.1 Examples of arithmetic sequences using procedural variation 1
3.1 Distributive reasoning sequences eliciting addition or subtraction of multiplicands 26
3.2 Scalar reasoning sequences eliciting doubling or quadrupling of previous products 26
3.3 Distributive reasoning intervention: ladders activity 29
3.4 Distributive reasoning intervention: speed quiz activity 30
3.5 Scalar reasoning intervention: ladders activity 31
3.6 Scalar reasoning intervention: speed quiz activity 32
4.1 Distribution of pre-test total correct scores 35
4.2 Mean correct scores in pre-test tasks 36
4.3 Pre-test mean correct scores between genders in each task 40
4.4 Pre-test mean correct scores by intervention group 45
4.5 Post-test mean correct scores by intervention group 45
4.6 Estimated marginal means for each group across post-test tasks 49
4.7 Examples of children's workings on distributive addition tasks 51
4.8 Examples of children's workings on distributive subtraction tasks 52
4.9 Examples of children's workings on scalar doubling tasks 53
4.10 Examples of children's workings on scalar tripling tasks 54
4.11 Examples of children's workings on scalar quadrupling tasks 55

List of Appendices

Appendix A: CUREC Documents and Approval 73
CUREC Application Form 73
Letter to Headteachers 83
School Consent Form 85
Letter to Parents 86
Participation Certificate 88
CUREC Approval Email 89
Appendix B: Pre- and Post-Test 90
Instructions and Practice Question 91
Distributive Addition Tasks 92
Distributive Subtraction Tasks 93
Scaling Doubling Tasks 94
Scaling Tripling and Quadrupling Tasks 95
Appendix C: Distributive Reasoning Intervention 96
Interventions Overview 96
Ladders Activity 97
Team Challenge 99
Speed Quiz 100
Card Matching Game 102
Appendix D: Scalar Reasoning Intervention 103
Interventions Overview 103
Ladders Activity 104
Team Challenge 106
Speed Quiz 107
Card Matching Game 109
Appendix E: Times Tables (Control) Intervention 110
Interventions Overview 110
Times Tables Codebreaking Activity 111
144 Square Challenge 112

1. Introduction

1.1 Procedural Variation

Procedural variation is a pedagogical approach which uses systematically varied sequences of questions to develop understanding of mathematical concepts as processes (Watson & Mason, 2006a; Sfard, 1991). Teaching using procedural variation is the opposite of rote learning as it uses repeated yet varied questions to deepen understanding (Ko & Marton, 2004). In their influential paper on teaching with variation, Gu, Huang and Marton (2004) emphasise the importance of highlighting the essential invariant features of a mathematical concept through varying the non-essential aspects and discuss the fundamental importance of procedural variation:

> During the process of solving problems, procedural variation manifests a series of concepts or propositions for scaffolding or the strategies and experiences for doing activities, thus fostering students' methods of solving problems with multiple stages or multiple options; in the process of the formation of cognitive structure, a hierarchical system of experience or strategies is formed by creating the procedural variation (2004, p. 324).

Figure 1.1 exemplifies the use of procedural variation in sequences of arithmetic questions. In the first two examples, the answer remains invariant, drawing learners' attention to the essential feature of the varying relations between the numbers. The third example, however, demonstrates how the invariant essential features of the base 10 place value system can be seen through the variation of the multipliers and products. Finally, the fourth example uses procedural variation to highlight the invariant relationship between divisor and quotient; when the divisor is doubled, the quotient is halved.

49 + 51 = 100	159 – 79 = 80	0.4 x 6 = 2.4	64 ÷ 2 = 32
39 + 61 = 100	160 – 80 = 80	4 x 6 = 24	64 ÷ 4 = 16
29 + 71 = 100	161 – 81 = 80	40 x 6 = 240	64 ÷ 8 = 8
19 + 81 = 100	162 – 82 = 80	40 x 60 = 2400	64 ÷ 16 = 4

Figure 1.1. Examples of arithmetic sequences using procedural variation (c.f. Morgan, 2015; Voigt, 1995).

1.2 Aims of the Present Study

Al-Murani, Kilhamn, Morgan and Watson (2018) describe the use of variation pedagogy as a 'paradigm shift' in the teaching of primary mathematics. By varying the non-essential aspects, teachers can draw pupils' attention to the invariant critical features of a concept. To date, however, only Al-Murani et al. (2018) have considered the use of variation theory in UK primary mathematics. In common with other studies into the variation theory of learning, their small-scale observational study only focused on the qualitative description of variation pedagogy, as opposed to its empirical verification.

Despite the widespread promulgation of variation theory in the UK, therefore, no rigorous experimental studies have been conducted to provide empirical data on its use in primary mathematics. The focus of this study on the use of procedural variation originated with my personal experience as a specialist primary mathematics teacher. Although I have found procedural variation pedagogy to be effective and beneficial in my own practice, I am surprised at the paucity of empirical evidence for its promulgation.

This dissertation will evaluate current evidence for the use of procedural variation whilst highlighting the absence of empirical studies in primary mathematics. The aim of this dissertation is to provide a significant original contribution to primary mathematics research by utilising a quasi-experimental approach to investigate the effects of procedural variation on the distributive and scaling reasoning skills of Year 3 pupils.

The effects of procedural variation pedagogy on the multiplicative reasoning skills of Year 3 pupils will be investigated through the following research questions.

1. Are Year 3 children capable of recognising and using the relationships between sequences of multiplication questions?
2. Does teaching using procedural variation improve Year 3 children's understanding of these relationships?

1.3 Structure of the Dissertation

This first chapter has defined the concept of procedural variation and has discussed the aims and research questions of the present study. The second chapter reviews the current literature on the theory of variation; evaluating multiple definitions of the concept, analysing qualitative studies and highlighting the current paucity of empirical evidence. Furthermore, this section justifies the choice of multiplicative reasoning as a relevant context for this research and defines the concepts of distributive and scalar reasoning. The third chapter outlines the methodology by briefly reiterating the aims of the study and explaining the choice of a quasi-experimental intervention study

design. The methodology provides detailed information on the sample and recruitment, the rationale of the interventions and the process of data collection. The fourth chapter analyses the results generated by the pre- and post-tests with reference to the research questions. Finally, the fifth chapter discusses the overall conclusions, implications for mathematics pedagogy, strengths and limitations of the study and recommendations for future research.

2. Literature Review

2.1 The Variation Theory of Learning

2.1.1 The Chinese tradition. The concept of procedural variation originates in the Chinese tradition of learning studies which use classroom observations to generate inductive theories of good practice. Gu (1991, 1994) conducted 50 quasi-experiments in the Qingpu district of Shanghai to analyse the features of effective mathematical teaching and identified the pedagogy of *bianshi*, translated as 'changing' or 'varying', to be of fundamental importance to the understanding of mathematical concepts. Gu provided the seminal definition of *bianshi* pedagogy, outlining the aims of teaching through variation:

> It intends to illustrate the essential features of a concept by demonstrating various visual materials and instances, or to highlight essential characteristics of a concept by varying non-essential features. The goal of using variation is to help students understand the essential features of a concept by differentiating them from non-essential features and further develop a scientific concept. (1991, p. 186)

The theory of *bianshi* is based on the Chinese maxim: 'only by comparing can one distinguish' (Gu, Huang, & Gu, 2017). By varying the non-essential features of a concept across multiple examples, attention can be drawn to an invariant essential feature of the concept.

Gu (1991; Gu et al., 2004) used the qualitative observational data from the Qingpu learning studies to postulate that *bianshi* variation pedagogy enhances students' mathematical understanding in a variety of ways. The process of varying multiple examples enables students to understand a specific procedure or principle by highlighting the invariant critical features. Students become aware of these features by contrasting different examples and generalising to subsequent calculations. In addition, through the process of discerning these variant and invariant features, students develop a deeper understanding and fluency with the underlying principles of a concept.

Another fundamental aspect of *bianshi* theory is the concept of *pu dian* which can be translated as *scaffolding* (Gu et al., 2004). Ausubel (1968) hypothesises that learning is based on the foundation of an *anchoring knowledge point*. In a study of Shanghai lessons on geometry, Gu (1994) analysed the progressive complexity of *anchoring knowledge* for 7^{th}, 8^{th} and 9^{th} grade pupils and their increased understanding over time of the positional relationships of lines and circles. Where the *potential distance* between the new learning and their prior knowledge was too great, however, students failed to make progress. From a Vygotskyan perspective, tasks must be within a pupil's *zone of proximal development* for learning to occur (Vygotsky, 1978). *Pu dian*, similar to Western theories of

scaffolding, requires educators to carefully design sequences of varied tasks to scaffold pupils' understanding of mathematical concepts by building on their prior knowledge (Wood, Bruner, & Ross, 1976).

2.1.2 The Variation Theory of Learning. In contrast to the inductive approach of Chinese *bianshi* theory, Marton and Booth (1997) take an interpretivist phenomenographic approach in their Variation Theory of Learning. They postulate that learning requires a change in the subjective experience of the individual. Learning requires awareness of *dimensions of variation*; the discernment of variations against an invariant background.

Al-Murani (2007, p. 9) elucidates the concept of *dimensions of variation* through the following example. If one had only ever been exposed to green apples and then saw red apples, the colour would vary while the fruit would remain invariant. In this instance, the *dimension of variation* is colour. In contrast, however, if one saw green apples and then green pears, the *dimension of variation* would be the type of fruit. Variation theory assumes that a new concept can only be learnt through the discernment of specific contrasts against a background of sameness.

Marton & Booth (1997) provide the seminal account of the variation theory of learning within the phenomenographic tradition. In particular, they emphasise how variation is essential for the discernment of new phenomena. Variation theory contends that 'when some aspect of a phenomenon or an event varies while another aspect or other aspects remain invariant, the varying aspect will be discerned' (Bowden & Marton, 1998, p. 35). It is not possible to learn about the 'greenness of green' without contrasting it with other colours and similarly, it is not possible to learn about the quantity of three without comparing it with numbers that are not three (Gu et al., 2004; Holmqvist Olander & Nyberg, 2014). In order to understand the 'critical features' of a new concept, therefore, pedagogical tasks must utilise a pattern of variation and invariance to develop students' understanding.

Pang, Bao and Ki (2017) contrast the Chinese *bianshi* theory with the variation theory of learning. Whereas *bianshi* teaching emphasises 'sameness' between tasks, variation theory focuses to a greater extent on differences against this background of 'sameness'. Marton and Häggström (2017), however, reject the distinction between the Chinese and phenomenographic traditions by emphasising how both approaches utilise variation to contrast essential and non-essential aspects of a construct. Indeed, they assert that 'nobody can teach mathematics without using *some* pattern of variation and

invariance, although it can be done more or less systematically and with more or less awareness' (2017, p. 403). Both traditions provide valuable contributions to the new field of research into teaching maths through the use of variation and invariance (Marton & Häggström, 2017). Despite the vast theoretical literature on variation theory, however, very few empirical studies have been conducted to evaluate the unsubstantiated assumption that variation is fundamental to mathematical learning.

2.1.3 Conceptual and procedural variation. Sfard (1991) suggests a dichotomy between conceiving of mathematical concepts as objects or, operationally, as processes. In their pre-eminent paper on variation theory, Gu et al. (2004) draw an important distinction between two different types of variation: conceptual and procedural. Conceptual variation is the exemplification of a single concept through the use of a variety of concrete and visual resources (Gu et al., 2017; Gu et al., 2004). Focusing on a particular object of learning, students are provided with a variety of concrete and pictorial representations to enable them to discern the invariant features and develop an abstract understanding (Bruner, 1966). In their study of a Shanghai Grade 2 lesson, Pang, Bao and Ki (2017) highlight the multiple representations of addition through an abacus, number line and place value grid as a clear example of conceptual variation. Conceptual variation, therefore, uses varied visual examples and perspectives to highlight the invariant features of a specific object of learning.

In contrast, procedural variation develops students' understanding of mathematical concepts as processes (Gu et al., 2004). By varying a mathematical procedure in a specific way, the students' attention is drawn to underlying connections and relationships. Procedural variation uses carefully selected sequences of varied problems to enable students to build on their prior knowledge, make new connections and develop their understanding of underlying mathematical processes (Gu et al., 2017; Pang et al., 2017). Gu et al. (2004, p. 325) argue that conceptual variation involves the 'static' comparison of objects whereas procedural variation is 'dynamic' as it focuses on mathematical processes and enables students to make connections and generalise from their prior knowledge to new understanding. While conceptual variation focuses on the visualisation of a concept from multiple perspectives, procedural variation demonstrates the 'process for formation of the concept stage by stage' (Gu et al., 2004, p. 321).

2.1.4 The purpose of procedural variation. The purpose of procedural variation is to enable students to develop their understanding of the concept whilst also allowing them to use deduction to generalise to new situations. Gu et al. (2004) state that

procedural variation can be utilised in three different ways to develop understanding of mathematical concepts as processes. It can be used to vary or extend a specific problem by changing the quantities or results. In addition, procedural variation can develop multiple ways to solve one problem by highlighting the different processes involved across varied methods. Furthermore, it can also be used to demonstrate multiple applications of a specific method across a variety of similar problems (Gu et al., 2004).

Procedural variation, therefore, requires carefully designed mathematical activities which are used as a pedagogical tool to 'scaffold' learners to make connections between processes (Zhang, Wang, Huang, & Kimmins, 2017; Wood et al., 1976). Indeed, Marton and Pang (2006) contend that all mathematics activities, to a greater or lesser extent, inherently use variation across multiple examples. Taking a constructivist approach, Gu et al. (2004, p. 324; Piaget, 1954) argues that by systematically focusing on the procedural variation of a sequence of calculations, the underlying processes are 'internalized into an important part of cognitive structure'. Similar to the Chinese *bianshi* concept of *pu dian*, procedural variation uses 'progressively unfolding mathematics activities' to enable students to see patterns and construct systematic connections between concepts (Gu et al., 2004, p. 319; Zhang et al., 2017). Despite these expansive theoretical assertions, however, no empirical studies have actually been conducted to empirically evaluate the theory of procedural variation in educational practice.

2.1.5 Patterns of variation and invariance. The simultaneous discernment of both variant and invariant features is fundamental to procedural variation (Chik & Lo, 2004). Indeed, Marton and Pang (2006, 2013) stipulate three essential components of variation pedagogy: an invariant essential concept; varied non-essential features; and the simultaneous discernment of these varied and invariant aspects. In his discussion on the necessary conditions of learning, Marton (2015) emphasises how both 'sameness' and 'difference', or invariant and variant features, are a prerequisite for learning. Using the example of colour, Marton (2015) illustrates how we can only develop an understanding of the 'critical aspects' of one colour by contrasting it with another. Drawing on the phenomenographic tradition, learning can only occur through a change in the individual's awareness; their discernment of the variant features against an invariant 'background of sameness' (Ko & Marton, 2004; Marton & Pang, 2006, 2013). It is the simultaneous discernment of these variant and invariant features, therefore, which constitutes learning (Marton & Booth, 1997).

Building on Piagetian constructivist theories (Inhelder & Piaget, 1958; Piaget, 1954), Dienes' (1971) theory of variability encompasses four principles specific to the construction of mathematical schema. Of particular relevance to the theory of procedural variation, his Mathematical Variability Principle states that children should experience mathematical concepts through the greatest possible number of variables. By providing the maximum amount of variation across different tasks, the essential features of the concept will be learnt. Gu et al. (2004) elucidates this principle through the example of the mathematical variability of parallelograms. Children should be provided with a variety of examples of parallelograms with differing dimensions and angles as long as the essential invariant feature of opposite parallel sides is retained. Dienes' theory of variability introduces a specific pattern of variation and invariance: the generalisation of an invariant concept across varying examples.

Marton, Runesson and Tsui (2004) identify four specific patterns of variation and invariance: contrast, generalisation, separation and fusion. Marton and Pang (2006) argue that contrast is a necessary condition of learning, with examples of the concept being juxtaposed with non-examples. In the previous example of parallelograms, contrasting a variety of shapes would enable the essential features of the parallelogram to be discerned. Subsequently, learners would be able to generalise their knowledge about the invariant features of parallelograms to future examples. Generalisation, therefore, requires a focus on the invariant concept across different varied examples, enabling students to identify its critical features and to generalise to new tasks (Marton, 2015).

In contrast, separation uses an opposite approach where students instead focus on the varying features across examples, with the invariant feature remaining constant. An example of the separation theory occurs in the concept of cardinality, where children learn about an invariant quantity through repeated exposure to separate concrete examples (Gu et al., 2004). The fourth pattern of variation and invariance, fusion, is more complex as it focuses on learners' simultaneous awareness of two or more dimensions of variation and their discernment of the relationship between these varying and invariant features (Marton et al., 2004). Equivalent fractions provide an example of fusion; learners must focus on the changes to both the numerator and denominator. Two or more aspects of a concept are simultaneously varied, allowing learners to contrast these varying features with invariant ones and make connections across calculations (Gu et al., 2004). The fusion pattern of variation and invariance is predominantly demonstrated through

sequences of mathematical calculations and methods and is therefore of particular relevance to the present study on multiplicative reasoning.

Building on these four patterns of variation, Leung (2017) postulates four Principles of Acquiring Invariance. Leung emphasises the relationship between variation and invariance which enables pupils to discern invariant mathematical patterns. Leung's first principle of difference and similarity corresponds with Marton's (2015) concepts of contrast and generalisation. Leung, however, elaborates on the idea of generalisation through two further principles. His 'sieving' and 'shifting' principles emphasise how learners may be required to 'sieve' through varied examples to seek hidden invariant features. Furthermore, they may be required to 'shift' their focus across different examples to discern 'generalized invariance'. The final co-variation principle, however, is of particular importance to the present study as it focuses on the co-varying or 'fusing' of different variables, enabling learners to seek a pattern or invariant relationship across calculations (Leung, 2017). In the context of multiplicative reasoning, teaching using procedural variation may draw learners' attention to the co-variation of the multiplier and the product and enable them to discern an invariant relationship between these two variables.

2.1.6 The object of learning. A further construct underlying the theory of procedural variation is the *space of learning*. Marton, Runesson and Tsui state that 'a *space of learning* comprises any number of dimensions of variation and denotes the aspects of a situation, or the phenomena embedded in that situation, that can be discerned due to the variation present in the situation' (2004, p. 21). Variation pedagogy requires the teacher to decide the *object of learning*; the specific phenomenon which is being taught (Bowden & Marton, 1998). Through simultaneous awareness of the variant and invariant features, learners will apprehend the 'critical aspects' of that *object of learning* (Marton, 2015).

Kullberg and Runesson (2013) have analysed the use of variation theory in classrooms and emphasise the importance of three interrelated concepts: the *intended, enacted* and *lived objects of learning*. Simon and Tzur (2004) theorise that teaching is contingent upon a *hypothetical learning trajectory* in which teachers plan a lesson with an *intended object of learning*. The *enacted object of learning,* however, can be quite distinct from this, encompassing the teacher's actions, their interaction with the students and students' interactions with the tasks and each other (Mok, 2017). Finally, taking a

phenomenographic approach, the *lived object of learning* is the student's subjective experience of the lesson (Chik & Lo, 2004; Marton et al., 2004).

Current research in the field of variation theory has predominantly focused on qualitative analysis of lessons using this theoretical framework, as it enables researchers to contrast the teacher's intentions with their actual use of variation theory and to discuss the students' learning (Kullberg, Runesson Kempe, & Marton, 2017). It is also, however, of particular relevance to the present study which uses an intervention study design to investigate the effects of procedural variation in primary mathematics. In this paper, the *intended object of learning* is the development of pupils' multiplicative reasoning skills through the use of procedural variation. By delivering interventions with a consistent *enacted object of learning* using procedural variation, it is possible to quantitatively analyse the *lived object of learning* through pupils' performance on the pre- and post-tests.

2.1.7 Variation pedagogy. Mason (2017) has critiqued the overcomplication of the variation theory of learning and queries whether the intricacies of the theory are beneficial to teaching. Instead, Watson and Mason (2005, 2006a, 2006b) propose the construct of *variation pedagogy* which concentrates on the fundamental role of variation in developing mathematical understanding. *Variation pedagogy* encompasses all the concepts heretofore discussed, including procedural variation, scaffolding theory and the object of learning. It is, therefore, particularly relevant to this dissertation and provides an apropos construct for the design of interventions using procedural variation pedagogy.

Watson and Mason (2005; Watson, 2017) see mathematics as a constructive activity where abstract concepts are understood through an *enacted object of learning* using variation and invariance. Mok (2017, p. 188) stipulates that 'discernment is made possible when variation concerning the critical aspects or features is created either in the design of the content (or the tasks) of the lesson or in the interaction between class participants (i.e., teacher and students, students and students)'. The concept of *variation pedagogy* emphasises the pedagogical choices of the teacher in constructing learning opportunities. Drawing on Simon and Tzur's (2004) theory of the 'hypothetical learning trajectory', Watson and Mason (2006a, p. 92) suggest that effective pedagogy identifies a specific *intended object of learning* with lessons being designed around a 'hypothetical (learner) response'. Consequently, the teacher's choice of activities and questions and the pupils' engagement with them during the actual lesson is the *enacted object of learning*.

It is this use of variation theory in the planning and delivery of lessons which constitutes *variation pedagogy* (Watson and Mason, 2006a).

Mason (2017) sees this *enacted object of learning* to be of fundamental importance in developing pupils' understanding of a general invariant concept through particular varying examples:

> Seeing the general through the particular, and the particular in the general are fundamental acts of attention which need to be invoked, supported, promoted, and internalized if learners are going to internalize actions (procedures) and appreciate and comprehend both concepts and procedures (2017, p. 434).

Using a 'pedagogy of variations' enables educators to draw students' attention to specific variant and invariant features of a concept and the 'underlying dependency relationships' which connect them (Watson, 2017, p. 96). Linked to Leung's (2017) principle of co-variation, 'dependency relationships' focus on how a change in one variable simultaneously affects another variable in the calculation. Once a particular 'dependency relationship' has been discerned, such as the invariant inverse relationship between addition and subtraction, students can then generalise this knowledge across varying calculations (Watson, 2017; Watson et al., 2016).

Variation pedagogy, therefore, requires the careful construction of activities and examples which enable students to focus on the simultaneous relationship between variant and invariant features. In relation to procedural variation, however, Mason (2017) and Watson et al. (2016) stipulate the importance of sequential attention; using sequences of calculations to systematically reveal an underlying invariant relationship. Watson and Mason (2006a, 2006b) discuss how a sequence of varied questions can be seen as a single mathematical object. Through analysing the structure of a sequence of questions, students 'see patterns, form expectations, express these as conjectures about patterns and generalise' (Watson & Mason, 2006b, p. 4). Rittle-Johnson and Star (2009, p. 529) have argued that the comparison of two or more examples is an important aspect of concept formation as it enables students to ask: 'what changes and what stays the same?' The pedagogical use of procedural variation to design sequences of questions, therefore, 'offers a structured and structural approach to exposing underlying mathematical form' (Watson & Mason, 2006a, p. 97).

Current qualitative analysis of the use of *variation pedagogy* has almost exclusively focused on secondary mathematics, taking an inductive grounded theory approach to infer the potential effects of variation on learning (Lincoln & Guba, 1985).

Watson and Mason (2005, 2006a, 2006b) present a variety of examples of the use of variation theory in secondary mathematics in diverse topics such as the sine, cosine and tangent rules of trigonometry; algebraic multiple linear equations; and quadratic graphs. It is a surprising feature of the literature on variation theory that it almost exclusively focuses on secondary mathematics despite the potentially equal relevance of *variation pedagogy* to primary-aged pupils. From the youngest of ages, mathematics teaching should not promote rote learning through arbitrary examples, it should deepen understanding through procedurally varied sequences of questions (Ko & Marton, 2004; Marton, Dall'Alba, & Tse, 1992). The following section on learning studies, therefore, will highlight the current deficit in empirical evidence for the variation theory of learning, particularly in the primary age group, and emphasise the significant original contribution of this paper.

2.2 Current Research in the Field of Variation Theory

2.2.1 Contrasting research paradigms. Marton and Pang (2013, p. 27) have stated that 'there are no teaching experiments' in this field and highlight the paucity of empirical evidence for the variation theory of learning within a positivist research paradigm. Similarly, Kullberg et al. (2017) have criticised the lack of empirical data in support of the theory and argue that current papers only provide 'empirical illustrations' using qualitative observations. There has been a disproportionate focus on the theory of variation, including procedural variation, to the exclusion of empirical studies to evaluate its efficacy (Cai & Nie, 2007).

Situated within the interpretivist research paradigm, studies of variation theory focus on the social construction of knowledge. The ontological assumption that social phenomena are constructed results in a phenomenological epistemology where qualitative observation is used to describe the use of variation theory (Bryman, 2016; Lincoln, Lynham, & Guba, 2018). This epistemological approach generates qualitative data upon which only inductive *post hoc* analysis can be conducted (De Vaus, 2001). It is the contention of this dissertation, however, that research into procedural variation can be conducted within a positivist research paradigm where reality can be apprehended through empirical testing of quantitative data.

This section will introduce the concept of *learning studies* and discuss their predominant use in secondary mathematics education. Furthermore, it will evaluate specific research papers which have the greatest relevance to the present study; those

which specifically focus on the use of variation pedagogy in primary mathematics or those which have used experimental pre- and post-test research designs.

2.2.2 Learning studies. The dominant methodology in the field of variation theory is the learning study. Drawing on Japanese teaching practice, Stigler and Hiebert (1999) initially proposed the concept of a *lesson study* in which a group of teachers plan a lesson around a specific *object of learning*; the lesson is taught by one teacher while the others observe; after which the group reconvenes to discuss learning outcomes. Drawing on this conceptual framework, Marton and Pang (2013) define a *learning study* as the experimental usage of a *lesson study* with a particular focus on the *intended* and *enacted objects of learning*, combined with an analysis of the students' experience or the *lived object of learning* (Marton et al., 2004). Learning studies are usually conducted in three cycles of planning, teaching, observation and feedback; each cycle is then adapted in response to feedback and resultant changes in pupil outcomes are analysed. Researchers and teachers engage in constant dialogue about the three *objects of learning* to improve teaching across the three cycles and to analyse the effects of deliberate usage of variation pedagogy. Over the past two decades, it has been estimated that approximately 1000 learning studies have been conducted, many of which have utilised the conceptual framework of variation theory (Marton & Pang, 2013).

Learning studies of variation theory, however, have almost exclusively been conducted in secondary schools. Mok (2006, 2013, 2017) has conducted a significant number of learning studies analysing the teaching of algebra in secondary schools in Shanghai. In her 2006 study, videos of lessons on algebraic linear equations were analysed from the perspective of variation theory. Mok emphasised the use of the 'contrast' form of variation where examples and counter-examples were used to scaffold learners' understanding of the critical features of linear equations. Observational data are used *post hoc* to infer a beneficial effect of variation pedagogy on pupil's understanding. Multiple similar learning studies have been conducted in secondary schools on diverse topics such as simultaneous equations, geometry and mathematical proofs (Al-Murani, 2003; Huang, Barlow, & Prince, 2016; Huang & Leung, 2004; Leung, 2017; Kullberg et al., 2017; Marton & Pang, 2006; Pang, Marton, Bao, & Ki, 2016; Peng, Li, Nie, & Li, 2017; Runesson & Kullberg, 2017). Despite the theorised relevance of variation pedagogy across all age groups, however, surprisingly few learning studies have been conducted within the primary age group of 5-11 years.

2.2.3 Learning studies in primary mathematics. The small minority of learning studies that have analysed the use of procedural variation in the 5-11 age group are of particular relevance to the present study. Holmqvist Olander and Nyberg (2014) analysed the use of variation pedagogy to teach halving and doubling to 6-7 year-olds (n = 51). As part of a Swedish teacher professional development project, a learning study was conducted where variation pedagogy was implemented, evaluated and adapted in three classrooms. A pre- and post-test was used to analyse the effects of using variation to highlight the invariant reciprocal relationship between doubling and halving. The results indicated a significant difference between pre- and post-test for one group ($p = .015$) with near significance for the other two groups at the delayed post-test ($p = .055$, $p = .054$). It must be emphasised that this was not an experimental design; it was a descriptive study of the usage of procedural variation in different teachers' lessons. There were considerable differences in the *enacted object of learning* across the three groups with no clear independent variable. Whilst this study did generate some quantitative data on the general use of procedural variation in primary mathematics, the researchers acknowledged its limitations as a small-scale study with no control group for statistical comparison.

The topic of fractions has been the predominant context for research into procedural variation in primary mathematics. In particular, Runesson and Mok (2004, 2005) undertook a comparative learning study of the teaching of fractions to 10-13 year-olds in Sweden and Hong Kong. They used observational data to identify and compare the different uses of variation to teach fractions in the two countries. In Sweden, the five teachers in the study focused on procedural variation of both the numerator and denominator to demonstrate equivalence (e.g. 1/3 = 2/6). In contrast, the Hong Kong teachers used both conceptual variation in the form of diagrams and procedural variation of the denominators to reveal multiple *dimensions of variation*. The Hong Kong learning study, therefore, revealed that teachers particularly used procedural variation to juxtapose multiple examples of varied fractions to enable pupils to compare fractions with different denominators.

