The efficiency of fair division*

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Abstract

In this paper we study the impact of fairness on the efficiency of allocations. We consider three different notions of fairness, namely proportionality, envy-freeness, and equitability for allocations of divisible and indivisible goods and chores. We present a series of results on the price of fairness under the three different notions that quantify the efficiency loss in fair allocations compared to optimal ones. Most of our bounds are either exact or tight within constant factors. Our study is of an optimistic nature and aims to identify the potential of fairness in allocations.

1 Introduction

Fair division (or fair allocation) dates back to the ancient times and has found applications such as border settlement in international disputes, greenhouse gas emissions reduction, allocation of mineral riches in the ocean bed, inheritance, divorces, etc. In the era of the Internet, it appears regularly in distributed resource allocation and cost sharing in communication networks.

We consider allocation problems in which a set of *goods* or *chores* has to be allocated among several players. Fairness is an apparent desirable property in these situations and means that each player gets a *fair share*. Depending on what the term "fair share" means, different notions of fairness can be defined. An orthogonal issue is *efficiency* that refers to the total happiness of the players. An important notion that captures the minimum efficiency requirement from an allocation is that of *Pareto-efficiency*; an allocation is Pareto-efficient if there is no other allocation that is strictly better for at least one player and is at least as good for all the others.

Model and problem statement. We consider two different allocation scenarios, depending on whether the items to be allocated are goods or chores. In both cases, we distinguish between *divisible* and *indivisible* items.

The problem of allocating divisible goods is better known as *cake-cutting*. In instances of cake-cutting, the term *cake* is used as a synonym of the whole set of goods to be allocated. Each player has a *utility function* on each piece of the cake corresponding to the happiness of the player if she is allocated the particular piece; this function is non-negative and additive. We assume that the utility of each player for the whole cake is 1. Divisibility means that the cake can be cut in arbitrarily small pieces which can then be allocated to the players. In instances with indivisible goods, the utility function of a player is

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defined over sets of items; again, utilities are non-negative and additive and the utility of each player for the whole set of items is 1. Each item cannot be cut in pieces and has to be allocated as a whole to some player. Given an allocation, the utility of a player is simply the sum of her utilities over the (pieces of) items she receives. An allocation with n players is proportional if the utility of each player is at least $1/n$. It is envy-free if the utility of a player is not smaller than the utility she would have when exchanging the (pieces of) items she gets with the items of any other player. It is equitable if the utilities of all players are equal. An allocation is *optimal* if it maximizes the total utility of all players, i.e., each (piece of) item is allocated to the player that values it the most (ties are broken arbitrarily).

In instances with divisible chores, each player has a *disutility function* for each piece of the cake which denotes the regret of the player when she is allocated the particular piece. Again, the disutility functions are non-negative and additive and the disutility of a player for the whole cake is 1. The case of indivisible chores is defined accordingly; indivisibility implies that an item cannot be cut into pieces and has to be allocated as a whole to some player. Given an allocation, the disutility of a player is simply the sum of her disutilities over the (pieces of) items she receives. An allocation with n players is proportional if the disutility of each player is at most $1/n$. It is envy-free if the disutility of a player is not larger than the disutility she would have when exchanging the (pieces of) items she gets with the items of any other player. It is equitable if the disutilities of all players are equal. An allocation is optimal if it minimizes the total disutility of all players, i.e., each (piece of) item is allocated to the player that values it the least (ties are broken arbitrarily).

Note that envy-freeness implies proportionality. Furthermore, instances with divisible items always have proportional, envy-free, or equitable allocations. It is not hard to see that this is not always the case for instances with indivisible items. Furthermore, in any case, there are instances in which no optimal allocation is fair.

Models similar to ours have been considered in the literature; the focus has been on the design of protocols for achieving proportionality, envy-freeness, and equitability or on the design of approximation algorithms in settings where fulfilling the fairness objective exactly is impossible. However, the related literature seems to have neglected the issue of efficiency. Although several attempts have been made to characterize fair division protocols in terms of Pareto-efficiency [9], the corresponding results are almost always negative. Most of the existing protocols do not even provide Pareto-efficient solutions and this seems to be due to the limited amount of information they use for the utility functions of the players. Recall that in the case of divisible goods and chores, complete information about the utility or disutility functions of the players may not be compactly representable. Furthermore, Pareto-efficiency is rather unsatisfactory since it may imply that an allocation is very far from optimal.

Instead, in the current paper we are interested in quantifying the decrease of efficiency due to fairness (*price of fairness*). Our study has an optimistic nature and aims to identify the efficiency loss in the most efficient fair allocation. We believe that such a study is well-motivated since the knowledge of tight bounds on the price of fairness may detect whether a fair allocation can be improved. In many settings, complete information about the utility functions of the players is available (e.g., in a divorce) and computing an efficient and fair allocation may not be infeasible. Fair allocations can be thought of as counterparts of equilibria in strategic games; hence, our work is similar in spirit to the line of research that studies the price of stability in games [1].

In order to capture the price of fairness, we define the price of proportionality, envy-freeness, and equitability. Given an instance I for the allocation of goods, its price of proportionality (resp., envy-freeness, resp., equitability) is defined as the ratio of the total utility of the players in the optimal allocation for I over the total utility of the players in the best proportional (resp., envy-free, resp., equitable) allocation for I. Similarly, if I is an instance for the allocation of chores, its price of proportionality (resp., envy-freeness, resp., equitability) is defined as the ratio of the total disutility of the players in the best proportional (resp., envy-free, resp., equitable) allocation for I over the total disutility of the players in the optimal allocation for I. The price of proportionality (resp., envy-freeness, resp., equitability) of a class $\mathcal{I}$ of instances is then the maximum price of proportionality (resp., envy-freeness, resp., equitability) over all instances of $\mathcal{I}$. The classes of instances considered in this paper are defined by the number of players, the type of items (whether they are goods or chores), and their divisibility properties (whether they are divisible or indivisible). We remark that, in order for the price of proportionality, envy-freeness, and equitability to be well-defined, in the case of indivisible items, we assume that the class of instances contains only those ones for which proportional, envy-free, and equitable allocations, respectively, do exist.

Related work. Research on fair division originated in the 1940s with a focus on protocols for achieving fairness objectives in cake-cutting [27] (i.e., for divisible goods). Since then, the problem of achieving a proportional allocation with the minimum number of operations has received much attention and is now well-understood [16, 17, 9, 25, 31]. The problem of achieving envy-freeness has been proven to be much more challenging [23, 8, 28]; in fact, under the most common computational model of cut and evaluation queries [25], no algorithm with bounded running time is known for more than 3 players. Very recently, envy-freeness was proved to be a harder property to achieve than proportionality [24, 29, 15]. Better solutions exist for different computational models (e.g., moving knife algorithms [10, 26]). Equitability seems to be a harder goal; the objective that all players must get the same utility is computationally costly to achieve.

