COMPACTLY SUPPORTED RADIAL BASIS FUNCTIONS: HOW AND WHY ? *

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Abstract. Compactly supported basis functions are widely required and used in many applications. We explain why radial basis functions are preferred to multi-variate polynomials for scattered data approximation in high-dimensional space and give a brief description on how to construct the most commonly used compactly supported radial basis functions—the Wendland functions and the new found missing wendland functions. One can construct a compactly supported radial basis function with required smoothness according to the procedure described here without sophisticated mathematics. Very short programs and extended tables for compactly supported radial basis functions are supplied.

Key words. Compactly supported radial basis functions, high dimensional approximation, scattered data approximation, Wendland functions, missing Wendland functions.

AMS subject classifications. 00A02, 26A33, 33C90, 41A05, 41A30, 41A63, 65D05, 97N50

1. Introduction. Recent years have witnessed that radial basis functions are powerful tools for scattered data approximation in high dimensional space. Radial basis functions have been successfully applied in many applications, from 3D surface reconstruction [6] to geodesy, geography, hydrology and digital terrain modelling [14] [15][16][26]; form sampling [29], signal processing [1][27], machine learning [17][30] to neural networks and artificial intelligence [11][25][28], as well as to kinds of mesh-free methods for solving PDEs [8][9][10][19][20][42][44][33]. Although these application arise from various disciplines, they share the same fundamental mathematical problem: interpolation—finding a function $s(\mathbf{x})$ which could interpolate observations $f_1, f_2, \dots, f_n$ on related data points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, i.e. $s(\mathbf{x}_i) = f_i$,for $i = 1, 2, \dots, n$, where $\mathbf{x}_i \in \mathbb{R}^d, i = 1, 2, \dots, n$. We shall see that this problem in high dimensional space is not trivial.

We aim to approximate $s(\mathbf{x})$ by a combination of simple functions, say, $s(\mathbf{x}) = \sum_{j=1}^{n} \alpha_j \phi_j(\mathbf{x})$. We call $\phi_j(\mathbf{x})$ a basis function. For a given set of basis functions, we can determine the weight α_j for each basis function by solving the following linear system

$$\begin{pmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \cdots & \phi_n(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \cdots & \phi_n(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{x}_n) & \phi_2(\mathbf{x}_n) & \cdots & \phi_n(\mathbf{x}_n) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}. \tag{1.1}$$

One may ask the following question: what kind of basis functions to be choose? does the linear system (1.1) have a unique solution? Is the linear systems easy to solve? We shall answer these question step by step.

2. Why radial basis functions in $\mathbb{R}^d$. In one dimensional space, commonly-used basis functions come from polynomial space of degree at most $n-1$. We can, for example, chose $\phi_j(x) = x^{j-1}$, $j = 1, \dots, n$. If the n interpolation points are

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distinct, then then the linear system (1.1) always has an unique solution, since it is a non-singular Vandermonde linear system. However, the Mairhuber-Curitis theorem [43, p.19][21][22] says that uniqueness of the solution to (1.1) can not always be guaranteed for multi-variate polynomial interpolation in high dimensional space. Such uncertainty was possibly first noted and proven by Haar [13][21, p.610]. He pointed out that the linear system can be singular even for some distinct points in $\mathbb{R}^d$,$d > 2$. His arguments are based on the following basic facts of linear algebra: (a) uniqueness of the solution to (1.1) is equivalent to the determinant of the interpolation matrix being non-zero; (b) the determinant of a matrix is continuous function of its elements; and (c) exchanging two rows of a matrix will change the sign of its determinant. Based on these facts, one can find two points, say, $\mathbf{x}_1, \mathbf{x}_2$ and construct two distinct curves $\xi_1(t), \xi_2(t)$ connecting these two points such that $\xi_1(0) = \mathbf{x}_1, \xi_1(1) = \mathbf{x}_2$, $\xi_2(0) = \mathbf{x}_2, \xi_2(1) = \mathbf{x}_1$;where the two curves have no other common points and do not intersect with the remaining $n-2$ interpolation points. When t goes from 0 to 1, the first two rows in (1.1) are continuously exchanged. Thus the determinant of the matrix will change sign. Therefore, there must be some $t \in [0, 1]$ such that the determinant is zero. Such uncertainty on uniqueness of multivariate polynomial interpolation in high dimensional space is quite different from uni-variate polynomial interpolation and might be another myth of polynomial interpolation [38]. It motivates us to find non-polynomial basis functions.

