Spin(7)-manifolds and Calibrated Geometry

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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2012

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Abstract

In this thesis we study Spin(7)-manifolds, that is Riemannian 8-manifolds with torsion-free Spin(7)-structures, and Cayley submanifolds of such manifolds. We use a construction of compact Spin(7)-manifolds from Calabi–Yau 4-orbifolds with antiholomorphic involutions, due to Joyce, to find new examples of compact Spin(7)-manifolds. We search the class of well-formed quasismooth hypersurfaces in weighted projective spaces for suitable Calabi–Yau 4-orbifolds. We consider anti-holomorphic involutions induced by the restriction of an involution of the ambient weighted projective space and we classify anti-holomorphic involutions of weighted projective spaces.

We consider the moduli problem for Cayley submanifolds of Spin(7)-manifolds and show that there is a fine moduli space of unobstructed Cayley submanifolds. This result improves on the work of McLean in that we consider the global issues of how to patch together the local result of McLean. We also use the work of Kriegl and Michor on 'convenient manifolds' to show that this moduli space carries a universal family of Cayley submanifolds.

Using the analysis necessary for the study of the moduli problem of Cayleys we find examples of compact Cayley submanifolds in any compact Spin(7)-manifold arising, using Joyce's construction, from a suitable Calabi–Yau 4-orbifold with antiholomorphic involution. For the analysis to work, we need to show that a given Cayley submanifold is unobstructed. To show that particular examples of Cayley submanifolds are unobstructed, we relate the obstructions of complex surfaces in Calabi–Yau 4-folds as complex submanifolds to the obstructions as Cayley submanifolds.

Acknowledgements

I would like to thank my supervisor Prof. Dominic Joyce for his guidance throughout this project.

I was supported by a Foley–Béjar scholarship from Balliol College, Oxford, an EPSRC Research Studentship and an Easter Week Commemoration Scholarship from the Department of Education and Skills, Ireland, throughout my studies.

Contents

Chapter 1

Introduction

To any connected Riemannian manifold (M, g) we can associate the *holonomy group of* (M, g), which we will denote $\mathrm{Hol}(g)$. The group $\mathrm{Hol}(g)$ is the group of parallel transport maps around piecewise-smooth loops based at a point $p \in M$. Berger's classification result [9] gives a list of possible holonomy groups of simply-connected, irreducible, and nonsymmetric Riemannian manifolds. The classification gives five infinite families $\mathrm{SO}(n)$, $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, and $\mathrm{Sp}(m)\mathrm{Sp}(1)$, and two exceptional groups G_2 and $\mathrm{Spin}(7)$.

Calibrated geometry was introduced by Harvey and Lawson in their seminal paper [33]. A *calibration* on a Riemannian manifold (M, g) is a closed m-form ϕ such that $\phi|_V \leq \mathrm{vol}_V$ for all oriented m-planes $V \subset T_pM$ and all $p \in M$. We say an oriented submanifold $N \subset M$ is *calibrated by* ϕ, or simply *calibrated*, if $\phi|_N = \mathrm{vol}_N$. Calibrated submanifolds are necessarily minimal and volume-minimizing if compact.

Suppose (M, g) is a Riemannian manifold of dimension n and $\mathrm{Hol}(g)$, regarded as a subgroup of $\mathrm{O}(n)$, fixes an element $\phi \in \Lambda^m(\mathbb{R}^n)^*$. Then ϕ determines a m-form on M, which is parallel with respect to the Levi-Civita connection. Any form, which is parallel with respect to a torsion-free connection, is closed and by rescaling ϕ we can ensure that ϕ is a calibration. The groups appearing in the classification of Riemannian holonomy groups $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, $\mathrm{Sp}(m)\mathrm{Sp}(1)$, G_2, and $\mathrm{Spin}(7)$ all fix one or more forms. Therefore any Riemannian manifold with holonomy in the list above admits one or more calibrations. In fact, other than the group $\mathrm{U}(m)$, each of the groups in the list can be defined as the stabilizer group of a collection of forms.

In this way the subjects of manifolds with special holonomy and calibrated geometry are linked.

As hinted in the title, in this thesis we will be concerned primarily with Riemannian manifolds (M, g), whose holonomy is contained in Spin(7), and the calibrated submanifolds of such manifolds. The group Spin(7) can be defined as the stabilizer of a particular 4-form $\Omega_0 \in \Lambda^4(\mathbb{R}^8)^*$, defined by Harvey and Lawson. We define a Spin(7)-manifold (M, Ω, g) to include a choice of parallel 4-form Ω such that at each point $p \in M$ there exists an isometric identification of T_pM with $\mathbb{R}^8$, such that Ω_p is identified with Ω_0. Submanifolds calibrated by Ω are known as *Cayley submanifolds*.

Kähler manifolds have holonomy contained in $\mathrm{U}(m)$ and examples of compact Kähler manifolds are numerous. Any smooth hypersurface in complex projective space, with the metric induced by the Fubini–Study metric, is an example of a Kähler manifold. The Calabi conjecture allows us to reduce the existence of metrics with holonomy $\mathrm{SU}(m)$ to a problem in complex geometry. In particular any hypersurface of degree $n+1$ in $\mathbb{CP}^n$ admits a Ricci-flat Kähler metric. Note that the metric given by the Calabi conjecture is generally not that induced by the Fubini–Study metric.

Although Berger's classification result was proved in 1955, it was not until the 1980's until the existence of metrics with holonomy equal to G_2 and Spin(7) was proved by Bryant [13]. Bryant and Salamon [14] constructed examples of complete Riemannian manifolds with holonomy G_2 and Spin(7) and Joyce [37–39] constructed the first examples of compact Riemannian manifolds with holonomy G_2 and Spin(7). Taylor [90] and Kim and Jang [52] have used the results of Joyce [39] to find more examples of compact Riemannian manifolds with holonomy Spin(7).

The group SU(4) is a subgroup of Spin(7) and this means that any Calabi–Yau 4-fold can be given the structure of a Spin(7)-manifold. A complex surface in a Calabi–Yau 4-fold is a Cayley submanifold, considering the Calabi–Yau 4-fold as a Spin(7)-manifold. In this way we can find examples of compact Cayley submanifolds in Spin(7)-manifolds. The complete Riemannian manifold with holonomy Spin(7) constructed by Bryant and Salamon contains a Cayley submanifold diffeomorphic to

S^4, however it is more difficult to find examples of compact Cayley submanifolds of the compact manifolds with holonomy Spin(7) constructed by Joyce.

Joyce [41, Prop. 10.8.6] shows that the connected components of the fixed point locus of involutions preserving the Spin(7)-structure of a Spin(7)-manifold is either a Cayley submanifold or an isolated point. One can use this fact to find examples of compact Cayley submanifolds of compact Spin(7)-manifolds. We will find new examples by investigating the deformation theory of compact Cayley submanifolds.

McLean [77] studied the deformation theory of many examples of calibrated submanifolds and showed that certain classes of calibrated submanifolds were unobstructed and therefore an infinitesimal deformation integrates to a deformation. Due to McLean's work, one may ask whether calibrated submanifolds are unobstructed but, in general, the answer is no. For example complex submanifolds may be obstructed. Cayley submanifolds may also be obstructed.

Although we will focus on compact Cayley submanifolds we should note that there have been some studies of non-compact Cayley submanifolds of $\mathbb{R}^8$. Lotay [69] and Fox [26] studied non-compact Cayley submanifolds of $\mathbb{R}^8$, which are '2-ruled'. Lotay in particular gives explicit examples of such non-compact Cayley submanifolds. Fox relates 2-ruled Cayley submanifolds to pseudoholomorphic curves in $\widetilde{\mathrm{Gr}}(2,8)$, which carries a non-integrable complex structure related to the identification of $\widetilde{\mathrm{Gr}}(2,8)$ as a homogeneous space $\widetilde{\mathrm{Gr}}(2,8) \cong \mathrm{Spin}(7)/\mathrm{U}(3)$. Any 2-ruled Cayley submanifold is an example of an *asymptotically conical* Cayley submanifold. These are Cayley submanifolds of $\mathbb{R}^8$, which asymptotically approach a cone away from the origin in a specified way. We will not study the class of asymptotically conical Cayley submanifolds in this thesis although they will reappear in §7.4 in our discussion of potential future work.

1.1 Main results

In [40] Joyce gives a second construction of compact Riemannian manifolds with holonomy Spin(7). The ingredients for this construction are a Calabi–Yau 4-orbifold and an antiholomorphic involution, which acts in a suitable manner. We search the class of hypersurfaces in weighted projective spaces for suitable Calabi–Yau 4-orbifolds with antiholomorphic involution.

We restrict attention to 'well-formed' and 'quasismooth' hypersurfaces of weighted projective spaces because, in that case, we can easily test whether the canonical bundle is trivial. The construction of Joyce requires that the Calabi–Yau 4-orbifold has singularities modelled on $\mathbb{C}^4/\mathbb{Z}_4$ with $\mathbb{Z}_4$ acting by diagonal multiplication. We say that such a singularity is *of type* $\frac{1}{4}(1,1,1,1)$, and we will define other types of singularities later in the text. Furthermore, the construction requires that the fixed points of the antiholomorphic involution are the singularities of type $\frac{1}{4}(1,1,1,1)$. Any other singularities of the hypersurface must admit a crepant resolution, so that the singularities can be resolved without affecting the triviality of the canonical bundle.

The main classification result is the following:

Theorem 3.2.6. *The weights* $a_0, \ldots, a_5$ *such that*

(i) A generic hypersurface X_d *of degree* $d = \sum_i a_i$ *in* $\mathbb{CP}^5_{a_0,\ldots,a_5}$ *is well-formed and quasismooth;*

(ii) X_d *has isolated singularities of the type* $\frac{1}{4}(1,1,1,1)$*;*

(iii) $\mathbb{CP}^5_{a_0,\ldots,a_5}$ *admits an antiholomorphic involution,* τ*, whose fixed point locus intersects a* τ*-invariant* X_d *at the isolated singularities of type* $\frac{1}{4}(1,1,1,1)$*;*

(iv) Any other singularities of X_d *admit crepant resolutions;*

are listed in Table 3.2 on page 48.

We also determine the Betti numbers of the Spin(7)-manifolds, which are results of this construction. The Betti numbers are listed in Table 3.4 on page 53.

McLean's study of the deformation theory of calibrated submanifolds was of a purely local nature. We will use the theory of 'convenient manifolds' developed by Kriegl and Michor [63] to discuss some global properties of the set of compact Cayley submanifolds of a Spin(7)-manifold. Our work will rely heavily on the fact that

Kriegl and Michor have shown that the set of compact submanifolds of a manifold, of a given diffeomorphism type, can be given a convenient manifold structure. The convenient manifold structure on this set is natural, in an appropriate sense which we will describe. We will also express the expected dimension of the moduli space of Cayley submanifolds in terms of familiar invariants of the submanifold. The main result is the following:

Theorem 6.0.1. *Let* (M, Ω, g) *be a manifold with* $\mathrm{Spin}(7)$*-structure,* N *a compact oriented 4-manifold and* $\alpha \in H_4(M, \mathbb{Z})$*. Suppose the set,* $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$*, of unobstructed embedded Cayley submanifolds* $f \colon N \to M$ *with homology class* $[f(N)] = \alpha$ *is nonempty. Then there exists a manifold, with underlying set* $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$*, of dimension*

$$\frac{1}{2}(\sigma(N) + \chi(N)) - \alpha \cdot \alpha.$$

Furthermore $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$ *is the base of a universal family of unobstructed Cayley submanifolds,* $f \colon N \to M$*, of* M *of type* N *such that* $[f(N)] = \alpha$*.*

The naturality of the manifold structure on the set $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$ is expressed in the existence of a universal family with base $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$, or in more categorical or algebro-geometric terms, the manifold $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$ can be said to represent an appropriate functor.

Finally we will find examples of compact Cayley submanifolds in the compact manifolds with holonomy Spin(7) given by Joyce's construction [40]. The following theorem produces examples of compact Cayley submanifolds in any compact Spin(7)-manifold constructed using the methods in [40].

Theorem 7.3.4. *Let* (Z, Ω, g) *be a* $\mathrm{Spin}(7)$*-orbifold with singularities modelled on* $\mathbb{R}^8/\Gamma$ *where* Γ *acts as in Example 2.4.1,* M^t *and* (Ω^t, g^t) *the manifold with* $\mathrm{Spin}(7)$*-structure defined in [41, §15.2.2] and* $\tilde{\Omega}^t$ *the torsion-free* $\mathrm{Spin}(7)$*-structure on* M^t *given by [41, Th. 13.6.1].*

Let $Y \subset \mathbb{CP}^3$ *be a smooth* β*-invariant hypersurface and* $N = Y/\langle\beta\rangle$*. Let* $p_i \in Z$ *be a singular point of* Z *and let* $f_i^t \colon N \to M^t$ *be as in Definition 7.3.1.*

Then there exists $\kappa' > 0$ *such that if* $t \in (0, \kappa']$, *there exists* $\tilde{f}_i^t \colon N \to M^t$, *isotopic to* $f_i^t \colon N \to M^t$, *which is Cayley with respect to* $\tilde{\Omega}^t$.

In Proposition 7.3.5 we determine the dimension of the moduli space of compact Cayley submanifolds near the Cayley submanifolds constructed by Theorem 7.3.4 in terms of the degree d of the embedding. We will also discuss a second construction of compact Cayley submanifolds, Theorem 7.3.7, which gives examples of compact Cayley submanifolds in some of those Spin(7)-manifolds found earlier in the thesis. The methods we will use to find these examples, in particular the relationship between complex and Cayley obstructions, Corollary 7.2.5, can be used to find many more examples of compact Cayley submanifolds in the compact Spin(7)-manifolds given by Joyce's constructions.

1.2 Chapter overview

The thesis is naturally divided in two between the study of Spin(7)-manifolds and the study of Cayley submanifolds of Spin(7)-manifolds. Chapters 2 and 3 are focused on Spin(7)-manifolds while the rest of the thesis deals primarily with Cayley submanifolds.

In Chapter 2 we will review Riemannian holonomy groups and introduce Spin(7)-manifolds. We will give an overview of Joyce's constructions of manifolds with special holonomy from Ricci-flat orbifolds. If (M, g) is a Ricci-flat Riemannian orbifold and $p \in M$ is a singularity modelled on $\mathbb{R}^n/\Gamma$, the aim is to desingularize (M, g) by gluing together M with a Riemannian manifold, which is asymptotic to a quotient $\mathbb{R}^n/\Gamma$ for $\Gamma \subset \mathrm{SO}(n)$ in a suitable sense, known as ALE or QALE spaces, and then to perturb the metric to a Ricci flat metric. This is a balancing act between how well the ALE space approximates the original metric in a neighbourhood of the singularity and the curvature of the metric obtained in the gluing procedure. The theorems of Joyce show us that, in certain cases, this gluing procedure works. We will also discuss the Calabi conjecture since this allows us to find many examples of Ricci-flat manifolds using algebraic techniques. Finally we will see various methods of finding explicit examples of complex manifolds, which satisfy the conditions of the Calabi conjecture, and therefore admit Ricci-flat Kähler metrics.

In Chapter 3 we find examples of Calabi–Yau 4-orbifolds together with antiholomorphic involutions, which satisfy the conditions for a construction of Spin(7)-manifolds of Joyce. In [40], Joyce finds examples of appropriate Calabi–Yau 4-orbifolds as hypersurfaces in weighted projective spaces. We will find all examples of appropriate Calabi–Yau 4-orbifolds, which arise as generic hypersurfaces in weighted projective spaces. In the process we will classify antiholomorphic involutions of weighted projective spaces, up to automorphism. We will then use techniques from toric geometry to find the Betti numbers of the Spin(7)-manifolds that arise from the construction.

In Chapter 4 we will introduce the second major objects of study in the thesis, that of Cayley submanifolds. We will cover some basic material on non-linear differential operators on compact manifolds, Dirac operators and the Atiyah–Singer Index Theorem as these will be necessary as we describe the moduli space of compact Cayley submanifolds.

In Chapter 5 we investigate the 'moduli problem' of compact submanifolds of a manifold. By 'moduli problem' we mean the question of whether there is any geometric structure on the set of all compact submanifolds of a manifold, which should be natural in some sense. The geometric structure on the set should be related to a notion of smooth family of compact submanifolds. We will use the category of convenient manifolds, introduced by Kriegl and Michor, to investigate this problem. The chapter will consist mostly of a review of material from the book by Kriegl and Michor on the subject of convenient manifolds [63]. Our contribution will be to relate the convenient manifold structure on the set of compact submanifolds with a suitable notion of smooth family of compact submanifolds.

In Chapter 6 we investigate the moduli problem of compact Cayley submanifolds of a Spin(7)-manifold. The deformation theory of Cayley submanifolds was first investigated by McLean [77] and the study is local in nature. The results of Chapter 5 will allow us to give a more global treatment of the problem. However we will restrict attention to those Cayley submanifolds that are unobstructed. Finally we will calculate the dimension of the moduli space of Cayley submanifolds near an unobstructed Cayley submanifold. This is an index calculation and we will make use of the Atiyah–Singer Index Theorem and other material from Chapter 4 in this Chapter.

In Chapter 7 we will find examples of compact Cayley submanifolds in the compact manifolds with holonomy Spin(7), which were found in Chapter 3. The strategy we will follow is to find an unobstructed Cayley submanifold in the ALE space, which we use to desingularize the Spin(7)-orbifold, and then to perturb the Cayley submanifold so that it is Cayley with respect to the torsion-free Spin(7)-structure on the desingularized Spin(7)-manifold. The main obstacle in following this strategy is to show that a given Cayley submanifold is unobstructed. In the case of complex submanifolds of Calabi–Yau 4-folds, we will relate the complex obstructions to the obstructions as a Cayley submanifold. We will then use complex algebraic geometry to show that certain Cayley submanifolds are unobstructed. We will finish by discussing various directions for future research involving Cayley submanifolds, most of which rely on a good theory of Cayley submanifolds with conical singularities and asymptotically conical Cayley submanifolds, neither of which we will cover in this thesis.

Chapter 2

Manifolds with special holonomy

In this chapter we give an overview of Riemannian holonomy groups, Riemannian manifolds with special holonomy, and constructions of examples of such Riemannian manifolds.

Berger's classification result, Theorem 2.1.2, is the fundamental theorem in the theory of Riemannian holonomy. The group Spin(7) appears as an exceptional case in this classification and motivates the study of Spin(7)-manifolds, which, in terms of holonomy groups, are Riemannian manifolds (M, g) with holonomy contained in Spin(7), together with a choice of embedding of $\mathrm{Hol}(g) \hookrightarrow \mathrm{Spin}(7) \subset \mathrm{GL}(8, \mathbb{R})$.

We will cover the known constructions of compact manifolds with holonomy Spin(7), due to Joyce. The two constructions of Joyce both make use of ALE manifolds to desingularize Spin(7)-orbifolds and we will give an idea of how these constructions work.

The second construction of compact manifolds with holonomy Spin(7) requires Calabi–Yau orbifolds as input and so in the last section we will cover various algebro-geometric methods for producing lots of examples of Calabi–Yau manifolds and orbifolds. In particular we will discuss hypersurfaces in weighted projective spaces and, more generally, in toric varieties.

2.1 Riemannian holonomy groups

In this section we introduce the notion of the holonomy group of a Riemannian manifold. We will show how the group Spin(7) has a special role to play in the classification of such groups. Let (M, g) be a connected Riemannian manifold. We will associate to (M, g) an algebraic object, which depends on the global structure of the Riemannian manifold. This algebraic object, which will be a group, will hopefully capture interesting information about the Riemannian manifold.

The fundamental theorem of Riemannian geometry gives us the Levi-Civita connection as a natural object to play with in our study of Riemannian geometry. Recall the Levi-Civita connection is the unique torsion-free connection on the tangent bundle of M preserving the metric.

Given a connection we can define parallel transport of vectors along paths in M and in particular to any loop based at $p \in M$ we can associate an element of $\mathrm{GL}(T_pM)$ given by parallel transport along the loop. The piecewise-smooth loops (based at p) carry a group structure given by concatenation, and parallel transport respects this group structure. This motivates the definition of a holonomy group.

Definition 2.1.1. The *holonomy group* $\mathrm{Hol}(g)$ of a connected Riemannian manifold (M, g) is the group generated by parallel transport maps around piecewise-smooth loops based at a point.

Note that we have not made reference to the basepoint in our definition. If M is an n-dimensional manifold then for any point $p \in M$ we can identify $\mathrm{GL}(T_pM)$ with $\mathrm{GL}(n, \mathbb{R})$. Using this identification, the holonomy group can be regarded as a subgroup of $\mathrm{GL}(n, \mathbb{R})$, up to conjugation since the identification of T_pM and $\mathbb{R}^n$ is not canonical.

If $q \in M$ is a different point then by using parallel transport along a curve joining p and q, which we can do since M is connected, we can identify T_pM and T_qM. If we also choose a non-canonical identification of T_qM with $\mathbb{R}^n$, the effect of parallel transport along the curve joining p and q is to change the holonomy group, as a subgroup of

$\mathrm{GL}(n,\mathbb{R})$, by conjugation. The net result is that we can suppress the basepoint and regard the holonomy group as a conjugacy class of subgroups of $\mathrm{GL}(n,\mathbb{R})$.

Remark. We can easily define a holonomy group for any connection on the tangent bundle of a manifold. In fact a lot of interesting geometric structures can be characterized by the existence of a torsion-free connection, whose holonomy group is a subgroup of an interesting Lie group. For example a complex manifold admits a torsion-free connection with holonomy group contained in $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$.

2.1.1 Classification of Riemannian holonomy groups

The major guiding result in the study of Riemannian holonomy is the classification result of Berger [9]. We will need some technical definitions in order to state the classification.

Firstly we will assume that the manifold in question is simply connected. Next we say a Riemannian manifold, (M,g), is *locally reducible* if in a neighbourhood of any point $p \in U \subset M$ is isometric to a product $(U,g) \cong (U_1 \times U_2, g_1 \times g_2)$ with $\dim U_1$, $\dim U_2 > 0$ and *irreducible* if (M,g) is not locally reducible. A metric is *locally symmetric* if $\nabla R = 0$, where R is the curvature of the metric, and *nonsymmetric* if it is not locally symmetric. Note that this is a nonstandard and unmotivated definition of the term nonsymmetric, and we direct the reader to [12, Ch. 7.F] for an introduction to the theory of Riemannian symmetric spaces.

Theorem 2.1.2 (Berger [9]). *Suppose M is a simply-connected manifold of dimension n and g is a Riemannian metric on M, which is irreducible and nonsymmetric. Then one of the following cases holds:*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$,

(iii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$,

(iv) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,

(v) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Each of the cases can be understood as related to one of the four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and the octonions $\mathbb{O}$. Cases (ii) and (iii) are related to complex geometry. Manifolds with holonomy $\mathrm{U}(n)$ are Kähler and any smooth projective variety gives an example of a Kähler manifold. The case of $\mathrm{SU}(n)$ is much more difficult as a manifold with holonomy $\mathrm{SU}(n)$ is necessarily Ricci-flat. The solution of the Calabi conjecture by Yau gives us a method for proving the existence of compact manifolds admitting metrics with holonomy $\mathrm{SU}(n)$, and they are sometimes called 'Calabi–Yau manifolds' in their honour. We will use the term Calabi–Yau manifold for a related but slightly different notion, see §2.2.1. Joyce uses the Calabi conjecture as a stepping stone to finding compact manifolds with holonomy Spin(7).

The next two cases of $\mathrm{Sp}(m)$ and $\mathrm{Sp}(m)\mathrm{Sp}(1)$ are related to the quaternions, and Riemannian manifolds with holonomy contained in those groups are called *hyperkähler* and *quaternionic Kähler* respectively. Hyperkähler manifolds are also Ricci-flat and as a result compact examples are difficult to find.

Finally we have the exceptional cases of G_2 and Spin(7), which only occur in dimensions 7 and 8. Berger's theorem dates from 1955, and it took nearly 40 years to find examples of compact manifolds with holonomy G_2 and Spin(7).

Bryant [13] in 1987 used the theory of exterior differential systems to show the existence of many metrics with holonomy G_2 and Spin(7) on small balls in $\mathbb{R}^7$ and $\mathbb{R}^8$, respectively. Then Bryant and Salamon [14] constructed examples of complete metrics with holonomy G_2 and Spin(7) on non-compact manifolds, which were vector bundles over manifolds of dimensions 3 and 4.

In 1994–5 Joyce [37–39] constructed examples of compact manifolds with holonomy G_2 and Spin(7) from quotients of flat tori by finite groups. Kovalev [56] gave a second construction of compact manifolds with holonomy G_2 from Fano 3-folds.

We will be interested in a second construction of compact manifolds with holonomy Spin(7) by Joyce [40]. The construction will be sketched in Section 2.4, but the essential ingredients will be a Calabi–Yau 4-orbifold with an antiholomorphic involution.

2.2 Spin(7)-manifolds

The material of this section is entirely from [41]. We will give a definition of the Lie group Spin(7) as a subgroup of $\mathrm{GL}(8,\mathbb{R})$ and describe how it relates to other subgroups, in particular SU(4).

There is a very close relation between the holonomy group of a Riemannian manifold and the set of parallel tensors on the manifold. In many cases each determines the other.

Proposition 2.2.1. *Let* (M,g) *be a connected Riemannian manifold of dimension* n*. Let* V *be a representation of* $\mathrm{O}(n)$ *and* E *the associated bundle over* M*. Then the parallel sections of* E *are in one-to-one correspondence with trivial subrepresentations of* V *as a representation of* $\mathrm{Hol}(g)$*.*

Spin(7) can be defined as the simply-connected double cover of $SO(7)$. We however will define it as the stabilizer group of a certain 4-form on $\mathbb{R}^8$, which will determine an embedding of Spin(7) in $\mathrm{GL}(8,\mathbb{R})$, and hence the irreducible 8-dimensional representation, to which Berger's theorem refers.

Definition 2.2.2. Let $\mathbb{R}^8$ have coordinates $(x_1,\ldots,x_8)$. Let $\mathrm{d}x_{ijkl}$ denote the 4-form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l$. We define the *Cayley form*, Ω_0, by

$$\begin{aligned}\Omega_0 = {}& \mathrm{d}x_{1234} + \mathrm{d}x_{1256} + \mathrm{d}x_{1278} + \mathrm{d}x_{1357} - \mathrm{d}x_{1368} - \mathrm{d}x_{1458} - \mathrm{d}x_{1467} \\ & + \mathrm{d}x_{5678} + \mathrm{d}x_{3478} + \mathrm{d}x_{3456} + \mathrm{d}x_{2468} - \mathrm{d}x_{2457} - \mathrm{d}x_{2367} - \mathrm{d}x_{2358}.\end{aligned}$$

Spin(7) is the subgroup of $\mathrm{GL}(8,\mathbb{R})$ preserving Ω_0.

The 4-form above can be motivated by the structure of the octonions. The relationship between the octonions and the Cayley form can be found in, for example, [33, Section IV.1.C]. It should be noted that the Cayley form given above differs from that in [33] by an orientation-preserving permutation of the coordinates and an overall change in sign.

Since we have defined Spin(7) as the stabilizer group of the Cayley form by Proposition 2.2.1, if (M, g) is a connected oriented Riemannian 8-manifold with $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$, then M admits a parallel 4-form Ω such that for any $p \in M$ there exists an oriented isometry $T_pM \to \mathbb{R}^8$, which takes Ω_p to Ω_0. If $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ then the form Ω is uniquely determined by g.

Definition 2.2.3. A Spin(7)-*manifold* is a triple (M, Ω, g) where (M, g) is a connected oriented Riemannian 8-manifold and Ω is a parallel, with respect to the Levi-Civita connection, 4-form such that for any $p \in M$ there exists an oriented isometry $T_pM \to \mathbb{R}^8$, which takes Ω_p to Ω_0.

We can break up the condition of being a Spin(7)-manifold into a topological condition, and one which expresses an integrability condition of sorts.

Definition 2.2.4. Let M be an oriented 8-manifold. A Spin(7)-*structure on* M is a pair (Ω, g) where g is a Riemannian metric and for any $p \in M$ there exists an oriented isometry $T_pM \to \mathbb{R}^8$, which takes Ω_p to Ω_0.

Definition 2.2.5. Let M be an oriented Riemannian 8-manifold. The *bundle of admissible forms*, denoted by $\mathcal{A}M$, is the bundle whose fibre at $p \in M$ consists of elements $\Omega_p \in \Lambda^4 T_p^* M$ such that there exists an oriented isomorphism from $T_pM \to \mathbb{R}^8$ which takes Ω_p to Ω_0.

If M is an oriented 8-manifold then a Spin(7)-structure is equivalent to a section of $\mathcal{A}M$. The condition for the existence of a section of $\mathcal{A}M$ is a topological one.

Definition 2.2.6. Let M be a manifold. The *frame bundle of* M, denoted FM, is the principal $\mathrm{GL}(n, \mathbb{R})$-bundle whose fibre at $p \in M$ is the set of ordered bases of T_pM. Equivalently, the fibre of FM at p is the set of linear isomorphisms $T_pM \to \mathbb{R}^n$.

Definition 2.2.7. Let M be an n-dimensional manifold and $G \subset \mathrm{GL}(n, \mathbb{R})$ a Lie subgroup. A G-*structure on* M is a subbundle $F_GM \subset FM$ such that the restriction of the $\mathrm{GL}(n, \mathbb{R})$ action to G makes F_GM into a principal G-bundle on M.

Suppose M admits a Spin(7)-structure as per Definition 2.2.4. We define $F_{\mathrm{Spin}(7)}M \subset FM$ to be the set of linear isomorphisms $T_pM \to \mathbb{R}^8$ taking Ω_p to Ω_0. Since we have defined the Lie group Spin(7) as the stabilizer of the form Ω_0, the bundle $F_{\mathrm{Spin}(7)}M$ is preserved by the action of Spin(7) and makes $F_{\mathrm{Spin}(7)}M$ into a principal Spin(7)-bundle. Therefore the pair (Ω, g) uniquely determines (and is uniquely determined by) a Spin(7)-structure as per Definition 2.2.7 and so we use the terms interchangeably. The existence of a Spin(7)-structure is a topological property of M as the following result shows:

Proposition 2.2.8 ([67, Th. 10.7]). *Let M be an oriented 8-manifold. M admits a* Spin(7)*-structure if and only if $w_2(M) = 0$ and*

$$p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0.$$

For M to be a Spin(7)-manifold it must satisfy an extra integrability condition as the following proposition shows from [41, Prop. 10.5.3]:

Proposition 2.2.9. *Let M be an oriented 8-manifold with* Spin(7)*-structure (Ω, g). Then $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$ and Ω is the induced 4-form if and only if $\mathrm{d}\Omega = 0$. In this case we say the* Spin(7)*-structure (Ω, g) is* torsion-free*.*

We use the term *torsion-free* in the previous definition because it can be shown [25, Th. 5.3] that if (Ω, g) is a torsion-free Spin(7)-structure then $\nabla\Omega = 0$ where ∇ is the Levi-Civita connection. Hence the Levi-Civita connection is a torsion-free connection with holonomy contained in Spin(7).

Remark. In general we say a G-structure is torsion-free if there is a torsion-free connection on the tangent bundle preserving the G-structure. The obstruction to finding such a torsion-free connection is called the *intrinsic torsion* of the G-structure. For example the intrinsic torsion of a $\mathrm{GL}(n, \mathbb{C})$-structure is determined by the Nijenhuis tensor, and vanishes if and only if the Nijenhuis tensor vanishes.

2.2.1 Relation with SU(4)

There is an inclusion of Lie groups $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$, which we can describe in terms of the forms that each of these groups stabilize. The Lie group SU(4) can be defined as the stabilizer of a metric, a Kähler form ω_0 and holomorphic volume form θ_0. If we let (z_1, z_2, z_3, z_4) be coordinates on $\mathbb{C}^4$ we can write ω_0, θ_0 as

$$\omega_0 = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_4 \wedge \mathrm{d}\bar{z}_4) \text{ and } \theta_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4.$$

In a similar manner to a Spin(7)-manifold, we define a Calabi–Yau 4-fold as a quadruple (X, g, ω, θ) consisting of a Kähler manifold of complex dimension 4 (X, g, ω) and a holomorphic $(4,0)$-form θ such that $|\theta|_g \equiv 4$. It can be shown that for any $p \in X$ there exists an isometry $T_pX \to \mathbb{C}^4$ taking (ω_p, θ_p) to (ω_0, θ_0).

Proposition 2.2.10. *Let* (X, g, ω, θ) *be a Calabi–Yau* 4*-fold. Define a* 4*-form by* $\Omega = \frac{1}{2}\omega \wedge \omega + \operatorname{Re}\theta$*, then* (Ω, g) *is a torsion-free* Spin(7)*-structure on* X*.*

Proof. Let $p \in X$ and identify ω_p and θ_p with the standard forms on $\mathbb{C}^4$. Identifying $\mathbb{C}^4$ with $\mathbb{R}^8$ via $z_j = x_{2j-1} + ix_{2j}$ and comparing the expressions for Ω_p and Ω_0 we see that (Ω, g) defines a Spin(7)-structure. Since (X, g, ω, θ) is a Calabi-Yau manifold we have $\mathrm{d}\omega = \mathrm{d}\theta = 0$, which implies $\mathrm{d}\Omega = 0$. □

Proposition 2.2.10 describes a particular embedding of $\mathrm{SU}(4) \hookrightarrow \mathrm{Spin}(7)$. Let (X, g, ω, θ) be a Calabi–Yau 4-fold. Then the 4-form $\Omega_\phi = \frac{1}{2}\omega \wedge \omega + \operatorname{Re}(e^{i\phi}\theta)$ for $\phi \in (0, 2\pi)$ also defines a torsion-free Spin(7)-structure on X and a different embedding of $\mathrm{SU}(4) \hookrightarrow \mathrm{Spin}(7)$.

2.3 Kummer constructions of compact Ricci-flat manifolds

The Calabi conjecture is an essential result in the study of compact Ricci-flat manifolds since we can relate the existence of a Ricci-flat metric, which is a question in Riemannian geometry, to that of the triviality of the canonical bundle, which is a problem in complex geometry. The Calabi conjecture was posed by Calabi in 1954 and proved by Yau in 1976 [91,92]. We refer the reader to [49, Ch. 6] for a discussion and proof.

Theorem 2.3.1 (Calabi Conjecture)**.** *Let* (X, J) *be a compact complex manifold, with Kähler metric* g *and Kähler form* ω*. Let* ρ *be a positive* $(1,1)$*-form such that* $[\rho] = 2\pi c_1(X)$*. Then there exists a Kähler metric* g' *on* (X, J) *with Kähler form* ω' *such that* $[\omega'] = [\omega]$ *and* ρ *is the Ricci form of* g'*.*

Now suppose (X, J) is a compact complex manifold admitting Kähler metrics such that the canonical bundle is trivial. Then $c_1(X) = 0$ and hence choosing $\rho = 0$ we can use the Calabi conjecture to give the existence of a Ricci-flat metric.

The Kummer construction is a description of a complex surface, which has a trivial canonical bundle, and hence by the Calabi conjecture carries a Ricci-flat metric.

Example 2.3.2. Consider a complex torus $\mathbb{C}^2/\Lambda \cong T^4$ where Λ is a lattice in $\mathbb{C}^2$ of rank 4. The canonical bundle of $\mathbb{C}^2/\Lambda$ is trivial. Define a map $\sigma\colon T^4 \to T^4$ by $(z_1, z_2) + \Lambda \mapsto (-z_1, -z_2) + \Lambda$. The map σ fixes the 16 points $\{(z_1, z_2) + \Lambda : (z_1, z_2) \in \frac{1}{2}\Lambda\}$ and each of the quotient singularities of T^4/σ is modelled on $\mathbb{C}^2/\{\pm 1\}$.

We can resolve each of these singularities by blowing-up and it can be shown that the canonical bundle of the resulting complex surface X is also trivial.

Unfortunately the Calabi conjecture does not give an idea of what the metrics on X look like. Page [82] describes a metric on X which approximates the Ricci-flat metric given by the Calabi conjecture. The metric is flat outside small neighbourhoods of the singularities and close to the singularities the metric is approximated by an asymptotically locally Euclidean (or ALE) metric, a term which we will define in the next section.

Joyce [37–39] based his constructions of compact manifolds with holonomy G_2 and Spin(7) from flat tori on the Kummer construction. He constructs manifolds with special holonomy by dividing flat tori by finite groups and then constructs metrics by gluing in approximate metrics close to the singular sets, and finally shows that one can perturb the metric slightly to get a Ricci-flat metric.

We will review orbifolds, as manifolds divided by finite groups naturally have this structure. We will then proceed to describe ALE manifolds, which will be used to desingularize the orbifolds.

2.3.1 Orbifolds

Orbifolds are similar to manifolds in the sense that they are topological spaces with extra structure, but orbifolds are modelled on quotients of open sets of $\mathbb{R}^n$ by finite groups.

Definition 2.3.3. An *orbifold* of dimension n is a Hausdorff topological space M locally modelled on open sets of $\mathbb{R}^n/\Gamma$ for finite subgroups $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$, such that if $1 \neq \gamma \in \Gamma$, then the subspace V_γ of $\mathbb{R}^n$ fixed by γ has $\dim V_\gamma \leq n-2$.

For a more precise definition see, for example, [19]. If Γ is a finite group acting on a manifold M then M/Γ is an example of an orbifold. Orbifolds are not always in this form of a quotient of a manifold by a finite group.

We say a point p in M is an *orbifold point with orbifold group* Γ if M is locally modelled on $\mathbb{R}^n/\Gamma$ for $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$ finite, where p is identified with the origin and with Γ non-trivial. We say a point p in M is *nonsingular* if it is not an orbifold point. Note that the set of nonsingular points in M is dense and is a manifold.

Definition 2.3.4. A *Riemannian metric g on an orbifold M* is a Riemannian metric in the usual sense on the nonsingular part of M, and where M is locally isomorphic to $\mathbb{R}^n/\Gamma$, the metric g can be identified with the quotient of a Γ-invariant Riemannian metric defined on an open neighbourhood of 0 in $\mathbb{R}^n$. We define the *holonomy group* $\mathrm{Hol}(g)$ of (M, g) to be the holonomy group of the restriction of g to the nonsingular part of M.

If p is an orbifold point of M with orbifold group Γ then we have an inclusion of groups $\Gamma \subseteq \mathrm{Hol}(g)$, induced by the limit of parallel transport maps around small loops at p as the radius of the loop tends to zero. Therefore for an orbifold to have holonomy (for example) Spin(7) we must have that each orbifold group Γ lies in Spin(7).

We can similarly define forms on orbifolds as forms which locally lift to Γ-invariant forms. With these definitions in place we can define Spin(7)-orbifolds and Calabi–Yau orbifolds as in analogy with the case of manifolds.

Many results for manifolds carry over with small modifications to orbifolds. In particular the Calabi conjecture holds for compact Kähler orbifolds. As a consequence we have the following theorem [41, Th. 6.5.6], which we can use to find Calabi–Yau metrics on orbifolds.

Theorem 2.3.5. *Let X be a compact complex orbifold with $c_1(X) = 0$ admitting Kähler metrics. Then there is a unique Ricci-flat Kähler metric in every Kähler class on X.*

2.3.2 ALE manifolds

Definition 2.3.6. Suppose Γ is a finite subgroup of $\mathrm{SO}(n)$ which acts freely on $\mathbb{R}^n \backslash \{0\}$. An *asymptotically locally Euclidean manifold* or *ALE manifold* asymptotic to $\mathbb{R}^n/\Gamma$ is a triple $(M_\Gamma, g_\Gamma, \pi)$ such that (M_Γ, g_Γ) is a non-compact Riemannian manifold and $\pi\colon M_\Gamma \to \mathbb{R}^n/\Gamma$ is a proper, continuous map such that there exists a compact set $S \subset M_\Gamma$ and a constant $R > 0$ such that the restriction $\pi\colon M_\Gamma \setminus S \to \{x \in \mathbb{R}^n : |x| > R\}/\Gamma$ is a diffeomorphism and

$$|\nabla^k(\pi_*(g_\Gamma) - g_0)|_{g_0} = O(r^{-n-k}) \text{ on } \{x \in \mathbb{R}^n : |x| > R\}/\Gamma$$

for all $k \geq 0$, where g_0 is the flat metric on $\mathbb{R}^n/\Gamma$ and ∇ is the Levi-Civita connection of g_0.

