The Duality Between Two-Index Potentials and the Non-Linear σ Model in Field Theory

Ioannis P. Zois

Exeter College
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Michaelmas Term 1996

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy at the University of Oxford

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Abstract

We try to give a mathematical interpretation of the "generalised" gauge transformation (infinitesimal version)

$$\delta B = d_A \xi$$

firstly observed in string theory but subsequently also in 4-dimensional space-time context. In the above formula, B is a Lie algebra valued 2-form, the gauge parameter ξ is a Lie algebra valued 1-form and d_A is the exterior covariant derivative with respect to a *flat* connection 1-form A which is defined through B. The Lie algebra corresponds to a non-abelian Lie group of symmetries. What we actually argue here is that this symmetry expresses a kind of *"cohomological freedom"* for some *twisted* de Rham cohomology and despite the original name "generalised gauge symmetry" used in physics literature it is in fact of different origin, namely, it is not related to "change of trivialisation" of the bundle. The twist is made by the adjoint flat bundle.

As an application of the *generalised Index theorem for elliptic complexes* we prove that the Euler number of this twisted de Rham complex by the flat connection A is equal to the Euler number of the twisted de Rham complex by the trivial flat connection, even if the flat connection A is gauge inequivalent to the trivial flat connection (this distinction makes sense only if the base space of the bundles we consider is *not* simply connected). Thus we have essentially proved that the Freedman-Townsend invariance is a special case of Grothendieck's *stability* equivalence relation used in the definition of the K^0 group. Moreover we calculate this Euler number for the case of actual physical interest, namely when our base space is $\mathbf{R^4}$. This is useful because it determines the dimension of the space of classical observables of our theory which constitutes the basis to construct the Hilbert space of the quantum observables, following the Geometric Quantization scheme.

Next, since this duality between the two-index potential theory and the non-linear σ model is also present in loop space formulation, we pass on to loop

spaces where we give a more mathematically rigorous proof of Polyakov's basic variational formula for loop spaces which was originally written down by direct immitation—without any further justification whatsoever— of an analogous result for *Lattice* Yang-Mills theory due to Wilson. (Their motivation came from studying quark confinement). We use the theory of *iterated integrals* and the induced isomorphism between the Hochschild homology of the algebra of forms on our original manifold and the de Rham cohomology of the loop space of our manifold (this isomorphism is true only if the manifold is simply-connected).

Finally we use Polyakov's observation that the non-linear σ model is "equivalent", in a certain sense, to a flat principal bundle to construct *a new invariant* using the pairing between *(even) KK-Theory* and *cyclic homology*. The reason for this invariant being something non-trivial is that the flat foliation of a principal bundle provides the simplest interesting example of a *genuinly* non-commutative space. An attempt to get a *flavour* of this invariant in physical context allowed us to discuss some possible interesting physical consequences. Moreover the *odd* pairing between KK-Theory and cyclic homology gives, after a renormalisation proceedure, the η invariant of our twisted de Rham differential, which is related to global anomalies as well as to regularisation techniques and calculation of determinants for certain path integrals arising in Topological Quantum Field Theories and also gives a more *refined* invariant than the index which can actually distinguish between different classes of gauge equivalent flat connections.

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Chapter 1

Introduction

1.1 Background

According to our present understanding of nature, gauge theories seem to be of indisputable importance since at least 3 out of the 4 known fundamental interactions are thought to be of this kind. The gauge principle explains interactions as exchange of spin-1 particles, i.e. the gauge potentials. For gravity the situation is not so clear, some people think of the metric as giving rise to the gauge potential in this case (while others think that the connection is more fundamental). However, the metric is a symmetric rank 2 tensor. So the picture is that spin-1 gauge particles, i.e. connection 1-forms in the geometric context, carry electroweak and strong interactions, whereas, in some sense symmetric rank 2 tensor gauge fields carry gravitational interactions. So the question is what is the role that antisymmetric rank 2 tensor gauge fields might play, and why one should be interested in them?

In order to explain our motivation clearly, we would like to mention the cases, in historical order, where these fields arise.

First of all, it was P.A.M. Dirac [1] who considered a whole family of higher spin abelian gauge fields, known as "Dirac hierarchies". The motivation of Dirac to study these objects is not clear to us, but at least they appear in the physical literature since 1930's.

Secondly, there is string theory. There, one has really all sorts of particles, of any spin and hence spin-2 gauge fields as well. However these particles appear quite naturally in [2] where one tries to see what the analogue of electromagnetism would be for string like objects. It is shown that if one considers 1-dimensional objects, then the analogue of the usual electromagnetic potential A_μ will be an abelian spin-2 gauge field.

Thirdly, spin-2 gauge fields appear in some formulations of supergravity [3]-[7]. For example such fields appeared quite naturally in the dimensional reduction procedure which led to N=8 extended supergravity theory [6], and in a superspace

formulation [7] of N=8 when all spin-0 particles were described by antisymmetric tensor fields. Moreover these theories have some unusual features that make them interesting models for field theories, for example an interacting version of this theory is shown to be the dual, at least classically, of non-linear σ models in 4-dimensions [8, 9, 10].

Recently those fields also appeared in [11] where they played a rather important and mysterious role in proving that under S-duality transformation, the electromagnetic Lagrangian over curved space-time transforms as a modular form. Moreover they are also related to the confinement of quarks [12].

Finally, 2-form potentials also appear in loop space formulation of Yang-Mills theories and dualities observed there [13]. The original formulation of Yang-Mills theories as non-linear σ models on loop space was due to Polyakov [14].

In this present work we shall be mainly interested in the duality between 2-form potentials and non-linear σ models in dimension 4 as well as with the loop space dualities and how these fields appear quite naturally there as well.

We would like to say something more about 4-dimensional non-linear σ models. These theories are non-renormalisable but they are thought to be of great importance in the phenomenology of particle physics, on account of the fact that they describe the dynamics of pions (and mesons in general) at energies which are small compared with the inverse confinement radious of QCD. The Higgs sector of the standard model is another example where 4-dimensional σ models appear as some approximation in the limit of a very large Higgs mass, where the Goldstone bosons of the symmetry breaking mechanism may be described by a non-linear σ model [15, 16, 17]. The connection in this case is made via the equivalence theorem [15, 18, 19, 20] which states that, for a given energy regime, the high energy amplitude for a process with external longitudinally polarised vector bosons is equal to the amplitude of the process in which the external vector bosons are replaced by the corresponding Goldstone bosons. Therefore the study of the dual of the 4-dimensional σ model has some physical significance to the Standard Model.

In an attempt to make a more general connection with string theory, let us mention that the 2-dimensional non-linear σ model is the low energy limit of string theory. One of the most striking features of string theory is that it possesses a dual symmetry which typically reveals itself in two ways: the first one is known as "T-duality" and is a generic feature of theories with compactified spatial dimensions. The simplest example of such duality manifests itself in the fact that the spectrum obtained when one dimension is compactified on a circle of radius R is found to be indistinguisable from that obtained when the compactification takes place on a circle of radius 1/R [21]. For example, if one considers a superstring theory in 10 dimensions, then under the usual compactification scheme this 10-dimensional space-time manifold is compactified down to a 4-dimensional manifold and a 6-dimensional Calabi-Yau space. In this setting, T-duality is related to *mirror symmetry* which simply states that the theory is

indistinguishable if the compactification goes down to a Calabi-Yau space or its mirror image, i.e. another Calabi-Yau space whose $h^{1,2}$ Hodge number equals the $h^{2,1}$ Hodge number of the original Calabi-Yau space. The second type of duality is known as "S-duality" and interchanges the strong and weak coupling limits of string theory, i.e. the theory is invariant under modular transformations acting on

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

where θ and g are the theta angle parametrising the space of vacua and the coupling constant respectively.

This remarkably generalises the strong-weak (or electric-magnetic) duality conjectured many years ago by Olive and Montonen [22] and shown by Osborne [23] to hold, if at all, for N=4 supersymmetric gauge theories only.

Recently, dramatic new evidence for the validity of this conjecture has emerged from the work of A. Sen [24] and furthermore a version of the Montonen-Olive duality was found by Seiberg and Witten in N=2 supersymmetric gauge theory in 4 dimensions [25].

These two dualities are however discrete symmetries of the spectrum of the string theory. The question which naturally arises is can this duality be implemented at the level of the 2-dimensional non-linear σ model, which, as we said earlier, is thought to provide the low energy limit of string theory? Indeed, the dual theory of an arbitrary σ model with an abelian isometry was constructed by Buscher [26]. There is in fact an algorithm for constructing the dual theory of a σ model which has the advantage of being applicable to any dimensions. The algorithm consists of gauging a symmetry (an isometry) of the action by introducing non-propagating gauge fields and whose field strength is forced to vanish by means of a Lagrange multiplier, the original theory being regained if one integrates out the Lagrange multiplier. On the other hand, if one integrates out the gauge fields instead, one obtains the dual theory where the Lagrange multiplier is now a dynamical field [27].

1.2 Summary of Thesis

It was P.A.M. Dirac in the 1930's who for the first time considered a *hierarchy* of *abelian* gauge potentials: A_μ, $A_{\mu\nu}$, $A_{\mu\nu\lambda}$,... In geometry language these are just real valued 1-forms, 2-forms, 3-forms, etc. Dirac of course considered 4-dimensional space-time, he was a wise man. This was exactly the case considered by string theorists many years later, not necessarily on 4-dimensional space-time, arround '70's, when they realised that the quantization of string includes particles of any spin. It is really very straightforward to write down the (abelian) gauge

transformations of these fields, simply by imitating the transformations for the electromagnetic vector potential, namely:

$$\delta A_\mu = \partial_\mu \Lambda$$

$$\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

$$\delta A_{\mu\nu\lambda} = \sum_{cycl} \partial_\mu \Lambda_{\nu\lambda}$$

where the gauge parameters are respectively Λ, Λ_μ and $\Lambda_{\mu\nu}$, being real valued 0-forms, 1-forms and 2-forms respectively.

The gauge invariant fields derived from these potentials are respectively:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu\lambda} = \sum_{cycl} \partial_\mu A_{\nu\lambda}$$

$$F_{\mu\nu\lambda\rho} = \sum_{cycl} \partial_\mu A_{\nu\lambda\rho}$$

The most natural question then for one to ask is: *What would the* non-abelian *version of these transformations be like?* This is a question asked by string theorists because in Dirac's era, non-abelian gauge theories were not known yet. (Let us emphasise what we mentioned above, namely that higher spin gauge potentials appear in two contexts in physics: string theory context where the dimension of space-time is not 4 and ordinary 4-dimensional space-time world, as in Dirac's case. We shall be mainly interested in the second context but we shall also mention higher dimensional space-time cases. In fact we shall see that the dimension of space-time plays a rather crucial role).

The answer is very well-known for the spin-1 potentials from non-abelian gauge theories due to the pioneering work of Yang-Mills in the '50's. However, for the higher spin gauge potentials the question is highly non-trivial! One can very easily see this if one considers the, spin-2 case, say. From now we shall denote the 2-index potential by B which is a 2-form. We know from the abelian version that the gauge parameter now has to be a Lie algebra valued 1-form, call it ξ. If we naively try to imitate the spin-1 case, where as it is very well-known one has:

$$\delta A = d_A \Lambda := d\Lambda + A \wedge \Lambda$$

one should write down an expression like

$$\delta B = d_B \xi = d\xi + B \wedge \xi$$

The first term is perfectly all right, but the second is a *3-form*! Thus one can see that if one wants to have a 2-form as a second term, one has either to replace

the gauge parameter ξ with a zero form like Λ above which in some sense must be "derived" from our original gauge parameter ξ keeping B on the formula, or one has to replace B by a "derived" 1-form like A above and keeping the original gauge parameter ξ in the second term. Moreover, an expression involving the potential B like the one written above has the additional problem that one takes a *covariant derivative* with respect to a 2-form, meaning something like a *parallel transport along a surface*. What should this mean in geometric terms, if it means something?

The answer to the covariant derivative picture is given by the *Frolicher-Nijenhuis bracket* (denoted $[,]_{FN}$) which is a generalisation of the usual bracket and it applies to tangent-valued forms of *any* degree (see [28] and [29]).

More concretely assuming we are on a manifold M with $T(M)$ its tangent bundle and $\wedge T^*(M)$ exterior powers of the cotangent bundle, the formula giving the Frolicher-Nijenhuis bracket is

$$[a \otimes u, b \otimes v]_{FN} := a \wedge b \otimes [u, v] + a \wedge L_u b \otimes v +$$

$$+(-1)^{rs} b \wedge L_v a \otimes u - (-1)^r (v \lrcorner a) \wedge db \otimes u - (-1)^{rs+s} (u \lrcorner b) \wedge da \otimes v$$

where $a \in \wedge^r T^*(M)$, $b \in \wedge^s T^*(M)$, $u, v \in T(M)$, L_v denotes Lie derivative along the vector field v and $\lrcorner$ is just the contraction or evaluation of a form on a vector.

For 1-forms it gives the usual result. Yet as far as we understand this is just an algebraic generalisation whose geometrical meaning is not very clear. Moreover, if one wishes to interpret this covariant derivative as some kind of *parallel transport along a surface*, one can see that one should use (the Poincare dual of) a surface-forming 2-cycle. This is an integrability condition and for this reason we shall see why *flat* connection 1-forms *derived* from the 2-form B appear quite naturally in [8].

However, as if this was not enough, one faces another equally important problem: in geometric language the gauge parameters are the matrices which change the local trivialisation of the bundle, thus they *have* to be Lie algebra (we discuss the infinitesimal version of gauge transformations) valued *0-forms*. A Lie algebra valued *1-form* makes absolutely no sense in the geometric context, if one wants to interpret gauge transformations geometrically as local change of trivialisation.

Let us make a small *digression* here to mention what we think is an important point: these transformations for the higher spin gauge potentials, even in the abelian case where they may appear very straightforward to write down, involve we think an important generalisation: the gauge parameters are no longer functions (0-forms) with suitable values but they are *forms*. Let us recall that gauge transformations of first kind-or global gauge transformations-is just the Lie algebra of the corresponding group considered (infinitesimally). When one "gauges" the symmetry, algebraically this means "tensoring the Lie algebra with the algebra of functions of our space-time manifold", hence the gauge parameter is now a Lie algebra valued function (0-form). But if one has Lie algebra valued

forms as gauge parameters, this means that we now tensor the Lie algebra with the algebra of forms on the manifold (which contains functions as a subalgebra). So one can think of this as a gauge transformations of "3rd kind" maybe. Now this situation is very non-trivial because one cannot *exponentiate* Lie algebra valued forms in order to get the *finite* version of these transformations. In fact we tried to find the finite version of the transformations appearing in [8], but we did not succeed. The situation seems as being a generalisation of the problems one has in supersymmetry (to be denoted SUSY in the sequel). It is clearly the *grading* of the algebra that causes the problems. In SUSY one has a $Z/2$ graded algebra, here we have an algebra graded by the dimension of the space. In SUSY people developed the notion of *supermanifolds* and *super-Lie groups* in order to overcome this difficulty. Here it seems that one needs something more general like *graded manifolds* or *graded-Lie groups*, namely spaces isomorphic to a graded algebra.

From the above description it is we think very clear that this problem is indeed interesting and non-trivial. The first question above concerning the non-abelian generalisation of the gauge transformations for the *spin-2* gauge potential was given in [8] for a 4-dimensional space-time. Our task was to answer the second question, namely to interpret this generalised gauge invariance mathematically being sure that it also incorporates the extra features of this generalised symmetry given in [8]. It is clear we think from the above analysis that a "change of trivialisation" explanation does not work in this case, so we had to look for something else.

The answer given in [8] covers *only* the spin-2 gauge potentials' case and not the higher than 2 spin cases. Moreover it is for 4-dimensional space-time. This point however can be generalised to include higher dimensional space-times, namely the generalisation is also valid for the string theory context, if one considers the *dual* potential instead (which will not be spin-2 but spin-(D-2)), where D is the dimension of space-time considered. This turns out to be a useful trick in physics, it was also used in [11] for instance very recently to prove that under duality transformations the electromagnetic Lagrangian on a general 4-manifold transforms as a modular form.

We mentioned above that higher spin gauge potentials appear in string theory after the quantization of the string (along with an infinite number of particles of any spin). We would like to explain in what sense the spin-2 gauge potentials appear in 4-dimensional context. (In this context the situation for gauge particles of spin higher than 2 is not clear, we do not even know what was Dirac's physical motivation to consider these hierarchies in the abelian case whereas if one believes in string theory, these particles,having any spin possible, "are there", as quantum excitations of the string). Anyway the point is that in the abelian case, spin-2 gauge potentials offer a dual description of the Klein-Gordon field and the generalisation given in [8] states that the non-abelian case gives a dual description of the non-linear σ model, which is very good feature because the non-linear σ

model is a generalisation of the Klein-Gordon field. This is the *spin-0, spin-2 duality* and it holds only in 4 dimensions. Moreover one also has the SUSY and supergravity contexts mentioned in the previous section where these spin-2 gauge potentials appear, but these involve higher dimensional space-times as it is always the case for supergravity for instance.

Let us now mention the rather unusual features of the non-abelian generalisation of the transformations for the spin-2 gauge potentials given in [8]: The first one is that one constructs a *flat* connection 1-form out of the spin-2 potential B which is what enters in the gauge transformation (so one does not have the problem with the degrees of the forms explained above). This is a key-point in the proof of the equivalence to the non-linear σ model in [8].

It also enables us to establish a connection from non-linear σ models which are very common in physics with *non-commutative geometry*. This is one of the most important points of this work because we are then able to define new-we believe-invariants for non-linear σ models that depend only on the base space and not on the target space. These invariants are to begin with, *complex numbers*. It is, as far as we know, the only invariants known for non-linear σ models and a rare example of complex invariants known in physics. However, the physical meaning of these invariants is not at all clear. We indicate some relation, after proper renormalisation, with Atiyah's well-known η invariants for self-adjoint operators which by Witten were related to global anomalies and to regularised determinants for Topological Quantum Field Theories in dimension 3. Another striking difference with ordinary geometry is that because we are using higher algebraic K-theory, there are many K-groups to do pairings with and not only two (in the complex case, as it is well-known one has mod 8 Bott periodicity in the real case) as in topological K-Theory (which is the ordinary geometry case). Thus one can in principle get more invariants.

The other "tricky" feature of the transformations appearing in [8] is that the algebra of these transformations is actually *abelian* despite the fact that one starts with a non-abelian structure group. This point actually, along with the one mentioned above, indicates that the usual "change of trivialisation" description of ordinary gauge transformations does not work here. Anyway, we find that one can interpret these transformations as describing a *cohomological* freedom, the cohomology being the so called *twisted* de Rham cohomology. In addition, this description also explains a feature of the B-Theory Lagrangian not mentioned in [8], namely that this Lagrangian respects *global* ordinary gauge transformations but not the local ones. We also calculated the *Euler number* of the corresponding *elliptic complex* of the twisted de Rham cohomology which by definition is the alternating sum of the dimensions of its cohomology groups. We thus obtained some more infiormation for this complex for our case, not appearing in [39]. In fact we proved as an application of the Atiyah-Singer Index Theorem (as formulated for the more general context of elliptic complexes over closed manifolds) that the Euler number of the twisted de Rham complex equals the Euler number of the

complex with differential $d \otimes 1$, namely the de Rham differential coupled to the trivial flat connection.

Then we moved on to study the loop space version of the Freedman-Townsend generalised gauge invariance as was formulated in [13]. But there, we were intrigued by the basic formula written down by Polyakov [14], concerning the formulation of Yang-Mills theories on loop spaces. Since we could not understand it, we could not actually move on to the B-Theory on loop spaces. Instead we tried, and we think we succeeded, in giving a more mathematically rigorous proof of Polyakov's basic variational formula which is essential for *everything* one attempts to do in loop spaces. (Hence we were not very far from our original motivation in that sense). The main point there is the isomorphism given by the *iterated integral map* between the *Hochschild homology* of the algebra of forms on our manifold with the *de Rham cohomology* of the *loop space* of our original manifold.

Chapter 2

Non-linear σ model

In this chapter we shall describe the non-linear σ model (denoted for brevity as "nlσm" in the sequel) in detail, since it would be extensively used in our work. The nlσm are very important in theoretical physics. A large number of physical systems, in fact the most interesting ones are either nlσm or they can be formulated as such. To mention only a few, first of all the Polyakov Lagrangian for string theory is a lσm from a 2-dim surface (this is actually the world sheet, the surface swept out by a string, a 1-dim object, in time) and target space (see below) the Minkowski space $\mathbf{R^d}$ where d=26 (or 10 assuming supersymmetry). Because the target space is linear, this is in fact a linear σ model. (For this treatment as well as for a 3+1 generalisation, namely 3-branes moving in time, see [30]).

Another important example is Yang-Mills theories. Polyakov gave a nlσm description in terms of loop spaces for pure Yang-Mills theory. We shall be particularly interested in that approach in a subsequent chapter.

But before we embark on describing the nlσm in detail, let us summarise some general facts about field theories.

Fields ϕ in a field theory are considered to be sections of a bundle E with fiber F over a base M. The fiber space is commonly called "target space" in physical literature for the fields ϕ. In familiar examples, M is the space-time manifold (assumed to be flat with Minkowski or Eucledian metric in most cases of field theory), and for scalar (spin-0) fields the space F is a linear space, for fermionic (spinor) fields E is a spin bundle and for Yang-Mills theory with gauge group G, E is an affine bundle whose sections are connections on a principal G-bundle P over M. In general, the base space has a metric and the target space is also assumed to have some geometrical structure, for instance because ultimately one is actually interested in quantising the theory, one needs a Poisson algebra structure and hence one needs also a symplectic form to be defined on the total space of the bundle considered. But at this point we are not interested in the quantisation scheme yet.

For the nlσm theory, we specialise to the cases where fields are smooth maps

from the 4-dimensional Minkowski space M to the target space F which is assumed to be a Riemannian space.

The dynamics of the fields are determined by the action functional with the following Lagrangian:

$$L = -\frac{1}{2}|d\phi|^2 \tag{2.1}$$

where ϕ are the scalar fields (spin-0) which can be represented as sections of the trivial bundle $\Xi = M \times F$, but they can also be considered as coordinates of the fiber space F. This expression makes sense as both M (Minkowski space) and F (Riemannian space) come equipped with a metric. In physical theories F is called the *target space* and it is often assumed to be a Lie group or a coset space. In this case the model is called a *(principal) chiral model*. We shall discuss these special cases later, but for the moment let us assume that F is simply a Riemannian manifold with metric g.

In physical theories formula (2.1) above is usually written as:

$$L = -\frac{1}{2}g_{ij}\partial^a\phi^i\partial_a\phi^j \tag{2.2}$$

where $(a = 0, 1, 2, 3)$ is a space-time index, i.e. an index on base manifold M, $(i, j = 1, 2, ..., dimF)$ are indices on the fiber (target) space F and g_{ij} are the components of the metric on F.

Field equations, i.e. Euler-Lagrange equations, follow from extremising the above action functional with respect to ϕ and they just state that the maps ϕ are harmonic, i.e. they satisfy the Laplace-Beltrami equation:

$$\partial^a\partial_a\phi^i + \Gamma^i_{jk}\partial^a\phi^j\partial_a\phi^k = 0 \tag{2.3}$$

where Γ^i_{jk} are the Christoffel symbols for the metric g on F. If g on F is flat then the theory is called "free" and in suitable coordinates ϕ on F the equations (2.3) are linear and in this case the general solution is a superposition of plane waves.

Let us now say some things about symmetries in physics. We shall consider only continuous symmetries here. These symmetries are most often given as infinitesimal transformations of the fields themselves. In general one distinguishes between two types of continuous symmetries of the Lagrangian (modulo boundary terms): 1.) space-time symmetries, i.e. symmetries of the base space M. In our case we ignore gravity, so we simply assume that our theory is Lorentz invariant, as it is common in quantum field theory, 2.) internal symmetries.

Given two or more infinitesimal symmetry transformations of the fields ϕ, say δ_1 and δ_2, the commutator

$$[\delta_1, \delta_2]\phi := \delta_1(\delta_2\phi) - \delta_2(\delta_1\phi)$$

generates another symmetry. The algebra of symmetry transformations may be finite or infinite dimensional. In many cases of interest it is finite dimensional as

in the nlσm where one has the algebra of isometries of the metric on the target space as the algebra of symmetries. Even if the algebra of the symmetries is infinite dimensional when acting on arbitrary fields ϕ, it may happen that it acts as a finite dimensional algebra on fields that are solutions of the field equations, i.e. in the nlσm case on harmonic maps. In this case the transformations are said to form an "on-shell" realisation of the finite algebra. The distinction between on-shell and "off-shell" realisations of an algebra is important in supersymmetry. Whereas for on-shell realisations the transformations $\delta\phi$ depend on the detailed form of the action functional, for off-shell realisations the transformation laws are general and typically leave families of actions invariant [9].

We just mention that equation (2.1)or (2.2) is invariant under the Lorentz group. So much for space-time symmetries of the non-linear σ model. We shall concentrate on internal symmetries of our theory now.

The Lagrangian (2.1) above is invariant under the isometries of the target space F (with respect to the given metric g of course). In the special case where F is a Lie group and g is a left invariant metric on F, then the algebra of isometries is exactly the corresponding Lie algebra of the Lie group and the number of commuting Killing vectors is just the rank of the Lie group. The form of the (internal) transformations that the Lagrangian is invariant under, is (infinitesimally):

$$\delta\phi^i = X^A K_A \phi^i \tag{2.4}$$

where X^A are constant parameters, hence we are talking up to now about global internal symmetries, and $K_A = k^i_A \partial_i$ are the genarators of the algebra of isommetries, i.e. they satisfy Killing's equation

$$k^i_{A;j} + k^j_{A;i} = 0$$

where ; denotes covariant derivative with respect to the metric g and the index (A=1,2,...,dim of algebra of isommetries). For a connected Riemannian space of dimension n we know that the algebra of isometries can be of dimension at most $\frac{1}{2}n(n+1)$. In the case of the maximum dimension, the manifold is a space of constant curvature.

If the target space F is a compact Lie group, then we know that there is an invariant Riemannian metric on the group which is both left and right invariant. If the group is also semi-simple, then the Killing form of the Lie algebra is negative definite, so we can just choose the metric to be the negative of the Killing form, [31]. If we make this particular choice of metric, which is natural, then the Lagrangian of the nlσm can be writen as [14]:

$$L = \frac{1}{2e_0^2} Tr(\partial_\mu U^{-1} \partial^\mu U) \tag{2.5}$$

where U is a group element represented as a matrix and e_0 is a constant depending on the representation and the normalisation. For convenience, we shall write (2.5)

above as

$$L = Tr(dg^{-1}dg) \tag{2.6}$$

Let us now discuss another situation that often arises in physics: *"gauging the symmetry"*. The meaning of this notion is that one promotes a global (or rigid) symmetry to a local one. In the nlσm case of interest here, this simply means that in the transformation formula for the fields ϕ (2.4) above we now want the quantities X_A to be space-time dependent, i.e. functions of space-time coordinates. In the geometric context this means that we actually want an associated bundle structure E over the base manifold M with fiber F where the group G is the group of isometries of the metric we have on the target space F. The bundle E is actually an associated bundle to the principal bundle P over the base space M with structure group G. In the nlσm case in physics,as we mentioned earlier, the target space F is actually a Lie group equipped with a left invariant metric, or usually with the canonical invariant Riemannian metric, so in this case one needs only the principal bundle structure. This is so because if one considers say, a left invariant metric on a Lie group, then the group of isometries of this metric is the group itself. It is very clear that the nlσm essentially depends on the metric we choose on the target space and if the target space also happens to be a Lie group too, the only way to make use of this additional property is to choose a left invariant metric on it so that the group of isometries with respect to that metric coincides with the group itself. If we do not pick a left invariant metric and the target space is again a Lie group, then we would be in the situation where we have the principal bundle P mentioned above over the base space M with structure group G the isometries of the metric on F and the associated to P bundle E will be also a principal bundle since its fiber F will be a Lie group, but this will be useless, since the symmetries we are actually using will be the group of isometries G and not F itself. Let us also comment that because we consider the base space being flat, i.e. topologically $\mathbf{R^4}$, the principal bundle P we are considering is actually trivial, $P = M \times G$, since all bundles over contractible base spaces are trivial.