If we choose $\phi_j(\mathbf{x}) = \phi(\mathbf{x} - \mathbf{x}_j)$ for a function ϕ, note that the basis function ϕ_j is a translation of ϕ involving the interpolation point $\mathbf{x}_j$. When we switch two rows in the interpolation matrix, two columns and two basis functions will also be switched. Therefore, the determinant of the interpolation matrix will keep the same sign. Such basis functions have the potential to avoid the singularity of the linear system (1.1). Possibly, the simplest such basis function is $\phi(\mathbf{x}) = \|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ which has *radial symmetry*. In this case, $\phi_j(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_j\|_2$, and the interpolation matrix is a distance matrix in $\mathbb{R}^d$. The distance matrix for n distinct points is always invertible, proved by Schoenberg who was motivated by proving what given n length in $\mathbb{R}^d$ can serve as the length of a simplex(in $\mathbb{R}^2$ a simplex is a triangle) [35]. Precisely, a distance matrix has 1 positive eigenvalue and $n-1$ negative eigenvalues, if the n points are distinct [36, p.792]. Therefore a distance matrix is *almost negative definite* (only 1 positive eigenvalue). It seems that Schoenberg's results did not draw much attention until Micchelli [24] proved that a class of *radial* basis functions can always guarantee invertible interpolation matrices. This builds up a solid mathematical foundation for using radial basis functions as powerful tools for scattered data approximation in high-dimensional space.

Micchelli's work is motivated by proving a conjecture, which can be interpreted as the interpolation matrix in (1.1) with $\phi_j(\mathbf{x}) = \sqrt{1 + \|\mathbf{x} - \mathbf{x}_j\|_2}$ is invertible. His proof is based on some results of distance geometry, conditionally positive definite functions and special functions that are beyond our discussion. But his results are encouraging: interpolation matrices with some radial basis functions are independent of the distribution of the interpolation points, provided that the n points are distinct. Such a result makes radial basis functions good candidates for *scattered* data approximation in $\mathbb{R}^d$. (Otherwise on the regular tensor like mesh, one may choose, for example, Fourier basis.)

Our next problem is whether the linear system (1.1) is easy to solve. In high-dimensional space $\mathbb{R}^d$, the linear systems (1.1) often involves many unknowns, for example, when reconstructing a 3D surface from point clouds. Therefore, the sparsity of interpolation matrices is important and thus compactly supported radial basis functions are most useful. Moreover, the linear system is also expected to have some useful property like positive definiteness, which makes the linear systems is relative ease to solve.

3. Construction of compactly supported radial basis functions. It is not difficult to construct compactly supported functions if there are no other requirements, such as like smoothness and *positive definiteness*. (A radial basis function is said to be *positive definite* if it can guarantee a positive definite interpolation matrix in (1.1).) For example the truncated power functions, which are also called Askey's power functions, given by

$$\phi_\ell(\mathbf{x}) = (1 - \|\mathbf{x}\|_2)_+^\ell = \begin{cases} (1 - \|\mathbf{x}\|_2)^\ell & \text{for } 1 - \|\mathbf{x}\|_2 \geq 0; \\ 0 & \text{for } 1 - \|\mathbf{x}\|_2 \leq 0, \end{cases} \tag{3.1}$$

have a compact support in the disc $\|\mathbf{x}\|_2 \leq 1$ [2]. But they don't have any continuous derivatives at $\|\mathbf{x}\|_2 = 0$ and $\|\mathbf{x}\|_2 = 1$, even when ℓ is large, i.e. $\phi_\ell \in C^0$. (See Figure 3.1(a)).

It is well known that an integral operator can transform a function to a smoother one. Consider $\varphi(t) = (1 - |t|)_+$, where $t \in \mathbb{R}$, and define $h(r) = \int_\infty^r \varphi(t)dt = \int_{-1}^r \varphi(t)dt$. We can verify $h(r)$ has *both* compact support in $[-1, 1]$ and a continuous first-order derivative $\varphi(r)$. For an even function $\varphi(t)$, in practice, we can only consider an integral operator on the right-half real line, and then extend it to the whole space.