If $(M_\Gamma, g_\Gamma, \pi)$ is an ALE manifold asymptotic to $\mathbb{R}^8/\Gamma$ and $(M_\Gamma, \Omega_\Gamma, g_\Gamma)$ is a Spin(7)-manifold, we say $(M_\Gamma, \Omega_\Gamma, g_\Gamma, \pi)$ is an *ALE* Spin(7)*-manifold* if also the following asymptotic estimates hold

$$|\nabla^k(\pi_*(\Omega_\Gamma) - \Omega_0)|_{g_0} = O(r^{-8-k}) \text{ on } \{x \in \mathbb{R}^8 : |x| > R\}/\Gamma,$$

for all $k \geq 0$, where (Ω_0, g_0) is the standard flat Spin(7)-structure on $\mathbb{R}^8/\Gamma$. We can similarly define ALE Calabi–Yau manifolds.

Note that if $(M_\Gamma, \Omega_\Gamma, g_\Gamma, \pi)$ is an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/\Gamma$ then so is $(M^t_\Gamma, \Omega^t_\Gamma, g^t_\Gamma, \pi^t)$ for $t > 0$ where $M^t_\Gamma = M_\Gamma$, $\Omega^t_\Gamma = t^4\Omega_\Gamma$, $g^t_\Gamma = t^2 g_\Gamma$ and $\pi^t(x) = t\pi(x)$. When t is small, M^t_Γ approximates $\mathbb{R}^8/\Gamma$ better in the sense that on the set $\{x \in \mathbb{R}^n : |x| > tR\}/\Gamma$ we have the estimate

$$|(\pi^t)_*(\Omega^t_\Gamma) - \Omega_0|_{g_0} = O(t^8 r^{-8}).$$

The following example of a series of ALE Calabi–Yau manifolds is from Calabi [15] but the case of $n = 2$ was first discovered by Eguchi and Hanson [24].

Example 2.3.7. Consider $\mathbb{C}^n/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts as diagonal multiplication by an nth root of unity. Let (X, π) be the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at 0, which is isomorphic to $K_{\mathbb{CP}^{n-1}}$ the canonical bundle over $\mathbb{CP}^{n-1}$. Let r be the radius function on $\mathbb{C}^n/\mathbb{Z}_n$ and define $f\colon \mathbb{C}^n/\mathbb{Z}_n \to \mathbb{R}$ by

$$f = \sqrt[n]{r^{2n}+1} + \frac{1}{n}\sum_{j=0}^{n-1} \xi^j \log\left(\sqrt[n]{r^{2n}+1} - \xi^j\right),$$

where ξ is an nth root of unity. The $(1,1)$-form ω on X defined by $\omega = \mathrm{dd}^c\pi^*(f)$ can be shown to extend to a smooth, closed, positive (1,1)-form on all of X. Calabi shows that the Kähler metric defined by ω is complete and Ricci-flat and Joyce [42] shows that there exists a holomorphic volume form θ so that (X, g, ω, θ) is an ALE Calabi–Yau manifold.

Gibbons and Hawking [31] gave explicit examples of ALE Calabi–Yau manifolds asymptotic to $\mathbb{C}^2/\mathbb{Z}_k$ for all $k \geq 2$. Kronheimer [64] constructed and classified all ALE Calabi–Yau manifolds asymptotic to $\mathbb{C}^2/G$ for G a finite subgroup of SU(2).

With Kronheimer's work, the case of $n = 2$ was fully understood. To understand higher dimensional ALE Calabi–Yau manifolds we will need a definition. Recall that a *resolution*, (X, π), consists of a normal nonsingular variety X with a proper birational morphism $\pi\colon X \to \mathbb{C}^n/G$.

Definition 2.3.8. A resolution $\pi\colon X \to \mathbb{C}^n/G$ is *crepant* if $c_1(X) = 0$.

Joyce [42] shows that a version of the Calabi conjecture holds for ALE Kähler manifolds, and deduces the following result:

Theorem 2.3.9. *Let G be a nontrivial finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^n \setminus \{0\}$, and (X, π) a crepant resolution of $\mathbb{C}^n/G$. Then each Kähler class of ALE Kähler metrics on X contains a unique Ricci-flat ALE Kähler metric g.*

We can use this theorem to find ALE Calabi–Yau metrics on crepant resolutions of quotient singularities. Joyce also proves a more general result about the existence of Ricci-flat metrics on Quasi-ALE Calabi–Yau manifolds [43], but we will not need those results in this thesis.

2.3.3 Resolving orbifolds using ALE manifolds

Suppose (M, g) is a Riemannian orbifold with holonomy contained in G, where G is either $\mathrm{Spin}(7)$ or $\mathrm{SU}(m)$, for some m, so M is a $\mathrm{Spin}(7)$ or a Calabi–Yau orbifold. Suppose further that the singular locus of M consists of a point $p \in M$ with local model $\mathbb{R}^n/\Gamma$ and suppose $(M_\Gamma, g_\Gamma, \pi)$ is an ALE manifold asymptotic to $\mathbb{R}^n/\Gamma$ with holonomy contained in G. We want to use M_Γ to desingularize the orbifold M by patching together the metric on M_Γ with that on M around the singular point to obtain a manifold $\tilde{M}$ with holonomy G.

Recall that if $(M_\Gamma, g_\Gamma, \pi)$ is an ALE manifold then by scaling by a factor t we get a one parameter family of ALE manifolds $(M_\Gamma^t, g_\Gamma^t, \pi^t)$ with $M_\Gamma^t = M_\Gamma$, $g_\Gamma^t = t^2 g_\Gamma$ and projection map $\pi^t(x) = t\pi(x)$. Due to the original estimate on M_Γ of the metric we have the estimate

$$|\pi_*^t(g_\Gamma^t) - g_0|_{g_0} = O(t^n r^{-n}). \tag{2.1}$$

From this estimate we see that for a fixed radius the metric g_Γ^t approaches the flat metric as $t \to 0$.

Let U be a neighbourhood of $p \in M$ such that the exponential map based at p identifies U with a ball of radius $2R > 0$ in T_pM. Let $\phi\colon U \to \mathbb{R}^n/G$ denote the inverse of the exponential map followed by an isometric identification of T_pM with $\mathbb{R}^n/G$. In exponential normal coordinates, the metric vanishes to order two at the origin [68, Lem. 5.5], and hence we have the estimate

$$|\phi_*(g) - g_0|_{g_0} = O(r^2). \tag{2.2}$$

We now wish to patch together the metric in a neighbourhood of p with the scaled metric on M^t_Γ. Let B_r denote the ball of radius r around the origin in $\mathbb{R}^n/G$. Fix $\alpha \in (0,1)$. We define open sets in $U^t \subset M$ and $U^t_\Gamma \subset M^t_\Gamma$ for $t \in (0,1]$ by

$$U^t = \phi^{-1}(B_{t^\alpha R}) \qquad\qquad \text{and} \qquad\qquad U^t_\Gamma = (\pi^t)^{-1}(B_{2t^\alpha R}).$$

For $t \in (0,1]$, let M^t denote the quotient of the disjoint union $(M \setminus \overline{U^t}) \sqcup U^t_\Gamma$ by the equivalence relation $x \sim y$ if $\phi(x) = \pi^t(y)$ in the annulus $B_{2t^\alpha R} \setminus \overline{B}_{t^\alpha R}$.

We will define a metric on M^t by patching the metrics over $B_{2t^\alpha R} \setminus \overline{B}_{t^\alpha R}$ using a partition of unity. Using the estimates (2.1) and (2.2) we can estimate the difference

$$|\phi_*(g) - (\pi^t)_*(g^t_\Gamma)|_{g_0} = O(t^{2\alpha}) + O(t^{n(1-\alpha)})$$

in the annular region. This difference is minimized when $\alpha = \frac{n}{n+2}$ and tends to 0 as $t \to 0$. Let $\eta\colon [0,\infty) \to [0,1]$ be a smooth function with $\eta(x) = 0$ for $x \leq R$ and $\eta(x) = 1$ for $x \geq 2R$. We define a metric on M^t by setting $g^t = g$ on $M^t \setminus U^t_\Gamma$, $g^t = g^t_\Gamma$ on $U^t_\Gamma \setminus (U^t_\Gamma \cap U^t)$ and on the annular region by

$$g^t = \eta(t^{-\alpha}r)\phi_*(g) + (1 - \eta(t^{-\alpha}r))(\pi^t)_*(g^t_\Gamma).$$

The metric g^t does not have holonomy contained in G but it is close to having holonomy in G, in the sense that the torsion of the G-structure is $O(t^\beta)$ for some positive power β and hence vanishes as $t \to 0$. The hard work is then to show that, for t sufficiently small, the metric g^t can be perturbed to a nearby metric $\tilde{g}^t$ with holonomy contained in G.

Note that in the constructions of compact manifolds with special holonomy we need to be more careful about how we patch together the G-structures. See [41, §15.2.2] for an example.

2.4 Constructions of Spin(7)-manifolds

The two known constructions of compact manifolds with holonomy Spin(7) are both based upon resolving orbifold singularities by gluing in ALE Spin(7) manifolds (or quasi-ALE Spin(7)-manifolds, which we will not define) around the singular sets.

The first construction [39] and [41, Ch.s 13,14] is inspired by the Kummer construction. We start with a flat 8-dimensional torus which carries a flat Spin(7)-structure and then divide by a finite group which preserves this structure.

Joyce then constructs an approximately Ricci-flat metric on a manifold obtained by gluing in ALE manifolds, scaled by a parameter t, around the singular sets. Finally he shows that for small enough t we can deform the Spin(7)-structure to a torsion-free Spin(7)-structure.

In this thesis we will be interested in Joyce's second construction of Spin(7)-manifolds [40] and [41, Ch. 15], which follows the same general plan of dividing by a finite group and then resolve the singularities, but starts with a Calabi–Yau 4-orbifold and not a flat 8-dimensional torus.

2.4.1 Construction from Calabi–Yau 4-orbifolds

In Example 2.4.1 we will give an ALE Spin(7)-manifold with holonomy $\mathrm{SU}(4) \rtimes \mathbb{Z}_2$. We will then use this ALE Spin(7)-manifold to desingularize Spin(7)-orbifolds which arise as the quotient of Calabi–Yau 4-folds by an antiholomorphic involution.

Example 2.4.1. Identify $\mathbb{R}^8$ with $\mathbb{C}^4$ using the complex coordinates

$$(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8).$$

We define $\alpha, \beta \colon \mathbb{C}^4 \to \mathbb{C}^4$, elements of Spin(7), by

$$\begin{aligned} \alpha &: (z_1, z_2, z_3, z_4) \longmapsto (iz_1, iz_2, iz_3, iz_4), \\ \beta &: (z_1, z_2, z_3, z_4) \longmapsto (\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3). \end{aligned} \tag{2.3}$$

Then $\Gamma = \langle \alpha, \beta \rangle$ is a non-abelian subgroup of Spin(7) of order 8 isomorphic to the quaternion group $\langle i, j, k \rangle \subset \mathbb{H}^*$.

The cyclic subgroup $\langle\alpha\rangle \cong \mathbb{Z}_4$ of Γ is contained in a copy of $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$ and in Example 2.3.7 we described an ALE Calabi–Yau manifold asymptotic to $\mathbb{C}^4/\mathbb{Z}_4$. Let us denote this ALE Calabi–Yau manifold by X, and let π denote the projection $\pi\colon X \to \mathbb{C}^4/\mathbb{Z}_4$. The antiholomorphic involution β lifts to a free antiholomorphic map $\beta\colon X \to X$ on the resolution, and we can arrange that β acts on the Kähler form and holomorphic volume form by

$$\beta^*(\omega) = -\omega \qquad \text{and} \qquad \beta^*(\theta) = \bar{\theta}.$$

The antiholomorphic involution therefore preserves the metric and the admissible form given by $\Omega = \frac{1}{2}\omega\wedge\omega + \mathrm{Re}(\theta)$. Since β acts freely on X, the quotient $M_\Gamma = X/\langle\beta\rangle$ is a manifold and M_Γ carries a torsion-free Spin(7)-structure as β preserves Ω. The map π descends to a continuous map $\pi\colon M_\Gamma \to \mathbb{R}^8/\Gamma$, and furthermore M_Γ has a ALE Ricci-flat metric asymptotic to $\mathbb{R}^8/\Gamma$ with holonomy contained in Spin(7). Note that M_Γ does not carry a complex structure, and hence the holonomy is not contained in SU(4). In fact the holonomy of M_Γ is $\mathrm{SU}(4) \rtimes \mathbb{Z}_2$.

With respect to the complex coordinates

$$(w_1, w_2, w_3, w_4) = (-x_1 + ix_3, x_2 + ix_4, -x_5 + ix_7, x_6 + ix_8) \tag{2.4}$$

the roles of α and β are reversed and by a similar argument we can find another possible resolution of $\mathbb{R}^8/\Gamma$. The importance of this observation is that although the two resolutions have the same holonomy groups as Lie groups, the embeddings into Spin(7) are different.

Suppose X is a complex 4-orbifold admitting metrics with holonomy SU(4). Suppose X has isolated singularities $\{p_1, \ldots, p_k\}$ modelled on $\mathbb{C}^4/\mathbb{Z}_4$, where the generator of $\mathbb{Z}_4$ acts as α in equation (2.3) and that X admits an antiholomorphic involution τ with fixed points the finite set $\{p_1, \ldots, p_k\}$, which acts on the Kähler form and holomorphic volume form as $\tau^*(\omega) = -\omega$ and $\tau^*(\theta) = \bar{\theta}$. Then as in the previous example, τ preserves the admissible form $\Omega = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\theta)$. In this case however since τ has fixed points and X had orbifold singularities to begin with, the quotient $M = X/\langle\tau\rangle$ is a Spin(7)-orbifold and not a manifold.

Joyce shows [40, Prop. 5.3] that the orbifold singularities of M are all of the form of $\mathbb{R}^8/\Gamma$ where Γ is the group described in Example 2.4.1. We can use one of the two ALE spaces given in Example 2.4.1 to construct an approximately torsion-free Spin(7)-structure (Ω_t, g_t) on a manifold, which converges to the singular Spin(7)-structure of M when a parameter t tends to 0. Then using similar techniques to the torus case, Joyce proves that, for t sufficiently small, one can perturb the Spin(7)-structure (Ω_t, g_t) to a torsion-free Spin(7)-structure $(\tilde{\Omega}_t, \tilde{g}_t)$.

The precise statement of the construction of Spin(7)-manifolds from Calabi–Yau 4-orbifolds with antiholomorphic involutions is given in the following theorem of Joyce:

Condition 2.4.2. Let X be a compact complex 4-orbifold with $c_1(X) = 0$, admitting Kähler metrics. Let τ be an antiholomorphic involution on X. We require that X have isolated singularities $\{p_1 \ldots, p_k\}$, with $k \geq 1$, modelled on $\mathbb{C}^4/\mathbb{Z}_4$ as described above and that the fixed point set of τ is $\{p_1, \ldots, p_k\}$. We also require that $X \setminus \{p_1, \ldots, p_k\}$ is simply-connected and $h^{2,0}(X) = 0$.

Theorem 2.4.3 ([41, Th. 15.2.15])**.** *Suppose X satisfies Condition 2.4.2. Let $\tilde{M}$ be the resulting compact 8-manifold defined in [40, Def. 15.2.8]. Then there exist torsion-free* Spin(7)*-structures (Ω, g) on $\tilde{M}$. We can choose the resolutions of the singularities so that* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$*.*

2.4.2 Analytic estimates of Spin(7)-structures

Joyce uses the following theorem to show the existence of torsion-free Spin(7)-structures on compact manifolds. The parameter t is a scaling parameter introduced when we desingularize the Spin(7)-orbifold using ALE (or QALE) spaces. By decreasing the parameter t, the error due to the gluing is decreased but the curvature also increases. The theorem shows that there are certain rates at which we can perform the gluing. We state the theorem with a slightly improved estimate on the difference $\tilde{\Omega} - \Omega$, which we will need later.

Theorem 2.4.4 ([41, Th. 13.6.1]). *Let λ, μ, ν be positive constants, $k \in \mathbb{N}$ and $\alpha \in (0, 1/5)$. Then there exist positive constants κ, K such that whenever $0 < t \leq \kappa$, the following is true.*

Let M be a compact 8-manifold, and (Ω, g) a Spin(7)*-structure on M. Suppose that ϕ is a smooth 4-form on M with $\mathrm{d}\Omega + \mathrm{d}\phi = 0$, and*

(i) $\|\phi\|_{L^2} \leq \lambda t^{13/3}$ and $\|\mathrm{d}\phi\|_{L^{10}} \leq \lambda t^{7/5}$,

(ii) the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \mu t$, and

(iii) the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq \nu t^{-2}$.

Then there exists a smooth, torsion-free Spin(7)*-structure $(\tilde{\Omega}, \tilde{g})$ on M with $\|t^{-4}\tilde{\Omega} - t^{-4}\Omega\|_{C^{0,\alpha}} \leq K t^{1/3}$, where the norms are calculated with respect to $t^{-2}\tilde{g}$.*

Proof. The only addition to [41, Th. 13.6.1] is the estimate

$$[t^{-4}\tilde{\Omega} - t^{-4}\Omega]_\alpha \leq K' t^{1/3}.$$

The Sobolev space L^{10}_1 embeds into $C^{0,\alpha}$ for $0 < \alpha < 1/5$ in dimension 8. Following the proof of the Sobolev embedding theorem in [5, §2.23] and the proof of [41, Th. S1] we can prove that there exists a constant C_1 depending on μ and ν such that for $\chi \in L^{10}_1(\Lambda^4 T^*M) \cap L^2(\Lambda^4 T^*M)$

$$[\chi]_\alpha \leq C_1(t^{1/5-\alpha}\|\nabla\chi\|_{L^{10}} + t^{-4-\alpha}\|\chi\|_{L^2}).$$

We write $\tilde{\Omega} - \Omega = \eta - F(\eta)$ where $\eta \in C^\infty(\Lambda^4_-)$ and $F(\eta) \in C^\infty(\Lambda^4_{27})$. From [41, Prop. 13.7.1] we have the estimates $\|\eta\|_{L^2} \leq 4\lambda t^{13/3}$ and $\|\nabla\eta\|_{L^{10}} \leq C_3 t^{2/15}$ for a constant C_3 depending on λ, μ, ν. Furthermore from the estimates of $F(\eta)$ in [41, Eq. (10.21), Eq. (10.22)] there exist constants ϵ_2, ϵ_3 such that

$$\|F(\eta)\|_{L^2} \leq \epsilon_2 \|\eta\|^2_{L^2}$$

and

$$\begin{aligned}\|\nabla F(\eta)\|_{L^{10}} &\leq \epsilon_3 \||\eta|^2|\mathrm{d}\Omega| + |\eta||\nabla\eta|\|_{L^{10}}\\ &\leq \epsilon_3(\|\eta\|^2_{C^0}\|\mathrm{d}\phi\|_{L^{10}} + \|\eta\|_{C^0}\|\nabla\eta\|_{L^{10}})\\ &\leq C_4 t^{7/15}\end{aligned}$$

for some constant C_4 depending on λ, μ, ν. We now have constants C_5 and C_6, depending on λ, μ, ν such that

$$\|\tilde{\Omega} - \Omega\|_{L^2} \leq \|\eta\|_{L^2} + \|F(\eta)\|_{L^2} \leq C_5 t^{13/3}$$
$$\|\nabla(\tilde{\Omega} - \Omega)\|_{L^{10}} \leq \|\nabla\eta\|_{L^{10}} + \|\nabla F(\eta)\|_{L^{10}} \leq C_6 t^{2/15}.$$

Therefore we have an estimate of

$$[\tilde{\Omega} - \Omega]_\alpha \leq K' t^{1/3-\alpha}$$

and by scaling by appropriate powers of t we get the estimate we want. □

In the proof of [41, Th. 13.6.1], Joyce uses elliptic regularity and the 'bootstrap method' to show that $\tilde{\Omega}$ is smooth. The elliptic regularity used is an interior estimate and, following the final part of the proof of [41, Th. 13.6.1] we can prove the following:

Theorem 2.4.5. *There exists a positive constant ε such that the following is true. Let (M, Ω, g) be a* Spin(7)*-manifold, $U \subset\subset M$. Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. There exists a constant K, depending on ε, Ω, U, k, and α such that if $\tilde{\Omega}$ is a torsion-free* Spin(7)*-structure on M with $\pi^4_1(\tilde{\Omega} - \Omega) = \pi^4_7(\tilde{\Omega} - \Omega) = 0$ and $\|\tilde{\Omega} - \Omega\|_{C^{0,\alpha}} < \varepsilon$ then*

$$\|(\tilde{\Omega} - \Omega)|_U\|_{C^k} \leq K \|\tilde{\Omega} - \Omega\|_{C^{0,\alpha}}.$$

2.5 Constructions of Calabi–Yau manifolds

Calabi–Yau manifolds play an essential role in the construction outlined in the previous section. They have enjoyed great attention due to their position in an intersection of subjects. Calabi–Yau manifolds lie at the intersection of geometric analysis, algebraic geometry and modern theoretical physics. There are many interesting conjectures about their properties such as mirror symmetry, which relates two or more Calabi–Yau manifolds in a very non-trivial way.

As we have said, the Calabi conjecture reduces the difficult problem of the existence of a Ricci-flat metric to that of a problem in algebraic geometry. The study of

Calabi–Yau manifolds has greatly benefited from the wide range of tools available to modern algebraic geometers.

The Kummer construction of Example 2.3.2 was generalized by Roan [85] to construct higher dimensional varieties with trivial canonical bundle. Candelas, Lynker and Schimmrigk [16] found examples of Calabi–Yau 3-folds by looking in the class of hypersurfaces in weighted projective spaces. Batyrev [6] studied Calabi–Yau hypersurfaces in toric varieties, and showed that toric varieties, which are determined by reflexive polytopes, contain Calabi–Yau hypersurfaces as anticanonical divisors. Kreuzer and Skarke [58] gave a classification scheme for reflexive polytopes in any dimension and used this to classify reflexive polytopes giving rise to Calabi–Yau varieties of dimensions 2 and 3 [59, 60]. There are even more general constructions of Calabi–Yau varieties as complete intersections in toric varieties, but we will not discuss these in this thesis.

We will also use algebraic geometry to find examples of complex 4-orbifolds which satisfy Condition 2.4.2. The class of hypersurfaces in weighted projective spaces will be our hunting ground for appropriate complex 4-orbifolds, but we will use some techniques due to the toric construction of Batyrev to understand how to resolve singularities of the hypersurfaces and to determine topological invariants.

In the next section we will review weighted projective spaces, their singularities, and hypersurfaces contained in weighted projective spaces. We will then proceed to give a short overview of toric geometry and the reflexive polytope construction of Batyrev.

2.5.1 Weighted projective spaces

This subsection is mainly a review of [34].

Definition 2.5.1. Let $a_0, \dots, a_n$ be positive integers with $\gcd(a_0, \dots, a_n) = 1$. The *weighted projective space* $\mathbb{CP}^n_{a_0,\dots,a_n}$ with weights $a_0, \dots, a_n$ is the quotient of $\mathbb{C}^{n+1} \backslash \{0\}$ by the action of $\mathbb{C}^*$ given by

$$\lambda : (z_0, \dots, z_n) \longmapsto (\lambda^{a_0} z_0, \dots, \lambda^{a_n} z_n).$$

In general $\mathbb{CP}^n_{a_0,\dots,a_n}$ will have singularities where the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is not free. The stabilizer groups of these points are finite and so we can treat $\mathbb{CP}^n_{a_0,\dots,a_n}$ as an orbifold. We can also treat $\mathbb{CP}^n_{a_0,\dots,a_n}$ as a singular algebraic variety by considering $\mathbb{CP}^n_{a_0,\dots,a_n}$ as Proj of a graded ring. This will be a useful viewpoint for us because it shows the similarities between weighted projective spaces and the usual straight projective space.

Let R be the graded ring $\mathbb{C}[z_0,\dots,z_n]$ where z_i has weight a_i. R has a direct sum decomposition $R = \bigoplus_d R_d$ into its graded pieces. Elements of R_d will be called *weighted homogeneous polynomials of degree* d but we will soon drop the term weighted and leave it as understood.

We can treat $\mathbb{CP}^n_{a_0,\dots,a_n}$ as a variety as $\mathrm{Proj}(R)$. From generalities on taking Proj of graded rings, see [32, Prop. 5.11], we have that a finitely generated graded R-module determines a coherent sheaf of $\mathcal{O}_{\mathbb{CP}^n_{a_0,\dots,a_n}}$-modules. In particular the module $R(m)$ determines a sheaf, which we will denote $\mathcal{O}(m)$ for brevity.

We should be careful when distinguishing between the two viewpoints. For example we have the following result, which holds only when considering $\mathbb{CP}^n_{a_0,\dots,a_n}$ as a variety.

Lemma 2.5.2 ([34, Cor. 5.9]). *Let* $a_0,\dots,a_n$ *be positive integers with* $\gcd(a_0,\dots,a_n) = 1$. *Let* $q = \gcd(a_1,\dots,a_n)$. *Then* $\mathbb{CP}^n_{a_0,\dots,a_n} \cong \mathbb{CP}^n_{a_0,a_1/q,\dots,a_n/q}$ *as varieties.*

Definition 2.5.3. We say $\mathbb{CP}^n_{a_0,\dots,a_n}$ is *well-formed* if

$$\gcd(a_0,\dots,a_{i-1},a_{i+1},\dots,a_n) = 1 \text{ for each } i.$$

By Lemma 2.5.2 any weighted projective space $\mathbb{CP}^n_{a_0,\dots,a_n}$ is isomorphic to a well-formed weighted projective space as a variety. The condition of being well-formed is related to the structure of the singularities of $\mathbb{CP}^n_{a_0,\dots,a_n}$. The singularities of $\mathbb{CP}^n_{a_0,\dots,a_n}$ are all cyclic quotient singularities. We say a cyclic quotient singularity $\mathbb{C}^n/\mathbb{Z}_m$ is *of type* $\frac{1}{m}(a_1,\dots,a_n)$ if $\mathbb{Z}_m$ acts on $\mathbb{C}^n$ as

$$(z_1,\dots,z_n) \xrightarrow{\xi} (\xi^{a_1}z_1,\dots,\xi^{a_n}z_n)$$

where $\xi^m = 1$.

For any subset $I = \{i_0 \ldots, i_k\} \subset \{0, \ldots, n\}$ we define $S_I = \{[z_0, \ldots, z_n] : z_j = 0, \forall j \notin I\} \subset \mathbb{CP}^n_{a_0,\ldots,a_n}$. Now suppose $\gcd(a_{i_0}, \ldots, a_{i_k}) = m \neq 1$, then a generic point $p \in S_{i_0,\ldots,i_k}$ is an orbifold point modelled on the singularity $\mathbb{C}^k \times \mathbb{C}^{n-k}/\mathbb{Z}_m$. Let $\{i_{k+1}, \ldots, i_n\} = \{0, \ldots, n\} \backslash I$ then the singularity $\mathbb{C}^{n-k}/\mathbb{Z}_m$ is of type $\frac{1}{m}(a_{i_{k+1}}, \ldots, a_{i_n})$.

Example 2.5.4. Consider the weighted projective space $\mathbb{CP}^3_{1,2,3,6}$. The singular locus consists of the union of the two curves $S_{1,3} \cup S_{2,3}$, which intersect at the singular point S_3. The singularity at a generic point of $S_{1,3}$ is modelled on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as $(z_0, z_2) \mapsto (-z_0, -z_2)$. The singularity at a generic point of $S_{1,3}$ is modelled on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_3$, where $\mathbb{Z}_3$ acts as $(z_0, z_1) \mapsto (\xi z_0, \xi^{-1} z_1)$ and $\xi^3 = 1$. Finally we have a nonisolated singular point at S_3, which is modelled on $\mathbb{C}^3/\mathbb{Z}_6$, where $\mathbb{Z}_6$ acts as $(z_0, z_1, z_2) \mapsto (\zeta z_0, \zeta^2 z_1, \zeta^3 z_2)$ and $\zeta^6 = 1$.

From the description of singularities above we see that the condition of being well-formed is equivalent to $\mathbb{CP}^n_{a_0,\ldots,a_n}$ having only singularities in complex codimension greater than 1.

Hypersurfaces in weighted projective spaces

Let $f \in R_d$ be a weighted homogenous polynomial of degree d. The zero set of f is a well defined hypersurface in $\mathbb{CP}^n_{a_0,\ldots,a_n}$, which by definition is of *degree d*. It can be shown that any hypersurface of weighted projective spaces is determined by such a weighted homogeneous polynomial [21, Th. 3.7]. This allows us to easily describe the properties of a hypersurface in terms of a defining homogeneous polynomial.

Recall that a projective variety is smooth if the affine cone is smooth away from the origin. We shall define quasismoothness in a similar way. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \mapsto \mathbb{CP}^n_{a_0,\ldots,a_n}$ be the projection.

Definition 2.5.5. Let $X \subset \mathbb{CP}^n_{a_0,\ldots,a_n}$ be an algebraic variety. We say X is *quasismooth* if $\pi^{-1}(X)$ is smooth.

An algebraic variety X is quasismooth if X only has singularities coming from the orbifold singularities of $\mathbb{CP}_{a_0,\ldots,a_n}$. We will restrict our attention to quasismooth hypersurfaces because we can understand their singularities easily in terms of those of the ambient weighted projective space. Regarding $\mathbb{CP}^n_{a_0,\ldots,a_n}$ as an orbifold, quasismoothness of X is equivalent to being a sub-orbifold, see [7, Prop. 3.5].

Note that a generic hypersurface of a fixed degree in a particular weighted projective space is not necessarily quasismooth.

Example 2.5.6. Consider the graded ring $R = \mathbb{C}[z_0, z_1, z_2]$ where the weights are $1, 2, 2$ respectively, then $\mathrm{Proj}(R) = \mathbb{CP}^2_{1,2,2}$. A generic weighted homogeneous polynomial of degree 3 is of the form $f = \lambda_1 z_0 z_1 + \lambda_2 z_0 z_2 + \lambda_3 z_0^3$. The hypersurface $X_3 = V(f)$ is not quasismooth since the affine variety defined by f is singular along the set $\{(z_0, z_1, z_2) \in \mathbb{C}^3 : \ z_0 = 0 \text{ and } \lambda_1 z_1 + \lambda_2 z_2 = 0\}$.

The conditions for the generic hypersurface of degree d to be quasismooth are described below.

Theorem 2.5.7 ([34, Th. 8.1])**.** *The generic hypersurface X_d of degree d in $\mathbb{CP}^n_{a_0,\ldots,a_n}$ is quasismooth if and only if either $a_i = d$ for some i, i.e. X_d is a linear cone, or for every nonempty subset $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$ either*

(i) there exists a monomial $z_{i_0}^{d_0} \cdots z_{i_k}^{d_k}$ of degree d; or

(ii) for $j = 0, \ldots, k$ there exist monomials $z_{i_0}^{d_{0,j}} \cdots z_{i_k}^{d_{k,j}} z_{e_j}$ of degree d, where the $e_j \notin \{i_0, \ldots, i_k\}$ are distinct.

Recall that if $X_d \subset \mathbb{CP}^n$ is a smooth hypersurface of degree d, i.e. defined by a homogeneous polynomial of degree d, then the adjunction formula gives us that $K_{X_d} = \mathcal{O}(d - n - 1)|_{X_d}$. We would like a similar result for weighted projective spaces so that we could test the triviality of the canonical sheaf easily.

Fortunately we have such a result for a large class of hypersurfaces, namely those which are quasismooth and well-formed.

Definition 2.5.8. Let $X \subset \mathbb{CP}^n_{a_0,\ldots,a_n}$ be a hypersurface. We say X is *well-formed* if $\mathbb{CP}^n_{a_0,\ldots,a_n}$ is well-formed and X does not contain a codimension 2 singular set of $\mathbb{CP}_{a_0,\ldots,a_n}$.

Iano-Fletcher gives the following criterion for well-formedness for generic hypersurfaces.

Proposition 2.5.9 ([34, Prop. 6.10])**.** *The generic hypersurface of degree d in $\mathbb{CP}^n_{a_0,\ldots,a_n}$ is well-formed if and only if*

(i) $\gcd(a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_{j-1},a_{j+1},\ldots,a_n) \mid d$ *for all i,j and*

(ii) $\gcd(a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_n) = 1$ *for all i.*

Proposition 2.5.10 ([34, Prop. 6.14])**.** *Let $X_d \subset \mathbb{CP}^n_{a_0,\ldots,a_n}$ be a well-formed and quasismooth hypersurface of degree d. The canonical sheaf of X_d is $K_{X_d} \cong \mathcal{O}(d-n-1)|_{X_d}$, in particular K_{X_d} is trivial if $d = n+1$.*

Proposition 2.5.10 gives us an easy test to ensure that the hypersurface X_d has a trivial canonical bundle. However the hypersurface will not in general be smooth but only quasismooth. Resolutions can eliminate these singularities but this process may affect the triviality of the canonical bundle. Crepant resolutions are those resolutions that do not change the canonical bundle.

Definition 2.5.11. Let X be a Gorenstein variety so that the sheaf of sections of the canonical bundle is an invertible. A resolution $\pi\colon \tilde{X} \to X$ is called *crepant* if $\pi^*(\mathcal{O}(K_X)) \cong \mathcal{O}(K_{\tilde{X}})$.

Crepant resolutions play an important part in finding examples of Calabi–Yau manifolds because it is often easier to find singular Calabi–Yau varieties.

2.5.2 Toric geometry

Toric varieties are another important source of ambient spaces for Calabi–Yau manifolds. Toric geometry appears to be quite strange to someone who has only seen projective spaces and projective varieties but they are extremely useful as a large playground for ideas. In the next section we will explain some of the basics of toric geometry. We direct the reader to [28] to a good introduction to the subject. After some basic definitions we will see how Calabi–Yau manifolds appear as hypersurfaces in toric varieties and their connection to reflexive polytopes.

A very short introduction to toric geometry

Definition 2.5.12. A *toric variety* is a normal variety, P, with an action of a torus, T, with a dense open orbit.

By the work of Cox [21] we can view a toric variety also as a (categorical) quotient

$$(\mathbb{C}^n - Z)/G$$

of the complement of an affine variety by an abelian algebraic group G.

Toric varieties may be determined by combinatorial data consisting of a fan, which we will describe. Toric geometry is useful because of this very explicit description of the variety and we can determine many properties of the variety and its subvarieties from this combinatorial data. We will attempt to give the main definitions and some examples.

Let M and N be dual lattices and let $M_{\mathbb{Q}}$ and $N_{\mathbb{Q}}$ denote $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N \otimes_{\mathbb{Z}} \mathbb{Q}$ respectively.

Definition 2.5.13. A *strongly convex rational polyhedral cone* is a convex cone $\sigma \subset N_{\mathbb{Q}}$, generated by a finite number of vectors in N, such that $\sigma \cap (-\sigma) = \emptyset$.

We will use the word cone to mean strongly convex rational polyhedral cone from now on. To any cone $\sigma \subset N_{\mathbb{Q}}$ we associate the dual cone $\sigma^\vee \subset M_{\mathbb{Q}}$ by

$$\sigma^\vee = \{m \in M_{\mathbb{Q}} | \langle m, v\rangle \geq 0 \text{ for all } v \in \sigma\}$$

where $\langle m, v\rangle$ is the pairing of dual vector spaces.

The set $\sigma^\vee \cap M$ carries a semigroup structure induced by the group structure of M. Let $\mathbb{C}[\sigma^\vee \cap M]$ denote the semigroup algebra, then this algebra is finitely generated and hence $\mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ is an affine variety. We denote this affine variety associated to the cone σ by U_σ.

Example 2.5.14. Suppose N and M are of rank n and consider the cone $\sigma = \{0\}$. The dual cone $\sigma^\vee$ is all of $M_\mathbb{Q}$. Choosing a basis for M we see that the semigroup algebra $\mathbb{C}[M]$ is isomorphic to $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ and hence the affine variety U_σ is isomorphic to the n-dimensional torus $(\mathbb{C}^*)^n$.

We say a cone $\tau \subset \sigma$ is a *face* of σ if for some $u \in \sigma^\vee \cap M$

$$\tau = \{v \in \sigma | \langle u, v\rangle = 0\}.$$

It can be shown that an inclusion of a face $\tau \subset \sigma$ induces a corresponding inclusion of affine varieties $U_\tau \subset U_\sigma$.

If two cones intersect along a face of each cone, we can use the inclusion maps to glue together the affine varieties associated to each cone. This is the basis for the definition of a fan and the construction of the toric variety associated to a fan.

Definition 2.5.15. A *fan* is a collection of cones which is closed under taking intersections and faces.

Example 2.5.16. Let Σ be the 1-dimensional fan consisting of $\{0\}$, $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}_{\leq 0}$. We have seen that $U_{\{0\}} = \mathrm{Spec}(\mathbb{C}[x_1, x_1^{-1}])$. From the definition, one can verify that $U_{\mathbb{Q}_{\geq 0}} = \mathrm{Spec}(\mathbb{C}[x_1])$ and $U_{\mathbb{Q}_{\leq 0}} = \mathrm{Spec}(\mathbb{C}[x_1^{-1}])$. The inclusion maps $U_{\{0\}} \hookrightarrow U_{\mathbb{Q}_{\geq 0}}$ and $U_{\{0\}} \hookrightarrow U_{\mathbb{Q}_{\leq 0}}$ are induced by the obvious inclusions of $\mathbb{C}[x_1]$ and $\mathbb{C}[x_1^{-1}]$ into $\mathbb{C}[x_1, x_1^{-1}]$. The resulting toric variety P_Σ is the projective line $\mathbb{CP}^1$.

The previous example can be generalized to projective spaces of any dimension. Let $e_1, \ldots, e_n$ be a basis for a lattice $\mathbb{Z}^n$ and let $e_0 = -e_1 - \cdots - e_n$. Let Σ be the fan whose top dimensional cones consist of $\mathrm{span}(e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)$ for $i = 0, \ldots, n$. Then P_Σ can be shown to be isomorphic to $\mathbb{CP}^n$.

Example 2.5.17. Weighted projective spaces are also examples of toric varieties.

Let $a_0, \ldots, a_n$ be positive integers with $\gcd(a_0, \ldots, a_n) = 1$. Let $\overline{N}$ be generated by $e_0, \ldots, e_n$ and let $N = \overline{N}/\mathbb{Z}\cdot(a_0e_0 + \cdots + a_ne_n)$. N is a lattice since $\gcd(a_0, \ldots, a_n) = 1$.

$\mathbb{CP}^n_{a_0,\dots,a_n}$ is isomorphic to the toric variety associated to the fan whose n dimensional cones are given by $\mathrm{span}(e_0,\dots,e_{i-1},e_{i+1},\dots,e_n)$ for $i=0,\dots,n$ in $N_{\mathbb{Q}}$.

Definition 2.5.18. A fan Σ' is a refinement of a fan Σ if any cone $\sigma \in \Sigma$ can be written as a union of cones in Σ'.

Refinements of fans play an important role because a refinement Σ' of Σ induces a proper birational map $\pi\colon P_{\Sigma'} \to P_{\Sigma}$. If a toric variety P_{Σ} has singularities, we can find a refinement Σ' such that $\pi\colon P_{\Sigma'} \to P_{\Sigma}$ is a resolution [28, Section 2.6].

