Particle systems and stochastic PDEs on the half-line

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A thesis submitted for the degree of

Doctor of Philosophy

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Abstract

The purpose of this thesis is to develop techniques for analysing interacting particle systems on the half-line. When the number of particles becomes large, stochastic partial differential equations (SPDEs) with Dirichlet boundary conditions will be the natural objects for describing the dynamics of the population's empirical measure.

As a source of motivation, we consider systems that arise naturally as models for the pricing of portfolio credit derivatives, although similar applications are found in mathematical neuroscience, stochastic filtering and mean-field games.

We will focus on a stochastic McKean–Vlasov system in which a collection of Brownian motions interact through a correlation which is a function of the proportion of particles that have been absorbed at level zero. We prove a law of large numbers where the limiting object is the unique solution to (the weak formulation of) the loss-dependent SPDE:

$$dV_t\,(x) = \frac{1}{2}\partial_{xx}V_t\,(x)\,dt - \rho\,(L_t)\,\partial_x V_t\,(x)\,dW_t, \quad V_t\,(0) = 0,$$

where $L_t = 1 - \int_0^\infty V_t\,(x)\,dx$, V is a density process on the half-line and W is a Brownian motion. The correlation function is assumed to be piecewise Lipschitz, which encompasses a natural class of credit models.

The first of our theoretical developments is to introduce the kernel smoothing method in the dual of the first Sobolev space, H^{-1}, with the aim of proving uniqueness results for SPDEs. A benefit of this approach is that only first order moment estimates of solutions are required, and in the particle setting this translates into studying the particles at an individual level rather than as a correlated collection. The second idea is to extend Skorokhod's M_1 topology to the space of processes that take values in the tempered distributions. The benefit we gain is that monotone functions have zero modulus of continuity under this topology, so the loss process, L, is easy to control.

As a final example, we consider the fluctuations in the convergence of a basic particle system with constant correlation. This gives rise to a central limit theorem, for which the limiting object is a solution to an SPDE with random transport and an additive idiosyncratic driver acting on the first derivative terms. Conditional on the systemic random variables, this driver is a space-time white noise with intensity controlled by the empirical measure of the underlying system. The SPDE has insufficient regularity for us to work in any Sobolev space higher than H^{-1}, hence we have an example of where our extension to the kernel smoothing method is necessary.

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Contents

CHAPTER 1

Introduction

We will begin by giving an overview of the problems studied in this thesis and their practical motivation. Only heuristic arguments and qualitative descriptions of the main results will be presented; full technical details are reserved until the relevant chapter is reached. A summary of all results can be found in Section 1.8.

1.1. The basic particle system

Before commencing with any background material or descriptions of results, we will first lay out the basic modelling framework for this thesis. We will add more advanced features later in this chapter. For now, it is worth remembering that the purpose of these models is to describe an interacting system of defaultable entities.

In the simplest system, each particle, X^i, in a collection of $N \in \mathbf{N}$ particles is modelled as a correlated Brownian motion:

$$X_t^i = X_0^i + \rho W_t + \sqrt{1-\rho^2} W_t^i. \tag{1.1.1}$$

Here, $W, W^1, W^2, \dots$ are independent standard Brownian motions and the parameter $\rho \in [0, 1)$ is called the *correlation*. We will refer to W as the *systemic Brownian motion*.

Remark 1.1.1 (Modelling assumptions)**.** Throughout, the initial values, X_0^i, are assumed to be independent (of each other and the Brownian motions) and identically distributed with law ν_0 that is support on some interval $[x_\star, x^\star]$ with $0 < x_\star < x^\star < \infty$. We will consider the system on a finite time interval $[0, T]$. All processes are assumed to live on some filtered probability space, $(\Omega, \{\mathcal{F}_t\}, \mathbf{P})$, satisfying the usual conditions.

Remark 1.1.2 (Initial condition)**.** The assumption that ν_0 is supported on $[x_\star, x^\star]$ is not essential, but does simplify some later calculations which describe the behaviour of the system at the boundary and at infinity (in particular, those in Section 2.2). It would be possible to replace our assumption by more general decay conditions on ν_0 at zero and infinity, at the cost of complicating later analysis. Such modifications would not alter the qualitative behaviour of the system, so we will not investigate more general initial conditions.

To introduce killing into the system (which models default in the context of credit derivatives), we denote the first hitting time of zero of the i^{th} particle by

$$\tau^i := \inf \left\{ t > 0 : X_t^i \leq 0 \right\}.$$

In order to study the population of surviving particles we introduce the *empirical process*:

$$\nu_t^N := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{t<\tau^i} \delta_{X_t^i}.$$

Here, $\delta_{X_t^i}$ is the Dirac measure at the point $X_t^i \in \mathbf{R}$, hence ν^N is a stochastic process taking values in the space of sub-probability measures on $\mathbf{R}$. For a subset $S \subseteq (0, \infty)$, $\nu_t^N(S)$ equals the proportion of particles in the set S at time t that have not yet been killed at the origin:

$$\nu_t^N(S) = \frac{\# \left\{ 1 \leq i \leq N : X_t^i \in S \text{ and } t < \tau^i \right\}}{N}.$$

Setting $S = (0, \infty)$ gives the proportion of surviving particles, and this inspires the definition of the *loss process*:

$$L_t^N := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\tau^i \leq t} = 1 - \nu_t^N(0, \infty).$$

(See Figure 1.1.1.)

The large population limit. It is straightforward to study the limiting behaviour as N becomes large. Convergence at the process level can be found in [**BHH**$^+$**11**]. Since the particles are conditionally i.i.d. given the systemic Brownian motion, W, Birkoff's Ergodic Theorem [**Dur10**] implies that

$$\nu_t^N\big(S\big) \to \mathbf{P}\big(X_t^1 \in S; t < \tau^1 \big| \, W\big) =: \nu_t\big(S\big), \quad \text{with probability 1,} \tag{1.1.2}$$

as $N \to \infty$. We call the measure-valued process ν the *limit empirical process.*

Remark 1.1.3 (Conditioning on W)**.** It is clear that the above definition of $\nu_t\big(S\big)$ is unchanged if we instead condition on the filtration of W up to time t. Throughout this thesis we will use the notation $(\cdot \,|W)$ to indicate that we are conditioning on the sigma algebra generated by the entire trajectory of W.

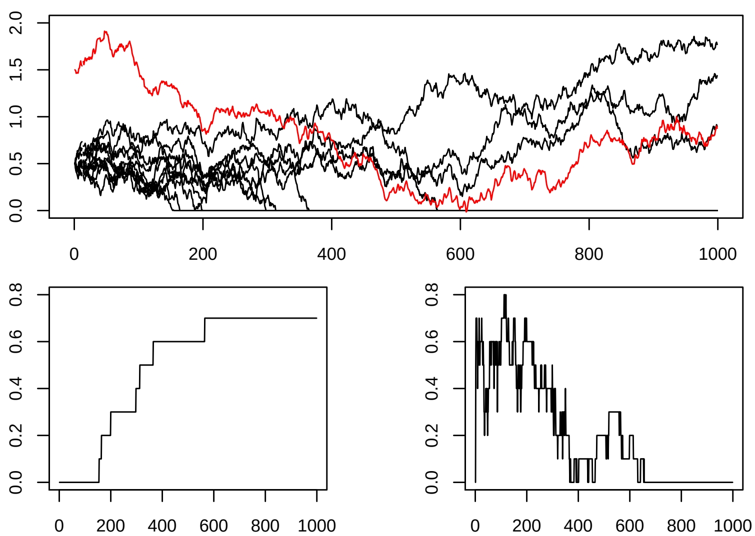


FIGURE 1.1.1. This is a realisation of (1.1.1) for $N = 10$. In each diagram time is plotted on the horizontal axis. In the top diagram we have the trajectories of $X^i_{t\wedge\tau^i}$ in black and the systemic Brownian motion, W, in red (a value of 1.5 is added to W for readability). The bottom left shows the evolution of L^N and the bottom right shows the evolution of $\nu^N(0, 0.5)$.

Conditional on $\sigma(W)$, the particles behave as independent Brownian motions with speed $\sqrt{1-\rho^2}$ and rough drift $t \mapsto \rho W_t$. In line with this, we find in [**BHH⁺11**, Theorem 1.1] that the limit empirical process satisfies the following version of the heat equation with a stochastic transport term:

$$\nu_t(\phi) = \nu_0(\phi) + \frac{1}{2}\int_0^t \nu_s(\partial_{xx}\phi)\,ds + \rho\int_0^t \nu_s(\partial_x\phi)\,dW_s, \tag{1.1.3}$$

for test functions ϕ in the space

$$C^{\text{test}} := \{\phi \in \mathscr{S} : \phi(0) = 0\}.$$

Here, $\mathscr{S}$ denotes the space of rapidly decreasing functions on $\mathbf{R}$ (see Section 2.1) and $\nu_t(\phi) = \int_0^\infty \phi d\nu_t$. We refer to (1.1.3) as the *basic SPDE* (a rigorous definition of the notion of a solution is given in Definition 2.0.3).

Since ν is supported on the positive half-line, C^{test} encodes the Dirichlet boundary condition. Thus ν behaves as a heat flow in a semi-infinite bar with random spatial transport due to the systemic Brownian motion. Setting $\rho = 0$ yields the deterministic heat equation in a semi-infinite bar (see Figure 1.1.2).

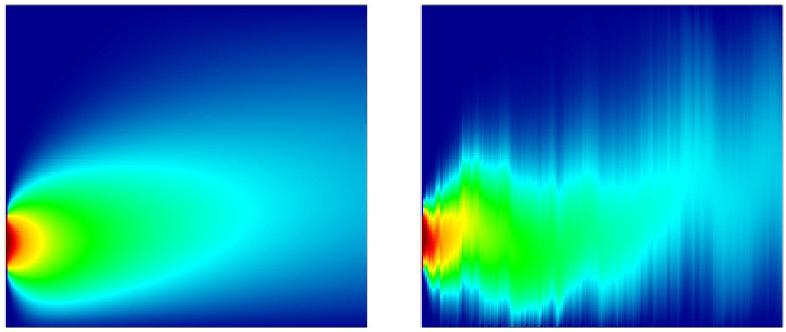


FIGURE 1.1.2. In each diagram space is plotted along the vertical axis and time along the horizontal axis. The colour at each pixel gives the value of the density of ν at that choice of (t, x); blue represents value 0 and red represents value 1. In both cases the initial condition is a step function $x \mapsto \mathbf{1}_{a<x<b}$, for some $0 < a < b$. On the left we have $\rho = 0$, which is the deterministic heat equation. On the right $\rho = 0.5$, and we see the effects of the random transport term. In both cases there is zero drift; the apparent upward trend is due to survivorship bias caused by the absorbing boundary at zero.

The basic SPDE is analysed fully in [**BHH⁺11**] and the limit empirical process is shown to be the unique solution. Our aim is to build more sophisticated interactions into this framework.

1.2. Practical motivation

Our motivating application is the development of portfolio credit derivative models. As a particular example of such a derivative product, we will focus on the collateralised debt obligation (CDO) — the reader is referred to [**Sch03a**] for a thorough treatment of the subject.

In an approximate sense, a CDO on a portfolio of N credit risky assets is a non-linear option on the loss process, L^N, with payoff function of the form

$$\Psi\left(L_T^N\right) = \begin{cases} 0, & \text{if } L_T < a \\ L_T - a, & \text{if } a \leq L_T \leq d \\ d - a, & \text{if } L_T > d. \end{cases}$$

Here, $a < d$ are called the *attachment* and *detachment points* and (a, d) a *tranche*. (This is a simplification, and, in particular, ignores the CDO term structure;

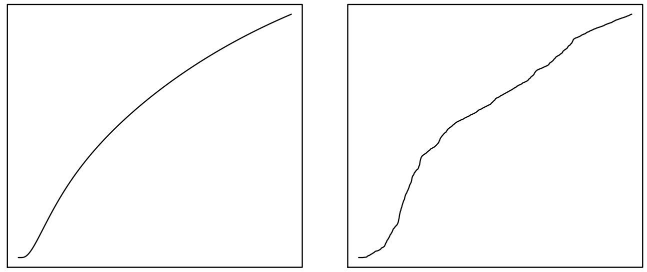


FIGURE 1.2.1. Here we have $t \mapsto L_t$ for the two cases in Figure 1.1.2. Time is on the horizontal axis and the value of the loss process on the vertical axis. Notice the roughness for $\rho = 0.5$ on the right.

again see [**Sch03a**] for full details.) Market participants can buy and sell protection on CDO tranches for various choices of a and d. The fee paid for buying one unit of protection (received for selling protection) is called the *spread*; the larger the expected loss, the larger the spread.

In [**BHH$^+$11**] each entity is assigned a distance-to-default process, modelled by X^i from (1.1.1) with an addition systemic drift term, and default of this entity is modelled as the first time its distance-to-default process hits zero. The idea is to use the law of large numbers approximation

$$L^N \approx L := (1 - \nu_t(0, +\infty))_{t\in[0,T]}$$

to price the CDO and other such contingent claims (see Figure 1.2.1). As the law of large numbers is at the process level, this approximation is dynamic in the sense that it can be used for derivatives with term structure.

Models such as this, in which a default is represented by some stochastic process crossing a threshold, are called *structural models* and were introduced in the context of portfolio derivatives by [**Zho01**] and [**HW01**]. Their origins trace back to [**Mer74**] and [**BC76**] for single-name derivatives.

Large portfolio approximations were first introduced in [**Vas91**] for a model which features no dynamics. Popular approaches include copula based modelling [**Li00, MN02, PD02, FM03, CLV04**] and reduced form models in which the individual hazard rates are modelled directly [**DG01, Mor06, DGT09, EGG10**].

The need for a better model. Since the CDO payoff is non-linear, pricing a collection of tranches requires knowledge of the entire distribution of the

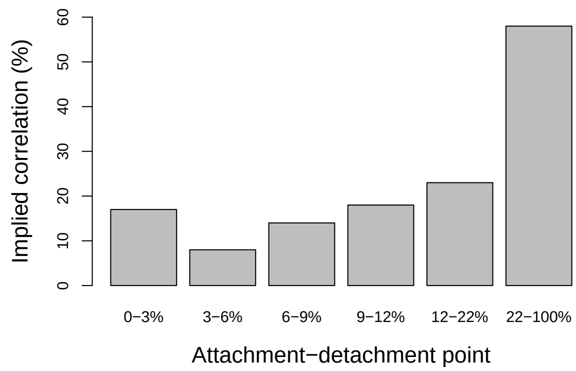


FIGURE 1.2.2. Implied correlation for each tranche for the data set from [**BHH+11**, Figure 2, 7 year maturity]. This is an example of correlation skew.

model's loss process. However, in [**BHH+11**, Section 5] the authors show that the correlation parameter ρ cannot be chosen so as to simultaneously solve

$$\mathrm{model\,spread}\,(\rho) = \mathrm{traded\,spread}$$

for every observed tranche spread in their data set. In other words, solving this equation (which actually might not even be possible) for each tranche leads to different values of ρ, and this phenomenon is known as *correlation skew* or *correlation smile* (see Figure 1.2.2).

This problem is common for Gaussian models; the risk processes are insufficient for producing large losses and so a large correlation is required to match the spreads for senior tranches [**Gor03, Sch03a, Fin05**]. Empirical studies highlight fluctuating historical periods of relatively few corporate defaults followed by periods of default clustering in which many entities default in close proximity [**DDKS07, AGS10**]. Key modelling concerns, therefore, include extreme events and feedback and contagion effects, as well as the need to have a framework that is dynamic. Of course, heavy-tail phenomena are not solely found in the credit space and are common throughout quantitative finance [**Con01, Sch03b**].

There is a large body of literature which aims to address the drawbacks of Gaussian credit models. Some modelling examples include the addition of jump processes and heavy-tailed distributions [**LM03, SO05, FJOS10, WY13**], introducing stochastic parameters and inhomogeneity [**AS05, LG05, BGL07**] and incorporating contagion effects [**GW05, GW06, DRST09, GSSS15**].

Models close to our own framework include [**BR12**], in which a jump process is added to the systemic factor, but in a discretised version of the system, and [**Jin10**], in which the particles are taken to be general diffusions. We will not consider systems with jump processes in this thesis, although replacing the systemic Brownian motion in our system with a more general Lévy process would be a natural extension and would lead to an SPDE driven by a Lévy process, a topic on which there is relevant literature [**CK12a, CK12b, Kim14**]. In Section 5.9 we will explain how the work of [**Jin10**] can be incorporated into our models.

The model we will study is inspired by Figure 1.2.2: we will keep the basic structure of [**BHH$^+$11**], but allow the correlation to be a function of the loss in the system, rather than just a constant. In practice one could then calibrate the model to fit every tranche spread, say by a using a piecewise constant correlation whereby, as the loss process attains the level of a given attachment point, the correlation value can be chosen to match the market spread of that tranche. We give a precise formulation of this model in the next section, but for now we notice that the coefficients determining the evolution of the system are themselves influenced by the (conditional law of the) evolution of the system.

1.3. A loss-dependent correlation model

Our first aim when studying the loss-dependent system is to isolate the relevant phenomenon for the problem. To this end, we will initially restrict our attention to the simplified system below, in which each particle interacts through a systemic Brownian motion with a loss-dependent correlation. As remarked, we show how this model can be expanded to include more general driving coefficients in Section 5.9.

With fixed $N \in \mathbf{N}$, we define *the loss-dependent correlation system* to be

$$\begin{cases} X_t^{i,N} := X_0^i + \int_0^t \rho\big(L_s^N\big) dW_s + \int_0^t \sqrt{1 - \rho\big(L_s^N\big)^2} dW_s^i \\ \tau^{i,N} := \inf\big\{t > 0 : X_t^{i,N} \leq 0\big\} \\ \nu_t^N := \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{t<\tau^{i,N}} \delta_{X_t^{i,N}} \\ L_t^N := \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\tau^{i,N} \leq t}. \end{cases} \tag{1.3.1}$$

There are no technicalities to check in writing this model down: we just start the system with correlation $\rho(0)$, stop at the first time a particle is killed (which is not instantaneously, with probability 1), continue the system with correlation $\rho(1/N)$ and repeat in this fashion. We must also make the dependence on

N explicit and we need some regularity conditions on the correlation function (otherwise see Remark 5.0.8):

Definition 1.3.1 (Assumption for $\rho(\cdot)$). Throughout we will assume that $\rho : [0,1] \to [0,1)$ is càdlàg, that there exists some $\rho_{\max} < 1$ such that

$$0 \leq \rho(x) \leq \rho_{\max}, \quad \text{for every } x \in [0,1]$$

and that ρ is Lipschitz on the intervals $[x_i, x_{i+1})$ for some

$$0 = x_0 < x_1 < x_2 < \ldots < x_n = 1 \quad \text{and} \quad n \in \mathbf{N}.$$

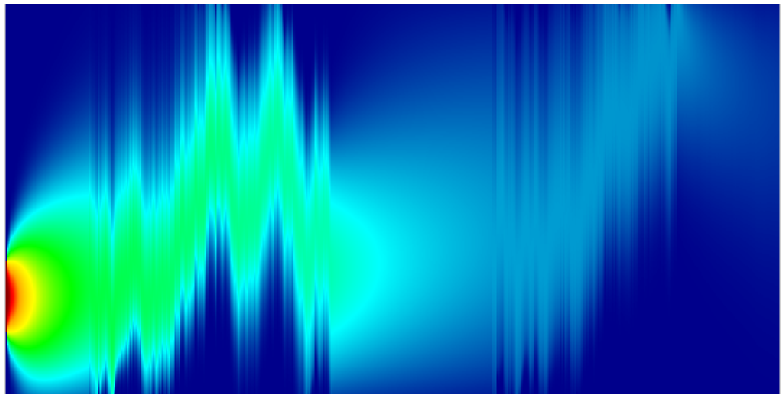
Remark 1.3.2 (Why discontinuous ρ). From Figure 1.2.2, a natural idea for calibrating this model is to assume that ρ is piecewise constant on each tranche and then to vary these constant values to match the market spreads for each tranche. As such, we want this example to be included in the class of models for which our theoretical results apply, hence we must allow ρ to have at least finitely many discontinuities.

McKean–Vlasov systems. Each particle in (1.3.1) interacts with the entire population through the coefficients of the random drivers, hence we have an example of a McKean–Vlasov system. There is a large amount of literature on these systems [**Kac56, Kac59, McK75, Oel84, Szn84, MR87, Gär88**] and Sznitman provides an excellent introduction [**Szn91**].

There are numerous examples of applications of McKean–Vlasov systems. To name a few: the modelling of large collections of neurons and threshold hitting times for membrane potential levels in mathematical neuroscience [**FTC09, BFFT12, DGLP15, LS14, IT14, DIRT14, DIRT15**], the modelling of a large number of non-cooperative agents in mean-field games [**HCM06, LL07, CD13, CD13, CCD15**], filtering theory [**BC09, CX10**], mathematical genetics [**DG14**] and portfolio credit modelling [**DRST09, SSG14**].

We will follow a standard strategy to show that (1.3.1) converges and to characterise the limiting law:

(i) Establish tightness of the system (in a suitable topology),
(ii) Characterise the limit points as weak solutions of a non-linear evolution equation,
(iii) Establish uniqueness of solutions for this equation,
(iv) Conclude all limiting laws agree, and hence that we have convergence.



$$\rho(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{5}\right) \cup \left[\frac{2}{5}, \frac{3}{5}\right) \cup \left[\frac{4}{5}, 1\right] \\ \frac{9}{10} & \text{if } x \in \left[\frac{1}{5}, \frac{2}{5}\right) \cup \left[\frac{3}{5}, \frac{4}{5}\right). \end{cases}$$

FIGURE 1.3.1. The heat plot as in Figure 1.1.2 for the loss-dependent system with the above correlation function. Markers are added to show the times at which the loss function, L, reaches levels $1/5$, $2/5$, $3/5$ and $4/5$. Notice the corresponding three periods of smooth heat flow between the two periods of highly correlated motion.

As the systemic randomness cannot be averaged-out through a limiting procedure, our eventual limits will be stochastic, and so we keep track of the joint law of the triple

$$\left(\nu^N, L^N, W\right)_{N\geq 1}. \tag{1.3.2}$$

Theorem 4.0.4 establishes tightness of these laws (in the topology which we discuss in Section 1.5) and Theorem 5.0.3 then implies realisations of any limiting law, (ν^*, L^*, W), are solutions of the *loss-dependent SPDE*:

$$\begin{cases} \nu_t^*(\phi) = \nu_0(\phi) + \frac{1}{2}\int_0^t \nu_s^*(\partial_{xx}\phi)\, ds + \int_0^t \rho(L_s^*)\, \nu_s^*(\partial_x\phi)\, dW_s \\ L_t^* = 1 - \nu_t^*(0,\infty), \end{cases} \tag{1.3.3}$$

where $\phi \in C^{\text{test}}$. We then prove that, for every W, this equation has a unique strong solution (under some regularity conditions — Definition 5.0.1), and therefore there can only be at most one limiting law. Hence we have the law of large numbers (Theorem 5.0.7): the sequence (1.3.2) converges in law to that of the unique solution of the loss-dependent SPDE.

It is important to recognise that the loss-dependent SPDE is a non-linear equation in the pair (ν^*, L^*). The dynamics of the solution are that of a heat flow with random spatial transport, however the magnitude of these fluctuations now depends on the mass that has been absorbed at the zero boundary (see Figure 1.3.1). An important step towards the uniqueness proof in Chapter 5 is to consider the linear problem of solving the loss-dependent SPDE for ν^* with a fixed L^*, which is a much easier problem. As a consequence we also find that the limiting system can be represented as the conditional law of a single particle with stochastic McKean–Vlasov dynamics:

$$\begin{cases} X_t = X_0 + \int_0^t \rho\left(\mathbf{P}(\tau \leq s |\, W)\right) dW_s + \int_0^t \sqrt{1 - \rho\left(\mathbf{P}(\tau \leq s |\, W)\right)^2} dW_s^{\perp} \\ \tau = \inf\left\{t > 0 : X_t \leq 0\right\} \\ \nu_t\left(\phi\right) = \mathbf{E}\left[\phi\left(X_t\right) \mathbf{1}_{t<\tau} |\, W\right], \end{cases}$$

where $W^{\perp}$ is an independent Brownian motion (Theorem 5.0.9). This is analogous to the definition in (1.1.2).

Systems with non-deterministic limits are common in the literature [**DV95, KX99**]. The novelty in our system comes from the fact that the correlation function interacts with the population through its boundary behaviour and that the correlation function can have (finitely many) jumps. In [**CKL14**] a similar system to ours is considered, however the authors are not able to prove uniqueness for the resulting evolution equation. Discontinuous coefficients are considered in [**Chi94**] in the deterministic setting and on the whole space.

Informally, the reason we can allow ρ to be discontinuous is because there is sufficient independence between the particles to ensure that the limiting loss process is strictly increasing (Proposition 5.2.3). This then allows us to prove uniqueness incrementally on time intervals (under the clock $t \mapsto L_t$) for which the correlation function is Lipschitz (Section 5.7). However, the complications caused by the boundary effects are more difficult to remedy, and the following two sections are devoted to explaining the theoretical developments in Chapters 2 and 4 which overcome these issues.

1.4. The kernel smoothing method in H^{-1}

The theory of existence and uniqueness of solutions for SPDEs is a well-established and active area of research [**DKM$^+$09, Hai09**], and most relevant to our setting are [**Kry94, KL98, KL99**]. Here, our approach is to adapt the

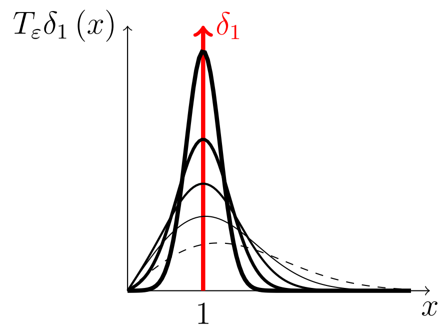


FIGURE 1.4.1. Here we are smoothing the measure δ_1 to produce $T_\varepsilon\delta_1$ for various values of ε (the heavier the curve, the smaller the value of ε). Straightforward calculation gives $T_\varepsilon\delta_1(x) = G_\varepsilon(1, x)$ and so $\|T_\varepsilon\delta_1\|_{L^2(0,\infty)} \asymp \varepsilon^{-\frac{1}{4}}$, hence we do not have L^2 control of δ_1, as expected.

kernel smoothing method from [**BHH$^+$11**]. The idea is to smooth a candidate solution, ν, of an SPDE by pushing it forward with the absorbing Gaussian heat kernel for a small time $\varepsilon > 0$:

$$\nu_t \mapsto T_\varepsilon\nu_t := \int_0^\infty (2\pi\varepsilon)^{-\frac{1}{2}} \left(e^{-\frac{((\cdot)-x_0)^2}{2\varepsilon}} - e^{-\frac{((\cdot)+x_0)^2}{2\varepsilon}} \right) \nu_t(dx_0) \in C^\infty(\mathbf{R}). \tag{1.4.1}$$

The potentially exotic object ν_t is therefore transformed into the tractable smooth function $T_\varepsilon\nu_t$, which can be manipulated classically. If it is the case that a suitable norm of $T_\varepsilon\nu_t$ can be controlled as $\varepsilon \to 0$, then we can deduce control of ν_t in that same norm (see Section 2.6 for full details). The method originated from [**KX99**] and [**Kot95**], where the Gaussian kernel without absorption is used on the whole space, and in [**BHH$^+$11**] the authors work in the space $L^2(0, \infty)$. (See Figure 1.4.1.)

This technique can then be used to prove uniqueness by smoothing the basic SPDE, an operation which is described rigorously in Sections 2.7, 2.8 and 2.9. As a heuristic, we obtain:

$$dT_\varepsilon\nu_t(x) = \frac{1}{2}\partial_{xx}T_\varepsilon\nu_t(x)\,dt - \rho\partial_x T_\varepsilon\nu_t(x)\,dW_t + \text{remainder}_{t,\varepsilon}(x),$$

for $x > 0$, for some remainder process. For this technique to work it is important that our smoothed solution $T_\varepsilon\nu_t$ vanishes at the origin, and this is why the absorbing heat kernel appears in (1.4.1). We can then apply Itô's formula to the smoothed SPDE, integrate over $(0, \infty) \times \Omega$ and apply some standard estimates

to obtain
(1.4.2)
$$\mathbf{E}\left\|T_\varepsilon \nu_t\right\|_{L^2}^2 + c\mathbf{E}\int_0^t \left\|\partial_x T_\varepsilon \nu_s\right\|_{L^2}^2 ds \le \left\|T_\varepsilon \nu_0\right\|_{L^2}^2 + \mathbf{E}\int_0^t \left\|\mathrm{remainder}_{s,\varepsilon}\right\|_{L^2}^2 ds.$$
As the basic SPDE is linear and the initial condition of the difference of two solutions is zero, this estimate shows that the smoothed difference of two solutions is controlled by the norm of the reminder term.

In [**BHH+11**] it is shown the remainder in (1.4.2) vanishes, provided we have
$$\mathbf{E}\int_0^t \left|\nu_s(0,\varepsilon)\right|^2 ds = O\big(\varepsilon^{3+\gamma}\big), \quad \text{as } \varepsilon \to 0,$$
for some $\gamma > 0$. It is shown in [**BHH+11**] that this second moment has the same growth exponent as the probability of a two-dimensional Brownian motion in a wedge being in an ε-neighborhood of the apex without having exited the wedge (Theorem 3.0.2). When the correlation is constant, explicit formulae are available for the law of a Brownian motion in a wedge [**Iye85, Met10, LS13b, LS13a**] (in Section 3.0.2 we present an alternative method which avoids heavy computations), however these methods break down when ρ is more general, hence we need to adapt the method.

As inequality (1.4.2) grants control over the first derivative of the solution in terms of the (zeroth derivative of) the initial condition, this hints that there is scope for weakening our method. The key idea in Chapter 2 is to introduce the anti-derivative to translate our estimates into the H^{-1} space, the dual of the first Sobolev space $H_0^1(0,\infty)$. As a result we require only the estimate
$$\mathbf{E}\int_0^t \left|\nu_s(0,\varepsilon)\right|^2 ds = O\big(\varepsilon^{1+\gamma}\big), \quad \text{as } \varepsilon \to 0,$$
for some $\gamma > 0$, to deduce that the corresponding remainder term vanishes. This is an improvement of order ε^2 in the required control.

Although estimating square moments for the loss-dependent system is difficult, if ν is a sub-probability measure (or, more broadly, the weak limit of sub-probability measures, like ν^N) then we have the trivial bound
$$\mathbf{E}\int_0^t \left|\nu_s(0,\varepsilon)\right|^2 ds \le \mathbf{E}\int_0^t \nu_s(0,\varepsilon)\, ds.$$
Hence we only need to estimate first moments of the mass near the boundary, which is a much simpler task as the correlations between the underlying particles do not need to be taken into account. Indeed, any process ν that has a

bounded density will automatically have $O(\varepsilon)$ behaviour (from the volume of $(0,\varepsilon)$), therefore we only require the boundary to have an additional $O(\varepsilon^\gamma)$ effect on the decay of the function, for any arbitrarily small γ, which is a very weak requirement. Since the particles in the loss-dependent system behave as Brownian motions when considered individually, for this system we have $\gamma = 1$, which is sufficient.

1.5. Skorokhod's M_1 topology and the tempered distributions

In his 1956 paper, [**Sko56**], Skorokhod introduced four topologies on the space of càdlàg paths that take values in a Banach space. The most well-known of these is the J_1 topology. It is common to refer to the space of càdlàg paths equipped with this topology as the Skorokhod space.

For our applications, however, it will be easier to work with the weaker M_1 topology. In [**Sko56**] an error is made in the topology's definition, but a comprehensive and modern overview is presented in [**Whi02**]. The main benefit the M_1 topology offers is that monotone functions vanish under the associated modulus of continuity (which is a proxy for the variation in a sequence of paths and features heavily in practical conditions for establishing compactness and tightness — see Theorems 4.1.8, 4.1.11 and 4.1.12). As the loss process in (1.3.1) is a monotone increasing function, the practicability of this topology in our setting, and to any system whose dynamics are driven by a monotone process, is evident. Such examples can be found in queuing theory [**PW10, Whi00, Whi01**], the study of maxima [**Kri14**], moving averages [**AT92**] and order statistics [**DMT11**] in functional settings and mathematical neuroscience [**DIRT15**], to give a non-exhaustive list. An overview of the relevant facts about the M_1 topology with Banach range spaces is presented in Section 4.1.

As the empirical processes, $\left(\nu^N\right)_{N\geq 1}$, take values in the space of finite-measures, we are not covered by the Banach-space-valued case above. We could extend the range space to the space of measures, as is well-known for the J_1 topology [**EK86**], however this excludes studying the fluctuations in convergence, as we do in the next section, because there we have to consider sequences of signed measures. We follow the line of [**Itô83**] and embed our processes in the space of tempered distributions, which is the dual of the Schwarz space of rapidly decreasing functions, $\mathscr{S}$ (we review these spaces in Sections 2.1 and 4.3).

The space of tempered-distributions, $\mathscr{S}'$, is an example of a countably Hilbertian nuclear space and the J_1 topology has been defined for the duals of such spaces by [**Mit83b**]. There it is shown that if $\left(\xi^N(\phi)\right)_{N\geq 1}$ is a tight sequence in $(D_{\mathbf{R}}, J_1)$ for every test function ϕ, then the sequence $\left(\xi^N\right)_{N\geq 1}$ is tight in $(D_{\mathscr{S}'}, J_1)$. The practical consequence is that tightness can be checked test-function-by-test-function, rather than working in the much larger space $\mathscr{S}'$. For the J_1 topology, this result has been extended to completely regular spaces in [**Jak86**]. In [**PR14**] the M_1 topology is considered for infinite-dimensional Banach range spaces, but where the space is equipped with the weak topology. An advantage of working in $\mathscr{S}'$ is that strong and weak convergence coincide. It should also be noted that there are other relevant non-Skorokhod topologies on the càdlàg paths with similar compactness criteria [**MZ84, Jak97**].

In Chapter 4 we extend the definition of the M_1 topology to $D_{\mathscr{S}'}$ and prove the equivalent tightness and convergence results (Theorems 4.0.1 and 4.0.3). To establish tightness of the sequence $\left(\nu^N\right)_{N\geq 1}$ from (1.3.1), all that we then require is to check that $\left(\nu^N(\phi)\right)_{N\geq 1}$ is tight as a real-valued process in the M_1 topology. The challenge we face is to control the discontinuity at the origin, and to do so we notice that we can write

$$\phi\big(X_t^{i,N}\big)\mathbf{1}_{t<\tau^{i,N}} = \phi\big(X_{t\wedge\tau^{i,N}}^{i,N}\big) - \phi\big(0\big)\mathbf{1}_{t\geq\tau^{i,N}},$$

and therefore

$$\nu_t^N(\phi) = \frac{1}{N}\sum_{i=1}^{N}\phi\big(X_{t\wedge\tau^{i,N}}^{i,N}\big) - \phi\big(0\big)L_t^N.$$

The advantage this offers is that the first term is continuous (in t), so can be controlled with standard moment estimates [**EK86**, Theorem 8.8], and the second term is monotone, so has zero contribution to the M_1 modulus of continuity. A full argument is presented in Section 4.7, and this establishes tightness of the sequence (1.3.2) in Theorem 4.0.4.

1.6. The central limit theorem

In [**GSS14**], the fluctuations in convergence of the large portfolio model [**GSSS15**] are analysed and a central limit theorem is established. In our context, this corresponds to finding a limiting process F such that

$$F^N := \sqrt{N}\big(L^N - L\big) \to F, \quad \text{as } N \to \infty$$

where the convergence is weak on the space of real-valued càdlàg paths (with the J_1 topology — we will not need the M_1 topology in this section). For the constant correlation model, Donsker's theorem [**Bil99**, Theorem 14.3] gives that

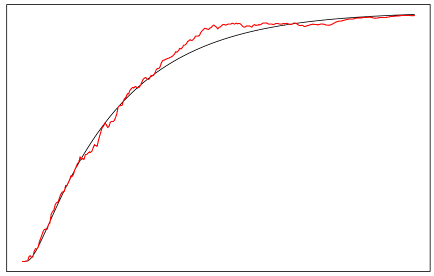


FIGURE 1.6.1. The black curve is the realisation of the loss function for the deterministic system ($\rho = 0$) — time runs along the horizontal axis. The red curve gives the value of the right hand-side of (1.6.1) for $N = 100$, which gives a 10% correction factor.

F is distributed as a Brownian bridge, B, independently of W and run at speed L:

$$\sqrt{N}\big(L^N - L\big) \to \big(B_{L_t}\big)_{t\in[0,T]}, \quad \text{as } N \to \infty.$$

The authors of **[GSS14]** provide numerical results that suggest the pricing approximation

$$L^N \approx L + \frac{1}{\sqrt{N}} B_L \tag{1.6.1}$$

might lead to significantly improved practical performance. Hence the Brownian bridge term can be seen as a stochastic correction. (Of course, the process on the right-hand side of (1.6.1) is not monotone, so there are limitations to the extend to which this is a sensible approximation. See Figure 1.6.1).

In Chapter 6 we will study the fluctuations in convergence at the level of the empirical processes for the basic system where ρ is constant. Specifically, in Theorem 6.0.5 we find that the fluctuation processes converge weakly

$$f^N := \sqrt{N}\left(\nu^N - \nu\right) \to f, \quad \text{as } N \to \infty,$$

on the space $D_{\mathscr{S}'}$ (again, with respect to the J_1 topology). By keeping track of the joint law of f^N, ν^N and W (and an idiosyncratic process J^N, see the beginning of Chapter 6) we find that f satisfies the *fluctuation SPDE*:

$$f_t(\phi) = f_0(\phi) + \frac{1}{2}\int_0^t f_s(\partial_{xx}\phi)ds + \rho\int_0^t f_s(\partial_x\phi)dW_s + \sqrt{1-\rho^2}J_t(\partial_x\phi),$$

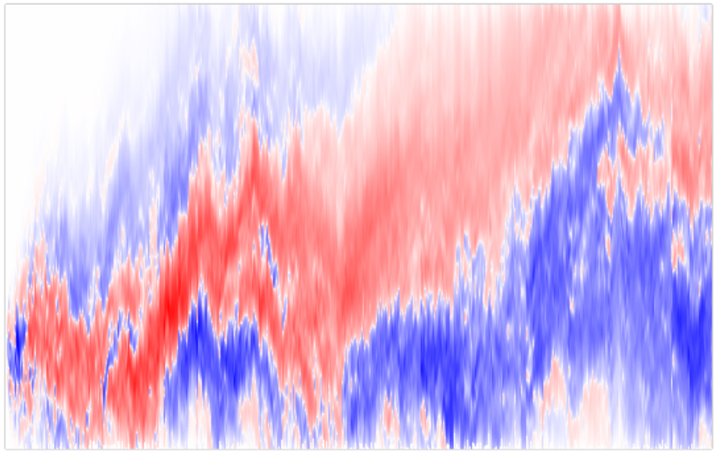


FIGURE 1.6.2. The fluctuation process, f, is the distributional spatial derivative of the process shown in the heat plot above. Again, time is plotted on the horizontal axis and space on the vertical axis. The white pixels represent value zero, the red pixels a positive value and the blue pixels a negative value, on a continuous colour scale. This is a realisation for $\rho = 0.5$.

where J is a Gaussian process when conditioned on $\sigma(W)$, with zero mean and covariance

$$\mathrm{cov}_{t,s}(\phi, \psi) = \mathbf{E}\left[J_t(\phi) J_s(\psi) \middle| W \right] = \int_0^{t \wedge s} \nu_u(\phi\psi)\, du, \quad \phi, \psi \in \mathscr{S},$$

hence J is a space-time white noise with spatial intensity controlled by ν (see Figure 1.6.2). This is a typical central limit theorem for SPDEs [**Met87, FM97, KX04**].

As the fluctuation SPDE is linear, the kernel smoothing method from Chapter 2 can be applied to deduce uniqueness of solutions (Theorem 6.0.4). Although this analysis is simple, in Section 6.4 we find that in order for our techniques to be valid we require

$$\mathbf{E} \int_0^T \left| f_t(0, \varepsilon) \right|^2 dt = O\left(\varepsilon^{1+\gamma} \right),$$

for some $\gamma > 0$. (Here we are abusing notation when writing $f_t(0, \varepsilon)$, since f_t is a distribution, not a measure. This suffices for a heuristic argument; full details are given in Section 6.4.) From the covariance structure we also find

$$\mathbf{E} \int_0^T \left| f_t(0, \varepsilon) \right|^2 dt = \mathbf{E} \int_0^T \left| \nu_t(0, \varepsilon) \right| dt = O\left(\varepsilon^2 \right), \tag{1.6.2}$$

which is sufficient. However, if the L^2 method were applicable we would need

$$\mathbf{E}\int_0^T \left|f_t\left(0,\varepsilon\right)\right|^2 dt = O\big(\varepsilon^{3+\gamma}\big)\,,$$

and this contradicts (1.6.2). Hence the fluctuation SPDE cannot be analysed with the L^2 method, and so we have an example of where the H^{-1} method is necessary.

1.7. Future research directions

As the framework and the theoretical techniques in this thesis are quite general, there is scope for extending the range of models to which our approach can be applied. In the context of portfolio credit modelling, two natural extensions have been mentioned already. The first is to allow particles to evolve as general diffusions, which is explored in Sections 2.10 and 5.9. The second is to incorporate more general systemic dynamics, which corresponds to considering SPDEs on the half-line with general transport drivers. As mentioned, an obvious class of systemic drivers to consider is that of the Lévy processes.

Obtaining a fluctuation result for the general loss-dependent system is a natural progression, and this would transfer the results of [**GSS14**] into our structural framework. Another possibility is to model the coefficients driving the particle dynamics as stochastic processes, analogous to stochastic volatility models in equity derivative pricing.

Contagion effects could be incorporated along the lines of [**DIRT14**] and [**DIRT15**]. There a finite particle system is considered in which the event of a particle hitting a given threshold triggers a proportional jump in the other particles towards the boundary. In our framework, the large population limit corresponds to the heat equation in a semi-infinite bar with a transport term pushing the solution towards the boundary at a rate which is proportional to the rate of mass lost from the half-line.

These systems incorporate the notion of positive feedback, and it has been shown from a PDE perspective that it is possible to select model parameters in such a way that the solution blows-up in a finite time [**CCP11**]. More specifically, the loss in the system begins evolving in a continuously differentiable manner, however at the blow-up time the system exhibits a jump and a macroscopic proportion of the mass is lost instantaneously. In the context of credit modelling, this behaviour represents systemic failure. Relevant theoretical challenges include adding a systemic driver and to analyse the system in such a way as to naturally continue the system after a blow-up has occured.

1.8. Thesis overview

Each subsequent chapter begins with a short summary of the work in each section and with the statements of the main definitions and results for that chapter.

In Chapter 2 we develop the kernel smoothing method which will be used throughout. The rigorous definition of the basic SPDE is given (Definition 2.0.3) and we prove that the limit empirical process of the basic system is its unique solution (Theorems 2.0.6 and 2.0.7). In Section 2.10 we show how the method can be extended to include spatially-dependent coefficients.

In Chapter 3 we show how the kernel smoothing methods can be used to describe the precise regularity of the limit empirical process at the Dirichlet boundary (Theorem 3.0.1). This is a condensed version of the paper [**Led14**], although we give an alternative proof of the main probabilistic estimate (Theorem 3.0.2).

In Chapter 4 we extend Skorokhod's M_1 topology to processes taking values in the tempered distributions. Our main results characterise tightness and weak convergence in this topology with respect to the action of processes on test functions (Theorems 4.0.1 and 4.0.3). We then use these tools to show that the loss-dependent correlation system is tight (Theorem 4.0.4).

In Chapter 5 we prove that the limit points of the loss-dependent correlation system satisfy the loss-dependent SPDE (Theorem 5.0.3), which we define rigorously in Definition 5.0.1. By adapting the kernel smoothing method to piecewise Lipschitz correlation functions, we demonstrate uniqueness of solutions for the loss-dependent SPDE (Theorem 5.0.5). Consequently we establish the law of large numbers for the particle system (Theorem 5.0.7) and the weak existence and uniqueness to the corresponding stochastic McKean–Vlasov problem (Theorem 5.0.9). In Section 5.9 we show how the results can be extended to allow the particle to be general correlated diffusions.

In Chapter 6 we establish tightness of the fluctuations in the convergence of the empirical processes for the basic system and make sense of the idiosyncratic noise as a conditionally Gaussian process (Theorem 6.0.1). This allows us to give a rigorous definition of the fluctuation SPDE (Definition 6.0.2) and to show that the limit points of the fluctuation system are solutions (Theorem 6.0.3). The kernel smoothing method then allows us to prove uniqueness of solutions (Theorem 6.0.4) and this immediately implies the central limit theorem for the empirical processes of the basic system (Theorem 6.0.5).

CHAPTER 2

The kernel smoothing method

We will outline the kernel smoothing method and use it to prove existence and uniqueness of solutions for the basic limiting SPDE. The method will be explored in further detail in Chapter 3 and the existence and uniqueness results will be applied directly in Chapters 5 and 6. This chapter also serves as a simple introduction to the basic theory before more complicated features are added later. In Chapter 5, it will be convenient to allow the correlation to be more general than was specified in Chapter 1, so throughout this chapter we will assume the following condition.

Definition 2.0.1 (Assumptions on ρ)**.** In this chapter, we will assume $t \mapsto \rho_t$ is a (stochastic) adapted càdlàg process that takes values in $[0,1]$, is $\sigma(W)$-measurable and is such that there exists $\rho_{\max} < 1$ for which

$$0 \leq \rho_t \leq \rho_{\max}, \quad \text{for every } t \in [0,T],$$

with probability 1. Each particle then evolves as

$$X_t^i = X_0^i + \int_0^t \rho_s dW_s + \int_0^t \sqrt{1-\rho_s^2} dW_s^i.$$

Remark 2.0.2 (Particles are Brownian motions)**.** Under Definition 2.0.1 we have $[X^i] \equiv t$, so when regarded individually, the particles have law of a Brownian motion, by Lévy's characterisation.

First, we should make rigorous the notion of a solution to the basic SPDE. Here, $D_{\mathscr{S}'}$ denotes the space of càdlàg processes taking values in the tempered distributions, $\mathscr{S}'$, which we describe in Section 2.1.

Definition 2.0.3 (Solution to the limit SPDE)**.** Let $f \in \mathscr{S}'$ be deterministic and W be a Brownian motion. An adapted process, ξ, taking values in the space $D_{\mathscr{S}'}$ is said to be a *solution to the basic SPDE with respect to* (f, W) if all the following hold:

(i) (Dynamics) With probability 1,

$$\xi_t(\phi) = f(\phi) + \frac{1}{2}\int_0^t \xi_s(\partial_{xx}\phi)\, ds + \int_0^t \rho_s \xi_s(\partial_x \phi)\, dW_s,$$

for all $t \in [0, T]$ and $\phi \in C^{\text{test}} := \{\phi \in \mathscr{S} : \phi(0) = 0\}$,

(ii) (Finite moments) For all $\phi \in \mathscr{S}$

$$\mathbf{E}\int_0^T |\xi_t(\phi)|^4\, dt < \infty,$$

(iii) (Tail estimate) For every $c > 0$ and $\phi_\lambda \in \mathscr{S}$ supported on (λ, ∞) with $\|\phi_\lambda\|_\infty \leq 1$

$$\mathbf{E}\int_0^T |\xi_t(\phi_\lambda)|^4\, dt = o(\exp\{-c\lambda\}), \quad \text{as } \lambda \to \infty,$$

(iv) (Boundary estimate) There exists a $\gamma > 0$ such that for every $\phi_\varepsilon \in \mathscr{S}$ supported on $(-\varepsilon, \varepsilon)$ with $\|\phi_\varepsilon\|_\infty \leq 1$

$$\mathbf{E}\int_0^T |\xi_t(\phi_\varepsilon)|^2\, dt = O\left(\varepsilon^{1+\gamma}\right), \quad \text{as } \varepsilon \to 0,$$

(v) (Vanishing on negatives) With probability 1, for every $t \in [0, T]$

$$\xi_t(\phi) = 0, \quad \text{whenever } \phi \in \mathscr{S} \text{ with support}(\phi) \subseteq (-\infty, 0).$$

Remark 2.0.4 (Linear combination of solutions are solutions)**.** It is important for our uniqueness arguments that Definition 2.0.3 is stable under linear combinations. It is clear that (i) and (v) hold for a linear combination of solutions (with the initial condition of the linear combination equal to the linear combination of the initial conditions), and conditions (ii), (iii) and (iv) hold by applying Proposition A.0.1.

Conditions (ii)-(v) guarantee we have sufficient regularity to carry out the practical calculations in Section 2.8, and there we use a number of technical corollaries that follow immediately from these conditions and are derived in Section 2.7.

The second and third conditions ensure that solutions do not grow large. Condition (iv) relates to the Dirichlet boundary. It is used in Lemma 2.7.3 to show that the error introduced at zero by smoothing the solution is negligible. For the empirical process, as mentioned in the introduction, estimating the second moment behaviour is challenging. However, ν takes values in the sub-probability measures, so we have the simple bound

$$\mathbf{E}\int_0^T \left|\nu_t\left(\phi_\varepsilon\right)\right|^2 dt \le \mathbf{E}\int_0^T \left|\nu_t\left(\phi_\varepsilon\right)\right| dt.$$

The right-hand side can be written in terms of the law of a single killed Brownian motion (Proposition 2.2.2), so is easy to control. The final condition is necessary as we are not incorporating boundary effects into the test space, $\mathscr{S}'$, directly.

Remark 2.0.5 (Boundary improvement)**.** As has been mentioned in Chapter 1, the equivalent condition to (iv) in [**BHH$^+$11**] is

$$\mathbf{E}\int_0^T \left|\xi_t\left(\phi_\varepsilon\right)\right|^2 dt = O\left(\varepsilon^{3+\gamma}\right),$$

for some $\gamma > 0$. Therefore our H^{-1} method leads to an improvement of a factor of ε^2 in the required regularity at the boundary.

In Section 2.2, we show that the limit empirical process (1.1.2) satisfies our regularity conditions. An explanation as to why the basic SPDE is a natural limiting object is given in Section 2.3, and follows from studying the microscopic dynamics of the particle population. In Section 2.4 we describe a conditional expectation approach for demonstrating condition (i), and as consequence we obtain:

Theorem 2.0.6 (Existence)**.** *Let ν_0 be a probability measure supported on a finite interval away from the origin (as in Remark 1.1.1) and let W be a standard Brownian motion. Then the limit empirical process,*

$$\nu_t\left(\phi\right) := \mathbf{E}\left[\phi\left(X_t^1\right)\mathbf{1}_{t<\tau^1}\middle| W\right],$$

solves the basic SPDE with respect to (ν_0, W).

Notice that ν is constructed in terms of W directly, hence we have the existence of *strong* solutions. The remainder of the chapter is devoted to demonstrating the uniqueness of solutions. In Section 2.5 we give an intuition as to why H^{-1} is the natural space to embed our processes in. The key concepts in the kernel smoothing method are introduced in Section 2.6 and then the practical energy estimation is done in Section 2.8, after some integrability facts are established in Section 2.7. The proof of the following result completes this chapter in Section 2.9:

Theorem 2.0.7 (Uniqueness)**.** *Let (ν_0, W) be as in Theorem 2.0.6. Then there exists at most one solution to the basic SPDE with respect to (ν_0, W) in a strong sense: if ξ^1 and ξ^2 are two solutions, then, with probability 1,*

$$\xi_t^1(\phi) = \xi_t^2(\phi), \quad \textit{for every } \phi \in \mathscr{S} \textit{ and } t \in [0,T].$$

(In other words, $\xi^1 = \xi^2$ as elements of $D_{\mathscr{S}'}$, with probability 1.) By Theorem 2.0.6, this unique solution is the limit empirical process.

In Section 2.10 we will extend the kernel smoothing method to allow the coefficients in our SPDE to have spatial dependence. These SPDEs describe the limiting population dynamics for particle systems consisting of general diffusions. The results will be used in Section 5.9.

2.1. Tempered-distribution-valued processes

Tempered-distribution-valued processes are studied in more detail in Chapter 4, where their full power will be exploited. For now we give a brief overview.

The space of tempered distributions, $\mathscr{S}'$, is defined as the topological dual of the rapidly decreasing functions:

Definition 2.1.1 (Rapidly decreasing functions)**.** The *space of rapidly decreasing functions* is defined to be

$$\mathscr{S} := \left\{\phi \in C^\infty(\mathbf{R}) : \|\phi\|_{\alpha,\beta} < \infty, \quad \text{for all integers } \alpha, \beta \geq 0\right\},$$

where we have the semi-norms

$$\|\phi\|_{\alpha,\beta} := \sup_{x\in\mathbf{R}} \left|x^\alpha \partial_x^\beta \phi(x)\right|.$$

We assign $\mathscr{S}$ the topology generated by this collection of semi-norms, which is metrised by

$$\mathrm{dist}_{\mathscr{S}}(x,y) := \sum_{\alpha,\beta\geq 0} 2^{-(\alpha+\beta)}\left(\|x-y\|_{\alpha,\beta} \wedge 1\right).$$

[**KX95**, Chapter 1]. (The dual space is then defined with respect to this topology.)

An example of a rapidly decreasing function (which we will be using often) is the *Gaussian heat kernel*:

$$x \mapsto \psi_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{2\varepsilon}\right\}, \tag{2.1.1}$$

where $x \in \mathbf{R}$ and $\varepsilon > 0$. Also note that $C_0^\infty(\mathbf{R}) \subseteq \mathscr{S}$.

There are a number of relevant topologies that can be assigned to $\mathscr{S}'$, but we shall defer an explanation until Chapter 4. For now we will just say we assign the *strong topology* to $\mathscr{S}'$, and then we can define càdlàg paths:

Definition 2.1.2 (Space of $\mathscr{S}'$-valued càdlàg paths)**.** Let $D_{\mathscr{S}'}$ denote the collection of functions mapping $[0,T]$ to $\mathscr{S}'$ that are right continuous and have left limits with respect to the strong topology on $\mathscr{S}'$.

Here, we do not need to be precise about the strong topology as we have the following result, due to Mitoma:

Theorem 2.1.3 (Sample continuity, [**Mit83a**])**.** *Let ξ be an $\mathscr{S}'$-valued process such that, for every $\phi \in \mathscr{S}$, the $\mathbf{R}$-valued process*

$$\xi(\phi) = (\xi_t(\phi))_{t\in[0,T]}$$

has a càdlàg version. Then ξ has a version taking values in $D_{\mathscr{S}'}$.

This is a stochastic result, so of course it applies to deterministic processes. It also illustrates why we use $D_{\mathscr{S}'}$: properties of càdlàg tempered-distribution-valued processes can generally be verified test function at a time. The reason for this is that $\mathscr{S}$ is a countably Hilbertian nuclear space (this is explored in Chapter 4).

From the definition of the empirical process,

$$\nu_t^N(\phi) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{t<\tau^i}\phi\left(X_t^i\right),$$

and the fact that X^i is continuous and $t \mapsto \mathbf{1}_{t<\tau^i}$ is càdlàg, it is immediately clear that ν^N is a $D_{\mathscr{S}'}$-valued process (rather, has a càdlàg version, but we will not make a distinction). The same reasoning, plus the dominated convergence theorem, applies to

$$\nu_t(\phi) = \mathbf{E}\left[\mathbf{1}_{t<\tau^1}\phi\left(X_t^1\right)\middle| W\right].$$

Remark 2.1.4 (The structural theorem)**.** It might seem that $\mathscr{S}'$ could be very large and exotic. However, the structural theorem, [**RS81**, Theorem V.10], tells us that every tempered distribution is the distributional derivative (of some order, possibly zero) of a polynomially bounded continuous function.

2.2. The limit empirical process satisfies the regularity conditions

Here, we will show that the limit empirical process, ν, satisfies conditions (ii)-(v) of Definition 2.0.3. Condition (v) is immediate from the definition of ν in (1.1.2), as is condition (ii) since

$$\nu_t(\phi) \leq \|\phi\|_\infty .$$

To check (iii) and (iv) all that we will use is that X^1 has the law of a Brownian motion (Remark 2.0.2) and that, because ν_t is a sub-probability measure,

$$\mathbf{E}\left[\left|\nu_t^N(\phi)\right|^p\right] \leq \mathbf{E}\left|\nu_t^N(\phi)\right| \tag{2.2.1}$$

whenever $p \geq 1$ and $\|\phi\|_\infty = 1$. If we take any subset $S \subseteq (0,\infty)$, then the linearity of expectation and that the particles have identical laws gives

$$\mathbf{E}\nu_t^N(S) = \mathbf{E}\Big[\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{X_t^i \in S, t<\tau^i}\Big] = \mathbf{P}\left(X_t^1 \in S, t < \tau^1\right) = \mathbf{E}\nu_t(S). \tag{2.2.2}$$

By setting $S = (\lambda, \infty)$ into (2.2.2) and disregarding the absorption at the origin we get

$$\mathbf{E}\nu_t^N(\lambda,\infty) \leq \mathbf{P}\left(X_t^1 > \lambda\right),$$

and then the proceeding tail estimate follows:

Proposition 2.2.1 (Tail estimate). *For any $N \geq 1$, $\lambda > 2x^\star$ and $t \in [0,T]$,*

$$\mathbf{E}\nu_t^N(\lambda,+\infty) \leq \frac{2\sqrt{T}}{\lambda}\exp\left\{-\frac{\lambda^2}{8T}\right\}.$$

(Recall that $[x_\star, x^\star]$ is assumed to be the support of ν_0, as defined in Remark 1.1.1.)

Proof. By conditioning on the initial value of X^1 we have

$$\begin{aligned}\mathbf{P}\left(X_t^1 > \lambda\right) &= \int_0^\infty \Phi\big(-t^{-\frac{1}{2}}(\lambda - x_0)\big)\nu_0(dx_0)\\ &\leq \Phi\big(-t^{-\frac{1}{2}}(\lambda - x^\star)\big)\\ &\leq \frac{\sqrt{t}}{\lambda - x^\star}\exp\Big\{-\frac{(\lambda - x^\star)^2}{2t}\Big\},\end{aligned}$$

where we have used Proposition A.0.3. Since the right-hand side is increasing in t, we can replace t by T, then we are done noting that

$$\lambda - x^\star > \lambda/2, \quad \text{whenever } \lambda > 2x^\star.$$

□

It is now clear from (2.2.1), (2.2.2) and Proposition 2.2.1 that we have (iii).