The general idea to construct compactly supported radial basis functions with a given smoothness is to use an integral operator acting on the truncated power function ϕ_ℓ and to adjust the size of ℓ to ensure positive definiteness. Positive definiteness can be checked by finding the Fourier transform of the radial basis function (see Appendix A for further details). Due to the radial symmetry, one can reduce almost all the operations to univariate operation sand only consider an integral operator on the right-half real line; and then extend the operator to the real line and generalize to higher dimensional space. The key question is what kind of integral operator should we choose for high-dimensional problems.

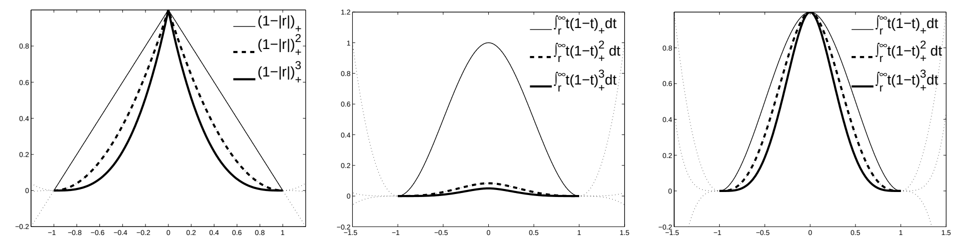
Among several precious authors using compactly supported radial basis functions, for example, [4][5][12][45], the most popular ones are the Wendland's functions—*compactly supported radial basis functions of minimal degree*. While the most interesting ones might be the missing Wendland functions. We shall discuss how to construct them.

3.1. Construction of the Wendland's functions. Consider the following integral operator

$$(\mathcal{I}\phi)(r) := \int_r^\infty t\phi(t)dt, \text{ for } r \geq 0. \tag{3.2}$$

was first introduced and studied by Wu, in the context of constructing compactly supported radial basis functions [45]. However, he started with very smooth functions in $\mathbb{R}$ and got less smooth functions in higher-dimensional space $\mathbb{R}^d$. Wendland uses this operator in a more elegant way. By repeatedly applying $\mathcal{I}$ on Askey's truncated power functions $\phi_\ell(r) = (1 - r)_+^\ell$, Wendland gets the functions

$$\phi_{d,k}(r) = \mathcal{I}^k \phi_\ell, \text{ where } \ell = \lfloor d/2 \rfloor + k + 1, \text{ and } \phi_\ell = (1 - r)_+^\ell. \tag{3.3}$$



(a) ϕ_ℓ where $\ell = 1, 2, 3$. (b) $\mathcal{I}(\phi_\ell), \ell = 1, 2, 3$. (c) scaled $\mathcal{I}(\phi_\ell), \ell = 1, 2, 3$.

FIG. 3.1. *Smoothing functions using the operator $\mathcal{I}$. Functions in (b) and (c) are even extensions of $\mathcal{I}(\phi_\ell)$.*

$\phi_{d,k}(r)$ can be easily computed with the help of mathematical software and can be represented in the general form

$$\phi_{d,k}(r) = \mathcal{I}^k \phi_\ell = \phi_{\ell+k} p_{k,\ell}(r) = (1-r)_+^{\lfloor d/2 \rfloor + k+1+k} p_{k,\ell}(r), \tag{3.4}$$

where $p_{k,\ell}(r)$ is a polynomial of degree k whose coefficients depend on ℓ. We define $p_{0,\ell} = 1$. We provide a Maple program and an extended table for Wendland's functions in the appendix.

