Revisiting Differential Categories: New Results on the Foundation of Differentiation

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Dedications

Je dédie cette thèse à ma famille et surtout à mes deux grand-papas : Henry-Paul Lemay et Yvan Pacaud. Vous me manquez tous et je vous aime.

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Epigraph

“ *I really need to restructure my life so I can spend more time reading abstracts and less time punching dinosaurs.* ”

-Atomic Robo, *The Shadow From Beyond Time, Issue 5*,
by Brian Clevinger (Author) and Scott Wegener (Illustrator)

Abstract

The theory of differential categories uses category theory to provide the mathematical foundations of differentiation in both mathematics and computer science. Differential categories are successful because they capture both the classical limit definition of differentiation and the more algebraic synthetic definition of differentiation. As such, the theory of differential categories has been able to formalize various aspects of differential calculus, from the very basic foundational aspects of differentiation, such as the notion of derivations from commutative algebras, to the more complex notions of differential geometry, such as the tangent bundle of a smooth manifold. The theory of differential categories has a rich literature and has been used to study differentiation in a variety of fields from mathematics, such as commutative algebra and differential geometry, as well as computer science, such as in machine learning and differentiable programming languages. This thesis is split into three parts, each relating to the theory of differential categories.

The first part of this thesis addresses a long-standing problem amongst the differential category theory community. Differential categories can either be axiomatized in terms of a deriving transformation or in terms of a codereliction. It has long been known that coderelictions are equivalent to deriving transformations that satisfy an extra rule called the ∇-rule. However, by definition, not every deriving transformation needs to satisfy the ∇-rule. As such, this left open the question of whether for categorical models of differential linear logic, coderelictions and deriving transformations were distinct notions of differentiation. However, we show that for a monoidal coalgebra modality, it turns out that every deriving transformation automatically satisfies the ∇-rule and that, in fact, it is equivalent to the product rule. Therefore, for a monoidal coalgebra modality, there is a bijective correspondence between coderelictions and deriving transformations. Thus, there is only one notion of differentiation in linear logic.

The second part of this thesis introduces the notion of a linearizing combinator which abstracts linearization in the Abelian functor calculus. We explain how a linearizing combinator provides an alternative axiomatization of a Cartesian differential category. Indeed, every Cartesian differential category comes equipped with a canonical linearizing combinator obtained by differentiation at zero. Conversely, a differential combinator can be constructed when one has a system of partial linearizing combinators in each context. Thus, while linearizing combinators do provide an alternative axiomatization of Cartesian differential categories, an explicit notion of partial linearization is required. The ability to form a system of partial linearizing combinators from a total linearizing combinator, while not being possible in general, is possible when the setting is Cartesian closed.

The third part of this thesis introduces differential exponential maps in Cartesian differential categories, which generalize the exponential function e^x from classical differential calculus. Every differential exponential map induces a commutative rig, which we call a differential exponential rig, and conversely, every differential exponential rig induces a differential exponential map. In particular, differential exponential maps can be defined without the need of limits, converging power series, or unique solutions of certain differential equations – which most Cartesian differential categories do not necessarily have. That said, we do explain how every differential exponential map does provide solutions to certain differential equations, and conversely how in the presence of unique solutions, one can derivative a differential exponential map.

Each part of this thesis is an important result to the theory of differential categories.

Contents

Chapter 1

Introduction

The topic of this thesis is in the field of **category theory**, specifically on the **theory of differential categories**. As the name suggests, the theory of differential categories uses category theory to provide and study the foundations and applications of differentiation in a variety of contexts from all fields of mathematics, computer science, and beyond. This thesis is a collection of the following three journal papers (for convenience, an open-source link of a pdf version is provided for each):

(i) [7] Blute, R. F., Cockett, J. R. B., Lemay, J-S. P., and Seely, R. A. G. (2020) **Differential categories revisited**. Applied Categorical Structures. 28: 171-235.

`https://arxiv.org/pdf/1806.04804.pdf`

(ii) [64] Lemay, J-S. P. (2021) **Exponential Functions for Cartesian Differential Categories**. Applied Categorical Structures. 29: 95-140.

`https://arxiv.org/pdf/1911.04790.pdf`

(iii) [29] Cockett, J. R. B., and Lemay, J-S. P. (2020) **Linearizing Combinators**. Preprint and submit to a journal.

`https://arxiv.org/pdf/2010.15490.pdf`

It is also worth mentioning that the first journal paper [7] is a full story version and extension of the following conference paper:

(iv) [26] Cockett, J. R. B., and J-S. P. Lemay. (2017) **There is only one notion of differentiation**. In the proceedings in Formal Structures for Computation and Deduction (FSCD) 2017. Leibniz International Proceedings in Informatics. 84: 13:1–13:21.

`https://drops.dagstuhl.de/opus/volltexte/2017/7716/pdf/LIPIcs-FSCD-2017-13.pdf`

Keen-eyed readers will note that two-thirds of these journal papers are in fact collaborations with multiple co-authors. In mathematics and theoretical computer science, authors are listed alphabetically and there is no designated lead author. While these journal papers are indeed collaborations, I was the lead investigator on these projects, and so I was responsible for all the mathematical proofs, as well as writing the entirety of the papers' main body. As such, everything found in this thesis was written by me, specifically all the multiple proofs, calculations, and string diagrams. That said, I am extremely thankful to Professors Blute, Cockett, and Seely, the fathers of differential categories, and I am very grateful to have had the chance to work with them.

This introduction chapter provides some history on differential categories, as well as an overview of the "world of differential". We end this chapter with a short outline of this thesis and an important discussion on the notational conventions used throughout this thesis. This introduction chapter is kept somewhat short since each chapter begins with a lengthy introduction which explain their story.

1.1 A Short History of Differential Categories

Differential calculus is arguably the most important, famous, and successful area of mathematics with applications throughout out all of science, economics, and engineering. As the well-known story goes, differential calculus was developed independently by Gottfried Leibniz and Isaac Newton in the late 17th century and early 18th century. In mathematics, the differential and its generalizations are used in a wide variety of fields such as multivariable calculus, differential geometry, algebraic geometry, synthetic differential geometry, and commutative algebra. On the other hand in computer science, the differential and its generalizations appear in numerous fields such as machine learning, discrete calculus, programming languages, and the λ-calculus.

Category theory [67] is the study of mathematical structure and has become the primary tool to express foundations and axiomatics. Category theory has been used throughout pure mathematics and theoretical computer science and has recently expanded its applications in other fields thanks in part to the "applied category theory" community. Of particular interest to this thesis is of course the use of category theory to study the foundations of differential calculus. Since calculus is revered for its place in applied mathematics, one rarely considers its foundations. The foundational perspective broadens our views and understanding of calculus: why it works and why it behaves the way it does. A foundational description of calculus provides a way of transporting ideas from calculus, such as differentiation and integration, to settings where there is no sensible notion of limit and where the differential is defined synthetically. The theory of differential categories, which we describe in detail in the next section, uses category theory to provide the mathematical foundations of differentiation in both mathematics and computer science. Differential categories are successful because they capture both the classical limit definition of differentiation and the more algebraic synthetic definition of differentiation. As such, the theory of differential categories has been able to formalize various aspects of differential calculus, from the very basic foundational aspects of differentiation, such as the notion of derivations from commutative algebras, to the more complex notions of differential geometry, such as the tangent bundle of a smooth manifold.

The story of differential categories does not in fact start from category theory or other classical mathematical fields such as differential geometry or commutative algebra. The prologue to this story actually originates from computer science, more specifically linear logic. In [34, 35], Ehrhard provided models of linear logic based on Köthe sequence spaces and finiteness spaces. Ehrhard noted that in these models there was a natural notion of a differential operator. From this observation, Ehrhard and Regnier introduced the differential λ-calculus [37], differential proof nets [38], and differential linear logic [36]. The key observation here was that differentiation from a logic point of view allowed one to count the number of times a variable was being used by computing how many times it took differentiating in terms of said variable before obtaining zero. In particular, being linear in the logic sense means using an argument exactly once, and this indeed corresponds to the mathematical point of view of linearity, since the second derivative of a linear function is zero. Since categorical semantics of linear logic and the λ-calculus are well studied, it was then natural to generalize the differential operators for their respective categorical model settings. This is where the story of differential categories begins.

Blute, Cockett, and Seely first introduced differential categories [8] to provide the categorical semantics of differential linear logic. Then a few years later, Blute, Cockett, and Seely introduced Cartesian differential categories [9] to provide the categorical semantics of the differential λ-calculus. However, the definitions of differential categories and Cartesian differential categories are in fact strictly weaker than a categorical model of differential linear logic and the differential λ-calculus. As such, for the differential linear logic side of the story, Fiore proposed in [39] an alternative axiomatization for differentiation of categorical models of linear logic. This left open the question of whether Fiore's approach was distinct and stronger than the Blute, Cockett, and Seely approach. Unfortunately, this caused a slight rift between the European and Canadian differential category communities. Luckily, as shown in [7, 26], it turns out that these two approaches were in fact the same, and peace was eventually made. On the other hand, for the differential λ-calculus side of the story, their categorical models were studied in greater detail by Manzonetto in [69], Cockett and Gallagher in [18], and also by Bucciarelli, Ehrhard, and Manzonetto [14].

Cockett, Gallagher, and Cruttwell then introduced differential restriction categories [22] which generalized Cartesian differential categories by allowing the notion of partially defined functions. At this point, differential categories correspond to differentiation in commutative algebra, Cartesian differential categories correspond to multivariable differential calculus over Euclidean spaces, and differential restriction categories correspond to multivariable differential calculus over open subsets. So the next logical step was to develop the theory that captures differential calculus over smooth manifolds. It turns out that this had already been done long before even differential linear logic. Indeed, Rosický in [81] introduced the abstraction of the tangent bundle over smooth manifolds in the context of category theory. Years later, Cockett and Cruttwell slightly generalized Rosický's approach and introduced tangent categories [19], which in particular allowed for models from computer science and provided a direct link with synthetic differential geometry. This is, so far, the last of the four chapters of the theory of differential categories.

The theory of differential categories now has a rich literature and has been used to study differentiation in a variety of fields. As of the writing of this, while it is true that differential categories have not yet been popularized in mainstream mathematics, they are trending in that direction. Differential categories are now well respected amongst category theorists, being the focus of recent category theory conferences and seminars, as well as having particularly gained interest amongst researchers in the renowned Australian category theory community, such as work done by Garner [40, 41] and Leung [65]. Differential categories have also found interest in higher category theory such as in [4, 5]. There has also recently been explicit applications of differential categories, specifically tangent categories, to differential geometry, which has allowed for new (and sometimes cleaner) formulations and deeper understandings in the field. Evidence of this can be found in the works of MacAdam [68], Burke and MacAdam [15], and Lucyshyn-Wright [66]. On the other hand, the theory of differential categories has definitely found its place in computer science. In particular, differential categories have been picked up by Sprunger and Katsumata in their work on causal computations [85], by Abadi and Plotkin in their work on differentiable programming languages [1], by Alvarez-Picallo and Ong in their work on incremental computation [2], by Laird, Manzonetto, and McCusker on their work in game theory [57], and lastly for its usage in machine learning with the introduction of *reverse* differential categories [23]. Furthermore, the theory of differential categories has recently been given a code in the 2020 version of the Mathematical Sciences Classification System. As such, the theory of differential categories is an active and healthy field of research with a bright future moving forward.

1.2 The World of Differential Categories

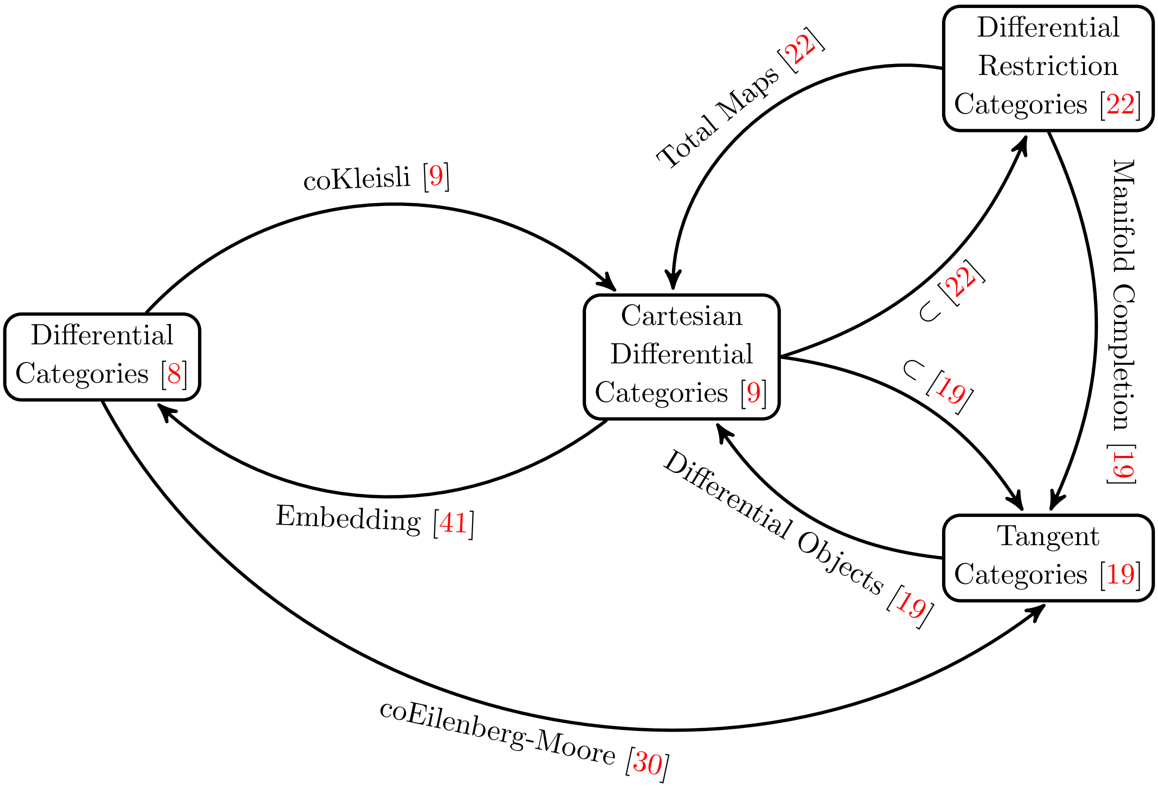


Figure 1.1: The world of differential categories and how it's all connected.