We must mention however, that although in QED or the standard model or QCD, one starts with a global fermionic symmetry and by gauging it, one gets the interactions of fermions with the gauge fields (the connection 1-forms introduced), in the nlσm the reason why one might wish to gauge the algebra of isometries is related to string theory, because as we said above the Polyakov string Lagrangian is a nlσm Lagrangian and moreover one wants to get real physics as a low energy limit of string theory. Hence, gauging this symmetry is the only way to get the usual gauge groups one has in real physics (e.g. $U(2)$ for electroweak theory). Geometrically speaking this means that we must introduce a connection 1-form on our bundle.

Anyway, if one wants to gauge the symmetry in the nlσm, the action functional (2.2) is made invariant under the local (2.4) symmetries if we replace partial

derivatives with covariant derivatives with respect to some connection 1-form which will have its values in the Lie algebra (in this case the algebra of isometries of the target space F, which under the special choice of F being a Lie group with the special metrics discussed above, will be just the corresponding Lie algebra).

The important observation here is that the gauged nlσm, considering the general case with target space F and isometry group G, under a special choice of connection 1-form, can be reduced to a nlσm with target space $F_0 = F/G$. (For more details see [9]). This special connection, call it A, is obtained by extremising the gauged action with respect to the connection 1-form introduced via the covariant derivative. The result one finds is:

$$A_a^B = A_i^B \partial_a \phi^i := g_{ij} k_C^j H^{BC} \partial_a \phi^i \tag{2.7}$$

where

$$H^{BC} := (g_{ij} k_B^i k_C^j)^{-1} \tag{2.8}$$

Substituting (2.7) to the gauged action, one gets the following action for the nlσm with target space F_0:

$$L = -\frac{1}{2}(g_{ij} - g_{im} g_{jn} k_A^m k_B^n H^{AB}) \partial^a \phi^i \partial_a \phi^j \tag{2.9}$$

Chapter 3

The 2-index potential

3.1 Free Theory

We suppose that space-time is 4-dimensional Minkowski space M.

As an outgrowth of work in string theory it was noted some time ago that a real 2-form B , i.e. a spin-2 potential, subject to the infinitesimal transformation

$$\delta B = d\xi \tag{3.1}$$

with ξ an arbitrary real 1-form and d the exterior differential, can be used to describe spinless, massless particles [32, 2, 33, 34]. We shall refer to this fact in the sequel as the "spin-0 , spin-2 duality".

To see that clearly, let us recall the Lagrangian for the real massless Klein-Gordon field ϕ:

$$L = \frac{1}{2}|d\phi|^2 \tag{3.2}$$

where ϕ is a real valued 0-form on M. The Euler-Lagrange equations are simply

$$\Box\phi = 0 \tag{3.3}$$

where $\Box$ is the D'Alembertian.

Let us now see what happens in the dual case. We start with a real valued 2-form B and construct the following Lagrangian:

$$L = -\frac{1}{8}|dB|^2 \tag{3.4}$$

The Euler-Lagrange equations then are:

$$d * dB = 0 \tag{3.5}$$

while

$$d^2 B = 0 \tag{3.6}$$

holds identically. Equation (3.5) implies, using Poincare lemma for the closed 1-form $*dB$, that

$$*dB = d\phi \tag{3.7}$$

for some real valued 0-form ϕ. Substituting (3.7) to (3.6) and after some elementary manipulations we get:

$$\Box\phi = 0 \tag{3.8}$$

Hence we recover the Klein-Gordon equation.

Before going on, let us make some comments:

1. The fact that space-time is flat is crucial because we use Poincare lemma to show that the scalar field ϕ exists. The relation between the spin-2 and the spin-0 fields is equation (3.7).

2. The term "duality" is justified from the following fact: we know that duality interchanges topology and dynamics (see [9] for a clear exposition). In our setting, which is quite trivial for this simple case, topology is just the identities $d^2B = 0$ (same for ϕ), (in a Yang-Mills theory, topology is expressed by the Bianchi identity-no monopoles present- which is a second order equation like our case here), whereas dynamics are the Euler-Lagrange equations in each case. We see therefore that under duality, (in this case it is Poincare lemma, i.e. equation (3.7)), equation (3.6) for instance which is topology for the spin-2 field B, becomes (3.8) which is dynamics for the spin-0 field. Similarly, via (3.7), (3.5) (dynamics for the spin-2 field) is equivalent to $d^2\phi = 0$ (topology for the spin-0 field). For more details see [9].

3. If one uses the geometric language, spin-n gauge fields are described as n-forms over space-time with values in an apropriate space, usually some vector space. In general, for a D-dimensional space-time manifold, the dual of a spin-n field will be a spin-$(D-n-2)$ field. The term -2 comes from the fact that in the Lagrangian what appears is the field and not the n-index potential itself and this is so for both the original field and for its dual. It is clear then that this duality described above between spin-0 and spin-2 fields holds only for 4-dimensional space-time. Moreover let us point out here that $D-n-2$ must be positive. This is a crucial observation in [12].

3.2 Mathematical interpretation of the "generalised" gauge symmetry

We shall now present *our* argument which is **crucial** in understanding this generalised gauge invariance and which actually demonstrates that the origin of the symmetry (3.1) is **completely different** from what is thought to be in physics literature: for the case of a 4-manifold, we know that real scalar fields have 1 degree of freedom. *But* for its dual, one has an anti-symmetric rank 2 tensor

gauge field with 6 degrees of freedom. This fact is a striking difference between this duality and the electric-magnetic duality in abelian gauge theory or other dualities appearing in physics, where the original fields and their duals are either of the same spin, or more general, they belong to vector spaces which are isomorphic (of the same dimension). This fact then implies that the 2-form B in our case above will posses some extra symmetry, since intrinsically it has "more freedom". This is exactly the origin of transformation (3.1) above and we observe that this is an *extra* symmetry of the Lagrangian of the dual field (3.4) and that there is no analogue of this symmetry for the spin-0 Lagrangian (3.2). This observation is also important for the following reason: since the dimension of the vector spaces involved does not remain constant, this actually indicates that a *generalised homology theory* must be involved, where by "generalised" we mean a homology theory that satisfies all but the dimension axiom of the Eilenberg-Steenrod axioms for homology theories. *(Co)bordism Theory* and *K-Theory* are the most well known examples of such theories. We think that by the end of this work the role that these theories play indeed in our case will become apparent.

Another way to say the same thing is that B in (3.4) is defined up to d exact terms, since it is its field strength which appears in the Lagrangian. But ϕ, being a 0-form, there is nothing to add to it and also its field strength appears in the Lagrangian as well. This is again due to the fact that the degrees of freedom are different.

We note also that the forms used above are real valued and that no internal symmetries are introduced, i.e. there is no principal bundle over space-time manifold M.

3.3 Interacting Theory

As in [8] we now want to make something more interesting; so we ask what would a self-interacting theory of fields B look like?

In order to interpret this statement in a geometric context, there are some really non trivial problems. What one has in mind is the example of Yang-Mills theory, where due to the use of a non-abelian structure group G, the gauge potentials (connection 1-forms on a principal G-bundle P over space-time manifold M) interact with each other due to the term

$$A \wedge A$$

in the covariant derivative with respect to the gauge potential A. However, that would immediately suggest that the 2-form B in our case should be thought of as a connection on some G-bundle over M. In addition to this geometrical problem, there is also a physical one: the reason why gauge potentials are introduced in physics is that one wishes to maintain invariance of the Lagrangian under local gauge transformations. But we have not assumed any internal symmetry at all,

at least not yet. Moreover the fields B exist anyway as the duals of the scalar fields ϕ.

Trying to resolve this problem, let us see what physics has to suggest [8]:

Let us assume that B is assigned to the adjoint representation of some nice Lie group G. In order to construct an interacting theory for B, this must couple either "non-minimally" through the 1-form G, where we call

$$G := *dB \tag{3.9}$$

or minimally to a conserved anti-symmetric rank 2 tensor current $J^{\mu\nu}$. The charge associated with such a current necessarily vanishes, and cannot describe any global symmetry. Nevertheless one might construct such currents and attempt to couple them consistently. Currents can be constructed either from conventional fields, such as the current

$$\frac{1}{4}(\bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\nu\partial^\mu\psi + \partial^\mu\bar{\psi}\gamma^\nu\psi)$$

which is conserved for free Dirac fields ψ, or from antisymmetric tensor fields themselves, which is the path followed in [8]. Specifically one notes that given a set of fields B, the current $|G|^2$ is conserved by virtue of the free field equation of motion (3.5) above. One uses this observation to construct iteratively a non-abelian extension of the "free" theory described above. Then one gets the following Lagrangian:

$$L = -\frac{1}{8}(B \wedge *F_A - A^2) \tag{3.10}$$

where we define now the connection 1-form A as follows (see the last section of this chapter for a discussion on the existence of A):

$$A := *d_A B \tag{3.11}$$

and

$$F_A = dA + A\wedge A, \tag{3.12}$$

the field strength (curvature) of the connection 1-form A.

This Lagrangian is then invariant under the following (infinitesimal) transformations:

$$\delta B = d_A\xi \tag{3.13}$$

where now ξ is an arbitrary 1-form with values in the same space as B, i.e. in the Lie algebra g of the Lie group G.

The Euler-Lagrange equations for the field B are now:

$$*F_A = 0 \tag{3.14}$$

One can use (3.11) and (3.12) above to get the explicit Euler-Lagrange equations for B itself, but we use this more convenient expression.

The key observation here is that if B satisfies the Euler-Lagrange equations, then A must be a flat connection, or in physical terminology A must be "pure gauge", since (3.14) implies that

$$F_A = 0$$

This is in fact what gives the first indication that there may be a relation to the non-linear σ model.

The "duality" equation in the interacting theory, i.e. the analogue of equation (3.7), is (3.11), or

$$g^{-1}dg = *d_A B \tag{3.15}$$

using the standard expression for flat connection 1-forms A (since our space is topologically $\mathbf{R^4}$)

$$A = g^{-1}dg$$

for g a group valued 0-form (see also the paragraph below where we demonstrate the equivalence to the nlσm).

The most important fact about the flat connection A for our purpose is that its exterior covariant derivative d_A is actually a differential, since (assume w is an arbitrary Lie algebra valued form)

$$d_A^2 w = d_A d_A w = d_A(dw + A \wedge w) = d(dw + A \wedge w) + A \wedge (dw + A \wedge w) =$$

$$= d^2 w + dA \wedge w - A \wedge dw + A \wedge dw + A \wedge A \wedge w = 0 + 0 + (dA + A \wedge A) \wedge w = F_A \wedge w = 0$$

since A is flat (has zero curvature).

¿From (3.13) we deduce that under two infinitesimal transformations

$$\delta_1 \delta_2 B = d_A \xi_1 + d_A \xi_2$$

so that

$$[\delta_1, \delta_2] B = 0$$

(where [,] is the commutator). Hence the algebra of these transformations is actually *abelian*. Furthermore, because of (3.11) the transformation (3.13) will be satisfied if A does not transform at all, namely

$$\delta A = 0 \tag{3.16}$$

The origin of transformations (3.13) becomes *very straightforward* if one considers our observation in the previous section. Using the terminology of [35], these transformations simply mean that the space of classical observables consists of the cohomology groups defined by a "twisted" deRham complex of the total space P of our principal bundle where the differential is now d_A with respect to the flat connection 1-form A defined through B via (3.11) above. Let us recall

that in usual Yang-Mills theories or in the geometric quantisation context, one considers as the space of classical observables the direct sum of the groups of the equivariant cohomology of P. Again, of course these transformations express the fact that the vector spaces of 0-forms and 2-forms are not isomorphic, the only difference with the free theory case being that now the forms are Lie algebra valued and the differential now is d_A instead of the usual d. In the sequel in order to distinguish it from the usual deRham complex we shall be refering to this as the "twisted" deRham complex. We shall study this complex extensively in Chapter 4.

The above comments concerning the real origins of these transformations also show why the algebra is actually abelian. We mention this point again because in [8] where these transformations appeared for the first time, they are considered to be gauge transformations and this actually suggests strong principal bundle isomorphisms of perhaps some generalised kind. However from the above disscusion, it is we think quite clear that these transformations are of a different origin and they should therefore be interpreted differently.

Remark:

Let us close this section by the following remark: one might be interested in the finite version of the transformations (3.13), or even of the free theory case (3.1). One would then have to exponentiate these expressions in order to obtain a Lie group element. But in this case one runs into great trouble because the gauge parameter ξ is a Lie algebra valued 1-form and not a Lie algebra valued 0-form as in Yang-Mills theories say. And the exponent of a 1-form can be obtained only as formal power series up to the dimension of the underlying manifold and clearly this cannot give a Lie group element. So we do not think that obtaining the finite version of these transformations is something which even in principle can be done. The situation is very similar to what happens in supersymmetry, where moving to the finite version of the transformations involves the development of the notion of super Lie groups. If we adopt the modern formulation of supersymmetry, this is just a graded Lie algebra. Our case here is similar and hence one should expect similar problems.

Another way of saying, using Connes' setting, is that we are actually interested in a *generalisation* of the group of gauge transformations $M_n(C^\infty(P))$ to the group $M_n(\Omega(P))$ where M_n denotes the $n \times n$ invertible matrices with entries from the appropriate algebra.

The precise mathematical statement then is that: *the Freedman-Townsend invariance is a special case of Grothendieck's* stability *property for* K^0. This is we think an important observation which essentially says that whereas ordinary gauge invariance in physics corresponds to isomorphism between vector bundles (i.e. the first equivalence relation used in the definition of K^0), this new generalised gauge invariance corresponds to the second! We *prove* this claim at the

end of section 4.3.1.

3.4 Relation to non-linear σ model

Following [8], in order to see explicitly the equivalence of this self-interacting theory with the non-linear σ model as we did the equivalence of the free theory with the real scalar field theory, let us introduce a matrix notation using an antihermitian representation with basis T_i of the Lie algebra, i.e.

$$[T^i, T^j] = f^{ijk}T^k \tag{3.17}$$

and

$$Tr(T^iT^j) = -C\delta^{ij} \tag{3.18}$$

with C a positive constant which is representation dependent. Defining matrix variables $B_{\mu\nu} = B^i_{\mu\nu}T^i$ and $A_\mu = A^i_\mu T^i$ we can write the Lagrangian (3.10) as:

$$L = \frac{1}{8C}Tr(\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma} - A_\mu A^\mu) \tag{3.19}$$

The constraint

$$F_{\mu\nu}(x) = 0 \tag{3.20}$$

implies that

$$A_\mu(x) = \frac{1}{g}U^{-1}(x)\partial_\mu U(x) \tag{3.21}$$

where

$$U(x) = exp(i2gT^i\Pi^i(x)) \tag{3.22}$$

and the $\Pi^i(x)$ are the group parameters. After substitution of the last two equations into the expression for the Lagrangian, one finds

$$L = \frac{1}{8Cg^2}Tr(\partial_\mu U^{-1}\partial^\mu U) \tag{3.23}$$

which is one of the familiar forms of the principal chiral model.

One can also reverse the above argument in order to go backwards, namely starting with the nlσm Lagrangian (2.6) we can recover the Lagrangian of the dual theory (3.19) as follows: Let us define the following object

$$A := g^{-1}dg \tag{3.24}$$

which is a Lie algebra element (now g is actually the group element U above). It looks like a flat connection, and in fact it can be thought of as one, but this is not neccessary. If we substitute (3.24) into (2.6) we get

$$L = TrA^2 \tag{3.25}$$

which is another form of the nlσm Lagrangian. If we think of A above as a connection 1-form, then by definition it is flat, i.e.

$$F_A := d_A A = 0 \tag{3.26}$$

Then the equation of motion in this case reads

$$d * A = 0 \tag{3.27}$$

which is equivalent to (2.3) bearing in mind the specific choice of metric. Now equation (3.26) above can be thought of as a constraint. Following the method of Lagrange multipliers, it can be thus inserted in the Lagrangian (3.25) coupled to a 2-form B (it has to be a 2-form, i.e. something skew with two indices, since this is also the case for the curvature form F_A), which will be the Lagrange multiplier. Hence we obtain the Lagrangian (3.19) which is the Lagrangian of the dual "B-theory". So we actually show how one can go from the nlσm to the dual theory. Hence the theories are equivalent, relying of course on specific choices. Because of the specific choice of metric the algebra of global symmetries in both theories is the same, namely the Lie algebra of the Lie group we started with.

The above argument demonstrates the classical equivalence of the anti-symmetric rank 2 tensor theory to the principal chiral non-linear σ model . As a check on the argument a calculation done in [5] of the tree approximation scattering amplitude using the Lagrangian (3.10) and the non-linear σ model Lagrangian shows that the results agree.

3.5 Comments

In this final section of this chapter we would like to make some comments.

The first one concerns equation (3.11).We must justify that this can be solved uniquely with respect to A, given B. In addition one has also to see if the resulting 1-form A *is* actually a connection 1-form (or if it can be thought of as one).

In [8], the existence of a unique solution is guaranteed if the coupling constant g in the interacting theory is sufficiently small. This is so because the existence and uniqueness of A depends essentially on the invertibility of a quantity of the form $1 - t$, where t is a function of gB. We know from the binomial series expansion that if t is sufficiently small, (i.e. for arbitrary B one needs g to be small enough), then one has that:

$$\frac{1}{1-t} = 1 + t + t^2 + ... \tag{3.28}$$

To be more specific, an equivalent form of the Lagrangian (3.10) is the following (see [8]):

$$L = -\frac{1}{8} G^{\mu i} \tilde{K}^{ij}_{\mu\nu} G^{\nu j}$$

with

$$K^{\mu\nu ij} := g^{\mu\nu}\delta^{ij} - gf^{ijk}\epsilon^{\mu\nu\rho\sigma}B^k_{\rho\sigma}$$

and

$$\tilde{K}^{ik}_{\mu\rho}K^{\rho\nu kj} = \delta^\nu_\mu\delta^{ij}$$

where g is the coupling constant, $g^{\mu\nu}$ are the components of the (Minkowski) metric (essentially 1), f^{ijk} are the structure constants of the Lie algebra and

$$G^{\mu i} := \frac{1}{3}\epsilon^{\mu\nu\rho\sigma}B^i_{\nu\rho,\sigma}$$

As to the second question, if A is a connection 1-form, in the physics context there is nothing we need to say.

Let us now see the above issues in the geometric context. We shall answer the second question first.

Any bundle over $\mathbf{R^4}$ is topologically trivial, so taking the trivial connection to define a basis α, one has that the other connections are in 1:1 correspondence with the 1-forms on $\mathbf{R^4}$ of adjoint type. Thus, A can indeed be thought of as a connection 1-form.

Concerning the solution of equation (3.11) with respect to A, given B, using the basis α introduced before, the equation can be reduced to:

$$(* - B\wedge)A = dB \tag{3.29}$$

As $*$ is an isomorphism from 1-forms to 3-forms, the solution will depend on the eigenvalues of the operator from Ω^1 to itself defined by "wedging with B and then take the dual". If 1 is not an eigenvalue, or better if B is small, then it will be an isomorphism and there will be a solution.

This is essentially the same result, if we replace the B in the geometric context (which must be small), with the product gB of the physics context, which is indeed small, for arbitrary B, provided that the coupling constant g is small enough as we pointed out earlier.

So, to sum up, the result is that for sufficiently small coupling constant, for each field B (2-form), there is a unique corresponding field A (1-form), which can always be thought of as a connection 1-form. In addition, if B satisfies the Euler-Lagrange equations, then the connection 1-form A has to be flat. The general case is not easy and we cannot say anything about it. One should note however that in the duality relation of the interacting theory (3.15), we do not use the Poincare lemma for the existence of the scalar field g above, as we did in the free theory case. This may indicate that the result, same or something similar, holds for non-flat space-time as well.

The final comment is about the question whether the "gauge invariance" (3.13) of the interacting theory is trivial or not. As it is pointed out in [36], in the quantum level, in dimension greater than 4, (the case of the dimension being

less than 4 is not interesting), the only consistent local interactions that can be added to the free Lagrangian are those which do not modify the free theory gauge invariance. In dimension 4 however, there exist consistent local interactions that deform the gauge symmetry of the free theory in a non-trivial way, which yield exactly the Lagrangian and the gauge symmetry of the interacting theory described in [8] and studied here. This is a very recent result, indicating that these transformations are indeed something non-trivial.

We would like now to comment on possible generalisations of this theory. The first thing to ask is what happens if one considers a 4-dimensional manifold as space-time which is no longer flat but has a Riemannian metric g on it. We do not want to include gravity as well, the metric is not treated as a dynamical variable of the theory so we do not have to modify the twisted de Rham differential in order to make it "compatible with the metric". In physical terminology we want to do field theory in a "curved background". The Lagrangian will not change—except of the measure which would be the Wiener measure and it will include a factor of the square root of the metric—neither will the gauge transformations of the fields B. This is of course the great advantage of describing expressions using forms in a coordinate free way. The Euler-Lagrange equations will also remain unaltered. However there is one question which has to be resolved and this is the existence of the flat connection A, namely the existence and uniqueness of solution of equation (3.13). This equation, given B, will now depend on the metric as well. So it is possible that given B, one might have to put some restrictions on the metrc in order to obtain a unique A, which if it exists, by virtue of the Euler-Lagrange equations which will not change, will have to be flat.

Let us emphasise that we are using two notions of duality here: one is the usual from geometry notion of duality which is just the Hodge star operator and we shall be refering to this as the Hodge duality; if our space has dimension D, then the Hodge dual of a k-form will be a $(D-k)$-form. The second is the duality in the physics sense, see for instance [9], which we shall be refering to as (simply) duality. In this context, if space-time has dimension D, then the dual of a spin-k field (a k-form) will be a spin-$(D-2-k)$ field, namely a $(D-2-k)$-form.

The next question is what will happen if one moves to a space-time which is of arbitrary dimension but still $\mathbf{R}^d$, for some d. This question is actually very useful because as we mentioned in the Introduction this new type of gauge symmetry appeared for the first time in string theory and in this context the dimensionality of space-time may be different from four.

In this case one first of all loses the spin-0, spin-2 duality, at first. If our space-time has dimension D say, then the dual of a spin-n gauge potential is a spin-$(D-n-2)$ potential. Hence the dual of a scalar field will be a $(D-2)$-spin field. But this is exactly the degree of $*B$ in dimension D, so the duality is not completely lost but rather one will have to consider the Hodge dual of the

potential B instead. In this case one has

$$A = \delta_A B$$

where δ_A is the codifferential of d_A.

This is very similar to the situation described in [11]. The Lagrangian, as well as the Euler-Lagrange equations plus the gauge invariance will be exactly the same but one will have to replace the field B with its Hodge dual $*B$. This was also disscused in [8]. This is actually the case in string theory, namely the Hodge dual of the two-indexed potential is the dual of the scalar field.

Another possibility is to couple the fields B with ordinary vector potentials. This has also been considered in [8]. The resulting theory describes interactions between the B potentials with gauge bosons. It actually turns out though that this is equivallent, classically, to a theory of massive gauge fields.

It is also possible to include supersymmetry in the B-theory. In [8] a supersymmetric version of the B-theory was constructed assuming $N = 1$ SUSY and 2-dimensional space-time (equivalently 2+2 dimensional superspace). The theory was proved to be equivalent to $N = 1$ SUSY 2-dimensional nlσm.

And now, to close this chapter, let us mention something about the classical solution space of the Euler-Lagrange equations of the B-theory.

We saw that if B satisfies the Euler-Lagrange equations, it uniquely determines a flat connection which in turn is written down using some group valued 0-form. So *the solution space of the fields* B mod the Freedman-Townsend invariance *is in one to one correspondence with* flat *connection 1-forms*. This is another equivalent way of looking at this symmetry. So the question is "how many flat connections does one have"?

The answer to this question is of great importance in geometry and in the case of manifolds of dimension 3 and 4 very fundamental. In the former case is directly related to Floer homology and the Casson invariant (as well as to the Jones polynomial) for 3-manifolds. In dimension 4 it is related to the properties of moduli spaces, the construction of Donaldson polynomial invariants and especially with the proof that they are metric independent. Anyway, there is a general result saying that:

Theorem :

Gauge equivalence classes of flat connections are in one to one correspondence with conjugacy classes of irreducible representations of the fundamental group of the manifold into the structure group of the principal bundle considered.

(For a very clear exposition of this result as well as other results about flat connections we refer to [37]).

It is important to see that if the space is simply connected, then by the above theorem, all flat connections are "gauge equivalent", namely they can be obtained by the trivial connection -essentially the Maurer-Cartan form of the structure Lie

group- by a suitable choice of trivialisation (gauge fixing). And in any case the gauge transform of a flat connection is always a flat connection.

Let us also emphasise another point which will be of great importance later.

It is clear we think from the above disscusion that a nlσm with target space a compact and semi-simple Lie group endowed with the canonical invariant Riemannian metric is actually equivalent to giving a flat principal bundle over space time with structure group the group manifold. The fact that the nlσm requires the maps to be harmonic simply means that the flat connection is also divergenceless. The equivalence becomes manifest from equations (3.23), (3.24), (3.25). This will give the starting point for entering the exciting field of non-commutative geometry.

Chapter 4

Cohomology

In the previous chapter we explained how one could think of this new type of gauge invariance as providing initially a different space of classical observables. The most convenient way to describe observables in physics is via some cohomology groups. Indeed, in the previous chapter we gave such an interpretation by stating that the space of classical observables associated to this new gauge invariance is the direct sum of the cohomology groups of some "twisted" version of the usual de Rham complex of forms with values in an appropriate vector space, where the differential d has been replaced by the differential d_A, where A is a flat connection 1-form on a principal bundle arising naturally in our case by physical arguments. This is in fact the key observation describing this theory as the dual theory of the 4-dimensional nlσm. The vector space where the forms might take their values will often be assumed to be the corresponding Lie algebra of the structure Lie group of the principal bundle, however in general it may be any vector space carrying a representation of the structure group and this will be determined by the actual "physical" fields we want to consider each time. In the sequel we shall refer to this new gauge invariance as the "twisted" version of the usual gauge invariance appearing in Yang-Mills theories.

The main purpose of this chapter will be to make some calculations involving this twisted de Rham cohomology in order to obtain a better understanding of it and to try to relate it with the usual cohomologies appearing on a principal bundle which correspond to the classical observables of Yang-Mills theory.

Before starting our endeavour, we would like to make clear the following point: this kind of duality we described earlier in chapter 2, depends crucially on the fact that space-time, i.e. the base space of the principal bundle under consideration, is 4-dimensional and flat, namely $\mathbf{R^4}$ (see remarks in chapter 2). But now where we come to study this setting purely mathematically, we encounter the peculiarities of $\mathbf{R^4}$, i.e. that it has trivial cohomology, apart from H^0, and that it is not compact. Moreover there is also the inconvenience of the metric being Minkowski. What we shall do, at least for most of this chapter, will be to choose as our base space the nicest possible 4-manifold for our purpose. The

same will hold for the structure Lie group: in fact what we would always have subconciously in mind will be SU(2). After developing the theory we shall go back to flat space-time and see what can be said for this case. After all this is not too bad because in Physics, despite the fact that space-time is non-compact, we often make assumptions like "fields falling off at infinity" at a suitable scale, so we can think of $\mathbf{R^4}$ being compactified to S^4. This is also the case for instantons for instance, where due to the above assumption and to a conformal invariance argument, one works with S^4 instead.