With ϕ_ε as in condition (iv), notice that we have

$$\mathbf{E}\left|\nu_t\left(\phi_\varepsilon\right)\right|^2 \leq \mathbf{E}\left[\nu_t\left(0,\varepsilon\right)^2\right] \leq \mathbf{E}\nu_t\left(0,\varepsilon\right).$$

and so condition (iv) is verified by the following result:

Proposition 2.2.2 (Boundary estimate)**.** *Let B be a standard Brownian motion started from initial law ν_0 (satisfying the assumptions of Theorem 2.0.6) with first hitting time*

$$\tau := \inf\left\{t > 0 : B_t \leq 0\right\}.$$

Then

$$\sup_{t\in[0,T]} \mathbf{P}\left(B_t < \varepsilon, t < \tau\right) = O\left(\varepsilon^2\right), \quad \text{as } \varepsilon \to 0.$$

In particular

$$\mathbf{E}\int_0^T |\nu_t(0,\varepsilon)|^2 dt \leq \mathbf{E}\int_0^T \nu_t(0,\varepsilon) dt = \int_0^T \mathbf{P}(B_t < \varepsilon, t < \tau) dt = O(\varepsilon^2).$$

Proof. Let $G\left(x_0,\cdot\right)$ be the transition kernel for a Brownian motion started at x_0 and killed at the origin:

$$G_t\left(x_0,x\right) = \left(2\pi t\right)^{-\frac{1}{2}}\left[\exp\left\{-\frac{\left(x-x_0\right)^2}{2t}\right\} - \exp\left\{-\frac{\left(x+x_0\right)^2}{2t}\right\}\right].$$

Conditioning on the initial value gives

$$\mathbf{P}\left(B_t < \varepsilon, t < \tau\right) = \int_0^\varepsilon \int_0^\infty G\left(x_0,x\right)\nu_0\left(dx_0\right)dx.$$

Following the calculation in [**BHH$^+$11**, Theorem 3.3], we have

$$\begin{aligned} G_t\left(x_0,x\right) &= \left(2\pi t\right)^{-\frac{1}{2}}\exp\left\{-\frac{\left(x-x_0\right)^2}{2t}\right\}\left(1-\exp\left\{-\frac{2xx_0}{t}\right\}\right) \\ &\leq c_1 x x_0 t^{-\frac{3}{2}}\exp\left\{-\frac{\left(x-x_0\right)^2}{2t}\right\} \end{aligned}$$

with $c_1 > 0$ a numerical constant. So for $x_\star \leq x_0 \leq x^\star$ and $x < \varepsilon < x^\star/2$

$$G_t\left(x_0,x\right) \leq c_1\varepsilon x^\star t^{-\frac{3}{2}}\exp\left\{-\frac{x_\star^2}{8t}\right\}.$$

As a function of $t > 0$, the right-hand side is bounded by a constant (depending only on $x_\star$ and $x^\star$) multiple of ε. Integrating over $x \in (0,\varepsilon)$ then gives the result.

The final statement follows from (2.2.2). □

To summarise this section, we have:

Proposition 2.2.3 (Regularity conditions are satisfied by ν)**.** *The empirical process, ν, satisfies conditions (ii)-(v) of Definition 2.0.3.*

2.3. Dynamics of the finite particle system

To derive the limiting SPDE, begin by considering the dynamics of $\phi\left(X_t^i\right)$ for any $\phi \in C^{\text{test}}$. Since ϕ is smooth, Itô's formula is applicable and gives

$$\phi\left(X_t^i\right) = \phi\left(X_0^i\right) + \frac{1}{2}\int_0^t \partial_{xx}\phi\left(X_s^i\right) ds + \int_0^t \rho_s \partial_x \phi\left(X_s^i\right) dW_s + \int_0^t \sqrt{1-\rho_s^2}\partial_x\phi\left(X_s^i\right) dW_s^i$$

To relate this equation to the empirical process,

$$\nu_t^N(\phi) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{t<\tau^i}\phi\left(X_t^i\right),$$

we could multiply throughout by the indicator $\mathbf{1}_{t<\tau^i}$ and take an average over $i \in \{1, 2, \ldots, N\}$. However, this is not a natural operation, as the indicator function cannot be taken inside the integrals. Instead we stop the process at τ^i and use the fact

$$\mathbf{1}_{t<\tau^i}\phi\left(X_t^i\right) = \phi\left(X_{t\wedge\tau^i}^i\right),$$

since $\phi(0) = 0$, and

$$\int_0^{t\wedge\tau^i} f_s dg_s = \int_0^t \mathbf{1}_{s<\tau^i} f_s dg_s, \quad \text{for general } f, g$$

to obtain

$$\mathbf{1}_{t<\tau^i}\phi\left(X_t^i\right) = \phi\left(X_0^i\right) + \frac{1}{2}\int_0^t \mathbf{1}_{s<\tau^i}\partial_{xx}\phi\left(X_s^i\right) ds + \int_0^t \rho_s \mathbf{1}_{s<\tau^i}\partial_x\phi\left(X_s^i\right) dW_s + \int_0^t \sqrt{1-\rho_s^2}\mathbf{1}_{s<\tau^i}\partial_x\phi\left(X_s^i\right) dW_s^i. \tag{2.3.1}$$

Taking an average over i now gives:

Proposition 2.3.1 (Dynamics of the finite particle system)**.** *With probability 1, we have the evolution equation*

$$\nu_t^N(\phi) = \nu_0^N(\phi) + \frac{1}{2}\int_0^t \nu_s^N(\partial_{xx}\phi) ds + \int_0^t \rho_s \nu_s^N(\partial_x\phi) dW_s + I_t^N(\phi), \tag{2.3.2}$$

for all $N \geq 1$, $t \in [0, T]$ and $\phi \in C^{\text{test}}$, where we define the idiosyncratic driver *to be*

$$I_t^N(\phi) := \frac{1}{N} \sum_{i=1}^{N} \int_0^t \sqrt{1-\rho_s^2} \mathbf{1}_{s<\tau^i} \partial_x \phi\left(X_s^i\right) dW_s^i.$$

We almost have the SPDE, except for the presence of the idiosyncratic driver. Notice, however, that this term is equal to an average of mean-zero martingales with no cross-variation, hence we get elimination in the limit:

Proposition 2.3.2 (Vanishing of the idiosyncratic driver)**.** *For every $\phi \in C^{\text{test}}$*

$$\mathbf{E}\Big[\sup_{t\in[0,T]} \left|I_t^N(\phi)\right|^2\Big] \to 0, \quad \text{as } N \to \infty.$$

Proof. A straightforward computation using the independence of the W^i's reveals

$$\left[I_\cdot^N(\phi)\right]_t = \frac{1}{N} \int_0^t \left(1-\rho_s^2\right) \nu_s^N\left((\partial_x \phi)^2\right) ds \leq \frac{1-\rho_{\max}^2}{N} \|\partial_x \phi\|_\infty^2 T,$$

which gives the result. □

These two results show why stochastic PDEs are the natural limiting object.

2.4. The empirical process solves the basic SPDE; proof of Theorem 2.0.6

At this stage we could use the fact that ν^N converges weakly to ν and Proposition 2.3.2 to establish ν as a solution to the SPDE. However, as the dependency structure in the model is straightforward, we do not have to take this approach and instead we can take a conditional expectation with respect to W over the Equation (2.3.2). This short-cut will also be useful in Chapter 5, Section 5.5. We need one preparatory result:

Lemma 2.4.1 (Interchanging stochastic integration and conditional expectation)**.** *To be clear, with $\left\{\mathcal{F}_t\right\}$ the filtration of the space, let $\left\{\mathcal{F}_t^W\right\}$ be the natural filtration of W. Let H be a real-valued $\left\{\mathcal{F}_t\right\}$-adapted process with*

$$\mathbf{E} \int_0^T H_s^2 ds < \infty.$$

Then, with probability 1,

$$\mathbf{E}\left[\int_0^t H_s dW_s \middle| \mathcal{F}_t^W\right] = \int_0^t \mathbf{E}\left[H_s \middle| \mathcal{F}_s^W\right] dW_s$$

and

$$\mathbf{E}\left[\int_0^t H_s dW_s^1 \middle| \mathcal{F}_t^W\right] = 0$$

for every $t \in [0, T]$.

Proof. As we can multiply H_s by $\mathbf{1}_{s<t}$, it suffices to take $t = T$. First, suppose that H is a basic process, that is

$$H_u = Z\mathbf{1}_{s_1<u\leq s_2},$$

where $s_1 < s_2 \leq T$ are real numbers and Z is $\mathcal{F}_{s_1}$-measurable. Then

$$\begin{aligned}
\mathbf{E}\left[\int_0^T H_s dW_s \middle| \mathcal{F}_T^W\right] &= \mathbf{E}\left[Z\left(W_{s_2} - W_{s_1}\right)\middle| \mathcal{F}_T^W\right] \\
&= \mathbf{E}\left[Z\middle| \mathcal{F}_{s_1}^W\right]\left(W_{s_2} - W_{s_1}\right) \\
&= \int_0^T \mathbf{E}\left[Z\middle| \mathcal{F}_s^W\right] \mathbf{1}_{s_1<s\leq s_2} dW_s \\
&= \int_0^T \mathbf{E}\left[H_s\middle| \mathcal{F}_s^W\right] dW_s
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\left[\int_0^T H_s dW_s^1 \middle| \mathcal{F}_t^W\right] &= \mathbf{E}\left[Z\left(W_{s_2}^1 - W_{s_1}^1\right)\middle| \mathcal{F}_t^W\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[Z\left(W_{s_2}^1 - W_{s_1}^1\right)\middle| \sigma\left(\mathcal{F}_T^W, \mathcal{F}_{s_1}\right)\right]\middle| \mathcal{F}_t^W\right] \\
&= \mathbf{E}\left[Z\mathbf{E}\left[\left(W_{s_2}^1 - W_{s_1}^1\right)\middle| \sigma\left(\mathcal{F}_T^W, \mathcal{F}_{s_1}\right)\right]\middle| \mathcal{F}_t^W\right] \\
&= \mathbf{E}\left[Z\mathbf{E}\left[W_{s_2}^1 - W_{s_1}^1\right]\middle| \mathcal{F}_T^W\right] = 0,
\end{aligned}$$

where we have used the fact that $W_{s_2}^1 - W_{s_1}^1$ is independent of $\sigma\left(\mathcal{F}_T^W, \mathcal{F}_{s_1}\right)$ since W^1 and W are independent and W^1 has independent increments. So the result holds in this case and immediately extends to linear combinations of basic processes. The usual density argument then allows us to extend the result to all required H. □

To verify condition (i) of Definition 2.0.3, take a conditional expectation with respect to $\sigma(W)$ over (2.3.1) with $i = 1$ and use Lemma 2.4.1. Recall that ρ is assumed to be $\sigma(W)$-measurable hence

$$\mathbf{E}\left[\rho_s \mathbf{1}_{s<\tau^i} \partial_x \phi\left(X_s^i\right)\middle| W\right] = \rho_s \nu_s\left(\partial_x \phi\right).$$

This completes the proof of Theorem 2.0.6, as the other conditions are checked in Proposition 2.2.3. □

2.5. Empirical measures and H^{-1}

Before introducing the kernel smoothing method, we will pause to explain why H^{-1} is the natural space to work in when studying empirical measures. To be precise, $H_0^1(0,\infty)$ is defined to be the closure of $C_0^\infty(0,\infty)$ in the norm

$$\|\phi\|_{H_0^1(0,\infty)}^2 := \int_0^\infty \left[\phi(x)^2 + \partial_x \phi(x)^2\right] dx,$$

and then H^{-1} is the dual space of $H_0^1(0,\infty)$ with dual norm

$$\|f\|_{H^{-1}} := \sup_{\|\phi\|_{H_0^1(0,\infty)}=1} |f(\phi)|.$$

(For an overview, see [**Eva10**, Section 5.9.1].)

It is well-known (see [**HS09**, Example 6.8] for instance) that the Dirac delta measure satisfies

$$\delta_0 \in H^s(\mathbf{R}^n) \quad \text{if and only if} \quad s < -\frac{n}{2}.$$

Since ν^N is a linear combination of delta masses and we are working in one dimension, the empirical processes live in every Sobolev space with exponent less than $-1/2$. Hence H^{-1} is the first Sobolev space with integral exponent that contains ν^N.

This explanation relies on Fourier analytic techniques, however we can give an elementary proof of $\delta_0 \in H^{-1}(\mathbf{R})$ by following the reasoning in establishing Morrey's inequality [**Eva10**, Section 5.6.2]: if $\phi \in C_0^\infty(\mathbf{R})$

$$\delta_0(\phi) = \phi(0) = \phi(y) + \int_y^0 \partial_x \phi(x)\, dx, \quad \text{for any } y \in \mathbf{R}.$$

Therefore by Cauchy–Schwarz

$$|\delta_0(\phi)| \le |\phi(y)| + |y|^{\frac{1}{2}} \|\partial_x \phi\|_{L^2(\mathbf{R})},$$

and so integrating over $y \in (0,1)$

$$\begin{aligned} |\delta_0(\phi)| &\le \int_0^1 |\phi(y)|\, dy + \frac{2}{3}\|\partial_x \phi\|_{L^2(\mathbf{R})} \\ &\le \|\phi\|_{L^2(\mathbf{R})} + \frac{2}{3}\|\partial_x \phi\|_{L^2(\mathbf{R})} \le 2\|\phi\|_{H^1(\mathbf{R})}, \end{aligned}$$

thus $\delta_0 \in H^{-1}(\mathbf{R})$.

2.6. Kernel smoothing

We begin by noting that any tempered distribution, ζ, can be approximated by its convolution with the heat kernel:

$$\zeta \star \psi_\varepsilon(x) := \zeta(\psi_\varepsilon(x-\cdot)) \in C^\infty(\mathbf{R}), \quad \int_{\mathbf{R}} \phi(x)\, \zeta \star \psi_\varepsilon(x)\, dx \to \zeta(\phi),$$

as $\varepsilon \downarrow 0$, for every $\phi \in \mathscr{S}$ [**HS09**, Section 4.2]. Throughout we will use the notation ψ_ε to denote the Gaussian heat kernel from (2.1.1).

Our strategy for smoothing a general solution is to set the smoothing kernel into the limit SPDE. To do this we need to control for the boundary condition, so we introduce a kernel, $G_\varepsilon(x,y)$, that vanishes at the origin. This choice ensures that the functon $y \mapsto G_\varepsilon(x,y)$ is an element of the test space, C^{test}, for every x, which enables us to insert this function into the SPDE to obtain the smoothed SPDE approximation in Section 2.8.

Definition 2.6.1 (Absorbing kernel, smoothed density)**.** For $x, y \in \mathbf{R}$ and $\varepsilon > 0$ we define the *absorbing heat kernel* to be

$$\begin{aligned} G_\varepsilon(x,y) &:= \psi_\varepsilon(x-y) - \psi_\varepsilon(x+y) \\ &= (2\pi\varepsilon)^{-\frac{1}{2}} \left[\exp\left\{ -\frac{(x-y)^2}{2\varepsilon} \right\} - \exp\left\{ -\frac{(x+y)^2}{2\varepsilon} \right\} \right]. \end{aligned}$$

For $\zeta \in \mathscr{S}'$, and fixed parameter $\varepsilon > 0$, we define *the smoothed distribution* to be the $C^\infty(\mathbf{R})$ function

$$T_\varepsilon \zeta(x) := \zeta(G_\varepsilon(x,\cdot)).$$

Notice that $G_\varepsilon(0,y) \equiv 0$ so that $T_\varepsilon\zeta(0) = 0$. If $\zeta : \mathbf{R} \to \mathbf{R}$ is a function we will abuse notation and write

$$T_\varepsilon \zeta(x) := \int_{\mathbf{R}} G_\varepsilon(x, x_0)\, \zeta(x_0)\, dx_0.$$

As we are choosing to build the boundary conditions into C^{test} rather than the range space $\mathscr{S}'$, in order for $T_\varepsilon\zeta$ to approximate ζ, this distribution must have no mass on the negative half-line. To be clear we have:

Proposition 2.6.2 (Approximation)**.** *If $\phi \in C_0^\infty(0,\infty)$ and $\zeta \in \mathscr{S}'$ is supported on $[0,\infty)$, that is*

$$\zeta(f) = 0, \quad \textit{whenever } f \in \mathscr{S} \textit{ has support in } (-\infty, 0),$$

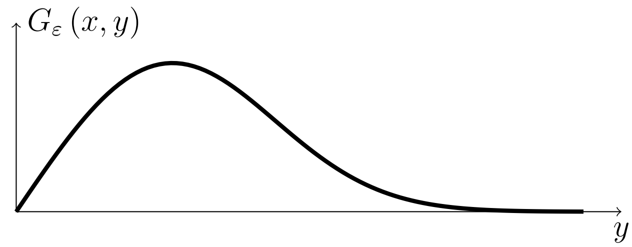


FIGURE 2.6.1. The absorbing heat kernel from Definition 2.6.1. For fixed $x \in \mathbf{R}$ and $\varepsilon > 0$, the function $y \mapsto G_\varepsilon(x,y)$ is plotted.

then

$$\int_{\mathbf{R}} T_\varepsilon \zeta(x)\, \phi(x)\, dx \to \zeta(\phi), \quad \text{as } \varepsilon \to 0.$$

Proof. Let us denote

$$\tilde{\phi}(x) := \phi(-x) \quad \text{and} \quad \tilde{\zeta}(\phi) = \zeta(\tilde{\phi}),$$

then

$$\begin{aligned}\int_{\mathbf{R}} T_\varepsilon \zeta(x)\, \phi(x)\, dx &= \int_{\mathbf{R}} \big(\psi_\varepsilon \star \zeta(x) - \psi_\varepsilon \star \tilde{\zeta}(x)\big) \phi(x)\, dx \\ &\to \zeta(\phi) - \tilde{\zeta}(\phi)\end{aligned}$$

which gives the result since $\tilde{\zeta}(\phi) = 0$ for ϕ supported in $(0, \infty)$. □

The following process will be very important in Section 2.8:

Definition 2.6.3 (Remainder term)**.** For $\zeta \in \mathscr{S}'$, the *remainder process* is defined to be

$$R_\varepsilon \zeta(x) := 2\zeta(\psi_\varepsilon(x + \cdot)).$$

The anti-derivative. Our next step is to introduce the *anti-derivative*, which, for $f : [0, \infty) \to \mathbf{R}$ integrable, we define to be

$$\partial_x^{-1} f(x) := -\int_x^\infty f(z)\, dz.$$

The fundamental theorem of calculus gives:

Lemma 2.6.4 (Manipulating the anti-derivative)**.** *Let $f \in C^\infty(0, \infty)$ be integrable. Then*

$$\partial_x^{-1} \partial_x f = f = \partial_x \partial_x^{-1} f.$$

The next result gives a connection between the anti-derivative and the H^{-1} norm, which we exploit in the next section.

Lemma 2.6.5 (Cauchy–Schwarz)**.** *Let $f \in C^{\infty}(0,\infty)$ be integrable and $\phi \in C_0^{\infty}(0,\infty)$. Then*

$$\left|\int_0^{\infty} f(x)\,\phi(x)\,dx\right| \le \left\|\partial_x^{-1} f\right\|_{L^2(0,\infty)} \left\|\partial_x \phi\right\|_{L^2(0,\infty)}.$$

Proof. Since ϕ has compact support and $\partial_x^{-1} f$ is well-defined, integration by parts gives

$$\int_0^{\infty} f(x)\,\phi(x)\,dx = -\int_0^{\infty} \partial_x^{-1} f(x)\,\partial_x \phi(x)\,dx,$$

and then the result follows by Cauchy–Schwarz. □

The lim inf **propositions.** The following results provide the connection between kernel smoothing and the energy estimates developed in the next section. These propositions show that if we can control the smooth approximation to a distribution then we can control the underlying distribution. Here, *to control* means to bound the distribution's H^{-1} norm.

Proposition 2.6.6. *If $\zeta \in \mathscr{S}'$ is such that*

$$\liminf_{\varepsilon\to 0} \left\|\partial_x^{-1} T_\varepsilon \zeta\right\|_{L^2(0,\infty)} < \infty,$$

then $\zeta \in H^{-1}$ with

$$\|\zeta\|_{H^{-1}} \le \liminf_{\varepsilon\to 0} \left\|\partial_x^{-1} T_\varepsilon \zeta\right\|_{L^2(0,\infty)}.$$

Proof. Take $\phi \in C_0^{\infty}(0,\infty)$, then Lemma 2.6.5 and Proposition 2.6.2 give

$$\begin{aligned}|\zeta(\phi)| &= \liminf_{\varepsilon\to 0} \left|\int_0^{\infty} \phi(x)\,T_\varepsilon \zeta(x)\,dx\right| \\ &\le \liminf_{\varepsilon\to 0} \left\|\partial_x^{-1} T_\varepsilon \zeta\right\|_{L^2(0,\infty)} \left\|\partial_x \phi\right\|_{L^2(0,\infty)}\end{aligned}$$

as required, since $\|\partial_x \phi\|_{L^2(0,\infty)} \le \|\phi\|_{H^1(0,\infty)}$. □

As a consequence we have the following result for $\mathscr{S}'$-valued processes:

Proposition 2.6.7. *If ξ is a random variable taking values in $D_{\mathscr{S}'}$ and*

$$\liminf_{\varepsilon\to 0} \mathbf{E}\left\|\partial_x^{-1} T_\varepsilon \xi_t\right\|^2_{L^2(0,\infty)} = 0, \quad \textit{for every } t \in \mathbf{T},$$

where $\mathbf{T}$ is a dense subset of $[0,T]$, then, with probability 1,

$$\xi_t(\phi) = 0, \quad \textit{for every } t \in [0,T] \textit{ and } \phi \in C_0^{\infty}(0,\infty).$$

Proof. By Fatou's lemma, for each fixed $t \in \mathbf{T}$

$$\liminf_{\varepsilon\to 0} \left\|\partial_x^{-1} T_\varepsilon \xi_t\right\|_{L^2(0,\infty)} = 0,$$

with probability 1. Therefore, with probability 1 $\xi_t(\phi) = 0$ for every $\phi \in C_0^\infty(0,\infty)$ and every $t \in \mathbf{T} \cap \mathbf{Q}$, simultaneously. We can then extend this to all $t \in [0,T]$ since ξ is càdlàg. ☐

The following result for the L^2 case will come in useful in Chapter 5:

Proposition 2.6.8. *If $f \in L^2(0,\infty)$ then*

$$\|T_\varepsilon f\|_{L^2(0,\infty)} \leq \|f\|_{L^2(0,\infty)}, \quad \textit{for every } \varepsilon > 0,$$

and

$$\|T_\varepsilon f\|_{L^2(0,\infty)} \to \|f\|_{L^2(0,\infty)}, \quad \textit{as } \varepsilon \to 0.$$

Proof. As f is a function we can apply Cauchy–Schwarz:

$$\begin{aligned} |T_\varepsilon f(x)|^2 &= \left| \int_0^\infty G_\varepsilon(x_0, x) f(x_0)\, dx_0 \right|^2 \\ &\leq \int_0^\infty G_\varepsilon(x_0,x) f(x_0)^2\, dx_0 \cdot \int_0^\infty G_\varepsilon(x_0,x)\, dx_0. \end{aligned}$$

Now $G_\varepsilon(x_0,x)$ is a defective density, therefore

$$\|T_\varepsilon f\|_{L^2(0,\infty)} \leq \int_0^\infty \left\{ \int_0^\infty G_\varepsilon(x_0,x)\, dx \right\} f(x_0)^2\, dx_0 \leq \|f\|_{L^2(0,\infty)}.$$

This inequality together with Proposition 2.6.6 (with the proof modified to include f and the L^2 norm, which is straightforward) we conclude

$$\|f\|_{L^2(0,\infty)} \leq \liminf_{\varepsilon\to0} \|T_\varepsilon f\|_{L^2(0,\infty)} \leq \limsup_{\varepsilon\to0} \|T_\varepsilon f\|_{L^2(0,\infty)} \leq \|f\|_{L^2(0,\infty)},$$

which completes the proof. ☐

2.7. Some integrability calculations

Here, we collect a number of integrability results which will be useful in Section 2.8. The reader may wish to skip this section and use it as a reference. The following lemmas are consequences of conditions (ii)–(v) of Definition 2.0.3 and of Corollary 4.4.9, which is proved independently in Chapter 4.

Lemma 2.7.1 (Semi-norm control on moments)**.** *Let $\xi \in D_{\mathscr{S}'}$ satisfy condition (ii) of Definition 2.0.3. Then for every $1 \leq p \leq 4$ there exists $\alpha, \beta \in \mathbf{N} \cup \{0\}$ and $0 < c < \infty$ such that*

$$\mathbf{E}\Big[\int_0^T |\xi_t(\phi)|^p\, dt\Big]^{\frac{1}{p}} \leq c\,\|\phi\|_{\alpha,\beta} = c \cdot \sup_{y\in\mathbf{R}} \left| y^\beta \partial_y^\alpha \phi(y) \right|, \quad \textit{for every } \phi \in \mathscr{S}.$$

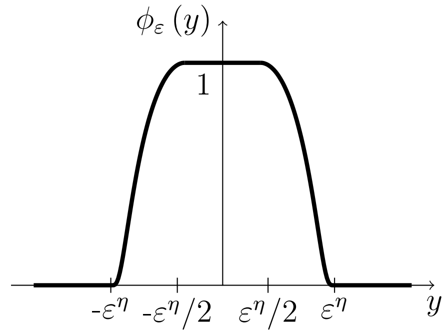


FIGURE 2.7.1. A possible example of the smooth cut-off function, ϕ_ε, from the proof of Lemma 2.7.3.

Proof. Condition (ii) of Definition 2.0.3 implies that

$$\mathbf{E}\Big[\int_0^T |\xi_t(\phi)|^p \, dt\Big] < \infty,$$

for every $1 \le p \le 4$ by Hölder's inequality. The result then follows from Corollary 4.4.9. $\square$

Lemma 2.7.2 (Derivative bound)**.** *For every $n \in \mathbf{N} \cup \{0\}$ and $0 < c_1 < 1$, there exists $0 < c_2 < \infty$ such that*

$$\left|\partial_y^n \psi_\varepsilon(x+y)\right| \le c_2 \left|\psi_\varepsilon(c_1(x+y))\right|, \quad \textit{for every } x, y \in \mathbf{R},$$

where ψ_ε is the Gaussian heat kernel defined in (2.1.1).

Proof. Straightforward computation reveals

$$\partial_y^\alpha \psi_\varepsilon(x+y) = p\left(\varepsilon^{-1}(x+y)\right) \phi\left(\varepsilon^{-1}(x+y)\right),$$

where ϕ is the standard normal density and p is some polynomial depending only on α. Proposition A.0.5 completes the proof. $\square$

Lemma 2.7.3 (Vanishing remainder)**.** *Let $\xi \in D_{\mathscr{S}'}$ satisfy conditions (ii), (iv) and (v) of Definition 2.0.3, then*

$$\mathbf{E}\int_0^T \|R_\varepsilon \xi_t\|^2_{L^2(0,\infty)} \, dt \to 0, \quad \textit{as } \varepsilon \to 0.$$

(Recall that the remainder term, $R_\varepsilon \xi_t$, was introduced in Definition 2.6.3.)

Proof. Introduce a free parameter $\eta > 0$, which will be specified later. Fix an arbitrary $C^\infty(\mathbf{R})$ function that satisfies

$$\phi_\varepsilon(y) := \begin{cases} 1, & \text{if } |y| < \varepsilon^\eta/2 \\ 0, & \text{if } |y| > \varepsilon^\eta \\ \in (0,1), & \text{if } \varepsilon^\eta/2 \le |y| \le \varepsilon^\eta, \end{cases}$$

and define $f_{x,\varepsilon}, g_{x,\varepsilon}, h_{x,\varepsilon} \in \mathscr{S}$ by

$$f_{x,\varepsilon}(y) := \phi_\varepsilon(y)\,\psi_\varepsilon(x+y), \quad g_{x,\varepsilon}(y) := (1-\phi_\varepsilon(y))\,\psi_\varepsilon(x+y)\,\mathbf{1}_{y>0}$$
$$\text{and} \quad h_{x,\varepsilon}(y) := (1-\phi_\varepsilon(y))\,\psi_\varepsilon(x+y)\,\mathbf{1}_{y<0}.$$

(See Figure 2.7.1.) Since $h_{x,\varepsilon}$ is supported on the negative reals, condition (v) of Definition 2.0.3 gives

$$R_\varepsilon\xi_t(x) = 2\xi_t(\psi_\varepsilon(x+\cdot)) = 2\xi_t(f_{x,\varepsilon}) + 2\xi_t(g_{x,\varepsilon}),$$

and so (using Proposition A.0.1)

$$\mathbf{E}\int_0^T \|R_\varepsilon\xi_t\|^2_{L^2(0,\infty)}\,dt \le 8\mathbf{E}\int_0^T\int_0^\infty |\xi_t(f_{x,\varepsilon})|^2\,dxdt + 8\mathbf{E}\int_0^T\int_0^\infty |\xi_t(g_{x,\varepsilon})|^2\,dxdt. \tag{2.7.1}$$

First, we can use Lemma 2.7.1 and that $g_{x,\varepsilon}$ is supported on $(\varepsilon^\eta/2,\infty)$ to get

$$\mathbf{E}\int_0^T\int_0^\infty |\xi_t(g_{x,\varepsilon})|^2\,dxdt \le c^2\,\|g_{x,\varepsilon}\|^2_{\alpha,\beta} \le q_1(\varepsilon^{-1}) \sup_{y>\varepsilon^\eta/2} \left|y^\beta\partial_y^\alpha\psi_\varepsilon(x+y)\right|^2,$$

where q_1 is some polynomial arising from the differentiation of the normal density function. Applying Lemma 2.7.2 with $c_1 = 1/2$ gives

$$\mathbf{E}\int_0^T |\xi_t(g_{x,\varepsilon})|^2\,dt \le c_2 q_1(\varepsilon^{-1}) \sup_{y>\varepsilon^\eta} \left|y^\beta\psi_\varepsilon((x+y)/2)\right|^2 \le q_2(\varepsilon^{-1}) \sup_{y>\varepsilon^\eta}\left|y^\beta\exp\left\{-y^2/4\varepsilon\right\}\right|^2 \exp\left\{-x^2/2\varepsilon\right\},$$

for some constant c_2 and where the extra factors of ε^{-1} are absorbed into the polynomial q_2. Integrating over x and applying Proposition A.0.5 over y gives

$$\mathbf{E}\int_0^T\int_{-\infty}^\infty |\xi_s(g_{x,\varepsilon})|^2\,dxdt = O\big(q_2(\varepsilon^{-1})\,\varepsilon^{\frac12}\exp\left\{-\varepsilon^{2\eta-1}/2\right\}\big), \tag{2.7.2}$$

which is $o(1)$ provided $\eta < 1/2$.

Since $f_{x,\varepsilon}$ is supported on $(-\varepsilon^{\eta}, \varepsilon^{\eta})$ and $\|f_{x,\varepsilon}\|_{\infty} \leq (2\pi\varepsilon)^{-\frac{1}{2}}$, by normalising and using condition (iv) we have

$$\mathbf{E}\int_0^T |\xi_t(f_{x,\varepsilon})|^2\, dt = \varepsilon^{-1} O\left(\varepsilon^{\eta(1+\gamma)}\right) = O\left(\varepsilon^{\eta(1+\gamma)-1}\right),$$

independently of x in the range $x > 0$. So we have

$$\mathbf{E}\int_0^T \int_0^{\varepsilon^{\eta}} |\xi_t(f_{x,\varepsilon})|^2\, dxdt = O\left(\varepsilon^{\eta(2+\gamma)-1}\right), \tag{2.7.3}$$

which is $o(1)$ provided $\eta > 1/(2+\gamma)$.

Now suppose we have $x > \varepsilon^{\eta}$. The same work leading up to (2.7.2) is applicable and gives

$$\begin{aligned}\mathbf{E}\int_0^T |\xi_t(f_{x,\varepsilon})|^2\, dt &\leq q_2\left(\varepsilon^{-1}\right) \sup_{y<\varepsilon^{\eta}} \left|y^{\beta} \exp\left\{-y^2/4\varepsilon\right\}\right|^2 \exp\left\{-x^2/2\varepsilon\right\} \\ &\leq q_2\left(\varepsilon^{-1}\right) \exp\left\{-x^2/2\varepsilon\right\},\end{aligned}$$

for sufficiently small ε. By integrating over $x \in (\varepsilon^{\eta}, \infty)$ we have

$$\mathbf{E}\int_0^T \int_{\varepsilon^{\eta}}^{\infty} |\xi_t(f_{x,\varepsilon})|^2\, dxdt \leq \varepsilon^{\frac{1}{2}} q_2\left(\varepsilon^{-1}\right) \Phi\left(-\varepsilon^{\eta-1/2}\right), \tag{2.7.4}$$

which is $o(1)$ provided $\eta < 1/2$ (recall Proposition A.0.3).

Inserting (2.7.2), (2.7.3) and (2.7.4) into (2.7.1) completes the proof since it is possible to choose an η such that

$$\frac{1}{2+\gamma} < \eta < \frac{1}{2}.$$

$\square$

Lemma 2.7.4 (Integrability)**.** *For $n = 0, 1$ and 2 and fixed $\varepsilon > 0$ we have*

$$\int_{-\infty}^{\infty} \exp\{b|x|\} F\left(\mathbf{E}\int_0^T |\partial_x^n \xi_t(\psi_\varepsilon(x-\cdot))|^2\, dt\right) dx < \infty,$$

for any $b > 0$ and $F : [0,\infty) \to [0,\infty)$ satisfying $F(x) = O(x^a)$, as $x \downarrow 0$ for some $a > 0$.

Proof. Fix n and ε. Take $x > 0$ and an arbitrary $C^{\infty}(\mathbf{R})$ function, ϕ_x, that satisfies

$$\phi_x(y) = \begin{cases} 1, & \text{if } |y| < x/2+1 \\ \in (0,1), & \text{if } x/2 \leq |y| \leq x/2+1 \\ 0, & \text{if } |y| > x/2. \end{cases}$$

Define the functions

$$f_x(y) := (1-\phi_x(y))\,\partial_y^n \psi_\varepsilon(x-y)\,\mathbf{1}_{y>0}, \quad g_x(y) := \phi_x(y)\,\partial_y^n \psi_\varepsilon(x-y)$$
$$\text{and} \quad h_x(y) := (1-\phi_x(y))\,\partial_y^n \psi_\varepsilon(x-y)\,\mathbf{1}_{y<0},$$

so that f_x is supported on $(x/2,\infty)$, g_x is supported on $(-x/2,x/2)$ and h_x is supported on $(-\infty,-x/2)$. By condition (v) of Definition 2.0.3 we have no contribution from h_x:

$$\xi_t\left(\partial_y^n \psi_\varepsilon(x-\cdot)\right) = \xi_t(f_x) + \xi_t(g_x)\,. \tag{2.7.5}$$

By setting $c_1 = 1/2$ in Lemma 2.7.2 we obtain

$$\|f_x\|_\infty \leq c_2 \sup_{y\in\mathbf{R}} |\psi_\varepsilon((x-y)/2)| = c_3,$$

where c_2 and c_3 are numerical constants (c_3 depending on ε). Hence condition (iii) of Definition 2.0.3 gives

$$\mathbf{E}\int_0^T |\xi_t(f_x)|^2\,dt = o(\exp\{-cx\}), \quad \text{as } x\to\infty,$$

for any $c > 0$. So for large x

$$\exp\{bx\}\,F\left(\mathbf{E}\int_0^T |\xi_t(f_x)|^2\,dt\right) = O(\exp\{-(ac-b)x\}), \tag{2.7.6}$$

which is integrable provided $c > a^{-1}b$.

By Lemma 2.7.1 we have $\alpha,\beta\in\mathbf{N}$ and $c_4 > 0$ such that

$$\begin{aligned}\mathbf{E}\int_0^T |\xi_t(g_x)|^2\,dt &\leq c_4 \sup_{y<x/2}\left|y^\beta \partial_y^\alpha\left((1-\phi_x(y))\,\partial_y^n\psi_\varepsilon(x-y)\right)\right|^2 \\ &\leq p_1(|x|)\sup_{|y|<x/2}\left|\sum_{0\leq k\leq\alpha} a_k \partial_y^{n+k}\psi_\varepsilon(x-y)\right|^2,\end{aligned}$$

where p_1 is a polynomial (with coefficients perhaps depending on ε). Applying Lemma 2.7.2 to each term in the sum gives

$$\begin{aligned}\mathbf{E}\int_0^T |\xi_t(g_x)|^2\,dt &\le p_2(|x|)\sup_{|y|<x/2}\psi_\varepsilon((x-y)/2)^2\\ &= p_2(|x|)\,\psi_\varepsilon(x/4)^2\\ &= o(\exp\{-cx\}), \quad \text{for large } x,\end{aligned}$$

for any $c>0$, and where p_2 a further polynomial. Again we obtain

$$\exp\{bx\}\,F\left(\mathbf{E}\int_0^T |\xi_t(g_x)|^2\,dt\right) = O(\exp\{-(ac-b)x\}),$$

and together with (2.7.6) and (2.7.5) we have

$$\int_0^\infty \exp\{bx\}\,F\left(\mathbf{E}\int_0^T \left|\xi_t\left(\partial_y^n\psi_\varepsilon(x-\cdot)\right)\right|^2 dt\right)dx<\infty.$$

Integrability on $x<0$ follows by symmetry, hence we have the result. $\square$

2.8. Smoothing the SPDE; energy estimation

We now suppose that ξ is a solution to the basic SPDE with respect to (ν_0, W) from the statement of Theorem 2.0.6. Since $y\mapsto G_\varepsilon(x,y)$ is a rapidly decreasing function and vanishes at $y=0$, it is an element of C^{test}, so we can set this function into the SPDE for ξ to obtain

$$\begin{aligned}T_\varepsilon\xi_t(x) = \xi_t(G_\varepsilon(x,\cdot)) = \nu_0(G_\varepsilon(x,\cdot)) &+ \frac{1}{2}\int_0^t \xi_s(\partial_{yy}G_\varepsilon(x,\cdot))\,ds\\ &+\int_0^t \rho_s\xi_s(\partial_y G_\varepsilon(x,\cdot))\,dW_s.\end{aligned}$$

We aim to rearrange this equation to make $T_\varepsilon\xi_t$ the subject and therefore need the following computational lemma:

Lemma 2.8.1 (Switching derivatives)**.** *Let $\zeta\in\mathscr{S}'$. For all $x,y\in\mathbf{R}$ and $\varepsilon>0$ we have*

(i) $\partial_y G_\varepsilon(x,y) = -\partial_x G_\varepsilon(x,y) - 2\partial_x\psi_\varepsilon(x+y)$,
(ii) $\partial_{yy}G_\varepsilon(x,y) = \partial_{xx}G_\varepsilon(x,y)$,
(iii) $\zeta(\partial_y G_\varepsilon(x,\cdot)) = -\partial_x T_\varepsilon\zeta(x) - \partial_x R_\varepsilon\zeta(x)$, *where the remainder, $R_\varepsilon\zeta$, was described in Definition 2.6.3,*
(iv) $\zeta(\partial_{yy}G_\varepsilon(x,\cdot)) = \partial_{xx}T_\varepsilon\zeta(x)$.

Proof. (i) and (ii) are just calculations, and then (iii) and (iv) follow by interchanging the derivative with respect to x with the distribution, ζ. □

Using this lemma, we arrive at the *smoothed SPDE*:

$$T_\varepsilon \xi_t(x) = T_\varepsilon \nu_0(x) + \frac{1}{2}\int_0^t \partial_{xx} T_\varepsilon \xi_s(x)\, ds - \int_0^t \rho_s \partial_x T_\varepsilon \xi_s(x)\, dW_s, \\ - \int_0^t \rho_s \partial_x R_\varepsilon \xi_s(x)\, dW_s \tag{2.8.1}$$

which holds, with probability 1, for all $t \in [0,T]$ and $x \in (0,\infty)$. At this stage we have obtained a strong SPDE from the original weak formulation, however we see that the smoothing introduces the remainder process, and controlling this term is now critical for the method to succeed, hence we will eventually require Lemma 2.7.3. From our construction $T_\varepsilon \xi_t$ is smooth in space and from equation (2.8.1) we see that $T_\varepsilon \xi_t$ is continuous in time.

We would like to take an anti-derivative over (2.8.1) to arrive at a H^{-1} energy estimate. To do this we integrate over the spatial variable on the range (x,∞) and interchange this integral with the time and stochastic integrals. For this final step to be justified we need some Fubini-type integrability results:

Lemma 2.8.2 (Taking the anti-derivative)**.** *We have:*

(i) With probability 1,

$$\int_0^\infty |T_\varepsilon \nu_0(x)|\, dx < \infty,$$

(ii) With probability 1,

$$\int_0^\infty |T_\varepsilon \xi_t(x)|\, dx < \infty, \quad \textit{for almost all } t \in (0,T),$$

(iii) For $n = 1$ and 2,

$$\int_0^\infty \left(\mathbf{E}\int_0^T |\partial_x^n T_\varepsilon \xi_t(x)|^2\, dt\right)^{\frac{1}{2}} dx < \infty,$$

(iv)

$$\int_0^\infty \left(\mathbf{E}\int_0^T |\partial_x R_\varepsilon \xi_t(x)|^2\, dt\right)^{\frac{1}{2}} dx < \infty.$$

Hence the deterministic and stochastic Fubini theorems [**Ver12**, (1.4)] *are applicable and we deduce that for almost all* (ω, t)

$$\partial_x^{-1} T_\varepsilon \xi_t(x) = \partial_x^{-1} T_\varepsilon \nu_0(x) + \frac{1}{2}\int_0^t \partial_x T_\varepsilon \xi_s(x)\, ds - \int_0^t \rho_s T_\varepsilon \xi_s(x)\, dW_t - \int_0^t \rho_s R_\varepsilon \xi_s(x)\, dW_s, \tag{2.8.2}$$

for every $x > 0$.

Proof. (i). Let $x > x_\star$, then

$$0 \le T_\varepsilon \nu_0(x) \le \int_0^\infty (2\pi\varepsilon)^{-\frac{1}{2}} \exp\left\{-\frac{(x-y)^2}{2\varepsilon}\right\} \nu_0(dy)$$
$$\le (2\pi\varepsilon)^{-\frac{1}{2}} \exp\left\{-\frac{(x-x_\star)^2}{2\varepsilon}\right\},$$

which is integrable.

(ii). By the deterministic Fubini theorem, it suffices to show that

$$\int_0^\infty \int_0^T \mathbf{E}\left|T_\varepsilon \xi_t(x)\right| dt dx < \infty,$$

which is implied by

$$\int_0^\infty \int_0^T \mathbf{E}\left[\left|T_\varepsilon \xi_t(x)\right|^2\right] dt dx < \infty. \tag{2.8.3}$$

Since

$$T_\varepsilon \xi_t(x) = \xi_t(\psi_\varepsilon(x-\cdot)) - \xi_t(\psi_\varepsilon(x+\cdot)), \tag{2.8.4}$$

Proposition A.0.1 and a change of variable give

$$\int_0^\infty \int_0^T \mathbf{E}\left[\left|T_\varepsilon \xi_t(x)\right|^2\right] dt dx \le 2\int_0^\infty \int_0^T \mathbf{E}\Big[\left|\xi_t(\psi_\varepsilon(x-\cdot))\right|^2 + \left|\xi_t(\psi_\varepsilon(x+\cdot))\right|^2\Big] dt dx$$
$$= 2\int_{-\infty}^\infty \int_0^T \mathbf{E}\left[\left|\xi_t(\psi_\varepsilon(x-\cdot))\right|^2\right] dt dx.$$

Therefore (2.8.3) follows by Lemma 2.7.4 with $n = 0$, $c = 0$ and $F(x) \equiv x$.

(iii). By the argument in (ii), and using the simple inequality $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$, we have

$$\int_0^\infty \left(\mathbf{E} \int_0^T |\partial_x T_\varepsilon \xi_t (x)|^2 \, dt \right)^{\frac{1}{2}} dx \leq \int_{-\infty}^\infty \left(\mathbf{E} \int_0^T |\xi_t (\psi_\varepsilon (x - \cdot))|^2 \, dt \right)^{\frac{1}{2}} dx,$$

and then the result follows by Lemma 2.7.4 with $n = 0$, $c = 0$ and $F(x) \equiv x^{1/2}$.

(iv). Similar to (iii). □

We can now apply the standard technique of energy estimation to (2.8.2). Squaring $\partial_x^{-1} T_\varepsilon \xi_t (x)$ and applying Ito's formula gives

$$\begin{aligned} \left(\partial_x^{-1} T_\varepsilon \xi_t (x) \right)^2 = \left(\partial_x^{-1} T_\varepsilon \nu_0 (x) \right)^2 &+ \int_0^t \partial_x^{-1} T_\varepsilon \xi_s (x) \cdot \partial_x T_\varepsilon \xi_s (x) \, ds \\ &+ \int_0^t \rho_s^2 \left(T_\varepsilon \xi_s (x) + R_\varepsilon \xi_s (x) \right)^2 ds + M_t (x), \end{aligned} \tag{2.8.5}$$

where M is given by

$$M_t (x) = -2 \int_0^t \rho_s \partial_x^{-1} T_\varepsilon \xi_s (x) \cdot \left(T_\varepsilon \xi_s (x) + R_\varepsilon \xi_s (x) \right) dW_s.$$

Our strategy is to integrate (2.8.5) over $(0, \infty) \times \Omega$, but for this we need some further integrability conditions, which we summarise in the following result.

Lemma 2.8.3 (More integrability results)**.** *We have:*

(i) $\mathbf{E} \int_0^T \int_0^\infty |T_\varepsilon \xi_t (x)|^2 dx dt < \infty$ *and* $\mathbf{E} \int_0^T \int_{-\infty}^\infty |\partial_x T_\varepsilon \xi_t (x)|^2 dx dt < \infty$,

(ii) $\mathbf{E} \int_0^T \int_0^\infty |R_\varepsilon \xi_t (x)|^2 \, dx dt < \infty$,

(iii) $\mathbf{E} \int_0^T \int_0^\infty |\partial_x^{-1} T_\varepsilon \xi_t (x)|^2 \, dx dt < \infty$,

(iv) $\mathbf{E} \int_0^t \int_0^\infty \rho_s \partial_x^{-1} T_\varepsilon \xi_s (x) \cdot \partial_x T_\varepsilon \xi_s (x) dx ds = -\mathbf{E} \int_0^t \rho_s \| T_\varepsilon \xi_s \|_{L^2(0,\infty)}^2 ds$, *for every* $t \in [0, T]$,

(v) For every $t \in [0, T]$ *and almost all* $x > 0$, $\mathbf{E} M_t (x) = 0$.

Proof. (i) & (ii). These follow immediately from Lemma 2.7.4 with $c = 0$ and $F(x) \equiv x$.

(iii). By introducing an exponential weighting, the Cauchy–Schwarz inequality gives

$$\left| \partial_x^{-1} T_\varepsilon \xi_t (x) \right|^2 \leq \int_x^\infty e^{-z} dz \cdot \int_0^\infty e^z \left| T_\varepsilon \xi_t (z) \right|^2 dz = e^{-x} \int_0^\infty e^z \left| T_\varepsilon \xi_t (z) \right|^2 dz.$$

Therefore we have

$$\mathbf{E} \int_0^T \int_0^\infty \left| \partial_x^{-1} T_\varepsilon \xi_t (x) \right|^2 dx dt \leq \int_0^\infty e^{-x} dx \cdot \mathbf{E} \int_0^T \int_0^\infty e^z \left| T_\varepsilon \xi_t (z) \right|^2 dz dt,$$

and the result follows by Lemma 2.7.4 with $c = 1$ and $F(x) \equiv x$ (and by using the method in (2.8.4), but we will omit this argument for the remainder of this proof).

(iv). The deterministic Fubini theorem allows us to interchange the space and time integrals since Cauchy–Schwarz gives

$$\mathbf{E}\int_0^t\int_0^\infty \left|\partial_x^{-1}T_\varepsilon\xi_s(x)\cdot\partial_x T_\varepsilon\xi_s(x)\right|dxds \leq \left(\mathbf{E}\int_0^t\int_0^\infty \left|\partial_x^{-1}T_\varepsilon\xi_s(x)\right|^2 dxds\right)^{\frac{1}{2}} \times \left(\mathbf{E}\int_0^t\int_0^\infty \left|\partial_x T_\varepsilon\xi_s(x)\right|^2 dxds\right)^{\frac{1}{2}},$$

and the right-hand side is finite by (i) and (iii). Once we do not have to worry about the order of integration, integrating by parts gives

$$\int_0^\infty \partial_x^{-1}T_\varepsilon\xi_s(x)\cdot\partial_x T_\varepsilon\xi_s(x)\,dx = -\left\|T_\varepsilon\xi_s\right\|^2_{L^2(0,\infty)},$$

since $T_\varepsilon\xi_s(0) = 0$, which completes the result.

(v). We will show that, for almost all $x > 0$, $M_\cdot(x)$ is a square-integrable martingale. Notice that this will be the case if

$$\mathbf{E}\int_0^T\int_0^\infty \left|\partial_x^{-1}T_\varepsilon\xi_t(x)\right|^2\left|T_\varepsilon\xi_t(x) + R_\varepsilon\xi_t(x)\right|^2 dxdt < \infty,$$

which, by Cauchy–Schwarz, is implied by

$$\mathbf{E}\int_0^T\int_0^\infty \left|\partial_x^{-1}T_\varepsilon\xi_t(x)\right|^4 dxdt < \infty \quad \text{and}$$
$$\mathbf{E}\int_0^T\int_0^\infty \left|T_\varepsilon\xi_t(x) + R_\varepsilon\xi_t(x)\right|^4 dxdt < \infty.$$

The work in (iii) and Proposition A.0.1 give that the above are implied by

$$\mathbf{E}\int_0^T\int_0^\infty e^x\left|T_\varepsilon\xi_t(x)\right|^2 dxdt < \infty, \quad \mathbf{E}\int_0^T\int_0^\infty \left|T_\varepsilon\xi_t(x)\right|^4 dxdt < \infty$$
$$\text{and} \quad \mathbf{E}\int_0^T\int_0^\infty \left|R_\varepsilon\xi_t(x)\right|^4 dxdt < \infty,$$

and these conditions hold by Lemma 2.7.4. □

First taking an expectation over (2.8.5) for x in the full subset of $(0,\infty)$ on which condition (v) of Lemma 2.8.3 holds, then integrating over these x and using conditions (i)-(iv) gives

$$\begin{aligned}\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_t\right\|^2_{L^2(0,\infty)} &= \mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_0\right\|^2_{L^2(0,\infty)} - \mathbf{E}\int_0^t \|T_\varepsilon\xi_s\|^2_{L^2(0,\infty)}\,ds\\ &\quad + \rho^2_{\max}\mathbf{E}\int_0^t \|T_\varepsilon\xi_s + R_\varepsilon\xi_s\|^2_{L^2(0,\infty)}\,ds.\end{aligned}$$

By applying Young's inequality to the final term above:

$$\begin{aligned}(T_\varepsilon\xi_s + R_\varepsilon\xi_s)^2 &\leq \left(1+\frac{1-\rho^2_{\max}}{\rho^2_{\max}}\right)(T_\varepsilon\xi_s)^2 + \left(1+\frac{\rho^2_{\max}}{1-\rho^2_{\max}}\right)(R_\varepsilon\xi_s)^2\\ &= \frac{1}{\rho^2_{\max}}\left(T_\varepsilon\xi_s\right)^2 + \frac{1}{1-\rho^2_{\max}}\left(R_\varepsilon\xi_s\right)^2,\end{aligned}$$

we obtain:

Proposition 2.8.4 (Energy estimate)**.** *Let ξ be a solution to the basic SPDE. For all $t \in [0,T]$*

$$\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_t\right\|^2_{L^2(0,\infty)} \leq \mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_0\right\|^2_{L^2(0,\infty)} + \frac{\rho^2_{\max}}{1-\rho^2_{\max}}\mathbf{E}\int_0^t \|R_\varepsilon\xi_s\|^2_{L^2(0,\infty)}ds.$$

2.9. Proof of Theorem 2.0.7

With Proposition 2.8.4 it is straightforward to prove the uniqueness theorem. Take two solutions to the basic SPDE. By Remark 2.0.4 their difference, Δ, which is in $D_{\mathscr{S}'}$, also solves the basic SPDE. However, $\Delta_0 = 0$, therefore

$$\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\Delta_t\right\|^2_{L^2(0,\infty)} \leq \frac{\rho^2_{\max}}{1-\rho^2_{\max}}\mathbf{E}\int_0^t \|R_\varepsilon\Delta_s\|^2_{L^2(0,\infty)}\,ds, \tag{2.9.1}$$

so by Lemma 2.7.3

$$\liminf_{\varepsilon\to 0}\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\Delta_t\right\|^2_{L^2(0,\infty)} = 0, \quad \text{for almost all } t \in [0,T].$$

Proposition 2.6.7 applied to Δ now completes the proof. □

Remark 2.9.1. Inequality (2.9.1) illustrates the importance of the condition $\rho_{\max} < 1$.

2.10. Incorporating spatially-dependent coefficients

Here, we will explain how the kernel smoothing method can be used when the basic SPDE has coefficients that are functions of the spatial variable. This will be helpful in Section 5.9 and the reader may wish to defer this section until then.

When results follow by analogy from those in previous sections of this chapter, we will state them without proof.

To be precise, we define:

Definition 2.10.1 (Solution). Let μ, σ_M and σ be adapted processes, let $f \in \mathscr{S}'$ be deterministic and let W be a Brownian motion. An adapted process, ξ, taking values in the space $D_{\mathscr{S}'}$ is said to be a *solution to the spatially-dependent basic SPDE with respect to* (f, W) if all the following hold:

(i) (Coefficient assumptions) With probability 1:

$$\mu(t,\cdot), \sigma_M(t,\cdot), \sigma(t,\cdot) \in C^\infty(\mathbf{R}), \quad \text{for all } t \in [0,T],$$

there exists $c_{\max} > 0$ such that

$$|\partial_x^n \mu(t,x)|, |\partial_x^n \sigma_M(t,x)|, |\partial_x^n \sigma(t,x)| \le c_{\max}, \quad \text{for all } t \in [0,T], n \ge 0, x \in \mathbf{R}$$

and there exists $\delta_0 > 0$ such that

$$\sigma(t,x)^2 - \sigma_M(t,x)^2 \ge \delta_0, \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbf{R},$$

(ii) (Dynamics) With probability 1,

$$\xi_t(\phi) = f(\phi) + \int_0^t \xi_s\left(\mu(s,\cdot)\,\partial_x\phi\right) ds + \frac{1}{2}\int_0^t \xi_s\left(\sigma(s,\cdot)^2\,\partial_{xx}\phi\right) ds$$
$$+ \int_0^t \xi_s\left(\sigma_M(s,\cdot)\,\partial_x\phi\right) dW_s,$$

for all $t \in [0,T]$ and $\phi \in C^{\text{test}} := \{\phi \in \mathscr{S} : \phi(0) = 0\}$,

(iii) (Finite moments) For all $\phi \in \mathscr{S}$

$$\mathbf{E}\int_0^T |\xi_t(g(t,\cdot)\,\phi)|^4\, dt < \infty,$$

whenever g is equal to μ, σ_M, σ^2 or the constant 1,

(iv) (Tail estimate) For every $c > 0$ and $\phi_\lambda \in \mathscr{S}$ supported on (λ, ∞) with $\|\phi_\lambda\|_\infty \le 1$

$$\mathbf{E}\int_0^T |\xi_t(g(t,\cdot)\,\phi_\lambda)|^4\, dt = o\left(\exp\{-c\lambda\}\right), \quad \text{as } \lambda \to \infty,$$

whenever g is equal to μ, σ_M, σ^2 or the constant 1,

(v) (Boundary estimate) There exists a $\gamma > 0$ such that for every $\phi_\varepsilon \in \mathscr{S}$ supported on $(-\varepsilon, \varepsilon)$ with $\|\phi_\varepsilon\|_\infty \leq 1$

$$\mathbf{E} \int_0^T |\xi_t(\phi_\varepsilon)|^2 \, dt = O\left(\varepsilon^{1+\gamma}\right), \quad \text{as } \varepsilon \to 0,$$

(vi) (Vanishing on negatives) With probability 1, for every $t \in [0, T]$ and $\phi \in \mathscr{S}$

$$\xi_t(\phi) = 0, \quad \text{whenever support}(\phi) \subseteq (-\infty, 0),$$

(vii) (Spatial regularity) There exist constants $c_{\mathrm{reg}} > 0$ and $\delta > 0$ such that whenever g is equal to μ, σ^2 or σ_M

$$\mathbf{E}\int_0^T |\xi_t((g(t,\cdot) - g(t,x))\,\phi)|^2 dt \leq c_{\mathrm{reg}} \mathrm{Leb}\left(\mathrm{support}(\phi)\right)^\delta \sup_{y \in \mathbf{R}} |(x-y)\,\phi(y)|^2,$$

for all $\phi \in \mathscr{S}$ and $x \in \mathbf{R}$. Here, Leb denotes the Lebesgue measure on $\mathbf{R}$.

Notice that, since the coefficients are smooth, their product with any element of $\mathscr{S}$ is again an element of $\mathscr{S}$. Hence the integrands in (ii) are well-defined. It is necessary to rewrite the conditions from Definition 2.0.3 with the coefficients inside the argument of the solution to account for the joint randomness between the solution and the coefficients. By applying the reasoning in Lemma 2.7.1 to condition (iii), we have that there exists constants $\alpha, \beta \in \mathbf{N}$ and $c > 0$ such that

$$\mathbf{E} \int_0^T |\xi_t(g(t,\cdot)\,\phi)|^4 \, dt \leq c \sup_{y \in \mathbf{R}} \left| y^\alpha \partial_y^\beta \phi(y) \right|, \quad \text{for all } \phi \in \mathscr{S}, \tag{2.10.1}$$

whenever g is equal to μ, σ_I^2,σ_M or the constant 1.

It is should be noted that if W and $W^\perp$ are independent Brownian motions and X is the diffusion

$$dX_t = \mu(t, X_t)\, dt + \sigma_M(t, X_t)\, dW_t + \sigma_I(t, X_t)\, dW_t^\perp,$$

then the conditional law

$$\nu_t(\phi) = \mathbf{E}\left[\phi(X_t)\, \mathbf{1}_{t<\tau} \middle| W\right], \quad \tau = \inf\{t > 0 : X_t \leq 0\},$$

satisfies Definition 2.10.1 with $\sigma^2 = \sigma_M^2 + \sigma_I^2$ (provided the diffusion coefficient are sufficient regular). Of course this motivates the study of this SPDE, and we will explore this fact further in Section 5.9. Note that the ellipticity condition in (i) ensures that the energy estimation method from Section 2.8 can be transferred to this setting. Specifically, the condition ensures that we can eliminate the effect of the higher-order derivative terms in the smoothed SPDE calculations — see

(2.10.9). Heuristically, at the level of the individual diffusions, the condition ensures there is sufficient independence between the particles for the evolution equation to be smoothed-out in the large population limit.