The theory of differential categories was developed in four stages with each stage formalizing a different aspect of the theory of differentiation. For the full definition, the curious readers can see the stage's introductory paper.

- **Differential categories** [8] formalize the basic algebraic properties of the derivative by generalizing the notion of derivations from commutative algebra. A differential category is a monoidal category with a comonad ! equipped with a *deriving transformation* $\mathsf{d} : !(A) \otimes A \longrightarrow !(A)$ which behaves like a derivation (in the opposite sense) on $!(A)$. The axioms of a deriving transformation include that the derivative of a constant function is zero, the product rule, that the derivative of a linear function is itself, and the chain rule. For example, the opposite category of the category of vector spaces over a field $\mathbb{K}$ is a differential category where $!(V) = \mathsf{Sym}(V)$ is the free symmetric algebra over V, and where the deriving transformation is defined by the standard differentiation of polynomials. Differential categories are studied in detail in Chapter 3.

- **Cartesian differential categories** [9] formalize the theory of multivariable differential calculus by axiomatizing the directional derivative. A Cartesian differential category is a category with finite products that comes equipped with a *differential combinator* D which for every map $f : A \longrightarrow B$ produces its derivatives $\mathsf{D}[f] : A \times A \longrightarrow B$. The differential combinator axiomatizes the basic properties of the directional derivative such as the chain rule, linearity in its second arguments, and symmetry of the mixed partial derivatives. The canonical example of a Cartesian differential category is the category of Euclidean spaces $\mathbb{R}^n$ and smooth functions between them, where the differential combinator is induced by the directional derivative of smooth scalar functions. Cartesian differential categories are studied in detail in Chapter 4 and Chapter 5.

- **Differential restriction categories** [22] instead formalize the theory of multivariable differential calculus over open subsets and differentiation of partially defined functions. Categorically, the notion of partial functions is captured by *restriction categories* [25] where for every map f there is a characterization of its domain of definition. A differential restriction category is a Cartesian differential category which is also a restriction category such that the differential combinator and the restriction structure are compatible. The category of open subsets of Euclidean spaces $U \subseteq \mathbb{R}^n$ and smooth functions between them, such as $\frac{1}{x}$ and $\mathsf{ln}(x)$, is a differential restriction category.

- **Tangent categories** [19] formalize tangent bundle structure and the theory of differential calculus over smooth manifolds, which has been used to formalize concepts from differential geometry, algebraic geometry, and synthetic differential geometry. A tangent category is a category equipped with an endofunctor T, whose functorality captures the chain rule, and natural transformations that axiomatizes the basic properties of the tangent bundle such as the natural projection, the zero vector field, the canonical lift, and the canonical flip. The canonical example is the category of smooth manifolds where $\mathsf{T}(M)$ is the classical tangent bundle over M. We briefly discuss tangent categories in Chapter 5.

As shown by the arrows in Figure 1 above, one of the most fascinating and important aspects of the theory of differential categories is that these four stages are connected by various constructions.

- A famous and important result in linear logic is that the coKleisli category of a categorical model of linear logic is a categorical model of the λ-calculus. This result still holds on the differential side of the story. Explicitly, the coKleisli category of a differential category is a Cartesian differential category [9, Proposition 3.2.1], which we discuss in detail in Chapter 4. Therefore, a smooth map from A to be B is interpreted as a coKleisli map $!A \longrightarrow B$. This implies that from a basic algebraic model of differentiation, one can construct a more complex model of multivariable differential calculus. As characterized by Blute, Cockett, and Seely, Cartesian differential *storage* categories [10] are precisely the Cartesian differential categories which are coKleisli categories of differential categories.

- Conversely, it is always possible to embed a Cartesian differential category in the coKleisli category of a differential category [41, Theorem 8.7].

- While the coKleisli category of a differential category has been well studied, the coEilenberg-Moore category has received far less attention. It turns out that the dual of the coEilenberg-Moore category is always a tangent category [30, Theorem 22], and that under a mild limit condition, the coEilenberg-Moore category is a representable tangent category [30, Theorem 27]. As such, the comonad coalgebras can be interpreted as generalized smooth manifolds. This implies that from a basic algebraic model of differentiation, one can construct a more complex model of differential geometry.

- For a differential restriction category, its subcategory of maps which are totally defined is a Cartesian differential category. Conversely, every Cartesian differential category is a differential restriction category where every map is totally defined.

- Every Cartesian differential category is a tangent category [19, Proposition 4.7] where the tangent bundle functor is defined on objects as $\mathsf{T}(A) = A \times A$ and on maps used the differential combinator D. The chain rule for the differential combinator implies the functorality of this functor T. We discuss this briefly in Chapter 5.

- If the object of a tangent category should be interpreted as smooth manifolds, then the differential objects [19, Definition 4.8] are those which generalize the Euclidean spaces $\mathbb{R}^n$. In particular, the tangent bundle of a differential object A is precisely of the form $\mathsf{T}(A) = A \times A$. For any tangent category, its category of differential objects is a Cartesian differential category [19, Theorem 4.11].

- Applying the manifold construction [19, Section 6] to a differential restriction category to obtain a tangent category corresponds to the fact that smooth manifolds are defined as atlases of open subsets of Euclidean spaces.

Keen-eyed readers may have noticed that certain arrows are missing between some of the stages. Completion of the "map of differential categories" is one of the main goals of the differential category community. Many researchers are currently working towards this goal and hopefully, the map will be complete in the near future.

1.3 Thesis Outline

As mentioned above, the main theoretical chapters of this thesis are based on the three journal papers above. Explicitly, Chapter 2 and Chapter 3 are based on [7], Chapter 4 is based on [29], and lastly Chapter 5 is based on [64]. In terms of the theory of differential categories, Chapter 2 and Chapter 3 have to do with differential categories, while Chapter 4 and Chapter 5 switch gears and work with Cartesian differential categories.

Chapter 2 revisits the various notions of coalgebra modalities, which are key components of categorical models of linear logic. This provides definitions and examples of key notions for this thesis including comonoids, comonads, coalgebra modalities, monoidal coalgebra modalities, Seely isomorphisms, bialgebra modalities, and, most importantly, additive bialgebra modalities. The main result of this chapter is showing that additive bialgebra modalities are equivalent to monoidal coalgebra modalities, which is a key result for Chapter 3. This chapter also introduces the notation and conventions used for symmetric monoidal categories and their graphical calculus.

Chapter 3 revisits the axiomatization of a differential category. This chapter reviews the definition of both deriving transformations and coderelictions, as well as providing a long list of interesting examples. This chapter also reviews the bijective correspondence between coderelictions and deriving transformation that satisfy the extra axiom called the ∇-rule for bialgebra modalities. The main result of this chapter is that for an additive bialgebra modality, every deriving transformation satisfies the ∇-rule. Therefore, for an additive bialgebra modality (or equivalently a monoidal coalgebra modality) there is a bijective correspondence between coderelictions and deriving transformations. As such, this shows that the Canadian and European approaches to differential categories were, in fact, the same. Thus, there is only one notion of differentiation in linear logic.

Chapter 4 introduces the notion of a linearizing combinator, which provides an alternative axiomatization of a Cartesian differential category. This chapter also reviews Cartesian differential categories, as well as providing numerous examples and reviewing how the coKleisli category of a differential category is a Cartesian differential category. The main result of this chapter is that for a Cartesian left additive category, there is a bijective correspondence between differential combinators and a system of linearizing combinators, which captures the ability for partial linearization. In the closed setting, this bijective correspondence extends to include total linearization.

Chapter 5 introduces differential exponential maps in Cartesian differential categories, which generalize the exponential function from classical differential calculus. Every differential exponential map induces a commutative rig, called a differential exponential rig, and conversely, every differential exponential rig induces a differential exponential map. In particular, differential exponential maps can be defined without the need of limits, converging power series, or unique solutions of certain differential equations – which most Cartesian differential categories do not necessarily have. That said, this chapter also explains how every differential exponential map does provide solutions to certain differential equations, and conversely how in the presence of unique solutions, one can derivative a differential exponential map. This chapter also studies differential exponential maps in the coKleisli categories of differential categories.

1.4 Conventions

For the remainder of the thesis, we switch to using the academic “we” instead of using “I”.

We assume a knowledge of basic category theory including the definitions and properties of categories, functors, natural transformation, duality and various kinds of maps (such as isomorphism, epimorphisms, etc.). Very importantly, we use diagrammatic order for composition: this means that the composite map $fg : A \longrightarrow C$ is the map which first does $f : A \longrightarrow B$ then $g : B \longrightarrow C$. We denote identity maps simply as $1 : A \longrightarrow A$ and natural transformations as $\eta : FA \longrightarrow GA$, so we omit the subscript $\_{A}$ as to not overload notation. Any other notational convention (such as for monoidal categories, products, etc.) will be introduced when appropriate throughout the thesis. A list of important symbols and terms can be found in the index, at the end of this thesis after the bibliography.

Chapter 2

Coalgebra Modalities Revisited

This chapter is based on [7, Sections 2,3,7,8 & Appendix A, B, C, D], as well as borrowing some bits from [61]. As such, the author would like to thank their coauthors Rick Blute, Robin Cockett, and Robert A. G. Seely, as well as the anonymous referee from "Applied Categorical Structures" for very helpful and constructive comments in their review. In particular, this chapter also introduces the conventions and notations for symmetric monoidal categories, their graphical calculus, and the various notions of coalgebra modalities.

Linear logic, as introduced by Girard [42], is a resource-sensitive logic which due to its flexibility admits multiple different fragments and a wide range of applications. A categorical model of the multiplicative fragment of intuitionistic linear logic (MILL) [6, 43] is a symmetric monoidal closed category. The multiplicative and exponential fragments of intuitionistic linear logic (MELL) adds in the exponential modality which is a unary connective !, read as either "of course" or "bang", admitting four structural rules [6, 72]: promotion, dereliction, contraction, and weakening.

$$\frac{\Gamma,\ A \vdash B}{\Gamma,\ !A \vdash B} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \qquad \frac{\Gamma,\ !A,\ !A \vdash B}{\Gamma,\ !A \vdash B} \qquad \frac{\Gamma \vdash B}{\Gamma,\ !A \vdash B}$$

In terms of the categorical semantics of intuitionistic linear logic [6, 71, 72, 82, 83], the exponential modality ! is interpreted as a monoidal coalgebra modality [7, 10] (Definition 2.2.3), also known as a linear exponential comonad [82], which in particular is a symmetric monoidal comonad, capturing the promotion and dereliction rules, such that for each object A, the exponential $!A$ comes equipped with a natural cocommutative comonoid structure, capturing the contraction and weakening rules. Categorical models of MELL are known as linear categories [6, 71, 72], which are symmetric monoidal closed categories with monoidal coalgebra modalities. There are many examples of monoidal coalgebra modalities throughout linear logic literature [50, Section 2.4]. An important source of examples of monoidal coalgebra modalities are those for which $!A$ is also the cofree cocommutative comonoid over A, as shown by Lafont in his PhD thesis [56]. Monoidal coalgebra modalities with this extra couniversal property are known as free exponential modalities [73] and models of linear logic with free exponential modalities are known as Lafont categories [72].