Building on this observational study, Kullberg and Runesson (2013; c.f. Hino, 2017) analysed the use of variation pedagogy to teach 9-10 year-old pupils (n = 59) how to find unit and non-unit fractions of amounts. While pre- and post-tests were conducted, there was no control group for comparison and the data collected were merely used to illustrate an improvement in understanding following the teaching of the concept. The

researchers analysed ‘what was made available for the pupils to learn’ through variation of numerators and denominators and discussed what they ‘seemed to have experienced’ during the 10 lessons (p. 559). In common with most of the other studies cited herein, therefore, this learning study used qualitative analysis of the teachers’ intentions, lesson observations and pupils’ output to inductively infer the purported efficacy of variation pedagogy in mathematics.

2.2.4 Critique of the learning study design. Marton and Pang (2006, p. 219) have argued that ‘there cannot be any true teaching experiments.’ Nuthall (2005), however, has emphasised the importance of empirical evidence for educational theory:

> It would be unethical to expect a teacher to use a new method if the results of research on that method had never been independently replicated. It would be similarly unethical to expect a teacher to believe a theoretical explanation of the teaching process if that explanation had been based on results of a single, unreplicated study (2005, p. 900).

Kullberg et al. (2017) used a learning study design with 12 different secondary mathematics teachers to analyse ‘what is made possible to learn’ through the use of variation theory. They conclude that variation and invariance ‘make it possible for learning to happen’ and yet describe their study as an ‘empirical illustration’ and they themselves highlight the lack of empirical data in the field of variation theory (2017, p. 566). Indeed, they argue that pedagogical recommendations cannot be derived from the variation theory of learning as ‘this has not been studied and is not within the scope of this paper' (2017, p. 566).

Runesson and Mok (2004) have stipulated a specific limitation of the learning study design; it does not actually focus on what the students have learnt. This can be a particularly difficult construct to measure and it cannot be inferred through merely focusing on what the teachers would like pupils to learn (Runesson & Kullberg, 2017). Marton and Pang (2013) particularly criticise the inductive *post hoc* methodology of learning studies, arguing that they provide inconclusive evidence for variation theory. Due to the predominant use of the learning study design to evaluate variation theory, there is a paucity of empirical data to support its assertions and this situation can only be rectified through experimental designs (Marton & Pang, 2013).

2.2.5 An experimental approach to variation theory. A small minority of studies on variation theory have used quasi-experimental designs. Kullberg (2010) adapted the learning study design by pre-emptively specifying the independent variable of variation pedagogy to be taught, prior to the lesson observations. Kullberg identified

specific patterns of variation and invariance used in the teaching of decimals and negative numbers and observed cycles of three lessons taught to 11-13-year-olds (n = 247) in an attempt to evaluate the effects of variation pedagogy on learning. Despite an analysis of variance (ANOVA) being conducted on the pre-test results to examine mean differences between groups, no such analysis was conducted on the post-test results. Instead, pre- and post-test means and standard deviations were compared to demonstrate an improvement in the pupils' learning. Schoenfeld (2006) has stated that the use of quantitative data to examine an education theory constitutes a 'design experiment'. This quasi-experimental approach, however, had two main methodological limitations. The lack of a control group meant that no comparison could be made with teaching that did not use variation pedagogy. Furthermore, Kullberg discussed how the *enacted object of learning* seen in lesson observations was inconsistent and often did not match the *intended object of learning*. It is unclear, therefore, which patterns of variation were used by the teachers, which critical feature were discerned by the pupils and how this affected their mathematical understanding.

Marton and Pang (2013) used a quasi-experimental approach to test whether activities using patterns of variation and invariance improve mathematical understanding. Marton and Pang used a pre- and post-test to analyse the effects of procedural variation on 9-10-year-olds' understanding of supply and demand (n=78). The children were split into two groups; 'same goods' and 'different goods'. The 'same goods' task used procedural variation in which the amount of money varied whilst the number of goods remained the same. In contrast, the 'different goods' design varied all aspects of the question, meaning that specific invariant features would be less easily discerned.

Pre- and post-test results demonstrated a significant difference between the two groups ($\chi^2 = 10.36$, p = 0.03; effect size = 0.32) with the relative frequencies of correct answers differing by as much as 24.2% in favour of the 'same goods' group (Marton & Pang, 2013, p. 37). By developing a pedagogical tool using procedural variation and analysing the results of the same pre- and post-test, the researchers were able to derive empirical evidence to support variation theory. Despite their novel use of an experimental approach, however, there were methodological limitations to this study. In particular, both of the experimental groups received tasks which used the construct of variation meaning that there was no control group. Furthermore, this was a small-scale study based on five test questions which generated very little data for sophisticated statistical analysis.

To date, however, this is the only study which has used a pre- and post-test design and comparison groups to analyse the use of procedural variation in primary mathematics.

2.2.6 An intervention study of variation theory. Al-Murani (2007) has emphasised the lack of empirical studies in this field and used an unprecedented quasi-experimental intervention study design to evaluate variation theory. The Dimensions of Variation intervention focused on the systematic procedural use of variation to improve the algebraic understanding of approximately 300 Year 7 and 8 pupils. In order for pupils to discern the variant and invariant algebraic features, a 'series of progressively developing examples' was used to enable students to 'identify structurally critical features of a concept' (2007, p. 12). Over 14 months, six classes received the intervention teaching whilst another four classes were used as a 'business as usual' control.

Qualitative data from lesson observations demonstrated that the intervention teachers systematically used patterns of variation and invariance to enable students to infer relationships between algebraic equations to a much greater extent than the control group. Quantitative data from a pre-, post- and delayed post-test were collected and t-tests and an analysis of covariance (ANCOVA) were conducted using the pre-test results as a covariate. Al-Murani found that Dimensions of Variation programme may have had a significant effect for the duration of the intervention yet the differences between groups were not maintained at the delayed post-test.

Although no conclusive evidence was found for the long-term effects of teaching with variation, Al-Murani suggests that further experimental studies should be conducted with the axiology of improving classroom practice. It is the aim of the present study, therefore, to replicate a similar intervention study design to evaluate the use of procedural variation in the neglected field of primary mathematics.

2.3 Variation Theory in Primary Mathematics

2.3.1 A 'paradigm shift'? Al-Murani, Kilhamn, Morgan and Watson (2018) use examples of variation pedagogy to illustrate a 'paradigm shift' in the teaching of primary mathematics in England. In particular, they analyse primary teachers' use of variation pedagogy as part of a new 'maths mastery' approach to teaching. Video excerpts of three maths lessons were used to identify and qualitatively analyse the *dimensions of variation* used by the teachers. In a Year 6 lesson, invariant L-shapes were used across multiple varying shapes to enable pupils to identify the critical features of perpendicular lines. In contrast, the focus of a Year 3 lesson on multiplication was on the variant representations of multiplication, against the background of invariant numbers and questions. The third

example of a Year 1 lesson focused on the varied visual presentations of subtraction with the intention for pupils to discern the concept of 'part-part-whole'.

In his previous research in the field of secondary mathematics, Al-Murani (2007) emphasised the paucity of empirical evidence for variation theory. Al-Murani et al. (2018) have now provided the first study on the use of variation theory in British primary mathematics. While the qualitative analysis of these three lessons is interesting, however, it does not provide sufficient evidence for the assertion of a 'paradigm shift' in primary mathematics. This dissertation, therefore, is the first study to use an experimental design to quantitatively analyse the effects of variation pedagogy in primary mathematics.

2.3.2 The use of procedural variation in primary mathematics. This section situates the present study against the background of the UK 'maths mastery' curriculum, outlines the current use of procedural variation in British mathematics teaching and reiterates the lack of evidence for its nationwide implementation.

In an effort to replicate East Asian success in comparative international tests of mathematical attainment (Mullis, Martin, & Foy, 2016; OECD, 2013), the UK Department for Education (DfE, 2013) have recently implemented a new Mathematics curriculum which promotes a 'maths mastery' approach. In particular, 'maths mastery' teaching encourages pupils to make connections between concepts and 'become fluent in the fundamentals of mathematics, including through varied and frequent practice' (2013, p. 3). Funded by the DfE, the National Centre for Excellence in the Teaching of Mathematics (NCETM, 2014, pp. 1-2) have particularly emphasised the importance of procedural variation; teaching which focuses on 'developing procedural fluency' through the 'carefully designed variation' of tasks' in order to 'reveal the underlying mathematical structure to pupils'.

The concept of procedural variation, therefore, is consistently promoted to teachers as a research-based mathematics pedagogy by the nationwide NCETM Maths Hubs (Morgan, 2015). As an experienced primary Maths specialist teacher, I have attended multiple conferences and training seminars where procedural variation is promulgated as high-quality primary mathematics pedagogy. To date, however, no experimental studies have been conducted to evaluate the efficacy of this theory in primary mathematics. The aim of this paper, therefore, is to provide a significant original contribution by using a quasi-experimental intervention study to evaluate the effects of using procedural variation in primary mathematics; specifically in the context of multiplicative reasoning.

2.4 Multiplicative Reasoning

2.4.1 Operational invariants. Multiplicative reasoning combines varied and invariant features and is a particularly germane area for the evaluation of procedural variation. Building on Piagetian schema theory (Piaget, 1953, 1954), Vergnaud stipulates three interrelated psychological sets that that are necessary for the development of multiplicative reasoning (Vergnaud, 1988, 1994, 1997). Understanding of new mathematical concepts requires a set of meaningful situations; a set of invariant properties or relationships; and a set of symbolic representation. Of particular relevance to variation theory is Vergnaud's set of operational invariants: 'categories, objects, properties, relationships, theorems-in-action are invariants' (1988, p. 142).

In his theory of the multiplicative conceptual field, Vergnaud (1994, p. 58) emphasises how effective teaching should draw attention to 'invariance across situations' to 'help students identify invariants and recognise them as mathematical objects.' Pupils use the invariant properties of concepts to develop 'concepts-in-action' and use them in 'theorems-in-action' to solve multiplication questions (Vergnaud, 1997). Vergnaud further specifies that teachers 'should try to identify the different invariants involved in the schemes used by individuals to master the different aspects of these situations, even though most of these invariants may be totally implicit' (1997, p. 6). Vergnaud's emphasis on the explicit teaching of the invariant features of multiplication reveals the particular relevance of the construct of multiplicative reasoning for the analysis of procedural variation pedagogy.

2.4.2 Multiplicative reasoning. Mathematical reasoning requires not only the understanding of quantities but also the relationships between those quantities (Thompson, 1993). In particular, both additive and multiplicative reasoning require children to identify and use *invariants*; the logical principles or relationships underlying specific concepts (Nunes et al., 2007; Wheeler, Inhelder, & Piaget, 1965). Multiplicative reasoning requires children to understand the logical principle of the one-to-many correspondence; as a direct ratio or inverse relation between two quantities, or through the formation of a third quantity using two quantities (Nunes & Bryant, 1996, 2009, 2015). Multiplication questions, therefore, must always involve a fixed ration between two or more quantities (Nunes, Bryant, & Watson, 2009). Despite emphasising how multiplicative reasoning involves this new *invariant* intensive function, Nunes and Bryant (1996) also highlight the extrapolation of additive reasoning to multiplication questions through the schema of 'repeated addition'. Indeed, Behr, Harel, Post and Lesh (1994)

emphasise the pedagogical connections which can be made between additive and multiplicative conceptual fields. Multiplicative reasoning can either be conceptualised distributively as 'repeated addition' (and the inverse of subtraction) or as scalar 'lots' of a quantity.

2.4.3 Distributive reasoning. Inhelder and Piaget (1958) theorise that children initially solve multiplication problems using additive reasoning. Nunes et al. (2009), however, critique the separate teaching of the four arithmetical concepts and stipulate that children should be taught additive and multiplicative reasoning concomitantly. Distributive reasoning is the use of additive reasoning to solve multiplication questions; using a multiplying scheme of incremental addition of 'iterable units' (Steffe, 1988, 1994). Watson, Jones and Pratt (2013) discuss how children initially conceptualise multiplication distributively, with concepts such as 'times tables' being taught through repeated addition of 'lots' of a number. The present study, therefore, analyses whether procedural variation can be used in the familiar context of distributive reasoning to enable children to discern the *invariant* addition of multiples of the multiplicand.

2.4.4 Scalar reasoning. Confrey (1994) critiques the predominant teaching of multiplication as repeated addition as opposed to focusing on scaling or magnification of quantities. Scalar reasoning is the exponential use of a scalar function to relate two quantities through an *invariant* multiplicative action. For example, in the geometric sequence 1, 2, 4, 8, 16, a scalar factor of 2 is used to double the quantity each time (1994, p. 312). Furthermore, Smith and Confrey (1994, p. 338) reiterate the importance of developing children's scalar multiplicative reasoning so that they recognise the functional *invariant* of a 'constant (multiplicative) ratio'.

In more complex multiplication questions, such as those used in the present study, the scalar factor is a third variable which describes the covariation between two variables, or two sets of variables (Nunes & Bryant, 1996). The covariation of two quantities requires children to analyse the 'structural similarity' to identify the *invariant* scalar ratio which applies to both quantities, and even across a sequence of multiplications (Lesh, Post, & Behr, 1988, p. 101; Lamon, 1994). Scalar multiplicative reasoning, therefore, requires children to use *protoratio* reasoning to identify the invariant scalar relation between two variables (Nunes & Bryant, 1996; Singer, Kohm, & Resnick, 1997).

Watson et al. (2013) discuss how young children's scalar reasoning may be restricted to doubling and halving. Nunes and Bryant (2009) promote using children's informal knowledge to develop understanding of the scalar relations between quantities.

The present study, therefore, uses procedural variation within the familiar context of doubling before extending pupils' scalar reasoning to tripling and quadrupling.

2.4.5 Multiplicative reasoning and procedural variation. Warren (2005, p. 305) states that 'the power of mathematics lies in relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning'. It is the contention of this paper, that procedural variation enables children to see these patterns and enhances their multiplicative reasoning skills. Both distributive and scalar multiplicative reasoning require pupils to understand the *invariant* relations or patterns between quantities. These two aspects of multiplicative reasoning, therefore, are highly relevant constructs for the evaluation of procedural variation.

2.5 The Present Study

Nunes et al. (2009) have highlighted the need for experimental studies into how students' quantitative reasoning can be developed. Furthermore, Nunes and Bryant (2015, p. 756) emphasise the dearth of mathematical reasoning intervention studies with more research particularly required into 'the understanding of necessary and contextual multiplicative relations … [and] the ability to generate correspondences and ratios'. The present study, therefore, utilises a quasi-experimental intervention study design to evaluate whether procedural variation affects pupils' distributive and scalar reasoning skills.

This research was conducted with 7-8 year-old children with the axiology of determining whether procedural variation is an effective pedagogical tool to develop multiplicative reasoning skills. The intervention groups constitute the independent variable. Two experimental groups receive interventions using procedural variation to develop their distributive or scalar reasoning skills. For comparison, an active control group receive teaching about times tables with no use of procedural variation. The dependent variable is pupils' performance on the post-test, with the pre-test results being used as a covariate, in order to ascertain the effects of teaching with procedural variation.

The effects of procedural variation pedagogy on the multiplicative reasoning skills of Year 3 pupils are analysed through the following research questions.

1. Are Year 3 children capable of recognising and using the relationships between sequences of multiplication questions?
2. Does teaching using procedural variation improve Year 3 children's understanding of these relationships?

Procedural variation uses varied sequences of questions to develop children's understanding of the invariant features of a mathematical concept. The fundamental assumption of procedural variation, therefore, is that children are capable of discerning the variant and invariant relationships between sequences of calculations. The first research question investigates whether Year 3 pupils can infer and use the distributive and scalar relationships between sequences of multiplication questions in the pre-test. Following the interventions, the second research question analyses whether teaching using procedural variation can improve children's recognition of these distributive and scalar relationships.

3. Methodology

3.1 Design

The vast majority of research into variation pedagogy in mathematics has utilised a qualitative learning study approach (e.g. Huang, Barlow, & Prince, 2016; Marton & Pang, 2006; Runesson & Mok, 2004) . In this study, however, it was decided to use a quasi-experimental pre- and post-test intervention study. A major advantage of this design is the unprecedented opportunity to generate quantitative empirical data to analyse whether primary-aged children can discern the relationships between multiplication questions and whether variation pedagogy has any effect on their multiplicative reasoning skills.

Research was conducted in each class over a period of three days, with a total of eighteen days of research across the six different Year 3 classes. The pre-test was administered to all of the pupils before they were randomly split into the three intervention groups. Each group received three 30-minute sessions from the same specialist teacher over a period of three days. Finally, the same pre-test was then administered again as a post-test to the whole class. The re-administration of the same test before and after the interventions allowed results to be compared both within children and between the two intervention groups and the active control group.

3.2 Ethics

This project was approved by the Central University Research Ethics Committee in accordance with procedure 15 (group B activities) in which the researcher may engage in pedagogic activity to trial teaching innovations. Children were taught age-appropriate multiplication interventions by an experienced Maths specialist primary-age group teacher. Procedure 15 states that opt-out consent may be used when the research does not impinge upon pupils' learning and is not significantly different from their normal experience. As the interventions were based on the Year 3 programme of study (DfE, 2013) and were designed by an experienced teacher, they would supplement normal Year 3 Maths lessons and might even provide potentially beneficial effects for the pupils' mathematical understanding. The study was designed to ensure minimal disruption of the normal classroom learning environment with activities similar to those taught in Year 3 Maths lessons. CUREC approval was, therefore, granted for an opt-out consent procedure in which headteachers gave consent for the project to take place and parents were given the opportunity to withdraw their child with no detrimental consequences for their learning.

Data were collected from schools in the form of the pre- and post-tests which were kept securely in a locked cupboard. In order to meet General Data Protection Regulation requirements, each pupil was assigned a random ID number meaning that all digitised data was fully anonymised. Neither the schools nor the individual children can in any way be identified in this dissertation. Appendix A contains all relevant documents including the CUREC approval.

3.3 Recruitment

Participants were recruited from three Oxford primary schools; two located in the city centre and one in a nearby village. All three schools were rated 'Good' or 'Outstanding' by Ofsted and had an average socioeconomic background, based on the proportion of pupils receiving additional governmental 'pupil premium' funding. The schools were initially contacted by an email with a letter of invitation and consent form (Appendix A). Follow up telephone calls and meetings were then arranged to answer questions about the project, organise dates and to obtain pupil data to create the randomised groups.

Parent information letters were then circulated to all of the Year 3 parents to provide full details about the project and to give the opportunity for consent to be withdrawn (Appendix A). Furthermore, the aims and procedure of the study were explained in child-friendly language to each class with multiple opportunities given for the children to ask questions, to withdraw their assent or for the teacher to withdraw pupils from the study. At the end of the study, copies of the intervention planning and resources were shared with teachers and each school received a certificate of participation.

3.4 Sample and Groups

While randomised controlled tests have been described as the 'gold standard' for experimental research, limitations of time, resources and logistics necessitated the use of convenience sampling, clustered within three different schools, resulting in a quasi-experimental approach (Bickman & Reich, 2009; Cohen, Manion, & Morrison, 2018). In total, a large valid sample of 184 pupils was recruited from six Year 3 classes across the three schools. Due to data protection regulations, the exact ages of the pupils could not be obtained yet all the pupils were between the ages of 7 and 8 years-old and there was a roughly equal gender balance (male = 91, female = 93). Attrition was low with consent being withdrawn for three pupils prior to the study and only four absences during the

study. Only students who took the pre- and post-test and who were present for all three interventions were included within the data set.

Within each class, random numbers were used to allocate pupils to the three randomised groups: distributive reasoning, scalar reasoning and times tables as an active control. As gender could be an extraneous variable, the sample was initially split by gender before random numbers were used to allocate them to a group. This stratified random sampling provided a roughly equal balance between genders in each group so that this potentially extraneous variable could be investigated. The stratification of gender, commensurate with the use of randomisation, was intended to minimise baseline differences between groups insofar as was possible. Table 3.1 details the gender distribution and size of each experimental group.

Table 3.1

Gender Distribution Across Intervention Groups

	Distributive	Scaling	Control
Male	31	31	29
Female	31	31	31
Total	62	62	60

3.5 Measures

The same pre- and post-test was administered to the whole class prior to and after the interventions (Appendix B). The test consisted of four different tasks, each designed to elicit a different type of multiplicative reasoning from the children:

1. Distributive: addition
2. Distributive: subtraction
3. Scaling: doubling
4. Scaling: tripling and quadrupling

The four tasks each contained four sequences of multiplication questions which aimed to elicit the relevant type of multiplicative reasoning. Across all four tasks, therefore, there were sixteen sequences of questions, with a total of 64 answer boxes for the children to complete.

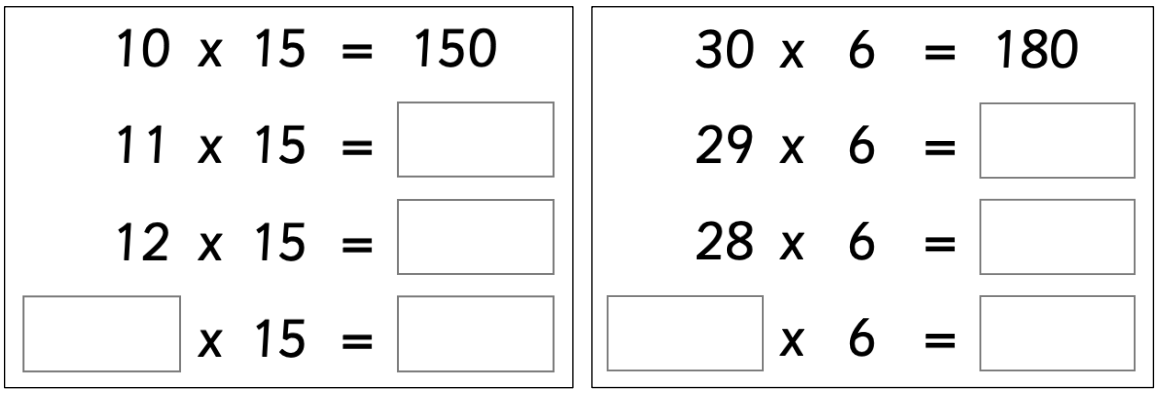


Figure 3.1. Distributive reasoning sequences eliciting addition or subtraction of multiplicands.

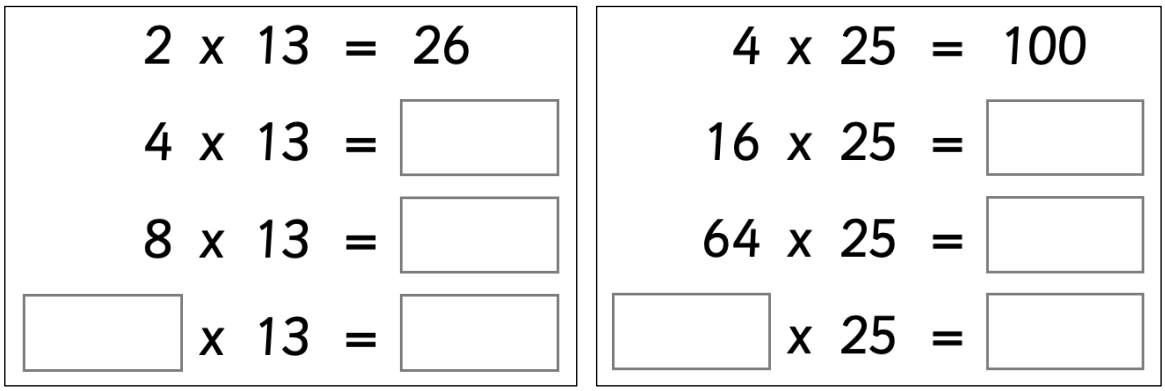


Figure 3.2. Scalar reasoning sequences eliciting doubling or quadrupling of previous products.

Figures 3.1 and 3.2 are examples of four sequences from the pre- and post-test requiring either distributive or scaling reasoning. Each sequence had an initial 'clue' with a question and answer. Children could then use this clue to find the answer to the two subsequent multiplication questions. Finally, children were asked to provide both the multiplier and product for the final multiplication question so that it 'matched the pattern', allowing them to further demonstrate their understanding of the relationships between the sequence of multiplications. A 'workings out box' was also provided for each question, allowing for qualitative analysis of the children's methods to supplement the quantitative data.

Each sequence of multiplication questions was carefully designed so that children could apply distributive or scalar multiplicative reasoning to the initial clue to find the answers. Throughout the test, all of the multiplicands were kept invariant so that the

children could focus on the variant multipliers and discern the relations between the calculations. The answer to the first easier multiplication was provided as a 'clue' which could then be manipulated to find the answer to subsequent multiplications. In the distributive sequences, the multipliers either increased or decreased by one, eliciting the addition or subtraction of one ‘lot’ of the multiplicand to the previous product. In the scalar sequences, the multipliers were doubled, tripled or quadrupled, meaning that children could apply this pattern to the clue to find the subsequent answers.

Pupils were given 30-minutes to attempt all four tasks and were provided with ‘working out boxes’ within the test paper. The same instructions script and practice question were used across the classes to ensure consistency in the test administration and to promote internal validity. Each question followed a pattern with which the children were familiarised through these instructions:

> Your challenge is to fill in the empty boxes in the sequence by spotting the pattern.
> The first answer has been given to you as a clue. Try to use the clue to help you find the answer to the next two multiplications.
> Then, complete the final multiplication question so that it matches the pattern.

3.6 Reliability and Validity

A reliability analysis was conducted on the items of the pre- and post-test to assess the internal consistency of the measure. The Cronbach’s alpha value was excellent across the 64 test items with $\alpha = .965$ on the pre-test and $\alpha = .973$ on the post-test. Despite being a newly created test, therefore, this measure was highly reliable (Kline, 2000).

In order to promote content and construct validity, questions were carefully created with one variant and one invariant variable (Watson & Mason, 2006a). Within each question, the multiplicand remains invariant whilst the multipliers are systematically varied to enable children to discern and use the distributive and scaling multiplicative relations to find the products. Furthermore, Bloom (1956) specifies that ‘creation’ of questions is a ‘higher order thinking skill’. The final question in each multiplication sequence, therefore, is designed to enable children to create their own question to match the pattern and to demonstrate their understanding of the invariant pattern between the calculations. Content validity was further measured through analysis of the ‘workings out box’ where children were given the opportunity to demonstrate their reasoning.

Internal validity requires the control of confounding variables in order for the relationship between the independent and dependent variables to be examined (De Vaus, 2001). As a Maths specialist teacher, all interventions were taught by myself to ensure high internal validity with a reliable and consistent 'dose' across the entire cohort. Internal validity was promoted through the almost exact replication of test procedures and interventions across the six classes. Confounding variables such as children's behaviour were controlled insofar as was possible, commensurate with the naturalistic setting of the research.

Indeed, a major strength of this study is the high ecological validity with children being taught this innovative pedagogy by an experienced primary teacher in a naturalistic classroom environment. Furthermore, Nuthall (2004) has emphasised the importance of pragmatic validity in educational research; the extent to which pedagogical theory can be implemented in classroom practice. The present study evaluates the use of procedural variation within a naturalistic setting, with the axiology of developing mathematics pedagogy. The external validity of the study, the extent to which the results can be generalised to other settings, is therefore promoted both through the naturalistic design of the study and the large sample size (Gomm, 2008). Both the ecological and pragmatic validity of the present study allow for the realistic evaluation of how the pedagogy of procedural variation can be utilised within the classroom.

3.7 Interventions

Each of the three groups received three 30-minute interventions focusing on either distributive or scaling reasoning, or times tables as an active control. Over the five weeks of research, a total of 54 intervention sessions were taught. As an experienced Maths specialist teacher, I specifically designed all of the interventions to be within a Year 3 pupil's 'zone of proximal development' (Vygotsky, 1978; Wood et al., 1976) based on the Year 3 Mathematics Programme of Study (DfE, 2013). The children already had prior knowledge of addition, subtraction and times tables which they could generalise to the new concepts of distributive and scaling reasoning. In particular, multiplication questions were carefully designed to avoid the multiplication of a pair of two-digit numbers, instead a combination of one and two-digit numbers at an appropriate level of difficulty for Year 3 pupils. Appendices C, D and E contain the detailed intervention planning and full resources for each of the three interventions.

3.7.1 Distributive reasoning. The distributive reasoning intervention focused on the procedural variation of sequences of multiplication questions where pupils were required to add or subtract 'lots' of the multiplicand. The children were taught to use an initial multiplication question as a 'clue' in order to infer the distributive relations between a sequence of calculations. In each sequence, the multiplier increased or decreased by one whilst the multiplicand remained invariant. A combination of teacher-child conversation, questioning and independent activities was used to encourage children to discern the variant and invariant features of the calculations. In particular, emphasis was placed on manipulating the initial 'clue' by using addition or subtraction of the multiplicand. This distributive reasoning enabled children to use the initial 'easier' calculation to quickly answer the sequence of multiplication questions without having to resort to a formal written method.