The background references for this chapter are [31], [38] and [39].

4.1 Cohomologies arising naturally on a principal bundle

4.1.1 Cohomologies of real valued forms

We fix the real numbers as our field, so everything considered will be real.

Let (P, π, M, G) be a principal bundle over a closed 4-manifold M with structure group G assumed to be compact and connected. We also assume that G acts on P from the left. We denote by L_a left translations by an element a of G and we define an action of G in the Lie algebra of the vector fields on P by

$$aX = (L_a)_* X \tag{4.1}$$

where X is a vector field on P and L_{a*} is the induced left translations of a on the vector fields of P. Denoting by L_a^* the induced left translations of the element a of G on the graded algebra (over the reals) of real valued differential forms $\Omega(P)$ of P, we can define analogously an action of G on $\Omega(P)$ in the obvious way, namely

$$a\phi = L_a^* \phi \tag{4.2}$$

A differential form ϕ on P is called *(left) invariant* (with respect to the action of G) if it satisfies the relation:

$$L_a^* \phi = \phi \tag{4.3}$$

for all a $\in G$. The invariant forms form a graded subalgebra of $\Omega(P)$ which will be denoted $\Omega_I(P)$. Since L_a^* commutes with the usual exterior derivative d acting on forms (being the differential for the de Rham complex), it follows that the subalgebra $\Omega_I(P)$ is stable under d, i.e. $d(\Omega_I^*(P)) \subseteq \Omega_I^{*+1}(P)$.

Let us denote by $i(X)$ the substitution operator acting on differential forms on P, namely it evaluates a form on the vector field X of P (the usual "hook"), and by $l(X)$ the Lie derivative along the vector field X on P.

A differential form ϕ on P is called *horizontal* (with respect to the action of G) if it satisfies the relation

$$i(h)\phi = 0 \tag{4.4}$$

for all elements h of the Lie algebra g of the Lie group G. To be more precise by h we mean the vertical vector field X_h on P generated by the element h of the Lie algebra via the Lie algebra homomorphism between the Lie algebra of the group G and the Lie algebra of the vertical (or fundamental) vector fields on P, but as usual, we identify them. Since each $i(h)$ is an antiderivation, the horizontal forms form a graded subalgebra of $\Omega(P)$ denoted by $\Omega_H(P)$ (H for horizontal). This subalgebra is stable under the Lie derivative but *not* stable under the differential d.

Similarly, the differential forms satisfying the relation

$$l(h)\phi = 0 \tag{4.5}$$

$\forall h \in g$, where here also we mean X_h, the vertical vector field on P generared by h, form another graded subalgebra $\Omega_l(P)$ of $\Omega(P)$ and since d commutes with the Lie derivative $l(h)$, the subalgebra $\Omega_l(P)$ is stable under d.

The intersection of algebras $\Omega_H(P) \cap \Omega_l(P)$ will be denoted by $\Omega_{H,l}(P)$ and this consists of forms which are horizontal and have vanishing Lie derivative along the *vertical* (or *fundamental*) vector fields of P. Here we identify (or we *extend*) the elements of the Lie algebra with the vertical vector fields on P generated by them. The important point here is that this graded subalgebra *is* stable under d.

(For the proofs of the four propositions below see the references):

Proposition 1:

If the group G is connected, then $\Omega_I(P) = \Omega_l(P)$. In general, one has that $\Omega_I(P) \subset \Omega_l(P)$.

The intersection of the graded subalgebras $\Omega_H(P) \cap \Omega_I(P)$ will be denoted by $\Omega_B(P)$ and are called *basic* forms. The basic forms *are* stable under d, as follows from Proposition 1 above and from the fact that the subalgebra $\Omega_{H,l}(P)$ is stable under d.

Proposition 2:

If the structure group G is connected, then one has an isomorphism of graded algebras $\Omega_B(P) \cong \Omega(M)$ between the basic forms of the total space P of the bundle and the forms of the base space M.

This induces an isomorphism on cohomology as well, by the projection map π, namely

Proposition 3:

In the situation of Proposition 2 above, one has an induced isomorphism in cohomology $H_B(P) \cong H(M)$, where $H_B(P)$ is the cohomology groups computed

from the complex $(\Omega_B(P), d)$ and $H(M)$ is the usual de Rham cohomology of the manifold M and d is the usual differential.

One also has the following result, since as we said invariant forms are stable under the differential d:

Proposition 4:

For structure group G compact and connected, one has an isomorphism between cohomology groups $H_I(P) \cong H(P)$, where $H(P)$ is the usual de Rham cohomology of the total space P and $H_I(P)$ is the cohomology computed from the following complex of invariant forms on P: $(\Omega_I(P), d)$ where d is the usual differential.

4.1.2 Cohomologies of forms with values in a finite dimensional vector space

Before dealing with the main topic of this subsection, let us make some general comments.

The main point of the above discussion is to find out how many *subcomplexes* of the usual de Rham complex we can naturally consider. The differential is always the usual d. So, to sum up, one can consider actually *four* subcomplexes, namely those graded subalgebras of $\Omega(P)$ which are stable under d:
1.) the invariant forms $\Omega_I(P)$,
2.) those with vanishing Lie derivative $\Omega_l(P)$,
3.)the intersection of 2.) with the horizontal forms $\Omega_{h,l}(P)$
and finally
4.) the basic forms.
In a nice situation, i.e. if G is compact and connected, one has some identifications, namely: 1.)=2.) where the cohomology groups actually turn out to be the same as the ordinary de Rham cohomology groups of the manifold P and 3.)=4.) coming essentialy from the 1.)=2.) identification, where the cohomology groups actually coincide with the ordinary de Rham cohomology of the *base* space M.

So, at the cohomological level, which, anyway, is what we are interested in, assuming the "nice" situation for possible physical applications, we finally end up with two cases, one with cohomology of the total space and one with cohomology of the base space.

Now we turn our disscusion to Physics. The classical observables must be *gauge invariant quantities*,i.e. quantities "constant" along the gauge orbits. Since the vertical vector fields on the total space P are tangent to the gauge orbits, the interpretation of this physical requirement in the mathematical language is that they must have vanishing Lie derivative with respect to fundamental vector fields. So in principal, we are interested in the subcomplex 2.) above.

But there is another thing which must be taken into account: because of gauge invariance, physical quantities must be such that they can be "brought down" (pulled-back by a section or gauge fixing) to space-time, i.e. the base manifold. After all for physics, the total space ("internal space" in physical terminology) is just a fictitious space, a mathematically convenient way to describe things. Physics "occurs" in space-time. That suggests that we must consider the second cohomology case, the subcomplex of the basic forms.

But unfortunatelly things are more complicated. The above disscusion would be suitable for "first quantized theories", in the case where the total space P would be the phase space of some system, which would also have to be a symplectic manifold in this case. After reductions, we would end up with a constraint space and our space of classical observables would be just the H^0 cohomology group of this space (gauge invariant functions).

We are interested in second quantized theories though, because this is the correct picture physically. In this case the fields in fact take their values in another vector space which, to avoid mathematical difficulties that we are not concerned at present with, we shall take to be finite dimensional here. Hence we need to study the cohomologies of forms with values in a finite dimensional vector space, W say.

We start with differential forms on P with values in the finite dimensional vector space W carrying a representation ρ of G.

Let W be a finite dimensional vector space. The cross-sections of the bundle $\wedge^r \tau_P^* \otimes P^W$, where $P^W := P \times W$ is the trivial vector bundle over P with fiber W and τ_P^* is the cotangent bundle of P, are called *r-forms on P with values in W*. These modules (graded algebras) will be denoted by $\Omega(P;W)$. There is a canonical isomorphism $\Omega(P;W) \cong \Omega(P) \otimes W$, so these spaces are naturally identified. One then has the operators $l(X)\otimes 1$, $i(X)\otimes 1$, $d\otimes 1$ and $L_a^*\otimes 1$ acting on $\Omega(P;W)$ but for convenience we shall omit the $\otimes 1$ factor. These are of course the extensions of the Lie derivative, substitution operator, exterior differentiation and left translations induced by the elements a of the Lie group G, respectively (where X is a vector field on P).

Now suppose that we want to use the representation ρ of the Lie group G on the vector space W. Then each element $a \in G$ determines the operator $\rho(a)_*$ in the space $\Omega(P;W)$ of W-valued differential forms given by (for an arbitrary r-form $\phi \in \Omega(P;W)$):

$$(\rho(a)_*\phi)(x; X_1, ..., X_r) = \rho(a)(\phi(x; X_1, ..., X_r)) \tag{4.6}$$

where $x \in P$ a point on the bundle and $X_1, ..., X_r \in T_x(P)$ are vectors on P over the point x. We denote $\rho(a)_*$ simply by $\rho(a)$.

A left action of the group G on the space $\Omega(P;W)$ is defined by

$$a\phi = (\rho(a) \circ L_a^*)\phi = (L_a^* \otimes \rho(a))\phi \tag{4.7}$$

where $a \in G$ and $\phi \in \Omega(P;Q)$ and where we have identified $\Omega(P;W)$ with $\Omega(P) \otimes W$.

A W-valued form ϕ is called *equivariant* with respect to the representation ρ if $a\phi = \phi$, $a \in G$. This is equivalent to the condition $L_a^*\phi = \rho(a)^{-1}\phi$. The space of equivariant forms is denoted $\Omega_E(P;W)$ and it is stable under d.

Proposition 5:

An equivariant differential form ϕ satisfies the relation

$$l(h)\phi = -\rho'(h)\phi$$

where ρ' is the induced representation of g in W and $h \in g$. If G is connected, this condition is equivalent to equivariance.

Let us mention that if we choose W to be just $\mathbf{R}$ and $\rho = 1$, then equivariant forms are just the invariant real valued forms. Moreover we could choose the group G to act on $\Omega(P;W)$ just as $L_a^* \otimes 1$ and in this case we can forget the representation altogether. If we choose the adjoint representation, i.e. W=g, and $\rho = Ad$, then an equivariant form ϕ of adjoint type would have to satisfy

$$L_a^*\phi = (Ada^{-1})\phi \tag{4.8}$$

for $a \in G$ and if G is connected, one has the equivalent condition of Proposition 5:

$$l(h)\phi = -(adh)\phi$$

where $h \in g$.

The notion of invariant (equivariant resp.) forms is just a simple case of a more general situation of *invariant sections* (*equivariant* resp.) of a general vector bundle over P, i.e. not just the vector bundle of differential forms on P.

A W-valued form $\phi \in \Omega(P;W)$ is called *horizontal* if $i(h)\phi = 0$, where $h \in g$. The horizontal forms form a graded subspace of $\Omega(P;W)$, it will be denoted by $\Omega_H(P;W)$ and just like their real-valued case analogues are not stable in general under d. The isomorphism $\Omega(P;W) \cong \Omega(P) \otimes W$ restricts to an isomorphism $\Omega_H(P;W) \cong \Omega_H(P) \otimes W$.

On the other hand, a W-valued form ϕ is called *invariant* if it satisfies the condition $L_a^*\phi = \phi$, where $a \in G$. Thus ϕ is invariant if and only if $\phi \in \Omega_I(P) \otimes W$. The space of invariant W-valued forms will be denoted by $\Omega_I(P;W)$ and it is stable under d as the above isomorphism suggests. Assuming that G is connected, these forms coincide with the W-valued forms with vanishing Lie derivative along vertical vector fields, so we do not treat them separately, it is just a straightforward generalisation of the real valued case: just tensor with W and linearly

extend $l(h)$ to $l(h) \otimes 1$ as mentioned above.

The intersection of the subalgebras $\Omega_I(P;W) \cap \Omega_H(P;W)$ will be called *basic* and will be denoted by $\Omega_B(P;W)$. Basic W-valued forms are stable under d, just like in the real valued case. (Of course in the G connected case we assume, this space coincides with forms with vanishing Lie derivative along vertical vector fields and which are horizontal).

Finally the space $\Omega_H(P;W) \cap \Omega_E(P;W)$ is called the space of ρ-*basic* W-valued forms and it is denoted by $\Omega_{B,\rho}(P;W)$ where the suffix ρ is used to remind us that they depend on the representation ρ of G onto W. These ρ-basic W-valued forms are *not* stable under d but they are stable under d_A, the exterior covariant derivative with respect to a connection 1-form A on the principal bundle P. If we choose ρ to be the trivial representation, then equivariant forms are just the invariant forms and in this case the ρ-basic forms are just the basic forms. The typical example of a basic form is the curvature of a connection 1-form (with values in the Lie algebra and assuming the adjoint representation), whereas the connection 1-forms are equivariant forms (again with values in the Lie algebra, assuming the adjoint representation).

So, to sum up, if we consider W-valued forms on P and assume G connected, we can obtain naturally three subcomplexes of the original de Rham complex where every vector space $\Omega(P)$ has been tensored with W and we have extended d to the operator $d \otimes 1$ to act on them:
1.) equivariant forms, provided one uses a representation of G on W and elements of G act on "both factors" of the tensor product (see (3.7) above),
2.)invariant forms and
3.) basic forms.
It seems that the appropriate cohomology to use for describing the space of classical observables in Yang-Mills theories is 3.). Because we assume G connected, the subcomplex of W-valued forms with vanishing Lie derivative along vertical vector fields has been identified with 2.) and those which in addition are horizontal, have been identified with 3.).

4.1.3 Equivariant Cohomology

There is another cohomology which is of interest in our disscusion here and this the so-called *equivariant* cohomology. The main reference for this subsection is [40].

We shall begin our discussion in a slightly more general setting. Let P be a G-space, namely a manifold P with a smooth action of a compact Lie group G on it. The *equivariant* cohomology of this G-space P is by definition the ordinary cohomology (de Rham cohomology) of the space P_G obtained from the

fixed universal G-bundle EG, by the mixing construction

$$P_G = EG \times_G P$$

where G acts on the right on EG and on the left on P and the notation means that we identify $(pg, q) \sim (p, gq)$ for $p \in EG$, $q \in P$ and $g \in G$. Thus P_G is the bundle with fiber P over the classifying space BG associated to the universal bundle $EG \to BG$ and π will denote the corresponding natural projection $\pi : P_G \to BG$. We also have a natural map $\sigma : P_G \to P/G$ where P/G is the quotient space. In general σ is not a fibering. In fact it is easy to see that the fiber $\sigma^{-1}(Gp)$ over the orbit Gp is the quotient of E by the stabilizer G_p of p, so that $\sigma^{-1}(Gp)$ is the classifying space of G_p, namely $\sigma^{-1}(Gp) = EG/G_p \simeq BG_p$.

Under reasonable assumptions on G and P, which are certainly satisfied if G is a compact group acting smoothly on P, this map σ induces a homotopy equivalence whenever G acts freely, so that $P_G \simeq P/G$ in that case. On the other hand P_G is a better functorial construction and turns out to be the proper homotopy theoretic quotient of P by G. In any case, the equivariant cohomology, denoted $H_G(P)$ is defined by $H_G(P) := H(P_G)$ and constitutes a contravariant functor from G-spaces to modules over the base ring $H_G = H_G(point) = H(BG)$. This module structure is induced by the projection π, while the map σ defines a natural arrow

$$\sigma^* : H(P/G) \to H_G(P)$$

which is an *isomorphism* if G acts freely. For a compact connected Lie group G and working over the reals, the base ring is simply a polynomial ring $H_G = \mathbf{C}[u_1, ..., u_l]$ for some l.

The main issue here is that if P is a principal bundle over M with structure group G, then by definition G acts freely and the quotient space P/G is just M. So in this specific case which we are mainly interested in, one has that the equivariant cohomology of P is just the ordinary de Rham cohomology of the base space M, i.e. $H(P/G) = H(M) \cong H_G(P)$ since in this case the map σ^* described above is an isomorphism.

This immediately suggests, if one considers our previous disscusion, that the equivariant cohomology of P is just the de Rham cohomology of the basic real valued forms of P (when G is compact and connected), since in this case they both coincide with the ordinary de Rham cohomology of the base space M. However, there is also another somehow different de Rham model for equivariant cohomology which we shall describe.

Let g denote the Lie algebra of G and write $W(g)$ for the Weil algebra, namely the tensor product

$$W(g) = \wedge g^* \otimes Sg^*$$

of the exterior algebra and the symmetric algebra of the dual g^* of g. In the geometric context of a smooth principal bundle P with structure group G and base

space M, which is the case of greatest interest to us, the elements X of g appear quite naturally as vector fields corresponding to the infinitesimal left translations of P in the direction of X. Hence $X \in g$ acts naturally on the de Rham complex $\Omega(P)$ of real-valued differenrial forms on P, both by the substitution operation $i(X)$ and by the Lie derivative $l(X)$.

It finally turns out that an infinitesimal de Rham model for equivariant cohomology, denoted $\Omega_g(P)$, is the following:

$\Omega_g(P)$=$\{$basic complex of $\Omega(P) \otimes W(g)\}$

One then can prove the following

Theorem 1:

If G is a compact connected group acting smoothly on the manifold P, there is a natural isomorphism:

$$H(\Omega_g(P)) \cong H_G(P)$$

When G acts freely and P is a principal bundle over M, then the above is just the ordinary de Rham cohomology of the base space M.

We would like to close this subsection with another variant of the above infinitesimal de Rham model for equivariant cohomology. As we mentioned earlier, basic means vanishing Lie derivative with respect to Lie algebra elements and horizontal. One can then consider directly invariant forms first and then modify the usual differential d to the following one (called "Cartan differential"), namely

$$d_G = d + \sum_{r=1}^{dimg} i(X_r)\xi^r$$

where X_r is a basis of the Lie algebra and ξ^r its dual basis.

Then one can see that

$$d_G^2 = \sum_{r=1}^{dimg} l(X_r)\xi^r$$

which of course vanishes on invariant forms. It is clear that the substitution operator with respect to the basis of the Lie algebra added to the differential actually imposes the horizontality condition in order to vanish, so we do indeed get the basic forms.

This "deformed" differential was originally due to Cartan. Its relation to Witten's deformed differential for the case where $G = U(1)$ appearing in [41] as well as its relation to equivariant cohomology for the abelian case is explained in [40].

4.1.4 Conclusions

Starting this subsection we would like to make the following remark: probably our notation is not the best possible, yet there is something we have to clarify; one must distinguish between "equivariant cohomology" of P, whose infinitesimal de Rham model is the subcomplex of the "basic forms on P with values in the Weil algebra of the Lie algebra of G" (where *no* representation of G itself is needed) from the one hand, and between the usual d-cohomology of "equivariant forms" on P where a representation of G in the appropriate space of values is needed.

Now we would like to argue why we think that the right cohomology to describe the space of classical observables in Yang-Mills theories is equivariant cohomology (i.e. essentially basic forms with values in the Weil algebra). One reason is because this cohomology is representation independent, unlike the usual d-cohomology of equivariant forms on P (with suitable values), so it may be used generally, without mentioning the kind of fields considered (it is well knowm that in Yang-Mills theories one has two kinds of fields: matter fields transforming under the fundamental representation of the structure group considered as well as gauge fields—connection 1-forms—transforming under the adjoint representation). But then one may say that the field strength-curvature 2-form-as well as the gauge fields are actually equivariant forms. But the point here is that neither of them is an observable! (neither classical nor quantum). Parenthetically we would like to mention however that in abelian gauge theories like electromagnetism, the field strength (curvature) contains the electric and magnetic fields which are observables. The second reason is that the equivariant cohomology as we mentioned earlier descends to the quotient space, i.e. space time, when the group is compact and connected and the manifold P on which it acts is actually a principal bundle. This is of course something desirable in Physics, since it is the base space which is of interest—that is spacetime—and not some fictitious space P, which is only a theoretical construction.

We must mention however that the classical observables are something intrinsically ambiguous and one can simply argue about them. For example in electromagnetism the only *experimental fact* one has is the difference measured in the Dirac phase factor, from the Bohm-Aharonov experiment. But this is a *quantum* observable, as it should be, because electromagnetism *is* a quantum theory (as we know it today). The classical description anyway is something artificial, used mainly for our convenience. So it is not possible to say what is really a classical observable, since we cannot "find" such a thing, to measure it and see if it is gauge invariant. And of course we cannot find such a thing because they do not exist since the theory *is* quantum. We would like to emphasise however that classical electromagnetism (even Newtonian mechanics) are good approximations in many cases.

4.2 Twisted de Rham complex

Let us begin this section by recalling the following fact: if M is a real manifold and W a finite dimensional real vector space, the space of W-valued differential forms is denoted $\Omega(M;W)$. If we extend the usual differential d to $d\otimes 1$, then the cohomology groups $H(M;W)$ of the "tensored complex" $(\Omega(M;W), d\otimes 1)$ are defined. If the vector space W has dimension n, then one has that

$$H(M;W) \cong H(M)\otimes W \cong \oplus_{i=1}^{n} H(M)$$

where $H(M)$ is the usual de Rham cohomology of M. This relation comes directly from the *Kunneth theorem* (see e.g. [68]) which says that if C, D are two chain complexes over a principal ideal domain Λ and if one of C, D is flat, then there is a natural short exact sequence

$$\oplus_{p+q=n} H_p(C)\otimes_\Lambda H_q(D) \rightarrowtail H_n(C\otimes_\Lambda D) \twoheadrightarrow \oplus_{p+q=n-1} Tor_1^\Lambda(H_p(C), H_q(D))$$

For the Kunneth theorem one can look at any homological algebra or algebraic topology text, we used [68] where we refer the reader for more details and explanations concerning the notation; the version we presented here is the more general one, in geometry and topology one is mainly interested in singular chain complexes.

In our case the torsion groups vanish because the tensor-over the reals- bifunctor is exact due to the fact that all modules over any field are free and as we know, torsion groups in Kunneth type of formulea measure exactly the deviation of the tensor-over an arbitrary ring-bifanctor from being exact. Thus we have this simple relation in our case. Alternatively we can use the picture of sections of the vector bundle $\wedge\tau_M^*\otimes M^W$, where M^W is the trivial vector bundle $M\times W$ over M with fiber W and τ_M^* is the cotangent bundle of M to describe the W-valued forms on M, as we did in the previous section. The descriptions are identical.

But now we would like to do something more interesting: Let E be any vector bundle over M with fiber W. Define the space of *E-valued*, that is *bundle-valued* differential forms on M, denoted $\Omega(M,E)$, to be the sections of the vector bundle $\wedge\tau_M^*\otimes E$. For these bundle valued differential forms on M, *no natural extension of the de Rham complex is possible, unless one first is given a way of differentiating the sections of* E. This actually means that such a naive extension like $d\otimes 1$ used before for the case of forms with values in a finite dimensional vector space would not do. Now our vector bundle E may not be trivial, unlike the previous situation where E was just the trivial bundle $M^W = M\times W$. Somehow one has to take the possible non-trivial topology of E into account. The only thing to do then is to define a *covariant derivative* on E which will tell us how to differentiate the sections of E in a non-trivial way, i.e. taking the topology of E into account as well. *But* this is not enough, because this "covariant derivative" must also satisfy the "nillpotency" condition, namely its square must be zero, since we actually

want to use this operator as a part of our deformed, or *twisted* differential in the de Rham complex of bundle valued forms. We know that the square of the covariant derivative is *not* zero in general, unless the connection is *flat*.

So, in order to generalise our de Rham complex to bundle-valued forms where the bundle may have non-trivial topology, we need to introduce a covariant derivative on this vector bundle. The fact that in addition we want an operator which is also a differential forces the connection 1-form used to define this covariant derivative to be flat.

The above setting goes under the name *twisted* (or deformed) de Rham complex of the manifold M. The vector bundle E over M with fiber W equipped with a flat covariant derivative is called a *flat* vector bundle. But we want to note that the strict definition of a flat vector bundle, at least as it appears in [39], is that it has a trivialisation relative to which the transition functions are locally constant (this is called a *locally constant* trivialisation). Since the existence of a covariant derivative on a vector bundle is equivalent to the existence of a connection 1-form on the associated principal bundle, this means that we are exactly in the situation we want: i.e. to study the d_A cohomology in the case where one has a flat connection 1-form on a principal bundle.

Having identified the twisted de Rham complex with what we actually want to investigate, we shall follow the vector bundle language because it is more convenient.

It is not so hard to see that the twisted de Rham complex depends on the chosen trivialisation.(see for instance [39]). The following proposition states up to what extend this is actually true:

Proposition 6:

The twisted cohomology is invariant under refinements of open covers. More precisely, let $\{(U_a, \phi_a)\}_{a\in I}$ be a locally constant trivialisation for E. Suppose $\{V_b\}_{b\in J}$ is a refinement of $\{U_a\}_{a\in I}$ and the coordinates maps ψ_b on $V_b \subset U_a$ are the restrictions of ϕ_a. Then the two twisted complexes $\Omega^*_\phi(M, E)$ and $\Omega^*_\psi(M, E)$ are identical and so are their cohomology groups

$$H^*_\phi(M, E) = H^*_\psi(M, E)$$

The proof is straightforward.

The following proposition allows one to compare two twisted de Rham cohomologies in some special cases:

Proposition 7:

Let E be a flat vector bundle of rank n over M and (g_{ab}), (h_{ab}) the transition functions for E relative to two locally constant trivialisations ϕ and ψ with the same open cover U_a. If there exist locally constant functions

$$\lambda_a : U_a \to G$$

such that

$$g_{ab} = \lambda_a h_{ab} \lambda_b^{-1}$$

where G is the Lie group acting on the fibers of E, then one has an isomorphism

$$\Omega_\phi(M, E) \cong \Omega_\psi(M, E)$$

which induces an isomorphism in cohomology, namely

$$H_\phi(M, E) \cong H_\psi(M, E)$$

where the suffices ϕ, ψ indicate the relative locally constant trivialisations.

In the case where the bundle E is in fact trivial, not just flat, then one has:

Proposition 8:
If E is a trivial rank n vector bundle over M with group G, fiber W and ϕ a trivialisation of E given by n global sections, then one has:

$$H_\phi(M, E) = H(M; W) = H(M) \otimes W = \oplus_{i=1}^{n} H(M)$$

This Proposition simply states the fact that for a trivial bundle, the d_A cohomology coincides with the d cohomology of forms on M with values in a finite vector space (the fiber of E).

Let us now try to interpret the above statements in physics and try to see what they can say about the specific case we are actually interested in.

To begin with, Proposition 8 above immediately indicates that for our case of interest in Physics, i.e. flat space-time, because $\mathbf{R^n}$ is contractible, all bundles over it are in fact trivial. Hence for our physical case, *the d_A cohomology coincides with the usual d cohomology of vector valued forms*. So we get nothing new as space of classical observables for our new gauge invariance.

However, we would like to point out some

Remarks :

1. The d_A cohomology is trivialisation dependent. In physical terminology, fixing a trivialisation means fixing the gauge. *This is in perfect accordance with the fact that the Lagrangian of our B-theory (3.10) is* not *gauge invariant due to the term A^2*. By gauge here we mean local gauge transformations.

2. Proposition 7 above indicates that *the d_A cohomology survives under global gauge transformations*. This is of course true for our Lagrangian (3.10). Moreover

it is also in accordance with the fact that the B-theory is equivalent—under specific choices of metrics— to the *ungauged* nlσm.