Wendland's functions $\phi_{d,k}(r)$ defined in (3.3) are polynomials of $r = \|\mathbf{x}\|_2$ and have the following properties [41][43, p.128, Theorem 9.13, p.160, Theorem 10.35]:

PROPOSITION 3.1. *The Wendland's functions $\phi_{d,k}(r)$ are polynomials of degree $\lfloor d/2 \rfloor + 3k + 1$ on $\mathbb{R}^d$, positive definite and compactly supported in $r \in [0, 1]$, i.e. $\phi_{d,k}(r) \in C^{2k}(\mathbb{R}^d)$.*

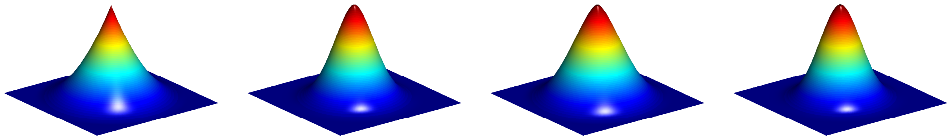
PROPOSITION 3.2. *For each k, $\phi_{d,k}(r)$ possesses continuous derivatives up to order $2k$ in $\mathbb{R}^d$, it possesses $2k$ continuous derivatives around zeros and $k + \ell - 1 = 2k + \lfloor d/2 \rfloor$ continuous derivatives around 1.*

PROPOSITION 3.3. *For any given space $\mathbb{R}^d$ and smoothness $2k$, the degree of $\phi_{d,k}$ is the minimal number to guarantee positive definiteness and the smoothness.*

PROPOSITION 3.4. *For $d \geq 3$, and k non-negative integer, $\phi_{d,k}$ is a reproducing kernel in Hilbert space, which is norm-equivalent to the Sobolev space $\mathcal{H}^{d/2+k+1/2}(\mathbb{R}^d)$*

Due to Proposition 3.3, the Wendland functions are called *compactly supported radial basis functions of minimal degree*. Proposition 3.4, which is called the reproducing property (jargon here is not important), suggests that there must be some missing Wendland functions in perhaps the most interesting case $\mathbb{R}^2$. We may also ask why the integral operator in (3.2) is other than the simplest one $\int_r^\infty \phi(t)dt$. The choice of the integral operator is determined by the fact the new function needs to be positive definite in certain $\mathbb{R}^d$. This requires a positive Fourier transform after some *dimension walk*[40][43, p.120]. No other simpler integral operator than $\mathcal{I}$ with such properties has been found. But another more general integral operator, used to simplify the multi-variate Fourier transform for radial functions in $\mathbb{R}^d$ [34], can be used to construct missing Wendland functions.

3.2. Construction of the missing Wendland functions. CSRBFs which can reproduce the Sobolev space $\mathcal{H}^{d/2+k+1/2}(\mathbb{R}^d)$ for even d and half-integer k have only been found recently [32]. Such functions are called the missing Wendland functions. The missing Wendland functions are constructed by using a more general integral



(a) $\phi_{2,0}$ (b) $\phi_{2,1}$ (c) $\Psi_{2,1/2}$ (d) $\Psi_{2,3/2}$

FIG. 3.2. *Scaled Wendland functions and Missing Wendland functions in* $\mathbb{R}^2$.

operator $\mathcal{I}_\alpha$, as mentioned above, which is given by

$$\mathcal{I}_\alpha(f)(t) := \int_t^\infty f(s)\frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}ds, \tag{3.5}$$

acting this is applied to a modified version of the truncated power function, $a_\mu(s) := (1-\sqrt{2s})_+^\mu$, so that

$$\mathcal{I}_\alpha(a_\mu)(t) = \int_t^\infty (1-\sqrt{2s})_+^\mu \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}ds. \tag{3.6}$$

A generalized function is defined by $\Psi_{\mu,\alpha}(r) := \mathcal{I}_\alpha(a_\mu)(r^2/2)$. The operator $\mathcal{I}_\alpha$ is a scaled integral operator which is closely connected with fractional derivatives, and was used to simplify the multivariate Fourier transform for radial functions [34]. Here α can be half-integer. The function $\Psi_{\mu,\alpha}(r)$ is given by

$$\Psi_{\mu,\alpha}(r) = \int_{r^2/2}^\infty (1-\sqrt{2s})_+^\mu \frac{(s-r^2/2)^{\alpha-1}}{\Gamma(\alpha)}ds = \int_r^1 t(1-t)^\mu \frac{(t^2-r^2)^{\alpha-1}}{\Gamma(\alpha)2^{\alpha-1}}dt. \tag{3.7}$$

In particular, when $\alpha = 1$

$$\Psi_{\mu,1} = \int_r^1 t(1-t)^\mu dt = \int_r^\infty t(1-t)_+^\mu dt = \mathcal{I}(\phi_\mu)(r). \tag{3.8}$$

