Non-linear anisotropic hyperelasticity for granular materials

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Abstract

We describe a non-linear anisotropic hyperelastic model appropriate for geomaterials, deriving the full stress-strain response from strain energy or complementary energy functions. Specific forms of the functions are chosen so that the stiffness and compliance matrices have the appropriate minor symmetries. The model employs two material parameters to describe basic volumetric and shear response, one to express nonlinearity of stiffness as a function of mean stress, and two more (together with the directions of the principal axes of anisotropy) to express the degree of anisotropy. The model is modular, so that non-linearity and anisotropy can be included separately or in combination. For specific parameter settings it reduces to simpler cases such as linear isotropic elasticity. Because the model employs hyperelasticity, thermodynamic acceptability is ensured and all appropriate cross-coupling terms are included between the shear and volumetric behaviour.

1 Introduction

1.1 A note on notation

In common with most work in continuum mechanics, this paper makes extensive use of tensors. In that field, two main forms of notation are current: the bold face (component-free) notation and the index notation. Both have advantages and drawbacks. To facilitate moving between the two forms we give, whenever appropriate, first the component-free form and then in square brackets […] the index form. We restrict ourselves to a Cartesian reference system. Apart from the exception noted below, we adopt the tensile positive convention usual in continuum mechanics, rather than compressive positive usual in geomechanics.

Considering the Cartesian basis $\boldsymbol{e}_i, i = 1\ldots3$ and adopting the summation convention over repeated indices, a second order tensor $\mathbf{a}$ can be written $\mathbf{a} = a_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$. Without ambiguity, in component-free form the tensor can be represented by $\mathbf{a}$ and in index form by a_{ij}. Products between two second order tensors $\mathbf{a}$ and $\mathbf{b}$ are defined by $\mathbf{a}:\mathbf{b} = a_{ij}b_{ij}$ (contraction), $\mathbf{ab} = a_{ij}b_{jk}\boldsymbol{e}_i\boldsymbol{e}_k$ (inner product), $\mathbf{a}\otimes\mathbf{b} = a_{ij}b_{kl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$ (tensor product) and $\mathbf{a}\overline{\underline{\otimes}}\mathbf{b} = \frac{1}{2}\left(a_{ik}b_{jl} + a_{il}b_{jk}\right)\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$ (symmetric product). The trace of a second order tensor is $\mathrm{tr}(\mathbf{a}) = a_{ii}$. The second order identity tensor is written $\mathbf{I} = \delta_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$ with δ_{ij} denoting the Kronecker delta ($\delta_{ij} = 1, i = j;\ \ \delta_{ij} = 0, i \neq j$). A prime is used to denote the deviatoric part of a tensor $\mathbf{a}' = \mathbf{a} - \frac{1}{3}\mathrm{tr}(\mathbf{a})\mathbf{I}$ [$a'_{ij} = a_{ij} - \frac{1}{3}a_{kk}\delta_{ij}$]. The strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ are both symmetric, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{T}}$ [$\varepsilon_{ij} = \varepsilon_{ji}$], $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ [$\sigma_{ij} = \sigma_{ji}$]. Further details of tensor notation, including definitions of invariants and their derivatives, are given in Table 1, and definitions of 4$^{\mathrm{th}}$ order unit tensors, used later in this paper, are given in Table 2. We follow principally the notation set out by Chaves (2013), see also Holzapfel (2000) and Bigoni (2012).

This paper is only concerned with the relationship between the stresses and strains (the constitutive relationship), and not with the definitions used for these variables. However, the stress and strain must each be properly defined tensors, and must form a work-conjugate pair such that the rate of work input per unit volume to a material element is $\dot{W} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = \sigma_{ij}\dot{\varepsilon}_{ij}$. Note though that the physical interpretation of "linearity" depends on the choice made for these variables: for instance a linear relationship between the second Piola-Kirchhoff stress and the Green-Lagrange strain tensor would not imply a linear stress-strain relationship if alternative definitions were used.

The triaxial variables (defined as compressive positive, following the usual convention in geomechanics) are related to the invariants through $p = -\frac{1}{3}\text{tr}(\boldsymbol{\sigma}) = -\frac{1}{3}\sigma_{ii}$ (mean stress), $q = \sqrt{\frac{3}{2}\text{tr}\left(\boldsymbol{\sigma}'^2\right)} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ji}}$ (deviatoric stress), $\varepsilon_v = -\text{tr}(\boldsymbol{\varepsilon}) = -\varepsilon_{ii}$ (volumetric strain) and $\varepsilon_s = \sqrt{\frac{2}{3}\text{tr}\left(\boldsymbol{\varepsilon}'^2\right)} = \sqrt{\frac{2}{3}\varepsilon'_{ij}\varepsilon'_{ji}}$ (deviatoric strain). Where it is appropriate we give in braces {…} the form of relevant expression in triaxial variables (see Table 3).

In the following, all stresses referred to are to be understood as "effective stresses" in soil mechanics terms, that is to say they are the total stress minus the pore fluid pressure. However, we omit the prime notation that is usually employed to indicate effective stress. Instead the prime is used (as is conventional in continuum mechanics) to indicate the deviator of a tensor.

1.2 Soils and elasticity

At very small strains the deformation behaviour of soils can be modelled as "elastic" (*i.e.* fully recoverable), and at larger strains elasticity theory plays a central role as a component of elastic-plastic theories of nonlinear behaviour. However, the elastic behaviour of most soils does not conform to simple linear isotropic elasticity. Two important complications need to be taken into account: anisotropy and nonlinearity.

Anisotropy

Anisotropy arises from the fact that the elastic properties of a soil may be different in different directions. In the theory of elastic crystals this is usually addressed by consideration of a variety of symmetries. For amorphous materials such as soils, however, it is more appropriate to describe anisotropy in a different way, and a widely accepted technique is to use a "fabric tensor" to define the degree of anisotropy. This tensor is usually chosen as a symmetric second order tensor, for instance $\mathbf{a}$ [a_{ij}], and we adopt that approach here. As a result, the model we describe belongs to the class of orthotropic materials (materials with three mutually perpendicular planes of symmetry), although it does not encompass the full generality of orthotropy. For many cases the model will reduce to the class of transverse isotropy (*i.e.* those materials possessing a single, usually vertical, axis of symmetry, also called cross-anisotropic), but again it does not encompass the full generality of such models. Implicit in the approach we describe here are some relationships between the moduli in the general orthotropic case, as discussed by Lodge (1955). In spite of these limitations, however, our approach is probably sufficient to describe the anisotropy of most soils and geomaterials, and it has the advantage that it employs very few parameters.

As the fabric tensor is intended to convey information about anisotropy, it can be normalised so that it excludes purely isotropic information. One way to do this, without loss of generality, is to assign an arbitrary value to the trace of the tensor $\text{tr}(\mathbf{a})$ [a_{ii}], which is an isotropic property of the tensor and therefore conveys no information about directionality. In this case we could choose the trace so that it is the same as the trace of the unit tensor in three dimensions $\text{tr}(\mathbf{a}) = \text{tr}(\mathbf{I}) = 3$ [$a_{ii} = \delta_{ii} = 3$]. Thus the fabric tensor could conveniently be written $\mathbf{a} = \mathbf{I} + \mathbf{a}'$ [$a_{ij} = \delta_{ij} + a'_{ij}$], where

$\mathbf{a}' = \mathbf{a} - \frac{1}{3}\mathrm{tr}(\mathbf{a})\mathbf{I}$ [$a'_{ij} = a_{ij} - \frac{1}{3}a_{kk}\delta_{ij}$] is the deviator of the fabric tensor. Zysset and Curnier (1995) use a variant of this approach in which they define a fabric tensor of the form $g\mathbf{I} + \mathbf{G}$ [$g\delta_{ij} + G_{ij}$] where g is a scalar and $\mathbf{G}$ [G_{ij}] a traceless tensor. However, we consider that the inclusion of the scalar variable g is unnecessary, as clearly it cannot convey any information about the directional features of anisotropy: its function can in effect be absorbed into other isotropic properties.

An alternative, which we pursue below, is to assign an arbitrary value to the determinant of the fabric tensor, and in this case it is convenient to choose $\det(\mathbf{a}) = I_{3a} = \frac{1}{6}\left(2\,\mathrm{tr}\left(\mathbf{a}^3\right) - 3\,\mathrm{tr}\left(\mathbf{a}^2\right)\mathrm{tr}(\mathbf{a}) + \mathrm{tr}^3(\mathbf{a})\right) = \det(\mathbf{I}) = 1$ [$\det(a_{ij}) = I_{3a} = \frac{1}{6}\left(2a_{ij}a_{jk}a_{ki} - 3a_{ij}a_{ji}a_{kk} + a_{ii}a_{jj}a_{kk}\right) = \det(\delta_{ij}) = 1$]. This approach was taken by Lodge (1955).

Bigoni and Loret (1999) choose a third option, constraining the anisotropy tensor (which they call $\mathbf{B}$ [B_{ij}]) by $\mathrm{tr}\left(\mathbf{B}^2\right) = \mathrm{tr}\left(\mathbf{I}^2\right) = 3$ [$B_{ij}B_{ji} = \delta_{ij}\delta_{ji} = 3$]. We emphasise though that the choice of which constraint to apply to the value of the anisotropy tensor is arbitrary, and simply governed by analytical convenience. For instance, Amorosi *et al*. (2018) pursue the option $\mathrm{tr}\left(\mathbf{a}^2\right) = 3$.

We are not concerned here how the fabric tensor might arise, and in particular whether it represents inherent anisotropy (fundamental to the structure of the material) or induced anisotropy (which can evolve as a result of the history of inelastic deformation). In the latter case some evolution law would need to be defined to link changes of the fabric tensor to the evolving history, but that too is not our concern here. We are concerned with how the degree of anisotropy can be taken into account rigorously within elasticity theory, once any degree of anisotropy is known.

Nonlinearity

Nonlinearity (at small strain) arises usually from the dependence of both the bulk and shear moduli on the mean compressive stress (pressure), most usually modelled by expressing the moduli as power functions of the pressure. This phenomenon should not be confused with the nonlinearity of the secant shear stiffness $G_s = \tau/\gamma$ when expressed as a function of the shear strain amplitude, giving rise to the well-known S-shaped curve in a G_s *v.* $\log\gamma$ plot. The latter arises entirely due to dissipative (elastic-plastic) processes, and is not due to the underlying elastic properties of the soil. The use of the variable G_s to quantify this strain-dependent "stiffness" is somewhat misleading, as it does not represent truly "elastic" properties of the material.

Given the above observations, it is surprising that no widely adopted procedures exist for addressing the combined anisotropy and nonlinear elasticity of soils. Gajo and Bigoni (2008) present one of the few attempts to combine nonlinearity and anisotropy. Unlike our approach below, they allow for separate power functions to be applied to the volumetric and deviatoric components, but we are unaware of data that supports this necessity, and their approach has the disadvantage that the inverse of the stiffness matrix (*i.e.* the compliance matrix) cannot be written explicitly. Aspects of the experimental data that support different models are discussed by Amorosi *et al.* (2019).

Our purpose here is to set out a new approach, to be used either stand-alone to describe the elasticity of soils at very small strains, or as the elastic component of a more complex elastic-plastic theory. Our starting point is isotropic linear elasticity, which we first extend independently to the nonlinear case and to anisotropy, and then deal with the combined case. Unlike our previous work on nonlinearity (Houlsby, Amorosi and Rojas, 2005), where we began by considering simplified stress states and then extended the analysis to general stresses, we shall start here by stating the general case and then simplify this to some special cases, notably the triaxial test. Much of this paper is devoted to setting out the defining equations in some detail. Some implications of the new model and comparisons with experimental data are discussed by Amorosi *et al.* (2019), where we also compare and contrast our model with theoretical models developed by other authors, for anisotropy, for nonlinearity and for the combination of the two phenomena.

2 Linear isotropic elasticity

We restate well-known results in this Section merely to set out our terminology and methodology.

Elasticity theories must be consistent with the laws of thermodynamics, and it is widely accepted that this can only be guaranteed if the material can be described as hyperelastic; that is to say that the stresses $\boldsymbol{\sigma}$ [σ_{ij}] {p, q} are the differential of an elastic strain energy $E(\boldsymbol{\varepsilon})$ [$E(\varepsilon_{ij})$] {$E(\varepsilon_v, \varepsilon_s)$} with respect to the strains: $\boldsymbol{\sigma} = \frac{\partial E}{\partial \boldsymbol{\varepsilon}}$ [$\sigma_{ij} = \frac{\partial E}{\partial \varepsilon_{ij}}$] {$p = \frac{\partial E}{\partial \varepsilon_v}$, $q = \frac{\partial E}{\partial \varepsilon_s}$}. An equivalent statement is that the strains are the differential of the complementary energy $C(\boldsymbol{\sigma})$ [$C(\sigma_{ij})$] {$C(p,q)$} with respect to the stresses: $\boldsymbol{\varepsilon} = \frac{\partial C}{\partial \boldsymbol{\sigma}}$ [$\varepsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}}$] {$\varepsilon_v = \frac{\partial C}{\partial p}$, $\varepsilon_s = \frac{\partial C}{\partial q}$}, where the strain energy and complementary energy are related through the Legendre transform $E + C = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ [$E + C = \sigma_{ij}\varepsilon_{ij}$] {$E + C = p\varepsilon_v + q\varepsilon_s$}. When considering isothermal problems, the strain energy can be identified with the Helmholtz free energy, $E = f$, and the complementary energy with the (negative) Gibbs free energy, $C = -g$, see Houlsby and Puzrin (2006).

The stiffness matrix is then obtained by further differentiation, writing:

$$\left.\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \frac{\partial^2 E}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \mathbb{D} : \dot{\boldsymbol{\varepsilon}} \\
\dot{\sigma}_{ij} &= \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} = \frac{\partial^2 E}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} \dot{\varepsilon}_{kl} = D_{ijkl} \dot{\varepsilon}_{kl}
\end{aligned}\right\} \quad \ldots(1)$$

Or equivalently the compliance matrix is defined through:

$$\left.\begin{aligned}
\dot{\boldsymbol{\varepsilon}} &= \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial^2 C}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \mathbb{C} : \dot{\boldsymbol{\sigma}} \\
\dot{\varepsilon}_{ij} &= \frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} \dot{\sigma}_{kl} = \frac{\partial^2 C}{\partial \sigma_{kl} \sigma_{ij}} \dot{\sigma}_{kl} = C_{ijkl} \dot{\sigma}_{kl}
\end{aligned}\right\} \quad \ldots(2)$$

When written in the form above, the stiffness (or compliance) matrix has $3 \times 3 \times 3 \times 3 = 81$ components. However, because of the symmetry of the stress and strain tensors, each has just 6 independent components, and the relationship between their increments can therefore be written in the form of a 6×6 matrix with just 36 components. There is therefore an ambiguity in the way the stiffness matrix can be expressed: multiple choices of the form of $\mathbb{D}$ [D_{ijkl}] can result in the same incremental stress-strain response. This ambiguity is almost universally resolved by requiring $\mathbb{D}$ [D_{ijkl}] to possess the "minor symmetries" $D_{jikl} = D_{ijkl}$ and $D_{ijlk} = D_{ijkl}$ as well as the "major symmetry" $D_{klij} = D_{ijkl}$ which arises from the existence of a strain energy potential and the independence of the order of differentiation of E. We adopt this canonical form of $\mathbb{D}$ $[D_{ijkl}]$, and when all the symmetries are applied the number of independent components reduces to 21.

Because $\mathbb{D}$ $[D_{ijkl}]$ is obtained by differentiation of E, the choice of the canonical form for the stiffness matrix imposes a corresponding requirement on the functional form of E, which we address in the next section.

2.1 Strain energy form

The strain energy for a linear isotropic elastic material is often written as:

$$\begin{aligned} E &= \frac{\lambda}{2}\mathrm{tr}^2(\boldsymbol{\varepsilon}) + \mu\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^2\right) = \frac{K}{2}\mathrm{tr}^2(\boldsymbol{\varepsilon}) + G\,\mathrm{tr}\left(\boldsymbol{\varepsilon}'^2\right) \\ &= \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ji} = \frac{K}{2}\varepsilon_{ii}\varepsilon_{jj} + G\varepsilon'_{ij}\varepsilon'_{ji} \end{aligned} \qquad \text{...(3)}$$

Where K is the bulk modulus, G the shear modulus, $\lambda = K - \frac{2}{3}G$ and $\mu = G$ are Lamé's parameters.