3. We know that for every principal bundle over a contractible space M equipped with a flat connection, (equivalently for associated flat vector bundle), there exists a trivialisation, i.e. in physical terminology one can suitably fix the gauge such, that makes this principal bundle isomorphic to the trivial bundle and takes the flat connection to the trivial connection.(e.g. see [31]). In terms of this statement, one can say that the result of Proposition 8 above is in a sense "expected" and of course in physics one has always the opportunity of the most convenient gauge choice, if the theory is gauge invariant! The B-theory is not though.

In addition to the above known Propositions, we can deduce using a K-theoretic argument, essentially an application of the Atiyah-Singer Index theorem for elliptic complexes, the following:

Proposition 9:

The d_A cohomologies have the same Euler number (where the Euler number is equal by definition to the alternating sum of dimensions of the cohomology groups) regardless of the flat connection 1-form A used.

Remark:

This result of ours is essentially a generalisation of Proposition 8 above because in the special case where the base space is simply connected, then the choice of a flat connection 1-form A is equivalent to gauge fixing and hence in this case using Proposition 8 we get in addition a nice formula for the Euler number of the twisted de Rham complex involving the Euler number of the base space itself and the rank of the flat bundle we twist with.

Proof :

The proof will be given at the end of next section because we have to explain some of the underlying theory first, namely *Index Theory* in the more general setting of *elliptic complexes*.

Before closing this section let us comment on a rather intriguing issue: in Yang-Mills theories it is clear that in order to get the space of classical observables, one has to consider the suitable subcomplex of forms on the *total* space of the principal bundle with values in a suitable finite dimensional vector space or equivalently with values in a trivial vector bundle over the total space. This is in a sense obvious there because the Lie group acts on the total space, so the induced actions involve forms on the same space where the Lie group acts. Yet in our case, with the B-theory we considerd forms on the *base* space with values in a flat vector bundle over the base space, where this flat vector bundle is a bundle associated to the flat principal bundle we had as initial data. The justification of this choice comes from the following argument: let ρ be a representation of

the structure group G on a finite dimensional vector space W. Then we want to consider the space of ρ-basic forms of the total space P, denoted $\Omega_{B,\rho}(P;W)$. One can see that a ρ-basic W-valued r-form on P, say ϕ, can be regarded as an assignement to each $m \in M$, where M is the base space, a multilinear skew-symmetric mapping Φ_m of $T_m(M) \times ... \times T_m(M)$ (r times) into the vector space $\pi_E^{-1}(m)$ which is the fiber over m of the associated vector bundle E to P with typical fiber W. Namely, one can define:

$$\Phi_m(X_1, ..., X_r) = u(\phi(X_1^*, ..., X_r^*))$$

where u is a point on P such that $\pi(u) = m$, $X_i \in T_m(M)$ and X_i^* is any vector at u such that $\pi(X_i^*) = X_i$, for each i=1,...,r. $\phi(X_1^*, ..., X_r^*)$ is then an element of the standard fiber W and u is a linear mapping of W onto $\pi_E^{-1}(m)$, so that $u(\phi(X_1^*, ..., X_r^*))$ is an element of $\pi_E^{-1}(m)$. It can be easily verified that this element is independent of the choice of u and X_i^*. Conversely, given a skew-symmetric multilinear mapping

$$\Phi : T_m(M) \times ... \times T_m(M) \to \pi_E^{-1}(m)$$

for each $m \in M$, a ρ-basic form ϕ of degree r on P with values in W can be defined by

$$\phi(X_1^*, ..., X_r^*) = u^{-1}(\Phi_m(\pi(X_1^*), ..., \pi(X_r^*)))$$

where $m = \pi(u)$ and $X_i^* \in T_u(P)$. Thus, since the modules are the same, we also know that ρ-basic forms are stable under d_A and if A is flat, we have a complex and from the above disscusion we see that this complex coincides with the "twisted" de Rham complex of M by the flat vector bundle E associated to P with typical fiber W. In our physical situation, we use the adjoint representation of G on its Lie algebra. Hence, essentially, we still use forms on the total space but in a more convenient formalism, that of associated vector bundles over the base space.

4.3 Elliptic Complexes

We think we should start by giving a brief presentation of Index Theory as first formulated by Atiyah and Singer.

Let X be a closed, oriented, smooth, n-dimensional manifold and let E, F be two smooth complex vector bundles over X. We shall consider linear differential operators, although most of what we have to say also holds for pseudo-differential operators, of the form

$$D : \Gamma(E) \to \Gamma(F)$$

i.e. linear operators defined on the spaces Γ of smooth sections and expressible locally by a matrix of partial derivatives. Let TX be the tangent or the cotangent

bundle of X with projection π', SX be the unit sphere bundle in TX (relative to some Riemannian metric), $\pi : SX \to X$ the projection. Then associated with D there is a vector bundle homomorphism

$$\sigma(D) : \pi^*E \to \pi^*F$$

which is called the *symbol* of D and in terms of local coordinates it is obtained from D by replacing

$$\frac{\partial}{\partial x_j}$$

by

$$i\xi_j$$

in the highest order terms of D (ξ_j is the jth coordinate in the cotangent bundle). D is called *elliptic* if the symbol is invertible, or equivalently if $\sigma(D)$ is an isomorphism. (This of course implies that E and F have the same rank).

We shall try to demonstrate explicitly how the symbol homomorphism is given. Let $s \in \Gamma(E)$, g a smooth real valued function on X and λ a real parameter. Then $e^{-i\lambda g}De^{i\lambda g}s \in \Gamma(F)$ will be a polynomial of order k in λ

$$e^{-i\lambda g}De^{i\lambda g}s = \lambda^k p_k(g,s) + ... + p_0(g,s)$$

whose coefficients depend only on D, s and g. In fact the correspondence $s \mapsto p_j(g,s)$ is a differential operator of degree $k-j$ and $p_0(g,s) = ds$. Moreover the value of the leading coefficient $p_k(g,s)$ at a point $x \in X$ depends only on $s(x)$ and on the value of the differential of g at x. Finally the assignement

$$s|x \mapsto i^{-k}p_k(g,s)_x$$

is linear in s and so defines for each cotangent vector $\eta \in T_x^*X$ a linear function

$$\sigma_D(\eta) : E_x \to F_x$$

This is the leading symbol map of D at the cotangent vector η and as η ranges over TX, these maps combine smoothly to yield a bundle homomorphism

$$\sigma(D) : \pi^*E \to \pi^*F$$

One of the basic properties of the elliptic operators is that $Ker(D)$ and $Coker(D) := \Gamma(F)/D(\Gamma(E))$ are both finite dimensional. The *index* $ind(D)$ is defined to be the integer

$$ind(D) := dimKer(D) - dimCoker(D)$$

(Note: The above definition is actually the definition of the "analytical" index).

From the above one sees that the symbol defines a class in $K(TX)$ (see below) and thus the (analytical) index is just a homomorphism

$$K(TX) \to \mathbf{Z}$$

where $\mathbf{Z}$ are the integers.

We would like to mention one important point here: if the bundles E and F are trivial of rank m, then $\sigma(D)$ is essentially a map $SX \to GL(m, \mathbf{C})$ (S denotes some sphere bundle over X). This means that the index is something like a generalisation of the "winding" number (or the "degree") of continuous maps between spheres.

If $D^* : \Gamma(F) \to \Gamma(E)$ is the formal adjoint of D with respect to some metrics on X, E and F, then D^* is also elliptic and one has that $Coker(D) \cong Ker(D^*)$.

The problem is to express $ind(D)$ in terms of $\sigma(D)$. This is due to the following fact:

Proposition 1:

1.) $ind(D)$ depends only on the homotopy class of $\sigma(D)$.

2.) $ind(D)$ is invariant under continuous deformations of D.

One also has the following property of the symbol:

Proposition 2:

$\sigma(D_1 \oplus D_2) = \sigma(D_1) + \sigma(D_2)$

For the proofs see the references.

More precisely what was actually done was to associate to $\sigma(D)$ a cohomological invariant denoted by $ch(D)$, and then to give an explicit formula for $ind(D)$ in terms of $ch(D)$ and of the Pontrjagin classes of X.

First one has to give a proof of the index theorem in the K-theory context. To do so, one first introduces a set of axioms that general index functions must satisfy. Next one also defines a "topological" index and verifies that this topological index and the analytical index defined above, both satisfy the set of axioms about general index functions. Then one proves a uniqueness theorem and in virtue of this uniqueness theorem one sees that the two indices coincide. This was done in [42]. One then tries to write down an explicit formula for the index of the operator using the symbol of the operator and topological data from the manifold X. This was done in [43].

Now the topological index is defined as follows: we choose a smooth embedding $i : X \to V$, where V is some Euclidian space and we let N be an open tubular neighborhood of X in V. Then N is a real vector bundle over X and TN can be identified with $\pi'^*(N \otimes_R \mathbf{C})$, the complexification of N lifted up to TX

by the projection π' of TX to X. We define a homomorphism

$$i_! : K(TX) \to K(TV)$$

to be the composition of the Thom homomorphism

$$\phi : K(TX) \to K(TN)$$

together with the natural homomorphism

$$K(TN) \to K(TV)$$

induced by the open inclusion $TN \hookrightarrow TV$. Finally letting $j : A \to V$ be the inclusion of the origin A in V, we also have

$$j_! : K(TA) \to K(TV)$$

But $TA = A$ is a point, so $K(TA) = \mathbf{Z}$ and $j_!$ is the periodicity isomorphism. Thus we can define the "topological" index

$$(j_!)^{-1} \circ i_! : K(TX) \to \mathbf{Z}$$

which as one can prove is independent of the choice of the embedding $X \to V$.

The actual formula is derived as follows: first one observes that tensor product of vector bundles makes $K(X)$ a commutative ring. An important cohomological invariant is provided by the Chern character which is a ring homomorphism

$$ch : K(X) \to H^*(X; \mathbf{Q})$$

of $K(X)$ into the rational cohomology ring.

Two vector bundles E,F over a space Y with an isomorphism σ on a subspace Y' of Y define a "difference" element $d(E, F, \sigma) \in K(Y/Y')$ where Y/Y' is Y with Y' pinched to a point. Hence if we denote by BX the unit ball bundle of TX, an elliptic operator D defines an element

$$d(p^*E, P^*F, \sigma(D)) \in K(BX/SX)$$

where $p : BX \to X$ is the projection and hence an element

$$ch(d(E, F, \sigma(D))) \in H^*(BX/SX)$$

Using the Thom isomorphism

$$\phi_* : H^k(X; Q) \cong H^{n+k}(BX/SX; Q)$$

where $dimX = n$, we obtain finally the element

$$\phi_*^{-1} ch(d(E, F, \sigma(D))) \in H^*(X; Q)$$

which we shall denote simply $ch(D)$.

Next we would like to recall the definition of the *Todd class* J. For any n-dimensional complex vector bundle ξ we define $J(\xi)$ by putting

$$J(\xi) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}}$$

where $ch(\xi) = \sum_{i=1}^{n} e^{x_i}$ and the x_i are of degree 2. The elementary symmetric functions of the x_i are the Chern classes $c_i(\xi)$, so that $J(\xi)$ is given by a power series in the $c_i(\xi)$. For real vector bundle η we define

$$J(\eta) = J(\eta \otimes_R \mathbf{C})$$

so that $J(\eta)$ will be given by a power series in the Chern classes of $\eta \otimes_R \mathbf{C}$, i.e. the Pontrjagin classes of η. For a smooth manifold X we define $J(X) = J(TX)$. Thus if the Pontrjagin classes of X are as usual regarded as the elementary symmetric functions in y_j^2, the Todd class is given by

$$J(X) = \prod_{j} \frac{y_j}{1 - e^{-y_j}} \frac{-y_j}{1 - e^{y_j}}$$

For any $a \in H^*(X; \mathbf{Q})$ we denote by $a[X]$ the value of the top dimensional component of a on the fundamental homology class of X. Then one has:

Theorem 1:
For any elliptic differential operator D on a compact manifold X as above, one has

$$ind(D) = \langle ch(D)J(X), [X] \rangle$$

Remarks

1. With some additional conditions, i.e. if the operator considered admits what are called "coercive" boundary conditions, the above theorem can be extended to manifolds with boundary.
2. One can deduce as a special case that the index of a *differential* operator on a closed manifold of *odd* dimension is zero.
3. The whole setting can be generalised to obtain an equivariant version of the above theorem when a Lie group G acts on X. Then one uses the equivariant K-functor K_G.
4. One can also generalise the above picture and get an analogous result if one considers a family of operators acting on X, parametrised by a compact space M. In this case the index of the family will not be an integer like the index of a single operator acting on X, but an element of $K(M)$.

We now describe the situation that arises in our case, namely another generalisation of the above using *elliptic complexes* over a closed manifold X rather than two vector bundles E and F over X.

So let M be a closed manifold, let $E_0, E_1, ..., E_N$ be a sequence of smooth vector bundles over M and let

$$d_i : \Gamma(E_i) \to \Gamma(E_{i+1})$$

be a sequence of differential operators, where Γ denotes the set of smooth sections of the relevant bundle. The d_i's are linear maps given locally by a matrix of partial differential operators with smooth coefficients.

The sequence

$$0 \to \Gamma(E_0) \to \Gamma(E_1) \to ... \to \Gamma(E_N) \to 0$$

is called a *complex* if $d^2 = 0$, namely $d_{i+1} \circ d_i = 0$ for $i = 0, 1, ..., N-2$. We shall often omit the grading and refer to such a complex as simply $\Gamma(E)$.

If d_i is of order k, then the terms of order k define in an invariant manner the *leading symbol* or *principal symbol* $\sigma(d_i)$ of d_i. If TM is the (co)tangent bundle of M with projection π, one has a corresponding sequence of symbols $\sigma(E)$

$$0 \to \pi^* E_0 \to \pi^* E_1 \to ... \to \pi^* E_N \to 0$$

where $\pi^* E_i$ is the pull-back of the bundle E_i over TM. The complex $\Gamma(E)$ is called *elliptic* if the corresponding sequence of symbols $\sigma(E)$ is exact outside the zero section of TM. (Note that one need not assume that all d_i have the same order.)

Example : "The de Rham complex"

The most well known examples of elliptic complexes are the de Rham and the Dolbeault complexes. Since we shall need the de Rham complex in what follows, we shall describe it in as much detail as we need.

Given a k-form $\omega \in \Gamma(\wedge^k)$ one has from the above discussion:

$$i^{-1} e^{-i\lambda g} d e^{i\lambda g} \omega = \lambda dg \wedge \omega + d\omega$$

Hence $\sigma_d(\eta) = \{$*exterior multiplication by* $\eta\}$.

The same argument gives for the formal adjoint d^* with respect to some metric, that $\sigma(d_i^*) : \pi^* E_{i+1} \to \pi^* E_i$ is given by *"exterior multiplication by* η^**"*, where now η^* is the dual element of η, which, since η is a 1-form, should be a vector field and in this case exterior multiplication should be thought of as something like "contraction" (or "hook") with respect to η^*. That operation indeed reduces the degree of the forms by one as it should.

Now given an elliptic complex as above over a closed manifold M which we shall call ξ, one can define the i-th cohomology group as usual:

$$H^i(\xi) := \frac{Ker d_i}{Im d_{i-1}}$$

Each $H^i(\xi)$ is a complex vector space. Since the base manifold is closed, these are finite dimensional and one defines the *Euler* number $\chi(\xi)$ of the *complex* ξ to be:

$$\chi(\xi) = \sum_{i=1}^{N} (-1)^i dim_C H^i(\xi)$$

Obviously, for $N = 1$ we simply get the index of d_0.

By an *endomorphism* T of an elliptic complex ξ as above, we mean a sequence of linear maps

$$T_i : \Gamma(E_i) \to \Gamma(E_i)$$

such that $d_i T_i = T_{i+1} d_i$. Such an endomorphism induces endomorphisms $H^i T$ of $H^i(\xi)$. Since these cohomology groups are finite dimensional, we can define Trace $H^i T$, and hence the *Lefschetz number*,

$$L(T) = \sum_{i=1}^{N} (-1)^i Trace H^i T$$

This a generalisation of the Euler number since for T=1 the identity, then

$$L(1) = \chi(\xi) = \sum_{i=1}^{N} (-1)^i dim H^i(\xi)$$

If moreover $N = 1$, we simply get the index of d_0.
If we now suppose that $f : M \to M$ is a smooth map, and let $\phi_i : f^* E_i \to E_i$ be smooth bundle homomorphisms, then we can define linear maps

$$T_i : \Gamma(E_i) \to \Gamma(E_i)$$

as the composition

$$\Gamma(E_i) \to \Gamma(f^* E_i) \to \Gamma(E_i)$$

Thus, if $s \in \Gamma(E_i)$, the section $T_i s$ is defined by

$$T_i s(m) = \phi_i s(f(m))$$

where $m \in M$. The point to note is that $s(f(m)) \in E_{i,f(m)}$, but that ϕ_i takes us back to $E_{i,m}$. If further $d_i T_i = T_{i+1} d_i$, then the T_i defines an endomorphism of the elliptic complex ξ. An endomorphism of this type, namely arising from a smooth map of the base space, will be called a *geometric* endomorphism. The bundle homomorphism $\phi_i : f^* E_i \to E_i$ is just a family of linear maps

$$\phi_{i,A} : E_{i,f(A)} \to E_{i,A}$$

and hence at a fixed point A of f we have

$$E_{i,A} = E_{i,f(A)}$$

If the smooth map f has a fixed point A, we call this point *simple* if $det(1 - df_A) \neq 0$, where df_A is the induced map on the tangent space over A. This is equivalent to requiring that the graph of f and the diagonal intersect transversally at (A, A) in $M \times M$. Thus a simple fixed point is an isolated fixed point. Hence if all fixed points are simple, since M is compact, they are finite in number. One then has a *Lefschetz fixed point theorem* which is a generalisation of the index theorem we stated previously, namely:

Theorem 2:

Let ξ be an elliptic complex over M as above, let T be a geometric endomorphism defined by a map $f : M \to M$ with only simple points and bundle homomorphisms ϕ_i. Then the Lefschetz number $L(T)$ is given by the formula

$$L(T) = \sum_A \nu(A)$$

where the summation is over the fixed points of f and $\nu(A)$ is given by

$$\nu(A) = \frac{\sum(-1)^i Trace\phi_{i,A}}{|det(1 - df_A)|}$$

If ξ is the de Rham complex, then one has a natural choice for ϕ_i, namely the i-th exterior power of df. In this case any f defines a natural geometric endomorphism T and in virtue of the de Rham theorem, the H^iT are just the endomorphisms H^if induced by f on $H^i(M; C)$. Thus in this case we find that $\nu(A) = signdet(1 - df_A) = +1$ or -1 and hence we recover the classical Lefschetz theorem which as we know expresses under suitable circumstances the number of the fixed points of f in terms of the transformation induced by f on the cohomology of M.

Let us now "assemble" the bundle. Consider the single operator

$$D : \Gamma(E^{ev}) \to \Gamma(E^{odd})$$

where $E^{ev} = \oplus_{ieven} E_i$ and $E^{odd} = \oplus_{iodd} E_i$ defined as $D := d \oplus d^*$, or in detail

$$D(u_0, u_2, ...) = (d_0u_0 + d_1^*u_2, d_2u_2 + d_3^*u_4, ...)$$

Since $d^2 = 0$, $(d^*)^2 = 0$ as well, so

$$DD^* = \oplus_{ieven} \Delta_i$$

and

$$D^*D = \oplus_{iodd} \Delta_i$$

where Δ_i is the Laplacian on $\Gamma(E_i)$, namely

$$\Delta_i = d_{i-1}d_{i-1}^* + d_i^*d_i$$

The exactness of the symbol complex $\sigma(E)$ *off* the zero section implies that

$$\sigma(\Delta_i) = \sigma_{i-1}\sigma_{i-1}^* + \sigma_i^*\sigma_i \in Hom(\pi^*E_i, \pi^*E_i)$$

is an isomorphism (off the zero section). Hence Δ_i and D are elliptic. Using regularity properties of elliptic operators it follows as in usual Hodge theory, that

$$KerD = \oplus_{ieven} h^i$$

and that

$$CokerD = \oplus_{iodd} h^i$$

where h^i stands for the space of "harmonic" sections of the bundle E_i, namely $Ker\Delta_i$. Hence we have constructed an elliptic operator D such that

$$ind(D) = \chi(\xi)$$

The above construction is known as the "assembling" of the bundle. Since every elliptic complex by the above construction gives rise to an elliptic operator D acting from the "even" bundle of the complex over M to the "odd" bundle of the complex over M, everything we mentioned earlier applies to elliptic complexes as well. Note however that had we defined D to act from $\Gamma(\oplus_{alli} E_i)$ to itself, then this operator would have been self-adjoint and its index would have to vanish, which is a rather uninteresting situation. However this is something we shall use later to study the η invariants.

We also want to mention that the elliptic complexes have been further generalised to *perfect, s-perfect and h-perfect* complexes recently by Quillen, (cf [44]).

4.3.1 The Proof of Proposition 9 and a crucial Remark

Now, with all the above in mind, one can see that our proposition is an application of the index theorem for elliptic complexes. Namely, consider the following twisted de Rham complex over our base 4-manifold M, supposed to be space-time, assumed closed: we start with a principal bundle P over M with structure group G assumed compact and connected. A flat connection was given on P, call it A. Form the associated vector bundle adP over M which is also flat, with fiber the Lie algebra g of G. Fix a trivialisation on adP and consider the twisted de Rham complex $(\wedge^i T^*M \otimes adP, d_A)$, where T^*M is the cotangent bundle of M and clearly $d_A = d + A\wedge$ is the exterior covariant derivative with respect to the flat connection 1-form A. The operator d_A is a differential since the connection 1-form A is flat and it depends on the chosen trivialisation since A does in the above formula. If we now get another flat connection 1-form A', we get another differential $d_{A'}$. Hence we have two elliptic complexes over M with modules sections of bundles $\wedge^i T^*M \otimes adP$ and two differentials, namely d_A, $d_{A'}$ and we want to compare their Euler numbers. The symbol of these two operators d_A

and $d_{A'}$ is the same, namely $i\eta$, where η is a Lie algebra valued 1-form on M. We then assemble the bundles and get two operators D_A and $D_{A'}$ which have the same symbol $i\eta - i\eta^*$ by virtue of Proposition 2 of this section. Applying the index formula on D_A and $D_{A'}$, namely Theorem 1 of this section, we see immediately that the indices coincide since they act on the same manifold M and they have the same symbol. But the index is by definition the Euler number of the corresponding elliptic complex. Hence we deduce that the alternating sum of the cohomology groups of the two twisted de Rham complexes is the same.

Note that neither did we make any assumption on M being simply connected nor did we assume anything about the flat connection 1-forms A and A' being gauge equivalent. This is so because the index depends only on the fundamental class of M and the symbol of the operator (and does not depend at all on the bundles used).

Moreover, by Proposition 8 of the previous section we can compute the index in the case where M *is simply connected* by simply multiplying the Euler number of M by the dimension of the Lie algebra g of the structure group G. This is done by taking A' to be the trivial flat connection and then using Proposition 8 (which essentially identifies this cohomology with the cohomology of $d \otimes 1$). Clearly our result is more general.

We would like to mention that we"twisted" the de Rham complex of M by the bundle adP because in the physical context we consider, we assumed that the fields B are related to the adjoint representation of the structure group G. The result of course holds in full generality for twisted de Rham complexes by flat vector bundles.

In the case of the base space being $\mathbf{R^4}$, then one has that the alternating sum of the dimensions of the twisted de Rham cohomology groups by a flat adjoint bundle adP is just the dimension of the Lie group=dimension of its Lie algebra=rank of vector bundle adP. This is so because $\mathbf{R^4}$ has trivial cohomology except of dimension 0 in which case one has $H^0(\mathbf{R^4}) = \mathbf{R}$.

Crucial Remark:

Since we have now proved our result about the index of twisted de Rham differential, we would like to present another argument related to our statement at the end of section 3.3 that the Freedman-Townsend invariance is actually a special case of Grothendieck's stability equivalence relation in the definition of K^0 group. The argument we shall present here essentially *proves* that claim.

It is mentioned in the original paper by Atiyah and Singer that *homotopic classes of principal symbols of pseudodifferential elliptic operators on a closed manifold say M are in one to one correspondence with elements of $K^0(TM)$*, where TM is the tangent bundle of M. One can have an equivalent picture us-

ing *K-homology* as formulated by Baum and Douglas [58] instead of K-theory, using the analytic K-homology group $K_0(M)$. Anyway, the point here is that since clearly the Freedman-Townsend invariance *is definitely not* an ordinary gauge transformation (i.e. moving to an isomorphic bundle) and since in Proposition 9 above we proved that the symbol is actually the same, namely we have *not* changed K-class, then Freedman-Townsend invariance *has to be in the stable class* of our original bundle. (This is true because K^0 is defined as stable isomorphism classes of complex vector bundles over the manifold considered, as it is well-known).

4.3.2 Index and Physics

We would like to indicate some relation with physics concerning the Fredholm index of an elliptic operator following [41]. In the above discussion we were interested in the Euler number of the twisted de Rham complex because the usual procedure for Geometric Quantization is to build the Hilbert space by considering at first stage the direct sum of some cohomology groups. This is exactly equivalent to calculating the Euler number of the corresponding elliptic complex.

However, there is also another use of the index of an elliptic operator in physics. Let us consider for instance that one has some symmetry in the classical level with a generator Q. Let us also assume that the Hamiltonian of the system depends only on Q, for example this is the case in chiral supersymmetry, namely

$$H = \sum_i Q_i^2$$

This is actually the case we consider here for simplicity but the argument can be applied to more general cases. Thus in chiral supersymmetry then, since H is the sum of squares of hermitian operators Q_i, where Q_i's are the supersymmetry charges, the energy of any state is positive or zero. A state can have zero energy only if it is annihilated by each of the Q, since if

$$H|0\rangle = 0$$

then

$$0 = \langle 0|H|0\rangle = \sum_i \langle 0|Q_i^2|0\rangle = \sum_i |Q_i|0\rangle|^2$$

which is possible only if

$$Q_i|0\rangle = 0$$

If conversely a state is annihilated by the Q_i, then its energy is zero because

$$Q_i|0\rangle = 0$$

implies

$$H|0\rangle = \sum_i Q_i^2|0\rangle = 0$$

If there is a supersymmetry invariant state, namely a state annihilated by the Q_i, then it is automatically the true vacuum state, since it has zero energy and any state which is not invariant under supersymmetry has positive energy. Thus if a supersymmetric state exists, it is the vacuum state and supersymmetry is not spontaneously broken. Only if there does not exist a state invariant under supersymmetry the supersymmetry is spontaneously broken. In the quantum level, a possible anomaly may spoil the symmetry. In order to investigate this issue, one considers the equation

$$Q|f\rangle = 0$$

where Q is the operator(s) generating the symmetry (supersymmetry in our case) and f is an element of the (quantum) Hilbert space. One then wishes to know whether the above equation has a solution in the Hilbert space of the quantum theory. This question in general is difficult. Index theory however can partly give an indirect answer: one can very easily see that if the index of the operator Q is *non-zero*, then the above equation definitely *has* a solution (more than one maybe, but for physics, proving existence of only one solution is enough). But if the index *is* zero, then one *cannot* tell, because the operator has either *no* zero eigenvalues in our Hilbert space, which means that the above equation has no solution and hence symmetry is broken *or* the operator has same number of positive and negative chirality solutions which of course give index zero but the symmetry is not broken.