Kernel smoothing. Throughout this subsection suppose that ξ satisfies Definition 2.10.1. Proceed by fixing $x > 0$ and passing the test function $\phi(\cdot) = G_\varepsilon(x, \cdot)$ to the basic SPDE:

$$\begin{aligned} dT_\varepsilon\xi_t(x) &= \xi_t(\mu(t,\cdot)\,\partial_y G_\varepsilon(x,\cdot))\,dt + \frac{1}{2}\xi_t\left(\sigma(t,\cdot)^2\,\partial_y G_\varepsilon(x,\cdot)\right)dt \\ &\quad + \xi_t(\sigma_M(t,\cdot)\,\partial_y G_\varepsilon(x,\cdot))\,dW_t, \end{aligned}$$

where we use ∂_y to denote that the derivative acts on the second variable in G_ε. By applying Lemma 2.8.1, and noting that the derivative ∂_x does not interact with the spatial variables in the coefficients, we obtain

$$\begin{aligned} dT_\varepsilon\xi_t(x) &= -\partial_x\xi_t(\mu(t,\cdot)\,G_\varepsilon(x,\cdot))\,dt + \frac{1}{2}\partial_{xx}\xi_t\left(\sigma(t,\cdot)^2\,G_\varepsilon(x,\cdot)\right)dt \\ &\quad - \partial_x\xi_t(\sigma_M(t,\cdot)\,G_\varepsilon(x,\cdot))\,dW_t - \partial_x R^1_\varepsilon\xi_t(x)\,dt - \partial_x R^2_\varepsilon\xi_t(x)\,dW_t, \end{aligned}$$

where the remainder terms $R^1_\varepsilon\xi_t$ and $R^2_\varepsilon\xi_t$ are as defined in Lemma 2.10.2. From the conditions in Definition 2.10.1, the analogous estimates and arguments to those in Section 2.7 and Lemmas 2.8.2 and 2.8.3 allow us to take the anti-derivative and to arrive at

$$\begin{aligned} d\partial_x^{-1}T_\varepsilon\xi_t(x) &= -\xi_t(\mu(t,\cdot)\,G_\varepsilon(x,\cdot))\,dt + \frac{1}{2}\partial_x\xi_t\left(\sigma(t,\cdot)^2\,G_\varepsilon(x,\cdot)\right)dt \\ &\quad - \xi_t(\sigma_M(t,\cdot)\,G_\varepsilon(x,\cdot))\,dW_t - R^1_\varepsilon\xi_t(x)\,dt - R^2_\varepsilon\xi_t(x)\,dW_t. \end{aligned}$$

Now, the coefficients are within the argument of the solution, so we require the following result in order to gain control in terms of $T_\varepsilon\xi$:

Lemma 2.10.2 (Error processes vanish)**.** *Define the error processes*

$$\begin{aligned} e^1_{\varepsilon,t}(x) &:= \xi_t(\mu(t,\cdot)\,G_\varepsilon(x,\cdot)) - \mu(t,x)\,\xi_t(G_\varepsilon(x,\cdot)) \\ e^2_{\varepsilon,t}(x) &:= \xi_t(\sigma_M(t,\cdot)\,G_\varepsilon(x,\cdot)) - \sigma_M(t,x)\,\xi_t(G_\varepsilon(x,\cdot)) \\ e^3_{\varepsilon,t}(x) &:= \xi_t\left(\sigma(t,\cdot)^2\,G_\varepsilon(x,\cdot)\right) - \sigma(t,x)^2\,\xi_t(G_\varepsilon(x,\cdot)) \end{aligned}$$

and the remainder processes

$$R^1_\varepsilon\xi_t(x) := 2\xi_t(\mu(t,\cdot)\,\psi_\varepsilon(x+\cdot)), \qquad R^2_\varepsilon\xi_t(x) := 2\xi_t(\sigma_M(t,\cdot)\,\psi_\varepsilon(x+\cdot))$$

Then, as $\varepsilon \to 0$,

$$\mathbf{E}\int_0^T \left\|e^i_{\varepsilon,t}\right\|^2_{L^2(0,\infty)} dt \to 0, \quad \mathbf{E}\int_0^T \left\|R^i_\varepsilon \xi_t\right\|^2_{L^2(0,\infty)} dt \to 0 \quad \textit{for all } i.$$

Proof. The proof is the same in the three cases, so we only consider μ, whereby

$$\mathbf{E}\Big[\left|e^1_{\varepsilon,t}(x)\right|^2\Big] = \mathbf{E}\Big[\left|\xi_t\big((\mu(t,\cdot) - \mu(t,x))\,G_\varepsilon(x,\cdot)\big)\right|^2\Big].$$

By letting $\eta > 0$ be a free parameter, we can consider the argument on the range $(x - \varepsilon^\eta, x + \varepsilon^\eta)$ and its complement by introducing a smooth cut-off function, ϕ_η, satisfying (for fixed x)

$$\phi_\eta(y) = \begin{cases} 1, & \text{if } |y - x| \le \varepsilon^\eta \\ 0, & \text{if } |y - x| \ge 2\varepsilon^\eta \\ \in (0,1), & \text{if } \varepsilon^\eta < |y - x| < 2\varepsilon^\eta. \end{cases}$$

From Proposition A.0.1, we have

$$\begin{aligned}\mathbf{E}\Big[\left|e^1_{\varepsilon,t}(x)\right|^2\Big] \le\; & 2\mathbf{E}\Big[\left|\xi_t\big((\mu(t,\cdot) - \mu(t,x))\,G_\varepsilon(x,\cdot)\,\phi_\eta\big)\right|^2\Big] \\ & + 2\mathbf{E}\Big[\left|\xi_t\big((\mu(t,\cdot) - \mu(t,x))\,G_\varepsilon(x,\cdot)\,(1-\phi_\eta)\big)\right|^2\Big].\end{aligned}$$

If $|y - x| \ge 2\varepsilon^\eta$, then for every $n \ge 0$

$$G_\varepsilon(x,\cdot) \le c_1 \varepsilon^{-1/2} \exp\left\{-2\varepsilon^{2\eta-1}\right\},$$

with $c_1 > 0$ a numerical constant, so if we choose $\eta < 1/2$ then (2.10.1) and the derivative bounds from condition (i) of Definition 2.10.1 give that for any $\alpha > 0$

$$\int_0^T \mathbf{E}\Big[\left|e^1_{\varepsilon,t}(x)\right|^2\Big]dt \le 2\int_0^T \mathbf{E}\Big[\left|\xi_t\big((\mu(t,\cdot) - \mu(t,x))\,G_\varepsilon(x,\cdot)\,\phi_\eta\big)\right|^2\Big]dt + O(\varepsilon^\alpha),$$

as $\varepsilon \to 0$, uniformly in $x \ge 0$. By applying condition (vii) of Definition 2.10.1 we obtain

$$\begin{aligned}\int_0^T \mathbf{E}\Big[\left|e^1_{\varepsilon,t}(x)\right|^2\Big]dt &\le 2c\,(8\varepsilon^\eta)^\delta \sup_{y:|x-y|\le 2\varepsilon^\eta} \left|(x-y)\,G_\varepsilon(x,y)\right|^2 + O(\varepsilon^\alpha) \\ &= c_2 \varepsilon^{(2+\delta)\eta - 1} + O(\varepsilon^\alpha).\end{aligned}$$

Therefore if we choose η to satisfy

$$\eta = \frac{1}{2+\delta} + \alpha,$$

with $\alpha > 0$ sufficiently small so that $\eta < 1/2$, then we have

$$\int_0^T \mathbf{E}\left[\left|e^1_{\varepsilon,t}(x)\right|^2\right]dt = O(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0,$$

uniformly in $x \geq 0$, and hence

$$\int_0^T \int_0^{\varepsilon^{-\alpha/2}} \mathbf{E}\left[\left|e^1_{\varepsilon,t}(x)\right|^2\right]dxdt = \varepsilon^{-\alpha/2}O(\varepsilon^\alpha) = o(1). \tag{2.10.2}$$

For the spatial region $x > \varepsilon^{-\alpha/2}$, introduce a smooth cut-off function, ϕ_x, satisfying

$$\phi_x(y) = \begin{cases} 1, & \text{if } y \geq x \\ 0, & \text{if } y \leq x/2 \\ \in (0,1), & \text{if } x/2 < y < x. \end{cases}$$

Then the rapidly decreasing function

$$y \mapsto (\mu(t,y) - \mu(t,x))\, G_\varepsilon(x,y)\, \phi_x(y)$$

is supported on $(x, +\infty)$ and has maximum value $c_2\varepsilon^{-1/2}$, for a numerical constant $c_2 > 0$, therefore condition (iv) of Definition 2.10.1 gives

$$\int_0^T \mathbf{E}\left[\left|\xi_t\left((\mu(t,\cdot) - \mu(t,x))\, G_\varepsilon(x,\cdot)\, \phi_x\right)\right|^2\right]dt = \varepsilon^{-1}o(\exp\{-x\}), \quad \text{as } x \to \infty, \tag{2.10.3}$$

where we have normalised by the maximum value, hence the factor of ε^{-1}. On the region $(0, x/2)$, the map

$$y \mapsto (\mu(t,y) - \mu(t,x))\, G_\varepsilon(x,y)\,(1 - \phi_x(y))$$

and its derivatives are bounded by

$$p\left(\varepsilon^{-1/2}\right)\exp\left\{-x^2/8\varepsilon\right\},$$

for some polynomial, p, therefore (2.10.1) implies

$$\begin{aligned} \int_0^T \mathbf{E}\left[\left|\xi_t\left((\mu(t,\cdot) - \mu(t,x))\, G_\varepsilon(x,\cdot)\, \phi_x\right)\right|^2\right]dt &= O\big(p\big(\varepsilon^{-1/2}\big)\exp\big\{-x^2/8\varepsilon\big\}\big) \\ &= p\big(\varepsilon^{-1/2}\big)O\big(\exp\big\{-x\big\}\big) \end{aligned} \tag{2.10.4}$$

(by introducing another smooth cut-off function, which we will omit, to eliminate the mass on $y < 0$ when applying condition (vi)). Hence, (2.10.3) and (2.10.4) give a polynomial, q, with numerical coefficients such that

$$\int_0^T \int_{\varepsilon^{-\alpha/2}}^\infty \mathbf{E}\big[\left|e^1_{\varepsilon,t}(x)\right|^2\big] dxdt \le q\left(\varepsilon^{-1/2}\right)\int_{\varepsilon^{-\alpha/2}}^\infty e^{-x}dx,$$
$$= q\left(\varepsilon^{-1/2}\right)\exp\left\{-\varepsilon^{-\alpha/2}\right\} \to 0,$$

which gives the result when combined with (2.10.2).

The remainder terms vanish by the work in Lemma 2.7.3 (which remains valid, as it only relies on condition (v) of Definition 2.10.1, which is unchanged from Definition 2.0.3). □

Definition 2.10.3 ($\bar{o}(1)$). To simplify the notation in the following calculation, we will write

$$f_{t,\varepsilon} = \bar{o}(1)$$

if $(f_{t,\varepsilon})_{t\in[0,T]}$ is a function-valued stochastic process satisfying

$$\mathbf{E}\int_0^T \|f_{t,\varepsilon}\|^2_{L^2(0,\infty)}\,dt \to 0, \quad \text{as } \varepsilon \to 0.$$

Notice that if $f_{t,\varepsilon}, g_{t,\varepsilon} = \bar{o}(1)$ then the triangle inequality gives

$$t \mapsto \lambda^1(t,\cdot)\,f_{t,\varepsilon} + \lambda^2(t,\cdot)\,g_{t,\varepsilon} = \bar{o}(1),$$

whenever λ^1 and λ^2 are processes that are bounded in absolute value by a deterministic constant over the region $(t,x) \in [0,T]\times(0,\infty)$. Also notice that for any process h, by applying Young's inequality with any free parameter $\eta > 0$, we have

$$\mathbf{E}\int_0^T \|h_{t,\varepsilon} + \bar{o}(1)\|^2_{L^2(0,\infty)}\,dt \le (1+\eta)\,\mathbf{E}\int_0^T \|h_{t,\varepsilon}\|^2_{L^2(0,\infty)}\,dt + o(1), \tag{2.10.5}$$

as $\varepsilon \to 0$.

In short-hand notation, we have

$$d\partial_x^{-1}T_\varepsilon\xi_t = -\left(\mu T_\varepsilon\xi_t + R^1_\varepsilon\xi_t + e^1_{\varepsilon,t}\right)dt + \frac{1}{2}\partial_x\left(\sigma^2 T_\varepsilon\xi_t + e^3_{\varepsilon,t}\right)dt$$
$$- \left(\sigma_M T_\varepsilon\xi_t + R^2_\varepsilon\xi_t + e^2_{\varepsilon,t}\right)dW_t,$$

and with the notation of Definition 2.10.3 and Lemma 2.10.2 we can simplify to get

$$d\partial_x^{-1}T_\varepsilon\xi_t = -\left(\mu T_\varepsilon\xi_t + \bar{o}(1)\right)dt + \frac{1}{2}\partial_x\left(\sigma^2 T_\varepsilon\xi_t + e^3_{\varepsilon,t}\right)dt - \left(\sigma_M T_\varepsilon\xi_t + \bar{o}(1)\right)dW_t. \tag{2.10.6}$$

(Note, we cannot replace $e^3_{\varepsilon,t}$ by $\bar{o}(1)$ above as the derivative, ∂_x, is acting on it and we do not have control of $e^3_{\varepsilon,t}$ in the H^1 norm.) Applying Itô's formula to $(\partial_x^{-1}T_\varepsilon\xi_t)^2$ gives

(2.10.7)
$$d\left(\partial_x^{-1}T_\varepsilon\xi_t\right)^2 = -2\partial_x^{-1}T_\varepsilon\xi_t\left(\mu T_\varepsilon\xi_t + \bar{o}(1)\right)dt + \partial_x^{-1}T_\varepsilon\xi_t\partial_x\left(\sigma^2 T_\varepsilon\xi_t + e^3_{\varepsilon,t}\right)dt$$
$$- 2\partial_x^{-1}T_\varepsilon\xi_t\left(\sigma_M T_\varepsilon\xi_t + \bar{o}(1)\right)dW_t + \left(\sigma_M T_\varepsilon\xi_t + \bar{o}(1)\right)^2 dt.$$

Integrating over $\Omega \times (0,\infty)$ (and using Lemma 2.8.3) gives

$$\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_t\right\|_2^2 = \left\|\partial_x^{-1}T_\varepsilon f\right\|_2^2 + 2\mathbf{E}\int_0^t\int_0^\infty \partial_x^{-1}T_\varepsilon\xi_s\left(\mu T_\varepsilon\xi_s + \bar{o}(1)\right)dxds$$
$$+ \mathbf{E}\int_0^t\int_0^\infty \partial_x^{-1}T_\varepsilon\xi_s\partial_x\left(\sigma^2 T_\varepsilon\xi_s + e^3_{\varepsilon,t}\right)dxds$$
$$+ \mathbf{E}\int_0^t \left\|\sigma_M T_\varepsilon\xi_s + \bar{o}(1)\right\|_2^2 ds.$$

Evaluating the third term on the right-hand side (by using integration by parts and noting that $e^3_{\varepsilon,s}(0) = 0$) and applying Young's inequality and (2.10.5) with parameter $\eta > 0$ to the second and fourth terms we have

(2.10.8)
$$\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_t\right\|_2^2 \le \left\|\partial_x^{-1}T_\varepsilon f\right\|_2^2 + \eta^{-1}\mathbf{E}\int_0^t \left\|\partial_x^{-1}T_\varepsilon\xi_s\right\|_2^2 ds$$
$$+4\eta(1+\eta)\mathbf{E}\int_0^t\left\|\mu T_\varepsilon\xi_s\right\|_2^2 ds - \mathbf{E}\int_0^t\int_0^\infty T_\varepsilon\xi_s\left(\sigma^2 T_\varepsilon\xi_s + e^3_{\varepsilon,s}\right)dxds$$
$$+ (1+\eta)\mathbf{E}\int_0^t \left\|\sigma_M T_\varepsilon\xi_s\right\|_2^2 ds + o(1).$$

We can expand the third term on the right-hand side of (2.10.8) and apply Cauchy–Schwarz to get

$$\mathbf{E}\int_0^t\int_0^\infty T_\varepsilon\xi_s\left(\sigma^2 T_\varepsilon\xi_s + e^3_{\varepsilon,s}\right)dxds = \mathbf{E}\int_0^t\left\|\sigma T_\varepsilon\xi_s\right\|_2^2 ds + \mathbf{E}\int_0^t\int_0^\infty T_\varepsilon\xi_s \cdot e^3_{\varepsilon,s}dxds$$
$$\le \mathbf{E}\int_0^t\left\|\sigma T_\varepsilon\xi_s\right\|_2^2 ds + \eta\mathbf{E}\int_0^t\left\|T_\varepsilon\xi_s\right\|_2^2 ds + o(1),$$

where the final line is due to Young's inequality. Hence we have

(2.10.9)

$$\mathbf{E}\big\|\partial_x^{-1}T_\varepsilon\xi_t\big\|_2^2 \le \big\|\partial_x^{-1}T_\varepsilon f\big\|_2^2 + \eta^{-1}\mathbf{E}\int_0^t \big\|\partial_x^{-1}T_\varepsilon\xi_s\big\|_2^2\,ds$$
$$-\,\mathbf{E}\int_0^t\int_0^\infty\big(\sigma^2-(1+\eta)\sigma_M^2-\eta-4\eta\,(1+\eta)\mu^2\big)|T_\varepsilon\xi_s|^2dxds+o(1)\,.$$

By condition (i) of Definition 2.0.3, we can fix $\eta>0$ sufficiently small to guarantee the third term on the right-hand side of (2.10.9) is negative, whence

$$\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_t\right\|_2^2 \le \left\|\partial_x^{-1}T_\varepsilon f\right\|_2^2 + \mathbf{E}\int_0^t \eta^{-1}\left\|\partial_x^{-1}T_\varepsilon\xi_s\right\|_2^2 ds + o\left(1\right).$$

Fatou's lemma then implies that

$$\liminf_{\varepsilon\to0}\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_t\right\|_2^2 \le \limsup_{\varepsilon\to0}\left\|\partial_x^{-1}T_\varepsilon f\right\|_2^2 + \eta^{-1}\int_0^t \liminf_{\varepsilon\to0}\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_s\right\|_2^2 ds,$$

and so Gronwall's lemma and the linearity of the SPDE gives:

Theorem 2.10.4 (Regularity and uniqueness)**.** *If ξ satisfies Definition 2.10.1, then there is some numerical constant $\eta>0$ such that for every $t\in[0,T]$*

$$\liminf_{\varepsilon\to0}\mathbf{E}\left\|\partial_x^{-1}T_\varepsilon\xi_t\right\|_2^2 \le \left(1+e^{\eta^{-1}T}\right)\limsup_{\varepsilon\to0}\left\|\partial_x^{-1}T_\varepsilon f\right\|_2^2.$$

Hence, for a given initial condition, there is at most one process satisfying Definition 2.10.1.

CHAPTER 3

Boundary regularity

This chapter is a condensed version of the paper [**Led14**] and explores the regularity of the limit empirical process near the Dirichlet boundary. Throughout, we will assume that ν_0 has a density, V_0, that is smooth and compactly supported away from the origin and the correlation, ρ, is constant and less than one.

To measure the regularity of ν near the origin, we introduce the weighted L^2 norms

$$\|f\|_{L^2_a(0,\infty)} := \left(\int_0^\infty w_a(x)\,|f(x)|^2\,dx\right)^{1/2}, \quad a \in \mathbf{R}, \tag{3.0.1}$$

where we have the *weight function*

$$w_a(x) = x^a e^{-x}. \tag{3.0.2}$$

Varying a allows us to suppress or exaggerate the degeneracy of f at zero. The main theorem of this chapter can then be stated as:

Theorem 3.0.1 (Sharp boundary regularity)**.** *On a full subset of $\Omega \times [0,T]$, ν_t has an n^{th} order weak derivative, $\partial_x^n \nu_t : (0,\infty) \to \mathbf{R}$, for every $n \in \mathbf{N} \cup \{0\}$ — that is*

$$\zeta(\partial_x^n \phi) = (-1)^n \int_0^\infty \partial_x^n \nu_t(x)\,\phi(x)\,dx, \quad \textit{for every } \phi \in C_0^\infty(0,\infty)$$

— and these weak derivatives satisfies

$$\mathbf{E}\int_0^T \|\partial_x^n \nu_t\|^2_{L^2_a(0,\infty)}\,dt \begin{cases} < \infty, & \textit{if } a > a_c(n) \\ = \infty, & \textit{if } a < a_c(n) \end{cases},$$

where we have the critical value

$$a_c(n) = 2n - 2 - \gamma.$$

The parameter $\gamma \in (0,1]$ is an explicit function of the correlation parameter:

$$\gamma = \pi/\alpha - 1, \qquad \textit{where } \alpha \in [\pi/2, \pi] \textit{ is such that } \cos\alpha = -\rho^2.$$

Our strategy is to introduce derivatives and weights into the energy estimate and kernel smoothing technique from the previous chapter, which we do in Sections 3.2 and 3.3. Consequently, we find that the level of regularity in Theorem 3.0.1 is determined by the behaviour of

$$\mathbf{E}\int_0^T \|\partial_x^n R_\varepsilon \nu_t\|^2_{L^2_a(0,\infty)}\,dt, \quad \text{for small } \varepsilon. \tag{3.0.3}$$

In Section 3.4 we show how n and a must be balanced in order for this quantity to remain finite as ε vanishes. We will see that this is strongly connected to the second moment of the mass near the boundary, and so we require the following estimate:

Theorem 3.0.2 (Second moment boundary estimate)**.** *There exists positive constants $\varepsilon_0 > 0$ and $c_1 < c_2$ such that,*

$$c_1\varepsilon^{3+\gamma} \leq \mathbf{E}\left[|\nu_t(0,\varepsilon)|^2\right] \leq c_2\varepsilon^{3+\gamma}, \quad \textit{for all } t \in [0,T], 0 < \varepsilon < \varepsilon_0,$$

where $\gamma \in (0,1]$ is defined in Theorem 3.0.1.

The above inequality is proved in [**BHH$^+$11**, Lemma 3.5], however we will present an alternative method in Section 3.1 using time-reversibility of Brownian motion and the growth of harmonic functions in a wedge-shaped region. We will not prove the lower bound, instead we refer the reader to [**Jin10**, Lemma 5.1]. In both these references, the moment estimate is shown to be equivalent to the probability that a two-dimensional Brownian motion falls in the ε-apex of a wedge without having exited that domain. Explicit formulae are available for the law of this process [**Iye85, Met10, LS13b, LS13a**]. The method breaks down when the correlation is non-constant, because the corresponding two-dimensional process no longer has a tractable law. Hence the results in this chapter apply only to the basic model. In later chapters, weaker techniques are required to analyse the more complicated systems and this is reflected in the smaller decay exponent required in Definition 2.0.3, condition (iv).

In deriving the energy estimates we find that in order to gain control of $\|\nu\|_{a,n}$, we are required to control $\|\nu\|_{a-2,n-1}$. This allows us to employ an inductive proof of Theorem 3.0.1 in Section 3.6. However, for the base case we will take $n = -1$, which requires the introduction of an anti-derivative in Section 3.5. It is convenient to consider this case as the starting point because the norm $\|\nu\|_{a,-1}$ can be written in terms of the function $x \mapsto \nu_t(0,x)$, whose boundary behaviour we know explicitly from Theorem 3.0.2.

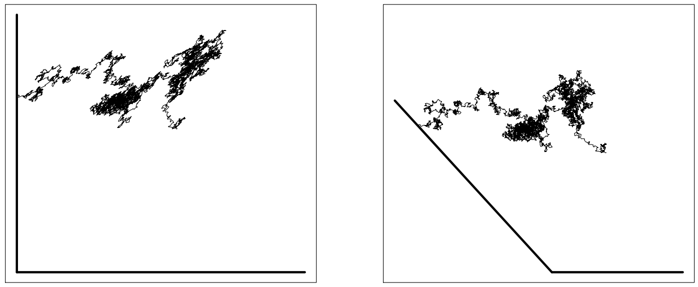


FIGURE 3.1.1. On the left is a Brownian path with correlated components stopped on exiting the quadrant. On the right is the image of the path under the map S_α. Here, we obtain uncorrelated components, but the domain is mapped to the wedge $\measuredangle_\alpha$.

Remark 3.0.3 (Regularity away from origin)**.** Theorem 3.0.1 gives that each ν_t has weak derivatives of all orders, hence ν has a denisty process taking values in $C^\infty(0,\infty)$. That is, the density is smooth away from the absorbing boundary.

3.1. The apex probability; proof of Theorem 3.0.2

Since the X^i are identically distributed and conditionally independent, for any measurable $S \subseteq (0,\infty)$ we have

$$\begin{aligned}\mathbf{E}\left[|\nu_t(S)|^2\right] &= \mathbf{E}\left[\mathbf{P}\left(X_t^1 \in S, t < \tau^1 \middle| W\right)\mathbf{P}\left(X_t^2 \in S, t < \tau^2 \middle| W\right)\right]\\ &= \mathbf{P}\left(X_t^1, X_t^2 \in S, t < \tau^1 \wedge \tau^2\right).\end{aligned}$$

Therefore to calculate the second moment of $\nu_t(S)$ we must consider the law of a pair of correlated particles killed upon exiting the positive quadrant. Note that the correlation is given by

$$\left[X^1, X^2\right] \equiv \rho^2 t.$$

By applying the following skew transformation, we can remove the correlation between the particles at the expense of altering our domain:

$$S_\alpha : (0,\infty)\times(0,\infty) \to \measuredangle_\alpha, \quad S_\alpha(x_1, x_2) := (x_1 + x_2\cos\alpha, x_2)$$

where $\measuredangle_\alpha \subseteq \mathbf{R}\times\mathbf{R}$ denotes the wedge of angle α defined in polar coordinates by

$$\measuredangle_\alpha = \{(r,\theta) : r > 0, 0 < \theta < \alpha\}.$$

If we choose $\alpha \in [\pi/2, \pi]$ such that $\cos\alpha = -\rho^2$ and define

$$B_t^1 := \left(1-\rho^2\right)^{-1} S_\alpha\left(X_t^1, X_t^2\right)_1 \quad \text{and} \quad B_t^2 := S_\alpha\left(X_t^1, X_t^2\right)_2,$$

then a simple calculation gives

$$\left[B^1\right] \equiv t \equiv \left[B^2\right] \quad \text{and} \quad \left[B^1, B^2\right] \equiv 0,$$

so we conclude that $\boldsymbol{B} = (B^1, B^2)$ is a Brownian motion and

$$\left(X_t^1, X_t^2\right) \in (0,\infty) \times (0,\infty) \quad \text{if and only if} \quad \boldsymbol{B}_t \in \measuredangle_\alpha.$$

Furthermore

$$S_\alpha\left((0,\varepsilon)\times(0,\varepsilon)\right) \subseteq \operatorname{apex}_\alpha\left(c_\rho\varepsilon\right) := \left\{(r,\theta) : 0 < r < c_\rho\varepsilon, 0 < \theta < \alpha\right\},$$

where c_ρ is some numerical constant depending only on ρ. Therefore we conclude:

Proposition 3.1.1 (Apex probability)**.** *For every $\varepsilon > 0$ and $t \in [0,T]$*

$$\mathbf{E}\left[\left|\nu_t(0,\varepsilon)\right|^2\right] = \mathbf{P}\left(\boldsymbol{B}_t \in \operatorname{apex}_\alpha\left(c_\rho\varepsilon\right), t < \tau\right), \tag{3.1.1}$$

where $\tau = \inf\left\{t > 0 : \boldsymbol{B}_t \notin \measuredangle_\alpha\right\}$.

Since each X_0^i is independent with bounded smooth density V_0 supported on a finite interval away from the origin, it follows that $\boldsymbol{B}_0$ has a bounded smooth density supported on some bounded region whose closure is contained in the interior of $\measuredangle_\alpha$. To be precise, if we denote the density of $\boldsymbol{B}_0$ by f_0, then

$$\operatorname{support}(f_0) = S_\alpha\left(\operatorname{support}\left(V_0 \otimes V_0\right)\right),$$

see Figure 3.1.2.

Reversing time. We will exploit the time-reversibility of Brownian motion to rewrite (3.1.1) in terms of the law of a Brownian motion started near the apex of $\measuredangle_\alpha$. The argument presented here is a modification of the proof of Hunt's switching identity in [**Ber98**, Chapter II, Proposition 1].

First, let $S \subseteq \mathbf{R}\times\mathbf{R}$ and $\tau(x) := \inf\left\{s > 0 : x + \boldsymbol{B}_s \notin \measuredangle_\alpha\right\}$. We will write $\mathbf{P}_x$ to indicate that we are conditioning on the event $\boldsymbol{B}_0 = x$. By a simple change of variable we have

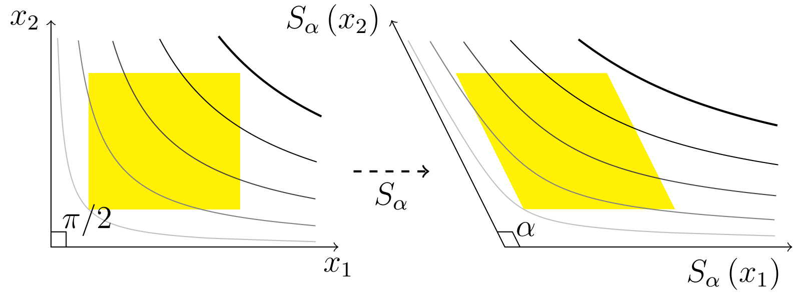


FIGURE 3.1.2. The action of the map S_α from the positive quadrant to $\measuredangle_\alpha$. The yellow region on the left is the support of $V_0 \otimes V_0$, and on the right it is the support of f_0. The curves are the contour lines of the harmonic map h in the two regions; the darker the line, the larger the value of h.

$$\begin{aligned}
\mathbf{P}\left(\boldsymbol{B}_t \in S, t<\tau\right) &= \int_{\mathbf{R}\times\mathbf{R}} \mathbf{P}_x\left(\boldsymbol{B}_t \in S, t<\tau\right) f_0\left(x\right) dx \\
&= \int_{\mathbf{R}\times\mathbf{R}} \mathbf{E}_x\left[\chi_S\left(\boldsymbol{B}_t\right) \mathbf{1}_{t<\tau} f_0\left(x\right) dx\right] \\
&= \mathbf{E}_0\left[\int_{\mathbf{R}\times\mathbf{R}} \chi_S\left(x+\boldsymbol{B}_t\right) \mathbf{1}_{t<\tau(x)} f_0\left(x\right) dx\right] \\
&= \mathbf{E}_0\left[\int_{\mathbf{R}\times\mathbf{R}} \chi_S\left(y\right) \mathbf{1}_{t<\tau(y-\boldsymbol{B}_t)} f_0\left(y-\boldsymbol{B}_t\right) dy\right] \\
&= \int_S \mathbf{E}_0\left[\mathbf{1}_{t<\tau(y-\boldsymbol{B}_t)} f_0\left(y-\boldsymbol{B}_t\right)\right] dy,
\end{aligned}$$

where χ_S denotes the characteristic function of the set S.

Now, with t fixed, define the Brownian motion $\hat{\boldsymbol{B}}_s := \boldsymbol{B}_{t-s} - \boldsymbol{B}_t$ (one easily checks that $\hat{\boldsymbol{B}}_0 = 0$ and $\hat{\boldsymbol{B}}$ has independent increments with the correct joint distribution) and define the exit time

$$\hat{\tau}\left(x\right) := \inf\left\{s>0 : x+\hat{\boldsymbol{B}}_s \notin \measuredangle_\alpha\right\}.$$

Notice that

$$\begin{gathered}
t < \tau\left(y-\boldsymbol{B}_t\right) = \inf\left\{s>0 : y-\boldsymbol{B}_t+\boldsymbol{B}_s \notin \measuredangle_\alpha\right\} \\
\text{if and only if} \quad t < \hat{\tau}\left(y\right),
\end{gathered}$$

and therefore

$$\mathbf{P}\left(\boldsymbol{B}_t \in S, t<\tau\right) = \int_S \mathbf{E}_0\left[\mathbf{1}_{t<\hat{\tau}(y)} f_0\left(y+\hat{\boldsymbol{B}}_t\right)\right] dy.$$

However, since the processes $\hat{\boldsymbol{B}}$ and $\boldsymbol{B}$ have the same law and, because f_0 vanishes on $\measuredangle_\alpha$,

$$\mathbf{1}_{t<\tau(y)} f_0\big(y+\boldsymbol{B}_t\big) = f_0\big(y+\boldsymbol{B}_{t\wedge\tau(y)}\big),$$

we conclude:

Proposition 3.1.2 (Switching identity)**.** *For every $t \in [0,T]$ and measurable $S \subseteq \mathbf{R}\times\mathbf{R}$*

$$\mathbf{P}\left(\boldsymbol{B}_t \in S, t<\tau\right) = \int_S \mathbf{E}_y\left[f_0\left(\boldsymbol{B}_{t\wedge\tau}\right)\right] dy.$$

A positive harmonic function. We complete the proof by bounding f_0 by a harmonic function and applying a martingale argument to Proposition 3.1.2.

Begin by defining the function $h : \overline{\measuredangle_\alpha} \to [0,\infty)$ given in polar coordinates by

$$h\left(r,\theta\right) = r^{\pi/\alpha}\sin\left(\pi\theta/\alpha\right).$$

A straightforward calculation verifies that

$$\begin{cases} \partial_{xx}h + \partial_{yy}h = 0, & \text{in } \measuredangle_\alpha \\ h = 0, & \text{on } \partial\measuredangle_\alpha. \end{cases}$$

Since the closure of the support of f_0 is contained within the interior of $\measuredangle_\alpha$ (recall Figure 3.1.2) it is possible to find a constant $h_0 > 0$ such that

$$h\left(x\right) \geq h_0, \quad \text{for every } x \in \overline{\text{support}\left(f_0\right)}.$$

Therefore, setting $K_1 := h_0^{-1}\left\|f_0\right\|_\infty$ gives

$$f_0\left(x\right) \leq K_1 h\left(x\right), \quad \text{for every } x \in \measuredangle_\alpha,$$

and so we can conclude that

$$\mathbf{P}\left(\boldsymbol{B}_t \in S, t<\tau\right) \leq K_1 \int_S \mathbf{E}_y\left[h\left(\boldsymbol{B}_{t\wedge\tau}\right)\right] dy. \tag{3.1.2}$$

Itô's formula implies that

$$h\left(\boldsymbol{B}_t\right) = h\left(\boldsymbol{B}_0\right) + \int_0^t \nabla h\left(\boldsymbol{B}_s\right)\cdot d\boldsymbol{B}_s,$$

which is a martingale since $|\nabla h| = O\left(r^{\pi/\alpha-1}\right)$, as $r\to\infty$, therefore

$$\begin{aligned} \mathbf{E}\int_0^t \left|\nabla h\left(\boldsymbol{B}_s\right)\right|^2 ds &\leq K_2 \int_0^t\int_0^\infty r^{2\pi/\alpha-2}\left(2\pi s\right)^{-1}\exp\left\{-r^2/2s\right\} r\,dr\,ds \\ &\leq K_3 \int_0^t s^{\pi/\alpha-3/2}ds \cdot \int_0^\infty \exp\left\{-r^2/2s\right\} r^{\pi/\alpha}dr, \end{aligned}$$

and this is finite since $\pi/\alpha \geq 1$, and here K_2 and K_3 are numerical constants. This allows us to evaluate the expectation in (3.1.2) and gives:

Proposition 3.1.3 (Harmonic bound)**.** *There exists a numerical constant $K_1 > 0$ (depending only on V_0 and ρ), such that for every $t \in [0,T]$ and measurable $S \subseteq \mathbf{R} \times \mathbf{R}$*

$$\mathbf{P}\left(\boldsymbol{B}_t \in S, t < \tau\right) \leq K_1 \int_S h\left(y\right) dy.$$

By setting $S = \text{apex}_\alpha\left(c_\rho \varepsilon\right)$ and noting that $h\left(y\right) \leq y^{\pi/\alpha}$ and

$$\text{volume}\left(\text{apex}_\alpha\left(c_\rho \varepsilon\right)\right) \leq \pi c_\rho^2 \varepsilon^2,$$

we obtain

$$\mathbf{P}\left(\boldsymbol{B}_t \in \text{apex}_\alpha\left(c_\rho \varepsilon\right), t < \tau\right) = O\big(\varepsilon^{2+\pi/\alpha}\big),$$

and so Proposition 3.1.1 completes the proof of Theorem 3.0.2. □

3.2. Kernel smoothing for higher order derivatives

First we will extend Proposition 2.6.6 for a deterministic tempered distribution:

Proposition 3.2.1. *Let $\zeta \in \mathscr{S}'$, $n \in \mathbf{N} \cup \{0\}$ and $a \in \mathbf{R}$. If*

$$\liminf_{\varepsilon \to 0} \left\|\partial_x^n T_\varepsilon \zeta\right\|_{L_a^2(0,\infty)} < \infty,$$

then ζ has an n^{th} order weak derivative, $\partial_x^n \zeta : (0,\infty) \to \mathbf{R}$, such that

$$\zeta\left(\partial_x^n \phi\right) = (-1)^n \int_0^\infty \partial_x^n \zeta\left(x\right) \phi\left(x\right) dx, \quad \textit{for every } \phi \in C_0^\infty\left(0,\infty\right),$$

and

$$\left\|\partial_x^n \zeta\right\|_{L_a^2(0,\infty)} \leq \liminf_{\varepsilon \to 0} \left\|\partial_x^n T_\varepsilon \zeta\right\|_{L_a^2(0,\infty)}.$$

Proof. Firstly note that if $\phi \in C_0^\infty\left(0,\infty\right)$ then $\sqrt{w_a}\phi \in C_0^\infty\left(0,\infty\right)$. So by Proposition 2.6.2

$$\begin{aligned} \zeta\left(\partial_x^n\left(\sqrt{w_a}\phi\right)\right) &= \lim_{\varepsilon \to 0} \int_0^\infty \partial_x^n\big(\sqrt{w_a\left(x\right)}\phi\left(x\right)\big) T_\varepsilon \zeta\left(x\right) dx \\ &= (-1)^n \lim_{\varepsilon \to 0} \int_0^\infty \sqrt{w_a\left(x\right)}\phi\left(x\right) \partial_x^n T_\varepsilon \zeta\left(x\right) dx, \end{aligned} \tag{3.2.1}$$

therefore the linear functional $f : C_0^\infty\left(0,\infty\right) \to \mathbf{R}$ defined by

$$f\left(\phi\right) := \zeta\left(\partial_x^n\left(\sqrt{w_a}\phi\right)\right)$$

satisfies

$$|f(\phi)| \leq \liminf_{\varepsilon\to 0} \|\partial_x^n T_\varepsilon \zeta\|_{L_a^2(0,\infty)} \|\phi\|_{L^2(0,\infty)}, \tag{3.2.2}$$

by applying Cauchy–Schwarz to (3.2.1) — recall from (3.0.1) that $L^2(0,\infty) = L_0^2(0,\infty)$ by definition. Therefore the Hahn–Banach Theorem [**Bha09**, Chapter 4, Theorem 3] allows us to extend f to an element of the dual of $L^2(0,\infty)$, then the Riesz representation theorem [**RS81**, Theorem II.4] gives $u \in L^2(0,\infty)$ such that

$$f(\phi) = \int_0^\infty \phi(x)\, u(x)\, dx, \quad \text{for every } \phi \in L^2(0,\infty).$$

In particular we have

$$\zeta(\partial_x^n \phi) = \int_0^\infty \phi(x)\, w_a(x)^{-\frac{1}{2}}\, u(x)\, dx, \quad \text{for every } \phi \in C_0^\infty(0,\infty),$$

hence we have the required weak derivative by setting

$$\partial_x^n \zeta := (-1)^n\, w_a^{-\frac{1}{2}} u \in L_a^2(0,\infty).$$

Inequality (3.2.2) then completes the proof since we have

$$\|\partial_x^n \zeta\|_{L_a^2(0,\infty)} = \sup_{\|\phi\|_{L^2(0,\infty)}=1} \int_0^\infty \sqrt{w_a}\,\partial_x^n \zeta(x)\,\phi(x)\, dx = \sup_{\|\phi\|_{L^2(0,\infty)}=1} f(\phi).$$

□

With the deterministic result in place, it is now straightforward to extend to $\Omega \times [0,T]$ for the limit empirical process:

Proposition 3.2.2 (Generalised lim inf proposition)**.** *Let $n \in \mathbf{N} \cup \{0\}$ and $a \in \mathbf{R}$. If*

$$\liminf_{\varepsilon\to 0} \mathbf{E} \int_0^T \|\partial_x^n T_\varepsilon \nu_t\|_{L_a^2(0,\infty)}^2\, dt < \infty,$$

then ν is n-times differentiable for almost all $(\omega,t) \in \Omega \times [0,T]$, that is there exists $\partial_x^n \nu_t : (0,\infty) \to \mathbf{R}$ such that

$$\nu_t(\partial_x^n \phi) = (-1)^n \int_0^\infty \partial_x^n \nu_t(x)\, \phi(x)\, dx, \quad \textit{for every } \phi \in C_0^\infty(0,\infty),$$

and we have

$$\mathbf{E} \int_0^T \|\partial_x^n \nu_t\|_{L_a^2(0,\infty)}^2\, dt < \infty.$$

Proof. By Fatou's lemma we have

$$\mathbf{E}\left[\liminf_{\varepsilon\to 0}\int_0^T \|\partial_x^n T_\varepsilon \nu_t\|^2_{L^2_a(0,\infty)}\,dt\right] < \infty,$$

so we can apply Proposition 3.2.1 on a full subset of $\Omega\times[0,T]$, which gives the existence of $\partial_x^n \nu_t$ and that

$$\|\partial_x^n \nu_t\|^2_{L^2_a(0,\infty)} \le \liminf_{\varepsilon\to 0}\|\partial_x^n T_\varepsilon \nu_t\|^2_{L^2_a(0,\infty)}.$$

Integrating this inequality over the full subset and applying Fatou's lemma completes the proof. □

Finally, our assumptions on ν_0 gives us control over $T_\varepsilon\nu_0$ regardless of the choice of (a,n). Consequently, the regularity in Theorem 3.0.1 is not determined by the initial condition, rather it is determined by the behaviour of the remainder term as given later in Proposition 3.4.1.

Proposition 3.2.3 (Initial condition)**.** *With ν_0 satisfying the assumption at the start of the chapter (that is, ν_0 has a density $V_0\in C_0^\infty(0,\infty)$) we have*

$$\liminf_{\varepsilon\to 0}\|\partial_x^n T_\varepsilon \nu_0\|^2_{L^2_a(0,\infty)} < \infty, \quad n\ge 0 \text{ and } a>-1.$$

Proof. Straightforward manipulation gives

$$\begin{aligned}
|\partial_x^n T_\varepsilon \nu_0(x)| &= \left|\int_0^\infty \partial_x^n G_\varepsilon(x,y)\,V_0(y)\,dy\right| \\
&\le \left|\int_0^\infty \partial_x^n \psi_\varepsilon(x-y)\,V_0(y)\,dy\right| + \left|\int_0^\infty \partial_x^n \psi_\varepsilon(x+y)\,V_0(y)\,dy\right| \\
&= \left|\int_0^\infty \psi_\varepsilon(x-y)\,\partial_y^n V_0(y)\,dy\right| + \left|\int_0^\infty \psi_\varepsilon(x+y)\,\partial_y^n V_0(y)\,dy\right|.
\end{aligned}$$

Applying the Cauchy–Schawarz inequality gives

$$\begin{aligned}
|\partial_x^n T_\varepsilon \nu_0(x)|^2 &\le \int_0^\infty \psi_\varepsilon(x-y)\,dy\cdot\int_0^\infty \psi_\varepsilon(x-y)\left|\partial_y^n V_0(y)\right|^2 dy \\
&\quad + \int_0^\infty \psi_\varepsilon(x+y)\,dy\cdot\int_0^\infty \psi_\varepsilon(x+y)\left|\partial_y^n V_0(y)\right|^2 dy \\
&\le \int_{x_\star}^{x^\star} \left[\psi_\varepsilon(x-y)+\psi_\varepsilon(x+y)\right]\left|\partial_y^n V_0(y)\right|^2 dy.
\end{aligned}$$

Notice that w_a is bounded (independently of ε) on $x \in (x_\star/2, \infty)$, so by substituting in the above inequality and integrating over $x > x_\star/2$

$$\int_{x_\star/2}^{\infty} w_a\left(x\right) \left|\partial_x^n T_\varepsilon \nu_0\left(x\right)\right|^2 dx \leq c_1(a) \left\|\partial_x^n V_0\right\|_{L_0^2(0,\infty)}^2,$$

for some constant $c_1(a)$. If $x < x_\star/2$ and $y > x_\star$, then $|x-y| < x_\star/2$, so

$$\begin{aligned}\int_0^{x_\star/2} w_a\left(x\right) \left|\partial_x^n T_\varepsilon \nu_0\left(x\right)\right|^2 dx &\leq 2\psi_\varepsilon\left(x_\star/2\right) \int_0^{x_\star/2} w_a\left(x\right) dx \int_{x_\star}^{x^\star} \left|\partial_y^n V_0\left(y\right)\right|^2 dy \\ &= o\left(1\right), \quad \text{as } \varepsilon \to 0,\end{aligned}$$

since both integrals are finite and

$$\psi_\varepsilon\left(x_\star/2\right) = \left(2\pi\varepsilon\right)^{-1/2} \exp\left\{-x_\star^2/8\varepsilon\right\}.$$

$\square$

3.3. Energy estimation with weighted norms

To be in a position to apply the results of the previous section, let us return to the calculations in Section 2.8 and generalise the absorbing Gaussian heat kernel by defining:

Definition 3.3.1 (Alternating heat kernel)**.** For $n \in \mathbf{N} \cup \{0\}$ define

$$\begin{aligned}G_\varepsilon^n\left(x,y\right) &:= \psi_\varepsilon\left(x-y\right) + \left(-1\right)^{n+1} \psi_\varepsilon\left(x+y\right) \\ &= \left(2\pi\varepsilon\right)^{-\frac{1}{2}} \left[\exp\left\{-\frac{\left(x-y\right)^2}{2\varepsilon}\right\} + \left(-1\right)^{n+1} \exp\left\{-\frac{\left(x+y\right)^2}{2\varepsilon}\right\}\right],\end{aligned}$$

for $x, y \in \mathbf{R}$ and $\varepsilon > 0$. Hence, for n even G_ε^n is the absorbing heat kernel and for n odd G_ε^n is the reflecting heat kernel.

Similar to Lemma 2.8.1, we have the following computational lemma which allows us to exchange derivatives:

Lemma 3.3.2 (Switching derivatives)**.** *Let $\zeta \in \mathscr{S}'$. For all $x, y \in \mathbf{R}$, $n \in \mathbf{N} \cup \{0\}$ and $\varepsilon > 0$ we have*

(i) $\partial_y^n G_\varepsilon^n(x,y) = (-1)^n \partial_x^n G_\varepsilon(x,y)$,
(ii) $\zeta\left(\partial_y^n G_\varepsilon^n(x,\cdot)\right) = (-1)^n \partial_x^n T_\varepsilon \zeta(x)$,
(iii) $\partial_y^n G_\varepsilon^n(x,0) = 0$,
(iv) $\partial_y^{n+1} G_\varepsilon^n(x,y) = -\left(-1\right)^n \partial_x^{n+1} G_\varepsilon(x,y) - 2\left(-1\right)^n \partial_x^{n+1} \psi_\varepsilon(x+y)$,
(v) $\zeta\left(\partial_y^{n+1} G_\varepsilon^n\left(x,\cdot\right)\right) = -\left(-1\right)^n \partial_x^{n+1} T_\varepsilon \zeta(x) - \left(-1\right)^n \partial_x^{n+1} R_\varepsilon \zeta(x)$,
(vi) $\partial_y^{n+2} G_\varepsilon^n(x,y) = \left(-1\right)^n \partial_x^{n+2} G_\varepsilon(x,y)$,

(vii) $\zeta\left(\partial_y^{n+2}G_\varepsilon^n(x,\cdot)\right) = (-1)^n\partial_x^{n+2}T_\varepsilon\zeta(x)$.

Proof. For (i) and (ii), notice that

$$\begin{aligned}\partial_y^n G_\varepsilon^n(x,y) &= \partial_y^n\left[\psi_\varepsilon(x-y) + (-1)^{n+1}\psi_\varepsilon(x+y)\right]\\ &= (-1)^n\psi_\varepsilon^{(n)}(x-y) + (-1)^{n+1}\psi_\varepsilon^{(n)}(x+y)\\ &= (-1)^n\partial_x^n\left[\psi_\varepsilon(x-y) - \psi_\varepsilon(x+y)\right] = (-1)^n\partial_x^n G_\varepsilon(x,y),\end{aligned}$$

where, in the second line, $\psi_\varepsilon^{(n)}$ denotes the n^{th} derivative of ψ_ε, then it is clear that this line implies (iii).

For the next derivative

$$\begin{aligned}\partial_y^{n+1}G_\varepsilon^n(x,y) &= (-1)^{n+1}\psi_\varepsilon^{(n+1)}(x-y) + (-1)^{n+1}\psi_\varepsilon^{(n+1)}(x+y)\\ &= -(-1)^n\partial_x^n\left[\psi_\varepsilon(x-y) + \psi_\varepsilon(x+y)\right]\\ &= -(-1)^n\partial_x^n\left[\psi_\varepsilon(x-y) - \psi_\varepsilon(x+y)\right] - 2(-1)^n\partial_x^n\psi_\varepsilon(x+y),\end{aligned}$$

which gives (iv) and (v). The proof of (vi) and (vii) follows similarly. ☐

From condition (ii) we see that $y \mapsto \partial_y^n G_\varepsilon^n(x,y)$ is an element of C^{test}, for every value of $x > 0$. Setting this function into the basic SPDE, applying Lemma 3.3.2 and cancelling the factor of $(-1)^n$ gives, with probability 1,

$$\begin{aligned}\partial_x^n T_\varepsilon\nu_t(x) = \partial_x^n T_\varepsilon\nu_0(x) + \frac{1}{2}\int_0^t \partial_x^{n+2}T_\varepsilon\nu_s(x)\,ds - \rho\int_0^t \partial_x^{n+1}T_\varepsilon\nu_s(x)\,dW_s,\\ -\rho\int_0^t \partial_x^{n+1}R_\varepsilon\nu_s(x)\,dW_s,\end{aligned} \tag{3.3.1}$$

for every $x > 0$, $t \in [0,T]$ and $n \geq 0$. (In principle we have just differentiated (2.8.1) n times with respect to x. Doing so would require us to check that the derivative and stochastic integral can be interchanged. Our method avoids the need to verify that.)

By applying Itô's formula to the square of $\partial_x^n T_\varepsilon\nu_\cdot(x)$ we obtain

$$\begin{aligned}\left(\partial_x^n T_\varepsilon\nu_t(x)\right)^2 = \left(\partial_x^n T_\varepsilon\nu_0(x)\right)^2 + \int_0^t \partial_x^n T_\varepsilon\nu_s(x)\cdot\partial_x^{n+2}T_\varepsilon\nu_s(x)\,ds\\ + \rho\int_0^t\left(\partial_x^{n+1}T_\varepsilon\nu_s(x) + \partial_x^{n+1}R_\varepsilon\nu_s(x)\right)^2 ds + M_t(x),\end{aligned} \tag{3.3.2}$$

where M is given by

$$M_t(x) = -2\rho\int_0^t \partial_x^n T_\varepsilon\nu_s(x)\cdot\left(\partial_x^{n+1}T_\varepsilon\nu_s(x) + \partial_x^{n+1}R_\varepsilon\nu_s(x)\right)dW_s.$$

Our aim is to multiply (3.3.2) throughout by w_a and integrate over $(0,\infty)\times\Omega$. In order to do this we need the following integrability results, which are analogous to those in Lemma 2.8.3. To simplify notation, for the rest of this chapter we will write $\|f\|_{a,n}$ to denote $\|\partial_x^n f\|_{L_a^2(0,\infty)}$

Lemma 3.3.3 (Integrability results)**.** *For every $n \geq 0$ and $a > a_c(n) + 2$ we have:*

(i) $\mathbf{E}\int_0^T\int_0^\infty w_a(x)\left|\partial_x^n T_\varepsilon \nu_t(x)\right|^2 dxdt < \infty$,
(ii) $\mathbf{E}\int_0^T\int_0^\infty w_a(x)\left|\partial_x^{n+1} R_\varepsilon \nu_t(x)\right|^2 dxdt < \infty$,
(iii) For every $t \in [0,T]$ and almost all $x > 0$, $\mathbf{E}M_t(x) = 0$,
(iv) There exists a constant $\theta(a) > 0$ such that for every $t \in [0,T]$

$$\int_0^\infty w_a(x)\,\partial_x^n T_\varepsilon\nu_t(x)\,\partial_x^{n+2}T_\varepsilon\nu_t(x)\,dx$$
$$\leq -\|T_\varepsilon\nu_t\|_{a,n+1}^2 + \theta(a)\big(\|T_\varepsilon\nu_t\|_{a-2,n}^2 + \|T_\varepsilon\nu_t\|_{a-1,n}^2 + \|T_\varepsilon\nu_t\|_{a,n}^2\big).$$

Proof. Since w_a has an exponential tail and is integrable at the origin (because $a > -1$), the proof of (i), (ii) and (iii) follows by the same work as in Lemma 2.8.3.

The new result is (iv). To prove this, first note that

$$\partial_x w_a(x) = a w_{a-1}(x) - w_a(x)$$

and

$$\partial_{xx} w_a(x) = a(a-1)\,w_{a-2}(x) - 2a w_{a-1}(x) + w_a(x)\,.$$

Also note that for a general $f : [0,\infty) \to \mathbf{R}$ integration by parts gives

$$\int_0^\infty w_a f \partial_{xx} f dx = -\|f\|_{a,1}^2 - \int_0^\infty \partial_x w_a f \partial_x f dx,$$

provided $w_a(0+)f(0+)\partial_x f(0+) = 0$ and

$$\int_0^\infty \partial_x w_a f \partial_x f dx = -\int_0^\infty \partial_{xx} w_a f^2 dx - \int_0^\infty \partial_x w_a f \partial_x f dx$$
$$= -\frac{1}{2}\int_0^\infty \partial_{xx} w_a f^2 dx,$$

provided $\partial_x w_a(0+)f(0+)^2 = 0$. So if both boundary conditions hold we have

$$\int_0^\infty w_a f \partial_{xx} f dx = -\|f\|_{a,1}^2 + \frac{a(a-1)}{2}\|f\|_{a-2,0}^2 - a\|f\|_{a-1,0}^2 + \frac{1}{2}\|f\|_{a,0}^2$$
$$\leq -\|f\|_{a,1}^2 + \theta(a)\left(\|f\|_{a-2,0}^2 + \|f\|_{a-1,0}^2 + \|f\|_{a,0}^2\right)$$

where it is sufficient to take

$$\theta(a) = \frac{1}{2}\left|a(a-1)\right| + |a| + \frac{1}{2}.$$

Therefore we have the result if we can verify the two boundary conditions for the case $f = \partial_x^n T_\varepsilon \nu_t$.

First note that if $n \geq 1$, then

$$a > a_c(1) + 2 = 2 - \gamma > 1,$$

so

$$w_a(0+) = 0 \quad \text{and} \quad \partial_x w_a(0+) = 0,$$

which is sufficient. In the case $n = 0$, $f = T_\varepsilon \nu_t$ and Proposition A.0.6 gives

$$T_\varepsilon \nu_t(x) = O(x), \quad \text{as } x \to 0,$$

which is sufficient as

$$w_a(x) = O\big(x^{-\gamma}\big), \quad \partial_x w_a(x) = O\big(x^{-1-\gamma}\big),$$

because $a > a_c(0) + 2 > -\gamma$. (Note that the big O's depend on ε, but this is not a problem as ε is held fixed in this result.) $\square$

Applying the above lemma gives

$$\begin{aligned}\mathbf{E}\left\|T_\varepsilon \nu_t\right\|_{a,n}^2 \leq \left\|T_\varepsilon \nu_0\right\|_{a,n}^2 &- \mathbf{E}\int_0^t \left\|T_\varepsilon \nu_s\right\|_{a,n+1}^2 ds \\ &+ \rho\mathbf{E}\int_0^t \left\|T_\varepsilon \nu_s + R_\varepsilon \nu_s\right\|_{a,n+1}^2 ds \\ &+ \theta(a)\,\mathbf{E}\int_0^t \Big(\left\|T_\varepsilon \nu_s\right\|_{a-2,n}^2 + \left\|T_\varepsilon \nu_s\right\|_{a-1,n}^2 + \left\|T_\varepsilon \nu_s\right\|_{a,n}^2\Big) ds.\end{aligned}$$

Young's inequality lets us estimate the final term; with an arbitrary $\lambda \in (\rho, 1)$ we have

$$\begin{aligned}\left(T_\varepsilon \nu_s + R_\varepsilon \nu_s\right)^2 &\leq \left(1 + \frac{\lambda - \rho}{\rho}\right)\left(T_\varepsilon \nu_s\right)^2 + \left(1 + \frac{\rho}{\lambda - \rho}\right)\left(R_\varepsilon \nu_s\right)^2 \\ &\leq \frac{\lambda}{\rho}\left(T_\varepsilon \nu_s\right)^2 + \frac{\lambda}{\lambda - \rho}\left(R_\varepsilon \nu_s\right)^2,\end{aligned}$$

hence

$$(1-\lambda)\,\mathbf{E}\int_0^t \|T_\varepsilon\nu_s\|_{a,n+1}^2\,ds \le \|T_\varepsilon\nu_0\|_{a,n}^2 + \frac{\lambda\rho}{\lambda-\rho}\mathbf{E}\int_0^t \|R_\varepsilon\nu_s\|_{a,n+1}^2\,ds$$
$$+\,\theta\,(a)\sum_{i=0}^{2}\mathbf{E}\int_0^t \|T_\varepsilon\nu_s\|_{a-i,n}^2\,ds.$$

By replacing n by $n-1$ and using the fact that $a_c\,(n-1)=a_c\,(n)-2$, we have the following:

Proposition 3.3.4 (Energy estimate)**.** *Let $n\ge 1$ and $a>a_c\,(n)$, then there exists a constant $c\,(a)\ge 0$ such that*

$$\mathbf{E}\int_0^t \|T_\varepsilon\nu_s\|_{a,n}^2\,ds \le c\,(a)\,\|T_\varepsilon\nu_0\|_{a,n-1}^2 + c\,(a)\,\mathbf{E}\int_0^t \|R_\varepsilon\nu_s\|_{a,n}^2\,ds$$
$$+\,c\,(a)\sum_{i=0}^{2}\mathbf{E}\int_0^t \|T_\varepsilon\nu_s\|_{a-i,n-1}^2\,ds,$$

for all $t\in[0,T]$ and $\varepsilon>0$.

3.4. Controlling the remainder process

Here, we will show how a and n affect the quantity

$$\mathbf{E}\int_0^T \|R_\varepsilon\nu_t\|_{a,n}^2\,dt$$

and how this is connected to the boundary estimate in Theorem 3.0.2.

Since ν_t is a measure, we can write out $R_\varepsilon\nu_t$ explicitly and with that we obtain

$$\|R_\varepsilon\nu_t\|_{a,n}^2 = 4\int\limits_{(0,\infty)^3} w_a(x)\,\partial_x^n\psi_\varepsilon(x+y_1)\,\partial_x^n\psi_\varepsilon(x+y_2)\,\nu_t(dy_1)\,\nu_t(dy_2)\,dx.$$

Recalling that

$$\psi_\varepsilon\,(z)=(2\pi\varepsilon)^{-\frac{1}{2}}\exp\left\{-\frac{z^2}{2\varepsilon}\right\},$$

we have a polynomial of degree n with numerical coefficients such that

$$\partial_x^n\psi_\varepsilon\,(z)=p_n\big(\varepsilon^{-1}z\big)\psi_\varepsilon\,(z)\,.$$

Writing $|p_n|$ for the polynomial obtained by replacing the coefficients in p_n by their absolute values, we see from the above that

$$\|R_\varepsilon \nu_t\|_{a,n}^2 \le 4 \int\limits_{(0,\infty)^3} w_a(x)\,|p_n|\big(\varepsilon^{-1}(x+y_1)\big)\,|p_n|\big(\varepsilon^{-1}(x+y_1)\big) \times \psi_\varepsilon(x+y_1)\,\psi_\varepsilon(x+y_2)\,\nu_t(dy_1)\,\nu_t(dy_2)\,dx \tag{3.4.1}$$

Our strategy is to split the integral into the region $(0,\varepsilon^\eta)^3$ and its complement. Here, η is a free parameter which we will specify later, but for now we will fix $\eta < 1$.