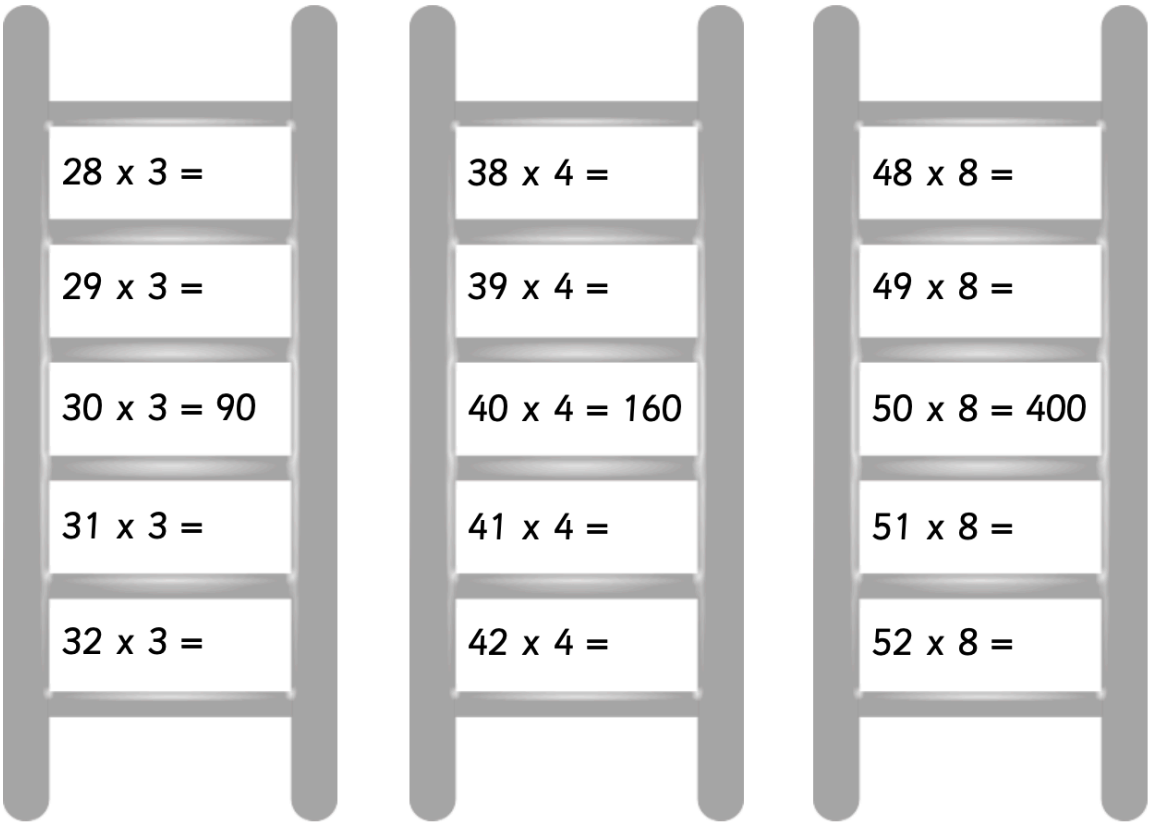


Figure 3.3. Distributive reasoning intervention: ladders activity.

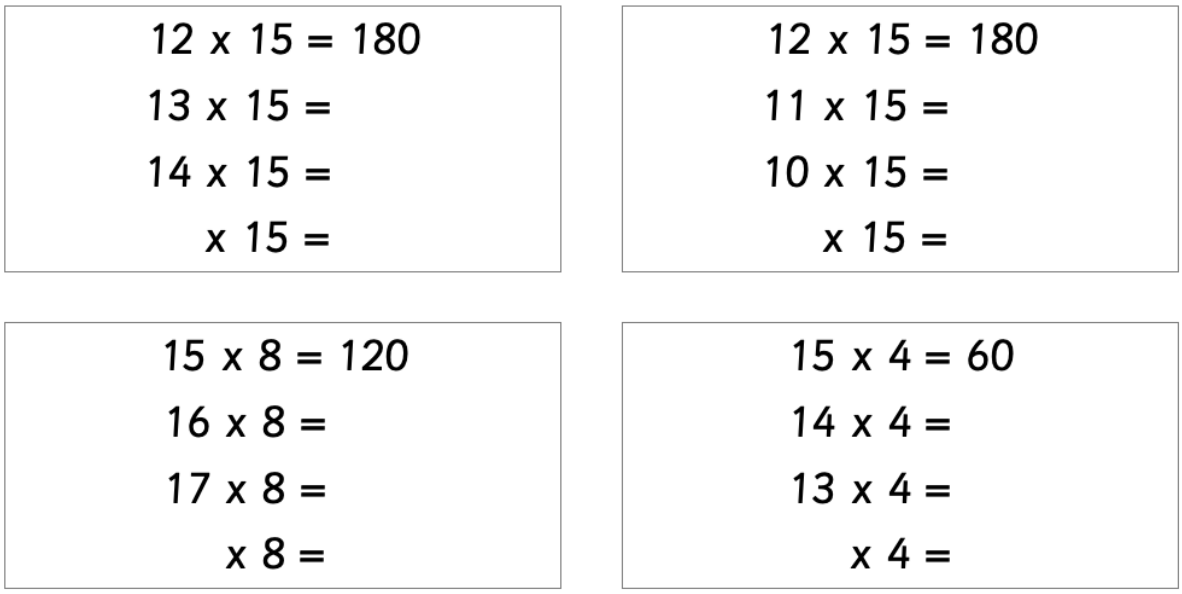


Figure 3.4. Distributive reasoning intervention: speed quiz activity.

Figures 3.3 to 3.6 are examples of resources used in the distributive and scaling interventions, where questions were systematically designed to highlight the variance and invariance between calculations. Rittle-Johnson & Star (2009) emphasise the importance of comparing two or more examples to enable children to analyse similarities and differences. The *intended object of learning* is for pupils to notice the variant and invariant features and to generalise to the invariant distributive relations between calculations, namely the addition and subtraction of the multiplicand (Marton & Tsui, 2004). In addition, Kullberg, Kempe, & Marton (2017) discuss how the *enacted object of learning*, the way in which interventions are taught and experienced further enables children to discern these relations. These resources generated discussion between teacher and pupils which promoted multiplicative reasoning about the relations between calculations and how these could be used to solve the subsequent calculations.

Vergnaud (1994) particularly emphasises the use of arrows to draw pupils' attention to multiplicative functions. In each session, therefore, arrows were drawn between each calculation and each side was labelled with '+ one lot'. In the distributive intervention, the other side of the calculation was labelled with plus or minus the multiplicand. In the scaling intervention, arrows were labelled with 'double', 'triple' or 'quadruple' to further draw the pupils' attention to the variant and invariant features in each sequence and to enable them to answer subsequent questions.

3.7.2 Scalar reasoning. The scalar reasoning intervention used procedural variation of a sequence of multiplication questions to promote children's multiplicative reasoning. Similar to the distributive intervention, the scalar reasoning intervention also prompted children to use an initial 'clue' to solve more complex multiplication questions. In this intervention, however, the teaching and activities focused on noticing whether there was a scalar relationship between each of the multiplication questions.

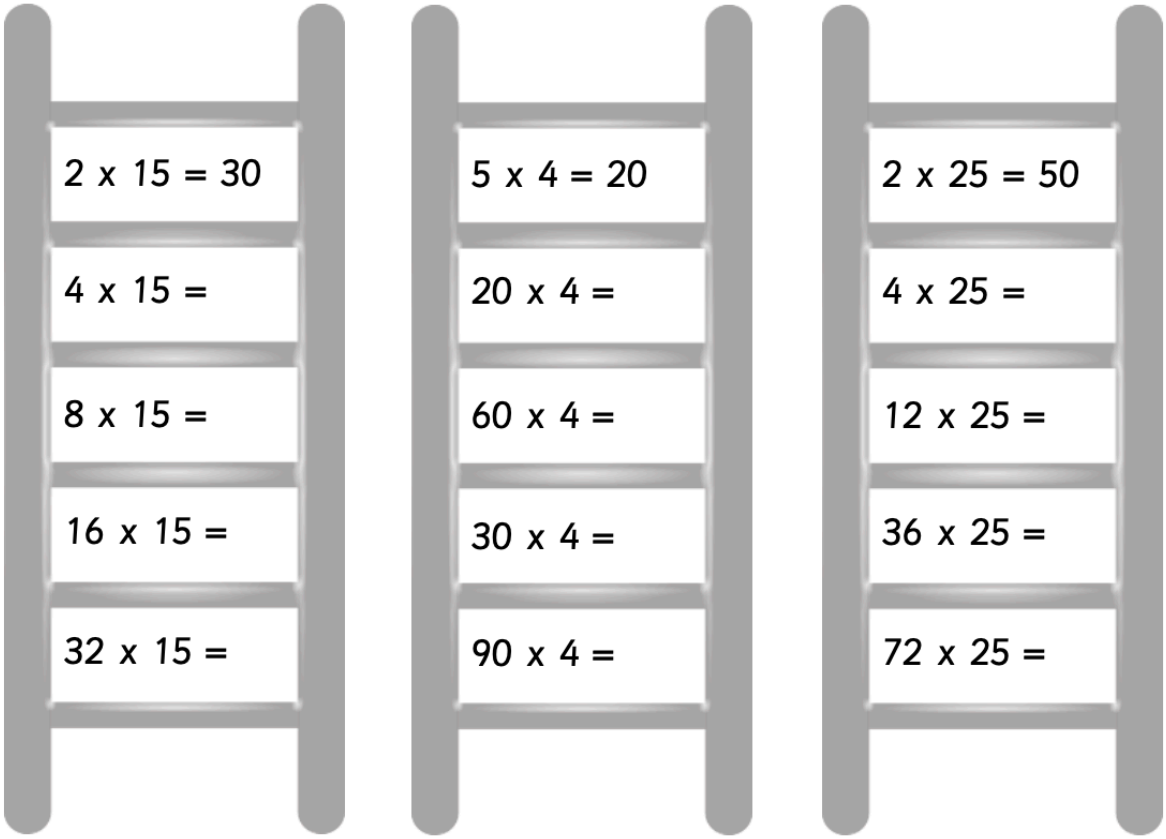


Figure 3.5. Scalar reasoning intervention: ladders activity.

Figures 3.5 and 3.6, in comparison with Figures 3.3 and 3.4 from the distributive intervention, highlight the similarities and differences between the two interventions. While the multiplicand is still invariant, the children were required to analyse whether there was a doubling, tripling or quadrupling relationship between the multipliers. Having identified the relationship between the questions, the children were taught to use this pattern to deduce whether they should multiply the previous product by two, three or four. Apart from the substantial difference of focusing either on distributive or scalar reasoning, the teaching approach and activities were the very similar across the two experimental groups. The similarity of the interventions allows for specific analysis of the

effects of procedural variation pedagogy on children's distributive or scaling reasoning skills whilst reducing the likelihood of any other confounding variables.

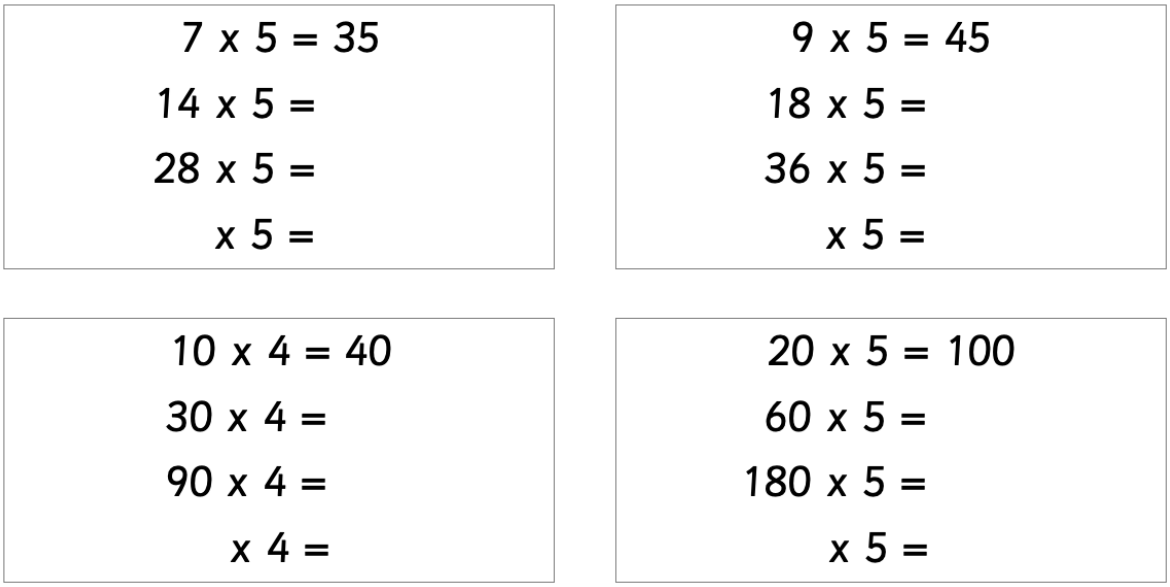


Figure 3.6. Scalar reasoning intervention: speed quiz activity.

3.7.3 Active control. Internal validity was promoted through the use of an active control group in which pupils were given a variety of times tables and activities. In order to ensure that additional time spent learning about multiplication was not a confounding variable, these pupils were also given three 30-minute intervention sessions, with the same teacher as the other interventions, yet focusing only on times tables. These interventions were carefully designed to promote the children's multiplication skills whilst ensuring that no distributive or scalar reasoning was taught.

3.8 Pilot Study

Prior to the main study, a pilot study was conducted with a Year 4 class (8-9 year-olds, n = 30). Originally, it was hypothesised that the multiplicative reasoning required in the interventions would be more appropriate for this age group. Results from the pre-test, however, exhibited ceiling effects with 43% of pupils scoring marks of 90% or more. Furthermore, although the interventions were interesting and accessible to the pupils, many of the children demonstrated, through their calculations and comments, that they already understood many of the distributive and scaling relationships between the multiplication questions. Following consultation with the Year 3 and 4 teachers, I decided to move the study from the Year 4 to Year 3 age group (7-8 year-olds) in order to obviate

these ceiling effects. While the interventions content would be significantly more challenging for this younger group, it would provide greater opportunities to analyse the effects of the interventions and to discern whether variation pedagogy has an effect on children's multiplicative reasoning.

4. Results

The results from the pre- and post-tests comprise over 20,000 data entries. This chapter analyses these data with respect to the following research questions.

1. Are Year 3 children capable of recognising and using the relationships between sequences of multiplication questions?
2. Does teaching using procedural variation improve Year 3 children's understanding of these relationships?

The first section uses data from the pre-test to assess whether children can infer and use the relationships between sequences of multiplications. Furthermore, it examines the distribution of children's errors and discusses common misconceptions in children's multiplicative reasoning.

The second section considers whether Gender should be a variable in the post-test inferential analysis by evaluating whether there is a significant interaction between Gender and Task in the pre-test.

The third section investigates the effects of procedural variation teaching on children's multiplicative reasoning. Pre- and post-test data are analysed to evaluate whether there are main effects of Group and Task and whether there is a Group x Task interaction.

The fourth section supplements these quantitative data with qualitative analysis of children's use of distributive and scalar reasoning in their workings on the post-test.

Finally, the fifth section provides a summary of the findings with respect to the two main research questions.

4.1 Children's Multiplicative Reasoning in the Pre-Test

4.1.1 Descriptive statistics. Figure 4.1 shows the approximately normal distribution of the pre-test total scores across all 184 participants. Each child's score indicates the number of correct answers they gave. There was a full range of scores between 1 and the maximum score of 64 with a mean of 22.36 (SD = 14.33). The difficulty of the pre-test resulted in a higher frequency of low scores with a positive skew of .687 and some positive kurtosis of .276. This skewness and kurtosis, however, is within the appropriate bounds for a normal distribution (Field, 2018).

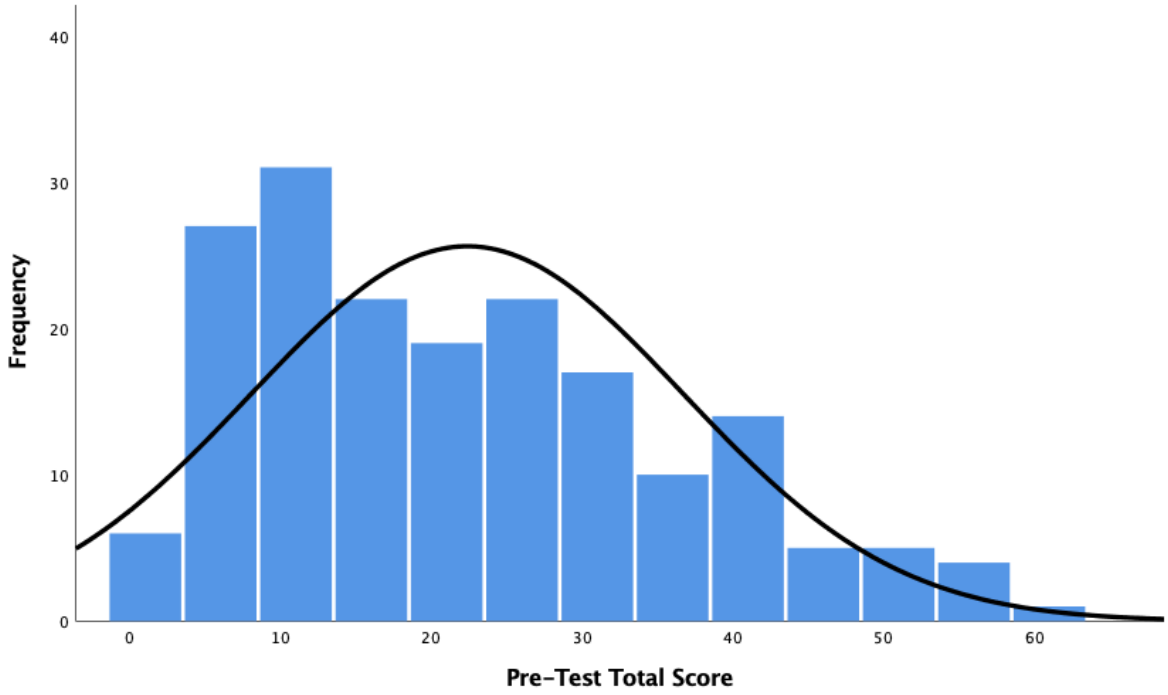


Figure 4.1. Distribution of pre-test total correct scores. Maximum total score of 64.

Table 4.1

Means and Standard Deviations of Correct Scores in Pre-Test Tasks

Tasks	*M*	*SD*
Distributive		
Addition	9.61	4.78
Subtraction	7.93	5.16
Scaling		
Doubling	3.79	4.86
Tripling and Quadrupling	1.04	2.89
Pre-Test Total	22.36	14.33

Note. *M* = mean, *SD* = standard deviation. Maximum score of 16 on each task.

Table 4.1 shows the pre-test means and standard deviations for all four tasks where 16 is the maximum score which can be obtained in each task. These data are presented graphically in Figure 4.2 with error bars representing 95% confidence intervals of the mean. The higher mean scores for the distributive tasks indicate that Year 3 children are more confident with distributive reasoning. Mean scores were highest for the addition and subtraction tasks which may be due to the children's familiarity with times

tables, which require addition of 'lots' of the multiplicand. Mean scores for the scaling tasks, however, were much lower with many children failing to recognise the doubling, tripling or quadrupling relationship between multiplication questions. The lack of overlap of the 95% confidence interval error bars in Figure 4.2 indicates significant differences between the tasks, particularly between distributive and scaling tasks and justifies the inclusion of Task as a within subjects factor on the post-test analysis.

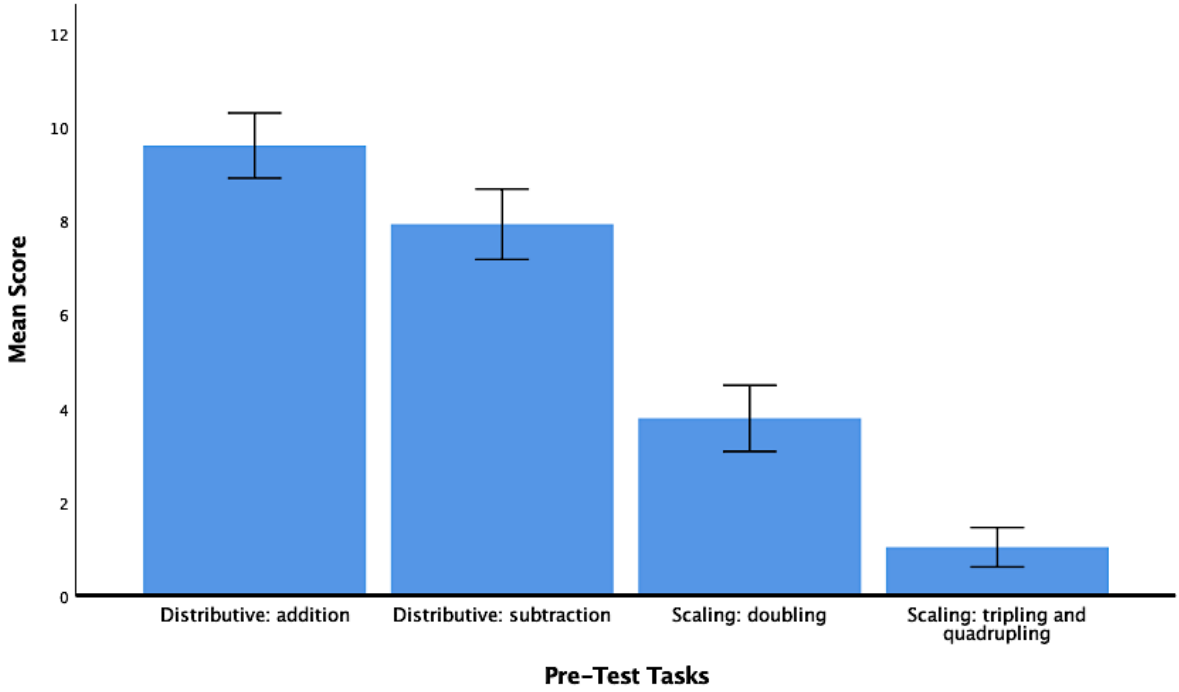


Figure 4.2. Mean correct scores in pre-test tasks. Error bars represent 95% confidence intervals of the mean. Maximum score of 16 on each task.

4.1.2 Conceptual and arithmetic errors. Analysis of the children's incorrect answers in the pre-test reveals both arithmetic inaccuracies and specific conceptual errors in their multiplicative reasoning. This section focuses on each child's first answer following the initial correct 'clue', in order to evaluate specific conceptual errors as opposed to arithmetic errors carried through from a preceding answer.

The question numbers used in Tables 4.2 and 4.3 denote where each question occurs within a particular sequence, with Question 2.1 referring to the first answer in the second sequence of questions. Table 4.2 presents the frequencies of specific types of errors in the distributive addition task. While a significant proportion of children gave the correct answer, analysis of the most frequent incorrect answers allows specific conceptual errors to be identified.

Table 4.2

Types of Answers in the Distributive Addition Task

	Question 1.1 10 x 15 = 150 11 x 15 = ___			Question 2.1 20 x 7 = 140 21 x 7 = ___		
Type of Answer	Answer	*n*	%	Answer	*n*	%
Correct Answer	165	79	42.9	147	119	64.7
Addition of 1	151	14	7.6	141	17	9.2
Addition of 10	160	6	3.3	150	7	3.8
Addition of multiplier	161	7	3.8	161	1	0.5
Addition of units only	155	23	12.5	-	-	-
Other Answer	-	55	29.9	-	40	21.7

Note. Dashes have been used where the error is not relevant or to denote multiple other answers.

The most common error in the distributive addition task was the addition of 1 as opposed to a whole 'lot' of the multiplicand. For example, 9.2% of children gave the answer of 141 to Question 2.1 by adding 1 onto the clue of 20 x 7 = 140 instead of adding the correct multiplicand of 7. These children recognise that the multiplier has increased by one and therefore add on one to the product, revealing an underlying misconception of the one-to-many correspondence of multiplication. Similarly, a smaller proportion of around 3% of the children committed a related error, the addition of 10 instead of the correct amount. Furthermore, 3.8% of the children on Question 1.1 added on the multiplier instead of the multiplicand, indicating a further conceptual error with the one-to-many correspondence.

In contrast to these underlying misconceptions, some children made a specific arithmetic error in Question 1.1 with 12.5% giving the answer of 155 instead of 165. These children added on 5 instead of 15 and this 'addition of units only' suggests that the children had some distributive understanding yet omitted to also add on the tens-digit. There may, therefore, be a dichotomy in children's errors between significant conceptual errors and common arithmetic inaccuracies where young children may forget one necessary element of the calculation.

Table 4.3

Types of Answers in the Distributive Subtraction Task

	Question 1.1 10 x 13 = 130 9 x 13 = ___			Question 2.1 20 x 8 = 160 19 x 8 = ___		
Type of Answer	Answer	*n*	%	Answer	*n*	%
Correct Answer	117	74	40.2	152	80	43.5
Addition instead of subtraction	143	10	5.4	168	12	6.5
Subtraction of units only	127	15	8.2	-	-	-
Subtraction of 1	129	4	2.2	159	4	2.2
Subtraction of 10	120	4	2.2	150	2	1.1
Other Answer	-	77	41.8	-	86	46.7

Note. Dashes have been used where the error is not relevant or to denote multiple other answers.

Table 4.3 analyses the types of errors made in the distributive subtraction task. A small proportion of children made similar errors to the addition task; subtraction of 1 or 10 instead of the multiplicand. As stated previously, these errors indicate a basic lack of understanding of the one-to-many correspondence of multiplication with these children failing to understand the concept of 'lots' of the multiplicand.

Where the children attempted to use distributive reasoning, however, two further arithmetic errors occurred in the subtraction task. Having completed the previous two pages of the pre-test which elicited distributive addition (see Appendix B), around 5% of the children used addition of the multiplicand instead of subtraction. For example, having been given the clue of 20 x 8 = 160 and being asked to find 19 x 8, they added on the multiplicand of 8 to give the answer of 168 instead of subtracting 8 to give 152. This error suggests that these children understand the concept of distributive reasoning, yet they failed to concentrate on the decreasing value of the multiplier.

Similar to the error of only adding the units in the addition task, 8.2% of children made the error of only subtracting the units in Question 1.1 of the subtraction task. Having been given the clue of 10 x 13 = 130, the following question in the sequence was 9 x 13. These children subtracted 3 to give an answer of 127 yet forgot to also subtract

the 10 to give the correct answer of 117. As suggested previously, these children may have had some understanding of the distributive relationship between the questions, yet they forgot to subtract both the tens and units from the previous answer. This error, therefore, may not be a mistake in multiplicative reasoning. Instead, it may demonstrate the common arithmetic difficulty of processing multiple pieces of information contemporaneously.

The children scored significantly lower on the scaling tasks, both on the doubling task and particularly on the tripling and quadrupling task. This result was not unexpected and revealed a general lack of understanding of the scalar relationships between calculations amongst this young age group. Despite the wide variety of incorrect answers, it is possible to discern some specific errors and misconceptions in children's reasoning.

In Question 1.1 of the doubling task, children were given the clue of 2 x 13 = 26 and were asked to find the answer to 4 x 13. Although 39.7% gave the correct answer of 52, the most common error made by 6% of the children was the addition of one 'lot' of 13, giving the answer of 39. This error was made by a similar percentage of children across all four sequences of questions and indicates that they may have wrongly applied distributive addition to these questions. These children may have failed to notice the scalar relationships between questions and instead they have assumed that the multipliers are only increasing by one.

On the most difficult scaling tasks, tripling and quadrupling, there was the greatest variety of answers with no discernible patterns in children's errors. As opposed to the distributive reasoning tasks, therefore, fewer specific errors could be identified on the scaling tasks due to children's general lack of understanding of scalar multiplicative reasoning.

4.2 Gender and Multiplicative Reasoning

4.2.1 Descriptive statistics. Gender was stratified across intervention groups as it was predicted that this could be a significant confounding variable. In order to test this prediction, pre-test scores were analysed with respect to gender. Interestingly, there was a large difference in total pre-test scores with a mean of 25.18 (SD = 15.61) for boys and 19.61 (*SD* = 12.44) for girls. Mean scores within each task were also analysed to see whether this difference was maintained across tasks.

Table 4.4

Means and Standard Deviations of Correct Scores in Pre-Test Tasks between Genders

	Distributive Tasks				Scaling Tasks			
	Addition		Subtraction		Doubling		Tripling and Quadrupling	
	M	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
Male	9.86	4.58	8.79	5.13	4.92	5.51	1.60	3.63
Female	9.37	4.97	7.09	5.06	2.68	3.84	0.48	1.77

Note. *M* = mean, *SD* = standard deviation. Maximum score of 16 on each task.