It turns out that $\Psi_{\ell,1}(r)$ is simply the operator $\mathcal{I}$ defined in (3.2) acting on the truncated power functions $\phi_\ell(t)$. We shall see that $\Psi_{\mu,\alpha}$ are generalized Wendland functions which include more than the Wendland functions discussed in section 3.1. The operator $\mathcal{I}_\alpha$ and functions $\Psi_{\mu,\alpha}$ have the following properties:

PROPOSITION 3.5.

$$\mathcal{I}_\alpha \circ \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta} \quad \textit{and} \quad \mathcal{I}_\alpha^k = \mathcal{I}_{k\alpha}. \tag{3.9}$$

PROPOSITION 3.6. *For all non-negative integers* $\mu \in \mathbb{N}$ *and all half-integer* $\alpha = n+1/2, n \in \mathbb{N}$*, the generalized Wendland function defined in* (3.7) *is positive definite on* $\mathbb{R}^d$*, if* $\mu \geq \lfloor d/2+\alpha \rfloor + 1$.

Proposition 3.6 is similar to Proposition 3.1, which ensures the positive definiteness of the linear system (1.1). For details, readers are referred to [34] and [32]. Using Proposition 3.5 and the construction process in the last section, we can derive the relationship between Wendland functions and generalized Wendland functions. Define $\tilde{\phi}_{d,k,\alpha} := \mathcal{I}_\alpha^k(a_\mu)(r^2/2)$, where $\mu = \lfloor d/2 \rfloor + k\alpha + 1$, for $k = 1, 2, 3, 4, \ldots$, then by

Proposition 3.5, we see that

$$\tilde{\phi}_{d,1,\alpha} = \mathcal{I}_\alpha(a_\mu)(r^2/2) = \Psi_{\mu,\alpha} = \Psi_{\lfloor d/2 \rfloor+\alpha+1,\alpha}(r), \tag{3.10}$$

$$\tilde{\phi}_{d,2,\alpha} = \mathcal{I}_\alpha^2(a_\mu)(r^2/2) = \mathcal{I}_{2\alpha}(a_\mu)(r^2/2) = \Psi_{\mu,2\alpha} = \Psi_{\lfloor d/2 \rfloor+2\alpha+1,2\alpha}(r), \tag{3.11}$$

$$\tilde{\phi}_{d,k,\alpha} = \mathcal{I}_\alpha^k(a_\mu)(r^2/2) = \mathcal{I}_{k\alpha}(a_\mu)(r^2/2) = \Psi_{\mu,2\alpha} = \Psi_{\lfloor d/2 \rfloor+k\alpha+1,k\alpha}(r). \tag{3.12}$$

From (3.8), we can show that:

Proposition 3.7. *(***Schaback***) For non-negative integers k, the Wendland functions of minimum degree defined in* (3.3) *and the generalized Wendland functions defined in* (3.7) *have the following relationship:*

$$\phi_{d,k} = \Psi_{\lfloor d/2 \rfloor+k+1,k} \tag{3.13}$$

More generally, we can apply different operator $\mathcal{I}_\alpha$ in different steps, for example $\mathcal{I}_\beta\mathcal{I}_\alpha(a_\mu)(r^2/r) = \Psi_{\mu,\alpha+\beta}$. If we want a positive definite function, according to Proposition 3.6, we have to adjust the size of μ so that $\mu > \lfloor d/2+\alpha+\beta \rfloor + 1$. The generalized Wendland functions can be computed by a 6-line Maple program that is given in the Appendix.

Schaback also proves that [32, p.75 Collollary 1]:

Proposition 3.8. *For integers* $m \geq 1, n \geq 0, d = 2m$, $\Psi_{\mu,n+1/2}$ *reproduce a Hilbert space which is isomorphic to Sobolev space* $\mathcal{H}^{m+n+1}(\mathbb{R}^d) = \mathcal{H}^{d/2+\alpha+1/2}(\mathbb{R}^d)$, *where* $\alpha = n + 1/2$

Here d can be 2. For such functions, μ is an integer and $\alpha = n + 1/2$ is a half integer. These are called the missing Wendland functions. This result extends Wendland's result given in Proposition 3.4 in which it requires $d \geq 3$.