Denoting the integrand in (3.4.1) by I_ε, notice that for $x, y_1, y_2 < \varepsilon^\eta$ we have

$$I_\varepsilon \le w_a(x)\,|p_n|\left(2\varepsilon^{-(1-\eta)}\right)^2 \psi_\varepsilon(0)^2 \le c_3 \varepsilon^{-2n(1-\eta)-1} w_a(x),$$

for a numerical constant $c_3 > 0$ and small ε, since $\psi_\varepsilon(0) = O\big(\varepsilon^{-1/2}\big)$ and the term of highest degree in the polynomial dominates. Hence we have

$$\mathbf{E} \int\limits_{(0,\varepsilon^\eta)^3} I_\varepsilon \nu_t(dy_1)\,\nu_t(dy_2)\,dx \le 4c_3\varepsilon^{-2n(1-\eta)-1}\mathbf{E}\left[\nu_t(0,\varepsilon^\eta)^2\right]\int_0^{\varepsilon^\eta} w_a(x)\,dx,$$

and provided that we restrict $a > -1$ so that the final term is integrable, Theorem 3.0.2 gives

$$\mathbf{E}\int_0^T \int\limits_{(0,\varepsilon^\eta)^3} I_\varepsilon \nu_t(dy_1)\,\nu_t(dy_2)\,dxdt = O\left(\varepsilon^{(a+2n+4+\gamma)\eta-(2n+1)}\right). \tag{3.4.2}$$

Now consider the remaining part of the integral where one of x, y_1 and y_2 must be not less than ε^η. In this case we have

$$x^2 + y_1^2 + y_2^2 \ge \varepsilon^{2\eta}.$$

By applying Proposition A.0.5 with $c_1 = 1/2$ to the polynomial terms in I_ε, we have some $c_4 > 0$ and $k \in \mathbf{N}$ such that

$$\begin{aligned} I_\varepsilon &\le c_4\varepsilon^{-k} w_a(x)\exp\Big\{-\frac{(x+y_1)^2}{8\varepsilon}\Big\}\exp\Big\{-\frac{(x+y_2)^2}{8\varepsilon}\Big\} \\ &\le c_4\varepsilon^{-k} w_a(x)\exp\Big\{-\frac{x^2+y_1^2}{8\varepsilon}\Big\}\exp\Big\{-\frac{x^2+y_2^2}{8\varepsilon}\Big\} \\ &\le c_4\varepsilon^{-k}\exp\left\{-\varepsilon^{-(1-2\eta)}/8\right\} w_a(x)\exp\left\{-x^2/8\varepsilon\right\}. \end{aligned}$$

Writing $D = (0,\infty)^3 \setminus (0,\varepsilon^{\eta})^3$, since ν_t is a sub-probability measure we have

$$\begin{aligned} \mathbf{E}\int_0^T \int_D I_\varepsilon \nu_t(dy_1)\,\nu_t(dy_2)\,dxdt &\le c_5 \varepsilon^{-k} \exp\left\{-\varepsilon^{-(1-2\eta)}/8\right\} \\ &\quad \times \int_0^\infty w_a(x) \exp\Big\{-\frac{x^2}{8\varepsilon}\Big\} dx \\ &= c_5 \varepsilon^{-k+(a+1)/2} \exp\left\{-\varepsilon^{-(1-2\eta)}/8\right\} \\ &\quad \times \int_0^\infty x^a \exp\Big\{-\frac{x^2}{8}\Big\} dx, \end{aligned} \tag{3.4.3}$$

and provided $a > -1$, the final integral term is $o(1)$, as $\varepsilon \to 0$, so by combining with (3.4.2) we have

$$\mathbf{E}\int_0^T \int_{(0,\infty)^3} I_\varepsilon \nu_t(dy_1)\,\nu_t(dy_2)\,dxdt = O\left(\varepsilon^{(a+2n+4+\gamma)\eta-(2n+1)}\right), \tag{3.4.4}$$

since $\eta < 1/2$ forces the exponential term in (3.4.3) to be negligible.

All that remains is to choose η so that (3.4.4) vanishes. For this we need

$$\frac{2n+1}{a+2n+4+\gamma} < \eta < \frac{1}{2},$$

which is possible if and only if

$$a > 2n-2-\gamma = a_c(n).$$

Therefore we conclude:

Proposition 3.4.1 (Vanishing remainder)**.** *If $n \ge 1$ and $a > a_c(n)$, then*

$$\mathbf{E}\int_0^T \|R_\varepsilon \nu_t\|_{a,n}^2\, dt \to 0, \quad \text{as } \varepsilon \to 0.$$

3.5. Another anti-derivative

We have the ingredients to prove all the cases for $n \ge 1$. However, to start the inductive argument in the proceeding section we need to consider the case $n = 0$. This requires the introduction of another type of anti-derivative which will be different to that seen in Chapter 2. Here, we will take the fixed integration limit at zero rather than at infinity:

Definition 3.5.1 (Another anti-derivative)**.** For all $x, y \in \mathbf{R}$ and $\varepsilon > 0$ we will write

$$G_\varepsilon^{(-1)}(x,y) := \int_0^x G_\varepsilon(x_0,y)\,dx_0, \quad T_\varepsilon^{(-1)}\nu_t(x) := \nu_t\left(G_\varepsilon^{(-1)}(x,\cdot)\right),$$

so that

$$T_\varepsilon^{(-1)}\nu_t(x) = \int_0^x T_\varepsilon \nu_t(x_0)\, dx_0.$$

The anti-derivative is useful as its L_a^2 norm can be written explicitly in terms of the function $x \mapsto \nu_t(0,x)$:

Proposition 3.5.2 (L_a^2 control on the anti-derivative)**.** *For every $a > a_c(-1) = -4-\gamma$ we have*

$$\liminf_{\varepsilon\to 0} \mathbf{E}\int_0^T \|T_\varepsilon \nu_t\|_{a,-1}^2 < \infty,$$

where we have the natural notation

$$\|T_\varepsilon \nu_t\|_{a,-1}^2 = \int_0^\infty w_a(x)\left|T_\varepsilon^{(-1)}\nu_t(x)\right|^2 dx.$$

Proof. Let us write $a = -4-\gamma+\delta$, for some $\delta > 0$. By splitting the range of y-integration at $y = 2x$ and using the fact that $G_\varepsilon(x,\cdot)$ is a defective density we have

$$\begin{aligned} T_\varepsilon^{(-1)}\nu_t(x) &= \int_0^\infty \int_0^x G_\varepsilon(x_0,y)\, dx_0 \nu_t(dy) \\ &\leq \nu_t(0,2x) + \int_{2x}^\infty \int_0^x G_\varepsilon(x_0,y)\, dx_0 \nu_t(dy). \end{aligned} \tag{3.5.1}$$

The first term is independent of ε and Theorem 3.0.2 gives a numerical constant, $c_1 > 0$, such that

$$\begin{aligned} \mathbf{E}\int_0^T \int_0^\infty w_a(x)\,\nu_t(0,2x)^2\, dx dt &\leq c_1 \int_0^\infty x^{-4-\gamma+\delta} x^{3+\gamma} e^{-x} dx \\ &= c_1 \int_0^\infty x^{-1+\delta} e^{-x} dx < \infty. \end{aligned}$$

So, writing $I_{t,\varepsilon}(x)$ for the integral term in (3.5.1), it suffices to show that

$$\liminf_{\varepsilon\to 0} \mathbf{E}\int_0^T \|I_{t,\varepsilon}\|_{L_a^2}^2 < \infty.$$

Applying Proposition A.0.6 we have a numerical constant, $c_2 > 0$, such that

$$I_{t,\varepsilon}(x) \leq c_2 \varepsilon^{-\frac{3}{2}} \int_{2x}^\infty \int_0^x x_0 y \exp\left\{-\frac{(y-x_0)^2}{2\varepsilon}\right\} dx_0 \nu_t(dy).$$

If $0 < x_0 < x$ and $y > 2x$, then $y - x_0 > (y-2x) + x$ and so

$$I_{t,\varepsilon}(x) \le c_2 \varepsilon^{-\frac{3}{2}} \exp\Big\{-\frac{x^2}{2\varepsilon}\Big\} \cdot \int_{2x}^{\infty} y \exp\Big\{-\frac{(y-2x)^2}{2\varepsilon}\Big\} \nu_t(dy) \cdot \int_0^x x_0 dx_0$$
$$\le \frac{c_2}{2} \varepsilon^{-\frac{3}{2}} x^2 \exp\Big\{-\frac{x^2}{2\varepsilon}\Big\} \cdot \int_{2x}^{\infty} y \exp\Big\{-\frac{(y-2x)^2}{2\varepsilon}\Big\} \nu_t(dy).$$

Therefore we have

$$\|I_{t,\varepsilon}\|_{L_a^2}^2 \le c_3 \varepsilon^{-3} \int_0^\infty x^{-\gamma+\delta} e^{-x} \left(\int_{2x}^\infty F_{t,\varepsilon}(x,y)\, \nu_t(dy) \right)^2 dx,$$

where we are denoting

$$F_{t,\varepsilon}(x,y) := y \exp\Big\{-\frac{x^2}{2\varepsilon}\Big\} \exp\Big\{-\frac{(y-2x)^2}{2\varepsilon}\Big\}.$$

Suppose that $x < \varepsilon^\eta$, where $\eta > 0$ is a free parameter. Then we have

$$F_{t,\varepsilon}(x,y) \le \begin{cases} 3\varepsilon^\eta, & \text{if } y < 3\varepsilon^\eta \\ y \exp\{-\varepsilon^{2\eta-1}\}, & \text{if } y \ge 3\varepsilon^\eta \end{cases},$$

so using the fact that $\int_0^\infty y d\nu_t \le \mathbf{E} X_t^1 \le x^\star < \infty$, we obtain

$$\int_{2x}^\infty F_{t,\varepsilon}(x,y)\, \nu_t(dy) \le c_4 \varepsilon^\eta \nu_t(0, 3\varepsilon^\eta) + c_4 \exp\big\{-\varepsilon^{2\eta-1}\big\},$$

where $c_4 > 0$ is a numerical constant. If $x \ge \varepsilon^\eta$, then

$$F_{t,\varepsilon}(x,y) = O\big(\exp\big\{-\varepsilon^{2\eta-1}\big\}\big),$$

so by combining both cases and fixing $\eta < 1/2$ gives

$$\mathbf{E} \int_0^T \|I_{t,\varepsilon}\|_{L_a^2}^2 dt \le c_3 \varepsilon^{-3} \left(\varepsilon^{2\eta} \mathbf{E}\left[\nu_t(0, 3\varepsilon^\eta)^2 \right] + O\big(\exp\big\{-2\varepsilon^{2\eta-1}\big\}\big) \right)$$
$$\times \int_0^{\varepsilon^\eta} x^{-\gamma+\delta} e^{-x} dx$$
$$+ c_3 \varepsilon^{-3} O\big(\exp\big\{-2\varepsilon^{2\eta-1}\big\}\big) \int_{\varepsilon^\eta}^\infty x^{-\gamma+\delta} e^{-x} dx$$
$$= O\left(\varepsilon^{(6+\delta)\eta - 3}\right) + o(1).$$

The proof is then complete, as the final expression is $o(1)$ provided we choose

$$\frac{3}{6+\delta} < \eta < \frac{1}{2}.$$

$\square$

To be able to use Proposition 3.5.2, we would like to set $G_\varepsilon^{(-1)}(x,\cdot)$ into the basic SPDE and follow the usual energy manipulation. Notice, however, that $G_\varepsilon^{(-1)}(x,\cdot) \notin C^{\text{test}}$, since the function is non-negative and increasing (so cannot belong to $\mathscr{S}$). Restricting the test functions to those in C^{test} is only necessary for the analysis in later chapters where solutions take values in $\mathscr{S}'$, rather than the space of sub-probability measures. Since ν is a finite measure, the derivation in Sections 2.3 and 2.4 only requires that test functions, ϕ, are elements of $C^2(\mathbf{R})$ with $\phi(0) = 0$ and

$$\mathbf{E}\int_0^T \left|\partial_x^n \phi\left(X_t^1\right)\right|^2 dt < \infty, \quad \text{for } n = 0, 1, 2,$$

which are easy conditions to check for the case $\phi(\cdot) = G_\varepsilon^{(-1)}(x,\cdot)$. Hence, at this point it is a legal operation to pass $G_\varepsilon^{(-1)}(x,\cdot)$ as a test function to the basic SPDE.

We make a note of the following computations, which differ from those in Lemma 3.3.2.

Lemma 3.5.3. *For every $x, y > 0$, $t \in [0,T]$ and $\varepsilon > 0$ we have:*

(i) $\nu_t\big(G_\varepsilon^{(-1)}(x,\cdot)\big) = T_\varepsilon^{(-1)}\nu_t(x)$,
(ii) $\partial_y G_\varepsilon^{(-1)}(x,y) = -G_\varepsilon(x,y) - \int_0^x 2\partial_x\psi_\varepsilon(x_0+y)\,dx_0$,
(iii) $\nu_t\big(\partial_y G_\varepsilon^{(-1)}(x,\cdot)\big) = -T_\varepsilon\nu_t(x) - \int_0^x \partial_x R_\varepsilon\nu_t(x_0)\,dx_0$,
(iv) $\partial_{yy} G_\varepsilon^{(-1)}(x,y) = \partial_x G_\varepsilon(x,y) + 2\partial_y\psi_\varepsilon(y)$,
(v) $\nu_t\big(\partial_{yy} G_\varepsilon^{(-1)}(x,\cdot)\big) = \partial_x T_\varepsilon\nu_t(x) + 2\nu_t(\partial_y\psi_\varepsilon(y))$.

Proof. (i) is trivial. The second point is given by

$$\begin{aligned}\partial_y G_\varepsilon^{(-1)}(x,y) &= \int_0^x \partial_y G_\varepsilon(x_0,y)\,dx_0 \\ &= \int_0^x \left(-\partial_x G_\varepsilon(x_0,y) - 2\partial_x\psi_\varepsilon(x_0+y)\right)dx_0 \\ &= -G_\varepsilon(x,y) - 2\int_0^x \partial_x\psi_\varepsilon(x_0+y)\,dx_0.\end{aligned}$$

(iii)-(v) follow similarly. ∎

By the above lemma we arrive at the smoothed SPDE

$$T_\varepsilon^{(-1)}\nu_t\left(x\right) = T_\varepsilon^{(-1)}\nu_0\left(x\right) + \frac{1}{2}\int_0^t \big(\partial_x T_\varepsilon \nu_s\left(x\right) + 2\nu_s\left(\partial_y \psi_\varepsilon\left(y\right)\right)\big) ds$$
$$- \rho \int_0^t \left(T_\varepsilon \nu_s\left(x\right) + \int_0^x \partial_x R_\varepsilon \nu_s\left(x_0\right) dx_0\right) dW_s,$$

and so we obtain the equivalent to (3.3.2):

$$\big(T_\varepsilon^{(-1)}\nu_t\left(x\right)\big)^2 = \big(T_\varepsilon^{(-1)}\nu_0\left(x\right)\big)^2 + \int_0^t T_\varepsilon^{(-1)}\nu_s\left(x\right) \cdot \partial_x T_\varepsilon \nu_s\left(x\right) ds$$
$$+ 2\int_0^t T_\varepsilon^{(-1)}\nu_s\left(x\right) \cdot \nu_s\left(\partial_y \psi_\varepsilon\left(y\right)\right) ds$$
$$+ \rho^2 \int_0^t \left(T_\varepsilon \nu_s(x) + \int_0^x \partial_x R_\varepsilon \nu_s(x_0)\, dx_0\right)^2 ds + M_t(x)\,,$$

where M is given by

$$M_t\left(x\right) = -2\rho \int_0^t T_\varepsilon^{(-1)}\nu_s\left(x\right) \cdot \left(T_\varepsilon \nu_s\left(x\right) + \int_0^x \partial_x R_\varepsilon \nu_s\left(x_0\right) dx_0\right) dW_s.$$

Notice, however, that $T_\varepsilon^{(-1)}\nu_s \geq 0$ and $\partial_y \psi_\varepsilon(y) \leq 0$ for $y \geq 0$, so we can simplify to the inequality

$$\big(T_\varepsilon^{(-1)}\nu_t\left(x\right)\big)^2 \leq \big(T_\varepsilon^{(-1)}\nu_0\left(x\right)\big)^2 + \int_0^t T_\varepsilon^{(-1)}\nu_s\left(x\right) \cdot \partial_x T_\varepsilon \nu_s\left(x\right) ds$$
$$+ \rho^2 \int_0^t \left(T_\varepsilon \nu_s(x) + \int_0^x \partial_x R_\varepsilon \nu_s(x_0)\, dx_0\right)^2 ds + M_t\left(x\right). \tag{3.5.2}$$

As in Section 3.3, our strategy is to multiply through by w_a and integrate over $(0,\infty)\times\Omega$. It is clear that the equivalent of Lemma 3.3.3 holds; the only new fact we need to check is that the integrands decay sufficient quickly at the origin for the integration by parts to be valid:

Lemma 3.5.4. *The results of Lemma 3.3.3 (excluding (ii)) hold for the case $n = -1$ and $a > a_c\left(0\right) = -2-\gamma$.*

Proof. The only change in the argument is that we need

$$T_\varepsilon^{(-1)}\nu_t\left(x\right) T_\varepsilon \nu_t\left(x\right) = o\left(x^{-2-\gamma+\delta}\right) \quad \text{and} \quad \big(T_\varepsilon^{(-1)}\nu_t\left(x\right)\big)^2 = o\left(x^{-3-\gamma+\delta}\right),$$

as $x \to 0$. By Proposition A.0.6 we have

$$G_\varepsilon\left(x, y\right) \leq c\left(\varepsilon\right) x_0 y,$$

hence

$$T_\varepsilon \nu_t (x) \leq c(\varepsilon) x \int_0^\infty y \nu_t (dy) = O(x),$$

so $T_\varepsilon^{(-1)} \nu_t (x) = O(x^2)$ and this suffices. $\square$

With this result in place we arrive at:

Proposition 3.5.5 (Energy estimate)**.** *Let $a > a_c(0) = -2-\gamma$, then there exists a constant $c(a) \geq 0$ such that*

$$\text{(3.5.3)} \quad \mathbf{E}\int_0^t \|T_\varepsilon \nu_s\|_{a,0}^2 \, ds \leq c(a) \|T_\varepsilon \nu_0\|_{a,-1}^2 + c(a)\, \mathbf{E}\int_0^t \left\| \int_0^\cdot \partial_x R_\varepsilon \nu_s (x_0)\, dx_0 \right\|_{L_a^2}^2 ds$$

$$+ c(a) \sum_{i=0}^{2} \mathbf{E}\int_0^t \|T_\varepsilon \nu_s\|_{a-i,-1}^2 \, ds,$$

for all $t \in [0,T]$ and $\varepsilon > 0$. Furthermore we can conclude

$$\liminf_{\varepsilon \to 0} \mathbf{E} \int_0^T \|T_\varepsilon \nu_s\|_{a,0}^2 \, ds < \infty.$$

Proof. To demonstrate the final statement, write $a = -2 - \gamma + \delta$ and suppose that

$$K_{\bar\delta}(f) := \int_0^\infty x_0^{-\gamma+\bar\delta} f(x_0)^2 \, dx_0 < \infty$$

with $\bar\delta < \delta$. Then the Cauchy–Schwarz inequality gives

$$\left(\int_0^x f(x_0) dx_0 \right)^2 \leq \int_0^x x_0^{\gamma-\bar\delta} dx_0 \int_0^x x_0^{-\gamma+\bar\delta} f(x_0)^2 dx_0 \leq K_{\bar\delta}(f)\, x^{1+\gamma-\bar\delta},$$

and so

$$\left\| \int_0^\cdot f(x_0)\, dx_0 \right\|_{L_a^2}^2 \leq K(f) \int_0^\infty x^{-1+\delta-\bar\delta} e^{-x} dx \leq \left(\delta - \bar\delta\right)^{-1} K_{\bar\delta}(f).$$

Now, from Propositions 3.4.1 and 3.2.3 we can see that

$$\mathbf{E}\int_0^T K_{\bar\delta}(R_\varepsilon \nu_t)\, dt \quad \text{and} \quad \mathbf{E}\int_0^T K_{\bar\delta}(T_\varepsilon \nu_0)\, dt$$

have finite limit infimums as $\varepsilon \to 0$. (Note we can drop the factor of e^{-x} from the weights in the proofs of these results.) Hence the first two terms on the right-hand side of (3.5.3) have finite limit infimum, and so do the terms in the summation, by Proposition 3.5.2. This completes the proof. $\square$

3.6. Proof of Theorem 3.0.1

The case $a > a_c(n)$. First take $n \geq 1$. By Propositions 3.2.3, 3.5.5 and 3.4.1, we have

$$\liminf_{\varepsilon \to 0} \mathbf{E} \int_0^T \|T_\varepsilon \nu_t\|_{a,n}^2 \, dt < \infty \quad \text{if} \quad \liminf_{\varepsilon \to 0} \mathbf{E} \int_0^T \|T_\varepsilon \nu_t\|_{a-2,n-1}^2 \, dt < \infty.$$

Therefore, by induction and Proposition 3.2.2, we need only check the case $n = 0$, so Proposition 3.5.5 completes the proof for $a > a_c(n)$.

The case $a < a_c(n)$. We now know that, on a full subset of (ω, t), ν_t has weak derivatives of all orders. As we have local integrability (because w_a is locally bounded away from zero and $L^1_{\text{loc}} \subseteq L^2_{\text{loc}}$), for every $n \geq 1$ and $0 < x < y < 1$ we can write

$$\begin{aligned} \partial_x^{n-1} \nu_t(x) &= \partial_x^{n-1} \nu_t(y) + \int_y^x \partial_x^n \nu_t(z) \, dz. \\ &\leq \partial_x^{n-1} \nu_t(y) + \|\nu_t\|_{a,n,1} \left| \int_y^x w_a(z)^{-1} \, dz \right|^{1/2}, \end{aligned} \tag{3.6.1}$$

where the second line is due to Cauchy–Schwarz and where we are introducing the notation

$$\|\nu_t\|_{a,n,1} = \left(\int_0^1 w_a(x) \left| \partial_x^n \nu_t(x) \right|^2 dx \right)^{1/2}.$$

If we assume that $0 < x < y < 1$ and $a > 1$ then we can reduce (3.6.1) to

$$\begin{aligned} \partial_x^{n-1} \nu_t(x) &\leq \partial_x^{n-1} \nu_t(y) + e^{|a|} \|\nu_t\|_{a,n} \int_x^1 z^{-a} dz \\ &\leq \partial_x^{n-1} \nu_t(y) + \frac{e^{|a|}}{a-1} \|\nu_t\|_{a,n} \, x^{-(a-1)}. \end{aligned}$$

Following the method from Section 2.5, multiplying by w_b, for some $b \in \mathbf{R}$, and integrating over $y \in (1/2, 1)$ gives

$$I_b \partial_x^{n-1} \nu_t(x) \leq \int_{1/2}^1 w_b(y) \, \partial_x^{n-1} \nu_t(y) \, dy + \frac{e^{|a|} I_b}{a-1} \|\nu_t\|_{a,n} \, x^{-(a-1)},$$

where $I_b = \int_{1/2}^1 w_b(y) \, dy$, which is finite and non-zero. Again applying Cauchy–Schwarz gives

$$\left| \partial_x^{n-1} \nu_t(x) \right| \leq I_b^{-1/2} \|\nu_t\|_{n,b} + \frac{e^{|a|}}{a-1} \|\nu_t\|_{a,n,1} \, x^{-(a-1)},$$

and if $\delta > 0$, $x^{-1+\delta}$ is integrable near zero, so multiplying throughout by $x^{a-2+\delta}$ and integrating gives

$$\mathbf{E}\int_0^T \|\nu_t\|^2_{a-2+\delta,n-1,1}\,dt \le c_1(a,b,\delta)\left(\mathbf{E}\int_0^T \|\nu_t\|^2_{a,n,1}\,dt + \mathbf{E}\int_0^T \|\nu_t\|^2_{n,b}\,dt\right), \tag{3.6.2}$$

with $c_1(a,b,\delta)$ a finite numerical constant.

Returning to the proof of the blow-up case for Theorem 3.0.1, suppose for a contradiction that $a < a_c(n)$ and

$$\mathbf{E}\int_0^T \|\nu_t\|^2_{a,n}\,dt < \infty.$$

Note that it suffices to assume $a = a_c(n) - n\delta$, where $\delta > 0$ is small. First consider the case $n \ge 2$, which forces $a_c(n) \ge 2-\gamma$, and by taking δ small enough we can assume $a \ge 1$. The above analysis gives

$$\mathbf{E}\int_0^T \|\nu_t\|^2_{a-2+\delta,n-1,1}\,dt < \infty,$$

by taking any $b > a_c(n)$. We can then repeat the argument until we reach the first derivative, whence

$$\mathbf{E}\int_0^T \|\nu_t\|^2_{-\gamma-\delta,1,1}\,dt < \infty.$$

So we have reduced the case $n \ge 2$ to the case $n = 1$.

Now consider the $n = 1$ case. Notice that

$$\mathbf{E}\int_0^\varepsilon \partial_x^0 \nu_t(z)\,dz = \mathbf{E}\nu_t(0,\varepsilon) = O\big(\varepsilon^2\big)$$

so we must have $\partial_x^0\nu_t(0+) = 0$, with probability 1. We can thus set $y = 0$ in (3.6.1) and this gives

$$\left|\partial_x^0\nu_t(x)\right|^2 \le \|\nu_t\|^2_{-\gamma-\delta,1,1}\int_0^x w_{-\gamma-\delta}(z)^{-1}\,dz \le c_1(\delta)\,\|\nu_t\|^2_{-\gamma-\delta,1,1}\,x^{1+\gamma+\delta},$$

for $x < 1$, where $c_1(\delta)$ is a numerical constant. Now, for $c_2(\delta)$ another constant we have

$$\begin{aligned}
\mathbf{E}\int_0^T \nu_t(0,\varepsilon)^2\,dt &= \mathbf{E}\int_0^T \left|\int_0^\varepsilon \partial_x^0\nu_t(x)\,dx\right|^2 dt \\
&\le \mathbf{E}\int_0^T \int_0^\varepsilon dx \cdot \int_0^\varepsilon c(\delta)\,\|\nu_t\|^2_{-\gamma-\delta,1,1}\,x^{1+\gamma+\delta}dxdt \\
&\le c_2(\delta)\,\varepsilon^{3+\gamma+\delta}\mathbf{E}\int_0^T \|\nu_t\|^2_{-\gamma-\delta,1,1}\,dxdt,
\end{aligned}$$

which is a contradiction of the lower inequality in Theorem 3.0.2. Hence we can conclude the result for $n \geq 1$.

The remaining case is $n = 0$, however this follows from the last calculation since $a = -2 - \gamma - \delta$ and

$$\begin{aligned}
\mathbf{E}\int_0^T \nu_t\left(0,\varepsilon\right)^2 dt &= \mathbf{E}\int_0^T \left|\int_0^\varepsilon \partial_x^0 \nu_t\left(x\right) dx\right|^2 dt \\
&\leq \mathbf{E}\int_0^T\int_0^\varepsilon x^{2+\gamma+\delta} dx \cdot \int_0^\varepsilon x^{-2-\gamma-\delta}\left|\int_0^\varepsilon \partial_x^0 \nu_t\left(x\right) dx\right|^2 dx dt \\
&\leq c_3(\delta)\,\varepsilon^{3+\gamma+\delta}\mathbf{E}\int_0^T \left\|\nu_t\right\|^2_{-\gamma-\delta,1} dx dt,
\end{aligned}$$

for another constant $c_3(\delta)$. ∎

CHAPTER 4

Skorokhod's M_1 topology for tempered-distribution-valued processes

The purpose of this chapter is to introduce the M_1 topology on the space, $D_{\mathscr{S}'}$, of càdlàg processes taking values in the tempered distributions and use this topology to establish tightness for the loss-dependent correlation system, (1.3.1).

After a review of the M_1 topology for processes taking values in a Banach space (Section 4.1), we illustrate why the topology is convenient by establishing tightness for the loss process (Section 4.2). The key fact is that monotone functions have zero modulus of continuity in the M_1 topology (Proposition 4.1.7).

The space of rapidly decreasing functions, $\mathscr{S}$, is an example of a countably Hilbertian nuclear space, and we give an overview of these spaces in Section 4.3. As it is this property of $\mathscr{S}$ that makes the dual space, $\mathscr{S}'$, a useful space to work with, therefore throughout the remainder of this chapter we will state our results for a general countably Hilbertain nuclear space, E.

In Section 4.4 we define the M_1 topology on the space $D_{E'}$ (Definition 4.4.3) and address measurability issues for the topological space $(D_{E'}, M_1)$. Having established the relevant facts, we then prove the main convergence results in Section 4.6. The first of these states that tightness of a sequence of processes in $(D_{E'}, M_1)$ can be checked test-function-by-test-function:

Theorem 4.0.1 (Tightness)**.** *Let $(\mu_n)_{n\geq 1}$ be a sequence of probability measures on $D_{E'}$ and let π^{ϕ} be the canonical projection*

$$\pi^{\phi} : D_{E'} \to D_{\mathbf{R}}, \quad \pi^{\phi}(x) := (x_t(\phi))_{t\in[0,T]} =: x(\phi).$$

If the sequence of probability measures $\left(\mu_n \circ \left(\pi^{\phi}\right)^{-1}\right)_{n\geq 1}$ is tight on $(D_{\mathbf{R}}, M_1)$ for every $\phi \in E$, then $(\mu_n)_{n\geq 1}$ is tight on $(D_{E'}, M_1)$.

A sequence of probability measures, $(\mu_n)_{n\geq 1}$, is said to converge weakly to a probability measure μ on $(D_{E'}, M_1)$ if

$$\int_{D_{E'}} f(x)\,\mu_n(dx) \to \int_{D_{E'}} f(x)\,\mu(dx), \quad \text{for every } f \in C_b(D_{E'}, M_1),$$

as $n \to \infty$, where $C_b(D_{E'}, M_1)$ denotes the space of continuous and bounded functions from $(D_{E'}, M_1)$ to $\mathbf{R}$. As the space $(D_{E'}, M_1)$ is completely regular, Theorem 4.0.1 is useful since it allows us to extract weakly convergent subsequences of a tight sequence:

Proposition 4.0.2 (Tightness implies relative compactness)**.** *If the sequence of probability measures, $(\mu_n)_{n\geq 1}$, is tight on $(D_{E'}, M_1)$, then it is weakly relatively compact — that is every subsequence contains a further subsequence, $(\mu_{n_k})_{k\geq 1}$, that is weakly convergent on $(D_{E'}, M_1)$ to some probability measure.*

Once tightness is established for a sequence of processes one only needs to check marginal convergence to establish weak convergence to the process level:

Theorem 4.0.3 (Identification from marginals)**.** *Let $\mu, \mu_1, \mu_2, \dots$ be probability measures on $D_{E'}$. If $(\mu_n)_{n\geq 1}$ is tight on $(D_{E'}, M_1)$ and*

$$\mu_n \circ \left(\pi^{\phi_1,\dots,\phi_k}_{t_1,\dots,t_k}\right)^{-1} \to \mu \circ \left(\pi^{\phi_1,\dots,\phi_k}_{t_1,\dots,t_k}\right)^{-1}, \quad \textit{weakly on } \mathbf{R}^k,$$

whenever $k \geq 1$, $\phi_1, \phi_2, \dots, \phi_k \in E$ and $t_1, t_2, \dots, t_k \in \mathrm{cont}(\mu)$, where

$$\mathrm{cont}(\mu) = \{t \in [0,T] : \mu\{x \in D_{E'} : x_{t-} = x_t\} = 1\},$$

then $(\mu_n)_{n\geq 1}$ converges weakly to μ on $(D_{E'}, M_1)$.

With the above three general results established, we show tightness for the loss-dependent correlation system in Section 4.7.

Theorem 4.0.4 (Tightness of (1.3.1))**.** *The sequence of triples*

$$\left(\nu^N, L^N, W\right)_{N\geq 1}$$

from (1.3.1) is tight on the product space

$$(D_{\mathscr{S}'}, M_1) \times (D_{\mathbf{R}}, M_1) \times (C_{\mathbf{R}}, U)$$

with the product topology, where $C_{\mathbf{R}}$ is the space of continuous functions on $[0,T]$ and U is the uniform topology. As a consequence, the sequence is relatively compact in the product topology.

Remark 4.0.5 (Other range spaces)**.** The loss-dependent system could be studied in the space of finite measures, rather than $\mathscr{S}'$. Although the space of finite measures is not linear, it is convex and this is enough to define the notion of an ordered graph as in Definition 4.1.2, and hence construct the M_1 topology. We have chosen to work with $\mathscr{S}'$ to present a broad framework that can be used for problems outside of those in this thesis. This level of generality allows us to perform operations such as multiplying by arbitrary (not necessarily non-negative) smooth functions and taking derivatives. The latter is important in establishing fluctuation results like the central limit theorem of [**Itô83**] and [**SSG14**]. Another option is to work in a functional range space and use the classical theory of the M1 topology with Banach range space. This approach requires estimates to be derived in the norm of the range space, which can be difficult when working with empirical processes. In [**PR14**] the authors discuss various M1 topologies induced by altering the mode of convergence in the range space. In our setting we do not have to address these alternatives and all practical computations are carried out after projecting down to one dimension.

4.1. An overview of the M_1 topology for processes taking values in a Banach space

We will follow Skorokhod's original set-up from his classic paper [**Sko56**] and Whitt's recent account [**Whi02**]. The topology we use is what Whitt refers to as the strong M_1 or SM_1 topology.

Definition 4.1.1 (D_X)**.** Let X be a Banach space with norm $\|\cdot\|_X$. We denote by D_X the space of càdlàg functions from $[0,T]$ to X (that is, the collection of functions $x : [0,T] \to X$ such that for every $t \in [0,T]$ there exists $z \in X$ for which

$$\|x(t+s) - x(t)\|_X \to 0 \quad \text{and} \quad \|x(t-s) - z\|_X \to 0$$

as $s \downarrow 0$.)

There are a number of natural topologies on D_X; we choose the one that encompasses what we want it to mean when we say two elements of D_X are *close*. The simplest topology is that arising from the uniform norm, but, unless we are working with continuous functions, this norm is of little use — it is too strong. To weaken the notion of closeness we introduce the J_1 and M_1 topologies on D_X, which are best visualised through Figure 4.1.1. We will not describe the better known J_1 topology in detail (the reader can see [**Whi02**]), but we note that it is coarser than the uniform topology and finer that the M_1 topology.

To define the M_1 topology, our strategy is to take an element $x \in D_X$ and join up its graph at its points of discontinuity to form a closed continuous graph, γ_x, in $X \times [0,T]$. We can then define the distance between elements in D_X through

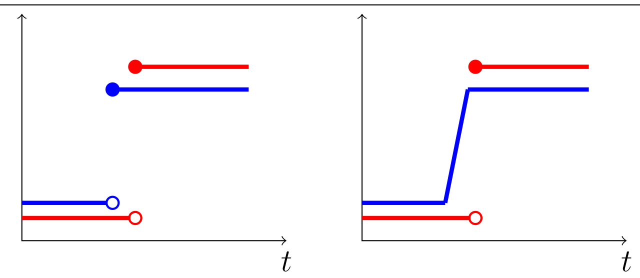


FIGURE 4.1.1. On the left the paths are close in the J_1 and M_1 topologies but not the uniform topology, U. On the right the paths are close in M_1, but not J_1 or U.

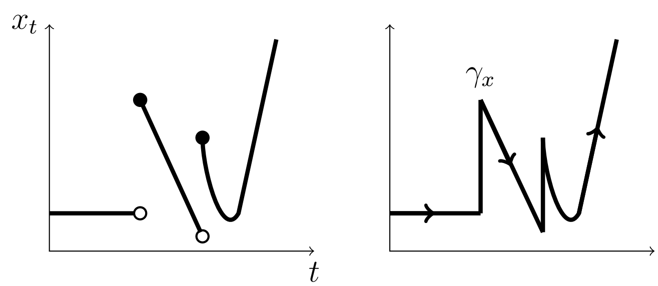


FIGURE 4.1.2. On the left is an example of a path $x \in D_X$. On the right is its graph $\gamma_x \subseteq X \times [0,T]$ with the natural graph ordering indicated.

(a suitable notion of) the distance between their graphs. To this end, we call the set

$$[z_1, z_2]_X := \{(1-\lambda)\, z_1 + \lambda z_2 : \lambda \in [0,1]\}$$

the *interval* between two points z_1 and z_2 in X. So for $x \in D_X$, its *graph*, γ_x, is defined to be the closed set

$$\gamma_x := \{(z,t) \in X \times [0,T] : z \in [x\,(t-)\,, x\,(t)]_X\}\,, \tag{4.1.1}$$

where we use the standard notation for left limits: $x\,(t-) = \lim_{s\uparrow t} x\,(s)$. This immediately becomes clear from Figure 4.1.2.

Since an interval in X has a natural ordering ($(1-\lambda)\, z_1 + \lambda z_2$ is closer to z_1 than to z_2 if λ is closer to 0 than 1) we can define an ordering on γ_x:

Definition 4.1.2 (Graph ordering)**.** For a pair of points in γ_x, we say that $(z_1, t_1) \leq (z_2, t_2)$ if either:

(i) $t_1 < t_2$,

(ii) Or $t_1 = t_2$ and z_1 is closer to $x(t_1-)$ than to z_2 in the sense that for the (unique) $\lambda_1, \lambda_2 \in [0,1]$ satisfying

$$z_1 = (1-\lambda_1)\, x(t_1-) + \lambda_1 x(t_1) \quad \text{and} \quad z_2 = (1-\lambda_2)\, x(t_1-) + \lambda_2 x(t_1)$$

we have $\lambda_1 \leq \lambda_2$. (Again, Figure 4.1.2 is helpful.)

Remark 4.1.3. Condition (ii) is phrased differently to [**Whi02**]. Our description is equivalent, but here we use only the linearity of X and do not refer to the norm $\|\cdot\|_X$. Hence the notion of a graph extends to functions taking values in a topological vector space, which will be important in Section 4.4.

Definition 4.1.4 (Parametric representation)**.** A *strong parametric representation* of the graph γ_x is a continuous function $s \mapsto (z(s), t(s))$ that maps $[0,T]$ onto γ_x and is non-decreasing with respect to the graph ordering on γ_x. (As mentioned in [**Whi02**, Remark 12.3.4], this monotonicity is incorrectly stated in [**Sko56**].) The collection of all such strong parametric representations of γ_x is denoted Γ_x.

We can now define the M_1 *metric* on D_X:

Definition 4.1.5 (M_1 metric, d_{X,M_1})**.** Let the *distance* between a pair of parametric representations, (u_1, r_1) and (u_2, r_2), be given by

$$\mathrm{dist}_X\left[(u_1, r_1); (u_2, r_2)\right] := \sup_{t\in[0,T]} \left\{ \|u_1(t) - u_2(t)\|_X \vee |r_1(t) - r_2(t)| \right\}.$$

Then the *Skorokhod* M_1 *metric* on D_X is defined to be

$$d_{X,M_1}(x,y) := \inf_{(u_1,r_1)\in\Gamma_x, (u_2,r_2)\in\Gamma_y} \mathrm{dist}_X\left[(u_1, r_1); (u_2, r_2)\right], \quad x, y \in D_X.$$

Theorem 12.3.1 of [**Whi02**] confirms that d_{X,M_1} is a metric. The topology induced by d_{X,M_1} on D_X is called the *Skorokhod* M_1 *topology.*

It is helpful to note that if (x_n) converges to $x \in D_X$ with respect to $d_{M_1,X}$ then we have local uniform convergence at all continuity points of x:

Lemma 4.1.6 (Local uniform convergence, [**Whi02**, Lemma 12.5.1])**.** *Let $x \in D_X$ and suppose $t \in [0,T]$ is such that $x(t-) = x(t)$. If $d_{X,M_1}(x_n, x) \to 0$, then*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{0\vee(t-\delta)\leq t_1, t_2 \leq (t+\delta)\wedge T} \|x_n(t_1) - x(t_2)\|_X = 0,$$

and so $x_n(t) \to x(t)$, as $n \to \infty$.

The modulus of continuity. During practical computations we work with the *modulus of continuity* function on D_X:

$$w_{X,M_1}(x;\delta) := \sup_{t\in[0,T]} \sup_{(t_1,t_2,t_3)\in \mathrm{Trip}_{t,\delta}} H_X\left[x(t_1), x(t_2), x(t_3)\right], \tag{4.1.2}$$

for $x \in D_X$ and $\delta > 0$, where we abbreviate the set of triples

$$\mathrm{Trip}_{t,\delta} := \{(t_1,t_2,t_3) : 0 \vee (t-\delta) \leq t_1 < t_2 < t_3 \leq (t+\delta) \wedge T\}$$

and define

$$H_X(v_1,v_2,v_3) := \inf_{\lambda\in[0,1]} \|v_2 - \lambda v_1 - (1-\lambda)v_3\|_X = \|v_2 - [v_1,v_3]_X\|_X,$$

that is the distance in X between the line $[v_1,v_3]_X$ and the point v_2.

As an easy, but crucial, example consider a monotone function, f: it is always the case that $f(t_2)$ falls between $f(t_1)$ and $f(t_3)$, so we conclude the following result.

Proposition 4.1.7 (Monotone functions vanish under $w_{\mathbf{R},M_1}$)**.** *Let $f : [0,T] \to \mathbf{R}$ be monotonic. Then $w_{\mathbf{R},M_1}(f;\delta) = 0$, for every $\delta > 0$.*

This proposition is the reason why the M_1 topology is useful in our setting. We will soon see that the modulus of continuity controls the convergence in D_X, so since the loss process is an increasing function, $w_{\mathbf{R},M_1}$ is well-adapted to our setting.

The modulus of continuity allows us to characterise the compact subsets of D_X. We take the following result from Section 2.7 of [**Sko56**].

Theorem 4.1.8 (Characterisation of compactness, [**Sko56**])**.** *Let X be a Banach space. Then a subset $A \subseteq D_X$ is compact in (D_X, M_1) if and only if both the following hold:*

(i) There exists a compact subset $B \subseteq X$ such that $x(t) \in B$ for all $x \in A$ and $t \in [0,T]$,

(ii) We have

$$\lim_{\delta\to 0} \sup_{x\in A} \Big\{ w_{X,M_1}(x;\delta) + \sup_{t\in(0,\delta)} \|x(t) - x(0)\|_X + \sup_{t\in(T-\delta,T)} \|x(T) - x(t)\|_X \Big\} = 0$$

The space D_X is not complete with respect to d_{X,M_1}, however (D_X, M_1) is a Polish space:

Proposition 4.1.9 ((D_X, M_1) is a Polish space)**.** *There exists a metric on D_X that is topologically equivalent to d_{X,M_1} and with respect to which the space is complete and separable.*

Proof. Whitt provides a construction of a complete metric equivalent to d_{X,M_1} in [**Whi02**, Theorem 12.8.1], so we shall not repeat it. If a space is separable with respect to some metric, then it is separable with respect to any equivalent metric. So it suffices to show that D_X is separable with respect to d_{X,M_1}. However, the usual Skorokhod metric, d_{X,J_1}, defined in [**Whi02**, Section 3.3], satisfies $d_{X,M_1} \leq d_{X,J_1}$ [**Whi02**, Theorem 12.3.2] and D_X is separable with respect to d_{X,J_1} [**KX95**, Theorem 2.4.2]. It therefore follows that D_X is separable with respect to d_{X,M_1}, which completes the proof. $\square$

Measures on the Skorokhod space. The Borel σ-algebras generated by any of the Skorokhod topologies coincide with the usual Kolmogorov σ-algebra [**Whi02**, Theorem 11.5.2]. Without confusion, we will denote this σ-algebra by $\mathscr{B}(D_X)$. A probability measure, μ, on D_X maps sets in $\mathscr{B}(D_X)$ to $[0,1]$ and a random variable on D_X is a measurable map from $(\Omega, \mathcal{F})$ to $(D_X, \mathscr{B}(D_X))$. Hence, whenever discussing random variable or probability measures on D_X we need not refer to the underlying topology.

In the previous subsection we characterised compact sets in D_X, so we are now equipped to tackle the issues of weak convergence and tightness. Recall the following definition:

Definition 4.1.10 (Weak convergence, tightness)**.** We say (μ_n) is *tight on* (D_X, M_1) if for every $\varepsilon > 0$ there exists a compact subset K_ε of (D_X, M_1) such that

$$\mu_n(K_\varepsilon) \geq 1 - \varepsilon, \quad \text{for every } n \geq 1.$$

As (D_X, M_1) is a Polish space it is a Hausdorff space, and so tightness implies relative compactness, [**KX95**, Theorem 2.2.1], and we have Proposition 4.0.2 and Theorem 4.0.3 with E' replaced by X.

In practice, the following result is useful for checking tightness in the Banach space setting:

Theorem 4.1.11 (Tightness criteria I, [**KX95**])**.** *Let X be a Banach space and $\mu_1, \mu_2, \ldots$ be probability measures on $(D_X, \mathscr{B}(D_X))$. Then (μ_n) is tight on (D_X, M_1) if both the following hold:*

(i) *For every $\varepsilon > 0$ there exists a compact subset $B \subseteq X$ such that*

$$\mu_n \{x \in D_{E'} : x(t) \in B \quad \forall t \in [0,T]\} \geq 1 - \varepsilon, \quad \textit{for every } n \geq 1,$$

(ii) *For every $\varepsilon > 0$ and $\eta > 0$ there exists $\delta > 0$ such that*

$$\mu_n \Big\{ x \in D_{E'} : w_{X,M_1}(x;\delta) + \sup_{t \in (0,\delta)} \|x(t) - x(0)\|_X + \sup_{t \in (T-\delta,T)} \|x(T) - x(t)\|_X \geq \eta \Big\} < \varepsilon,$$

for every $n \geq 1$.

By the reasoning in [**EK86**, Theorem 7.2, Corollary 7.4] we have the equivalent conditions:

Theorem 4.1.12 (Tightness criteria II)**.** *Let X be a Banach space and $\mu_1, \mu_2, \ldots$ be probability measures on $(D_X, \mathscr{B}(D_X))$. Let $\mathbf{T} \subseteq [0,T]$ be dense and for $S \subseteq X$ let*

$$N(S;\varepsilon) = \{x \in X : \|x - s\|_X < \varepsilon \textit{ for some } s \in S\}.$$

Then (μ_n) is tight on (D_X, M_1) if both the following hold:

(i) *For every $\varepsilon > 0$ and $t \in \mathbf{T}$ there exists a compact subset $B \subseteq X$ such that*

$$\liminf_{n\to\infty} \mu_n \{x \in D_{E'} : x(t) \in N(B;\varepsilon)\} \geq 1 - \varepsilon,$$

(ii) *For every $\eta > 0$*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mu_n \Big\{ x \in D_{E'} : w_{X,M_1}(x;\delta) + \sup_{t \in (0,\delta)} \|x(t) - x(0)\|_X + \sup_{t \in (T-\delta,T)} \|x(T) - x(t)\|_X \geq \eta \Big\} = 0.$$

4.2. Tightness of the loss processes

Since L^N is an increasing function we know that $w_{\mathbf{R},M_1}(L^N;\delta) = 0$ from Proposition 4.1.7. It is also clear that condition (i) of Theorem 4.1.11 is satisfied since, by construction,

$$L_t^N \in [0,1], \quad \text{for every } N \geq 1 \text{ and } t \in [0,T].$$

Now for $\eta > 0$ fixed, we have

$$\mathbf{P}\Big(\sup_{t \in (0,\delta)} \left| L_t^N - L_0^N \right| \geq \eta \Big) \leq \eta^{-1} \mathbf{E}\Big[L_\delta^N - L_0^N \Big]. \tag{4.2.1}$$

By writing the increment of the loss process as

$$L_t^N - L_s^N = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{s<\tau^{i,N}\leq t},$$

linearity of expectation gives

$$\mathbf{E}\left[L_t^N - L_s^N\right] = \mathbf{P}\left(s < \tau^{1,N} \leq t\right).$$

When considered individually, every $X^{1,N}$ has the law of a Brownian motion (independently of N), hence we have

$$\mathbf{E}\left[L_t^N - L_s^N\right] \to 0, \quad \text{uniformly in } N,$$

as $t - s \to 0$, because the law of the first hitting time of a Brownian motion is absolutely continuous. Applying this to (4.2.1) and the corresponding inequality at T we have established (ii) of Theorem 4.1.11. Therefore we have proved:

Proposition 4.2.1. *The sequence of loss processes, $\left(L^N\right)_{N\geq 1}$, for the loss-dependent correlation system (1.3.1) is tight in $(D_{\mathbf{R}}, M_1)$.*

4.3. An overview of countably Hilbertian nuclear spaces

The material here is based on Chapter 1 of Kallianpur and Xiong [**KX95**] and the very readable account of Becnel and Sengupta [**BS09**].

Frèchet spaces. Suppose that E is a linear space and

$$\mathcal{C} = \{p_\lambda : \lambda \in \Lambda\}$$

is a collection of semi-norms on E, where Λ is some index set. A subset $U \subseteq E$ is said to be a *neighbourhood of the point* $x_0 \in E$ if there exists $n \geq 1$, $\lambda_j \in \Lambda$ and $\varepsilon_j > 0$ such that

$$U = \left\{x \in E : p_{\lambda_j}(x - x_0) < \varepsilon_j, \quad \text{for } j = 1, 2, \ldots, n\right\}.$$

A subset $G \subseteq E$ is then said to be *open* if, for every $x_0 \in G$, there exists a neighbourhood, U, of x_0 such that $U \subseteq G$. The collection of all such G's forms a topology, $\mathcal{T}$, on E, and we say that $\mathcal{T}$ is the *topology generated by the collection* $\mathcal{C}$. From Definition 2.1.1, if $E = \mathscr{S}$ the relevant topology is the one generated by

$$\mathcal{C} = \left\{\|\cdot\|_{\alpha,\beta} : \alpha, \beta \geq 0\right\}.$$

Definition 4.3.1 (Frèchet space)**.** A topological space $(E, \mathcal{T})$ is called a *Frèchet space* if all the following hold:

(i) $\mathcal{T}$ is generated by a *countable* collection of semi-norms, $\mathcal{C} = \{p_\lambda : \lambda \in \Lambda\}$,
(ii) For every $x_0 \neq 0$ there exists $\lambda_0 \in \Lambda$ such that $p_{\lambda_0}(x_0) > 0$,
(iii) $(E, \mathcal{T})$ is *sequentially complete*: every Cauchy sequence in E (that is a sequence (x_n) with $x_n - x_m \to 0$, as $n, m \to \infty$) converges to a point in E.

A sequence, (x_n), in the Frèchet space E converges to $x \in E$ if and only if

$$p_\lambda (x_n - x) \to 0, \quad \text{for every } \lambda \in \Lambda.$$

This is immediate from the fact that $(E, \mathcal{T})$ is metrized by

$$d(x, y) := \sum_{j=1}^{\infty} 2^{-j} \left[p_j (x - y) \wedge 1\right],$$

where we have chosen some arbitrary enumeration of the countable index set Λ. Hence, $(E, \mathcal{T})$ is a complete metric space when equipped with d.

Countably Hilbertian spaces. On a vector space E, we say that the family of semi-norms $\mathcal{C} = \{p_\lambda : \lambda \in \Lambda\}$ is *compatible* if, for every $\lambda_1, \lambda_2 \in \Lambda$, whenever (x_n) is a Cauchy sequence with respect to both p_{λ_1} and p_{λ_2} and $p_{\lambda_1}(x_n) \to 0$, then we also have $p_{\lambda_2}(x_n) \to 0$.

A separable Frèchet space, $(E, \mathcal{T})$, is called *countably Hilbertian* if its topology is generated by an increasing sequence of compatible Hilbertian semi-norms

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots .$$

The space $(\mathscr{S}, \mathcal{T})$ is a countably Hilbertian space, that is, there is such a sequence of increasing Hilbertian semi-norms on $\mathscr{S}$ which generates the same topology as that generated by $\left\{\|\cdot\|_{\alpha,\beta} : \alpha, \beta \geq 0\right\}$. (An explicit construction using the Hermite polynomials is given in [**KX95**, Section 1.3].)

For every $n \geq 0$, $(E, \|\cdot\|_n)$ is a pre-Hilbert space and we denote its *completion* by E_n [**KX95**, Theorem 1.1.4]. From [**KX95**, Theorem 1.3.1], $(E_n)_{n \geq 0}$ is a decreasing sequence of Hilbert spaces, that is

$$E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots ,$$

and

$$E = \bigcap_{n=0}^{\infty} E_n.$$

Nuclear spaces. Let $T : X \to Y$ be a linear operator between two separable Hilbert spaces. Take any complete orthonormal system, $\{x_i\}$, of X. The norm

$$\|T\| := \left(\sum_{i=1}^{\infty} \|Tx_i\|_Y^2 \right)^{1/2}$$

does not depend on the choice of $\{x_i\}$, and we say that T is *Hilbert–Schmidt* if $\|T\| < \infty$.

Returning to our countably Hilbertian space E, we say that E is also *nuclear* if, for every $n \geq 0$, there exists $m > n$ such that the canonical injection $E_m \hookrightarrow E_n$ is Hilbert–Schmidt. That is, for every $n \geq 0$ there exists an $m > n$ such that for any complete orthonormal system, $\{e_i^m\}_{i\geq 1}$, of E_m

$$\sum_{i=1}^{\infty} \|e_i^m\|_n^2 < \infty.$$

Theorem 1.3.2 of [**KX95**, Theorem 1.3.2] confirms that $\mathscr{S}$ is nuclear.

Any Hilbert–Schmidt operator is a compact operator (the image of any bounded set is pre-compact). Therefore, with $m > n$ above, if $B \subseteq E_m$ is bounded then B is pre-compact in $(E_n, \|\cdot\|_n)$.

The dual space of E**.** We denote the topological dual space of $(E, \mathcal{T})$ by E', that is the collection of all $\mathcal{T}$-continuous linear maps from E to $\mathbf{R}$. The dual, E_n', of $(E_n, \|\cdot\|_n)$ is a Hilbert space with norm

$$\|f\|_{-n} := \sup_{\|x\|_n \leq 1} |f(x)|.$$

Using the Riesz Representation Theorem to identify E_0' with E_0, we have

$$\|\cdot\|_0 \geq \|\cdot\|_{-1} \geq \|\cdot\|_{-2} \geq \cdots, \; E_0 \subseteq E_1' \subseteq E_2' \subseteq \cdots \quad \text{and} \quad E' = \bigcup_{n=0}^{\infty} E_n'.$$

Remember, when $E = \mathscr{S}$, E' is the space of tempered distributions.

We have several natural choices of topology for E'. The *weak topology*, $\mathcal{T}_w$, is the coarsest vector topology with respect to which all the maps $x \mapsto f(x)$ are continuous for every $f \in E'$. The *inductive limit topology*, $\mathcal{T}_i$, is the finest locally convex vector topology such that all the inclusion maps, $\iota_n : E_n' \hookrightarrow E'$, are continuous. The *strong topology*, $\mathcal{T}_s$, is the topology generated by the base of sets

$$N(B;\varepsilon) = \left\{ f \in E' : \sup_{x\in B} |f(x)| < \varepsilon \right\}, \quad B \subseteq E \text{ bounded}, \varepsilon > 0.$$

Here, B being bounded means that for every open neighbourhood U of 0 in E, there exists $r \in \mathbf{R}$ such that $B \subseteq rU$. In other words, the strong topology is generated by the family of semi-norms

$$\left\{p_B(f) := \sup_{x \in B} |f(x)|\right\}_{B \in \mathcal{B}} \tag{4.3.1}$$

with $\mathcal{B}$ the collection of bounded sets of E. We know the following facts:

Proposition 4.3.2 ([**Bec06**, Theorem 4.16])**.** *We have the following:*

(i) $\mathcal{T}_s = \mathcal{T}_i$,

(ii) A sequence in E' is weakly convergent if and only if it is strongly convergent,

(iii) The weak and strong topologies of the dual of an infinite-dimensional nuclear space are not metrizable.

From this point we will work only with the strong topology on E', and we will write the corresponding space as $(E', \mathcal{T}_{E'})$. Result (ii) of Proposition 4.3.2 is another example of where we have equivalence between strong and weak properties.

Since the dual operator of a compact operator is also compact, the nuclear structure of E gives the following useful result:

Proposition 4.3.3. *Let $m > n$ be such that $\iota : E_m \hookrightarrow E_n$ is Hilbert–Schmidt. If $B \subseteq E_n'$ is bounded, then B is pre-compact in $\left(E_m, \|\cdot\|_{-m}\right)$.*

4.4. Defining the M_1 topology for E'-valued processes

Analogous to Definition 2.1.2, we have:

Definition 4.4.1 ($D_{E'}$)**.** The space $D_{E'}$ is defined to be the space of functions from $[0, T]$ to E' that are right-continuous and have left limits with respect to the strong topology on E'.

And likewise we still have Mitoma's helpful result:

Theorem 4.4.2 (Sample continuity, [**Mit83a**])**.** *Let ξ be an E'-valued process such that, for every $\phi \in E$, the $\mathbf{R}$-valued process*

$$\xi(\phi) = (\xi_t(\phi))_{t \in [0,T]}$$

has a càdlàg version. Then ξ has a version in $D_{E'}$.

Unlike in Section 4.1, when the range space was a Banach space, the path space $D_{E'}$ is no longer metrizable. Therefore some thought must be given to generalise the notion of the M_1 topology and to do so we use a *projective limit topology* — that is, a topology that arises as a limit of a sequence of pseudometrics.

First, observe that the definition of the graph of an element $x \in D_X$ in (4.1.1) and Definitions 4.1.2 and 4.1.4 only rely on X being a linear space, hence we can take $X = E'$.

Let $\{p_B : B \in \mathcal{B}\}$ be the family of semi-norms (4.3.1) that generates the strong topology on E'. Fix $B \in \mathcal{B}$, then, analogous to the definition of the M_1 metric for a Banach space, we can define the map $d_{B,M_1} : D_{E'} \times D_{E'} \to [0, \infty)$ by

$$d_{B,M_1}(x, y) := \inf_{(u_1,r_1)\in\Gamma_x,(u_2,r_2)\in\Gamma_y} \mathrm{dist}_B\left[(u_1, r_1); (u_2, r_2)\right],$$

where we have the semi-norm

$$\mathrm{dist}_B\left[(u_1, r_1); (u_2, r_2)\right] := \sup_{t\in[0,T]} \left\{p_B(u_1(t) - u_2(t)) \vee |r_1(t) - r_2(t)|\right\}.$$

The proofs of Lemma 12.3.1 and Lemma 12.3.2 of [**Whi02**] are unchanged if we replace the spatial norms by the semi-norm p_B, so we conclude that d_{B,M_1} satisfies the triangle inequality. It is straightforward to see that d_{B,M_1} is symmetric in its arguments, hence d_{B,M_1} is a *pseudometric*: d_{B,M_1} has all the properties of a metric except $d_{B,M_1}(x, y) = 0$ need not imply $x = y$. We then equip $D_{E'}$ with the projective limit topology:

Definition 4.4.3 (M_1 topology for $D_{E'}$)**.** We define the M_1 topology on $D_{E'}$ to be topology generated by the collection of pseudometrics $\{d_{B,M_1} : B \in \mathcal{B}\}$. That is, the topology with neighbourhoods

$$\left\{x \in D_{E'} : d_{B_j,M_1}(x, x_0) < \varepsilon_j, j = 1, 2, \ldots, n\right\}, \quad n \geq 1, \varepsilon_j > 0, B_j \in \mathcal{B}.$$

We will occasionally denote this topology by $\mathcal{T}_{E',M_1}$.

With this construction it is straightforward to check that $(D_{E'}, M_1)$ is completely regular, which is helpful for the proof of Proposition 4.0.2.