Let us also mention that as Witten observed, the de Rham complex of a Riemannian manifold M is equivalent to the canonical quantization of a 1-dimensional $N = 2$ SUSY nlσm with target space the manifold M (which is equivalent to 2-dim Quantum mechanics, again with $N = 2$ SUSY). In fact Witten uses the differential $d_t = e^{-ht}de^{ht}$ where t is a parameter (which will become the 1-dimension we have in the equivalent nlσm picture) and h is a Morse function on M. However this deformation does not alter the cohomology groups (hence neither the index). In this setting, our twisted de Rham complex should correspond to a "twisted" nlσm, without being able to specify what this might mean more concretely (maybe target space being the total space of the flat bundle over M). This is an important observation which later, after Atiyah's relation with Floer homology and 3-dimensional $N = 2$ SUSY QM lead Witten to $N = 2$ 4-dimensional Topological Quantum Field Theory and to a different, non-rigorous approach to calculating the Donaldson polynomial invariants for 4-manifolds and recently to the Seiberg-Witten invariants for smooth 4-manifolds.

Chapter 5

Loop spaces

We mentioned in the Introduction that the spin-2, spin-0 duality is also manifest in loop space formalism. In this chapter we shall be interested in the loop spaces. However the problem we are actually interested in is not directly related to this duality in loop spaces but it is a more general, rather fundamental problem concerning formulation of Yang-Mills theories as nlσ models on loop spaces. We begin by some motivaton about " why one should make his life more difficult than it really is by moving on to infinite dimensional spaces and trying to do field theory on them".

5.1 Introduction and some motivation

We shall begin our discussion about loop spaces by trying to explain the reasons for being interested in the loop space formalism of various field theories. The motivation is coming both from physics and mathematics:
1. It is a well-known fact that the Bohm-Aharonov experiment demonstrates clearly that the significant observable quantity in gauge theories is the Dirac Phase factor. Moreover, in non-perturbative QCD where one exploits heavily the conformal invariance assuming that the fermions (quarks) are massless, which by the way is the only non-perturbative quantum field theory quantitative attempt known for the moment, one also uses only products of Wilson loops in the quantum level, which are of course the Dirac phase factors for this special case. This demonstrates that the use of loops as dynamical variables is physically the most correct one.

2. One can work keeping manifest gauge invariance throughout.

3. As we shall see below, Polyakov formulated gauge theories as nlσm on the loop space. His motivation came from an attempt to obtain analogous to Wilson's formulae on a lattice for the confinement of quarks, in the continuoum

limit. This formalism gives an extra $U(1)$ symmetry (appart from the original gauge symmetry) and much better integrability properties. In fact in the rather special case of 2+1 space time, Polyakov showed that the gauge theory when formulated on the loop space is completely integrable, something which is far from true for ordinary space-time formalism, even in 2+1 dimensions. One in principle should be able to generalise this for 3+1 dimensions, meaning that there is not any "no-go" theorem of any kind forbidding that. Of course this is not true in general for the space-time formalism.

4. When one is interested in monopoles , the patching condition essentially dictates the use of loop variables. Otherwise the problem becomes extremely complicated.

We now present some mathematical facts about loop spaces which, we believe have some deeper significance:

5. It is known from the heat equation proof of the index theorem that the Wiener measure on the loop space of a manifold is related to the heat equation on the manifold. Witten used this fact to describe a way of computing the index of the Dirac operator on a manifold using an equivariantly closed de Rham current on the loop space of the manifold. This is related to 1+1 SUSY nlσm in physics. Bismut generalised the result for twisted Dirac operators. Atiyah, elaborating on this observation by Witten, suggested possible ways to generalise the Duistermaat-Heckman formula for oscilating integrals in infinite dimensional spaces. This could be of some importance for QFT.

6. There are two other rather interesting results; we shall mention them here but give exact statements below. Assume we have a simply connected compact manifold. The first one asserts that the Hochschild homology of the algebra of forms on the manifold is isomorphic to the de Rham cohomology of the loop space, whereas the second relates the cyclic homology of the algebra of forms on the manifold to the S^1 equivariant cohomology of the loop space. We shall try to explain that this actually may indicate a relation between the generalised gauge invariance we are studying here and non-commutative geometry. This is not rigorous, however one cannot avoid thinking that on the one hand in our generalised gauge invariance, we actually use Lie algebra valued 1-forms as gauge parameters instead of the usual 0-forms (infinitesimally), thus moving essentially from the algebra of functions of the manifold to the algebra of forms on it. On the other hand, the flat foliation is actually a very interesting example of genuine non-commutative geometry case, where the cyclic homology replaces ordinary de Rham cohomology in the pairings with K-group elements in order to obtain invariants. We hope this will be made clearer after the non-commutative chapter.

7. Another reason, not directly related to our problem, is General Relativity

and Ashtekar variables. Loop spaces are also important there.

However, there is a price to pay: one moves to an infinite dimensional space and we know that Field theory suffers from infinities even in 4-dimensional space times, hence if one uses loop spaces this will make things much worse. Yet the hope mainly lies on the most powerful tool of Field theory, namely symmetries and as we explained above in loop spaces one has extra symmetries and better integrability properties anyway.

5.2 Yang-Mills theory as a nlσm on Loop space

In this section we shall present an idea due to Polyakov which describes Yang-Mills theories as nlσm on the loop space of our space time manifold. We are talking about based loops at a point x_0 of our manifold. The main references are [13] ,[14] ,[41]

In establishing the above formulation, the basic role is played by a well-known object, the element of the holonomy group. The mathematical context is that of a principal bundle X over flat 4-dim space time M. We assume a structure Lie-group G acting on X being compact and connected. We are also given an arbitrary connection 1-form A on X. First we recall two basic facts about holonomy: 1. If one gauge transforms the connection 1-form (keeping of course the loop fixed), then the holonomy gets conjugated by this gauge group element. 2. The holonomy depends only on the homotopy class of the loop iff the connection 1-form is *flat*.

Following standard physics terminology we write the holonomy element h as

$$h(c) = Pexp \int_c A$$

where c is some loop and P stands for the *Dyson ordering* along this loop, see for instance [46] for a precise expression of the Dyson ordering. We now consider h as a chiral field and in mathematical language h is a zero form on the loop space of M with values in the group G, we introduce a connection on the loop space by the formula

$$F_\mu(s,c) := \frac{\delta h}{\delta x^\mu(s)} h^{-1}$$

where

$$\frac{\delta}{\delta x^\mu(s)}$$

is the loop derivative.

In the above expression the loop c is parametrised by the function $x^\mu(s)$ and clearly the index μ takes the values 0,...,3. The above defined connection should be reparametrisation invariant because $h(c)$ is and hence one must have

$$\frac{dx^\mu(s)}{ds}F_\mu(s,c)=0$$

From the definition of the connection on the loop space one can deduce that this connection is flat, namely

$$\frac{\delta F_\mu(s,c)}{\delta x^\nu(s_1)}-\frac{\delta F_\nu(s_1,c)}{\delta x^\mu(s)}+[F_\mu(s,c),F_\nu(s_1,c)]=0$$

We would like to note here that strictly speaking this is not a connection on the G-bundle over the loop space since it is defined explicitly in terms of the given holonomy and there is no notion of "gauge transformation" on $F_\mu(s,c)$ itself. However for convenience we shall refer to it as a connection.

The important result is that the Yang-Mills equations take a simple form in terms of the connection on the loop space, namely

$$\frac{\delta F_\mu(s,c)}{\delta x^\mu(s)}=0$$

So to sum up, one has two important results when formulating Yang-Mills equations on loop spaces:

1. The connection 1-form on the loop space is *flat* even if the space-time connection one starts with is not.

2. The Yang-Mills equations reduce to a "divergenceless-like" condition for the connection 1-form on loop space. (Actually the word "like" is very important, one cannot define a Hodge star on the loop space forms since the de Rham complex is not bounded above—due to infinite dimensionality—although one has a naturally induced metric on the loop space if the manifold itself has a metric, see below).

For the proofs and explanation of the functional logarithmic derivatives we refer to the main references at the begining of the section. The two results mentioned above are based on the following variational formula:

$$\delta h(c)=\int_0^1 dsP\left(exp\int_0^s A_\mu dx^\mu\right)\Omega_{\mu\nu}(c(s))\frac{dx^\nu(s)}{ds}P\left(exp\int_s^1 A_\mu dx^\mu\right)\delta x^\mu(s)$$

where Ω is the curvature of the connection A on space time.

This formula was written down by simple inspection by Polyakov and the reasoning can be seen by drawing some pictures, see for instance [13]. The motivation came from analogous formulae on *Lattice* Yang-Mills theory due to Wilson. They are related to the confinement problem of quarks. Our main attempt in this chapter is to try to derive this formula rigorously. Then all the results follow. We would like to emphasise the following point: *gauge fields with non-zero*

field strength in ordinary space define a chiral field with zero curvature in loop space. This is the reason why this formalism has much better integrability properties than the ordinary space time formalism. Moreover, the most mysterious feature of the basic variational formula above is the appearence of the curvature $\Omega_{\mu\nu}$ of the connection A on ordinary space time. This will appear naturally—in mathematical terms—in our derivation of the above formula.

In order to present our result, we need to understand loop spaces better. Then we shall present the main results of the theory of *"iterated integrals"* which give actually the right de Rham cohomology for loop spaces. This is the main mathematical tool we shall use, in order to derive Polyakov's basic variational formula more rigorously.

5.3 Loop spaces

The main references here are: [47], [48], [49], [52], [51], [53].

Consider a finite dimensional compact orientable Riemannian manifold M. Then by definition the loop space of M is the following infinite dimensional manifold

$$LM := Map(T, M)$$

consisting of all maps from the circle T to our manifold. Our description of LM will not be absolutelly rigorous, we ignore analytic issues.

Thus a point on LM is by definition a smooth map $\phi : T \to M$ and the tangent space $T_\phi LM$ of LM at ϕ can be identified with the space of sections of the vector bundle $\phi^*(TM)$, the tangent bundle TM of M pulled-back to T by ϕ. The metric on M defines a metric on $\phi^*(TM)$ and hence by integration over T we get an inner product on the space of sections. This defines a pre-Hilbert space structure on $T_\phi LM$. Next we introduce the Levi-Civita connection ∇ on M. This induces a connection on the bundle $\phi^*(TM)$ and hence (evaluating along the tangent vector to the loop) a covariant derivative operator ∇_ϕ. This is a skew-adjoint operator on the space of sections $T_\phi LM$ and hence, using the inner product, it defines a skew-bilinear form on $T_\phi LM$. As we now vary the point $\phi \in LM$ we get a 2-form on LM. One can prove that it is closed, the proof based crucially on the use of the Levi-Civita connection on M. However this form is not non-degenerate in general, this is so only at the points ϕ for which ∇_ϕ has a zero eigenvalue, i.e. a tangent vector to M which is covariantly constant along the loop ϕ. The Hamiltonian function H associated to the obvious action of the circle can nonetheless still be defined as follows: recall that the energy E of a loop ϕ is defined as

$$E(\phi) = \frac{1}{2}\int_T |d\phi|$$

Computing the derivative of E in the direction of a tangent vector $\xi \in T_\phi$ we get

$$(dE, \xi) = \int_T < \frac{d\phi}{dt}, \nabla_\phi \xi >$$

which establishes that $E = H$. This allows one to get an analogue of the Duistermaat-Heckman formula in infinite dimensions.

The lesson here therefore is that the loop space of any Riemannian manifold is *almost* a symplectic manifold and in fact most of the "symplectic" things can be done on the loop space (if one can overcome the infinite dimensions!).

The orientability of LM can be understood as follows: we have the natural evaluation map $f : T \times LM \to M$; pulling back by f^* and then integrating over T induces a homomorphism

$$a : H^2(M; \mathbf{Z}_2) \to H^1(LM; \mathbf{Z}_2)$$

The image of the second Stiefel-Whitney class of M is then the obstruction to orientability of LM. In particular, if M is spin, then LM is orientable. The converse holds if M is also simply connected.

We would like to mention another fact which is not relevant for our immediate discusion but it is useful to know and it is actually one of the main motivations to study loop spaces in general: the Wiener integration on the loop space LM is related to the heat equation on M, thus giving, another way to calculate the index of elliptic operators on M using data from LM. One however must be careful to distinguish between the Wiener measure (using Riemannian structure) and the Liouville meassure (using symplectic structure) in the case of LM. They are related via the Pfaffian. This fact is also useful in physics, in SUSY nlσm.(cf [41]).

5.4 Iterated integrals

The iterated integral is a rather complicated object. In order to demonstrate the fact that these "creatures" living in the Platonic ideal world do indeed project down to something useful in the "real" world, we would like to begin our discussion by considering the following situation: Consider the following ODE

$$\frac{d\phi(t)}{dt} = a(t)\phi(t)$$

where $a(t)$ is a given function and we want to solve this in the interval [0,1] given the initial condition $\phi(0) = 1$. Of course there are ways to solve this. But one may ask the following question, along with the Italian mathematician Volterra in the beginning of our century: is it possible to calculate the single *value* $\phi(1)$ *without* solving the equation with respect to the function $\phi(t)$?

The answer is *yes* and the formula is the following:

$$\phi(1) = \sum_{k=0}^{\infty} \int_{\Delta_k} a(t_1)...a(t_k)dt_1...dt_k$$

where Δ_k is the standard k-simplex

$$\{(t_1, ..., t_k) \in \mathbf{R}^k : 0 \leq t_1 \leq ... \leq t_k \leq 1\}$$

The above is a sum of iterated integrals, namely each term k *is* an iterated integral.

So we see that iterated integrals were not invented by an "abstract" mathematical mind, but they are solutions to this rather pragmatic real problem, quite useful for applied mathematics also we think.

One can easily notice that the above equation suitably generalised, actually gives the correct expression for parallel transport of a vector field (replacing ϕ above) given a connection 1-form A (replacing $a(t)$ above), namely essentially the covariant derivative. And now we think the whole thing starts to take shape. In the above setting then, assuming [0,1] parametrising a circle, the value $\phi(1)$ is exactly the holonomy (more precisely the final vector which is the holonomy times the initial vector as matrices). This is actually the observation that made us think about relating Polyakov's heuristic formulae on loop spaces with iterated integrals. Let us however, before giving formal definitions, write down an iterated integral explicitly: suppose we are in Euclidian space $\mathbf{R^n}$. If $w = w_i(x)dx^i$ and $v = v_i(x)dx^i$ are two real valued 1-forms on $\mathbf{R^n}$ and suppose $a : [0, 1] \to \mathbf{R^n}$ is a path, then the "twice" iterated integral of the forms w and v is by definition the following expression:

$$\int_0^1 [\int_0^t w_i(x(a(t)))dx^i(a(t))]v_j(x(a(t)))dx^j(a(t))$$

We now pass directly to the definitions on the loop spaces. We shall begin with some general facts about spaces carrying smooth circle actions. (We write them specifically for the loop spaces, but they hold in general for spaces carrying smooth circle actions).

Let M be a finite dimensional smooth manifold and LM the free loop space of M, namely the space of all smooth maps from T to M. This loop space may be given the structure of an infinite dimensional manifold modeled on a Frechet space. The circle group acts smoothly by rotating loops, namely $(\phi_t c)(s) = c(s+t)$ where $c(s)$ is a loop and ϕ_t is a smooth 1-parameter group of diffeomorphisms with period 1 which describes the smooth circle action. This natural circle action on LM defines several operators on the space of differential forms on LM. The first is the contraction with the vector field which is tangent to the loop, namely the generator of the T-action, which will be denoted by i. Then there is an averaging operator A defined via

$$A(w) = \int_0^1 \phi_t^* w dt$$

Furthermore there is a sequence of operators p_k defined as follows:

$$p_k : \Omega(LM)^{\otimes k} \to \Omega(LM)$$

where $1 \leq k < \infty$

The explicit formula is the following: given a form w on LM, let $w(t)$ be the form $\phi_t^*(w)$ on LM; then one has:

$$p_k(w_1, ..., w_k) = \int_{\Delta_k} iw_1(t_1) \wedge ... \wedge iw_k(t_k) dt_1 ... dt_k$$

where Δ_k is the standard k-simplex

$$\{(t_1, ..., t_k) \in \mathbf{R}^k | 0 \leq t_1 \leq ... \leq t_k \leq 1\}$$

We shall now give the key property of the maps p_k :

Proposition 1 :

If $\epsilon_i = |w_1| + ... + |w_k| - i$, then

$$\begin{aligned} dp_k(w_1, ..., w_k) = &- \sum_{i=1}^{k} (-1)^{\epsilon_i - 1} p_k(w_1, ..., dw_i, ...w_k) \\ &- w_1 p_{k-1}(w_2, ..., w_k) \\ &- \sum_{i=1}^{k-1} (-1)^{\epsilon_i} p_{k-1}(w_1, ..., w_i w_{i+1}, ..., w_k) \\ &+ (-1)^{\epsilon_{k-1}} p_{k-1}(w_1, ..., w_{k-1}) w_k \end{aligned}$$

This formula simply states the fact that

$$\sum_{k=0}^{\infty} p_k$$

is a Hochschild cocycle (for the definition see next page) on the differential graded algebra (DGA for short) $\Omega(LM)$ with coefficients in $\Omega(LM)$ itself.

The proof of Proposition 1 above is by direct computation using the explicit formula we gave for the maps p_k above (cf [48]).

One can observe that p_1 in particular is equal to $iA = Ai$ and that its square is zero. Moreover it anticommutes with the de Rham differential d. Hence one

actually has a *mixed complex* $(\Omega(LM), d, p_1)$. This observation will be important later.

We now pass to the iterated integrals. Notice first the shifting of the degree between forms on M and forms on LM by the following example: if w is a 1-form on M, then $\int_\gamma w$ is a function on LM, where $\gamma \in LM$. Iterated integrals generalise this idea. If w is a form on M, let $w(t)$ be the form $e_t^*(w)$ on LM, namely the pull back of w via the evaluation map $e_t : LM \to M$ given by evaluating loops at time t. Given forms $w_0, w_1, ..., w_k$ on M, the iterated integral

$$\sigma(w_0, ..., w_k)$$

is a form on LM of total degree $|w_0| + ... + |w_k| - k$ defined by the formula

$$\sigma(w_o, ..., w_k) = \int_{\Delta_k} w_0(0) \wedge iw_1(t_1) \wedge ... \wedge iw_k(t_k) dt_1 ... dt_k$$

where Δ_k is the standard k-simplex and i is the contraction operator with the tangent vector to the loop. There is something we would like to emphasise here: the difference of the iterated integrals written above with ordinary multiple integrals is mainly that in the iterated integral one integrates over k-simplices and not over cartesian products as for k-tuple integrals and that the *order* of integration is the main thing: it cannot be changed! (In ordinary k-tuple integrals one may integrate whatever function is more convenient).

We can rewrite the above formula for the iterated integral using the maps p_k, namely

$$\sigma(w_0, ..., w_k) = w_0 p_k(w_1(0), ..., w_k(0))$$

One can then build a model for the forms on LM using iterated integrals. We begin by recalling the definition of the **cyclic bar complex** of the algebra $\Omega(M)$. Let $C(\Omega(M))$ be the direct sum

$$\sum_{k=0}^{\infty} \Omega(M) \otimes s\Omega(M)^{\otimes k}$$

Here s is the suspension functor on graded vector spaces, that is the functor which simply reduces degree by 1. In general, the cyclic bar complex of any associative algebra comes naturally equipped with two "differentials", the *Hochschild* differential b_0 defined via

$$b_0(w_0, ..., w_k) = -\sum_{i=0}^{k-1} (-1)^{\epsilon_i} (w_0, ..., w_{i-1}, w_i w_{i+1}, w_{i+2}, ..., w_k)$$
$$+(-1)^{(|w_k|-1)\epsilon_{k-1}} (w_k w_0, w_1, ..., w_{k-1})$$

and *Connes'* differential B defined via

$$B(w_0,...,w_k)=\sum_{i=0}^{k}(-1)^{(\epsilon_{i-1}+1)(\epsilon_k-\epsilon_{i-1})}(1,w_i,...,w_k,w_0,...,w_{i-1})$$

$$-\sum_{i=0}^{k}(-1)^{(\epsilon_{i-1}+1)(\epsilon_k-\epsilon_{i-1})}(w_i,...,w_k,w_0,...,w_{i-1},1)$$

However, in our case we are interested in the cyclic bar complex of the algebra $\Omega(M)$ which is itself also a DGA with de Rham differential d. In the bar complex then one has an extension of this d, still denoted d, given via

$$d(w_0,...,w_k)=-\sum_{i=0}^{k}(-1)^{\epsilon_{i-1}}(w_0,...,w_{i-1},dw_i,w_{i+1},...,w_k)$$

As before, $\epsilon_i=|w_0|+...+|w_i|-i$

N.B. A piece of warning here: these formulas refer to the cyclic bar complex of $\Omega(M)$, that is forms on M, whereas the formula of Proposition 1 above refers to forms on the loop space LM. One must not confuse the d on $\Omega(LM)$ (Proposition 1) with the above d of the cyclic bar complex $C(\Omega(M))$

Now we combine the de Rham differential d on $C(\Omega(M))$ with the Hochschild differential b_0 on $C(\Omega(M))$ to get a single differential b:

$$b=d+b_0$$

We shall refer to this cohomology $(C(\Omega(M)),b)$ as the *Hochschild cohomology* of $\Omega(M)$ Now one also has a *mixed complex* for the cyclic bar complex $C(\Omega(M))$, namely $(C(\Omega(M)),b,B)$. The total differential $b+B$ gives the *cyclic cohomology* of $\Omega(M)$

And now we are ready to state the main results relating iterated integrals, forms on loop space LM and the cyclic bar complex of the algebra of forms on M:

Theorems :

1. The iterated integral map σ induces a map between the two mixed complexes

$$(C(\Omega(M)),b,B)\to(\Omega(LM),d,p_1)$$

This simply means that $p_1\sigma=\sigma B$ and $\sigma b=d\sigma$

2. If M is simply connected, one has an isomorphism between the de Rham cohomology of the loop space and the Hochschild cohomology of the algebra of forms on M induced by the iterated integral map σ, namely

$$\sigma:(C(\Omega(M)),b)\to(\Omega(LM),d)$$

is an isomorphism in cohomology.

3. "Essentially" the cyclic cohomology of $\Omega(M)$ is isomorphic to the T-equivariant cohomology of LM. (The word "essentially" means that we ignore the complications that lead to the correct Jones' variant HC^-_{-*} functor, see [49]).

4. Under the map σ, the shuffle product on the normalised cyclic bar complex $N(\Omega(M))$ of $\Omega(M)$ is carried into the wedge product on $\Omega(LM)$. (The shuffle product is the usual one from algebraic topology, for the definition see any text in algebraic topology, say for instance [50]).

5. The forms on LM which are images of the iterated integral map are basic with respect to the action of the Lie pair $(vect[0,1], Diff[0,1])$ (see beow). In particular this means that they are *reparametrisation invariant* under reparametrisations of the loops.

For the proof of these results see the references.

In the above, the *normalised* cyclic bar complex $N(\Omega(M))$ is just the quotient of $C(\Omega(M))$ by the kernel of the iterated integral map σ.

Now a *Lie pair* (g, G) in general consists of a Lie algebra g together with a Lie group G with the requirement that there is an exponential map from g to G and that G acts on g in a manner compatible with the exponential map. The trouble arises because $vect[0,1]$ is not quite the Lie algebra of $Diff[0,1]$. A Lie pair acts on a manifold M in general if we are given a homomorphism from G to $Diff(M)$ and a homomorphism from g to the Lie algebra of vector fields on M satisfying the natural compatibility relations. If a Lie pair acts on M we say that a form on M is *basic* if it is invariant under the action of G and its interior product with any vector field coming from g is zero, namely it it horizontal.

We just want to end this section by mentioning that many of the above generalise to actions of arbitrary compact Lie groups G acting on manifolds. One then gets *equivariant* versions of the above results. This is actually what we need in our physical problem, because we consider gauge theories and *Lie algebra valued* forms whereas all the above discussion refers to *real* valued forms. Fortunately, if one starts with a principal bundle P over a base manifold M with structure group G, the loop space LP also is a principal G-bundle over LM in a natural way, see [51]. This actually implies that the generalisations are straightforward. We would not like to give all the precise definitions and results for this setting because this will make our notation more complicated without actually adding any new information. If one wishes, one can find most of these things in the references we give.

5.5 The Proof

To begin with, in physics people usually work with *based* loops, which means that one picks a point $x \in M$ and considers loops having this point as starting and ending point. We shall denote this space $LM(x)$. The reason for this is that one can compose loops easily in this way. Clearly one has that

$$LM = M \times LM(x)$$

This is not actually important in our treatment because we shall use formulae valid for the bigger space of free loops.

First one writes the holonomy element h as an infinite sum of iterated integrals by expanding the Dyson ordering (see [46]). We use the formula of the σ map in terms of the maps p_k where we conventionally define $p_0 = 1$ and we assume the 0th form $w_0(0)$ appearing in the formula to be equal to the constant form 1 and thus we omit it, since there is no integration on that either.

In more concrete terms then, since the Dyson ordering is by definition the following expression (c is a loop)

$$h = Pexp(\int_c A) := \sum_{j=0}^{\infty} \int_0^1 dt_j \int_0^{t_j} dt_{j-1} ... \int_0^{t_2} A(\frac{dx^\mu}{dt}(t_j))...$$

$$...A(\frac{dx^\mu}{dt}(t_2))A(\frac{dx^\mu}{dt}(t_1))$$

then one has that

$$h = \sum_{n=o}^{\infty} p_n(A_1, ..., A_n)$$

where we simplify the notation slightly by writing A_i instead of $A_i(0)$, $1 \leq i \leq n$. In our notation the index i states the *"position"* of the form A in the iterated integral and *it does not mean that we have different forms*, we only have one form A, which is the connection 1-form on the principal bundle pulled back to the base by a section (gauge fixing). Recall that because this is the holonomy element, namely an element of the structure group G, we know that the above converges and that the "internal" (Lie algebra) indices of A give exactly a group element because we exponentiate. This is also the reason why we have an infinite sum of iterated integrals and not just one iterated integral.

Anyway, then one goes on by looking at Polyakov's variational formula and it is not hard to suspect that this "looks like" taking the d of the holonomy element h. The d enters the sum and hits every individual term p_n. This is actually a slightly tricky point but the interchange of d and $\sum$ is justified because both sides make sense. In fact by definition the d of the p_n is actually a sum of p_n's, applied to different forms (see formula of Proposition 1 above). Moreover recall that as mentioned in [48], the sum $\sum_{n=0}^{\infty} p_n$ is a Hochschild cocycle no matter

what forms it is applied to and also recall that each iterated integral is finite, hence the sum is well defined (converges):

$$dh = d\sum_{n=0}^{\infty} p_n(A_1, ..., A_n) =$$

$$= \sum_{n=o}^{\infty} dp_n(A_1, ..., A_n) =$$

$$= dp_0 + dp_1(A_1) + dp_2(A_1, A_2) + dp_3(A_1, A_2, A_3) + dp_4(A_1, A_2, A_3, A_4) + ...$$

We now take each term separately and applying the formula of Proposition 1 above we get:

$$dp_0 = 0$$

$$dp_1(A_1) = p_1(dA_1)$$

$$dp_2(A_1, A_2) = p_2(dA_1, A_2) + p_2(A_1, dA_2) - p_1(A_1 \wedge A_2)$$

$$dp_3(A_1, A_2, A_3) = p_3(dA_1, A_2, A_3) + p_3(A_1, dA_2, A_3)$$

$$+p_3(A_1, A_2, dA_3) - p_2(A_1 \wedge A_2, A_3) - p_2(A_1, A_2 \wedge A_3)$$

$$\begin{aligned} dp_4(A_1, A_2, A_3, A_4) &= p_4(dA_1, A_2, A_3, A_4) \\ +p_4(A_1, dA_2, A_3, A_4) &+ p_4(A_1, A_2, dA_3, A_4) \\ +p_4(A_1, A_2, A_3, dA_4) &- p_3(A_1 \wedge A_2, A_3, A_4) \\ -p_3(A_1, A_2 \wedge A_3, A_4) &- p_3(A_1, A_2, A_3 \wedge A_4) \end{aligned}$$

...