Optimization problems with objectives related to fairness have been studied in the recent literature. Lipton et al. [22] studied envy minimization with indivisible goods (where envy-freeness may not be guaranteed). Among other results, they showed how to compute allocations with bounded envy in polynomial time. They also present algorithms that compute allocations that approximate the minimum *envy-ratio*; the envy ratio of a player p for a player q is the utility of player p for the items allocated to player q over p's utility for the items allocated to her. Complexity considerations about envy-freeness for indivisible goods and more general non-additive utilities are presented in [7]. The papers [13, 14] study the problem of achieving envy-free and efficient allocations in distributed settings and when the allocation of items is accompanied by monetary side payments (in this case, envy-freeness is always a feasible goal [30]).

Another fairness objective that has been extensively considered recently for indivisible goods is *max-min fairness*. Here, the objective is to compute an allocation in which the utility of the least happy player is maximized. The problem was studied by Bezáková and Dani [6] and Golovin [18] who obtained approximation algorithms that provably return a solution that is always a factor of $O(n)$ within the optimal value. The problem was popularized by Bansal and Sviridenko [4] as the *Santa Claus problem*, where Santa Claus aims to distribute presents to the kids so as to maximize the happiness of the least happy kid. Subsequently, Asadpour and Saberi [2] presented an $O(\sqrt{n}\log^3 n)$-approximation algorithm for this problem. See [5] and [12] for some very recent related results.

Fair division with chores is discussed in [9]. A related optimization problem is scheduling on unrelated machines [21] where the objective is to compute an allocation that minimizes the disutility of the most unhappy player. Different notions of fairness have also been studied for other scheduling and resource allocation problems [11, 20, 19].

Overview of results. In this paper we provide upper and lower bounds on the price of proportionality, envy-freeness, and equitability in fair division with divisible and indivisible goods and chores. Our work reveals an almost complete picture. In all subcases except the price of envy-freeness with divisible goods and chores, our bounds are either exact or tight within a small constant factor.

Table 1 summarizes our results for fair division of goods. For divisible goods, the price of proportionality is very close to 1 (i.e., $8-4\sqrt{3}\approx 1.072$) for two players and $\Theta(\sqrt{n})$ in general. The upper bound for two players follows by a detailed analysis that takes into account the properties of the best proportional and the optimal allocation; this proof structure is adapted in order to prove the upper bounds on the price of equitability with divisible goods and the price of proportionality with divisible chores. Instead, the upper bound for the general case of n players is constructive; its proof follows by defining a proportional allocation starting from the optimal one. The price of equitability is slightly worse for two players (i.e., $9/8$) and $\Theta(n)$ in general. Our lower bound for the price of proportionality implies the same lower bound for the price of envy-freeness; while a very simple upper bound of $n-1/2$ completes the picture for divisible goods. For indivisible goods, we present an exact bound of $n-1+1/n$ on the price of proportionality while we show that the price of envy-freeness is $\Theta(n)$ in this case. Although our upper bounds follow by very simple arguments, the lower bounds use quite involved constructions. The price of equitability is proved to be finite only for the case of two players.

	Divisible goods			Indivisible goods		
Price of	LB	UB	$n=2$	LB	UB	$n=2$
Proportionality	$\Omega(\sqrt{n})$	$O(\sqrt{n})$	$8-4\sqrt{3}$	$n-1+1/n$	$n-1+1/n$	$3/2$
Envy-freeness	$\Omega(\sqrt{n})$	$n-1/2$		$\frac{3n+7}{9}-O(1/n)$	$n-1/2$	
Equitability	$\frac{(n+1)^2}{4n}$	n	$9/8$	∞	∞	2

Table 1: Summary of our results (lower and upper bounds) for fair division of goods.

Table 2 summarizes our results for fair division of chores. For divisible chores, the price of proportionality is $9/8$ for two players and $\Theta(n)$ in general while the price of equitability is exactly n. For indivisible chores, we present an exact bound of n on the price of proportionality while both the price of envy-freeness and the price of equitability are infinite. These last results imply that in the case of indivisible chores, envy-freeness and equitability are usually incompatible with efficiency.

	Divisible chores			Indivisible chores		
Price of	LB	UB	$n=2$	LB	UB	$n=2$
Proportionality	$\frac{(n+1)^2}{4n}$	n	$9/8$	n	n	2
Envy-freeness	$\frac{(n+1)^2}{4n}$	∞		∞	∞	
Equitability	n	n	2	∞	∞	∞

Table 2: Summary of our results (lower and upper bounds) for fair division of chores.

The rest of our paper is structured as follows. We present our results for divisible goods in Section 2 and for indivisible goods in Section 3, while the case of chores is considered in Section 4. Finally, we conclude in Section 5.

2 Fair division with divisible goods

In this section, we focus on divisible goods. As a warm-up, we begin with simple upper bounds for the price of envy-freeness and equitability.

Lemma 1 *For n players and divisible goods, the price of envy-freeness is at most $n-1/2$ and the price of equitability is at most n.*

Proof. Consider an instance and a corresponding optimal allocation. If this allocation is envy-free or equitable, then the price of envy-freeness or equitability, respectively, is 1. In the following, we assume that this is not the case.

An envy-free allocation is also proportional; so the total utility of the players in any envy-free allocation is at least 1. Since the optimal allocation is not envy-free, at least one player is envious, and has utility over the pieces of the cake she receives less than $1/2$. So, the total utility in the optimal allocation is at most $n - 1/2$.

Now consider the allocation in which each negligibly small piece of the cake is shared equally among the n players. This is an equitable allocation of total utility equal to 1 while the optimal allocation has total utility at most n. ■

We continue with a tight (up to constant factors) result for the price of proportionality.

Theorem 2 *For n players and divisible goods, the price of proportionality is $\Theta(\sqrt{n})$.*

Proof. Consider an instance with n players and let $\mathcal{O}$ denote the optimal allocation and OPT be the total utility of $\mathcal{O}$. We partition the set of players into two sets, namely L and S, so that if a player obtains utility at least $1/\sqrt{n}$ in $\mathcal{O}$, then she belongs to L, otherwise she belongs to S. Clearly, $OPT \leq |L| + |S|/\sqrt{n}$. We now describe how to obtain a proportional allocation $\mathcal{A}$; we distinguish between two cases depending on $|L|$.