Proposition 4.4.4. *$(D_{E'}, M_1)$ is a completely regular space.*

Proof. By [**KX95**, Theorem 2.1.1], we need only check that for every pair of elements in $D_{E'}$ such that $x^1 \neq x^2$, there exists $B \in \mathcal{B}$ such that $d_{B,M_1}(x^1, x^2) > 0$. Since x^1 and x^2 aren't equal, there exists $t \in [0, T]$ and $\phi \in E$ such that

$$x^1_t(\phi) \neq x^2_t(\phi). \tag{4.4.1}$$

Now take any neighbourhood of the origin in E,

$$U = \left\{ \psi \in E : p_{\lambda_j}(\psi) < \varepsilon_j, j = 1, 2, \ldots, n \right\},$$

where $\{p_\lambda\}_{\lambda \in \Lambda}$ are the semi-norms generating the topology of E. If we define

$$c_1 := \frac{\max_{1 \le j \le n} \left\{ p_{\lambda_j}(\phi) \right\}}{2 \min_{1 \le j \le n} \{\varepsilon_j\}} < \infty,$$

then

$$p_{\lambda_j}\left(c_1^{-1}\phi\right) = c_1^{-1} p_{\lambda_j}(\phi) \le \varepsilon_j/2 < \varepsilon_j,$$

and so $c_1^{-1}\phi \in U$, which implies $\{\phi\} \subseteq c_1 U$. Therefore $\{\phi\}$ is a bounded set in E. Thus $d_{\{\phi\},M_1}$ is one of our pseudometrics and, from (4.4.1), $d_{\{\phi\},M_1}(x^1, x^2) > 0$. $\square$

It is also helpful to know that we have continuity of the canonical projections and inclusions:

Proposition 4.4.5 (Continuity of projections and inclusions)**.** *We have the following:*

(i) For every $\phi \in E$, the projection map

$$\pi^\phi : (D_{E'}, M_1) \to (D_{\mathbf{R}}, M_1), \quad \pi^\phi(x) := (x_t(\phi))_{t \in [0,T]} =: x(\phi)$$

is continuous,

(ii) For every $n \ge 0$, the inclusion map $\iota_n : \left(D_{E'_n}, M_1\right) \to (D_{E'}, M_1)$ is continuous.

Proof. (i) It suffices to show that the pre-image of

$$U = \{g \in D_{\mathbf{R}} : d_{\mathbf{R},M_1}(f, g) < \varepsilon\}$$

is open in $D_{E'}$ for all $f \in D_{\mathbf{R}}$ and $\varepsilon > 0$. Now

$$\left(\pi^\phi\right)^{-1}\left(U\right) = \left\{ y \in D_{E'} : d_{\mathbf{R},M_1}\left(f, \pi^\phi\left(y\right)\right) < \varepsilon \right\},$$

and if this set is empty then it is open and we are done, so suppose we can take $x \in \left(\pi^\phi\right)^{-1}\left(U\right)$. We will then be done if we can show a $(D_{E'}, M_1)$-neighbourhood of x is contained in $\left(\pi^\phi\right)^{-1}\left(U\right)$.

Set $\eta := d_{\mathbf{R},M_1}\left(f, \pi^\phi\left(x\right)\right) < \varepsilon$ and define

$$\begin{aligned} N &= \left\{ y \in D_{E'} : d_{\mathbf{R},M_1}\left(\pi^\phi\left(x\right), \pi^\phi\left(y\right)\right) < \varepsilon - \eta \right\} \\ &= \left\{ y \in D_{E'} : d_{\{\phi\},M_1}\left(x, y\right) < \varepsilon - \eta \right\}. \end{aligned}$$

For any $z \in N$ we have

$$d_{\mathbf{R},M_1}\big(f,\pi^{\phi}(z)\big) \le d_{\mathbf{R},M_1}\big(f,\pi^{\phi}(x)\big) + d_{\mathbf{R},M_1}\big(\pi^{\phi}(x),\pi^{\phi}(z)\big) < \varepsilon,$$

so $z \in \left(\pi^{\phi}\right)^{-1}\big(U\big)$, and therefore we are done since $\{\phi\}$ is a bounded set in E, for every $\phi \in E$.

(ii) The argument is similar to (i). Fix $n \ge 0$ and $x \in D_{E'}$. Take an open neighbourhood of x, say

$$U = \{y \in D_{E'} : d_{B,M_1}(x,y) < \varepsilon\},$$

for some $\varepsilon > 0$ and B a bounded subset of E. If $\iota_n^{-1}(U)$ is empty we are done, so suppose $x_0 \in \iota_n^{-1}(U)$. Since $E \subseteq E_n$, there exists a constant $c_2 > 0$ such that for every $f \in E_n'$

$$p_B(f) \le \sup_{\phi\in B} \|\phi\|_n \|f\|_{-n} \le c_2 \|f\|_{-n},$$

and hence

$$d_{B,M_1}(x_0,y) \le c_2 d_{E_n',M_1}(x_0,y).$$

Therefore

$$\left\{y \in E_n' : d_{E_n',M_1}(x_0,y) < \varepsilon/c_2\right\}$$

is an open neighbourhood of f and is contained in $\iota_n^{-1}(U)$. This shows $\iota_n^{-1}(U) \in \mathcal{T}_{E_n',M_1}$, which completes the proof. ☐

The special structure of E' transfers to $D_{E'}$ and we have the following strong result:

Proposition 4.4.6. $D_{E'} = \bigcup_{n=0}^{\infty} D_{E_n'}$.

To prove this result we use the following lemma .

Lemma 4.4.7 (A Baire category result, [**KX95**, Lemma 1.3.1])**.** *If $V : E \to [0,\infty)$ satisfies all of the following:*

(i) V is lower semicontinous*: $V(\phi) \le \liminf_{n\to\infty} V(\phi_n)$, whenever $(\phi_n) \to \phi$ in E,*
(ii) $V(\phi+\psi) \le V(\phi) + V(\psi)$, for all $\phi,\psi \in E$,
(iii) $V(a\phi) = |a| V(\phi)$ for all $a \in \mathbf{R}$ and $\phi \in E$,

then V is continuous and there exists $\theta > 0$ and $r \ge 0$ such that

$$V(\phi) \le \theta \|\phi\|_r, \quad \textit{for every } \phi \in E.$$

Proof of Proposition 4.4.6. The non-trivial inclusion is

$$D_{E'} \subseteq \bigcup_{n=0}^{\infty} D_{E'_n},$$

so fix $x \in D_{E'}$. It is straightforward to check the conditions of Lemma 4.4.7 hold for the map

$$V(\phi) := \sup_{t\in[0,T]} |x_t(\phi)|.$$

Therefore take $\theta > 0$ and $r \geq 0$ as in the conclusion of the lemma and let $q > r$ be such that the inclusion map from $E_q \hookrightarrow E_r$ is Hilbert–Schmidt. Let $\{e_i^q\}_i$ be a complete orthonormal system for E_q. Then

$$\|x_t\|_{-q}^2 = \sum_{i=1}^{\infty} |x_t(e_i^q)|^2 \leq \theta \sum_{i=1}^{\infty} \|e_i^q\|_r^2, \quad \text{for every } t,$$

and this completes the proof since the final sum is finite. □

Since càdlàg functions on compact sets are bounded [**Whi02**, Corollary 12.2.3], an immediate consequence of Proposition 4.4.6 and [**KX95**, Lemma 1.3.4] is:

Corollary 4.4.8 (Bounded on compact sets)**.** *If $x \in D_{E'}$, then there exists $r > 0$ and $0 < c < \infty$ such that*

$$\sup_{t\in[0,T]} \|x_t\|_{-r} \leq c.$$

In particular, if $x \in D_{\mathscr{S}'}$, then there exists $\alpha, \beta \in \mathbf{N} \cup \{0\}$ and $0 < c < \infty$ such that

$$\sup_{t\in[0,T]} |x_t(\phi)| \leq c\,\|\phi\|_{\alpha,\beta} = c \cdot \sup_{y\in\mathbf{R}} \left|y^\alpha \partial_y^\beta \phi(y)\right|, \quad \text{for all } \phi \in \mathscr{S}.$$

This is a deterministic result, so for a random process $\xi \in D_{E'}$ we may have a different r and c for every $\omega \in \Omega$. However, we can define

$$V(\phi) := \mathbf{E}\left[\int_0^T |\xi_t(\phi)|^p \, dt\right]^{1/p}, \quad \text{for } 1 \leq p < \infty,$$

then Fatou's Lemma guarantees condition (i) of Lemma 4.4.7 remain valid, and (ii) and (iii) hold trivially. It may be the case that $V(\phi) = \infty$ (indeed it is simple to produce such an example), so we need a caveat to deduce a stochastic equivalent of Corollary 4.4.8:

Corollary 4.4.9 (Moments bounded on compact sets). *If ξ is a random variable taking values in $D_{E'}$, then for every $1 \leq p < \infty$ such that*

$$\mathbf{E}\left[\int_0^T \left|\xi_t(\phi)\right|^p dt\right]^{1/p} < \infty$$

there exists $\alpha, \beta \in \mathbf{N} \cup \{0\}$ and $c < \infty$ (depending on ξ and p) such that

$$\mathbf{E}\left[\int_0^T \left|\xi_t(\phi)\right|^p dt\right]^{1/p} \leq c \left\|\phi\right\|_{\alpha,\beta} = c \cdot \sup_{y \in \mathbf{R}} \left|y^\alpha \partial_y^\beta \phi(y)\right|.$$

Remark 4.4.10 ($p = \infty$). It is clear that Corollary 4.4.9 holds with $p = \infty$, whereby the integral is replaced by a supremum over $[0, T]$.

Measurability. We have an analogous result to Section 4.1, where the Kolmogorov σ-algbera coincides with the Borel σ-algbera of $(D_{E'}, M_1)$. Here, we have to take into account spatial projections as well as time projections:

Proposition 4.4.11 (Measurability). *Let $\mathscr{B}(D_{E'})$ denote the Borel σ-algebra generated by the M_1 topology on $D_{E'}$ and $\mathscr{K}(D_{E'})$ be the Kolmogorov σ-algebra generated by the projection maps of the form*

$$\pi_{t_1,\ldots,t_n}^{\phi_1,\ldots,\phi_n} : D_{E'} \to \mathbf{R}^n, \quad \pi_{t_1,\ldots,t_n}^{\phi_1,\ldots,\phi_n}(x) = \left(x_{t_1}(\phi_1), \ldots, x_{t_n}(\phi_n)\right).$$

Then $\mathscr{B}(D_{E'}) = \mathscr{K}(D_{E'})$.

Proof. The inclusion $\mathscr{K} \subseteq \mathscr{B}$ is straightforward: without loss of generality take $n = 1$, then we can write $\pi_t^\phi = \pi_t \circ \pi^\phi$ with the canonical projections

$$\pi_t : (D_{\mathbf{R}}, M_1) \to \mathbf{R} \quad \text{and} \quad \pi^\phi : (D_{E'}, M_1) \to (D_{\mathbf{R}}, M_1).$$

As remarked in Section 4.1, [**Whi02**, Theorem 11.5.2] gives that π_t is measurable and we have that π^ϕ is measurable because it is continuous by Proposition 4.4.5 (i).

For the inclusion $\mathscr{B} \subseteq \mathscr{K}$, let $\iota_n : D_{E'_n} \to D_{E'}$ be the canonical inclusion map and $U \in \mathcal{T}_{E',M_1}$. Proposition 4.4.6 gives

$$U = \bigcup_{n \geq 0} \iota_n^{-1}(U), \tag{4.4.2}$$

and from Proposition 4.4.5 we know that $\iota_n : \left(D_{E'_n}, M_1\right) \to (D_{E'}, M_1)$ is continuous and therefore measurable, so

$$\iota_n^{-1}(U) \in \mathscr{B}\left(D_{E'_n}, M_1\right) = \sigma\left(\pi_t : D_{E'_n} \to E'_n\right),$$

where these σ-algebras are equal by [**EK86**, Proposition 7.1]. (E'_n is separable by construction.) Since $E'_n \subseteq E'$, we have

$$\sigma\left(\pi_t : D_{E'_n} \to E'_n\right) \subseteq \sigma\left(\pi_t : D_{E'} \to E'\right).$$

Now $\mathscr{B}(E') = \sigma\left(\pi^\phi : E' \to \mathbf{R}\right)$ by [**KX95**, Theorem 3.1.1], therefore

$$\sigma\left(\pi_t : D_{E'} \to E'\right) = \mathscr{K}$$

and so $\iota_n^{-1}(U) \in \mathscr{K}$. The countable union in (4.4.2) gives $U \in \mathscr{K}$, which completes the proof. $\square$

4.5. Two technical lemmas

Before we can consider the limiting behavior of random variables on the space $D_{E'}$, we need to develop the following results, which are computationally heavy. The first is a simple geometric argument and an adaptation of [**HS79**, Lemma A.28].

Lemma 4.5.1 (A geometric argument). *Fix $M \in \mathbf{N}$ and $\varepsilon > 0$, and let B be the closed unit ball in $\mathbf{R}^M$. Then there exists a finite set $\Theta = \{\theta_1, \ldots, \theta_k\} \subseteq \mathbf{R}^M$ such that $\|\theta_i\|_{\mathbf{R}^M} = 1$ and for all $v_1, v_2 \in B$ satisfying*

$$\min_{\lambda \in [0,1]} \|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M} \geq \varepsilon,$$

there is an $i \in \{1, 2, \ldots, k\}$ for which

$$\|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M} \leq 2\varepsilon^{-1} \left|(\lambda v_1 + (1-\lambda) v_2) \cdot \theta_i\right|,$$

for every $\lambda \in [0,1]$. Here, $\|\cdot\|_{\mathbf{R}^M}$ is the Euclidean norm and k depends on M and ε.

Proof. For $v_1, v_2 \in B$ and let $\lambda_0 = \lambda_0(v_1, v_2)$ be a minimiser of

$$\lambda \mapsto \|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M}$$

on $[0,1]$. (In other words, choose $\lambda_0 \in [0,1]$ so that $\lambda_0 v_1 + (1-\lambda_0) v_2$ is as close to the origin as possible.) Define

$$S = \left\{(v_1, v_2) \in B \times B : \|\lambda_0 v_1 + (1-\lambda_0) v_2\|_{\mathbf{R}^M} \geq \varepsilon\right\},$$

then for $(v_1, v_2) \in S$ set $\theta(v_1, v_2)$ to be the unit vector in the direction of the pair's minimiser:

$$\theta(v_1, v_2) = \frac{\lambda_0 v_1 + (1-\lambda_0) v_2}{\|\lambda_0 v_1 + (1-\lambda_0) v_2\|_{\mathbf{R}^M}}.$$

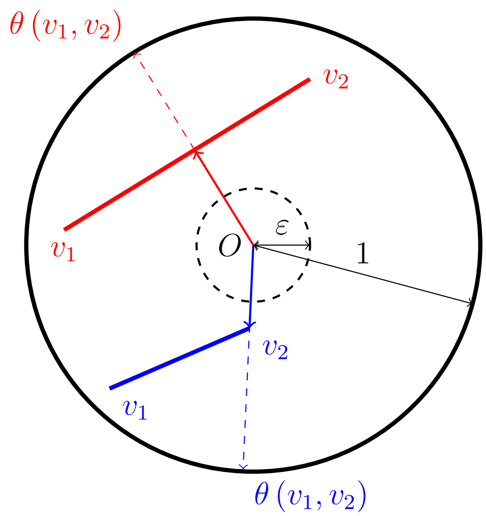


FIGURE 4.5.1. Two examples of the vectors v_1 and v_2 from Lemma 4.5.1 and the unit vector, $\theta(v_1, v_2)$, in the direction of the shortest distance from O to the line $[v_1, v_2]$.

By considering the plane through 0, v_1 and v_2 (see Figure 4.5.1), we have that, for every $\lambda \in [0,1]$,

$$
\begin{aligned}
(\lambda v_1 + (1-\lambda) v_2) \cdot \theta(v_1, v_2) \geq \varepsilon \geq \varepsilon \|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M} \\
> \frac{\varepsilon}{2} \|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M}.
\end{aligned}
$$

Hence there is an open neighbourhood, $N(v_1, v_2)$, in $\mathbf{R}^M \times \mathbf{R}^M$ about (v_1, v_2) such that for all $(w_1, w_2) \in N(v_1, v_2)$

$$
\varepsilon \|\lambda w_1 + (1-\lambda) w_2\|_{\mathbf{R}^M} \leq (\lambda w_1 + (1-\lambda) w_2) \cdot \theta(v_1, v_2).
$$

Now S is a compact set with open cover $\{N(v_1, v_2)\}_{(v_1, v_2) \in S}$, therefore we can take a finite sub-cover, $\{N(v_1^i, v_2^i)\}_{i=1,\dots k}$. To complete the proof we take $\Theta = \{\theta(v_1^i, v_2^i)\}_{i=1,\dots k}$. $\square$

This lemma allows us to replace supremums in the proceeding proof with maximums over finite sets, which are much more tractable objects to work with.

The next lemma is an adaption of the work in [**Mit83b**, Proposition 2.2] to replace the J_1 topology with the M_1 topology.

Lemma 4.5.2 (Controlling the modulus of continuity)**.** *Let $p > n$ be such that the inclusion $E_p \hookrightarrow E_n$ is Hilbert–Schmidt, and let $A \subseteq D_{E'}$ be such that*

$$
\sup_{x \in A} \sup_{t \in [0,T]} \|x_t\|_{-n} =: c < \infty.
$$

Then for every $\varepsilon > 0$, there exists $k \geq 1$ and $\phi_1, \phi_2, \ldots, \phi_k \in E$ such that

$$\sup_{x \in A} w_{E_p', M_1}(x; \delta) \leq 2c\varepsilon^{-1} \max_{i=1,\ldots k} \sup_{x \in A} w_{\mathbf{R}, M_1}(x(\phi_i); \delta) + 4c\varepsilon,$$

for every $\delta > 0$.

Proof. Recall that we have the definition

$$w_{E_p', M_1}(x; \delta) = \sup_{t, \mathrm{Trip}_{t,\delta}} \inf_{\lambda \in [0,1]} \|x_{t_2} - \lambda x_{t_1} - (1-\lambda) x_{t_3}\|_{-p}.$$

Our first aim is to approximate the elements of E on the right-hand side above with finite-dimensional vectors. For readability, let us introduce the abbreviation

$$y^\lambda = y(\lambda; x; t_1, t_2, t_3) := x_{t_2} - \lambda x_{t_1} - (1-\lambda) x_{t_3}.$$

Begin by recalling that

$$\|y^\lambda\|_{-p}^2 = \sum_{i=1}^{\infty} \left|y^\lambda(e_i^p)\right|^2 \tag{4.5.1}$$

and, for $x \in A$,

$$\left|y^\lambda(e_i^p)\right| \leq \|e_i^p\|_n \|y^\lambda\|_{-n} \leq 2c\|e_i^p\|_n,$$

where the second inequality is due to the triangle inequality. Choose $M = M(\varepsilon, p, n, s^*)$ large enough so that

$$\sum_{i=M+1}^{\infty} \|e_i^p\|_n^2 \leq \varepsilon.$$

Combining these inequalities gives

$$\|y^\lambda\|_{-p}^2 \leq \sum_{i=1}^{M} \left|y^\lambda(e_i^p)\right|^2 + 2c\varepsilon.$$

So we have approximated the tail of (4.5.1) and now focus on the finitely many remaining terms.

Define the following vectors in $\mathbf{R}^M$:

$$v_1 := \begin{pmatrix} (x_{t_2} - x_{t_1})(e_1^p) \\ (x_{t_2} - x_{t_1})(e_2^p) \\ \vdots \\ (x_{t_2} - x_{t_1})(e_M^p) \end{pmatrix}, \quad v_2 := \begin{pmatrix} (x_{t_2} - x_{t_3})(e_1^p) \\ (x_{t_2} - x_{t_3})(e_2^p) \\ \vdots \\ (x_{t_2} - x_{t_3})(e_M^p) \end{pmatrix}$$

Then we can rewrite the bound on the modulus of continuity as

$$w_{E_p',M_1}(x;\delta) \leq \sup_{t,\mathrm{Trip}_{t,\delta}} \inf_{\lambda\in[0,1]} \|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M} + 2c\varepsilon. \tag{4.5.2}$$

With ε and M as defined in this proof, take $\Theta = \{\theta_1, \theta_2, \ldots, \theta_k\}$ from Lemma 4.5.1. From the triangle inequality and our assumption on A we have

$$\|v_1(x;t_1,t_2)\|_{\mathbf{R}^M}, \|v_2(x;t_1,t_2)\|_{\mathbf{R}^M} \leq 2c,$$

so $(2c)^{-1} v_1$ and $(2c)^{-1} v_2$ are in the unit ball of $\mathbf{R}^M$. Therefore we can find an $i \in \{1,2,\ldots,k\}$ such that

$$\inf_{\lambda\in[0,1]} \|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M} \leq 2c\varepsilon^{-1} \inf_{\lambda\in[0,1]} |\theta_i \cdot (\lambda v_1 + (1-\lambda) v_2)|, \tag{4.5.3}$$

provided we don't have

$$\inf_{\lambda\in[0,1]} \|\lambda v_1 + (1-\lambda) v_2\|_{\mathbf{R}^M} < 2c\varepsilon.$$

However, both of these situations can be combined to give the following bound which holds for all $x \in A$

$$w_{E_p',M_1}(x;\delta) \leq 2c\varepsilon^{-1} \sup_{t,\mathrm{Trip}_{t,\delta}} \max_{i=1,\ldots k} \inf_{\lambda\in[0,1]} |\theta_i \cdot (\lambda v_1 + (1-\lambda) v_2)| + 4c\varepsilon. \tag{4.5.4}$$

In other words, either we can bound the right-hand side of (4.5.2) by $2c\varepsilon$, or (4.5.3) holds for some i, and therefore certainly holds for the maximum over all $i \in \{1,2,\ldots,k\}$. Therefore, the sum of these upper bounds is obviously an upper bound.

Now for each $i \in \{1,2,\ldots,k\}$ define

$$\phi_i := \sum_{j=1}^{M} \theta_i^j e_j^p \in E_p,$$

where θ_i^j is the j^{th} coordinate of θ_i. Then (4.5.4) becomes

$$\begin{aligned} w_{E_p',M_1}(x;\delta) &\leq 2c\varepsilon^{-1} \sup_{t,\mathrm{Trip}_{t,\delta}} \max_{i=1,\ldots k} \inf_{\lambda\in[0,1]} \left|y^{\lambda}(\phi_i)\right| + 4c\varepsilon. \\ &\leq 2c\varepsilon^{-1} \max_{i=1,\ldots k} w_{\mathbf{R},M_1}(x(\phi_i);\delta) + 4c\varepsilon, \end{aligned}$$

by switching the supremum and maximum. Taking a supremum over $x \in A$ and switching this supremum with the maximum gives the result. □

4.6. Proof of Theorem 4.0.1, Proposition 4.0.2 and Theorem 4.0.3

We begin by characterising compactness in $(D_{E'}, M_1)$, and we follow the methods from [**KX95**].

Proposition 4.6.1 (The compact subsets of $(D_{E'}, M_1)$)**.** *Let $A \subseteq D_{E'}$. The following are equivalent:*

(i) A is compact in $(D_{E'}, M_1)$,
(ii) For every $\phi \in E$, $\pi^{\phi}(A) = \{x(\phi) : x \in A\}$ is compact in $(D_{\mathbf{R}}, M_1)$,
(iii) There exists $p \in \mathbf{N}$ such that $A \subseteq D_{E'_p}$ and A is compact in $\left(D_{E'_p}, M_1\right)$.

Proof. Proposition 4.4.5 gives (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). We now show (ii) $\Rightarrow$ (iii).

Since $\pi^{\phi}(A)$ is compact for every $\phi \in E$, Theorem 4.1.8 with $X = \mathbf{R}$ allows us to define $V : E \to [0, \infty)$ by

$$V(\phi) := \sup_{x \in A} \sup_{t \in [0,T]} |x_t(\phi)| < \infty.$$

It is straightforward to see that V satisfies Lemma 4.4.7, so there exists $\theta > 0$ and $r \geq 0$ such that

$$V(\phi) \leq \theta \|\phi\|_r, \quad \text{for every } \phi \in E.$$

Take $p > n > r$ such that both inclusions $E_p \hookrightarrow E_n \hookrightarrow E_r$ are Hilbert–Schmidt. Let $\{e_i^p\}_{i \geq 1}$ and $\{e_i^n\}_{i \geq 1}$ be orthonormal bases of E_p and E_n. Notice that

$$\sup_{x \in A} \sup_{t \in [0,T]} \|x_t\|_{-n}^2 \leq \sum_{j=1}^{\infty} \sup_{x \in A} \sup_{t \in [0,T]} \left|x_t\left(e_j^n\right)\right|^2 \leq \theta^2 \sum_{j=1}^{\infty} \left\|e_j^n\right\|_r^2 =: c^2 < \infty.$$

Since $\|\cdot\|_{-p} \leq \|\cdot\|_{-n}$ we conclude $A \subseteq D_{E'_p}$ and, for every $x \in A$ and $t \in [0, T]$,

$$x_t \in \left\{f \in E'_n : \|f\|_{-n} \leq c\right\},$$

which is a bounded subset of E'_n, so is a pre-compact subset of E'_p. Therefore, by taking the closure, we have verified condition (i) of Theorem 4.1.8 for A.

It remains to control the modulus of continuity of A in E'_p. Fix an $\varepsilon > 0$ and take $\phi_1, \phi_2, \ldots, \phi_k$ from Lemma 4.5.2. Since k is finite, Theorem 4.1.8 and our hypothesis allow us to deduce that

$$\lim_{\delta \to 0} \max_{i=1,\ldots,k} \sup_{x \in A} w_{\mathbf{R}, M_1}\left(x\left(\phi_i\right); \delta\right) = 0.$$

Therefore Lemma 4.5.2 gives

$$\lim_{\delta\to 0}\sup_{x\in A} w_{E'_p,M_1}(x;\delta) \leq 4c\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, we conclude

$$\lim_{\delta\to 0}\sup_{x\in A} w_{E'_p,M_1}(x;\delta) = 0.$$

Reapplying the methods in the proof of Lemma 4.5.2 and the above reasoning, it is straightforward to also deduce

$$\lim_{\delta\to 0}\sup_{x\in A}\Big\{\sup_{t\in(0,\delta)}\|x_t - x_0\|_{E'_p} + \sup_{t\in(T-\delta,T)}\|x_T - x_t\|_{E'_p}\Big\} = 0,$$

and so we have verified condition (ii) of Theorem 4.1.8 for $A \subseteq D_{E'_p}$, which guarantees A is compact in $\left(D_{E'_p}, M_1\right)$. $\square$

Since E'_p is a Hilbert space, it is immediate that we now have:

Corollary 4.6.2. *Every compact subset of $(D_{E'}, M_1)$ is metrizable.*

And this corollary implies Proposition 4.0.2:

Proof of Proposition 4.0.2. From [**SF76**, Section 3, Definition 2, Example 1] we know that each $D_{E'_p}$ is a topological Radon space since it is a Polish space. Therefore [**SF76**, Section 3, Example 4] gives that $D_{E'}$ is a topological Radon space, hence every probability measure on $D_{E'}$ is a Radon measure. As we know that $D_{E'}$ is completely regular (Proposition 4.4.4), [**SF76**, Section 5, Theorem 2] together with Corollary 4.6.2 gives the result. $\square$

Next, we prove our main tightness result:

Proof of Theorem 4.0.1. Fix $\varepsilon > 0$. Since Lemma 2.5.1 of [**KX95**] uses only condition (i) of Theorem 4.1.8, and so does not depend on the choice of Skorokhod topology, we can deduce that there exists $c > 0$ and $q > 0$ such that the set

$$K_0 := \Big\{x \in D_{E'} : \sup_{t\in[0,T]}\|x_t\|_{-q} \leq M\Big\}$$

satisfies

$$\mu_n(K_0) \geq 1 - \varepsilon/2, \quad \text{for every } n \geq 1.$$

By applying the argument in the first half of the proof of Proposition 4.6.1 with A replaced by K_0 (and n by q), we deduce that there is a $p > q$ for which $E_p \hookrightarrow E_q$ is Hilbert–Schmidt and $K_0 \subseteq D_{E'_p}$ with every $x \in K_0$ taking values in some compact subset, B, of E'_p.

Let $\{e_i^p\}_{i\geq 1}$ be an orthonormal basis of E_p. By hypothesis, for every $i \geq 1$, there exists a compact subset $\tilde{K}_i$ of $(D_{\mathbf{R}}, M_1)$ such that

$$\mu_n \circ \left(\pi^{e_i^p}\right)^{-1}\left(\tilde{K}_i\right) \geq 1 - \varepsilon/2^{i+1}, \quad \text{for every } n \geq 1.$$

If we define

$$K_i := \left(\pi^{e_i^p}\right)^{-1}\left(\tilde{K}_i\right) \subseteq D_{E'}, \quad K_\infty := K_0 \cap \bigcap_{i=1}^{\infty} K_i$$

then we have

$$\mu_n\left(K_\infty\right) \geq 1 - \varepsilon \sum_{i=0}^{\infty} 2^{-(i+1)} = 1 - \varepsilon, \quad \text{for every } n \geq 1,$$

and that every $x \in K_\infty$ is in $x \in K_0$, so x takes values in the compact subset $B \subseteq D_{E_p'}$. The work in the second half of the proof of Proposition 4.6.1 with $A = K_\infty$ allows us to deduce that

$$\lim_{\delta\to 0} \sup_{x\in K_\infty} \left\{ w_{E_p', M_1}\left(x;\delta\right) + \sup_{t\in(0,\delta)} \left\|x_t - x_0\right\|_X + \sup_{t\in(T-\delta,T)} \left\|x_T - x_t\right\|_X \right\} = 0.$$

Therefore by Theorem 4.1.8, K_∞ is a compact subset of $\left(D_{E_p'}, M_1\right)$, and so is also a compact subset of $(D_{E'}, M_1)$, by Proposition 4.6.1. This confirms tightness.

As the Borel σ-algebra of $(D_{E'}, M_1)$ coincides with Kolmogorov σ-algebra generated by the projection maps, the proof of Proposition 5.1 of **[Mit83b]** gives Theorem 4.0.3. $\square$

4.7. Tightness of the loss-dependent correlation system; proof of Theorem 4.0.4

Here, we use the technology we have developed to prove Theorem 4.0.4. To accommodate the boundary effect, we introduce the continuous process

$$\bar{\nu}_t^N := \nu_t^N + L_t^N \delta_0 = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}_{t\wedge\tau^{i,N}}},$$

then

$$\bar{\nu}_t^N\left(\phi\right) = \nu_t^N\left(\phi\right) + \phi\left(0\right) L_t^N, \quad \text{for } \phi \in \mathscr{S}.$$

It is straightforward to apply the triangle inequality to Definition 4.1.2 to get

$$\begin{aligned} H_{\mathbf{R}}\left(\nu_{t_1}^N\left(\phi\right), \nu_{t_2}^N\left(\phi\right), \nu_{t_3}^N\left(\phi\right)\right) &\leq \left|\bar{\nu}_{t_1}^N\left(\phi\right) - \bar{\nu}_{t_2}^N\left(\phi\right)\right| + \left|\bar{\nu}_{t_2}^N\left(\phi\right) - \bar{\nu}_{t_3}^N\left(\phi\right)\right| \\ &\quad + \left|\phi\left(0\right)\right| H_{\mathbf{R}}\left(L_{t_1}^N, L_{t_2}^N, L_{t_3}^N\right). \end{aligned} \tag{4.7.1}$$

The final term vanishes by Proposition 4.1.7 and we can exploit the continuity of ν^N to control the increment terms:

Lemma 4.7.1 (Increment moments)**.** *For every $\phi \in \mathscr{S}$, there exists a constant $c = c(\phi) > 0$ such that for all $s, t \in [0, T]$ and $N \geq 1$*

$$\mathbf{E}\left[\left|\bar{\nu}_t^N(\phi) - \bar{\nu}_s^N(\phi)\right|^4\right] \leq c\,(t-s)^2.$$

Hence for all $N \geq 1$, $0 \leq t_1 \leq t_2 \leq t_3 \leq T$ and $\varepsilon > 0$

$$\mathbf{P}\left(H_{\mathbf{R}}\left(\nu_{t_1}^N(\phi), \nu_{t_2}^N(\phi), \nu_{t_3}^N(\phi)\right) \geq \varepsilon\right) \leq 2c\varepsilon^{-4}(t_3 - t_1)^2.$$

Proof. By definition

$$\bar{\nu}_t^N(\phi) - \bar{\nu}_s^N(\phi) = \frac{1}{N}\sum_{i=1}^N \left[\phi\big(X_{t\wedge\tau^{i,N}}^{i,N}\big) - \phi\big(X_{s\wedge\tau^{i,N}}^{i,N}\big)\right].$$

Since $\phi \in \mathscr{S}$, ϕ must be Lipschitz (with constant $\|\phi\|_{0,1}$) and therefore

$$\left|\bar{\nu}_t^N(\phi) - \bar{\nu}_s^N(\phi)\right| \leq \frac{\|\phi\|_{0,1}}{N}\sum_{i=1}^N \left|X_{t\wedge\tau^{i,N}}^{i,N} - X_{s\wedge\tau^{i,N}}^{i,N}\right|.$$

By Proposition A.0.2 we have

$$\begin{aligned}\mathbf{E}\Big[\left|\bar{\nu}_t^N(\phi) - \bar{\nu}_s^N(\phi)\right|^4\Big] &\leq \frac{\|\phi\|_{0,1}^4}{N}\sum_{i=1}^N \mathbf{E}\Big[\left|X_{t\wedge\tau^{i,N}}^{i,N} - X_{s\wedge\tau^{i,N}}^{i,N}\right|^4\Big] \\ &= \|\phi\|_{0,1}^4\,\mathbf{E}\Big[\left|X_{t\wedge\tau^{1,N}}^{1,N} - X_{s\wedge\tau^{1,N}}^{1,N}\right|^4\Big],\end{aligned}$$

where the final equality occurs because the particles are identically distributed. Since $X^{1,N}$ is a Brownian motion, the Burkholder–Davis–Gundy inequality [**RW00**] gives a numerical constant $C > 0$ such that

$$\begin{aligned}\mathbf{E}\Big[\left|\bar{\nu}_t^N(\phi) - \bar{\nu}_s^N(\phi)\right|^4\Big] &\leq C\,\|\phi\|_{0,1}^4\,\mathbf{E}\Big[\left|t\wedge\tau^{1,N} - s\wedge\tau^{1,N}\right|^2\Big] \\ &\leq C\,\|\phi\|_{0,1}^4\,(t-s)^2,\end{aligned}$$

for all $s, t \in [0, T]$. Taking $c = C\,\|\phi\|_{0,1}^4$ gives the first statement.

From (4.7.1) and Markov's inequality we deduce that

$$\begin{aligned}\mathbf{P}\left(H_{\mathbf{R}}\left(\nu_{t_1}^N(\phi), \nu_{t_2}^N(\phi), \nu_{t_3}^N(\phi)\right) \geq \varepsilon\right) \leq \varepsilon^{-4}\Big(&\mathbf{E}\left[\left|\bar{\nu}_{t_1}^N(\phi) - \bar{\nu}_{t_2}^N(\phi)\right|^4\right] \\ &+ \mathbf{E}\left[\left|\bar{\nu}_{t_2}^N(\phi) - \bar{\nu}_{t_3}^N(\phi)\right|^4\right]\Big).\end{aligned}$$

Therefore we are done by using the above result and the convexity of $x \mapsto x^2$. $\square$

This lemma is useful for establishing condition (ii) of Theorem 4.1.11 and from this we can prove:

Proposition 4.7.2. $\left(\nu^N\right)_{N\geq 1}$ *is tight on* $(D_{\mathscr{S}'}, M_1)$.

We make use of the following result, which is helpful in practical calculations involving the M_1 topology.

Theorem 4.7.3 (Avram–Taqqu condition, **[AT89]**). *Let* ξ *be a* $D_{\mathbf{R}}$*-valued stochastic process. If there exists* $a, b, c > 0$ *such that*

$$\mathbf{P}\left(H_{\mathbf{R}}\left(\xi_{t_1}, \xi_{t_2}, \xi_{t_3}\right) \geq \eta\right) \leq c\eta^{-a}\left(t_3 - t_1\right)^{1+b},$$

for all $0 \leq t_1 \leq t_2 \leq t_3 \leq T$ *and* $\eta > 0$*, then there is some constant* $C = C(a, b, c)$ *such that*

$$\mathbf{P}\left(w_{\mathbf{R},M_1}(\xi;\delta) \geq \eta\right) \leq C\eta^{-a}\delta^b, \quad \textit{for all } \varepsilon, \delta > 0.$$

Proof of Proposition 4.7.2. By Theorem 4.0.1, we are done if we can show $\left(\nu^N(\phi)\right)_{N\geq 1}$ is tight on $(D_{\mathbf{R}}, M_1)$ for every $\phi \in \mathscr{S}$. Theorem 4.7.3 gives

$$\mathbf{P}\left(w_{\mathbf{R},M_1}\left(\nu^N(\phi);\delta\right) \geq \eta\right) = O(\delta), \quad \text{independent of } N, \tag{4.7.2}$$

from Lemma 4.7.1. To establish (ii) of Theorem 4.1.11, we also need to control the end-point terms. For the initial point notice that

(4.7.3)

$$\begin{aligned}
\mathbf{P}\Big(\sup_{t\in(0,\delta)}\left|\nu_t^N(\phi)-\nu_0^N(\phi)\right|\geq\eta\Big) &\leq \mathbf{P}\Big(\sup_{t\in(0,\delta)}\left|\bar{\nu}_t^N(\phi)\right|\geq\eta\Big)+\mathbf{P}\Big(\sup_{t\in(0,\delta)}\left|\phi(0)\right|L_t^N\geq\eta\Big)\\
&\leq \mathbf{P}\Big(\frac{1}{N}\sum_{i=1}^N\sup_{t\in(0,\delta)}\left|\phi\big(X^{i,N}_{t\wedge\tau^{i,N}}\big)-\phi\big(X^{i,N}_{0\wedge\tau^{i,N}}\big)\right|\geq\eta\Big)\\
&\qquad+\mathbf{P}\big(|\phi(0)|L_\delta^N\geq\eta\big)\\
&\leq \eta^{-1}\mathbf{E}\Big[\sup_{t\in(0,\delta)}\left|\phi\big(X^{1,N}_{t\wedge\tau^{1,N}}\big)-\phi\big(X^{1,N}_{0\wedge\tau^{1,N}}\big)\right|\Big]\\
&\qquad+\eta^{-1}|\phi(0)|^{-1}\mathbf{E}L_\delta^N,
\end{aligned}$$

where, in the final line, we have applied Markov's inequality and the fact that the particles are identical in law. As

$$\mathbf{E}L_\delta^N = \mathbf{P}\big(\tau^{1,N}\leq\delta\big)$$

and the particle behave as Brownian motions when considered individually, the final line in (4.7.3) is bounded by $\eta^{-1}o(1)$ as $\delta \to 0$, uniformly in N. This gives

$$\mathbf{P}\Big(\sup_{t\in(0,\delta)}\big|\nu_t^N(\phi) - \nu_0^N(\phi)\big| \geq \eta\Big) = o(1)\,, \quad \text{uniformly in } N,$$

as $\delta \to 0$, and similar work gives the corresponding condition at T, so together with (4.7.2) we have verified (ii) of Theorem 4.1.11.

Condition (i) of Theorem 4.1.11 is straightforward: since ν^N is a sub-probability measure we have

$$\big|\nu_t^N(\phi)\big| \leq \|\phi\|_\infty\,, \quad \text{for every } N \text{ and } t.$$

Therefore we have that $\big(\nu^N(\phi)\big)_{N\geq 1}$ is tight, as required. $\square$

We can now complete our main proof. Let μ_1, μ_2 and μ_3 denote the laws of ν^N, L^N and W, and let μ denote the joint law of the triple. Then with K_ε^i a compact set such that

$$\mu_i\left(K_\varepsilon^i\right) > 1 - \varepsilon,$$

(as in Definition 4.0.4) we have that $K_\varepsilon := K_\varepsilon^1 \times K_\varepsilon^2 \times K_\varepsilon^3$ is compact in the product topology [**Mun00**, Theorem 26.5] and

$$\mu\big(\left(K_\varepsilon\right)^c\big) \leq \mu_1\big(\left(K_\varepsilon^1\right)^c\big) + \mu_2\big(\left(K_\varepsilon^2\right)^c\big) + \mu_3\big(\left(K_\varepsilon^3\right)^c\big) \leq 3\varepsilon.$$

Therefore the laws of $\left(\nu^N, L^N, W\right)$ are tight. (That is, marginal tightness implies joint tightness.) Relative compactness follows since the product space is a product of completely regular spaces, so is completely regular [**Mun00**, Theorem 33.2]. $\square$

CHAPTER 5

Loss-dependent correlation

Now that we have developed sufficient machinery to handle the loss-dependent correlation system (1.3.1), we will apply our methods to prove the system converges to the unique solution of the following non-linear SPDE:

Definition 5.0.1 (Solution)**.** Let f be a probability measure on $(0,\infty)$ and W a standard Brownian motion. A pair of processes (ξ, L) taking values in $D_{\mathscr{S}'} \times D_{\mathbf{R}}$ is said to be a *solution to the loss-dependent SPDE with respect to* f *and* W if the following hold with probability 1:

(i) For every $t \in [0,T]$, ξ_t is a finite measure,
(ii) L takes values in $[0,1]$, is initially zero and is increasing,
(iii) For every $t \in [0,T]$ and $\phi \in C^{\text{test}}$

$$\xi_t(\phi) = f(\phi) + \frac{1}{2}\int_0^t \xi_s(\partial_{xx}\phi)\,ds + \int_0^t \rho(L_s)\,\xi_s(\partial_x\phi)\,dW_s,$$

(iv) For every $t \in [0,T]$, $L_t = 1 - \xi_t(0,\infty)$,
(v) The regularity conditions (ii)-(v) of Definition 2.0.3 hold.

The first two conditions are necessary to enable us to consider the total mass of ξ on the half-line and, since ρ is a function on $[0,1]$, to ensure that $t \mapsto \rho(L_t)$ is a well-defined càdlàg process (the composition of a càdlàg function with an increasing càdlàg function is càdlàg). Conditions (iii) and (iv) capture the loss-dependent dynamics and constitute the non-linearity in the system. The final condition ensures that we can apply the analysis developed in Chapter 2.

Remark 5.0.2 (Model parameters)**.** Recall the assumptions on ρ from Definition 1.3.1. As usual, ν_0 will be assumed to be supported on a finite interval away from the boundary. For this chapter, however, we will also assume ν_0 has a density $V_0 \in L^2(0,\infty)$. In Proposition 5.5.5, this assumption allows us to deduce that limiting empirical processes have an L^2 density at all times with norm bounded by that of the initial condition. This cannot be used directly to prove uniqueness, however it forms part of the uniqueness argument in Section 5.6 where it is used to control higher-order terms in the energy estimate — see (5.6.2).

Our strategy will be to take a realisation, $\big(\nu^*, L^*, W\big)$, of one of the limiting laws of the system $\big(\nu^N, L^N, W\big)_{N\geq 1}$, using Theorem 4.0.4, and to show that Definition 5.0.1 is satisfied:

Theorem 5.0.3 (Existence)**.** *Every limit point, (ν^*, L^*, W), is a solution to the loss-dependent SPDE with respect to (ν_0, W).*

Remark 5.0.4. Whenever we are considering a general limit point, we will use the notation $\big(\nu^*, L^*, W\big)$ to indicate that we do not yet know whether or not this is the unique limit point.

As we are considering weak limit points, checking conditions (i)-(v) of Definition 5.0.1 requires the use of the continuous mapping theorem (see Section 5.1). In Sections 5.2 and 5.3 we are able to recover (ii) and (v) by applying a number of technical probability estimates at the microscopic particle level. In Section 5.4 a martingale argument is used to establish (iii).

An important point to notice is that, although the loss-dependent SPDE is non-linear in the pair (ξ, L), the SPDE in (iii) is linear in ξ — it is just the basic SPDE from Definition 2.0.3 with $\rho_t = \rho(L_t)$. In Section 5.5 we construct the tagged particle system:

$$\mu_t^*(S) := \mathbf{P}\left(X_t^* \in S, t < \tau^* \middle| W, L^*\right),$$

where, for $W^\perp$ an independent Brownian motion, we have

$$\begin{cases} X_t^* = X_0^* + \int_0^t \rho(L_s^*)\,dW_s + \int_0^t \sqrt{1-\rho(L_s^*)^2}\,dW_s^\perp \\ \tau^* = \inf\{t > 0 : X_t^* \leq 0\}, \end{cases} \tag{5.0.1}$$

and we show that μ^* solves the basic SPDE with respect to $\rho_t = \rho(L_t^*)$. Linearity then allows us to apply the methods in Chapter 2 and deduce $\nu^* = \mu^*$ (see Remark 5.5.3). This representation immediately implies conditions (i) and (iv).

After establishing Theorem 5.0.3, the goal is to prove uniqueness of solutions:

Theorem 5.0.5 (Uniqueness)**.** *Let W be a Brownian motion. Then there exists at most one solution to the loss-dependent SPDE with respect to (ν_0, W). That is, if (ξ^1, L^1) and (ξ^2, L^2) satisfy Definition 5.0.1, then with probability 1*

$$\xi_t^1(\phi) = \xi_t^2(\phi) \quad \textit{and} \quad L_t^1 = L_t^2, \qquad \textit{for all } t \in [0, T], \phi \in \mathscr{S}.$$

Remark 5.0.6 (Weak existence, strong uniqueness)**.** As we are taking realisations of the limit points to construct solutions in Theorem 5.0.3, we cannot define a solution for an a priori fixed Brownian motion. Hence we only have weak existence. On the other hand, we see that Theorem 5.0.5 gives uniqueness in the strong (pathwise) sense.

If (ν^*, L^*, W) realises a limiting law, then Theorems 5.0.3 and 5.0.5 give that ν^* and L^* are determined by W. An immediate consequence is that the limiting laws are determined by the law of the Brownian motion, so are all identical and hence:

Theorem 5.0.7 (Law of large numbers)**.** *The sequence $\left(\nu^N, L^N, W\right)_{N\geq 1}$ converges in law on the product space*

$$(D_{\mathscr{S}'}, M_1) \times (D_{\mathbf{R}}, M_1) \times (C_{\mathbf{R}}, U)$$

to the law of the unique solution of the loss-dependent SPDE with respect to (ν_0, W), for some Brownian motion W.

Remark 5.0.8 (Pathological ρ)**.** We cannot choose ρ arbitrarily and expect to recover weak convergence. As an example, let

$$\rho(x) = \begin{cases} p^{-1}, & \text{if } x = kp^{-n} \text{ for some } p \text{ prime, } n \in \mathbf{N} \text{ and } 1 \leq k \leq p^n - 1 \\ 0, & \text{otherwise.} \end{cases}$$

For $N = p^n$, L^N is supported on $\{kp^{-n}\}_{0\leq k\leq p^n}$, hence ν^N behaves as the basic system with constant correlation $\rho = p^{-1}$, so we have

$$\nu^{p^n} \Rightarrow \nu\big|_{\rho=p^{-1}}, \quad \text{in } D_{\mathscr{S}'},$$

as $n \to \infty$. Hence there is a distinct limit point for every prime, so weak convergence fails for this example.

To prove Theorem 5.0.5, we consider the system on time intervals on which L_t remains in a region where $\rho(\cdot)$ is Lipschitz. We can then apply the kernel smoothing method (plus some extra details) to deduce uniqueness on this interval (Section 5.6). Continuity then allows us to propagate uniqueness onto the next such time interval and repeating this argument gives global uniqueness (Section 5.7).

In Section 5.8 we show how Theorem 5.0.5 can be applied to (5.0.1) to prove existence and uniqueness of the following stochastic McKean–Vlasov problem:

Theorem 5.0.9 (Stochastic McKean–Vlasov dynamics)**.** *There exists a Brownian motion W such that for any independent Brownian motion, $W^{\perp}$, there exists* *a process $X \in D_{\mathbf{R}}$ satisfying the stochastic McKean–Vlasov equation*

$$\begin{cases} X_t = X_0 + \int_0^t \rho(\mathbf{P}(\tau \leq s|\, W))dW_s + \int_0^t \sqrt{1 - \rho\left(\mathbf{P}(\tau \leq s|\, W)\right)^2} dW_s^\perp \\ \tau = \inf\left\{t > 0 : X_t \leq 0\right\}. \end{cases}$$

(Here, X_0 has law ν_0 and is independent of all other processes.) Furthermore, the law of (X, W) is unique and

$$\nu_t\left(\phi\right) = \mathbf{E}\left[\phi\left(X_t\right)\mathbf{1}_{t<\tau}|\, W\right].$$

Finally, in Section 5.9 we extend the results from this chapter to the case when the particles evolve as correlated diffusions, rather than just correlated Brownian motions.

5.1. Continuous mappings for the M_1 topology

To recover information about the weak limit points, we need to apply the continuous mapping theorem. In this section we present a number of results which will aid us in doing so.

Theorem 5.1.1 (Continuous mapping theorem, [**Whi02**, Theorem 3.4.3])**.** *Let $\xi^N \Rightarrow \xi$ be random variables in a separable metric space, X. Let $g : X \to Y$ be measurable, where Y is another separable metric space, and define*

$$\mathrm{disc}\left(g\right) := \left\{x \in X : g \text{ is discontinuous at } x\right\}.$$

If $\mathbf{P}\left(\xi \in \mathrm{disc}\left(g\right)\right) = 0$, then $g\left(\xi^N\right) \Rightarrow g\left(\xi\right)$, in Y.

Remark 5.1.2 (Notation $\Rightarrow, \to$)**.** Throughout this chapter we will use the notation $\xi^N \Rightarrow \xi$ to denote weak convergence (in the relevant topological space) and $\xi^N \to \xi$ to denote strong convergence. Hopefully, this should always be clear from the context.

Our general strategy in the forthcoming sections will be to project $\xi \in D_{\mathscr{S}'}$ to $\xi\left(\phi\right) \in D_{\mathbf{R}}$, since the spatial canonical projection is continuous (Proposition 4.4.5), and then work in $D_{\mathbf{R}}$. An immediate danger is that the M_1 topology has some awkward properties, for example addition

$$+ : D_{\mathbf{R}} \times D_{\mathbf{R}} \to D_{\mathbf{R}}$$

is not M_1 continuous [**Whi02**, Corollary 12.7.1]. We avoid these problems by further projecting our process to $\mathbf{R}$ by the canonical time projection, $x \mapsto x_t$. This map is not unconditionally continuous (it is only continuous at paths that are continuous at t), so some care must be taken. The following result shows that there is only a small set of deterministic times at which a discontinuity occurs with positive probability:

Proposition 5.1.3 (Countably many discontinuities, [**Bil99**, Section 13])**.** *Let ξ be a stochastic process in $D_{\mathbf{R}}$ and define*

$$\operatorname{disc}(\xi) = \{t \in [0,T] : \mathbf{P}(\xi_t - \xi_{t-} \neq 0) > 0\}.$$

Then $\operatorname{disc}(\xi)$ *is at most countable.*

The following is an immediate and straightforward consequence, but we will write a careful proof:

Corollary 5.1.4 (Weak marginal convergence on a cocountable set)**.** *Suppose $\xi^N \Rightarrow \xi$ in $(D_{\mathbf{R}}, M_1)$, then there exists a deterministic subset $\mathbf{T} \subseteq [0,T]$ whose complement is countable and is such that*

$$\xi_t^N \Rightarrow \xi_t \quad \text{in } \mathbf{R}, \text{ for every } t \in \mathbf{T}.$$

Proof. Take $\mathbf{T} = [0,T] \setminus \operatorname{disc}(\xi)$ and $\pi_t : D_{\mathbf{R}} \to \mathbf{R}$ to be the usual projection map. Then, by [**Whi02**, Section 12.4], π_t is continuous at all paths $x \in D_{\mathbf{R}}$ which are continuous at t, therefore

$$\mathbf{P}(\xi \in \operatorname{disc}(\pi_t)) \leq \mathbf{P}(\xi_{t-} \neq \xi_t),$$

which vanishes if $t \in \mathbf{T}$. Thus we have the convergence on $\mathbf{T}$, which is cocountable by Proposition 5.1.3. □

The following corollary is useful for integrals:

Corollary 5.1.5 (Continuous mapping for integrals)**.** *Suppose we have a finite collection of sequences of weakly convergent random variables in $(D_{\mathbf{R}}, M_1)$, say:*

$$\left(\xi^{N,i}\right)_{N\geq 1} \Rightarrow \xi^{\infty,i}, \quad \text{for each } 1 \leq i \leq k,$$

for some $k \geq 1$. If $f : \mathbf{R}^k \to \mathbf{R}$ is continuous and bounded then

$$\int_0^T f\big(\xi_t^{N,1}, \xi_t^{N,2}, \ldots, \xi_t^{N,k}\big)dt \Rightarrow \int_0^T f\big(\xi_t^{\infty,1}, \xi_t^{\infty,2}, \ldots, \xi_t^{\infty,k}\big)dt, \quad \text{in } \mathbf{R},$$

as $N \to \infty$.

Proof. Apply Corollary 5.1.4 to [**Whi02**, Theorem 11.5.1]. □

5.2. Recovering properties of limiting loss functions

Here, we will show that any limiting loss function satisfies the requirements of Definition 5.0.1 (ii) and any limiting loss function is strictly increasing whenever there is non-zero mass left in the system. This second condition is needed to apply the continuous mapping theorem in Corollary 5.2.4. It implies that any limiting loss function is strictly increasing at discontinuity points of ρ, so occupies the discontinuity set for only finitely many discrete times. Heuristically, the result ensures that the system never gets stuck in bad states where the continuous mapping theorem fails.

First, we prove that L^* has the correct range and is increasing, however the argument we present is insufficient for showing that this monotonicity is strict. (This is because the argument uses the application of the portmanteau theorem in (5.2.1), which requires the inequality inside the probabilities to be strict.)

Proposition 5.2.1 (Correct range and weak monotonicity)**.** *If (ν^*, L^*, W) realises a limiting law, then, with probability 1*

$$L_t^* \in [0,1], \quad \textit{for every } t \in [0,T]$$

and $t \mapsto L_t^*$ is increasing.

Proof. Assume $(N_k)_{k\geq 1}$ is a subsequence leading to the limiting law. From [**Whi02**, Lemma 13.4.1] we know that

$$x \mapsto \sup_{s\in[0,\cdot]} |x(s)|$$

is a continuous map from $(D_{\mathbf{R}}, M_1)$ to itself. By Corollary 5.1.4, we have a sequence of times $(T_n) \uparrow T$ such that for every n

$$\sup_{t\in[0,T_n]} \left| L_t^{N_k} - \frac{1}{2} \right| \Rightarrow \sup_{t\in[0,T_n]} \left| L_t^* - \frac{1}{2} \right|, \quad \text{in } \mathbf{R},$$

hence the portmanteau theorem [**Bil99**, Theorem 2.1] gives

$$\mathbf{P}\Big(\sup_{t\in[0,T_n]} \left| L_t^* - \frac{1}{2} \right| > \frac{1}{2} \Big) \leq \liminf_{k\to\infty} \mathbf{P}\Big(\sup_{t\in[0,T_n]} \left| L_t^{N_k} - \frac{1}{2} \right| > \frac{1}{2} \Big) = 0.$$

Letting $n \to \infty$ and applying the monotone convergence theorem gives the result.

For $s, t \in [0,T]$, define the projection map

$$\pi_{s,t} : D_{\mathbf{R}} \to \mathbf{R} \times \mathbf{R}, \quad \pi_{s,t}(x) := (x_s, x_t).$$

By Lemma 5.1.4, there exists a cocountable subset $\mathbf{T} \subseteq [0,T]$ for which

$$\pi_{s,t}\big(L^{N_k}\big) \Rightarrow \pi_{s,t}\big(L^*\big), \quad \text{in } \mathbf{R}\times\mathbf{R}, \text{ for all } s,t \in \mathbf{T}$$

(the union of cocountable sets is cocountable). By continuity of subtraction as a map from $\mathbf{R}\times\mathbf{R}$ to $\mathbf{R}$ (not as map on processes) we therefore have

$$L_t^{N_k} - L_s^{N_k} \Rightarrow L_t^* - L_s^*, \quad \text{in } \mathbf{R}, \text{ for all } s,t \in \mathbf{T}.$$

Again we can apply the portmanteau theorem [**Bil99**, Theorem 2.1]: if $s < t$ are elements of $\mathbf{T}$, then

$$\mathbf{P}\big(L_t^* - L_s^* < 0\big) \le \liminf_{k\to\infty} \mathbf{P}\big(L_t^{N_k} - L_s^{N_k} < 0\big) = 0. \tag{5.2.1}$$

Since $\mathbf{T}$ is cocountable, we can take a countable dense subset $\mathbf{T}_0 \subseteq \mathbf{T}$ such that, with probability 1,

$$L_t^* \ge L_s^*, \quad \text{for all } t > s \text{ in } \mathbf{T}_0, \tag{5.2.2}$$

As L^* is càdlàg, (5.2.2) extends from $\mathbf{T}_0$ to $[0,T]$. □

With monotonicity established, it is straightforward to conclude that $L_0^* = 0$:

Proposition 5.2.2 (Initially zero)**.** *For any limit point* (ν^*, L^*, W)*,* $L_0^* = 0$ *with probability 1.*

Proof. Since L^* is increasing we have

$$\mathbf{P}\left(L_0^* > 0\right) = \lim_{\varepsilon\downarrow 0} \mathbf{P}\left(L_0^* \ge \varepsilon\right) \le \lim_{\varepsilon\downarrow 0} \mathbf{P}\left(L_t^* \ge \varepsilon\right),$$

for every $t > 0$. With $\mathbf{T}$ as in the previous proof, if $t \in \mathbf{T}$ then we have

$$\mathbf{P}\big(L_t^* \ge \varepsilon\big) \le \liminf_{k\to\infty} \mathbf{P}\big(L_t^{N_k} \ge \varepsilon\big) \le \liminf_{k\to\infty} \varepsilon^{-1}\mathbf{E}L_t^{N_k},$$

by [**Bil99**, Theorem 2.1] and Markov's inequality. Linearity of expectation gives

$$\mathbf{E}L_t^{N_k} = \mathbf{P}\big(\tau^{1,N} \le t\big) = \int_0^\infty \Phi\big(-t^{-\frac{1}{2}}x_0\big)\nu_0\left(dx_0\right) \le \Phi\big(-t^{-\frac{1}{2}}x_\star\big),$$

where we are also using that $X^{1,N}$ behaves as a Brownian motion. Choosing $t \in (\varepsilon, 2\varepsilon) \cap \mathbf{T}$ (which is non-empty since $\mathbf{T}$ is dense) yields the result since

$$\mathbf{P}\left(L_0^* > 0\right) \le \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\Phi\big(-\varepsilon^{-\frac{1}{2}}x_\star\big) = 0$$

(recall Proposition A.0.3). □

We have to work harder than in Proposition 5.2.1 to demonstrate that the monotonicity of L^* is strict. In the following proof, we exploit the indepedence between the particles to show that, for the limiting system, in any non-zero amount of time a non-zero proportion of the particles must hit the absorbing boundary.

Proposition 5.2.3 (Strict monotonicity)**.** *If (ν^*, L^*, W) is a limit point, then*

$$L^*_{t+h} - L^*_t > 0, \qquad \textit{whenever } L^*_t < 1 \textit{ and } h > 0,$$

with probability 1.