The missing Wendland functions $\Psi_{\mu,\alpha}$ involve two non-polynomial terms, and can be written as

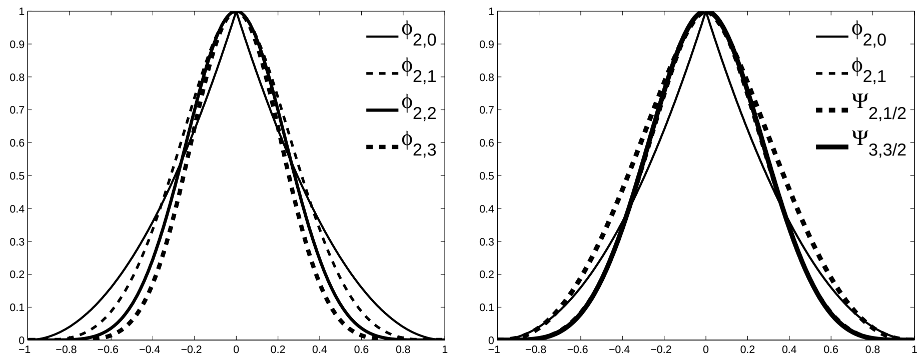
$$\Psi_{\mu,\alpha}(r) = \mathcal{P}_{\mu,\alpha} \log(\frac{r}{1+\sqrt{1-r^2}}) + \mathcal{Q}_{\mu,\alpha}\sqrt{1-r^2}, \tag{3.14}$$

where, $\mathcal{P}_{\mu,\alpha}$ and $\mathcal{Q}_{\mu,\alpha}$ are polynomials in r^2. For a detailed derivation and property of $\mathcal{P}_{\mu,\alpha}$ and $\mathcal{Q}_{\mu,\alpha}$, the reader is directed to [32]. Several missing Wendland functions of interest are listed in Table 4.2.

See Figure 3.3 for the comparison between the Wendland functions and the missing Wendland functions in $\mathbb{R}^2$. For more details on the Wendland and the missing Wendland functions, one can refer to a recent paper [18].

3.3. Construction by convolution and others. Provided some CSRBFs have been found, we can construct a class of CSRBFs by convolution. This is based on two facts: a function is positive definite if its Fourier transform is positive definite (see Appendix A); and the Fourier transform of two functions' convolution is the product of the Fourier transforms of the two functions, namely, if $h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$, then the Fourier $\hat{h}(\xi) = \hat{f}(\xi)\hat{h}(\xi)$. Therefore, any two positive definite radial basis function give another positive definite basis functions(not necessary be radial); if there is one compacted supported, the resulting function is compactly supported.

And also, we can construct positive definite compactly supported basis function on a square. For example, if $\phi_1(x), \phi_2(x)$ are positive definite with a compact support $[-1,1]$, then $\phi(x,y) = \phi_1(x)\phi_2(y)$ is positive definite with a compact support on



(a) Wendland functions in $\mathbb{R}^2$ (b) Missing Wendland functions in $\mathbb{R}^2$

Fig. 3.3. *Wendland functions and missing Wendland functions $\mathbb{R}^2$. The missing Wendland function $\Psi_{3,3/2}$ is very similar to the Wendland function $\phi_{2,1}$ and overlaps it.*

$[-1,1]\times[-1,1]$, but ϕ is not radial symmetric. For compactly supported basis function on a general polygon, the reader is referred to box-spline[7].

As seen, Wendland functions and missing Wendland functions are only finite smooth, a natural and interesting question is whether there are some *positive definite* CSRBFs in $\mathbb{R}^d$ with infinite smoothness. Schaback points out this is a open problem [31]. If there is no positive definiteness constraint, the known Mollifier given by

$$\phi(\mathbf{x}) = \begin{cases} e^{-\frac{1}{1-\|\mathbf{x}\|^2}} & \text{if } \|\mathbf{x}\| \le 1; \\ 0 & \text{if } \|\mathbf{x}\| \ge 1. \end{cases} \tag{3.15}$$

is infinitely differentiable with compact support. But we are pretty sure that even the Mollifier is positive definite on a lower-dimensional space, it must not be positive definite on some higher-dimensional space $\mathbb{R}^d$, because it has been proven that a continuous CSRBF can not positive definite on every $\mathbb{R}^d$ [43, p.120].