Table 4.4 shows the consistent disparity between genders within each task and these data are presented graphically in Figure 4.3 which illustrates how boys outperform girls on each task. It was therefore decided to conduct a mixed design analysis of variance (ANOVA) to investigate whether there was a significant difference between genders and whether gender interacted with the different tasks.

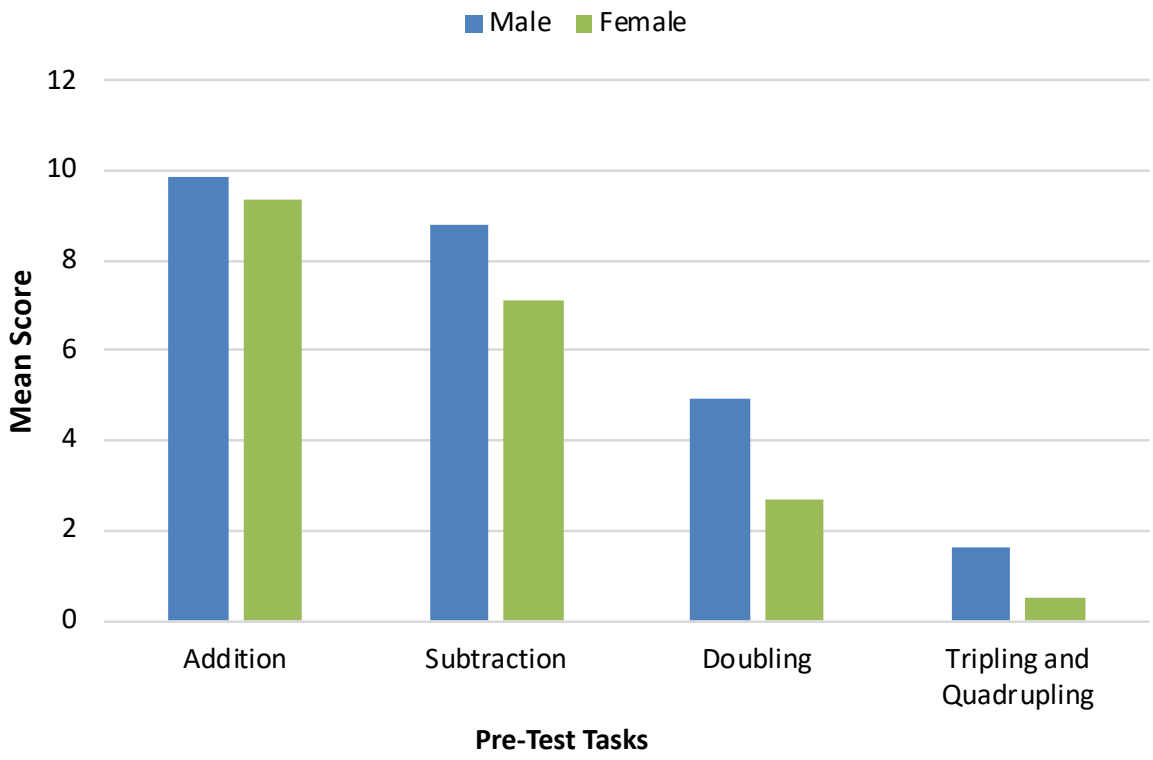


Figure 4.3. Pre-test mean correct scores between genders in each task. Maximum score of 16 on each task.

4.2.2 Parametric assumptions of the ANOVA. The between subjects variable of gender was dichotomous, meaning that the parametric assumption of independence of groups was met. The dependent variable of pre-test scores was measured at the continuous level and Levene's test of homogeneity of variance was not significant ($p > .05$). Although the Shapiro-Wilk test of normality was significant across all test variables ($df = 91$, $p < .05$), this test is extremely susceptible to small deviations from normality in large sample sizes ($n = 184$) (Field, 2018). Visual inspection of histograms and Q-Q plots, however, indicated roughly normal distributions. Mean and median values for each test variable were approximately equal and skewness and kurtosis were within tolerable bounds (Hair, Black, Babin, & Anderson, 2019). Ghasemi and Zahediasl (2012) state that such minor violations of normality often occur in larger sample sizes (> 40) yet this should not preclude the use of parametric tests. Indeed, Lunney (1970) emphasises that the ANOVA test is particularly robust to violations of normality when group sizes are equal, as in the present study (male = 91, female = 93).

4.2.3 Mixed ANOVA between genders and within tasks. A mixed Gender (2) x Task (4) ANOVA was conducted to investigate whether there was a significant difference between genders and whether there was an interaction between gender and task type. Furthermore, this ANOVA also evaluated whether the difference in mean scores between the four tasks was significant.

Mauchly's test of sphericity indicated that the assumption of sphericity had been violated, $\chi^2(5) = 40.62$, $p < .001$) and as $\epsilon > .75$, the degrees of freedom were corrected using Huynh-Feldt estimates of sphericity ($\epsilon = .89$) (Field, 2018, p. 461).

Table 4.5 shows that there was a significant between subjects effect of Gender ($F_{[1, 182]} = 7.17$, $p = .008$, $\eta_p^2 = .038$) with a small partial eta-squared effect size of less than .06 (Cohen, 1988). This indicates that Gender has a small yet significant moderating effect on children's multiplicative reasoning. Planned contrasts using the Bonferroni correction to reduce the likelihood of a Type I error showed an estimated marginal mean difference of 1.391 between male and female subjects ($p = .008$).

Gender is a significant variable in this study with the descriptive statistics of Section 4.2.1 demonstrating how boys consistently outperform girls in their multiplicative reasoning. This result justified the methodological decision to stratify the sample by gender. The sampling procedure minimised the effects of this potentially confounding variable by randomly allocating the same number of male and female

subjects to each group. As a result, there was an exactly equal balance of genders in both experimental groups.

Table 4.5

Mixed ANOVA Between Genders and Within Tasks on Pre-Test Total Correct Score

Source	Sum of Squares	Degrees of Freedom	Mean Square	*F*	*p*	Partial η^2
Between Subjects						
(Intercept)	5766.66	1	5766.66	464.58	.000***	.719
Gender	88.96	1	88.96	7.17	.008**	.038
Error	2259.08	182	12.41			
Within Subjects						
Tasks	8381.17	2.67	3138.14	281.20	.000***	.607
Gender * Tasks	78.71	2.67	29.47	2.64	.056	.014
Error	5424.51	486.08	11.16			

*p < .05. **p<.01. ***p<.001.

Table 4.5 also shows that there was a significant main effect of Task ($F_{[2.67, 486.08]} = 281.20$, $p < .001$, $\eta_p^2 = .607$). The partial eta-squared value is greater than .1379 and indicates that there is a large effect size of task type (Cohen, 1988). This result justifies the use of Task as a within subjects factor in the post-test analyses.

There was not, however, a significant interaction between Gender and Task ($F_{[2.67, 486.08]} = 2.64$, $p = .056$, $\eta_p^2 = .014$). Due to the lack of significant interaction between gender and tasks, and the equal balance of genders in the intervention groups, it was therefore decided not to include gender in the post-test analyses of intervention effects.

4.3 Post-Test Differences between Intervention Groups.

4.3.1 Descriptive statistics. Pre- and post-test group means on the distributive and scaling tasks are presented in Tables 4.6 and 4.7 and graphically in Figures 4.4 and 4.5. The maximum score possible on each task was 16. Despite randomisation, there were some differences in baseline means between groups. To facilitate comparison between intervention groups, therefore, mean difference scores between pre- and post-test results have also been calculated. In order to further control for these baseline differences, pre-test scores are used as a covariate in the inferential analysis of Section 4.3.3.

Table 4.6

Means and Standard Deviations of Correct Scores in Pre- and Post-Test Distributive Tasks

	Distributive Tasks			
	Addition		Subtraction	
Group	*M*	*SD*	*M*	*SD*
Distributive				
Pre	10.16	4.70	8.19	4.83
Post	13.55	3.41	10.98	4.63
Difference	+ 3.39		+ 2.79	
Scaling				
Pre	10.27	4.62	8.61	5.39
Post	10.24	5.22	8.98	5.65
Difference	- 0.03		+ 0.37	
Control				
Pre	8.35	4.84	6.95	5.17
Post	10.25	4.46	7.55	5.05
Difference	+ 1.90		+ 0.60	

Note. Difference scores have been calculated by finding the difference between pre- and post-test means. *M* = mean, *SD* = standard deviation. Maximum score of 16 on each task.

Table 4.6 enables comparison of pre- and post-test means within each intervention group on the distributive tasks. The distributive intervention group made substantial mean gains of + 3.39 and + 2.79 on the addition and subtraction tasks respectively. In contrast with the relatively small improvements made by the other two groups, this suggests that there was a specific effect of the distributive intervention. The improvements made in this group suggest that teaching using procedural variation improved children's understanding of the distributive relationships between multiplication questions.

Table 4.7

Means and Standard Deviations of Correct Scores in Pre- and Post-Test Scaling Tasks

	Scaling Tasks			
	Doubling		Tripling and Quadrupling	
Group	*M*	*SD*	*M*	*SD*
Distributive				
Pre	3.66	4.56	1.05	2.58
Post	5.39	5.69	1.90	3.65
Difference	+ 1.73		+ 0.85	
Scaling				
Pre	4.76	5.74	1.45	3.75
Post	8.68	6.12	4.42	5.91
Difference	+ 3.92		+ 2.98	
Control				
Pre	2.92	4.00	0.60	2.04
Post	3.68	4.89	1.07	2.72
Difference	+ 0.77		+ 0.47	

Note. Difference scores have been calculated by finding the difference between pre- and post-test means. *M* = mean, *SD* = standard deviation. Maximum score of 16 on each task.

Table 4.7 shows pre- and post-test means within each intervention group on the scaling tasks. The scaling intervention group made considerable mean gains of + 3.92 and + 2.98 on the doubling and tripling and quadrupling tasks respectively. The distributive group have made slightly larger gains than the control group which may indicate a small transfer of procedural variation understanding to the scaling tasks. Nevertheless, the larger gains made by the scaling group appear to indicate a specific effect of procedural variation teaching on children's understanding of the scalar relationships between sequences of questions.

Although task type is a discrete categorical variable, the decision was made to present the mean scores for each group using a line graph instead of a bar graph. The use of lines for each group allows comparison of the differences in trajectories between pre- and post-test and facilitates analysis of the interactions between interventions and tasks.

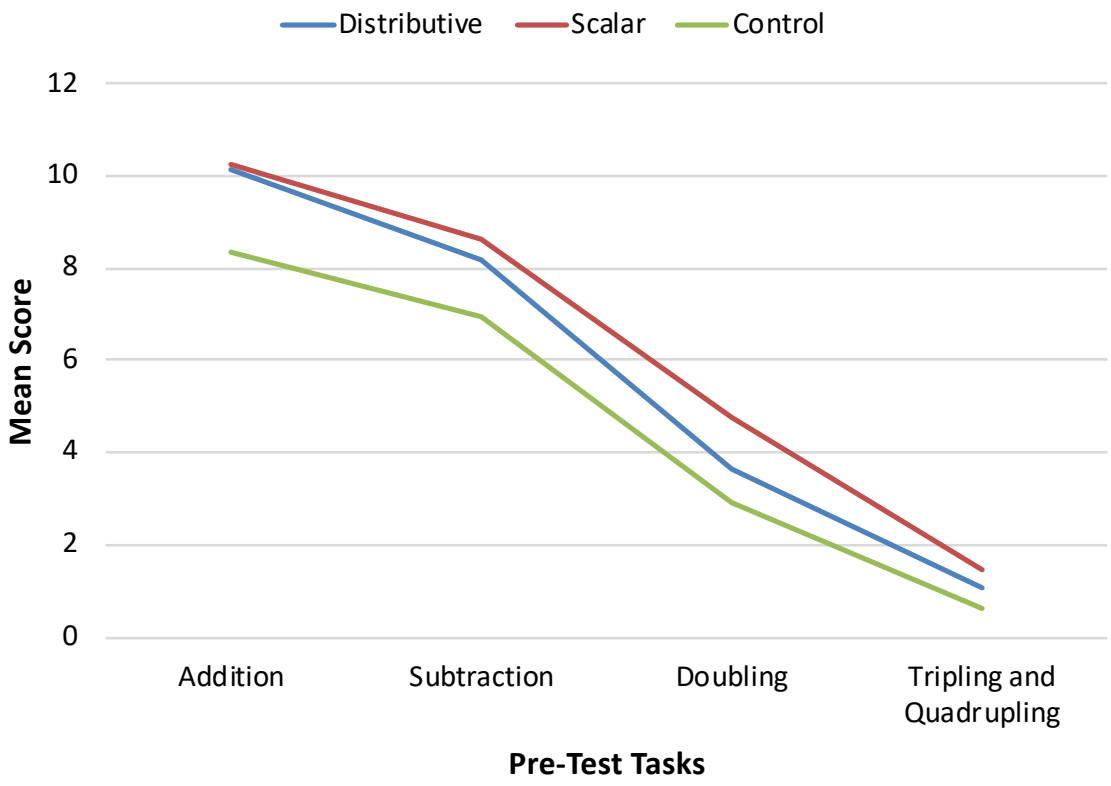


Figure 4.4. Pre-test mean correct scores by intervention group.

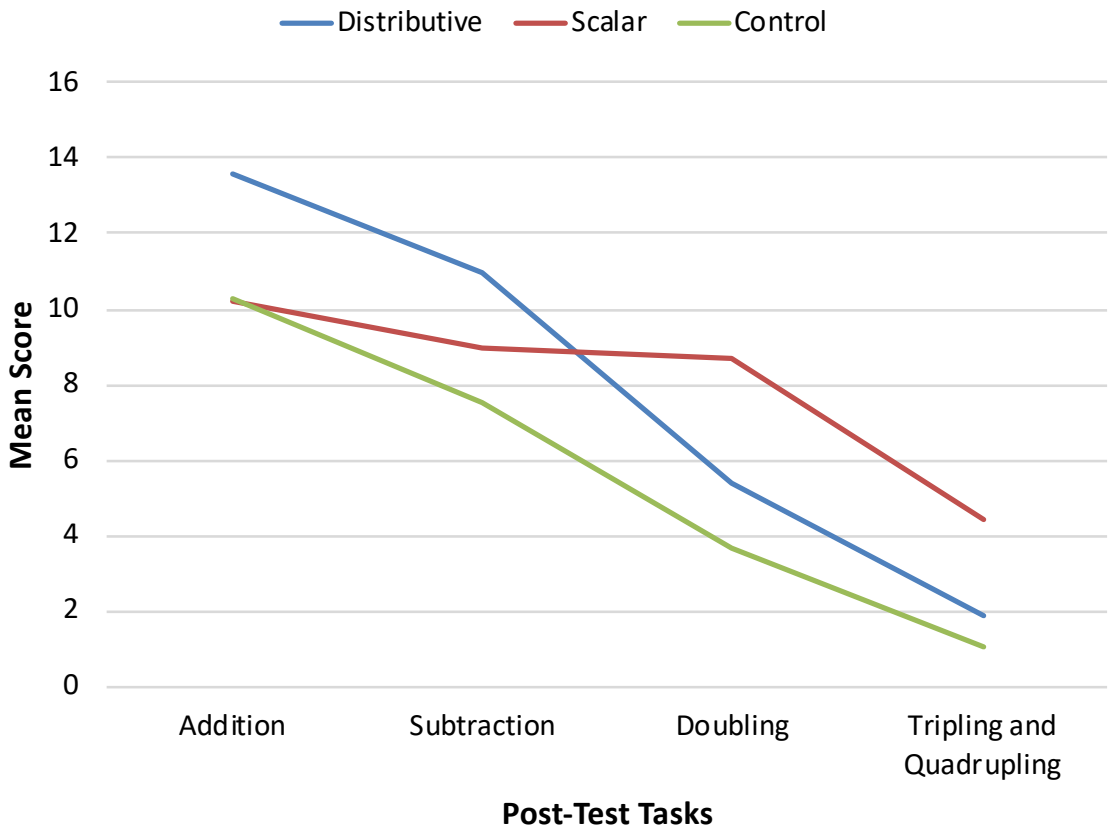


Figure 4.5. Post-test mean correct scores by intervention group.

Figure 4.4 demonstrates the similar baseline performance of all groups in each of the four tasks. In particular, the parallel downward trajectories demonstrate how all three groups experienced similar difficulties on the more complex tasks. In comparison, Figure 4.5 presents the post-test mean scores with clear interactions between intervention groups and tasks. While the control group maintains a similar gradient to the pre-test scores, the distributive and scalar reasoning lines have substantially different trajectories and are no longer parallel to the control group. The distributive group score more highly on the distributive addition and subtraction tasks whereas the scaling group show an improvement on tasks requiring scalar reasoning. Furthermore, the intersection of the distributive and scalar lines further demonstrates the specific interaction between each intervention and the distributive and scalar reasoning tasks. It was therefore decided to conduct an analysis of covariance (ANCOVA) to further investigate these interactions between intervention groups and tasks using pre-test scores as a covariate to control for baseline differences.

4.3.2 Parametric Assumptions of the ANCOVA. The between subjects factor of intervention group and the within subjects factor of task type are both categorical variables. The intervention groups were completely independent with no crossover in content. The within subjects factor of task type is also categorical and consisted of four related tasks which all participants received. The dependent variable of participants' scores was measured at the continuous level and there were no significant outliers. The random allocation of participants to experimental groups meant that the pre-test covariate was not correlated with the experimental outcome ($p = .11$). Independence of the covariate and the dependent variables across experimental groups was established through inspection of the homogeneity of regression slopes which were almost exactly parallel.

The Shapiro-Wilk test of normality was significant across the 12 different Group x Task conditions ($df = 62$, $p < .05$) yet, as stated previously, this test is very sensitive to minor deviations from normality in large samples ($n = 184$) (Field, 2018). Normality of the dependent variable between groups and within tasks was also evaluated through Q-Q plots which showed approximately normal distributions. Levene's test of equality of error variances was not significant ($p > .05$) for the distributive tasks yet it was significant for the scaling tasks ($p < .05$) indicating some heterogeneity in variances between groups on these tasks. Keppel (2004) argues that Levene's test can be adversely affected by small differences in group variances and instead postulates that the ratio between the largest

and smallest variances should not exceed 9. Using this criterion, $F_{max} = 5.05$ meaning that the robustness of the measure is maintained. The ANCOVA measure is particularly robust to small violations of parametric assumptions, especially when the sample sizes across all 12 conditions are almost exactly equal, and it was therefore appropriate to conduct the following analyses (Field, 2018; Lunney, 1970).

4.3.3 Mixed ANCOVA between groups and within tasks. A mixed Group (3) x Task (4) ANCOVA was conducted with the pre-test total score used as a covariate to control for baseline differences. As Mauchly's test of sphericity indicated that the assumption of sphericity had been violated, $\chi^2(5) = 34.69$, $p < .001$) with $\epsilon > .75$, the Huynh-Feldt estimates of sphericity were used ($\epsilon = .91$) (Field, 2018, p. 461).

Table 4.8

Mixed ANCOVA Between Groups and Within Tasks on Post-Test Scores

Source	Sum of Squares	Degrees of Freedom	Mean Square	*F*	*p*	Partial η^2
Between Subjects						
Intercept	157.68	1	157.68	35.44	.000***	.164
Pre-Test Covariate	2193.38	1	2193.377	492.92	.000***	.733
Group	50.98	2	25.49	5.73	.004**	.060
Error	800.96	180	4.45			
Within Subjects						
Task	2731.67	2.73	1002.14	95.79	.000***	.347
Covariate x Task	237.71	2.73	87.21	8.34	.000***	.044
Group x Task	988.17	5.45	181.26	17.33	.000***	.161
Error	5133.21	490.65	10.46			

*p < .05. **p<.01. ***p<.001.

Table 4.8 shows that the pre-test covariate significantly predicts post-test scores ($F_{[1, 180]} = 492.92$, $p < .001$, $\eta_p^2 = .733$) with a large partial eta-squared effect size greater than .14 (Cohen, 1988). After controlling for this covariate, a significant main effect of Group was found ($F_{[2, 180]} = 5.73$, $p = .004$, $\eta_p^2 = .060$) with a medium effect size (Cohen, 1988).

There was a significant main effect of Task ($F_{[2.73, 490.65]} = 95.79$, $p < .001$, $\eta_p^2 = .347$) indicating that there was a significant difference between tasks with a particularly large partial eta-squared effect size (Cohen, 1988). There was also a significant interaction between Group and Task ($F_{[5.45, 490.65]} = 5.73$, $p < .001$, $\eta_p^2 = .161$) indicating a specific effect of intervention group on task type with a large effect size greater than .14 (Cohen, 1988).

Planned contrasts using estimated marginal means were conducted to investigate the main effect of Group. In comparison with the control group, custom hypothesis tests revealed significant estimated mean differences of + 1.27 for the distributive intervention ($p = .001$) and + .90 for the scaling intervention ($p = .022$). There was, therefore, a significant difference between both the distributive and scaling interventions and the control group.

When the Bonferroni correction was applied to these mean differences, however, only the distributive group remained significant ($p = .003$) whilst the scaling difference was not significant ($p = .065$). While the Bonferroni correction protects from the likelihood of Type I errors, it is a conservative adjustment which can be vulnerable to Type II errors (Field, 2018). In order to evaluate whether there was a significant difference between intervention groups, therefore, further analysis of the significant Group x Task interaction was conducted.

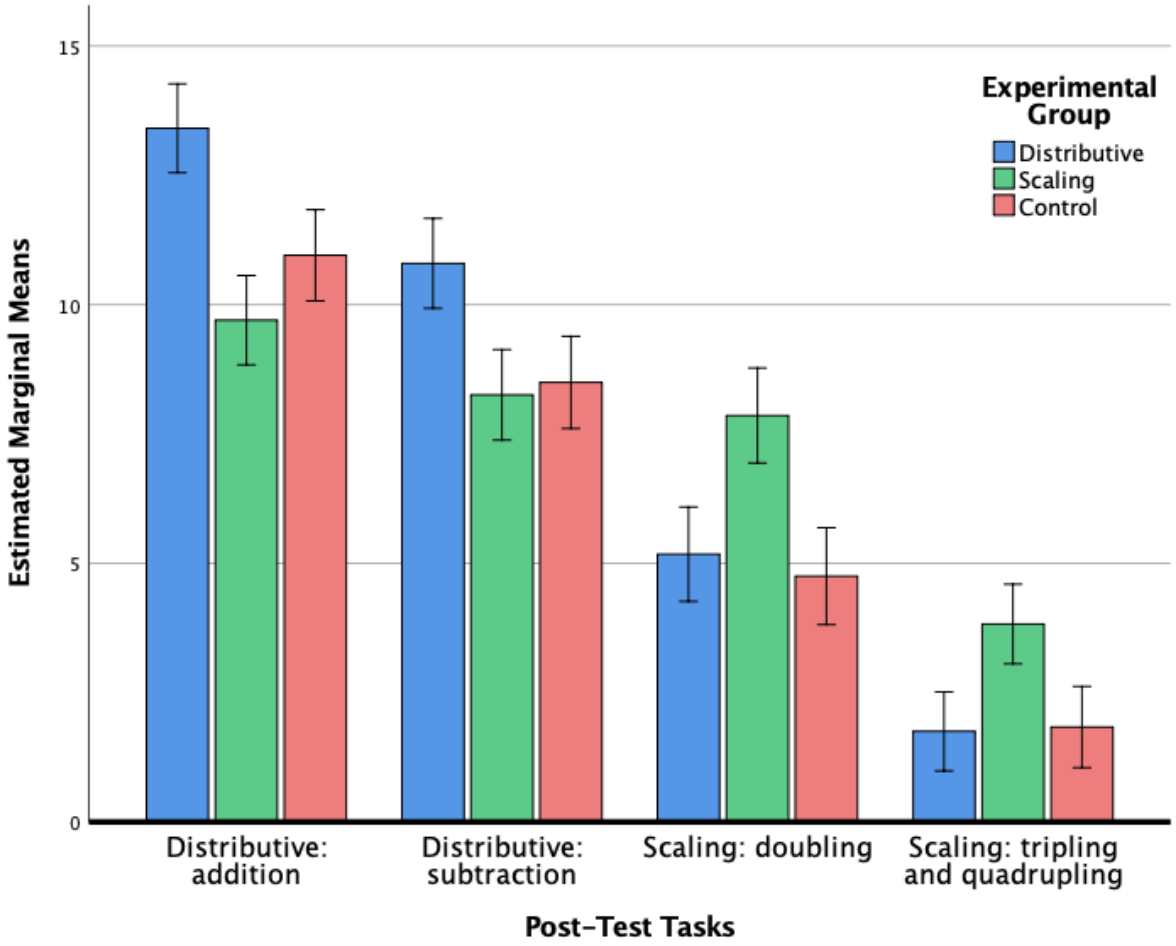


Figure 4.6. Estimated marginal means for each group across post-test tasks. Error bars represent 95% confidence intervals of the mean. The covariate of the pre-test total score is evaluated at the mean value of 22.36.

Estimated marginal means were used to investigate the nature of the significant Group x Task interaction. Figure 4.6 presents the estimated marginal means for each group across the four tasks. The distributive group outperformed the scaling and control group on the distributive tasks with estimated marginal mean scores of 13.41 and 10.80 on the addition and subtraction tasks respectively. The lack of overlap of the 95% confidence interval error bars indicates a significant difference between the distributive group and other two groups on the distributive tasks.

A similar pattern can be observed for the scaling group with estimated marginal mean scores of 7.86 and 3.83 on the doubling and tripling and quadrupling tasks. In contrast, the distributive and control groups have much lower estimated marginal means on the scaling tasks. The lack of overlap of the 95% confidence interval error bars further indicates that a significant difference between the scaling group and the other two groups

on the scaling tasks. Analysis of the Group x Task interaction, therefore, suggests that the distributive and scaling interventions had specific effects on the children's distributive and scalar reasoning.

4.4 Qualitative Analysis of the Children's Workings

The significant interactions between intervention group and tasks suggests that teaching using procedural variation specifically improves children's distributive or scalar reasoning. Qualitative analysis of children's workings on the post-test has also been conducted to supplement the quantitative data. Although it is difficult to apprehend children's internal reasoning processes, examples of their use of distributive and scalar reasoning in the post-test provide qualitative evidence for the effects of procedural variation teaching. The multiple examples presented herein demonstrate how teaching using procedural variation can enable children to recognise and use the relationships between sequences of multiplications.

4.4.1 Distributive addition. Figure 4.7 provides a range of examples of children's use of distributive reasoning on the addition tasks. Labelled arrows between calculations were consistently used in both experimental interventions to draw children's attention to the invariant distributive or scalar relationships. The children's independent use of these labelled arrows to denote the invariant '+15' and '+7' relationships between calculations therefore indicates their use of distributive reasoning. Figure 4.7 also presents examples of children's identification of the invariant distributive relationship between calculations and their use of mental addition to calculation the answer. Even when minor arithmetic errors have been made, it is evident that children have recognised the distributive relationship and have attempted to use addition to calculate their answer.