However, for the reasons discussed above in relation to the symmetries of the stiffness matrix, we define $\boldsymbol{\varepsilon}^{\mathrm{sym}} = \frac{1}{2}\left(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{\mathrm{T}}\right)$ $[\varepsilon_{ij}^{\mathrm{sym}} = \frac{1}{2}\left(\varepsilon_{ij} + \varepsilon_{ji}\right)$ $]$ and then write the strain energy in the form:

$$\begin{aligned} E &= \frac{\lambda}{2}\mathrm{tr}^2(\boldsymbol{\varepsilon}) + \mu\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^{\mathrm{sym}^2}\right) = \frac{K}{2}\mathrm{tr}^2(\boldsymbol{\varepsilon}) + G\,\mathrm{tr}\left(\boldsymbol{\varepsilon}'^{\mathrm{sym}^2}\right) \\ &= \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{jj} + \mu\left(\varepsilon_{ij}^{\mathrm{sym}}\varepsilon_{ji}^{\mathrm{sym}}\right) = \frac{K}{2}\varepsilon_{ii}\varepsilon_{jj} + G\varepsilon_{ij}'^{\mathrm{sym}}\varepsilon_{ji}'^{\mathrm{sym}} \end{aligned} \qquad \text{...(4)}$$

The above rewriting may seem to be unnecessary pedantry, because the symmetry of the strain tensor means that the rewriting does not change the numerical value of the strain energy. However, when twice differentiated with respect to strain, the alternative form results in different entries in the $\mathbb{D}$ $[D_{ijkl}]$ matrix. We write the strain energy in this form for purely formal purposes so that when twice differentiated it gives the required canonical form of the stiffness matrix, respecting the minor symmetries. We note that these niceties can be avoided by use of the Voigt notation in which stress and strain are represented by 6×1 vectors rather than 3×3 tensors.

Later, to facilitate the development of the non-linear form we introduce a reference pressure p_r, and dimensionless bulk and shear stiffness coefficients k and g, and we write:

$$\begin{aligned} E &= \frac{p_r}{2}\left(k\,\mathrm{tr}^2(\boldsymbol{\varepsilon}) + 2g\,\mathrm{tr}\left(\boldsymbol{\varepsilon}'^{\mathrm{sym}^2}\right)\right) \\ &= \frac{p_r}{2}\left(k\varepsilon_{ii}\varepsilon_{jj} + 2g\varepsilon_{ij}'^{\mathrm{sym}}\varepsilon_{ji}'^{\mathrm{sym}}\right) \end{aligned} \qquad \text{...(5)}$$

In triaxial variables we can also write:

$$E = \frac{K}{2}\varepsilon_v^2 + \frac{3G}{2}\varepsilon_s^2 = \frac{p_r}{2}\left(k\varepsilon_v^2 + 3g\varepsilon_s^2\right) \qquad \text{...(6)}$$

Note that Poisson's ratio is given by $\nu = \frac{3k-2g}{6k+2g}$, or alternatively $\frac{g}{k} = \frac{3(1-2\nu)}{2(1+\nu)}$. For a typical Poisson's ratio of $\nu \approx 0.2$ for a granular material, the latter yields $\frac{g}{k} \approx 0.75$.

Noting the results $\frac{\partial \mathrm{tr}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \mathbf{I}$ [$\frac{\partial \varepsilon_{ii}}{\partial \varepsilon_{kl}} = \delta_{kl}$] and $\frac{\partial \mathrm{tr}\left(\boldsymbol{\varepsilon}^{\mathrm{sym}^2}\right)}{\partial \boldsymbol{\varepsilon}} = 2\boldsymbol{\varepsilon}^{\mathrm{sym}}$ [$\frac{\partial\left(\varepsilon_{ij}^{\mathrm{sym}}\varepsilon_{ji}^{\mathrm{sym}}\right)}{\partial \varepsilon_{kl}} = 2\varepsilon_{lk}^{\mathrm{sym}} = 2\varepsilon_{kl}^{\mathrm{sym}}$], we immediately obtain by differentiation:

$$\left.\begin{aligned}
&\boldsymbol{\sigma} = \frac{\partial E}{\partial \boldsymbol{\varepsilon}} = \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\left(\boldsymbol{\varepsilon}^{\mathrm{sym}}\right) = K\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2G\left(\boldsymbol{\varepsilon}'^{\mathrm{sym}}\right) \\
&\sigma_{ij} = \frac{\partial E}{\partial \varepsilon_{ij}} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}'^{\mathrm{sym}} = K\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}'^{\mathrm{sym}} \\
&p = \frac{\partial E}{\partial \varepsilon_v} = K\varepsilon_v, \quad q = \frac{\partial E}{\partial \varepsilon_s} = 3G\varepsilon_s
\end{aligned}\right\} \quad \text{...(7)}$$

so that $\mathrm{tr}(\boldsymbol{\sigma}) = 3K\,\mathrm{tr}(\boldsymbol{\varepsilon})$ [$\sigma_{ii} = 3K\varepsilon_{ii}$] {$p = K\varepsilon_v$} and $\boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon}'^{\mathrm{sym}}$ [$\sigma_{ij}' = 2G\varepsilon_{ij}'^{\mathrm{sym}}$] {$q = 3G\varepsilon_s$}.

We can also immediately write the incremental forms:

$$\dot{\boldsymbol{\sigma}} = K\,\mathrm{tr}(\dot{\boldsymbol{\varepsilon}})\mathbf{I} + 2G\dot{\boldsymbol{\varepsilon}}'^{\mathrm{sym}} = \left(K - \frac{2}{3}G\right)\mathrm{tr}(\dot{\boldsymbol{\varepsilon}})\mathbf{I} + 2G\dot{\boldsymbol{\varepsilon}}^{\mathrm{sym}} = \left(\left(K - \frac{2}{3}G\right)\mathbf{I}\otimes\mathbf{I} + 2G\mathbf{I}\underline{\overline{\otimes}}\mathbf{I}\right) : \dot{\boldsymbol{\varepsilon}}$$

$$= \left(K\overline{\overline{\mathbb{I}}} + 2G\mathbb{P}^{\mathrm{sym}}\right) : \dot{\boldsymbol{\varepsilon}} = \mathbb{D} : \dot{\boldsymbol{\varepsilon}}$$

$$\dot{\sigma}_{ij} = K\dot{\varepsilon}_{kk}\delta_{ij} + 2G\dot{\varepsilon}_{ij}'^{\mathrm{sym}} = \left(K\delta_{ij}\delta_{kl} + 2G\left(\tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \tfrac{1}{3}\delta_{ij}\delta_{kl}\right)\right)\dot{\varepsilon}_{kl} = D_{ijkl}\dot{\varepsilon}_{kl}$$

$$\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}K & 0\\ 0 & 3G\end{bmatrix}\begin{bmatrix}\dot{\varepsilon}_v\\ \dot{\varepsilon}_s\end{bmatrix} \quad \text{...(8)}$$

where $\mathbb{D}$ [D_{ijkl}] is the incremental stiffness matrix, a fourth order tensor (see Table 2 for definitions of 4$^{\mathrm{th}}$ order unit tensors and projection tensors).

2.2 Complementary energy form

Alternatively one may write the complementary energy variously as:

$$C = \frac{-\lambda\,\mathrm{tr}^2(\boldsymbol{\sigma})}{4\mu(3\lambda + 2\mu)} + \frac{\mathrm{tr}\left(\boldsymbol{\sigma}^{\mathrm{sym}^2}\right)}{4\mu} = \frac{\mathrm{tr}^2(\boldsymbol{\sigma})}{18K} + \frac{\mathrm{tr}\left(\boldsymbol{\sigma}'^{\mathrm{sym}^2}\right)}{4G} = \frac{1}{2p_r}\left(\frac{\mathrm{tr}^2(\boldsymbol{\sigma})}{9k} + \frac{\mathrm{tr}\left(\boldsymbol{\sigma}'^{\mathrm{sym}^2}\right)}{2g}\right)$$

$$= \left(\frac{1}{K} - \frac{3}{2G}\right)\frac{\sigma_{ii}\sigma_{jj}}{18} + \frac{\sigma_{ij}^{\mathrm{sym}}\sigma_{ji}^{\mathrm{sym}}}{4G} = \frac{\sigma_{ii}\sigma_{jj}}{18K} + \frac{\sigma_{ij}'^{\mathrm{sym}}\sigma_{ji}'^{\mathrm{sym}}}{4G} = \frac{1}{2p_r}\left(\frac{\sigma_{ii}\sigma_{jj}}{9k} + \frac{\sigma_{ij}'^{\mathrm{sym}}\sigma_{ji}'^{\mathrm{sym}}}{2g}\right) \quad \text{...(9)}$$

$$= \frac{p^2}{2K} + \frac{q^2}{6G} = \frac{1}{2p_r}\left(\frac{p^2}{k} + \frac{q^2}{3g}\right)$$

Because of the rather complex form that C takes in terms of the Lamé parameters, we prefer the form using the bulk and shear moduli and we do not pursue the use of Lamé parameters further. We can immediately obtain:

$$\left.\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{\partial C}{\partial \boldsymbol{\sigma}} = \frac{1}{9K}\operatorname{tr}(\boldsymbol{\sigma})\mathbf{I} + \frac{1}{2G}\boldsymbol{\sigma}'^{\text{sym}} \\
\varepsilon_{ij} &= \frac{\partial C}{\partial \sigma_{ij}} = \frac{1}{9K}\sigma_{kk}\delta_{ij} + \frac{1}{2G}\sigma_{ij}'^{\text{sym}} \\
\varepsilon_v &= \frac{\partial C}{\partial p} = \frac{p}{K}, \quad \varepsilon_s = \frac{\partial C}{\partial q} = \frac{q}{3G}
\end{aligned}\right\} \quad \text{...(10)}$$

so that $\operatorname{tr}(\boldsymbol{\varepsilon}) = \frac{1}{3K}\operatorname{tr}(\boldsymbol{\sigma})$ [$\varepsilon_{ii} = \frac{1}{3K}\sigma_{ii}$] {$\varepsilon_v = p/K$} and $\boldsymbol{\varepsilon}' = \frac{1}{2G}\boldsymbol{\sigma}'^{\text{sym}}$ [$\varepsilon_{ij}' = \frac{1}{2G}\sigma_{ij}'^{\text{sym}}$] { $\varepsilon_s = q/3G$ }.

The incremental forms follow in a similar way to Eq. 8, leading for instance to $\dot{\boldsymbol{\varepsilon}} = \left(\frac{1}{9K}\overline{\overline{\mathbb{I}}} + \frac{1}{2G}\mathbb{P}^{sym}\right) : \dot{\boldsymbol{\sigma}} = \mathbb{C} : \dot{\boldsymbol{\sigma}}$.

3 Isotropic nonlinear elasticity

In the following we only consider compressive stress states, *i.e.* those in which the mean effective stress or pressure $p = -\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma}) = -\frac{1}{3}\sigma_{ii}$ is positive.

3.1 Complementary energy form

Houlsby *et al.* (2005) describe a simple extension of linear isotropic elasticity to the case of non-linear (pressure-dependent moduli). Their model involves raising the complementary energy to an appropriate power. Here we implement this by first defining p_o as the positive root of:

$$\begin{aligned}
p_o^2 &= \frac{\operatorname{tr}^2(\boldsymbol{\sigma})}{9} + \frac{k(1-n)}{2g}\operatorname{tr}\left(\boldsymbol{\sigma}'^{\text{sym}^2}\right) \\
&= \frac{\sigma_{ii}\sigma_{jj}}{9} + \frac{k(1-n)}{2g}\sigma_{ij}'^{\text{sym}}\sigma_{ji}'^{\text{sym}} \\
&= p^2 + \frac{k(1-n)}{3g}q^2
\end{aligned} \quad \text{...(11)}$$

Note that either (a) on the isotropic stress axis $\boldsymbol{\sigma}' = 0$ [$\sigma_{ij}' = 0$] {$q = 0$}, or (b) for $n = 1$ it follows that $p_o = -\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma}) = -\frac{1}{3}\sigma_{kk} = p$.

It follows that:

$$\left.\begin{aligned}
p_o\frac{\partial p_o}{\partial \boldsymbol{\sigma}} &= \tfrac{1}{9}\operatorname{tr}(\boldsymbol{\sigma})\mathbf{I} + \frac{k(1-n)}{2g}\boldsymbol{\sigma}'^{\text{sym}} \\
p_o\frac{\partial p_o}{\partial \sigma_{ij}} &= \tfrac{1}{9}\sigma_{kk}\delta_{ij} + \frac{k(1-n)}{2g}\sigma_{ij}'^{\text{sym}} \\
p_o\frac{\partial p_o}{\partial p} &= p, \quad p_o\frac{\partial p_o}{\partial q} = \frac{k(1-n)}{3g}q
\end{aligned}\right\} \quad \text{...(12)}$$

We then write the complementary energy for $0 \le n < 1$ as:

$$C = p_r \frac{1}{k(1-n)(2-n)} \left(\frac{p_o}{p_r} \right)^{2-n} \qquad \text{...(13)}$$

In the above expressions the factors $(1-n)$ and $(2-n)$ appear so that, on the isotropic axis, the incremental bulk and shear stiffnesses take the simple forms:

$$\frac{K}{p_r} = k\left(\frac{p}{p_r} \right)^n \quad \text{and} \quad \frac{G}{p_r} = g\left(\frac{p}{p_r} \right)^n \qquad \text{...(14)}$$

However, note that for stress states not on the isotropic axis, the incremental response involves a form of "stress induced anisotropy" and cannot simply be expressed through bulk and shear moduli. It is straightforward to show that the linear case is recovered by setting $n = 0$.

The above expression results in the reference point for zero strain being, as is conventional, at zero stress. Unfortunately this requirement means that the above expression cannot be extended to the important limiting case $n = 1$, which represents moduli proportional to pressure, as this results in strains that are infinite at zero stress. If, however, the reference point for zero strain is moved to the point $\boldsymbol{\sigma} = -p_r\mathbf{I}$ [$\sigma_{ij} = -p_r\delta_{ij}$] {$p = p_r, q = 0$} on the isotropic axis, then the limiting case can be considered. This shift of datum point is achieved by setting the switch parameter $N = 1$ in the following modified expression for the complementary energy for $0 \le n < 1$:

$$C = N\left(\frac{p_r}{k(2-n)} - \frac{p}{k(1-n)} \right) + \frac{p_r}{k(1-n)(2-n)} \left(\frac{p_o}{p_r} \right)^{2-n} \qquad \text{...(15)}$$

The previous case is recovered by setting $N = 0$. Note that the change of origin for strain has no effect on the incremental stiffness values. Equation 15 is still not valid for $n = 1$, and this special case is dealt with in the Section 3.2 below.