Proof. Assume $(L^{N_k})_{k\geq 1}$ is a subsequence converging in law to that of L^*. Using Corollary 5.1.4, take a countable dense subset $\mathbf{T}_0 \subseteq [0, T]$ for which

$$L^{N_k}_t \Rightarrow L^*_t, \qquad \text{in } \mathbf{R},$$

for every $t \in \mathbf{T}_0$. It then suffices to show that

$$\mathbf{P}(L^*_{t+h} - L^*_t = 0, L^*_t < 1) = 0, \qquad \text{for all } t, t+h \in \mathbf{T}_0, \tag{5.2.3}$$

since the result extends at all $t, t+h \in [0, T]$ by the fact that L^* is càdlàg. With $t, t+h \in \mathbf{T}_0$ fixed, to show (5.2.3) it suffices to prove

$$\mathbf{P}(L^*_{t+h} - L^*_t = 0, L^*_t < r) = 0, \qquad \text{for every } r \in [0, 1),$$

which is implied by

$$\lim_{\delta\to 0} \mathbf{P}(L^*_{t+h} - L^*_t < \delta, L^*_t < r) = 0, \qquad \text{for every } r \in [0, 1).$$

From [**Bil99**, Theorem 2.1], the result will therefore be complete if we can show

$$\limsup_{\delta\to 0} \limsup_{k\to\infty} \mathbf{P}(L^{N_k}_{t+h} - L^{N_k}_t < \delta, L^{N_k}_t < r) = 0, \qquad \text{for every } r \in [0, 1),$$

and this forms the remainder of the work in this proof. Our strategy is as follows:

(i) Consider only particles that have survived up to time t and fall in the range $(0, a)$, for some $a = a(\delta)$,
(ii) Remove the common driver between the particles on the interval $(t, t+h)$,
(iii) Apply a time-change argument to extract the independence between the particles,
(iv) Apply the law of large numbers to send $N_k \to \infty$,
(v) Choose $a = a(\delta)$ appropriately and send $\delta \to 0$.

Let $a = a(\delta)$ be the function which we will specify later, but for now assume that $a(\delta) \to \infty$, as $\delta \to 0$. Also let $c_1 = (1-r)/2 > 0$. By considering the location of the particles at time t, if $L_t^{N_k} < r$ and $\nu_t^{N_k}(a, +\infty) < c_1$ then

$$\nu_t^{N_k}(0, a) = 1 - L_t^{N_k} - \nu_t^{N_k}(a, +\infty) > 1 - r - c_1 = c_1.$$

We therefore have the bound

$$\begin{aligned} \mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta, L_t^{N_k} < r) &\leq \mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta, \nu_t^{N_k}(0, a) > c_1) + \mathbf{P}(\nu_t^{N_k}(a, +\infty) > c_1) \\ &\leq \mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta, \nu_t^{N_k}(0, a) > c_1) + o(1), \end{aligned} \tag{5.2.4}$$

where the second line is due to Markov's inequality and the tail estimate in Proposition 2.2.1, and the $o(1)$ term vanishes as $a \to \infty$, uniformly in N_k. Consider the first term on the right-hand side of (5.2.4) and define the $\mathcal{F}_t$-measurable random set of indices

$$\mathbf{I} = \{1 \leq i \leq N : X_t^{i,N} \in (0, a) \text{ and } \tau^{i,N} > t\}.$$

By partitioning on the value of $\mathbf{I}$

$$\mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta, \nu_t^{N_k}(0, a) > c_1) \leq \sum_{|\mathbf{i}| > c_1 N_k} \mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta, \mathbf{I} = \mathbf{i}).$$

As the particles are identically distributed, by symmetry we have

$$\begin{aligned} \mathbf{P}(L_{t+h}^{N_k} - &L_t^{N_k} < \delta, \nu_t^{N_k}(0, a) > c_1) \\ &\leq \sum_{|\mathbf{i}| > c_1 N_k} \binom{N_k}{|\mathbf{i}|} \mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta, \mathbf{I} = [|\mathbf{i}|]) \\ &\leq \sum_{|\mathbf{i}| > c_1 N_k} \binom{N_k}{|\mathbf{i}|} \mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta \mid \mathbf{I} = [|\mathbf{i}|]) \mathbf{P}(\mathbf{I} = [|\mathbf{i}|]) \end{aligned} \tag{5.2.5}$$

where, for $m \in \mathbf{N}$, we use the notation $[m] = \{1, 2, \ldots, m\}$. With $m \in \mathbf{N}$ fixed, we can write

$$\begin{aligned} &\mathbf{P}(L_{t+h}^{N_k} - L_t^{N_k} < \delta \mid \mathbf{I} = [m]) \\ &\quad = \mathbf{P}(\#\{1 \leq i \leq N : X_t^{i,N_k} + \inf_{u \in [0,h]} (I_u^{i,N_k} + M_u^{N_k}) < 0\} < N_k \delta \mid \mathbf{I} = [m]), \end{aligned}$$

where we introduce the processes

$$M_u^{N_k} := \int_t^{t+u} \rho(L_s^{N_k}) dW_s, \qquad I_u^{i,N_k} := \int_t^{t+u} \sqrt{1 - \rho(L_s^{N_k})^2} dW_s^i,$$

which gives the upper bound

$$\begin{aligned}&\mathbf{P}(L^{N_k}_{t+h}-L^{N_k}_t<\delta\mid \mathbf{I}=[m])\\&\qquad\le \mathbf{P}(\#\{1\le i\le m:\inf_{u\in[0,h]}I^{i,N_k}_u<-a-\sup_{u\in[0,h]}M^{N_k}_u\}<N_k\delta\mid \mathbf{I}=[m]).\end{aligned}$$

By Markov's inequality and Doob's maximal inequality (which is applicable because M^{N_k} and I^{i,N_k} are $(\mathcal{F}_{t+s})_{s\ge 0}$-martingales and $\mathbf{I}$ is $\mathcal{F}_t$-measurable)

$$\begin{aligned}\mathbf{P}(\sup_{u\in[0,h]}M^{N_k}_u>a\mid \mathbf{I}=[m])&\le c_2a^{-2}\mathbf{E}[\sup_{u\in[0,h]}|M^{N_k}_u|^2\mid \mathbf{I}=[m]\,]\\&\le c_2a^{-2}\int_t^{t+h}\mathbf{E}[|\rho(L^{N_k}_u)|^2\mid \mathbf{I}=[m]\,]du\le c_2ha^{-2},\end{aligned}$$

where $c_2>0$ is a universal constant and where we have used $\rho\le 1$. Therefore the upper bound can be written

$$\begin{aligned}&\mathbf{P}(L^{N_k}_{t+h}-L^{N_k}_t<\delta\mid \mathbf{I}=[m])\\&\qquad\le \mathbf{P}(\#\{1\le i\le m:\inf_{u\in[0,h]}I^{i,N_k}_u<-2a\}<N_k\delta\mid \mathbf{I}=[m])+o(1),\end{aligned}\tag{5.2.6}$$

and again the $o(1)$ term is uniform in N_k. In the next step we apply a time-change argument to remove the dependence on $\mathbf{I}$. Define

$$v^{N_k}(s):=\inf\{u>0:\int_t^{t+u}(1-\rho(L^{N_k}_{u_0}))du_0>s\},$$

then $B^{i,N_k}:=I^{i,N_k}_{v^{N_k}(\cdot)}$ is a martingale (with respect to the appropriate time-change of $(\mathcal{F}_{t+s})_{s>0}$) and satisfies $[B^{i,N_k},B^{j,N_k}]_t=\mathbf{1}_{i=j}t$, so has the law of an m-dimensional Brownian motion under $\mathbf{P}(\,\cdot\mid \mathbf{I}=[m])$. Therefore (5.2.6) becomes

$$\begin{aligned}&\mathbf{P}(L^{N_k}_{t+h}-L^{N_k}_t<\delta\mid \mathbf{I}=[m])\\&\qquad\le \mathbf{P}(\#\{1\le i\le m:\inf_{v^{N_k}(u)\in[0,h]}B^{i,N_k}_u<-2a\}<N_k\delta\mid \mathbf{I}=[m])+o(1)\\&\qquad\le \mathbf{P}(\#\{1\le i\le m:\inf_{u\in[0,(1-\rho^2_{\max})h]}B^{i,N_k}_u<-2a\}<N_k\delta\mid \mathbf{I}=[m])+o(1)\\&\qquad\le \mathbf{P}(\#\{1\le i\le m:B^{i,N_k}_{(1-\rho^2_{\max})h}<-2a\}<N_k\delta\mid \mathbf{I}=[m])+o(1),\end{aligned}$$

where we have used the fact $s\le v^{N_k}(s)\le(1-\rho^2_{\max})^{-1}s$, so

$$s\le(1-\rho^2_{\max})h\qquad\text{implies}\qquad v^{N_k}(s)\le h.$$

By taking Z^i to be i.i.d. standard normal random variables (under $\mathbf{P}(\cdot)$)

$$\mathbf{P}(L^{N_k}_{t+h} - L^{N_k}_t < \delta \mid \mathbf{I} = [m]) \leq \mathbf{P}(\frac{1}{N_k}\sum_{i=1}^{m} \mathbf{1}_{Z^i < -c_3 a} < \delta) + o(1),$$

where $c_3 = 2(1-\rho^2_{\max})^{-1/2}h^{-1/2}$. Notice that the right-hand side is a decreasing function in m. Setting the above inequality into (5.2.5) with m equal to the minimal case $m = c_1 N_k$ gives

$$\begin{aligned}\mathbf{P}(L^{N_k}_{t+h} &- L^{N_k}_t < \delta, \nu^{N_k}_t(0,a) > c_1) \\ &\leq \Big\{\mathbf{P}(\frac{1}{N_k}\sum_{i \leq c_1 N_k} \mathbf{1}_{Z^i < -c_3 a} < \delta) + o(1)\Big\} \sum_{|\mathbf{i}| > c_1 N_k} \binom{N_k}{|\mathbf{i}|} \mathbf{P}(\mathbf{I} = [|\mathbf{i}|]) \\ &\leq \mathbf{P}(\frac{1}{N_k}\sum_{i \leq c_1 N_k} \mathbf{1}_{Z^i < -c_3 a} < \delta) + o(1).\end{aligned}$$

Returning to (5.2.4) and applying the weak law of large numbers gives

$$\limsup_{N_k \to \infty} \mathbf{P}(L^{N_k}_{t+h} - L^{N_k}_t < \delta, L^{N_k}_t < r) \leq \mathbf{1}_{p(a) \leq \delta / c_1} + o(1),$$

where $p(\delta) = \mathbf{P}(Z^1 < -c_3 a) = \Phi(-c_3 a)$, with Φ the standard normal c.d.f. To complete the proof we make the choice

$$a(\delta) := \sqrt{2c_3^{-2}\log(\delta^{-1/2})},$$

then $a(\delta) \to \infty$, as $\delta \to 0$, and Corollary A.0.4 implies

$$p(a(\delta))\delta^{-1} \geq \frac{1}{2c_3 a(\delta)}\phi(c_3 a(\delta))\delta^{-1} = \frac{1}{2c_3 a(\delta)}\delta^{-1/2} \to \infty, \qquad \text{as } \delta \to 0,$$

hence

$$\limsup_{\delta \to 0} \limsup_{N_k \to \infty} \mathbf{P}(L^{N_k}_{t+h} - L^{N_k}_t < \delta, L^{N_k}_t < r) = 0.$$

$\square$

Since ρ is càdlàg, now that we know L^* is strictly increasing we can conclude that $t \mapsto \rho(L^*_t)$ is also càdlàg and that the set of times when L^* takes a value in the discontinuity set of ρ is null. We therefore have:

Corollary 5.2.4. *If $(\nu^{N_k}, L^{N_k}, W) \Rightarrow (\nu^*, L^*, W)$ then for every $\phi \in C^{\text{test}}$ and $t \in [0,T]$*

$$\int_0^t \rho(L^{N_k}_s)\nu^{N_k}_s(\phi)ds \Rightarrow \int_0^t \rho(L^*_s)\nu^*_s(\phi)ds, \quad \textit{in } \mathbf{R}.$$

Proof. Let $l \in D_{[0,1]}$ be strictly increasing and

$$\mathbf{T}_{\mathrm{disc}} := \{t \in [0,T] : l_t \in \mathrm{disc}(\rho)\}.$$

Then we know that $\mathbf{T}_{\mathrm{disc}}$ is a finite set, since $\mathrm{disc}(\rho)$ is a finite set. Therefore

$$\int_0^t \rho\big(l_s\big) x_s ds = \int_{[0,t]\setminus \mathbf{T}_{\mathrm{disc}}} \rho\big(l_s\big) x_s ds,$$

for any $x \in D_{\mathbf{R}}$, so the work in Corollary 5.1.5 and Theorem 13.2.3 of [**Whi02**] give that the map

$$\big(x, l\big) \in D_{\mathbf{R}} \times D_{[0,1]} \mapsto \int_0^t \rho\big(l_s\big) x_s ds \in \mathbf{R}$$

is continuous at all $l \in D_{[0,1]}$ that are strictly increasing. By the continuity of spatial projections, we have

$$\nu_s^{N_k}\big(\phi\big) \Rightarrow \nu_s^*\big(\phi\big),$$

so we conclude the result.

Spatial projections are continuous, so we have

$$\nu^{N_k}\big(\phi\big) \Rightarrow \nu^*\big(\phi\big), \quad \text{in } \big(D_{\mathbf{R}}, M_1\big).$$

Take $\mathbf{T}$ to be the cocountable set of times for which we have

$$\big(\nu_t^{N_k}\big(\phi\big), L_t^{N_k}\big) \Rightarrow \big(\nu_t^*\big(\phi\big), L_t^*\big), \quad \text{in } \mathbf{R} \times \mathbf{R}$$

and $L_t^* \notin \mathrm{disc}(\rho)$. Then □

5.3. The regularity conditions

This is a short section in which we verify that condition (v) of Definition 5.0.1 holds for a general limit point $\big(\nu^*, L^*, W\big)$.

To do this, fix a subsequence, $(N_k)_{k\geq 1}$, along which the limit is attained. For any $\phi \in \mathscr{S}$ we have

$$\nu^{N_k}(\phi) \Rightarrow \nu^*(\phi), \quad \text{in } (D_{\mathbf{R}}, M_1),$$

by Proposition 4.4.5. Because ν^{N_k} is a sub-probability measure, we know that

$$\big|\nu_t^{N_k}(\phi)\big| \leq \big\|\phi\big\|_\infty, \quad \text{for all } k \geq 1, t \in [0,T], \phi \in \mathscr{S},$$

so it is immediate that, for any $p \geq 1$, the sequence

$$\left(\int_0^T \left|\nu_t^{N_k}(\phi)\right|^p dt\right)_{k\geq 1}$$

is uniformly integrable. By [**Bil99**, Theorem 3.5] and Corollary 5.1.5 we conclude that

$$\mathbf{E}\int_0^T \left|\nu_t^*(\phi)\right|^p dt = \lim_{k\to\infty} \mathbf{E}\int_0^T \left|\nu_t^{N_k}(\phi)\right|^p dt.$$

Each $X^{i,N}$ is a Brownian motion, so we can apply the calculations from Section 2.2 to recover conditions (ii), (iii) and (iv) of Definition 2.0.3. (Note that it might be that $T \notin \mathbf{T}$ from Corollary 5.1.5. If this is the case then take $T_n \uparrow T$ with $T_n \in \mathbf{T}$ and apply the monotone convergence theorem.)

Condition (v) follows from the fact that each ν^{N_k} is supported on $(0,\infty)$, therefore if $\phi \in \mathscr{S}$ is supported on $(-\infty, 0)$ then $\nu_t^{N_k}(\phi) = 0$ for every t, so

$$\mathbf{E}\int_0^T \left|\nu_t^*(\phi)\right| dt = \lim_{k\to 0} \mathbf{E}\int_0^T \left|\nu_t^{N_k}(\phi)\right| dt = 0,$$

which suffices.

5.4. The martingale approach

To show that limit points solve the loss-dependent SPDE, one strategy could be to consider the evolution equation as a map from $D_{\mathscr{S}'} \times D_{\mathbf{R}}$ to $D_{\mathbf{R}}$:

$$\mathrm{SPDE}^{\phi}(f,l)(t) := f_t(\phi) - \nu_0(\phi) - \frac{1}{2}\int_0^t f_s(\partial_{xx}\phi)ds - \int_0^t \rho(l_s)\, f_s(\partial_x\phi)dW_s,$$

and to show that, for every $\phi \in C^{\text{test}}$, this function vanishes when we pass a limit point:

$$\mathrm{SPDE}^{\phi}(\nu^*, L^*) \equiv 0.$$

The presence of the stochastic integral, however, would complicate the application of the continuous mapping theorem. We therefore decompose the SPDE into smaller processes and use a martingale method.

Definition 5.4.1 (Martingale components)**.** For a fixed test function $\phi \in C^{\text{test}}$ define:

(i) $M^\phi : D_{\mathscr{S}'} \to D_{\mathbf{R}}$,

$$M^{\phi}(f)(t) := f_t(\phi) - \nu_0(\phi) - \frac{1}{2}\int_0^t f_s(\partial_{xx}\phi)\,ds$$

(ii) $S^{\phi} : D_{\mathscr{S}'} \times D_{\mathbf{R}} \to D_{\mathbf{R}}$,

$$S^{\phi}(f,l)\,(t) := M^{\phi}(f)\,(t)^2 - \int_0^t \rho\,(l_s)^2\, f_s\,(\partial_x\phi)^2\, ds$$

(iii) $C^{\phi} : D_{\mathscr{S}'} \times D_{\mathbf{R}} \times C_{\mathbf{R}} \to D_{\mathbf{R}}$,

$$C^{\phi}(f,l,w)\,(t) := M^{\phi}(f)\,(t) \cdot w\,(t) - \int_0^t \rho\,(l_s)\, f_s\,(\partial_x\phi)\, ds.$$

These processes capture the dynamics of the loss-dependent SPDE:

Proposition 5.4.2 (Martingale approach)**.** *Let $\xi \in D_{\mathscr{S}'}$ and $L \in D_{\mathbf{R}}$ be random processes satisfying (ii) and (v) of Definition 5.0.1, and let W be a Brownian motion. If the processes*

$$M^{\phi}(\xi)\,, \quad S^{\phi}(\xi, L) \quad \textit{and} \quad C^{\phi}(\xi, L, W)$$

are square-integrable martingales for every $\phi \in C^{test}$, then (ξ, L, W) satisfies condition (iii) of the Definition 5.0.1.

Proof. Under the martingale assumption we have

$$\left[M^{\phi}(\xi)\right]_t = \int_0^t \rho(L_s)^2 \xi_s(\partial_x\phi)^2 ds, \qquad \left[M^{\phi}(\xi), W\right]_t = \int_0^t \rho(L_s)\, \xi_s(\partial_x\phi) ds.$$

and therefore

$$\left[M^{\phi}(\xi) - \int_0^{\cdot} \rho\,(L_s)\, \xi_s\,(\partial_x\phi)\, dW_s\right]_t = 0, \quad \text{for every } t \geq 0.$$

(Note the stochastic integral above is well-defined as ρ is bounded and condition (v) ensures we have sufficient integrability.) Hence we conclude

$$M^{\phi}(\xi) - \int_0^{\cdot} \rho\,(L_s)\, \xi_s\,(\partial_x\phi)\, dW_s = 0,$$

for every t, with probability 1. Taking a countable dense collection of test functions gives the result. $\square$

It now suffices to show that a general limit point,

$$\left(\nu^{N_k}, L^{N_k}, W\right) \Rightarrow \left(\nu^{*}, L^{*}, W\right),$$

satisfies the hypotheses of Proposition 5.4.2. We begin by noting the following:

Lemma 5.4.3. *For every $\phi \in C^{\text{test}}$ we have a deterministic cocountable subset of times, $\mathbf{T} \subseteq [0,T]$, on which*

$$M^{\phi}\big(\nu^{N_k}\big)(t) \Rightarrow M^{\phi}\big(\nu^{*}\big)(t), \quad S^{\phi}\big(\nu^{N_k}, L^{N_k}\big)(t) \Rightarrow S^{\phi}\big(\nu^{*}, L^{*}\big)(t)$$
$$C^{\phi}\big(\nu^{N_k}, L^{N_k}, W\big)(t) \Rightarrow C^{\phi}\big(\nu^{*}, L^{*}, W\big)(t), \quad \text{in } \mathbf{R}.$$

Furthermore, these sequences are uniformly integrable and the limits are square-integrable.

Proof. Weak convergence follows from [**Bil99**, Theorem 2.7] along with Proposition 4.4.5, Corollary 5.1.4, Corollary 5.1.5 and Corollary 5.2.4. Uniform integrability and square integrability follow from the work in Section 5.3. □

With this lemma, consider $M^{\phi}\big(\nu^{N_k}\big)$. As the work in Section 2.3 is unchanged, we have

$$M^{\phi}\big(\nu^{N_k}\big)(t) = \int_0^t \rho\left(L_s^{N_k}\right)\nu_s^{N}\left(\partial_x\phi\right)dW_s + I_t^{N_k}(\phi) \tag{5.4.1}$$

with

$$I_t^{N_k}(\phi) = \frac{1}{N_k}\sum_{i=1}^{N_k}\int_0^t \sqrt{1-\rho\big(L_s^{N_k}\big)^2}\,\nu_s^{N_k}\left(\partial_x\phi\right)dW_s^i.$$

This gives that $M^{\phi}\big(\nu^{N_k}\big)$ is a martingale, so, for any $k \geq 1$, $s,t \in \mathbf{T}$, $s_1,\ldots,s_k \in \mathbf{T} \cap [0,s]$ and $f_1,\ldots,f_k : \mathbf{R} \to \mathbf{R}$ bounded and continuous and F defined by

$$F\left(\nu^{N_k}\right) := \left(M^{\phi}\big(\nu^{N_k}\big)(t) - M^{\phi}\big(\nu^{N_k}\big)(s)\right)\prod_{i=1}^{k} f_i\left(M^{\phi}\big(\nu^{N_k}\big)(s_i)\right),$$

we have

$$\mathbf{E}F\left(\nu^{N_k}\right) = 0.$$

By the assumptions on f, we have that $F\left(\nu^{N_k}\right)$ is a uniformly integrable sequence and converges weakly to $F\left(\nu^{*}\right)$ in $\mathbf{R}$, by Lemma 5.4.3 and [**Bil99**, Theorem 2.7]. Therefore we conclude

$$\mathbf{E}F\big(\nu^{*}\big) = \lim_{k\to\infty}\mathbf{E}F\big(\nu^{N_k}\big) = 0,$$

hence $M^{\phi}\big(\nu^{N_k}\big)$ is a martingale.

Similarly for S^ϕ, we can define

$$G\left(\nu^{N_k}, L^{N_k}\right) := \left(S^\phi\big(\nu^{N_k}, L^{N_k}\big)(t) - S^\phi\big(\nu^{N_k}, L^{N_k}\big)(s)\right) \times \prod_{i=1}^{k} f_i\left(S^\phi\big(\nu^{N_k}, L^{N_k}\big)(s_i)\right).$$

By applying Itô's formula to (5.4.1) we see that

$$S^\phi\big(\nu^{N_k}, L^{N_k}\big)(t) = S^\phi\big(\nu^{N_k}, L^{N_k}\big)(0) + \text{martingale term} + 2\left[I^{N_k}(\phi)\right]_t.$$

From Proposition 2.3.2 we know that, as $k \to \infty$,

$$\mathbf{E}\left[I^{N_k}(\phi)\right]_t = O\left(N_k^{-1}\right),$$

therefore

$$\mathbf{E}G\left(\nu^{N_k}, L^{N_k}\right) = O\left(N_k^{-1}\right),$$

and so we again conclude

$$\mathbf{E}G\left(\nu^*, L^*\right) = 0,$$

and that $S^\phi(\nu^*, L^*)$ is a martingale.

The result for C^ϕ follows similarly, so we deduce:

Proposition 5.4.4. *Every limit point satisfies condition (iii) of Definition 5.0.1.*

As we have already verified (ii) and (v) in the preceding two sections, we have:

Corollary 5.4.5 (Linearised SPDE)**.** *For any limit point, $\big(\nu^*, L^*, W\big)$, ν^* solves the basic SPDE (that is, satisfies Definition 2.0.3) with the correlation function $\rho_t = \rho\left(L_t^*\right)$.*

5.5. The tagged particle representation; proof of Theorem 5.0.3

Fixing a general limit point, $\big(\nu^*, L^*, W\big)$, we now show that the conditional law of the tagged particle in (5.0.1) gives a representation for ν^*.

First observe that X^* has the law of a Brownian motion, so the work in Sections 2.2, 2.3 and 2.4 is applicable and gives:

Proposition 5.5.1 (Linearised SPDE)**.** *The law of the tagged particle process, μ^*, solves the basic SPDE (that is, satisfies Definition 2.0.3) with respect to the correlation process $\rho_t = \rho\left(L_t^*\right)$.*

We know from Corollary 5.4.5 that ν^* is also a solution, so Theorem 2.0.7 gives:

Corollary 5.5.2 (Representation). *With probability 1,*

$$\nu_t^*(\phi) = \mu_t^*(\phi), \quad \text{for every } \phi \in \mathscr{S} \text{ and } t \in [0,T].$$

Therefore we can regard ν^* as a (sub-probability) measure-valued process, and from now on we will do so without referring to μ^*. Immediately we conclude (i) of Definition 5.0.1.

Remark 5.5.3 (Identifying ν^* with μ^*). Corollary 5.5.2 does not make ν_t^* a measure, rather it states that ν_t^* agrees with the measure μ_t^* on $\mathscr{S}$. The precise argument is then as follows: every limit point gives rise to a measure-valued process, and it is this measure-valued process which satisfies the loss-dependent SPDE. Notice that the uniqueness result (Theorem 5.0.5) is unaffected by this identification; if the measure-valued processes generated by two candidate solutions agree, then the solutions must agree on $\mathscr{S}$ and vice versa. This reasoning justifies associating ν^* and μ^* for the remainder of this chapter.

Although we can now make sense of ν^* as measure-valued, we cannot conclude statements of the form

$$\nu^{N_k} \Rightarrow \nu^* \text{ in } (D_{\mathscr{S}'}, M_1) \quad \text{implies} \quad \nu^{N_k}(S) \Rightarrow \nu^*(S) \text{ in } \mathbf{R}$$

for a set $S \subseteq \mathbf{R}$, since the the characteristic function χ_S is not an element of $\mathscr{S}$, so we cannot apply the usual weak convergence results. To conclude (iv) of Definition 5.0.1 we need some care:

Proposition 5.5.4 (Loss and surviving mass). *If (ν^*, L^*, W) is a limit point, then it satisfies condition (iv) of Definition 5.0.1.*

Proof. It is enough to show that

$$\mathbf{E}\int_0^T |1 - L_t^* - \nu_t^*(0,\infty)|^2 dt = 0.$$

However, we cannot apply the continuous mapping theorem to $\nu_t^*(0,\infty)$ directly, so we introduce $\lambda > 0$ and a cut-off function, $\phi_\lambda \in \mathscr{S}$, supported on $(-\lambda, \lambda)$.

Then

$$\begin{aligned}\mathbf{E}\int_0^T \left|1-L_t^*-\nu_t^*\left(0,\infty\right)\right|^2 dt \leq 2\mathbf{E}\int_0^T \left|1-L_t^*-\nu_t^*\left(\phi_\lambda\right)\right|^2 dt \\ + 2\mathbf{E}\int_0^T \left|\nu_t^*\left(\left[\lambda,\infty\right)\right)\right|^2 dt \\ = 2\lim_{k\to\infty}\mathbf{E}\int_0^T \left|1-L_t^{N_k}-\nu_t^{N_k}\left(\phi_\lambda\right)\right|^2 dt \\ + 2\lim_{k\to\infty}\mathbf{E}\int_0^T \left|\nu_t^{N_k}\left(\left[\lambda,\infty\right)\right)\right|^2 dt \\ = 4\lim_{k\to\infty}\mathbf{E}\int_0^T \left|\nu_t^{N_k}\left(\left[\lambda,\infty\right)\right)\right|^2 dt,\end{aligned}$$

by Corollary 5.1.5. Letting $\lambda \to \infty$ and using Lemma 2.2.1 gives the result. □

The density process. We can deduce that ν^* has a density process by a simple Radon–Nikodym argument. This will be helpful in the next section, where it simplifies some estimates.

Proposition 5.5.5 (Existence of a density)**.** *Let (ν^*, L^*, W) be a limit point. For every $t \in [0,T]$, ν_t^* has a density, V_t^*:*

$$\nu_t^*\left(\phi\right) = \int_{\mathbf{R}} \phi\left(x\right) V_t^*\left(x\right) dx, \quad \textit{for all bounded measurable } \phi : \mathbf{R} \to \mathbf{R}.$$

Furthermore, if ν_0 has a density $V_0 \in L^2\left(0,\infty\right)$ then, for every $t \in [0,T]$, $V_t \in L^2\left(0,\infty\right)$ and

$$\left\|V_t\right\|_{L^2(0,\infty)} \leq \left\|V_0\right\|_{L^2(0,\infty)}.$$

Proof. From the tagged particle representation, if we disregard killing at the origin we have the simple inequality

$$\nu_t^*\left(S\right) = \mathbf{P}\left(X_t^* \in S; t < \tau^* \middle| L^*, W\right) \leq \mathbf{P}\left(X_t^* \in S \middle| L^*, W\right).$$

Conditional on $\sigma\left(L^*, W\right)$, the marginal X_t^* is of the form

$$\begin{aligned}X_t^* &= X_0 + \int_0^t \rho\left(L_s^*\right) dW_s + \int_0^t \sqrt{1-\rho\left(L_s^*\right)^2}\, dW_s^\perp \\ &\overset{\mathrm{d}}{=} X_0 + \mathrm{Normal}\left(m,\sigma^2\right),\end{aligned}$$

where

$$m = \int_0^t \rho\left(L_s^*\right) dW_s, \quad \sigma^2 = \int_0^t \left(1-\rho\left(L_s^*\right)^2\right) ds \geq \left(1-\rho_{\max}^2\right) t.$$

Therefore, if we write

$$K(x_0, x) = \left(2\pi\sigma^2\right)^{-\frac{1}{2}} \exp\Big\{-\frac{(x - m - x_0)^2}{2\sigma^2}\Big\},$$

then we have

$$\nu_t^*(S) \leq \int_{x\in S}\int_0^\infty K(x_0, x)\,\nu_0(dx_0)\,dx.$$

So if S is Lebesgue-null then the right-hand side vanishes, in which case ν_t^* is absolutely continuous and V_t^* exists by the Radon–Nickodym theorem [**CK04**, Theorem 7.3].

Furthermore, for almost all $x \in (0, \infty)$

$$\begin{aligned}
|V_t(x)|^2 &\leq \left|\int_0^\infty K(x_0, x)\,V_0(x_0)\,dx_0\right|^2 \\
&\leq \int_0^\infty K(x_0, x)\,dx_0 \cdot \int_0^\infty K(x_0, x)\,|V_0(x_0)|^2\,dx_0 \\
&\leq \int_0^\infty K(x_0, x)\,|V_0(x_0)|^2\,dx_0,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and that $K(\cdot, x)$ integrates to 1 over the real-line. Integrating over $x \in (0, \infty)$ and using the previous fact once more completes the proof. □

5.6. Uniqueness for Lipschitz ρ; beginning the proof of Theorem 5.0.5

We now have sufficient machinery to tackle the main theorem. First we restrict to the case when $\rho(\cdot)$ is a Lipschitz function, say

$$|\rho(x) - \rho(y)| \leq \rho_{\text{lip}}\,|x - y|, \quad \text{for all } x, y \in [0, 1],$$

where $\rho_{\text{lip}} \geq 0$ is a finite constant. (For brevity we will write $\|\cdot\|_2$ in the following calculations to denote the usual norm on $L^2(0, \infty)$.)

For a fixed W, suppose that we have two pairs of processes (ν^1, L^1) and (ν^2, L^2) that satisfy Definition 5.0.1 for the same initial condition and define their difference

$$\Delta := \nu^1 - \nu^2 \in D_{\mathscr{S}'}.$$

Taking the difference of the two smoothed SPDEs satisfied by the solutions (that is, (2.8.1) for ν^1 and ν^2) gives

$$dT_\varepsilon\Delta_t = \frac{1}{2}\partial_{xx}T_\varepsilon\Delta_t dt - \left[\rho\left(L_t^1\right)\partial_x T_\varepsilon\nu_t^1 - \rho\left(L_t^2\right)\partial_x T_\varepsilon\nu_t^2 + \partial_x r_t^\varepsilon\right] dW_t,$$

where

$$r_t^\varepsilon(x) := \rho\left(L_t^1\right) R_\varepsilon \nu_t^1(x) - \rho\left(L_t^2\right) R_\varepsilon \nu_t^2(x).$$

By Remark 2.0.4, Δ satisfies conditions (ii)-(v) of Definition 2.0.3, so we can apply the work in Section 2.8 to arrive at

$$\begin{aligned}
\mathbf{E}\left\|\partial_x^{-1} T_\varepsilon \Delta_t\right\|_2^2 &= -\mathbf{E}\int_0^t \left\|T_\varepsilon \Delta_s\right\|_2^2 ds \\
&\quad + \mathbf{E}\int_0^t \left\|\rho\left(L_s^1\right) T_\varepsilon \nu_s^1 - \rho\left(L_s^2\right) T_\varepsilon \nu_s^2 + r_s^\varepsilon\right\|_2^2 ds \\
&\leq -\mathbf{E}\int_0^t \left\|T_\varepsilon \Delta_s\right\|_2^2 ds + \left(1+\eta_1^{-1}\right)\mathbf{E}\int_0^t \left\|r_s^\varepsilon\right\|_2^2 ds \qquad (5.6.1)\\
&\quad + \left(1+\eta_1\right)\mathbf{E}\int_0^t \left\|\rho\left(L_s^1\right) T_\varepsilon \nu_s^1 - \rho\left(L_s^2\right) T_\varepsilon \nu_s^2\right\|_2^2 ds
\end{aligned}$$

where we have applied Young's inequality with a free parameter $\eta_1 > 0$.

We can write

$$\rho\left(L_s^1\right) T_\varepsilon \nu_s^1 - \rho\left(L_s^2\right) T_\varepsilon \nu_s^2 = \rho\left(L_s^1\right) T_\varepsilon \Delta_s + \left(\rho\left(L_s^1\right) - \rho\left(L_s^2\right)\right) T_\varepsilon \nu_s^2,$$

therefore another application of Young's inequality gives

$$\begin{aligned}
\left\|\rho\left(L_s^1\right) T_\varepsilon \nu_s^1 - \rho\left(L_s^2\right) T_\varepsilon \nu_s^2\right\|_2^2 &\leq \left(1+\eta_2\right)\rho_{\max}\left\|T_\varepsilon \Delta_s\right\|_2^2 \\
&\quad + \left(1+\eta_2^{-1}\right)\left\|T_\varepsilon \nu_s^2\right\|_2^2 \left|\rho\left(L_s^1\right) - \rho\left(L_s^2\right)\right|^2 \\
&\leq \left(1+\eta_2\right)\rho_{\max}\left\|T_\varepsilon \Delta_s\right\|_2^2 \\
&\quad + \left(1+\eta_2^{-1}\right)\left\|V_0\right\|_2^2 \rho_{\mathrm{lip}}\left|L_s^1 - L_s^2\right|^2,
\end{aligned}$$

with $\eta_2 > 0$ a free parameter and where we have used the Lipschitz property for $\rho(\cdot)$ and

$$\left\|T_\varepsilon \nu_s^2\right\|_2 \leq \left\|V_s\right\|_2 \leq \left\|V_0\right\|_2,$$

which follows from Propositions 2.6.8 and 5.5.5. The final term in (5.6.1) vanishes as $\varepsilon \to 0$ (by applying Lemma 2.7.3) so we have

$$\begin{aligned}
\mathbf{E}\left\|\partial_x^{-1} T_\varepsilon \Delta_t\right\|_2^2 &\leq -\left(1-\left(1+\eta_1\right)\left(1+\eta_2\right)\rho_{\max}\right)\mathbf{E}\int_0^t \left\|T_\varepsilon \Delta_s\right\|_2^2 ds \qquad (5.6.2)\\
&\quad + \left(1+\eta_1\right)\left(1+\eta_2^{-1}\right)\left\|V_0\right\|_2^2 \rho_{\mathrm{lip}}\mathbf{E}\int_0^t \left|L_s^1 - L_s^2\right|^2 ds + o(1).
\end{aligned}$$

Now fix η_1 and η_2 small enough so that

$$c_1 := 1 - \left(1+\eta_1\right)\left(1+\eta_2\right)\rho_{\max} > 0,$$

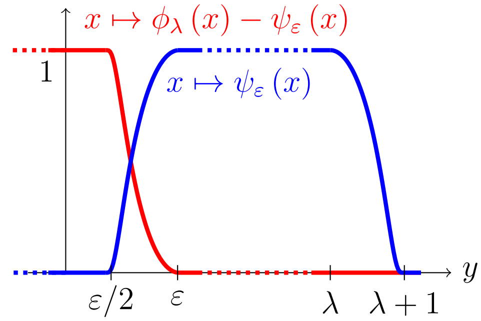


FIGURE 5.6.1. The smooth cut-off functions from the proof of Lemma 5.6.1.

which is possible since $\rho_{\max} < 1$. Then, by rearranging (5.6.2), taking a limit infimum and applying Proposition 2.6.6 we have

$$\mathbf{E}\left\|\Delta_t\right\|_{H^{-1}}^2 + c_1\mathbf{E}\int_0^t \left\|\Delta_s\right\|_{L^2(0,\infty)}^2 ds \le c_2\mathbf{E}\int_0^t \left|L_s^1 - L_s^2\right|^2 ds, \tag{5.6.3}$$

where

$$c_2 = \left(1+\eta_1\right)\left(1+\eta_2^{-1}\right)\left\|V_0\right\|_2^2 \rho_{\text{lip}}.$$

To relate the difference in the loss functions to $\left\|\Delta_t\right\|_{H^{-1}}^2$ (in preparation for applying Gronwall's inequality) we need the following lemma:

Lemma 5.6.1. *There exists constants $c_3, c_4, c_5 > 0$ such that $c_3 < c_1$ and*

$$c_2\mathbf{E}\int_0^t \left|L_s^1 - L_s^2\right|^2 ds \le c_3\mathbf{E}\int_0^t \left\|\Delta_s\right\|_{L^2(0,\infty)}^2 ds + (c_4\lambda + c_5)\mathbf{E}\int_0^t \left\|\Delta_s\right\|_{H^{-1}}^2 ds$$
$$+\, o\left(\exp\left\{-c\lambda\right\}\right),$$

uniformly in $t \in [0,T]$, as $\lambda \to \infty$, for any $c > 0$.

Proof. From Definition 5.0.1 condition (iv) we can write

$$\begin{aligned} L_s^1 - L_s^2 &= \nu_s^2(0,\infty) - \nu_s^1(0,\infty) \\ &= \Delta_s\left(1-\phi_\lambda\right) - \Delta_s\left(\phi_\lambda\right), \end{aligned}$$

where ϕ_λ is any smooth function satisfying

$$\phi_\lambda(x) \begin{cases} = 1, & \text{if } |x| \le \lambda \\ = 0, & \text{if } |x| \ge \lambda+1 \\ \in (0,1), & \text{if } \lambda < |x| < \lambda+1. \end{cases}$$

By condition (iii) of Definition 2.0.3 we have

$$\mathbf{E}\int_0^t \left|L_s^1 - L_s^2\right|^2 ds \leq 2\mathbf{E}\int_0^t \left|\Delta_s\left(\phi_\lambda\right)\right|^2 ds + o\left(\exp\left\{-c\lambda\right\}\right). \tag{5.6.4}$$

Now, ϕ_λ does not vanish at the origin, so we cannot yet insert the dual norm of Δ_t. Instead we split ϕ_λ into

$$\phi_\lambda = \psi_\varepsilon + \left(\phi_\lambda - \psi_\varepsilon\right),$$

where ψ_ε is any smooth function satisfying

$$\psi_\varepsilon \begin{cases} = 1, & \text{if } x \leq \varepsilon/2 \\ = 0, & \text{if } x \geq \varepsilon \\ \in (0,1), & \text{if } x \in (\varepsilon/2, \varepsilon) \end{cases}$$

(see Figure 5.6.1). Notice that it is no loss of generality to assume that

$$\left\|\partial_x \psi_\varepsilon\right\|_\infty \leq c_3 \varepsilon^{-1} \quad \text{and} \quad \left\|\partial_x \phi_\lambda\right\|_\infty \leq c_3,$$

for some numerical constant c_3.

With this construction (and $\varepsilon < \lambda$) we have $\psi_\varepsilon \in L^2(0,\infty)$, $\phi_\lambda - \psi_\varepsilon \in H_0^1(0,\infty)$ and, for $x > 0$

$$\begin{aligned} &\left|\psi_\varepsilon(x)\right| \leq \mathbf{1}_{x<\varepsilon}, \quad \left|\phi_\lambda(x) - \psi_\varepsilon(x)\right| \leq \mathbf{1}_{0<x<\lambda+1} \\ \text{and} \quad &\left|\partial_x \phi_\lambda(x) - \partial_x \psi_\varepsilon(x)\right| \leq c_1 \varepsilon^{-1} \mathbf{1}_{0<x<\varepsilon} + c_1 \mathbf{1}_{\lambda<x<\lambda+1}. \end{aligned}$$

Therefore

$$\left\|\psi_\varepsilon\right\|_{L^2(0,\infty)}^2 \leq \varepsilon \quad \text{and} \quad \left\|\phi_\lambda - \psi_\varepsilon\right\|_{H^1(0,\infty)}^2 \leq \lambda + 1 + c_3^2 \varepsilon^{-1} + c_3^2,$$

so we can conclude

$$\begin{aligned} \left|\Delta_t\left(\phi_\lambda\right)\right|^2 &\leq 2\left|\Delta_t\left(\psi_\varepsilon\right)\right|^2 + 2\left|\Delta_t\left(\phi_\lambda - \psi_\varepsilon\right)\right|^2 \\ &\leq 2\left\|\psi_\varepsilon\right\|_{L^2(0,\infty)}^2 \left\|\Delta_t\right\|_{L^2(0,\infty)}^2 + 2\left\|\phi_\lambda - \psi_\varepsilon\right\|_{H^1(0,\infty)}^2 \left\|\Delta_t\right\|_{H^{-1}}^2 \\ &\leq 2\varepsilon \left\|\Delta_t\right\|_{L^2(0,\infty)}^2 + 2\left(\lambda + 1 + c_3^2\varepsilon^{-1} + c_3^2\right) \left\|\Delta_t\right\|_{H^{-1}}. \end{aligned}$$

By setting this inequality into (5.6.4) we have the result by fixing ε sufficiently small. □

Applying the above lemma to (5.6.3) gives

$$\mathbf{E}\left\|\Delta_t\right\|_{H^{-1}}^2 \leq \int_0^t \left(c_4\lambda + c_5\right) \mathbf{E}\left\|\Delta_s\right\|_{H^{-1}}^2 ds + o\left(\exp\left\{-c\lambda\right\}\right),$$

for any $c > 0$, therefore Gronwall's lemma implies

$$\mathbf{E}\left\|\Delta_t\right\|_{H^{-1}}^2 = o\left(\exp\left\{-c\lambda\right\}\right) \times \exp\left\{c_4\lambda T + c_5 T\right\} = o\left(1\right),$$

as $\lambda \to \infty$, provided we take $c > (c_4 + c_5)\,T$. Since λ is arbitrary, we conclude

$$\mathbf{E}\left\|\Delta_t\right\|_{H^{-1}}^2 = 0, \quad \text{for every } t \in [0, T],$$

which is enough to prove uniqueness, since Δ is càdlàg.

5.7. Uniqueness for piecewise Lipschitz ρ; proof of Theorem 5.0.5 continued

To complete the proof we need to extend the argument from the previous section to piecewise Lipschitz ρ. To consider ρ on intervals on which it is Lipschitz, let

$$0 = x_0 < x_1 < x_2 < \cdots < x_n \leq 1$$

be as in Definition 1.3.1. For two candidate solutions (ν^1, L^1) and (ν^2, L^2), define the stopping times

$$\sigma_i^1 := \inf\left\{t > 0 : L_t^1 > x_i\right\} \wedge T, \quad \sigma_i^2 := \inf\left\{t > 0 : L_t^2 > x_i\right\} \wedge T.$$

Let $\sigma := \sigma_1^1 \wedge \sigma_1^2$, which is again a stopping time. The argument in the previous section remains valid if we replace every instance of t by $t \wedge \sigma$. This is because the only probabilistic techniques used are to take the expectation of a martingale (unchanged as a stopped martingale is a martingale) and the estimate on

$$\mathbf{E}\int_0^{t\wedge\sigma} \left\|r_s^\varepsilon\right\|^2 ds$$

(it is no problem to now replace the upper limit in the integral by T, since $\sigma \leq T$). Therefore we conclude that $\Delta_t = 0$ and $L_t^1 = L_t^2$ for every $t \in [0, \sigma)$, and that $\sigma = \sigma_1^1 = \sigma_1^2$.

Since we only have right-continuity, we cannot immediately conclude that $\Delta_\sigma = 0$, however since $\Delta_t\left(\phi\right)$ satisfies the SPDE, it can be written as continuous processes (namely, the right-hand side of the SPDE) so must be continuous. Therefore, with probability 1,

$$\Delta_\sigma\left(\phi\right) = 0, \quad \text{for every } \phi \in C^{\text{test}}.$$

We can now repeat the above argument on the interval $(\sigma, \sigma_2^1 \wedge \sigma_1^2)$, on which ρ is Lipschitz, since we can condition on $\mathcal{F}_\sigma$ (using the strong Markov property of the drivers) and the new initial condition, $\Delta_\sigma(\phi)$, is again zero. Since we have finitely many jumps in ρ, we have uniqueness for the whole interval $[0,T]$. This completes the proof. $\square$

5.8. Stochastic McKean–Vlasov dynamics; proof of Theorem 5.0.9

For existence, take W such that we have a solution (ν, L, W) to the loss-dependent SPDE. Strong uniqueness implies that (ν, L) is $\sigma(W)$-measurable, hence we can drop the dependency on L in the conditional expectation in the definition of (5.0.1). This gives a solution to the problem.

For uniqueness, if we have a particle with the prescribed dynamics, then by the work in Section 5.5 its conditional law satisfies Definition 5.0.1. Therefore Theorem 5.0.5 fixes this conditional law, which completes the proof. $\square$

5.9. Extending to general diffusions

In this section we will give an overview of how the results from this chapter can be extended to allow the particles to evolve as general correlated diffusions. There are no new techniques presented in this section, although some additional work has to be done to account for the fact that the individual particles no longer behave as Brownian motions. Whenever a result follows from earlier work by analogy, we will omit its full proof.

We are going to consider the model

$$\begin{cases} X_t^{i,N} := X_0^i + \int_0^t \mu\big(s, X_s^{i,N}, L_s^N\big)ds + \int_0^t \sigma_M\big(s, X_s^{i,N}, L_s^N\big)dW_s \\ \qquad\qquad + \int_0^t \sigma_I\big(s, X_s^{i,N}, L_s^N\big)dW_s^i \\ \tau^{i,N} := \inf\big\{t > 0 : X_t^{i,N} \leq 0\big\} \\ \nu_t^N := \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{t<\tau^{i,N}}\delta_{X_t^{i,N}} \\ L_t^N := \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\tau^{i,N}\leq t}, \end{cases} \tag{5.9.1}$$

with the following assumptions:

Definition 5.9.1 (Diffusion coefficients)**.** Throughout this section we will assume that there exist constants $c_{\min}, c_{\max}, c_{\text{lip}}, \delta_0 > 0$ such that the maps

$$\mu, \sigma_M, \sigma_I : [0,T] \times \mathbf{R} \times [0,1] \to \mathbf{R}$$

satisfy:

(i) (Smoothness) For all $t \in [0,T]$ and $l \in [0,1]$

$$\mu(t,\cdot,l),\ \sigma_M(t,\cdot,l),\ \sigma_I(t,\cdot,l) \in C^\infty(\mathbf{R}),$$

(ii) (Boundedness) For all $t \in [0,T]$, $x \in \mathbf{R}$, $l \in [0,1]$ and $n \in \mathbf{N} \cup \{0\}$

$$|\partial_x^n \mu(t,x,l)|,\ |\partial_x^n \sigma_M(t,x,l)|,\ |\partial_x^n \sigma_I(t,x,l)| \leq c_{\max},$$

(iii) (Lipschitz in space) For all $t \in [0,T]$, $x, y \in \mathbf{R}$ and $l \in [0,1]$

$$|\mu(t,x,l)-\mu(t,y,l)|, |\sigma_M(t,x,l)-\sigma_M(t,y,l)|, |\sigma_I(t,x,l)-\sigma_I(t,y,l)| \leq c_{\mathrm{lip}}|x-y|,$$

(iv) (Piecewise Lipschitz in loss) There exists $n \geq 1$ and

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

such that

$$\big|\mu(t,x,l)-\mu\big(t,x,\bar{l}\,\big)\big|, \big|\sigma_M(t,x,l)-\sigma_M\big(t,x,\bar{l}\,\big)\big|, \big|\sigma_I(t,x,l)-\sigma_I\big(t,x,\bar{l}\,\big)\big| \leq c_{\mathrm{lip}}\big|l-\bar{l}\big|,$$

whenever $l, \bar{l} \in [x_i, x_{i+1})$, $t \in [0,T]$ and $x \in \mathbf{R}$

(v) (Non-zero speed) For all $t \in [0,T]$, $x \in \mathbf{R}$ and $l \in [0,1]$

$$\sigma(t,x,l)^2 := \sigma_M(t,x,l)^2 + \sigma_I(t,x,l)^2 \geq c_{\min},$$

(vi) (Non-degenerate) For all $t \in [0,T]$, $x \in \mathbf{R}$ and $l \in [0,1]$

$$\sigma_I(t,x,l)^2 > \delta_0.$$

The system in (5.9.1) is well-defined because we can use the stopping argument: run the system with $L^N = 0$ initially, the diffusions then exist by standard SDE theory [**Øks03**, Chapter 5], stop the system at the first time one of the diffusions hits zero, restart the system with $L^N = 1/N$ and repeat the argument.

We will show that, as $N \to \infty$, limit points of the system must satisfy the following version of the loss-dependent SPDE:

Definition 5.9.2 (Solution)**.** Let f be a probability measure on $(0,\infty)$, let W be a standard Brownian motion and let μ, σ_M and σ_I be as in Definition 5.9.1, and write $\sigma^2 = \sigma_M^2 + \sigma_I^2$. A pair of processes, (ξ, L), taking values in $D_{\mathscr{S}'} \times D_{\mathbf{R}}$ is said to be a *solution to the loss-dependent SPDE with spatially-dependent coefficients with respect to f and W* if the following hold with probability 1:

(i) For every $t \in [0,T]$, ξ_t is a finite measure,
(ii) L takes values in $[0,1]$, is initially zero and is strictly increasing,

(iii) For every $t \in [0,T]$ and $\phi \in C^{\text{test}}$

$$\xi_t(\phi) = f(\phi) + \int_0^t \xi_s\left(\mu(t,\cdot,L_t)\,\partial_x\phi\right) ds + \frac{1}{2}\int_0^t \xi_s\left(\sigma(t,\cdot,L_t)^2\,\partial_{xx}\phi\right) ds$$
$$+ \int_0^t \xi_s\left(\sigma_M(t,\cdot,L_t)\,\partial_x\phi\right) dW_s,$$

(iv) For every $t \in [0,T]$, $L_t = 1 - \xi_t(0,\infty)$,

(v) The regularity conditions (iii)-(vii) of Definition 2.10.1 hold for the processes

$$\tilde{\mu}(t,x) := \mu\big(t,x,L_t^N\big),\ \tilde{\sigma}_M(t,x) := \sigma_M\big(t,x,L_t^N\big),\ \tilde{\sigma}(t,x) := \sigma\big(t,x,L_t^N\big).$$

Remark 5.9.3. Notice that if μ, σ_M and σ_I satisfy Definition 5.9.1, then $\tilde{\mu}$, $\tilde{\sigma}_M$ and $\tilde{\sigma}$ satisfy condition (i) of Definition 2.10.1.

Remark 5.9.4. As the limiting arguments used to construct solutions to the loss-dependent SPDE embed the emprical processes in $D_{\mathscr{S}'}$, the smoothness condition in Definition 5.9.1 is useful as it ensures that $g(t,\cdot,l)\phi \in \mathscr{S}$ whenever g is a coefficient and $\phi \in \mathscr{S}$. Hence the continuous mapping theorem can be applied directly (see Lemma 5.9.20), without adding additional arguments that use the fact that solutions are more regular (measure-valued) in order to allow g to be less regular. Such regularising arguments would allow us to consider coefficients which are only $C^2(\mathbf{R})$, however, because Lemma 5.9.8 requires second-order Taylor approximations on the coefficients, this would be the weakest possible assumption under the present theory.

Probabilistic estimates. Crucially, the exponent in the first moment estimate of the mass near the origin (Proposition 2.2.2) remains almost unchanged. This is because conditions (ii) and (v) of Definition 5.9.1 allow us to apply a time-change argument:

Proposition 5.9.5 (Boundary estimate)**.** *Assume we have coefficients satisfying Definition 5.9.1. For every fixed* $\delta > 0$

$$\mathbf{E}\int_0^T \nu_t^N(0,\varepsilon)\,dt = \int_0^T \mathbf{P}\big(X_t^{1,N} \in (0,\varepsilon)\,;t < \tau^{1,N}\big)dt = O\big(\varepsilon^{2-\delta}\big)$$

uniformly in $N \geq 1$*, as* $\varepsilon \to 0$.

Proof. Fix $N \geq 1$. The first equality follows by the linearity of expectation, the definition of ν^N and that the particles have identical laws.

For the second equality, suppose initially that $\mu \equiv 0$. Define

$$v(t) := \left[X^{1,N}\right]_t = \int_0^t \sigma\left(s, X_s^{1,N}, L_s^N\right)^2 ds,$$

which is a differentiable with

$$c_{\min} \leq \partial_t v(t) \leq c_{\max}. \tag{5.9.2}$$

Therefore v is invertible and

$$c_{\max}^{-1} \leq \partial_t v^{-1}(t) = \left(\partial_t v(t)\right)^{-1} \leq c_{\min}^{-1}. \tag{5.9.3}$$

By a change of variable we have

$$\int_0^T \mathbf{P}\big(X_t^{1,N} \in (0,\varepsilon); t < \tau^{1,N}\big) dt = \mathbf{E}\int_0^{v(T)} \mathbf{1}_{X_{v^{-1}(t)}^{1,N} \in (0,\varepsilon); v^{-1}(t) < \tau^{1,N}} \cdot \partial_t v^{-1}(t)\, dt$$

and standard theory [**KS91**, Theorem 4.6] gives that $X_{v^{-1}(\cdot)}^{1,N}$ is a Brownian motion (with respect to the time-changed filtration). Set $B_t := X_{v^{-1}(t)}^{1,N}$ and

$$\tau^B := \inf\left\{t > 0 : B_t \leq 0\right\},$$

then $\tau^B = v\big(\tau^{1,N}\big)$. Using (5.9.2) and (5.9.3) we have

$$\begin{aligned}
\int_0^T \mathbf{P}\big(X_t^{1,N} \in (0,\varepsilon); t < \tau^{1,N}\big) dt &= \mathbf{E}\int_0^{v(T)} \mathbf{1}_{B_t \in (0,\varepsilon); t < \tau^B} \cdot \partial_t v^{-1}(t)\, dt \\
&\leq c_{\min}^{-1}\mathbf{E}\int_0^{c_{\max}T} \mathbf{1}_{B_t \in (0,\varepsilon); t < \tau^B}\, dt \\
&\leq c_{\min}^{-1}\int_0^{c_{\max}T} \mathbf{P}\left(B_t \in (0,\varepsilon); t < \tau^B\right) dt \\
&= O\big(\varepsilon^2\big),
\end{aligned} \tag{5.9.4}$$

where the final line follows by Proposition 2.2.2.

For the non-zero drift case we use Girsanov's Theorem [**KS91**, Theorem 5.1]. For ease of notation, we will write

$$\begin{aligned}
d\bar{W}_s &:= \frac{\sigma_M\big(s, X_s^{1,N}, L_s^N\big)}{\sigma\big(s, X_s^{1,N}, L_s^N\big)} dW_s + \frac{\sigma_I\big(s, X_s^{1,N}, L_s^N\big)}{\sigma\big(s, X_s^{1,N}, L_s^N\big)} dW_s^i \\
M_t &:= \int_0^t \mu\big(s, X_s^{1,N}, L_s^N\big) ds \\
I_t &:= \int_0^t \sigma\big(s, X_s^{1,N}, L_s^N\big) d\bar{W}_s.
\end{aligned}$$

Define the change of measure

$$Z_t := \frac{d\mathbf{Q}}{d\mathbf{P}}\bigg|_{\mathcal{F}_t} = \exp\left\{-\int_0^t \frac{\mu}{\sigma}\big(s, X_s^{1,N}, L_s^N\big)d\bar{W}_s - \frac{1}{2}\int_0^t \frac{\mu^2}{\sigma^2}\big(s, X_s^{1,N}, L_s^N\big)ds\right\}.$$

Under $\mathbf{Q}$ we have that $X_t^{1,N}$ is a martingale, so for any $S \in \mathcal{F}_t$ and $p^{-1} + q^{-1} = 1$ we have

$$\begin{aligned}\mathbf{P}(S) &= \mathbf{E}_\mathbf{Q}\left[Z_t^{-1}\mathbf{1}_S\right] \leq \mathbf{E}_\mathbf{Q}\left[Z_t^{-p}\right]^{\frac{1}{p}}\mathbf{Q}(S)^{\frac{1}{q}} = \mathbf{E}_\mathbf{P}\left[Z_t^{1-p}\right]^{\frac{1}{p}}\mathbf{Q}(S)^{\frac{1}{q}} \\ &= \exp\Big\{c_p\int_0^t \frac{\mu^2}{\sigma^2}\big(s, X_s^{i,N}, L_s^N\big)ds\Big\}\mathbf{Q}(S)^{\frac{1}{q}} \\ &= \exp\big\{tc_p c_{\max} c_{\min}^{-1}\big\}\mathbf{Q}(S)^{\frac{1}{q}},\end{aligned}$$

for some constant c_p. Setting

$$S = \left\{X_t^{1,N} \in (0,\varepsilon)\,; t < \tau^{1,N}\right\}$$

and taking $q > 1$ as close to 1 as is required gives the result. $\square$

Remark 5.9.6. From the above proof, we see that by including a drift term we can no longer achieve a boundary estimate of order $O\big(\varepsilon^2\big)$, as we could in Proposition 2.2.2. However, since $\delta > 0$ can be taken arbitrarily small, we can get a decay exponent as close to 2 as we wish. We only require $O\big(\varepsilon^{1+\delta}\big)$ control for the kernel smoothing method (condition (iv) of Definition 2.0.3), so this is sufficient.

Notice that the argument in (5.9.4) is unchanged if we replace $(0,\varepsilon)$ with any subset of $\mathbf{R}$, hence the following is immediate:

Corollary 5.9.7. *For any $\delta \in (0,1)$ there exists $c_\delta > 0$ such that for every $S \subseteq \mathbf{R}$*

$$\mathbf{E}\int_0^T \nu_t^N(S)\,dt \leq c_\delta\left(\int_0^{c_\delta T}\mathbf{P}\big(B_t \in S; t < \tau^B\big)dt\right)^\delta,$$

uniformly in $N \geq 1$, where B is a standard Brownian motion with initial law ν_0 and

$$\tau^B := \inf\{t > 0 : B_t \leq 0\}.$$

Some technical lemmas. This subsection presents a collection of technical results that will be used in the proof of Proposition 5.9.15. The reader should skip this material on first reading and use only as a reference. Throughout, we will use the notation

$$\bar{\nu}_t^N := \sum_{i=1}^N \delta_{X_t^{i,N}}, \tag{5.9.5}$$

which is the empirical process without killing at the origin, and

$$\bar{T}_\varepsilon \zeta(x) := \int_{\mathbf{R}} \psi_\varepsilon(x-y)\zeta(dy), \qquad \text{for } x \in \mathbf{R} \text{ and } \zeta \in \mathscr{S}', \tag{5.9.6}$$

which is the corresponding smoothed kernel on the whole space. These objects form an important part of the work in the proof of Proposition 5.9.15.