We first consider the case $|L| \geq \sqrt{n}$; hence, $|S| \leq n - \sqrt{n}$. Then, for any negligibly small item that is allocated to a player $i \in L$ in $\mathcal{O}$, we allocate to i a fraction of $\sqrt{n}/n$ of the item, while we allocate to each player $i \in S$ a fraction of $\frac{n-\sqrt{n}}{n|S|} \geq 1/n$. Furthermore, for any negligibly small item that is allocated to a player $i \in S$ in $\mathcal{O}$, we allocate to each player $i \in S$ a fraction of $1/|S| > 1/n$. In this way, all players obtain a utility of at least $1/n$, while all items are fully allocated; hence, $\mathcal{A}$ is proportional. For every player $i \in L$, her utility in $\mathcal{A}$ is exactly $1/\sqrt{n}$ times her utility in $\mathcal{O}$, while every player $i \in S$ obtained a utility strictly less than $1/\sqrt{n}$ in $\mathcal{O}$ and obtains utility at least $1/n$ in $\mathcal{A}$. So, we conclude that the total utility in $\mathcal{A}$ is at least $1/\sqrt{n}$ times the optimal total utility.

Otherwise, let $|L| < \sqrt{n}$. Since $OPT \leq |L| + |S|/\sqrt{n}$, we obtain that $OPT < 2\sqrt{n} - 1$, while the total utility of any proportional allocation is at least 1. Hence, in both cases we obtain that the price of proportionality is $O(\sqrt{n})$. We continue by presenting a lower bound of $\Omega(\sqrt{n})$.

Consider the following instance with n players and $m < n$ items. Player i, for $i = 1, ..., m$, has utility 1 for item i and 0 for any other item, while player i, for $i = m + 1, ..., n$, has utility $1/m$ for any item. In the optimal allocation, item i, for $i = 1, ..., m$, is allocated to player i, and the total utility is m. Consider any proportional allocation and let x be the sum of the fractions of the items that are allocated to the last $n-m$ players. The total utility of these players is x/m. Clearly, $x \geq m(n-m)/n$, otherwise some of them would obtain a utility less than $1/n$ and the allocation would not be proportional. The first m players are allocated the remaining fraction of $m - x$ of the items and their total utility is at most $m - x$. The total utility of all players is $m - x + x/m \leq \frac{m^2+n-m}{n}$. We conclude that the price of proportionality is at least $\frac{mn}{m^2+n-m}$ which becomes more than $\sqrt{n}/2$ by setting $n = m^2$. ■

Since every envy-free allocation is also proportional, the lower bound on the price of proportionality also holds for envy-freeness. Interestingly, in the case of two players, there always exist almost optimal proportional allocations. Recall that in this case proportionality and envy-freeness are equivalent.

Theorem 3 *For two players and divisible goods, the price of proportionality (or envy-freeness) is exactly $8 - 4\sqrt{3} \approx 1.072$.*

Proof. Consider an optimal allocation $\mathcal{O}$ and a proportional allocation $\mathcal{E}$ that maximizes the total utility of the players. We partition the cake into four parts A, B, C, and D as follows:

- A is the part of the cake which is allocated to player 1 in both $\mathcal{O}$ and $\mathcal{E}$,
- B is the part of the cake which is allocated to player 2 in both $\mathcal{O}$ and $\mathcal{E}$,
- C is the part of the cake which is allocated to player 1 in $\mathcal{O}$ and to player 2 in $\mathcal{E}$, and

- D is the part of the cake which is allocated to player 1 in $\mathcal{E}$ and to player 2 in $\mathcal{O}$.

In the following, we use the notation $u_i(X)$ to denote the utility of player i for part X of the cake. Since $\mathcal{O}$ maximizes the total utility, we have $u_1(A) \geq u_2(A)$, $u_1(B) \leq u_2(B)$, $u_1(C) \geq u_2(C)$, and $u_1(D) \leq u_2(D)$. First observe that if $u_1(C) = u_2(C)$ and $u_1(D) = u_2(D)$, then $\mathcal{E}$ has the same total utility with $\mathcal{O}$. So, in the following we assume that this is not the case.

We consider the case $u_1(C) > u_2(C)$; the other case is symmetric. In this case, we also have that $u_1(D) = u_2(D) = 0$. Assume otherwise that $u_2(D) > 0$. Then, there must be a subpart X of C for which player 1 has utility x and player 2 has utility at most $x \cdot u_2(C)/u_1(C)$ and a subpart Y of D for which player 2 has utility x; note that since D is allocated to player 2 in $\mathcal{O}$, player 1 has utility at most x for Y.

Then, the allocation in which player 1 gets parts A, X, and $D - Y$ and player 2 gets parts B, $C - X$, and Y is proportional and has larger utility than $\mathcal{E}$.

Now, we claim that $u_2(A) = 1/2$. Clearly, since $\mathcal{E}$ is proportional, the utility of player 2 in $\mathcal{E}$ is at least $1/2$, i.e., $u_2(B) + u_2(C) \geq 1/2$. Since the utilities of player 2 sum up to 1 over the whole cake, we also have that $u_2(A) \leq 1/2$. If it were $u_2(A) < 1/2$, then we would have $u_2(B) + u_2(C) > 1/2$. Then, there would exist a subpart X of C for which player 2 has utility x for some $x \leq 1/2 - u_2(A)$ and player 1 has utility strictly larger than x. By allocating X to player 1 instead of player 2, we would obtain another proportional allocation with larger total utility.

Also, it holds that $u_2(A)/u_1(A) \leq u_2(C)/u_1(C)$. Otherwise, there would exist a subpart X of C for which player 1 has utility x and player 2 has utility $u_2(X)$ at most $x \cdot u_2(C)/u_1(C)$ and a subpart Y of A for which player 1 has utility x and player 2 has utility $u_2(Y)$ at least $x \cdot u_2(A)/u_1(A) > x \cdot u_2(C)/u_1(C) \geq u_2(X)$. By allocating the subpart X to player 1 and subpart Y to player 2, we would obtain another proportional allocation with larger total utility.

By the discussion above, we have $u_2(C) \geq \frac{u_1(C)}{2u_1(A)}$. We are now ready to bound the ratio of the total utility of $\mathcal{O}$ over the total utility of $\mathcal{E}$ which will give us the desired bound. We obtain that the price of proportionality is

$$\begin{aligned}
\frac{u_1(A)+u_2(B)+u_1(C)}{u_1(A)+u_2(B)+u_2(C)} &= \frac{u_1(A)+1/2-u_2(C)+u_1(C)}{u_1(A)+1/2} \\
&\leq \frac{u_1(A)+1/2-\frac{u_1(C)}{2u_1(A)}+u_1(C)}{u_1(A)+1/2} \\
&= \frac{u_1(A)+1/2+u_1(C)\left(1-\frac{1}{2u_1(A)}\right)}{u_1(A)+1/2} \\
&\leq \frac{u_1(A)+1/2+(1-u_1(A))\left(1-\frac{1}{2u_1(A)}\right)}{u_1(A)+1/2}
\end{aligned}$$

where the last inequality follows since $u_1(A) \geq u_2(A) = 1/2$ and $u_1(C) \leq 1 - u_1(A)$. The last expression is maximized to $8 - 4\sqrt{3}$ for $u_1(A) = \frac{1+\sqrt{3}}{4}$ and the upper bound follows.