For $0 \le n < 1$ the strains are derived as follows:

$$\boldsymbol{\varepsilon} = \frac{\partial C}{\partial \boldsymbol{\sigma}} = \frac{N\mathbf{I}}{3k(1-n)} + \frac{\partial C}{\partial p_o}\frac{\partial p_o}{\partial \boldsymbol{\sigma}} = \frac{N\mathbf{I}}{3k(1-n)} + \frac{1}{k(1-n)}\left(\frac{p_o}{p_r} \right)^{1-n} \frac{\partial p_o}{\partial \boldsymbol{\sigma}}$$
$$= \frac{N\mathbf{I}}{3k(1-n)} + \frac{1}{p_r}\left(\frac{p_o}{p_r} \right)^{-n} \left[\frac{\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}}{9k(1-n)} + \frac{\boldsymbol{\sigma}'^{\mathrm{sym}}}{2g} \right]$$

$$\varepsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}} = \frac{N\delta_{ij}}{3k(1-n)} + \frac{\partial C}{\partial p_o}\frac{\partial p_o}{\partial \sigma_{ij}} = \frac{N\delta_{ij}}{3k(1-n)} + \frac{1}{k(1-n)}\left(\frac{p_o}{p_r} \right)^{1-n} \frac{\partial p_o}{\partial \sigma_{ij}}$$
$$= \frac{N\delta_{ij}}{3k(1-n)} + \frac{1}{p_r}\left(\frac{p_o}{p_r} \right)^{-n} \left[\frac{\sigma_{kk}\delta_{ij}}{9k(1-n)} + \frac{\sigma_{ij}'^{\mathrm{sym}}}{2g} \right] \qquad \text{...(16)}$$

From which follows:

$$\left.\begin{aligned}
\mathrm{tr}(\boldsymbol{\varepsilon}) &= \frac{1}{k(1-n)}\left(N+\frac{\mathrm{tr}(\boldsymbol{\sigma})}{3p_r}\left(\frac{p_o}{p_r}\right)^{-n}\right)\\
\varepsilon_{kk} &= \frac{1}{k(1-n)}\left(N+\frac{\sigma_{kk}}{3p_r}\left(\frac{p_o}{p_r}\right)^{-n}\right)\\
\varepsilon_v &= \frac{1}{k(1-n)}\left(\frac{p}{p_r}\left(\frac{p_o}{p_r}\right)^{-n}-N\right)
\end{aligned}\right\} \quad ...(17)$$

$$\left.\begin{aligned}
\boldsymbol{\varepsilon}' &= \frac{1}{2gp_r}\left(\frac{p_o}{p_r}\right)^{-n}\boldsymbol{\sigma}'^{\mathrm{sym}}\\
\varepsilon'_{ij} &= \frac{1}{2gp_r}\left(\frac{p_o}{p_r}\right)^{-n}\sigma'^{\mathrm{sym}}_{ij}\\
\varepsilon_s &= \frac{1}{3gp_r}\left(\frac{p_o}{p_r}\right)^{-n}q
\end{aligned}\right\} \quad ...(18)$$

Further differentiation leads to:

$$\left.\begin{aligned}
\dot{\boldsymbol{\varepsilon}} &= \frac{\partial\boldsymbol{\varepsilon}}{\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}} = \frac{\partial^2 C}{\partial\boldsymbol{\sigma}\otimes\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}\\
&= \frac{1}{p_r}\left(\frac{p_o}{p_r}\right)^{-n}\left\{\begin{array}{l}\dfrac{\overline{\overline{\mathbb{I}}}}{9k(1-n)}+\dfrac{\mathbb{P}^{sym}}{2g}-\\ \dfrac{k(1-n)n}{p_o^2}\left[\dfrac{\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}}{9k(1-n)}+\dfrac{\boldsymbol{\sigma}'^{\mathrm{sym}}}{2g}\right]\otimes\left[\dfrac{\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}}{9k(1-n)}+\dfrac{\boldsymbol{\sigma}'^{\mathrm{sym}}}{2g}\right]\end{array}\right\}:\dot{\boldsymbol{\sigma}} = \mathbb{C}:\dot{\boldsymbol{\sigma}}\\
\dot{\varepsilon}_{ij} &= \frac{\partial\varepsilon_{ij}}{\partial\sigma_{kl}}\dot{\sigma}_{kl} = \frac{\partial^2 C}{\partial\sigma_{kl}\sigma_{ij}}\dot{\sigma}_{kl}\\
&= \frac{1}{p_r}\left(\frac{p_o}{p_r}\right)^{-n}\left\{\begin{array}{l}\left[\dfrac{\delta_{ij}\delta_{kl}}{9k(1-n)}+\dfrac{1}{2g}\left(\tfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\tfrac{1}{3}\delta_{ij}\delta_{kl}\right)\right]-\\ \dfrac{k(1-n)n}{p_o^2}\left[\dfrac{\sigma_{mm}\delta_{ij}}{9k(1-n)}+\dfrac{\sigma'^{\mathrm{sym}}_{ij}}{2g}\right]\left[\dfrac{\sigma_{nn}\delta_{kl}}{9k(1-n)}+\dfrac{\sigma'^{\mathrm{sym}}_{kl}}{2g}\right]\end{array}\right\}\dot{\sigma}_{kl} = C_{ijkl}\dot{\sigma}_{kl}\\
\begin{bmatrix}\dot{\varepsilon}_v\\ \dot{\varepsilon}_s\end{bmatrix} &= \frac{1}{p_r}\left(\frac{p_o}{p_r}\right)^{-n}\begin{bmatrix}\dfrac{3gp^2+kq^2}{3gkp_o^2} & \dfrac{-nqp}{3gp_o^2}\\ \dfrac{-nqp}{3gp_o^2} & \dfrac{1}{3gp_o^2}\left(p^2+\dfrac{k}{3g}(1-n)^2q^2\right)\end{bmatrix}\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix}
\end{aligned}\right\} \quad ...(19)$$

Exploiting (from Eq. 11) the fact that $\frac{1}{(1-n)}\left(1-\frac{n\,\mathrm{tr}^2(\boldsymbol{\sigma})}{9p_o^2}\right)=\frac{1}{p_o^2}\left(\frac{\mathrm{tr}^2(\boldsymbol{\sigma})}{9}+\frac{k}{2g}\mathrm{tr}\left(\boldsymbol{\sigma}'^{\mathrm{sym}^2}\right)\right)$, we can rewrite the compliance as:

$$\mathbb{C}=\frac{1}{p_r}\left(\frac{p_o}{p_r}\right)^{-n}\left\{\begin{array}{l}\left(\frac{\mathrm{tr}^2(\boldsymbol{\sigma})}{9k}+\frac{\mathrm{tr}\left(\boldsymbol{\sigma}'^{\mathrm{sym}^2}\right)}{2g}\right)\frac{\overline{\overline{\mathbb{I}}}}{9p_o^2}+\frac{\mathbb{P}^{sym}}{2g}\\-\frac{n}{p_o^2}\frac{\mathrm{tr}(\boldsymbol{\sigma})}{18g}\left[\boldsymbol{\sigma}'^{\mathrm{sym}}\otimes\mathbf{I}+\mathbf{I}\otimes\boldsymbol{\sigma}'^{\mathrm{sym}}\right]-\frac{k(1-n)n}{4g^2p_o^2}\boldsymbol{\sigma}'^{\mathrm{sym}}\otimes\boldsymbol{\sigma}'^{\mathrm{sym}}\end{array}\right\}$$

$$C_{ijkl}=\frac{1}{p_r}\left(\frac{p_o}{p_r}\right)^{-n}\left\{\begin{array}{l}\left(\frac{\sigma_{mm}\sigma_{nn}}{9k}+\frac{\sigma_{mn}'^{\mathrm{sym}}\sigma_{nm}'^{\mathrm{sym}}}{2g}\right)\frac{\delta_{ij}\delta_{kl}}{9p_o^2}+\frac{1}{2g}\left(\tfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\tfrac{1}{3}\delta_{ij}\delta_{kl}\right)\\-\frac{n}{p_o^2}\frac{\sigma_{mm}}{18g}\left[\sigma_{ij}'^{\mathrm{sym}}\delta_{kl}+\delta_{ij}\sigma_{kl}'^{\mathrm{sym}}\right]-\frac{k(1-n)n}{4g^2p_o^2}\sigma_{ij}'^{\mathrm{sym}}\sigma_{kl}'^{\mathrm{sym}}\end{array}\right\} \quad ...(20)$$

It may readily be confirmed that the above expression respects the minor symmetries of the compliance matrix.

Specifically, on the isotropic axis (and only on this axis), $\boldsymbol{\sigma}'=0\ [\sigma_{ij}'=0]\ \{q=0\}$, for which $p_o=p$, and the incremental form reduces to:

$$\dot{\boldsymbol{\varepsilon}}=\frac{1}{p_r}\left(\frac{p_o}{p_r}\right)^{-n}\left[\frac{\overline{\overline{\mathbb{I}}}}{9k}+\frac{\mathbb{P}^{sym}}{2g}\right]\dot{\boldsymbol{\sigma}}$$

$$\dot{\varepsilon}_{ij}=\frac{1}{p_r}\left(\frac{p}{p_r}\right)^{-n}\left\{\frac{\delta_{ij}\delta_{kl}}{9k}+\frac{1}{2g}\left(\tfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\tfrac{1}{3}\delta_{ij}\delta_{kl}\right)\right\}\dot{\sigma}_{kl} \quad ...(21)$$

$$\left.\begin{array}{l}\mathrm{tr}(\dot{\boldsymbol{\varepsilon}})=\frac{1}{3kp_r}\left(\frac{p}{p_r}\right)^{-n}\mathrm{tr}(\dot{\boldsymbol{\sigma}})\\ \dot{\varepsilon}_{kk}=\frac{1}{3kp_r}\left(\frac{p}{p_r}\right)^{-n}\dot{\sigma}_{kk}\\ \dot{\varepsilon}_{v}=\frac{1}{kp_r}\left(\frac{p}{p_r}\right)^{-n}\dot{p}\end{array}\right\} \quad ...(22)$$

$$\left.\begin{aligned}
\dot{\boldsymbol{\varepsilon}}' &= \frac{1}{2gp_r}\left(\frac{p}{p_r}\right)^{-n}\dot{\boldsymbol{\sigma}}'^{\text{sym}} \\
\dot{\varepsilon}'_{ij} &= \frac{1}{2gp_r}\left(\frac{p}{p_r}\right)^{-n}\dot{\sigma}'^{\text{sym}}_{ij} \\
\dot{\varepsilon}_s &= \frac{1}{3gp_r}\left(\frac{p}{p_r}\right)^{-n}\dot{q}
\end{aligned}\right\} \qquad \text{...(23)}$$

Showing that on the isotropic axis only the incremental bulk and shear moduli take the form as given in Eqs. 14.

3.2 Complementary energy form: special case *n* = 1

It can be shown that in the limit for $\text{tr}(\boldsymbol{\sigma})<0$ $[\sigma_{ii}<0]$ $\{p>0\}$ and $N=1$, as $n \Rightarrow 1$, Eq. 15 approaches asymptotically the expression:

$$\begin{aligned}
C &= \frac{p_r}{k} - \frac{\text{tr}(\boldsymbol{\sigma})}{3k}\left[\log\left(\frac{-\text{tr}(\boldsymbol{\sigma})}{3p_r}\right)-1\right] - \frac{3\,\text{tr}\left(\boldsymbol{\sigma}'^{\text{sym}^2}\right)}{4g\,\text{tr}(\boldsymbol{\sigma})} \\
&= \frac{p_r}{k} - \frac{\sigma_{ii}}{3k}\left[\log\left(\frac{-\sigma_{jj}}{3p_r}\right)-1\right] - \frac{3\sigma'^{\text{sym}}_{ij}\sigma'^{\text{sym}}_{ji}}{4g\sigma_{kk}} \qquad \text{...(24)} \\
&= \frac{p_r}{k} + \frac{p}{k}\left[\log\left(\frac{p}{p_r}\right)-1\right] + \frac{q^2}{6gp}
\end{aligned}$$

And the equivalent expressions to Eqs. 16-19 and 21-23 are given by Eqs. 25-31:

$$\left.\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{\partial C}{\partial \boldsymbol{\sigma}} = -\frac{1}{3k}\log\left(\frac{-\text{tr}(\boldsymbol{\sigma})}{3p_r}\right)\mathbf{I} - \frac{3\boldsymbol{\sigma}'^{\text{sym}}}{2g\,\text{tr}(\boldsymbol{\sigma})} + \frac{3\,\text{tr}\left(\boldsymbol{\sigma}'^{\text{sym}^2}\right)\mathbf{I}}{4g\,\text{tr}^2(\boldsymbol{\sigma})} \\
\varepsilon_{ij} &= \frac{\partial C}{\partial \sigma_{ij}} = -\frac{1}{3k}\log\left(\frac{-\sigma_{kk}}{3p_r}\right)\delta_{ij} - \frac{3\boldsymbol{\sigma}'^{\text{sym}}_{ij}}{2g\sigma_{kk}} + \frac{3\boldsymbol{\sigma}'^{\text{sym}}_{mn}\boldsymbol{\sigma}'^{\text{sym}}_{nm}\delta_{ij}}{4g\sigma_{kk}\sigma_{ll}}
\end{aligned}\right\} \qquad \text{...(25)}$$

$$\left.\begin{aligned}
\text{tr}(\boldsymbol{\varepsilon}) &= -\frac{1}{k}\log\left(\frac{-\text{tr}(\boldsymbol{\sigma})}{3p_r}\right) + \frac{9\,\text{tr}\left(\boldsymbol{\sigma}'^{\text{sym}^2}\right)}{4g\,\text{tr}^2(\boldsymbol{\sigma})} \\
\varepsilon_{kk} &= -\frac{1}{k}\log\left(\frac{-\sigma_{kk}}{3p_r}\right) + \frac{9\sigma'^{\text{sym}}_{mn}\sigma'^{\text{sym}}_{nm}}{4g\sigma_{kk}\sigma_{ll}} \\
\varepsilon_v &= \frac{1}{k}\log\left(\frac{p}{p_r}\right) - \frac{q^2}{6gp^2}
\end{aligned}\right\} \qquad \text{...(26)}$$

$$\left.\begin{aligned}
\boldsymbol{\varepsilon}' &= -\frac{3\boldsymbol{\sigma}'^{sym}}{2g\,\mathrm{tr}(\boldsymbol{\sigma})} = \frac{\boldsymbol{\sigma}'^{sym}}{2gp} \\
\varepsilon'_{ij} &= -\frac{3\sigma'^{sym}_{ij}}{2g\sigma_{kk}} = \frac{\sigma'^{sym}_{ij}}{2gp} \\
\varepsilon_s &= \frac{q}{3gp}
\end{aligned}\right\} \qquad ...(27)$$

Further differentiation gives:

$$\left.\begin{aligned}
\dot{\boldsymbol{\varepsilon}} &= \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} = \frac{\partial C}{\partial \boldsymbol{\sigma}\otimes\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}} \\
&= \frac{1}{p}\left\{\left(\frac{1}{9k}+\frac{\mathrm{tr}\left(\boldsymbol{\sigma}'^{sym^2}\right)}{2g\,\mathrm{tr}^2(\boldsymbol{\sigma})}\right)\bar{\bar{\mathbb{I}}}+\frac{\mathbb{P}^{sym}}{2g}+\frac{1}{2g\,\mathrm{tr}(\boldsymbol{\sigma})}\left(\boldsymbol{\sigma}'^{sym}\otimes\mathbf{I}+\mathbf{I}\otimes\boldsymbol{\sigma}'^{sym}\right)\right\}:\dot{\boldsymbol{\sigma}} = \mathbb{C}:\dot{\boldsymbol{\sigma}} \\
\dot{\varepsilon}_{ij} &= \frac{\partial\varepsilon_{ij}}{\partial\sigma_{kl}}\dot{\sigma}_{kl} = \frac{\partial C}{\partial\sigma_{kl}\sigma_{ij}}\dot{\sigma}_{kl} \\
&= \frac{1}{p}\left\{\begin{aligned}&\left(\frac{1}{9k}+\frac{\sigma'^{sym}_{mn}\sigma'^{sym}_{nm}}{2g\sigma_{pp}\sigma_{qq}}\right)\delta_{ij}\delta_{kl} + \\ &\frac{1}{2g}\left[\tfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\tfrac{1}{3}\delta_{ij}\delta_{kl}\right]+\frac{1}{2g\sigma_{mm}}\left(\sigma'^{sym}_{ij}\delta_{kl}+\delta_{ij}\sigma'^{sym}_{kl}\right)\end{aligned}\right\}\dot{\sigma}_{kl} = C_{ijkl}\dot{\sigma}_{kl} \\
\begin{bmatrix}\dot{\varepsilon}_v \\ \dot{\varepsilon}_s\end{bmatrix} &= \frac{1}{p}\begin{bmatrix}\frac{1}{k}+\frac{q^2}{3gp^2} & \frac{-q}{3gp} \\ \frac{-q}{3gp} & \frac{1}{3g}\end{bmatrix}\begin{bmatrix}\dot{p} \\ \dot{q}\end{bmatrix}
\end{aligned}\right\} \qquad ...(28)$$

It can be seen that the compliance matrix can be obtained simply by substituting $n=1$ and $p_o = p$ into Eq. 20.