We think that these terms are enough to demonstrate what happens. We would like to stress that because we have only *one* differential form, namely A, in different positions in the iterated integrals, the position being indicated by the index, (something which will be crucial later), the terms on the second line and on the fourth line in the formula of Proposition 1 mutually cancel. We are thus left with the terms coming from lines 1 and 3 from the formula. Now the iterated integral, as a self-respecting integral, *is linear in the ith slot*, hence if *all* the remaining part of the iterated integral is the same, these slots can be *added*. This is indeed the case here (because we have only the form A in different positions), so if we take the curvature 2-form F of the connection 1-form A to be $F = dA - A \wedge A$, we have the folowing formula:

$$dh = p_1(F) +$$
$$+p_2(F_1, A_2) + p_2(A_1, F_2) +$$
$$+p_3(F_1, A_2, A_3) + p_3(A_1, F_2, A_3) + p_3(A_1, A_2, F_3)$$
$$+p_4(F_1, A_2, A_3, A_4) + p_4(A_1, F_2, A_3, A_4) + p_4(A_1, A_2, F_3, A_4) + p_4(A_1, A_2, A_3, F_4) + ... =$$

$$= \sum_{k=1}^{\infty} (\sum_{j=1}^{k} p_k(A_1, ..., A_{j-1}, F_j, A_{j+1}, ..., A_k))$$

Now this formula has a very attractive feature: *the curvature F appears naturally*. The deep reason is of course Theorem 2 above: the curvature has two terms, the dA term comes from the d part and the $A \wedge A$ part comes from the b_0 part of the Hochschild differential $b = d + b_0$. And it is the Hochschild cohomology of the cyclic bar complex which is isomorphic with the de Rham cohomology of the loop space.

So far so good. But what does our formula involving sums have to do with Polyakov's variational formula?

In fact, they are the same. To see that, we shall rewrite Polyakov's formula using iterated integrals. Then to see that they coincide, it is actually an exercise in simplicial homology.

We must try to understand Polyakov's formula first. The key point in his formula is that he actually "breaks" the loop at a point s and then integrates over all s, namely from 0 to 1. We know how to write the path order exponent using iterated integrals. However one now must distinguish between two simplices because we have broken the loop.

In all of our discussion above, the integrals were taken over the standard simplices over the interval [0,1], namely Δ_1 is the interval [0,1] and then Δ_k was $\{(s_1, ..., s_k) \in \mathbf{R}^k | 0 \leq s_1 \leq ... \leq s_k \leq 1\}$. We shall continue to keep this notation for the simplices over the interval [0,1]. The same holds for the maps p_k.

Now we break the loop at the point s, so we must, in addition, have two extra classes of simplices:

I. One will be denoted Δ_k^s and the Δ_1^s will simply be the interval [0,s] and the general Δ_k^s will be

$$\{(s_1, ..., s_k) \in \mathbf{R}^k | 0 \leq s_1 \leq ... \leq s_k \leq s\}$$

The corresponding maps p_k will be accordingly denoted p_k^s

II. The other will be denoted Δ_k^1 and the Δ_1^1 will simply be the interval [s,1] and the general Δ_k^1 will be

$$\{(s_1, ..., s_k) \in \mathbf{R}^k | s \leq s_1 \leq ... \leq s_k \leq 1\}$$

The maps will be denoted p_k^1 in this case.

Now with the above notation, Polyakov's variation can be written as:

$$\delta h = \int_0^1 ds(\sum_{n=0}^{\infty} p_n^s(A_1, ..., A_n))iF(s)(\sum_{n=0}^{\infty} p_n^1(A_1, ..., A_n)$$

where we supress the indices and remember that the integral ds refers to the F factor, indicated as $F(s)$.

If we expand the above, we get:

$$\delta h = \int_0^1 ds(1 + p_1^s(A_1) + p_2^s(A_1, A_2) + p_3^s(A_1, A_2, A_3) + ...)iF(s)\times$$

$$\times(1 + p_1^1(A_1) + p_2^1(A_1, A_2) + p_3^1(A_1, A_2, A_3) + ...) =$$

$$= \int_0^1 ds(iF(s) + iF(s)p_1^1(A_1) + iF(s)p_2^1(A_1, A_2) + iF(s)p_3^1(A_1, A_2, A_3) + ...$$

$$+p_1^s(A_1)iF(s)+p_1^s(A_1)iF(s)p_1^1(A_1)+p_1^s(A_1)iF(s)p_2^1(A_1, A_2)+p_1^s(A_1)iF(s)p_3^1(A_1, A_2, A_3)+...$$

$$+p_2^s(A_1, A_2)iF(s) + p_2^s(A_1, A_2)iF(s)p_1^1(A_1) + p_2^s(A_1, A_2)iF(s)p_2^1(A_1, A_2)$$

$$+p_2^s(A_1,A_2)iF(s)p_3^1(A_1,A_2,A_3)+...$$

$$+p_3^s(A_1,A_2,A_3)iF(s)+p_3^s(A_1,A_2,A_3)iF(s)p_1^1(A_1)+p_3^s(A_1,A_2,A_3)iF(s)p_2^1(A_1,A_2)+...$$

$$+...)$$

Now here comes algebraic topology to say that, in our notation

$$\int_0^1 ds\Delta_i^s * \Delta_j^1 = \Delta_{i+j+1}$$

where the extra vertex s goes in the $(i+1)$-st slot, namely using sets we can write symbolically:

$$\int_0^1 ds\{0 \leq s_1 \leq ... \leq s_i \leq s\} * \{s \leq s_{i+1} \leq ... \leq s_{i+j} \leq 1\} =$$

$$= \{0 \leq s_1 \leq ... \leq s_i \leq s \leq s_{i+1} \leq ... \leq s_{i+j} \leq 1\}$$

which is exactly an $(i+j+1)$ simplex over [0,1], namely Δ_{i+j+1} with the extra vertex s in the $(i+1)$-st slot. It is very crucial in which slot the extra vertex will go.

For convenience we present a simple example with Δ_1^s joint to Δ_1^1 and then integrating s from 0 to 1. The result is Δ_3. More explicitly, one has Δ_1^s is $0 \leq s_1 \leq s$ and Δ_1^1 is $s \leq s_2 \leq 1$, hence:

$$\int_0^1 ds(\int_0^s ds_1)(\int_s^1 ds_2) =$$

$$= \int_0^1 dss(1-s) = \frac{1}{6}$$

which is exactly the volume of the standard 3-simplex

$$\Delta_3 = \{(s_1,s,s_2) \in \mathbf{R}^3 | 0 \leq s_1 \leq s \leq s_2 \leq 1\}$$

Note moreover that the extra vertex s goes in the 2nd slot (i+1=1+1=2 in this simple case).

With the above in mind, the formula gives exactly our expression for dh, namely:

$$\delta h = p_1(F_1) + p_2(F_1,A_2) + p_3(F_1,A_2,A_3) + ...$$

$$+p_2(A_1,F_2)+p_3(A_1,F_2,A_3)+p_4(A_1,F_2,A_3,A_4)+p_5(A_1,F_2,A_3,A_4,A_5)+...$$

$$+p_3(A_1,A_2,F_3)+p_4(A_1,A_2,F_3,A_4)+p_5(A_1,A_2,F_3,A_4,A_5)+...$$

$$+p_4(A_1,A_2,A_3,F_4)+p_5(A_1,A_2,A_3,F_4,A_5)+...$$

$$+...$$

$$=\sum_{k=1}^{\infty}(\sum_{j=1}^{k}p_k(A_1,...,A_{j-1},F_j,A_{j+1},...,A_k))=dh$$

QED

We would like to emphasise that the reason for the appearence of the curvature in the formula is Theorem 2.

Let us now comment briefly on Polyakov's results:

With our formalism, the flat connection on the loop space is just

$$F=h^{-1}dh$$

where h^{-1} is expressed exactly like h using sum of iterated integrals but now the vertices of the simplices will be in the opposite order, namely

$$0\leq s_n\leq ...\leq s_1\leq 1$$

One can see that this is exactly the standard expression for flat connections for finite dimensional bundles. The proof that the above defined connection is flat now becomes trivial, exactly like the finite dimensional case.

Similarly the analogue of Yang-Mills equations for loop space simplifies drastically.

Finally, by virtue of Theorem 5 quoted in the previous section, all the expresions are reparametrisation invariant since we use iterated integrals.

Chapter 6

Non-Commutative Geometry

As we explained in the end of Chapter 3, the B-Theory, being equivalent to a nlσm with target space a (Lie) group manifold and the metric on the target space being the Killing-Cartan form (essentially the Trace) of the group, gives rise naturally to a flat principal bundle over space-time with structure group the group manifold. The solutions of the Euler-Lagrange equations for the nlσm are just the harmonic maps from the space-time manifold to the target space. Considering these harmonic maps as group valued 0-forms on the base space, one constructs the flat connection 1-forms via

$$A = g^{-1}dg$$

where g is a harmonic form. Of course all flat connections can be written as above for some group-valued 0-form g on a simply connected manifold. (However we shall use non-simply connected manifolds in this Chapter in a crucial way but this does not affect the relation to physics. It just becomes more complicated). The fact that the group-valued 0-form g is harmonic means then that A is divergence-less. (See Chapter 3 where we mention all these in detail). The point we want to make clear here is that nlσm (with the above restrictions concerning target space and metric) give rise to flat principal bundles. If one wants to "gauge" the nlσm, then the bundle will no longer be flat. If on the other hand one just has a Riemannian manifold as target space (namely **not** a Lie group with the natural invariant metric on it), then the same fact holds, the only difference being that in this case one has to use the group of isometries of the metric of the target space as providing the structure group for the flat principal bundle.

Nlσmodels in physics are very common indeeed, in fact the most successful and interesting theories such as Yang-Mills theories, or string theories are, or can be thought of as, nlσm as we explained in various points previously. The fact that nlσm gives rise to flat principal bundles becomes now very interesting because flat principal bundles are the most common and possibly the simplest example of genuine **"non-commutative"**, or "quantum" spaces. (We shall explain in the sequel the word "genuine" we used above).

There is also another reason for being interested in non-commutative geometry, denoted *ncg* for brevity in the sequel, coming from a different point of view. In Chapter 4 we saw that the (Fredholm) Index cannot distinguish between the operators d_A and $d_{A'}$ where A' is another flat connection, not neccessarily gauge equivalent to the flat connection A. (Remember that if the base space is simply connected, all flat connections are gauge equivalent as we mentioned before). So one has to look for another invariant to do that. The idea then was to look at the η invariant of self adjoint operators. (Of course these operators are not self adjoint but we can use a trick when assembling the bundles of the corresponding elliptic complexes to get self adjoint operators for each one of them). It turned out that this invariant *partly does the job*, namely it can distinguish d_A and $d_{A'}$ *only* if A and A' are *not* gauge equivalent. The relation between the η invariant and ncg was suggested to us by Dr John Roe through the work of Douglas, Harder and Kaminker. Essentially the idea is that one thinks of the operator in question as defining a K-cycle over an algebra. The Chern character then gives a cyclic homology class which after renormalisation, when evaluated on the right element of the algebra, gives the η invariant of the operator.

Going back to non-commutative geometry, let us mention that it is a truly remarkable new theory. We shall try to give at least a very brief outline of its origins and try to explain the main ideas behind it. Then we shall study only the example of *"flat foliations"* on principal bundles which is our case of interest here. Yet we think that the connection we establish between a very common physical model, the nlσm, and non-commutative geometry is important and as far as we know, not being studied at all. Non-commutative geometry has a lot of new things to say. For instance, one can obtain new topological invariants that do not have an analogue in ordinary geometry. These invariants should correspond to something in Physics. This is not merely a hope, we have actually something more specific at this point. But for the moment, non-commutative geometry in our context is largely *"an answer seeking for a question"*.

The remaining references in the bibliography refer to this chapter.

6.1 Rationale

The starting point of non-commutative geometry is Gelfand's theorem which essentially states that the category of compact topological spaces with homoeomorphisms and the opposite of the category of commutative unital C^* algebras (we fix the ground field to be the complex numbers $\mathbf{C}$ with unit-preserving *-homomorphisms are equivalent. The equivalence is given naturally by the C and Sp functors, namely to any compact topological space X one associates the unital C^*-algebra $C(X)$ of continuous complex valued functions on X. Conversely,

to any unital C^*-algebra A one associates the compact topological space $Sp(A)$, the *spectrum* of A, consisting of unit-preserving homomorphisms of A into the complex numbers equipped with the topology of point-wise convergence in A. Finally, to a continuous map $f : X \to Y$ there corresponds the homomorphism $C(f) : C(Y) \to C(X)$ that sends $h \in C(Y)$ to $hf \in C(X)$, where the operation meant is just composition. One then has that $Sp(C(X)) = X$ and $C(Sp(A)) = A$.

One then can very naturally ask, *"what happens then with non-commutative C^* algebras"?* Is it possible that they might correspond to **"non-commutative"**—whatever this might mean—spaces?

The answer to the above extended duality between C^* algebras and spaces in the non-commutative case is **yes** at least for the case of interest to us here, namely *foliated manifolds*. In particular we shall be mainly interested in what are known as *flat foliations* of bundles.

Let us make some general remarks about manifold topology and geometry; there are different points of view to study manifolds, the following three being the ones most commonly used: measure theory, topology and geometry. The same holds true for ncg. We shall mainly be interested in the second point of view, and will occasionally touch on the third.

It is clear we think from what we mentioned above that ncg primarily deals with C^* algebras; in the commutative case it gives as a special case the same results as ordinary geometry whereas in the nc case it gives new results. The question whether there is a topological space realising the nc C^*-algebra situation in general, is not clear to us but this is always true, due to Gelfand's theorem, for the commutative case. It is also clear what happens for the case of interest here, namely foliations in general. However it seems that for nc C^*-algebras it is more appropriate to look at the space of irreducible representations of C^*-algebras to Hilbert spaces to find the missing analogues (replacing the spectrum in the commutative case), but this, as we said, is not very clear at the moment for the general case.

The great achievement of ncg is that it can generalise both K-homology and de Rham cohomology of manifolds to the nc setting, hence one can *do geometry* on nc C^*-algebras. The first generalisation is made via various "algebraic" versions of K-homology and the second via "cyclic homology". In order to clarify further this statement, let us briefly summarise some fundamental results from geometry on manifolds.

When we say *geometry* in the case of manifolds, (we are oversimplifying somewhat here), we essentially mean the Chern character. As it is well known, given a (closed) manifold M, this is a map from the K-groups of M to the de Rham cohomology groups of M, namely:

$$ch : K^i(M) \to H^*(M)$$

where $i = 0, 1$. This map has to satisfy certain axioms. All the theory of charac-

teristic classes, classifying bundles over M, are essentially encoded there. Even index theory relies heavily on the Chern character. Moreover in *topological K-Theory for spaces*, which is the case here, due to Bott periodicity, one has only *two* K-groups, namely K^0, consisting of stable isomorphism classes of complex vector bundles over our space M, and K^1, which is *the* K^0 *of the suspension of* M. (Obviously we consider complex vector bundles over M, in the real case we have mod 8 Bott periodicity).

When we are dealing with C^* algebras then, *doing geometry* means that we would like to construct something analogous, namely a map from "some" K-groups of the C^* algebra considered to "some" homology groups of our C^* algebra satisfying the same axioms as the usual Chern character for manifolds. One then could also have analogous index theorems. However, in order that our new theory is a proper generalisation of ordinary geometry, due to Gelfand's theorem mentioned above, this new Chern character must coincide with the usual one in the commutative C^*-algebra case.

Yet in this "algebraic" setting things are more complicated. To exhibit the difficulties clearer, let us mention that now there are actually *two* K-Theories one can use:

I.) *K-Theory of* B^**-algebras.* (A C^*-algebra is by definition a B^*-algebra satisfying extra conditions). Let B be a B^*-algebra. In this case one has also two groups, namely K_0 which by definition is the "Grothendiek group" G of

$$K_0(B) := G\pi_0 P(B)$$

where

$$P(B) := \lim_{n\to\infty} ProjM_n(B)$$

An equivalent description is the following:

$$K_0(B) = \pi_1 GL(B)$$

If $B = C(M)$ for some space M, then this is exactly the topological K-group $K^0(M)$ described above. This is essentially Serre-Swan theorem (see below). Moreover,

$$K_1(B) := \pi_0 GL(B)$$

Again, if $B = C(M)$ for some space M, one can prove that one gets exactly the topological group $K^1(M)$.

II.) *Quillen's higher algebraic K-Theory.* This applies to unital rings, hence to B^*-algebras as well. In this case, assuming R a unital ring, one defines the 0th algebraic K-group to be the group of stable isomorphism classes of finitely generated projective modules over R. Serre-Swan theorem then tells us that if

$R = C(M)$ for some space M, then this is exactly the topological K^0-group of the space M (which as we said above is the same as the K_0-group of B^*-algebras if one sees $C(M)$ as a B^*-algebra). The exact correspondence is described below:

To the vector bundle E one associates the $C(M)$-module $\Gamma(E)$, namely continuous sections of E. Conversely, given a finitely generated projective module E over $C(M)$, one associates the bundle over M with fiber $E \otimes_{C(M)} \mathbf{C}$.

The higher algebraic K-groups were defined by Quillen for general unital rings R as:

$$K_i(R) := \pi_i(BGL(R)^+), i \geq 1$$

where R is any unital ring and $BGL(R)$ is the classifying space of the (inductive limit of the) general linear group with entries from R, π_i stands for the homotopy groups and "+" is Quillen's famous *plus construction*. We would also like to mention that there is an attempt to construct the analogue of these higher algebraic K-groups for spaces and this is what is called *Waldhausen K-Theory*. We shall not use it here.

N.B. Recently Quillen also defined algebraic K-groups for *non-unital* rings using the category

$$m(R) := m(R')/\{M \text{ is an R-module st } \exists n \text{ with } R^n M = 0\}$$

where R' is the ring R with a unit attached to it. The category we are dividing by is a Serre subcategory of the category $m(R')$, cf [45].

Now the above two K-Theories which can be applied to C^*-algebras are *not* the same. They both give the *same* K_0 but *different* K_1*'s!* Moreover Bott periodicity holds for the first, hence one has essentially only two K-groups, but it does not hold for Quillen's K-Theory, where one has *many* K-groups.

Since both theories give the same K_0 group and since from Serre-Swan theorem mentioned above, in the commutative case we get Atiyah's topological K^0, we do not have any difficulty to choose either of these K-Theories at this level. But the K_1's are different and since as we mentioned above it is the K-Theory of B^*-algebras that gives Atiyah's topological K^1 in the commutative case, we choose the K-Theory of B^*-algebras as the non-commutative analogue to use in our Chern character. Moreover, unlike Quillen's theory, it also poseses Bott periodicity as well, thus making it more like Atiyah's topological K-Theory for spaces. Let us mention however that it is *exactly this feature* which will make us use Quillen's algebraic K-Theory later to get *more* invariants!

Let us now pass to the second ingredient of the Chern character, namely cohomology. There is the following result which actually dictates that the right (co)homology is actually *cyclic homology*:

If M is a closed manifold, let $C(M)$ denote the C^*-algebra of continuous func-

tions on M. Then the (periodic) cyclic homology of $C(M)$ is isomorphic to the deRham cohomology of M (mod 2).

The above result clearly says that one should use cyclic homology as the second ingredient for the new Chern character, since this homology gives the correct picture in the commutative case. The actual construction of the Chern character will be given later.

Let us close this section with some

Remarks:

1. In ordinary Index theory, one uses pairings between *topological* K-theory and *de Rham (co)homology* of the manifold in order to get the index, (or equivalently this Index theory pairing can be simply described as the pairing between K-theory and K-homology if one adopts the more advanced description of K-homology of Baum-Douglas), roughly speaking K^0-groups correspond to elliptic operators and K^1-groups correspond to self adjoint operators on the manifold considered. But due to Bott periodicity "there are no other" topological K-groups. Yet this is in striking contrast with ncg where one can alternatively use Quillen's Higher Algebraic K-theory and thus pair *higher algebraic K-groups* with *cyclic homology* classes. And higher algebraic K-groups are *many*, no Bott periodicity in this case, not only two! Thus one can get more invariants! The meaning of these invariants is not well studied and we shall try to give them some physical meaning as well.

2. Since we shall be primarily concerned with K_0 and K_1 groups, we can use the more convenient *bivariant KK-Theory*, the first "K" is for "Kasparov" and the second is for "K-Theory (or Connes' deformed *E-Theory*). We shall explain these theories later as we shall use them. These theories are more in the spirit of K-Theory of B^*-algebras mentioned above than Quillen's higher algebraic K-Theory.

3. One can very easily see that if ncg is supposed to give a generalisation of ordinary geometry, it should give the same results in the commutative case. This is indeed the case from the above two main results stated, namely Serre-Swan theorem and the isomorhpism between cyclic homology and deRham cohomology in the commutative case.

4. We would also like to mention that there is an important difference between topological and C^*-algebra K-Theory: in the later one cannot define Adams operations.

We think we gave a "flavour" of the generalisations of ncg and that the theory is very well adapted to agree with ordinary geometry in the commutative case. We are not going to deal with ncg in general though because it is a huge subject. From now on we shall concentrate on our special case of nc space, which, as we mentioned, is the flat foliations of bundles. We think then we should give a detailed description of this setting.

6.2 Foliations

Let V be a smooth manifold and TV its tangent bundle. A smooth subbundle F of TV is called *integrable* iff one of the following equivalent conditions is satisfied:

1. Every $x \in V$ is contained in a submanifold W of V such that

$$T_y(W) = F_y$$

where T_y denotes the tangent space over y.

2. Every $x \in V$ is in the domain $U \subset V$ of a submersion $p : U \to \mathbf{R}^q$ (q=codim F) with $F_y = ker(p_*)_y \forall y \in U$.

3. $C^\infty(V, F) = X \in C^\infty(V, TV); X_x \in F_x \forall x \in V$ is a Lie subalgebra of the Lie algebra of vector fields on V.

4. The ideal $J(F)$ of smooth differential forms which vanish on F is stable under differentiation:$d(J) \subset J$

The condition 3. is simply Frobenius' Theorem and 4. its dual.

Any 1-dimensional subbundle F of TV is integrable, but for $dimF \geq 2$ the condition is non-trivial; for instance if V is the total space of a principal bundle with compact structure group, then we know that the subbundle of *vertical* vectors is always integrable, but the *horizontal* subbundle is integrable iff the connection is **flat**.

A *foliation* of V is given by an integrable subbundle F of V. The *leaves* of the foliation are the maximal connected submanifolds L of V with $T_x(L) = F_x \forall x \in L$, and the partition of V into leaves $V = \cup L_a$ where $a \in A$ is characterised geometrically by its "local triviality": every point $x \in V$ has a neighborhood U and a system of local coordinates (x^j), j=1,...,dimV which is called *foliation chart* , so that the partition of U into connected components of leaves, called *plaques* (they are the leaves of the restriction of the foliation on U), corresponds to the partition of $\mathbf{R}^{dimV} = \mathbf{R}^{dimF} \times \mathbf{R}^{codimF}$ into the parallel affine subspaces $\mathbf{R}^{dimF} \times pt$.

Very simple examples indicate that the leaves L may **not** be compact even if the manifold V is and that the space of leaves $X := V/F$ may **not** be Hausdorff for the quotient topology. The "rational torus" is such an example.

As before we now consider a principal bundle P with structure (Lie) group G (assumed compact and connected) over a compact manifold M. The total space P has automatically a foliation induced by the fibration: the leaves are the fibers which are isomorphic to the structure group G and the space of leaves is just the base space M. We shall be refering to this foliation as the *vertical* foliation of the principal bundle. Actually there are *two* natural foliations on the total space

of a principal bundle: one with leaf being the fiber and another one with the leaf being the orbit of the group action through a point. But for principal bundles, they are isomorphic, so we identify them. Clearly, the dimension of this foliation is equal to the dimension of the group G, the integrable subbundle of TP being in this case the vertical subbundle. The codimension is equal to the dimension of the base space.

Now if in addition a flat connection is given on our principal bundle, we have another foliation of the total space which we shall be referring to as the *horizontal* or *flat* foliation. We shall study this foliation extensively in the following section. The dimension of this foliation equals the dimension of the base space and the codimension equals the dimension of the group. From this one can see that the vertical and the horizontal foliations of a principal bundle are *transverse* to each other.

Now the vertical foliation behaves very well; everything is compact and Hausdorff, as were the spaces we started with to build our bundle. However, the horizontal foliation can suffer from various "pathological" deffects and for this reason it is interesting from the ncg point of view. Let us study it in greater detail.

6.2.1 Flat foliation of a principal bundle

To begin with, a flat connection on a principal bundle P with structure group G and base space M, corresponds to reduction of the structure group from G to a subgroup isomorphic to a normal subgroup of the fundamental group of the base space $\pi_1(M)$. Moreover, as we mentioned in the last section of Chapter 3, a (gauge equivalence class of a) flat connection also defines a (conjugacy class of a) representation

$$H : \pi_1(M) \to G$$

If we identify the fundamental group with the group of covering translations of the universal covering $\tilde{M}$ of M we get an action $\varkappa$ of $\pi_1(M)$ on $\tilde{M} \times G$ defined as follows:

$$\varkappa : \pi_1(M) \times (\tilde{M} \times G) \to (\tilde{M} \times G)$$

$$(\gamma, (\tilde{m}, g)) \mapsto (\gamma(\tilde{m}), H(\gamma)(g))$$

where we use the obvious notation $\gamma \in \pi_1(M), g \in G, \tilde{m} \in M$. This action gives a commutative diagram:

$$\begin{array}{ccc} \tilde{M} \times G & \xrightarrow{pr} & \tilde{M} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle q} \\ P' = (\tilde{M} \times G)/\varkappa & \xrightarrow[p]{} & M \end{array} \tag{6.1}$$

where pr is the canonical projection, π is the quotient map by $\varkappa$, p is uniquely induced by pr and q is just the map from the universal covering space to the original space.

This construction is called *suspension of the representation* H. One can prove that the map π is a covering map and that if $\digamma := ImH$ is endowed with the induced topology, then $\xi_H = (P', p, M)$ is a fiber bundle with fiber G, total space P', base M, projection p and structure group $\digamma$.

To study the geometric properties of suspensions we introduce a new topology on the total space P' of ξ_H. We denote by G^δ the set G supplied with the *discrete* topology. Then the action $\varkappa$ of $\pi_1(M)$ on $\tilde{M} \times G^\delta$ remains continuous and the map $\pi : \tilde{M} \times G^\delta \to P'$ induces on P' a new topology which is **finer** than its manifold topology. We denote by P^δ the set P' supplied with this topology. The topology on $\tilde{M} \times G^\delta$ and the topology P^δ are called the *leaf topologies*. Then the suspension diagram below is a commutative diagram of covering maps:

$$\begin{array}{ccc} \tilde{M} \times G^\delta & \xrightarrow{pr} & \tilde{M} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle q} \\ P^\delta & \xrightarrow[p]{} & M \end{array} \tag{6.2}$$

The topological space P^δ is not connected unless the fiber is contractible. A connected component of P^δ is called a leaf of ξ_H. Each point $x = \pi(\tilde{m}, g) \in P'$ belongs to exactly one leaf which is denoted L_x and equals $\pi(\tilde{M} \times g)$. The leaves are injectively immersed submanifolds of P' but in general not embedded. They are transverse to the fibers of ξ_H. Conjugate representations H and H' give suspension bundles ξ_H and $\xi_{H'}$ which are isomorphic.