It turns out, however, that axiomatizing differentiation [8] only requires the slightly weaker notion of a coalgebra modality (Definition 2.1.5), which drops the requirement that the underlying comonad is symmetric monoidal. Linear logicians will often overlook mere coalgebra modalities due to the lack of Seely isomorphisms [6, 10, 83]. Indeed, in the presence of finite products, it is a well-known result that a monoidal coalgebra modality is equivalent to a coalgebra modality with Seely isomorphisms (Definition 2.2.7), and so, in particular, $!(A \times B) \cong !A \otimes !B$. In this case, linear categories with finite products are known as monoidal storage categories [7, 10] or (new) Seely categories [6, 71, 72]. As such, there is a tendency to assume that the only important coalgebra modalities are the monoidal coalgebra modalities, i.e., those which arise through linear logic. While it is certainly true that monoidal coalgebra modalities have had the greater majority of the attention of researchers and in the literature, this does not mean that "mere" coalgebra modalities are not worthy of attention. Some compelling examples of non-monoidal coalgebra modalities include the one induced by free $\mathcal{C}^\infty$-rings [33, Remark 5.16] and the one induced by free Rota-Baxter algebras [7, Example 4].

The underlying structure of a differential category is that of an additive symmetric monoidal category (Definition 2.3.1), which is essentially a symmetric monoidal category where one can add parallel maps. As we will see in the next chapter, for coalgebra modalities, the differential operator is captured by a natural transformation called a deriving transformation. In [8], it was observed that if a coalgebra modality came equipped with a natural bialgebra structure, then the differential structure could instead be described via a natural transformation called a codereliction. Such coalgebra modalities with a natural bialgebra structure are called bialgebra modalities (2.3.5). Thus for a bialgebra modality, coderelictions are equivalent to deriving transformations that satisfy an extra axiom called the ∇-rule. This latter rule was originally thought to be a completely independent requirement. It turns out that if one slightly strengthens the notion of a bialgebra modality, the ∇-rule is in fact implied. These stronger bialgebra modalities are called *additive* bialgebra modalities (Definition 2.3.7), which are the main novel contribution of this chapter. Briefly, additive bialgebra modalities are bialgebra modalities such that the additive structure and the bialgebra structure are compatible via bialgebra convolution. For an additive bialgebra modality, every deriving transformation automatically satisfies the ∇-rule, and therefore, in this case, deriving transformations and coderelictions are equivalent. Thus additive bialgebra modalities are of fundamental importance to the theory of differential categories.

Additive bialgebra modalities and monoidal coalgebra modalities are closely related. In fact, it turns out that in the presence of additive structure, additive bialgebra modalities are equivalent to monoidal coalgebra modalities (Theorem 2.3.25), which is the main result of this chapter. In the presence of finite (bi)products, additive bialgebra modalities are then also equivalent to coalgebra modalities with Seely isomorphisms (Theorem 2.3.27). With regards to the differential story, it follows that for a monoidal coalgebra modality, deriving transformations and coderelictions are again equivalent. This observation of equivalence between deriving transformations and coderelictions in certain cases required precisely the axioms of an additive bialgebra modality, which are somewhat hidden when one considers only the axioms of a monoidal coalgebra modality. The proof that a monoidal coalgebra modality is an additive bialgebra modality is mostly simple, as there are not too many extra required identities to check. In fact, it is quite possible that this result was known but the proof was folklore, in the sense that it had never been written down in detail. The converse direction, proving that an additive bialgebra modality is a monoidal coalgebra modality is definitely a new observation since the notion of an additive bialgebra modality is novel. Furthermore, this direction requires much work since the constructions are more complex and there are quite a few extra identities one must check in detail. To help with these proofs, we make use of the graphical calculus for symmetric monoidal categories [7, 54, 84] and use string diagrams for our calculations. This has the advantage over long strings of algebraic equations since this provides a clearer picture of each step that is being taken. That said, these calculations may still seem as simply using brute force and thus quite tedious. Indeed, we do not claim that these proofs are optimized or even the most elegant. The merit of this direct proof is that it must at least be done once (which is a suitable task for some sacrificial graduate student), providing the result with absolute certainty, as well as recording a complete demonstration of this equivalence.

Chapter Outline: Section 2.1 reviews the definitions of comonads and coalgebra modalities, as well as introducing the notation and conventions used for symmetric monoidal categories and the graphical calculus. Section 2.2 reviews the definitions of symmetric monoidal comonads, monoidal coalgebra modalities, and the Seely isomorphisms. Section 2.3 is the main section of this chapter, which introduces additive bialgebra modalities and proves that they are in bijective correspondence with monoidal coalgebra modalities for additive symmetric monoidal categories. This section also discusses the biproduct completion of an additive symmetric monoidal category and how additive bialgebra modalities/monoidal coalgebra modalities lift to the biproduct completion. Lastly, Section 2.4 provides a construction of a family of non-monoidal coalgebra modalities induced by a monoidal coalgebra modality. For an additive bialgebra modality, this construction results in a family of non-additive bialgebra modalities.

2.1 Coalgebra Modalities

In this section we review the notion of a coalgebra modality [8], one of the key components of a differential category. In brief, coalgebra modalities are comonads such that each cofree coalgebra comes equipped with a natural cocommutative comonoid structure. As mentioned above, coalgebra modalities are strictly weaker structure than what is required for a categorical model of MELL, for that one requires a *monoidal* coalgebra modality, which we will review in the next section. However, coalgebra modalities are sufficient to axiomatize differentiation.

Throughout this thesis we shall make extensive use of the graphical calculus [54] for symmetric monoidal categories as this makes proofs easier to follow. Specifically, we will use the same string diagrams and conventions used in [7]. As such, our diagrams are to be read from top to bottom and we shall often omit labeling wires with objects. We refer the reader to [84] for an introduction to the graphical calculus in monoidal categories and its variations, and to [7] for the graphical calculus of a differential category, which in turn is based on the graphical calculus found in [8]. For most equalities, we provide both a commutative diagrams and string diagram presentation, so that readers wishing to skip proofs may choose which they prefer to read the stories as. Also, whenever possible, we try and avoid modifying the size of the string diagrams as much as possible. That said, certain string diagrams are quite large and so we are forced to resize them. As a result, the formatting of certain pages may seem quite strange, as the template tries to accommodate these numerous string diagrams of various sizes. We hope that this won't be too much of an inconvenience for the reader. And if so, we refer the reader to the journal paper version of this story [7].

We will be working with coalgebra modalities which in particular involves an endofunctor !, and so as in [7,8] we will use functor boxes when dealing with string diagrams involving the endofunctor. A mere map $f : A \longrightarrow B$ will be encased in a circle while $!(f) : !A \longrightarrow !B$ will be encased in a box:

If only to introduce notation and provide a simple graphical calculus example, we now recall the definition of a comonad.

Definition 2.1.1 *A* ***comonad*** *[72, Section 6.8] on a category* $\mathbb{X}$ *is a triple* $(!, \delta, \varepsilon)$ *consisting of a functor* $! : \mathbb{X} \longrightarrow \mathbb{X}$ *and two natural transformations:*

$$\delta : !A \longrightarrow !!A \qquad\qquad !A \longrightarrow A$$

such that the following diagrams commute:

$$\begin{array}{ccc} !A & \xrightarrow{\delta} & !!A \\ {\scriptstyle\delta}\downarrow & \searrow^{=} & \downarrow{\scriptstyle\varepsilon} \\ !!A & \xrightarrow[!(\varepsilon)]{} & !A \end{array} \qquad\qquad \begin{array}{ccc} !A & \xrightarrow{\delta} & !!A \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ !!A & \xrightarrow[!(\delta)]{} & !!!A \end{array} \tag{2.1}$$

We call δ *the comonad comultiplication and* ε *the comonad counit.*

In the graphical calculus, the comonad identities are drawn as follows:

(2.2)

while the naturality of δ and ε are drawn as follows:

(2.3)

Note that rather then use a double box for $!!(f)$ we simply write a box with $!(f)$. This is to not overload our string diagrams. That said, in certain proof we will sometimes be forced to work with boxes in boxes.

To every comonad, there are two important categories associated to it: the coEilenberg-Moore category and the coKleisli category. While neither is necessarily fundamental to the story of this chapter, the coEilenberg-Moore category may help provide some intuitions regarding coalgebra modalities. For a comonad $(!, \delta, \eta)$, recall that a **!-coalgebra** is a pair (A, ω) consisting of an object A and a map $\omega : A \longrightarrow !A$ such that the well-known necessary compatibility diagrams commute. For each object A, the cofree !-coalgebra over A is the !-coalgebra $(!A, \delta)$. The category of !-coalgebras and !-coalgebra morphisms is denoted $\mathbb{X}^!$ and is called the coEilenberg-Moore category of the comonad $(!, \delta, \varepsilon)$. There is also an obvious forgetful functor $\mathsf{U}^! : \mathbb{X}^! \longrightarrow \mathbb{X}$. We will review the coKleisli category of a comonad in Chapter 4. For now, it is simply worth mentioning that the coKleisli category is equivalent to the full subcategory of cofree !-coalgebras.

As previously mentioned, the underlying category of a differential category is that of a symmetric monoidal category [67, Chapter VII.]. Recall that a symmetric monoidal category is a septuple $(\mathbb{X}, \otimes, k, \alpha, \lambda, \rho, \sigma)$ consisting of a category $\mathbb{X}$, a functor $\otimes : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ called the monoidal product, an object k of $\mathbb{X}$ called the monoidal unit, and four natural isomorphisms: $\alpha : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$ (called the associativity isomorphisms), $\lambda : k \otimes A \xrightarrow{\cong} A$ and $\rho : A \otimes k \xrightarrow{\cong} A$ (called the unit isomorphisms), and $\sigma : A \otimes B \xrightarrow{\cong} B \otimes A$ (called the symmetry isomorphism); and such that the well known coherences hold (see for example [72, Section 4]). However, following the conventions in most papers on (co)differential categories, in this thesis, we will work in a symmetric *strict* monoidal category, that is, the associativity and unit isomorphisms of the monoidal product are strict equalities, so we have that $A \otimes k = A = k \otimes A$ and $(A \otimes B) \otimes C = A \otimes B \otimes C = A \otimes (B \otimes C)$. From now on, we will also simply denote symmetric monoidal categories by their underlying base category $\mathbb{X}$. Here are now some of the main examples of symmetric monoidal categories which we will consider throughout this paper.

Example 2.1.2 Let REL be the category of sets and relations, where recall that the objects are sets and the maps are relations between them, that is, a relation from a set X to a set Y, denoted $R : X \longrightarrow Y$, is a subset $R \subseteq X \times Y$. Then REL is a symmetric monoidal category where the monoidal product is given by the Cartesian product of sets, $X \otimes Y = X \times Y$, and where the monoidal unit is a chosen singleton $\{*\}$. It is important to note that the Cartesian product of sets is not a product (in the categorical sense) for REL.

Example 2.1.3 Let $\mathbb{K}$ be a field, and let $\mathsf{VEC}_{\mathbb{K}}$ be the category of all $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps between them. Then $\mathsf{VEC}_{\mathbb{K}}$ is a symmetric monoidal category where the monoidal product is given by the standard algebraic tensor product of vector spaces and the monoidal unit is the field itself $\mathbb{K}$.

Example 2.1.4 The opposite category of any symmetric monoidal category is again a symmetric monoidal category in the obvious way. Therefore, $\mathsf{VEC}_{\mathbb{K}}^{op}$ and REL^{op} are both symmetric monoidal categories. It is interesting to note that REL is self-dual, that is, REL^{op} is isomorphic to REL.

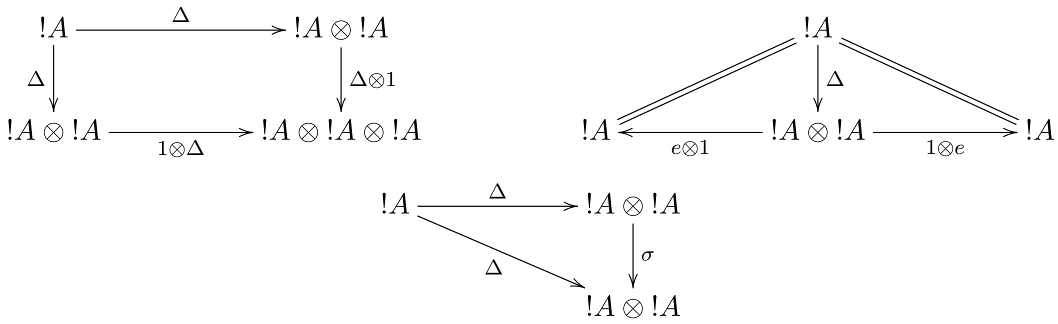
As we have mentioned, coalgebra modalities are comonads ! on symmetric monoidal categories such that for each object A, $!A$ comes equipped with a natural cocommutative comonoid structure.