On the isotropic axis (and only on this axis), $\boldsymbol{\sigma}' = 0$ $[\sigma'_{ij} = 0]$ $\{q = 0\}$ we obtain:

$$\left.\begin{aligned}
\dot{\boldsymbol{\varepsilon}} &= \frac{1}{p}\left[\frac{1}{9k}\bar{\bar{\mathbb{I}}}+\frac{1}{2g}\mathbb{P}^{sym}\right]:\dot{\boldsymbol{\sigma}} \\
\dot{\varepsilon}_{ij} &= \frac{1}{p}\left[\frac{1}{9k}\delta_{ij}\delta_{kl}+\frac{1}{2g}\left(\tfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\tfrac{1}{3}\delta_{ij}\delta_{kl}\right)\right]\dot{\sigma}_{kl}
\end{aligned}\right\} \qquad ...(29)$$

$$\left.\begin{aligned}\mathrm{tr}(\dot{\boldsymbol{\varepsilon}}) &= \frac{\mathrm{tr}(\dot{\boldsymbol{\sigma}})}{3kp}\\ \dot{\varepsilon}_{kk} &= \frac{\dot{\sigma}_{kk}}{3kp}\\ \dot{\varepsilon}_{v} &= \frac{\dot{p}}{3kp}\end{aligned}\right\} \qquad \text{...(30)}$$

$$\left.\begin{aligned}\dot{\boldsymbol{\varepsilon}}' &= \frac{\dot{\boldsymbol{\sigma}}'^{\mathrm{sym}}}{2gp}\\ \dot{\varepsilon}'_{ij} &= \frac{\dot{\sigma}'^{\mathrm{sym}}_{ij}}{2gp}\\ \dot{\varepsilon}_{s} &= \frac{\dot{q}}{3gp}\end{aligned}\right\} \qquad \text{...(31)}$$

3.3 Strain energy form

Alternatively, to derive the response from the elastic strain energy, one can write that r_o is the positive root of:

$$\begin{aligned}r_o^2 &= \left(N-k(1-n)\mathrm{tr}(\boldsymbol{\varepsilon})\right)^2 + 2gk(1-n)\mathrm{tr}\left(\boldsymbol{\varepsilon}'^{\mathrm{sym}^2}\right)\\ &= \left(N-k(1-n)\varepsilon_{ii}\right)^2 + 2gk(1-n)\varepsilon'^{\mathrm{sym}}_{ij}\varepsilon'^{\mathrm{sym}}_{ji}\\ &= \left(N+k(1-n)\varepsilon_v\right)^2 + 3gk(1-n)\varepsilon_s^2\end{aligned} \qquad \text{...(32)}$$

On the isotropic strain axis $\boldsymbol{\varepsilon}'=0$ $[\varepsilon'_{ij}=0]$ $\{\varepsilon_s=0\}$, $r_o=N-k(1-n)\mathrm{tr}(\boldsymbol{\varepsilon})=N-k(1-n)\varepsilon_{kk}=$ $=N+k(1-n)\varepsilon_v$, and for $n=1$, $r_o=N$.

It follows that:

$$\left.\begin{aligned}r_o\frac{\partial r_o}{\partial \boldsymbol{\varepsilon}} &= k(1-n)\left[-\left(N-k(1-n)\mathrm{tr}(\boldsymbol{\varepsilon})\right)\mathbf{I}+2g\boldsymbol{\varepsilon}'^{\mathrm{sym}}\right]\\ r_o\frac{\partial r_o}{\partial \varepsilon_{ij}} &= k(1-n)\left[-\left(N-k(1-n)\varepsilon_{kk}\right)\delta_{ij}+2g\varepsilon'^{\mathrm{sym}}_{ij}\right]\\ r_o\frac{\partial r_o}{\partial \varepsilon_v} &= k(1-n)\left(N+k(1-n)\varepsilon_v\right),\quad r_o\frac{\partial r_o}{\partial \varepsilon_s}=3gk(1-n)\varepsilon_s\end{aligned}\right\} \qquad \text{...(33)}$$

and then for $0\le n<1$ we write:

$$E=\frac{p_r}{k(2-n)}\left[r_o^{(2-n)/(1-n)}-N\right] \qquad \text{...(34)}$$

Differentiation, following the pattern for the complementary energy case, leads directly to expressions for stress in terms of strain. For $0\le n<1$:

$$\left.\begin{aligned}\boldsymbol{\sigma} &= \frac{\partial E}{\partial \boldsymbol{\varepsilon}} = \frac{\partial E}{\partial r_o}\frac{\partial r_o}{\partial \boldsymbol{\varepsilon}} = \frac{p_r}{k(1-n)} r_o^{1/(1-n)} \frac{\partial r_o}{\partial \boldsymbol{\varepsilon}} \\ &= p_r r_o^{n/(1-n)}\left[-\left(N - k(1-n)\mathrm{tr}(\boldsymbol{\varepsilon})\right)\mathbf{I} + 2g\boldsymbol{\varepsilon}'^{\mathrm{sym}}\right] \\ \sigma_{ij} &= \frac{\partial E}{\partial \varepsilon_{ij}} = \frac{\partial E}{\partial r_o}\frac{\partial r_o}{\partial \varepsilon_{ij}} = \frac{p_r}{k(1-n)} r_o^{1/(1-n)} \frac{\partial r_o}{\partial \varepsilon_{ij}} \\ &= p_r r_o^{n/(1-n)}\left[-\left(N - k(1-n)\varepsilon_{kk}\right)\delta_{ij} + 2g\varepsilon_{ij}'^{\mathrm{sym}}\right]\end{aligned}\right\} \quad \text{...(35)}$$

From which follows:

$$\left.\begin{aligned}\mathrm{tr}(\boldsymbol{\sigma}) &= -3p_r r_o^{n/(1-n)}\left(N - k(1-n)\mathrm{tr}(\boldsymbol{\varepsilon})\right) \\ \sigma_{kk} &= -3p_r r_o^{n/(1-n)}\left(N - k(1-n)\varepsilon_{kk}\right) \\ p &= p_r r_o^{n/(1-n)}\left(N + k(1-n)\varepsilon_v\right)\end{aligned}\right\} \quad \text{...(36)}$$

$$\left.\begin{aligned}\boldsymbol{\sigma}' &= 2gp_r r_o^{n/(1-n)}\boldsymbol{\varepsilon}'^{\mathrm{sym}} \\ \sigma_{ij}' &= 2gp_r r_o^{n/(1-n)}\varepsilon_{ij}'^{\mathrm{sym}} \\ q &= 3gp_r r_o^{n/(1-n)}\varepsilon_s\end{aligned}\right\} \quad \text{...(37)}$$

And furthermore we can derive the relationship:

$$\frac{p_o}{p_r} = r_o^{1/(1-n)} \quad \text{...(38)}$$

Further differentiation gives:

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \frac{\partial^2 E}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \\ &= p_r r_o^{n/(1-n)}\left\{\begin{aligned}&\left[k(1-n)\overline{\overline{\mathbb{I}}} + 2g\mathbb{P}^{\mathrm{sym}}\right] + \\ &\frac{kn}{r_o^2}\left[\left(k(1-n)\mathrm{tr}(\boldsymbol{\varepsilon}) - N\right)\mathbf{I} + 2g\boldsymbol{\varepsilon}'^{\mathrm{sym}}\right] \otimes \left[\left(k(1-n)\mathrm{tr}(\boldsymbol{\varepsilon}) - N\right)\mathbf{I} + 2g\boldsymbol{\varepsilon}'^{\mathrm{sym}}\right]\end{aligned}\right\} : \dot{\boldsymbol{\varepsilon}} = \mathbb{D} : \dot{\boldsymbol{\varepsilon}}\end{aligned}$$

$$\begin{aligned}\dot{\sigma}_{ij} &= \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl} = \frac{\partial^2 E}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}}\dot{\varepsilon}_{kl} \\ &= p_r r_o^{n/(1-n)}\left\{\begin{aligned}&\left[k(1-n)\delta_{ij}\delta_{kl} + 2g\left(\tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \tfrac{1}{3}\delta_{kl}\delta_{ij}\right)\right] + \\ &\frac{kn}{r_o^2}\left[\left(k(1-n)\varepsilon_{mm} - N\right)\delta_{ij} + 2g\varepsilon_{ij}'^{\mathrm{sym}}\right]\left[\left(k(1-n)\varepsilon_{nn} - N\right)\delta_{kl} + 2g\varepsilon_{kl}'^{\mathrm{sym}}\right]\end{aligned}\right\}\dot{\varepsilon}_{kl} = D_{ijkl}\dot{\varepsilon}_{kl}\end{aligned}$$

...(39)

$$\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix} = p_r r_o^{n/(1-n)} \begin{bmatrix} \dfrac{k}{r_o^2}\left(r_o^2 - 3gkn(1-n)\varepsilon_s^2\right) & \dfrac{3gkn\varepsilon_s\left(N+k(1-n)\varepsilon_v\right)}{r_o^2} \\ \dfrac{3gkn\varepsilon_s\left(N+k(1-n)\varepsilon_v\right)}{r_o^2} & \dfrac{3g}{r_o^2}\left(r_o^2+3gkn\varepsilon_s^2\right) \end{bmatrix} \begin{bmatrix}\dot{\varepsilon}_v\\ \dot{\varepsilon}_s\end{bmatrix}$$

$$= p_r\left(\frac{p_o}{p_r}\right)^n \begin{bmatrix} \dfrac{k}{p_o^2}\left(p^2+\dfrac{k}{3g}(1-n)^2 q^2\right) & \dfrac{knqp}{p_o^2} \\ \dfrac{knqp}{p_o^2} & \dfrac{3gp^2+kq^2}{p_o^2} \end{bmatrix} \begin{bmatrix}\dot{\varepsilon}_v\\ \dot{\varepsilon}_s\end{bmatrix}$$

On the isotropic strain axis only, $\boldsymbol{\varepsilon}'=0$ [$\varepsilon'_{ij}=0$] {$\varepsilon_s=0$}, the incremental form reduces to:

$$\left.\begin{aligned} \dot{\boldsymbol{\sigma}} &= p_r r_o^{n/(1-n)}\left\{k\bar{\bar{\mathbb{I}}}+2g\mathbb{P}^{\text{sym}}\right\}:\dot{\boldsymbol{\varepsilon}} \\ \dot{\sigma}_{ij} &= p_r r_o^{n/(1-n)}\left\{k\delta_{ij}\delta_{kl}+2g\left(\tfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\tfrac{1}{3}\delta_{kl}\delta_{ij}\right)\right\}\dot{\varepsilon}_{kl} \end{aligned}\right\} \quad \text{...(40)}$$

$$\left.\begin{aligned} \operatorname{tr}(\dot{\boldsymbol{\sigma}}) &= 3kp_r r_o^{n/(1-n)}\operatorname{tr}(\dot{\boldsymbol{\varepsilon}}) \\ \dot{\sigma}_{kk} &= 3kp_r r_o^{n/(1-n)}\dot{\varepsilon}_{kk} \\ \dot{p} &= kp_r r_o^{n/(1-n)}\dot{\varepsilon}_v \end{aligned}\right\} \quad \text{...(41)}$$

$$\left.\begin{aligned} \dot{\boldsymbol{\sigma}}' &= 2gp_r r_o^{n/(1-n)}\dot{\boldsymbol{\varepsilon}}'^{\text{sym}} \\ \dot{\sigma}'_{ij} &= 2gp_r r_o^{n/(1-n)}\dot{\varepsilon}'^{\text{sym}}_{ij} \\ \dot{q} &= 3gp_r r_o^{n/(1-n)}\dot{\varepsilon}_s \end{aligned}\right\} \quad \text{...(42)}$$

where we note that on the isotropic axis $r_o^{n/(1-n)} = (p_o/p_r)^n = (p/p_r)^n$.

3.4 Strain energy form: special case *n* = 1

For $N=1$, in the limiting case $n \Rightarrow 1$ it can be shown that Eq. 34 approaches asymptotically:

$$\begin{aligned} E &= \frac{p_r}{k}\left[\exp\left(-k\operatorname{tr}(\boldsymbol{\varepsilon})+gk\operatorname{tr}\left(\boldsymbol{\varepsilon}'^{\text{sym}^2}\right)\right)-1\right] \\ &= \frac{p_r}{k}\left[\exp\left(-k\varepsilon_{ii}+gk\varepsilon'^{\text{sym}}_{ij}\varepsilon'^{\text{sym}}_{ji}\right)-1\right] \\ &= \frac{p_r}{k}\left[\exp\left(k\varepsilon_v+\frac{3gk}{2}\varepsilon_s^2\right)-1\right] \end{aligned} \quad \text{...(43)}$$

The equivalents to Eqs. 35-37 and 39-42 become Eqs. 44 to 50:

$$\left.\begin{aligned} \boldsymbol{\sigma} &= \frac{\partial C}{\partial \boldsymbol{\varepsilon}} = p_r \exp\left(-k\operatorname{tr}(\boldsymbol{\varepsilon})+gk\operatorname{tr}\left(\boldsymbol{\varepsilon}'^{\text{sym}^2}\right)\right)\left[-\mathbf{I}+2g\boldsymbol{\varepsilon}'^{\text{sym}}\right] \\ \sigma_{ij} &= \frac{\partial C}{\partial \varepsilon_{ij}} = p_r \exp\left(-k\varepsilon_{kk}+gk\varepsilon'^{\text{sym}}_{kl}\varepsilon'^{\text{sym}}_{lk}\right)\left[-\delta_{ij}+2g\varepsilon'^{\text{sym}}_{ij}\right] \end{aligned}\right\} \quad \text{...(44)}$$

$$\left.\begin{aligned}
\operatorname{tr}(\boldsymbol{\sigma}) &= -3p_r \exp\left(-k\operatorname{tr}(\boldsymbol{\varepsilon}) + gk\operatorname{tr}\left(\boldsymbol{\varepsilon}'^{\text{sym}^2}\right)\right) \\
\sigma_{kk} &= -3p_r \exp\left(-k\varepsilon_{kk} + gk\varepsilon'_{ij}\varepsilon'_{ji}\right) \\
p &= p_r \exp\left(k\varepsilon_v + \frac{3gk}{2}\varepsilon_s^2\right)
\end{aligned}\right\} \quad \text{...(45)}$$

$$\left.\begin{aligned}
\boldsymbol{\sigma}' &= 2gp_r \exp\left(-k\operatorname{tr}(\boldsymbol{\varepsilon}) + gk\operatorname{tr}\left(\boldsymbol{\varepsilon}'^{\text{sym}^2}\right)\right)\boldsymbol{\varepsilon}'^{\text{sym}} = 2gp\boldsymbol{\varepsilon}'^{\text{sym}} \\
\sigma'_{ij} &= 2gp_r \exp\left(-k\varepsilon_{kk} + gk\varepsilon'^{\text{sym}}_{kl}\varepsilon'^{\text{sym}}_{lk}\right)\varepsilon'^{\text{sym}}_{ij} = 2gp\varepsilon'^{\text{sym}}_{ij} \\
q &= 3gp_r \exp\left(k\varepsilon_v + \frac{3gk}{2}\varepsilon_s^2\right)\varepsilon_s = 3gp\varepsilon_s
\end{aligned}\right\} \quad \text{...(46)}$$

Further differentiation (and substituting the solution for p) gives:

$$\left.\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \frac{\partial C}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \\
&= p\left\{k\left[-\mathbf{I} + 2g\boldsymbol{\varepsilon}'^{\text{sym}}\right] \otimes \left[-\mathbf{I} + 2g\boldsymbol{\varepsilon}'^{\text{sym}}\right] + 2g\mathbb{P}^{\text{sym}}\right\} : \dot{\boldsymbol{\varepsilon}} = \mathbb{D} : \dot{\boldsymbol{\varepsilon}} \\
\dot{\sigma}_{ij} &= \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl} = \frac{\partial f}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}}\dot{\varepsilon}_{kl} \\
&= p\left\{\begin{array}{l} k\left[-\delta_{ij} + 2g\varepsilon'^{\text{sym}}_{ij}\right]\left[-\delta_{kl} + 2g\varepsilon'^{\text{sym}}_{kl}\right] + \\ 2g\left[\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{1}{3}\delta_{kl}\delta_{ij}\right] \end{array}\right\}\dot{\varepsilon}_{kl} = D_{ijkl}\dot{\varepsilon}_{kl} \\
\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} &= p\begin{bmatrix} k & 3gk\varepsilon_s \\ 3gk\varepsilon_s & 3g\left(1 + 9g^2k\varepsilon_s^2\right) \end{bmatrix}\begin{bmatrix} \dot{\varepsilon}_v \\ \dot{\varepsilon}_s \end{bmatrix} = p\begin{bmatrix} k & \dfrac{kq}{p} \\ \dfrac{kq}{p} & 3g + k\dfrac{q^2}{p^2} \end{bmatrix}\begin{bmatrix} \dot{\varepsilon}_v \\ \dot{\varepsilon}_s \end{bmatrix}
\end{aligned}\right\} \quad \text{...(47)}$$

Noting that for $N = 1$,as $n \Rightarrow 1$, $r_o^{n/(1-n)} \Rightarrow \exp\left(-k\operatorname{tr}(\boldsymbol{\varepsilon})\right)$, it can be seen that the stiffness matrix can be obtained by substituting $n = 1$, $r_o = 1$ and $p_r r_o^{n/(1-n)} = p = p_r r_o^{1/(1-n)}$ into Eq. 39.