Reflexive polytopes and Calabi–Yau varieties

Batyrev's construction of Calabi–Yau varieties as hypersurfaces in toric varieties makes use of the theory of reflexive polytopes. Let M be an n-dimensional lattice and $\Delta \subset M$ be an n-dimensional lattice polytope, i.e. a polytope with vertices in M, and suppose Δ contains the origin. We associate a toric variety to Δ by taking cones over the maximal faces of Δ as described in the following proposition:

Proposition 2.5.19. *For every k-dimensional face $\Theta \subset \Delta$ let $\check{\sigma}(\Theta) \subset M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ be the cone over Θ, $\check{\sigma}(\Theta) = \{\lambda x \in M_{\mathbb{Q}} : x \in \Theta$ and $\lambda \in \mathbb{Q}\}$, and let $\sigma(\Theta) \subset N_{\mathbb{Q}}$ be the $(n-k)$-dimensional dual cone. Then the collection of dual cones $\Sigma(\Delta) = \{\sigma(\Theta) : \Theta \subset \Delta\}$ is a fan, and hence determines a toric variety P_{Δ}.*

A polytope Δ determines not only a toric variety P_{Δ} but also a choice of ample invertible sheaf $\mathcal{O}_{\Delta}(1)$ on P_{Δ} by [6, Prop. 2.1.5].

Definition 2.5.20. Let M be a lattice and $\Delta \subset M$ a lattice polytope of the same dimension as M containing the origin. We define the *dual polytope* $\Delta^* \subset N_{\mathbb{Q}}$ as the set

$$\Delta^* = \{x \in N_{\mathbb{Q}} : \langle y, x\rangle \geq -1 \text{ for all } y \in \Delta\}.$$

We say a lattice polytope Δ is *reflexive* if Δ^* is also a lattice polytope, i.e. if the vertices of Δ^* lie in N and not just $N_{\mathbb{Q}}$.

The relevance of reflexive polytopes to Calabi–Yau orbifolds is described in the following result from [6, Th. 4.1.9].

Theorem 2.5.21. *Let Δ be an integral polytope and P_Δ the corresponding projective toric variety. The following conditions are equivalent.*

(i) the ample invertible sheaf $\mathcal{O}_\Delta(1)$ on P_Δ is anticanonical;

(ii) Δ is reflexive.

Then a generic section of the sheaf $\mathcal{O}_\Delta(1)$ determines a (possibly singular) Calabi–Yau hypersurface, by an application of the adjunction formula. Hence we see that we can associate a family of Calabi–Yau hypersurfaces to any reflexive polytope.

The fact that Δ is reflexive if and only if Δ^* is reflexive is the basis of a symmetry between the families of Calabi–Yau orbifolds, which relates in particular their Hodge numbers. This is an example of mirror symmetry.

If $X \subset P_\Delta$ is a Calabi–Yau hypersurface then a crepant resolution of the singularities of P_Δ will induce a crepant resolution of the singularities of X. Batyrev describes how to resolve the singularities of P_Δ and in the process desingularize all of the Calabi–Yau hypersurfaces.

A subdivision of the polytope Δ^* will determine a refinement of the fan of P_Δ and hence a partial resolution of P_Δ. A subdivision of Δ^* is called a *(maximal) triangulation* if every lattice point in Δ^* is the vertex of some simplex in the subdivision. A triangulation of Δ^* will determine a maximal partial crepant resolution of P_Δ and hence X. A triangulation is called *projective* if the associated resolution is. The existence of projective triangulations is guaranteed by [29, Prop. 4] and the resulting resolution will, very importantly, be crepant by [6, Th. 2.2.24]. We denote the resolved toric variety by $\tilde{P}_\Delta$.

In the case of a hypersurface of dimension 3, this maximal partial crepant resolution resolves all the singularities of the hypersurface. However in dimension 4 there may remain isolated quotient singularities which do not admit crepant resolutions. This is an issue which we will return to later on in our discussion of singularities of hypersurfaces in weighted projective spaces.

Chapter 3

Examples of compact manifolds with holonomy Spin(7)

In this chapter we will find examples of compact manifolds with holonomy Spin(7) constructed from Calabi–Yau 4-orbifolds with antiholomorphic involutions using the construction of Joyce [40]. We will search the class of well-formed quasismooth hypersurfaces in weighted projective spaces for suitable Calabi–Yau 4-orbifolds and we will consider only antiholomorphic involutions induced by the restriction of involutions on the ambient weighted projective space. The main classification result is Theorem 3.2.6. We will then determine topological invariants of the resulting Spin(7)-manifolds using methods from toric geometry. The topological invariants are listed in Table 3.4.

This work originally appeared in [20]. Note that the author missed several examples of suitable weighted projective spaces in the published version of the paper.

3.1 Antiholomorphic involutions of weighted projective spaces

The main result of this section is the classification of antiholomorphic involutions of weighted projective spaces, Theorem 3.1.3. It is shown in [84] that for standard projective space $\mathbb{CP}^n$ the number of conjugacy classes of antiholomorphic involutions is either 1 or 2 depending on whether n is odd or even respectively. If n is odd, then the only antiholomorphic involution of $\mathbb{CP}^n$ up to conjugacy is the following:

$$[z_0,\dots,z_n]\longmapsto[\overline{z}_0,\dots,\overline{z}_n], \tag{3.1}$$

which fixes $\mathbb{RP}^n\subset\mathbb{CP}^n$. If n is even we also have the involution

$$[z_0,\dots,z_n]\longmapsto[\overline{z}_1,-\overline{z}_0,\dots,\overline{z}_n,-\overline{z}_{n-1}], \tag{3.2}$$

which acts freely and hence is certainly not conjugate to the standard involution.

The case of weighted projective cases is somewhat similar. The standard involution is always well defined on any weighted projective space, but the existence of nonstandard involutions depends on the weights as well as the dimension.

3.1.1 Automorphisms of weighted projective spaces

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ defining a weighted projective space with weights $a_0,\dots,a_n$. We want to decompose $\mathbb{C}^{n+1}$ by the action of $\mathbb{C}^*$ into a direct sum of vector spaces on which $\mathbb{C}^*$ acts by the same weight.

Explicitly, we relabel the collection of weights $w_1<\cdots<w_m$ and let k_i be the number of times w_i appears in the sequence $a_0,\dots,a_n$. We decompose $\mathbb{C}^{n+1}=\bigoplus_{i\in I}W_i$ where $\mathbb{C}^*$ acts on W_i with weight w_i, $\dim(W_i)=k_i$ and $I=\{1,\dots,m\}$.

Any automorphism of $\mathbb{C}^{n+1}\setminus\{0\}$ extends to an automorphism of $\mathbb{C}^{n+1}$ for $n>0$ by Hartogs' theorem. Let $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ denote the $\mathbb{C}^*$-equivariant automorphisms of $\mathbb{C}^{n+1}$. We can also describe $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ as the centralizer of $\mathbb{C}^*$ in $\mathrm{Aut}(\mathbb{C}^{n+1})$. Any $\mathbb{C}^*$-equivariant morphism of $\mathbb{C}^{n+1}$ descends to an automorphism of $\mathbb{CP}^n_{a_0,\dots,a_n}$ and the converse can be shown to hold. This follows from the work of Cox [21, §4] on the automorphism group of simpicial toric varieties.

Cox shows that there is an exact sequence

$$0\longrightarrow\mathbb{C}^*\longrightarrow\mathrm{Aut}_{\mathbb{C}^*}\mathbb{C}^{n+1}\longrightarrow\mathrm{Aut}(\mathbb{CP}^n_{a_0,\dots,a_n})\longrightarrow\mathrm{Aut}(A_{n-1}(\mathbb{CP}^n_{a_0,\dots,a_n})),$$

where $A_{n-1}(\mathbb{CP}^n_{a_0,\dots,a_n})$ denotes the group of Weil divisors modulo rational equivlanence. The group $A_{n-1}(\mathbb{CP}^n_{a_0,\dots,a_n})$ is generated by a single element and hence the automorphism group $\mathrm{Aut}(A_{n-1}(\mathbb{CP}^n_{a_0,\dots,a_n}))$ is $\mathbb{Z}_2$. The map $\mathrm{Aut}(\mathbb{CP}^n_{a_0,\dots,a_n})\to$

$\mathrm{Aut}(A_{n-1}(\mathbb{CP}^n_{a_0,\dots,a_n}))$ is induced by direct image of divisors and therefore the image lies in the trivial subgroup. Therefore, by the exact sequence, the map $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \to \mathrm{Aut}(\mathbb{CP}^n_{a_0,\dots,a_n})$ is surjective.

A $\mathbb{C}^*$-equivariant morphism $F\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is determined by a collection of polynomials $(F_{i,j})$ where $i \in I$, $1 \leq j \leq k_i$ and $F_{i,j}$ is of degree w_i. Each polynomial $F_{i,j}$ can be decomposed

$$F_{i,j} = A_{i,j} + f_{i,j}$$

into a linear part and a non-linear, i.e. weighted homogenous quadratic and higher, part.

Example 3.1.1. Consider the graded ring $R = \mathbb{C}[z_0, z_1, z_2]$ where z_0, z_1, z_2 have weights $1, 1, 2$ respectively. $\mathbb{C}^3$ splits as $\mathbb{C}^3 = W_1 \oplus W_2$ as representations of $\mathbb{C}^*$ where $\mathbb{C}^*$ acts on W_1 with weight 1 and on W_2 with weight 2.

A $\mathbb{C}^*$-equivariant morphism $F\colon \mathbb{C}^3 \to \mathbb{C}^3$ is determined by a collection $F_{1,1}, F_{1,2}, F_2$ where $F_{1,1}, F_{1,2}$ are linear functions of z_0, z_1 and F_2 is a sum of a linear multiple of z_2 and a homogeneous quadratic polynomial in z_0, z_1.

For each $i \in I$ we define A_i to be the matrix formed from the rows $(A_{i,j})_{1\leq j\leq k_i}$. The morphism F is invertible on W_1 if A_1 is invertible since there are no polynomials with degree less than w_1. An inductive argument gives us that F is invertible if and only if each A_i is.

The map that sends a morphism F to the corresponding collection of linear maps A_i is a surjective homomorphism

$$\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \longrightarrow \prod_{i\in I} \mathrm{GL}(W_i).$$

The kernel of this homomorphism is the set of morphisms of the form $F_{i,j} = z_{i,j} + f_{i,j}$ where $(z_{i,j})_{1\leq j\leq k_i}$ are coordinates on W_i. Let us denote this kernel by H. We have an inclusion of groups $\prod_{i\in I} \mathrm{GL}(W_i) \hookrightarrow \mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ and hence the short exact sequence

$$0 \longrightarrow H \longrightarrow \mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \longrightarrow \prod_{i\in I} \mathrm{GL}(W_i) \longrightarrow 0$$

is right split and we have $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \cong H \rtimes \prod_{i\in I} \mathrm{GL}(W_i)$.

Each element of $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ determines an automorphism of $\mathbb{CP}^n_{a_0,\ldots,a_n}$, but in order to determine an automorphism uniquely we must take a quotient by the diagonal action of $\mathbb{C}^*$ on $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$. More explicitly, consider the homomorphism

$$\begin{aligned} \mathbb{C}^* &\hookrightarrow \mathrm{GL}(W_1) \times \cdots \times \mathrm{GL}(W_m) \\ \lambda &\longmapsto (\lambda^{w_1}, \lambda^{w_2}, \ldots, \lambda^{w_m}) \end{aligned}$$

which is an embedding since $\gcd(w_1, \ldots, w_m) = 1$. Then the quotient of $\mathrm{Aut}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ by this subgroup is isomorphic to $\mathrm{Aut}(\mathbb{CP}^n_{a_0,\ldots,a_n})$.

The linear structure on polynomials gives a linear structure to the group H. Note that with respect to this linear structure that the origin is the identity mapping. The group $\prod_{i\in I} \mathrm{GL}(W_i)$ acts on H via the adjoint action, which we will denote $\mathrm{Ad}_A\colon H \to H$ for $A \in \prod_{i\in I} \mathrm{GL}(W_i)$. With respect to this linear structure the adjoint action of $\prod_{i\in I} \mathrm{GL}(W_i)$ on H is linear. H decomposes as a vector space as $H = \bigoplus_{i\in I} H_i$ where H_i consists of the morphisms in H with $f_{i',j} = 0$ for $i' \neq i$.

We can describe some of the group structure on H using the order on I given by $i < i'$ if $w_i < w_{i'}$. For $f, g \in H$, let i be such that $f_{i',j} = 0$ $\forall i' < i$, then $(gf)_{i,j} = g_{i,j} + f_{i,j}$. In particular $(f^{-1})_{i,j} = -f_{i,j}$.

3.1.2 Classification of antiholomorphic involutions

Any invertible antiholomorphic map can be written as the composition of the standard antiholomorphic involution coming from complex conjugation on $\mathbb{C}^{n+1}$, which we will denote by c, followed by an automorphism of $\mathbb{CP}^n_{a_0,\ldots,a_n}$ so let us write $\widetilde{\mathrm{Aut}}(\mathbb{CP}^n_{a_0,\ldots,a_n})$ for $\mathrm{Aut}(\mathbb{CP}^n_{a_0,\ldots,a_n}) \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by c.

By the discussion above we can write any antiholomorphic involution of $\mathbb{CP}^n_{a_0,\ldots,a_n}$ as a composition $f \circ A \circ c$ where $f \in H$ and $A \in \prod_{i\in I} \mathrm{GL}(W_i)$. We can ignore H when we consider antiholomorphic involutions up to conjugation by elements of $\mathrm{Aut}(\mathbb{CP}^n_{a_0,\ldots,a_n})$ due to the following lemma.

Lemma 3.1.2. *Let* $\tau = (f, A, c) \in \widetilde{\mathrm{Aut}}(\mathbb{CP}^n_{a_0,\ldots,a_n})$ *be an antiholomorphic involution. Then* τ *is conjugate to* $(0, A, c)$ *in* $\widetilde{\mathrm{Aut}}(\mathbb{CP}^n_{a_0,\ldots,a_n})$.

Proof. Since τ is an involution we have

$$\begin{aligned}(f, A, c)(f, A, c) &= (f \operatorname{Ad}_A(\bar{f}), A\bar{A}, 1)\\ &= (1, 1, 1),\end{aligned}$$

up to the action of $\mathbb{C}^*$. Let $f = f_1 + \cdots + f_m$ be the direct sum decomposition of f corresponding to the decomposition of H under the action of $\prod_{i \in I} \mathrm{GL}(W_i)$. Let i be minimal in I, with respect the order of weights, such that $f_i \neq 0$. Since i is minimal and the action of Ad_A on H fixes each H_j we have $(f \operatorname{Ad}_A \bar{f})_i = f_i + \operatorname{Ad}_A(\bar{f}_i)$. Recalling that the origin is the identity with respect to this linear structure we have $f_i + \operatorname{Ad}_A(\bar{f}_i) = 0$.

Let $h \in H$ be such that $h_i = -\frac{1}{2} f_i$ and $h_j = 0$ for $j < i$. Then $(h\tau h^{-1})_j = 0$ for $j < i$ and

$$\begin{aligned}(h\tau h^{-1})_i &= h_i + f_i - \operatorname{Ad}_A(\bar{h}_i)\\ &= -\frac{1}{2} f_i + f_i + \frac{1}{2} \operatorname{Ad}_A(\bar{f}_i)\\ &= \frac{1}{2}(f_i + \operatorname{Ad}_A(\bar{f}_i))\\ &= 0,\end{aligned}$$

where we have used the linearity of the action of $\prod_{i \in I} \mathrm{GL}(W_i)$ on H and the discussion of the group structure of H. Now by induction on i we find that $(0, A, c)$ is in the conjugacy class of τ. □

Theorem 3.1.3. *The number of conjugacy classes of antiholomorphic involutions of* $\mathbb{CP}^n_{a_0,\ldots,a_n}$ *is either* 1 *or* 2*. Let* k_j*,* w_j *be defined in terms of* $a_0, \ldots, a_n$ *as above. Then* $\mathbb{CP}^n_{a_0,\ldots,a_n}$ *admits a non-standard antiholomorphic involution if and only if* $w_i k_i$ *is even for each* i*.*

Proof. Let $\tau \in \widetilde{\mathrm{Aut}}(\mathbb{CP}^n_{a_0,\ldots,a_n})$ be an antiholomorphic involution. By Lemma 3.1.2 we have that τ is conjugate to $(0, A, c)$ so we will assume τ is of this form. For τ to be an involution we must have $A\bar{A} = 1$ up to the action of $\mathbb{C}^*$. Decomposing A according to this action we must have

$$A_j \overline{A}_j = \lambda^{w_j} \tag{3.3}$$

for some $\lambda \in \mathbb{C}^*$. Taking trace shows that λ^{w_j} is real for each j so λ is real since the weights are relatively prime. By an action of $\mathbb{R}^* \subset \mathbb{C}^*$ we can ensure that $|\lambda| = 1$. Taking determinants of equation (3.3) then implies that $\lambda^{w_j k_j} = 1$.

We know that there are exactly two antiholomorphic involutions of $\mathbb{CP}^{k_j}$ for k_j even and one for k_j odd up to conjugation and scale. For the standard antiholomorphic involution we have $A_j\overline{A}_j = 1$ and for the non-standard involution we have $A_j\overline{A}_j = -1$.

For A_j to be non-standard we must have $\lambda^{w_j} = -1$ so w_j must be odd and k_j must be even. In this case $\lambda = -1$ and since $(-1)^{w_i k_i} = 1$ we must have $w_i k_i$ even for each i. □

The standard antiholomorphic involution on $\mathbb{CP}^n$ has a fixed point locus of (real) dimension n so the fixed point locus of the involution restricted to a generic hypersurface will not consist of isolated points. We therefore must consider only weighted projective spaces which admit non-standard involutions.

Let τ be a non-standard antiholomorphic involution and let w_j, k_j be defined as before. The fixed point locus of $\tau \circ c$ is of (complex) dimension

$$\sum_{j:w_j \in 2\mathbb{Z}} k_j - 1.$$

Since we want τ to have isolated fixed points when acting on a generic hypersurface we therefore require that $\sum_{j:w_j\in 2\mathbb{Z}} k_j = 2$.

For the case we are interested in, namely $\mathbb{CP}^5_{a_0,\ldots,a_n}$ the discussion above imposes conditions on the allowed sets of weights. In order for $\mathbb{CP}^5_{a_0,\ldots,a_n}$ to admit an antiholomorphic involution whose fixed locus has (real) dimension 1 we must have, without loss of generality, $a_0 = a_1$ and $a_2 = a_3$, all of which are odd, and a_4, a_5 both even. The action of τ on $\mathbb{CP}^5_{a_0,\ldots,a_5}$ can be given as

$$\tau : [z_0, z_1, z_2, z_3, z_4, z_5] \longmapsto [\overline{z}_1, -\overline{z}_0, \overline{z}_3, -\overline{z}_2, \overline{z}_4, \overline{z}_5]. \tag{3.4}$$

From now on we will assume τ is of this form.

3.2 Suitable ambient weighted projective spaces

Recall from Condition 2.4.2 that we require our Calabi–Yau 4-fold to have singularities of type $\frac{1}{4}(1,1,1,1)$. We also have stated previously that we must ensure that any other singularities of the 4-fold admit crepant resolutions and there is one more issue which we must be careful of: The antiholomorphic involution must lift to the resolution of any unwanted singularities of the 4-fold. We will deal with these three issues in turn in this section.

3.2.1 Desired singularities

Let X_d be a generic hypersurface of degree $d = \sum_i a_i$ in $\mathbb{CP}^5_{a_0,\ldots,a_5}$ and let τ be as in equation (3.4). Recall that we want the action of τ on X_d to fix singular points of type $\mathbb{C}^4/\mathbb{Z}_4$. The fixed point locus of τ is contained in $S_{4,5}$ so we should find hypersurfaces with singularities of the correct type in $S_{4,5}$.

Suppose the weights $a_0, \ldots, a_5$ have been chosen so that X_d is well-formed and quasismooth. The isolated singularities of type $\frac{1}{4}(1,1,1,1)$ can occur in two ways, either X_d transversely intersects the singular locus $S_{4,5}$ at a generic point and $\gcd(a_4, a_5) = 4$ or X_d contains a point S_4 or S_5 with $a_4 = 4$ or $a_5 = 4$ respectively.

For X_d to intersect a generic point of $S_{4,5}$ transversely we must have that there exist at least two monomials $z_4^{d_4} z_5^{d_5}$ of degree d. These singularities are of the type $\frac{1}{4}(1,1,1,1)$ if $a_k \equiv 1 \mod 4$ for $k \neq 4,5$ or $a_k \equiv 3 \mod 4$ for $k \neq 4,5$.

X_d contains the point S_4 if $a_4 \nmid d$ so that there does not exist a monomial $z_4^{d_4}$ of degree d. For X_d to be quasismooth we must have that $a_4 \mid d - a_j$ for some j so there exists a monomial $z_4^{d_4} z_j$ of degree d. As before in order for this singularity to be of the type $\frac{1}{4}(1,1,1,1)$ we must have $a_k \equiv 1 \mod 4$ for $k \neq 4, j$ or $a_k \equiv 3 \mod 4$ for $k \neq 4, j$. However a_4 and a_5 are both even so therefore we must have $j = 5$.

If $\gcd(a_4, a_5) = 2$ then for X_d to be quasismooth we must have that either X_d intersects $S_{4,5}$ transversely at generic points with singularities modelled on $\mathbb{C}^4/\mathbb{Z}_2$ or we have a monomial $z_4 z_5$ of degree d. We eliminate the first possibility because the singularity of type $\frac{1}{2}(1,1,1,1)$ does not admit a crepant resolution and we eliminate the second because $d = \sum_i a_i$. Hence $\gcd(a_4, a_5) = 4$ and X_d does not contain S_4 or S_5. We summarize our results in the following proposition:

Proposition 3.2.1. *Suppose the generic hypersurface of degree* $d = \sum_i a_i$ *in* $\mathbb{CP}^5_{a_0,\ldots,a_5}$ *is well-formed, quasismooth, has isolated singularities of type* $\frac{1}{4}(1,1,1,1)$*, and* $\mathbb{CP}^5_{a_0,\ldots,a_5}$ *admits an antiholomorphic involution, which fixes only these points in* X_d*, then without loss of generality the weights* $a_0, \ldots, a_5$ *satisfy*

(i) $a_0 = a_1$ *and* $a_2 = a_3$*, and*

(ii) $\gcd(a_4, a_5) = 4$*, and*

(iii) $a_i \equiv 1 \mod 4$ *for* $0 \leq i \leq 3$ *or* $a_i \equiv 3 \mod 4$ *for* $0 \leq i \leq 3$*, and*

(iv) $a_4 | d$ *and* $a_5 | d$*.*

3.2.2 Undesired singularities

Recall that the maximal partial crepant resolution may not resolve all singularities and we may have singularities of codimension 4 which do not admit any crepant resolutions. We will use methods from [22] to determine whether a given cyclic quotient singularity of codimension 4 admits a crepant resolution.

Consider a cyclic quotient singularity of the type $\frac{1}{m}(a_1, a_2, a_3, a_4)$. We can describe this as an affine toric variety. Let $N = \mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{m}(a_1, a_2, a_3, a_4)$ be a lattice and $\sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ the cone spanned by the unit vectors $e_1 = (1,0,0,0), \ldots, e_4 = (0,0,0,1)$. The affine toric variety associated to the cone σ is isomorphic to the cyclic quotient singularity of type $\frac{1}{m}(a_1, a_2, a_3, a_4)$.

The set of elements of age i, σ_i, is defined to be the convex hull in N of the elements $\{ie_1, ie_2, ie_4, ie_4\} \in N$. The following theorem, from [22, Th. 6.1], gives a necessary condition for the cyclic quotient singularity to admit a crepant resolution.

Theorem 3.2.2. *Let* $\mathbb{C}^n/G$ *be a quotient singularity, where* $G \subset \mathrm{SL}(n, \mathbb{C})$ *is a finite abelian group. If* $\mathbb{C}^n/G$ *admits a crepant resolution, then the set of elements of age 1,* σ_1*, is a minimal generating set for* σ *over* $\mathbb{Z}$*.*

Theorem 3.2.2 gives us a necessary condition for a given cyclic quotient singularity to admit a crepant resolution. This condition is sufficient for all singularities of codimension 4 where the cyclic group has order less than 39 and is sufficient in all but 10 cases for quotient singularities of codimension 4 with cyclic group of order less than 100 [22].

Example 3.2.3. Consider the isolated cyclic quotient singularity of type $\frac{1}{2}(1,1,1,1)$. The elements of age 1 are $e_1,\dots,e_4$. The element $\frac{1}{2}(1,1,1,1)\in\sigma$ cannot be written as a sum of $e_1,\dots,e_4$ with integer coefficients. Therefore the elements of age 1 do not form a generating set for σ over $\mathbb{Z}$ and hence the singularity of type $\frac{1}{2}(1,1,1,1)$ does not admit a crepant resolution.

Example 3.2.4. The generic hypersurface, X_{84}, of degree 84 in $\mathbb{CP}^5_{1,1,21,21,12,28}$ is a well-formed quasismooth Calabi–Yau hypersurface. The singularities of X_{84} consist of the curves $X_{84}\cap S_{2,3,4}$ and $X_{84}\cap S_{2,3,5}$, which intersect in the 4 points $\{p_1,\dots,p_4\} = X_{84}\cap S_{2,3}$. The singularities of X_{84} at each of the p_i are of type $\frac{1}{21}(1,1,7,12)$.

Let $N=\mathbb{Z}^4+\mathbb{Z}\cdot\frac{1}{21}(1,1,7,12)$ and $\sigma\subset N_{\mathbb{Q}}$ the cone spanned by $e_1=(1,0,0,0),\dots,e_4=(0,0,0,1)$. The elements of age 1, which are listed in Table 3.1, are a minimal generating set for σ and hence the singularity $\mathbb{C}^4/\mathbb{Z}_{21}$ admits a crepant resolution.

$(1,0,0,0)$,	$(0,1,0,0)$,	$(0,0,1,0)$,	$(0,0,0,1)$,	$\frac{1}{21}(1,1,7,12)$,
$\frac{1}{21}(2,2,14,3)$,	$\frac{1}{21}(3,3,0,15)$,	$\frac{1}{21}(4,4,7,6)$,	$\frac{1}{21}(6,6,0,9)$,	$\frac{1}{21}(7,7,7,0)$,
$\frac{1}{21}(9,9,0,3)$				

Table 3.1: Elements of age 1 in the cone σ, which defines the cyclic quotient singularity of type $\frac{1}{21}(1,1,7,12)$.

3.2.3 Equivariant resolutions of singularities

We will consider only resolutions such that the antiholomorphic involution lifts. We could require that the involution lifts to only the hypersurface but we will instead consider the case when the involution lifts to the resolution of the whole toric variety.

Let M, N be dual lattices, $\Delta \subset M$ a lattice polytope and let P_Δ bet the associated toric variety. Let τ be an antiholomorphic involution of P_Δ. In the cases we consider, the antiholomorphic involution τ can be decomposed into three parts

$$\tau = t \cdot \tau_m \cdot c, \tag{3.5}$$

where c denotes the standard antiholomorphic involution on the toric variety, τ_m is an element of the torus, which acts on P_Δ, and t is a morphism of P_Δ induced by an involution of the lattice N, which fixes the polytope Δ^* (and Δ).

The involution τ will lift to the resolution if the triangulation of Δ^* is invariant under the action of t, by which we mean that t sends a simplex in the triangulation to another simplex in the triangulation. Unfortunately we cannot always find such an invariant triangulation as the following example shows.

Example 3.2.5. Let $e_0 = (0, 0, -2)$, $e_1 = (1, 1, 1)$, $e_2 = (1, -1, 1)$, $e_3 = (-1, -1, 1)$, and $e_4 = (-1, 1, 1)$ be points in $\mathbb{R}^3$. Let N be the lattice generated by e_1, e_2, e_3 and let Δ^* denote the reflexive polytope given by the convex hull of the set $\{e_0, \dots, e_4\}$. The polytope Δ^* can be pictured as a cone over a square with vertices e_1, e_2, e_3, e_4. Let t be the lattice isomorphism defined by $t(e_1) = e_2$, $t(e_2) = e_1$ and $t(e_3) = e_4$, which is reflection in a plane in $\mathbb{R}^3$.

There are two triangulations of the polytope Δ^*. One contains the edge joining e_1 and e_3 and the other contains the edge joining e_2 and e_4. Neither of these triangulations is invariant under the action of t and we see that the map induced by t on the toric variety does not lift to any resolution. It is interesting to note that the polytope is not simplicial and hence the associated toric variety is not an orbifold. We have not found an example of a simplicial reflexive polytope with an involution t, which does not admit a t-invariant projective triangulation.

Projective triangulations of the lattice polytope, Δ^*, are in one-to-one correspondence with the faces of an associated polytope, known as the secondary polytope [30, Ch. 7]. For Δ^* to admit a t-invariant triangulation, we require that t must fix a face of the secondary polytope, or equivalently the fixed point set of t intersects the interior of a face of the secondary polytope. The secondary polytope sits inside the vector space $A_{n-1}(\tilde{P}_\Delta) \otimes \mathbb{Q}$, where $\tilde{P}_\Delta$ is any maximal partial crepant resolution of P_Δ. The Chow group $A_{n-1}(\tilde{P}_\Delta)$ is determined by the exact sequence [28]

$$0 \longrightarrow M \longrightarrow \sum_{\rho \in \Delta^* \setminus \{0\}} \mathbb{Z} \cdot e_\rho \longrightarrow A_{n-1}(\tilde{P}_\Delta) \longrightarrow 0,$$

where $m \in M \mapsto \sum_{\rho \in \Delta^* \setminus \{0\}} \langle m, \rho \rangle e_\rho$. The lattice isomorphism t acts on M and Δ^* and hence on $A_{n-1}(\tilde{P}_\Delta)$. If we tensor this sequence with $\mathbb{Q}$ we can check whether such a t-invariant maximal crepant resolution exists. In fact for all of the cases we are interested in, t fixes the whole secondary polytope and hence τ will lift to any maximal crepant resolution of P_Δ.

We are now in a position to determine whether a particular weighted projective space contains a suitable Calabi–Yau 4-orbifold as a well-formed quasismooth hypersurface.

Theorem 3.2.6. *The weights $a_0, \dots, a_5$ such that*

(i) A generic hypersurface X_d of degree $d = \sum_i a_i$ in $\mathbb{CP}^5_{a_0,\dots,a_5}$ is well-formed and quasismooth;

(ii) X_d has isolated singularities of the type $\frac{1}{4}(1,1,1,1)$;

(iii) $\mathbb{CP}^5_{a_0,\dots,a_5}$ admits an antiholomorphic involution, τ, whose fixed point locus intersects a τ-invariant X_d at the isolated singularities of type $\frac{1}{4}(1,1,1,1)$;

(iv) Any other singularities of X_d admit crepant resolutions;

are listed in Table 3.2.

Proof. Lynker, Schimmrigk and Wißkirchen [75] determined the complete set of weighted projective spaces of dimension 5 such that the generic hypersurface of degree $d = \sum_i a_i$ is quasismooth. The list of weights can be found at `http://thp.uni-bonn.de/Supplements/cy.html`.

Propositions 2.5.9 and 3.2.1 translate conditions (i)–(iii) into numerical conditions on the weights $a_0, \dots, a_5$. We use a computer programme to search the list of 1,100,055 sets of weights given by Lynker to get a list of 34 sets of weights, for which conditions (i)–(iii) apply.

Then we use Theorem 3.2.2 to test whether any undesired singularities of the generic hypersurface admit crepant resolutions. This test eliminates the weights which are not listed in Table 3.2. □

1, 1, 1, 1, 4, 4	1, 1, 1, 1, 4, 8	1, 1, 1, 1, 8, 12	1, 1, 5, 5, 8, 20
1, 1, 9, 9, 4, 4	1, 1, 13, 13, 4, 8	1, 1, 21, 21, 4, 16	1, 1, 21, 21, 12, 28
1, 1, 37, 37, 8, 28	1, 1, 53, 53, 20, 32	1, 1, 69, 69, 16, 52	3, 3, 3, 3, 4, 8
3, 3, 15, 15, 4, 8	3, 3, 19, 19, 4, 12	3, 3, 23, 23, 8, 12	3, 3, 31, 31, 4, 24
3, 3, 31, 31, 12, 16	3, 3, 47, 47, 8, 36	3, 3, 55, 55, 24, 28	3, 3, 59, 59, 20, 36
3, 3, 79, 79, 16, 60	5, 5, 13, 13, 4, 4	5, 5, 25, 25, 4, 16	7, 7, 11, 11, 8, 44
21, 21, 49, 49, 4, 24			

Table 3.2: The admissible weights of the ambient weighted projective spaces of Calabi–Yau 4-orbifolds. The weighted projective spaces with weights listed in the first row appear as ambient spaces for Calabi–Yau 4-orbifolds in [39].

3.3 Topological invariants of constructed Spin(7)-manifolds

In this section we will determine the Betti numbers of the Spin(7)-manifolds, which result from generic hypersurfaces in weighted projective spaces with weights listed in Table 3.2, using toric geometry. Note that we have corrected some errors in [20], which were due to mistakes in calculations.

For the moment let us assume that we can determine the Hodge numbers of the Calabi–Yau 4-orbifold and understand the action of the antiholomorphic involution on its cohomology. We can determine the Betti numbers of the resulting Spin(7)-manifold as follows.

Proposition 3.3.1. *Suppose* (X, τ) *satisfies Condition 2.4.2 and let* $M = X/\langle\tau\rangle$ *and* $\tilde{M}$ *be the resulting* Spin(7)*-manifold. Suppose* M *has* k *singularities modelled on*

$\mathbb{R}^8/\Gamma$ *as in Section 2.4.1. Then the Betti numbers of* $\tilde{M}$ *are*

$$\begin{aligned} b^2(\tilde{M}) &= b^2(M), & b^4_+(\tilde{M}) &= \frac{1}{2}(h^{2,2}(X)+k) - b^2(M) + 1, \\ b^3(\tilde{M}) &= \frac{1}{2}b^3(X), & b^4_-(\tilde{M}) &= h^{3,1}(X) + b^2(X) - b^2(M) + k - 1. \end{aligned}$$

Proof. Let $h^{p,p}_\tau(X)$ be the dimension of the τ-invariant part of $H^{p,p}(X)$. Noting that in all of the cases we will discuss we have $h^{2,0}(X) = h^{3,0}(X) = 0$ and $h^{4,0}(X) = 1$ since X has $\mathrm{Hol}(X) = SU(4)$, the Betti numbers of M can then be expressed as

$$\begin{aligned} b^2(M) &= h^{1,1}_\tau(X), & b^4_+(M) &= h^{2,2}_\tau(X) - h^{1,1}(X) + h^{1,1}_\tau(X) + 2, \\ b^3(M) &= h^{2,1}(X), & b^4_-(M) &= h^{3,1}(X) + h^{1,1}(X) - h^{1,1}_\tau(X) - 1. \end{aligned}$$

Applying the Lefschetz fixed point theorem we find that

$$k = 2 + 4h^{1,1}_\tau(X) - 2h^{1,1}(X) + 2h^{2,2}_\tau(X) - h^{2,2}(X),$$

which we use to eliminate $h^{2,2}_\tau(X)$ from the expressions for the Betti numbers of M.

The ALE Spin(7) manifolds that we use to resolve the quotient singularities of M have Betti numbers $b^1 = b^2 = b^3 = b^4_+ = 0$ and $b^4_- = 1$ hence, by an application of Mayer-Vietoris, the Betti numbers of $\tilde{M}$ satisfy

$$\begin{gathered} b^j(\tilde{M}) = b^j(M) \text{ for } j = 1,\, 2,\, 3 \\ b^4_+(\tilde{M}) = b^4_+(M) \text{ and } b^4_-(\tilde{M}) = b^4_-(M) + k. \end{gathered}$$

Combining these facts gives us the result. □

From Proposition 3.3.1 we see that to determine the Betti numbers of $\tilde{M}$ it suffices to know the Hodge numbers of the orbifold X and to understand how τ acts on $H^{1,1}(X)$. We will use techniques from toric geometry both to determine the Hodge numbers and to understand the action of τ.

3.3.1 Hodge numbers of Calabi–Yau hypersurfaces

The Hodge numbers of well-formed quasismooth Calabi–Yau hypersurfaces in weighted projective spaces, after resolving the singularities, were calculated by Lynker et.

al [75] and Kreuzer verified that they match the Hodge numbers given by Batyrev and Dais [6, 8]. The list of weights and corresponding Hodge numbers can be found at `http://hep.itp.tuwien.ac.at/~kreuzer/CY` or at `http://thp.uni-bonn.de/Supplements/cy.html`. To understand how the antiholomorphic involution acts on $H^{1,1}$ we will describe how Batyrev and Dais arrive at their formulae for $h^{1,1}$.

Let $X \subset P_\Delta$ denote a Calabi–Yau hypersurface in the toric variety determined by the reflexive polytope Δ and τ an antiholomorphic involution of P_Δ. Let $\tilde{P}_\Delta$ denote the maximal crepant resolutions of P_Δ and $\tilde{X}$ the induced resolution of X. Note that in the cases we will consider $\tilde{X}$ will be smooth and does not satisfy Condition 2.4.2 since it has no singularities but we may choose not to resolve some of the singularities so as to leave fixed points of the involution τ.

The basic idea is that $h^{1,1}(\tilde{X})$ counts the components of the intersection of the resolution with the union of all irreducible toric divisors in $\tilde{P}_\Delta$. We denote the torus in $\tilde{P}_\Delta$ by T and let Y denote the intersection of $\tilde{X}$ with the union of all irreducible T-invariant divisors in $\hat{P}_\Delta$. We have a short exact sequence of cohomology groups

$$0 \longrightarrow H^6_c(\tilde{X}) \longrightarrow H^6_c(Y) \longrightarrow H^7_c(\tilde{X} \setminus Y) \longrightarrow 0,$$

upon which τ acts. By Poincaré duality we have that $b^2(\tilde{X}) = \dim(H^6_c(\tilde{X}))$ and hence $b^2_\tau(\tilde{X})$ is the difference of the dimension of the τ-invariant parts of $H^6_c(Y)$ and $H^7_c(\tilde{X} \setminus Y)$. Using a Lefchetz-type theorem by Danilov and Khovanskiĭ [23] for affine hypersurfaces in algebraic tori we have that $H^7_c(\tilde{X} \setminus Y) \cong H^9_c(T)$ is an isomorphism. Given the description of τ as in equation (3.4), it is easy to check that the dimension of the τ-invariant part of $H^9_c(T)$ is 3.

The irreducible T-invariant divisors in $\tilde{P}_\Delta$ are indexed by the set $\Delta^* \setminus \{0\}$. Let $\rho \in \Delta^* \setminus \{0\}$ and D_ρ be the corresponding T-invariant divisor. If ρ is contained in the interior of a face of codimension ≥ 3, the intersection $D_\rho \cap \tilde{X}$ is irreducible, while if ρ is contained in the interior of a face of codimension 2, D_ρ intersects $\tilde{X}$ in multiple components, the number of which is determined by the polytope.

Let t be defined in terms of τ as in equation (3.5). If $D_\rho \cap \tilde{X}$ is irreducible, the action of τ is determined by whether t fixes $\rho \in \Delta^* \setminus \{0\}$ or not. If τ fixes ρ, ρ does not contribute to $b^2_\tau(\tilde{X})$ and a pair ρ_1, ρ_2 of T-invariant divisors, which are swapped by τ, contribute 1 to $b^2_\tau(\tilde{X})$.

If $D_\rho \cap \tilde{X}$ has multiple components then the contribution to $b^2_\tau(\tilde{X})$ also depends on the multiplicative part of the involution. If $D_\rho \cap \tilde{X}$ is the result of blowing up $2n$ singularities, which are not fixed by τ then τ acts freely on $D_\rho \cap \tilde{X}$ and contributes n to $b^2_\tau(\tilde{X})$. If $D_\rho \cap \tilde{X}$ is the result of blowing up m singularities of the form $\frac{1}{4}(1,1,1,1)$, by deforming X, we can arrange so that τ fixes $m - 2k$ points in X and swaps $2k$ in pairs for $0 \leq 2k < m$.