In order to prove the lower bound, it suffices to consider a cake consisting of two parts A and B. Player 1 has utility $u_1(A) = 1$ and $u_1(B) = 0$ and player 2 has utility $u_2(A) = \sqrt{3} - 1$ and $u_2(B) = 2 - \sqrt{3}$. The utilities of each player within the parts are uniform. ■

We now study the price of equitability and show that when the number of players is large, equitability may provably lead to less efficient allocations. The next lower bound matches the upper bound of Lemma 1 concerning equitability within a constant factor.

Theorem 4 *For n players and divisible goods, the price of equitability is at least $\frac{(n+1)^2}{4n}$.*

Proof. We distinguish between the cases of odd and even n.

In the first case, there are $(n+1)/2$ items. Player i for $i = 1, ..., (n+1)/2$, has utility 1 for item i and utility 0 for any other item. Player i, for $i = (n+3)/2, ..., n$, has utility $2/(n+1)$ for any item. In the optimal allocation, each player i, for $i = 1, ..., (n+1)/2$, gets item i and the total utility is $(n+1)/2$. We complete the proof of this case by showing that no equitable allocation in which each player has a utility $\chi > 2/(n+1)$ exists. Assume otherwise; then, the total utility of the last $(n-1)/2$ players is $(n-1)\chi/2 > (n-1)(n+1)$. Thus, there exists at least one item g such that a fraction of at least $(n-1)\chi/2$ of g is allocated to the last $(n-1)/2$ players, and, therefore, player g cannot obtain a utility larger than $1-(n-1)\chi/2 < 2/(n+1)$; this contradicts the equitability assumption. So, the total utility of any equitable allocation is at most $2n/(n+1)$ and the proof of this case is complete.

In the case of even n, there are n items and player i, for $i = 1, ..., n/2$, has utility 1 for item i and utility 0 for any other item. Player i, for $i = n/2+1, ..., n$, has utility $1/n$ for any item. In the optimal allocation, player i, for $i = 1, ..., n$, obtains item i and the total utility is $(n+1)/2$. Again, we show that no equitable allocation in which each player has utility $\chi > 2/(n+1)$ exists. Assume otherwise; then the total utility of the first $n/2$ players is $n\chi/2$ and, hence, a total fraction of $n\chi/2$ of the first $n/2$ items has been allocated to them. This leaves a total fraction of $n(2-\chi)/2$ items to be allocated to the last $n/2$ players. Clearly, in any such allocation there is a player g with $n/2+1 \leq g \leq n$ with utility at most $(2-\chi)/n < \chi$. This again contradicts the equitability assumption. So, the total utility of any equitable allocation is again at most $2n/(n+1)$ and the proof of this case is complete. ■

Our last result of this section concerns the simplest case with $n = 2$, for which we present a matching upper bound on the price of equitability. The proof is along similar lines with the proof of Theorem 3.

Theorem 5 *For two players and divisible goods, the price of equitability is* $9/8$*.*

Proof. Consider an optimal allocation $\mathcal{O}$ and an equitable allocation $\mathcal{E}$ that maximizes the total utility of the players. We partition the cake into four parts A, B, C, and D as follows:

- A is the part of the cake which is allocated to player 1 in both $\mathcal{O}$ and $\mathcal{E}$,
- B is the part of the cake which is allocated to player 2 in both $\mathcal{O}$ and $\mathcal{E}$,
- C is the part of the cake which is allocated to player 1 in $\mathcal{O}$ and to player 2 in $\mathcal{E}$, and
- D is the part of the cake which is allocated to player 1 in $\mathcal{E}$ and to player 2 in $\mathcal{O}$.

In the following, we use the notation $u_i(X)$ to denote the utility of player i for part X of the cake. Since $\mathcal{O}$ maximizes the total utility, we have $u_1(A) \geq u_2(A)$, $u_1(B) \leq u_2(B)$, $u_1(C) \geq u_2(C)$, and $u_1(D) \leq u_2(D)$. First observe that if $u_1(C) = u_2(C)$ and $u_1(D) = u_2(D)$, then $\mathcal{E}$ has the same total utility with $\mathcal{O}$. So, in the following we assume that this is not the case.

We consider the case $u_1(C) > u_2(C)$; the other case is symmetric. In this case, we also have that $u_1(D) = u_2(D) = 0$. Assume otherwise that $u_2(D) > 0$. Then, there is a subpart X of C and a subpart Y of D such that $u_1(X) > u_2(X)$ and $0 < u_1(X) + u_2(X) = u_1(Y) + u_2(Y) \leq \min\{u_1(C)+u_2(C), u_1(D)+u_2(D)\}$. Also, $u_2(Y) \geq u_1(Y)$ since Y is allocated to player 2 in $\mathcal{O}$. Equivalently, we have that $u_1(X) - u_1(Y) = u_2(Y) - u_2(X)$. We also claim that $u_1(X) - u_1(Y) > 0$. Assume otherwise; then, we would also have $u_2(Y) \geq u_1(Y) \geq u_1(X) > u_2(X)$ which implies that $u_2(Y) - u_2(X) > 0$ and contradicts the above equality. Hence, $u_1(X) - u_1(Y) = u_2(Y) - u_2(X) > 0$ and the allocation in which player 1 gets part X instead of Y and player 2 gets part Y instead of X is also equitable and has larger total utility than $\mathcal{E}$.

Since $\mathcal{E}$ is equitable, we have $u_1(A) = u_2(B) + u_2(C)$. Since the utilities of player 2 sum up to 1, this implies that $u_2(A) = 1 - u_1(A)$. Since $u_1(A) \geq u_2(A)$, we also have that $u_1(A) \geq 1/2$.

Also, it holds that $u_2(A)/u_1(A) \leq u_2(C)/u_1(C)$. Otherwise, since $u_1(A), u_1(C) > 0$, there would exist a subpart X of C and a subpart Y of A such that $u_1(X), u_1(Y) > 0$, $\frac{u_2(Y)}{u_1(Y)} > \frac{u_2(X)}{u_1(X)}$ which implies that $u_2(Y) > 0$ and $\frac{u_1(X)}{u_1(Y)} > \frac{u_2(X)}{u_2(Y)}$, and $0 < u_1(X) + u_2(X) = u_1(Y) + u_2(Y) \leq \min\{u_1(A) + u_2(A), u_1(C) + u_2(C)\}$. The equality in the last expression is equivalent to $u_1(X) - u_1(Y) = u_2(Y) - u_2(X)$. We also claim that $u_1(X) - u_1(Y) > 0$. Assume otherwise; then we would also have $1 \geq \frac{u_1(X)}{u_1(Y)} > \frac{u_2(X)}{u_2(Y)}$ which implies that $u_2(Y) - u_2(X) > 0$ and contradicts the above equality. Hence, $u_1(X) - u_1(Y) = u_2(Y) - u_2(X) > 0$ and the allocation in which player 1 gets part X instead of Y and player 2 gets part Y instead of X is also equitable and has larger total utility than $\mathcal{E}$.