On the isotropic strain axis only, $\boldsymbol{\varepsilon}' = 0$ $[\varepsilon'_{ij} = 0]$ $\{\varepsilon_s = 0\}$:

$$\left.\begin{aligned}
\dot{\boldsymbol{\sigma}} &= p\left\{k\overline{\overline{\mathbb{I}}} + 2g\mathbb{P}^{\text{sym}}\right\} : \dot{\boldsymbol{\varepsilon}} \\
\dot{\sigma}_{ij} &= p\left\{k\delta_{ij}\delta_{kl} + 2g\left[\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{1}{3}\delta_{ij}\delta_{kl}\right]\right\}\dot{\varepsilon}_{kl}
\end{aligned}\right\} \quad \text{...(48)}$$

$$\left.\begin{aligned}
\operatorname{tr}(\dot{\boldsymbol{\sigma}}) &= 3kp\dot{\boldsymbol{\varepsilon}} \\
\dot{\sigma}_{kk} &= 3kp\dot{\varepsilon}_{kk} \\
\dot{p} &= kp\dot{\varepsilon}_v
\end{aligned}\right\} \quad \text{...(49)}$$

$$\left.\begin{aligned}\dot{\boldsymbol{\sigma}}' &= 2gp\dot{\boldsymbol{\varepsilon}}'^{\text{sym}}\\ \dot{\sigma}'_{ij} &= 2gp\dot{\varepsilon}'^{\text{sym}}_{ij}\\ \dot{q} &= 3gp\dot{\varepsilon}_s\end{aligned}\right\} \qquad \text{...(50)}$$

It can be confirmed that the results in Sections 3.1 and 3.2 are exactly equivalent to those in 3.3 and 3.4.

4 Anisotropy

The following argument follows very closely that of Zysset and Curnier (1995), except that here we employ a positive-definite symmetric second order fabric tensor $\mathbf{B}$ $[B_{ij}]$, whereas they used a traceless second order tensor $\mathbf{G}$ $[G_{ij}]$ in combination with a scalar g.

If linear elasticity is to be defined so that anisotropy depends on a symmetric second order fabric tensor $\mathbf{B}$ $[B_{ij}]$, then it can be argued from the theory of tensor invariants (see for instance Spencer (1971)) that the general form of the strain energy function must be of the form:

$$\begin{aligned}E &= c_1\,\text{tr}^2(\boldsymbol{\varepsilon}) + c_2\,\text{tr}(\boldsymbol{\varepsilon}^2) + c_3\,\text{tr}^2(\boldsymbol{\varepsilon}\mathbf{B}) + c_4\,\text{tr}(\boldsymbol{\varepsilon}^2\mathbf{B}) + c_5\,\text{tr}^2(\boldsymbol{\varepsilon}\mathbf{B}^2)\\ &\quad + c_6\,\text{tr}\left((\boldsymbol{\varepsilon}\mathbf{B})^2\right) + c_7\,\text{tr}(\boldsymbol{\varepsilon})\text{tr}(\boldsymbol{\varepsilon}\mathbf{B}) + c_8\,\text{tr}(\boldsymbol{\varepsilon})\text{tr}(\boldsymbol{\varepsilon}\mathbf{B}^2) + c_9\,\text{tr}(\boldsymbol{\varepsilon}\mathbf{B})\text{tr}(\boldsymbol{\varepsilon}\mathbf{B}^2)\end{aligned} \qquad \text{...(51)}$$

The above may be compared with the very similar equation 4 of Zysset and Curnier (1995). We follow them in preferring the use of $\text{tr}\left((\boldsymbol{\varepsilon}\mathbf{B})^2\right)$ $[\varepsilon_{ij}B_{jk}\varepsilon_{kl}B_{li}]$ rather than the possible alternative of $\text{tr}(\boldsymbol{\varepsilon}^2\mathbf{B}^2)$ $[\varepsilon_{ij}\varepsilon_{jk}B_{kl}B_{li}]$, see also Spencer (1971) and Boehler (1987). Each of the coefficients $c_1 \ldots c_9$ may themselves be functions of $\text{tr}(\mathbf{B})$ $[B_{ii}]$, $\text{tr}(\mathbf{B}^2)$ $[B_{ij}B_{ji}]$, and $\text{tr}(\mathbf{B}^3)$ $[B_{ij}B_{jk}B_{ki}]$. However, this form admits a bewildering variety of possibilities, which Zysset and Curnier (1995) effectively went on to restrict by making further assumptions.

Although they express it in slightly different terms, Gajo and Bigoni (2008) suggest a restricted form of the above expression, choosing for the linear case just two terms. We have followed their terminology in using $\mathbf{B}$ $[B_{ij}]$ here for the fabric tensor:

$$E = c_3\,\text{tr}^2(\mathbf{B}\boldsymbol{\varepsilon}) + c_6\,\text{tr}\left((\mathbf{B}\boldsymbol{\varepsilon})^2\right) \qquad \text{...(52)}$$

(Note that $\text{tr}(\boldsymbol{\varepsilon}\mathbf{B}) = \text{tr}(\mathbf{B}\boldsymbol{\varepsilon})$ *etc.*) If c_3, c_6 are taken as constants and if $\mathbf{B} = \mathbf{I}$ $[B_{ij} = \delta_{ij}]$, then linear isotropic elasticity is recovered. Gajo and Bigoni (2008) note that this form of strain energy function leads to the anisotropic linear elastic form described by Bigoni and Loret (1999).

Equation (52) is clearly inspired by the concept that the strain $\boldsymbol{\varepsilon}$ $[\varepsilon_{ij}]$ in the strain energy expression for an isotropic material can simply be replaced by the product of the strain and the anisotropy tensor $\mathbf{B}\boldsymbol{\varepsilon}$ $[B_{ik}\varepsilon_{kj}]$ in order to define the strain energy for an anisotropic material. However, we consider this analogy should be approached with caution. Whilst both the strain $\boldsymbol{\varepsilon}$ $[\varepsilon_{ij}]$ and the anisotropy $\mathbf{B}$ $[B_{ij}]$ are symmetric tensors, the product $\mathbf{B}\boldsymbol{\varepsilon}$ $[B_{ik}\varepsilon_{kj}]$ is not. To treat the unsymmetric tensor $\mathbf{B\varepsilon}$ $[B_{ik}\varepsilon_{kj}]$ as if it were directly analogous to the symmetric tensor $\boldsymbol{\varepsilon}$ $[\varepsilon_{ij}]$ is open to question. We emphasise though that this concern does not invalidate Eq. 52, which is a perfectly correct special case of Eq. 51. Our concern relates solely to the implicit interpretation that the symmetric $\boldsymbol{\varepsilon}$ $[\varepsilon_{ij}]$ can be generalised to the non-symmetric $\mathbf{B\varepsilon}$ $[B_{ik}\varepsilon_{kj}]$. In the following we therefore pursue an approach originally due to Lodge (1955) in which a symmetric equivalent strain is defined.

Whilst $\mathbf{B\varepsilon}$ $[B_{ik}\varepsilon_{kj}]$ is not symmetric, $\mathbf{a\varepsilon a}^{\mathrm{T}}$ $[a_{ik}\varepsilon_{kl}a_{jl}]$ is symmetric, where $\mathbf{a}$ $[a_{ij}]$ is an alternative definition of an anisotropy tensor, also symmetric. We therefore follow Lodge (1955) in defining a symmetric equivalent strain $\bar{\boldsymbol{\varepsilon}} = \mathbf{a\varepsilon a}^{\mathrm{T}}$ $[\bar{\varepsilon}_{ij} = a_{ik}\varepsilon_{kl}a_{jl}]$ and note that $\boldsymbol{\varepsilon} = \mathbf{a}^{-1}\bar{\boldsymbol{\varepsilon}}\mathbf{a}^{-\mathrm{T}}$ $[\varepsilon_{ij} = a_{ik}^{-1}\bar{\varepsilon}_{kl}a_{jl}^{-1}]$.

Furthermore, if we define $\mathbf{B} = \mathbf{a}^2$ $[B_{ij} = a_{ik}a_{kj}]$, it follows that:

$$\left.\begin{aligned} &\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}\right) = \mathrm{tr}\left(\mathbf{a\varepsilon a}^{\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{a}^2\boldsymbol{\varepsilon}\right) = \mathrm{tr}\left(\mathbf{B\varepsilon}\right) = \mathrm{tr}\left(\bar{\bar{\boldsymbol{\varepsilon}}}\right) \\ &\bar{\varepsilon}_{ii} = a_{ij}\varepsilon_{jk}a_{ik} = a_{ki}a_{ij}\varepsilon_{jk} = B_{kj}\varepsilon_{jk} = \bar{\bar{\varepsilon}}_{ii} \end{aligned}\right\} \qquad \text{...(53)}$$

where $\bar{\bar{\varepsilon}} = \mathbf{B}\varepsilon$ $[\bar{\bar{\varepsilon}}_{ij} = B_{ik}\varepsilon_{kj}]$, and

$$\left.\begin{aligned} &\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}^2\right) = \mathrm{tr}\left(\left(\mathbf{a\varepsilon a}^{\mathrm{T}}\right)^2\right) = \mathrm{tr}\left(\mathbf{a\varepsilon a}^{\mathrm{T}}\mathbf{a\varepsilon a}^{\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{aa\varepsilon aa\varepsilon}\right) = \mathrm{tr}\left(\left(\mathbf{B\varepsilon}\right)^2\right) = \mathrm{tr}\left(\bar{\bar{\boldsymbol{\varepsilon}}}^2\right) \\ &\bar{\varepsilon}_{il}\bar{\varepsilon}_{li} = a_{ij}\varepsilon_{jk}a_{lk}a_{lm}\varepsilon_{mn}a_{in} = a_{ni}a_{ij}\varepsilon_{jk}a_{kl}a_{lm}\varepsilon_{mn} = B_{nj}\varepsilon_{jk}B_{km}\varepsilon_{mn} = \bar{\bar{\varepsilon}}_{nk}\bar{\bar{\varepsilon}}_{kn} \end{aligned}\right\} \qquad \text{...(54)}$$

Note that the above results do not imply that $\bar{\boldsymbol{\varepsilon}}$ $[\bar{\varepsilon}_{ij}]$ is equal to $\bar{\bar{\boldsymbol{\varepsilon}}}$ $[\bar{\bar{\varepsilon}}_{ij}]$; they only imply that certain invariants of these tensors are equal.

As a result of Eqs. 53 and 54 it follows that Eq. 52 is exactly equivalent to:

$$E = c_3\,\mathrm{tr}^2\left(\mathbf{a\varepsilon a}^{\mathrm{T}}\right) + c_6\,\mathrm{tr}\left(\left(\mathbf{a\varepsilon a}^{\mathrm{T}}\right)^2\right) = c_3\,\mathrm{tr}^2\left(\bar{\boldsymbol{\varepsilon}}\right) + c_6\,\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}^2\right) \qquad \text{...(55)}$$

Which allows direct comparison with Eq. 3, suggesting that the symmetric $\bar{\boldsymbol{\varepsilon}}$ $[\bar{\varepsilon}_{ij}]$ can indeed be treated as an equivalent strain. In the following it will be convenient at different times to make use of either $\bar{\boldsymbol{\varepsilon}}$ $[\bar{\varepsilon}_{ij}]$ or $\bar{\bar{\boldsymbol{\varepsilon}}}$ $[\bar{\bar{\varepsilon}}_{ij}]$.

Considering the eigendecomposition of a_{ij}:

$$\left.\begin{aligned} &\mathbf{a} = \mathbf{q\lambda q}^{-1} \\ &a_{ij} = q_{ik}\lambda_{kl}q_{lj}^{-1} \end{aligned}\right\} \qquad \text{...(56)}$$

It follows that

$$\left.\begin{aligned} &\mathbf{B} = \mathbf{a}^2 = \mathbf{q\lambda q}^{-1}\mathbf{q\lambda q}^{-1} = \mathbf{q\lambda}^2\mathbf{q}^{-1} = \mathbf{q\Lambda q}^{-1} \\ &B_{ij} = a_{ik}a_{kj} = q_{il}\lambda_{lm}q_{mk}^{-1}q_{kn}\lambda_{np}q_{pj}^{-1} = q_{il}\lambda_{lm}\lambda_{mp}q_{pj}^{-1} = q_{il}\Lambda_{lp}q_{pj}^{-1} \end{aligned}\right\} \qquad \text{...(57)}$$

with $\mathbf{\Lambda}=\boldsymbol{\lambda}^2$ [$\Lambda_{ij}=\lambda_{ik}\lambda_{kj}$].

so that a_{ij} and B_{ij} have the same eigenvectors, and the eigenvalues of B_{ij} are simply the squares of those of a_{ij}. If we choose the normalisation $\det(\mathbf{a})=1$ we can take the eigenvalues of $\mathbf{a}$ as $\left(a,b,a^{-1}b^{-1}\right)$ and those of $\mathbf{B}$ as $\left(A,B,A^{-1}B^{-1}\right)=\left(a^2,b^2,a^{-2}b^{-2}\right)$. Anisotropy is therefore defined by the eigenvectors, which give the directions of the orthotropic axes, and two independent dimensionless factors, either a,b or A,B.

Define also $\bar{\boldsymbol{\sigma}}=\mathbf{a}^{-1}\boldsymbol{\sigma}\mathbf{a}^{-\mathrm{T}}$ [$\bar{\sigma}_{ij}=a_{ik}^{-1}\sigma_{kl}a_{jl}^{-1}$] and note that $\boldsymbol{\sigma}=\mathbf{a}\bar{\boldsymbol{\sigma}}\mathbf{a}^{\mathrm{T}}$ [$\sigma_{ij}=a_{ik}\bar{\sigma}_{kl}a_{jl}$]. It follows that, noting the symmetry of $\mathbf{a}$, $\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{\varepsilon}}=\mathrm{tr}\left(\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}\right)=\mathrm{tr}\left(\mathbf{a}^{-1}\boldsymbol{\sigma}\mathbf{a}^{-\mathrm{T}}\mathbf{a}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{a}^{\mathrm{T}}\right)=\mathrm{tr}\left(\mathbf{a}^{-1}\boldsymbol{\sigma}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{a}\right)=\mathrm{tr}\left(\mathbf{a}\mathbf{a}^{-1}\boldsymbol{\sigma}\boldsymbol{\varepsilon}^{\mathrm{T}}\right)=\boldsymbol{\sigma}:\boldsymbol{\varepsilon}$ [$\bar{\sigma}_{ij}\bar{\varepsilon}_{ij}=a_{ik}^{-1}\sigma_{kl}a_{jl}^{-1}a_{im}\varepsilon_{mn}a_{jn}=\sigma_{ij}\varepsilon_{ij}$], so that the quantities $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are work-conjugate in the same way as $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$.

Alternatively, and noting that $\boldsymbol{\varepsilon}=\mathbf{B}^{-1}\bar{\bar{\boldsymbol{\varepsilon}}}$ [$\varepsilon_{ij}=B_{ik}^{-1}\bar{\bar{\varepsilon}}_{kj}$], define $\bar{\bar{\boldsymbol{\sigma}}}=\mathbf{B}^{-1}\boldsymbol{\sigma}$ [$\bar{\bar{\sigma}}_{ij}=B_{ik}^{-1}\sigma_{kj}$] and note that $\boldsymbol{\sigma}=\mathbf{B}\bar{\bar{\boldsymbol{\sigma}}}$ [$\sigma_{ij}=B_{ik}\bar{\bar{\sigma}}_{kj}$]. Given the symmetry of $\mathbf{B}$, it follows that $\bar{\bar{\boldsymbol{\sigma}}}:\bar{\bar{\boldsymbol{\varepsilon}}}=\mathrm{tr}\left(\bar{\bar{\boldsymbol{\sigma}}}\bar{\bar{\boldsymbol{\varepsilon}}}^{T}\right)=\mathrm{tr}\left(\mathbf{B}^{-1}\boldsymbol{\sigma}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\right)=\mathrm{tr}\left(\mathbf{B}\mathbf{B}^{-1}\boldsymbol{\sigma}\boldsymbol{\varepsilon}^{\mathrm{T}}\right)=\boldsymbol{\sigma}:\boldsymbol{\varepsilon}$ [$\bar{\bar{\sigma}}_{ij}\bar{\bar{\varepsilon}}_{ij}=B_{ik}^{-1}\sigma_{kj}B_{il}\varepsilon_{lj}=\sigma_{ij}\varepsilon_{ij}$], so that the quantities $\bar{\bar{\boldsymbol{\sigma}}}$ and $\bar{\bar{\boldsymbol{\varepsilon}}}$ are also work-conjugate in the same way as $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$.