Definition 2.1.5 *A **coalgebra modality** [7, Definition 1] on a symmetric monoidal category is a quintuple $(!, \delta, \varepsilon, \Delta, e)$ consisting of a comonad $(!, \delta, \varepsilon)$ and two natural transformations:*

$$\Delta : !A \longrightarrow !A \otimes !A \qquad\qquad e : !A \longrightarrow k$$

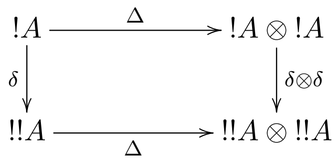
such that:

(i) $(!A, \Delta, e)$ *is a cocommutative comonoid, that is, the following diagrams commute:*



(2.4)

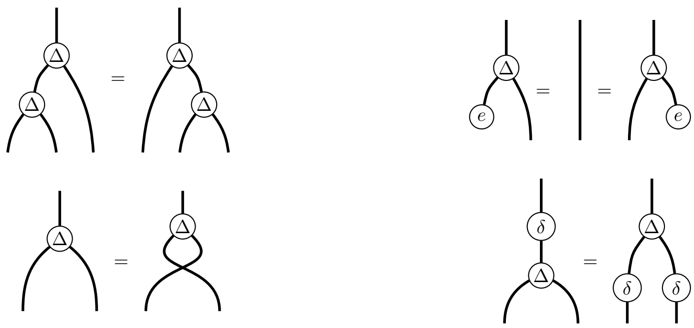
(ii) δ *preserves the comultiplication, that is, the following diagram commutes:*



(2.5)

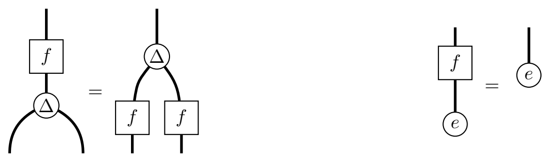
We call Δ *the comultiplication and* e *the counit.*

In the graphical calculus, the coalgebra modality identities are drawn as follows:



(2.6)

while the naturality of Δ and e are drawn as follows:



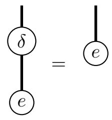
(2.7)

Note that requiring that Δ and e be natural transformations is equivalent to asking that for each map $f : A \longrightarrow B$, $!(f) : !A \longrightarrow !B$ is a comonoid morphism. We use this fact to show that δ is also a comonoid morphism.

Lemma 2.1.6 *[7, Lemma 1] For any coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$, δ also preserves the counit e, that is, the following diagram commutes:*

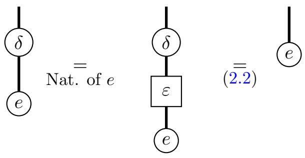
$$\begin{array}{ccc} !A & \xrightarrow{\delta} & !!A \\ & \searrow^{e} & \downarrow{\scriptstyle e} \\ & & K \end{array}$$

which is drawn as follows in the graphical calculus:



Therefore, δ is a comonoid morphism.

PROOF: By the naturality of e and the comonad identities, we obtain that:



As such, since δ preserves both the comultiplication and the counit, it follows that δ is a comonoid identity. □

CoKleisli maps of coalgebra modalities, that is, maps of type $f : !A \longrightarrow B$, are of particular interest as they should be thought of as *smooth* maps. This terminology is of no coincidence. Indeed, as we will see in the next chapter, in a differential category, the differentiable maps are precisely the coKleisli maps, and they are (in a certain way) infinitely differentiable and hence smooth. A subclass of these smooth maps are the *linear* maps which are coKleisli maps of the form $\varepsilon g : !A \longrightarrow B$ for some map $g : A \longrightarrow B$. On the other hand, what can we say about the coEilenberg-Moore category of a coalgebra modality? It turns out that every !-coalgebra of a coalgebra modality comes equipped with a cocommutative comonoid structure. Indeed if (A, ω) is a !-coalgebra, then the triple $(A, \Delta^\omega, \mathsf{e}^\omega)$ is a cocommutative comonoid [61, Section 4.1] where $\Delta^\omega : A \longrightarrow A \otimes A$ and $\mathsf{e}^\omega : A \longrightarrow k$ are defined as follows:

$$\begin{aligned} \Delta^\omega &:= \; A \xrightarrow{\omega} !A \xrightarrow{\Delta} !A \otimes !A \xrightarrow{\varepsilon \otimes \varepsilon} A \otimes A \\ \mathsf{e}^\omega &:= \; A \xrightarrow{\omega} !A \xrightarrow{\mathsf{e}} k \end{aligned} \tag{2.8}$$

It is important to point out that $(A, \Delta^\omega, \mathsf{e}^\omega)$ is in general only a cocommutative comonoid in the base category $\mathbb{X}$ and not in the coEilenberg-Moore category $\mathbb{X}^!$, since the latter does not necessarily have a monoidal product. Also notice that since δ is a comonoid morphism, when applying this construction to a cofree !-coalgebra $(!A, \delta)$ we recover Δ and e, that is, $\Delta^\delta = \Delta$ and $\mathsf{e}^\delta = \mathsf{e}$.

We conclude this section with our two main examples of coalgebra modalities which we will use as running examples throughout this chapter.

Example 2.1.7 REL (with the symmetric monoidal structure of Example 2.1.2) has a coalgebra modality induced by finite multisets. Recall that a finite multiset (also known as a finite bag) of a set X is a function $f : X \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, such that the set $\mathsf{supp}(f) := \{x \in X |\ f(x) \neq 0\}$ is finite. The empty multiset is the function $0 : X \longrightarrow \mathbb{N}$ which maps everything to zero, $0(x) = 0$. The union of multisets corresponds to point-wise addition: so for finite multisets $f : X \longrightarrow \mathbb{N}$ and $g : X \longrightarrow \mathbb{N}$, their sum is the finite multiset $f + g : X \longrightarrow \mathbb{N}$ defined as $(f+g)(x) = f(x)+g(x)$. Furthermore, for each $x \in X$ there an associated characteristic function which is the finite multiset $\eta_x : X \longrightarrow \mathbb{N}$ which maps $y \in X$ to 1 if $x = y$ and 0 otherwise. Now define a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ where:

(i) The endofunctor $! : \mathsf{REL} \longrightarrow \mathsf{REL}$ maps a set X to the set of all finite multisets of X (including the empty one), that is, on objects the functor ! is defined as follows:

$$!X = \{f : X \longrightarrow \mathbb{N} |\ |\mathsf{supp}(f)| < \infty\}$$

while for relation $R : X \longrightarrow Y$, $!(R) : !X \longrightarrow !Y$ is the relation which relates a bag of X to a bag of Y of the same size and such that the elements of the bags are related by R:

$$!(R) = \{(f, g)\} |\ \forall x \in \mathsf{supp}(f) \exists ! y \in \mathsf{supp}(g).\ (f(x), g(y)) \in R\} \subseteq !X \times !(Y)$$

(ii) The comonad counit $\varepsilon : !X \longrightarrow X$ is the relation which relates characteristic functions to their corresponding element in X:

$$\varepsilon = \{(\eta_x, x) |\ x \in X\} \subseteq !X \times X$$

(iii) The comonad comultiplication $\delta : !X \longrightarrow !!X$ is the relation which relates a finite multiset f to every finite multiset F of finite multisets whose sum is equal f:

$$\delta = \{(f, F) |\ f \in !X, F \in !!X \text{ s.t. } \sum_{g \in \mathsf{supp}(F)} g = f\} \subseteq !X \times !(!X)$$

Note that the sum is well-defined since $\mathsf{supp}(F)$ is finite.

(iv) The comultiplication $\Delta : !X \longrightarrow !X \times !X$ is the relation which relates a finite multiset f to every pair of finite multisets (g, f) whose sum is equal to f:

$$\Delta = \{(f, (g, h)) |\ f, g, h \in !X \text{ s.t. } f = g + h\} \subseteq !X \times (!X \times !X)$$

(v) The counit $e : !X \longrightarrow \{*\}$ is the relation which relates the empty multiset to the single element:

$$e = \{(0, *)\} \subseteq !X \times \{*\}$$

It is worth pointing out that $!X$ is the (co)free (co)commutative (co)monoid over X in REL. For more details on this coalgebra modality, see [8, Section 2.5.1].

Example 2.1.8 The dual notion of a coalgebra modality is that an **algebra modality** [12, Definition 3.8], which is essentially a monad such that every free algebra comes equipped with a natural commutative monoid structure. Explicitly, an algebra modality on a symmetric monoidal category $\mathbb{X}$ is quintuple $(\mathsf{S}, \mu, \eta, \nabla, u)$ consisting of an endofunctor $\mathbb{S} : \mathbb{X} \to \mathbb{X}$ and four natural transformations:

$$\mu : \mathsf{SS}A \to \mathsf{S}A \qquad \eta : A \to \mathsf{S}A$$
$$\nabla : \mathsf{S}A \otimes \mathsf{S}A \to \mathsf{S}A \qquad u : k \to \mathsf{S}A$$

such that (S, μ, η) is a monad, so the dual diagrams of (2.1) commute, and for each object A, $(\mathsf{S}A, \nabla, u)$ is a commutative monoid, so the dual diagrams of (2.4) commute, and μ preserves the multiplication, so the dual diagram of (2.5) commutes. Note that by the dual of Lemma 2.1.6, it follows that μ is a monoid morphism. For every S-algebra (A, ν), the triple $(A, \nabla^\nu, \mathsf{u}^\nu)$ is a commutative monoid [12, Theorem 2.12] where ∇^ω and u^ω are defined as follows:

$$\nabla^\nu := \; A \otimes A \xrightarrow{\eta \otimes \eta} \mathsf{S}A \otimes \mathsf{S}A \xrightarrow{\nabla} \mathsf{S}A \xrightarrow{\nu} A$$
$$\mathsf{u}^\nu := \; K \xrightarrow{\mathsf{u}} \mathsf{S}A \xrightarrow{\nu} A$$

In summary, if $(\mathsf{S}, \mu, \eta, \nabla, u)$ is an algebra modality on $\mathbb{X}$, then $(\mathsf{S}, \mu, \eta, \nabla, u)$ is a coalgebra modality on $\mathbb{X}^{op}$.

Example 2.1.9 For a fixed field $\mathbb{K}$, $\mathsf{VEC}_\mathbb{K}$ (with the symmetric monoidal structure of Example 2.1.3) has an algebra modality induced by **symmetric algebras** [58, Section 8, Chapter XVI]. For a $\mathbb{K}$-vector space V, let $S_n(V)$ be the subspace of $V^{\otimes^n}$ generated by the tensor symmetries:

$$v_1 \otimes ... \otimes v_n - v_{\tau(1)} \otimes ... \otimes v_{\tau(n)}$$

for all $v_i \in V$ and all n-permutations τ. Define the n-th symmetric tensor power of V as $\mathsf{Sym}_n(V) := V^{\otimes^n}/S_n(V)$ and let $v_1 \otimes_s \ldots \otimes_s v_n$ be the equivalence class of $v_1 \otimes \ldots \otimes v_n$ in $\mathsf{Sym}_n(V)$, which we refer as pure symmetric tensors. Define $\mathsf{Sym}(V)$, called the symmetric algebra over V, as follows:

$$\mathsf{Sym}(V) = \bigoplus_{n=0}^{\infty} \mathsf{Sym}_n(V) = \mathbb{K} \oplus V \oplus \mathsf{Sym}_2(V) \oplus ...$$

It is well known that $\mathsf{Sym}(V)$ is the free commutative $\mathbb{K}$-algebra over V. In particular, if X is a basis of V, then $\mathsf{Sym}(V) \cong \mathbb{K}[X]$ as $\mathbb{K}$-algebras (where $\mathbb{K}[X]$ is the polynomial ring over X). Now define an algebra modality $(\mathsf{Sym}, \eta, \mu, \nabla, u)$ where:

(i) The functor $\mathsf{Sym} : \mathsf{VEC}_\mathbb{K} \to \mathsf{VEC}_\mathbb{K}$ which maps V to its symmetric algebra $\mathsf{Sym}(V)$, while for a $\mathbb{K}$-linear map $f : V \to W$, $\mathsf{Sym}(f) : \mathsf{Sym}(V) \to \mathsf{Sym}(W)$ is defined on pure symmetric tensors as follows:

$$\mathsf{Sym}(f)(v_1 \otimes_s \ldots \otimes_s v_n) = f(v_1) \otimes_s \ldots \otimes_s f(v_n)$$

which we extend by linearity.

(ii) The monad unit $\eta_V : V \longrightarrow \mathsf{Sym}(V)$ is the injection map of V into $\mathsf{Sym}(V)$:

$$\eta(v) = v$$

(iii) The monad multiplication $\mu : \mathsf{Sym}(\mathsf{Sym}(V)) \longrightarrow \mathsf{Sym}(V)$ on pure symmetric tensors simply removes the brackets and concatenates pure symmetric tensors together:

$$\mu\left((v_1 \otimes_s \ldots \otimes_s v_n) \otimes_s \ldots \otimes_s (w_1 \otimes_s \ldots \otimes_s w_m)\right) = v_1 \otimes_s \ldots \otimes_s v_n \otimes_s \ldots \otimes_s w_1 \otimes_s \ldots \otimes_s w_m$$

which we then extend by linearity.

(iv) The multiplication $\nabla : \mathsf{Sym}(V) \otimes \mathsf{Sym}(V) \longrightarrow \mathsf{Sym}(V)$ is concatenation of pure symmetric tensors:

$$\nabla_V\left((v_1 \otimes_s ... \otimes_s v_n) \otimes (w_1 \otimes_s ... \otimes_s w_m)\right) = v_1 \otimes_s ... \otimes_s v_n \otimes_s w_1 \otimes_s ... \otimes_s w_m$$

which we then extend by linearity.