Lemma 5.9.8 (Interchanging coefficients and distributions)**.** *Let $g = g(t,x)$ be a function that satisfies*

$$x \mapsto g(t,x) \in C^2(\mathbf{R}), \text{ for all } t \in [0,T], \qquad \sup_{t,x} |\partial_x^n g(t,x)| < \infty, \text{ for } n = 0,1,2.$$

Define the processes

$$\begin{aligned} e_{\varepsilon,t}^{N,g}(x) &:= \bar{\nu}_t^N(g(t,\cdot)\partial_x\psi_\varepsilon(x-\cdot)) - g(t,x)\partial_x\bar{\nu}_t^N(\psi_\varepsilon(x,\cdot)) + \partial_x g(t,x) h_{\varepsilon,t}^N(x) \\ h_{\varepsilon,t}^N(x) &:= \bar{\nu}_t^N((x-\cdot)\partial_x\psi_\varepsilon(x-\cdot)), \end{aligned}$$

then

$$\mathbf{E}\int_0^T \|e_{\varepsilon,t}^{N,g}\|_{L^2(\mathbf{R})}^2 dt \to 0, \qquad \text{as } \varepsilon \to 0,$$

uniformly in N, and there exists a numerical constant, $C > 0$, such that

$$|h_{\varepsilon,t}^N(x)| \le C\bar{\nu}_t^N(\psi_{2\varepsilon}(x-\cdot)), \qquad \text{for all } N \ge 1, t \in [0,T], x \in \mathbf{R} \text{ and } \varepsilon > 0.$$

Proof. For the process $e_{\varepsilon,t}^{N,g}$, notice that a second-order Taylor expansion gives

$$\begin{aligned} |e_{\varepsilon,t}^{N,g}(x)| &\le \int_{\mathbf{R}} |g(t,y) - g(t,x) - (y-x)\partial_x g(t,x)||\partial_x\psi_\varepsilon(x-y)|\bar{\nu}_t^N(dy) \\ &\le \frac{1}{2}\int_{\mathbf{R}} |\partial_{xx} g(t,x)||x-y|^3\varepsilon^{-1}|\psi_\varepsilon(x-y)|\bar{\nu}_t^N(dy). \end{aligned} \tag{5.9.7}$$

With fixed $\varepsilon > 0$, let $\lambda = \lambda(\varepsilon)$ be a function which we specify later that satisfies $\lambda = \lambda(\varepsilon) \to \infty$ as $\varepsilon \to 0$. For any $x \in \mathbf{R}$, by splitting the above integral on $|y - x| < \varepsilon^\eta$ and its complement we have

$$|e_{\varepsilon,t}^{N,g}(x)| \le c_1\varepsilon^{3(\eta-\frac{1}{2})}\bar{\nu}_t^N(x-\varepsilon^\eta, x+\varepsilon^\eta) + c_1\varepsilon^{-3/2}\exp\{-\varepsilon^{2\eta-1}/2\}.$$

By Corollary 5.9.7 applied to $\bar{\nu}^N$ (with the absorption at zero dropped from the conclusion) we have

$$\mathbf{E}\int_0^T \|e^{N,g}_{\varepsilon,t}\|^2_{L^2(-\lambda,\lambda)}dt = \lambda(\varepsilon)O(\varepsilon^{(3+\delta)\eta-3/2} + \varepsilon^{-3/2}\exp\{-\varepsilon^{2\eta-1}/2\}). \tag{5.9.8}$$

Now consider the range $|x| \geq \lambda$. Decomposing the y-integral in (5.9.7) on the range $|y| < |x|/2$ and its complement gives

$$\begin{aligned}|e^{N,g}_{\varepsilon,t}(x)| &\leq \frac{1}{2}\|\partial_{xx}g\|_\infty \int_{\mathbf{R}} |x-y|^3\varepsilon^{-1}\psi_\varepsilon(x-y)\bar{\nu}^N_t(dy)\\ &\leq c_2\varepsilon^{-1}|x|^3\psi_\varepsilon(x/2) + c_2\varepsilon^{-3/2}\bar{\nu}^N_t((-\infty,-|x|/2)\cup(|x|/2,+\infty)),\end{aligned}$$

with $c_2 > 0$ another universal constant. Therefore the uniform tail bound gives

$$\mathbf{E}\int_0^T \|e^{N,g}_{\varepsilon,t}\|^2_{L^2((-\lambda,\lambda)^c)}dt = O(\varepsilon^{1/2} + \varepsilon^{-3/2}\int_{\lambda(\varepsilon)}^\infty e^{-x}dx) = O(\varepsilon^{1/2} + \varepsilon^{-3/2}e^{-\lambda(\varepsilon)}). \tag{5.9.9}$$

Summing (5.9.8) and (5.9.9) and taking any η in the range

$$\frac{3}{2(3+\delta)} < \eta < \frac{1}{2}$$

and fixing $\lambda(\varepsilon) = \log(\varepsilon^{-1})$ gives

$$\mathbf{E}\int_0^T \|e^{N,g}_{\varepsilon,t}\|^2_{L^2(\mathbf{R})}dt \to 0, \qquad \text{as } \varepsilon \to 0.$$

The second result follows from noticing that

$$|z\partial_x\psi_\varepsilon(z)| = \varepsilon^{-1}z^2\psi_\varepsilon(z) = \sqrt{2}\varepsilon^{-1}z^2e^{-z^2/4\varepsilon}\psi_{2\varepsilon}(z),$$

and that the function $z \mapsto ze^{-z}$ is bounded. $\square$

Lemma 5.9.9 (Interchanging integrals for $\bar{\nu}^N$)**.** *For all $n, m \geq 0$, $\varepsilon > 0$ and $t \in [0,T]$*

$$\int_{\mathbf{R}}\Big(\int_0^t \mathbf{E}[|\partial_x^n\bar{T}_\varepsilon\bar{\nu}^N_s(x)\cdot\partial_x^m\bar{T}_\varepsilon\bar{\nu}^N_s(x)|^2]ds\Big)^{1/2}dx < \infty,$$

hence the stochastic Fubini theorem [**Ver12**, (1.4)] *gives*

$$\int_{\mathbf{R}}\int_0^t \partial_x^n\bar{T}_\varepsilon\bar{\nu}^N_s(x)\cdot\partial_x^m\bar{T}_\varepsilon\bar{\nu}^N_s(x)dW_sdx = \int_0^t\int_{\mathbf{R}} \partial_x^n\bar{T}_\varepsilon\bar{\nu}^N_s(x)\cdot\partial_x^m\bar{T}_\varepsilon\bar{\nu}^N_s(x)dxdW_s.$$

Proof. By applying Young's inequality and concavity of $z \mapsto \sqrt{z}$, it suffices to show that

$$\int_{\mathbf{R}}\Big(\int_0^t \mathbf{E}[|\partial_x^n\bar{T}_\varepsilon\bar{\nu}^N_s(x)|^4]ds\Big)^{1/2}dx < \infty$$

First notice that

$$\partial_x^n \bar{T}_\varepsilon \bar{\nu}_s^N(x) = \bar{\nu}_s^N(\partial_x^n \psi_\varepsilon(x-\cdot)) = \bar{\nu}_s^N(P_n(\varepsilon^{-1}(x-\cdot))\psi_\varepsilon(x-\cdot)),$$

where P_n is a polynomial of degree n. Since $\bar{\nu}_s^N$ is a probability measure, Hölder's inequality gives

$$\mathbf{E}[|\partial_x^n \bar{T}_\varepsilon \bar{\nu}_s^N(x)|^4] \leq \mathbf{E}\int_{\mathbf{R}} |P_n(\varepsilon^{-1}(x-y))|^4 \psi_\varepsilon(x-y)^4 \bar{\nu}_s^N(dy). \tag{5.9.10}$$

For any value of x, the integrand above is bounded (recall that ε is fixed). Hence it suffices to bound the right-hand side of (5.9.10) in terms of x only for large values of $|x|$. Splitting the y-integral on the region $|y| < x/2$ and its complement gives the bound

$$\mathbf{E}[|\partial_x^n \bar{T}_\varepsilon \bar{\nu}_s^N(x)|^4] \leq c_\varepsilon \mathbf{E}\bar{\nu}_s^N((x/2,+\infty)\cup(-\infty,-x/2)) + c_\varepsilon \exp\{-x^2/2\varepsilon\},$$

where c_ε depends only on ε and is bounded by some power of $\varepsilon^{-1/2}$. Therefore

$$\mathbf{E}[|\partial_x^n \bar{T}_\varepsilon \bar{\nu}_s^N(x)|^4] = O(c_\varepsilon \exp\{-x/8\varepsilon\}),$$

and this suffices to complete the proof. □

Lemma 5.9.10 (Idiosyncratic integral). *Define the process*

$$\bar{I}_t^N(\phi) := \frac{1}{N}\sum_{i=1}^N \int_0^t \sigma_I(s, X_s^{i,N}, L_s^N)\partial_x\phi(X_s^{i,N})dW_s^i.$$

There exists a function $f_1 = f_1(\varepsilon)$ depending only on ε for which

$$\int_{\mathbf{R}} [\bar{I}_t^N(\psi_\varepsilon(x-\cdot))]dx \leq \frac{f_1(\varepsilon)}{N}, \qquad \textit{for all } N \geq 1, t\in[0,T] \ \textit{and } \varepsilon > 0.$$

Proof. Since the W^i's are independent, the quadratic variation evaluates to

$$[\bar{I}_t^N(\psi_\varepsilon(x-\cdot))] = \frac{1}{N^2}\sum_{i=1}^N \int_0^t \sigma^I(s, X_s^{1,N} L_s^N)^2 \psi_\varepsilon(x - X_s^{i,N})^2 ds.$$

The result is then complete by noting that

$$\int_{\mathbf{R}} \psi_\varepsilon(x - X_s^{i,N})^2 dx = \int_{\mathbf{R}} \psi_\varepsilon(x)^2 dx$$

is purely a function of ε. □

Lemma 5.9.11 (An integration by parts calculation). *Let $f, g \in C^1(\mathbf{R})$ be bounded with bounded first derivatives. Assume also that f and $\partial_x f$ vanish at*

$\pm\infty$ and that g and $\partial_x g$ are bounded. Then

$$\int_{\mathbf{R}} g(x)f(x)\partial_x f(x)dx = -\frac{1}{2}\int_{\mathbf{R}} \partial_x g(x) f(x)^2 dx.$$

Proof. This is an easy application of integration by parts. □

Lemma 5.9.12. *For every $\eta > 0$ there exists a constant $C_\eta > 0$ such that*

$$\mathbf{E}\sup_{u\in[0,T]}\Big|2\int_0^u\int_{\mathbf{R}}\bar{T}_\varepsilon\bar{\nu}_s^N(\sigma_s^M\partial_x\bar{T}_\varepsilon\bar{\nu}_s^N+\partial_x\sigma_s^M h_{\varepsilon,s}^N+e_{\varepsilon,s}^{N,\sigma^M})dxdW_s\Big|$$
$$\leq \eta\mathbf{E}\sup_{s\in[0,t]}\left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2^2+C_\eta\mathbf{E}\int_0^t\left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2^2 ds+C_\eta\mathbf{E}\int_0^t\left\|e_{\varepsilon,s}^{N,\sigma^M}\right\|_2^2 ds.$$

Proof. For every fixed ε, the integrand above is a rapidly decaying function of x, hence the stochastic integral is a martingale, so the Burkholder–Davis–Gundy inequality [**RW00**, Theorem IV.42.1] gives a univeral constant, $c_1 > 0$, for which the left-hand side above is bounded by

$$2c_1\mathbf{E}\Big[\Big(\int_0^t\Big(\int_{\mathbf{R}}\bar{T}_\varepsilon\bar{\nu}_s^N(\sigma_s^M\partial_x\bar{T}_\varepsilon\bar{\nu}_s^N+\partial_x\sigma_s^M h_{\varepsilon,s}^N+e_{\varepsilon,s}^{N,\sigma^M})dx\Big)^2 ds\Big)^{1/2}\Big].$$

By Lemma 5.9.11, this is equal to a constant multiple of

$$\mathbf{E}\Big[\Big(\int_0^t\Big(\int_{\mathbf{R}}\bar{T}_\varepsilon\bar{\nu}_s^N(-\partial_x\sigma_s^M\bar{T}_\varepsilon\bar{\nu}_s^N+\partial_x\sigma_s^M h_{\varepsilon,s}+e_{\varepsilon,s}^{2,\sigma^M})dx\Big)^2 ds\Big)^{1/2}\Big],$$

which, by Hölder's inequality, is bouneded by a constant multiple of

$$\mathbf{E}\Big[\Big(\int_0^t\left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2^2\left\|-\partial_x\sigma_s^M\bar{T}_\varepsilon\bar{\nu}_s^N+\partial_x\sigma_s^M h_{\varepsilon,s}^N+e_{\varepsilon,s}^{N,\sigma^M}\right\|_2^2 ds\Big)^{1/2}\Big]$$
$$\leq \mathbf{E}\Big[\sup_{s\in[0,t]}\left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2\Big(\int_0^t\left\|-\partial_x\sigma_s^M\bar{T}_\varepsilon\bar{\nu}_s^N+\partial_x\sigma_s^M h_{\varepsilon,s}^N+e_{\varepsilon,s}^{N,\sigma^M}\right\|_2^2 ds\Big)^{1/2}\Big].$$

The result then follows by applying Young's inequality with free parameter η, using the boundedness of the coefficients and using the last part of Lemma 5.9.8. □

Lemma 5.9.13. *There exists a function $f_2 = f_2(\varepsilon)$ such that*

$$\mathbf{E}\sup_{u\in[0,t]}\Big|2\int_{\mathbf{R}}\int_0^u\bar{T}_\varepsilon\bar{\nu}_s^N d\bar{I}_s^N(\psi_\varepsilon(x-\cdot))dx\Big|\leq\frac{f_2(\varepsilon)}{N},$$

for all $N \geq 1$, $t \in [0,T]$ and $\varepsilon > 0$.

Proof. First notice that the process can be written as

$$\int_{\mathbf{R}}\int_0^u \bar{T}_\varepsilon \bar{\nu}_s^N d\bar{I}_s^N(\psi_\varepsilon(x-\cdot))dx = -\frac{1}{N}\sum_{i=1}^N \int_{\mathbf{R}}\int_0^u \sigma^M \bar{T}_\varepsilon \bar{\nu}_s^N \partial_x \psi_\varepsilon(x - X_s^{i,N})dW_s^i\,dx.$$

By repeating the work in Lemma 5.9.9 we can switch the order of integration and apply the Birkholder–Davis–Gundy inequality to obtain an upper bound that is a constant multiple of

$$\frac{1}{N}\int_0^t\int_{\mathbf{R}} \mathbf{E}[|\bar{T}_\varepsilon \bar{\nu}_s^N \partial_x \psi_\varepsilon(x - X_s^{1,N})|^2]dxds,$$

and the same reasoning as in Lemma 5.9.10 gives the result. □

Lemma 5.9.14 (Initial regularity)**.** *If $V_0 \in L^2(0,\infty)$, then*

$$\liminf_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\big\|\bar{T}_\varepsilon \nu_0^N\big\|^2_{L^2(\mathbf{R})} \le \big\|V_0\big\|^2_{L^2(0,\infty)}.$$

Proof. First,

$$(\bar{T}_\varepsilon \nu_0^N(x))^2 = \frac{1}{N^2}\sum_{i,j=1}^N \psi_\varepsilon(x - X_0^i)\psi_\varepsilon(x - X_0^j),$$

therefore using the fact that the initial values are i.i.d.

$$\begin{aligned}\mathbf{E}\big\|\bar{T}_\varepsilon \nu_0^N\big\|^2_{L^2(\mathbf{R})} &\le \frac{1}{N}\big\|\psi_\varepsilon\big\|^2_\infty + \mathbf{E}\int_{\mathbf{R}} \psi_\varepsilon(x - X_0^1)\psi_\varepsilon(x - X_0^2)dx \\ &= \frac{1}{N}\big\|\psi_\varepsilon\big\|^2_\infty + \big\|\bar{T}_\varepsilon V_0\big\|^2_{L^2(\mathbf{R})}.\end{aligned}$$

The result then follows by Proposition 2.6.8, with the absorbing kernel replaced by ψ_ε. □

Marginal regularity. As a single particle no longer behaves as a Brownian motion, we require more sophisticated arguments to deduce the regularity properties of the marginal densities. The following asymptotic result plays the role of the arguments in Section 5.5:

Proposition 5.9.15 (Asymptotic marginal L^2 regularity)**.** *If ν_0 has a density $V_0 \in L^2(0,\infty)$, then*

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{t\in[0,T]}\big\|T_\varepsilon \nu_t^N\big\|^2_{L^2(0,\infty)} \\ &\qquad\qquad \le \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{t\in[0,T]}\big\|\bar{T}_\varepsilon \nu_t^N\big\|^2_{L^2(\mathbf{R})} < \infty,\end{aligned}$$

where $\bar{T}_\varepsilon$ is as defined in (5.9.6).

Proof. The first inequality follows from the fact that $T_\varepsilon\zeta \le \bar{T}_\varepsilon\zeta$, whenever ζ is a non-negative measure. Since $\nu^N \le \bar{\nu}^N$ (as defined in (5.9.5)), it suffices to show

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{t\in[0,T]}\left\|\bar{T}_\varepsilon\bar{\nu}_t^N\right\|_{L^2(0,\infty)}^2 < \infty.$$

This allows us to work on the whole real line, rather than the half-line, so difficult boundary effects are avoided.

By repeating the particle-level calculations from Section 2.3, we have

$$\begin{aligned} d\bar{\nu}_t^N(\phi) &= \bar{\nu}_t^N\big(\mu\big(t,\cdot,L_t^N\big)\partial_x\phi\big)dt + \frac{1}{2}\bar{\nu}_t^N\big(\sigma\big(t,\cdot,L_t^N\big)^2\partial_{xx}\phi\big)dt \\ &\quad + \bar{\nu}_t^N\big(\sigma_M\big(t,\cdot,L_t^N\big)\partial_x\phi\big)dW_t + d\bar{I}_t^N(\phi), \end{aligned}$$

for any $\phi \in \mathscr{S}$, since there are no boundary effects in the dynamics of $\bar{\nu}^N$, so we no longer have to restrict to test functions satisfying $\phi(0) = 0$. Here, $\bar{I}^N$ is as defined in Lemma 5.9.10. Setting the function $y \mapsto \psi_\varepsilon(x-y)$ into the evolution equation and interchanging derivatives gives

$$\begin{aligned} d\bar{T}_\varepsilon\bar{\nu}_t^N(x) &= -\partial_x\bar{\nu}_t^N(\mu_t\psi_\varepsilon(x-\cdot))dt + \frac{1}{2}\partial_{xx}\bar{\nu}_t^N(\sigma_t^2\psi_\varepsilon(x-\cdot))dt \\ &\quad - \partial_x\bar{\nu}_t^N(\sigma_t^M\psi_\varepsilon(x-\cdot))dW_t + d\bar{I}_t^N(\psi_\varepsilon(x-\cdot)). \end{aligned}$$

With the notation of Lemma 5.9.8 we have

$$\partial_x\bar{\nu}_t^N(g_t\psi_\varepsilon(x-\cdot)) = g_t(x)\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N(x) + \varepsilon\partial_x g_t(x)\partial_x h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,g}(x),$$

and substituting into the previous equation gives

$$\begin{aligned} d\bar{T}_\varepsilon\bar{\nu}_t^N &= -(\mu_t\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N + \partial_x\mu_t h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,\mu})dt + \frac{1}{2}\partial_x(\sigma_t^2\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N + \partial_x\sigma_t^2 h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,\sigma^2})dt \\ &\quad - (\sigma_t^M\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N + \partial_x\sigma_t^M h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,\sigma^M})dW_t + d\bar{I}_t^N(\psi_\varepsilon(x-\cdot)), \end{aligned}$$

where the dependency on x has been omitted for clarity. Applying Itô's formula to the square $(\bar{T}_\varepsilon\bar{\nu}_t^N)^2$ gives

$$\begin{aligned} d(\bar{T}_\varepsilon\bar{\nu}_t^N)^2 &= -2\bar{T}_\varepsilon\bar{\nu}_t^N(\mu_t\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N + \partial_x\mu_t h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,\mu})dt \\ &\quad + \bar{T}_\varepsilon\bar{\nu}_t^N\partial_x(\sigma_t^2\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N + \partial_x\sigma_t^2 h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,\sigma^2})dt \\ &\quad - 2\bar{T}_\varepsilon\bar{\nu}_t^N(\sigma_t^M\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N + \partial_x\sigma_t^M h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,\sigma^M})dW_t + 2\bar{T}_\varepsilon\bar{\nu}_t^N d\bar{I}_t^N(\psi_\varepsilon(x-\cdot)) \\ &\quad + (\sigma_t^M\partial_x\bar{T}_\varepsilon\bar{\nu}_t^N + \partial_x\sigma_t^M h_{\varepsilon,t}^N + e_{\varepsilon,t}^{N,\sigma^M})^2dt + d[\bar{I}_t^N(\psi_\varepsilon(x-\cdot))]. \end{aligned}$$

Our strategy now is to integrate over $x \in \mathbf{R}$, take a supremum over $t \in [0,T]$ and then take expectation. For the first of these we appeal to Lemma 5.9.9, the final part of Lemma 5.9.8 and Young's inequality with free parameter η to obtain

$$\begin{aligned}\left\|\bar{T}_\varepsilon\bar{\nu}_t^N\right\|_2^2 \le \left\|\bar{T}_\varepsilon\bar{\nu}_0^N\right\|_2^2 + c_\eta\int_0^t \left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2^2 ds + c_\eta\int_0^t \left\|\bar{T}_{2\varepsilon}\bar{\nu}_s^N\right\|_2^2 ds\\ + c_\eta\int_0^t \left\|e_{\varepsilon,s}^{2,\mu}\right\|_2^2 + \left\|e_{\varepsilon,s}^{2,\sigma^2}\right\|_2^2 + \left\|e_{\varepsilon,s}^{2,\sigma^M}\right\|_2^2 ds\\ - \int_0^t\int_{\mathbf{R}}(\sigma_s^2 - (1+\eta)(\sigma_s^M)^2 - \eta - \eta\mu_s^2)(\partial_x\bar{T}_\varepsilon\bar{\nu}_s^N)^2 dxds\\ - 2\int_0^t\int_{\mathbf{R}}\bar{T}_\varepsilon\bar{\nu}_s^N(\sigma_s^M\partial_x\bar{T}_\varepsilon\bar{\nu}_s^N + \partial_x\sigma_s^M h_{\varepsilon,s} + e_{\varepsilon,s}^{2,\sigma^M})dxdW_s\\ + 2\int_{\mathbf{R}}\int_0^t \bar{T}_\varepsilon\bar{\nu}_s^N d\bar{I}_s^N(\psi_\varepsilon(x-\cdot))dx + \int_{\mathbf{R}}[\bar{I}_t^N(\psi_\varepsilon(x-\cdot))]dx,\end{aligned}$$

where $c_\eta > 0$ is a constant depending on η. The final term is bounded by $f_1(\varepsilon)/N$ from Lemma 5.9.10, independently of t. Also, the coefficient on the third line evaluates to

$$(\sigma_s^I)^2 - \eta((\sigma_s^M)^2 + 1 + \mu_s),$$

so it is possible to choose $\eta > 0$ small enough such that this expression is not smaller than zero (recall Definition 2.10.1). Therefore we arrive at the simplification

$$\begin{aligned}\left\|\bar{T}_\varepsilon\bar{\nu}_t^N\right\|_2^2 \le \left\|\bar{T}_\varepsilon\bar{\nu}_0^N\right\|_2^2 + c_\eta\int_0^t \left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2^2 ds + c_\eta\int_0^t \left\|\bar{T}_{2\varepsilon}\bar{\nu}_s^N\right\|_2^2 ds\\ + c_\eta\int_0^t \left\|e_{\varepsilon,s}^{2,\mu}\right\|_2^2 + \left\|e_{\varepsilon,s}^{2,\sigma^2}\right\|_2^2 + \left\|e_{\varepsilon,s}^{2,\sigma^M}\right\|_2^2 ds\\ - 2\int_0^t\int_{\mathbf{R}}\bar{T}_\varepsilon\bar{\nu}_s^N(\sigma_s^M\partial_x\bar{T}_\varepsilon\bar{\nu}_s^N + \partial_x\sigma_s^M h_{\varepsilon,s} + e_{\varepsilon,s}^{2,\sigma^M})dxdW_s\\ + 2\int_{\mathbf{R}}\int_0^t \bar{T}_\varepsilon\bar{\nu}_s^N d\bar{I}_s^N(\psi_\varepsilon(x-\cdot))dx + \frac{f_1(\varepsilon)}{N},\end{aligned}$$

Taking a supremum up to time t and then expectation, and applying Lemma 5.9.12 with $\eta = 1/2$ and Lemma 5.9.13 gives

$$\begin{aligned}\mathbf{E}\sup_{s\in[0,t]}\left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2^2 \le c\mathbf{E}\left\|\bar{T}_\varepsilon\bar{\nu}_0^N\right\|_2^2 + c\mathbf{E}\int_0^t \left\|\bar{T}_\varepsilon\bar{\nu}_s^N\right\|_2^2 ds + c\mathbf{E}\int_0^t \left\|\bar{T}_{2\varepsilon}\bar{\nu}_s^N\right\|_2^2 ds\\ + c\mathbf{E}\int_0^t \left\|e_{\varepsilon,s}^{2,\mu}\right\|_2^2 + \left\|e_{\varepsilon,s}^{2,\sigma^2}\right\|_2^2 + \left\|e_{\varepsilon,s}^{2,\sigma^M}\right\|_2^2 ds + c\frac{f(\varepsilon)}{N},\end{aligned}$$

where $f = \max(f_1, f_2)$ and $c > 0$ is a further universal constant. Using the bound

$$\int_0^t \left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2 ds \leq t \sup_{s\in[0,t]} \left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2$$

and applying Lemma 5.9.8 gives

$$\begin{aligned}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{s\in[0,t]}\left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2 \\ \leq c\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\left\|\bar{T}_\varepsilon \bar{\nu}_0^N\right\|_2^2 + 2ct\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{s\in[0,t]}\left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2.\end{aligned}$$

Using Proposition 2.6.8 (with the absorbing kernel replaced by ψ_ε) and taking $t = (4c)^{-1}$ gives

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{s\in[0,(4c)^{-1}]}\left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2 \leq 2c\left\|V_0\right\|_2^2. \tag{5.9.11}$$

The proof is completed by propagating the argument onto $[(4c)^{-1}, 2(4c)^{-1}]$. To do this, apply the same work as above but started from $s = (4c)^{-1}$, rather than $s = 0$. We then obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{s\in[(4c)^{-1},2(4c)^{-1}]}\left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2 \\ \leq 2c\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{s\in[0,(4c)^{-1}]}\left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2 \leq (2c)^2\left\|V_0\right\|_2^2,\end{aligned}$$

so by taking an integer $M \geq 4cT$

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{E}\sup_{s\in[0,T]}\left\|\bar{T}_\varepsilon \bar{\nu}_s^N\right\|_2^2 \leq (2c)^M\left\|V_0\right\|_2^2 < \infty,$$

as required. □

A direct consequence of the above result is that we can control the maximum asymptotic mass of ν^N contained in any interval of $\mathbf{R}$ throughout the lifetime of the process:

Proposition 5.9.16. *There exists a constant $c > 0$ such that for every interval $I \subseteq \mathbf{R}$*

$$\limsup_{N\to\infty}\mathbf{E}\Big[\sup_{t\in[0,T]}\nu_t^N(I)\Big] \leq c\mathrm{Leb}(I)^{\frac{1}{2}},$$

where Leb *denotes the Lebesgue measure on* $\mathbf{R}$*.*

Proof. By the argument at the beginning of the proof of Proposition 5.9.15, we can work with the process $\bar{\nu}^N$, rather than ν^N. Given $I = (a, b)$, let

$$J := (a - (b-a), a + (b-a)),$$

so that

$$\mathrm{Leb}(J) = 3\mathrm{Leb}(I).$$

Then, by an elementary argument, there exists ε_0 such that whenever $\varepsilon < \varepsilon_0$

$$\frac{1}{2}\chi_I \leq \bar{T}_\varepsilon \chi_J,$$

with χ denoting the usual characteristic function. Therefore

$$\bar{\nu}_t^N(I) \leq 2\int_{\mathbf{R}} \bar{T}_\varepsilon \chi_J(x)\, \bar{\nu}_t^N(dx) = 2\int_J \bar{T}_\varepsilon \bar{\nu}_t^N(x)\, dx \leq 2\mathrm{Leb}(J)^{\frac{1}{2}} \left\|\bar{T}_\varepsilon \bar{\nu}_t^N\right\|_2,$$

hence the result follows from Proposition 5.9.15. □

Tightness. We only have to make minor changes to the arguments in Chapter 4 to retrieve the corresponding tightness results. First, we consider the loss function:

Proposition 5.9.17. *The sequence of loss functions, $\left(L^N\right)_{N\geq 1}$, is tight in $\left(D_{\mathbf{R}}, M_1\right)$.*

Proof. As L^N is monotone, by Proposition 4.1.7 and Theorem 4.1.11, we only need to check that for every $\eta > 0$

$$\limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\in(0,\delta)} \left|L_t^N\right| \geq \eta\Big) \to 0 \quad \text{and}$$
$$\limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\in(0,\delta)} \left|L_T^N - L_{T-t}^N\right| \geq \eta\Big) \to 0$$

as $\delta \to 0$. By applying Markov's inequality, we have the simple bounds

$$\mathbf{P}\Big(\sup_{t\in(0,\delta)} \left|L_t^N\right| \geq \eta\Big) \leq \eta^{-1}\mathbf{E}L_\delta^N = \eta^{-1}\mathbf{P}\big(\tau^{1,N} \leq \delta\big) \quad \text{and}$$
$$\mathbf{P}\Big(\sup_{t\in(0,\delta)} \left|L_T^N - L_{T-t}^N\right| \geq \eta\Big) \leq \eta^{-1}\mathbf{E}\big[L_T^N - L_{T-\delta}^N\big] = \eta^{-1}\mathbf{P}\big(T - \delta < \tau^{1,N} \leq T\big).$$

Fixing an arbitrary time $t \in [0, T]$, by conditioning on the value of $X_{t-\delta}$ we have

(5.9.12)
$$\begin{aligned}\mathbf{P}\big(t-\delta<\tau^{1,N}\leq t\big) &= \int_0^\infty \mathbf{P}\big(t-\delta<\tau^{1,N}\leq t\big| X_{t-\delta}^{1,N}=x_0\big)\mathbf{P}\big(X_{t-\delta}^{1,N}\in dx_0\big)\\ &\leq \mathbf{P}\big(X_{t-\delta}^{1,N}\in(0,\lambda), t-\delta<\tau^{1,N}\big)+\mathbf{P}\big(t-\delta<\tau^{1,N}\leq t\big| X_{t-\delta}^{1,N}=\lambda\big),\end{aligned}$$

for any $\lambda > 0$. By linearity of expectation we have

$$\mathbf{E}\nu_{t-\delta}^{N}(0,\lambda) = \mathbf{P}\big(X_{t-\delta}^{1,N} \in (0,\lambda), t-\delta < \tau^{1,N}\big).$$

The second term in (5.9.12) can be estimated by using Doob's maximal inequality: for $\lambda < c_{\max}\delta$

$$\begin{aligned}
\mathbf{P}\big(t-\delta < \tau^{1,N} \leq t \big| X_{t-\delta}^{1,N} = \lambda\big) &\leq \mathbf{P}\Big(\sup_{t-\delta<s\leq t}\Big|\int_{t-\delta}^{t}\sigma^2 d\bar{W}_s + \int_{t-\delta}^{t}\mu ds\Big| \geq \lambda\Big)\\
&\leq \mathbf{P}\Big(\sup_{t-\delta<s\leq t}\Big|\int_{t-\delta}^{t}\sigma^2 d\bar{W}_s\Big| + c_{\max}\delta \geq \lambda\Big)\\
&\leq (\lambda - c_{\max}\delta)^{-2}\,\mathbf{E}\Big[\sup_{t-\delta<s\leq t}\Big|\int_{t-\delta}^{t}\sigma^2 d\bar{W}_s\Big|^2\Big]\\
&\leq c\delta\,(\lambda - c_{\max}\delta)^{-2},
\end{aligned}$$

with $c > 0$ a universal numerical constant. So from (5.9.12) we have

$$\mathbf{P}\big(t-\delta < \tau^{1,N} \leq t\big) \leq \mathbf{E}\sup_{t\in[0,T]}\nu_t^N(0,\lambda) + c\delta\,(\lambda - c_{\max}\delta)^{-2}.$$

By taking $\lambda = \delta^{1/4}$ and applying Proposition 5.9.16 we have

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbf{P}\big(t-\delta < \tau^{1,N} \leq t\big) = 0,$$

and this suffices to complete the proof. □

To prove that $\left(\nu^N\right)_{N\geq 1}$ is a tight sequence in $(D_{\mathscr{S}'}, M_1)$, first note that the argument in (4.7.1) remains valid, so we need only check the following results to verify that the work in Section 4.7 extends to general diffusions:

Lemma 5.9.18 (Increment moments)**.** *There exists a constant $c > 0$ such that for all $0 < s < t$ and $N \in \mathbf{N}$*

$$\mathbf{E}\big[\big|X_{t\wedge\tau^{i,N}}^{i,N} - X_{s\wedge\tau^{i,N}}^{i,N}\big|^4\big] \leq c\,(t-s)^2.$$

Proof. We begin by introducing the short-hand notation:

$$\begin{aligned}
M_t &:= \int_0^t \mu\big(s, X_s^{i,N}, L_s^N\big)ds\\
I_t &:= \int_0^t \sigma_M\big(s, X_s^{i,N}, L_s^N\big)dW_s + \int_0^t \sigma_I\big(s, X_s^{i,N}, L_s^N\big)dW_s^i.
\end{aligned}$$

By Proposition A.0.2

(5.9.13)

$$\mathbf{E}\big[\big|X_{t\wedge\tau^{i,N}}^{i,N} - X_{s\wedge\tau^{i,N}}^{i,N}\big|^4\big] \leq 8\mathbf{E}\big[\big|M_{t\wedge\tau^{i,N}} - M_{s\wedge\tau^{i,N}}\big|^4\big] + 8\mathbf{E}\big[\big|I_{t\wedge\tau^{i,N}} - I_{s\wedge\tau^{i,N}}\big|^4\big].$$

Condition (ii) of Definition 5.9.1 gives

$$|M_{t\wedge\tau^{i,N}} - M_{s\wedge\tau^{i,N}}|^4 \leq c_{\max}^4 (t-s)^4$$

and by the Burkholder–Davis–Gundy inequality [**RW00**, Theorem IV.42.1]

$$\mathbf{E}\Big[\Big|I_{t\wedge\tau^{i,N}} - I_{s\wedge\tau^{i,N}}\Big|^4\Big] \leq c\mathbf{E}\Big[\Big|\int_s^t \sigma\big(u, X_u^{i,N}, L_u^N\big)^2 du\Big|^2\Big] \leq c\big(2c_{\max}^2\big)^2\big(t-s\big)^2,$$

where $c > 0$ is a universal constant. We then have the result by (5.9.13). □

To conclude, we have:

Theorem 5.9.19 (Tightness). *The sequence $\big(\nu^N, L^N, W\big)$ from (5.9.1) with coefficients satisfying Definition 5.9.1 is tight in the product space*

$$(D_{\mathscr{S}'}, M_1) \times (D_{\mathbf{R}}, M_1) \times (C_{\mathbf{R}}, U)\,.$$

Checking the regularity conditions. Here, we will verify that for a general limit point, (ν^*, L^*, W), condition (v) of Definition 5.9.2 is satisfied. First we need one technical lemma:

Lemma 5.9.20. *Fix $t \in [0,T]$, $\phi \in \mathscr{S}$ and g to be one of μ, σ_M or σ^2. Then the map*

$$(\xi, l) \in \big(D_{\mathscr{S}'}, M_1\big) \times \big(D_{[0,1]}, M_1\big) \mapsto \int_0^t \xi_s\left(g\left(s, \cdot, l_s\right)\phi\right) ds \in \mathbf{R}$$

is continuous at all points $l \in D_{[0,1]}$ that are strictly increasing.

Proof. Begin by fixing (ξ, l) with l satisfying the hypotheses and introduce the short-hand notation

$$G_s^l\left(x\right) = g\left(s, x, l_s\right)\phi\left(x\right) \in \mathbf{R}.$$

For any $\big(\bar{\xi}, \bar{l}\big) \in D_{\mathscr{S}'} \times D_{[0,1]}$ we have

(5.9.14)
$$\begin{aligned}\Big|\int_0^t \big(\xi_s\big(G_s^l\big) - \bar{\xi}_s\big(G_s^{\bar{l}}\big)\big)ds\Big| &\leq \int_0^t \big|\xi_s\big(G_s^l\big) - \bar{\xi}_s\big(G_s^{\bar{l}}\big)\big|ds\\ &\leq \int_0^t \big|\xi_s\big(G_s^l - G_s^{\bar{l}}\big)\big|ds + \int_0^t \big|\big(\xi_s - \bar{\xi}_s\big)\big(G_s^{\bar{l}}\big)\big|ds.\end{aligned}$$

By Corollary 4.4.8 there exists $\alpha, \beta \in \mathbf{N}\cup\{0\}$ and $c > 0$ such that

$$\Big|\int_0^t \big(\xi_s\big(G_s^l\big) - \bar{\xi}_s\big(G_s^{\bar{l}}\big)\big)ds\Big| \leq c\int_0^t \big\|G_s^l - G_s^{\bar{l}}\big\|_{\alpha,\beta} ds + \int_0^t \big|\big(\xi_s - \bar{\xi}_s\big)\big(G_s^{\bar{l}}\big)\big|ds.$$

Applying the Leibniz rule gives

$$\begin{aligned}\left\|G_s^l - G_s^{\bar{l}}\right\|_{\alpha,\beta} &= \sup_{x\in\mathbf{R}}\left|x^\alpha \partial_x^\beta\big(\big(g(s,x,l_s) - g(s,x,\bar{l}_s)\big)\phi(x)\big)\right| \\ &\leq \sup_{x\in\mathbf{R}}\left|x^\alpha \sum_{j=0}^{\beta}\binom{\beta}{j}\partial_x^{\beta-j}\big(g(s,x,l_s) - g(s,x,\bar{l}_s)\big)\partial_x^j\phi(x)\right| \\ &\leq \sum_{n=0}^{\beta}\binom{\beta}{j}\|\phi\|_{\alpha,j}\sup_{x\in\mathbf{R}}\left|\partial_x^{\beta-j}\big(g(s,x,l_s) - g(s,x,\bar{l}_s)\big)\right|.\end{aligned}$$

Recalling $x_1 < x_2 < \cdots < x_n$ from Definition 5.9.1, if l_s and $\bar{l}_s$ are both elements of the same interval $[x_i, x_{i+1})$, then we can apply the Lipschitz property in the loss variable to get

$$\left\|G_s^l - G_s^{\bar{l}}\right\|_{\alpha,\beta} \leq \sum_{j=0}^{\beta}\binom{\beta}{j}\|\phi\|_{\alpha,j}\cdot c_{\text{lip}}\left|l_s - \bar{l}_s\right| =: c_1\left|l_s - \bar{l}_s\right|.$$

However, if l_s and $\bar{l}_s$ are not necessarily in the same interval, then we can only deduce

$$\left\|G_s^l - G_s^{\bar{l}}\right\|_{\alpha,\beta} \leq \sum_{n=0}^{\beta}\binom{\beta}{j}\|\phi\|_{\alpha,j}\cdot 2c_{\max} =: c_2.$$

Since l is strictly increasing, the set of times

$$\mathbf{T}_{\neg\text{lip}} = \{s\in[0,t] : l_s = x_i,\ \text{for some } i = 1,2,\ldots,n\}$$

is finite. Therefore, the ε-neighborhood of $\mathbf{T}_{\neg\text{lip}}$, $N_\varepsilon\big(\mathbf{T}_{\neg\text{lip}}\big)$, has volume

$$\operatorname{Leb}\big(N_\varepsilon\big(\mathbf{T}_{\neg\text{lip}}\big)\big) \leq n\varepsilon.$$

So taking any $\varepsilon > 0$ gives

$$\begin{aligned}\int_0^t\left\|G_s^l - G_s^{\bar{l}}\right\|_{\alpha,\beta}ds &\leq \int_{[0,t]\setminus N_\varepsilon(\mathbf{T}_{\neg\text{lip}})}\left\|G_s^l - G_s^{\bar{l}}\right\|_{\alpha,\beta}ds + \int_{N_\varepsilon(\mathbf{T}_{\neg\text{lip}})}\left\|G_s^l - G_s^{\bar{l}}\right\|_{\alpha,\beta}ds \\ &\leq \int_{[0,t]\setminus N_\varepsilon(\mathbf{T}_{\neg\text{lip}})}\left\|G_s^l - G_s^{\bar{l}}\right\|_{\alpha,\beta}ds + c_2 n\varepsilon.\end{aligned}$$

Let $\mathbf{T}_{\text{cont}}$ denote the cocountable set of times at which l is continuous [**Whi02**, Corollary 12.2.1]. If $\bar{l} \to l$ in $\big(D_{[0,1]}, M_1\big)$, then by [**Whi02**, Lemma 12.5.1] we have

$$\bar{l}_s \to l_s, \quad \text{for all } s\in\mathbf{T}_{\text{cont}},$$

and so eventually l_s and $\bar{l}_s$ will be contained in the same interval for every

$$s \in \mathbf{T}_{\text{cont}} \cap [0,t] \setminus N_\varepsilon\big(\mathbf{T}_{\neg\text{lip}}\big)$$

— note that we require an ε of room to ensure that l_s is contained within an interval, and not on the boundary between two intervals. Since the integrands are bounded, the dominated convergence theorem and the fact that $\mathbf{T}_{\text{cont}}$ is a full set implies

$$\int_{[0,t]\setminus N_\varepsilon\left(\mathbf{T}_{\neg\text{lip}}\right)} \big\|G^l_s - G^{\bar{l}}_s\big\|_{\alpha,\beta} ds \le c_1 \int_{[0,t]\setminus N_\varepsilon\left(\mathbf{T}_{\neg\text{lip}}\right)} \big\|l_s - \bar{l}_s\big\|_{\alpha,\beta} ds \to 0,$$

as $\bar{l} \to l$, and therefore

$$\limsup_{\bar{l}\to l} \int_0^t \big\|G^l_s - G^{\bar{l}}_s\big\|_{\alpha,\beta} ds \le c_2 n\varepsilon.$$

But ε is arbitrary so we conclude

$$\int_0^t \big|\xi_s\big(G^l_s - G^{\bar{l}}_s\big)\big| ds \to 0. \tag{5.9.15}$$

Returning to (5.9.14), for every $\bar{l} \in D_{[0,1]}$

$$\big\|G^{\bar{l}}_s\big\|_{\alpha,\beta} \le 2c_{\max} \sum_{j=0}^{\beta} \binom{\beta}{j} \|\phi\|_{\alpha,j}\,, \quad \text{for any } \alpha,\beta,$$

therefore

$$\text{dist}_{\mathscr{S}}\big(G^{\bar{l}}_s, 0\big) \le 2c_{\max} \sum_{\alpha,\beta\ge 0} 2^{-(\alpha+\beta)} \min\Big(1, \sum_{j=0}^{\beta} \binom{\beta}{j} \|\phi\|_{\alpha,n}\Big),$$

where $\text{dist}_{\mathscr{S}}$ denotes the metric introduced in Definition 2.1.1. The dominated convergence theorem implies

$$\text{dist}_{\mathscr{S}}\big(rG^{\bar{l}}_s, 0\big) \le 2c_{\max} \sum_{\alpha,\beta\ge 0} 2^{-(\alpha+\beta)} \min\Big(1, r\sum_{j=0}^{\beta} \binom{\beta}{j} \|\phi\|_{\alpha,j}\Big) \to 0,$$

uniformly in s and $\bar{l}$, as $r \to 0$, since

$$rG^{\bar{l}}_s = g\left(s, \cdot, \bar{l}_s\right)(r\phi)\,.$$

Hence the set

$$B := \left\{G^{\bar{l}}_s : s \in [0,T]\,, \bar{l} \in D_{[0,1]}\right\}$$

is a bounded subset of $\mathscr{S}$ (for every open ball centered at the origin, we can find λ sufficiently small so that the ball contains λB). We conclude that

$$\left|\left(\xi_s - \bar{\xi}_s\right)\left(G_s^{\bar{l}}\right)\right| \leq p_B\left(\xi_s - \bar{\xi}_s\right) := \sup_{\phi \in B}\left|\left(\xi_s - \bar{\xi}_s\right)(\phi)\right|.$$

We will now adapt the proof of [**Whi02**, Lemma 12.5.1] to show that for every $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$d_{B,M_1}\left(\xi, \bar{\xi}\right) < \delta \quad \text{implies} \int_0^t p_B\left(\xi_s - \bar{\xi}_s\right) ds < \varepsilon,$$

which will complete the proof by (5.9.14), (5.9.15) and Definition 4.4.3.

First note that $\xi \in D_{\mathscr{S}'_n}$ for some $n \in \mathbf{N}$ (recall Section 4.3), therefore

$$p_B(\xi_s - \xi_u) \leq c \left\|\xi_s - \xi_u\right\|_{-n}, \quad \text{for all } s, u \in [0, T],$$

where $c > 0$ is some numerical constant. For $s \in [0, t]$ at which ξ is continuous, take $\delta_s > 0$ such that

$$\left\|\xi_{s+h} - \xi_s\right\|_{-n} < \varepsilon,$$

whenever $|h| < \delta_s$. If we take $\bar{\xi}$ such that $d_{B,M_1}\left(\xi, \bar{\xi}\right) < \delta_s/4$ then, by definition, we can find a pair of continuous parameterisations, (u, r) of γ_ξ and $(\bar{u}, \bar{r})$ of $\gamma_{\bar{\xi}}$, such that

$$\sup_{s_0 \in [0,t]} p_B(u(s_0) - \bar{u}(s_0)) \vee (r(s_0) - \bar{r}(s_0)) < \delta_s/2. \tag{5.9.16}$$

Now take $v \in [0, t]$ such that

$$(u(v), r(v)) = (\xi_s, s),$$

and v_1, v_2 such that

$$r(v_1) = s - \delta_s, \quad r(v_2) = s + \delta_s,$$

which, together with (5.9.16), then guarantees that we have

$$\bar{r}(v_1) < s - \delta_s/2, \quad \bar{r}(v_2) > s + \delta_s/2.$$

Therefore if we take any $s' \in (s - \delta_s, s + \delta_s)$ and find $\bar{v}$ such that $s' = \bar{r}(\bar{v})$ and $\bar{\xi}_{s'} = \bar{u}(\bar{v})$, then

$$\bar{r}(v_1) < \bar{r}(\bar{v}) < \bar{r}(v_2),$$

which forces $v_1 < \bar{v} < v_2$ (by monotonicity of $\bar{r}$) and so (by monotonicity of r)

$$s - \delta_s < r(\bar{v}) < s + \delta_s.$$

Hence we have

$$\begin{aligned}
p_B\big(\xi_{s'} - \bar{\xi}_{s'}\big) &\le p_B(\xi_{s'} - \xi_s) + p_B\big(\xi_s - \bar{\xi}_{s'}\big) \\
&\le c\,\|\xi_{s'} - \xi_s\|_{-n} + p_B(u\,(v) - \bar{u}\,(\bar{v})) \\
&\le c\varepsilon + p_B(u\,(v) - u\,(\bar{v})) + p_B(u\,(\bar{v}) - \bar{u}\,(\bar{v})) \\
&\le c\varepsilon + p_B\big(\xi_s - \xi_{r(\bar{v})}\big) + \delta_s/2 \le c\varepsilon + \delta_s.
\end{aligned}$$

Since it is no loss of generality to rescale ε and δ_s and to specify $\delta_s < \varepsilon$, we conclude: for almost every $s \in [0,t]$ there exist $\delta_s > 0$ such that

$$d_{B,M_1}\big(\xi, \bar{\xi}\,\big) < \delta_s \text{ implies } p_B\big(\xi_{s'} - \bar{\xi}_{s'}\big) < \varepsilon, \text{ whenever } s' \in (s - \delta_s, s + \delta_s)\,.$$

By the boundedness of càdlàg functions [**Whi02**, Corollary 12.2.3], we can take η sufficiently small so that we have

$$\int_0^t p_B\big(\xi_s - \bar{\xi}_s\big)\,ds \le \int_\eta^{t-\eta} p_B\big(\xi_s - \bar{\xi}_s\big)\,ds + \varepsilon,$$

whenever $d_{B,M_1}\left(\xi, \bar{\xi}\,\right) < 1$. Since the collection of intervals

$$\{(s - \delta_s, s + \delta_s)\}_{s\in(0,t)}$$

is an open cover of $[\eta, t-\eta]$, we can take a finite sub-cover

$$[\eta, t-\eta] \subseteq \{(s_i - \delta_{s_i}, s_i + \delta_{s_i})\}_{i=1,2,\ldots,k}$$

and set $\delta = \min_i \delta_{s_i} > 0$. Then for every $s \in [\eta, t-\eta]$ and $d_{B,M_1}\left(\xi, \bar{\xi}\,\right) < \delta$ we have

$$p_B\big(\xi_s - \bar{\xi}_s\big)\,ds < \varepsilon,$$

hence

$$\int_\eta^{t-\eta} p_B\big(\xi_s - \bar{\xi}_s\big)\,ds < t\varepsilon,$$

which completes the proof. □

With this result established, [**Bil99**, Theorem 3.4] allows us to deduce that, along an appropriate subsequence,

$$\mathbf{E}\int_0^T \left|\nu_t^*\left(g\left(s,\cdot,L_s^*\right)\phi\right)\right|^p dt \le \liminf_{k\to\infty} \mathbf{E}\int_0^T \left|\nu_t^{N_k}\left(g\left(s,\cdot,L_s^{N_k}\right)\phi\right)\right|^p dt \tag{5.9.17}$$

for all $p \ge 1$. (Note that we can take the p-norm because

$$\left|\int_0^t |f_s|^p\,ds - \int_0^t |g_s|^p\,ds\right| \le p\int_0^t \left(|f_s| \vee |g_s|\right)^{p-1} |f_s - g_s|\,ds,$$

which doesn't affect the proof in Lemma 5.9.20.) We now verify (iii)-(vii) of Definition 2.10.1 in order:

(iii) Every $\nu_t^{N_k}$ is a sub-probability measure, so

$$\nu_t^{N_k}\left(g\left(s,\cdot,L_s^{N_k}\right)\phi\right) \leq c_{\max}\|\phi\|_\infty,$$

thus the condition follows from (5.9.17).

(iv) Follows immediately from Corollary 5.9.7, Proposition 2.2.1 and (5.9.17).

(v) Follows immediately from Proposition 5.9.5 and (5.9.17).

(vi) is unchanged from the original loss-dependent system.

(vii) This final point requires more work. Notice that

$$\left|\nu_t^*\left(\left(g\left(t,\cdot,L_t^*\right)-g\left(t,x,L_t^*\right)\right)\phi\right)\right|^2 = \left|\nu_t^*\left(g\left(t,\cdot,L_t^*\right)\phi\right)-g\left(t,x,L_t^*\right)\nu_t^*\left(\phi\right)\right|^2,$$

so continuity at the process level is preserved and we have

$$\begin{aligned}
&\mathbf{E}\int_0^T \left|\nu_t^*\big(\big(g\big(t,\cdot,L_t^*\big)-g\big(t,x,L_t^*\big)\big)\phi\big)\right|^2 dt \\
&\quad\leq \liminf_{k\to\infty}\mathbf{E}\int_0^T \left|\nu_t^{N_k}\big(\big(g\big(t,\cdot,L_t^{N_k}\big)-g\big(t,x,L_t^{N_k}\big)\big)\phi\big)\right|^2 dt \\
&\quad\leq \liminf_{k\to\infty}\mathbf{E}\int_0^T \left|\nu_t^{N_k}\big(\big|\big(g\big(\big(t,\cdot,L_t^{N_k}\big)-g\big(t,x,L_t^{N_k}\big)\big)\big)\phi\big|\big)\right|^2 dt \\
&\quad\leq c_{\mathrm{lip}}\liminf_{k\to\infty}\mathbf{E}\int_0^T \nu_t^{N_k}\left(\left|\cdot-x\right|\phi\right)^2 dt \\
&\quad\leq c_{\mathrm{lip}}\sup_{y\in\mathbf{R}}\left|\left(x-y\right)\phi\left(y\right)\right|\liminf_{k\to\infty}\mathbf{E}\int_0^T \nu_t^{N_k}\left(\mathrm{support}\left(\phi\right)\right)dt.
\end{aligned}$$

From Corollary 5.9.7, for any $\delta<1$ there exists $c_\delta>0$ such that

$$\begin{aligned}
\mathbf{E}\int_0^T \nu_t^{N_k}\left(\mathrm{support}\left(\phi\right)\right)dt &\leq c_\delta\left(\int_0^{c_\delta T}\mathbf{P}\big(B_t\in S;t<\tau\big)dt\right)^\delta \\
&\leq c_\delta\left(\int_0^{c_\delta T}\mathbf{P}\big(B_t\in S\big)dt\right)^\delta.
\end{aligned}$$

The density B_t is bounded by $\left(2\pi t\right)^{-1/2}$ so

$$\mathbf{E}\int_0^T \nu_t^{N_k}\left(\mathrm{support}\left(\phi\right)\right)dt \leq c'_\delta\left(\int_0^{c_\delta T}t^{-1/2}\mathrm{Leb}\left(S\right)dt\right)^\delta \leq c''_\delta\mathrm{Leb}\left(S\right)^\delta,$$

where $c'_\delta, c''_\delta>0$ are constants, and this competes the proof. So we conclude that limit points satisfy condition (v) of Definition 5.9.2.

Limit points solve the loss-dependent SPDE. We will state without proof that condition (ii) of Definition 5.9.2 holds for any limit point. This is because the results in Section 5.2 remain valid for (5.9.1) by applying the time-change argument from the proof of Proposition 5.9.5.

To show condition (iii) of Definition 5.9.2, first notice that the corresponding particle-level calculation from Section 2.3 gives

$$\begin{aligned} d\nu_t^N(\phi) &= \frac{1}{2}\nu_t^N\big(\sigma\big(t,\cdot,L_t^N\big)^2\partial_{xx}\phi\big)dt + \nu_t^N\big(\mu\big(t,\cdot,L_t^N\big)\partial_x\phi\big)dt \\ &\quad + \nu_t^N\big(\sigma_M\big(t,\cdot,L_t^N\big)\partial_x\phi\big)dW_t + dI_t^N(\phi), \end{aligned} \tag{5.9.18}$$

where

$$I_t^N(\phi) := \frac{1}{N}\sum_{i=1}^N \int_0^t \sigma_I\left(s, X_s^{i,N}, L_s^N\right)\partial_x\phi\left(X_s^{i,N}\right)dW_s^i,$$

which satisfies

$$\left[I_\cdot^N(\phi)\right]_t \le T c_{\max}^2 \left\|\partial_x\phi\right\|_\infty^2 N^{-1} \to 0.$$

The martingale argument in Section 5.4 is now applicable, although we need to set the martingale components from Definition 5.4.1 to the corresponding one from (5.9.18) and to apply Lemma 5.9.20.

Similar to Section 5.5, we define a tagged particle with the following dynamics:

$$\begin{cases} dX_t^* = \mu\left(t, X_t^*, L_t^*\right)dt + \sigma_M\left(t, X_t^*, L_t^*\right)dW_t + \sigma_I\left(t, X_t^*, L_t^*\right)dW_t^\perp \\ \tau^* = \inf\left\{t > 0 : X_t^* < 0\right\}, \end{cases}$$

with $W^\perp$ an independent Brownian motion, and the corresponding conditional law

$$\tilde{\nu}_t^*(S) := \mathbf{P}\left(X_t^* \in S, t < \tau^* \middle| L^*, W\right).$$

Then by the analogous work to Proposition 5.5.1, we conclude that ν^* and $\tilde{\nu}^*$ are both solutions to the linearised SPDE (Definition 2.10.1). By Theorem 2.10.4 we have that $\tilde{\nu}^* = \nu^*$ as elements of $D_{\mathscr{S}'}$, hence condition (i) of Definition 5.9.2 is satisfied. Condition (iv) then follows by the work in Proposition 5.5.4, so we have:

Theorem 5.9.21 (Existence)**.** *If (ν^*, L^*, W) realises a limiting law, then (ν^*, L^*) satisfies Definition 5.9.2 with respect to W and ν_0.*

Uniqueness. We will complete this section by showing the loss-dependent SPDE has unique solution when the coefficients satisfy Definition 5.9.1. We follow the methods from Sections 5.6 and 5.7, however a localisation argument is required in order to incorporate Proposition 5.9.15.

Theorem 5.9.22 (Uniqueness, law of large numbers). *Suppose $\big(\xi, L\big)$ and $\big(\tilde{\xi}, \tilde{L}\big)$ satisfy Definition 5.9.2 with respect to some W and f. Then, with probability 1*

$$\xi_t(\phi) = \tilde{\xi}_t(\phi), \quad \text{for all } \phi \in \mathscr{S} \text{ and } t \in [0, T].$$

Therefore if ν_0 has a density $V_0 \in L^2(0,\infty)$, then the sequence $\big(\nu^N, L^N, W\big)$ converges in law to the unique solution of the loss-dependent SPDE with spatially-dependent coefficients and with initial value ν_0.

Proof. The convergence result is implied by Theorem 5.9.21 and the first part of this theorem, so we only need to show uniqueness. Also, by repeating the argument in Section 5.7 as necessary, it is no loss of generality for us to assume that the coefficients are globally Lipschitz in loss (that is, to take $n = 1$ in Definition 5.9.1).