Let now $x = \pi(\tilde{m}, g)$. Then the representation

$$H_x : \pi_1(L_x) \to G$$

with image $\digamma_g$, is called the *holonomy* representation of the leaf L_x at the point x. The group $\digamma_g$ is the *holonomy group* of the leaf L_x at the point x. $\digamma_g$ is the isotropy group of $\digamma$ in $g \in G$. Moreover $\pi_1(L_x)$ is isomorphic to the isotropy group of $\pi_1(M)$ in the point $g \in G$, namely $\pi_1(L_x) \cong \{\gamma \in \pi_1(M) | H(\gamma)g = g\}$.

From the above discussion, we want to emphasise the "leaf topology" on the total space P' which is *finer* than the original manifold topology. We would also like to mention that the notion of suspension of a representation we used, is more

general, but for our case of interest here is exactly the same as a horizontal foliation of the total space P of our principal bundle. The suspension bundle ξ_H is exactly the reduction of P to another bundle with same fiber but with reduced structure group $\Gamma = ImH \subset G$, namely this is exactly what a flat connection does. It is the leaf topology which will give the non trivial things in the ncg setting.

We would also like to mention something else about *classifying spaces of flat bundles*.

Let G be a connected Lie group and let G^δ denote the same group with the discrete topology. The *Milnor join construction* for G defines a connected space BG which is the classifying space for principal G-bundles. The same construction applied to G^δ yields a connected topological space BG^δ which is an *Eilenberg-Maclane space* $K(G,1)$, namely $\pi_1(BG) = G$ and $\pi_j(BG) = 0$ for $j > 1$. The inclusion $i : G^\delta \to G$ induces a continuous map $Bi : BG^\delta \to BG$. As sets these two spaces are the same with the source having *finer* topology than the range. The difference in these two topologies is measured by introducing the homotopy fiber BG'. This is defined by first replacing Bi with a homotopy equivalent weak fibration over BG, then take for BG' the (homotopy class of the) fiber. The description then is just the construction of the *Puppe Sequence* for Bi (cf [67]).

Choose a base point in BG^δ and consider its image in BG. Then let $\Omega(BG)$ and $P(BG)$ denote the space of based loops and paths with initial point of BG respectively. Let e be the end point map of a path. Then one has a fibration

$$\Omega(BG) \hookrightarrow P(BG) \to BG$$

where the second map is e. Then define BG' via the homotopy pull-back diagram:

$$\begin{array}{ccc} \Omega(BG) & \longrightarrow & \Omega(BG) \\ \downarrow & & \downarrow \\ BG' & \longrightarrow & P(BG) \\ \downarrow & & \downarrow e \\ BG^\delta & \xrightarrow[Bi]{} & BG \end{array} \tag{6.3}$$

A principal G-bundle P over a manifold M is equivalent to giving an open covering for M and the transition functions. This data defines a continuous map $g_P : M \to BG$. If the transition functions are locally constant, namely if the bundle P is flat, then g_P can be factored through BG^δ as a continuous map.

A choice of transition functions which are locally constant is equivalent to specifying a flat G-structure on P. Hence P has a horizontal foliation whose

holonomy map $a : \pi_1(M) \to G$ defines the classifying map $Ba : M \to BG^\delta$. Conversely, given a continuous map $Ba : M \to BG^\delta$, there is induced a representation $a : \pi_1(M) \to G$ and a corresponding flat principal G-bundle $P_a = \tilde{M} \times_{\pi_1(M)} G$, where $\tilde{M}$ is the universal covering of M. The topological type of the G-bundle P_a is determined by the composition

$$g_a : M \to BG^\delta \to BG$$

The principal bundle is trivial iff g_a is homotopic to the constant map $M \to pt$. The choice of the homotopy is equivalent to specifying a global section on P_a.

6.2.2 Groupoids and C^*-algebras associated to Foliations

The next step is to associate the *holonomy groupoid* to any foliation. A groupoid is roughly speaking a small category with inverses, or more precisely

Definition 1:

A groupoid consists of a set Γ, a distinguished subset $\Gamma^{(0)}$ of Γ, two maps $r, s : \Gamma \to \Gamma^{(0)}$ and a law of composition

$$\circ : \Gamma^{(2)} := (\gamma_1, \gamma_2) \in \Gamma \times \Gamma; s(\gamma_1) = r(\gamma_2) \to \Gamma$$

such that:

1.$s(\gamma_1 \circ \gamma_2) = s(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$ $\forall(\gamma_1, \gamma_2) \in \Gamma^{(2)}$
2. $s(x) = r(x) = x \forall x \in \Gamma^{(0)}$
3.$\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma \forall \gamma \in \Gamma$
4. $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$
5. Each γ has a two sided inverse γ^{-1}, with $\gamma\gamma^{-1} = r(\gamma)$ and $\gamma^{-1}\gamma = s(\gamma)$

The maps r, s are called *range* and *source* maps.

In the category theory terminology, $\Gamma^{(0)}$ is the space of objects and $\Gamma^{(2)}$ is the space of morphisms.

Definition 2:

A *smooth* groupoid Γ is a groupoid together with a differentiable structure on Γ and $\Gamma^{(0)}$ such that the maps r, s are submersions and the object inclusion map $\Gamma^{(0)} \to \Gamma$ is smooth, as is the composition map $\Gamma^{(2)} \to \Gamma$

The notion of a $\frac{1}{2}$-*density* on a smooth manifold allows one to define in a canonical manner the convolution algebra of a smooth groupoid Γ. Specifically, given Γ, let $\Omega^{1/2}$ be the line bundle over Γ whose fiber $\Omega^{1/2}_\gamma$ at $\gamma \in \Gamma, r(\gamma) = x, s(\gamma) = y$, is the linear space of maps

$$\rho : \wedge^k T_\gamma(\Gamma^x) \otimes \wedge^k T_\gamma(\Gamma_y) \to \mathbf{C}$$

such that

$$\rho(\lambda\nu) = |\lambda|^{1/2}\rho(\nu)\forall\lambda \in \mathbf{R}$$

Here $\Gamma_y = \gamma \in \Gamma; s(\gamma) = y, \Gamma^x = \gamma \in \Gamma; r(\gamma) = x$, and $k = dimT_\gamma(\Gamma^x) = dimT_\gamma(\Gamma_y)$ are the dimensions of the fibers of the submersions $r : \Gamma \to \Gamma^{(0)}$ and $s : \Gamma \to \Gamma^{(0)}$.

Then we endow the linear space $C_c^\infty(\Gamma, \Omega^{1/2})$ of smooth compactly supported sections of $\Omega^{1/2}$ with the convolution product

$$(a * b)(\gamma) = \int_{\gamma_1\circ\gamma_2=\gamma} a(\gamma_1)b(\gamma_2)$$

$\forall a, b \in C_c^\infty(\Gamma, \Omega^{1/2})$ where the integral on the RHS makes sense since it is the integral of a 1-density (essentially a measure), namely $a(\gamma_1)b(\gamma_1^{-1}\gamma)$, on the manifold Γ^x, $x = r(\gamma)$.

One then can prove that if Γ is a smooth groupoid and $C_c^\infty(\Gamma, \Omega^{1/2})$ is the convolution algebra of smooth compactly supported $\frac{1}{2}$-densities with involution $*$, $f^*(\gamma) = \overline{f(\gamma^{-1})}$. Then for each $x \in \Gamma^{(0)}$, the following defines an involutive representation π_x of $C_c^\infty(\Gamma, \Omega^{1/2})$ in the Hilbert space $L^2(\Gamma_x)$:

$$(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1)\xi(\gamma_1^{-1}\gamma)$$

$\forall\gamma \in \Gamma_x, \xi \in L^2(\Gamma_x)$
The completion of $C_c^\infty(\Gamma, \Omega^{1/2})$ for the norm $||f|| = sup_{x\in\Gamma^{(0)}}||\pi_x(f)||$ is a C^*-algebra denoted $C_r^*(\Gamma)$.

Moreover one defines the C^*-algebra $C^*(\Gamma)$ as the completion of the involutive algebra $C_c^\infty(\Gamma, \Omega^{1/2})$ for the norm

$||f||_{max} = sup||\pi(f)||$; $\{\pi$ involutive Hilbert space representation of $C_c^\infty(\Gamma, \Omega^{1/2})\}$

After this general introduction to groupoids, now we pass to groupoids and C^*-algebras associated to foliations.

Let (V, F) be a foliated manifold of codim q. Given any $x \in V$ and a small enough open set W in V containing x, the restriction of the foliation F to W has as its leaf space an open set of $\mathbf{R^q}$ which we shall call a transverse neighborhood of x. In other words, this open set W/F is the set of plaques around x. Now given a leaf L of (V, F) and two points $x, y \in L$, any simple path γ from x to y on L uniquely determines a germ $h(\gamma)$ of a diffeomorphism from a transverse neighborhood of x to one of y. This depends only on the homotopy class of γ and is called the *holonomy* of the path γ. The *holonomy groupoid of a leaf* L

is the quotient of its fundamental groupoid by the equivalence relation which identifies two paths γ_1, γ_2 from x to y both in L iff $h(\gamma_1) = h(\gamma_2)$. Here by the fundamental groupoid of a leaf we mean the groupoid $\Gamma = L \times L$, r, s are the two projections, $\Gamma^{(0)} = L$ and the composition is $(x, y) \circ (y, z) = (x, z)$. (From this one can see that every space is a groupoid). The holonomy covering $\tilde{L}$ of a leaf L is the covering of L associated to the normal subgroup of its fundamental group $\pi_1(L)$ given by paths with trivial holonomy. The **holonomy groupoid** or **graph of the foliation** is the union Γ of the holonomy groupoids of its leaves. Given an element γ of Γ we denote by $s(\gamma) = x$ the origin of the path γ and by $r(\gamma) = y$ its end point, where r, s are the range and source maps as in the general case.

An element of Γ is thus given by two points $x = s(\gamma)$ and $y = r(\gamma)$ of V together with an equivalence class of smooth paths : the $\gamma(t)$, $t \in [0, 1]$ with $\gamma(0) = x$ and $\gamma(1) = y$, tangent to the bundle F, namely with $\frac{d\gamma}{dt} \in F_{\gamma(t)} \forall t \in \mathbf{R}$, identifying γ_1 and γ_2 as equivalent iff the holonomy of the path $\gamma_2\gamma_1^{-1}$ at the point x is the identity. The graph Γ has an obvious composition law. For γ_1 and γ_2 in Γ, the composition $\gamma_1 \circ \gamma_2$ makes sense if $s(\gamma_1) = r(\gamma_2)$. The groupoid Γ is by construction a (not neccessarily Hausdorff) manifold of dimension $dim\Gamma = dimV + dimF$.

Definition : The C^***-algebra of the foliation** is exactly the C^*-algebra of its graph, as described for arbitrary groupoids above.

For our foliations of interest, the graph Γ is the following: for the vertical foliation is just the manifold $P \times G$ whereas for the horizontal foliation is $P \times_a \tilde{M}$, where a is the representation from $\pi_1(M)$ to G induced by the flat connection 1-form A (via the holonomy). Moreover the distinguished subset $\Gamma^{(0)}$ in both cases is the manifold we want to foliate, namely P, the total space of our bundle in our case.

The C^*-algebras associated to our foliations are: for the vertical foliation is $C(M)$ tensored with compact operators which act as smoothing kernels along the leaves which in turn is strongly Morita equivalent to just $C(M)$, whereas for the horizontal foliation is strongly Morita equivalent to $C(P) \rtimes \pi_1(M)$. (Note: the representation of the fundamental group of the base onto the structure Lie group induced by the flat connection 1-form used enters the definition of the crossed product, see below). The first algebra is **commutative** (up to SME), but the second **is not!** It is for this reason that we can see now that ncg has an important role to play, in fact we are deeply in the ncg setting! The reason for this is the "leaf topology" described above which is finer than the manifold topology as we said. In this construction, the corresponding C^*-algebra "captures" all the topological data and hence we do K-theory on it, instead of the "space".

Let us now mention what is *strong Morita equivalence* (denoted SME for brevity in the sequel), what is the *crossed product* we wrote above and why we

are interested specifying our C^*-algebras only up to SME.

We shall answer the last question first because it is the simplest. Well, the reason is that there is a theorem saying that SME C^*-algebras have the same K-homology. We mentioned that the right framework for ncg is K-Theory which is a generalised homology theory ("generalised" means that it does not satisfy the dimension axiom, the last of the 5 Steenrod axioms for homology).

Let B, C be two C^*-algebras. A *strong Morita equivalence* $B \cong C$ is given by a pair (E_1, E_2) of C^*-bimodules such that

$$E_1 \otimes E_2 = 1_B \text{ and } E_2 \otimes E_1 = 1_C$$

A (B, C) *C^*-bimodule* E is a C^*- module E over C together with a *-homomorphism from B to $End_C(E)$.

A *C^*-module* E over C is a pre-C^*-module E over C for which $\| \ \|$ is a complete norm.

A *pre-C^*-module* E over C is a right C-module E endowed with a C-valued inner product. A *semi-norm* on E is defined by:

$$||\xi|| = ||\langle \xi, \xi \rangle||^{1/2}$$

$\forall \xi \in E$, where $||\langle \xi, \xi \rangle||$ is the C^*-algebra norm of $\langle \xi, \xi \rangle \in C$.

A *C-valued inner product* on a right C-module E is a C-valued sesquilinear form $\langle , \rangle$, conjugate linear in the first variable and such that:
1.$\langle \xi, \xi \rangle$ is a positive element of C for any $\xi \in E$.
2.$\langle \xi, n \rangle^* = \langle n, \xi \rangle \forall \xi, n \in E$.
3.$\langle \xi, nc \rangle = \langle \xi, n \rangle c \forall c \in C, \xi, n \in E$.

Now we come to the "crossed product" $\rtimes$. Let A be a C^*-algebra, G a locally compact group and $a : G \to Aut(A)$ a continuous action of G on A. (Recall here that for our horizontal foliation this action is given by the representation of the fundamental group of the base to the structure Lie group induced by the flat connection we use). Thus for each g in G, $a_g \in Aut(A)$ is a *-automorphism of A and for each $x \in A$ the map $g \mapsto a_g(x)$ is norm continuous. A covariant representation π of (A, a) is given by unitary representations π_A of A and π_G of G on a Hilbert space H such that:

$$\pi_G(g)\pi_A(x)\pi_G(g)^{-1} = \pi_A(a_g(x))$$

$\forall g \in G, x \in A$.
Then there exists a natural C^*-algebra, the *crossed product* $B = A \rtimes_a G$, whose

unitary representations correspond exactly to the covariant representations of (A, a). Indeed let dg be a left Haar measure on G and let us endow the linear space $G_c(G, A)$ of continuous compactly supported maps from G to A with the following involutive algebra structure:

$$(f_1 * f_2)(g) = \int f_1(g_1) a_{g_1}(f_2(g_1^{-1}g)) dg_1$$

$\forall f_j \in C_c(G, A), g \in G$

$$(f^*)(g) = \delta(g)^{-1} a_g(f(g^{-1})^*)$$

$\forall f \in C_c(G, A), g \in G$ where $\delta : G \to \mathbf{R}_+^*$ is the modular function of the not necessarily unimodular group G, namely the homomorphism from G to $\mathbf{R}_+^*$ defined by the equality

$$d(g^1) = \delta(g)^{-1} dg$$

These algebraic operations on $C_c(G, A)$ are uniquely prescribed in order to get an involutive representation $\widetilde{\pi}$ of $C_c(G, A)$ from a covariant representation π of (A,a) by the following formula:

$$\widetilde{\pi}(f) = \int \pi_A(f(g)) \pi_G(g) dg$$

$\forall f \in C_c(G, A)$

It is not difficult to check that the completion $B = A \rtimes_a G$ of $C_c(G, A)$ for the following norm is a C^*-algebra whose unitary representations correspond exactly to the covariant representations of (A, a):

$||f|| = sup||\widetilde{\pi}(f)||$; $\{\pi$ a covariant representation of (A,a)$\}$

We would like to mention that there is also a *von Neuman* algebra canonically associated to a foliation but this is related to measure theoretic aspects of foliations (and ncg in general) and we shall not be dealing with these here.

Now the next step is to associate a canonical *cycle* to each C^*-algebra (hence to each foliation).

6.3 Cyclic Homology

6.3.1 Some elements of cyclic homology

We gave a brief account of cyclic homology of algebras in the previous chapter where we presented the differentials b (Hochschild) and B (Connes'). The cyclic

homology is the total homology with differential $b + B$. We now give another important

Definition 1:

1. A *cycle* of dimension n is a triple $(\Omega, d, \int)$ where $\Omega = \oplus_{j=0}^{n} \Omega^j$ is a graded algebra over C, d is a differential on Ω's and $\int : \Omega^n \to C$ is a closed graded trace on Ω.

2. Let A be an algebra over $\mathbf{C}$. Then a cycle over A is given by a cycle $(\Omega, d, \int)$ and a homomorphism $\rho : A \to \Omega^0$.

A cycle over A of dimension n is essentially determined by its *character* which is the following $(n+1)$-linear functional on A:

$$\tau(a^0, ..., a^n) = \int \rho(a^0) d(\rho(a^1)) d(\rho(a^2)) ... d(\rho(a^n))$$

$\forall a^j \in A$

One can then prove that this is a *cyclic cocycle* of A, namely it defines a cohomology class in the cyclic homology of A and that the above is a necessary and sufficient statement.

One then can define a pairing between $K_0(A)$ with even cyclic homology of A and similarly for the odd case, namely:

Definition 2:

The following equality defines a bilinear pairing between K-theory and cyclic homology:

1. Even case: $K_0(A)$ and $HC^{ev}(A)$:

$$\langle [e], [\phi] \rangle := (m!)^{-1} (\phi \# Tr)(e, ..., e)$$

for $e \in K_0(A)$ using the idempotents' description and $\phi \in HC^{2m}(A)$ and where $\#$ is the *cup product* in cyclic homology (see for instance [30] for the precise definition).

2. Odd case: $K_1(A)$ and $HC^{odd}(A)$:

$$\langle [u], [\phi] \rangle = \frac{1}{\sqrt{2i}} 2^{-n} \Gamma(\frac{n}{2} + 1)^{-1} (\phi \# Tr)(u^{-1} - 1, u - 1, u^{-1} - 1, ..., u - 1)$$

where $u \in K_1(A)$ and $\phi \in HC^n(A)$.

6.3.2 Cyclic classes associated to foliations

We now pass to the *transverse fundamental class* naturally associated to a foliation. We shall work generally first.

Let (V, F) be a foliated manifold and $t = TV/F$ the transverse bundle of the foliation. The holonomy groupoid Γ of (V, F) acts in a natural way on t by the differential of the holonomy, thus for every $\gamma \in \Gamma$, $\gamma : x \to y$ determines a linear map $h(\gamma) : t_x \to t_y$. We denote this action by h. It is not in general possible to find a Euclidean metric on t which is invariant under the above action of Γ. Let g be an arbitrary smooth Euclidean metric on the real vector bundle t. Thus for $\xi \in t_x$ we let $||\xi||_g = (\langle \xi, \xi \rangle_g)^{1/2}$ be the corresponding norms and inner products and drop subscript g henceforth. Using g we define a C^*-module E on the C^*-algebra $C_r^*(V, F)$ of the foliation. Recall that $C_r^*(V, F)$ is the completion of the convolution algebra $C_c^\infty(\Gamma, \Omega^{1/2})$ which acts by right convolution on the linear space $C_c^\infty(\Gamma, \Omega^{1/2} \otimes r^*(t_{\mathbf{C}}))$:

$$(\xi f)(\gamma) = \int_{\Gamma^y} \xi(\gamma_1) f(\gamma_1^{-1}\gamma)$$

where $y = r(\gamma)$. Endowing the complexified bundle $t_{\mathbf{C}}$ with the inner product associated to g and anti-linear in the first variable, the following formula defines a $C_c^\infty(\Gamma, \Omega^{1/2})$-valued inner product

$$\langle \xi, n \rangle(\gamma) = \int_{\Gamma^y} \langle \xi(\gamma_1^{-1}), n(\gamma_1^{-1}\gamma) \rangle$$

for any $\xi, n \in C_c^\infty(\Gamma, \Omega^{1/2} \otimes r^*(t_{\mathbf{C}}))$ One then checks that the completion E of the space $C_c^\infty(\Gamma, \Omega^{1/2} \otimes r^*(t_{\mathbf{C}}))$ for the norm

$$||\xi|| = (||\langle \xi, \xi \rangle||_{C_r^*(V,F)})^{1/2}$$

becomes a C^*-module over $C_r^*(V, F)$. If one takes also the action h of Γ on t into account, with some extra effort one can make E into an (B, A)-bimodule. The first construction thus gives us an element E of $K_0(C_r^*(V, F))$ whereas the second gives E as an element of $KK_0(B, A)$, *Kasparov's bivariant K-Theory.* (Recall that the 0th Kasparov's bivariant K-group in this case consists of stable isomorphism classes of (B, A)-bimodules). We shall use this action h to define a left action of $C_c^\infty(\Gamma, \Omega^{1/2}$ on E by:

$$(f\xi)(\gamma) = \int_{\Gamma^y} f(\gamma_1) h(\gamma_1) \xi(\gamma_1^{-1}\gamma)$$

$\forall f \in C_c^\infty(\Gamma, \Omega^{1/2}), \xi \in E$

One then can prove that for any $f \in C_c^\infty(\Gamma, \Omega^{1/2})$ the above formula defines an endomorphism $\lambda(f)$ of the C^*-module E whose adjoint $\lambda(f)^*$ is given by

$$(\lambda(f)^*\xi)(\gamma) = \int_{\Gamma^y} f^\#(\gamma_1) h(\gamma_1) \xi(\gamma_1^{-1}\gamma)$$

where

$$f^{\#}(\gamma) := \widetilde{f}(\gamma^{-1})\Delta(\gamma)$$

and

$$\Delta(\gamma) = (h(\gamma)^{-1})^t h(\gamma)^{-1} \in End(t_{\mathbf{C}}(r(\gamma)))$$

This shows that unless the metric on t is Γ-invariant, the representation λ is not a *-representation, the subtle difference between $\lambda(f)^*$ and $\lambda(f^*)$ being measured by Δ. In particular λ is not in general bounded for the C^*-algebra norms on both $End_{C_r^*(V,F)}E$ and $C_r^*(V,F) \supset C_c^\infty(\Gamma, \Omega^{1/2})$. However λ is a closable homomorphism of C^*-algebras, namely, the closure of the graph of λ is the graph of a densely defined homomorphism. Then with the graph norm

$$||x||_\lambda = ||x|| + ||\lambda(x)||$$

the domain B of the closure $\widetilde{\lambda}$ of λ is a Banach algebra which is dense in the C^*-algebra $A = C_r^*(V,F)$. The C^*-module E is then a (B,A)-bimodule. This particular module E we constructed here will be the one of the two main ingredients which define the invariant we want.

Let us now return to our case. The vertical and the horizontal (or flat) foliations of the total space of our principal bundle P will be denoted (P,V) and (P,H) respectively. Note that they are transverse, hence one can be used as the transverse bundle of the other. Moreover they are both **integrable**, hence both d_V and d_H are differentials. One then does **not** have to complete the graded algebras.(cf[30]). One then has that there is a natural cycle for the algebra of each foliation, namely:

Vertical Foliation

The natural cycle canonically associated to the algebra $C_c^\infty(P\times G, \Omega^{1/2})$ of the vertical foliation consists of: 1. The graded algebra $C_c^\infty(P\times G, \Omega^{1/2}\otimes r^*(\wedge H^*))$ where H is the *horizontal* subbundle (i.e. the transverse bundle to the vertical foliation).

2. The differential $d = d_V + d_H + \theta$ where

$$d_H : C^\infty(P, \wedge^r V^* \otimes \wedge^s H^*) \to C^\infty(P, \wedge^r V^* \otimes \wedge^{s+1} H^*)$$

$$d_V : C^\infty(P, \wedge^r V^* \otimes \wedge^s H^*) \to C^\infty(P, \wedge^{r+1} V^* \otimes \wedge^s H^*)$$

θ means contraction with the section $\theta \in C^\infty(P, V\otimes\wedge^2 H^*)$, where θ is defined by $\theta(p_H(X), p_H(Y)) = p_V([X,Y])$ for any pair of horizontal vector fields $X, Y \in C^\infty(P,H)$ and (p_H, p_V) is the isomorphism $TP \to H\oplus V$ given by H.

3.The trace is defined via

$$\tau(w) = \int_{\Gamma^{(0)}} w$$

where Γ is the graph of the vertical foliation $\Gamma = P \times G$.

Similarly one defines a fundamental class for the horizontal foliation.

One then can define the character of these cycles which will be a class in cyclic homology of the appropriate algebra for each foliation (see for instance [30]). Let us mention here that the class in the cyclic homology does not depend on the choice of the transverse bundle of the foliation. We want to emphasise this point because one might think that the choice of the vertical foliation say, as transverse to the horizontal was *ad hoc*, but one can see that it really makes no difference: the cyclic cocycle depends on the foliation only and not on the choice of the transverse bundle which is just an intermediate tool to construct the cyclic cocycle corresponding to the foliation. We also want to mention that the cycle is of dimension equal to the codimension of the foliation.

Note that the cyclic cocycle we will get from the vertical foliation which is the commutative case, will be of dimension equal to the codimension of the vertical foliation which is equal to the dimension of the base space of our bundle. Moreover as we mentioned above, in this case the C^*-algebra of this foliation is SME to the algebra of functions on the base. This is a commutative C^*-algebra whose cyclic homology is exactly the de Rham cohomology of the base space. Hence we get a top cohomology class, which is in fact the fundamental class of our base space. Hence we recover ordinary geometry in the commutative case.

Since now we have a cyclic homology class, say ϕ of the algebra of the foliation, say A, we automatically have a map

$$K_i(A) \to C$$

given by pairing it with K-group elements ($i = 0, 1$ above) to get index theorems for *leafwise* elliptic operators or invariants for foliations if we pair this cyclic cocycle ϕ with the natural A-module E we constructed above for our foliation. Let us mention here that the analytic *Index* of an operator *elliptic along the leaves of a foliation* say (V, F), is an element of $K_0(C^*(V, F))$, being in fact a generalisation of the index of families of elliptic operators considered by Atiyah and Singer. (In the Atiyah-Singer case of families of elliptic operators one is dealing with the foliation induced by the fibration, which is the commutative geometry case). The operator *itself* which is elliptic along the leaves of the foliation is an element of $KK(C^*(V, F), C(V))$.

6.3.3 Invariant(s) for the nlσm

This is an important point because by pairing the C^*-module E we constructed previously naturally associated to the foliation considered with the cyclic cocycle naturally associated to the foliation, if we apply this to the horizontal foliation, we get a *complex number* which is a invariant for the nlσm. Namely one has:

$$\langle [E], [P/H] \rangle = (m!)^{-1}((P/H)\#Tr)(E, ..., E) \in \mathbf{C}$$

Moreover as far as we know it is the only known invariant for the nlσm and the only invariant known in physics which is, to begin with, a complex number. The nature of this invariant needs further study. It is not immediate that it is topological because in the construction of the module E (seen either as a C^*-module in which case it represents a K class or, taking the action of the graph Γ of the foliation into account as well, as a (B, A)-bimodule in which case it represents a KK-class) one uses a metric. However the K (or KK) class may be independent of this choice, but this requires a proof.