By the discussion above, we have $u_2(B) = u_1(A) - u_2(C)$ and $u_2(C) \geq u_1(C)\left(\frac{1}{u_1(A)} - 1\right)$. We are now ready to bound the ratio of the total utility of $\mathcal{O}$ over the total utility of $\mathcal{E}$ which will give us the desired bound. We obtain that the price of equitability is

$$
\begin{aligned}
\frac{u_1(A) + u_2(B) + u_1(C)}{u_1(A) + u_2(B) + u_2(C)} &= \frac{2u_1(A) + u_1(C) - u_2(C)}{2u_1(A)} \\
&\leq \frac{2u_1(A) + u_1(C) - u_1(C)\left(\frac{1}{u_1(A)} - 1\right)}{2u_1(A)} \\
&= \frac{2u_1(A) + u_1(C)\left(2 - \frac{1}{u_1(A)}\right)}{2u_1(A)} \\
&\leq \frac{2u_1(A) + (1 - u_1(A))\left(2 - \frac{1}{u_1(A)}\right)}{2u_1(A)} \\
&= \frac{3u_1(A) - 1}{2u_1(A)^2}.
\end{aligned}
$$

The last inequality follows since $u_1(A) \geq 1/2$ and $u_1(C) \leq 1 - u_1(A)$. The last expression is maximized to $9/8$ for $u_1(A) = 2/3$ and the theorem follows. ■

3 Fair division with indivisible goods

We now turn our attention to indivisible goods. Again, we begin this section by simple upper-bounds for the price of proportionality and envy-freeness.

Lemma 6 *For n players and divisible goods, the price of proportionality is at most $n - 1 + 1/n$ and the price of envy-freeness is at most $n - 1/2$.*

Proof. Consider an instance and a corresponding optimal allocation. If this allocation is proportional or envy-free, then the price of proportionality or envy-freeness, respectively, is 1. In the following, we assume that this is not the case.

In any proportional allocation, each player has utility at least $1/n$ on the items she receives and the total utility is at least 1. Since the optimal allocation is not proportional, some player has utility less than $1/n$ and the total utility in the optimal allocation is at most $n - 1 + 1/n$.

An envy-free allocation is also proportional; so the total utility of the players in any envy-free allocation is at least 1. Since the optimal allocation is not envy-free, at least one player is envious, and has utility over the items she receives less than $1/2$. So, the total utility in the optimal allocation is at most $n - 1/2$. ■

In the following, we present lower bounds which are either exact or tight within a constant factor.

Theorem 7 *For n players and indivisible goods, the price of proportionality is at least $n - 1 + 1/n$.*

Proof. Consider the following instance with n players and $2n - 1$ items. Let $0 < \epsilon < 1/n$. For $i = 1, ..., n - 1$, player i has utility ϵ for item i, utility $1 - 1/n$ for item $i + 1$, utility $1/n - \epsilon$ for item $n + i$ and utility 0 for all other items. The last player has utility $1/n - \epsilon$ for items $1, 2, ..., n - 1$, utility $1/n + (n - 1)\epsilon$ for item n, and utility 0 for all other items.

We argue that the only proportional allocation assigns items i and $n+i$ to player i for $i = 1, ..., n-1$, and item n to player n. To see that, notice that each player must be allocated at least one of the first n items, regardless of what other items she obtains, in order to be proportional. Since there are n players, each of them must be allocated exactly one of the first n items. Now, consider player n. It is obvious that she must be allocated item n, since she has utility strictly less than $1/n$ for any other item. The only available items (with positive utility) left for player $n - 1$ are items $n - 1$ and $2n - 1$, and it is easy to see that both of them must be allocated to her. Using the same reasoning for players $n - 2, n - 3, ..., 1$, we conclude that the only proportional allocation is the aforementioned one, which has total utility $1 + (n - 1)\epsilon$.

Now, the total utility of the optimal allocation is lower-bounded by the total utility of the allocation where player i gets items $i + 1$ and $n + i$, for $i = 1, ..., n - 1$, and player n gets the first item. The total utility obtained by this allocation is $(1 - 1/n + 1/n - \epsilon)(n - 1) + \frac{1}{n} - \epsilon = n - 1 + 1/n - n\epsilon$. By selecting ϵ to be arbitrarily small, the theorem follows. ■

We remark that in the construction in the proof of Theorem 7 we use instances with no envy-free allocation that cannot be used to prove bounds on the price of envy-freeness. The lower bound construction in the proof of Theorem 2 can be extended in order to yield a lower bound of $\Omega(\sqrt{n})$ for indivisible items as well. In the following we prove an even stronger and tight lower bound of $\Omega(n)$.

Theorem 8 *For n players and indivisible goods, the price of envy-freeness is at least $\frac{3n+7}{9} - O(1/n)$.*

Proof. We construct the following instance with $n \geq 5$ players. Let integers $\ell \geq 2$ and $k > \ell + 1$ be such that $n = k + \ell - 1$; we note that no such integers exist for $n < 5$. Furthermore, let the number of items m be such that $m = \ell(k + 1)$.

We denote by U the $m \times n$ matrix of utilities, where the entry in the i-th column and j-th row denotes the utility of player i for item j. Let U_1 be the $\ell k \times (k - 1)$ upperleft submatrix, U_2 be the $\ell k \times \ell$ upperright submatrix, U_3 be the $\ell \times (k - 1)$ lowerleft submatrix and U_4 be the $\ell \times \ell$ lowerright submatrix.

The utilities are defined as follows. Each entry in U_1 has value $\frac{1}{\ell(k+\ell-1)} + \epsilon$, while each entry in U_3 has value $\frac{\ell-1}{\ell(k+\ell-1)} - k\epsilon$, for some sufficiently small $\epsilon > 0$ (e.g., $\epsilon \leq 1/n^3$). Clearly, it holds that the sum of utilities for any of the first $k - 1$ players over the items is exactly 1. As far as submatrix U_2 is concerned, player i, for $i = k, \ldots, k + \ell - 1$, has utility $1/k - \epsilon$ for items $(i - k)k + 1, \ldots, (i - k + 1)k$ and 0 otherwise. Finally, each entry in U_4 has value $\frac{k\epsilon}{\ell}$. Clearly, it holds that the sum of utilities for any of the last ℓ players over the items is exactly 1.