(v) The unit $u : \mathbb{K} \longrightarrow \mathsf{Sym}(V)$ is the injection map of $\mathbb{K}$ into $\mathsf{Sym}(V)$:

$$u(1) = 1$$

which we then extend by linearity.

Viewing $\mathsf{Sym}(V)$ instead as the polynomial ring $\mathbb{K}[X]$, the above structure maps can be described in terms of polynomials as follows:

(i) The monad unit corresponds to mapping basis elements to their corresponding one variable monomial of degree 1:

$$\eta\left(\sum_{i=1}^{n} k_i x_i\right) = \sum_{i=1}^{n} k_i x_i$$

(ii) First note that the set of monomials over X (including the unit constant), $M[X]$, is a basis of $\mathbb{K}[X]$. Then monad multiplication corresponds to composing polynomials together:

$$\mu\Big(P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right)\Big) = P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right)$$

where on the left P is a polynomial in variable $p_i(\vec{x_i})$, while on the right it is interpreted as polynomial in variables in X.

(iii) The multiplication corresponds to multiplying polynomials together:

$$\nabla(p(\vec{x}) \otimes q(\vec{y})) = p(\vec{x})q(\vec{y})$$

(iv) The unit picks out the constant polynomials:

$$u(k) = k$$

As such, we see that μ and η correspond to polynomial composition, while ∇ and u correspond to polynomial multiplication. Then $(\mathsf{Sym}, \mu, \eta, \nabla, u)$ is an algebra modality on $\mathsf{VEC}_{\mathbb{K}}$, and thus induces a coalgebra modality on $\mathsf{VEC}_{\mathbb{K}}^{op}$. We note that this (co)algebra modality construction can be generalized to the category of modules over any commutative semiring. For more details on this algebra modality, see [8, Section 2.5.3].

It is interesting to point out that both the coalgebra modalities of Example 2.1.7 and Example 2.1.9 are both examples of **free exponential modalities** [73], which are coalgebra modalities with the added property that $!A$ is also the cofree cocommutative comonoid over A [61, Definition 4.4]. As a result, the coEilenberg-Moore category of a free exponential modality is equivalent to the category of cocommutative comonoids of the base symmetric monoidal category. We point out that apart from providing examples, free exponential modalities do not necessarily play a theoretical role in this thesis.

2.2 Monoidal Coalgebra Modalities

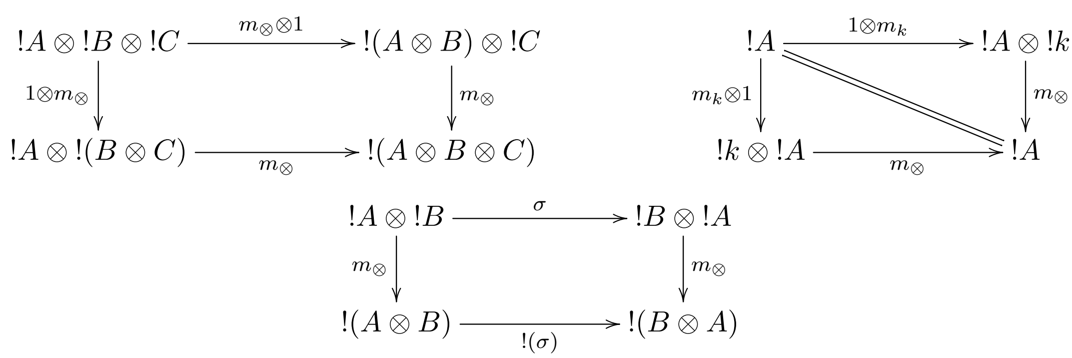
In this section, we now turn our attention to *monoidal* coalgebra modalities [10], which are also known as **linear exponential modalities** [82]. Symmetric monoidal closed categories with a monoidal coalgebra modality are categorical models of MELL [6]. As such, monoidal coalgebra modalities are more important and interesting to linear logicians than the weaker notions of simple coalgebra modalities. From a differential category theory point of view, monoidal coalgebra modalities allow for an equivalent axiomatization of the differential structure which is not possible for arbitrary coalgebra modalities.

Monoidal coalgebra modalities are coalgebra modalities whose underlying comonad is also a symmetric monoidal comonad. As such, we must first review the notions of symmetric monoidal (endo)functor and symmetric monoidal comonads.

Definition 2.2.1 *A* ***symmetric monoidal endofunctor*** *[61, Definition 3.8] on a symmetric monoidal category $\mathbb{X}$ is a triple $(!, m_\otimes, m_k)$ consisting of an endofunctor $! : \mathbb{X} \longrightarrow \mathbb{X}$, a natural transformation*

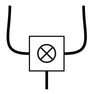
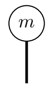
$$m_\otimes : !A \otimes !B \longrightarrow !(A \otimes B)$$

and a map $m_k : k \longrightarrow !k$ such that the following diagrams commute:

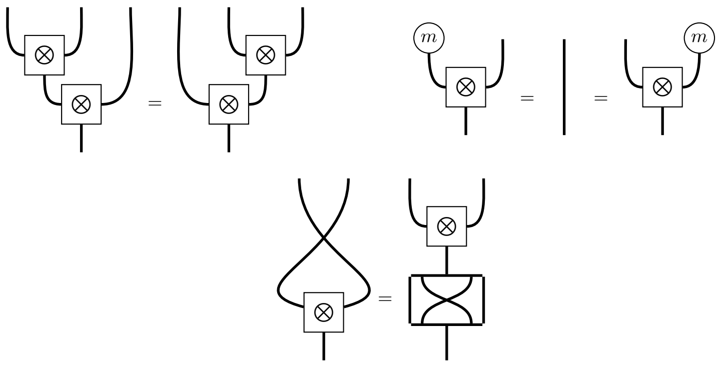


(2.9)

In the graphical calculus, $m_\otimes$ and m_k are drawn respectively as follows:

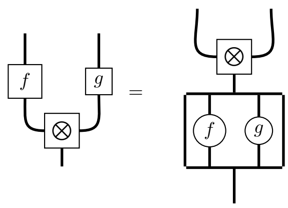
$m_\otimes =$  $m_k =$ 

And so the symmetric monoidal endofunctor identities are drawn as follows:



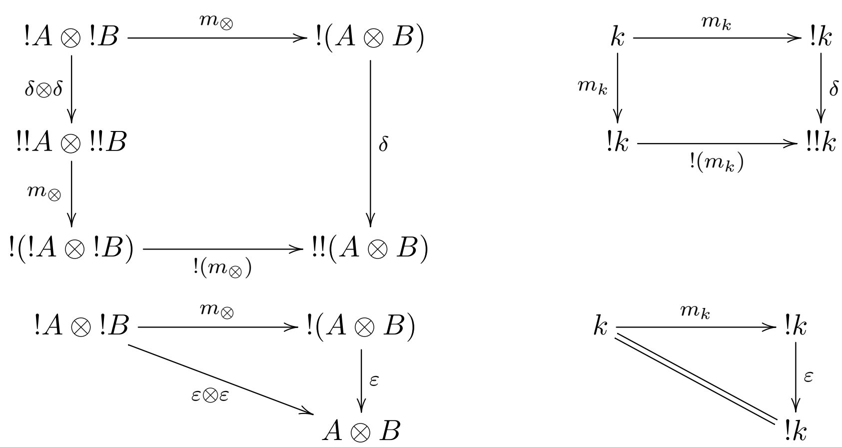
(2.10)

while the naturality of $m_\otimes$ is drawn as follows:



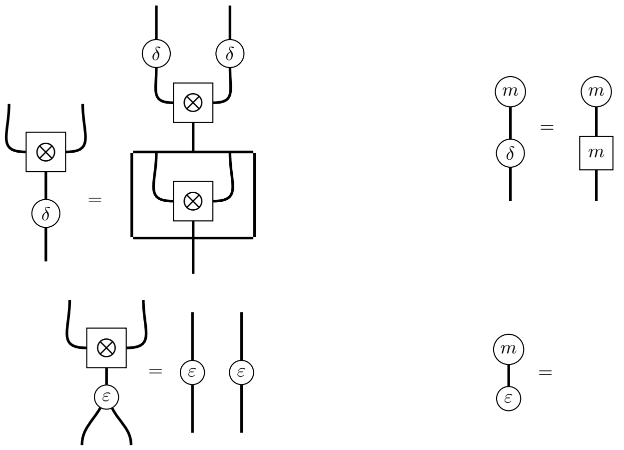
(2.11)

Definition 2.2.2 *A* ***symmetric monoidal comonad*** *[61, Definition 3.8] on a symmetric monoidal category $\mathbb{X}$ is a quintuple $(!, \delta, \varepsilon, m_\otimes, m_k)$ consisting of a comonad $(!, \delta, \varepsilon)$ and a symmetric monoidal endofunctor $(!, m_\otimes, m_k)$ and such that δ and ε are monoidal natural transformations, that is, the following diagrams commute:*



(2.12)

In the graphical calculus, that δ and ε are monoidal are drawn as follows:



(2.13)

The symmetric monoidal comonad coherences are precisely what is required so that the co-Eilenberg-Moore category be a symmetric monoidal category such that the forgetful functor preserves the symmetric monoidal structure strictly [74, 87]. Indeed, the tensor product of a pair of !-coalgebras (A, ω) and (B, ω') is defined as follows:

$$(A, \omega) \otimes (B, \omega') := \left(A \otimes B,\ A \otimes B \xrightarrow{\omega \otimes \omega'} !(A) \otimes !(B) \xrightarrow{\mathsf{m}_\otimes} !(A \otimes B) \right) \tag{2.14}$$

while the monoidal unit is (k, m_k).

A monoidal coalgebra modality is a coalgebra modality whose underlying comonad is a symmetric monoidal comonad and such that the comonoid structure is compatible with this extra symmetric monoidal structure.

Definition 2.2.3 *A* ***monoidal coalgebra modality*** *[7, Definition 2] on a symmetric monoidal category is a septuple consisting of a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ (Definition 2.1.5) and a symmetric monoidal comonad $(!, m, m_k, \delta, \varepsilon)$, and such that:*

(i) *Δ and e are monoidal transformations (or equivalently $m_\otimes$ and m_k are comonoid morphisms), that is, the following diagrams commute:*

$$\begin{array}{ccc} !A \otimes !B & \xrightarrow{m_\otimes} & !(A \otimes B) \\ \downarrow{\scriptstyle \Delta \otimes \Delta} & & \\ !A \otimes !A \otimes !B \otimes !B & & \downarrow{\scriptstyle \Delta} \\ \downarrow{\scriptstyle 1 \otimes \sigma \otimes 1} & & \\ !A \otimes !B \otimes !A \otimes !B & \xrightarrow[m_\otimes \otimes m_\otimes]{} & !(A \otimes B) \otimes !(A \otimes B) \end{array} \qquad \begin{array}{ccc} !A \otimes !B & \xrightarrow{m_\otimes} & !(A \otimes B) \\ \downarrow{\scriptstyle e \otimes e} & & \downarrow{\scriptstyle e} \\ K \otimes K & = & K \end{array}$$

$$\begin{array}{ccc} K & \xrightarrow{m_k} & !k \\ \| & & \downarrow{\scriptstyle \Delta} \\ K \otimes K & \xrightarrow[m_k \otimes m_k]{} & !k \otimes !k \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{m_k} & !k \\ & \searrow\!\!= & \downarrow{\scriptstyle e_K} \\ & & K \end{array} \tag{2.15}$$

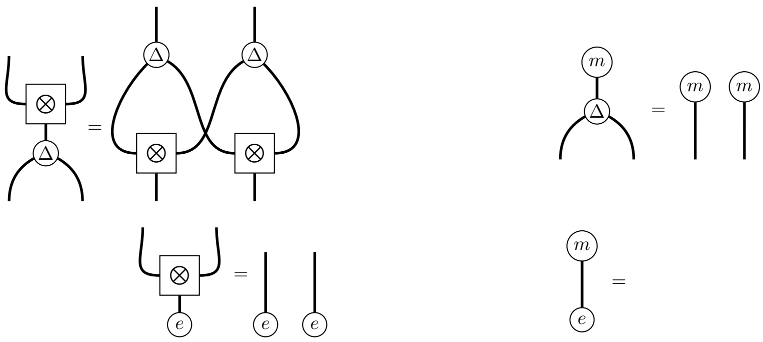
(ii) Δ *and* e *are* $!$*-coalgebra morphisms, that is, the following diagrams commute:*

$$\begin{array}{ccccc} !A & & \xrightarrow{\delta} & & !!A \\ \downarrow{\scriptstyle\Delta} & & & & \downarrow{\scriptstyle !(\Delta)} \\ !A \otimes !A & \xrightarrow[\delta\otimes\delta]{} & !!A \otimes !!A & \xrightarrow[m_\otimes]{} & !(!A \otimes !A) \end{array} \qquad \begin{array}{ccc} !A & \xrightarrow{\delta} & !!A \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle !(e)} \\ K & \xrightarrow[m_k]{} & !(K) \end{array} \tag{2.16}$$

A ***linear category*** *is a symmetric monoidal category with a monoidal coalgebra modality.*

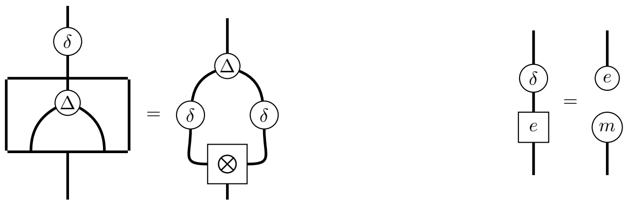
We should note that here we are using the term "linear category" in the sense of Blute, Cockett, and Seely as in [10], which is the same as Bierman's definition in [6] but which drops the closed structure requirement.