4. Conclusion. In this paper we have considered high-dimensional approximation problems. These problems are challenging because, as seen, some well-accepted results in one-dimensional space may not be valid in higher-dimensional space, and there are some challenging computational issues which beyond our discussion. Radial basis functions are good candidates for high-dimensional scattered data approximation because they can avoid singular interpolation matrix and there are simple and efficient ways to construct compactly supported radial basis functions with given smoothness. We want to emphasize that *"in almost every area of numerical analysis, sooner or later, the discussion comes down to approximation theory"*[37, p.605]; and radial basis function is one *"major newer topic "* in this fundamental area (compared with polynomial and rational minimax approximation et al) . Recent years have seen the are many advancement in this filed, but further research is still needed to make these methods more effective and applicable to an even broader range of real-life applications.

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Appendix .

A. Check positive definite functions by Fourier Tranform. Suppose $\phi_j(\mathbf{x}) = \phi(\mathbf{x} - \mathbf{x}_j)$, where $\phi(\mathbf{x})$ is radial symmetric and has an integrable Fourier transform $\hat{\phi}$. Then by inverse Fourier transform, one gets

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{\phi}(\omega) e^{i\mathbf{x}^T\omega} d\omega. \tag{4.1}$$

The linear systems (1.1) is positive definite is equivalent to the following quadratic form is always positive

$$\sum_{k,j=1}^{n} \alpha_k \bar{\alpha}_j \phi(\mathbf{x}_k - \mathbf{x}_j) = \frac{1}{(2\pi)^{d/2}} \sum_{j,k=1}^{N} \alpha_j \bar{\alpha}_k \int_{\mathbb{R}^d} \hat{\phi}(\omega) e^{i\omega^T(\mathbf{x}_j - \mathbf{x}_k)} d\omega \tag{4.2}$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{\phi}(\omega) \left| \sum_{j=1}^{n} \alpha_j e^{i\mathbf{x}_j\omega} \right|^2 d\omega. \tag{4.3}$$

Form (4.2) to (4.3), we need to separate $e^{i\omega^T(\mathbf{x}_j - \mathbf{x}_k)}$ as $e^{i\omega^T\mathbf{x}_j} e^{-i\omega^T\mathbf{x}_k}$ and use the relationship $\sum_{k=1}^{n} \bar{\alpha}_k e^{-i\omega^T\mathbf{x}_k} = \overline{\sum_{j=1}^{n} \alpha_j e^{i\omega^T\mathbf{x}_j}}$. According to (4.3), a function ϕ whose Fourier transform $\hat{\phi}$ is positive can guarantee a positive definite linear system (1.1), and thus is said to be *positive definite*. Using Fourier transform to characterization a positive definite function dates back to Mathias[23], Bochner [3][43, p.67], followed by von Neumann, Schoenberg [39] and many others; and it can serve a handy way to verify whether the linear system (1.1) is positive definite for given basis functions. Generally speaking, find a multi-variate Fourier transforms is not easy, but find Fourier transform for radial functions can be carried only on univariate operations due to Schaback and Wu's work [34].

B. Maple Program to compute the Wendland functions.

```
wd := proc (d, k, r)
local wd, kk;
wd := (1-r)^ (floor((1/2)*d)+k+1);
for kk form 1 by 1 to k do
wd := int(t*subs(r = t, wd), t = r .. 1)
end do;
return factor(wd)
end proc
```

Table 4.1 are computed by the above Maple Program.

C . Maple Program to compute the missing Wendland functions . The following program is a revised version of that in [32]

```
mswd := proc (mu, alpha, r)
local mswd;
mswd := t*(1-t)^ mu*(t^ 2-r^ 2)^ (alpha-1)/(GAMMA(alpha)*2^ (alpha-
    1));
mswd := int(mswd, t = r ..1);
return combine(simplify(mswd), ln)
end proc
```

We point out that the program does not work when both μ and α are half-integer, then $\Psi_{\mu,\alpha}$.