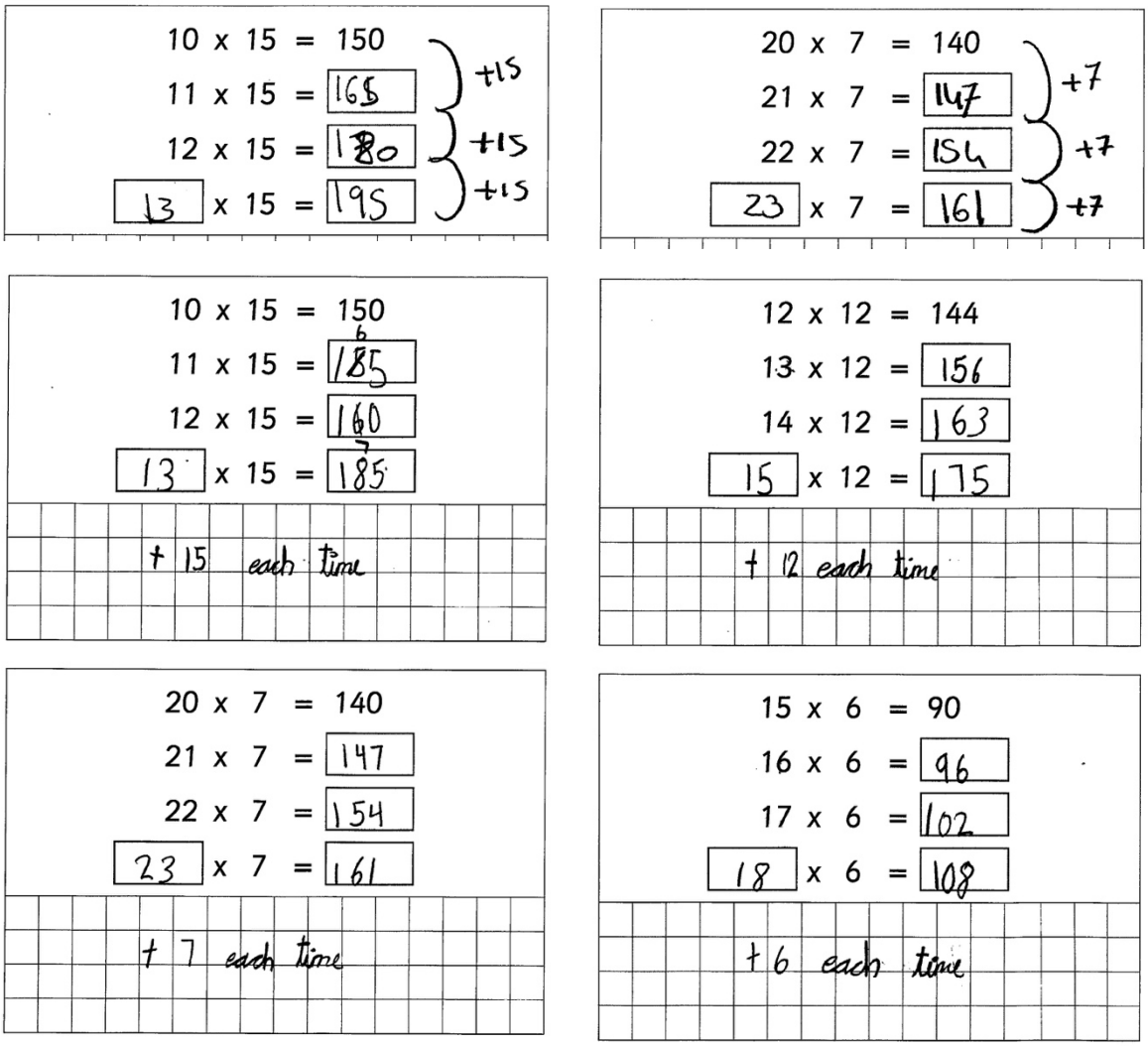


Figure 4.7. Examples of children's workings on distributive addition tasks.

4.4.2 Distributive subtraction. Figure 4.8 is a common example of children's identification of the invariant distributive relationship between the sequence of multiplications. Even though some of the answers are incorrect, the child has identified the invariant subtraction of the multiplicand and has attempted to use mental subtraction to find the subsequent answers. This example also reveals how an initial error made on Question 1.1 is carried through to Questions 1.2 and 1.4. Distributive reasoning is contingent upon arithmetic accuracy and one minor error can result in a sequence of incorrect answers.

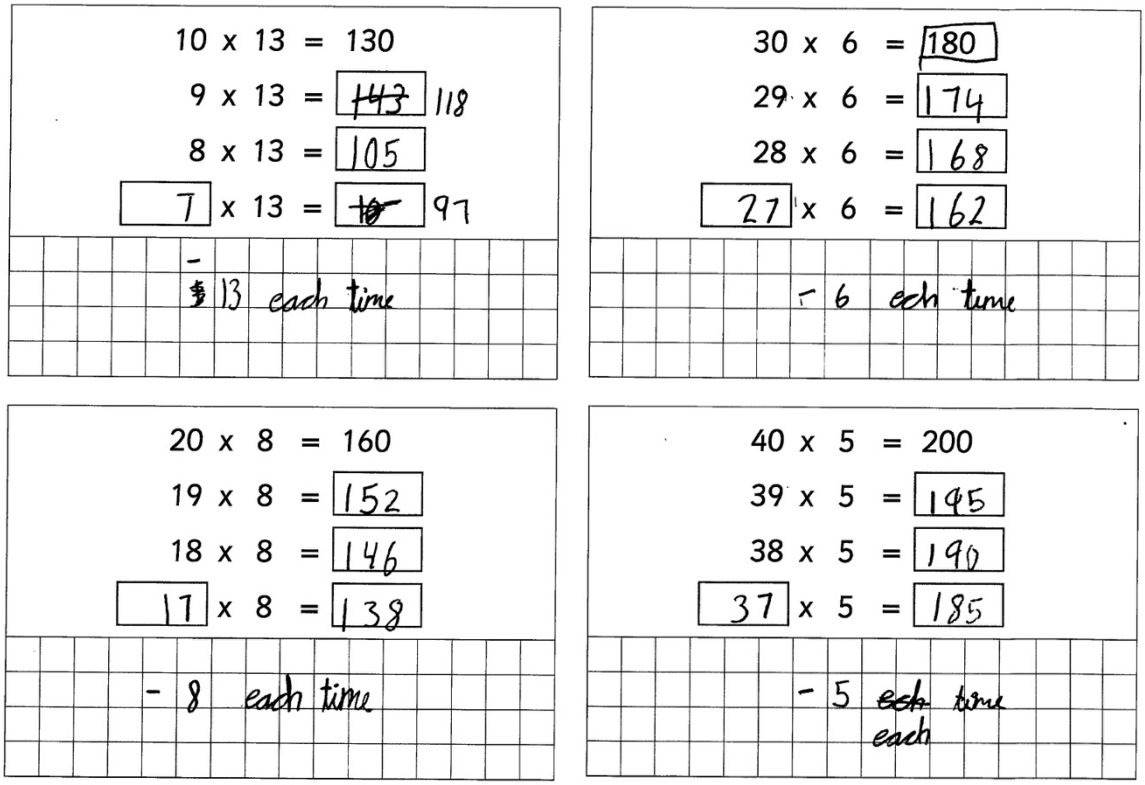


Figure 4.8. Examples of children's workings on distributive subtraction tasks.

4.4.3 Scalar doubling. Figure 4.9 demonstrates children's use of scalar doubling to solve complex multiplication questions. The absence of workings and identification of a 'double' pattern evidences how teaching using procedural variation can enable children to infer the scalar relationships between sequences of multiplications. Similarly, the use of column addition to double the product of the previous multiplication further indicates some children's use of scalar doubling.

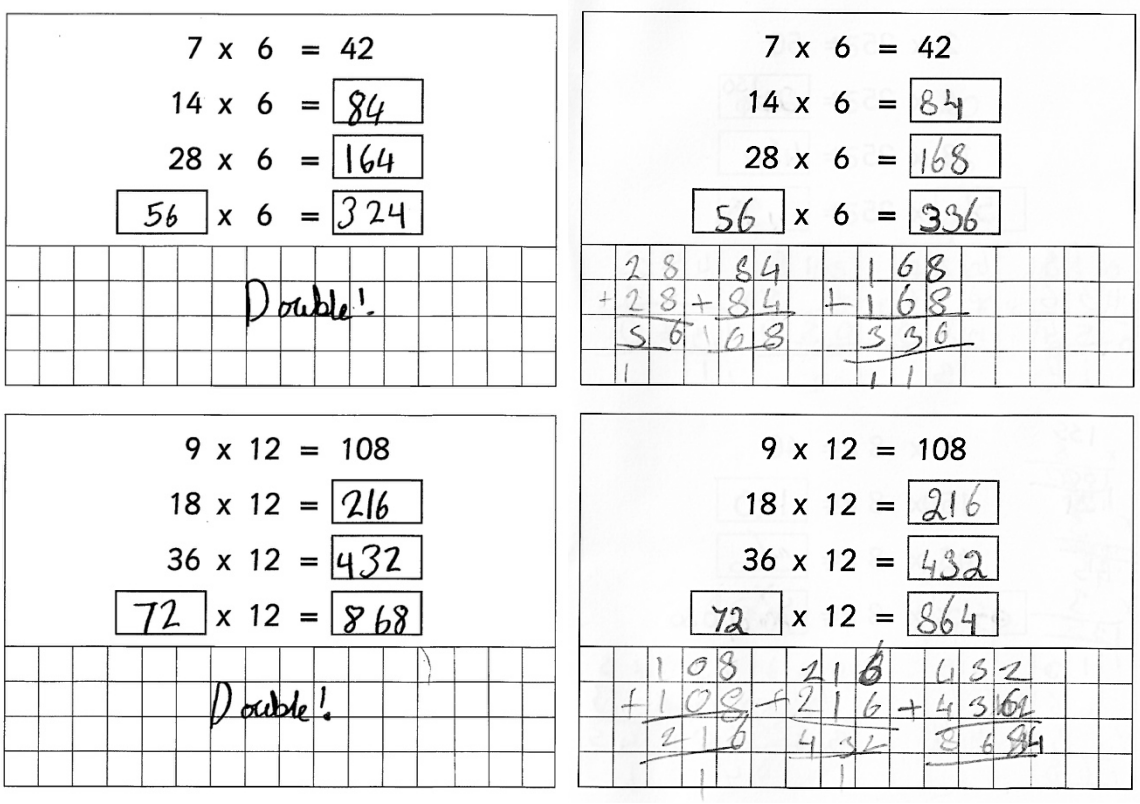


Figure 4.9. Examples of children's workings on scalar doubling tasks.

4.4.4 Scalar tripling and quadrupling. Many of the children found the scalar tripling and quadrupling task extremely difficult and either gave multiple incorrect answers or failed to answer the questions at all. Children who received the scalar reasoning interventions, however, attempted a greater number of these questions. Figures 4.10 and 4.11 present a variety of workings used by children who were taught scalar reasoning using procedural variation.

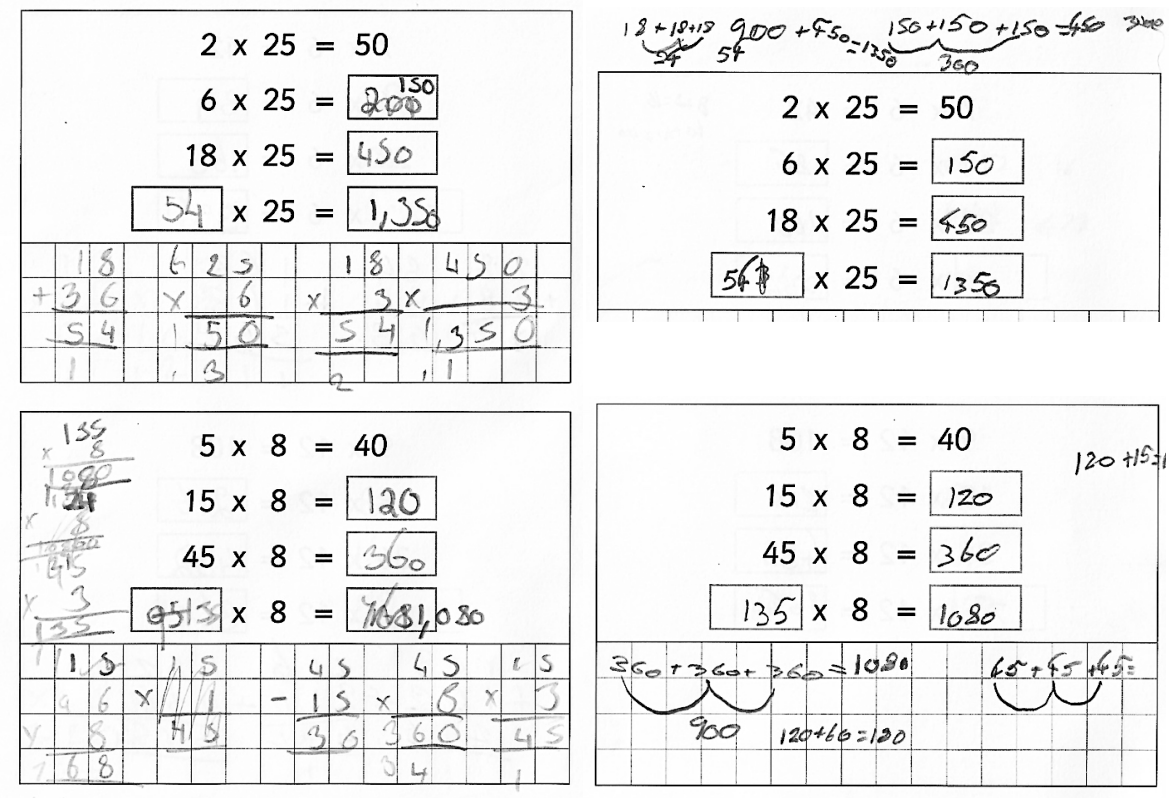


Figure 4.10. Examples of children's workings on scalar tripling tasks.

Figure 4.10 demonstrates children's scalar reasoning through their recognition of the tripling relationship between multiplications. Both children use scalar reasoning, either through column multiplication or the method of repeated addition. These workings reveal how the children combine their scalar understanding of the tripling relationship between multiplications with their own preferred strategies for calculating the answer. Similarly, in Figure 4.11, one child uses the scalar quadrupling relationship to multiply the previous product by 4. In contrast, the other child concludes that the pattern is 'qroople' (quadruple) and therefore mentally calculates the answer, albeit somewhat incorrectly.

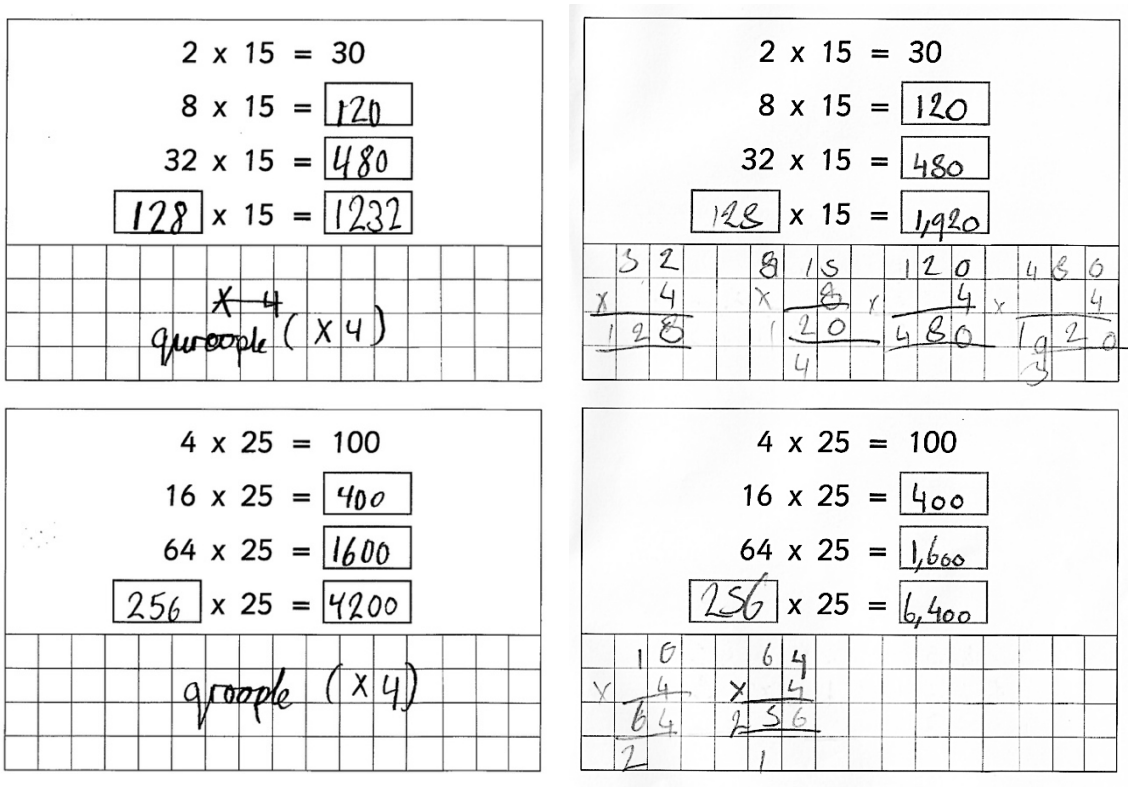


Figure 4.11. Examples of children's workings on scalar quadrupling tasks.

It is must be emphasised that these multiplication questions are extremely difficult for this age group. This qualitative analysis of children's post-test workings shows that children are capable of recognising these scalar tripling and quadrupling relationships. Their attempts to use scalar reasoning to answer these complex questions further substantiates the assertion that procedural variation teaching can improve children's understanding of the relationships between multiplication questions.

4.4.5 Further inference of children's reasoning processes. Despite the complexity of the multiplications, it is notable that many children in the intervention groups failed to show any workings at all yet still obtained the correct answer. In this age group, children would not usually be able to calculate these multiplications mentally and would be expected to use written calculations (DfE, 2013). It is hypothesised, therefore, that this lack of workings is more likely to be due to children's reasoning about the distributive and scalar relationships between multiplications than a sudden unexpected improvement in their mental arithmetic. Where children have not shown any workings, therefore, it is likely that the procedural variation of each sequence has enabled them to infer the distributive and scalar relationships between multiplications.

4.5 Summary of Results

The first research question investigates whether Year 3 children are capable of recognising and using the relationships between a sequence of procedurally varied multiplication questions.

1. The majority of Year 3 children are capable of identifying and using the distributive addition and subtraction relationships between questions, with high mean scores on these tasks.
2. These children have considerable difficulty, however, with recognising the scalar relationships between questions, with significantly lower mean scores on the doubling, tripling and quadrupling tasks.
3. Analysis of the children's incorrect answers reveals a dichotomy between arithmetic inaccuracies and conceptual errors in their multiplicative reasoning.
4. There was a significant main effect of Gender with boys consistently outperforming girls in mean scores across all four tasks.
5. There was not, however, a significant interaction effect between Gender and Task. Gender was not therefore included in the post-test analyses of intervention effects.

The second research question considers whether teaching using procedural variation can improve Year 3 children's understanding of the distributive and scalar relationships between multiplication questions.

1. There was a significant main effect of Group with higher estimated marginal means for the intervention groups in comparison to the control group.
2. There was a significant Group x Task interaction effect with estimated marginal means indicating a specific effect of the distributive and scaling interventions on their respective tasks.
3. Qualitative analysis of the children's post-test workings provides further evidence of their usage of distributive and scalar reasoning following specific teaching using procedural variation.

5. Discussion

5.1 Conclusions

The Variation Theory of Learning defines learning as the discernment of the invariant features of a concept through the variation of its non-essential features (Marton & Booth, 1997). Procedural variation uses systematically varied sequences of questions to draw children's attention to the invariant critical features of a mathematical concept (Gu et al., 2004). Although there is substantial literature on variation theory, however, there is a surprising paucity of empirical evidence to substantiate its theoretical assertions. Al-Murani et al. (2018) used qualitative examples of variation pedagogy to postulate a 'paradigm shift' in primary mathematics yet they provide no empirical data to demonstrate its efficacy. Despite the current promulgation of variation theory in UK primary schools, therefore, no experimental study has been conducted to investigate whether it is an effective pedagogy.

This dissertation has made an attempt to partially rectify this situation through the unprecedented use of an intervention study design to empirically evaluate whether procedural variation teaching improves Year 3 children's distributive and scalar multiplicative reasoning. The fundamental theoretical assumption of procedural variation is that children can discern the variant and invariant relationships between a sequence of questions. As a pedagogical approach, procedural variation assumes that using varied sequences of questions will enables pupils to understand the underlying invariant features of a mathematical concept. The following research questions have been used to evaluate these assumptions and to structure the conclusions of this dissertation:

1. Are Year 3 children capable of recognising and using the relationships between sequences of multiplication questions?
2. Does teaching using procedural variation improve Year 3 children's understanding of these relationships?

5.1.1 Children's use of procedural variation. The first research question uses the pre-test data to examine three interrelated assumptions of procedural variation. First, whether Year 3 children are capable of recognising how the multiplier and product vary, whilst the multiplicand remains invariant across a sequence of multiplication questions. Secondly, whether they can use these variant and invariant features to infer a distributive or scalar relationship between each question. Finally, whether they are able to use these distributive or scalar relationships to answer each question in an efficient way without resorting to a written multiplication method.

The pre-test results indicate that the majority of Year 3 children are capable of recognising the distributive relationships between questions. Procedural variation of the sequences enabled the children to see that as the multiplier increases or decreases by one, one 'lot' of the multiplicand is added or subtracted from the previous product. The children were able to apply distributive reasoning to the initial clue provided to give the correct answers for the three subsequent questions. The procedural variation of each sequence allowed the children to answer complex multiplication questions through efficient mental manipulation of the initial clue instead of extended written methods. It is hypothesised that Year 3 children's prior knowledge of times tables, which use the addition and subtraction of multiplicands, provided a familiar context for their fluent distributive reasoning on these tasks.

The children had significantly more difficulty, however, with recognising the scalar relationships of doubling, tripling and quadrupling. While a small minority of children were able to answer the doubling questions, very few were capable of completing the tripling and quadrupling task correctly. The mean scores for these tasks were considerably lower, particularly in the tripling and quadrupling task. Whereas the children had some prior understanding of the distributive relationships in times tables, most of them appeared to be completely unfamiliar with the scalar relationship between calculations. It is suggested, therefore, that some prior knowledge of distributive or scalar reasoning must be taught to Year 3 children in order for them to apprehend these relationships between a sequence of multiplications.

Analysis of the children's incorrect answers demonstrates some of the conceptual errors underlying their use of procedural variation. The errors on the distributive tasks reflect an underlying misconception of the nature of the one-to-many correspondence of multiplication, with some children merely adding or subtracting 1 or 10 instead of the correct multiplicand. The addition of the multiplicand on the subtraction task, however, is arguably an arithmetic error where children are using distributive reasoning yet are mistakenly choosing the wrong operation.

On the scaling tasks it was not possible to categorise conceptual errors due to the multifarious nature of the children's incorrect answers. Some of the children attempted to apply distributive reasoning to these tasks, adding one 'lot' of the multiplicand instead of using scalar multiplication. The majority of the children, however, either left these questions unanswered or made random guesses with no clear underlying reasoning processes. The majority of Year 3 children, therefore, are capable of using procedural

variation to infer and use the distributive relationships between multiplications. Very few of these children, however, are able to recognise and use the scalar relationships between multiplications.

One further interesting result was the significant difference between genders with boys consistently outperforming girls in mean scores across all four tasks. Despite the recruitment of a large sample (male = 91, female = 93) from six classes across three different schools, there appears to be a systematic difference between genders. This gender difference in mathematics attainment is not a unique result of the present study. In their empirical analysis of international comparative tests, Fryer and Levitt (2009) conclude that this mathematics gender gap is present in every strata of society. Furthermore, they concluded that it is difficult to explain why this gender difference is perpetuated across society and they could not identify any specific underlying causes of this disparity. It is not possible, therefore, to explain why there was a significant difference between genders in the present study and future research is required to explain the mechanisms underlying this gender gap.

5.1.2 The effects of procedural variation teaching. The second research question investigates whether teaching using procedural variation can improve Year 3 children's understanding of the distributive and scalar relationships between multiplication questions. Using the pre-test results as a covariate, the post-test results for the distributive and scalar reasoning groups were compared with the active control group to ascertain whether there was an effect of procedural variation teaching. Inferential analysis focused on whether there was a main effect of Group and, in particular, whether there was a specific interaction between Group and Task.

The ANCOVA demonstrated that there was not only a significant main effect of Group but also a significant interaction effect between Group and Task. Both the distributive and scalar reasoning groups performed significantly better on the post-test than the active control group. Comparison of the estimated marginal means indicated a specific effect of the distributive and scalar interventions on each of their respective tasks. The interaction between Group and Task, therefore, revealed that teaching using procedural variation significantly improved the Year 3 children's distributive or scalar reasoning with a large partial eta-squared effect size of .161 (Cohen, 1988).

As the first intervention study to empirically evaluate the use of procedural variation teaching, these results provide substantial evidence for the use of variation theory in mathematics pedagogy. Systematically varied sequences of multiplications were

used to draw the Year 3 pupils' attention to the invariant relationships between the questions. Procedural variation teaching enhanced the pupils' multiplicative reasoning by enabling them to recognise and use the distributive and scalar relationships between multiplications. Furthermore, these results were not confined to the intervention sessions, with the children being able to independently use their distributive and scalar reasoning in the post-test. Qualitative analysis of the children's workings further substantiated the assertion that procedural variation enabled the pupils to identify and use these relationships between related multiplication questions.

The significant effects of procedural variation teaching are particularly surprising as they occurred after only three 30-minute intervention sessions. The results from this study, therefore, indicate that procedural variation may be a highly effective pedagogy for enhancing children's multiplicative reasoning. The specific effect of procedural variation teaching on either distributive or scalar reasoning, however, indicates that providing procedurally varied questions without explanation is not sufficient to improve children's mathematical understanding. Procedural variation is only effective in the context of effective teaching which draws the pupils' attention to the specific variant and invariant features of a mathematical concept.

5.2 Implications for Mathematics Pedagogy

Procedural variation is currently promulgated as an effective pedagogy across UK primary schools by the NCETM (2014; Morgan, 2015) with its usage illustrating a 'paradigm shift' in primary mathematics teaching (Al-Murani et al., 2018). No experimental studies, however, have been conducted to evaluate these assertions.

The present study, therefore, provides unprecedented empirical evidence to substantiate the use of procedural variation pedagogy in primary mathematics. The significant improvements made by the intervention groups suggest that procedural variation has the potential to be a highly effective pedagogy. Carefully designed sequences of systematically varied questions can be used to elicit children's understanding of the critical invariant features of a mathematical concept (Watson & Mason, 2005, 2006a, 2006b).

The procedural variation of mathematical examples, however, is not solely sufficient to develop children's conceptual understanding. Instead, this pedagogy can only be used in the context of the effective teaching of a specific mathematical concept. The significant interaction between Group and Task in the present study reveals the specific effect of teaching using procedural variation on either distributive or scalar reasoning.

Teachers should first identify a particular *object of learning* before using procedural variation pedagogy to construct pupils' understanding of the critical features of the concept.

Despite these caveats, the results of the present study provide encouraging evidence for the use of procedural variation in primary mathematics teaching. After only three 30-minute sessions, the Year 3 pupils made significant improvements on tasks which required complex multiplicative reasoning. In the long-term, it is hypothesised that embedding procedural variation within mathematics teaching could develop children's understanding of the underlying relationships between mathematical concepts.

5.3 Strengths of the Present Study

A major strength of the present study is the use of an intervention study design to evaluate whether teaching using procedural variation can improve children's multiplicative reasoning. Previous learning study designs have predominantly used qualitative observational data to inductively infer the efficacy of variation theory. In contrast, this pre- and post-test design has allowed for the inferential analysis of over 20,000 items of quantitative data. This experimental design allowed for rigorous statistical analysis of a quantitative data to empirically evaluate the effects of procedural variation teaching on Year 3 children's multiplicative reasoning.

Furthermore, in contrast to the few other quasi-experimental studies of variation theory, the present study included not only the two intervention groups but also an active control group for comparison. Another strength of the present study, therefore, was the opportunity to empirically evaluate whether or not procedural variation pedagogy is beneficial for pupils. The significant results of this study provide the only empirical evidence to support the widespread promulgation of procedural variation with valuable implications for the teaching of primary mathematics.

The large sample size of 184 children, recruited from six classes across three different schools, is a further strength of the present study. This research project required 18 days of teaching in schools, with a total of 54 intervention sessions being delivered. Despite inevitable limitations of time, the methodological choice was made to conduct this medium-scale project in order to generate a substantial body of empirical data. The scale of the project allowed for rigorous statistical analysis with a reduced margin of error and high external validity. Furthermore, the study also had high internal and ecological validity, with all interventions being taught by the same specialist Maths teacher in naturalistic classroom settings.

5.4 Limitations of the Present Study

Despite the strengths of the present study, it is important to also consider its limitations. The methodological limitations of the present study were necessitated by the timescale available for the research. Due to time constraints, therefore, it was only possible to conduct short-term interventions comprising three 30-minute sessions. It is unclear whether longer interventions would have enabled the children to make greater gains in their understanding as a result of the procedural variation teaching. A further disadvantage of this short-term study was the lack of a delayed post-test which would have established whether the significant gains made by the intervention groups were maintained over time.

The timescale of this project also made it necessary to limit the content of the interventions, focusing specifically on distributive and scalar reasoning. If it had been possible to conduct longer interventions, the remit of the study could have been expanded to evaluate the use of procedural variation teaching across all four arithmetic operations, or to other areas of mathematics. Furthermore, the present study was confined to Year 3 pupils and it would have been interesting to compare the effects of procedural variation teaching across different age groups.

In order to recruit schools and reduce disruption to the children's normal lessons, it was necessary to only spend a short period of three days with each class. One further limitation of the present study, therefore, was the reduced number of tests which could be ethically administered to a Year 3 class over three days. It would have been advantageous to conduct a baseline measure of IQ which could have been used as an additional covariate. Furthermore, a broad test of mathematical ability could have been administered after the interventions to establish whether procedural variation teaching has a general effect across different mathematical topics.

5.5 Future Research

Despite the vast theoretical literature on variation theory, very few experimental studies have been conducted to evaluate its assertions. Although learning studies have been used to inductively infer the efficacy of variation pedagogy, none of these studies have used an experimental design with an active control group. Future research, therefore, should utilise experimental pre- and post-test designs with an active control group to empirically evaluate the assumption that variation theory is an effective mathematical pedagogy. Furthermore, longitudinal designs such as that used by Al-

Murani (2007) should also be used to investigate the long-term effects of teaching using variation.