In the graphical calculus, that Δ and e are both monoidal transformations is drawn as:



(2.17)

while the coherences that Δ and e are !-colagebra morphisms is drawn as follows:



(2.18)

There are multiple equivalent ways of defining a monoidal coalgebra modality, some of which can be found in [6, 71, 72, 82]. For example, the monoidal coalgebra modality coherences are precisely what is required so that the tensor product of the base category becomes a product in the coEilenberg-Moore category. Explicitly, $(A, \omega) \otimes (B, \omega')$, as defined in (2.14), is the product of !-coalgebras (A, ω) and (B, ω'), while (k, m_k) becomes a terminal object in the coEilenberg-Moore category. It is also interesting to point out that the diagonal map and unique map to the terminal object of (A, ω) are precisely the induced comultiplication Δ^ω and counit e^ω maps as defined in (2.8). As such, a monoidal coalgebra modality can equivalently be described as a comonad whose coEilenberg-Moore category has finite products and such that the canonical adjunction between the base category and the coEilenberg-Moore category is a symmetric monoidal adjunction.

Another characterization, which is of particular important to this thesis, is that in the presence of finite products, a monoidal coalgebra modality can equivalently be defined as a coalgebra modality that has the *Seely isomorphisms* [6, 10, 83]. For a category with finite products [67, Chapter III.Section 4], we denote the binary product of objects A and B by $A \times B$ with projection maps $\pi_0 : A \times B \longrightarrow A$ and $\pi_1 : A \times B \longrightarrow B$, pairing operation $\langle -, - \rangle$, thus $f \times g = \langle \pi_0 f, \pi_1 g \rangle$, and we denote the chosen terminal object as $\top$, with unique map to terminal object as $t : A \longrightarrow \top$.

Example 2.2.4 REL has finite products where the categorical product is given by the disjoint union of sets which we denote as $X \sqcup Y$, and where the terminal object is the empty set $\emptyset$.

Example 2.2.5 The dual notions of a product and a terminal are respectively that of a coproduct and an initial object [67, Chapter III.Section 3]. As such, if $\mathbb{X}$ has finite coproducts, then its dual $\mathbb{X}^{op}$ has finite products.

Example 2.2.6 $\mathsf{VEC}_{\mathbb{K}}$ has finite coproducts where the coproduct of $\mathbb{K}$-vector spaces is given by their direct sum $V \oplus W$, and where the initial object is the zero vector space 0. As such, $\mathsf{VEC}_{\mathbb{K}}^{op}$ has finite products.

Keen-eyed readers will note that, in fact, the finite (co)product structures of Example 2.2.4 and 2.2.6 are in fact finite biproduct structures! This will be important in the next section. That said, for this section, only the (co)product structure is necessary.

Definition 2.2.7 *In a symmetric monoidal category $\mathbb{X}$ with finite products $\times$ and terminal object $\top$, a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ has* ***Seely isomorphisms*** *[83] if the natural transformation $\chi : !(A \times B) \longrightarrow !A \otimes !B$ and the map $\chi_\top : !\top \longrightarrow K$ defined respectively as follows:*

$$\chi := \ !(A \times B) \xrightarrow{\Delta} !(A \times B) \otimes !(A \times B) \xrightarrow{!(\pi_0) \otimes !(\pi_1)} !A \otimes !B \qquad \chi_\top := \ !\top \xrightarrow{e} k \tag{2.19}$$

are isomorphisms, so $!\top \cong k$ and $!(A \times B) \cong !A \otimes !B$. We call χ and $\chi_\top$, and their inverses, the Seely maps. A ***monoidal storage category*** *(also sometimes known as a (new) Seely category [6, 71, 72]) is a symmetric monoidal category with finite products and a coalgebra modality which has Seely isomorphisms.*

As explained in [10], every coalgebra modality which has Seely isomorphisms is a monoidal coalgebra modality, where $m_\otimes$ is defined as follows:

$$m_\otimes := \ !A \otimes !B \xrightarrow{\chi^{-1}} !(A \times B) \xrightarrow{\delta} !!(A \times B) \xrightarrow{!(\chi)} !(!A \otimes !B) \xrightarrow{!(\varepsilon \otimes \varepsilon)} !(A \otimes B)$$

and m_k is defined as follows:

$$m_k := \ k \xrightarrow{\chi_\top^{-1}} !\top \xrightarrow{\delta} !!\top \xrightarrow{!(\chi_\top)} !k$$

Conversly, in the presence of finite products, every monoidal coalgebra modality has Seely isomorphisms [6] where the inverse of χ is defined as follows:

$$\chi^{-1} := \ !A \otimes !B \xrightarrow{\delta \otimes \delta} !!A \otimes !!B \xrightarrow{m_\otimes} !(!A \otimes !B) \xrightarrow{!(\langle \varepsilon \otimes e, e \otimes \varepsilon \rangle)} !(A \times B)$$

while the inverse of $\chi_\top$ is defined as follows:

$$\chi_\top^{-1}\ k \xrightarrow{m_k} !(K) \xrightarrow{!(t)} !\top$$

Therefore we obtain the following:

Theorem 2.2.8 *[10, Theorem 3.1.6] For a symmetric monoidal category with finite products, the following are equivalent:*

(i) *A coalgebra modality with Seely isomorphisms;*

(ii) *A monoidal coalgebra modality.*

Therefore, every monoidal storage category is a linear category and conversely, every linear category with finite products is a monoidal storage category.

We conclude this section by discussing examples of monoidal coalgebra modalities. There is no shortage of examples of monoidal coalgebra modalities since every categorical model of MELL admits a monoidal coalgebra modality. For example, Hyland and Schalk provide a nice list of such examples in [50, Section 2.4]. Interesting examples of coalgebra modalities that are not monoidal can be found in [7, Section 9]. In particular, in Section 2.4, we will construct a family of non-monoidal coalgebra modalities from a monoidal coalgebra modality.

Example 2.2.9 The coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ from Example 2.1.7 on REL has the Seely isomorphisms. Indeed, note that in the category of sets, the disjoint union is in fact the coproduct. Therefore, a finite multiset of type $X \sqcup Y \longrightarrow \mathbb{N}$ is equivalent to a pair of finite multisets $X \longrightarrow \mathbb{N}$ and $Y \longrightarrow \mathbb{N}$. Explicitly, given a finite multiset $f : X \sqcup Y \longrightarrow \mathbb{N}$, define the finite multisets $f_X : X \longrightarrow \mathbb{N}$ and $f_Y : Y \longrightarrow \mathbb{N}$ respectively as follows $f_X(x) = f(x)$ and $f_Y(y) = y$. Conversely, given a pair of finite multisets $g : X \longrightarrow \mathbb{N}$ and $h : Y \longrightarrow \mathbb{N}$, define the finite $g \sqcup h : X \sqcup Y \longrightarrow \mathbb{N}$ as $(g \sqcup h)(z) = g(z)$ if $z \in X$ and $(g \sqcup h)(z) = h(z)$ if $z \in Y$. Therefore, $!(X \sqcup Y) \cong !X \times !Y$. Similarly, since the empty is an initial object in the category of sets, there is only one finite multiset of type $\emptyset \longrightarrow \mathbb{N}$, namely the empty function $\varnothing$. Therefore, $!\emptyset \cong \{*\}$. In REL, the disjoint union is also the product and the empty set is also the terminal object, and thus we see that ! has Seely isomorphisms. Explicitly, the Seely isomorphisms are given as follows:

(i) $\chi : !(X \sqcup Y) \longrightarrow !X \times !Y$ and $\chi^{-1} : !X \times !Y \longrightarrow !(X \sqcup Y)$ are defined as follows:

$$\chi = \{(f, (f_X, f_Y)) |\ f \in !(X \sqcup Y)\} \subseteq !(X \sqcup Y) \times (!X \times !Y)$$

$$\chi^{-1} = \{((g, h), g \sqcup h) |\ g \in !X, h \in !Y\} \subseteq (!X \times !Y) \times !(X \sqcup Y)$$

(ii) $\chi_{\{*\}} : !\emptyset \longrightarrow \{*\}$ and $\chi_{\{*\}}^{-1} : !X \times !Y \longrightarrow !(X \sqcup Y)$ are defined as follows:

$$\chi_{\{*\}} = \{(\varnothing, *)\} \subseteq !\emptyset \times \{*\} \qquad \chi_{\{*\}}^{-1} = \{(*, \varnothing)\} \subseteq \{*\} \times !\emptyset$$

Therefore, it follows that this coalgebra modality is also a monoidal coalgebra modality where:

(i) $m_\otimes : !X \times !Y \longrightarrow !(X \times Y)$ is the relation which does the following:

$$m_\otimes = \left\{ \big((f,g),h\big) \mid f \in !X, g \in !Y, h \in !(X \times Y) \text{ s.t. } f(x) = \sum_{y \in Y} h(x,y) \text{ and } g(y) = \sum_{x \in X} h(x,y) \right\}$$
$$\subseteq (!X \times !Y) \times !(X \times Y)$$

Note that the sums are well-defined since $\mathsf{supp}(h)$ is finite. Furthermore, if $\big((f,g),h\big) \in m_\otimes$, then we have the following equality:

$$\sum_{x \in X} f(x) = \sum_{y \in Y} g(y) = \sum_{(x,y) \in X \times Y} h(x,y)$$

(ii) $m_{\{*\}} : \{*\} \longrightarrow !\{*\}$ is the relation which relates $*$ to every finite multiset of $\{*\}$:

$$m_{\{*\}} = \{(*, f) | \ f \in !\{*\}\} = \{*\} \times !\{*\}$$

Note that $!\{*\} \cong \mathbb{N}$, so every (finite) multiset corresponds to unique a natural number.

Therefore, $(!, \delta, \varepsilon, \Delta, e, m_\otimes, m_{\{*\}})$ is a monoidal coalgebra modality on REL.

Example 2.2.10 The dual notion of a monoidal coalgebra modality is that a **comonoidal algebra modality**, which is essentially an algebra modality whose underlying monad is a symmetric comonoidal monad which is compatible with the monoid structure. Explicitly, a comonoidal algebra modality on a symmetric monoidal category $\mathbb{X}$ is septuple $(\mathsf{S}, \mu, \eta, \nabla, u, n_\otimes, n_k)$ consisting of an algebra modality $(\mathsf{S}, \mu, \eta, \nabla, u)$ equipped with a natural transformation:

$$n_\otimes : \mathsf{S}(A \otimes B) \longrightarrow \mathsf{S}A \otimes \mathsf{S}B$$

and a map $\mathsf{S}k \longrightarrow k$ such that $(\mathsf{S}, \mu, \eta, n_\otimes, n_k)$ is a comonoidal monad, so that the dual diagrams of (2.9) and (2.12) commutate, as as well the dual diagrams of (2.15) and (2.16) commute. So if $(\mathsf{S}, \mu, \eta, \nabla, u, n_\otimes, n_k)$ is a comonoidal algebra modality on $\mathbb{X}$, then $(\mathsf{S}, \mu, \eta, \nabla, u, n_\otimes, n_k)$ is a monoidal coalgebra modality on $\mathbb{X}^{op}$. If $\mathbb{X}$ has finite coproducts, then a comonoidal algebra modality is equivalent to an algebra modality which has Seely isomorphisms, that is, the dual maps from Definition 2.2.7 are (natural) isomorphisms.