Moreover one can pair the fundamental cocycle of the foliation with elements of the *higher algebraic K-groups* of the C^*-algebra of the foliation. This is a completely new feature of ncg because in ordinary geometry, due to Bott periodicity, one has only the 0th and the 1st K-theory groups. The meaning of these invariants however is not clear, not even from the mathematical point of view, letting alone their physical meaning!

Another interesting thing one can do in particular for the horizontal foliation is to try to use *Waldhausen K-theory*. Given a compact space X with first homotopy group $\pi_1(X)$, the Waldhausen K-groups are roughly speaking Quillen's higher algebraic K-groups for the group ring $\mathbf{Z}(\pi_1(X))$. We think that since, as we saw above, in the horizontal foliation of a principal bundle the first homotopy group of the base space is crucial, Waldhausen K-groups might be useful in understanding these pairings.

6.3.4 Connes' conjecture

However there are even more things one can do: we mentioned above that what we presented was actually the *analytic* version of the K-theory of foliated manifolds. There is also a *topological* one which we shall try to give a very brief outline of. (In all which follows we are only considering the 0th algebraic K-group of the C^*-algebra of the foliation).

We know that the original Atiyah-Singer index theorem proves that the analytic and the topological indices of an elliptic operator coincide. If one has an operator acting on a closed manifold M, say, the homotopy class of the principal symbol of the operator defines a class in $K^0(TM)$. One then constructs two maps from this K-group to the integers, one is the analytic index and the other is the topological index. However one can proceed slightly differently: one can actually

construct two K-groups for a space, the analytic and the topological K-groups of the space and one map from each one of them to the integers, giving the analytic and the topological index. One then proves that these two K-groups are isomorphic which is the index theorem in this setting. This is the Boum-Douglas reformulation of K-theory, also called K-homology (cf [58] for more details).

Following these lines, given a foliation (V, F) with graph Γ, one can construct the $K_0(C^*(V, F))$ which is exactly the same group constructed above and which will be called the *analytic* K-group of the foliation. One then constructs another group, $K^*_{top}(\Gamma)$, which consists of equivalence classe of *geometric cycles*. (We shall give a general definition which applies to any smooth groupoid, we can get our case of interested by being restricted to the groupoids for the foliations we are intersted in). More specifically:

Definition 1:

Let Γ be a smooth groupoid. Then a Γ-*manifold* is given by:
1. A manifold P with a submersion $a : P \to \Gamma^{(0)}$
2. A right action of Γ on P:$P \times_a \Gamma \to P$, $(p, \gamma) \mapsto p \circ \gamma$ such that $(p \circ \gamma_1) \circ \gamma_2 = p \circ (\gamma_1\gamma_2)$, $\forall(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, where $P \times_a \Gamma = (p, \gamma) \in P \times \Gamma; a(p) = r(\gamma)$.

Such a Γ-manifold provides us with a contravariant functor from the small category Γ to the category of smooth manifolds. Moreover we shall say that a Γ-manifold is *proper* when the following map is proper:

$$P \times_a \Gamma \to P \times P$$

$(p, \gamma) \mapsto (p, p \circ \gamma)$.
One then calls a smooth groupoid *proper* if the map $\Gamma \to \Gamma^{(0)} \times \Gamma^{(0)}$ given by $\gamma \mapsto (r(\gamma), s(\gamma))$ is a proper map.

Now given a Γ-manifold P we get another Γ-manifold $T_\Gamma P$ by replacing each P_x, $x \in \Gamma^{(0)}$, with its tangent bundle TP_x. Thus the total space $P' = T_\Gamma P$ is the space of vectors tangent to P which belong to the kernel of the map a_*, the derivative of a. Hence we have:

Definition 2:

Let Γ be a smooth groupoid. Then a *geometric cycle* for Γ is given by a proper Γ-manifold P and an element $y \in K_0(C^*(T_\Gamma P \rtimes \Gamma))$.

If we devide the space of all geometric cycles by the following equivalence relation, we get the *topological* group $K^*_{top}(\Gamma)$:

$$(P_1, y_1) \sim (P_2, y_2)$$

iff there exists a proper Γ-manifold P and Γ-equivariant smooth maps $f_j : P_j \to P$ such that

$$(Tf_1)!(y_1) = (Tf_2)!(y_2)$$

where T is the tangent map and ! denotes the induced map in the *E-Theory*, namely if we consider f_j as defining a morphism from the functor $x \to P_{j,x}$ to the

functor $x \to P_x$, because of the properness of P_j's, we get induced tangent maps $(Tf_{j,x})!$ in the E-theory groups:

$$(Tf_{j,x})! \in E(TP_{j,x}, TP_x)$$

and correspondingly

$$(Tf_j)! \in E(C^*(T_\Gamma P_j \rtimes \Gamma), C^*(T_\Gamma P \rtimes \Gamma))$$

Let us recall here that Connes' bivariant *E-Theory* was constructed having in mind that the bivariant KK-functor was *not* half exact, although the usual K-functor is. The elements of bivariant E-theory groups are homotopy classes of asymptotic morphisms between the seperable C^*-algebras considered, stabilised by tensoring with the elementary C^*-algebra of compact operators on a seperable infinite dimensional Hilbert space. One then gets an analogue of Kasparov's tensor product which is much easier to handle.

One then has the following

Theorem:

The map

$$\mu : K^*_{top}(\Gamma) \to K_*(C^*_r(\Gamma))$$

is an isomorphism *for a smooth groupoid* Γ *coming from a foliation.*

Moreover, one also has in general

Connes' conjecture:

The map

$$\mu : K^*_{top}(\Gamma) \to K_*(C^*_r(\Gamma))$$

is an isomorphism for *any* smooth groupoid Γ.

Another comment we would like to make is that if we consider the foliation induced by the fibration on the total space of a fiber bundle, then substituting the graph Γ of this foliation into Connes' conjectural formula, we get the "Atiyah-Singer Index theorem for *families* of elliptic operators". Thus Connes' conjecture is a natural generalisation of ordinary index theory.

One open question is if one can use the inverse of the μ map in order to find the geometric cycle corresponding to the module E, living in the analytic K_0-group of our horizontal foliation we are considering. The hope is that maybe this will help us to understand better the topological invariant we get by pairing E with the fundamental class of our horizontal foliation and thus maybe to say something about its possible physical significance. In principle this can be done since the above isomorphism has been proved for foliations. Recall now the comment we made in section 1 of chapter 3 about this spin-0 spin-2 duality as being a case (unique in physics until now as far as we know) where a generalised homology theory appears to be needed. The geometric cycle then in this particular case would be another bundle (cobordant to the original total space since for CW complexes in general, and hence for manifolds in special, we know that all

generalised homology theories coincide) whose strong bundle isomorphisms (i.e. gauge transformations) should correspond to this generalised gauge symmetry. It may be possible then similarly to give a more geometric interpretation of SUSY as well.

6.4 η invariant and global anomalies

6.4.1 η invariant and a KK-Theoretic generalisation

Let us start this section by recalling that in Chapter 4 studying the twisted de Rham complex we saw that the cohomology groups obtained not only do they depend on the gauge equivalence class of the flat connection chosen but they actually depend on the particular flat connection chosen in its gauge equivalence class, namely they also depend on the chosen trivialisation of the bundle. They are *very sensitive* indeed.

Yet we proved as an appication of the Atiyah-Singer index theorem for elliptic complexes that the Euler number of the corresponding elliptic complexes with differentials d_A and $d_{A'}$ was the same for any pair of flat connections A and A', no matter if they are gauge equivalent or not.

Thus, since the index—being a primary invariant—could not distinguish these two compexes and since we knew that the cohomology groups of these two complexes are in general different (even if the connections A and A' are gauge equivalent), we had to look for another invariant which could possibly capture some—if not all—of the difference.

The next invariant that comes to mind is the η *invariant*.

Let D be a self-ajoint operator (we need this condition in order that its spectrum is discrete) on a closed manifold. Then one defines the η function with respect to variable s, denoted $\eta(D, s)$, of the operator D to be

$$\eta(D, s) := \sum_{\lambda_i \neq 0} sign(\lambda_i) |\lambda_i|^{-s}$$

where λ_i are the eigenvalues of the opreator D. One can see immediately that it really measures the *spectral assymetry* of the operator considered.

Then the η *invariant* is by definition the value of the above function at $s = 0$. It will be simply denoted by $\eta(D)$.

What actually turns out is that this invariant depends on the *gauge equivalence class* of the flat connection used to twist the de Rham differential. Hence, although it does not capture all the difference in cohomology groups, it is at least more "sensitive" than the index.

Of course one has to be more careful: how do we know that this sum is finite for instance? One can prove that the η function can be extended to the whole

complex plane and that it is *holomorphic*. The point $s = 0$ is a possible (simple) pole. In order that the η invariant is finite then, one has to prove that this pole is removable, namely $Res(0) = 0$. For the usual operators in Riemannian geometry this is indeed the case. However if one tries to prove that $Res(0) = 0$ for any self-adjoint operator acting on a closed manifold, namely that the η invariant is finite, one has to restrict himself to *odd* dimensional manifolds. In general, all one can prove is that it is a homotopy invariant. That was known from the papers by Atiyah-Patodi-Singer. Private communication with A.Connes indicated to us though that this difficulty can be overcome using type II_∞ factors (von Neumann algebras and measured foliations for operators transversaly elliptic and essentially self-adjoint on foliations). Yet we shall not need this when making connection with global anomalies, because as Witten pointed out, in this case one actually has to use Atiyah's trick to *shift parity*. Moreover, we are actually interested in the de Rham differential (and its "variants" in Riemannian setting), hence it is fine for these simple operators as pointed out above.

One can also introduce modifications of the η invariant. To begin with, one can add the kernel of the operator because by definition the η invariant disregards the zero eigenvalues. This leads to the ξ invariant. We shall be including this term to our η invariant but we shall still keep the notation η. Moreover we shall also consider the *twist* with a flat bundle. In the original Atiyah-Patodi-Singer terminology this is a "local coefficient system". This is a reasonable thing to do, since we are interested in the η invariant of the operator d_A. *Hence from now on, we shall always mean the twisted (*or reduced*) η invariant, including also the zero eigenvalues*. Clearly, the η invariant of d_A is by definition the reduced η invariant of d. One can also prove that the reduced η invariant depends on the *gauge equivalence class* of the flat connection A (or equivalently it depends only on the conjugacy class of the representation $a : \pi_1(M) \to G$, where M is our manifold and G is the structure group of the flat bundle over M we twist with), see for instance [64]. Moreover it still remains a homotopy invariant of the operator we consider, namely it depends only on the homotopy class of its principal symbol. So we can see from the above discussion that the η invariant will give different answer for different flat connections, which are not gauge equivalent, hence it is indeed more sensitive than the index but not as sensitive as the actual cohomology groups.

The above discussion was an analytic definition of the η invariant. But there is deep geometry behind it and in fact the motivation for its discovery was the attempt to find an analogue of the Atiyah-Singer index theorem (which applies to closed manifolds) for *manifolds with boundary*. It turns out, without giving any details here, that if one wants to extend the index theorem for an operator acting on a manifold with boundary, one will have *two* terms: one involving the interior of the manifold which will be the same as if the operator acted on this manifold without its boundary but in addition one also has a boundary term and this term involves the η invariant of the operator considered acting on the boundary. This

deep geometric formula will be used to give the connection with anomalies in physics. Moreover let us also mention that the *analytic torsion* of Ray-Singer, which is the η invariant of the Laplace operator on a closed Riemannian manifold, is closely related to the purely topological *Reidemeister torsion* of a CW complex.

The next question then is how one could possibly compute it by a *topological* formula. This is analogous to the Index theorem formula. One way is to use cobordisms and the Index theorem for manifolds with boundary. But this is rather complicated and one would actually seek for a more direct way, namely without having to consider that the η invariant comes from the boundary contribution of an index problem for manifolds with boundary. There may be other ways to do this as well, but here we shall use ncg. The main result will actually be an application of the pairings we introduced previously between cyclic homology and K-theory (odd case). We shall be very brief though, describing mainly the strategy and not all the constructions because they are quite involved relying on the references for details.

Recall what the problem is: we want to compute the η invariant of the d_A operator acting on a Riemannian manifold M of odd dimension m. Since we are given a flat connection, we also have a flat principal bundle P over M with structure group G. The flat connection A corresponds to a representation a of $\pi_1(M)$ to G. Hence we are given the following data: (d, a, M). To this we associate a flat principal bundle P over M with structure group G. From the physical problem we consider the associated flat vector bundle adP. The operator d is elliptic on M but not self-adjoint. We can get a self-adjoint operator as follows: the operator d_A is a differential in the twisted de Rham complex with modules the forms of M with values in the flat bundle adP. This is actually an elliptic complex as we said in Chapter 4. After assembling the bundle we get an operator $D_A = d_A \oplus d_A^*$ acting from even to odd degree forms. If we consider the same operator acting from the space of all forms to itself, we get a self-adjoint operator. We shall actually call this operator D and not D_A because the twisting is considered since we want to calculate the reduced η invariant. So D is the operator acting on real valued forms on M and it is equal to $D = d \oplus d^*$. Now this operator D has a unique lifting to P, still denoted D. The lifting however is not elliptic on P. The flat bundle P has two natural foliations: the vertical and the horizontal. The lifted operator D on P is leafwise elliptic; specifically it is transversally elliptic with respect to the vertical foliation and longitudinally elliptic with respect to the horizontal foliation of P. However there is a little subtlety here: an important difference between elliptic and leafwise elliptic operators is that the latter may have continum spectrum. In order to overcome this difficulty one uses another operator which has pure point spectrum, still denoted D which is a modification of the original operator D.

Both our foliations are measured: the vertical since the manifold M is compact, hence the metric can give a Borel measure and the horizontal gets a measure by using the Haar measure of the Lie group G.

Since the operator D is transversally elliptic to the vertical foliation, by using the Chern character we can associate to it a class of dimension $m+2$ in the cyclic cohomology of the C^*-algebra of the vertical foliation and we shall call this the *transverse cocycle* denoted by $C_D^{\pitchfork}$. Recall now that the C^*-algebra of the vertical foliation is SME to $C(M)$. We then renormalise this transverse cocycle to get a class in the cyclic cohomology of the algebra $C(P)$ of the same dimension, denoted $TC_D^{\pitchfork}$. The use of the lifting D was actually a tool to construct this class. Then the main theorem in [62] says that the pairing of the renormalised transverse cocycle with elements [u] of $K_1(C(P)) := K^1(P)$ gives the η invariant of the *original* operator D acting on M, namely:

$$\langle TC_D^{\pitchfork}, [u] \rangle = -\eta(D, a)$$

Let us note that similar things can be applied in order to get a longitudinal cyclic cocycle also in the cyclic homology of $C(P)$. One then can prove that, by using suitable approximate units in the renormalisation, the renormalised transverse cocycle equals the longitudinal cocycle. Moreover the pairings of $K_1(C(P))$ classes with the longitudinal cocycle give the measured index of the corresponding Toeplitz operators.

Let us give a generalisation of the η invariant:
For a topological space X, let $\Omega_m^U(X)$ denote the bordism group of oriented cycles $f : M \to X$, where M is a closed manifold of dimension m with an almost complex structure on TM. This determines a $spin_C$ structure on TM and hence a Dirac operator D acting on spinors on M. The stabilisation of $\Omega_m^U(X)$ under vector bundle modification yields the topological K-homology group $K_*^t(M)$ of Baum-Douglas (which is just $K^*(M)$ in Atiyah's K-Theory).

A cycle in $\Omega_m^U(BG')$, where BG' was the homotopy fiber (see section 6.2.1 where we discussed briefly classifying spaces for flat bundles), is equivalent to specifying a flat principal G-bundle P_a over M, where $a : \pi_1(M) \to G$, up to bordism of flat G-bundles. Then one can see that the η invariant (of the twisted Dirac operator) is preserved by stabilisation to the K-homology group $K_1^t(BG')$.

Fix now a cycle P_a in $\Omega_m^U(BG')$ with m odd and consider the horizontal foliation (P_a, F_a) on this bundle and BG' is the homotopy fiber (see section 6.2.1 about classifying spaces of flat bundles). The Dirac operator on M lifts to a leafwise operator along the leaves of the horizontal foliation, thus giving a class

$$[D] \in KK^1(C(P_a), C^*(P_a, F_a))$$

A class [u] in $K^1(G)$ is pulled-back by the trivialisation $\theta : P_a \to M \times G \to U_N$ to give a class

$$[u \circ \theta] \in KK^1(C, C(P_a))$$

Then the *Kasparov external product* defines a pairing

$$[u \circ \theta] \boxtimes [D] \in KK^0(C, C(P_a)) \cong K_0(C^*(P_a, F_a))$$

Then one has:
Definition :
The *KK-η invariant* of the cycle (P_a, D, θ) is the element

$$E(a,u) := [u \circ \theta] \boxtimes [D] \in K_0(C^*(P_a, F_a))$$

This generalisation of the η invariant is due to A. Connes.

6.4.2 η invariant and Quantum Field Theory

Let us now try to explain what is the relevance of the η invariant with physics. In general this has to do with anomalies and with regularised determinants of operators in certain Topological Quantum Field Theories. We shall briefly discuss global anomalies and how they might relate to our generalised gauge invariance.

Anomalies which arise in perturbation can ruin the consistency of quantum field theories. When these perturbative anomalies cancel, this means that the effective action is invariant under the gauge or coordinate tranformations that can be reached continously from the identity. One however must still ask whether the effective action is invariant under gauge or coordinate transformations that are *not continuously connected to the identity.* We shall be referring to gauge or coordinate transformations that cannot be reached continuously from the identity as *global* gauge or coordinate transformations and lack of invariance of the effective action under such transformations will be referred to as a *global anomaly.*

We can either consider coordinate transformations or gauge transformations and study global gravitational or internal anomalies respectively. We shall consider the former case.

If we have an action given on a manifold M, then we want to see whether the effective action is invariant under a parity-conserving diffeomorphism π, not continuously connected to the identity.

A diffeomorphism π on an n-dimensional manifold M can be studied by relating it to a manifold of dimension $n+1$. The most convenient choice is the manifold denoted $(M \times S^1)_\pi$ which is constructed from the "cylinder" $M \times I$, where I is the unit interval, and then gluing together the top and the bottom of $M \times I$ by identifying $(x, 0)$ with $(\pi(x), 1)$ for any $x \in M$. We can see then that if M is a 4-manifold, in order to study its diffeomorphisms, we have to consider an odd dimensional manifold of dimension 5 in this case.

Let us now try to be more specific about our case. Consider the B-theory action in a curved background with metric g. In order to investigate gravitational anomalies, we must make our theory generally covariant. This is done by replacing derivatives with "covariant" derivatives. We apply this technique and thus make our B-theory generally covariant. Under a local diffeomorphism π not continuously connected to the identity, we get a metric $\pi^* g$. Then we consider the family of metrics

$$g_t = (1-t)g + t\pi^* g$$

where t parametrises a circle, and the corresponding family of d_A operators coupled to this family of metrics. We thus have constructed the manifold $(M \times S^1)_\pi$. Then the simplest properties of $(M \times S^1)_\pi$ are invariants of a manifold N which has it for boundary. Hence the index of an operator on N, and as we saw in Chapter 4 the index is related to anomalies in general, since N is a manifold with boundary, will involve the η invariant of the operator considered as a boundary term.

This is all we would actually like to mention here because we do not have something more concrete than this rough relation between the η invariant and physics. This argument also holds for gauge transformations though.

We tried to exhibit that: *the "even" case pairings between K-homology and cyclic homology is related to new invariants for the nlσm whereas the "odd" case is related to the η invariants and global gravitation anomalies.*

6.4.3 Some physical remarks

We would like to present in this subsection some consequences in physics from the above mathematical discussion. This part of our work is mainly *suggestive* and could very well serve as a guide for future research.

Let us start by mentioning again that the invariant we have introduced gives non-trivial results only when *space-time is not simply connected*. This is so because the C^*-algebra associated to the flat foliation is non-commutative only if the first fundamental group of the base space is non-vanishing. We must admit however that the assumption that space-time is not simply connected is somehow beyond standard cosmology. We emphasise here that non-trivial homotopy groups play an important role in the existence of many *topological* objects in quantum field theory, like monopoles, instantons, vortex lines, dyons. However in the above cases what is actually involved is the homotopy (first or higher) group of some *configuration space* or *symmetry group* and *not* of space-time itself! For example, the Bohm-Aharonov effect exists because the configuration space of the electromagnetic potential is not simply connected. In general, non simply connected spaces involve the existence of some *holes*. This is quite fine if one talks about configuration space, it simply means that somewhere the field is not defined or that it is singular. Yet it is not clear to us at all what a *hole* in space-time means.

Having mentioned this objection, let us try to see if there is any role that our invariant might play in physics. Assuming then that space-time is not simply connected, let us see what are the consequences for *instantons*. The reason why we focus on instantons is that by definition a flat connection *is* an instanton since it trivially satisfies both the anti- and self-dual Yang-Mills equations, namely

$$F_A^+ = F_A^- = 0$$

where F_A^+ is the self-dual part of the curvature (field strength) of A and F_A^- is the anti-self dual part.

Let us assume then that we are on an $SU(2)$ bundle over a base space which, to start with, *is* simply connected. Then ordinary instanton theory from the *physical* point of view simply states that (see [69]):

Instantons describe solutions of gauge field equations in which as x_4 (i.e. time) evolves from $-\infty$ to $+\infty$, a vacuum belonging to a homotopy class evolves into another vacuum belonging to another homotopy class.

Let us recall that we are actually dealing with the boundary of our manifold which, assuming space-time to be $\mathbf{R^4}$ compactified, it is S^3 and we assume that our potential is non-zero but it becomes *pure gauge* over the boundary. We refer to [69] for all the relevant details. We would like to recall here that the homotopy group considered *classifies* $SU(2)$ *bundles over* S^4, namely we are talking about $\pi_3(SU(2)) = \pi_3(S^3) = \mathbf{Z}$. Hence different homotopy classes correspond to *topologically distinct* $SU(2)$ bundles over our manifold. For the above argument one has to know that if G is a connected Lie group, then G-bundles over S^n are characterised by $\pi_{n-1}(G)$. The proof uses the *homotopy sequence* for bundles. The link is established via Chern-Simons forms and Stokes' theorem where the key point is that Chern-Simons forms—unlike Chern classes—are *not* gauge invariant (for details see [69]). We mentioned also that the potential is not pure gauge, except the boundary region, hence there is *positive field energy*. The Yang-Mills (classical) vacuum is therefore infinitely degenerate, consisting of an infinite number of homotopically non-equivalent vacua, where each homotopy class represents a different topology for $SU(2)$ bundles over the boundary. The instanton solution represents then *a transition* from one vacuum to another. Physics enters the scene when we ask what is the *amplitude* for this transition. Classically of course it is *zero*, since there is an energy barrier in between two vacua. But because of quantum mechanics there is a *barrier penetration factor*. So if we adopt the semi-classical WKB approximation, in order to find this factor we have to solve Schrodinger's equation where *the potential is the 2nd Chern class c_2 of our bundle*. Let us remind that there are two important "coincidences" here: 1. because of the group being $SU(2)$, $c_2 = Tr(F_A \wedge F_A)$ and 2. because of self-duality, this expresion is also equal to the Yang-Mills Lagrangian (namely the energy, roughly speaking). Then one can compute that the penetration factor is

$$e^{-c_2}$$

The lesson here then is that the potential is of *topological* origin and the energy for this potential is provided by the positive energy density existing because our vector potential is only pure gauge over the boundary and not everywhere.

Having recalled the main points of instanton theory from the physical point of view, let us try to see what happens in our case. So we start with a *flat* $SU(2)$, say, bundle over a base manifold which is is *not* simply connected. One

can however assume that the boundary is the non-simply connected part in this discussion and try to see what are the consequences of this fact for instantons, not assuming flat bundles. We do not have a clear picture for the moment which setting is more appropriate. Anyway, we know the following fact from geometry:

Gauge equivalence classes of flat connections are in 1:1 correspondence with irreducible representations of the first homotopy group to $SU(2)$, which is the structure group in our case.

This fact then clearly means that there are flat connections which are *not* gauge equivalent. Hence we see clearly from the above fact that there is also *another degeneracy* of the classical vacua which is due to the non-trivial homotopy of the manifold considered. We would like to emphasise that this degeneracy is of a different origin than the previous one: *the previous was due to different topologies of bundles, this is due to non trivial homotopy of the base manifold (or its boundary).*

So the picture is that classical vacua are separated into distinct classes consisting of gauge equivalent vacua in each class, but one cannot go via a gauge transformation from one class to another. But for gauge equivalent vacua in one class, the instanton argument holds. The question is what happens if one wants to go from one class to another one, namely if one wants to relate gauge inequivalent vacua. So the *conjecture* here is that: *This invariant actually gives the "barrier penetration factor" between gauge inequivalent vacua*, which of course exist due to non-simply connectedness of the manifold we consider (which is space-time itself or its boundary). The expression, in analogy with ordinary instantons, would be something like

$$e^{-"invariant"}$$

which in this case must be a positive real number. However one could further ask if there is any relation with the *value* of the θ angle. Might it be the case that the invariant determines in which class the fields will fall and thus determine the value of θ? But what would be the rule? Why choose one class over another? This invariant will probably involve the flat connection A itself and not its curvature. Why then A should be such that the value is the one we see?

Similar cases arise in quantum gravity when it is considered as a *topological* quantum field theory [76]. Moreover another topic which may be related to this invariant are the so-called "ALE gravitational instantons" [77].

We shall now try to see if there is any possibility to find a more *topological* formula for this invariant which, recall, was defined purely algebraicly. Discussions with Dr John Roe and Prof. D. G. Quillen indicated that the topological formula for this invariant must involve *characteristic classes for foliations*, see [71], which is a natural generalisation of ordinary characteristic classes' theory. It is also possible that the Chern-Simons forms also play a role. However our invariant is defined for **any** foliation, yet the physical situation we have in mind (namely the nlσm), deals only with *flat* foliations.

Anyway, once started studying characteristic classes for flat foliations we noticed the following striking fact (which at the end may be irrelevant): In order to construct in particular characteristic classes for flat foliations, one uses expressions which are in some— but different than in Chern-Weil theory—sense linear functionals of the flat connection itself and not of its curvature (which by the way is a reasonable thing to start with, since the curvature vanishes). Trying to remember if there are any expressions involving the flat connection itself we thought of the non-linear σ model Lagrangian as formulated by Polyakov, namely

$$TrA^2$$

where A is the flat connection.

If there is anything deeper in this observation, the consequences concerning string theory for instance would be interesting. Yet this is not the end of the story, because *all extended objects' Lagrangian is a σ model Lagrangian*, namely the Lagrangian of a theory describing *d-branes* is a σ model Lagrangian from a $(d+1)$ dimensional manifold to some target space. The related subjects on branes, dualities and M-theory are presently of immense interest in physics.

We would like to end this section with three points: the first is that the σ model Lagrangian as formulated by Polyakov involving flat connections is *not* locally gauge invariant as we pointed out also in Chapter 3, hence if there is any characteristic class for flat foliations related to it must be in some sense *gauge fixed*. From the physical point of view this is perfectly all right since we *did not gauge* the σ model.

The second thing we would like to mention is that as we know from physics, if space-time is not simply connected, *anyons* may exist. This fact may be related somehow with our argument presented here, however a clear picture does not exist for the moment.

The final point is that in the above argument we presented, there are mainly three pieces: the first is the invariant defined in an abstract K-theoretic setting, the second is the theory of characteristic classes for flat foliations and the third is the Lagrangian A^2. There seems to be some kind of relation between the second and the third piece but the relation between the first and the second is only conjectured.

We may mention that the relation between foliations and **coarse geometry**, see for instance [72], is also of interest.

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