Example 3.3.2. Consider $Y_{12} \subset \mathbb{CP}^5_{1,1,1,1,4,4}$ to be the hypersurface defined by the weighted homogeneous polynomial $f = z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} + (z_4 - z_5)(z_4^2 + \lambda z_5^2)$. If $\lambda = 1$ then τ has a single fixed point in Y_{12} whereas if $\lambda = -1$ then τ fixes three points.

We have used the software PALP [61] to find the toric divisors and to determine the toric divisors fixed by τ.

Example 3.3.3. Let M be a lattice and consider the reflexive polytope Δ with weights 1, 1, 9, 9, 4, 4. Let $N = M^*$ and Δ^* the dual polytope of Δ. The points of $\Delta^* \setminus \{0\}$ correspond to toric divisors in P_Δ. There are exactly 11 of these, which are listed in Table 3.3 with respect to a particular basis of N. The antiholomorphic involution swaps the elements in the first row in pairs and leaves the other 7 invariant.

$(1, 0, 0, 0, 0)$,	$(0, 1, 0, 0, 0)$,	$(0, 0, 1, 0, 0)$,	$(-1,-9,-9,-4,-4)$
$(0, 0, 0, 1, 0)$,	$(0, 0, 0, 0, 1)$,	$(0,-7,-7,-3,-3)$,	$(0,-1,-1, 0, 0)$
$(0,-5,-5,-2,-2)$,	$(0,-3,-3,-1,-1)$,	$(0,-2,-2,-1,-1)$	

Table 3.3: T-invariant divisors in a maximal partial crepant resolution of the reflexive polytope with weights $1,1,9,9,4,4$.

The divisor $(0,-2,-2,-1,-1)$ is contained in the interior of the face with vertices $(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (-1,-9,-9,-4,-4)$ of codimension 2, which corresponds to the singularities of our Calabi–Yau hypersurface of type $\frac{1}{4}(1,1,1,1)$.

A generic Calabi–Yau hypersurface in $\tilde{P}_\Delta$ intersects the divisor with 7 irreducible components.

3.3.2 Betti numbers of constructed Spin(7)-manifolds

Using Proposition 3.3.1 and the results of the previous section to determine the Hodge numbers of the Calabi–Yau 4-orbifolds we can determine the Betti numbers of the Spin(7)-manifolds, which are constructed from hypersurfaces in weighted projective spaces.

Table 3.4 lists the weights of the weighted projective spaces and the Betti numbers of the Spin(7)-manifolds constructed from well-formed quasismooth hypersurfaces in those weighted projective spaces. We include the examples already given in [41, Ch. 15] for the sake of completeness, which appear as the first four rows.

The sets of Betti numbers realized by manifolds constructed in this thesis are all distinct from those of compact manifolds with holonomy Spin(7) already known. Also it should be noted that the example with $b^4 = 15118$ and $b^4_- = 5031$ has the largest known value of b^4 or b^4_- for a compact manifold with holonomy Spin(7).

The mirror symmetry of Calabi–Yau manifolds arising from Batyrev's construction is due to the fact that if Δ is a reflexive polytope, then Δ^* is also reflexive. The Calabi–Yau hypersurfaces determined by Δ and Δ^* are (conjecturally) related by mirror symmetry. In particular, Batyrev verifies that their Hodge numbers satisfy the expected relations. If Δ admits an involution then this involution will also act on Δ^* and so P_{Δ^*} will also admit a non-standard antiholomorphic involution. One might hope that one could construct Spin(7)-manifolds from the Calabi–Yau hypersurfaces in P_{Δ^*} and use the relation between hypersurfaces in P_Δ and P_{Δ^*} to deduce a 'mirror symmetry' between the Spin(7)-manifolds on either side. However the Calabi–Yau hypersurfaces in P_{Δ^*} will not, in general, have the type of singularities required by Condition 2.4.2 and we would not be able to desingularize the quotients using methods known to the author.

a_0	a_1	a_2	a_3	a_4	a_5	b^2	b^3	b^4_+	b^4_-
1	1	1	1	4	4	$0 \leq k \leq 1$	0	$1639 - k$	$807 - k$
1	1	1	1	4	8	0	0	3175	1575
1	1	1	1	8	12	0	0	7784	3879
1	1	5	5	8	20	0	6	2493	1237
1	1	9	9	4	4	$0 \leq k \leq 3$	0	$1415 - k$	$695 - k$
1	1	13	13	4	8	$0 \leq k \leq 2$	0	$1991 - k$	$983 - k$
1	1	21	21	4	16	$0 \leq k \leq 1$	0	$3927 - k$	$1951 - k$
1	1	21	21	12	28	6	0	2977	1473
1	1	37	37	8	28	0	0	5911	2943
1	1	53	53	20	32	0	0	6055	3015
1	1	69	69	16	52	0	0	10087	5031
3	3	3	3	4	8	$0 \leq k \leq 1$	0	$615 - k$	$295 - k$
3	3	15	15	4	8	$0 \leq k \leq 2$	0	$503 - k$	$239 - k$
3	3	19	19	4	12	$0 \leq k \leq 2$	36	$523 - k$	$267 - k$
3	3	23	23	8	12	$0 \leq k \leq 1$	55	$430 - k$	$230 - k$
3	3	31	31	4	24	$0 \leq k \leq 1$	45	$1012 - k$	$516 - k$
3	3	31	31	12	16	0	105	464	272
3	3	47	47	8	36	0	69	1100	572
3	3	55	55	24	28	0	162	681	409
3	3	59	59	20	36	0	116	827	459
3	3	79	79	16	60	0	117	1484	788
5	5	13	13	4	4	$0 \leq k \leq 5$	0	$295 - k$	$135 - k$
5	5	25	25	4	16	$0 \leq k \leq 2$	0	$487 - k$	$231 - k$
7	7	11	11	8	44	0	15	262	126
21	21	49	49	4	24	$8 \leq k \leq 11$	0	$263 - k$	$119 - k$

Table 3.4: The weights of the ambient weighted projective spaces of the Calabi–Yau 4-folds and Betti numbers of the resulting Spin(7)-manifolds.

Chapter 4

Cayley submanifolds

In this chapter we will change our focus to the subject of calibrated geometry, in particular Cayley submanifolds in Spin(7)-manifolds. Cayley submanifolds are the natural, in some sense, submanifolds to study in the realm of Spin(7)-manifolds.

We will introduce the notion of a calibration and motivate the study of Cayley submanifolds. We will then collect some useful results which we will need when discussing the deformation theory of Cayley submanifolds.

In the following chapters we will study the deformation theory of compact Cayley submanifolds of manifolds with Spin(7)-structure. The deformation theory will allow us to study the set of compact Cayley submanifolds of a given manifold with Spin(7)-structure. We will also investigate the problem of varying both the Spin(7)-structure and the Cayley submanifold simultaneously. Using these results, we will find examples of compact Cayley submanifolds in the compact Spin(7)-manifolds we constructed in the first half of this thesis.

4.1 Introduction to calibrated geometry

Calibrated geometry is motivated by the study of minimal submanifolds in Riemannian geometry. Recall that a submanifold is minimal if it is a stationary point of the volume functional on submanifolds. The term ‘minimal’ is a bit misleading since a minimal submanifold is a stationary point and not necessarily a local minimizer. The condition for a given submanifold to be minimal is a second-order p.d.e. on the inclusion map. One motivation for the theory of calibrated submanifolds is that the condition for a submanifold to be calibrated is first-order and every calibrated submanifold is minimal. In this way we have a first-order condition, namely the tangent space is calibrated at every point, that implies a second-order p.d.e., that the mean curvature vanishes. Harvey and Lawson introduced the notion of a calibration and gave many interesting examples of calibrations in their seminal paper [33]. We refer the reader to Joyce [49] for a survey of calibrated geometry and the relationship between calibrations and manifolds with special holonomy.

Definition 4.1.1. A closed k-form ϕ is a *calibration* on a Riemannian manifold (M, g) if for all $p \in M$ and for all $V \subset T_pM$ an oriented k-plane at p we have

$$\phi|_V \leq \mathrm{vol}_V,$$

where vol_V is the volume form on V induced by g.

Definition 4.1.2. Let (M, g) be a Riemannian manifold and ϕ a calibration on M. An oriented submanifold $N \subset M$ is called *ϕ-calibrated* or simply *calibrated* if $\phi|_N = \mathrm{vol}_N$.

We will now see the (well-known) proof that calibrated submanifolds are volume-minimizing in their homology class, and hence are minimal. Let (M, g) be a Riemannian manifold and ϕ a calibration on M. Let N and N' be k-dimensional compact submanifolds such that $[N] = [N']$ in $H_k(M, \mathbb{Z})$ and suppose N is ϕ-calibrated. Then the following inequality shows that N is minimal:

$$\mathrm{vol}(N) = \int_N \mathrm{vol}_N = \int_N \phi = \int_{N'} \phi \leq \int_{N'} \mathrm{vol}_{N'} = \mathrm{vol}(N').$$

By considering compactly supported deformations of N, we can show that N is minimal when N is a non-compact ϕ-calibrated submanifold.

Since a calibrated submanifold is minimal, we can consider deforming a calibrated submanifold as a minimal submanifold. In fact any deformation of a calibrated submanifold as a minimal submanifold is again calibrated as the following result shows. In this sense, the set of calibrated submanifolds is a clopen subset of the space of minimal submanifolds.

Proposition 4.1.3. *Let* (M, g) *be a Riemannian manifold and* ϕ *a calibration on* M*. Suppose* $N_t \hookrightarrow M$ *for* $t \in I$ *is a family of compact minimal submanifolds of* M*, where* $I \subset \mathbb{R}$ *is an interval containing 0, and suppose* N_0 *is* ϕ*-calibrated, then* N_t *is* ϕ*-calibrated for* $t \in I$*.*

Proof. Since N_t is a family of minimal submanifolds, the volume of N_t is constant in the family. Therefore

$$\mathrm{vol}(N_t) = \mathrm{vol}(N_0) = \int_{N_0} \mathrm{vol}_{N_0} = \int_{N_0} \phi = \int_{N_t} \phi$$

and hence N_t is ϕ-calibrated. □

Harvey and Lawson motivated the study of Spin(7)-manifolds because they (defined and) showed the Cayley form Ω_0 is a calibration and hence the form Ω is a calibration on a Spin(7)-manifold (M, Ω, g).

Definition 4.1.4. A *Cayley submanifold* of a Spin(7)-manifold, (M, Ω, g), is an Ω-calibrated submanifold of M.

We will now study the linearized problem of Cayley submanifolds as an essential prerequisite for the general non-linear problem. That is we will study the set of Ω_0-calibrated oriented 4-planes in $\mathbb{R}^8$. Let $\widetilde{\mathrm{Gr}}(m, n)$ denote the Grassmannian of oriented m-planes in $\mathbb{R}^n$.

Definition 4.1.5. We define the *Grassmannian of Cayley planes* as the set

$$\mathrm{Cay} = \{V \in \widetilde{\mathrm{Gr}}(4, 8) : \Omega_0|_V = \mathrm{vol}_V\}$$

of oriented 4-planes which are calibrated by Ω_0. We define the *Grassmannian of anti-Cayley planes*, denoted $\overline{\mathrm{Cay}}$ as the set of oriented 4-planes, which are Cayley with the opposite orientation, or equivalently those which are calibrated by $-\Omega_0$.

Harvey and Lawson [33] show that the action of Spin(7) is transitive on $\widetilde{\mathrm{Gr}}(m, 8)$ for $m \leq 3$ and on Cay but is not on $\widetilde{\mathrm{Gr}}(4, 8)$. Berndt and Tamaru [11, p.3436] show that the action of Spin(7) on $\widetilde{\mathrm{Gr}}(4, 8)$ is of cohomogeneity one with singular orbits Cay and $\overline{\mathrm{Cay}}$.

Example 4.1.6. If we identify $\mathbb{R}^8$ with $\mathbb{C}^4$ by $z_j = x_{2j-1} + ix_{2j}$ then the plane

$$V_\phi = \{(e^{i\phi/4}y_1, e^{i\phi/4}y_2, e^{i\phi/4}y_3, e^{i\phi/4}y_4) : y_j \in \mathbb{R}\}$$

is a 4-plane such that $\Omega_0|_{V_\phi} = \cos(\phi)\mathrm{vol}_{V_\phi}$ and any 4-plane is conjugate to V_ϕ for some $\phi \in [0, \pi]$ under the action of Spin(7). Note that V_ϕ is Cayley for $\phi = 0$ and anti-Cayley for $\phi = \pi$. We will use this example throughout this section to simplify the proofs.

The vector space $\Lambda^2(\mathbb{R}^8)^* = \Lambda^2_7 \oplus \Lambda^2_{21}$ splits as a direct sum of two representations of Spin(7) [86, Prop. 12.5]. Under the identification of $\Lambda^2(\mathbb{R}^8)^*$ with the skew-symmetric endomorphisms of $\mathbb{R}^8$ using the metric, $S^6 \subset \Lambda^2_7$ is identified as the complex structures on $\mathbb{R}^8$ which are compatible with Ω_0.

Harvey and Lawson [33, Th. 1.28] show that there exists a 4-form with values in Λ^2_7, which we denote τ_0, such that for $v_1, \ldots, v_4 \in \mathbb{R}^8$ we have the following

$$|\Omega_0(v_1, \ldots, v_4)|^2 + |\tau_0(v_1, \ldots, v_4)|^2 = |v_1 \wedge \cdots \wedge v_4|^2.$$

It follows that a 4-plane $V \in \widetilde{\mathrm{Gr}}(4, 8)$ is in $\mathrm{Cay} \cup \overline{\mathrm{Cay}}$ if and only if $\tau_0|_V = 0$ and V is in Cay if and only if $\tau_0|_V = 0$ and $\Omega_0|_V > 0$. The 4-form τ_0 defines a section of the rank 7 vector bundle $\det \otimes \Lambda^2_7$ over $\widetilde{\mathrm{Gr}}(4, 8)$ whose zero locus is $\mathrm{Cay} \cup \overline{\mathrm{Cay}}$. The Cayley Grassmannian is of codimension 4 in $\widetilde{\mathrm{Gr}}(4, 8)$ and hence the section τ_0 is not transverse to the zero section.

Harvey and Lawson [33, Th. 1.38] show that the isotropy group of a Cayley plane in Spin(7) is the group $(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$. We will label the factors of Sp(1) by $+$, $-$ and $\circ$. Let $V \subset \mathbb{R}^8$ be a Cayley plane and let us identify $V \cong \mathbb{H}$ and $\mathbb{R}^8 = V \oplus V^\perp$ with $\mathbb{H} \oplus \mathbb{H}$. Identifying Sp(1) with the group of unit quaternions, the action of an element $g = (q_+, q_-, q_\circ) \in H$ on $V \oplus V^\perp$ is given by

$$g(v, n) = (q_+ v \bar{q}_-, -kq_+ kn\bar{q}_\circ).$$

he introduction of the conjugation by k is an artefact of our choice of convention for the Cayley form.

The irreducible representation Λ^2_7 of Spin(7) splits as a sum $\Lambda^2_7 = \Lambda^2_+ \oplus E$ of representations of H, where Λ^2_+ is the adjoint representation of $\mathrm{Sp}(1)_+$, or equivalently the self-dual 2-forms, and E is the 4-dimensional representation of H given by

$$g\colon e \longmapsto q_- e \bar{q}_0,$$

where we identify E with $\mathbb{H}$.

Let $V \in \mathrm{Cay}$, the 4-form τ_0 induces an exact sequence

$$0 \longrightarrow T_V \,\mathrm{Cay} \hookrightarrow T_V \widetilde{\mathrm{Gr}}(4,8) \xrightarrow{D_V \tau_0} \Lambda^2_7 \otimes \Lambda^4 V^*$$

of vector spaces. The tangent space $T_V \widetilde{\mathrm{Gr}}(4,8)$ can be identified with $\mathrm{Hom}(V, V^\perp) \cong E \oplus \Lambda^2_+ \otimes E$ as H-representations. Since $\Lambda^2_+ \otimes E$ is an irreducible 12 dimensional representation of H, this allows us to identify $T_V \,\mathrm{Cay} \cong \Lambda^2_+ \otimes E$ and the cokernel of $D_V \tau_0$, i.e. the normal space to Cay in $\widetilde{\mathrm{Gr}}(4,8)$ at V, with $E \otimes \Lambda^4 V^*$.

It follows that if $W \in \widetilde{\mathrm{Gr}}(4,8)$ is sufficiently close to V then $W \in \mathrm{Cay}$ if and only if $\mathrm{proj}_E \circ \phi \circ \tau_0|_W = 0$, where ϕ is a local trivialization of the bundle det and $\mathrm{proj}_E\colon \Lambda^2_7 \to E$ is the projection.

Definition 4.1.7. Let (M, Ω, g) be an 8-manifold with Spin(7)-structure and $f\colon N \to M$ a Cayley submanifold. We define the *obstruction bundle of* $f\colon N \to M$, denoted E, to be the rank 4 vector bundle given by kernel of the restriction map

$$f^* \Lambda^2_7 \longmapsto \Lambda^2_+.$$

Note that we could define the obstruction bundle for submanifolds, whose tangent planes are not Cayley but only not anti-Cayley, that is $\Omega|_N > -\mathrm{vol}_N$.

4.2 Adapted frame bundles

Recall that a manifold (M, Ω, g) with Spin(7)-structure determines a principal subbundle of the frame bundle with structure group Spin(7), denoted $F_{\mathrm{Spin}(7)}M$ and, if $\mathrm{d}\Omega = 0$, a torsion-free connection on $F_{\mathrm{Spin}(7)}M$. A point in $F_{\mathrm{Spin}(7)}M$ is a point $p \in M$ and a choice of frame $e_1(p), \ldots, e_8(p)$ of T_pM such that Ω_p is in the standard form with respect to this frame.

Definition 4.2.1. Let (M, Ω, g) be a manifold with Spin(7)-structure and let $f \colon N \to M$ be a Cayley submanifold. We define the *adapted frame bundle of* $f \colon N \to M$, which we will denote $F_H^M N$, to be the subbundle of $f^* F_{\mathrm{Spin}(7)} M$ defined by

$$F_H^M N = \{(p, (e_1, \ldots, e_8)) \in f^* F_{\mathrm{Spin}(7)} M : e_1, \ldots, e_4 \in D_p f(T_p N)\}.$$

The bundle $F_H^M N$ is a principal H-bundle, where $H \cong (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$ is the isotropy group of a Cayley plane in Spin(7).

A connection on $F_{\mathrm{Spin}(7)} M$ induces a connection on $f^* F_{\mathrm{Spin}(7)} M$ and hence $F_H^M N$ by Proposition 4.2.2. This connection determines connections on all vector bundles associated to representations of H, for example TN, ν and the obstruction bundle E.

Proposition 4.2.2 ([53, Prop. II.6.4]). *Let M be a manifold, P_G a principal G-bundle on M and P_H a subbundle of P_G with fibre H and H a Lie subgroup of G. Suppose the Lie algebra of G admits an Ad_H-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.*

Then if θ is a connection one-form on P_G, the $\mathfrak{h}$-component of the restriction of θ to P_H is a connection one-form on P_H.

Lemma 4.2.3. *Let M be a manifold, G a Lie group, H_1, H_2, H_3 Lie subgroups of G, such that $H_3 = H_1 \cap H_2$, and let $\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ be their Lie algebras, respectively. Let*

$$\begin{tikzcd}
P_{H_1} \arrow[r] & P_G \\
P_{H_3} \arrow[u] \arrow[r] & P_{H_2} \arrow[u]
\end{tikzcd}$$

be a commutative diagram of inclusions of principal subbundles.

Suppose $\mathfrak{g}$ admits an Ad_{H_1}-invariant splitting $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{m}$ and suppose this splitting induces an ad_{H_3}-invariant splitting $\mathfrak{h}_2 = (\mathfrak{h}_1 \cap \mathfrak{h}_2) \oplus (\mathfrak{m} \cap \mathfrak{h}_2) = \mathfrak{h}_3 \oplus (\mathfrak{m} \cap \mathfrak{h}_2)$. Suppose θ_G is a connection one-form on P_G, which reduces to a connection on P_{H_2}, i.e. $\theta_G|_{P_{H_2}}$ has values in $\mathfrak{h}_2$.

Then the connection θ_{H_1} on P_{H_1}, induced by θ_G, reduces to a connection on P_{H_3} and is equal to the connection induced by $\theta_G|_{P_{H_2}}$ on P_{H_3}.

Proof. Let $X \in T_pP_{H_3}$. Then we have

$$\begin{aligned}\theta_{H_1}(X) &= \operatorname{proj}_{\mathfrak{h}_1}\theta_G(X)\\ &= \operatorname{proj}_{\mathfrak{h}_1}\theta_G|_{P_{H_2}}(X)\\ &= \operatorname{proj}_{\mathfrak{h}_3}\theta_G|_{P_{H_2}}(X).\end{aligned}$$

□

Corollary 4.2.4. *Let $f\colon N \to M$ be a Cayley submanifold of a* Spin(7)*-manifold. Then the connection on $F^M_H N$ induces the Levi-Civita connection on TN and the normal connection on ν.*

Proof. Let $V \subset \mathbb{R}^8$ be a Cayley plane. The stabilizer group of V in SO(8) is SO(4) × SO(4) and in Spin(7) is H, so, regarding Spin(7) as a subgroup of SO(8) and with respect to this embedding of SO(4) × SO(4), we have $H = (\mathrm{SO}(4)\times\mathrm{SO}(4))\cap\mathrm{Spin}(7)$.

Take $G = \mathrm{SO}(8)$, $H_1 = \mathrm{SO}(4)\times\mathrm{SO}(4)$, $H_2 = \mathrm{Spin}(7)$, $H_3 = H$ and $P_G = f^*F_{\mathrm{SO}(8)}M$, $P_{H_1} = F^M_{\mathrm{SO}(4)\times\mathrm{SO}(4)}N$, $P_{H_2} = f^*F_{\mathrm{Spin}(7)}M$, $P_{H_3} = F^M_H N$ in Lemma 4.2.3.

The $\mathrm{Ad}_{\mathrm{SO}(4)\times\mathrm{SO}(4)}$-invariant splitting

$$\begin{aligned}\mathfrak{so}(8) &= (\mathfrak{so}(4)\oplus\mathfrak{so}(4))\oplus(\mathfrak{so}(4)\oplus\mathfrak{so}(4))^\perp\\ &= \{x\in\mathfrak{so}(8) : x(V)\subset V\}\oplus\{x\in\mathfrak{so}(8) : x(V)\subset V^\perp \text{ and } x(V^\perp)\subset V\}\end{aligned}$$

decomposes further as Ad_H-representations into

$$\mathfrak{so}(8) = (\mathfrak{h}\oplus\Lambda_+)\oplus(E\oplus E\otimes\Lambda_+),$$

where $\mathfrak{h}$ is the adjoint representation and Λ_+ and E are as in §4.1. The Lie subalgebra $\mathfrak{spin}(7)$ decomposes as Ad_H-representations into

$$\mathfrak{spin}(7) = \mathfrak{h}\oplus E\otimes\Lambda_+.$$

Since $E\otimes\Lambda_+$ is an irreducible Ad_H-representation, which appears with multiplicity one in $\mathfrak{so}(8)$, we must have $\mathfrak{spin}(7)\cap(\mathfrak{so}(4)\oplus\mathfrak{so}(4))^\perp = \mathfrak{h}^\perp = (\mathfrak{spin}(7)\cap(\mathfrak{so}(4)\oplus\mathfrak{so}(4)))^\perp$. Therefore the $\mathrm{Ad}_{\mathrm{SO}(4)\times\mathrm{SO}(4)}$-invariant splitting of $\mathfrak{so}(8)$ induces an Ad_H-invariant splitting of $\mathfrak{spin}(7)$.

Then, by Lemma 4.2.3, the connection on $F_H^M N$ induces the same connection on $F^M_{\mathrm{SO}(4)\times\mathrm{SO}(4)}N$ as that induced by $f^* F_{\mathrm{SO}(8)}M$. Finally, the connection on $F^M_{\mathrm{SO}(4)\times\mathrm{SO}(4)}N$ induced by that on $f^* F_{\mathrm{SO}(8)}M$ is the Levi-Civita connection together with the normal connection, see [54, p. 4]. □

4.3 Differential operators on manifolds

In our study of Cayley submanifolds we will relate the calibrated condition to a non-linear elliptic partial differential equation. In this section we state some general results on non-linear differential operators on compact manifolds, which will be useful later. We will give more specific references as necessary but our main reference for this material is Palais [83]. Some of our definitions do not match exactly with Palais, in particular we avoid using jet bundles by choosing a connection, but it should be clear how they are equivalent.

Definition 4.3.1. Let (M, g) be a Riemannian manifold and E and F be Euclidean vector bundles on M with connections. A *smooth differential operator of order k from E to F* is a map $\mathcal{P}\colon \mathcal{U} \subset C^\infty(E) \to C^\infty(F)$ such that there exists a smooth map of fibre bundles over M

$$P\colon U \subset E \oplus (T^*M \otimes E) \oplus \cdots \oplus (S^k(T^*M) \otimes E) \longrightarrow F$$

defined on an open subset containing the zero section such that for any $s \in \mathcal{U} \subset C^\infty(E)$, $\mathcal{P}(s)$ is given by

$$\mathcal{P}(s) = P(s, \nabla s, \dots, \nabla^k s),$$

where $\mathcal{U} = \{s \in C^\infty(E) : (s(p), \nabla s(p), \dots, \nabla^k s(p)) \in U_p \ \forall p \in M\}$. In particular, $\mathcal{P}(s)$ only depends on the first k derivatives of s.

We similarly define a C^r *differential operator from E to F* to be a map $\mathcal{P}\colon \mathcal{U} \subset C^\infty(E) \to C^r(F)$ to be a map determined by a C^r map of fibre bundles P.

Note that one can define differential operators independent of a choice of connection using jet bundles, see [83, Ch. 2].

Definition 4.3.2. A differential operator is *linear* if it is a linear map of vector spaces and *non-linear* otherwise.

The *linearization of a differential operator* $\mathcal{P}\colon \mathcal{U} \subset C^\infty(E) \to C^\infty(F)$ *at* $s \in \mathcal{U}$ is the linear differential operator, denoted $D_s\mathcal{P}\colon C^\infty(E) \to C^\infty(F)$ determined by the linear map

$$D_{(s,\nabla s,\ldots,\nabla^k s)}P : \left(\bigoplus_{j=0}^{k} S^j(T^*M)\right) \otimes E \longrightarrow F,$$

where we use the canonical identification of the vertical tangent space of a vector bundle, $\pi\colon E \to M$, with the pullback of the vector bundle, $\pi^*E = E \oplus E$.

Definition 4.3.3. Let $\mathcal{P}\colon C^\infty(E) \to C^\infty(F)$ be a linear differential operator of order k and $P\colon \left(\bigoplus_{j=0}^{k} S^j(T^*M)\right) \otimes E \to F$ the corresponding linear map. The linear map P breaks up into components $P_j\colon S^j(T^*M) \otimes E \to F$. The *symbol* of $\mathcal{P}$ is the linear map $\sigma(\mathcal{P}) = P_k$.

Definition 4.3.4. A linear differential operator $\mathcal{P}\colon C^\infty(E) \to C^\infty(F)$ between vector bundles on a compact Riemannian manifold M is *elliptic* if for any $p \in M$ and $v \in T_pM$ such that $v \neq 0$, the map

$$\sigma(\mathcal{P})_p(v)\colon E_p \longrightarrow F_p$$

is a linear isomorphism.

We say a non-linear differential operator $\mathcal{P}\colon \mathcal{U} \subset C^\infty(E) \to C^\infty(F)$ is *elliptic at* $s \in C^\infty(E)$ if the linearization $D_s\mathcal{P}$ of $\mathcal{P}$ at s is elliptic. Note that since ellipticity is a pointwise condition, for a non-linear differential operator of order k, the ellipticity of $\mathcal{P}$ at $s \in C^k(E)$ is well-defined.

The theory of linear elliptic differential operators on compact manifolds is very well understood. For example we have the following result, which will follow from Theorem 4.3.17:

Theorem 4.3.5. *Let M be a compact Riemannian manifold and $\mathcal{P}\colon C^\infty(E) \to C^\infty(F)$ a linear elliptic differential operator between Euclidean vector bundles on M and let $\mathcal{P}^*\colon C^\infty(F) \to C^\infty(E)$ denote the formal adjoint of $\mathcal{P}$. Then*

$$\dim\ker\mathcal{P} < \infty \ \textit{and} \ \dim\ker\mathcal{P}^* < \infty.$$

The theory of linear elliptic operators proceeds in two steps. We first solve the solution in a weak sense and then use a regularity result to show that the solution is smooth. We will follow a similar plan for non-linear operators.

4.3.1 Smooth maps of Banach spaces

We will reformulate the theory of differential operators in terms of maps of Banach spaces. This will allow us to use the abstract results about Banach spaces to obtain existence in a weak sense of solutions to a non-linear differential equation. This material is from Lang [65].

Definition 4.3.6. Let X and Y be Banach spaces. Let U be open in X, $x \in U$ and $f\colon U \to Y$ a map. Following Lang [65, Ch. 13.2] we say f is *differentiable at x* if there exists a continuous linear map $Df_x\colon X \to Y$ and a map ϕ defined for $h \in X$ sufficiently small with values in Y such that

$$f(x+h) = f(x) + Df_x(h) + \phi(h)$$

and

$$\lim_{h\to 0} \frac{\|\phi(h)\|_Y}{\|h\|_X} = 0.$$

We say f is *differentiable* if it is differentiable for all $x \in U$ and f is C^1 if $Df\colon U \to L(X,Y)$ from U to the continuous linear maps from X to Y is continuous. We define the k-th derivative inductively as the map $D(D^{k-1}f)$ and we say f is C^k if $D^k f$ exists and is continuous and C^∞ or *smooth* if all derivatives exist and are continuous.

Remark. The open set $U \subset X$ is clearly not in general a Banach space but it is instead an example of a Banach manifold. We will develop this idea further when we discuss manifolds modelled on open subsets of convenient vector spaces in §5.1, where a more general notion of smooth map between topological vector spaces is discussed.

Just as in the finite-dimensional case, we have an Inverse Mapping Theorem for Banach spaces. The Implicit Mapping Theorem follows from the Inverse Mapping Theorem by a similar argument to the finite-dimensional case. We will use the Implicit

Mapping Theorem as a sort of existence theorem for solutions to non-linear differential equations.

Theorem 4.3.7 (Inverse Mapping Theorem [65, Ch. 14 Th.1.2])**.** *Let X and Y be Banach spaces and $U \subseteq X$ be open. Let*

$$f\colon U \longrightarrow Y$$

be a C^k map. Let $x \in U$ and assume that $D_x f\colon X \to Y$ is a continuous isomorphism. Then f is a local C^k-isomorphism at x, i.e. there locally exists a C^k inverse for f.

Corollary 4.3.8 (Implicit Mapping Theorem [65, Ch. 14 Th. 2.1])**.** *Let X, Y, Z be Banach spaces and $U \subseteq X$, $V \subseteq Y$ be open. Let*

$$f\colon U \times V \longrightarrow Z$$

be a C^k mapping, $(x_0, y_0) \in U \times V$, and assume that

$$Df_{(x_0,y_0)}\colon Y \longrightarrow Z$$

is an isomorphism of topological vector spaces and $f(x_0, y_0) = 0$. Then there exists an open neighbourhood $U_0 \subset U$ of x_0 and a unique C^k map $g\colon U_0 \to V$ such that $g(x_0) = y_0$ and

$$f(x, g(x)) = 0$$

for all $x \in U_0$.

Suppose $f\colon X \to Y$ is a smooth map such that $D_x f\colon X \to Y$ is surjective. To apply the previous theorem we would need that $\ker D_x f$ is a splitting subspace, which in general is not true. In most applications the kernel will be finite dimensional and therefore splits.

4.3.2 Banach spaces of sections of fibre bundles

We now wish to relate the general theory of Banach spaces and smooth maps between then with the differential operators we defined earlier.

Definition 4.3.9. Let (M, g) be a compact Riemannian manifold and E a vector bundle on M with a connection ∇ and inner-product on E. Let $\nabla^k \colon C^\infty(E) \to C^\infty(S^k(T^*M) \otimes E)$ denote the iterated derivative, using the connection on E and the Levi-Civita connection on M, followed by symmetrization.

We define the Banach space $C^k(E)$ to be the vector space of sections of E, which locally are C^k in any trivializing local chart of E with the norm

$$\|s\|_{C^k} = \sum_{j=0}^{k} \sup_M \|\nabla^j s\|.$$

In fact in analytical applications, the C^k spaces are not the appropriate Banach spaces to work with. We will instead work with Hölder spaces of sections.

Definition 4.3.10. Let (M, g) be a Riemannian manifold and E a vector bundle with inner-product and connection. Suppose the injectivity radius $\delta(g)$ of (M, g) is strictly positive. Let $d(x, y)$ denote the distance between $x, y \in M$, then if $d(x, y)$ is less than the injectivity radius $\delta(g)$, there exists a unique geodesic of length $d(x, y)$, $\gamma \colon [0, 1] \to M$, between x and y.

A section $s \colon M \to E$ is said to be *Hölder continuous with exponent* $\alpha \in (0, 1)$ if

$$[s]_\alpha = \sup_{\substack{x,y \in M \\ 0 < d(x,y) < \delta(g)}} \frac{|\Gamma_\gamma(s(x)) - s(y)|}{d(x, y)^\alpha}$$

is finite, where Γ_γ denotes parallel transport along γ.

The Banach space $C^{0,\alpha}(E)$ is the vector space of continuous, bounded sections of E such that s is Hölder continuous with exponent α with the norm

$$\|s\|_{C^{0,\alpha}} = \|s\|_{C^0} + [s]_\alpha.$$

The Banach space $C^{k,\alpha}(E)$ is the vector space of sections of E such that $s \in C^k(E)$ and $[\nabla^k s]_\alpha$ is finite with the norm

$$\|s\|_{C^{k,\alpha}} = \|s\|_{C^k} + [\nabla^k s]_\alpha.$$

Note that this definition of $C^{k,\alpha}(E)$ does not match that of Palais. Palais only considers $C^{k,\alpha}(E)$ as a Banachable space, that is without a choice of norm. We will define $C^{k,\alpha}(E)$ to include a choice of norm, which depends on an inner product and connection on E.

Differential operators between vector bundles determine maps between appropriate Banach spaces. The regularity of these maps follow from the regularity of the fibre bundle map determining the differential operator.

Theorem 4.3.11. *Let* $\mathcal{P}\colon \mathcal{U} \subset C^\infty(E) \to C^\infty(F)$ *be a* C^{p+q+2} *differential operator of order* k *between vector bundles on a compact Riemannian manifold* M *and* $P\colon U \subset (\bigoplus_{j=0}^k S^j(T^*M)) \otimes E \to F$ *the corresponding* C^{p+q+2} *map.*

Then there exists a C^q *map of Banach spaces, extending* $\mathcal{P}$,

$$\mathcal{P}^{k+p,\alpha}\colon \mathcal{U}^{k+p,\alpha} \longrightarrow C^{p,\alpha}(F),$$

where $\mathcal{U}^{k+p,\alpha} = \{u \in C^{k+p,\alpha}(E) : (s(x), \nabla s(x), \dots, \nabla^k s(x)) \in U \ \forall x \in M\}$.

Proof. The proof follows from that of [83, Th. 15.7], which covers the case of a smooth differential operator, by replacing [83, Lem. 9.9] with Lemma 4.3.12 and making suitable changes to [83, Th. 11.3]. □

Remark. The linearization $D_s\mathcal{P}$ of $\mathcal{P}$ at $s \in C^\infty(E)$ determines a linear map of Banach spaces $(D_s\mathcal{P})^{k+r,\alpha}\colon C^{k+r,\alpha}(E) \to C^{r,\alpha}(F)$. This linear map agrees with the derivative of $\mathcal{P}^{k+r,\alpha}$ at s, where we regard s as an element of $C^{k+r,\alpha}(E)$ by [83, Th. 17.3].

The following lemma is an analogue of [83, Lem. 9.9], which covered smooth maps and Sobolev spaces, for maps with lower regularity and Hölder spaces.

Lemma 4.3.12. *Let* D^n *denote the closed unit ball in* $\mathbb{R}^n$ *and* $\alpha \in (0,1]$. *Let* $f \in C^2(D^n \times \mathbb{R}^r, \mathbb{R})$ *and* $s \in C^{0,\alpha}(D^n, \mathbb{R}^r)$. *Let* $\phi(f,s)$ *denote the map from* D^n *to* $\mathbb{R}$ *defined by* $\phi(f,s)(x) = f(x, s(x))$. *Then*

(i) $\phi(f,s) \in C^{0,\alpha}(D^n, \mathbb{R})$ *and*

(ii) the map $C^{0,\alpha}(D^n, \mathbb{R}^r) \to C^{0,\alpha}(D^n, \mathbb{R})$, $s \mapsto \phi(f,s)$ *is continuous.*

Proof. The first part follows from the estimate

$$\begin{aligned}|f(x,s(x))-f(y,s(y))| &\leq \sup_{t\in[0,1]}|D_{\gamma_t}f|(|x-y|+|s(x)-s(y)|)\\ &\leq \sup_{\substack{(x,z)\in D^n\times\mathbb{R}^r\\ |z|\leq\|s\|_{C^0}}}|Df|(2^{1-\alpha}+[s]_\alpha)|x-y|^\alpha,\end{aligned}$$

where γ_t is the path joining $(x,s(x))$ to $(y,s(y))$.

For the second part, let $s_1,s_2\in C^{0,\alpha}(D^n,\mathbb{R}^r)$, $x,y\in D^n$ and let $\gamma_t^x\colon[0,1]\to D^n\times\mathbb{R}^r$ denote the path $(x,ts_1(x)+(1-t)s_2(x))$. Then since f is C^1 we have

$$f(x,s_1(x))-f(x,s_2(x))=\int_0^1 D_{\gamma_t^x}f(s_1(x)-s_2(x))\mathrm{d}t.$$

and therefore

$$\|\phi(f,s_1)-\phi(f,s_2)\|_{C^0}\leq\sup_{B_{s_1,s_2}}|Df|\|s_1-s_2\|_{C^0},$$

where $B_{s_1,s_2}=\{(x,z)\in D^n\times\mathbb{R}^r : |z|\leq\|s_1\|_{C^0}+\|s_1-s_2\|_{C^0}\}$.

We can also estimate

$$\begin{aligned}|f(x,&s_1(x))-f(x,s_2(x))-f(y,s_1(y))+f(y,s_2(y))|\leq\\ &\int_0^1|D_{\gamma_t^x}f(s_1(x)-s_2(x))-D_{\gamma_t^y}f(s_1(y)-s_2(y))|\mathrm{d}t\\ &\leq\int_0^1|D_{\gamma_t^x}f-D_{\gamma_t^y}f||s_1(x)-s_2(x)|+|D_{\gamma_t^y}f||s_1(x)-s_2(x)-s_1(y)+s_2(y)|\mathrm{d}t\\ &\leq\sup_{B_{s_1,s_2}}|D^2f|(|x-y|+|s_1(x)-s_2(x)-s_1(y)+s_2(y)|)|s_1(x)-s_2(x)|+\\ &\quad+\sup_{B_{s_1,s_2}}|D^2f||s_1(x)-s_1(y)||s_1(x)-s_2(x)|+\\ &\quad+\sup_{B_{s_1,s_2}}|Df||s_1(x)-s_2(x)-s_1(y)+s_2(y)|\\ &\leq\sup_{B_{s_1,s_2}}|D^2f|(2^{1-\alpha}+[s_1]_\alpha+[s_1-s_2]_\alpha)\|s_1-s_2\|_{C^0}|x-y|^\alpha+\\ &\quad+\sup_{B_{s_1,s_2}}|Df|[s_1-s_2]_\alpha|x-y|^\alpha.\end{aligned}$$

Therefore we have

$$\begin{aligned}\|\phi(f,s_1)-\phi(f,s_2)\|_{C^{0,\alpha}}\leq&\sup_{B_{s_1,s_2}}|Df|\|s_1-s_2\|_{C^{0,\alpha}}+\\ &+\sup_{B_{s_1,s_2}}|D^2f|(2^{1-\alpha}+[s_1]_\alpha+[s_1-s_2]_\alpha)\|s_1-s_2\|_{C^0}\end{aligned}$$

and hence $\phi(f,\cdot\,)$ is continuous. □

Remark. The operators $\phi(f,\cdot\,)$ are known in the analysis literature as *nonlinear superposition* or *Nemyskij operators.* There are results showing that one can relax the conditions on f and retain continuity of the operator, for example see the monograph by Appell and Zabrejko [3].