Example 2.2.11 The algebra modality $(\mathsf{Sym}, \mu, \eta, \nabla, u)$ from Example 2.1.9 on $\mathsf{VEC}_\mathbb{K}$ has the Seely isomorphisms. Indeed, it is a well known result from commutative algebra that $\mathsf{Sym}(V) \otimes \mathsf{Sym}(W) \cong \mathsf{Sym}(V \oplus W)$ and $\mathbb{K} \cong \mathsf{Sym}(0)$ [58]. Explicitly, we have that:

- $\chi : \mathsf{Sym}(V) \otimes \mathsf{Sym}(W) \longrightarrow \mathsf{Sym}(V \oplus W)$ and its inverse $\chi^{-1} : \mathsf{Sym}(V \oplus W) \longrightarrow \mathsf{Sym}(V) \otimes \mathsf{Sym}(W)$ are defined as follows on pure symmetric tensors:

$$\chi\left((v_1 \otimes_s \ldots \otimes_s v_n) \otimes (w_1 \otimes_s \ldots \otimes_s w_m)\right) = (v_1 \oplus 0) \otimes_s \ldots \otimes_s (v_n \oplus 0) \otimes_s (0 \oplus w_1) \otimes_s \ldots \otimes_s (0 \oplus w_m)$$

$$\chi^{-1}\left((v_1 \oplus w_1) \otimes_s \ldots \otimes_s (v_n \oplus w_n)\right) = \sum_{\substack{\{a_1,\ldots,a_i\} \cup \{a_{i+1},\ldots,a_n\} = \{1,\ldots,n\} \\ \{a_1,\ldots,a_i\} \cap \{a_{i+1},\ldots,a_n\} = \emptyset}} (v_{a_1} \otimes_s \ldots \otimes_s v_{a_i}) \otimes (w_{a_{i+1}} \otimes_s \ldots \otimes_s w_{a_n})$$

which we then extend by linearity.

- Note that $\mathsf{Sym}(0) = \mathbb{K} \oplus 0 \oplus 0 \oplus 0 \oplus \ldots$. Therefore, $\chi_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathsf{Sym}(0)$ and $\chi_{\mathbb{K}}^{-1} : \mathsf{Sym}(0) \longrightarrow \mathbb{K}$ are defined simply as:

$$\chi_{\mathbb{K}}(k) = k \qquad\qquad \chi_{\mathbb{K}}^{-1}(k) = k$$

Therefore, it follows that this algebra modality is also a comonoidal algebra modality where:

(i) $n_{\otimes} : \mathsf{Sym}(V \otimes W) \longrightarrow \mathsf{Sym}(V) \otimes \mathsf{Sym}(W)$ splits pure symmetric tensors appart:

$$n_{\otimes}\left((v_1 \otimes w_1) \otimes_s \ldots \otimes_s (v_n \otimes w_n)\right) = (v_1 \otimes_s \ldots \otimes_s v_n) \otimes (w_1 \otimes_s \ldots \otimes_s w_n)$$

which we then extend by linearity.

(ii) $n_{\mathbb{K}} : \mathsf{Sym}(\mathbb{K}) \longrightarrow \mathbb{K}$ simply multiplies pure symmetric tensors together:

$$n_{\mathbb{K}}(k_1 \otimes_s \ldots \otimes_s k_n) = k_1 k_2 \ldots k_n$$

which we then extend by linearity.

In terms of polynomials, we have that $\mathsf{Sym}(V \oplus W) = \mathbb{K}[X \sqcup Y]$ and $\mathsf{Sym}(V \otimes W) \cong \mathbb{K}[X \times Y]$, where X is a basis set of V and Y is a basis set of W, and also that $\mathsf{Sym}(\mathbb{K}) \cong \mathbb{K}[x]$ and $\mathsf{Sym}(0) \cong \mathbb{K}[\emptyset] = \mathbb{K}$. Therefore, we have that:

(i) On monomials, χ corresponds to multiplying monomials together while χ^{-1} corresponds to splitting apart monomials:

$$\chi\left((x_1^{p_1} \ldots x_n^{p_n}) \otimes (y_1^{q_1} \ldots y_m^{q_m})\right) = x_1^{p_1} \ldots x_n^{p_n} y_1^{q_1} \ldots y_m^{q_m}$$

$$\chi^{-1}(x_1^{p_1} \ldots x_n^{p_n} y_1^{q_1} \ldots y_m^{q_m}) = (x_1^{p_1} \ldots x_n^{p_n}) \otimes (y_1^{q_1} \ldots y_m^{q_m})$$

which we extend by linearity for arbitrary polynomials.

(ii) $\chi_{\mathbb{K}}$ and $\chi_{\mathbb{K}}^{-1}$ are the same as above, which amounts to picking out the constant polynomials, which in this case are the only possible polynomials.

(iii) $n_{\otimes}$ corresponds to splitting apart monomials:

$$n_{\otimes}\left((x_1, y_1)^{m_1}(x_2, y_2)^{m_2} \ldots (x_n, y_n)^{m_n}\right) = (x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}) \otimes (y_1^{m_1} y_2^{m_2} \ldots y_n^{m_n})$$

which we extend by linearity for arbitrary polynomials.

(iv) $n_{\mathbb{K}}$ corresponds to evaluating polynomials at 1:

$$n_{\mathbb{K}}(p(x)) = p(1)$$

Therefore, $(\mathsf{Sym}, \mu, \eta, \nabla, u, n_{\otimes}, n_k)$ is a comonoidal algebra modality, and so $(\mathsf{Sym}, \mu, \eta, \nabla, u, n_{\otimes}, n_k)$ is a monoidal coalgebra modality on $\mathsf{VEC}_{\mathbb{K}}^{op}$.

It is worth pointing out that every free exponential modality is also a monoidal coalgebra modality, as shown by Lafont in his Ph.D. thesis [56], and in the presence of finite products, free exponential modalities also have Seely isomorphisms. The monoidal coalgebra modality structure and Seely isomorphisms are constructed using the couniversal property of the cofree cocommutative comonoid. As such, a linear category whose monoidal coalgebra modality is a free exponential modality is called a **Lafont category** [72]. In fact, most examples of monoidal coalgebra modalities found in linear logic literature are free exponential modalities. Indeed, while free exponential modalities have been around since the beginning of linear logic with Girard's free exponential modality for coherence spaces [42], new free exponential modalities are still being constructed and studied, which shows the importance of these kinds models.

2.3 Additive Bialgebra Modalities

In this section we introduce the notion of an *additive* bialgebra modality, which is the main novel contribution of this chapter. In the presence of additive structure, additive bialgebra modalities are in bijective correspondence to monoidal coalgebra modalities (Theorem 2.3.25). In particular, in Section 3.3, we will show that for additive bialgebra modalities, deriving transformations and coderiction maps are equivalent. For the proofs in this section, we will use the full force of the graphical calculus.

The underlying categorical structure of a differential category is not only a symmetric monoidal category but that of an *additive* symmetric monoidal category. Indeed, the two of the basic properties of the derivative from classical differential calculus require addition: the Leibniz rule and the constant rule. Therefore we must first discuss additive structure, and so we begin this section by recalling additive structure by starting with the notion of an additive category. Here we mean "additive" in the Blute, Cockett, and Seely sense of the term [8], that is, enriched over commutative monoids. In particular, we do not assume negatives nor do we assume biproducts which differs from other definitions of an additive category found in the literature [67]. This allows for a wide range of examples from both mathematics and computer science.

Definition 2.3.1 *An **additive category** [7, Definition 3] is a commutative monoid enriched category, that is, a category in which each hom-set is a commutative monoid with an addition operation* $+$ *and a zero* 0*, and such that composition preserves the additive structure, that is:*

$$k(f+g)h = kfh + kgh \qquad\qquad k0h = 0$$

*An **additive symmetric monoidal category** [7, Definition 3] is a symmetric monoidal category which is also an additive category in which the tensor product is compatible with the additive structure in the sense that:*

$$k \otimes (f+g) \otimes h = k \otimes f \otimes h + k \otimes g \otimes h \qquad\qquad k \otimes 0 \otimes h = 0$$

It is worth mentioning that any category with finite biproducts is an additive category, and conversely that every additive category can be completed to a category with biproducts [67]. Similarly any symmetric monoidal category with distributive finite biproducts is an additive symmetric monoidal category, and conversely every additive symmetric monoidal category can be completed to an additive symmetric monoidal category with distributive biproducts. For this reason, it is possible to argue, such as in [39], that one should always assume a setting with finite biproducts. The problem is that arbitrary coalgebra modalities do not necessarily extend to the biproduct completion. However, monoidal coalgebra modalities induce monoidal coalgebra modalities on the biproduct completion – which we will discuss below. With all that said, if an additive symmetric monoidal category has finite (co)products, then said finite (co)product structure is in fact a finite biproduct structure which is distributive. This is the case for our main examples of additive symmetric monoidal categories.

Example 2.3.2 REL is an additive symmetric monoidal category where the sum of parallel relations $R : X \longrightarrow Y$ and $S : X \longrightarrow Y$, which recall are subsets $R, S \subseteq X \times Y$, is defined as their union $R + S := R \cup S$, while the zero map $0 : X \longrightarrow Y$ is the empty relation $0 := \emptyset$.

Example 2.3.3 $\mathsf{VEC}_{\mathbb{K}}$ is an additive symmetric monoidal category where the sum of $\mathbb{K}$-linear maps $f : V \longrightarrow W$ and $g : V \longrightarrow W$ is the standard sum of linear maps $f + g$ defined pointwise, $(f + g)(v) = f(v) + g(v)$, and where the zero map $0 : V \longrightarrow W$ is the $\mathbb{K}$-linear map which maps every element of V to the zero element of W.

Example 2.3.4 The opposite category of any additive symmetric monoidal category is again an additive symmetric monoidal category in the obvious way. Therefore, for example, $\mathsf{VEC}_{\mathbb{K}}^{op}$ is an additive symmetric monoidal category.

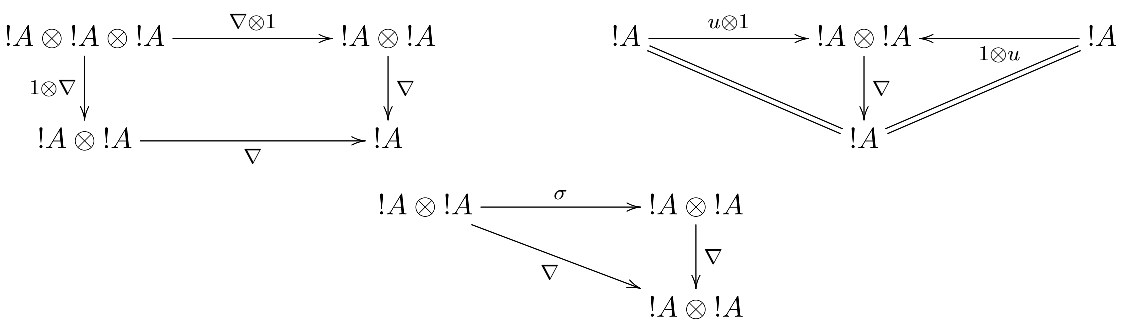
In the presence of additive structure, it is possible to talk about coalgebra modalities which come equipped with a natural bialgebra structure. Such coalgebra modalities are called bialgebra modalities [8].

Definition 2.3.5 *A* ***bialgebra modality*** *[7, Definition 4] on an additive symmetric monoidal category is a septuple* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *consisting of a coalgebra modality* $(!, \delta, \varepsilon, \Delta, e)$*, and two natural transformations:*

$$\nabla : !A \otimes !A \longrightarrow !A \qquad\qquad u : K \longrightarrow !A$$

such that:

(i) $(!A, \nabla, u)$ *is a commutative monoid, that is, the following diagrams commute:*



(2.20)

(ii) $(!A, \nabla, u, \Delta, e)$ *is a bialgebra, that is, the following diagrams commute:*

$$\begin{array}{ccccc}
!A \otimes !A & \xrightarrow{\Delta\otimes\Delta} & !A \otimes !A \otimes !A \otimes !A & & k \xrightarrow{u} !A \\
 & & \downarrow{\scriptstyle 1\otimes\sigma\otimes 1} & & {\scriptstyle u\otimes u}\searrow \quad \downarrow{\scriptstyle \Delta} \\
\downarrow{\scriptstyle \nabla} & & !A \otimes !A \otimes !A \otimes !A & & !A \otimes !A \\
 & & \downarrow{\scriptstyle \nabla\otimes\nabla} & & \\
!A & \xrightarrow[\Delta]{} & !A \otimes !A & &
\end{array}$$

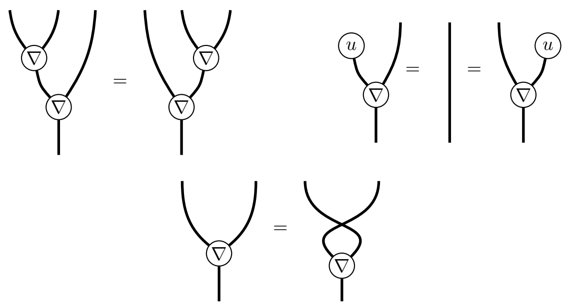
$$\begin{array}{ccc}
!A \otimes !A \xrightarrow{\nabla} !A & \qquad & k \xrightarrow{u} !A \\
{\scriptstyle e\otimes e}\searrow \quad \downarrow{\scriptstyle e} & & \searrow\!\!\!= \quad \downarrow{\scriptstyle e} \\
k & & k
\end{array} \tag{2.21}$$

(iii) ε *is compatible with* ∇ *in the sense that the following diagram commutes:*

$$\begin{array}{c}
!A \otimes !A \xrightarrow{\nabla} !A \\
{\scriptstyle \varepsilon\otimes e + e\otimes\varepsilon}\searrow \quad \downarrow{\scriptstyle \varepsilon} \\
A
\end{array} \tag{2.22}$$