Consider the allocation where each player $i \in \{k, \ldots, k + \ell - 1\}$ gets each item among the first $k\ell$ ones with strictly positive utility in U_2, while the last ℓ items are allocated (arbitrarily) to the first $k - 1$ players. This is an optimal allocation with total utility

$$OPT = \ell - \ell k\epsilon + \frac{\ell(\ell - 1)}{\ell(k + \ell - 1)} - \ell k\epsilon = \ell + \frac{\ell - 1}{k + \ell - 1} - 2\ell k\epsilon.$$

We now consider the envy-free allocation and we argue about some important properties concerning its structure. First, due to the utilities in the $m \times (k - 1)$ left submatrix of U, it is not hard to see that, for any players $i, j \in \{1, \ldots, k - 1\}$, player i gets the same number of items as player j, otherwise i

would envy j or vice versa. This means that none of these $k-1$ players gets any of the last ℓ items, since $k-1>\ell$, i.e., these items are not enough so that all those $k-1$ players receive the same number of these items.

Therefore, the last ℓ items are allocated to the last ℓ players. Moreover, assume that one of the last ℓ players (let i^* be this player) does not obtain any of the first $k\ell$ items. Then, since i^*'s utility is at most $\ell\frac{k\epsilon}{\ell}=k\epsilon$, she would be envious of any player that obtains one of the first $k\ell$ items that has positive value for i^*, since this value is $1/k-\epsilon$. So, we conclude that each of the last ℓ players must obtain at least one of the first $k\ell$ items, so in total the last ℓ players receive at least 2ℓ items. This leaves at most $\ell\left(k+1\right)-2\ell=\left(k-1\right)\ell$ items for the first $k-1$ players.

We now argue that each of the $k-1$ first players must obtain at least ℓ items; thus, each of them must obtain exactly ℓ items, since the number of items should be equally divided among them. Assume otherwise and let i be a player that receives at most $\ell-1$ items. Then, i's utility is at most $\frac{\ell-1}{\ell(k+\ell-1)}+\left(\ell-1\right)\epsilon$, which, for sufficiently small $\epsilon>0$ is less than $1/n$; clearly, i is envious.

Therefore, we consider the following allocation. Each player $i\in\{1,\dots,k-1\}$ gets ℓ items from the first $k\ell$ ones, so that each of these ℓ items has strictly positive utility for exactly one of the last ℓ players, and no item from the last ℓ ones. Furthermore, each player $i'\in\{k,k+\ell-1\}$ gets exactly one item among the first $k\ell$ ones for which she has a strictly positive utility, while all the last ℓ items are allocated to the last ℓ players. According to the properties above, this is the only envy-free allocation (up to relabeling players and items) and the total utility is

$$\begin{aligned} EF &= \frac{\ell\left(k-1\right)}{\ell\left(k+\ell-1\right)}+\ell\left(k-1\right)\epsilon+\ell/k-\ell\epsilon+\frac{k\epsilon}{\ell}\ell \\ &= \ell/k+\frac{k-1}{k+\ell-1}+\left(\ell\left(k-2\right)+k\right)\epsilon. \end{aligned}$$

Therefore, we obtain that the price of envy-freeness is

$$\rho = \frac{OPT}{EF}=\frac{\ell+\frac{\ell-1}{k+\ell-1}-2\ell k\epsilon}{\ell/k+\frac{k-1}{k+\ell-1}+\left(\ell\left(k-2\right)+k\right)\epsilon}=\frac{\ell+\frac{\ell-1}{k+\ell-1}}{\ell/k+\frac{k-1}{k+\ell-1}}-\epsilon'$$

for some $\epsilon'>0$ that can become arbitrarily small by selecting ϵ to be arbitrarily small. When n is odd, we set $k=(n+3)/2$ and $\ell=(n-1)/2$ and the price of envy-freeness becomes

$$\rho=\frac{n^3+3n^2-3n-9}{3n^2+2n+3}-\epsilon'.$$

When n is even, we set $k=n/2+2$ and $\ell=n/2-1$ and the price of envy-freeness becomes

$$\rho=\frac{n^3+3n^2-4n}{3n^2+2n+8}-\epsilon'.$$

In both cases, we obtain that

$$\rho=\frac{3n+7}{9}-O(1/n).$$

Finally, we note that for small values of n (i.e., $n=3$ or $n=4$) for which no k,ℓ satisfying $\ell\geq 2$ and $k>\ell+1$ exist, we can ignore the constraint that $k>\ell+1$ and construct the instance with the values $(k,\ell)=(2,2)$ for $n=3$ and $(k,\ell)=(3,2)$ for $n=4$. The lower bounds obtained are $7/4$ and $27/14$, respectively. ■

Unfortunately, equitability may lead to arbitrarily inefficient allocations of indivisible goods when the number of players is at least 3.

Theorem 9 *For n players and indivisible goods, the price of equitability is 2 for $n = 2$ and infinite for $n > 2$.*

Proof. For $n > 2$, let ϵ be an arbitrarily small positive number and consider the following instance with n players and n items. For $i = 1, ..., n-1$, player i has utility ϵ for item i, utility $1-\epsilon$ for item $i+1$, and utility 0 for all other items. Player n has utility $1-2\epsilon$ for item 1, utility ϵ for items $n-1$ and n, and utility 0 for all other items. The total utility of the optimal allocation is $n-(n+1)\epsilon$, which is obtained by allocating item $i+1$ to player i, for $i = 1, ..., n-1$, and allocating item 1 to player n. Clearly, this is not an equitable allocation and, furthermore, the only equitable allocation assigns item i to player i, for $i = 1, ..., n$. The total utility of this allocation is $n\epsilon$ and the price of equitability is $\Omega(1/\epsilon)$; the statement for $n > 2$ follows.

For $n = 2$, the upper bound on the price of equitability holds since the optimal total utility is at most 2 and in an equitable allocation, each player obtains a utility of at least $1/2$ (otherwise, the two players would exchange bundles). The lower bound consists of four items a, b, c, and d. Player 1 has utilities $1/2$, ϵ', $1/2-\epsilon'$ and 0, respectively, while player 2 has utilities $2\epsilon'$, $1/2$, 0, and $1/2-2\epsilon'$, respectively. In the optimal allocation, player 1 obtains items a and c, while player 2 obtains items b and d for a total utility of $2-3\epsilon'$. Clearly, in any equitable allocation, each player obtains utility exactly $1/2$, and, hence, the statement for two players follows by selecting ϵ' to be arbitrarily small. ■

4 Fair division with chores

In this section, we study the allocation of divisible and indivisible chores. The next theorem states a tight bound on the price of proportionality for divisible chores.