The very few experimental studies heretofore conducted into variation theory have concentrated on the secondary age group. The results of the present study, however, provide encouraging evidence for the beneficial effects of procedural variation teaching with a much younger age group. Although Al-Murani et al. (2018) have discussed the use of variation theory as illustrating a 'paradigm shift' in primary mathematics, they provide no quantitative data to substantiate this assertion. Future research, therefore, could provide a valuable contribution to primary mathematics by investigating the effects of variation pedagogy across the 5-11 age group.

In particular, procedural variation could be a significant pedagogical tool to develop young children's understanding of the invariant underlying relationships between mathematical concepts. The results of present study indicate the efficacy of procedural variation in the field of multiplicative reasoning. Future research could expand its focus to the use of procedural variation across all four arithmetic operations. Arithmetic is a fundamental component of primary mathematics teaching and is the foundation for children's mathematical fluency. Future studies could investigate whether systematically varied sequences of examples can be used to draw pupils' attention to specific arithmetic concepts.

This dissertation has provided a small yet significant contribution to the field of variation theory and three main directions for future research have been identified. First, greater use should be made of experimental designs to empirically evaluate the assertions of variation theory. Secondly, studies of variation pedagogy should no longer predominantly focus on the secondary age range. The present study establishes the potential relevance of variation theory to primary mathematics and further research should be conducted in this age group. Finally, this study has demonstrated the importance of procedural variation as a specific pedagogical component of variation theory. Future studies should evaluate whether teaching using procedural variation is effective across a wide range of mathematical concepts. If the findings of the present study could be replicated across age groups and mathematical topics, this would have significant implications for the use of procedural variation in mathematics education.

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CENTRAL UNIVERSITY RESEARCH ETHICS COMMITTEE (CUREC)

Form CUREC 1A Checklist for the Social Sciences and Humanities

The University of Oxford places a high value on the knowledge, expertise, and integrity of its members and their ability to conduct research to high standards of scholarship and ethics. The research ethics clearance procedures have been established to ensure that the University is meeting its obligations as a responsible institution.

They start from the presumption that all members of the University will take their responsibilities and obligations seriously and will ensure that their research involving human participants is conducted according to the established principles and good practice in their fields and in accordance, where appropriate, with legal requirements. Since the requirements of research ethics review will vary from field to field and from project to project, the University accepts that different guidelines and procedures will be appropriate.

- Please check **"Where and how to apply for ethical review"** and the **CUREC flowchart** first to see if you need ethics approval.
- Please complete this form using a word processor and email it, together with your supporting documents, to your **Departmental Research Ethics Committee (DREC)** (**if applicable**). If you don't have a DREC please email this form to **ethics@socsci.ox.ac.uk** using your official **ox.ac.uk email** address. **Only emailed applications will be accepted.**

WHAT THIS CHECKLIST IS DESIGNED FOR

This **CUREC 1A checklist** is designed largely for research that falls within the Divisions of Social Sciences and Humanities where ethical issues are relatively few and straightforward. Interviews, field work and oral history are also included in the CUREC process.

The **full CUREC 2 application** is only required where certain project characteristics (e.g. type of participants, or procedures) result in a more complex set of ethical issues. It is expected that only in a limited number of cases will it be necessary for researchers to complete a CUREC 2 application. The checklist below will direct you to a CUREC 2 application if needed.

WHAT THIS CHECKLIST WILL NOT ASSESS

This checklist does not cover research governance, satisfactory methodology, or compliance with the requirements of publishers when administering their tests or questionnaires. As **principal researcher (i.e. principal investigator)**, it is your responsibility to ensure that requirements in these areas are met.

CUREC does not review studies classed as **audit** (see Glossary and Decision Flowchart for CUREC on our website).

If your study involves **NHS patients, NHS staff / data / facilities, or human tissue**, please check the Decision Flowchart for NHS approval and contact the Clinical Trials and Research Governance (CTRG) team in the first instance.

Further information on the University's research ethics procedures is available from the CUREC website.

SECTION A: Filter for CUREC2 application

This section determines whether your study raises more complex issues which require the completion of a full application for ethical review, known as the CUREC 2 application.

(Please mark 'X' in the Yes/No column as appropriate to indicate your response.)

1. Are research participants classed as **people whose ability to give free and informed consent is in question**? (This may include those under 18 (though see "competent youths"), prisoners, or adults "at risk".) Your attention is drawn to the University's **Safeguarding Code of Practice** and its implications for researchers involving children or adults at risk, including the need for the work to be risk assessed and for researchers to undertake related training. (**Note**: If any of your participants are aged 16 or under, please answer 'Yes' here **and** also answer question 5 below.)	Yes X	No
2. By taking part in the research, will participants be at serious risk of criminal prosecution (e.g. by providing information on drug abuse or child abuse)?	Yes	No X
3. Does the research involve the **deception** of participants?	Yes	No X
4. Does your research raise issues relevant to the **Counter-Terrorism and Security Act (the Prevent duty)**, which seeks to prevent people from being drawn into terrorism? Please see advice on this on our **Best Practice Guidance web page**.	Yes	No X

If you have answered 'No' to all of the questions above please go to **Section B**. If you have answered 'Yes' to any question above continue to question 5 below.

5. Is your project covered by a CUREC **approved procedure** (formerly known as "CUREC Protocols")?	Yes X	No

If yes, please give research procedure number(s): 15 - Group B activities (teaching innovation)

If you answered '**Yes'** to **ANY** of questions **1-4**, **and** answered '**No'** to question **5**, **please stop completing this checklist and do not submit it for ethical review**. Instead, please complete the CUREC 2 application form from the CUREC website. Then submit the CUREC 2 form for ethical review.
If you answered 'Yes' to ANY of questions 1-3, **and** answered 'Yes' to question 5, please go on to **Section B**.

SECTION B: Contact details and project description (NB: must be typed not handwritten)

Contact details:

1. **Principal investigator /** supervisor (if student research (title and full name):	Professor Peter Bryant
2. Name of student (if student research):	Charlotte Moss
3. Degree programme, e.g. DPhil, BA, MPhil, BSc, MSc (if student research):	MSc Education (Child Development and Education)
4. Department or Institute name:	Department of Education
5. Address for correspondence	
6. University e-mail (**not** private email) and telephone:	peter.bryant@education.ox.ac.uk
7. Name and status of others taking part in the project	

SECTION B continued	
Project description:	
8. Title of research project:	Investigating the effects of procedural variation on multiplicative reasoning
9. List of location(s) where project will be conducted:	Oxfordshire, Gloucestershire and Warwickshire
10. If your research involves overseas travel or fieldwork and your department requires a travel risk assessment, will you have completed and returned a risk assessment form beforehand? (This has to be approved by your department before you travel. If you are travelling overseas, you are strongly advised to take out University travel insurance.)	Yes X No Not required in this instance (No overseas travel but I will be completing a risk assessment)
11. Anticipated duration of research project overall:	5 months or years (maximum 5)
12. Anticipated start and end dates of the research project involving human participants:	From: 18/03/2019 To: 30/07/2019 Please note that you will need ethics approval **before** you start your research. CUREC 1As may take up to 30 days to process.
13. External organisation funding the research (if applicable):	
14. Title and very brief and simple lay description of **research** (about 150 words), plus description (about 200 words) of the nature of participants.	
a) Title, brief description of research (150 words) in lay language. When describing the research, please include your methodology, how you are applying professional guidelines, and the use to which results/data will be put. **Please also declare any conflicts of interest here.**	
I wish to investigate whether children can solve multiplication questions by adding and subtracting or multiplying. The children will complete a 30 minute pre-test with pairs of multiplication questions before being randomly split into three groups. As a qualified primary teacher, I will be teaching three lessons to each group focusing on adding and subtracting, multiplying or times tables. The children will engage with exciting and stimulating activities to promote their multiplication skills, as part of their normal Maths curriculum. The activities will be whole-class learning about multiplication, individual worksheets, a card matching game and a speed quiz. After the three lessons, each group will complete the same test as before and will use the data to analyse whether the teaching has had any effects on their understanding. I will be consistently applying professional guidelines, ensuring all data is held under GDPR regulations and that all personal data is anonymised.	

SECTION B continued

b) Description of participants and **obtaining informed consent** (200 words). When describing participants, please include

- criteria for inclusion/exclusion
- method of recruitment
- processes for consent to participate

Please ensure you attach as separate documents (if applicable, in English translation):

- your recruitment and advertisement material e.g. a poster or brief invitation letter/ email
- information for participants to read (or hear) before they agree to take part e.g. written information sheets or (only if applicable) oral information scripts.
- a document to record informed consent. Templates for written consent forms and/or oral information scripts (in case of an oral consent process) are available from the CUREC website
- a guide to interview questions (this may be a list of questions to be asked, or a preliminary scope of questions), or a sample of other instruments (such as a sample questionnaire)
- (if relevant) debriefing document after participants have taken part

I plan to recruite Year 4 pupils (8-9 year olds) from local primary schools by contacting the headteachers and sending them a detailed letter (attached) to obtain their written consent. I will then be seeking informed consent from parents and guardians by sending out an extensive information letter(attached) explaining the nature of the study, how the children's data will be used and protected and their right to opt-out with no disadavantage to their child. Throughout the study, I will ensure that parents and staff are aware that they may withdraw their consent at any time.

I have chosen to obtain opt-out consent for this study as I am confident that it meets the description of Group B activities in the guidance for Procedure 15 which stipulates that opt-out consent may be used. I will, therefore, include all participants in the Year 4 class unless their consent has been withdrawn. In this project, I will be teaching lessons on multiplication to supplement the children's normal Maths curriculum (intervention overview attached). As a qualified primary teacher, I will be teaching Maths lessons very similar to the ones that children normally experience. The pre- and post-test that I will administer (attached) has been carefully written with child-friendly language and formatting. If any child displays any negative emotional response to the lessons or test, I will immediately speak to their teacher and ask them whether the child should be withdrawn from the study.

15. What are the ethical issues connected with your research and what steps have you taken to address them? Please do not answer 'none'. The committee needs to see evidence that you have identified potential ethical issues with respect to your research and have taken steps to address them. These issues could relate to:

- your own physical and psychological safety as a researcher (please see the University's and Social Science Division's Safety in Fieldwork guidance
- participant burdens and/or risks, and
- data protection/ confidentiality (please also see section 18).

For more guidance on ethical issues, please see
http://researchsupport.admin.ox.ac.uk/governance/ethics/resources

The two main ethical issues in my research are whether the pre- and post-test and intervention are likely to upset the children in any way and whether the loss of normal lesson time will in any way impinge upon their learning. I have carefully designed the pre- and post-test to be child-friendly using speech bubbles with positive language and a clear layout with Primary Sassoon font. As an experienced Year 4 teacher, I am aware of the capabilities of the age group and have designed both the test and intervention to be accessible for all children. In addition, the intervention itself will use child-friendly worksheets and games which would feature in any normal Maths lesson that the children would experience. The duration of the pre- and post-test and the three intervention sessions will be no longer than three normal Maths lessons for the children. As a Year 4 teacher, I have a thorough knowledge of the Year 4 National Curriculum Programme of Study and am confident that the children will learn at least as much as they would in a normal Maths lesson. The interventions are based on current research into children's understanding of multiplication and I hope they will be beneficial to the pupils whilst also being enjoyable and challenging.

I will share detailed information about the study with the headteacher, teachers and parents of the children. Although I have chosen for consent to be an opt-out process, I will still require consent from the headteacher and will provide an ongoing opportunity for parents to withdraw their children at any time throughout the study. I will also liaise throughout with the class teacher to ensure that no child is worried or anxious during the study.

A further major ethical issue in my research is the importance of data protection. Having received school-based training on the new GDPR, I am well aware of the importance of data security and confidentiality. I will be assigning each child a random ID number and will destroy the paper tests with the children's names on them as soon as I have entered the random number and test result into the computer. The paper tests will be kept in a locked suitcase before being transferred to a locked cabinet in the department and they will be kept for no longer than a week. Data stored on computers will be encrypted and only shared with myself and my supervisor. Neither the schools nor the individual children will be in any way identifiable in my dissertation.

With respect to my personal safety, I will be completing a risk assessment for this study and have classified it as low risk. I am experienced at working in schools and will enjoy teaching the short Maths lessons to the children. I will be travelling to and from the schools via public transport or in my own car and have no concerns for my personal safety.

Section B continued		
16. Will you obtain informed consent according to CUREC guidelines and good practice in your discipline before participation?	Yes X	No
If you have marked '**No**', please give a brief explanation and justification for this decision here:		
17. Will your research involve discussing sensitive issues? This could be information relating to race or ethnic origin, political opinions, religious beliefs, physical/mental health, trade union membership, sexual life or criminal activities.	Yes	No X
If you have marked '**Yes**', please make sure that you have included some **supporting information** (as directed in question 14 of this section) showing the range of questions covering these issues.		
18. **Management and handling of personal and other research data** Your management and handling of personal data and special category data of human participants, either directly or via a third party, will need to comply with the requirements of the General Data Protection Regulation (GDPR) and the new Data Protection Act, as set out in the University's Guidance on Data Protection and Research. In answering the questions below, please also consider the points raised in the Data Protection Checklist. For advice on research data management and security, please consult with the University's Research Data Team (**researchdata@ox.ac.uk)** and/or your local IT department and the University's web pages on research data management.		
a) Will your research involve the collection of **records of consent** (e.g. written forms, audio-recorded, or other recorded consent)? **If 'Yes', these will be classed as fully identifiable personal data (directly linked to an individual).**	Yes	No X
b) Will your research involve the collection of **other personal data?** **If 'Yes**', specify in what form(s) this will be stored:	Yes X	No
• Fully identifiable **temporarily for a maximum period of w eek (test papers in locked cabinet)**	Yes X	No
• Pseudonymised (potentially identifiable as data may be attributed to an individual if linkage information can be accessed elsewhere by researchers)	Yes	No X
• Fully anonymised (i.e. cannot be linked to an individual)	Yes X	No
c) Will any of the personal data you collect classify as **special category** data? **If 'Yes',** in what form(s) will this be stored:	Yes	No X
• Fully identifiable (directly linked to an individual)	Yes	No
• Pseudonymised (potentially identifiable as data may be attributed to an individual if linkage information can be accessed elsewhere by researchers)	Yes	No
• Fully anonymised (i.e. cannot be linked to an individual)	Yes	No
d) How will any personally identifiable data be collected, transferred and backed up? Please describe the arrangements for any physical transfer of personal data (including paper records and data captured electronically via portable media) from where it is collected to local storage.		

The children's names will be written on the front of the test papers, making them fully identifiable. I will, therefore, transfer these to the department in a locked suitcase and keep them in a locked filing cabinet. As soon as possible, and no longer than within seven days, I will mark the tests and type the data into a computer using a random number ID for each child. Once this process is complete, the linkage document of the children's names and the random numbers will be destroyed and the data at that point will be fully anonymised. The data will be kept in an encrypted document and will only be accessible to myself and my supervisor and shared using an encrypted memory drive.
e) Where, and for how long, will participants' personally identifiable data be stored during and after the study? (Please outline the procedures for ensuring confidentiality, eg security arrangements, anonymisation or pseudonymisation of such data. Please distinguish between records of consent and other forms of personally identifiable data stored)
The children's names will be written on the front of the test papers, making them fully identifiable. I will, therefore, transfer these to the department in a locked suitcase and keep them in a locked filing cabinet. I will create a spreadsheet allocating a random number to each child. Within a week, I will mark the tests and type the data into a computer using the random number ID for each child. Once this process is complete, the linkage spreadsheet of the children's names and the random numbers will be destroyed and the data will be fully anonymised. The anonymised data will be kept for the duration of the study and for no longer than until October 2019.
f) If storing pseudonymised data, please confirm that identifiers will be held separately from the research data and linked through a unique study number. Specify how and at what point the pseudonymisation will occur, how the linkage information will be stored and state whether or not (and when) the linkage will be destroyed.
The anonymisation will occur within the same week as the collection of the data. Children's names will be allocated a unique ID number and this linkage information will be within an encrypted spreadsheet. As soon as the tests are marked, the results will be typed into a computer using the unique study number. The computerised data, therefore, will be fully anonymised. As soon as the data entry is complete, the linkage information will be destroyed. At this point, the computerised data will be anonymous with no possible way to identify the individual.
g) Who will have access to the personally identifiable data? If personally identifiable data is to be shared with another organisation, how will it be transferred/disclosed securely?
The personally identifiable data will only be accessible to myself and my supervisor. At all times, it will be stored in a locked filing cabinet, or transferred to the department in a locked suitcase.
h) When and how will personally identifiable data be destroyed? (NB. Personally identifiable data should be destroyed when no longer required.)
The personally identifiable data of the children's names on the front of the test papers will be destroyed as soon as possible and within a week of data collection. As soon as the tests have been marked, data will be entered into a computer using a random ID number for each child. After the data has been entered into the computer, the linkage information and the original paper tests with the children's names will be destroyed.
i) How, where and for how long will other research data be stored after the study has finished? For more information about University and research funder retention policies, please see the University's web pages on research data management.
The anonymised research data will only be kept for the duration of the study and will be deleted after the final exam board have made their decision on the dissertation grade. This is estimated to be during September 2019 and the data will be kept no longer than until 31 October 2019.

SECTION C: Methods and procedures to be used	
Method used: Please ensure you have addressed any potential ethical issues related to these methods in Section 14 and in your Participant Information Sheet	**Please mark 'X'**
1. Analysis of existing records	
2. Snowball sampling (recruiting through contacts of existing participants)	
3. Use of casual or local workers e.g. interpreters	
4. Participant observation	
5. Covert observation	
6. Observation of specific organisational practices	
7. Participant completes questionnaire in hard copy	
8. Participant completes online questionnaire or other online task	
9. Using social media	
10. Participant performs paper and pencil task	X
11. Participant performs verbal or aural task (e.g. for linguistic study)	
12. Focus group	
13. Interview	
14. Audio recording of participant (you will generally need specific consent from participants for this)	
15. Video recording of participant (you will generally need specific consent from participants for this)	
16. Photography of participant (you will generally need specific consent from participants for this)	
17. Others (please specify):	X
The only method of data collection in my study is the collection of the pre- and post-test results to the written multiplication test (attached). As part of this procedure, however, I will be teaching three short multiplication lessons to each group using times tables games and card matching activities based on my three years experience as a primary school teacher. These do not form part of data collection as no data will be collected from the pupils during the three multiplication lessons but they are part of the procedure of the study. In order to assist the board with their decision, therefore, I have also attached an overview of the interventions that I will be conducting.	

SECTION D: Professional guidelines and training		
In this section, please mark 'X' against at least one of the following professional guidelines you aim to adhere to. You should use the principles listed in your chosen guideline(s) in conducting your own research. **Note**: this is not an exhaustive list.		**Please mark 'X'**
Research specialism/ methodology	**Association and guidance document**	
Anthropology	Association of Social Anthropologists of the UK and Commonwealth	
Criminology	http://www.britsoccrim.org/ethics/	
Education	British Educational Research Association Ethical Guidelines for Educational Research	X
Geography	Association of American Geographers Statement on Professional Ethics	
History	Oral History Society of the UK Ethical Guidelines	
Internet-based Research	British Psychological Society: Conducting Research on the Internet Association of Internet Researchers Ethics Guide Also see our Best Practice Guidance on internet-based research	
Law (Socio-Legal)	Socio-Legal Studies Association: Statement of Principles of Ethical Research	
Management	Academy of Management's Professional Code of Ethics	
Political Science	American Political Science Association (APSA) Guide to Professional Ethics in Political Science	
Politics	Political Studies Association. Guidelines for Good Professional Conduct	
Psychology	British Psychological Society Code of Ethics and Conduct	
Social Research	Social Research Association: Ethical Guidelines	
Sociology	The British Sociological Association: Statement of Ethical Practice	
Visual Research	ESRC National Centre for Research Methods Review Paper: Visual Ethics: Ethical Issues in Visual Research	
Other professional guidelines. Please specify the other guidelines used here*:*		

Please indicate what training in research ethics the researchers involved with this study have received, e.g. the title of the course and date completed (online training available at http://researchsupport.admin.ox.ac.uk/support/training/ethics). If no formal training has been undertaken, please indicate any discussions of research methodology between researchers and supervisors here.
I have attended the Department of Education lectures and seminars on research ethics and obtaining consent from participants. In addition, when working as a teacher, I was given up-to-date GDPR training and formal safeguarding training to enhance my understanding of the risks and responsibilities of data collection and working with children.

SECTION E: Signatures (The SSH IDREC Secretariat accepts either option below. If you have a DREC, check which signature option it prefers.)

- **Option 1:** 'Electronic signatures', i.e. email confirmations from a University of Oxford email address, can be accepted. Separate emails should come from each of the relevant signatories as outlined below, indicating acceptance of the relevant responsibilities. **Pasted images of signatures cannot be accepted in the sections below.**
- **Option 2:** Handwritten (wet-ink) signatures. Please scan them and the rest of the checklist pages to create a single PDF document and email through.

Please ensure this checklist is signed by:

For staff research:	**For student research**:
1. **Principal investigator**	1. **Principal investigator (project supervisor)**
2. **Head of Department (or nominee)**	2. **Head of Department (or nominee)**
	3. **Student researcher**

1. **Principal investigator signature/supervisor signature (if student research)**

I understand my responsibilities as **principal investigator** as outlined in the CUREC glossary and guidance on the CUREC website.

I declare that the answers above accurately describe the research as presently designed, and that a new checklist will be submitted should the research design change in a way which would alter any of the above responses so as to require completion of CUREC 2 (involving full scrutiny by an IDREC). I will inform the relevant IDREC if I cease to be the principal investigator on this project and supply the name and contact details of my successor if appropriate.

Signature: Peter Bryant

Print name (block capitals): PETER BRYANT **Date**: 15th February 2019

2. **Departmental endorsement signature**

I have read the research project application named above. On the basis of the information available to me, I:

(i) consider the principal investigator to be aware of her/his ethical responsibilities in regard to this research;
(ii) consider that any ethical issues raised have been satisfactorily resolved or are covered by relevant professional guidelines and/or CUREC approved procedures, and that it is appropriate for the research to proceed (noting the principal investigator's obligation to report should the design of the research change in a way which would alter any of the above responses so as to require completion of a CUREC 2 full application);
(iii) am satisfied that: the proposed project design and scientific methodology is sound; the project has been/will be subject to appropriate **peer review**; and is likely to contribute to existing knowledge and/or to the education and training of the researcher(s) and that it is in the **public interest**.

Signed by Head of Department or nominee *(example nominees for student research include the Director of Graduate Studies/ Director of Undergraduate Studies)*:

Signature: See CUREC approval email

3. **Student signature (if student research)**

I understand the questions and answers that have been entered above describing the research, and I will ensure that my practice in this research complies with these answers, subject to any modifications made by the principal investigator properly authorised by the CUREC system.

Signed by student: Charlotte Moss

Date: 10th February 2019

Print name (block capitals): CHARLOTTE MOSS

DEPARTMENT OF EDUCATION
15 Norham Gardens, Oxford, OX2 6PY

Telephone: 01865 284892
Researcher: charlotte.moss@education.ox.ac.uk
Supervisor: peter.bryant@education.ox.ac.uk

[Date]

Dear [Headteacher],

RE: Multiplication research project for the University of Oxford

I am writing to ask whether it would be possible for me to conduct a short research project at [school name] looking at how children in Year 3 respond to different ways of teaching multiplication. I am a student currently conducting research into multiplication teaching as part of my Master's at the Department of Education in the University of Oxford. Before my Master's, I worked as a primary school teacher for three years and was graded 'Outstanding' in all eight of the Teachers' Standards. I am extremely interested in developing pedagogy to enhance children's multiplication skills and this research project will focus on how children use times tables and reasoning to answer more complex multiplication questions.

Who is taking part in the study? What will they be doing?
I would like to work with the children in Year 3 during three of their Maths lessons and hope that the project will be beneficial to their multiplication skills. The children will complete a short multiplication test before being randomly split into three different groups. Each group will receive three 30-minute Maths lesson on times tables or other multiplication skills. As an experienced teacher, I will use a variety of exciting games and activities to challenge the children and develop their understanding. After the three lessons, the children will complete the same short multiplication test as before. I will use the results to compare how different ways of teaching multiplication can enhance children's progress.

How will the research be conducted?
The research would be conducted on three convenient days between the 28th April and the 28th June. I understand the pressures of time and staff in schools and am very flexible about when I come in to school. This research can be completed over three mornings, with all the children taking the short test before being split into the three groups. I would teach the first group's session for half an hour and then swap to teach the subsequent two sessions with the other two groups. Each session would take the format of a short Maths lesson, pitched at an appropriate level of challenge for Year 3 pupils, with lots of interactive and engaging activities to enhance their multiplication skills. I will only require a classroom and, if they are available, mini whiteboards and a projector. I do not require any additional staffing for this lesson as I am a qualified teacher (with DBS). I am extremely sensitive to any SEN and am keen to liaise with you to ensure that all the children are able to actively participate and enjoy these sessions.

How will this data be used? Will it be kept confidential?

This research has received ethical approval from the Central University Research Ethics Committee. Each child's personal data will be completely anonymised and all results from the study will be reported without any identifiable data being used about either the school or pupils. In accordance with data protection laws, any personal data collected will be stored in a locked office in the university department and will be destroyed as soon as the data has been anonymised. I will only be collecting data on the children's results on the test and this data will be stored completely anonymously through the use of random numbers and will only be accessed by myself and my supervisor. All results from the study will be reported without any identifiable data being used about either the school or pupils. This anonymous data will be held for the duration of the study and then this data will also be destroyed. Having had school-based training on GDPR regulations, I am extremely sensitive to the importance of data security and welcome any further questions that you may have.

Do the children have to participate?

As the content of these lessons is part of the Year 3 Maths Programme of Study, I have chosen an opt-out consent procedure for the parents of the children involved. With your permission, I would write to the Year 3 parents to tell them about the project. Participation will be completely voluntary and parents will be able to withdraw their participation at any time without any consequences. I have attached a letter which I would send to parents giving them the opportunity to withdraw their consent for pupils' data to be securely shared and for their children to participate in the study.

What are the benefits of taking part?

Multiplication is a particularly important skill in Year 3. These multiplication lessons are part of the children's normal Maths curriculum and should be beneficial and enjoyable for them. The children will participate in a variety of stimulating games and activities to enhance their learning. In addition, the school will receive a Certificate of Participation in educational research from the Department of Education.

If you would like your Year 3 class to participate, please complete the attached consent form and return it to me via email at your earliest convenience. If you have any queries about this research or would like to arrange a meeting to discuss it, you are very welcome to email me or my supervisor. I very much hope that your school will participate in this exciting project and look forward to hearing from you in the near future.

Yours sincerely,

Charlotte Moss BA(Hons) PGCE

DEPARTMENT OF EDUCATION
15 Norham Gardens, Oxford, OX2 6PY

Department phone: 01865 284892
Researcher: charlotte.moss@education.ox.ac.uk
Supervisor: peter.bryant@education.ox.ac.uk

Multiplication Skills in Year 3 Research Project - Consent Form

Department of Education, University of Oxford

Researcher: Charlotte Moss

Supervisor: Professor Peter Bryant

[Headteacher name]

[School name]

[School address]

Please delete as appropriate:

1. I have read all of the information and **do/do not** have any further questions about the research project.

2. Our school **does/does not** wish to participate in this research project.

If you would like any further information, or are interested in taking part, please provide details for a designated contact in school.

Name:

Email:

Telephone:

Thank you very much for completing this form.

DEPARTMENT OF EDUCATION
15 Norham Gardens, Oxford, OX2 6PY

Department phone: 01865 284892
Researcher: charlotte.moss@education.ox.ac.uk
Supervisor: peter.bryant@education.ox.ac.uk

Dear Parent/Guardian,

RE: Multiplication research project for the University of Oxford

I am writing to you about a short research project being conducted at [school name] in [month] this year. I am an Education Master's student at Oxford University and, as part of my course, I will be conducting research into different ways of teaching multiplication. Having worked as a primary school teacher for the past three years, I am particularly interested in ways to help children develop strong multiplication skills. The aim of this project will be to discover whether children can use their times tables knowledge to help them with more difficult questions and I would very much like to include your child in this project.

Who is taking part in the study? What will the children be doing?
I have considerable experience of teaching Maths in Key Stage 2 and will therefore be working with the Year 3 children during three of their Maths lessons. The children will complete a short multiplication test before being randomly split into three different groups. Each group will receive three 30-minute Maths lessons on times tables or other multiplication skills. I will be providing the children with exciting games and activities which will be accessible to all. After the three lessons, the children will complete the short multiplication test again so that I can see whether their skills have improved.

How will this data be used?
This research will be conducted following the university's ethical guidelines. Each child's personal data will be completely anonymised and all results from the study will be reported without any identifiable data being used about either the school or pupils. The results of the multiplication test will not be shared with the school and will only be accessible to myself and my supervisor. In accordance with data protection laws, any personal data collected will be stored in a locked office in the university department and will be destroyed at the end of the study. The project has received ethical approval from the Central University Research Ethics Committee.