REFERENCES

Table 4.1
Compactly supported functions of minimal degree

d	Wendland function $\phi_{d,k}(r), r = \|\mathbf{x}\|_2$	Smoothness
	$\phi_{1,0}(r) = (1-r)_+$	C^0
	$\phi_{1,1}(r) = (1-r)_+^3(1+3r)/12$	C^2
$d = 1$	$\phi_{1,2}(r) = (1-r)_+^5(3+15r+24r^2)/840$	C^4
	$\phi_{1,3}(r) = (1-r)_+^7(15+105r+285r^2+315r^3)/151200$	C^6
	$\phi_{1,4}(r) = (1-r)_+^9(105+945r+3555r^2+6795r^3+5760r^4)/51891840$	C^8
	$\phi_{3,0}(r) = (1-r)_+^2$	C^0
	$\phi_{3,1}(r) = (1-r)_+^4(1+4r)/20$	C^2
$d \leq 3$	$\phi_{3,2}(r) = (1-r)_+^6(3+18r+35r^2)/1680$	C^4
	$\phi_{3,3}(r) = (1-r)_+^8(15+120r+375r^2+480r^3)/332640$	C^6
	$\phi_{3,4}(r) = (1-r)_+^{10}(105+1050r+4410r^2+9450r^3+9009r^4)/121080960$	C^8
	$\phi_{5,0}(r) = (1-r)_+^3$	C^0
	$\phi_{5,1}(r) = (1-r)_+^5(1+5r)/30$	C^2
$d \leq 5$	$\phi_{5,2}(r) = (1-r)_+^7(3+21r+48r^2)/3024$	C^4
	$\phi_{5,3}(r) = (1-r)_+^9(15+135r+477r^2+693r^3)/665280$	C^6
	$\phi_{5,4}(r) = (1-r)_+^{11}(105+1155r+5355r^2+12705r^3+13440r^4)/259459200$	C^8
	$\phi_{7,0}(r) = (1-r)_+^4$	C^0
	$\phi_{7,1}(r) = (1-r)_+^6(1+6r)/42$	C^2
$d \leq 7$	$\phi_{7,2}(r) = (1-r)_+^8(3+24r+63r^2)/5040$	C^4
	$\phi_{7,3}(r) = (1-r)_+^{10}(15+150r+591r^2+960r^3+591r^2+960r^3)/1235520$	C^6
	$\phi_{7,4}(r) = (1-r)_+^{12}(105+1260r+6390r^2+16620r^3+19305r^4)/518918400$	C^8

Table 4.2
The missing Wendland functions

$\Psi_{\mu,\alpha}$	$\mathcal{H}^k(\mathbb{R}^d)$	function
$\Psi_{2,1/2}$	$\mathcal{H}^2(\mathbb{R}^2)$	$\frac{\sqrt{2}}{3\Gamma(1/2)}\left(3r^2\mathcal{L}+(2r^2+1)\mathcal{S}\right)$
$\Psi_{3,3/2}$	$\mathcal{H}^3(\mathbb{R}^2)$	$\frac{-\sqrt{2}}{480\Gamma(3/2)}\left((15r^6+90r^4)\mathcal{L}+(81r^4+28r^2-4)\mathcal{S}\right)$
$\Psi_{4,5/2}$	$\mathcal{H}^4(\mathbb{R}^2)$	$\frac{\sqrt{2}}{40320\Gamma(5/2)}\left((945r^8+2520r^6)\mathcal{L}+(256r^8+2639r^6+690r^4-136r^2+16)\mathcal{S}\right)$
$\Psi_{5,7/2}$	$\mathcal{H}^5(\mathbb{R}^2)$	$\frac{-\sqrt{2}}{5677056\Gamma(7/2)}\left(\mathcal{P}_{5,7/2}\mathcal{L}+\mathcal{Q}_{5,7/2}\mathcal{S}\right)$
		$\mathcal{P}_{5,7/2} = 3465r^{12}+83160r^{10}+13860r^8$
		$\mathcal{Q}_{5,7/2} = 37495r^{10}+160290r^8+33488r^6-724r^4+1344r^2-128$
		$\mathcal{L}(r) = \log\left(\frac{r}{1+\sqrt{1-r^2}}\right)$
		$\mathcal{S}(r) := \sqrt{1-r^2}$

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