Note the differentials (most easily derived by considering the subscript notation):

$$\left.\begin{aligned}&\frac{\partial\bar{\boldsymbol{\varepsilon}}}{\partial\boldsymbol{\varepsilon}}=\mathbf{a}\bar{\otimes}\mathbf{a}\\&\frac{\partial\bar{\varepsilon}_{ij}}{\partial\varepsilon_{kl}}=a_{im}\delta_{mk}\delta_{nl}a_{jn}=a_{ik}a_{jl}\end{aligned}\right\}\qquad\text{...(58)}$$

and

$$\left.\begin{aligned}&\frac{\partial\bar{\boldsymbol{\sigma}}}{\partial\boldsymbol{\sigma}}=\mathbf{a}^{-1}\bar{\otimes}\mathbf{a}^{-1}\\&\frac{\partial\bar{\sigma}_{ij}}{\partial\sigma_{kl}}=a_{im}^{-1}\delta_{mk}\delta_{nl}a_{jn}^{-1}=a_{ik}^{-1}a_{jl}^{-1}\end{aligned}\right\}\qquad\text{...(59)}$$

Now consider any isotropic model defined by $E=E_0(\boldsymbol{\varepsilon})$ [$E=E_0(\varepsilon_{ij})$] {$E=E_0(\varepsilon_v,\varepsilon_s)$}. We convert this to an equivalent anisotropic model by defining $E=E_0(\bar{\boldsymbol{\varepsilon}})$ [$E=E_0(\bar{\varepsilon}_{ij})$]. Alternatively, if we have defined the complementary energy $C=C_0(\boldsymbol{\sigma})$ [$C=C_0(\sigma_{ij})$] for an isotropic model, we convert this to the anisotropic case through $C=C_0(\bar{\boldsymbol{\sigma}})$ [$C=C_0(\bar{\sigma}_{ij})$]. Noting that $E+C=\boldsymbol{\sigma}:\boldsymbol{\varepsilon}=\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{\varepsilon}}$ [$E+C=\sigma_{ij}\varepsilon_{ij}=\bar{\sigma}_{ij}\bar{\varepsilon}_{ij}$], these definitions mean that the strain and complementary energies are proper Legendre transforms of each other in both the isotropic and anisotropic cases.

The differentials give:

$$\left.\begin{aligned}\boldsymbol{\sigma}&=\frac{\partial E}{\partial \boldsymbol{\varepsilon}}=\frac{\partial E_0}{\partial \bar{\boldsymbol{\varepsilon}}}:\frac{\partial \bar{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}}=\frac{\partial E_0}{\partial \bar{\boldsymbol{\varepsilon}}}:\mathbf{a}\,\bar{\otimes}\,\mathbf{a}=\mathbf{a}^{\mathrm{T}}\,\bar{\otimes}\,\mathbf{a}^{\mathrm{T}}:\frac{\partial E_0}{\partial \bar{\boldsymbol{\varepsilon}}}\\ \sigma_{ij}&=\frac{\partial E}{\partial \varepsilon_{ij}}=\frac{\partial E_0}{\partial \bar{\varepsilon}_{kl}}\frac{\partial \bar{\varepsilon}_{kl}}{\partial \varepsilon_{ij}}=\frac{\partial E_0}{\partial \bar{\varepsilon}_{kl}}a_{ki}a_{lj}=a_{ki}a_{lj}\frac{\partial E_0}{\partial \bar{\varepsilon}_{kl}}\end{aligned}\right\}\quad ...(60)$$

$$\left.\begin{aligned}\dot{\boldsymbol{\sigma}}&=\frac{\partial^2 E}{\partial \boldsymbol{\varepsilon}\otimes\partial \boldsymbol{\varepsilon}}:\dot{\boldsymbol{\varepsilon}}=\left(\mathbf{a}\,\bar{\otimes}\,\mathbf{a}:\frac{\partial^2 E_0}{\partial \bar{\boldsymbol{\varepsilon}}\otimes\partial \bar{\boldsymbol{\varepsilon}}}:\mathbf{a}\,\bar{\otimes}\,\mathbf{a}\right):\dot{\boldsymbol{\varepsilon}}=\left(\mathbf{a}^{\mathrm{T}}\,\bar{\otimes}\,\mathbf{a}^{\mathrm{T}}:\mathbb{D}^0:\mathbf{a}\,\bar{\otimes}\,\mathbf{a}\right):\dot{\boldsymbol{\varepsilon}}\\ \dot{\sigma}_{ij}&=\frac{\partial^2 E}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}}\dot{\varepsilon}_{kl}=\frac{\partial^2 E_0}{\partial \bar{\varepsilon}_{rs}\partial \bar{\varepsilon}_{pq}}\frac{\partial \bar{\varepsilon}_{pq}}{\partial \varepsilon_{ij}}\frac{\partial \bar{\varepsilon}_{rs}}{\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl}=\mathbb{D}^0_{pqrs}\frac{\partial \bar{\varepsilon}_{pq}}{\partial \varepsilon_{ij}}\frac{\partial \bar{\varepsilon}_{rs}}{\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl}=\left(a_{pi}a_{qj}\mathbb{D}^0_{pqrs}a_{rk}a_{sl}\right)\dot{\varepsilon}_{kl}\end{aligned}\right\}\quad ...(61)$$

$$\left.\begin{aligned}\boldsymbol{\varepsilon}&=\frac{\partial C}{\partial \boldsymbol{\sigma}}=\frac{\partial \bar{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}}:\frac{\partial C_0}{\partial \bar{\boldsymbol{\sigma}}}=\frac{\partial C_0}{\partial \bar{\boldsymbol{\sigma}}}:\mathbf{a}^{-1}\,\bar{\otimes}\,\mathbf{a}^{-1}=\mathbf{a}^{-\mathrm{T}}\,\bar{\otimes}\,\mathbf{a}^{-\mathrm{T}}:\frac{\partial C_0}{\partial \bar{\boldsymbol{\sigma}}}\\ \varepsilon_{ij}&=\frac{\partial C}{\partial \sigma_{ij}}=\frac{\partial C_0}{\partial \bar{\sigma}_{kl}}\frac{\partial \bar{\sigma}_{kl}}{\partial \sigma_{ij}}=\frac{\partial C_0}{\partial \bar{\sigma}_{kl}}a^{-1}_{ki}a^{-1}_{lj}=a^{-1}_{ki}a^{-1}_{lj}\frac{\partial C_0}{\partial \bar{\sigma}_{kl}}\end{aligned}\right\}\quad ...(62)$$

$$\left.\begin{aligned}\dot{\boldsymbol{\varepsilon}}&=\frac{\partial^2 C}{\partial \boldsymbol{\sigma}\otimes\partial \boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}=\frac{\partial \bar{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}}:\frac{\partial^2 C_0}{\partial \bar{\boldsymbol{\sigma}}\otimes\partial \bar{\boldsymbol{\sigma}}}:\frac{\partial \bar{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}=\left(\mathbf{a}^{-\mathrm{T}}\,\bar{\otimes}\,\mathbf{a}^{-\mathrm{T}}:\frac{\partial^2 C_0}{\partial \bar{\boldsymbol{\sigma}}\otimes\partial \bar{\boldsymbol{\sigma}}}:\mathbf{a}^{-1}\,\bar{\otimes}\,\mathbf{a}^{-1}\right):\dot{\boldsymbol{\sigma}}\\ \dot{\varepsilon}_{ij}&=\frac{\partial^2 C}{\partial \sigma_{ij}\partial \sigma_{kl}}\dot{\sigma}_{kl}=\frac{\partial^2 C_0}{\partial \bar{\sigma}_{rs}\partial \bar{\sigma}_{pq}}\frac{\partial \bar{\sigma}_{pq}}{\partial \sigma_{ij}}\frac{\partial \bar{\sigma}_{rs}}{\partial \sigma_{kl}}\dot{\sigma}_{kl}=\mathbb{C}^0_{pqrs}\frac{\partial \bar{\sigma}_{pq}}{\partial \sigma_{ij}}\frac{\partial \bar{\sigma}_{rs}}{\partial \sigma_{kl}}\dot{\sigma}_{kl}=\left(a^{-1}_{pi}a^{-1}_{qj}\mathbb{C}^0_{pqrs}a^{-1}_{rk}a^{-1}_{sl}\right)\dot{\sigma}_{kl}\end{aligned}\right\}$$

...(63)

So that the incremental stiffness and compliance matrices are readily determined. The above results can be slightly simplified by exploiting the symmetry of **a**.

4.1 Linear anisotropy

Considering then the case of linear isotropic elasticity, we modify Eq. 4 to:

$$\begin{aligned}E&=\frac{K}{2}\mathrm{tr}^2\left(\bar{\boldsymbol{\varepsilon}}\right)+G\,\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}'^{sym^2}\right)\\&=\frac{K}{2}\bar{\varepsilon}_{ii}\bar{\varepsilon}_{jj}+G\bar{\varepsilon}'^{sym}_{ij}\bar{\varepsilon}'^{sym}_{ji}\end{aligned}\quad ...(64)$$

Tensor representation theorems could be used to recast Eq. 64 as another special case of Eq. 51. It is intended in the same spirit as Eq. 52, but using the symmetric form of the equivalent strain for the same reasons as discussed above in relation to the minor symmetries of the resulting stiffness matrix.

We can then use Eqs. 7 and 8 together with 60 and 61 to derive the stress and the stiffness matrix, or we can do this explicitly:

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{\partial E}{\partial \boldsymbol{\varepsilon}} = \frac{\partial E}{\partial \bar{\boldsymbol{\varepsilon}}} : \mathbf{a} \bar{\otimes} \mathbf{a} = \left(K \operatorname{tr}(\bar{\boldsymbol{\varepsilon}}) \mathbf{I} + 2G \bar{\boldsymbol{\varepsilon}}'^{sym} \right) : \mathbf{a} \bar{\otimes} \mathbf{a} \\ &= \left[\mathbf{a}^{\mathrm{T}} \bar{\otimes} \mathbf{a}^{\mathrm{T}} : \left(K \bar{\bar{\mathbb{I}}} + 2G \mathbb{P}^{\mathrm{sym}} \right) : \mathbf{a} \bar{\otimes} \mathbf{a} \right] : \boldsymbol{\varepsilon} \\ \sigma_{ij} &= \frac{\partial E}{\partial \varepsilon_{ij}} = \frac{\partial E}{\partial \bar{\varepsilon}_{kl}} a_{ki} a_{lj} = \left(K \bar{\varepsilon}_{mm} \delta_{kl} + 2G \bar{\varepsilon}_{kl}'^{\mathrm{sym}} \right) a_{ki} a_{lj} \\ &= a_{ik} a_{jl} \left(K a_{mn} \varepsilon_{np} a_{mp} \delta_{kl} + 2G \left[\tfrac{1}{2} \left(a_{kn} \varepsilon_{np} a_{lp} + a_{ln} \varepsilon_{np} a_{kp} \right) - \tfrac{1}{3} a_{mn} \varepsilon_{np} a_{mp} \delta_{kl} \right] \right) \\ &= a_{ik} a_{jl} \left(K \delta_{kl} \delta_{qm} + 2G \left[\tfrac{1}{2} \left(\delta_{kq} \delta_{lm} + \delta_{km} \delta_{lq} \right) - \tfrac{1}{3} \delta_{kl} \delta_{qm} \right] \right) a_{qn} a_{mp} \varepsilon_{np} \end{aligned} \quad \text{...(65)}$$

From which we immediately derive $\mathbb{D} = \mathbf{a}^{\mathrm{T}} \bar{\otimes} \mathbf{a}^{\mathrm{T}} : \left(K \bar{\bar{\mathbb{I}}} + 2G \mathbb{P}^{\mathrm{sym}} \right) : \mathbf{a} \bar{\otimes} \mathbf{a}$, which can also be written $\mathbb{D} = \left(K - \frac{2G}{3} \right) \left(\mathbf{a}^2 \otimes \mathbf{a}^2 \right) + 2G \left(\mathbf{a}^2 \underline{\bar{\otimes}} \mathbf{a}^2 \right)$, which should be compared to $\mathbb{D} = \left(K - \frac{2G}{3} \right) (\mathbf{I} \otimes \mathbf{I}) + 2G \left(\mathbf{I} \underline{\bar{\otimes}} \mathbf{I} \right)$ in Eq. 8.

An alternative to the above is to define the complementary energy expression:

$$\begin{aligned} C &= \frac{\operatorname{tr}^2(\bar{\boldsymbol{\sigma}})}{18K} + \frac{\operatorname{tr}\left(\bar{\boldsymbol{\sigma}}'^2 \right)}{4G} \\ &= \frac{\bar{\sigma}_{ii} \bar{\sigma}_{jj}}{18K} + \frac{\bar{\sigma}'_{ij} \bar{\sigma}'_{ji}}{4G} \end{aligned} \quad \text{...(66)}$$

The derivation of the response follows very closely the development from the strain energy, with the roles of stress and strain interchanged and $\mathbf{a}$ $[a_{ij}]$ replaced by $\mathbf{a}^{-1}$ $[a_{ij}^{-1}]$.

5 Combined nonlinearity and anisotropy

In the above approach, both anisotropy and nonlinearity are achieved by applying modifications to the strain energy or complementary energy expressions for a linear isotropic elastic material. The two effects may simply be combined by applying both modifications. To avoid excessive repetition we pursue the derivation from the complementary energy only.

We start by defining $\bar{p} = -\frac{1}{3} \operatorname{tr}(\bar{\boldsymbol{\sigma}}) = -\frac{1}{3} \bar{\sigma}_{ii}$. Then define $\bar{p}_o$ as the positive root of:

$$\begin{aligned} \bar{p}_o^2 &= \frac{\operatorname{tr}^2(\bar{\boldsymbol{\sigma}})}{9} + \frac{k(1-n)}{2g} \operatorname{tr}\left(\bar{\boldsymbol{\sigma}}'^{\mathrm{sym}^2} \right) \\ &= \frac{\bar{\sigma}_{ii} \bar{\sigma}_{jj}}{9} + \frac{k(1-n)}{2g} \bar{\sigma}_{ij}'^{\mathrm{sym}} \bar{\sigma}_{ji}'^{\mathrm{sym}} \end{aligned} \quad \text{...(67)}$$

and note the result

$$\begin{aligned} \bar{p}_o \frac{\partial \bar{p}_o}{\partial \bar{\boldsymbol{\sigma}}} &= \tfrac{1}{9} \operatorname{tr}(\bar{\boldsymbol{\sigma}}) \mathbf{I} + \frac{k(1-n)}{2g} \bar{\boldsymbol{\sigma}}'^{\mathrm{sym}} \\ \bar{p}_o \frac{\partial \bar{p}_o}{\partial \bar{\sigma}_{ij}} &= \tfrac{1}{9} \bar{\sigma}_{kk} \delta_{ij} + \frac{k(1-n)}{2g} \bar{\sigma}_{ij}'^{\mathrm{sym}} \end{aligned} \quad \text{...(68)}$$

We then define for $0 \le n < 1$:

$$C = N\left(\frac{p_r}{k(2-n)} - \frac{\bar{p}}{k(1-n)}\right) + p_r \frac{1}{k(1-n)(2-n)}\left(\frac{\bar{p}_o}{p_r}\right)^{2-n} \quad \text{...(69)}$$

For $N=1$ and the limiting case $n=1$ we define:

$$\begin{aligned} C &= \frac{p_r}{k} - \frac{\mathrm{tr}(\bar{\boldsymbol{\sigma}})}{3k}\left[\ln\left(\frac{-\mathrm{tr}(\bar{\boldsymbol{\sigma}})}{3p_r}\right) - 1\right] - \frac{3\,\mathrm{tr}\left(\bar{\boldsymbol{\sigma}}'^{\mathrm{sym}^2}\right)}{4g\,\mathrm{tr}(\bar{\boldsymbol{\sigma}})} \\ &= \frac{p_r}{k} - \frac{\bar{\sigma}_{ii}}{3k}\left[\ln\left(\frac{-\bar{\sigma}_{jj}}{3p_r}\right) - 1\right] - \frac{3\bar{\sigma}'^{\mathrm{sym}}_{ij}\bar{\sigma}'^{\mathrm{sym}}_{ji}}{4g\bar{\sigma}_{kk}} \end{aligned} \quad \text{...(70)}$$

Further development follows exactly as set out in section 3, with every occurrence of $\boldsymbol{\varepsilon}$ $[\varepsilon_{ij}]$ replaced by $\bar{\boldsymbol{\varepsilon}}$ $[\bar{\varepsilon}_{ij}]$ and $\boldsymbol{\sigma}$ $[\sigma_{ij}]$ by $\bar{\boldsymbol{\sigma}}$ $[\bar{\sigma}_{ij}]$. The necessary conversion to true stresses and strains are achieved by $\boldsymbol{\varepsilon} = \mathbf{a}^{-1}\bar{\boldsymbol{\varepsilon}}\mathbf{a}^{-\mathrm{T}}$ $[\varepsilon_{ij} = a^{-1}_{ik}\bar{\varepsilon}_{kl}a^{-1}_{jl}]$ and $\boldsymbol{\sigma} = \mathbf{a}\bar{\boldsymbol{\sigma}}\mathbf{a}^{\mathrm{T}}$ $[\sigma_{ij} = a_{ik}\bar{\sigma}_{kl}a_{jl}]$. Note that all references to the isotropic axis now refer to isotropy of $\bar{\boldsymbol{\sigma}}$, not $\boldsymbol{\sigma}$, and in general these conditions do not coincide.