What are the benefits of taking part?
Multiplication is a particularly important skill in Year 3. These multiplication lessons are part of the children's normal Maths curriculum and should be beneficial and enjoyable

for them. The children will participate in a variety of stimulating games and activities to enhance their learning whilst contributing to the educational research of the university.

Does my child have to participate?

Participation in this research is completely voluntary. While the school has given consent for your child to participate, you may withdraw this consent at any time. If you choose to withdraw your child, this will have no negative consequences for their learning. I hope that you are happy for your child to participate and that they will enjoy being part of this research. If you are happy for your child to participate, you do not need to do anything at this point. If you would like to withdraw your consent, however, please complete the opt-out form below and return it to your child's teacher.

This is an exciting opportunity for your child to participate in educational research for the University of Oxford. I hope that you will discuss the project with them and that they will enjoy being part of this research. If you or your child have any questions about the research, you are very welcome to email me or my supervisor. I very much hope that your child will enjoy participating and I look forward to meeting them.

Yours faithfully,

Charlotte Moss BA(Hons) PGCE

Year 3 Multiplication Research Project - Withdrawal of Consent Form

UNIVERSITY OF OXFORD

Please only complete this form if you would not like your child to take part or if you have further questions about the project.

Please tick the appropriate box:

☐ I would not like my child to participate in this study.

☐ Although I am happy for my child to participate, I would like further information about this study. I understand that I can withdraw my consent at any time.

Child's Name:

Signature of Parent/Guardian:

Thank you for completing this form. If you have withdrawn your consent, there will be no negative consequences for your child's education. If you would like further information about this study, please feel free to email me or to speak to the class teacher.

Department of Education

University of Oxford

This is to certify that

_____ **Primary School**

participated in the educational research project:

Investigating the effects of procedural variation on multiplicative reasoning

for the University of Oxford in May 2019

_____ _____

Principal Researcher Student Researcher

CUREC Approval Email

From: Sandra Mathers <sandra.mathers@education.ox.ac.uk>
Subject: CUREC Submission
Date: 22 February 2019

Dear Charlotte and Peter

The effects of procedural variation on multiplicative reasoning

The above application has been considered on behalf of the Departmental Research Ethics Committee (DREC) in accordance with the procedures laid down by the University for ethical approval of all research involving human participants.

I am pleased to inform you that, on the basis of the information provided to DREC, the proposed research has been judged as meeting appropriate ethical standards, and accordingly, approval has been granted.

If your research involves participants whose ability to give free and informed consent is in question (this includes those under 18 and vulnerable adults), then it is advisable to read the following NSPCC professional reporting requirements for cases of suspected abuse

http://www.nspcc.org.uk/globalassets/documents/information-service/factsheet-child-abuse-reporting-requirements-professionals.pdf

Should there be any subsequent changes to the project which raise ethical issues not covered in the original application you should submit details to research.office@education.ox.ac.uk for consideration.

Good luck with your research study.

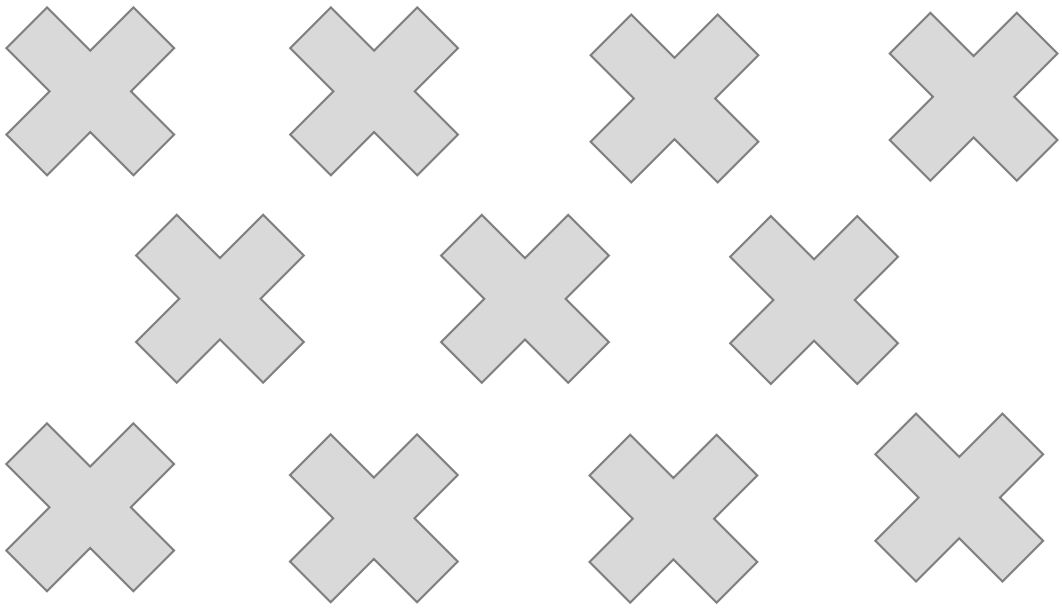
With kind regards,

Sandra Mathers
Department of Education

Appendix B: Pre- and Post-Test

Multiplication Challenge

First Name	
Surname	
School Name	
Date	



Instructions

In this booklet, there are sixteen multiplication challenges.

Your challenge is to fill in the empty boxes in the sequence by spotting the pattern.

The first answer has been given to you as a clue. Try to use the clue to help you find the answer to the next two multiplications.

Then, complete the final multiplication question so that it matches the pattern.

You will have 30 minutes to have a go at all the questions. Make sure you show all your working out in the method box.

Put your hand up if you have a question or would like a question read to you. Make sure you check all of your answers. When you have finished, put your hand up and an adult will collect your booklet.

Practice Question

$100 \times 8 = 800$

$101 \times 8 =$ ☐

$102 \times 8 =$ ☐

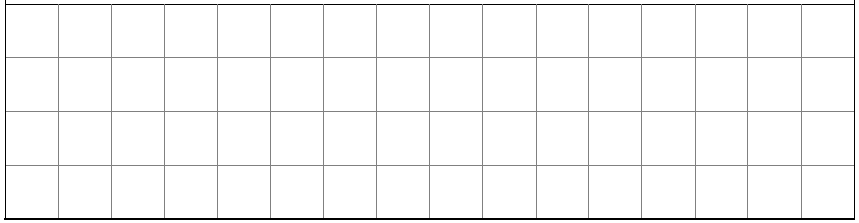
☐ $\times 8 =$ ☐

$10 \times 15 = 150$

$11 \times 15 =$ ☐

$12 \times 15 =$ ☐

☐ $\times 15 =$ ☐

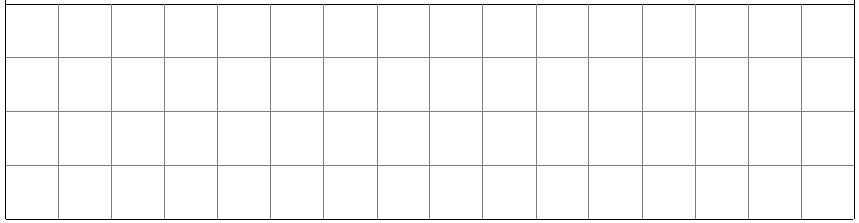


$12 \times 12 = 144$

$13 \times 12 =$ ☐

$14 \times 12 =$ ☐

☐ $\times 12 =$ ☐

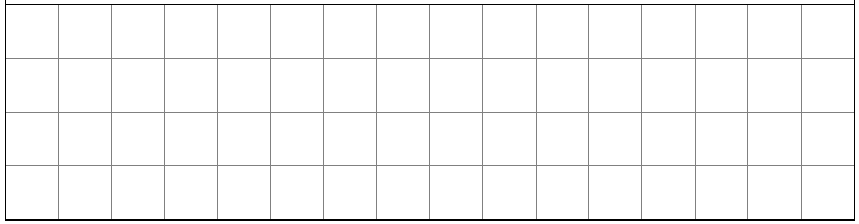


$20 \times 7 = 140$

$21 \times 7 =$ ☐

$22 \times 7 =$ ☐

☐ $\times 7 =$ ☐

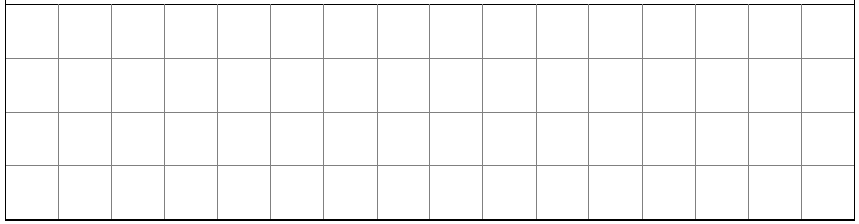


$15 \times 6 = 90$

$16 \times 6 =$ ☐

$17 \times 6 =$ ☐

☐ $\times 6 =$ ☐

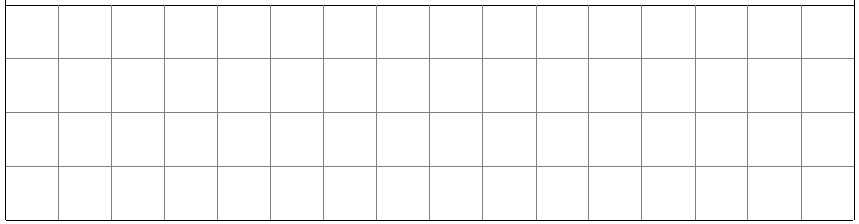


$10 \times 13 = 130$

$9 \times 13 =$ ☐

$8 \times 13 =$ ☐

☐ $\times 13 =$ ☐

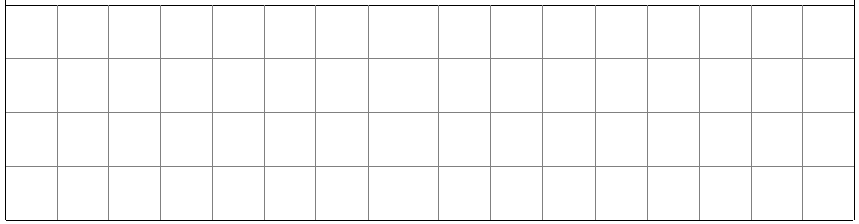


$30 \times 6 = 180$

$29 \times 6 =$ ☐

$28 \times 6 =$ ☐

☐ $\times 6 =$ ☐

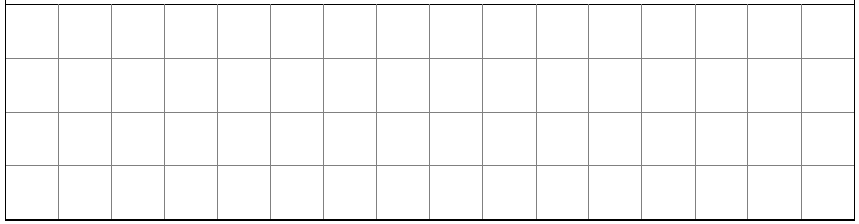


$20 \times 8 = 160$

$19 \times 8 =$ ☐

$18 \times 8 =$ ☐

☐ $\times 8 =$ ☐

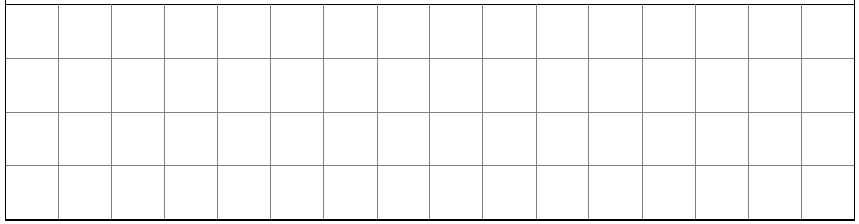


$40 \times 5 = 200$

$39 \times 5 =$ ☐

$38 \times 5 =$ ☐

☐ $\times 5 =$ ☐

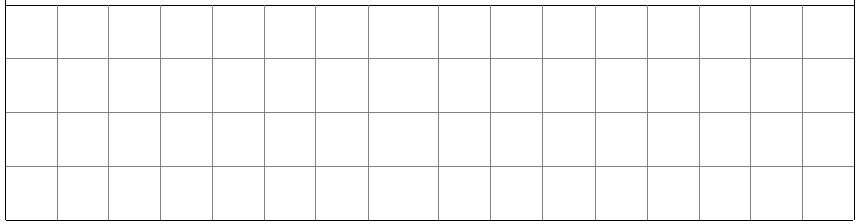


$2 \times 13 = 26$

$4 \times 13 = \square$

$8 \times 13 = \square$

$\square \times 13 = \square$

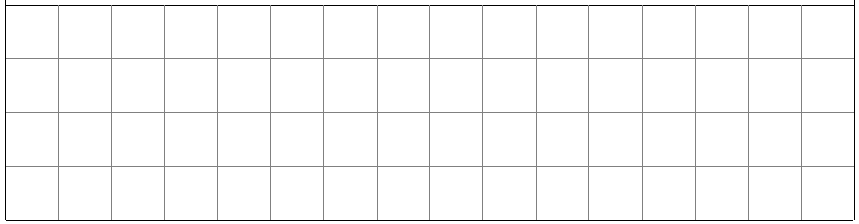


$7 \times 6 = 42$

$14 \times 6 = \square$

$28 \times 6 = \square$

$\square \times 6 = \square$

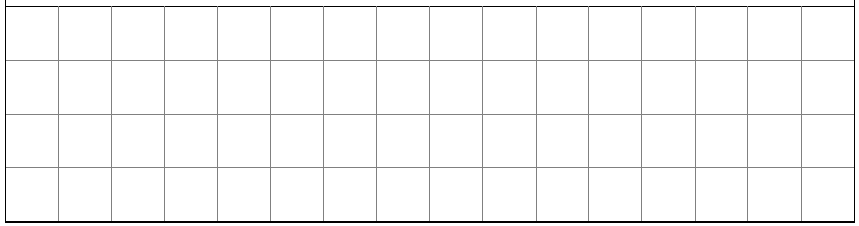


$8 \times 5 = 40$

$16 \times 5 = \square$

$32 \times 5 = \square$

$\square \times 5 = \square$

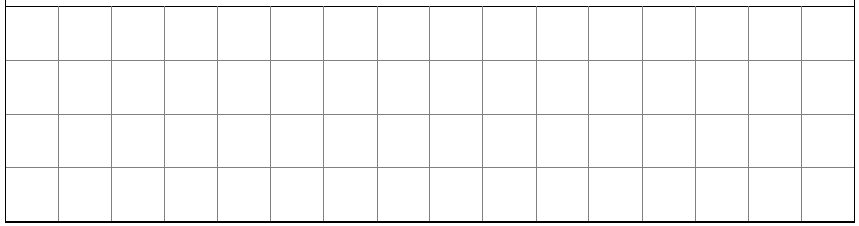


$9 \times 12 = 108$

$18 \times 12 = \square$

$36 \times 12 = \square$

$\square \times 12 = \square$

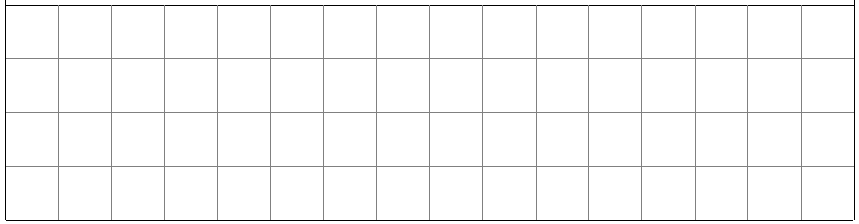


$2 \times 25 = 50$

$6 \times 25 =$ ☐

$18 \times 25 =$ ☐

☐ $\times 25 =$ ☐

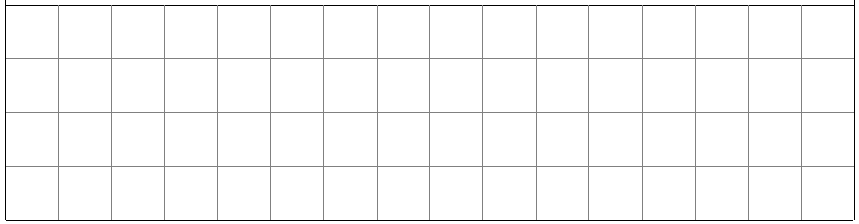


$2 \times 15 = 30$

$8 \times 15 =$ ☐

$32 \times 15 =$ ☐

☐ $\times 15 =$ ☐

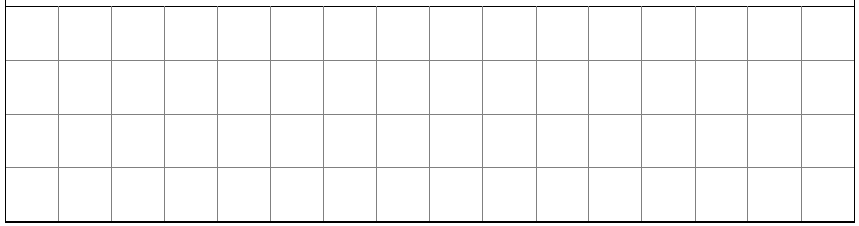


$5 \times 8 = 40$

$15 \times 8 =$ ☐

$45 \times 8 =$ ☐

☐ $\times 8 =$ ☐

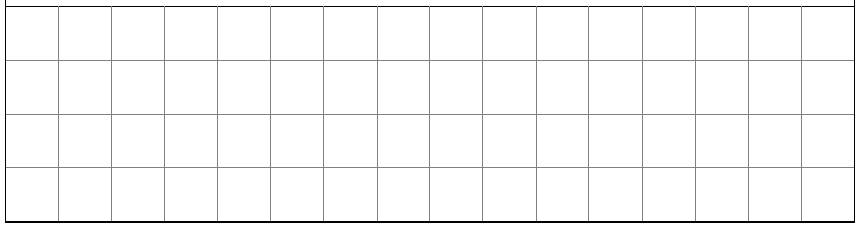


$4 \times 25 = 100$

$16 \times 25 =$ ☐

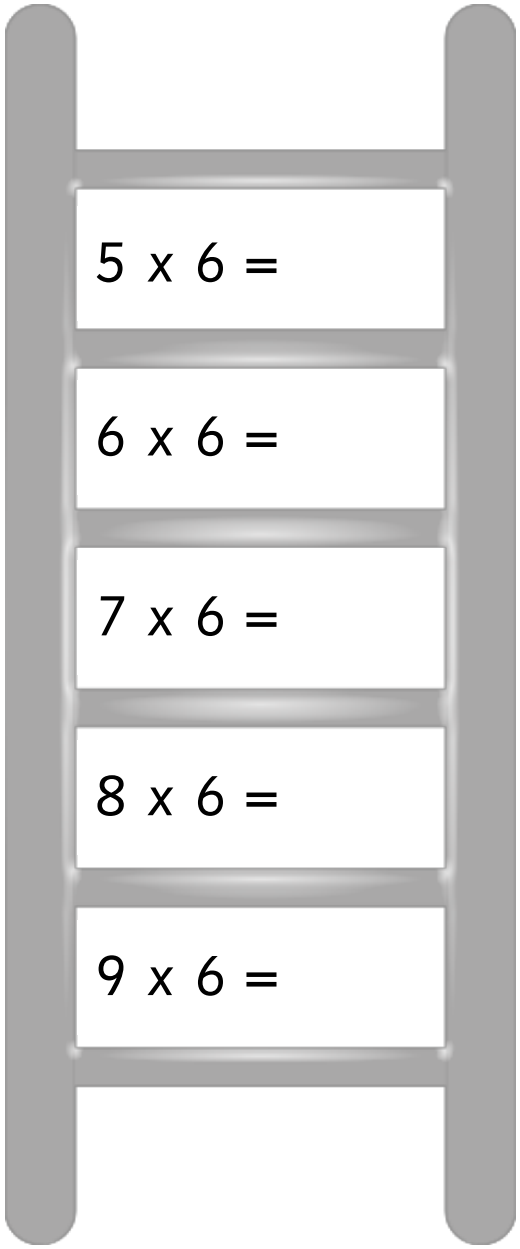
$64 \times 25 =$ ☐

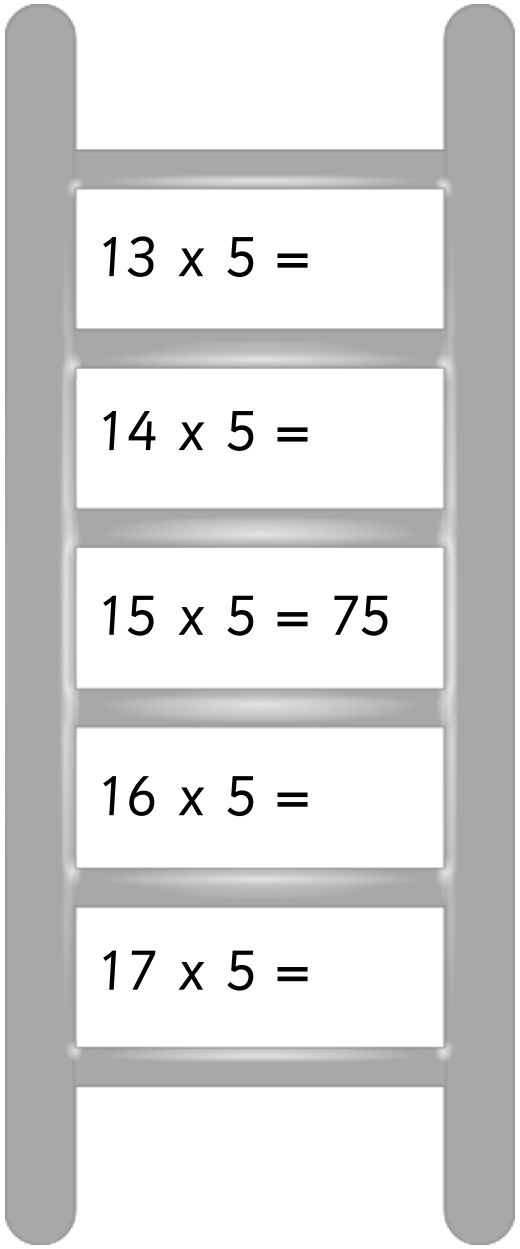
☐ $\times 25 =$ ☐

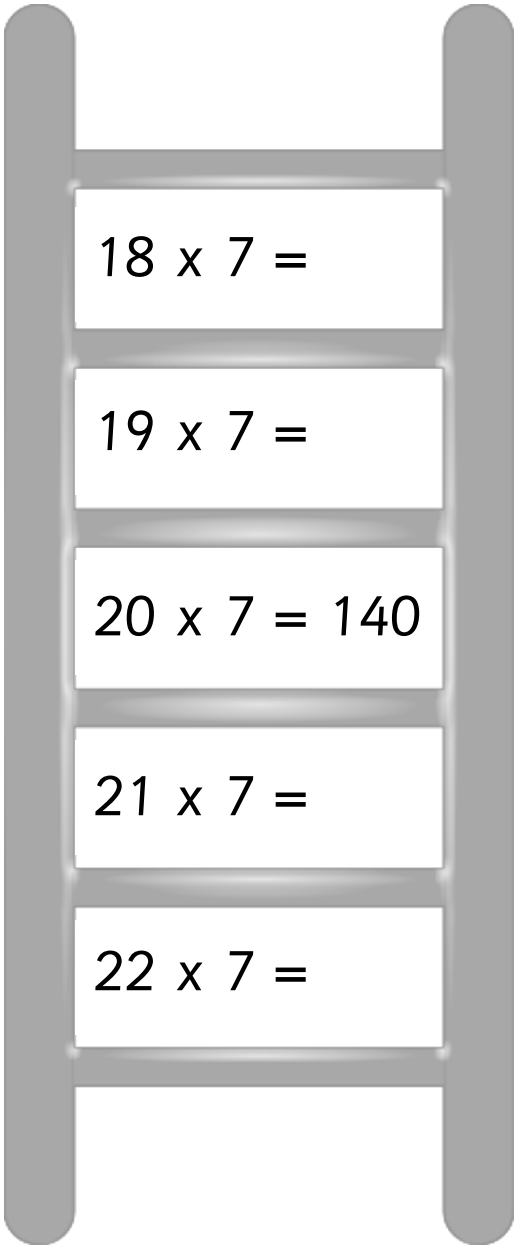


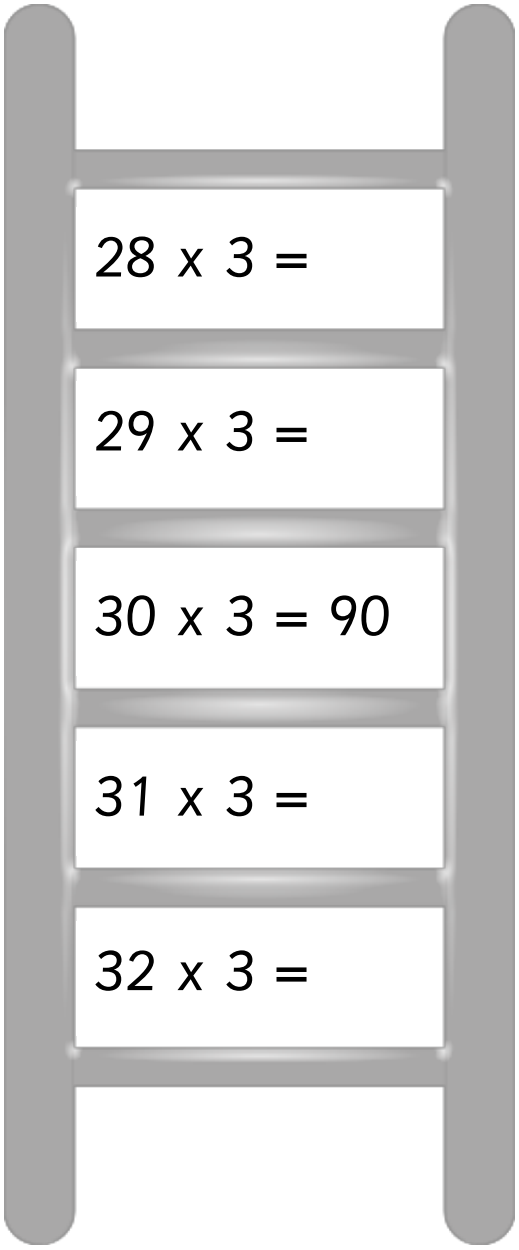
Appendix C: Distributive Reasoning Intervention

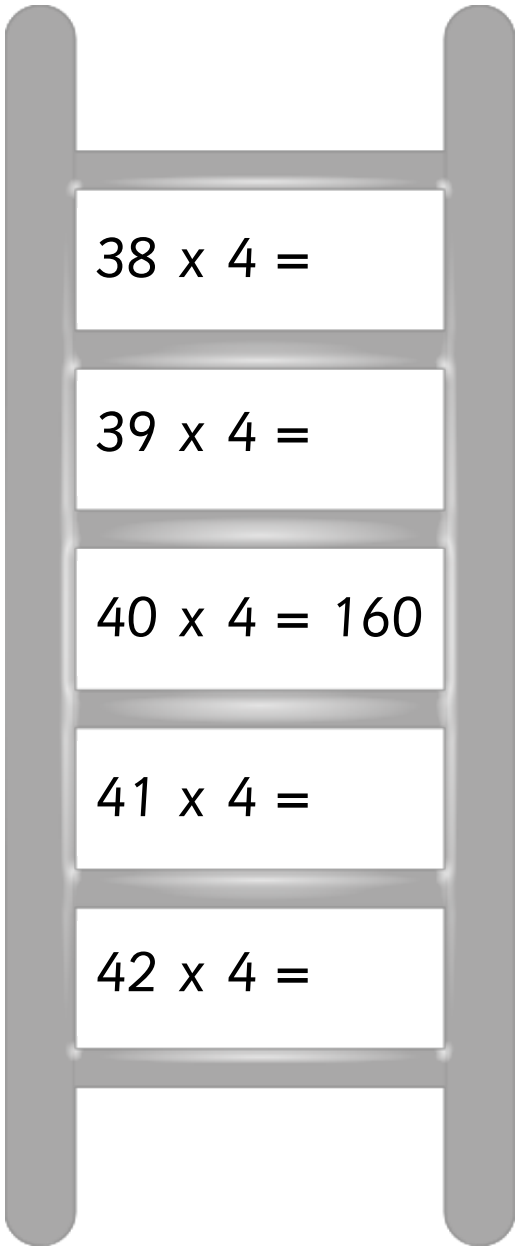
Group 1 - Distributive Multiplication		
	Group Activity	Individual Activity
Session 1	<u>Topic Introduction</u> *We are going to be learning how to find the answer to some tricky multiplication questions.* Write 11 x 14 on the board *What do I know that could help me with this question?* Give children time to think and listen to their answers before giving them 10 x 14 = 140 *How could we use this answer to help us?* Elicit the idea of using addition *What about this question?* 9 x 14 Elicit idea of using subtraction Ask the children to find two more examples *Which one of these could we find first to help us?* 18 x 4 19 x 4 **20 x 4** 21 x 4 22 x 4 49 x 3 **50 x 3** 51 x 3 52 x 3 53 x 3	<u>Ladders Worksheet</u> The children will be given a worksheet with questions on the rungs of a ladder. They will use distributive reasoning to add and subtract multiples from the one answer that is provided. I will circulate and work with the children to support them with their reasoning I will model answering the first question on the board *Can you see a pattern in these ladders?* *How could we use the clue that we've been given?* *Should we add or take away?* At the end, we will go through the answers together and discuss any misconceptions
Session 2	<u>Team Challenge</u> *We're going to have a team challenge where you work in teams to solve the question using the clue. To win the point, you need to explain how you got your answer. Can you get an extra point by creating another multiplication that matches the pattern?* The children will work in two teams, taking turns to use their clue to try to solve the question. I will also ask them to create another multiplication using distributive reasoning to get another point. I will elicit answers from the children and explain how to use distributive reasoning to find the answer to the multiplication questions.	<u>Speed Quiz</u> *What clues could you use to solve the questions?* *How quickly can you solve them?* I will work through an example on the board 30 x 4 = 120 29 x 4 31 x 4 I will give the children a sheet of multiplication questions with some answers already given to them. I will help them use addition and subtraction to find the other answers. I will challenge the children to do the sheet as quickly as possible. I will give formative feedback for the children to check their answers. We will then work together and discuss any questions the children found tricky
Session 3	<u>Card Matching Game</u> I will ask for a volunteer to help me demonstrate to the class and we will explain why the clue goes with the answer on the card. *Can you find your partner? Which clue is best for your question? Can you think of another multiplication that fits the pattern?* I will give each child a card with either a clue on it or a multiplication question. They will work together to find their partners and then explain why their clue is the right one. I will elicit explanations which use distributive reasoning and will write the children's ideas on the board.	<u>Whiteboard Race</u> I will make sure all the children have mini-whiteboards and pens. I will use a PowerPoint with lots of distributive multiplication questions and will challenge the children to write the answer as quickly as possible I will use the first question as a practice and will explain it to the children *How could I use that answer?* *Do I need to add or subtract?* *Can you think of the next one in the pattern?* 13 x 10 = 130 13 x 9 = ?