Theorem 10 *For n players and divisible chores, the price of proportionality is at most n and at least $\frac{(n+1)^2}{4n}$.*

Proof. The upper bound is obtained in a similar manner as in Theorem 1. Consider the optimal allocation. Clearly, if the optimal allocation is proportional, then the price of proportionality is 1. So, we can assume that at least one of the players has disutility strictly larger than $1/n$ and this bounds the total disutility of the optimal allocation from below. By definition, in any proportional allocation, the disutility of each of the n players is at most $1/n$. Hence, the total disutility of any proportional allocation is at most 1, which means that the price of proportionality is at most n.

We now present the lower bound. Consider the following instance with n players and 2 items, where players $1, \ldots, n-1$ have disutility 1 for item 1 and disutility 0 for item 2. Player n has disutility $2/(n+1)$ for item 1, and disutility $(n-1)/(n+1)$ for item 2. In the optimal allocation, item 1 is allocated to player n, whereas item 2 is allocated to one of the first $n-1$ players. The total disutility of this allocation is $2/(n+1)$.

In the best proportional allocation, player n will be allocated a fraction of item 1, such that her disutility is exactly $1/n$, and the rest of the players will in some way share the rest of item 1 as well as item 2 in such a way that they are proportional. In more detail, player n obtains a fraction of $(n+1)/2n$ of the first item and the rest of the players share the rest $(n-1)/2n$ fraction of the first item, as well as the whole item 2. The total disutility of this allocation is $(n+1)/2n$. We conclude that the price of proportionality is at least $\frac{(n+1)^2}{4n}$. ■

Since every envy-free allocation is also proportional, the lower bound on the price of proportionality also holds for envy-freeness. We also have a matching upper bound for proportionality (or envy-freeness) in the case $n = 2$.

Theorem 11 *For two players and divisible chores, the price of proportionality (or envy-freeness) is at most* $9/8$*.*

Proof. Consider an optimal allocation $\mathcal{O}$ and a proportional allocation $\mathcal{E}$ that minimizes the total disutility of the players. We partition the cake into four parts A, B, C, and D as follows:

- A is the part of the cake which is allocated to player 1 in both $\mathcal{O}$ and $\mathcal{E}$,
- B is the part of the cake which is allocated to player 2 in both $\mathcal{O}$ and $\mathcal{E}$,
- C is the part of the cake which is allocated to player 1 in $\mathcal{O}$ and to player 2 in $\mathcal{E}$, and
- D is the part of the cake which is allocated to player 1 in $\mathcal{E}$ and to player 2 in $\mathcal{O}$.

In the following, we use the notation $u_i(X)$ to denote the disutility of player i for part X of the cake. Since $\mathcal{O}$ minimizes the total disutility, we have $u_1(A) \leq u_2(A)$, $u_1(B) \geq u_2(B)$, $u_1(C) \leq u_2(C)$, and $u_1(D) \geq u_2(D)$. First observe that if $u_1(C) = u_2(C)$ and $u_1(D) = u_2(D)$, then $\mathcal{E}$ has the same total disutility with $\mathcal{O}$. So, in the following we assume that this is not the case.

We consider the case $u_1(C) < u_2(C)$; the other case is symmetric. In this case, we also have that $u_1(D) = u_2(D) = 0$. Assume otherwise that $u_1(D) > 0$. Then, there must be a subpart X of C for which player 2 has disutility x and player 1 has disutility at most $x \cdot u_1(C)/u_2(C)$ and a subpart Y of D for which player 1 has disutility x and player 2 has disutility at most x. Then, the allocation in which player 1 gets parts A, X, and $D - Y$ and player 2 gets parts B, $C - X$, and Y is proportional and has smaller disutility than $\mathcal{E}$.

Now, we claim that $u_1(A) = 1/2$. Clearly, since $\mathcal{E}$ is proportional, the disutility of player 1 in $\mathcal{E}$ is at most $1/2$, i.e., $u_1(A) \leq 1/2$. If it were $u_1(A) < 1/2$, then, there would exist a subpart X of C for which player 1 has disutility x for some $x \leq 1/2 - u_1(A)$ and player 2 has disutility strictly larger than x. By allocating X to player 1 instead of player 2, we would obtain another proportional allocation with smaller total disutility.

Also, observe that $u_1(A)/u_2(A) \leq u_1(C)/u_2(C)$. Otherwise, there would exist a subpart X of C for which player 2 has disutility x and player 1 has disutility $u_1(X)$ at most $x \cdot u_1(C)/u_2(C)$ and a subpart Y of A for which player 2 has disutility x and player 1 has disutility $u_1(Y)$ at least $x \cdot u_1(A)/u_2(A) > x \cdot u_1(C)/u_2(C) \geq u_1(X)$. By allocating X to player 1 and Y to player 2, we would obtain another proportional allocation with smaller total disutility.

By the discussion above, we have $u_1(C) \geq \frac{u_2(C)}{2u_2(A)}$ and, clearly, $u_2(B) = 1 - u_2(A) - u_2(C)$. We are now ready to bound the ratio of the total disutility of $\mathcal{E}$ over the total disutility of $\mathcal{O}$ which will give us the desired bound. We obtain that the price of proportionality is

$$\begin{aligned}
\frac{u_1(A)+u_2(B)+u_2(C)}{u_1(A)+u_2(B)+u_1(C)} &= \frac{u_1(A)+1-u_2(A)-u_2(C)+u_2(C)}{u_1(A)+1-u_2(A)-u_2(C)+u_1(C)} \\
&= \frac{3/2-u_2(A)}{3/2-u_2(A)-u_2(C)+u_1(C)} \\
&\leq \frac{3/2-u_2(A)}{3/2-u_2(A)-u_2(C)+\frac{u_2(C)}{2u_2(A)}} \\
&= \frac{3/2-u_2(A)}{3/2-u_2(A)-u_2(C)\left(1-\frac{1}{2u_2(A)}\right)} \\
&\leq \frac{3/2-u_2(A)}{3/2-u_2(A)-(1-u_2(A)\left(1-\frac{1}{2u_2(A)}\right)} \\
&= 3u_2(A)-2u_2(A)^2
\end{aligned}$$

where the last inequality follows since $u_2(A) \geq u_1(A) = 1/2$ and $u_2(C) \leq 1 - u_2(A)$. The last expression is maximized to $9/8$ for $u_2(A) = 3/4$ and the theorem follows. ■

We now turn our attention to the price of equitability and prove a tight bound.