We will also consider the variation of solutions when the operator itself is perturbed. We can regard a smooth map of fibre bundles $P\colon E \to F$ as a section of the pullback bundle $\pi_E^* F$ over E. We will restrict attention to operators defined on compact neighbourhoods of the zero section in $\bigoplus_{j=0}^k S^j(T^*M)\otimes E$. Hölder spaces of sections of vector bundles on compact manifolds with boundary can also be defined, and we refer the reader to [83] for more details.

Theorem 4.3.13. *Let E and F be vector bundles on a compact Riemannian manifold M and suppose $U \subset\subset (\bigoplus_{j=0}^k S^j(T^*M))\otimes E$ is a compactly contained subset such that $\overline{U}$ is a compact manifold with boundary. Then the map*

$$\begin{aligned} C^{p+q+2}(\pi_{\overline{U}}^* F)\times \mathcal{U}^{k+p,\alpha} &\longrightarrow \quad C^{p,\alpha}(F)\\ (P,u) &\longmapsto \;\; \mathcal{P}^{k+p,\alpha}(u) \end{aligned}$$

is a C^q map of Banach manifolds.

Proof. We can reduce the theorem to Lemma 4.3.14. □

Lemma 4.3.14. *Let D^n denote the open unit ball in $\mathbb{R}^n$ and $\alpha\in(0,1]$. Then the map*

$$\phi\colon C^2(\overline{D^n}\times\overline{D^r},\mathbb{R})\times C^{0,\alpha}(\overline{D^n},D^r)\longrightarrow C^{0,\alpha}(\overline{D^n},\mathbb{R})$$

$$\phi(f,s)(x) = f(x,s(s))$$

is continuous.

Proof. Using the estimates given in the proof of Lemma 4.3.12 we can show

$$\begin{aligned} \|\phi(f_1,s_1)-\phi(f_2,s_2)\|_{C^{0,\alpha}} &\le \|\phi(f_1,s_1)-\phi(f_1,s_2)\|_{C^{0,\alpha}} + \|\phi(f_1,s_2)-\phi(f_2,s_2)\|_{C^{0,\alpha}}\\ &\le \|Df_1\|_{C^0}\|s_1-s_2\|_{C^{0,\alpha}} + \|D^2 f_1\|_{C^0}(2^{1-\alpha}+[s_1]_\alpha+[s_1-s_2]_\alpha)\|s_1-s_2\|_{C^0}\\ &\quad + \|f_1-f_2\|_{C^0} + \|Df_1-Df_2\|_{C^0}(2^{1-\alpha}+[s_1]_\alpha+[s_1-s_2]_\alpha), \end{aligned}$$

and hence ϕ is continuous. □

4.3.3 Fredholm maps

Recall that we have claimed that the kernel and cokernel of a linear elliptic operator on a compact manifold are finite-dimensional. This property follows from the fact that elliptic operators are Fredholm.

Definition 4.3.15. A continuous linear map $T\colon X \to Y$ of Banach spaces is *Fredholm* if

(i) $\dim \ker T < \infty$ and $\dim \operatorname{coker} T < \infty$, and

(ii) $\operatorname{im} T \subset Y$ is closed.

Remark. Let $T\colon X \to Y$ be a Fredholm continuous linear map and $X = \ker T \oplus X'$, $Y = \operatorname{coker} T \oplus Y'$ be splittings of X and Y. Then T induces an isomorphism $T\colon X' \to Y'$ so up to an isomorphism of Banach spaces, T is of the form $T\colon \mathbb{R}^n \oplus E \to \mathbb{R}^m \oplus E$, sending $(x, e) \longmapsto (0, e)$.

Definition 4.3.16. The *index* of a Fredholm map $T\colon X \to Y$ is the difference

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T.$$

Theorem 4.3.17 ([49, §1.5])**.** *Let $\mathcal{P}\colon C^\infty(E) \to C^\infty(F)$ be a linear elliptic differential operator of order k between vector bundles on a compact Riemannian manifold. The induced linear map $\mathcal{P}^{k+r,\alpha}\colon C^{k+r,\alpha}(E) \to C^{r,\alpha}(F)$ is Fredholm.*

The natural generalization of Fredholm operators to the non-linear case is that of a Fredholm map.

Definition 4.3.18. A C^1 map of Banach spaces $f\colon U \subset X \to Y$ is *Fredholm* if $D_x f\colon X \to Y$ is Fredholm for all $x \in U$.

Note that being elliptic is an open property, and so we have the following corollary of Theorem 4.3.17:

Corollary 4.3.19. *Let (M,g) be a compact Riemannian manifold and $\mathcal{P}\colon \mathcal{U} \subset C^\infty(E) \to C^\infty(F)$ a differential operator of order k such that the linearization $D_0\mathcal{P}\colon C^\infty(E) \to C^\infty(F)$ is elliptic. Then there exists an open set $V \subset U \subset \bigoplus_{j=0}^k S^jT^*M \otimes E$, containing the zero section, such that*

$$\mathcal{P}^{k+r,\alpha}\colon \mathcal{V}^{k+r,\alpha} \subset C^{k+r,\alpha}(E) \longrightarrow C^{r,\alpha}(F)$$

is a Fredholm map.

Fredholm maps can be put into a particularly nice local form.

Theorem 4.3.20 (Local Representation Theorem)**.** *Let X and Y be Banach spaces, f be a smooth Fredholm map from an open neighbourhood V of 0 in X to Y, $f(0) = 0$, $\dim\ker D_0f = n$, and $\dim\operatorname{coker} D_0f = m$.*

Then there exists a linear space $E \subset X$, an open neighbourhood U of 0 in X, a smooth map $\phi_X\colon U \to \mathbb{R}^n \times E$, which is a diffeomorphism onto its image, and a continuous isomorphism $\phi_Y\colon Y \to \mathbb{R}^m \times E$ such that

(i) $\phi_X(0) = (0,0)$ *and* $\phi_Y(0) = (0,0)$,

(ii) the map

$$\phi_Y \circ f \circ \phi_X^{-1}\colon \phi_X(U) \longrightarrow \mathbb{R}^m \times E$$

has the form

$$\phi_Y \circ f \circ \phi_X^{-1}(a,b) = (g(a,b),b)$$

for a smooth map $g\colon \phi_X(U) \to \mathbb{R}^m$ and $D_0g = 0$.

Proof. This proof is based on [1, Th. 1.7]. Let $X = \ker D_0f \times E$ be a splitting of X. Then D_0f maps E isomorphically onto the image of D_0f. Let $\phi_Y\colon Y \to \operatorname{coker} D_0f \times E$ be a splitting of Y such that $\operatorname{proj}_E \circ \phi_Y \circ D_0f\colon E \to E$ is the identity.

Then $\phi_Y \circ f$ has the form

$$f(a,b) = (g(a,b), h(a,b))$$

for (a,b) in the domain of f, where $D_0h\colon E \to E$ is the identity and $D_0g = 0$.

Consider the map $F\colon X \to X$ defined on the domain of f by

$$F(a,b) = (a, h(a,b)).$$

Then D_0F is a continuous isomorphism and by the Inverse Mapping Theorem, Theorem 4.3.7, there exists a neighbourhood U of 0 in X and a local diffeomorphism ϕ_X defined on U such that $F \circ \phi_X^{-1}$ is the identity. Therefore $h \circ \phi_X^{-1}(a,b) = b$ and so the map $\phi_Y \circ f \circ \phi_X^{-1}$ satisfies the criteria of the theorem. □

4.3.4 Regularity

The other important aspect of elliptic operators is the regularity of the solutions. We will use the following general regularity result for non-linear elliptic operators.

Theorem 4.3.21. *Let $\mathcal{P}\colon \mathcal{U} \subset C^\infty(E) \to C^\infty(F)$ be a non-linear differential operator of order k between vector bundles on a compact Riemannian manifold and let $s \in \mathcal{U}^{k,\alpha}$ and $\mathcal{P}^{k,\alpha}(s) \in C^{r,\alpha}(F)$ for some $r > 0$ and $\alpha \in (0,1)$.*

Suppose $\mathcal{P}^{k,\alpha}$ is elliptic at s. Then $s \in C^{k+r,\alpha}(E)$.

Proof. This follows directly from the interior regularity results of Agmon et. al. [2, Th. 12.1] or Morrey [81, Th. 6.8.1]. □

4.4 Atiyah–Singer Index Theorem

The Atiyah–Singer Index Theorem relates an analytical quantity, namely the index of an elliptic differential operator, to a topological quantity. This topological quantity will be expressed in terms of characteristic classes and genera, so we will give a short overview of these terms so we can state the Atiyah–Singer Index Theorem. We will only state the Atiyah–Singer Index Theorem for Dirac operators, which are a special kind of first-order elliptic operator, but note that it can be stated for general elliptic operators (and even elliptic pseudo-differential operators). Our main reference for this section is Berline, Getzler and Vergne [10], where Dirac operators are emphasized and Lawson and Michelson [67].

4.4.1 Topological index

Characteristic classes and genera

In this section we define characteristic classes of principal bundles and relate them to invariants defined in terms of the curvature of their associated vector bundles. In this sense we already see the beginnings of a relationship between analytical objects and topological ones. We also define genera of principal bundles, which are combinations of characteristic classes which arise in a natural way in the Atiyah–Singer Index Theorem. Milnor [79] is the classic reference for characteristic classes.

Characteristic classes are topological invariants, and as such they can be defined from the structure of M as a topological space, so for this section M will refer to the underlying topological space of a manifold M. Note that we can define principal bundles in the category of topological spaces.

It is a result of Milnor that for any Lie group G there exists a topological space BG with principal G-bundle $EG \to BG$ such that any (topological) principal G-bundle on M can be realized as the pullback of this principal G-bundle on BG for some continuous map $f\colon M \to BG$. The topological space BG is called a *classifying space* for G and $EG \to BG$ is called the *universal principal G-bundle*, and EG is contractible.

Definition 4.4.1. Let G be a Lie group and R a commutative ring. Each non-zero element of $H^*(BG, R)$ is called a *universal characteristic class.*

Given a principal G-bundle on M, say P, determined by a continuous map $f\colon M \to BG$ and a non-zero class $\theta \in H^*(BG, R)$ we can define a class $\theta(P) \in H^*(M, R)$ by

$$\theta(P) = f^*\theta.$$

This class behaves naturally under maps $f\colon M \to N$ since it is defined as a pullback.

We will now give some examples of characteristic classes without proof which can be found in for example [80].

Example 4.4.2. The cohomology ring of BSO_n with $\mathbb{Q}$ coefficients is given by

$$\begin{aligned} H^*(BSO_{2n+1}, \mathbb{Q}) &= \mathbb{Q}[p_1, \dots, p_n] \\ H^*(BSO_{2n}, \mathbb{Q}) &= \mathbb{Q}[p_1, \dots, p_n, e]/ < e^2 - p_n > \end{aligned}$$

where $p_k \in H^{4k}(BSO_n, \mathbb{Q})$ is called the k^{th} *universal Pontryagin class* and $e \in H^{2n}(BSO_{2n}, \mathbb{Q})$ is called the *universal Euler class.*

Example 4.4.3. The cohomology ring of BU_n with $\mathbb{Z}$ coefficients is a polynomial ring

$$\mathbb{Z}[c_1, \dots, c_n],$$

where c_k is called the k^{th} *universal Chern class.* The cohomology ring of BSU_n is isomorphic to $H^*(BU_n, \mathbb{Z})/(c_1)$.

Chern–Weil theory

To any vector bundle on M, E, we can assign characteristic classes to it by taking the associated principal G-bundle, $P(E)$, where G is the structure group of the vector bundle and then taking characteristic classes of this principal G-bundle. For example if E is a complex vector bundle on M we define

$$c_k(E) = c_k(P(E)).$$

The Pontryagin classes, p_k, of a real vector bundle can also be defined in terms of the Chern classes as

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}).$$

There is a relationship between characteristic classes (or more precisely their image under the map induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$) and the curvature of connections on vector bundles. We will give a very brief sketch of this correspondence for the case of Chern classes.

Let E be a rank n complex vector bundle on M. Let ∇ be any connection on E and $R \in C^\infty(\Lambda^2 T^*M \otimes \mathrm{End}(E))$ the curvature of ∇. Then we define the *Chern forms*

$c_k(E) \in C^\infty(\Lambda^{2k} T^* M)$ by the following relation

$$\det\left(\frac{itR}{2\pi} + I\right) = \sum_{k=0}^{n} c_k(E) t^k.$$

It can be shown that the forms $c_k(E)$ are real and closed, $[c_k(E)] \in H^{2k}(M, \mathbb{R})$ is independent of the choice of connection on E and is the image of the k^{th} Chern class under the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. Expanding the above formula in powers of t we see that

$$c_k(E) = (-2\pi i)^{-k} \operatorname{tr}(\Lambda^k R).$$

Genera

Certain combinations of characteristic classes arise naturally in relation to the Atiyah–Singer Index Theorem. These combinations are defined in terms of a formal power series. Let $f \in \mathbb{Q}[[x]]$ be a formal power series in x. Then

$$\prod_{i=1}^{m} f(x_i)$$

is a symmetric formal power series in the variables x_i and, by the fundamental theorem of symmetric polynomials, can be written as a formal power series in elementary symmetric polynomials.

Definition 4.4.4. Let $f \in \mathbb{Q}[[x]]$ be a formal power series and E a rank m complex vector bundle over M. The *Chern genus associated to f* is the characteristic class given by

$$\Pi_f(E) = \prod_{i=1}^{m} f(x_i),$$

where x_i are formal variables subject to the conditions that $c_i(E)$ is the i^{th} elementary symmetric polynomial in the variables x_i.

Example 4.4.5. Consider the case when $f(x) = 1 + x$. Then

$$\begin{aligned}
\Pi_f(E) &= \prod_{i=1}^{m} (1 + x_i) \\
&= 1 + (x_1 + \cdots + x_m) + \cdots + x_1 x_2 \cdots x_m \\
&= 1 + c_1(E) + \cdots + c_m(E) \\
&= c(E)
\end{aligned}$$

is called the *total Chern class.*

We can similarly describe combinations of characteristic classes for real vector bundles.

Definition 4.4.6. Let f be a formal power series and E a rank m real vector bundle over M. The *Pontryagin genus associated to* f is the characteristic class given by

$$\Pi_f(E) = \prod_{i=1}^{m} f(x_i),$$

where x_i are formal variables subject to the conditions that $p_i(E)$ is the i^{th} elementary symmetric polynomial in the variables x_i.

Example 4.4.7. The $\widehat{A}$ genus of a real vector bundle is the Pontryagin genus associated to the power series

$$\frac{\sqrt{x}/2}{\sinh \sqrt{x}/2}.$$

The first few terms are given by

$$\widehat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots .$$

The genera we have defined all have the property that $\Pi_f(E \oplus F) = \Pi_f(E) \cup \Pi_f(F)$ and hence Π_f defines a monoid homomorphism from the monoid of vector bundles under direct sum to the cohomology of M.

The Chern character, defined below, has the property that $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ and $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cup \mathrm{ch}(F)$ and hence defines a semiring homomorphism from vector bundles to real cohomology.

Example 4.4.8. The Chern character of a complex vector bundle E, $\mathrm{ch}(E)$, is a characteristic class associated to the power series e^x but not in the manner we have discussed before. Instead we define for a rank m complex vector bundle E

$$\mathrm{ch}(E) = \sum_{i=1}^{m} e^{x_i}.$$

The first few terms are given by

$$\mathrm{ch}(E) = (\operatorname{rank} E) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots .$$

Since the Chern classes of E are related to the curvature of connections on E we should expect a relation between that of Chern genera and the curvature. For a complex vector bundle with curvature $R \in C^\infty(\Lambda^2 T^*M \otimes \operatorname{End} E)$ we can define the (differential geometer's) Chern character by

$$\operatorname{ch}(E) = \operatorname{tr}\exp\left(\frac{iR}{2\pi}\right).$$

It can be shown that $\operatorname{ch}(E)$ is a closed differential form and hence defines an element of $H^*(M, \mathbb{R})$, which we will also denote by $\operatorname{ch}(E)$. Furthermore, the class $\operatorname{ch}(E) \in H^*(M, \mathbb{R})$ can be shown to be independent of choice of connection on E and to be the image of the topologist's $\operatorname{ch}(E)$ under the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

4.4.2 Dirac operators

We will state the Atiyah–Singer Index Theorem for a class of first-order differential operators known as Dirac operators on manifolds of even dimension, since this is all we will need.

Definition 4.4.9. Let (M, g) be a Riemannian manifold and E a vector bundle on M. A linear differential operator $\mathcal{P}\colon C^\infty(E) \to C^\infty(E)$ is a *generalized Laplacian* if

$$\sigma(\mathcal{P})_p(\xi) = -g_p(\xi, \xi)\mathrm{id}_E$$

for any $p \in M$ and $\xi \in T^*M$, i.e. the symbol of $\mathcal{P}$ is the same as the standard Hodge Laplacian.

Proposition 4.4.10. *A generalized Laplacian $\mathcal{P}\colon C^\infty(E) \to C^\infty(E)$ is an elliptic operator.*

Proof. The symbol $\sigma(\mathcal{P})_p(\xi) = -g_p(\xi, \xi)$ is not invertible clearly only when $\xi = 0$. □

Definition 4.4.11. Let M be a manifold. A $\mathbb{Z}_2$*-graded vector bundle* on M is a vector bundle E and a splitting $E = E^0 \oplus E^1$. A linear map $A\colon C^\infty(E) \to C^\infty(E)$ is said to be *even* if A preserves the splitting and *odd* if A maps $C^\infty(E^0) \to C^\infty(E^1)$ and vice-versa.

Note that if E is a $\mathbb{Z}_2$-graded vector bundle on M then $\mathrm{End}(E)$ is naturally a $\mathbb{Z}_2$-graded vector bundle, where $\mathrm{End}(E)^0 = \mathrm{Hom}(E^0, E^0) \oplus \mathrm{Hom}(E^1, E^1)$ and $\mathrm{End}(E)^1 = \mathrm{Hom}(E^0, E^1) \oplus \mathrm{Hom}(E^1, E^0)$. This makes $C^\infty(\mathrm{End}(E))$ into a $\mathbb{Z}_2$-graded algebra. Clearly sections of $\mathrm{End}(E)^0$ are even and sections of $\mathrm{End}(E)^1$ are odd.

Example 4.4.12. The exterior complex $\bigoplus_{j=0}^{2n} \Lambda^j T^*M$ naturally carries a $\mathbb{Z}_2$-grading, with respect to which the exterior differential $\mathrm{d}\colon C^\infty(\Lambda^j T^*M) \to C^\infty(\Lambda^{j+1} T^*M)$ is odd.

Definition 4.4.13. Let (M, g) be a Riemannian manifold of dimension $2n$. Let E be a hermitian vector bundle on M. A linear differential operator $\mathcal{D}\colon C^\infty(E) \to C^\infty(E)$ is a *Dirac operator* if

(i) E is a $\mathbb{Z}_2$-graded vector bundle,

(ii) $\mathcal{D}$ is odd,

(iii) $\mathcal{D}^2\colon C^\infty(E) \to C^\infty(E)$ is a generalized Laplacian, and

(iv) $\mathcal{D}$ is self-adjoint.

Proposition 4.4.14. *A Dirac operator* $\mathcal{D}\colon C^\infty(E) \to C^\infty(E)$ *is elliptic.*

Proof. The square of $\mathcal{D}$ is elliptic so $\mathcal{D}$ is elliptic. □

Let $\mathcal{D}_i$ denote the restriction of $\mathcal{D}$ to $C^\infty(E^i)$. In terms of the decomposition $E = E^0 \oplus E^1$, since $\mathcal{D}$ is an odd operator, $\mathcal{D}$ has the form

$$\begin{pmatrix} 0 & \mathcal{D}_1 \\ \mathcal{D}_0 & 0 \end{pmatrix}$$

Since $\mathcal{D}_0$ is elliptic if M is compact then, by Theorem 4.3.17, $\mathcal{D}_0$ is Fredholm and the index of $\mathcal{D}_0$ is finite. The self-adjointness of $\mathcal{D}$ implies that the index of $\mathcal{D}_0$ could also be defined as

$$\operatorname{ind} \mathcal{D}_0 = \dim\ker \mathcal{D}_0 - \dim\ker \mathcal{D}_1.$$

Definition 4.4.15. The *index of a Dirac operator* $\mathcal{D}\colon C^\infty(E) \to C^\infty(E)$ on a compact manifold is the index of the elliptic differential operator $\mathcal{D}_0\colon C^\infty(E^0) \to C^\infty(E^1)$.

The symbol of a Dirac operator defines a linear map $c\colon T^*M \to \mathrm{End}(E)$ such that $c(\xi)c(\xi) = -g(\xi,\xi)$, which we call a *Clifford action.* We say a connection on E is *compatible with the Clifford action* if

(i) The connection preserves the splitting $E = E^0 \oplus E^1$ and

(ii) $\nabla c = 0$,

where ∇ is the connection induced by the Levi-Civita connection on M and the connection on E. The existence of such a connection is guaranteed by [10, Cor. 3.41].

Definition 4.4.16. Let (V, b) be a $\mathbb{R}$-vector space with symmetric bilinear form b. The Clifford algebra $\mathrm{Cl}(V, b)$ is the quotient

$$\mathcal{T}(V)/I(V, b)$$

of the tensor algebra $\mathcal{T}(V) = \bigoplus_{j\in\mathbb{N}} V^{\otimes j}$ over V by the ideal generated by elements $v \otimes v - b(v, v)$ for $v \in V$.

Note that Clifford algebras are $\mathbb{Z}_2$-graded algebras, the grading induced from the $\mathbb{Z}_2$-grading on $\mathcal{T}(V)$ given by $\mathcal{T}(V)^i = \bigoplus_{j\equiv i \pmod 2} V^{\otimes j}$. Clifford algebras have been classified and their representation theory is well understood. In particular, Clifford algebras are matrix algebras over the $\mathbb{R}$-algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We refer the reader to [67, Ch. 1] for more details. If the dimension of V is even the complex representations of $\mathrm{Cl}(V, b)$ are particularly easy to understand.

Suppose (V, b) is an oriented vector space of dimension $2n$ with inner product. Let $e^1, \ldots, e^n$ be an oriented orthonormal basis of V and we define an element of $\mathrm{Cl}(V, b)$ by $\omega = i^n e^1 \cdots e^n$. Then $\omega^2 = 1$ in $\mathrm{Cl}(V, b) \otimes_{\mathbb{R}} \mathbb{C}$ and induces a splitting of any complex representation of $\mathrm{Cl}(V, b)$. If E is a complex representation of $\mathrm{Cl}(V, b)$ then the choice of $\mathbb{Z}_2$-grading $E = E^0 \oplus E^1$ making E into a $\mathbb{Z}_2$-graded module of $\mathrm{Cl}(V, b)$ depends on the orientation of V.

Proposition 4.4.17. *Let (V, b) be an oriented $\mathbb{R}$-vector space of dimension $2n$ with inner product. There is a unique irreducible $\mathbb{Z}_2$-graded complex $\mathrm{Cl}(V, b)$-module up to isomorphism of dimension 2^n, which we will denote by S.*

If we let $\mathrm{Cl}(M, g)$ denote the bundle of Clifford algebras whose fibre at $p \in M$ is $\mathrm{Cl}(T_p^*M, g_p)$ then the linear map $c\colon T^*M \to \mathrm{End}(E)$ extends to an algebra homomorphism $c\colon \mathrm{Cl}(M, g) \to \mathrm{End}(E)$. Therefore any Dirac operator $\mathcal{D}\colon C^\infty(E) \to C^\infty(E)$ gives us a representation of a Clifford algebra. Since the Dirac operator is odd, $c(\xi)$ is odd for any $\xi \in C^\infty(T^*M)$ and $c\colon \mathrm{Cl}(M, g) \to \mathrm{End}(E)$ respects the $\mathbb{Z}_2$-grading. We let $\mathrm{End}_{\mathrm{Cl}(M,g)}(E)$ denote the bundle of algebras whose fibre at $p \in M$ is the centralizer of the image of $c_p\colon \mathrm{Cl}(M, g)_p \to \mathrm{End}(E_p)$. Note that $\mathrm{End}_{Cl(M,g)}(E)$ is also a $\mathbb{Z}_2$-graded algebra.

Proposition 4.4.18 ([10, Prop. 3.43])**.** *Let $\mathcal{D}$ be a Dirac operator on a vector bundle E over a Riemannian manifold M and ∇ a connection compatible with the Clifford action. Then the curvature of ∇ decomposes*

$$R = R^S + R^{E/S}$$

where R^S is determined by the Riemann curvature on M and the Clifford action and $R^{E/S} \in C^\infty(\Lambda^2T^*M \otimes \mathrm{End}_{\mathrm{Cl}(M,g)}(E)^0)$. We call $R^{E/S}$ the twisting curvature *of E.*

The reason we use the suggestive notation E/S is that if there exists a vector bundle S over M such that S_p is the unique irreducible representation of $\mathrm{Cl}(M, g)_p$ for all $p \in M$ then we can write E as a twist of S, $E = S \otimes W$, for some complex vector bundle W [10, Prop. 3.35]. In this case, modulo choosing a connection on S which is compatible with the Clifford action, the twisting curvature is simply the curvature of the induced connection on W.

Definition 4.4.19. Let M be a Riemannian manifold of dimension $2n$ and $\mathcal{D}$ a Dirac operator on a vector bundle E. Let ∇ be a connection compatible with the Clifford action and $R^{E/S}$ the twisting curvature of E. We define the *relative Chern character of E* by the formula

$$\mathrm{ch}(E/S) = 2^{-n}\,\mathrm{tr}_E \exp\left(\frac{iR^{E/S}}{2\pi}\right).$$

It can be shown to be independent of the choice of connection compatible with the Clifford action.

Theorem 4.4.20 (Atiyah–Singer [10, Th. 4.3])**.** *Let* M *be a compact Riemannian manifold of dimension* $2n$ *and* $\mathcal{D}\colon C^\infty(E) \to C^\infty(E)$ *a Dirac operator. The index of* $\mathcal{D}\colon C^\infty(E^0) \to C^\infty(E^1)$ *is given by*

$$\operatorname{ind}\mathcal{D} = \langle \widehat{A}(M)\operatorname{ch}(E/S), [M]\rangle.$$

Chapter 5

The moduli space of submanifolds

In this chapter we will discuss the moduli problem of compact submanifolds of a manifold. We will use this work in the study of the moduli problem of Cayley submanifolds in the following chapter. Let M and N be manifolds and say N is compact. Let $\mathrm{Emb}(N, M)$ denote the set of embeddings of N in M and let $\mathrm{Sub}(N, M)$ denote the quotient of $\mathrm{Emb}(N, M)$ by the free action of $\mathrm{Diff}(N)$. The set $\mathrm{Sub}(N, M)$ carries a Hausdorff topology induced by the Hausdorff metric on the space of embeddings. The question we pose in this chapter is whether the set $\mathrm{Sub}(N, M)$ carries any other natural geometric structure, and how to interpret this extra structure. For example, we should not expect the set of compact submanifolds of a manifold to be a finite-dimensional manifold because intuitively the set of compact submanifolds close to a given submanifold should be modelled on the infinite-dimensional vector space of smooth sections of the normal bundle.

Kriegl and Michor [63] have developed a theory of manifolds modelled on open subsets of certain topological vector spaces called 'convenient vector spaces' and show that the sets $\mathrm{Emb}(N, M)$ and $\mathrm{Sub}(N, M)$ carry the structure of 'convenient manifolds'. The main result of this chapter is Theorem 5.2.4, in which we interpret the set of smooth maps from a convenient manifold into $\mathrm{Sub}(N, M)$ in terms of families of compact submanifolds.

We will begin by introducing the category of convenient manifolds and give some of the major results. Section 5.1 will be almost entirely a review of material in [63]. We will then study the smooth structure on $\mathrm{Sub}(N, M)$ in terms of families. Finally we will discuss a more category-theoretic viewpoint on the problem, which may be more familiar to those with a background in algebraic geometry.

5.1 A convenient category of manifolds

In this section we introduce the category of convenient manifolds described by Kriegl and Michor [63], which are topological spaces with extra structure modelled on open subsets of particular topological vector spaces called convenient vector spaces. This category contains finite-dimensional manifolds as a full subcategory, which implies that any manifold can uniquely be given the structure of a convenient manifold.

The motivation for convenient manifolds comes from the study of mapping spaces of manifolds. If M and N are finite-dimensional manifolds with N compact, then the set of smooth maps $C^\infty(N, M)$ can be given the structure of a Fréchet manifold, that is a manifold modelled on open subsets of Fréchet spaces. The question then is whether this is the correct notion of 'smooth structure' on the set $C^\infty(N, M)$.

Recall [76, p. 106] that a category with products $\mathbf{C}$ is *cartesian closed* if for all objects b and c there exists an object c^b such that for any object a there exists a bijection

$$\mathrm{Hom}_{\mathbf{C}}(a \times b, c) \cong \mathrm{Hom}_{\mathbf{C}}(a, c^b),$$

which is natural in a. The object c^b should be thought of as the set of morphisms from b to c. We will see that the category of convenient vector spaces with smooth maps as morphisms is cartesian closed. This fact is the motivation for the study of convenient vector spaces and convenient manifolds.

The category of convenient manifolds is not cartesian closed but there are still some useful consequences that follow from the cartesian closedness of the category of the modelling vector spaces, in particular the following:

Theorem 5.1.22 (Exponential law [63, Th. 42.14])**.** *Let M, N, X be convenient manifolds with M, N finite-dimensional and N compact. Then there is a bijection*

$$C^\infty(X \times N, M) \cong C^\infty(X, C^\infty(N, M)).$$

The term ‘convenient’ is a reference to a paper by Steenrod [89], where he describes a subcategory of the category of topological spaces, which is cartesian closed. We will assume some basic knowledge of category theory and refer the reader to [76] for more details.

5.1.1 Convenient vector spaces

The desire to define smooth maps between infinite dimensional vector spaces has led to numerous versions of an infinite dimensional calculus. The theory of smooth maps between Banach spaces is the most well known and studied framework. Lang [66] gives a full account of differential geometry based on Banach spaces. The theory of smooth maps between convenient vector spaces was independently developed by Frölicher [27] and Kriegl [62]. The main motivation for the introduction of this category is its cartesian closedness properties. That is we wish the space of smooth maps between convenient vector spaces to also carry a convenient vector space structure, which should be natural in some sense.

We also want the notion of smooth map to reduce to the known definitions for finite dimensional and Banach spaces. In category-theoretic language we want the category of Banach spaces with smooth maps to be a full subcategory of the category of convenient vector spaces and smooth maps. We refer the reader to [63, Ch. I] for a more detailed historical overview.

A *locally convex vector space* is a topological vector space E, whose topology can be induced by a family of seminorms $\{p_\alpha\}_{\alpha\in A}$ such that $p_\alpha(x) = 0$ for all $\alpha \in A$ if and only if $x = 0$. The topology on E is generated by the sets $B_{\alpha,\varepsilon}(x_0) = \{x \in E : p_\alpha(x - x_0) < \varepsilon\}$ and is Hausdorff. We refer the reader to [35] or [63, Ch. 52] for more details about locally convex vector spaces.

Definition 5.1.1. Let E be a locally convex vector space. A map $c\colon \mathbb{R} \to E$ is *differentiable* at $t \in \mathbb{R}$ if

$$c'(t) = \lim_{h\to 0} \frac{c(t+h) - c(t)}{h}$$

exists. A map $c\colon \mathbb{R} \to E$ is C^1 if it is differentiable for all $t \in \mathbb{R}$ and $c'\colon \mathbb{R} \to E$ is continuous, and we say c is C^n if c is C^1 and c' is C^{n-1}. A map $c\colon \mathbb{R} \to E$ is C^∞ or *smooth* if all derivatives exist and are continuous. We will denote the set of smooth maps $c\colon \mathbb{R} \to E$ by $C^\infty(\mathbb{R}, E)$.

Definition 5.1.2. A *convenient vector space* is a locally convex vector space such that a map $c\colon \mathbb{R} \to E$ is smooth if and only if $l \circ c\colon \mathbb{R} \to \mathbb{R}$ is smooth for all continuous linear maps $l \in E^*$.

Example 5.1.3. Banach spaces and Fréchet spaces are examples of convenient vector spaces. In fact, by [63, Th. 2.14] and [35, Prop. 10.1.4], a metrizable locally convex vector space is convenient if and only if it is complete, and hence a convenient vector space is metrizable if and only if it is a Fréchet space.

Definition 5.1.4. Let E be a convenient vector space. The c^∞-topology on E is the final topology with respect to the set of smooth maps $C^\infty(\mathbb{R}, E)$. That is a subset $U \subset E$ is open in the c^∞-topology if and only if $c^{-1}(U) \subset \mathbb{R}$ is open for all $c \in C^\infty(\mathbb{R}, E)$.

Remark. Let E be a convenient vector space. In general the c^∞-topology on E will not agree with the locally convex topology on E. However this is the case if E is metrizable [63, Th. 4.11].

Example 5.1.5 ([63, Ex. 4.8])**.** On $E = \mathbb{R}^J$, the locally convex topology and the c^∞-topology do not agree so long as the cardinality of the set J is at least that of the continuum.

Definition 5.1.6. Let E, F be convenient vector spaces and $U \subset E$ a c^∞-open subset. A map $f\colon U \to F$ is *smooth* if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. We denote the set of smooth maps from U to F by $C^\infty(U, F)$.

Example 5.1.7. A continuous linear map $f\colon E \to F$ is an example of a smooth map of convenient vector spaces.

The smooth linear maps are exactly those which are continuous with respect to the c^∞-topology. Since in general this is finer than the locally convex topology on a convenient vector space, we must distinguish between continuous and smooth linear maps. This is necessary for the general theory of convenient vector spaces but is not an issue if the locally convex space is a Banach or Fréchet space since, in this case, the c^∞-topology agrees with the locally convex topology.

Frölicher [27, Th. 1] shows that a map of Banach spaces is smooth as a map of convenient vector spaces if and only if it is a smooth map of Banach spaces according to Definition 4.3.6.

5.1.2 Convenient manifolds

The definition of a manifold modelled on convenient vector spaces is similar to that of a finite-dimensional manifold.

Definition 5.1.8. An *atlas* on a set M is a set $\{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ of pairs, which are called *charts*, such that

(i) $U_\alpha \subset M$ for all $\alpha \in A$ and $\bigcup_{\alpha \in A} U_\alpha = M$,

(ii) for all $\alpha \in A$, $u_\alpha \colon U_\alpha \to u_\alpha(U_\alpha) \subset E_\alpha$ is a bijection onto an c^∞-open subset of a convenient vector space E_α,

(iii) for all $\alpha, \beta \in A$, the subset $u_\alpha(U_\alpha \cap U_\beta) \subset E_\alpha$ is c^∞-open,

(iv) for all $\alpha, \beta \in A$, the map $u_\alpha \circ u_\beta^{-1} \colon u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta)$ is smooth.

Let M be a set, there is a poset structure on the set of atlases of M ordered by inclusion. We call an atlas *maximal* if it is a maximal element of this poset.

Definition 5.1.9. A *convenient manifold* is a set with a maximal atlas.

Definition 5.1.10. A map $f \colon M \to N$ between convenient manifolds is called *smooth* if for any point $p \in M$ and any chart (U, u) of N such that $f(p) \in U$, there exists a chart (V, v) of M such that $p \in V$, $f(V) \subset U$ and $u \circ f \circ v^{-1} \colon v(V) \to u(U)$ is smooth.

Remark. A convenient manifold is naturally a topological space, with the final topology with respect to the set of charts. We call this topological space *the topological space underlying the convenient manifold.* We will require that a convenient manifold is *smoothly Hausdorff*, that is for any $p, q \in M$ such that $p \neq q$ there exists a smooth function $f \in C^\infty(M, \mathbb{R})$ such that $f(p) \neq f(q)$.

The (smooth) isomorphism class of the model convenient vector spaces E_α is locally constant on a convenient manifold. We say a convenient manifold is *pure* if the modelling convenient vector spaces are all isomorphic. We say a convenient manifold is *finite-dimensional* if the modelling convenient vector spaces are finite-dimensional. Note that a finite-dimensional convenient manifold is not necessarily a manifold since, by definition, the topological space underlying a manifold is second countable.

If we restrict the modelling vector spaces to Banach or Fréchet spaces, we can define categories of Banach manifolds and Fréchet manifolds. The theory of Banach manifolds is well-developed and we refer the reader to [66] for more details.

Initial and final maps

The notions of immersion and submersion are quite subtle in the setting of convenient manifolds since in general we have no Implicit Mapping Theorem as we do in the case of Banach spaces. Following Kriegl and Michor we define initial and final maps and we will show that initial maps have properties similar to embeddings.

Definition 5.1.11. A map of convenient manifolds $f\colon M \to N$ is *initial* if for any convenient manifold X, a map of sets $g\colon X \to M$ is smooth if and only if $f \circ g$ is smooth.

A map of convenient manifolds $f\colon M \to N$ is *final* if for any convenient manifold X, a map of sets $g\colon N \to X$ is smooth if and only if $g \circ f$ is smooth.

Initial and final maps are clearly always smooth and initial maps are necessarily injective. The composition of initial maps is again initial and similarly the class of final maps is closed under composition. The inclusion of an open set is always initial.

Since smoothness of maps can be tested by smooth curves, to check whether a map is initial it suffices to check in the case when $X = \mathbb{R}$.

It is easy to show that if $f\colon N \to M$ and $g\colon P \to M$ are initial maps such that $f(N) = g(P)$ then $f^{-1} \circ g\colon P \to N$ is a diffeomorphism. Let $N \subset M$ be a subset such that there exist initial maps $f_1\colon N_1 \to M$ and $f_2\colon N_2 \to M$ and open subsets $U_1 \subset M$ and $U_2 \subset M$ such that $f_i(N_i) = N \cap U_i$ for $i = 1, 2$, then $f_1^{-1}(U_2) \cong f_2^{-1}(U_1)$ and N carries a unique maximal atlas such that the inclusion map $N \to M$ is initial. We can generalize this argument to show that the condition that a set is the image of an initial map is local on the codomain.

Proposition 5.1.12. *Let M be a convenient manifold and $N \subset M$ a subset such that for every $x \in N$ there exists an open $U \subset M$ containing x and an initial map $f_U\colon N_U \to M$ such that $f_U(N_U) = N \cap U$. Then there exists a unique maximal atlas on N such that the inclusion map $N \to M$ is initial.*

Note that initial maps are not necessarily topological embeddings or immersions.

Example 5.1.13. If $h\colon \mathbb{R} \to \mathbb{R}$ is a function such that h^p and h^q are smooth, where $p, q \in N$ are relatively prime, then h is smooth [36]. Therefore the map $f\colon \mathbb{R} \to \mathbb{R}^2$ sending $f : t \mapsto (t^2, t^3)$ is initial but not an immersion.