The resulting model bears a superficial similarity to that of Gajo and Bigoni (2008) in that they use a strain energy which involves non-integer power functions of mixed invariants of the anisotropy tensor and elastic strain. However, their strain energy function involves multiples of the two invariants $\mathrm{tr}(\bar{\bar{\boldsymbol{\varepsilon}}})$ and $\mathrm{tr}(\bar{\bar{\boldsymbol{\varepsilon}}}^2)$ each separately raised to a power and then summed to give the strain energy function, *i.e.* of the form $A\left(-\mathrm{tr}(\bar{\bar{\boldsymbol{\varepsilon}}})\right)^n + B\left(\mathrm{tr}\left(\bar{\bar{\boldsymbol{\varepsilon}}}^2\right)\right)^m$, whereas ours involves the sum of multiples of the invariants, with this overall sum then raised to a power *i.e.* of the form $\left(A\left(-\mathrm{tr}(\bar{\bar{\boldsymbol{\varepsilon}}})\right)^2 + B\,\mathrm{tr}\left(\bar{\bar{\boldsymbol{\varepsilon}}}^2\right)\right)^m$. The two models are thus not equivalent, even if $2m = n$, as $a^m + b^m \neq (a+b)^m$. Houlsby *et al.* (2005, page 386) discuss, for the isotropic case, the merits of alternative ways of introducing non-linearity, and conclude that the method adopted here is the most versatile, as it allows explicit formulation of both the strain energy and the complementary energy.

6 Discussion and conclusion

We have described a hyperelastic model that can accommodate pressure-dependent nonlinearity (with stiffness a power function of mean stress) and anisotropy (through an anisotropy tensor). In Table 4 and 5 a summary of the necessary expressions is given for the derivation from complementary energy or strain energy respectively.

The model is fully defined by the following parameters:

p_r an arbitrary reference pressure (conveniently chosen, for instance, as atmospheric pressure, 100kPa)

k a dimensionless bulk modulus coefficient. Typical values for soils may vary quite widely, from say 50 to around 2000.

g a dimensionless shear modulus coefficient. Typically $g/k \approx 0.75$.

n a dimensionless exponent $0 \le n \le 1$ specifying the pressure-dependent nonlinearity. Stiffness is proportional to pressure to the power n. The value $n = 0$ corresponds to linear elasticity. For sands $n \approx 0.3 - 0.5$ and for clays data supports slightly higher values, and a commonly used assumption is that stiffness is proportional to pressure, implying $n = 1.0$.

N a switch parameter that determines whether the datum point for strain is at zero stress ($N = 0$) or at the reference pressure p_r ($N = 1$).

a $[a_{ij}]$ a symmetric second order anisotropy tensor. If we constrain this by $\det(\mathbf{a}) = 1$ [$\det(a_{ij}) = 1$], it is fully defined by five quantities, which may conveniently be taken as the (dimensionless) values of its principal components (eigenvalues) $(a, b, a^{-1}b^{-1})$ and the corresponding directions (eigenvectors) of its principal axes. Any positive values of a and b are allowable, but values relatively close to unity are expected.

Note that the most general linear orthotropic material requires specification of the directions of orthotropy and nine independent parameters, whereas our model only admits four (k, g, a, b). A general linear transverse isotropic material requires five parameters, whilst our model (with $a = b$) only admits three (k, g, a). Thus the choice that anisotropy can be defined through a symmetric second order tensor restricts the form of anisotropy that can be described, effectively imposing certain relationships between the parameters for the more general model (see Lodge, 1955).

Because it is based on a hyperelastic approach, the model guarantees thermodynamic acceptability. It is modular in form: nonlinearity and anisotropy can be incorporated separately or in combination. Simpler cases (for instance isotropic linear elasticity) are recovered simply by appropriate parameter settings. The model should find application either as appropriate for the description of granular materials at very small strains, or in describing the elastic strains as part of an elastic-plastic model. It has not been necessary in this work to distinguish between the various possible definitions of strain, as any of these would be appropriate provided that the corresponding work-conjugate stress is employed.

7 References

Amorosi, A., Rollo, F. and Houlsby, G.T. (2019) "A non-linear anisotropic hyperelastic formulation model for granular materials: comparison with existing models and validation", *Acta Geotechnica*, in press.

Bigoni, D. (2012) *Nonlinear Solid Mechanics*, Cambridge University Press

Bigoni, D. and Loret, B. (1999) "Effects of elastic anisotropy on strain localization and flutter instability in plastic solids", *J. Mech. Phys. Sol.*, 47, pp 1409-1426.

Boehler, J.P. (1987) *Applications of Tensor Functions in Solid Mechanics*, Springer Verlag, Vienna

Chaves, E.W.V. (2013) *Notes on Continuum Mechanics*, Springer

Gajo, A. and Bigoni, D. (2008) "A model for stress and plastic strain induced nonlinear, hyperelastic anisotropy in soils", *Int. J. Numer. Anal. Meth. Geomech.*, 32, pp 833–861.

Holzapfel, G.A. (2000) *Nonlinear Solid Mechanics*, John Wiley, Chichester

Houlsby, G.T., Amorosi, A. and Rojas, E. (2005) "Elastic moduli of soils dependent on pressure: a hyperelastic formulation", *Géotechnique*, 55(5), pp 383-392. doi:10.1680/geot.55.5.383.66021

Houlsby, G.T. and Puzrin, A.M. (2006) *Principles of Hyperplasticity: an Approach to Plasticity Theory Based on Thermodynamic Principles*, Springer Verlag, London, 375 pp

Lodge, A.S. (1955) "The transformation to isotropic form of the equilibrium equations for a class of anisotropic elastic solids", *Q. J. of Mech. and App. Math.*, 8, pp 211-255.

Spencer, A.J.M. (1971) "Theory of Invariants", in *Continuum Physics*, ed. Eringen, A., Vol. 1, pp 239-353, Academic Press, New York.

Zysset, P.K. and Curnier, A. (1995) "An alternative model for anisotropic elasticity based on fabric tensors", *Mechanics of Materials*, 21, pp 243-250.

8 Notation

a,b	Anisotropy factors
A,B	Anisotropy factors $A=a^2, B=b^2$
$\mathbf{a}, a_{ij}$	Anisotropy tensor
$\mathbf{B}, B_{ij}$	Anisotropy tensor $\mathbf{B}=\mathbf{a}^2, B_{ij}=a_{ik}a_{kj}$
$c_1 \ldots c_9$	Coefficients in general anisotropy expression
C	Complementary energy
$\mathbb{C}, C_{ijkl}$	Compliance matrix
$\mathbb{D}, D_{ijkl}$	Stiffness matrix
E	Strain energy
f	Helmholtz free energy
g	(1) Dimensionless shear modulus coefficient, (2) Gibbs free energy
G	Shear modulus
G_s	Secant shear stiffness
k	Dimensionless bulk modulus coefficient
K	Bulk modulus
n	Exponent in power-law relationship for stiffness
N	Switch parameter: $N=0$ datum for strain at zero stress, $N=1$ datum for strain at $p=p_r$ or $\bar{p}=p_r$ in the anisotropic case
p	Pressure, mean compressive effective stress
p_o	Stress function used in definition of nonlinearity
p_r	Reference pressure

q	Deviator stress in triaxial test
r_o	Strain function used in definition of nonlinearity
$\boldsymbol{\varepsilon}, \varepsilon_{ij}$	Strain tensor
$\bar{\boldsymbol{\varepsilon}}, \bar{\varepsilon}_{ij}$	Equivalent symmetric strain tensor in anisotropic model
$\bar{\bar{\boldsymbol{\varepsilon}}}, \bar{\bar{\varepsilon}}_{ij}$	Equivalent (but not symmetric) strain tensor in anisotropic model
$\varepsilon_v, \varepsilon_s$	Triaxial strain variables
γ	Engineering shear strain
λ	Lamé parameter
μ	Lamé parameter
ν	Poisson's ratio
$\boldsymbol{\sigma}, \sigma_{ij}$	Stress tensor
$\bar{\boldsymbol{\sigma}}, \bar{\sigma}_{ij}$	Equivalent symmetric stress tensor in anisotropic model
$\bar{\bar{\boldsymbol{\sigma}}}, \bar{\bar{\sigma}}_{ij}$	Equivalent (but not symmetric) stress tensor in anisotropic model
τ	Shear stress

Table 1: Second order tensors and their derivatives

	Component-free notation	Subscript notation	Differential (component-free notation)	Differential (subscript notation)
Tensor	$\mathbf{t}$	t_{ij}	$\frac{\partial \mathbf{t}}{\partial \mathbf{t}} = \mathbf{I} \overline{\otimes} \mathbf{I} = \mathbb{I}$	$\frac{\partial t_{ij}}{\partial t_{kl}} = \delta_{ik}\delta_{jl}$
Unit tensor	$\mathbf{I}$	δ_{ij} (Kronecker delta)		
Trace	$\mathrm{tr}(\mathbf{t})$	t_{kk}	$\frac{\partial}{\partial \mathbf{t}} \mathrm{tr}(\mathbf{t}) = \mathbf{I}$	$\frac{\partial t_{kk}}{\partial t_{ij}} = \delta_{ij}$
Trace of square	$\mathrm{tr}\left(\mathbf{t}^2\right)$	$t_{kl}t_{lk}$	$\frac{\partial}{\partial \mathbf{t}} \mathrm{tr}\left(\mathbf{t}^2\right) = 2\mathbf{t}^{\mathrm{T}}$	$\frac{\partial}{\partial t_{ij}}\left(t_{kl}t_{lk}\right) = 2t_{ji}$
Trace of cube	$\mathrm{tr}\left(\mathbf{t}^3\right)$	$t_{kl}t_{lm}t_{mk}$	$\frac{\partial}{\partial \mathbf{t}} \mathrm{tr}\left(\mathbf{t}^3\right) = 3\left(\mathbf{t}^2\right)^{\mathrm{T}}$	$\frac{\partial}{\partial t_{ij}}\left(t_{kl}t_{lm}t_{mk}\right) = 3t_{jk}t_{ki}$
Transpose	$\mathbf{t}^{\mathrm{T}}$	$t_{ij}^{\mathrm{T}} = t_{ji}$	$\frac{\partial \mathbf{t}^{\mathrm{T}}}{\partial \mathbf{t}} = \mathbf{I} \underline{\otimes} \mathbf{I} = \overline{\mathbb{I}}$	$\frac{\partial t_{ij}^{\mathrm{T}}}{\partial t_{kl}} = \delta_{jk}\delta_{il}$
Deviator	$\mathbf{t}' = \mathbf{t} - \frac{1}{3}\mathrm{tr}(\mathbf{t})\mathbf{I}$	$t'_{ij} = t_{ij} - \frac{1}{3}t_{kk}\delta_{ij} = \left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right)t_{kl}$	$\frac{\partial \mathbf{t}'}{\partial \mathbf{t}} = \mathbf{I} \overline{\otimes} \mathbf{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I} = \mathbb{P}$	$\frac{\partial t'_{ij}}{\partial t_{kl}} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$
Symmetric part	$\mathbf{t}^{\mathrm{sym}} = \frac{1}{2}\left(\mathbf{t} + \mathbf{t}^{\mathrm{T}}\right)$	$t_{ij}^{\mathrm{sym}} = t_{(ij)} = \frac{1}{2}\left(t_{ij} + t_{ji}\right)$	$\frac{\partial \mathbf{t}^{\mathrm{sym}}}{\partial \mathbf{t}} = \frac{1}{2}\left(\mathbf{I} \overline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{I}\right)$ $= \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} = \mathbb{I}^{\mathrm{sym}}$	$\frac{\partial t_{ij}^{\mathrm{sym}}}{\partial t_{kl}} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right)$
Skew part	$\mathbf{t}^{\mathrm{skew}} = \frac{1}{2}\left(\mathbf{t} - \mathbf{t}^{\mathrm{T}}\right)$	$t_{ij}^{\mathrm{skew}} = t_{[ij]} = \frac{1}{2}\left(t_{ij} - t_{ji}\right)$	$\frac{\partial \mathbf{t}^{\mathrm{skew}}}{\partial \mathbf{t}} = \frac{1}{2}\left(\mathbf{I} \overline{\otimes} \mathbf{I} - \mathbf{I} \underline{\otimes} \mathbf{I}\right) = \mathbb{I}^{\mathrm{skew}}$	$\frac{\partial t_{ij}^{\mathrm{skew}}}{\partial t_{kl}} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}\right)$
Inner product	$\mathbf{ab}$	$a_{ij}b_{jk}$		
Contraction	$\mathbf{a} : \mathbf{b}$	$a_{ij}b_{ij}$		
Tensor product	$\mathbf{a} \otimes \mathbf{b}$	$a_{ij}b_{kl}$		
	$\mathbf{a} \overline{\otimes} \mathbf{b}$	$a_{ik}b_{jl}$		

Table 1 continued …

	$\mathbf{a}\underline{\otimes}\mathbf{b}$	$a_{il}b_{jk}$		
Symmetric product	$\mathbf{a}\overline{\underline{\otimes}}\mathbf{b}$	$\frac{1}{2}\left(a_{ik}b_{jl}+a_{il}b_{jk}\right)$		
Inverse	$\mathbf{t}\mathbf{t}^{-1}=\mathbf{I}$	$t_{ij}t_{jk}^{-1}=\delta_{ik}$		
1st invariant	$I_{1t}=\mathrm{tr}(\mathbf{t})$	$I_{1t}=t_{kk}$	$\frac{\partial I_{1t}}{\partial \mathbf{t}}=\mathbf{I}$	$\frac{\partial I_{1t}}{\partial t_{ij}}=\delta_{ij}$
2nd invariant	$I_{2t}=\frac{1}{2}\left(\mathrm{tr}\left(\mathbf{t}^2\right)-\mathrm{tr}^2(\mathbf{t})\right)$	$I_{2t}=\frac{1}{2}\left(t_{ij}t_{ji}-t_{ii}t_{jj}\right)$	$\frac{\partial I_{2t}}{\partial \mathbf{t}}=\mathbf{t}^{\mathrm{T}}-\mathrm{tr}(\mathbf{t})\mathbf{I}$	$\frac{\partial I_{2t}}{\partial t_{ij}}=t_{ji}-I_{1t}\delta_{ij}$
3rd invariant	$I_{3t}=\frac{1}{6}\left(2\,\mathrm{tr}\left(\mathbf{t}^3\right)-3\,\mathrm{tr}\left(\mathbf{t}^2\right)\mathrm{tr}(\mathbf{t})+\mathrm{tr}^3(\mathbf{t})\right)$ $=\det(\mathbf{t})$	$I_{3t}=\frac{1}{6}\left(2t_{ij}t_{jk}t_{ki}-3t_{ij}t_{ji}t_{kk}+t_{ii}t_{jj}t_{kk}\right)$ $=\det\left(t_{ij}\right)$	$\frac{\partial I_{3t}}{\partial \mathbf{t}}=\left(\mathbf{t}^2\right)^{\mathrm{T}}-\mathrm{tr}(\mathbf{t})\mathbf{t}^{\mathrm{T}}-\mathrm{tr}\left(\mathbf{t}^2\right)\mathbf{I}$	$\frac{\partial I_{3t}}{\partial t_{ij}}=t_{jk}t_{ki}-I_{1t}t_{ji}-I_{2t}\delta_{ij}$
2nd invariant of deviator	$J_{2t}=I_{2t'}=\frac{1}{2}\mathrm{tr}\left(\mathbf{t}'^2\right)=\frac{1}{2}\left(\mathrm{tr}\left(\mathbf{t}^2\right)-\frac{1}{3}\mathrm{tr}^2(\mathbf{t})\right)$	$J_{2t}=I_{2t'}=\frac{1}{2}t'_{ij}t'_{ji}=\frac{1}{2}\left(t_{ij}t_{ji}-\frac{1}{3}t_{ii}t_{jj}\right)$	$\frac{\partial J_{2t}}{\partial \mathbf{t}}=\mathbf{t}'^{\mathrm{T}}$	$\frac{\partial J_{2t}}{\partial t_{ij}}=t'_{ji}$