Theorem 12 *For n players and divisible chores, the price of equitability is n.*

Proof. We begin by proving the upper bound. By starting from an optimal allocation, we will show how to compute an equitable allocation with at most n times larger disutility. Consider a piece a in the optimal allocation and let $\{u_1(a), u_2(a), \ldots, u_n(a)\}$ be the vector denoting the disutility that each player obtains if she gets a. Without loss of generality, let $u_1(a) = \min_i u_i(a)$ and $u_n(a) = \max_i u_i(a)$, i.e., piece a was allocated to player 1. If $u_1(a) = 0$, then in the equitable allocation item a is allocated to player 1, and all players obtain a disutility of 0 from this item. Otherwise, if $u_1(a) > 0$, then each player i is allocated a fraction $\chi\frac{u_n(a)}{u_i(a)}$ of piece a, where χ is such that $\sum_i \chi\frac{u_n(a)}{u_i(a)} = 1$. Clearly, every player obtains a disutility of $\chi u_n(a)$ from a. Furthermore, it holds that $\chi u_n(a) \leq u_1(a)$, since player 1 obtains at most the whole piece a; hence, a contributes to the equitable allocation a disutility of at most n times the disutility it contributes to the optimal allocation. By applying similar reasoning for all pieces of the optimal allocation, we can obtain the desired equitable allocation.

We now proceed to present the lower bound. Consider the following instance with n players and 2 items. Player i, for $i = 1, \ldots, n-1$, has disutility 1 for item 1 and disutility 0 for item 2, while player n has disutility ϵ for item 1, and disutility $1-\epsilon$ for item 2, for an arbitrarily small ϵ. In the optimal allocation, item 1 is allocated to player n, whereas item 2 is allocated to one of the first $n-1$ players. The total disutility of this allocation is ϵ.

In the best equitable allocation, we suppose without loss of generality that the second item is shared among the first $n-1$ players. Also, each of the first $n-1$ players must be allocated the same fraction of item 1; let that fraction be χ. Then, player n will be allocated fraction $1-(n-1)\chi$ of item 1. Hence, $\chi = (1-(n-1)\,\chi)\,\epsilon$, which implies that $\chi = \frac{\epsilon}{1+(n-1)\epsilon}$, and the total disutility of the best equitable allocation is $\frac{n\epsilon}{1+(n-1)\epsilon}$. Since the optimal disutility is ϵ, we conclude that the price of equitability is at least $\frac{n}{1+(n-1)\epsilon}$. The lower bound follows by selecting ϵ to be arbitrarily small. ■

Finally, we consider the case of indivisible chores. Although the price of proportionality is bounded, the price of envy-freeness and equitability is infinite.

Theorem 13 *For n players and indivisible chores, the price of proportionality is n.*

Proof. The upper bound for the price of proportionality with divisible items can be easily extended to the indivisible case. We continue by presenting the lower bound.

Consider the following instance with n players and $2n-1$ items. For $i = 1, ..., n-1$, player i has disutility ϵ for items $1, \ldots, n-1$, disutility $1/n + (n-1)\epsilon$ for item n, and disutility $1/n - 2\epsilon$ for any other item. The last player has disutility $1/n - \epsilon$ for items $1, ..., n-1$, disutility $1/n$ for item n, and disutility ϵ for items $n+1, ..., 2n-1$.

In the optimal allocation, player i, for $i = 1, ..., n-1$, obtains item i, and player n obtains items $n, n+1, ..., 2n-1$. The total disutility of the optimal allocation is $1/n + (2n-2)\epsilon$. It is not hard to see that in any proportional allocation, player i, for $i = 1, \ldots, n-1$, obtains exactly one of the first $n-1$ and exactly one of the last $n-1$ items, while player n obtains item n. The total disutility of this allocation is $1-(n-1)\epsilon$ and the theorem follows by letting ϵ be arbitrarily small. ■

Theorem 14 *For n players and indivisible chores, the price of envy-freeness (for $n \geq 3$) and equitability (for $n \geq 2$) is infinite.*

Proof. We begin with the case of envy-freeness. Consider the following instance with n players and $2n$ items. Let $\epsilon < 1/(2n)$. For $i = 1, \ldots, n-2$, player i has disutility $1/n$ for the first n items and disutility 0 for every other item. Player $n-1$ has disutility 0 for the first $n-1$ items, disutility ϵ for item n, disutility $1/n$ for items $n+1, \ldots, 2n-1$ and disutility $1/n - \epsilon$ for item $2n$. Finally, player n has disutility 0 for the first $n-1$ items, disutility $1/(2n)$ for items n and $2n$, and disutility $1/n$ for items $n+1, \ldots, 2n-1$.

Clearly, the optimal allocation has total disutility ϵ and is obtained by allocating items $n+1, \ldots, 2n$ to players $1, \ldots, n-2$, item n to player $n-1$, and items $1, \ldots, n-1$ either to player $n-1$, or to player n. In each case, player $n-1$ envies player n. Furthermore, the allocation in which player i, for $i = 1, \ldots, n$ is allocated items i and $i+n$ is envy-free. The remark that concludes this proof is that there cannot exist an envy-free allocation having negligible disutility (i.e., less than $1/(2n)$).

Now, we prove the lower bound regarding the price of equitability. Consider the following instance with n players and $n+2$ items. Player 1 has disutility $1/2$ for item 1, disutility $1/2 - \epsilon'$ for item 3, disutility ϵ' for item 4 and disutility 0 for all other items. Player 2 has disutility $\epsilon'/4$ for item 1, disutility $1/2 - \epsilon'/4$ for item 2, disutility $3\epsilon'/4$ for item 3, disutility $1/2 - 3\epsilon'/4$ for item 4, and disutility 0 for all other items. For $i = 3, \ldots, n$, player i has disutility $1/2$ for item $i+2$ and disutility $\frac{1}{2(n+1)}$ for all other items.

Clearly, the optimal allocation has total disutility $2\epsilon'$ and is obtained by allocating items 2 and 4 to player 1, items 1 and 3 to player 2 and items $5, \ldots, n+2$ to any of the first two players. Since the last $n-2$ players have strictly positive disutility for any item, in an equitable allocation the first two players must be allocated some of the first four items so that they have strictly positive disutility. It is not hard to see that in the only equitable allocation, player 1 gets items 1 and 2, player 2 gets items 3 and 4 (or vice versa), and player i gets item $i+2$, for $i = 3, \ldots, n$. Thus, each player has a disutility of $1/2$ and the theorem follows by letting ϵ' be arbitrarily small. ■

5 Conclusions

We have studied the impact of fairness on the efficiency of allocations by considering divisible and indivisible items, both for the case of goods and chores. We have considered different measures of fairness, like proportionality, envy-freeness and equitability, and our results provide a rather complete picture of the decrease of the efficiency in all cases.

Our work has essentially left open the correct bound for the price of envy-freeness in the cases of divisible goods and chores. Although equitability seems to be worse than the other two fairness properties as far as efficiency is concerned, it is not clear whether proportionality is better than envy-freeness in the case of divisible goods and chores. It is tempting to conjecture that this is not the case but we have been unable to prove such a claim.

An interesting variation of the model for divisible items that we consider in this paper is to include the restriction that each player is allocated a contiguous part of the cake. Bounds on the price of fairness for this model were recently obtained in [3].

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