Definition 5.1.14. Let M be a convenient manifold. A subset $N \subset M$ is a *weak submanifold* if there exists an atlas on N such that the inclusion map $N \hookrightarrow M$ is an initial map of convenient manifolds and a topological embedding of the underlying topological spaces.

We use the term weak submanifold because Kriegl and Michor use the term submanifold for the following, strictly stronger, notion of submanifold. A submanifold is an example of a weak submanifold.

Definition 5.1.15. A subset $N \subset M$ of a convenient manifold M is called a *submanifold* if for all $x \in N$ there exists a chart (U, u) of M such that $u(N \cap U) = u(U) \cap F_U$ where $F_U \subset E_U$ is a closed linear subspace of the convenient vector space E_U.

Bundles

If M and N are convenient manifolds then the cartesian product $M \times N$ can be given the structure of a convenient manifold which satisfies the usual universal property. We note that the topology of $M \times N$ is not necessarily the product topology, however this is the case if both M and N are metrizable [63, §27.3]. The definition of a fibre bundle is the same as in the case of finite-dimensional manifolds.

Definition 5.1.16. A *fibre bundle* is a tuple (E, M, π, F) where E, M and F are convenient manifolds and $\pi\colon E \to M$ is a smooth surjective map such that for any $p \in M$ there exists an open set $U \subset M$ containing p and a diffeomorphism ϕ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
& U, &
\end{tikzcd}$$

where proj_1 is the projection onto the first factor. We call the pair (U, ϕ) a *fibre bundle chart* for the fibre bundle (E, M, π, F).

We will occasionally use the notation $\pi\colon E \to M$ for a fibre bundle, where the fibre is understood.

Definition 5.1.17. Let (E, M, π_E, F) and $(E', N, \pi_{E'}, F')$ be fibre bundles. A *smooth bundle map* from (E, M, π_E, F) to $(E', N, \pi_{E'}, F')$ is a pair of smooth maps $\phi\colon E \to E'$, $f\colon M \to N$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "\pi_E"'] & E' \arrow[d, "\pi_{E'}"] \\
M \arrow[r, "f"] & N.
\end{tikzcd}$$

If (E, M, π, F) is a fibre bundle and $f\colon N \to M$ a smooth map, the *pullback* $f^*E = \{(n, e) \in N \times E : f(n) = \pi(e)\}$ is a fibre bundle over N with fibre F and the following diagram commutes:

$$\begin{tikzcd}
f^*E \arrow[r, "\tilde{f}"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\
N \arrow[r, "f"] & M,
\end{tikzcd}$$

where $f^*\pi$ and $\tilde{f}$ are the composition of the inclusion of f^*E in $N \times E$ followed by projection onto N and E, respectively. If (U, ϕ) is a fibre bundle chart for (E, M, π, F) then $(f^{-1}(U), (f^*\pi, \mathrm{proj}_2 \circ \phi \circ \tilde{f}))$ is a fibre bundle chart for $(f^*E, N, f^*\pi, F)$.

Lemma 5.1.18. *Let (E, M, π, F) be a fibre bundle, $f \in C^\infty(N, M)$ a smooth map, and $(f^*E, N, f^*\pi, F)$ the pullback fibre bundle.*

Then f^*E is a (categorical) pullback, i.e. if (X, g_1, g_2) is a triple for which the following diagram commutes, there exists a unique $u \in C^\infty(X, f^*E)$ completing the diagram.

$$\begin{tikzcd}
X \arrow[dr, dotted, "u"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& f^*E \arrow[r, "\tilde{f}"] \arrow[d, "f^*\pi"] & E \arrow[d, "\pi"'] \\
& N \arrow[r, "f"] & M.
\end{tikzcd}$$

Proof. Since f^*E is a pullback in the category of sets, there exists a map of sets $u \colon X \to f^*E$ satisfying the universal property. Let $x \in X$ and (U, ϕ) be a fibre bundle chart such that U contains $f(g_1(x))$ and let $V \subset X$ be an open subset containing x such that $(f \circ g_1)(V) \subset U$.

The composition of $u|_V$ with the diffeomorphism from $(f \circ f^*\pi)^{-1}(U)$ to $f^{-1}(U) \times F$ is $(g_1|_V, \mathrm{proj}_2 \circ \phi \circ g_2|_V)$, which is smooth by the universal property of products, and hence u is smooth. ☐

We can similarly define vector bundles (whose model fibres are necessarily convenient vector spaces). We call group objects in the category of convenient manifolds convenient Lie groups.

Definition 5.1.19. A *convenient Lie group* is a convenient manifold G and smooth maps

$$\mu \colon G \times G \longrightarrow G$$
$$^{-1} \colon G \longrightarrow G$$

making G into a group with group operation μ and inversion $^{-1}$.

If G is a convenient Lie group, we can define a principal G-bundle as in the finite-dimensional case. If (P, M, π, G) is a principal G-bundle, with G a convenient Lie group, and $G \times F \to F$ is a smooth action of G on a convenient manifold F then we can form an associated bundle, as in the finite-dimensional case.

Theorem 5.1.20 ([63, Th. 37.13]). *Let $\pi\colon P \to M$ be a principal G-bundle and let $l\colon G \times F \to F$ be a smooth left action of G on a manifold F. Consider the right action of G on $P \times F$ given by $g\colon (p, x) \mapsto (p.g, g^{-1}x)$. Then*

- *The space $P \times_G F$ of orbits of the action of G on $P \times F$ carries a unique smooth structure such that the quotient map $q\colon P \times F \to P \times_G F$ is a final smooth map.*
- *There exists a smooth map $\tilde{\pi}\colon P \times_G F \to M$, such that $(P \times_G F, M, \tilde{\pi}, F)$ is a fibre bundle, making the following diagram commute:*

$$\begin{tikzcd}
P \times F \arrow[r, "q"] \arrow[d, "\mathrm{proj}_1"'] & P \times_G F \arrow[d, "\tilde{\pi}"] \\
P \arrow[r, "\pi"] & M.
\end{tikzcd}$$

- *The smooth map $q_p\colon \{p\} \times F \to (P \times_G F)_{\pi(p)}$ is a diffeomorphism for all $p \in P$.*

5.1.3 Manifolds of mappings

The main motivation for our study of convenient manifolds comes from the results in this section. We should note that the following results may be stated in more general terms but we will only need them for a compact domain and so we will include this assumption in the statements to simplify them.

Theorem 5.1.21 ([63, Th. 42.1, Prop. 42.3, Th. 42.13]). *Let M and N be manifolds with N compact. Then the set $C^\infty(N, M)$ can be given the structure of a smooth convenient manifold such that*

(i) $C^\infty(N, M)$ is second countable and metrizable and

(ii) the evaluation map $\mathrm{ev}\colon C^\infty(N, M) \times N \to M$ is smooth.

Theorem 5.1.22 (Exponential law [63, Th. 42.14])**.** *Let M, N, be manifolds with N compact and let X be a convenient manifold. Then there is a bijection*

$$C^\infty(X \times N, M) \cong C^\infty(X, C^\infty(N, M)).$$

Remark. The bijection $C^\infty(X, C^\infty(N, M)) \xrightarrow{\sim} C^\infty(X \times N, M)$ sends $f \mapsto f^\wedge$, where $f^\wedge(x, p) = f(x)(p)$. The inverse sends $f \mapsto f^\vee$, where $f^\vee(x) = \big(p \mapsto f(x, p)\big)$.

Theorem 5.1.23 ([63, Th. 43.1])**.** *Let M be a compact manifold. The diffeomorphism group* $\operatorname{Diff}(M)$ *is an open submanifold of* $C^\infty(M, M)$ *and is a convenient Lie group.*

Theorem 5.1.24 ([63, Th. 44.1] and [78, §13])**.** *Let M, N be manifolds with N compact and* $\dim N \leq \dim M$*. The set of smooth embeddings* $\operatorname{Emb}(N, M)$ *is an open submanifold of* $C^\infty(N, M)$*. The quotient* $\operatorname{Sub}(N, M) = \operatorname{Emb}(N, M)/\operatorname{Diff}(N)$ *is a convenient manifold and* $\operatorname{Emb}(N, M)$ *is a smooth principal* $\operatorname{Diff}(N)$*-bundle over* $\operatorname{Sub}(N, M)$.

We call the convenient manifold $\operatorname{Sub}(N, M)$ the *moduli space of submanifolds of M of type N*. In the next section we will give an interpretation of smooth maps into $\operatorname{Sub}(N, M)$ in terms of families, which will use Theorem 5.1.22 in an essential way. We will describe the local charts for $\operatorname{Sub}(N, M)$ in terms of tubular neighbourhoods of submanifolds.

Definition 5.1.25. Let M be a manifold. A *submanifold of M* is an equivalence class of embeddings $i \colon N \to M$, where $i \colon N \to M$ is equivalent to $i' \colon N' \to M$ if there exists a diffeomorphism $\phi \colon N \to N'$ such that $i' = \phi \circ i$.

A *submanifold of M of type N* is an equivalence class of embeddings $i \colon N \to M$, where $i \colon N \to M$ is equivalent to $i' \colon N \to M$ if there exists a diffeomorphism $\phi \colon N \to N$ such that $i' = \phi \circ i$.

Definition 5.1.26. Let $i \colon N \to M$ be a submanifold of M, $\pi \colon \nu \to N$ the normal bundle to $i \colon N \to M$ and $0 \colon N \to \nu$ the zero section.

A *tubular neighbourhood* of N in M consists of an open neighbourhood V of the zero section in ν and an open embedding $\phi\colon V \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[dr, "\phi"] & \\
N \arrow[u, "0"] \arrow[r, "i"'] & M,
\end{tikzcd}$$

and the composition of $D_{(p,0)}\phi$ restricted to $\nu_p \subset T_{(p,0)}V$ with the projection $T_{i(p)}M \to \nu_p$ is the identity for all $p \in N$.

Let $i\colon N \to M$ be a compact submanifold of M and $V \subset \nu$, $\phi\colon V \to M$ a tubular neighbourhood of $i\colon N \to M$. Kriegl and Michor define charts, $(U(i), u)$, on $\mathrm{Sub}(N, M)$ as

$$U(i) = \{[j] \in \mathrm{Sub}(N, M) : j(N) \subset \phi(V) \text{ and } \pi \circ \phi^{-1} \circ j \in \mathrm{Diff}(N)\}$$
$$u([j]) = \phi^{-1} \circ j \circ (\pi \circ \phi^{-1} \circ j)^{-1} \in C^\infty(\nu).$$

If $[j] \in U(i)$ then it is clear that the homology classes $[i(N)] = [j(N)] \in H_*(M, \mathbb{Z})$ are the same. Therefore the function $\mathrm{Sub}(N, M) \to H_*(M, \mathbb{Z})$ sending an embedding i to the homology class $[i(N)]$ is continuous. If $\alpha \in H_*(M, \mathbb{Z})$ we let $\mathrm{Sub}(N, M)_\alpha$ denote the open submanifold of $i \in \mathrm{Sub}(N, M)$ such that $[i(N)] = \alpha$.

Proposition 5.1.27. *Let M, N be manifolds with N compact and* $\dim N \leq \dim M$. *The convenient manifold* $\mathrm{Sub}(N, M)$ *is second countable.*

Proof. The manifold topology on $C^\infty(N, M)$ agrees with the compact-open C^∞-topology and the Whitney C^∞-topology. By [63, Cor. 41.12] and the fact that second countability is preserved under countable projective limits, the compact-open C^∞-topology is second countable. Therefore $\mathrm{Emb}(N, M) \subset C^\infty(N, M)$ is second countable.

Recall that for metrizable convenient manifolds, the product topology agrees with the smooth topology of the product. The convenient manifold $\mathrm{Sub}(N, M)$ is modelled on Fréchet spaces and, by Theorems 5.1.21 and 5.1.23, the diffeomorphism group $\mathrm{Diff}(N)$ is metrizable. Therefore $\mathrm{Emb}(N, M)$ is a topological fibre bundle over $\mathrm{Sub}(N, M)$ and hence the projection is open. The topological space $\mathrm{Sub}(N, M)$ is the open image of the second countable space $\mathrm{Emb}(N, M)$ and hence is second countable. $\square$

5.2 Families of submanifolds

In this section we define the notion of families of submanifolds and we relate the smooth structure on the set $\mathrm{Sub}(N, M)$ with the notion of family. We will allow our families to be parameterized by convenient manifolds. This will be necessary if we wish to have a family parameterizing all submanifolds of M of type N, since one cannot expect this to be finite-dimensional.

Definition 5.2.1. Let M, N be manifolds. A *family of submanifolds of M of type N* is an equivalence class of tuples $(B, \mathcal{N}, \pi, i)$, where $\pi\colon \mathcal{N} \to B$ is a fibre bundle of convenient manifolds with fibre N, $i\colon \mathcal{N} \to M$ is a smooth map such that $i_b\colon \pi^{-1}(b) \to M$ is an embedding for each $b \in B$ and a family $(B, \mathcal{N}, \pi, i)$ is equivalent to $(B', \mathcal{N}', \pi', i')$ if there exists a diffeomorphism $\phi\colon \mathcal{N} \to \mathcal{N}'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M & \\
\mathcal{N} \arrow[ur, "i"] \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathcal{N}' \arrow[ul, "i'"'] \arrow[dl, "\pi'"] \\
 & B. &
\end{tikzcd}$$

We say $(B, \mathcal{N}, \pi, i)$ is a *family of submanifolds of M*, without reference to the type of the submanifold, to mean there exists a manifold N such that $(B, \mathcal{N}, \pi, i)$ is a family of submanifolds of M of type N.

Definition 5.2.2. Let M be a manifold and $(B, \mathcal{N}, \pi, i)$ be a family of submanifolds of M. Let $f\colon B' \to B$ be a smooth map.

We define the *pullback of the family along f* to be the family of submanifolds $(B', f^*\mathcal{N}, f^*\pi, f^*i)$, where $f^*\pi\colon f^*\mathcal{N} \to B'$ is the pullback of the fibre bundle $\pi\colon \mathcal{N} \to B$ and f^*i is the composition of i with the induced map from $f^*\mathcal{N}$ to $\mathcal{N}$.

Definition 5.2.3. Let $(B, \mathcal{N}, \pi, i)$ be a family of submanifolds of M of type N. We say $(B, \mathcal{N}, \pi, i)$ is *universal* if whenever $(B', \mathcal{N}', \pi', i')$ is a family of submanifolds of M of type N, there exists a unique smooth map $f\colon B' \to B$ such that $(B', \mathcal{N}', \pi', i')$ is the pullback of $(B, \mathcal{N}, \pi, i)$ along f.

Theorem 5.2.4. *Let M, N be finite-dimensional manifolds and suppose N is compact. Then there exists a universal family of compact submanifolds of M of type N with base* $\mathrm{Sub}(N, M)$*, which we denote* $(\mathrm{Sub}(N, M), \mathcal{U}(N, M), \pi_{\mathcal{U}}, j)$.

Proof. Recall, by Theorem 5.1.24, that the set of embeddings of N in M is a principal $\mathrm{Diff}(N)$-bundle $\pi_{\mathrm{Emb}}\colon \mathrm{Emb}(N, M) \to \mathrm{Sub}(N, M)$. The standard action of $\mathrm{Diff}(N)$ on N is smooth since $\mathrm{Diff}(N)$ is an open subset of $C^\infty(N, N)$ and the evaluation map $\mathrm{ev}\colon C^\infty(N, N) \times N \to N$ is smooth. Let $\pi_{\mathcal{U}}\colon \mathcal{U}(N, M) \to \mathrm{Sub}(N, M)$ denote the fibre bundle associated to the standard action of $\mathrm{Diff}(N)$ on N. The evaluation map $\mathrm{ev}\colon \mathrm{Emb}(N, M) \times N \to M$ is a smooth $\mathrm{Diff}(N)$-invariant map and since the quotient map from $\mathrm{Emb}(N, M) \times N \to \mathcal{U}(N, M)$ is final, ev descends to a smooth map $j\colon \mathcal{U}(N, M) \to M$, which is a fibrewise embedding. Therefore $(\mathrm{Sub}(N, M), \mathcal{U}(N, M), \pi_{\mathcal{U}}, j)$ is a family of submanifolds of M of type N.

Let $(B, \mathcal{N}, \pi, i)$ be a family of submanifolds of M of type N. We will first show that the data $(B, \mathcal{N}, \pi, i)$ induces a smooth map $f\colon B \to \mathrm{Sub}(N, M)$ and then that the family is equivalent to the pullback of the family $(\mathrm{Sub}(N, M), \mathcal{U}(N, M), \pi_{\mathcal{U}}, j)$ along f.

Let (U, ϕ) be a fibre bundle chart for $\pi\colon \mathcal{N} \to B$. The map $i \circ \phi^{-1}\colon U \times N \to M$ is smooth and since N is compact the map $(i \circ \phi^{-1})^\vee\colon U \to \mathrm{Emb}(N, M)$ is smooth by Theorem 5.1.22. Let $f_U\colon U \to \mathrm{Sub}(N, M)$ denote the composition of $(i \circ \phi^{-1})^\vee$ with the projection π_{Emb} from $\mathrm{Emb}(N, M)$ to $\mathrm{Sub}(N, M)$.

Now let (V, ψ) be a different fibre bundle chart for $\pi\colon \mathcal{N} \to B$. Then for any $b \in U \cap V$ the maps $(i \circ \phi^{-1})^\vee(b) \in \mathrm{Emb}(N, M)$ and $(i \circ \psi^{-1})^\vee(b) \in \mathrm{Emb}(N, M)$ differ by $(\mathrm{proj}_2 \circ \psi \circ \phi^{-1})^\vee(b) \in \mathrm{Diff}(N)$. The equivalence classes $f_U(b) = [(i \circ \phi^{-1})^\vee(b)]$ and $f_V(b) = [(i \circ \psi^{-1})^\vee(b)]$ in $\mathrm{Sub}(N, M)$ therefore agree. In other words, the maps f_U and f_V agree on the overlaps of their domains $U \cap V$. Therefore we can extend f_U

and f_V to a smooth map $f_{U\cup V}\colon U\cup V \to \mathrm{Sub}(N,M)$ such that $f_{U\cup V}|_U = f_U$ and $f_{U\cup V}|_V = f_V$. Now let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of B by trivializing open sets for the fibre bundle $\pi\colon \mathcal{N} \to B$. We can define $f_{U_\alpha}\colon U_\alpha \to \mathrm{Sub}(N,M)$ as before and since each f_{U_α} agrees with f_{U_β} on $U_\alpha \cap U_\beta$, there exists a smooth map $f\colon B \to \mathrm{Sub}(N,M)$ such that $f|_{U_\alpha} = f_{U_\alpha}$ for all $\alpha \in A$.

We now wish to show that $(B, \mathcal{N}, \pi, i)$ is equivalent to the pullback of the family $(\mathrm{Sub}(N,M), \mathcal{U}(N,M), \pi_{\mathcal{U}}, j)$ by f. That is we need a diffeomorphism $\xi\colon \mathcal{N} \to f^*\mathcal{U}(N,M)$ such that the following diagram commutes:

$$\begin{tikzcd}
& M & \\
\mathcal{N} \arrow[ur, "i"] \arrow[rr, "\xi"] \arrow[dr, "\pi"'] & & f^*\mathcal{U}(N,M) \arrow[ul, "f^*j"'] \arrow[dl, "f^*\pi_{\mathcal{U}}"] \\
& B. &
\end{tikzcd}$$

In fact if ξ simply commutes, it follows that ξ is a diffeomorphism. Suppose ξ commutes. Then for $b \in B$ we have a commutative diagram

$$\begin{tikzcd}
& M & \\
\pi^{-1}(b) \arrow[ur, "i_b"] \arrow[rr, "\xi_b"] & & (f^*\pi_{\mathcal{U}})^{-1}(b), \arrow[ul, "f^*j_b"']
\end{tikzcd}$$

with i_b and f^*j_b embeddings and therefore ξ is a fibrewise diffeomorphism. By Lemma 5.2.5, it follows that ξ is a diffeomorphism.

Now, since $f^*\mathcal{U}(N,M)$ is a categorical pullback, a smooth map $\xi\colon \mathcal{N} \to f^*\mathcal{U}(N,M)$ is uniquely determined by a pair of smooth maps ζ and η such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{N} \arrow[drr, bend left, "\zeta"] \arrow[dr, dotted, "\xi"] \arrow[ddr, bend right, "\eta"'] & & \\
& f^*\mathcal{U}(N,M) \arrow[r, "\tilde{f}"] \arrow[d, "f^*\pi_{\mathcal{U}}"] & \mathcal{U}(N,M) \arrow[d, "\pi_{\mathcal{U}}"'] \\
& B \arrow[r, "f"] & \mathrm{Sub}(N,M).
\end{tikzcd}$$

In order for ξ to be an equivalence, we require that $f^*\pi_{\mathcal{U}} \circ \xi = \pi$ and $(f^*j)\circ\xi = i$. Therefore we must have $\eta = \pi$. Furthermore the smooth map f^*j is defined as simply the composition $\tilde{f} \circ j$ and so the commutativity of the diagram

$$\begin{tikzcd} & M & \\ \mathcal{N} \arrow[ur, "i"] \arrow[rr, "\xi"] & & f^*\mathcal{U}(N,M) \arrow[ul, "f^*j"'] \end{tikzcd}$$

would follow from the commutativity of

$$\begin{tikzcd} & M & \\ \mathcal{N} \arrow[ur, "i"] \arrow[rr, "\zeta"] & & \mathcal{U}(N,M). \arrow[ul, "j"'] \end{tikzcd}$$

So it remains to find a smooth map $\zeta\colon \mathcal{N} \to \mathcal{U}(N,M)$ such that $\pi_{\mathcal{U}} \circ \zeta = f \circ \pi$ and $j \circ \zeta = i$. If such a smooth map ζ existed, then ξ, which is uniquely determined by ζ and π, would yield an equivalence between the families $(B, \mathcal{N}, \pi, i)$ and $(B, f^*\mathcal{U}(N,M), f^*\pi_{\mathcal{U}}, f^*j)$.

Let (U,ϕ) be as before and let $\zeta_U\colon \pi^{-1}(U) \to \mathcal{U}(N,M)$ denote the composition

$$\pi^{-1}(U) \xrightarrow{\phi} U \times N \xrightarrow{(i\circ\phi^{-1})^\vee \times \mathrm{id}_N} \mathrm{Emb}(N,M) \times N \longrightarrow \mathcal{U}(N,M),$$

where the final arrow is the quotient map induced by the action of $\mathrm{Diff}(N)$ on $\mathrm{Emb}(N,M) \times N$. Let (V,ψ) be another fibre bundle chart for $\pi\colon \mathcal{N} \to B$ and let $\zeta_V\colon \pi^{-1}(V) \to \mathcal{U}(N,M)$ be defined similarly. Let $p \in \pi^{-1}(U \cap V)$. We wish to show that $\zeta_U(p) = \zeta_V(p)$. The two points $((i \circ \phi^{-1})^\vee(\pi(p)), (\mathrm{proj}_2 \circ \phi)(p))$ and $((i \circ \psi^{-1})^\vee(\pi(p)), (\mathrm{proj}_2 \circ \psi)(p))$ in $\mathrm{Emb}(N,M) \times N$ are related by the action of the diffeomorphism $(\mathrm{proj}_2 \circ \psi \circ \phi^{-1})^\vee(\pi(p)) \in \mathrm{Diff}(N)$ on $\mathrm{Emb}(N,M) \times N$ and therefore the equivalence classes in $\mathcal{U}(N,M)$ agree. By definition of the maps ζ_U and ζ_V, this shows that they agree on $\pi^{-1}(U \cap V)$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B by trivializing open sets for the fibre bundle $\pi\colon \mathcal{N} \to B$. We can define $\zeta_{U_\alpha}\colon \pi^{-1}(U_\alpha) \to \mathcal{U}(N,M)$ as before and since each ζ_{U_α} agrees with the others on the overlaps, we can construct a smooth map $\zeta\colon \mathcal{N} \to \mathcal{U}(N,M)$ such that $\zeta|_{U_\alpha} = \zeta_{U_\alpha}$ for all $\alpha \in A$.

The commutativity $\pi_{\mathcal{U}} \circ \zeta = f \circ \pi$ follows from the commutativity of the following diagram, recalling the local definitions of f_U and ζ_U:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"] & U \times N \arrow[r, "(i\circ\phi^{-1})^\vee \times \mathrm{id}_N"] \arrow[d, "\mathrm{proj}_1"] & \mathrm{Emb}(N,M) \times N \arrow[r] \arrow[d, "\mathrm{proj}_1"] & \mathcal{U}(N,M) \arrow[d, "\pi_{\mathcal{U}}"] \\ U \arrow[r, "\mathrm{id}_U"] & U \arrow[r, "(i\circ\phi^{-1})^\vee"] & \mathrm{Emb}(N,M) \arrow[r, "\pi_{\mathrm{Emb}}"] & \mathrm{Sub}(N,M). \end{tikzcd}$$

Finally, the commutativity $j \circ \zeta = i$ follows from that of the following diagram:

$$\begin{tikzcd}
 & M \arrow[rr, "\mathrm{id}_M"] & & M & \\
\pi^{-1}(U) \arrow[ur, "i"] \arrow[r, "\phi"] & U \times N \arrow[u, "i\circ\phi^{-1}"'] \arrow[rr, "(i\circ\phi^{-1})^{\vee}\times\mathrm{id}_N"] & & \mathrm{Emb}(N,M) \times N \arrow[u, "\mathrm{ev}"] \arrow[r] & \mathcal{U}(N,M). \arrow[ul, "j"']
\end{tikzcd}$$

□

Lemma 5.2.5. *Let (E, B, π, F) and (E', B, π', F) be fibre bundles with F finite-dimensional and compact, and $f\colon E \to E'$ a smooth fibre bundle map such that $f_b\colon E_b \to E'_b$ is a diffeomorphism for all $b \in B$.*

Then f is a diffeomorphism.

Proof. Since f is a fibrewise diffeomorphism, f is bijective and hence has an inverse f^{-1}.

Let (U, ϕ) and (U, ϕ') be fibre bundle charts for E and E' respectively. Since f is a fibre bundle map, there exists a smooth map $\psi\colon U \times F \to F$ such that $\phi' \circ f \circ \phi^{-1} = (id_U, \psi)$. Since F is compact, $\psi^{\vee}\colon U \to C^{\infty}(F, F)$ is smooth, by Theorem 5.1.22, and since $\mathrm{Diff}(F)$ is an open submanifold of $C^{\infty}(F, F)$ by Theorem 5.1.23, $\psi^{\vee}\colon U \to \mathrm{Diff}(F)$ is smooth.

The inversion map ${}^{-1}\colon\ \mathrm{Diff}(F) \to \mathrm{Diff}(F)$ is smooth since $\mathrm{Diff}(F)$ is a convenient Lie group and therefore ${}^{-1}\circ\psi^{\vee}$ is smooth, and hence $({}^{-1}\circ\psi^{\vee})^{\wedge}\colon U\times F \to F$ is smooth. Therefore f^{-1} restricted to $(\pi')^{-1}(U)$ is smooth, and hence f^{-1} is smooth. □

5.3 Universal families and representable functors

In Theorem 5.2.4 we see that the smooth structure on $\mathrm{Sub}(N, M)$ is intimately linked with the notion of a family of smooth submanifolds, in that smooth maps from a manifold B to $\mathrm{Sub}(N, M)$ parameterize families of submanifolds of M of type N over B. In this sense, Theorem 5.2.4 gives meaning to the smooth structure on $\mathrm{Sub}(N, M)$.

The existence of a smooth structure on the set of submanifolds of M of type N and the interaction between smooth maps from a manifold B to $\mathrm{Sub}(N, M)$ and families of submanifolds of M of type N with base B can be rephrased in terms of whether a particular functor is representable.

We will recall the definition of a representable functor and give some examples from set theory and algebraic geometry. We will then show how the properties of $\mathrm{Sub}(N, M)$ are naturally expressed in this language.

Definition 5.3.1. Let $\mathbf{C}$ be a locally small category, $\mathbf{Set}$ be the category of sets and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. A *representation of* F is a pair (c, Φ) where c is an object of $\mathbf{C}$ and $\Phi\colon \mathrm{Hom}_{\mathbf{C}}(-, c) \to F$ is a natural isomorphism. A functor is *representable* if it has a representation.

Representations of functors are closely related to universal elements.

Definition 5.3.2. Let $\mathbf{C}$ be a category and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. A *universal element of the functor* F is a pair (c, x) consisting of an object c of $\mathbf{C}$ and a set $x \in F(c)$ such that for every pair (d, y) with $y \in F(d)$, there is a unique morphism $f\colon d \to c$ with $F(f)(y) = x$.

If (c, Φ) is a representation of a functor F, then $(c, \Phi(\mathrm{id}_c))$ is a universal element of F. Conversely a universal element of a functor from a locally small category to $\mathbf{Set}$ determines uniquely a representation [76, §III.2].

Example 5.3.3. Let X be a set. Consider the functor $F_X\colon \mathbf{Set}^{op} \to \mathbf{Set}$ which sends Y to the collection of families of subsets of X with base Y, namely the set $\{Z \subset X \times Y)\colon \mathrm{proj}_2\colon Z \to Y \text{ is surjective}\}$ and which sends $f\colon W \to Y$ to the pullback, that is the map sending $Z \subset X \times Y$ to the set $\{(x, w) \in X \times W : (x, f(w)) \in Z\}$.

Let $\Phi\colon \mathrm{Hom}_{\mathbf{Set}}(-, \mathcal{P}(X)) \to F_X$ be the natural transformation, which for any set Y sends $f \in \mathrm{Hom}_{\mathbf{Set}}(Y, \mathcal{P}(X))$ to the set $\{(x, y) \in X \times Y : x \in f(y)\}$. Then the pair $(\mathcal{P}(X), \Phi)$ is a representation of F_X. The pair $(\mathcal{P}(X), \{(x, y) \in X \times \mathcal{P}(X) : x \in y\})$ is the corresponding universal element of F_X.

Example 5.3.4. Let k be a field, $X \subset \mathbb{P}^n_k$ a projective scheme over k. We define a *flat family of subschemes of* X *over* S to be a closed subscheme $Z \subset X \times_k S$ such that Z is flat over S via the natural projection $\mathrm{proj}_2\colon X \times_k S \to S$. The function sending $s \in S$ to the Hilbert polynomial of Z_s is locally constant.

Let $P(t) \in \mathbb{Q}[t]$ be a integer-valued polynomial. We define the Hilbert functor $F^{\mathrm{Sch}}_{X,P(t)}\colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$, which sends a scheme S to

$$F^{\mathrm{Sch}}_{X,P(t)}(S) = \left\{ \begin{array}{c} \text{flat families of subschemes of } X \text{ over } S \\ \text{with fibres having Hilbert polynomial } P(t) \end{array} \right\}$$

and a morphism $f\colon T \to S$ to pullback along f. This functor is well defined since pullbacks preserve flatness.

Grothendieck showed that the functor $F^{\mathrm{Sch}}_{X,P(t)}$ is represented by a pair $(\mathrm{Hilb}^X_{P(t)}, \Phi)$, where $\mathrm{Hilb}^X_{P(t)}$ is a projective scheme, called the Hilbert scheme of X relative to $P(t)$. We refer the reader to, for example, [87, Ch. 4] for more details.

Example 5.3.5. Let M, N be manifolds and suppose N is compact. Let **conMan** be the category of convenient manifolds and let $F^{\mathrm{Man}}_{M,N}\colon \mathbf{conMan}^{\mathrm{op}} \to \mathbf{Sets}$ be the functor sending a convenient manifold B to

$$F^{\mathrm{Man}}_{M,N}(B) = \{\text{families of submanifolds of } M \text{ of type } N \text{ with base } B\}$$

and a smooth map $f\colon B' \to B$ to the pullback of families along f.

Then by Theorem 5.2.4, the pair $\big(\mathrm{Sub}(N,M), (\mathrm{Sub}(N,M), \mathcal{U}(N,M), \pi_{\mathcal{U}}, j)\big)$ is a universal element of the functor $F^{\mathrm{Man}}_{M,N}$ and hence determines a representation of $F^{\mathrm{Man}}_{M,N}$.

Chapter 6

The moduli space of Cayley submanifolds

In this chapter we will investigate the moduli problem of families of Cayley submanifolds of a Spin(7)-manifold. Let (M, Ω, g) be a manifold with Spin(7)-structure and N a compact 4-manifold. The question we will investigate is whether there is a universal family of Cayley submanifolds of M of type N. Note that we do not require the Spin(7)-structure (Ω, g) to be torsion-free. This does not affect the theory of Cayley submanifolds, in contrast to the deformation theory of coassociative submanifolds of G_2-manifolds or special Lagrangian submanifolds of Calabi–Yau manifolds, where various results hold only if certain forms are closed.

By the results of the previous chapter, we know that there is a universal family of submanifolds of M of type N with base $\mathrm{Sub}(N, M)$. A likely candidate for the base of a universal family would be the subset

$$\mathrm{Cay}(N, M) = \{f \colon N \to M \text{ is a Cayley submanifold}\} \subset \mathrm{Sub}(N, M).$$

In general $\mathrm{Cay}(N, M)$ is not a submanifold or weak submanifold of the convenient manifold $\mathrm{Sub}(N, M)$. We will consider the subset $\mathrm{Cay}^{\mathrm{sm}}(N, M) \subset \mathrm{Cay}(N, M)$ of unobstructed Cayley submanifolds, which is a weak submanifold of $\mathrm{Sub}(N, M)$. The main result of this chapter will be the following theorem:

Theorem 6.0.1. *Let (M, Ω, g) be a manifold with* Spin(7)*-structure, N a compact oriented 4-manifold and $\alpha \in H_4(M, \mathbb{Z})$. Suppose the set, $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$, of unobstructed*

Cayley submanifolds $f\colon N \to M$ *with homology class* $[f(N)] = \alpha$ *is nonempty. Then there exists a manifold, with underlying set* $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$*, of dimension*

$$\frac{1}{2}(\sigma(N) + \chi(N)) - \alpha \cdot \alpha.$$

Furthermore $\mathrm{Cay}^{\mathrm{sm}}(N, M)_\alpha$ *is the base of a universal family of unobstructed Cayley submanifolds,* $f\colon N \to M$*, of* M *of type* N *such that* $[f(N)] = \alpha$*.*

McLean [77] studied the local structure of the subset $\mathrm{Cay}(N, M)$ near a given Cayley submanifold $f\colon N \to M$. He described the subset as the zero locus of a non-linear elliptic differential operator, whose linearization is a Dirac operator. He also gave a formula for the index of this operator. The index of the linearization is the *virtual dimension of the moduli space of Cayley submanifolds at* $f\colon N \to M$. If $f\colon N \to M$ is unobstructed, the virtual dimension agrees with dimension given in Theorem 6.0.1.

We will give a more explicit description of this non-linear differential operator and express the index of the linearization in terms of more familiar topological invariants of the embedding $f\colon N \to M$. The study of the linearization will lead us to define unobstructed Cayley submanifolds and we will then use the theory of smooth maps of Banach spaces and convenient manifolds to show that the set is locally a weak submanifold. The results of the previous chapter will then allow us to deduce results about the global structure of this moduli space.

The methods used here to deduce Theorem 6.0.1 could easily be used to give similar results about the existence of universal families of unobstructed compact calibrated submanifolds of Calabi–Yau and G_2-manifolds.

6.1 Families of Cayley submanifolds

Definition 6.1.1. Let (M, Ω, g) be a manifold with Spin(7)-structure. A *family of Cayley submanifolds of* M is a family of submanifolds of M, $(B, \mathcal{N}, \pi, i)$, such that the image of $i_b\colon \mathcal{N}_b \to M$, with an appropriate choice of orientation, is a Cayley submanifold for all $b \in B$.

Note that the base of a family of Cayley submanifolds is, in general, a convenient manifold and hence not necessarily finite-dimensional.

Definition 6.1.2. Let (M, Ω, g) be a manifold with Spin(7)-structure and $f\colon N \to M$ a Cayley submanifold. A *family of deformations of* $f\colon N \to M$ is a family of Cayley submanifolds of M of type N, $(B, \mathcal{N}, \pi, i)$, and a point $0 \in B$, such that B is connected and $f\colon N \to M$ is equivalent to $i_0\colon \mathcal{N}_0 \to M$.

A *universal family of deformations of* $f\colon N \to M$ is a family of deformations of $f\colon N \to M$, $(B, \mathcal{N}, \pi, i)$, such that if $(B', \mathcal{N}', \pi', i')$ is a family of deformations of $f\colon N \to M$ then there exists an open neighbourhood $B'' \subset B'$ of $0 \in B'$ such that the restriction of the family to B'' is equivalent to the pullback of $(B, \mathcal{N}, \pi, i)$ by a unique smooth map $h\colon B'' \to B$.

Let (M, Ω, g) be a manifold with Spin(7)-structure and $f\colon N \to M$ a compact Cayley submanifold of M. Suppose $(B, \mathcal{N}, \pi, i)$ is a family of deformations of $f\colon N \to M$. Since $(B, \mathcal{N}, \pi, i)$ is a family of submanifolds, there is an induced map $h : B \to \mathrm{Sub}(N, M)$. Suppose (U, u) is a local chart of $\mathrm{Sub}(N, M)$ containing f, with modelling convenient vector space the sections of the normal bundle $C^\infty(\nu)$ of $f\colon N \to M$. The local chart (U, u) depends upon the choice of tubular neighbourhood of $f\colon N \to M$. Then, letting $B' = h^{-1}(U)$, we have a smooth map $\sigma = u \circ h : B' \to C^\infty(\nu)$. We call such a map a *normal parameterization of the family.*

6.2 Local structure of moduli space

Let (M, Ω, g) be a manifold with Spin(7)-structure and $f\colon N \to M$ a compact Cayley submanifold. We wish to study the local structure of the subset of Cayley submanifolds of $\mathrm{Sub}(N, M)$ near $f\colon N \to M$.

Recall from §4.1 that a submanifold $f\colon N \to M$ is Cayley if and only if $f^*\tau$ vanishes and $f^*\Omega > 0$, where τ is a 4-form with values in Λ^2_7. If $f\colon N \to M$ is Cayley, the bundle $f^*\Lambda^2_7$ splits as a sum $\Lambda^2_+ \oplus E$ of the self-dual 2-forms on N and a rank 4 vector bundle E from Definition 4.1.7, which we call the *obstruction bundle of* $f\colon N \to M$.

We will define a non-linear differential operator $\mathcal{P}\colon C^\infty(\nu) \to C^\infty(E)$, which morally should be thought of as the pullback of the form τ, whose solutions correspond to Cayley submanifolds. McLean [77] defines this operator also and determines its linearization.

Proposition 6.2.1. *Let* (M, Ω, g) *be a manifold with* $\mathrm{Spin}(7)$*-structure and* $f\colon N \to M$ *a compact Cayley submanifold. Let* $\pi : \nu \to N$ *be the normal bundle to* N *and let* $V \subset \nu$ *and* $\phi\colon V \to M$ *be a tubular neighbourhood of* N *in* M*. There exists a first-order non-linear elliptic differential operator,* $\mathcal{P}\colon \mathcal{U} \subset C^\infty(\nu) \to C^\infty(E)$*, on* N*, determined by a smooth map*

$$P\colon U \subset \nu \oplus (T^*N \otimes \nu) \longrightarrow E$$

defined on a neighbourhood of the zero section such that if $s \in C^\infty(\nu)$ *is such that* $(s_p, \nabla s_p) \in U_p$ *for all* $p \in N$ *then* $\phi \circ s\colon N \to M$ *is Cayley if and only if* $P(s, \nabla s) = 0$*.*

Proof. Parallel transport along the radial direction identifies $\phi^*\Lambda^2_7$ with $\pi^* f^* \Lambda^2_7$. This fibrewise trivialization allows us to make sense of $s^*\phi^*\tau \in C^\infty(\Lambda^4 T^*N \otimes \Lambda^2_7|_N)$, where $s \in C^\infty(\nu)$. Explicitly we have

$$(s^*\phi^*\tau)_p(X_1, \ldots, X_4) = \mathrm{proj}_{(\Lambda^2_7)_p} \circ \Gamma^{-1}_{\gamma_{s(p)}} \circ (\phi^*\tau)_{s(p)}(Ds(X_1), \ldots, Ds(X_4)),$$

for $X_1, \ldots, X_4 \in T_pN$ and where $\Gamma_{\gamma_{s(p)}}\colon (f^*\Lambda^2_7)_p \to (\phi^*\Lambda^2 T^*M|_N)_{s(p)}$ is parallel transport along the path $\gamma_{s(p)}$ sending $t \mapsto ts(p)$. The map $\phi \circ s\colon N \to M$ is Cayley if and only if $*s^*\phi^*\tau = 0 \in C^\infty(f^*\Lambda^2_7)$ where $*$ is the Hodge star.