Define the difference $\Delta := \xi^1 - \xi^2 \in D_{\mathscr{S}'}$ and begin by taking the difference of the evolution equations satisfied by the solutions, which yields

$$\begin{aligned} d\partial_x^{-1} T_\varepsilon \Delta_t = &-\big(\mu T_\varepsilon \xi_t - \tilde{\mu} T_\varepsilon \tilde{\xi}_t + \bar{o}(1)\big)dt + \frac{1}{2}\partial_x\big(\sigma^2 T_\varepsilon \xi_t - \tilde{\sigma}^2 T_\varepsilon \tilde{\xi}_t + e^3_{\varepsilon,t} - \bar{e}^3_{\varepsilon,t}\big)dt \\ &- \big(\sigma_M T_\varepsilon \xi_t - \tilde{\sigma}_M T_\varepsilon \tilde{\xi}_t + \bar{o}(1)\big)dW_t, \end{aligned}$$

where for the coefficients we have the short-hand notation

$$g = g\big(t, x, L_t\big), \quad \tilde{g} = \tilde{g}\big(t, x, \tilde{L}_t\big),$$

and where the notation $\bar{o}(1)$ is taken from Definition 2.10.3 and $e^3_{\varepsilon,t}$ and $\tilde{e}^3_{\varepsilon,t}$ are the analogous error processes for the two solutions to those in Lemma 2.10.2. By writing $\Delta(g) := g - \tilde{g}$ we have

$$g T_\varepsilon \xi_t - \tilde{g} T_\varepsilon \tilde{\xi}_t = \Delta(g)_t T_\varepsilon \xi_t + \tilde{g} T_\varepsilon \Delta_t,$$

and hence

$$\begin{aligned} d\partial_x^{-1} T_\varepsilon \Delta_t = &-\left(\Delta(\mu)_t T_\varepsilon \xi_t + \tilde{\mu} T_\varepsilon \Delta_t + \bar{o}(1)\right)dt \\ &+ \frac{1}{2}\partial_x\left(\Delta\big(\sigma^2\big)_t T_\varepsilon \xi_t + \tilde{\sigma}^2 T_\varepsilon \Delta_t + e^3_{\varepsilon,t} - \bar{e}^3_{\varepsilon,t}\right)dt \\ &- \left(\Delta(\sigma_M)_t T_\varepsilon \xi_t + \tilde{\sigma}_M T_\varepsilon \Delta_t + \bar{o}(1)\right)dW_t. \end{aligned} \tag{5.9.19}$$

We now introduce the localisation argument. Since ξ_t has the tagged particle representation,

$$\xi_t(S) = \mathbf{P}\left(X_t \in S, t < \tau \,|\, L, W\right), \quad S \subseteq \mathbf{R},$$

where for any independent Brownian motion, $W^\perp$, we have

$$\begin{cases} dX_t = \mu(t, X_t, L_t)\,dt + \sigma_M(t, X_t, L_t)\,dW_t + \sigma_I(t, X_t, L_t)\,dW_t^\perp \\ \tau = \inf\{t > 0 : X_t \leq 0\}. \end{cases}$$

We can therefore define the measure-valued process

$$\bar{\xi}_t(S) = \mathbf{P}\left(X_t \in S \,|\, L, W\right), \quad S \subseteq \mathbf{R}.$$

The work in the proof of Proposition 5.9.15 (but with the dependence on N omitted) gives

$$\liminf_{\varepsilon \to 0} \mathbf{E} \sup_{t\in[0,T]} \left\|T_\varepsilon \bar{\xi}_t\right\|_2^2 < \infty. \tag{5.9.20}$$

By the methods in Section 2.6 we conclude that $\bar{\xi}_t$ has a density process u_t which satisfies

$$\left\|u_t\right\|_2 \leq \liminf_{\varepsilon\to 0} \left\|T_\varepsilon \bar{\xi}_t\right\|_2, \quad \text{for every } t \in [0, T],$$

and so

$$\sup_{t\in[0,T]} \left\|u_t\right\|_2 \leq \liminf_{\varepsilon\to 0} \sup_{t\in[0,T]} \left\|T_\varepsilon \bar{\xi}_t\right\|_2 < \infty, \tag{5.9.21}$$

where we know that the upper bound is finite by applying Fatou's Lemma to (5.9.20). However, from Proposition 2.6.8 we have $\left\|T_\varepsilon \bar{\xi}_t\right\|_2 \leq \left\|u_t\right\|_2$, so we conclude

$$\sup_{t\in[0,T]} \left\|T_\varepsilon \bar{\xi}_t\right\|_2 \leq \sup_{t\in[0,T]} \left\|u_t\right\|_2, \quad \text{for every } \varepsilon > 0. \tag{5.9.22}$$

If we now define the increasing sequence of stopping times

$$T_n := \inf\Big\{s > 0 : \sup_{t\in[0,s]} \|u_t\|_2 > n\Big\} \wedge T,$$

then (5.9.22) gives that

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} \left\|T_\varepsilon \bar{\xi}_t\right\|_2 \leq n, \quad \text{whenever } t \leq T_n,$$

and (5.9.21) guarantees that $T_n \uparrow T$, as $n \to \infty$, with probability 1.

Returning to (5.9.19), by stopping the equation at time T_n and applying the calculations from (2.10.7) onwards (which remain valid provided that time integrals and expectation are not interchanged and using the fact that a stopped martingale is a martingale) we arrive at

$$\begin{aligned}\mathbf{E}\big\|\partial_x^{-1}T_\varepsilon\Delta_{t\wedge T_n}\big\|_2^2 &= \eta^{-1}\mathbf{E}\int_0^{t\wedge T_n}\big\|\partial_x^{-1}T_\varepsilon\Delta_s\big\|_2^2ds + 4\eta(1+\eta)\,\mathbf{E}\int_0^{t\wedge T_n}\|\Delta(\mu)_s\,T_\varepsilon\nu_s+\tilde\mu T_\varepsilon\Delta_s\|_2^2ds\\ &\quad - \mathbf{E}\int_0^{t\wedge T_n}\|\tilde\sigma T_\varepsilon\Delta_s\|_2^2\,ds - \mathbf{E}\int_0^{t\wedge T_n}\int_0^\infty\Delta\big(\sigma^2\big)_s\,T_\varepsilon\Delta_s\cdot T_\varepsilon\nu_s dxds\\ &\quad + \eta\mathbf{E}\int_0^{t\wedge T_n}\|T_\varepsilon\Delta_s\|_2^2ds+(1+\eta)\mathbf{E}\int_0^{t\wedge T_n}\|\Delta(\sigma_M)_sT_\varepsilon\nu_s+\tilde\sigma_MT_\varepsilon\Delta_s\|_2^2ds+o(1).\end{aligned}$$

Applying Young's inequality with $\eta>0$ gives

$$\|\Delta(g)_s\,T_\varepsilon\nu_s+\tilde gT_\varepsilon\Delta_s\|_2^2\le\big(1+4\eta^{-1}\big)\,\|\Delta(g)_s\,T_\varepsilon\nu_s\|_2^2+(1+\eta)\,\|\tilde gT_\varepsilon\Delta_s\|_2^2\,.$$

By the Lipschitz assumption on the loss variable of g we have that

$$\big|\Delta(g)_s\big|\le c_{\mathrm{lip}}\big|L_s-\tilde L_s\big|,$$

therefore if $s<T_n$

$$\begin{aligned}\big\|\Delta(g)_s\,T_\varepsilon\nu_s+\tilde gT_\varepsilon\Delta_s\big\|_2^2&\le\big(1+4\eta^{-1}\big)\,c_{\mathrm{lip}}\big|L_s-\tilde L_s\big|\big\|T_\varepsilon\nu_s\big\|_2^2+(1+\eta)\,\big\|\tilde gT_\varepsilon\Delta_s\big\|_2^2\\ &\le\big(1+4\eta^{-1}\big)\,nc_{\mathrm{lip}}\big|L_s-\tilde L_s\big|+(1+\eta)\,\big\|\tilde gT_\varepsilon\Delta_s\big\|_2^2.\end{aligned}$$

So we obtain a constant $c_\eta>0$ (depending on n and η) such that

$$\begin{aligned}\mathbf{E}\big\|\partial_x^{-1}T_\varepsilon\Delta_{t\wedge T_n}\big\|_2^2&\le c_\eta\mathbf{E}\int_0^{t\wedge T_n}\big\|\partial_x^{-1}T_\varepsilon\Delta_s\big\|_2^2ds+c_\eta\mathbf{E}\int_0^{t\wedge T_n}\big|L_s-\tilde L_s\big|^2ds+o(1)\\ &\quad-\mathbf{E}\int_0^{t\wedge T_n}\int_0^\infty\big(\tilde\sigma^2-\big(1+\eta\big)^2\tilde\sigma_M^2+\eta+4\eta\big(1+\eta\big)^2\tilde\mu^2\big)\big|T_\varepsilon\Delta_s\big|^2dxds\\ &\le c_\eta\mathbf{E}\int_0^t\big\|\partial_x^{-1}T_\varepsilon\Delta_{s\wedge T_n}\big\|_2^2ds+c_\eta\mathbf{E}\int_0^t\big|L_{s\wedge T_n}-\tilde L_{s\wedge T_n}\big|^2ds+o(1)\\ &\quad-\mathbf{E}\int_0^t\int_0^\infty\big(\tilde\sigma^2-\big(1+\eta\big)^2\tilde\sigma_M^2+\eta+4\eta\big(1+\eta\big)^2\tilde\mu^2\big)\big|T_\varepsilon\Delta_{s\wedge T_n}\big|^2dxds.\end{aligned}$$

For the above estimate, the work in Section 5.6 is now applicable and gives $\Delta=0$ on $[0,T_n)$, with probability 1. By sending $n\to\infty$ we get $\Delta=0$ on $[0,T)$, and this extends to $[0,T]$ by the continuity argument in Section 5.7. This completes the proof. □

CHAPTER 6

A central limit theorem

In this short chapter we show how our methods can be used to study the fluctuation processes,

$$f_t^N := \sqrt{N}\left(\nu_t^N - \nu_t\right) \in D_{\mathscr{S}'},$$

and their associated limit. We will work with the basic model in which the correlation, ρ, is a constant strictly less than 1. This assumption allows us to use the strong estimate in Theorem 3.0.2 for the proceeding analysis and to prove convergence in the stronger space $(D_{\mathscr{S}'}, J_1)$. The purpose of this chapter is to illustrate how we can analyse the system and limiting SPDE when the driving noise is rougher. Working with the constant correlation model greatly simplifies this chapter and we do not attempt to provide results for the general model in Chapter 5 (see Remark 6.0.6). We will also assume that ν_0 has a density V_0 which is bounded.

By considering the system at the particle level, in Section 6.5 we will see that the finite system has dynamics given by

$$f_t^N(\phi) = f_0^N(\phi) + \frac{1}{2}\int_0^t f_s^N(\partial_{xx}\phi)ds + \rho\int_0^t f_s^N(\partial_x\phi)dW_s + \sqrt{1-\rho^2}J_t^N(\partial_x\phi),$$

where the idiosyncratic driver, $J^N \in D_{\mathscr{S}'}$, is defined by

$$J_t^N(\phi) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \int_0^t \phi\left(X_s^i\right)\mathbf{1}_{s<\tau^i}dW_s^i.$$

As we have $\sqrt{N}$ scaling, this noise does not vanish in the limit $N \to \infty$, therefore to summarise the system we keep track of the quadruple $\left(\nu^N, f^N, W, J^N\right)$. In Section 6.3 we prove the sequence is tight, and this relies on estimates from Section 6.2 where we apply the central limit theorem for real-valued random variables to deduce marginal convergence for the system.

Theorem 6.0.1 (Tightness)**.** *The sequence* $\left(\nu^N, f^N, W, J^N\right)_{N\geq 1}$ *is tight, and hence relatively compact, in the product space*

$$(D_{\mathscr{S}'}, M_1) \times (D_{\mathscr{S}'}, M_1) \times (C_{\mathbf{R}}, U) \times (D_{\mathscr{S}'}, M_1)\,.$$

Furthermore, $\left(\nu^N, W, J^N\right)_{N\geq 1}$ *converges in law to the triple* (ν, W, J) *where we have the representation*

$$J_t = \int_0^t \Psi_s dW_s^{\perp},$$

with $W^{\perp}$ *a standard* $L^2(\mathbf{R})$*-cylindrical Brownian motion independent of* W *and*

$$\Psi_t : L^2(\mathbf{R}) \to L^2(\mathbf{R}), \quad \Psi_t(\phi)(x) := V_t(x)^{1/2}\phi(x)\,.$$

The process J is described in detail in Section 6.1. It captures the idiosyncratic noise introduced by scaling the fluctuations in convergence and, conditional on $\sigma(W)$, is a centered $\mathscr{S}'$-valued Gaussian process with covariance

$$\mathrm{cov}_{t,s}(\phi,\psi) = \mathbf{E}\left[J_t(\phi)\,J_s(\psi)\middle|\,W\right] = \int_0^{t\wedge s} \nu_u(\phi\psi)\,du, \quad \phi,\psi \in \mathscr{S}.$$

It can therefore be regarded as a stochastic space-time white noise, but with its spatial component weighted according to the $\sigma(W)$-random measure ν. This is intuitive; the larger the value $\nu_t(S)$, the greater the potential for the fluctuations to be large at the space-time location (S,t).

Again an SPDE arises as the limiting object for the system:

Definition 6.0.2 (Solution)**.** Let g be a probability measure on $(0,\infty)$, W a standard Brownian motion and ν the limit empirical process with respect to W. Define J with respect to W and ν as in Theorem 6.0.1. Then a stochastic process $\xi \in D_{\mathscr{S}'}$ is said to *solve the fluctuation SPDE with respect to* (g,ν,W,J) if the following hold with probability 1:

(i) For every $t \in [0,T]$ and $\phi \in C^{\mathrm{test}}$

$$\xi_t(\phi) = g(\phi) + \frac{1}{2}\int_0^t \xi_s(\partial_{xx}\phi)\,dt + \rho\int_0^t \xi_s(\partial_x\phi)\,dW_t + \sqrt{1-\rho^2}J_t(\partial_x\phi)\,,$$

(ii) The regularity conditions (ii)-(v) of Definition 2.0.3 are satisfied by ξ.

Theorem 6.0.3 (Existence)**.** *Let* (ν, f^*, W, J) *realise a limiting law of the system* $\left(\nu^N, f^N, W, J^N\right)_{N\geq 1}$*. (By Proposition 6.2.3 and the established convergence of* ν^N*, we can fix* (ν, W, J) *as in Theorem 6.0.1.) Then* f^* *solves the fluctuation SPDE with respect to* (f_0, ν, W, J)*, where* f_0 *is a centered Gaussian* $\mathscr{S}'$*-valued* *random variable with covariance*

$$\mathbf{E}\big[f_0(\phi)f_0(\psi)\big] := \nu_0(\phi\psi) - \nu_0(\phi)\nu_0(\psi), \quad \phi, \psi \in \mathscr{S}.$$

This result is proved in Section 6.5 after the regularity conditions are verified in Section 6.4. As the fluctuation SPDE is linear, the work in Chapter 2 gives:

Theorem 6.0.4 (Uniqueness)**.** *Let (g, ν, W, J) be as in Definition 6.0.2, then there exists at most one solution to the fluctuation SPDE with respect to this quadruple. That is, if ξ^1 and ξ^2 are two solutions, then with probability 1 we have*

$$\xi_t^1(\phi) = \xi_t^2(\phi), \quad \textit{for every } t \in [0, T] \textit{ and } \phi \in \mathscr{S}.$$

Since ν is determined by W, and therefore J is determined by W, from Theorem 6.0.4 we deduce that the law of any limit point is determined by W. Hence all limit laws agree and so we immediately deduce weak convergence:

Theorem 6.0.5 (Central limit theorem)**.** *The system*

$$\left(\nu^N, f^N, W, J^N\right)_{N \geq 1}$$

converges in law on the product space

$$(D_{\mathscr{S}'}, M_1) \times (D_{\mathscr{S}'}, M_1) \times (C_{\mathbf{R}}, U) \times (D_{\mathscr{S}'}, M_1)$$

to the law of a quadruple (ν, f, W, J), where (ν, W, J) is as in Definition 6.0.2 and f is the unique solution to the fluctuation SPDE with respect to (f_0, ν, W, J), with f_0 as in Theorem 6.0.3.

Remark 6.0.6 (The loss-dependent model)**.** To extend the work in this chapter to the loss-dependent model requires us to generalise the techniques from Chapter 5 for the fluctuation process. The present analysis uses the strong estimates from Theorem 3.0.2, which are only available to us in the constant correlation case. In particular, to generalise the M_1-tightness calculations we would need to derive equivalent conditions to those in [**Bil99**, Section 14] for the J_1 topology, however this an open problem [**Whi02**, Remark 7.2.2].

6.1. The idiosyncratic driver

The goal of this section is to understand the precise construction of the idiosyncratic process, J. Our main references are [**KX95**, Chapter 3] and [**Hai09**, Section 3], and also the paper [**Itô83**] in which our problem is studied on the whole space with $\rho = 0$.

Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $L^2(\mathbf{R})$. We have the following definition:

Definition 6.1.1 (Cylindrical Brownian motion)**.** We will say the family

$$\left\{B_t(\phi) : t \in [0,T], \phi \in L^2(\mathbf{R})\right\}$$

is a *standard* $L^2(\mathbf{R})$*-cylindrical Brownian motion* if the following hold:

(i) For every $\phi \in L^2(\mathbf{R})$, $\|\phi\|_2^{-1} B_t(\phi)$ is a one-dimensional standard Brownian motion,

(ii) For all $t \in [0,T]$, $\lambda_1, \lambda_2 \in \mathbf{R}$ and $\phi_1, \phi_2 \in L^2(\mathbf{R})$

$$B_t(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1 B_t(\phi_1) + \lambda_2 B_t(\phi_2),$$

with probability 1,

(iii) For every $\phi \in L^2(\mathbf{R})$, $B_\cdot(\phi)$ is a martingale with respect to

$$\mathcal{F}_t^B = \sigma\left(B_s(\psi) : s \le t, \psi \in L^2(\mathbf{R})\right).$$

Notice from [**KX95**, Theorem 3.2.1, Remark 3.2.1] that it is *not* possible to find an $L^2(\mathbf{R})$-valued process B' such that

$$B_t(\phi) = \langle B_t', \phi \rangle, \quad \text{for all } t, \phi.$$

However, we do have that

$$B_t(\phi) = \sum_{n=1}^{\infty} \langle \phi, e_n \rangle B_t(e_n),$$

where $\{e_n\}_{n\ge1}$ is any complete orthonormal system in $L^2(\mathbf{R})$ and that the one-dimensional Brownian motions $\{B_\cdot(e_n)\}_{n\ge1}$ are independent [**KX95**, Theorem 3.2.2]. So, alternatively, by starting with a sequence of independent Brownian motions on $\mathbf{R}$, we can construct a standard cylindrical Brownian motion through the above summation, which we note converges in square expectation, since

$$\mathbf{E}\Big[\sup_{t\in[0,T]} \Big(\sum_{n=m}^{m+k} \langle \phi, e_n \rangle B_t(e_n)\Big)^2\Big] \le c \sum_{n=m}^{m+k} \langle \phi, e_n \rangle^2 T \to 0,$$

as $m, k \to \infty$, where $c > 0$ is a numerical constant from Doob's maximal inequality.

Now consider the family of maps Ψ as defined in Theorem 6.0.1. From the proof of Proposition 5.5.5, we know that

$$V_t(x) \le \int_0^\infty K(x_0, x) V_0(dx_0),$$

where $K(x_0, x)$ is the Gaussian density, therefore by bounding V_0 by $\|V_0\|_\infty$ we have

$$\|V_t\|_\infty \le \|V_0\|_\infty, \quad \text{for every } t, \text{ with probability 1.}$$

Therefore we can confirm that Ψ_t maps $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ since

$$\|\Psi_t(\phi)\|_2^2 = \int_0^\infty V_t(x) |\phi(x)|^2 dx \le \|V_0\|_\infty \|\phi\|_2^2,$$

and so we also have that the operator norm is uniformly controlled:

$$\|\Psi_t\|_{\mathrm{Op}} \le \|V_0\|_\infty^{1/2}, \quad \text{for every } t \in [0, T],$$

with probability 1.

It is now clear that

$$\mathbf{E} \int_0^T \|\Psi_t\|_{\mathrm{Op}}^2 dt < \infty,$$

so we can define the stochastic integral of Ψ with respect to an independent standard cylindrical Brownian motion, $W^\perp$, which gives a continuous $\mathscr{S}'$-valued process satisfying

$$\left(\int_0^t \Psi_s dW_s^\perp \right)(\phi) = \sum_{n=1}^\infty \int_0^t \langle \Psi_s(\phi), e_n \rangle dW_s^\perp(e_n),$$

[**KX95**, Section 3.3]. This is the precise definition of J. Note also that $\left(\int_0^\cdot \Psi_s dW_s^\perp\right)(\phi)$ is a real-valued continuous process with covariation

$$[J_\cdot(\phi), J_\cdot(\psi)]_t = \int_0^t \left\langle V_s^{1/2}\phi, V_s^{1/2}\psi \right\rangle ds = \int_0^t \nu_s(\phi\psi) ds. \tag{6.1.1}$$

We conclude that our construction gives a process which is Gaussian when conditioned on $\sigma(W)$:

Proposition 6.1.2 (Conditionally Gaussian process). *With $W^\perp$ a cylindrical Brownian motion independent of W, we have that*

$$J := \int_0^\cdot \Psi_s dW_s^\perp$$

is a $\mathscr{S}'$-valued continuous process whose marginals,

$$(J_{t_1}(\phi_1), J_{t_2}(\phi_2), \dots, J_{t_k}(\phi_k)) \in \mathbf{R}^k, \quad t_1, \dots, t_k \in [0, T],\ \phi_1, \dots, \phi_k \in \mathscr{S},$$

are conditionally Gaussian given $\sigma(W)$ with mean zero and covariance

$$\operatorname{cov}_{t,s}(\phi,\psi) = \mathbf{E}\left[J_t(\phi)\,J_s(\psi)\middle|\,W\right] = \int_0^{t\wedge s}\nu_u(\phi\psi)\,du.$$

Proof. It is straightforward to see that J is centered and has the required covariance structure from (6.1.1).

To show the marginals are conditionally Gaussian, for $t \in [0,T]$ and $\phi \in \mathscr{S}$ we have that $J_t(\phi)$ is a square-integrable martingale with

$$\left[J_\cdot(\phi)\right]_t = \int_0^t \left\|V_s^{1/2}\phi\right\|_2^2 ds.$$

So for every $\lambda \in \mathbf{R}$ we have

$$\begin{aligned} e^{i\lambda J_t(\phi)} &= e^{i\lambda J_0(\phi)} + i\lambda\int_0^t e^{i\lambda J_s(\phi)}dJ_s(\phi) - \frac{1}{2}\lambda^2\int_0^t\left\|V_s^{1/2}\phi\right\|_2^2 e^{i\lambda J_s(\phi)}ds \\ &= i\lambda\sum_{n=1}^\infty I_n - \frac{1}{2}\lambda^2\int_0^t\left\|V_s^{1/2}\phi\right\|_2^2 e^{i\lambda J_s(\phi)}ds \end{aligned} \tag{6.1.2}$$

where the I_n's denote the summands

$$I_n = \int_0^t e^{i\lambda J_t(\phi)}\left\langle\Psi_s(\phi), e_n\right\rangle dW_s^\perp(e_n).$$

and where their infinite sum converges in square expectation.

From Lemma 2.4.1 we have

$$\mathbf{E}\Big[\sum_{n=1}^k I_n\Big|W\Big] = 0, \quad \text{for every } k\in\mathbf{N}$$

and we can extend this fact to $k=\infty$ by noting that

$$\begin{aligned} \mathbf{E}\Big[\mathbf{E}\Big[\sum_{n=1}^\infty I_n\Big|W\Big]^2\Big] &\le 2\mathbf{E}\Big[\mathbf{E}\Big[\sum_{n=1}^\infty I_n - \sum_{n=1}^k I_n\Big|W\Big]^2\Big] \\ &\qquad + 2\mathbf{E}\Big[\mathbf{E}\Big[\sum_{n=1}^k I_n\Big|W\Big]^2\Big] \\ &\le 2\mathbf{E}\Big[\Big(\sum_{n=1}^\infty I_n - \sum_{n=1}^k I_n\Big)^2\Big] \\ &\to 0, \quad \text{as } k\to\infty. \end{aligned}$$

Taking a conditional expectation over (6.1.2) therefore yields

$$m_t = m_0 - \frac{1}{2}\lambda^2\int_0^t\left\|V_s^{1/2}\phi\right\|_2^2 m_s ds, \quad m_t = \mathbf{E}\left[e^{i\lambda J_t(\phi)}\middle|\,W\right],$$

and solving for m_t gives

$$\mathbf{E}\left[e^{i\lambda J_t(\phi)}\middle|\, W\right] = \exp\Big\{-\frac{1}{2}\lambda^2\int_0^t \big\|V_s^{1/2}\phi\big\|_2^2 ds\Big\},$$

so $J_t(\phi)$ is conditionally Gaussian.

The result extends to all finite-dimensional margins in the usual way. By noting that we can rewrite any linear combination:

$$\sum_{j=1}^k \lambda_j J_{t_j}(\phi_i) = \sum_{i=1}^k \gamma_j\left(J_{t_j}(\psi_j) - J_{t_{j-1}}(\psi_j)\right) =: F_k,$$

for some other $\gamma_i \in \mathbf{R}$, $\psi_i \in \mathscr{S}$ and $t_0 := 0$, we have

$$\begin{aligned}\mathbf{E}\big[e^{iF_k}\big|W\big] &= \mathbf{E}\big[\mathbf{E}\big[e^{iF_k}\big|\mathcal{F}_{t_{k-1}}\big]\big|W\big]\\ &= \mathbf{E}\Big[e^{iF_{k-1}}\mathbf{E}\Big[e^{i\gamma_k\left(J_{t_k}(\psi_k)-J_{t_{k-1}}(\psi_k)\right)}\Big|W\Big]\Big|W\Big]\\ &= \exp\Big\{-\frac{1}{2}\gamma_k^2\int_{t_{k-1}}^{t_k}\big\|V_s^{1/2}\phi\big\|_2^2 ds\Big\}\mathbf{E}\big[e^{iF_{k-1}}\big|W\big],\end{aligned}$$

and repeating the calculation gives the required result. □

6.2. Marginal convergence

In this section we will show that the marginals of f^N and J^N converge in law. This requires only basic probabilistic methods, however the results will be necessary for establishing tightness in Section 6.3.

Begin by noting that, for any fixed $t \in [0,T]$, both $f_t^N(\phi)$ and $J_t^N(\phi) - J_s^N(\phi)$ are of the form

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N I_i,$$

with I_i conditionally i.i.d. and mean zero given $\sigma(W)$. Therefore the central limit theorem in $\mathbf{R}$ gives that the one-dimensional marginals converge in conditional distribution to a random variable of the form

$$\text{Normal}\Big(0, \mathbf{E}\big[(I_1)^2\,|W\,\big]\Big). \tag{6.2.1}$$

Hence we conclude:

Proposition 6.2.1 (Marginal convergence)**.** *For every $s < t \in [0,T]$ and $\phi \in \mathscr{S}$*

$$f_t^N(\phi) \Rightarrow Z^f \quad \textit{and} \quad J_t^N(\phi) - J_s^N(\phi) \Rightarrow Z^J, \quad \textit{in } \mathbf{R},$$

where, conditional on $\sigma(W)$,

$$Z^f \sim \text{Normal}\Big(0, \nu_t(\phi^2) - \nu_t(\phi)^2\Big) \quad \textit{and}$$

$$Z^J \sim \text{Normal}\Big(0, \int_s^t \nu_u(\phi^2)du\Big).$$

Proof. All that remains is to calculate the variance in (6.2.1) for both cases. For $f_t^N(\phi)$,

$$I_i = \mathbf{1}_{t<\tau^i}\phi(X_t^i) - \nu_t(\phi),$$

therefore

$$\begin{aligned}\mathbf{E}\big[(I_1)^2|W\big] &= \mathbf{E}\big[\mathbf{1}_{t<\tau^i}\phi(X_t^i)^2|W\big] + 2\mathbf{E}\big[\mathbf{1}_{t<\tau^i}\phi(X_t^i)|W\big]\nu_t(\phi) - \nu_t(\phi)^2 \\ &= \nu_t(\phi^2) - \nu_t(\phi)^2.\end{aligned}$$

For $J_t^N(\phi) - J_s^N(\phi)$

$$I_i = \int_s^t \mathbf{1}_{u<\tau^i}\phi\left(X_u^i\right) dW_u^i,$$

so this case is trivial. □

Remark 6.2.2 (Non-negative variance)**.** The Cauchy–Schwarz inequality guarantees that the stated conditional variance for Z_t^f in Proposition 6.2.1 is non-negative:

$$\nu_t(\phi)^2 \le \nu_t(\phi^2)\nu_t(\mathbf{R}) \le \nu_t(\phi^2),$$

since ν_t is a sub-probability measure.

As the increments of J^N are independent, we can deduce the full marginal convergence of J^N to J:

Proposition 6.2.3 (Full marginal convergence for J)**.** *For every $k \ge 1$, $t_1, t_2, \dots, t_k \in [0,T]$ and $\phi_1, \phi_2, \dots, \phi_k \in \mathscr{S}$ we have*

$$\left(J_{t_1}^N(\phi_1), \dots, J_{t_k}^N(\phi_k)\right) \Rightarrow \left(J_{t_1}(\phi_1), \dots, J_{t_k}(\phi_k)\right), \quad \textit{in } \mathbf{R}^k,$$

as $N \to \infty$.

Proof. As in the proof of Proposition 6.1.2, for any $\lambda_j \in \mathbf{R}$ we can rewrite

$$F_k^N := \sum_{i=1}^k \lambda_j J_{t_j}^N(\phi_j) = \sum_{i=1}^k \gamma_j\big(J_{t_j}^N(\psi_j) - J_{t_{j-1}}^N(\psi_j)\big)$$

for some $\gamma_j \in \mathbf{R}$, $\psi_j \in \mathscr{S}$ and $t_0 := 0$. The work in the final part of that proof also gives

$$\mathbf{E}\big[e^{iF_k^N}\big|W\big] = \prod_{j=1}^{k} \mathbf{E}\big[\exp\big\{i\gamma_j\big(J_{t_j}^N(\psi_j) - J_{t_{j-1}}^N(\psi_j)\big)\big\}\big|W\big].$$

Since the random variables on the right-hand side are bounded, they are uniformly integrable so Proposition 6.2.1 gives

$$\lim_{N\to\infty} \mathbf{E}\big[e^{iF_k^N}\big|W\big] = \prod_{j=1}^{k} \mathbf{E}\big[\exp\big\{i\gamma_j\left(J_{t_j}(\psi_j) - J_{t_{j-1}}(\psi_j)\right)\big\}\big|W\big],$$

which gives the result by the Cramér–Wold device [**Whi02**, Theorem 4.3.3]. $\square$

6.3. Tightness of the system; proof of Theorem 6.0.1

Here, we establish tightness of the system $\left(\nu^N, f^N, W, J^N\right)_{N\geq 1}$ as outlined in Theorem 6.0.1. Since we know $\left(\nu^N\right)_{N\geq 1}$ is tight from Chapter 4, it suffices to only check that $\left(f^N\right)_{N\geq 1}$ and $\left(J^N\right)_{N\geq 1}$ are tight in $(D_{\mathscr{S}'}, M_1)$, because marginal tightness implies joint tightness (as explained in the proof of Theorem 4.0.4 in Section 4.7).

In the following proofs, we will proceed by showing that, for every $\phi \in \mathscr{S}$, $J^N(\phi)$ and $f^N(\phi)$ form tight sequences in $D_{\mathbf{R}}$ with respect to the J_1 topology. As this topology is stronger than the M_1 topology, this implies the sequences are tight in $(D_{\mathbf{R}}, M_1)$. We then have the required results by Theorem 4.0.1.

To verify tightness in the J_1 topology, we will use Corollary 7.2 and Theorem 8.8 of [**EK86**, Chapter 3] and the work in the proof of Donsker's theorem for empirical processes from [**Bil99**, Theorem 14.3], which we are able to apply in this context as the interaction between the particles is simple.

First, the easier result:

Proposition 6.3.1. *The sequence* $\left(J^N\right)_{N\geq 1}$ *is tight in* $\left(D_{\mathscr{S}'}, M_1\right)$.

Before proceeding, we note the following elementary calculation, which we will use throughout this section:

Lemma 6.3.2 (Fourth moment)**.** *Let* $X_1, X_2, \ldots, X_k$ *be real-valued i.i.d. random variables with finite fourth moment and zero mean. Then*

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{k} X_i\Big)^4\Big] \leq k\mathbf{E}\left[X_1^4\right] + k^2\mathbf{E}\left[X_1^2\right]^2 \leq 2k^2\mathbf{E}\left[X_1^4\right].$$

Proof of Proposition 6.3.1. As above, fix $\phi \in \mathscr{S}$. For condition (a) of [**EK86**, Chapter 3, Corollary 7.2], notice that Proposition 6.2.1 gives

$$\lim_{N\to\infty} \mathbf{P}\left(\left|J_t^N(\phi)\right| > \lambda\right) \leq \mathbf{E}\Phi\Big(-\Big(\int_0^t \nu_s(\phi^2)ds\Big)^{-1}\lambda\Big)$$
$$\leq \Phi\big(-\lambda/T\|\phi\|_\infty^2\big) \to 0, \quad \text{as } \lambda \to \infty.$$

Therefore the compact subset $B = [-\lambda, \lambda] \subseteq \mathbf{R}$ satisfies our requirements.

For condition (b), since $J_\cdot^N(\phi)$ is a martingale with

$$\left[J_\cdot^N(\phi)\right]_t = \frac{1}{N}\sum_{i=1}^N \int_0^t \phi(X_s^i)^2 \mathbf{1}_{s<\tau^i} ds,$$

the Burkholder–Davis–Gundy inequality gives a universal constant $C > 0$ such that

$$\mathbf{E}\big[\big(J_t^N(\phi) - J_s^N(\phi)\big)^4\big] \leq C\,\|\phi\|_\infty^4\,(t-s)^2.$$

Hence we are done by applying [**EK86**, Chapter 3, Theorem 8.8]. $\square$

To show that $\left(f^N\right)_{N\geq 1}$ is tight, we first demonstrate the following result:

Proposition 6.3.3 (Loss process increments)**.** *There exists constants $c_1 > 0$ and $\beta \in (0,1)$ such that for every $s < t$ we have*

$$\mathbf{E}\left[(L_t - L_s)^2\right] \leq c_1\,(t-s)^{1+\beta}.$$

(The range allowable β depends on the parameter γ from Theorem 3.0.1; see the final line of the proceeding proof.) Therefore by Kolmogorov's continuity criterion [**EK86**, Chapter 3, Corollary 8.10], *L has a continuous version.*

Proof. Fix s and t. From the definition of L (HERE) we have

$$\begin{aligned}\mathbf{E}\left[(L_t - L_s)^2\right] &= \mathbf{E}\left[\mathbf{P}\left(s < \tau^1 \leq t\middle| W\right)\mathbf{P}\left(s < \tau^2 \leq t\middle| W\right)\right] \\ &= \mathbf{P}\left(s < \tau^1, \tau^2 \leq t\right).\end{aligned}$$

Writing $\mathbf{P}_{x_1,x_2}$ to indicate that we are conditioning on the event that $(X_s^1, X_s^2) = (x_1, x_2)$, we have

$$\begin{aligned}
\mathbf{E}\left[(L_t - L_s)^2\right] &= \mathbf{E}\int_0^\infty\int_0^\infty \mathbf{P}_{x_1,x_2}\left(s<\tau^1,\tau^2\le t\right)\nu_s\left(dx_1\right)\nu_s\left(dx_2\right) \\
&\le \mathbf{E}\int_{(t-s)^\eta}^\infty\int_{(t-s)^\eta}^\infty \mathbf{P}_{x_1,x_2}\big(s<\tau^1,\tau^2\le t\big)\,\nu_s(dx_1)\,\nu_s(dx_2) \\
&\quad + \mathbf{E}\big[\nu_s\left(0,(t-s)^\eta\right)^2\big] \\
&\le 2\mathbf{E}\int_{(t-s)^\eta}^\infty \mathbf{P}_{x_1}\big(s<\tau^1\le t\big)\,\nu_s(dx_1) + \mathbf{E}\big[\nu_s\left(0,(t-s)^\eta\right)^2\big]
\end{aligned}$$

where $\eta > 0$ is a free parameter and where we are using that the conditional independence of the particles gives

$$\mathbf{P}\left(X_s^1 \in dx_1, X_s^2 \in dx_2\right) = \mathbf{E}\left[\nu_s\left(dx_1\right)\nu_s\left(dx_2\right)\right].$$

If $x_1 > (t-s)^\eta$, then

$$\begin{aligned}
\mathbf{P}_{x_1}\big(s<\tau^1\le t\big) &= \mathbf{P}\big(x_0^1 + \inf_{0<h<t-s} B_h \le 0\big) \\
&= 2\Phi\big(-(t-s)^{-\frac{1}{2}}x_1\big) \le 2\Phi\big(-(t-s)^{-\left(\frac{1}{2}-\eta\right)}\big),
\end{aligned}$$

therefore we have the bound

$$\mathbf{E}\big[(L_t-L_s)^2\big] \le c\,(t-s)^{(3+\gamma)\eta} + 4\Phi\big(-(t-s)^{-\left(\frac{1}{2}-\eta\right)}\big),$$

where we have used Theorem 3.0.2 and where $c > 0$ is some numerical constant. The proof is now complete by choosing

$$\eta = \frac{1+\beta}{3+\gamma} < \frac{1}{2},$$

where β is any number satisfying

$$0 < \beta < \frac{1+\gamma}{2}.$$

□

Now, we have that the random variables $\{L_{\tau^i}\}_{i\ge1}$ are i.i.d. and uniform when conditioned on $\sigma(W)$, since

$$\mathbf{P}\left(\tau^i \le t\middle|\, W\right) = L_t,$$

by definition. Therefore if we define

$$F_t^N = \sqrt{N}\left(L_t^N - L_t\right) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\mathbf{1}_{\tau^i \le t} - L_t\right)$$

and $L^{-1}(t) = \inf\{s > 0 : L_s = t\}$, then

$$F_{L^{-1}(t)}^N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\mathbf{1}_{L_{\tau^i} \le t} - t\right)$$

and the work in the proof of [**Bil99**, Theorem 14.3] gives that for $t_1 < t < t_2$

$$\mathbf{E}\big[\big(F_{L^{-1}(t)}^N - F_{L^{-1}(t_1)}^N\big)^2\big(F_{L^{-1}(t_2)}^N - F_{L^{-1}(t)}^N\big)^2\big|W\big] \le 6\big(t - t_1\big)\big(t_2 - t\big).$$

We therefore recover that

$$\mathbf{E}\big[\big(F_t^N - F_{t_1}^N\big)^2\big(F_{t_2}^N - F_t^N\big)^2\big|W\big] \le 6\big(L_t - L_{t_1}\big)\big(L_{t_2} - L_t\big),$$

so Proposition 6.3.3 and the Cauchy–Schwarz inequality imply

$$\mathbf{E}\Big[\big(F_t^N - F_{t-h}^N\big)^2\big(F_{t+h}^N - F_t^N\big)^2\Big] \le c_2 h^{1+\beta}, \quad \text{for all } t, h > 0, \tag{6.3.1}$$

where $c_2 > 0$ is a numerical constant.

In the following result we use (6.3.1) in order to apply [**EK86**, Chapter 3, Theorem 8.8]:

Proposition 6.3.4. *The sequence* $\left(f^N\right)_{N\ge1}$ *is tight in* $\left(D_{\mathscr{S}'}, M_1\right)$.

Proof. Condition (a) of [**EK86**, Chapter 3, Theorem 7.2] follows from the same working as that for condition (i) in the proof of Proposition 6.3.1, so we will omit the details. We will then be done if we can verify that the conditions of [**EK86**, Chapter 3, Theorem 8.8] hold.

To do this, first notice that we can write

$$f_t^N(\phi) - f_s^N(\phi) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} I_i,$$

where

$$I_i = \phi\big(X_t^i\big)\mathbf{1}_{t<\tau^i} - \phi\big(X_s^i\big)\mathbf{1}_{s<\tau^i} - \nu_t\big(\phi\big) + \nu_s\big(\phi\big)$$

and this can be rewritten as

$$\begin{aligned} I_i = \phi\big(X_{t\wedge\tau^i}^i\big) - \phi\big(X_{s\wedge\tau^i}^i\big) - \phi\big(0\big)\big(\mathbf{1}_{\tau^i\le t} - \mathbf{1}_{\tau^i\le s}\big) \\ - \big(\bar{\nu}_t\big(\phi\big) - \bar{\nu}_s\big(\phi\big)\big) + \phi\big(0\big)\big(L_t - L_s\big). \end{aligned}$$

Therefore we have that

$$f_t^N(\phi) - f_s^N(\phi) = -\phi(0)\big(F_t^N - F_s^N\big) + \alpha_{s,t}^N, \tag{6.3.2}$$

where we define

$$\alpha_{s,t}^N := \frac{1}{\sqrt{N}}\sum_{i=1}^N K_i, \quad K_i := \phi\big(X^i_{t\wedge\tau^i}\big) - \phi\big(X^i_{s\wedge\tau^i}\big) - \big(\bar\nu_t(\phi) - \bar\nu_s(\phi)\big).$$

By applying (6.3.2), Proposition A.0.1 and Cauchy–Schwarz we have

$$\begin{aligned}
&\mathbf{E}\big[\big(f_{t+h}^N(\phi) - f_t^N(\phi)\big)^2\big(f_t^N(\phi) - f_{t-h}^N(\phi)\big)^2\big] \\
&\qquad = 4\,|\phi(0)|^4\,\mathbf{E}\big[\big(F_{t+h}^N - F_t^N\big)^2\big(F_t^N - F_{t-h}^N\big)^2\big] \\
&\qquad\quad + 4\,|\phi(0)|^2\,\mathbf{E}\big[\big(\alpha_{t,t+h}^N\big)^4\big]^{\frac12}\mathbf{E}\big[\big(F_t^N - F_{t-h}^N\big)^4\big]^{\frac12} \\
&\qquad\quad + 4\,|\phi(0)|^2\,\mathbf{E}\big[\big(\alpha_{t-h,t}^N\big)^4\big]^{\frac12}\mathbf{E}\big[\big(F_{t+h}^N - F_t^N\big)^4\big]^{\frac12} \\
&\qquad\quad\quad + 4\,|\phi(0)|^2\,\mathbf{E}\big[\big(\alpha_{t-h,t}^N\big)^4\big]^{\frac12}\mathbf{E}\big[\big(\alpha_{t+h,t}^N\big)^4\big]^{\frac12}.
\end{aligned}$$

From the calculation in Lemma 4.7.1 we know that

$$\mathbf{E}\big[\big(\alpha_{s,t}^N\big)^4\big] = O\big((t-s)^2\big), \quad \text{uniformly in } N,$$

so this fact and (6.3.1) gives

$$\begin{aligned}
&\mathbf{E}\big[\big(f_{t+h}^N(\phi) - f_t^N(\phi)\big)^2\big(f_t^N(\phi) - f_{t-h}^N(\phi)\big)^2\big] \\
&\qquad = \mathbf{E}\big[\big(F_t^N - F_{t-h}^N\big)^4\big]^{\frac12} O(h) \\
&\qquad\quad + \mathbf{E}\big[\big(F_{t+h}^N - F_t^N\big)^4\big]^{\frac12} O(h) + O\big(h^{1+\beta}\big).
\end{aligned} \tag{6.3.3}$$

Now, Lemma 6.3.2 implies

$$\mathbf{E}\big[\big(F_t^N - F_s^N\big)^4\big] \;\le\; 2\mathbf{E}\big[\big(\mathbf{1}_{\tau^i\le t} - L_t - \mathbf{1}_{\tau^i\le s} + L_s\big)^4\big],$$

and then Proposition A.0.2 with $N=2$ and $p=4$ implies

$$\begin{aligned}
\mathbf{E}\big[\big(F_t^N - F_s^N\big)^4\big] &\le 2^4\mathbf{E}\big[\big(\mathbf{1}_{\tau^i\le t} - \mathbf{1}_{\tau^i\le s}\big)^4\big] + 2^4\mathbf{E}\big[\big(L_t - L_s\big)^4\big], \\
&\le 2^4\mathbf{E}\big[\mathbf{1}_{\tau^i\le t} - \mathbf{1}_{\tau^i\le s}\big] + 2^4\mathbf{E}\big[L_t - L_s\big] \\
&= 2^5\mathbf{E}\big[L_t - L_s\big],
\end{aligned}$$

where we have used the fact that the random variables on the right-hand side have absolute value not greater than 1. By Cauchy–Schwarz and Proposition

6.3.3 we get

$$\mathbf{E}\big[\big(F_t^N - F_s^N\big)^4\big] \leq 2^5 \mathbf{E}\big[\big(L_t - L_s\big)^2\big]^{\frac{1}{2}} = O\big(\big(t-s\big)^{\frac{1+\beta}{2}}\big).$$

Setting this into (6.3.3) gives

$$\mathbf{E}\big[\big(f_{t+h}^N(\phi) - f_t^N(\phi)\big)^2\big(f_t^N(\phi) - f_{t-h}^N(\phi)\big)^2\big] = O\left(h^{1+\delta}\right),$$

with $\delta = \beta \wedge (1+\beta/4) > 0$. Therefore condition (b) of [**EK86**, Chapter 3, Theorem 8.8] is verified, so the proof is complete. □

By the discussion at the start of this section, we have Theorem 6.0.1. □

6.4. The regularity conditions

The results of the previous section make the regularity conditions very easy to check for a general limit point (ν, f^*, W, J). First, by fixing a sequence $(N_k)_{k\geq 1}$ along which the weak limit is obtained, observe that the portmanteau theorem [**Bil99**, Theorem 2.1] and Corollary 5.1.4 give a cocountable set of times $\mathbf{T} \subseteq [0, T]$ on which

$$\mathbf{E}\big[\big|f_t^*(\phi)\big|^p\big] \leq \liminf_{k\to\infty} \mathbf{E}\big[\big|f_t^{N_k}(\phi)\big|^p\big],$$

for any $p \geq 1$. By Proposition 6.2.1

$$\begin{aligned}\mathbf{E}\big[\big|f_t^*(\phi)\big|^p\big] \leq \mathbf{E}\big[\big|Z_t^f\big|^p\big] &\leq c(p)\,\mathbf{E}\big[\big|\nu_t\left(\phi^2\right) - \nu_t(\phi)^2\big|^{p/2}\big] \\ &\leq c(p)\,\mathbf{E}\big[\big|\nu_t\left(\phi^2\right)\big|^{p/2}\big]\end{aligned} \tag{6.4.1}$$

where $c(p)$ is the numerical constant

$$c(p) = \mathbf{E}\left[\left|\mathrm{Normal}\,(0,1)\right|^p\right],$$

and where in the final line we have used Remark 6.2.2.

Since $\nu_t(\phi^2) \leq \|\phi\|_\infty^2$, condition (ii) of Definition 2.0.3 is immediate. With ϕ_λ as in (iii), Cauchy–Schwarz gives

$$\begin{aligned}\mathbf{E}\int_0^T \big|f_t^*(\phi_\lambda)\big|^4 dt &\leq c(4)\int_0^T \mathbf{E}\big[\big|\nu_t\left(\phi_\lambda^2\right)\big|^2\big]dt \\ &\leq c(4)\,T^{\frac{1}{2}}\left(\int_0^T \mathbf{E}\big[\big|\nu_t\left(\phi_\lambda^2\right)\big|^4\big]dt\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore (iii) holds for f^*, since ϕ_λ^2 has the same properties as ϕ_λ and taking the square-root in the final line does not change the asymptotic behavior because we can vary the exponent c condition in (iii) for ν. Similarly if ϕ is supported on $(-\infty, 0)$ then so is ϕ^2, so again condition (v) remains valid.

Finally, we will show (iv). With ϕ_ε as specified in the condition, Proposition 6.2.1 gives

$$\mathbf{E}\int_0^T \left|f_t^*(\phi_\varepsilon)\right|^2 dt = \int_0^T \mathbf{E}\left[\nu_t(\phi_\varepsilon^2) - \nu_t(\phi_\varepsilon)^2\right] dt \leq \int_0^T \mathbf{E}\nu_t(0,\varepsilon) dt,$$

so we are done by Proposition 2.2.2.

Remark 6.4.1 (The H^{-1} method is necessary)**.** In order for the L^2 method from [**BHH$^+$11**] to be applicable, we require $O(\varepsilon^{3+\gamma})$ control in this final calculation. However, the above only gives control at the level of $\mathbf{E}\nu_t\,(0,\varepsilon)$, which is $O(\varepsilon^2)$. Furthermore, it is not the case that we have lost information in the bound in (6.4.1), since if we set $\phi = \chi_{(0,\varepsilon)}$ we have

$$\mathbf{E}\left|\nu_t\big(\chi^2_{(0,\varepsilon)}\big) - \nu_t\big(\chi_{(0,\varepsilon)}\big)^2\right| = \mathbf{E}\nu_t\big(0,\varepsilon\big) - \mathbf{E}\left[\nu_t\big(0,\varepsilon\big)^2\right],$$

and we know from Theorem 3.0.2 that this final term is bounded below by a constant multiple of $\varepsilon^{3+\gamma}$, so the expression is still only $O(\varepsilon^2)$. Therefore, in order to analyse the fluctuation SPDE with the kernel smoothing method, it is necessary to consider H^{-1} energy estimates.

6.5. Proof of Theorem 6.0.3 and Theorem 6.0.4

Existence. First, let us consider the initial condition in Theorem 6.0.3. The random variables $\{X_0^i\}_{i\geq 1}$ are assumed to be i.i.d. with law ν_0, so it is clear that any finite vector of marginals of the form

$$f_0^N(\phi) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \left[\phi\big(X_0^i\big) - \nu_0(\phi)\right],$$

converges weakly to a Gaussian vector, by the central limit theorem. The covariance can be calculated from

$$\begin{aligned}
\mathbf{E}\left[f_0^N(\phi) f_0^N(\psi)\right] &= \frac{1}{N}\sum_{i,j=1}^N \mathbf{E}\left[\big(\phi\big(X_0^i\big) - \nu_0(\phi)\big)\big(\psi\big(X_0^i\big) - \nu_0(\psi)\big)\right] \\
&= \mathbf{E}\left[\big(\phi\big(X_0^1\big) - \nu_0(\phi)\big)\big(\psi\big(X_0^1\big) - \nu_0(\psi)\big)\right] \\
&= \nu_0\big(\phi\psi\big) - \nu_0\big(\phi\big)\nu_0\big(\psi\big).
\end{aligned}$$

To show that condition (i) of Definition 6.0.2 is satisfied for a general limit point we will use the martingale method from Section 5.4. To capture the effect of the driver J^N, we need to introduce another component process, which we denote by I^ϕ.

Definition 6.5.1 (Martingale components)**.** For a fixed test function $\phi \in C^{\text{test}}$ define:

(i) $M^\phi : D_{\mathscr{S}'} \to D_{\mathbf{R}}$,

$$M^\phi(f)\,(t) := f_t\,(\phi) - f_0\,(\phi) - \frac{1}{2}\int_0^t f_s\,(\partial_{xx}\phi)\,ds$$

(ii) $S^\phi : D_{\mathscr{S}'} \times D_{\mathscr{S}'} \to D_{\mathbf{R}}$,

$$S^\phi(\mu, f)\,(t) := M^\phi(f)\,(t)^2 - \int_0^t \big(\rho^2 f_s\,(\partial_x\phi)^2 + \big(1-\rho^2\big)\mu_s\,(\partial_x\phi)^2\big)ds$$

(iii) $C^\phi : D_{\mathscr{S}'} \times C_{\mathbf{R}} \to D_{\mathbf{R}}$,

$$C^\phi(f, w)\,(t) := M^\phi(f)\,(t) \cdot w\,(t) - \rho \int_0^t f_s\,(\partial_x\phi)\,ds$$

(iv) $I^\phi : D_{\mathscr{S}'} \times D_{\mathscr{S}'} \times D_{\mathscr{S}'} \to D_{\mathbf{R}}$,

$$I^\phi(\mu, f, j)\,(t) := M^\phi(f)\,(t) \cdot j_t\,(\phi) - \sqrt{1-\rho^2}\int_0^t f_s\,(\partial_x\phi)\,ds.$$

Subtracting the basic SPDE for ν from equation (2.3.2) for ν^N and scaling by $\sqrt{N}$ we obtain

$$f_t^N(\phi) = f_0^N(\phi) + \frac{1}{2}\int_0^t f_s^N(\partial_{xx}\phi)ds + \rho\int_0^t f_s^N(\partial_x\phi)dW_s + \sqrt{1-\rho^2}\,J_t^N(\partial_x\phi),$$

which can be rewritten in terms of the martingales components as

$$M^\phi\big(f^N\big)\,(t) = M^\phi\big(f^N\big)\,(0) + \rho\int_0^t f_s^N(\partial_x\phi)dW_s + \sqrt{1-\rho^2}\,J_t^N(\partial_x\phi). \tag{6.5.1}$$

From this equation it is straightforward to obtain:

Lemma 6.5.2. *For every $\phi \in C^{\text{test}}$, the processes*

$$M^\phi\big(f^N\big),\ S^\phi\big(\nu^N, f^N\big),\ C^\phi\big(f^N, W\big)\ \text{and}\ I^\phi\big(\nu^N, f^N, J^N\big)$$

are all square-integrable martingales.

Proof. First, begin by noting that, since the summands in $f_t^N\,(\phi)$ are centered and i.i.d. given $\sigma\,(W)$, we have

$$\mathbf{E}\big[\big|f_t^N(\phi)\big|^2\big|W\big] = \mathbf{E}\big[\phi\big(X_t^1\big)^2\mathbf{1}_{t<\tau^1}\big|W\big] \le \|\phi\|_\infty^2.$$

It is then clear that all the processes are square-integrable.

From (6.5.1) $M^\phi\big(f^N\big)\,(t)$ is a martingale. Applying Itô's formula to the square of $M^\phi\big(f^N\big)$ gives

$$dS^\phi\big(\nu^N, f^N\big)\,(t) = 2M^\phi\big(f^N\big)\,(t)\, dM^\phi\big(f^N\big)\,(t)\,,$$

since

$$\big[M^\phi\big(f^N\big)\big]_t = \int_0^t \big(\rho^2 f_s^N\,(\partial_x\phi)^2 + \big(1-\rho^2\big)\nu_s\,(\partial_x\phi)^2\big)ds.$$

Therefore $S^\phi\big(\nu^N, f^N\big)$ is a martingale. The processes C^ϕ and I^ϕ are seen to be martingales by the same reasoning and noting that

$$\big[M^\phi\big(f^N\big)\,, W\big]_t = \rho\int_0^t f_s^N\,(\partial_x\phi)\,ds \quad \text{and}$$

$$\big[M^\phi\big(f^N\big)\,, J^N\,(\partial_x\phi)\big]_t = \sqrt{1-\rho^2}\int_0^t \nu_s^N\,\big((\partial_x\phi)^2\big)\,ds.$$

$\square$

With the above lemma, the work in Section 5.4 leading up to Proposition 5.4.4 is applicable, therefore we shall omit the calculations and just conclude:

Proposition 6.5.3. *Let (ν, f^*, W, J) be any weak limit point. For every $\phi \in C^{\text{test}}$ the processes $M^\phi(f^*)$, $S^\phi(\nu, f^*)$, $C^\phi(f^*, W)$ and $I^\phi(\nu, f^*, J)$ are all square-integrable martingales.*

This result implies that

$$\big[M^\phi(f^*)\big]_t = \int_0^t \big(\rho^2 f_s^*\,(\partial_x\phi)^2 + \big(1-\rho^2\big)\nu_s\,(\partial_x\phi)^2\big)ds,$$

$$\big[M^\phi(f^*), W\big]_t = \rho\int_0^t f_s^*\,(\partial_x\phi)\,ds \quad \text{and}$$

$$\big[M^\phi(f^*), J\,(\partial_x\phi)\big]_t = \sqrt{1-\rho^2}\int_0^t \nu_s\,\big((\partial_x\phi)^2\big)\,ds$$

and so we compute that the quadratic variation

$$\begin{aligned}
&\left[M^\phi(f^*) - \rho\int_0^{\cdot} f_s^*(\partial_x\phi)dW_s - \sqrt{1-\rho^2}J_{\cdot}(\partial_x\phi)\right]_t \\
&\quad = \big[M^\phi(f^*)\big]_t + \rho^2\int_0^t f_s^*\,(\partial_x\phi)^2\,ds + \big(1-\rho^2\big)\int_0^t \nu_s\,\big((\partial_x\phi)^2\big)ds \\
&\qquad - 2\rho\int_0^t f_s^*(\partial_x\phi)d\,\big[M^\phi(f^*), W\big]_t - 2\sqrt{1-\rho^2}\,\big[M^\phi(f^*), J_{\cdot}\,(\partial_x\phi)\big]_t
\end{aligned}$$

must be zero. Hence we have condition (i) of Definition 6.0.2. Condition (ii) was checked in the previous section, so we have completed the proof. $\square$

Uniqueness. Proving Theorem 6.0.4 is simple. Take two solutions ξ^1 and ξ^2 of the fluctuation SPDE with respect to some (g, ν, W, J). Define their difference, $\Delta \in D_{\mathscr{S}'}$, to be

$$\Delta_t(\phi) := \xi_t^1(\phi) - \xi_t^2(\phi).$$

By Remark 2.0.4, Δ satisfies condition (ii) of Definition 6.0.2. Also, by subtracting the linear evolution equations for ξ^1 and ξ^2 we get that

$$\Delta_t(\phi) = \frac{1}{2}\int_0^t \Delta_s(\partial_{xx}\phi)\, ds + \rho \int_0^t \Delta_s(\partial_x \phi)\, dW_s.$$

In other words, Δ solves the basic SPDE from Definition 2.0.3 with zero initial condition, hence $\Delta = 0$ by Theorem 2.0.7. $\square$

APPENDIX A

Elementary facts

Inequalities. In computations we make frequent use of the following basic inequality:

Proposition A.0.1. *For every $a, b \in \mathbf{R}$, $(a+b)^2 \leq 2a^2 + 2b^2$.*

And we also use the generalisation:

Proposition A.0.2. *Let $N \in \mathbf{N}$, $a_1, a_2, \ldots, a_N \in \mathbf{R}$ and $p > 1$. Then*

$$\left|\sum_{i=1}^{N} a_i\right|^p \leq N^{p-1} \sum_{i=1}^{N} |a_i|^p .$$

Proof. Apply Hölder's inequality in $\mathbf{R}^N$:

$$\left|\sum_{i=1}^{N} a_i\right| = |(1,1,\ldots,1)\cdot(a_1,a_2,\ldots,a_N)| \leq N^{1/q}\Big(\sum_{i=1}^{N} |a_i|^p\Big)^{1/p},$$

where q is the Hölder conjugate of p. We are done since $p/q = p-1$. $\square$

Gaussian tails. Throughout we will use the notation

$$\phi(x) = (2\pi)^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{2}\right\}, \quad \Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy.$$

Proposition A.0.3. *Let $x > 0$, then*

$$\big(x^{-1} - x^{-3}\big)\phi\big(x\big) \leq \Phi\big(-x\big) \leq x^{-1}\phi\big(x\big).$$

Proof. $\partial_x \phi(x) = -x\phi(x)$, so we have

$$\Phi(-x) = \int_x^{\infty} \phi(y)\, dy \leq x^{-1} \int_x^{\infty} y\phi(y)\, dy = x^{-1}\phi(x),$$

since in the range of integration $x^{-1}y \geq 1$.

For the lower bound, applying integration by parts gives

$$\Phi(-x) = \int_x^{\infty} y^{-1}\cdot y\phi(y)\, dy = x^{-1}\phi(x) - \int_x^{\infty} y^{-2}\phi(y)\, dy,$$

and then we are done by using $x^{-1}y \geq 1$ in the final integral. $\square$

Corollary A.0.4. *If $x > \sqrt{2}$, then*

$$\frac{1}{2}x^{-1}\phi\left(x\right) \leq \Phi\left(-x\right) \leq x^{-1}\phi\left(x\right).$$

Proposition A.0.5. *For every $0 < c_1 < 1$ and polynomial p, there exists $c_2 > 0$ such that*

$$\left|p\left(x\right)\phi\left(x\right)\right| \leq c_2\phi\left(c_1 x\right), \quad \textit{for every } x \in \mathbf{R}.$$

Proof. Without loss of generality, take $p\left(x\right) = x^k$ for any $k \in \mathbf{N}$

$$\frac{\left|p\left(x\right)\phi\left(x\right)\right|}{\phi\left(c_1 x\right)} = \left|x\right|^k \exp\left\{-\frac{\left(1-c_1\right)x^2}{2}\right\} \to 0,$$

as $\left|x\right| \to \infty$. □

Proposition A.0.6. *For every $x, y > 0$ and $\varepsilon > 0$ we have*

$$\exp\left\{-\frac{\left(x-y\right)^2}{2\varepsilon}\right\} - \exp\left\{-\frac{\left(x+y\right)^2}{2\varepsilon}\right\} \leq 2\varepsilon^{-1}xy\exp\left\{-\frac{\left(x-y\right)^2}{2\varepsilon}\right\}.$$

Proof. Factorise $\exp\left\{-\left(x-y\right)^2/2\varepsilon\right\}$ from the left-hand side and then apply the inequality $1 - e^{-z} \leq z$. □

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