Table 2: Fourth order unit tensors (Chaves, 2013), projection tensor (Holzapfel, 2000) and symmetric projection tensor

Component-free notation	Contraction with second order tensor	Index notation
$\mathbb{I}=\mathbf{I}\overline{\otimes}\mathbf{I}$	$\mathbb{I}:\mathbf{t}=\mathbf{t}$	$\mathbb{I}_{ijkl}=\delta_{ik}\delta_{jl}$
$\overline{\mathbb{I}}=\mathbf{I}\underline{\otimes}\mathbf{I}$	$\overline{\mathbb{I}}:\mathbf{t}=\mathbf{t}^{\mathrm{T}}$	$\overline{\mathbb{I}}_{ijkl}=\delta_{il}\delta_{jk}$
$\overline{\overline{\mathbb{I}}}=\mathbf{I}\otimes\mathbf{I}$	$\overline{\overline{\mathbb{I}}}:\mathbf{t}=\mathrm{tr}(\mathbf{t})\mathbf{I}$	$\overline{\overline{\mathbb{I}}}_{ijkl}=\delta_{ij}\delta_{kl}$
$\mathbb{I}^{\mathrm{sym}}=\frac{1}{2}\left(\mathbb{I}+\overline{\mathbb{I}}\right)$ $=\frac{1}{2}\left(\mathbf{I}\overline{\otimes}\mathbf{I}+\mathbf{I}\underline{\otimes}\mathbf{I}\right)=\mathbf{I}\overline{\underline{\otimes}}\mathbf{I}$	$\mathbb{I}^{\mathrm{sym}}:\mathbf{t}=\mathbf{t}^{\mathrm{sym}}=\frac{1}{2}\left(\mathbf{t}+\mathbf{t}^{\mathrm{T}}\right)$	$\mathbb{I}^{\mathrm{sym}}_{ijkl}=\frac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$
$\mathbb{I}^{\mathrm{skew}}=\frac{1}{2}\left(\mathbb{I}-\overline{\mathbb{I}}\right)$ $=\frac{1}{2}\left(\mathbf{I}\overline{\otimes}\mathbf{I}-\mathbf{I}\underline{\otimes}\mathbf{I}\right)$	$\mathbb{I}^{\mathrm{skew}}:\mathbf{t}=\mathbf{t}^{\mathrm{skew}}=\frac{1}{2}\left(\mathbf{t}-\mathbf{t}^{\mathrm{T}}\right)$	$\mathbb{I}^{\mathrm{skew}}_{ijkl}=\frac{1}{2}\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)$
$\mathbb{P}=\mathbb{I}-\frac{1}{3}\overline{\overline{\mathbb{I}}}$ $=\mathbf{I}\overline{\otimes}\mathbf{I}-\frac{1}{3}\mathbf{I}\otimes\mathbf{I}$	$\mathbb{P}:\mathbf{t}=\mathbf{t}'=\mathbf{t}-\frac{1}{3}\mathrm{tr}(\mathbf{t})\mathbf{I}$	$\mathbb{P}_{ijkl}=\delta_{ik}\delta_{jl}-\frac{1}{3}\delta_{ij}\delta_{kl}$
$\mathbb{P}^{\mathrm{sym}}=\frac{1}{2}\left(\mathbb{I}+\overline{\mathbb{I}}\right)-\frac{1}{3}\overline{\overline{\mathbb{I}}}$ $=\mathbf{I}\overline{\underline{\otimes}}\mathbf{I}-\frac{1}{3}\mathbf{I}\otimes\mathbf{I}$	$\mathbb{P}^{\mathrm{sym}}:\mathbf{t}=\mathbf{t}'^{\mathrm{sym}}$ $=\frac{1}{2}\left(\mathbf{t}+\mathbf{t}^{T}\right)-\frac{1}{3}\mathrm{tr}(\mathbf{t})\mathbf{I}$	$\mathbb{P}^{\mathrm{sym}}_{ijkl}=\frac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\frac{1}{3}\delta_{ij}\delta_{kl}$

Table 3: Triaxial variables. Following soil mechanics practice, a compressive positive convention is used for $p,q,\varepsilon_v,\varepsilon_s$, but all other stresses and strains use the tensile positive convention of continuum mechanics. The (tensile positive) axial and radial stresses are σ_a,σ_r and strains $\varepsilon_a,\varepsilon_r$.

Triaxial case	Invariants	Component-free notation	Index notation
$p=-\frac{1}{3}(\sigma_a+2\sigma_r)$	$p=-\frac{1}{3}I_{1\sigma}$	$p=-\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$	$p=-\frac{1}{3}\sigma_{ii}$
$q=-(\sigma_a-\sigma_r)$	$q^2=3J_{2\sigma}$	$q^2=\frac{3}{2}\mathrm{tr}\left(\boldsymbol{\sigma}'^2\right)$	$q^2=\frac{3}{2}\sigma'_{ij}\sigma'_{ji}$
$\varepsilon_v=-(\varepsilon_a+2\varepsilon_r)$	$\varepsilon_v=-I_{1\varepsilon}$	$\varepsilon_v=-\mathrm{tr}(\boldsymbol{\varepsilon})$	$\varepsilon_v=-\varepsilon_{ii}$
$\varepsilon_s=-\frac{2}{3}(\varepsilon_a-\varepsilon_r)$	$\varepsilon_s^2=\frac{4}{3}J_{2\varepsilon}$	$\varepsilon_s^2=\frac{2}{3}\mathrm{tr}\left(\boldsymbol{\varepsilon}'^2\right)$	$\varepsilon_s^2=\frac{2}{3}\varepsilon'_{ij}\varepsilon'_{ji}$

Table 4: Strain energy expressions (the switch parameter $N=0$ gives the origin for strain $\boldsymbol{\varepsilon}=0$ at $\boldsymbol{\sigma}=0$ [$\varepsilon_{ij}=0$ at $\sigma_{ij}=0$]; $N=1$ gives $\boldsymbol{\varepsilon}=0$ at $\bar{\boldsymbol{\sigma}}=-p_r\mathbf{I}$ [$\varepsilon_{ij}=0$ at $\bar{\sigma}_{ij}=-p_r\delta_{ij}$]).

<table>
<tr><td></td><td>Anisotropic
$\bar{\boldsymbol{\varepsilon}}=\mathbf{a}\boldsymbol{\varepsilon}\mathbf{a}^{\mathrm{T}}$ $\bar{\varepsilon}_{ij}=a_{im}\varepsilon_{mn}a_{jn}$
$\bar{\boldsymbol{\sigma}}=\dfrac{\partial E}{\partial \bar{\boldsymbol{\varepsilon}}}$ $\bar{\sigma}_{ij}=\dfrac{\partial E}{\partial \bar{\varepsilon}_{ij}}$
$\boldsymbol{\sigma}=\mathbf{a}\bar{\boldsymbol{\sigma}}\mathbf{a}^{\mathrm{T}}$ $\sigma_{ij}=a_{im}\bar{\sigma}_{mn}a_{jn}$</td><td>Isotropic</td></tr>
<tr><td>Linear</td><td>Substitute $n=0$ in general case, leading to
$E=\dfrac{p_r}{2}\left(k\,\mathrm{tr}^2\left(\bar{\boldsymbol{\varepsilon}}\right)+2g\,\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}'^{\mathrm{sym}^2}\right)\right)-Np_r\,\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}\right)$
$=\dfrac{p_r}{2}\left(k\bar{\varepsilon}_{ii}\bar{\varepsilon}_{jj}+2g\bar{\varepsilon}'^{\mathrm{sym}}_{ij}\bar{\varepsilon}'^{\mathrm{sym}}_{ji}\right)-Np_r\bar{\varepsilon}_{ii}$</td><td rowspan="3">Substitute:

$\mathbf{a}=\mathbf{I}$
so that $\bar{\boldsymbol{\varepsilon}}=\boldsymbol{\varepsilon}$
and $\boldsymbol{\sigma}=\dfrac{\partial E}{\partial \boldsymbol{\varepsilon}}$

$a_{ij}=\delta_{ij}$
so that $\bar{\varepsilon}_{ij}=\varepsilon_{ij}$
and $\sigma_{ij}=\dfrac{\partial E}{\partial \varepsilon_{ij}}$</td></tr>
<tr><td>Nonlinear</td><td>General case $0\le n<1$
$\bar{r}_o^2=\left(N-k\left(1-n\right)\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}\right)\right)^2+2gk\left(1-n\right)\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}'^{\mathrm{sym}^2}\right)$
$=\left(N-k\left(1-n\right)\bar{\varepsilon}_{ii}\right)^2+2gk\left(1-n\right)\bar{\varepsilon}'^{\mathrm{sym}}_{ij}\bar{\varepsilon}'^{\mathrm{sym}}_{ji}$
$E=\dfrac{p_r}{k(2-n)}\left[\bar{r}_o^{(2-n)/(1-n)}-N\right]$</td></tr>
<tr><td>Stiffness proportional to pressure,
$N=1$ only</td><td>Limit of general case as $n\to 1$
$E=\dfrac{p_r}{k}\left[\exp\left(-k\,\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}\right)+gk\,\mathrm{tr}\left(\bar{\boldsymbol{\varepsilon}}'^{\mathrm{sym}^2}\right)\right)-1\right]$
$=\dfrac{p_r}{k}\left[\exp\left(-k\bar{\varepsilon}_{ii}+gk\bar{\varepsilon}'^{\mathrm{sym}}_{ij}\bar{\varepsilon}'^{\mathrm{sym}}_{ji}\right)-1\right]$</td></tr>
</table>

Table 5: Complementary energy expressions (the switch parameter $N=0$ gives the origin for strain $\boldsymbol{\varepsilon}=0$ at $\boldsymbol{\sigma}=0$ [$\varepsilon_{ij}=0$ at $\sigma_{ij}=0$]; $N=1$ gives $\boldsymbol{\varepsilon}=0$ at $\bar{\boldsymbol{\sigma}}=-p_r\mathbf{I}$ [$\varepsilon_{ij}=0$ at $\bar{\sigma}_{ij}=-p_r\delta_{ij}$]).

<table>
<tr><td></td><td>Anisotropic
$\bar{\boldsymbol{\sigma}}=\mathbf{a}^{-1}\boldsymbol{\sigma}\mathbf{a}^{-\mathrm{T}}$ $\bar{\sigma}_{ij}=a^{-1}_{im}\sigma_{mn}a^{-1}_{jn}$
$\bar{\boldsymbol{\varepsilon}}=\dfrac{\partial C}{\partial\bar{\boldsymbol{\sigma}}}$ $\bar{\varepsilon}_{ij}=\dfrac{\partial C}{\partial\bar{\sigma}_{ij}}$
$\boldsymbol{\varepsilon}=\mathbf{a}^{-1}\bar{\boldsymbol{\varepsilon}}\mathbf{a}^{-\mathrm{T}}$ $\varepsilon_{ij}=a^{-1}_{im}\bar{\varepsilon}_{mn}a^{-1}_{jn}$</td><td>Isotropic</td></tr>
<tr><td>Linear $n=0$</td><td>Substitute $n=0$ in general case, leading to
$C=N\left(\dfrac{p_r}{2k}+\dfrac{\mathrm{tr}(\bar{\boldsymbol{\sigma}})}{3k}\right)+\dfrac{1}{2p_r}\left(\dfrac{\mathrm{tr}^2(\bar{\boldsymbol{\sigma}})}{9k}+\dfrac{1}{2g}\mathrm{tr}\left(\bar{\boldsymbol{\sigma}}'^{\mathrm{sym}^2}\right)\right)$
$=N\left(\dfrac{p_r}{2k}+\dfrac{\bar{\sigma}_{ii}}{3k}\right)+\dfrac{1}{2p_r}\left(\dfrac{\bar{\sigma}_{ii}\bar{\sigma}_{jj}}{9k}+\dfrac{1}{2g}\bar{\sigma}'^{\mathrm{sym}}_{ij}\bar{\sigma}'^{\mathrm{sym}}_{ji}\right)$</td><td rowspan="3">Substitute:

$\mathbf{a}^{-1}=\mathbf{I}$
so that $\bar{\boldsymbol{\sigma}}=\boldsymbol{\sigma}$
and $\boldsymbol{\varepsilon}=\dfrac{\partial C}{\partial\boldsymbol{\sigma}}$

$a^{-1}_{ij}=\delta_{ij}$
so that $\bar{\sigma}_{ij}=\sigma_{ij}$
and $\varepsilon_{ij}=\dfrac{\partial C}{\partial\sigma_{ij}}$</td></tr>
<tr><td>Nonlinear</td><td>General case $0\le n<1$
$\bar{p}_o^2=\dfrac{\mathrm{tr}^2(\bar{\boldsymbol{\sigma}})}{9}+\dfrac{k(1-n)}{2g}\mathrm{tr}\left(\bar{\boldsymbol{\sigma}}'^{\mathrm{sym}^2}\right)$
$=\dfrac{\bar{\sigma}_{ii}\bar{\sigma}_{jj}}{9}+\dfrac{k(1-n)}{2g}\bar{\sigma}'^{\mathrm{sym}}_{ij}\bar{\sigma}'^{\mathrm{sym}}_{ji}$
$C=N\left(\dfrac{p_r}{k(2-n)}+\dfrac{\mathrm{tr}(\bar{\boldsymbol{\sigma}})}{3k(1-n)}\right)+\dfrac{p_r}{k(1-n)(2-n)}\left(\dfrac{\bar{p}_o}{p_r}\right)^{2-n}$
$=N\left(\dfrac{p_r}{k(2-n)}+\dfrac{\bar{\sigma}_{ii}}{3k(1-n)}\right)+\dfrac{p_r}{k(1-n)(2-n)}\left(\dfrac{\bar{p}_o}{p_r}\right)^{2-n}$</td></tr>
<tr><td>Stiffness proportional to pressure,
$N=1$ only</td><td>Limit of general case as $n\to1$
$C=\dfrac{p_r}{k}-\dfrac{\mathrm{tr}(\bar{\boldsymbol{\sigma}})}{3k}\left[\ln\left(\dfrac{-\mathrm{tr}(\bar{\boldsymbol{\sigma}})}{3p_r}\right)-1\right]-\dfrac{3\,\mathrm{tr}\left(\bar{\boldsymbol{\sigma}}'^{\mathrm{sym}^2}\right)}{4g\,\mathrm{tr}(\bar{\boldsymbol{\sigma}})}$
$=\dfrac{p_r}{k}-\dfrac{\bar{\sigma}_{ii}}{3k}\left[\ln\left(\dfrac{-\bar{\sigma}_{jj}}{3p_r}\right)-1\right]-\dfrac{3\bar{\sigma}'^{\mathrm{sym}}_{ij}\bar{\sigma}'^{\mathrm{sym}}_{ji}}{4g\bar{\sigma}_{kk}}$</td></tr>
</table>