We define $\mathcal{P}\colon C^\infty(V) \to C^\infty(E)$ as the composition

$$s \longmapsto \mathrm{proj}_E \circ * \, s^*\phi^*\tau.$$

This is a non-linear differential operator determined by the smooth fibre bundle map

$$\begin{aligned} &P\colon V \oplus T^*N \otimes \nu \longrightarrow E \\ &\qquad (p, v, \alpha) \longmapsto \mathrm{proj}_{E_p} \circ \Gamma^{-1}_{\gamma_v} \circ (\phi^*\tau)_v(\alpha(e_1) + \Gamma_{\gamma_v} e_1, \ldots, \alpha(e_4) + \Gamma_{\gamma_v} e_4), \end{aligned} \tag{6.1}$$

where $p \in N$, $v \in V_p \subset \nu_p$, $\alpha \in T^*_pN \otimes \nu_p$, and $(e_1, \ldots, e_4)$ is an oriented orthonormal basis of T_pN.

For $p \in N$, the map $D_{(p,0,0)}P\colon T_p^*N \otimes \nu_p \to E_p$ is surjective. Therefore there exists a neighbourhood $U \subset V \oplus T^* \otimes \nu$ of the zero section such that $D_{(p,v,\alpha)}P\colon T_p^*N \otimes \nu_p \to E_p$ is surjective. Suppose $(p, v, 0) \in U$, then the set $Z_{(p,v)} = \{\alpha \in T_p^*N \otimes \nu_p : P(p, v, \alpha) = 0\} \cap U$ is a submanifold of $T_p^*N \otimes \nu_p$ of codimension 4. The set of $\alpha \in T_p^*N \otimes \nu_p$ such that the graph of α is a Cayley plane in T_pV is also of codimension 4 and is contained in $Z_{(p,v)}$ and hence, by making U smaller if necessary, we have that they agree.

Therefore if $s \in C^\infty(V)$ is a section such that $(s_p, \nabla s_p) \in U_p$ for all $p \in N$, $\phi \circ s\colon N \to M$ is Cayley if and only if $P(s, \nabla s) = 0$.

The description of the operator in equation (6.1) allows us to easily compute the linearization $D_0\mathcal{P}$ as

$$D_0\mathcal{P}(s) = * \operatorname{ad}_{\nabla s} \tau + \operatorname{proj}_E \circ * \nabla_s \tau, \tag{6.2}$$

where $\operatorname{ad}_\alpha(\tau)(X_1, \dots, X_4) = \sum_{i=1}^4 \tau(X_1, \dots, \alpha(X_i), \dots, X_4)$. McLean [77] studied the symbol of this operator and showed that $D_0\mathcal{P} : C^\infty(\nu) \to C^\infty(E)$ is a Dirac operator and hence elliptic. □

Since $D_0\mathcal{P}$ is elliptic, the kernel and the cokernel of the operator $D_0\mathcal{P}$ is finite-dimensional and $\ker D_0\mathcal{P} \subset C^\infty(\nu)$ is closed.

Corollary 6.2.2. *Let (M, Ω, g) be a manifold with* Spin(7)*-structure and $f\colon N \to M$ a compact Cayley submanifold. Suppose $(B, \mathcal{N}, \pi, i)$ is a family of deformations of $f\colon N \to M$ and let $\sigma\colon B \to C^\infty(\nu)$ be a normal parameterization of the family. Then $D_0\sigma(T_0B)$ lies in kernel of $D_0\mathcal{P}\colon C^\infty(\nu) \to C^\infty(E)$.*

Proof. Let $\mathcal{P}\colon \mathcal{U} \subset C^\infty(\nu) \to C^\infty(E)$ be as in Proposition 6.2.1. Let $B' = \sigma^{-1}(\mathcal{U})$ then since $(B, \mathcal{N}, \pi, i)$ is a family of Cayley submanifolds of M we have $\mathcal{P} \circ \sigma|_{B'} = 0$. Since the chain rule holds for convenient manifolds, we have $D_0\mathcal{P} \circ D_0\sigma = 0$, and hence $D_0\sigma(T_0B) \subset \ker D_0\mathcal{P}$. □

We call elements of the kernel $\ker D_0\mathcal{P}$ *infinitesimal deformations* of the Cayley submanifold $f\colon N \to M$. For a given infinitesimal deformation $s \in \ker D_0\mathcal{P}$ there may or may not be a family of deformations $(B, \mathcal{N}, \pi, i)$ of $f\colon N \to M$ such that $s \in D_0\sigma(T_0B)$. If this is the case we say that $s \in \ker D_0\mathcal{P}$ is *integrable.*

A condition that will guarantee that all (sufficiently small) infinitesimal deformations are integrable is that the compact Cayley submanifold is unobstructed.

Definition 6.2.3. Let (M, Ω, g) be a manifold with Spin(7)-structure and $f\colon N \to M$ a compact Cayley submanifold. Let ν be the normal bundle to $f\colon N \to M$ and let $\mathcal{P}\colon \mathcal{U} \subset C^\infty(\nu) \to C^\infty(E)$ be as in Proposition 6.2.1. We say $f\colon N \to M$ is *unobstructed* if $\operatorname{coker} D_0\mathcal{P} = 0$.

Proposition 6.2.4. *Let (M, Ω, g) be a manifold with* Spin(7)*-structure and $f\colon N \to M$ an unobstructed Cayley submanifold. Then there exists a universal family of deformations of $f\colon N \to M$, $(B, \mathcal{N}, \pi, i)$ such that $D_0\sigma\colon T_0B \to \ker D_0\mathcal{P}$ is a linear isomorphism for σ some normal parameterization of the family.*

Proof. Let (U, u) be a chart of $\operatorname{Sub}(N, M)$ containing $f\colon N \to M$ such that $\mathcal{P}\colon u(U) \subset C^\infty(\nu) \to C^\infty(E)$ is well-defined.

By assumption, $f\colon N \to M$ is unobstructed and so $\operatorname{coker} D_0\mathcal{P}^{1,\alpha} = 0$. Since $\mathcal{P}^{1,\alpha}$ is smooth, there exists an open set $V \subset C^{1,\alpha}(\nu)$ such that $\operatorname{coker} D_x\mathcal{P}^{1,\alpha}$ is zero for all $x \in V$. Furthermore we may assume that $D_x\mathcal{P}^{1,\alpha}$ is Fredholm for all $x \in V$ and, by elliptic regularity, $\operatorname{coker} D_x\mathcal{P}^{k,\alpha} = \operatorname{coker} D_x\mathcal{P}^{1,\alpha} = 0$ for all $x \in C^{k,\alpha}(\nu) \cap V$.

Now by the Implicit Mapping Theorem for Banach spaces we know that $B^{k,\alpha} = V \cap (\mathcal{P}^{k,\alpha})^{-1}(0)$ is a submanifold of $C^{k,\alpha}(\nu)$, in the sense of Definition 5.1.15, and hence the inclusion map $B^{k,\alpha} \hookrightarrow C^{k,\alpha}(\nu)$ is an initial map and a topological embedding. The continuous inclusion $C^{k,\alpha}(\nu) \hookrightarrow C^{1,\alpha}(\nu)$ induces a smooth inclusion $i_{k,\alpha}\colon B^{k,\alpha} \hookrightarrow B^{1,\alpha}$ and, by the regularity result of Theorem 4.3.21, this map is surjective.

Let $x \in B^{k,\alpha}$. The map $D_x i_{k,\alpha}\colon T_xB^{k,\alpha} \to T_xB^{1,\alpha}$ can be identified with the continuous linear inclusion $\ker D_x\mathcal{P}^{k,\alpha} \hookrightarrow \ker D_x\mathcal{P}^{1,\alpha}$, which by elliptic regularity for linear operators is surjective and hence an isomorphism. Since $B^{k,\alpha}$ are finite-dimensional, this implies that $i_{k,\alpha}$ are local diffeomorphisms and, since $i_{k,\alpha}$ are bijections, hence diffeomorphisms.

Let $B = B^{1,\alpha}$ and $\sigma_{k,\alpha}\colon B \to C^{k,\alpha}(\nu)$ denote the composition of $i_{k,\alpha}^{-1}$ with the inclusion $B^{k,\alpha} \hookrightarrow C^{k,\alpha}(\nu)$. The convenient vector space $C^\infty(\nu)$ is a limit of $C^{k,\alpha}(\nu)$ in the category of locally convex vector spaces with continuous maps. By [63, Lem. 3.8], $C^\infty(\nu)$ is a limit in **conMan** and hence the maps $\sigma_{k,\alpha}$ induce a smooth map $\sigma\colon B \to C^\infty(\nu)$, which is initial since one (in fact all) of the $\sigma_{k,\alpha}$ are initial. The space $C^\infty(\nu)$ is also a limit in the category of topological spaces since it carries the projective topology by definition. Therefore $\sigma\colon B \to C^\infty(\nu)$ is also an embedding. Furthermore $\sigma(B) = V \cap \mathcal{P}^{-1}(0)$ and $D_0\sigma\colon T_0B \to \ker D_0\mathcal{P}$ is an isomorphism by construction.

Let $(B, \mathcal{N}, \pi, i)$ be the family associated to the map $u^{-1} \circ \sigma\colon B \to \mathrm{Sub}(N, M)$. It remains to show that this family of deformations is universal. Let $(B', \mathcal{N}', \pi', i')$ be a family of deformations of $f\colon N \to M$ and let $h\colon B' \to \mathrm{Sub}(N, M)$ be the induced map of convenient manifolds. Let $B'' = h^{-1}(U \cap u^{-1}(V))$ and $(B'', \mathcal{N}'', \pi'', i'')$ be the restriction of the family to B''. Since B'' is the base of a family of deformations of Cayley submanifolds the image $(u \circ h)(B'') \subset C^\infty(\nu)$ must lie in the zero set of $\mathcal{P}$, i.e. $(u \circ h)(B'') \subset V \cap \mathcal{P}^{-1}(0) = \sigma(B)$. Since the map $\sigma\colon B \to C^\infty(\nu)$ is initial, there exists a smooth map $k\colon B'' \to B$ such that $h = u^{-1} \circ \sigma \circ k$. The family $(B'', \mathcal{N}'', \pi'', i'')$ is equivalent to the pullback of the universal family with base $\mathrm{Sub}(N, M)$ by h and therefore $(B'', \mathcal{N}'', \pi'', i'')$ is equivalent to the pullback of $(B, \mathcal{N}, \pi, i)$ by k. $\square$

Let $f\colon N \to M$ be an unobstructed compact Cayley submanifold. By the previous proposition, the universal family of deformations of $f\colon N \to M$ is finite-dimensional since $\ker D_0\mathcal{P}$ is finite-dimensional. So although the base of a family of deformations of $f\colon N \to M$ is in general a convenient manifold, and hence infinite-dimensional, the germ of a family of deformations is the pullback of a family over a finite-dimensional base.

Remark. If $f\colon N \to M$ is not necessarily unobstructed, the Local Representation Theorem, Theorem 4.3.20, shows that there is a germ of a smooth map $h\colon \ker D_0\mathcal{P} \to \operatorname{coker} D_0\mathcal{P}$ such that $D_0h = 0$ and $V \cap \mathcal{P}^{-1}(0) = h^{-1}(0)$ for a sufficiently small open set $V \subset C^\infty(\nu)$.

In general the set $h^{-1}(0) \subset \ker D_0\mathcal{P}$ will not be a manifold. However this is the local model of a C^∞-scheme [50], d-manifold [51] or derived manifold [88]. This shows that to capture the geometric structure on the whole set of Cayley submanifolds of M of type N, one should consider families, whose bases are more general than manifolds. We will not consider this problem in this thesis.

6.3 Dimension of moduli space

Let $f\colon N \to M$ be an unobstructed compact Cayley submanifold of a manifold (M, Ω, g) with Spin(7)-structure. By Proposition 6.2.4 we know that there is a universal family of deformations of $f\colon N \to M$ of dimension $\dim\ker D_0\mathcal{P} = \operatorname{ind} D_0\mathcal{P}$.

In this section we will express the index of $D_0\mathcal{P}$ in terms of familiar topological invariants of $f\colon N \to M$ using the Atiyah–Singer Index Theorem. For this section we will denote the map $D_0\mathcal{P}$ by $\mathcal{D}$.

McLean [77, Eq. (6.13)] expresses the index of $\mathcal{D}$ in terms of characteristic classes of a vector bundle on N related to ν and E but does not relate this to the self-intersection. Joyce [49, Eq. (12.12)] gives an incorrect formula for the index of $\mathcal{D}$ in terms of the signature, Euler characteristic and self-intersection.

Theorem 6.3.1. *Let $f\colon N \to M$ be a compact Cayley submanifold of a manifold with* Spin(7)*-structure. Then the index of $\mathcal{D}\colon C^\infty(\nu) \to C^\infty(E)$ is given by*

$$\operatorname{ind}\mathcal{D} = \frac{1}{2}(\sigma(N) + \chi(N)) - [N]\cdot[N], \tag{6.3}$$

where $\sigma(N)$ and $\chi(N)$ are the signature and Euler characteristic of N, respectively, and $[N]\cdot[N]$ is the self-intersection number of $f(N) \subset M$.

Symbol of $\mathcal{D}$

Recall from §4.2 that a Cayley submanifold $f\colon N \to M$ of a manifold with Spin(7)-structure naturally carries a principal bundle with structure group the isotropy group of a Cayley plane, which we denote by H. If the Spin(7)-structure is torsion-free, the adapted frame bundle carries a connection, which induces the Levi-Civita connection on TN and the normal connection on ν. We can describe the symbol of $\mathcal{D}$ explicitly in terms of representations of H. If the Spin(7)-structure is torsion-free, the operator $\mathcal{D}$ can be described in terms of the connection on $F_H^N M$ and the symbol.

The isotropy group of a Cayley plane in Spin(7) is the group $(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$. We will again label each factor by $+, -$ and $\circ$. Let us denote the representation

$$g(a) = q_+ a \overline{q}_-,$$

where $a \in \mathbb{H}$, by S_{+-} and similarly we will define representations $S_{+\circ}$ etc. The vector bundles TN, ν, and E are isomorphic to the bundles associated to the representations S_{+-}, $S_{+\circ}$ and $S_{-\circ}$ respectively.

The symbol of $\mathcal{D}$ is the map associated to the H-equivariant map from $S_{+-} \times S_{+\circ} \to S_{-\circ}$ given by

$$(v, n) \longmapsto \overline{v} k n.$$

The symbol of the adjoint of $\mathcal{D}$ is the map associated to the H-equivariant map from $S_{+-} \times S_{-\circ} \to S_{+\circ}$ given by

$$(v, e) \longmapsto k v e.$$

This description allows us to easily verify that $\mathcal{D}$ (together with its adjoint) is a Dirac operator.

We can locally lift the principal H-bundle $F_H^M N$ to a principal $\mathrm{Sp}(1)^3$-bundle. A global lift of $F_H^M N$ to a principal $\mathrm{Sp}(1)^3$-bundle exists if and only if N is spin, i.e. the second Stiefel–Whitney class of TN vanishes. Let S_+, S_-, and $S_\circ$ be locally defined complex vector bundles associated to the representations of each copy of Sp(1), where $q \in \mathrm{Sp}(1)$ acts by $q : a \mapsto a\bar{q}$ and with complex structure given by left multiplication by i. The $\mathrm{Sp}(1)^3$-equivariant, $\mathbb{C}$-bilinear map $S_+ \times S_- \to TN \otimes_{\mathbb{R}} \mathbb{C} \cong TN \oplus TN$ defined by $(a, b) \mapsto (\bar{a} j b, \bar{a} k b)$ extends by the universal property of the tensor product to a $\mathbb{C}$-linear map $S_+ \otimes_{\mathbb{C}} S_- \to TN \otimes_{\mathbb{R}} \mathbb{C}$ and it is easy to check that this is an isomorphism.

We can show that a similar result holds for ν and E and so (locally) we have isomorphisms of complex vector bundles

$$\nu \otimes_{\mathbb{R}} \mathbb{C} \cong S_+ \otimes_{\mathbb{C}} S_\circ \qquad \text{and} \qquad E \otimes_{\mathbb{R}} \mathbb{C} \cong S_- \otimes_{\mathbb{C}} S_\circ.$$

Therefore by the Atiyah–Singer Index Theorem, if we have a cover of the $F_H^M N$ bundle by a $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$-bundle, the index of $\mathcal{D}$ is

$$\operatorname{ind} \mathcal{D} = \langle \hat{A}(N) \operatorname{ch}(S_\circ), [N] \rangle. \tag{6.4}$$

In order to express this index in terms of more familiar topological invariants we will first review the relationship between $\mathrm{SO}(4)$ and $\mathrm{Sp}(1) \times \mathrm{Sp}(1)/\mathbb{Z}_2$.

Characteristic classes of Spin(4)

Recall that $\mathrm{Spin}(n)$ is defined to be the double cover of $\mathrm{SO}(n)$ and is simply connected for $n \geq 3$. In low dimensions we have 'accidental isomorphisms' between Spin groups and classical Lie groups. For example we have isomorphisms $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$ and $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$.

The isomorphism $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and the double cover $\phi \colon \mathrm{Spin}(4) \to \mathrm{SO}(4)$ can be easily understood in terms of quaternions. Recall that we can view $\mathrm{Sp}(1)$ as the group of unit quaternions. If $g = (q_+, q_-) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ we can describe a 4-dimensional representation, ϕ, of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ by

$$\phi(g)(a) = q_+ a \bar{q}_-,$$

where $a \in \mathbb{H}$. The standard inner product on $\mathbb{H}$ can be written as $(a, b) = \mathrm{Re}(a\bar{b})$ and it is easy to see that this action preserves this inner product and hence gives a map $\phi \colon \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{O}(4)$. It is also true that this map is in fact a double cover onto $\mathrm{SO}(4) \subset \mathrm{O}(4)$ and hence gives us the isomorphism $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ we described.

The covering map $\phi \colon \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ induces a map of characteristic classes, i.e. a map of cohomology of classifying spaces

$$B\phi^* \colon H^*(B\mathrm{SO}(4), \mathbb{Q}) \longrightarrow H^*(B(\mathrm{Sp}(1) \times \mathrm{Sp}(1)), \mathbb{Q}).$$

We now recall from Examples 4.4.2 and 4.4.3 that the cohomology rings of $B\mathrm{SO}(4)$ and $B\mathrm{Sp}(1)$ are given by

$$\begin{aligned} H^*(B\mathrm{SO}(4), \mathbb{Q}) &\cong \mathbb{Q}[p_1, p_2, e]/\langle p_2 - e^2 \rangle, \\ H^*(B\mathrm{Sp}(1), \mathbb{Q}) &\cong \mathbb{Q}[c_2], \end{aligned}$$

where p_1, p_2 and e are the Pontryagin and Euler classes of the fundamental representation of $\mathrm{SO}(4)$ and c_2 is the second Chern class of the fundamental complex representation of $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$. By the Künneth theorem we deduce that

$$H^*(B(\mathrm{Sp}(1) \times \mathrm{Sp}(1)), \mathbb{Q}) \cong \mathbb{Q}[c_2^+, c_2^-].$$

Lemma 6.3.2. *With notation as before we have*

$$\begin{aligned} B\phi^*(p_1) &= -2(c_2^+ + c_2^-), \\ B\phi^*(e) &= -c_2^+ + c_2^-. \end{aligned}$$

Proof. Let V be the fundamental representation of $\mathrm{SO}(4)$, and S_+ and S_- be the fundamental representations of each copy of $\mathrm{Sp}(1)$ as described before. Then we have an isomorphism of $\mathrm{Spin}(4)$-representations $V \otimes_{\mathbb{R}} \mathbb{C} = S_+ \otimes_{\mathbb{C}} S_-$ and so

$$p_1(V) = -c_2(V \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(S_+ \otimes_{\mathbb{C}} S_-) = -2(c_2(S_+) + c_2(S_-)).$$

Quaternionic conjugation is an orientation-reversing map which intertwines the representations ϕ and $\tilde{\phi}$ where $\tilde{\phi}(q_+, q_-) = \phi(q_-, q_+)$. Therefore we see that the effect of changing the orientation is to swap the two copies of $Sp(1)$. If we change the orientation of V, the Euler class changes sign and therefore we must have

$$B\phi^*(e) = \alpha(-c_2^+ + c_2^-), \tag{6.5}$$

for some $\alpha \in \mathbb{Q}$.

Now suppose the structure group of V reduces to $\mathrm{U}(2) \subset \mathrm{SO}(4)$, where the complex structure is given by left multiplication by i. Note that the orientation induced by the complex structure matches the standard orientation of $\mathbb{H}$, unlike that induced by right multiplication by i. For $g = (q_+, q_-) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ to commute with this complex structure we must have $q_+ \in \mathrm{U}(1)$ and hence the representation S_+ splits into a direct sum $L \oplus \bar{L}$ with respect to the complex structure of left multiplication by i. Therefore $c_2(S_+) = c_2(L \oplus \bar{L}) = -c_1(L) \cup c_1(L)$.

Furthermore we see that $V \cong L \otimes_{\mathbb{C}} S_-$ and therefore the Chern character formula gives

$$\begin{aligned}
\mathrm{ch}(V) &= \mathrm{ch}(L \otimes S_-) \\
&= \mathrm{ch}(L)\,\mathrm{ch}(S_-) \\
&= (1 + c_1(L) + \frac{1}{2} c_1(L) \cup c_1(L)) \cup (2 - c_2(S_-)) \\
&= 2 + 2c_1(L) + c_1(L) \cup c_1(L) - c_2(S_-)
\end{aligned}$$

and hence $c_1(V) = 2c_1(L)$ and

$$\begin{aligned}
e(V) &= c_2(V) \\
&= \frac{1}{2} c_1(V) \cup c_1(V) - c_1(L) \cup c_1(L) + c_2(S_-) \\
&= c_1(L) \cup c_1(L) + c_2(S_-) \\
&= -c_2(S_+) + c_2(S_-).
\end{aligned}$$

Since equation (6.5) must hold in general, we see that α must be 1. □

Proof of Theorem 6.3.1. Using Lemma 6.3.2 we can relate the Pontryagin and Euler classes of TN and ν to those of S_+, S_- and $S_\circ$ as follows

$$\begin{aligned}
p_1(TN) &= -2(c_2^+ + c_2^-), & p_1(\nu) &= -2(c_2^+ + c_2^\circ), \\
e(TN) &= -c_2^+ + c_2^-, & e(\nu) &= -c_2^+ + c_2^\circ.
\end{aligned}$$

We can express c_2° in terms of any 3 of $p_1(TN)$, $e(TN)$, $p_1(\nu)$ and $e(\nu)$. In particular we have

$$c_2^\circ = -\frac{1}{4} p_1(TN) - \frac{1}{2} e(TN) + e(\nu).$$

Also recall from the Hirzebruch signature theorem that the signature of an oriented compact 4-manifold is given by $\sigma(N) = \langle \frac{1}{3} p_1(TN), [N] \rangle$.

The index of $\mathcal{D}$ can now be given as

$$\begin{aligned}
\operatorname{ind}\mathcal{D} &= \langle \hat{A}(N)\operatorname{ch}(S_\circ), [N]\rangle \\
&= \langle (1 - \frac{1}{24}p_1(TN))(2 - c_2(S_\circ)), [N]\rangle \\
&= \langle (1 - \frac{1}{24}p_1(TN))(2 + \frac{1}{4}p_1(TN) + \frac{1}{2}e(TN) - e(\nu)), [N]\rangle \\
&= \langle \frac{1}{6}p_1(TN) + \frac{1}{2}e(TN) - e(\nu), [N]\rangle \\
&= \frac{1}{2}(\sigma(N) + \chi(N)) - [N]\cdot[N].
\end{aligned}$$

□

Proof of Theorem 6.0.1. The proof of Proposition 6.2.4 shows that $\mathrm{Cay}^{\mathrm{sm}}(N,M)$ is locally a finite-dimensional weak submanifold of $\mathrm{Sub}(N,M)$ of dimension $\dim\ker D_0\mathcal{P}$ and hence, by Proposition 5.1.12, is a weak submanifold of $\mathrm{Sub}(N,M)$. Furthermore $\mathrm{Cay}^{\mathrm{sm}}(N,M)$ is second countable since $\mathrm{Sub}(N,M)$ is second countable by Proposition 5.1.27. Let $\alpha \in H_4(M,\mathbb{Z})$ and suppose $f \in \mathrm{Cay}^{\mathrm{sm}}(N,M)_\alpha$. By Theorem 6.3.1 we know that the index of $D_0\mathcal{P}$ is constant on $\mathrm{Cay}^{\mathrm{sm}}(N,M)_\alpha$ since it is determined by the topology of N and the class $\alpha \in H_4(M,\mathbb{Z})$. By assumption the index of $D_0\mathcal{P}$ is equal to the dimension of $\mathrm{Cay}^{\mathrm{sm}}(N,M)_\alpha$ and hence is constant.

The pullback of the universal family on $\mathrm{Sub}(N,M)_\alpha$ by the initial inclusion map $\mathrm{Cay}^{\mathrm{sm}}(N,M)_\alpha \hookrightarrow \mathrm{Sub}(N,M)_\alpha$ is a universal family of unobstructed Cayley submanifolds of M of type N such that $[N] = \alpha$. □

Chapter 7

Examples of compact Cayley submanifolds

In this chapter we will find examples of compact Cayley submanifolds of compact manifolds with holonomy Spin(7). There are lots of examples of compact Cayley submanifolds of Spin(7)-manifolds, since any smooth compact complex surface in a Calabi–Yau 4-fold is an example of a compact Cayley submanifold, but for manifolds with holonomy Spin(7) the known examples are not so numerous. All examples of compact Cayley submanifolds of compact manifolds with holonomy Spin(7) in the literature arise as the fixed points of involutions preserving Ω.

Proposition 7.0.1 ([41, Prop. 10.8.6]). *Let M be an 8-manifold, (Ω, g) a torsion-free* Spin(7)*-structure on M, and $\tau\colon M \to M$ a nontrivial isometric involution with $\tau^*(\Omega) = \Omega$. Then each connected component of the fixed point set of τ is either a Cayley submanifold or a single point.*

Recall that the general strategy Joyce uses in his constructions of compact manifolds with holonomy Spin(7) is to glue together manifolds with torsion-free Spin(7)-structures, and then perturb the Spin(7)-structure to give a genuine torsion-free Spin(7)-structure on M. The strategy we will follow is to look for unobstructed compact Cayley submanifolds, which do not intersect any of the patching regions, and then show that if the perturbation of the Spin(7)-structure is small enough, we can perturb the Cayley submanifold also.

The analysis in this chapter will be straightforward. The main difficulty in applying this method is to show that a given Cayley submanifold is unobstructed. If the Cayley submanifold arises as a complex surface in a Calabi–Yau 4-fold, or a finite quotient of such a submanifold, we can relate the obstructions as a Cayley submanifold to those as a complex surface. We can then use techniques of complex geometry to show that a complex surface is unobstructed.

7.1 Existence of compact Cayley submanifolds

In this section we prove the necessary analytic results used later in the chapter.

Proposition 7.1.1. *Let M be an 8-manifold with* $\mathrm{Spin}(7)$*-structure* (Ω, g) *and let* $f\colon N \to M$ *be a compact Cayley submanifold. Let* $\pi\colon \nu \to N$ *be the normal bundle to* N*,* E *be the obstruction bundle of* $f\colon N \to M$ *as per Definition 4.1.7, and let* $V \subset \nu$ *and* $\phi\colon V \to M$ *be a tubular neighbourhood of* N *in* M *with* $\phi(V) \subset\subset M$*. Then there exists a* C^1 *map of Banach manifolds*

$$\mathcal{P}\colon \mathcal{U} \subset C^3(\mathcal{A}M) \times C^{1,\alpha}(\nu) \longrightarrow C^{0,\alpha}(E)$$

defined on an open neighbourhood $\mathcal{U}$ *of* $(\Omega, 0)$*, such that if* $\tilde{\Omega} \in C^\infty(\mathcal{A}M)$ *and* $s \in C^\infty(V)$*,* $\mathcal{P}(\tilde{\Omega}, s) = 0$ *if and only if* $\phi \circ s\colon N \to M$ *is Cayley with respect to* $\tilde{\Omega}$*.*

Proof. This follows from the proofs of Proposition 6.2.1 and Theorem 4.3.13. □

Theorem 7.1.2. *Let M be an 8-manifold with* $\mathrm{Spin}(7)$*-structure* (Ω, g) *and* $f\colon N \to M$ *an unobstructed compact Cayley submanifold. Let* $\pi\colon \nu \to N$ *be the normal bundle to* N*,* $V \subset \nu$ *and* $\phi\colon V \to M$ *be a tubular neighbourhood of* N *in* M*. Let* $\mathcal{P}$ *denote the* C^1 *map of Banach manifolds of Proposition 7.1.1 and let* $\mathcal{D}\colon C^{1,\alpha}(\nu) \to C^{0,\alpha}(E)$ *denote the restriction of the linearization of* $\mathcal{P}$ *to* $C^{1,\alpha}(\nu)$*. Then there exists* $\varepsilon_1, \varepsilon_2, C > 0$ *such that the following is true.*

Suppose $(\tilde{\Omega}, \tilde{g})$ *is a* $\mathrm{Spin}(7)$*-structure on* M *such that* $\|\tilde{\Omega} - \Omega\|_{C^3} \le \varepsilon_1$*. Then there exists a unique smooth section* s *of* ν *such that* $s \perp_{L^2} \ker \mathcal{D}$*,* $\|s\|_{C^{1,\alpha}} < \varepsilon_2$ *and* $\tilde{f} = \phi \circ s$ *is Cayley with respect to* $\tilde{\Omega}$*. Furthermore we have* $\|s\|_{C^{1,\alpha}} \le C\|\tilde{\Omega} - \Omega\|_{C^3}$*.*

Proof. Since $f\colon N \to M$ is unobstructed, $\mathcal{D}\colon C^{1,\alpha}(\nu) \to C^{0,\alpha}(E)$ is surjective and hence $D_0\mathcal{P}$ is also surjective. Let $C^{1,\alpha}(\nu) = \ker \mathcal{D} \oplus X$ be an L^2 splitting of $C^{1,\alpha}(\nu)$ with repsect to the induced metric on ν. Then, by the Implicit Mapping Theorem, there exists a germ of a C^1 map of Banach spaces $Q\colon C^3(\mathcal{A}M) \times \ker \mathcal{D} \to X$ such that, in a neighbourhood of $(0, 0)$, for $y \in \ker \mathcal{D}$ and $x \in X$, $\mathcal{P}(\tilde{\Omega}, y + x) = 0$ if and only if $x = Q(\tilde{\Omega}, y)$.

In particular there exists $\varepsilon_1, \varepsilon_2 > 0$ such that if $\|\tilde{\Omega} - \Omega\|_{C^3} < \varepsilon_1$, there exists a unique $s \in C^{1,\alpha}(\nu)$ such that $s \perp_{L^2} \ker \mathcal{D}$, $\|s\|_{C^{1,\alpha}} < \varepsilon_2$ and $\mathcal{P}(\tilde{\Omega}, s) = 0$. Also the map Q is C^1 and hence locally Lipschitz. Therefore there exists C, depending on ε_1, such that $\|s\|_{C^{1,\alpha}} \leq C\|\tilde{\Omega} - \Omega\|_{C^3}$.

For $\|\tilde{\Omega} - \Omega\|_{C^0}$ and $\|s\|_{C^1}$ sufficiently small, $D_{(\tilde{\Omega},s)}\mathcal{P}|_{C^{1,\alpha}(\nu)}$ is elliptic and, since the C^1 norm of s is bounded by the C^3 norm of $\tilde{\Omega} - \Omega$, by making ε_1 smaller, we can ensure that $D_{(\tilde{\Omega},s)}\mathcal{P}$ is elliptic. Then, by elliptic regularity, Theorem 4.3.21, s is smooth. Finally, by Proposition 7.1.1, we have that $\tilde{f} = \phi \circ s$ is Cayley with respect to $\tilde{\Omega}$. □

7.2 Obstructions of compact Cayley submanifolds

Suppose (X, g, ω, θ) is a Calabi–Yau 4-fold and let $\Omega = \frac{1}{2}\omega \wedge \omega + \operatorname{Re}\theta$. Recall from Proposition 2.2.10 that (Ω, g) is a torsion-free Spin(7)-structure on X.

If $Y \subset X$ is a complex surface, Y is Cayley with respect to Ω. In this section we will relate the deformation theory of Y as a complex submanifold with that of Y as a Cayley submanifold. In particular we will show that if Y is unobstructed as a complex submanifold then it is unobstructed as a Cayley submanifold.

7.2.1 Deformation theory of complex submanifolds

The deformation theory of compact complex submanifolds was first studied by Kodaira [55]. Suppose $Y \subset X$ is a compact complex submanifold of dimension n of a complex manifold X. Kodaira showed that if $H^1(Y, \nu)$ vanishes, there is a universal family of deformations of Y of dimension $\dim H^0(Y, \nu)$.

We should think of the map

$$\overline{\partial}\colon C^\infty(\nu) \longrightarrow C^\infty(\Lambda^{0,1} \otimes \nu)$$

as analogous to $\mathcal{D}$ in the Cayley case. In contrast to the Cayley case, the operator $\overline{\partial}$ is not elliptic, unless Y is a curve. In general, the operator $\overline{\partial}$ fits into an elliptic complex

$$0 \longrightarrow C^\infty(\nu) \xrightarrow{\overline{\partial}} C^\infty(\Lambda^{0,1} \otimes \nu) \xrightarrow{\overline{\partial}} C^\infty(\Lambda^{0,2} \otimes \nu) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty(\Lambda^{0,n} \otimes \nu) \longrightarrow 0,$$

which determines an elliptic operator in the usual way, given by $\overline{\partial} \oplus \overline{\partial}^*$ acting between the even and odd parts of the complex.

Suppose X is Kähler. Then the operator $\overline{\partial} \oplus \overline{\partial}^*$ (up to some constants) is a Dirac operator on $\bigoplus_i \Lambda^{0,i} \otimes \nu$, where the grading is by even and odd parts of the complex. This Dirac operator is uniquely determined by the Chern connections on TY and the normal bundle, and the symbol of the operator, see Berline et. al. [10, Prop. 3.6.7].

7.2.2 Complex surfaces as Cayley submanifolds

Lemma 7.2.1. *Let (X, g, ω, θ) be a Calabi–Yau 4-fold and $\Omega = \frac{1}{2}\omega \wedge \omega + \operatorname{Re}\theta$. Let $Y \subset X$ be a complex surface and let $F^X_{\mathrm{S(U(2)\times U(2))}}Y$ and $F^X_H Y$ be the adapted frame bundles of Y with respect to X as a complex and a Cayley submanifold, respectively.*

Then the connection on $F^X_H Y$ reduces to a connection on $F^X_{\mathrm{S(U(2)\times U(2))}}Y$.

Proof. The proof is the same as that of Corollary 4.2.4. □

Lemma 7.2.2. *Let (X, g, ω, θ) be a Calabi–Yau 4-fold and $\Omega = \frac{1}{2}\omega \wedge \omega + \operatorname{Re}\theta$. Let $Y \subset X$ be a complex surface. Let ν denote the normal bundle with its complex structure, $\nu_{\mathbb{R}}$ the underlying real vector bundle and E the obstruction bundle of Y as a Cayley submanifold as in Definition 4.1.7.*

There are parallel isomorphisms of complex vector bundles

$$(\nu_{\mathbb{R}})^{\mathbb{C}} \cong \nu \oplus (\Lambda^{0,2} \otimes \nu) \text{ and } E^{\mathbb{C}} \cong \Lambda^{0,1} \otimes \nu.$$

Proof. By Lemma 7.2.1, the statement reduces to one about representations of $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))$. Recall from §4.1 that the isotropy group H of a Cayley plane V acts on $\mathbb{R}^8$ as

$$g(v,n) = (q_+ v\bar{q}_-, -kq_+kn\bar{q}_\circ),$$

for $g = (q_+, q_-, q_\circ) \in H$ and $(v,n) \in \mathbb{H}\times\mathbb{H} \cong \mathbb{R}^8$. Let $S_+, S_-, S_\circ$ denote the fundamental complex representations of $\mathrm{Sp}(1)^3$, which is a 2-fold cover of H. Recall that as $\mathrm{Sp}(1)^3$ representations we have isomorphisms

$$V^{\mathbb{C}} \cong S_+ \otimes_{\mathbb{C}} S_-,\ (V^\perp)^{\mathbb{C}} \cong S_+ \otimes_{\mathbb{C}} S_\circ, \text{ and } E^{\mathbb{C}} \cong S_- \otimes_{\mathbb{C}} S_\circ.$$

The subgroup

$$\{(q_+, q_-, q_\circ) \in H : q_+ \in \mathbb{C}\}$$

is $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2)) \cong (\mathrm{U}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\mathbb{Z}_2$. The fundamental representation of $\mathrm{Sp}(1)_+$ splits as a sum $S_+ \cong L \oplus \overline{L}$ when restricted to $\mathrm{U}(1)_+$. As complex representations of $\mathrm{U}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ we have

$$V \cong L \otimes_{\mathbb{C}} S_- \quad\text{and}\quad V^\perp \cong \overline{L} \otimes_{\mathbb{C}} S_\circ.$$

Therefore

$$\begin{aligned}(V^\perp)^{\mathbb{C}} &\cong S_+ \otimes S_\circ & E^{\mathbb{C}} &\cong S_- \otimes S_\circ\\ &\cong (\overline{L}\otimes S_\circ) \oplus (L \otimes S_\circ) & &\cong S_- \otimes L \otimes \overline{L} \otimes S_\circ\\ &\cong V^\perp \oplus (L^2 \otimes V^\perp) & &\cong \overline{V}^* \otimes V^\perp.\\ &\cong V^\perp \oplus (\Lambda^2\overline{V}^* \otimes V^\perp) & & \end{aligned}$$

These isomorphisms descend to isomorphisms of representations of $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))$. □

Lemma 7.2.3. *Let (X, g, ω, θ) be a Calabi–Yau 4-fold and $\Omega = \frac{1}{2}\omega\wedge\omega + \mathrm{Re}\,\theta$. Let $Y \subset X$ be a complex surface.*

Via the isomorphisms $\nu^{\mathbb{C}} \cong \nu \oplus (\Lambda^{0,2}\otimes\nu)$ and $E^{\mathbb{C}} \cong \Lambda^{0,1}\otimes\nu$, the symbols of $\mathcal{D}^{\mathbb{C}}$ and $\overline{\partial}\oplus\overline{\partial}^*$ agree up to a constant.