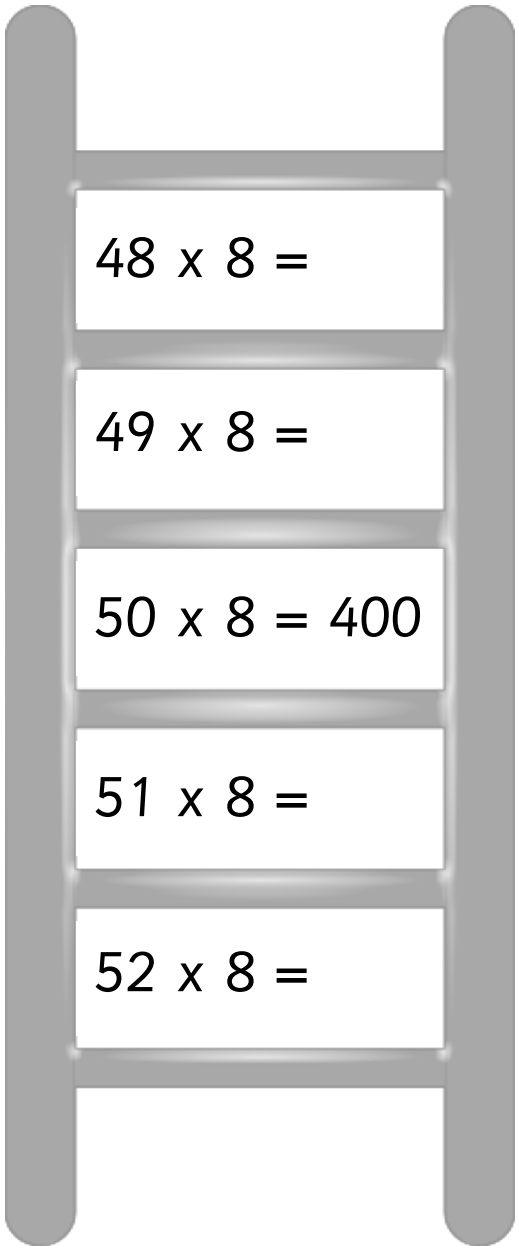

5 x 6 =
6 x 6 =
7 x 6 =
8 x 6 =
9 x 6 =


13 x 5 =
14 x 5 =
15 x 5 = 75
16 x 5 =
17 x 5 =


18 x 7 =
19 x 7 =
20 x 7 = 140
21 x 7 =
22 x 7 =


28 x 3 =
29 x 3 =
30 x 3 = 90
31 x 3 =
32 x 3 =


38 x 4 =
39 x 4 =
40 x 4 = 160
41 x 4 =
42 x 4 =


48 x 8 =
49 x 8 =
50 x 8 = 400
51 x 8 =
52 x 8 =

Distributive Team Challenge

10 x 17 = 170 11 x 17 =	12 x 25 = 300 13 x 25 =
12 x 25 = 300 11 x 25 =	10 x 13 = 130 9 x 13 =
15 x 11 = 165 16 x 11 =	20 x 18 = 360 21 x 18 =
20 x 13 = 260 19 x 13 =	15 x 8 = 120 14 x 8 =
25 x 12 = 300 26 x 12 =	30 x 16 = 480 31 x 16 =
30 x 8 = 240 29 x 8 =	25 x 3 = 75 24 x 3 =
40 x 15 = 600 41 x 15 =	40 x 11 = 440 39 x 11 =

Distributive Speed Quiz

Use the clues to help you answer as many questions as you can.

10 x 14 = 140 11 x 14 = 12 x 14 = x 14 =	10 x 13 = 130 9 x 13 = 8 x 13 = x 13 =
12 x 15 = 180 13 x 15 = 14 x 15 = x 15 =	12 x 15 = 180 11 x 15 = 10 x 15 = x 15 =
15 x 8 = 120 16 x 8 = 17 x 8 = x 8 =	15 x 4 = 60 14 x 4 = 13 x 4 = x 4 =
20 x 17 = 340 21 x 17 = 22 x 17 = x 17 =	20 x 12 = 240 19 x 12 = 18 x 12 = x 12 =
25 x 6 = 150 26 x 6 = 27 x 6 = x 6 =	25 x 4 = 100 24 x 4 = 23 x 4 = x 4 =

30 x 12 = 360
31 x 12 =
32 x 12 =
x 12 =

30 x 7 = 210
29 x 7 =
28 x 7 =
x 7 =

35 x 4 = 140
36 x 4 =
37 x 4 =
x 4 =

35 x 4 = 140
34 x 4 =
33 x 4 =
x 4 =

40 x 25 = 1000
41 x 25 =
42 x 25 =
x 25 =

40 x 5 = 200
39 x 5 =
38 x 5 =
x 5 =

Can you make up your own multiplication sequences?

x =
x =
x =
x =
x =

x =
x =
x =
x =
x =

Distributive Card Matching Game

20 x 12 = 240
10 x 17 = 170
20 x 14 = 280
30 x 12 = 360

10 x 25 = 250
20 x 6 = 120
30 x 5 = 150

21 x 12 =
11 x 17 =
21 x 14 =
31 x 12 =

9 x 25 =
19 x 6 =
29 x 5 =

Appendix D: Scalar Reasoning Intervention

<table>
<tr><th colspan="3">Group 2 - Scalar Multiplication</th></tr>
<tr><th></th><th>Group Activity</th><th>Individual Activity</th></tr>
<tr><td>Session 1</td><td><u>Topic Introduction</u>
We are going to be learning how to find the answer to some tricky multiplication questions.
Write 20 x 15 on the board What do I know that could help me with this question?
Give children time to think and listen to their answers before giving them 10 x 15 = 150
How could we use this answer to help us?
Elicit the idea of using multiplication - write on board
What about this question? 40 x 15
Give the children columns of questions to answer
How can I use the first answer to find this one?
What's the pattern? What changes each time?
4 x 5 8 x 5 16 x 5 2 x 15 6 x 15 18 x 15
7 x 5 70 x 5 700 x 5 6 x 5 18 x 5 36 x 5</td><td><u>Ladders Worksheet</u>
The children will be given a worksheet with questions on the rungs of a ladder. They will use scaling reasoning to multiply from the one answer that is provided. I will circulate and work with the children to support them with their reasoning

I will model answering the first question on the board
Can you see a pattern in these ladders?
How could we use the clue that we've been given?
What's the pattern? What do we need to multiply by?

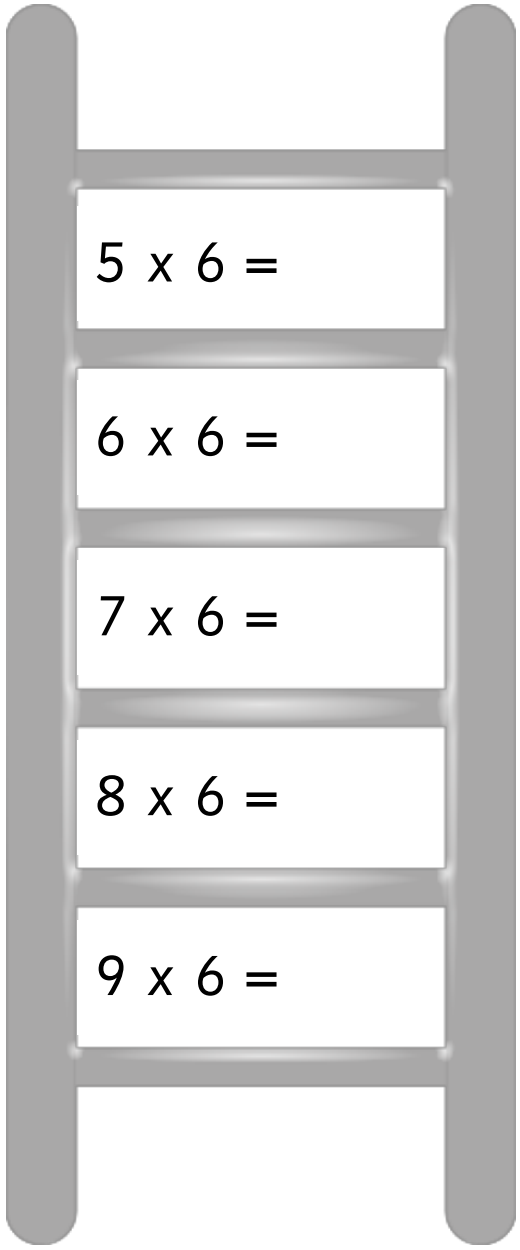
At the end, we will go through the answers together and discuss any misconceptions</td></tr>
<tr><td>Session 2</td><td><u>Team Challenge</u>
We're going to have a team challenge where you work in teams to solve the question using the clue. To win the point, you need to explain how you got your answer. Can you get an extra point by creating another multiplication that matches the pattern?

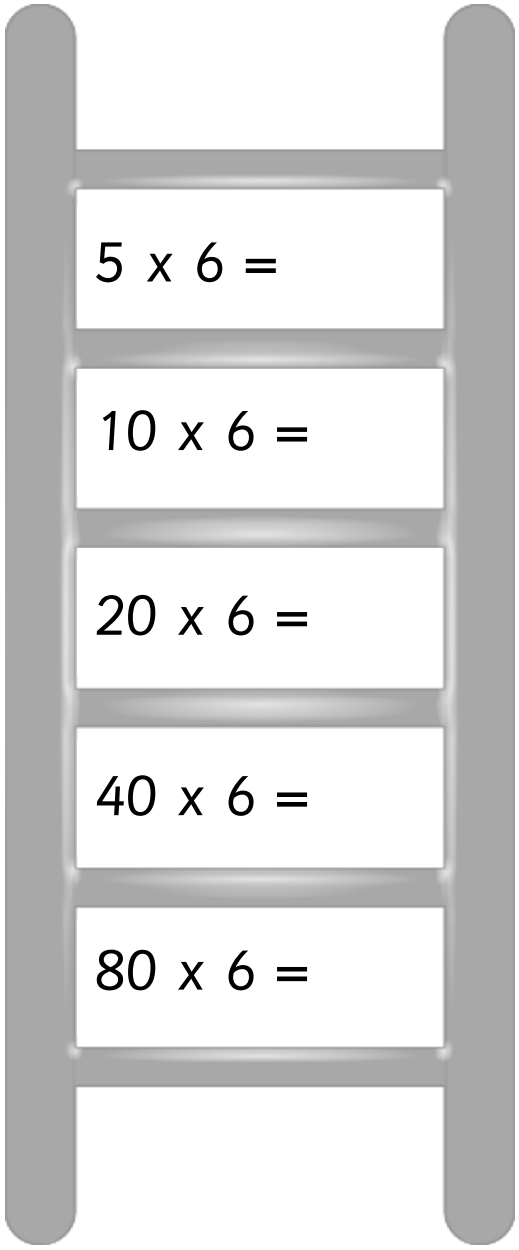
The children will work in two teams, taking turns to use their clue to try to solve the question. I will also ask them to create another multiplication using scaling reasoning to get another point.
I will elicit answers from the children and explain how to use scaling reasoning to find the answer to the multiplication questions.</td><td><u>Speed Quiz</u>
What clues could you use to solve the questions?
How quickly can you solve them?
I will work through examples on the board
3 x 12 = 36 30 x 12 = ?
5 x 6 = 15 x 6 = ?

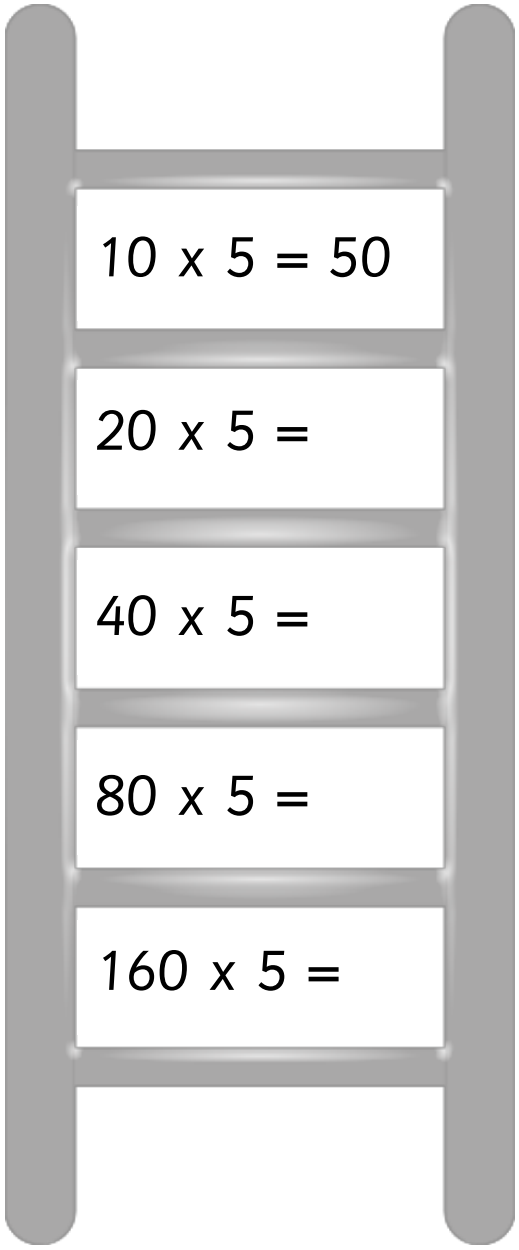
I will give the children a sheet of multiplication questions with some answers already given to them. I will help them use multiplication to find the other answers. I will challenge the children to do the sheet as quickly as possible. I will give formative feedback for the children to check their answers. We will then work together and discuss any questions the children found tricky</td></tr>
<tr><td>Session 3</td><td><u>Card Matching Game</u>
I will ask for a volunteer to help me demonstrate to the class and we will explain why the clue goes with the answer on the card.
Can you find your partner? Which clue is best for your question? Can you think of another multiplication that fits the pattern?

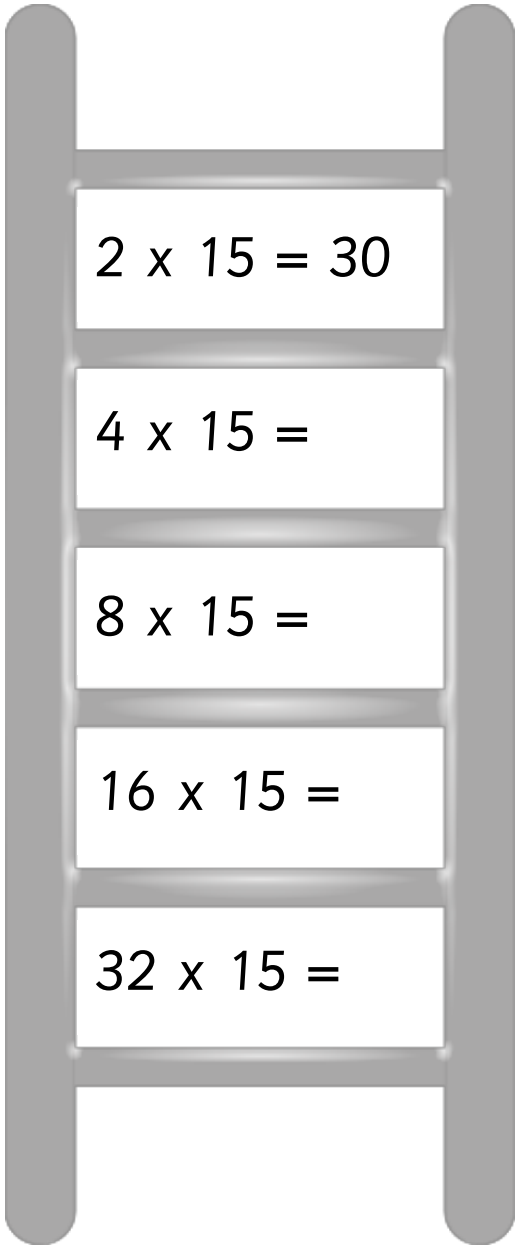
I will give each child a card with either a clue on it or a multiplication question. They will work together to find their partners and then explain why their clue is the right one. I will elicit explanations which use scaling reasoning and will write the children's ideas on the board.</td><td><u>Whiteboard Race</u>
I will make sure all the children have mini-whiteboards and pens. I will use a PowerPoint with lots of scaling multiplication questions and will challenge the children to write the answer as quickly as possible

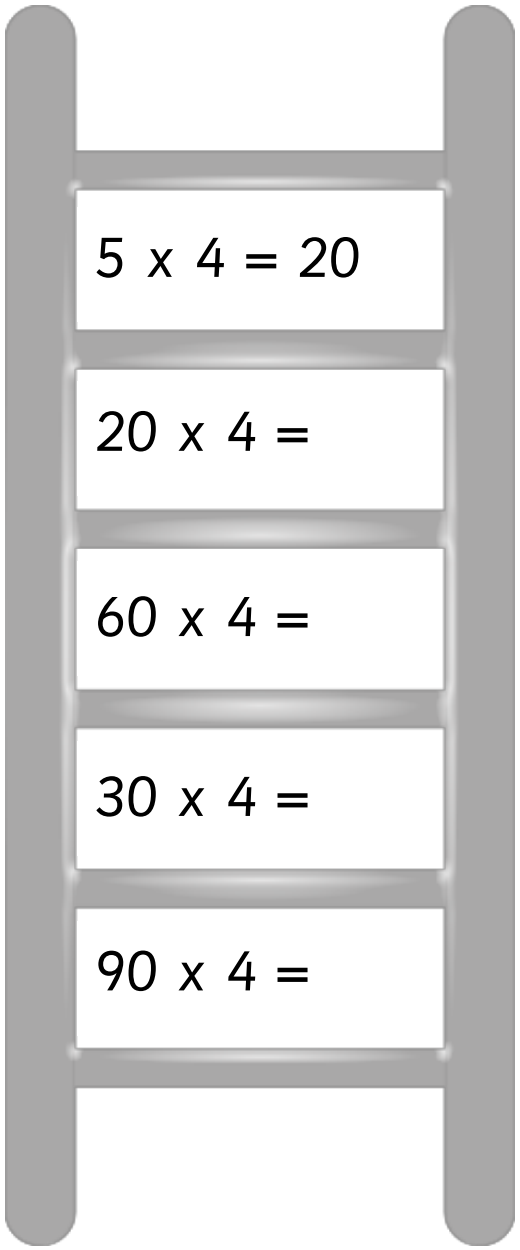
I will use the first question as a practice and will explain it to the children
How could I use that answer? Do I need to double or multiply by a different number?
Can you think of the next one in the pattern?
10 x 13 = 130
20 x 13 = ?</td></tr>
</table>

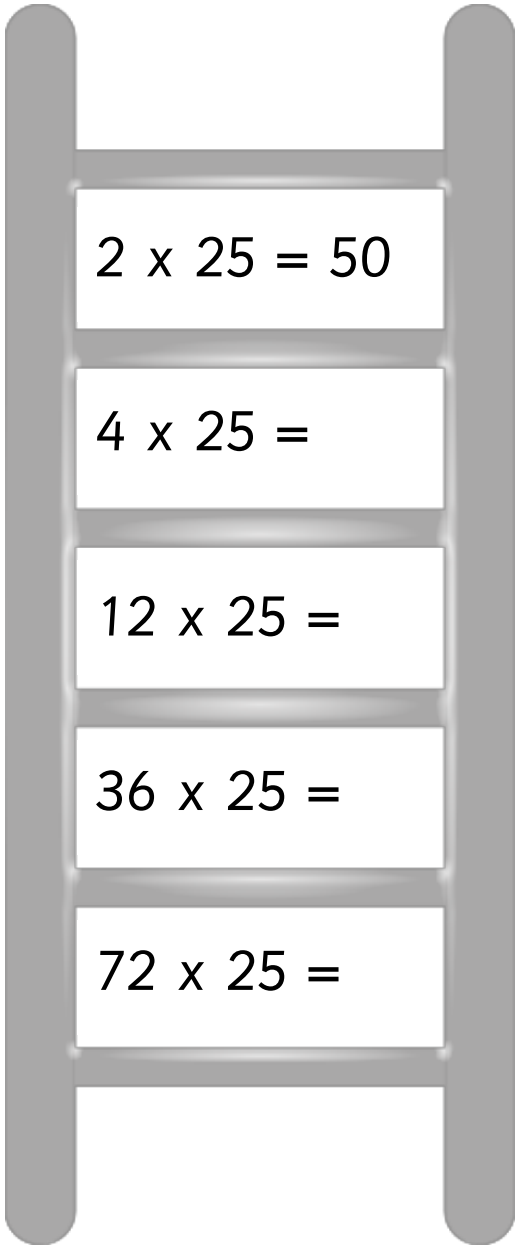

5 x 6 =
6 x 6 =
7 x 6 =
8 x 6 =
9 x 6 =


5 x 6 =
10 x 6 =
20 x 6 =
40 x 6 =
80 x 6 =


10 x 5 = 50
20 x 5 =
40 x 5 =
80 x 5 =
160 x 5 =


2 x 15 = 30
4 x 15 =
8 x 15 =
16 x 15 =
32 x 15 =


5 x 4 = 20
20 x 4 =
60 x 4 =
30 x 4 =
90 x 4 =


2 x 25 = 50
4 x 25 =
12 x 25 =
36 x 25 =
72 x 25 =

Scaling Team Challenge

2 x 15 = 30 4 x 15 =	12 x 11 = 132 24 x 11 =
8 x 5 = 40 16 x 5 =	8 x 25 = 200 16 x 25 =
4 x 13 = 52 8 x 13 =	7 x 13 = 91 14 x 13 =
10 x 12 = 120 30 x 12 =	10 x 11 = 110 40 x 11 =
5 x 11 = 55 15 x 11 =	2 x 16 = 32 6 x 16 =
5 x 12 = 60 20 x 12 =	2 x 25 = 50 8 x 25 =
6 x 7 = 42 18 x 7 =	4 x 13 = 52 12 x 13 =

Scaling Speed Quiz

Use the clues to help you answer as many questions as you can.

2 x 35 = 70 4 x 35 = 8 x 35 = x 35 =	3 x 14 = 42 6 x 14 = 12 x 14 = x 14 =
6 x 15 = 90 12 x 15 = 24 x 15 = x 15 =	8 x 25 = 200 16 x 25 = 32 x 25 = x 25 =
7 x 5 = 35 14 x 5 = 28 x 5 = x 5 =	9 x 5 = 45 18 x 5 = 36 x 5 = x 5 =
10 x 4 = 40 30 x 4 = 90 x 4 = x 4 =	20 x 5 = 100 60 x 5 = 180 x 5 = x 5 =
2 x 15 = 30 6 x 15 = 18 x 15 = x 15 =	4 x 25 = 100 12 x 25 = 36 x 25 = x 25 =

10 x 3 = 30	10 x 5 = 50
40 x 3 =	40 x 5 =
160 x 3 =	160 x 5 =
x 3 =	x 5 =

5 x 4 = 20	2 x 25 = 50
20 x 4 =	8 x 25 =
80 x 4 =	32 x 25 =
x 4 =	x 25 =

4 x 5 = 20	4 x 15 = 60
16 x 5 =	16 x 15 =
64 x 5 =	64 x 15 =
x 5 =	x 15 =

Can you make up your own multiplication sequences?

x =	x =
x =	x =
x =	x =
x =	x =
x =	x =

Scaling Card Matching Game

4 x 25 = 100
6 x 3 = 18
12 x 12 = 144
5 x 8 = 40

6 x 5 = 30
7 x 8 = 56
4 x 9 = 36

12 x 25 =
60 x 3 =
24 x 12 =
15 x 8 =

18 x 5 =
14 x 8 =
40 x 9 =

Appendix E: Times Tables (Control) Intervention

Group 3 (Control) - Times Tables		
	Group Activity	Individual Activity
Session 1	<u>Topic Introduction</u> *We are going to become times tables experts* Vote on which is the children's least favourite times table and ask them why it is the hardest <u>Fizz Buzz Game</u> I will ask the children if they know how to play this times tables game. We will replace multiples of specific times tables with the words *fizz* and *buzz* and will play a counting game in a circle.	<u>Code Breaking Worksheet</u> The children will be given a worksheet with the alphabet corresponding to different times table multiples. The children will come up with a message for their partner, translate it into the numbers and write times table questions for each number. Their partner will solve the times tables to find out the message. I will explain the activity and give the children an example codebreaking challenge to make sure that they understand the activity. At the end, I will ask the children to challenge me to code break a times table message.
Session 2	<u>Around the World Game</u> The children will stand in a circle and I will give two children a times table question. The child who answers correctly will then move to stand by the next child. The children will try to answer all of their questions correctly and go all the way around the circle. I will ask the children whether they have played the game before and will explain the rules. *Who can answer the times table first?* *Can you get all the way around the circle?*	<u>144 Square Challenge</u> *Can you fill in all the times tables?* *How quickly can you solve them?* I will explain the sheet to the children and check whether they have any questions. I will give the children a 12 x 12 grid with the numbers 1-12 printed in a column and along the top row. The children will use the grid to fill in all of their times tables as quickly as possible. I will give formative feedback for the children to check their answers. We will then work together and discuss any questions the children found tricky.
Session 3	<u>Times Tables Bingo</u> The children will draw a 3 x 3 grid and write multiples of a times table I will give the children times table questions and they will cross out the correct answer until one of the children shouts Bingo!	<u>Whiteboard Race</u> I will make sure all the children have mini-whiteboards and pens. I will use a powerpoint with lots of times tables questions and will challenge the children to write the answer as quickly as possible I will use the first question as a practice and will check all of the children know what to do.

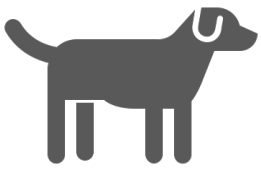
Times Table Codebreaking

a	b	c	d	e	f	g	h	i	j	k	l	m
42	24	12	32	48	25	64	27	36	14	30	72	81
n	o	p	q	r	s	t	u	v	w	x	y	z
28	21	18	20	16	54	56	63	80	35	44	15	9

Can you use your times tables knowledge to break the code?

Use the clue to circle the correct animal.






4 x 9	8 x 7		12 x 3	9 x 6		5 x 5	7 x 9	4 x 4	8 x 2	3 x 5

7 x 5	9 x 4	7 x 8	9 x 3		6 x 7		6 x 12	3 x 7	7 x 4	8 x 8

8 x 7	7 x 6	3 x 12	12 x 6		6 x 7	4 x 7	8 x 4

6 x 9	3 x 7	9 x 9	6 x 8	8 x 7	4 x 9	9 x 9	12 x 4	6 x 9

12 x 3	7 x 8		3 x 6	9 x 7	4 x 4	2 x 8	9 x 6

144 Square Challenge

x	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												

Time:_____

Time with penalties:_____