Proof. The symbols of $\mathcal{D}$ and $\overline{\partial}\oplus\overline{\partial}^*$ are both determined by $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))$-equivariant maps $V^*\otimes(V^\perp)^{\mathbb{C}}\to E^{\mathbb{C}}$. By considering representations of $\mathrm{U}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)$, we see that $\mathrm{Hom}(V^*\otimes(V^\perp)^{\mathbb{C}},E^{\mathbb{C}})$ contains a unique trivial subrepresentation and hence the symbols agree up a constant. □

Proposition 7.2.4. *Let (X,g,ω,θ) be a Calabi–Yau 4-fold and let Ω be the admissible form $\frac{1}{2}\omega\wedge\omega+\mathrm{Re}\,\theta$. Let $Y\subset X$ be a complex submanifold and let ν be the normal bundle to Y in X. Let E and $\mathcal{D}$ be defined with respect to the admissible form Ω.*

Then there are isomorphisms of complex vector bundles $\nu^{\mathbb{C}}\cong\nu\oplus(\Lambda^{0,2}\otimes\nu)$ and $E^{\mathbb{C}}\cong\Lambda^{0,1}\otimes\nu$ such that the following diagram commutes

$$\begin{tikzcd}
C^\infty(\nu^{\mathbb{C}}) \arrow[r, "\mathcal{D}"] \arrow[d, "\wr"] & C^\infty(E^{\mathbb{C}}) \arrow[d, "\wr"] \\
C^\infty(\nu\oplus(\Lambda^{0,2}\otimes\nu)) \arrow[r, "\overline{\partial}\oplus\overline{\partial}^*"] & C^\infty(\Lambda^{0,1}\otimes\nu).
\end{tikzcd}$$

Proof. By Lemmas 7.2.2 and 7.2.3 we have that the isomorphisms $\nu^{\mathbb{C}}\cong\nu\oplus(\Lambda^{0,2}\otimes\nu)$ and $E^{\mathbb{C}}\cong\Lambda^{0,1}\otimes\nu$ are parallel and identify the symbols of the operators $\mathcal{D}$ and $\overline{\partial}\oplus\overline{\partial}^*$. In other words, the operators $\mathcal{D}$ and $\overline{\partial}\oplus\overline{\partial}^*$ agree up to zero-order terms. The form Ω is closed and it follows that τ is parallel. Therefore the zero-order term in the description of $\mathcal{D}$ in equation (6.2) vanishes. The explicit description of $\overline{\partial}\oplus\overline{\partial}^*$ in Berline et. al. [10, Prop. 3.6.7] shows that $\overline{\partial}\oplus\overline{\partial}^*$ also has no lower order terms and therefore the operators agree, up to a constant factor. □

Corollary 7.2.5. *Let (X,g,ω,θ) be a Calabi–Yau 4-fold and let Ω be the admissible form $\frac{1}{2}\omega\wedge\omega+\mathrm{Re}\,\theta$. Let $Y\subset X$ be a compact complex submanifold.*

Then

$$\begin{aligned}
\dim_{\mathbb{R}}\ker\mathcal{D} &= \dim_{\mathbb{C}} H^0(Y,\nu)+\dim_{\mathbb{C}} H^2(Y,\nu)=\dim_{\mathbb{R}} H^0(Y,\nu)\\
\dim_{\mathbb{R}}\ker\mathcal{D}^* &= \dim_{\mathbb{C}} H^1(Y,\nu).
\end{aligned}$$

In particular Y is unobstructed as a Cayley submanifold with respect to Ω if and only if Y is unobstructed as a complex submanifold.

Remark. It is interesting to note that since $H^1(Y,\nu)$ can be identified with the complexification of $\ker \mathcal{D}^*$, it follows that $H^1(Y,\nu)$ carries a real structure. This real structure depends on the Kähler metric as well as the complex structure and holomorphic volume form.

There is a non-degenerate symmetric pairing

$$q\colon H^1(Y,\nu)\times H^1(Y,\nu)\longrightarrow \mathbb{C},$$

which only depends on the complex structure and choice of holomorphic volume form. The holomorphic volume form gives an isomorphism of holomorphic vector bundles $\nu\cong K_Y\otimes\nu^*$ and then Serre duality implies

$$H^1(Y,\nu)\cong H^1(Y,K_Y\otimes\nu^*)^*\cong H^1(Y,\nu)^*.$$

Let θ denote the holomorphic volume form. In terms of basic forms $\alpha\otimes v,\beta\otimes w\in C^\infty(\Lambda^{0,1}\otimes\nu)$, the pairing is given by

$$(\alpha\otimes v,\beta\otimes w)\longmapsto\int_Y \alpha\wedge\beta\wedge(v\wedge w)\lrcorner\,\theta,$$

which is clearly symmetric.

The real symmetric bilinear form $\mathrm{Re}(q)$ is of signature $(h^1(Y,\nu),h^1(Y,\nu))$. The real structure on $H^1(Y,\nu)$ gives a splitting into real and imaginary subspaces $H^1(Y,\nu)=H^1(Y,\nu)_{\mathrm{Re}}\oplus H^1(Y,\nu)_{\mathrm{Im}}$, which is also a splitting of $H^1(Y,\nu)$ into positive definite and negative definite subspaces with respect to $\mathrm{Re}(q)$.

7.2.3 Quotients of Cayley submanifolds

Recall in the construction of compact Spin(7)-manifolds from Calabi–Yau 4-orbifolds, that the building blocks of the Spin(7)-manifolds are quotients of non-compact Calabi–Yau manifolds. Therefore we need to consider what happens to Cayley submanifolds under a quotient by a finite group preserving the Spin(7)-structure.

Proposition 7.2.6. *Let M be an 8-manifold with* Spin(7)*-structure (Ω,g), Γ a group of freely acting isometries preserving Ω, and $N\subset M$ a Γ-invariant compact Cayley submanifold.*

Then there is an action of Γ *on* ν_N *and* E_N *such that under the identification of* Γ*-invariant sections of* ν_N *and* E_N *with sections of* $\nu_{N/\Gamma}$ *and* $E_{N/\Gamma}$ *we have*

$$(\ker \mathcal{D}_N)^\Gamma \cong \ker \mathcal{D}_{N/\Gamma} \text{ and } (\ker \mathcal{D}^*_N)^\Gamma \cong \ker \mathcal{D}^*_{N/\Gamma}.$$

In particular, if N *is unobstructed in* M*,* N/Γ *is unobstructed in* M/Γ*.*

Proof. We choose a Γ-invariant tubular neighbourhood of N in M. Since Γ preserves Ω, there is a natural action of Γ on $F_{\mathrm{Spin}(7)}M$. Furthermore since N is Γ-invariant, this action descends to $F^M_H N$ and any associated vector bundle, in particular the normal bundle ν_N and the obstruction bundle E_N. We can arrange that the non-linear differential operator given by Proposition 6.2.1 and its linearization at 0, which we denote $\mathcal{D}_N$, are Γ-equivariant with respect to this action. The action of Γ therefore descends to an action on $\ker \mathcal{D}_N$.

Any Γ-invariant section of ν_N descends to a section of $\nu_{N/\Gamma}$ and any section of $\nu_{N/\Gamma}$ lifts to a unique Γ-invariant section of ν_N. With this identification, the Γ-invariant elements of $\ker \mathcal{D}_N$ are identified with the elements of $\ker \mathcal{D}_{N/\Gamma}$ and similarly

$$(\ker \mathcal{D}^*_N)^\Gamma \cong \ker \mathcal{D}^*_{N/\Gamma}.$$

□

Corollary 7.2.7. *Let* (X, g, ω, θ) *be a Calabi–Yau 4-fold and* Ω *be the admissible 4-form* $\frac{1}{2}\omega\wedge\omega + \mathrm{Re}\,\theta$*. Suppose* τ *is a freely acting anti-holomorphic involution preserving* Ω *and let* $Y \subset X$ *be a* τ*-invariant compact complex surface.*

Then

$$\dim_{\mathbb{R}} \ker \mathcal{D}_{Y/\langle\tau\rangle} = \dim_{\mathbb{C}} H^0(Y,\nu) \text{ and } \dim_{\mathbb{R}} \ker \mathcal{D}^*_{Y/\langle\tau\rangle} = \frac{1}{2}\dim_{\mathbb{C}} H^1(Y,\nu).$$

Proof. The action of τ lifts to an action on ν and E. The action of τ on ν is $\mathbb{C}$-antilinear and hence the $\mathbb{C}$-linear extension of τ to $(\nu_{\mathbb{R}})^{\mathbb{C}} \cong \nu \oplus \overline{\nu} \cong \nu \oplus (\Lambda^{0,2}\otimes\nu)$ maps ν to $\Lambda^{0,2}\otimes\nu$ and vice versa. This involution gives a $\mathbb{C}$-linear isomorphism $H^0(Y,\nu) \to H^2(Y,\nu)$ and therefore

$$\dim_{\mathbb{R}} \ker \mathcal{D}_{Y/\langle\tau\rangle} = \dim_{\mathbb{R}} (\ker \mathcal{D}_Y)^\tau = \dim_{\mathbb{C}} (\ker \mathcal{D}^{\mathbb{C}}_Y)^\tau = \dim_{\mathbb{C}} H^0(Y,\nu).$$

The index of an elliptic operator is multiplicative under finite coverings, see for example [4, Eq. (1.2)]. In particular since Y is a two-fold covering we have

$$\begin{aligned}\operatorname{ind}\mathcal{D}_{Y/\langle\tau\rangle} &= \frac{1}{2}\operatorname{ind}\mathcal{D}_Y\\ \dim\ker\mathcal{D}_{Y/\langle\tau\rangle} - \dim\ker\mathcal{D}^*_{Y/\langle\tau\rangle} &= \frac{1}{2}(\dim H^0(Y,\nu) - \dim H^1(Y,\nu) + \dim H^2(Y,\nu))\end{aligned}$$

and therefore $\dim_{\mathbb{R}}\ker\mathcal{D}^*_{Y/\langle\tau\rangle} = \frac{1}{2}\dim_{\mathbb{C}} H^1(Y,\nu)$.

Note that $\dim_{\mathbb{C}} H^1(Y,\nu)$ is necessarily even since $\operatorname{ind}\mathcal{D}_{Y/\langle\tau\rangle}$ is an integer. It is essential that τ acts freely so that $Y \to Y/\langle\tau\rangle$ is a covering of manifolds. There is an index formula for non-free actions, which contains contributions from the fixed loci, but we will not need it here. □

7.3 Examples of compact Cayley submanifolds

Let (Z,Ω,g) be a Spin(7)-orbifold with singular points $p_1,\dots,p_k$ modelled on $\mathbb{R}^8/\Gamma$ of Example 2.4.1 and let $(M^t_\Gamma,\Omega^t_\Gamma,g^t_\Gamma,\pi^t)$ be the scaled version of the ALE Spin(7)-manifold of Example 2.4.1. Let $\zeta > 0$ be chosen such that, for each p_i, the exponential map is a diffeomorphism on the ball of radius 2ζ around p_i and let $U^t_\Gamma = (\pi^t)^{-1}(B_{t^{4/5}\zeta})$. Joyce [41, §15.2.2] defines a desingularization of Z, M^t and a Spin(7)-structure (Ω^t,g^t) on M^t, with the property that for each singular point p_i, there is an isometric embedding

$$\phi^t_i\colon U^t_\Gamma \longrightarrow M^t$$

such that $(\phi^t_i)^*\Omega^t = \Omega^t_\Gamma$.

Recall that M_Γ is the quotient of $K_{\mathbb{CP}^3}$ by a free antiholomorphic involution β and the ALE Spin(7)-structure on M_Γ is given by an ALE Calabi–Yau structure on $K_{\mathbb{CP}^3}$. Therefore if $Y \subset \mathbb{CP}^3$ is any β-invariant complex surface, $Y/\langle\beta\rangle \subset M_\Gamma$ is a Cayley submanifold.

Definition 7.3.1. Let (Z,Ω,g) be a Spin(7)-orbifold with singular points $p_1,\dots,p_k$ modelled on $\mathbb{R}^8/\Gamma$ and let M^t and (Ω^t,g^t) be the desingularization and Spin(7)-structure of [41, §15.2.2]. Let $Y \subset \mathbb{CP}^3$ be a smooth β-invariant complex surface, let $N = Y/\langle\beta\rangle$ and let p_i be a singularity of Z.

We define $f_i^t \colon N \to M^t$ to be the composition

$$N \hookrightarrow \mathbb{CP}^3/\langle\beta\rangle \hookrightarrow U_\Gamma^t \xrightarrow{\phi_i^t} M^t.$$

The submanifold $f_i^t \colon N \to M^t$ is Cayley with respect to the Spin(7)-structure Ω^t since $(\phi_i^t)^*\Omega^t = \Omega_\Gamma^t$.

We remark that the Fermat hypersurfaces of degree $2d$

$$\{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : z_0^{2d} + z_1^{2d} + z_2^{2d} + z_3^{2d} = 0\}$$

are examples of smooth β-invariant hypersurfaces. In fact a smooth β-invariant hypersurface is necessarily of even degree by the following lemma.

Lemma 7.3.2. *Suppose $Y \subset \mathbb{CP}^3$ is an irreducible β-invariant hypersurface. Then Y is of even degree.*

Proof. Let f be a defining homogeneous polynomial of Y. Let $\tilde{\beta} \colon \mathbb{C}^4 \to \mathbb{C}^4$ be a lift of β to $\mathbb{C}^4$ such that $\tilde{\beta}^2 = -1$ and let σ denote the map $f \mapsto \overline{\tilde{\beta}^* f}$.

Then, since Y is β-invariant, we must have

$$\sigma(f)^e = gf$$

for some $g \in \mathbb{C}[z_0, \ldots, z_3]$ and $e \in \mathbb{N}$. By assumption f is irreducible and it is easy to see that $\sigma(f)$ is also. Therefore

$$\sigma(f) = \lambda f$$

for some $\lambda \in \mathbb{C}^*$. Now $\sigma^2(f) = (-1)^d f = |\lambda|^2 f$ and hence $|\lambda| = 1$ and d is even. □

Remark. In fact this is also true for reducible hypersurfaces but we will not discuss this case since we only consider embedded Cayley submanifolds.

Lemma 7.3.3. *Let $Y \subset \mathbb{CP}^3$ be a smooth hypersurface of degree d. Then Y is unobstructed as a surface in $K_{\mathbb{CP}^3}$.*

Proof. The exact sequence of normal bundles

$$0 \longrightarrow \nu_{Y/\mathbb{CP}^3} \longrightarrow \nu_{Y/K_{\mathbb{CP}^3}} \longrightarrow \nu_{\mathbb{CP}^3/K_{\mathbb{CP}^3}}|_Y \longrightarrow 0 \tag{7.1}$$

gives us a long exact sequence in sheaf cohomology, part of which is

$$\ldots \longrightarrow H^1(Y, \nu_{Y/\mathbb{CP}^3}) \longrightarrow H^1(Y, \nu_{Y/K_{\mathbb{CP}^3}}) \longrightarrow H^1(Y, \nu_{\mathbb{CP}^3/K_{\mathbb{CP}^3}}|_Y) \longrightarrow \ldots . \tag{7.2}$$

The normal bundle to Y in $\mathbb{CP}^3$ is isomorphic to $\mathcal{O}_Y(d)$ since Y is a degree d hypersurface. The normal bundle $\nu_{\mathbb{CP}^3/K_{\mathbb{CP}^3}}$ is isomorphic to $K_{\mathbb{CP}^3} \cong \mathcal{O}(-4)$ since the bundle $TK_{\mathbb{CP}^3}$ splits when restricted to $\mathbb{CP}^3$. Therefore $\nu_{\mathbb{CP}^3/K_{\mathbb{CP}^3}}|_Y \cong \mathcal{O}_Y(-4)$. By Serre duality and the adjunction formula we have

$$\begin{aligned} H^1(Y, \nu_{\mathbb{CP}^3/K_{\mathbb{CP}^3}}|_Y) &\cong H^1(Y, K_Y \otimes \mathcal{O}_Y(-4)^*)^* \\ &\cong H^1(Y, \mathcal{O}_Y(d-4) \otimes \mathcal{O}_Y(4))^* \\ &\cong H^1(Y, \mathcal{O}_Y(d))^* \\ &\cong H^1(Y, \nu_{Y/\mathbb{CP}^3})^* \end{aligned}$$

We recall that $H^1(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(d)) = 0$ for all $d \in \mathbb{Z}$, see for example [32, Th. 5.1]. Then the long exact sequence associated to short exact sequence of sheaves on $\mathbb{CP}^3$

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^3} \longrightarrow \mathcal{O}_{\mathbb{CP}^3}(d) \longrightarrow \mathcal{O}_Y(d) \longrightarrow 0$$

shows that $H^1(\mathbb{CP}^3, \mathcal{O}_Y(d)) = 0$ and since the inclusion of Y in $\mathbb{CP}^3$ is a closed immersion we have $H^1(Y, \mathcal{O}_Y(d)) = H^1(\mathbb{CP}^3, \mathcal{O}_Y(d)) = 0$. Therefore $H^1(Y, \nu_{Y/\mathbb{CP}^3}) = H^1(Y, \nu_{\mathbb{CP}^3/K_{\mathbb{CP}^3}}) = 0$ and so by equation (7.2) we see $H^1(Y, \nu_{Y/K_{\mathbb{CP}^3}}) = 0$ and Y is unobstructed in $K_{\mathbb{CP}^3}$. □

Theorem 7.3.4. *Let (Z, Ω, g) be a* Spin(7)*-orbifold with singularities modelled on $\mathbb{R}^8/\Gamma$ where Γ acts as in Example 2.4.1, M^t and (Ω^t, g^t) the manifold with* Spin(7)*-structure defined in [41, §15.2.2] and $\tilde{\Omega}^t$ the torsion-free* Spin(7)*-structure on M^t given by [41, Th. 13.6.1].*

Let $Y \subset \mathbb{CP}^3$ be a smooth β-invariant hypersurface and $N = Y/\langle\beta\rangle$. Let $p_i \in Z$ be a singular point of Z and let $f_i^t \colon N \to M^t$ be as in Definition 7.3.1.

Then there exists $\kappa' > 0$ such that if $t \in (0, \kappa']$, there exists $\tilde{f}_i^t \colon N \to M^t$, isotopic to $f_i^t \colon N \to M^t$, which is Cayley with respect to $\tilde{\Omega}^t$.

Proof. Note that the volume of $f_i^t(N)$ tends to zero as $t \to 0$. To apply our existence results we will rescale the Spin(7)-structure so that the Spin(7)-structure is independent of t in a neighbourhood of $f_i^t(N)$.

Let $V \subset M_\Gamma^t$ be the subset $(\pi^t)^{-1}(B_{t\zeta})$. Note that $V \subset M_\Gamma^t$ is independent of t since $|\pi^t(x)| = t|\pi(x)|$. Furthermore $(\phi_i^t)^*\Omega^t = \Omega_\Gamma^t = t^4\Omega_\Gamma$ on V and therefore $t^{-4}(\phi_i^t)^*\Omega^t = \Omega_\Gamma$ is independent of t by definition. Theorem 7.1.2 gives ε such that if Ω is a torsion-free Spin(7)-structure on V such that $\|\Omega - \Omega_\Gamma\|_{C^3} < \varepsilon$ then there exists $\tilde{f}\colon N \to V$, isotopic to the inclusion $f\colon N \to V$, which is Cayley with respect to Ω.

Let $\tilde{\Omega}_\Gamma^t = t^{-4}(\phi_i^t)^*\tilde{\Omega}$ and let $\alpha \in (0, 1/5)$. Theorem 2.4.4 gives a κ, K such that if $t \in (0, \kappa]$ then $\|t^{-4}\tilde{\Omega}^t - t^{-4}\Omega^t\|_{C^{0,\alpha}} \leq Kt^{1/3}$. Although it is not stated in [41, Th. 13.6.1], it is apparent from the proof that the torsion-free Spin(7)-structure $(\tilde{\Omega}, \tilde{g})$ satisfies $\pi_1^4(\tilde{\Omega} - \Omega) = \pi_7^4(\tilde{\Omega} - \Omega) = 0$. This is required to make the problem elliptic. Then Theorem 2.4.5 gives ε', K' such that if $\|t^{-4}\tilde{\Omega}^t - t^{-4}\Omega^t\|_{C^{0,\alpha}} \leq \varepsilon'$ then $\|\tilde{\Omega}_\Gamma^t - \Omega_\Gamma\|_{C^3} \leq K'\|t^{-4}\tilde{\Omega}^t - t^{-4}\Omega^t\|_{C^{0,\alpha}}$.

Let $\kappa' = \min((\frac{\varepsilon'}{K})^3, (\frac{\varepsilon}{K'K})^3)$. Then for $t \in (0, \kappa']$ there exists $\tilde{f}\colon N \to V$, isotopic to $f\colon N \to V$, which is Cayley with respect to $\tilde{\Omega}_\Gamma^t$. Therefore $\tilde{f}_i^t = \phi_i^t \circ \tilde{f}$ is Cayley with respect to $\tilde{\Omega}^t$. □

Note that while we have studied Cayley submanifolds of M coming from β-invariant complex surfaces in the ALE Calabi–Yau manifold $K_{\mathbb{CP}^3}$, the same methods would immediately apply to show the existence of Cayley submanifolds corresponding to unobstructed τ-invariant complex surfaces in X, which do not intersect any of the singular points. We chose to study complex surfaces in $K_{\mathbb{CP}^3}$ simply because it immediately gives examples of smooth compact Cayley submanifolds in any Spin(7)-manifold constructed from a Spin(7)-orbifold with singularities modelled on $\mathbb{R}^8/\Gamma$.

7.3.1 Topological invariants

Let $f\colon N \to M$ be a compact Cayley submanifold constructed using Theorem 7.3.4. We now wish to determine some topological invariants of $f\colon N \to M$ and in particular the dimension of the moduli space of Cayley embeddings.

Proposition 7.3.5. *Let* (M, Ω, g) *be a* Spin(7)*-manifold constructed from a Calabi–Yau 4-orbifold with antiholomorphic involution satisfying Condition 2.4.2. Let* $Y \subset \mathbb{CP}^3$ *be a smooth* β*-invariant degree* d *hypersurface,* $N = Y/\langle\beta\rangle$*, and let* $f\colon N \to M$ *be a compact Cayley submanifold constructed using Theorem 7.3.4.*

Then the moduli space of compact Cayley submanifolds of M *of type* N *with homology class* $[f(N)]$ *is of dimension* $\frac{1}{6}d(11+6d+d^2)$*.*

Proof. Let $\alpha \in H^2(\mathbb{CP}^3, \mathbb{Z})$ be the first Chern class of $\mathcal{O}(1)$. From the normal bundle sequence

$$0 \longrightarrow TY \longrightarrow T\mathbb{CP}^3|_Y \longrightarrow \mathcal{O}(d)|_Y \longrightarrow 0$$

and the Whitney formula it follows that

$$c(Y) = (1 + (4-d)\alpha + (6 - 4d + d^2)\alpha^2)|_Y.$$

The Euler characteristic and signature of Y are given in terms of the Chern numbers as

$$\chi(Y) = c_2(Y) \quad \text{and} \quad \sigma(Y) = \frac{1}{3}(c_1(Y)^2 - 2c_2(Y)).$$

Using the normal bundle sequence (7.1) and the Whitney formula we can determine $c(\nu)$ in terms of α. This determines the self-intersection since

$$[Y]\cdot[Y] = c_2(\nu).$$

In terms of d we have

$$\chi(Y) = d(6-4d+d^2)\,, \quad \sigma(Y) = \frac{1}{3}d(4-d^2) \quad \text{and} \quad [Y]\cdot[Y] = 4d^2.$$

Finally using Theorem 6.3.1 and the fact that the index is multiplicative under finite coverings we find

$$\operatorname{ind}\mathcal{D}_N = \frac{1}{6}d(11+6d+d^2).$$

□

Remark. By Bertini's theorem, a generic degree d hypersurface is smooth and hence the smooth hypersurfaces form a dense open subset of the projective space $\mathbb{P}(H^0(\mathbb{CP}^3, \mathcal{O}(d)))$

of (complex) dimension $\binom{d+3}{3}-1 = \frac{1}{6}d(11+6d+d^2)$. The smooth β-invariant hypersurfaces of degree d form a dense open subset of a real projective space of real dimension $\frac{1}{6}d(11+6d+d^2)$, which agrees with the dimension calculation of Proposition 7.3.5.

From the Lefchetz hyperplane theorem we know that $\pi_1(Y) = 0$ and therefore $\pi_1(N) = \mathbb{Z}_2$. Since $b^1(N) = 0$ we can deduce both $b^2_+(N)$ and $b^2_-(N)$ from $\chi(N)$ and $\sigma(N)$. Both the Euler characteristic and the signature are multiplicative under finite coverings and hence we find

$$\begin{aligned} \chi(N) &= \frac{1}{2}d(6-4d+d^2) & \sigma(N) &= \frac{1}{6}d(4-d^2) \\ b^2_+(N) &= \frac{1}{6}(d^3-6d^2+11d-6) & b^2_-(N) &= \frac{1}{6}(2d^3-6d^2+7d-6). \end{aligned}$$

7.3.2 A variation on this construction

The method of desingularizing compact Spin(7)-orbifolds with singularities modelled on $\mathbb{R}^8/\Gamma$ described by Joyce in [41, §15.2.2], with suitable changes, gives a method of desingularizing any compact Spin(7)-orbifold with isolated singularities, provided we have an appropriate ALE Spin(7)-manifold for each singularity.

Let (Z, Ω, g) be a Spin(7)-orbifold with singularities $p_1, \ldots, p_k$ modelled on either $\mathbb{R}^8/\Gamma$ or $\mathbb{C}^4/\mathbb{Z}_4$. Let $(M^t_\Gamma, \Omega^t_\Gamma, g^t_\Gamma, \pi^t_\Gamma)$ be the scaled version of the ALE Spin(7)-manifold of Example 2.4.1. Let $(X^t_{\mathbb{Z}_4}, g^t_{\mathbb{Z}_4}, \omega^t_{\mathbb{Z}_4}, \theta^t_{\mathbb{Z}_4}, \pi^t_{\mathbb{Z}_4})$ be the scaled version of the ALE Calabi–Yau 4-manifold of Example 2.3.7 and let $M^t_{\mathbb{Z}_4} = X^t_{\mathbb{Z}_4}$ and $\Omega^t_{\mathbb{Z}_4} = \frac{1}{2}\omega^t_{\mathbb{Z}_4} \wedge \omega^t_{\mathbb{Z}_4} + \operatorname{Re}\theta^t_{\mathbb{Z}_4}$ so $(M^t_{\mathbb{Z}_4}, \Omega^t_{\mathbb{Z}_4}, g^t_{\mathbb{Z}_4}, \pi^t_{\mathbb{Z}_4})$ is an ALE Spin(7)-manifold asymptotic to $\mathbb{C}^4/\mathbb{Z}_4$ with its standard Spin(7)-structure. Following the construction of Joyce in [41, §15.2.2], we can define a resolution M^t of Z and a Spin(7)-structure with small torsion on M^t using M^t_Γ to resolve $\mathbb{R}^8/\Gamma$ and $M^t_{\mathbb{Z}_4}$ to resolve $\mathbb{C}^4/\mathbb{Z}_4$.

Let $\zeta > 0$ be chosen such that, for each p_i, the exponential map is a diffeomorphism on the ball of radius 2ζ around p_i and let $U^t_\Gamma = (\pi^t_\Gamma)^{-1}(B_{t^{4/5}\zeta})$ as before and $U^t_{\mathbb{Z}_4} = (\pi^t_{\mathbb{Z}_4})^{-1}(B_{t^{4/5}\zeta})$. The desingularization is defined so that for each singular point p_i modelled on $\mathbb{C}^4/\mathbb{Z}_4$, there is an isometric embedding

$$\phi^t_i\colon U^t_{\mathbb{Z}_4} \longrightarrow M^t$$

such that $(\phi_i^t)^*\Omega^t = \Omega^t_{\mathbb{Z}_4}$.

Definition 7.3.6. Let (Z, Ω_Z, g_Z) be a Spin(7)-orbifold with singular points $p_1, \dots, p_k$ modelled on either $\mathbb{R}^8/\Gamma$ or $\mathbb{C}^4/\mathbb{Z}_4$ and let M^t and (Ω^t, g^t) be the desingularization and Spin(7)-structure defined in analogy with [41, §15.2.2]. Let $Y \subset \mathbb{CP}^3$ be a smooth hypersurface and suppose the singularity p_i is locally modelled on $\mathbb{C}^4/\mathbb{Z}_4$.

We define $f_i^t \colon Y \to M^t$ to be the composition

$$Y \hookrightarrow \mathbb{CP}^3 \hookrightarrow U^t_{\mathbb{Z}_4} \xrightarrow{\phi_i^t} M^t.$$

In analogy with Theorem 7.3.4 we can prove the following:

Theorem 7.3.7. *Let (Z, Ω, g) be a* Spin(7)*-orbifold with singularities modelled on $\mathbb{R}^8/\Gamma$ or $\mathbb{C}^4/\mathbb{Z}_4$, M^t and (Ω^t, g^t) the manifold with* Spin(7)*-structure defined in analogy with [41, §15.2.2] and $\tilde{\Omega}^t$ the torsion-free* Spin(7)*-structure on M^t given by [41, Th. 13.6.1]. Let $Y \subset \mathbb{CP}^3$ be a smooth complex surface and suppose the singularity p_i is modelled on $\mathbb{C}^4/\mathbb{Z}_4$. Let $f_i^t \colon Y \to M^t$ be as in Definition 7.3.6.*

Then there exists $\kappa' > 0$ such that if $t \in (0, \kappa']$, there exists $\tilde{f}_i^t \colon Y \to M^t$, isotopic to $f_i^t \colon Y \to M^t$, which is Cayley with respect to $\tilde{\Omega}^t$.

From our previous topological calculations, we can find the dimension of the moduli space of Cayley submanifolds constructed using Theorem 7.3.7.

Proposition 7.3.8. *Let (M, Ω, g) be a* Spin(7)*-manifold constructed from a* Spin(7)*-orbifold with singularities modelled on $\mathbb{R}^8/\Gamma$ or $\mathbb{C}^4/\mathbb{Z}_4$ and suppose there is at least one singularity of type $\mathbb{C}^4/\mathbb{Z}_4$. Let $Y \subset \mathbb{CP}^3$ be a smooth degree d hypersurface, and let $f \colon Y \to M$ be a compact Cayley submanifold constructed using Theorem 7.3.7.*

Then moduli space of compact Cayley submanifolds of M of type Y with homology class $[f(Y)]$ is of dimension $\frac{1}{3}d(11 + 6d + d^2)$.

7.4 Future work and open problems

7.4.1 Examples of families of Cayley submanifolds

In this chapter we found examples of Cayley submanifolds by deforming a given Cayley submanifold as the Spin(7)-structure varies. The next step would be to consider how the moduli space $\mathrm{Cay}(N, M)$ varies as the Spin(7)-structure varies.

We have a good understanding of the hypersurfaces in $\mathbb{CP}^3$ and one could hope to use this to understand the moduli space of smooth Cayley submanifolds constructed using Theorem 7.3.4 or Theorem 7.3.7. In particular the moduli space of smooth hypersurfaces of $\mathbb{CP}^3$ of degree 1 is *compact* and diffeomorphic to $\mathbb{CP}^3$. If $\tilde{f}_i^t \colon N \to M^t$ is a Cayley submanifold constructed from a smooth hypersurface of degree 1 using Theorem 7.3.7, for t sufficiently small, the connected component of $\mathrm{Cay}(N, M^t)$ containing $\tilde{f}_i^t$ will be diffeomorphic to $\mathbb{CP}^3$.

7.4.2 AC and CS Cayley submanifolds

In general, families of Cayley submanifolds, whether they are obstructed or not, will have non-compact bases, and to compactify them we would need to add Cayley submanifolds with singular behaviour.

Joyce [44–48] developed a theory of special Lagrangian manifolds with isolated conical singularities (or CS special Lagrangians) and asymptotically conical special Lagrangian manifolds (or AC special Lagrangians). In these series of papers he proves an existence result, [47, Th. 5.3], which he uses to desingularize CS special Lagrangians.

Lotay [72] has proved an analogous existence result for coassociative submanifolds building on his earlier work on coassociative submanifolds with isolated conical singularities and AC coassociative submanifolds [70, 71].

The theory of CS and AC Cayleys will have a different flavour since Cayley submanifolds are not always unobstructed. Lotay [73] has also studied the deformation theory of AC associative submanifolds. The virtual dimension of the moduli space of deformations of AC associative submanifolds depends only on the link of the asymptotic cone of the submanifold. In the Cayley case, we should expect the virtual dimension of the moduli space of AC Cayley submanifolds to be the sum of a topological contribution, as in the compact case, and an analytic contribution from the cone at infinity. Lotay [74] has studied Cayley cones in $\mathbb{R}^8$ and classified those Cayley cones, whose links are the orbits of 3-dimensional Lie subgroups of Spin(7).

Suppose (X, τ) is a Calabi–Yau 4-orbifold with antiholomorphic involutions satisfying Condition 2.4.2. Let $Y \subset X$ be a τ-invariant complex surface with isolated conical singularities, which does not intersect any of the singular points of X. If one had a deformation theory of CS Cayleys one could hope to construct examples of CS Cayleys in the Spin(7)-manifolds constructed from (X, τ) by deforming $Y/\langle\tau\rangle$. The relevant papers in the special Lagrangian and coassociative cases are [46] and [70], respectively.

7.4.3 Simultaneous desingularizations

Suppose (M, Ω, g) is a Spin(7)-orbifold and $N \subset M$ a Cayley suborbifold, that is we can find orbifold charts on M such that N lifts to a Cayley submanifold. We could attempt to desingularize M and N at the same time by gluing in appropriate submanifolds of the ALE Spin(7)-manifolds used to desingularize M.

Chan [17,18] used the results of Joyce to study the simultaneous desingularization of Calabi–Yau 3-folds and special Lagrangian 3-folds with conical singularities. One would hope that, modulo a theory of CS and AC Cayleys, one could use similar techniques to perform such a simultaneous desingularization.

In particular let $(M_\Gamma, \Omega_\Gamma, g_\Gamma, \pi)$ be the ALE Spin(7)-manifold of Example 2.4.1 and suppose $V \subset \mathbb{R}^8$ is a Γ-invariant Cayley plane, which is necessarily complex with respect to either of the complex structures described in Example 2.4.1. Then there exists a Cayley submanifold $N_\Gamma \subset M_\Gamma$, asymptotic to V/Γ, which is given as the quotient of the proper transform of $V \subset \mathbb{C}^4/\Gamma$ by the involution β. Suppose (M, Ω, g) is a Spin(7)-orbifold with singularities modelled on $\mathbb{R}^8/\Gamma$ and suppose $N \subset M$ is a

Cayley suborbifold of M. Then we could attempt to use N_Γ to desingularize N at the same time as using M_Γ to desingularize M.

7.4.4 Fibrations of Spin(7)-manifolds

If one had a good understanding of Cayley submanifolds as described above then one could hope to construct a fibration of a Spin(7)-manifold by Cayley submanifolds, some of which would have isolated conical singularities. Kovalev [57] has proposed a construction of a fibration of a class of G_2-manifolds, whose generic fibre is a nonsingular coassociative submanifold. The outline below is due to Joyce and is mentioned in [49, §12.5.3].

Let $\Gamma \subset \text{Spin}(7)$ be the group of Example 2.4.1 and let M_Γ denote a Spin(7)-manifold asymptotic to $\mathbb{R}^8/\Gamma$. Recall that M_Γ is the quotient of $K_{\mathbb{CP}^3}$ by an anti-holomorphic involution β. The twistor fibration of $\mathbb{CP}^3$ is a fibration $\pi_{\mathbb{H}}\colon \mathbb{CP}^3 \to S^4$ of $\mathbb{CP}^3$ by complex lines. We can understand the map $\pi_{\mathbb{H}}\colon \mathbb{CP}^3 \to S^4$ as the map sending a complex line $l \in \mathbb{CP}^3$ to the unique quaternionic line $l \oplus jl \in \mathbb{HP}^1 \cong S^4$ containing l. The composition of the twistor fibration with the projection of $K_{\mathbb{CP}^3}$ to $\mathbb{CP}^3$ is a fibration $f\colon K_{\mathbb{CP}^3} \to S^4$ of $K_{\mathbb{CP}^3}$ by complex surfaces. The fibration f is β-invariant and descends to a fibration f_Γ of M_Γ by Cayley submanifolds.

Let us identify $\mathbb{R}^8$ with $\mathbb{H}^2$ so that Γ acts as the group of unit quaternions. The fibration of $\mathbb{H}^2 \setminus \{0\}$ by quaternionic lines is Γ invariant and hence descends to a fibration of $\mathbb{R}^8/\Gamma \setminus \{0\}$ by flat Cayley submanifolds. The restriction of the fibration f_Γ to $M_\Gamma \setminus \pi_\Gamma^{-1}(0)$ can be identified, using π_Γ, with this fibration of $\mathbb{R}^8/\Gamma \setminus \{0\}$. Recall that we can find a second ALE Spin(7)-manifold, M'_Γ, asymptotic to $\mathbb{R}^8/\Gamma$ by considering a different complex structure on $\mathbb{R}^8$, with which β acts holomorphically. The fibration of M'_Γ by Cayley submanifolds, away from $(\pi'_\Gamma)^{-1}(0)$, can also be identified with the fibration of $\mathbb{R}^8/\Gamma \setminus \{0\}$. In this way both ALE Spin(7)-manifolds admit fibrations which, away from a closed compact set, are equal to the same fibration of $\mathbb{R}^8/\Gamma \setminus \{0\}$.

Now suppose M is a Spin(7)-orbifold with singularities of the type $\mathbb{R}^8/\Gamma$, which admits a fibration by Cayley submanifolds such that the fibration is asymptotic to that of $\mathbb{R}^8/\Gamma \setminus \{0\}$ at a singular point. The idea would be to desingularize M using

M_Γ and at the same time patch together the fibration of M with that of M_Γ by Cayley submanifolds.

Some of the examples of Calabi–Yau 4-orbifolds in weighted projective spaces admit suitable fibrations. For example, let $X \subset \mathbb{CP}^5_{1,1,1,1,4,4}$ be a τ-invariant well-formed quasismooth hypersurface of degree 12 so that (X, τ) satisfies Condition 2.4.2. The restriction of the projection map $\pi \colon \mathbb{CP}^5_{1,1,1,1,4,4} \setminus \mathbb{CP}^1 \to \mathbb{CP}^3$ to $X \setminus \{p_1, \dots, p_k\}$, which maps $[z_0, \dots, z_5] \mapsto [z_0, \dots, z_3]$, defines a fibration of $X \setminus \{p_1, \dots, p_k\}$ by complex curves. If we compose this map with the twistor fibration of $\mathbb{CP}^3$ to S^4 we get a τ-invariant map $X \setminus \{p_1, \dots, p_k\} \to S^4$ such that the preimage of any point is a τ-invariant complex surface in X and hence the set of smooth points of the complex surface is a Cayley submanifold. This fibration descends to a fibration of $Z = X/\langle\tau\rangle$ whose generic fibres are Cayley suborbifolds.

We can holomorphically identify X in a neighbourhood of a singular point with a neighbourhood of the origin in $\mathbb{C}^4/\mathbb{Z}_4$ so that the fibration is isomorphic to the composition $(\mathbb{C}^4/\mathbb{Z}_4) \setminus \{0\} \to \mathbb{CP}^3 \to S^4$. After changing coordinates so that the metric osculates to order two at the singular point, we find that the fibration is asymptotic, in a suitable sense, to the fibration of $\mathbb{C}^4/\mathbb{Z}_4 \setminus \{0\}$. Then we could use either ALE Spin(7)-manifold to resolve the singularity while, at the same time, use methods similar to Chan [17, 18] to patch together the Cayley submanifolds.

There will necessarily be singular complex surfaces in the fibration of $X \setminus \{p_1, \dots, p_k\}$. We should expect that there is a codimension 2 subset of S^4 where the complex surfaces have isolated conical singularities, in particular modelled on the cone in $\mathbb{C}^3$ given by

$$ax^2 + by^2 + cz^2 = 0,$$

for $a, b, c \in (0, \infty)$. However isolated singularities, which are not locally modelled on a cone with an isolated singularity at the origin, appear in codimension 4 and, so far as the author knows, there are no appropriate analytical tools developed to handle these types of singularities at present.

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