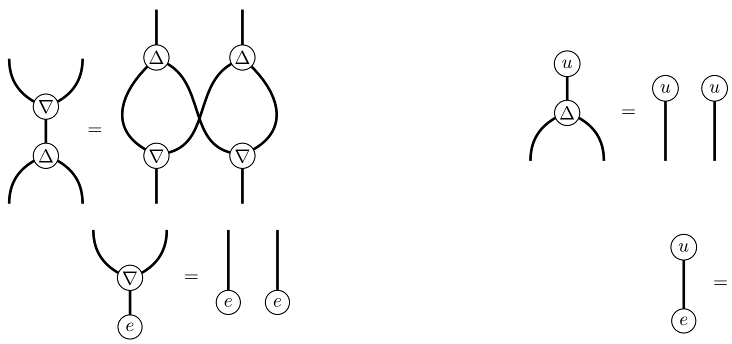
We call ∇ *the multiplication and* u *the unit.*

In the graphical calculus, the commutative monoid identities are drawn as follows:



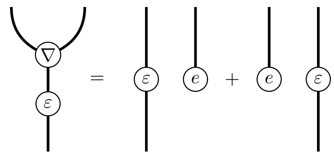
(2.23)

The bialgebra identities are drawn as follows:



(2.24)

The compatibility between ε and ∇ are drawn as follows:



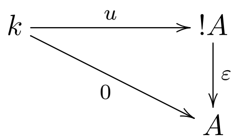
(2.25)

Lastly, the naturality of ∇ and u is drawn as follows:

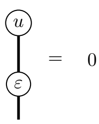
(2.26)

We again note that by the naturality of ∇, Δ, u and e, for every map f, $!(f)$ is both a monoid and comonoid morphism. In the original definition of a bialgebra modality in [8] it was also required that $u\varepsilon = 0$; however this is provable:

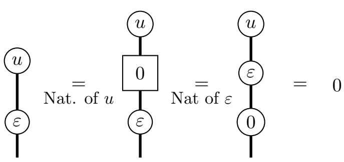
Lemma 2.3.6 *[7, Lemma 2] For a bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$*, the following diagram commutes:*



which in the graphical calculus is drawn as follows:



Proof: By the naturality of u and ε, and the additive structure we have the following:



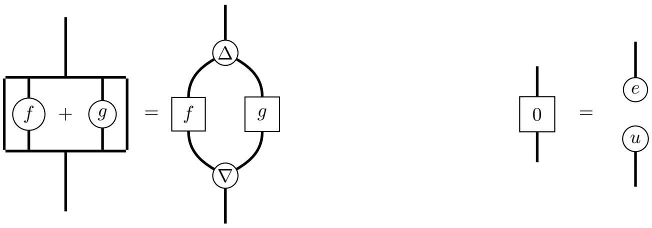
So we conclude that the desired equality holds. □

Additive bialgebra modalities are bialgebra modalities such that the additive structure of the category and the natural bialgebra structure of the bialgebra modality are compatible via bialgebra convolution.

Definition 2.3.7 An **additive bialgebra modality** [7, Definition 5] on an additive symmetric monoidal category is a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ which is compatible with the additive structure in the sense that the following diagrams commute (for any parallel maps f and g):

$$\begin{array}{ccc} !A & \xrightarrow{!(f+g)} & !B \\ {\scriptstyle \Delta}\downarrow & & \uparrow{\scriptstyle \nabla} \\ !A \otimes !A & \xrightarrow[!(f)\otimes !(g)]{} & !B \otimes !B \end{array} \qquad \begin{array}{ccc} !A & \xrightarrow{!(0)} & !B \\ & {\scriptstyle e}\searrow \quad \nearrow{\scriptstyle u} & \\ & k & \end{array} \tag{2.27}$$

In the graphical calculus, the additive bialgebra modality identities are drawn as follows:



(2.28)

Example 2.3.8 The coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ from Example 2.1.7 on REL is also an additive bialgebra modality where:

(i) The multiplication $\nabla : !X \times !X \longrightarrow !X$ is the dual relation of the comultiplication, that is, it relates a pair of finite multisets to their sum:

$$\nabla = \{((f, g), f + g)|\ f, g \in !X\} \subseteq (!X \times !X) \times !X$$

(ii) The unit $u : \{*\} \longrightarrow !X$ is the dual relation of the counit, that is, it relates the single element to the empty multiset:

$$u = \{(*, 0)\} \subseteq \{*\} \times !X$$

Therefore, $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ is an additive bialgebra modality on REL.

Example 2.3.9 The dual notion to an additive bialgebra modality would be a septuple $(\mathsf{S}, \mu, \eta, \nabla, u, \Delta, e)$ consisting an algebra modality $(\mathsf{S}, \mu, \eta, \nabla, u)$ and two natural transformations:

$$\Delta : \mathsf{S}A \longrightarrow \mathsf{S}A \otimes \mathsf{S}A \qquad\qquad e : \mathsf{S}A \longrightarrow k$$

such that the dual diagrams of (2.20), (2.21), (2.22), and (2.27) all commute. In particular, $(\mathsf{S}A, \nabla, u, \Delta, e)$ is a bialgebra. Unfortunately, due to the standard conventions of naming dual notions, it is not obvious what to call the dual of an additive bialgebra modality. Often, the dual notion is simply referred to also as an additive bialgebra modality. Therefore, for an additive symmetric monoidal category $\mathbb{X}$, if $(\mathsf{S}, \mu, \eta, \nabla, u, \Delta, e)$ is (the dual of) an algebra modality on $\mathbb{X}$, then $(\mathsf{S}, \mu, \eta, \nabla, u, \Delta, e)$ is an additive bialgebra modality on $\mathbb{X}^{op}$.

Example 2.3.10 It is well known that symmetric algebras have a canonical bialgebra structure. As such, the algebra modality $(\mathsf{Sym}, \mu, \eta, \nabla, u)$ from Example 2.1.9 on $\mathsf{VEC}_{\mathbb{K}}$ is also an additive bialgebra modality where:

(i) The comultiplication $\Delta : \mathsf{Sym}(V) \longrightarrow \mathsf{Sym}(V) \otimes \mathsf{Sym}(V)$ is defined as follows on pure tensors:

$$\Delta\left(v_1 \otimes_s \ldots \otimes_s v_n\right) = \sum_{\substack{\{a_1,\ldots,a_i\}\cup\{a_{i+1},\ldots,a_n\}=\{1,\ldots,n\} \\ \{a_1,\ldots,a_i\}\cap\{a_{i+1},\ldots,a_n\}=\emptyset}} \left(v_{a_1} \otimes_s \ldots \otimes_s v_{a_i}\right) \otimes \left(v_{a_{i+1}} \otimes_s \ldots \otimes_s v_{a_n}\right)$$

which we then extend by linearity.

(ii) The counit $e : \mathsf{Sym}(V) \longrightarrow \mathbb{K}$ is the "projection" map onto $\mathbb{K}$, so on pure tensors:

$$e(k) = k \qquad \text{if } k \in \mathbb{K}$$
$$e\left(v_1 \otimes_s \ldots \otimes_s v_n\right) = 0 \qquad \text{o.w.}$$

In terms of polynomials, if X is a basis for V, then $\mathsf{Sym}(V) \cong \mathbb{K}[X]$ and let $M[X]$ be the set of monomials. Then we have that:

(i) On monomials, the comultiplication splits up monomial into pairs of monomials which when multiplied together give the starting monomial:

$$\Delta(m(\vec{x})) = \sum_{\substack{m_0(\vec{x}),m_1(\vec{x})\in M[X] \\ m_0(\vec{x})m_1(\vec{x})=m(\vec{x})}} m_0(\vec{x}) \otimes m_1(\vec{x}) \qquad m(\vec{x}) \in M[X]$$

where the sum is well-defined since the indexing is finite, and where we extend by linearity for arbitrary polynomials.

(ii) The counit correspondends to evaluating polynomials at 0, which picks out their constant term:

$$e(p(\vec{x})) = p(\vec{0})$$

Therefore, $(\mathsf{Sym}, \mu, \eta, \nabla, \Delta, e)$ is (the dual of) an additive bialgebra modality on $\mathsf{VEC}_{\mathbb{K}}$, and so $(\mathsf{Sym}, \mu, \eta, \nabla, \Delta, e)$ is an additive bialgebra modality on $\mathsf{VEC}_{\mathbb{K}}^{op}$.

That Example 2.3.8 and Example 2.3.10 are indeed additive bialgebra modalities will follow from Theorem 2.3.25 below. Examples of bialgebra modalities which are not additive bialgebra modalities can be found in Section 2.4. In particular, we will construct a family of non-additive bialgebra modalities from an additive bialgebra modality.

We now turn our attention to the main objective of this chapter: explaining the relationship between additive bialgebra modalities and monoidal coalgebra modalities. Explicitly, we will show that for additive symmetric monoidal categories, monoidal coalgebra modalities correspond bijectively to additive bialgebra modalities. Once again, we would like to mention that these proofs may not be fully optimized or elegant. However, we take the pain of providing the full direct proofs in detail since additive bialgebra modalities are a new concept and thus is worth recording the complete proof somewhere in the literature.

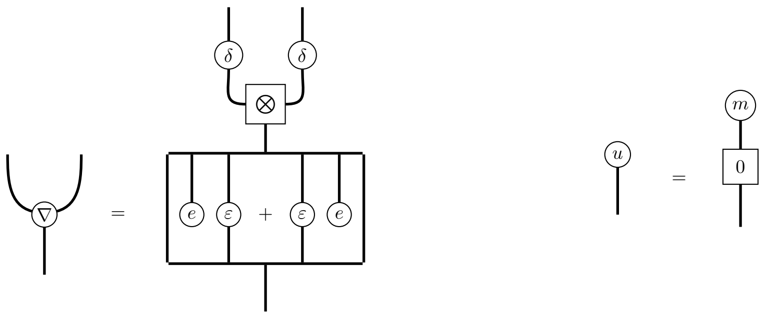
We start by explaining how every monoidal coalgebra modality is also an additive bialgebra modality.

Definition 2.3.11 *An **additive linear category** [7, Definition 6] is a linear category (Definition 2.2.3) which is also an additive symmetric monoidal category, that is, an additive symmetric monoidal category equipped with a monoidal coalgebra modality.*

The monoidal coalgebra modality of an additive linear category induces an additive bialgebra modality where ∇ and u are respectively:

$$\nabla := \ !A \otimes !A \xrightarrow{\delta\otimes\delta} !!A \otimes !!A \xrightarrow{m_\otimes} !(!A \otimes !A) \xrightarrow{!(\varepsilon\otimes e + e\otimes\varepsilon)} !A$$
$$u := \ K \xrightarrow{m_k} !K \xrightarrow{!(0)} !A \tag{2.29}$$

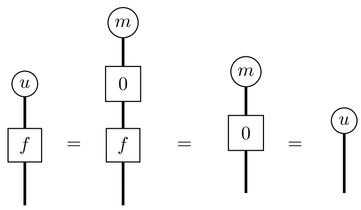
In the graphical calculus, these are drawn as:


(2.30)

We will now carefully prove in bite-size steps that this is indeed an additive bialgebra modality.

Lemma 2.3.12 *[7, Lemma 15] ∇ and u are natural transformations.*

Proof: By construction, ∇ is a natural transformation since it is the composition of natural transformation. The unit u on the other hand is not automatically a natural transformation by construction. Let $f : A \longrightarrow B$, and since $!(0)!(f) = !(0)$, we obtain the following equality:

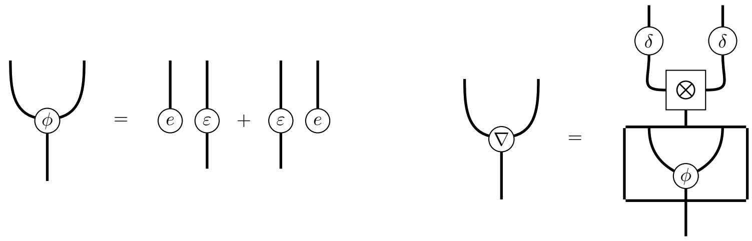


So we conclude that ∇ and u are natural transformations. □

For space and simplification, we define the natural transformation $\phi : !A \otimes !A \longrightarrow A$ as follows:

$$\phi := \ !A \otimes !A \xrightarrow{e\otimes\varepsilon + \varepsilon\otimes e} A$$

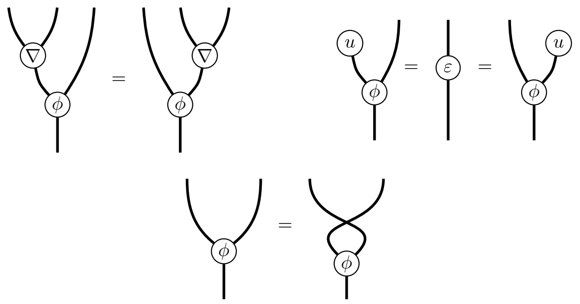
Therefore, ∇ can also be drawn as:



(2.31)

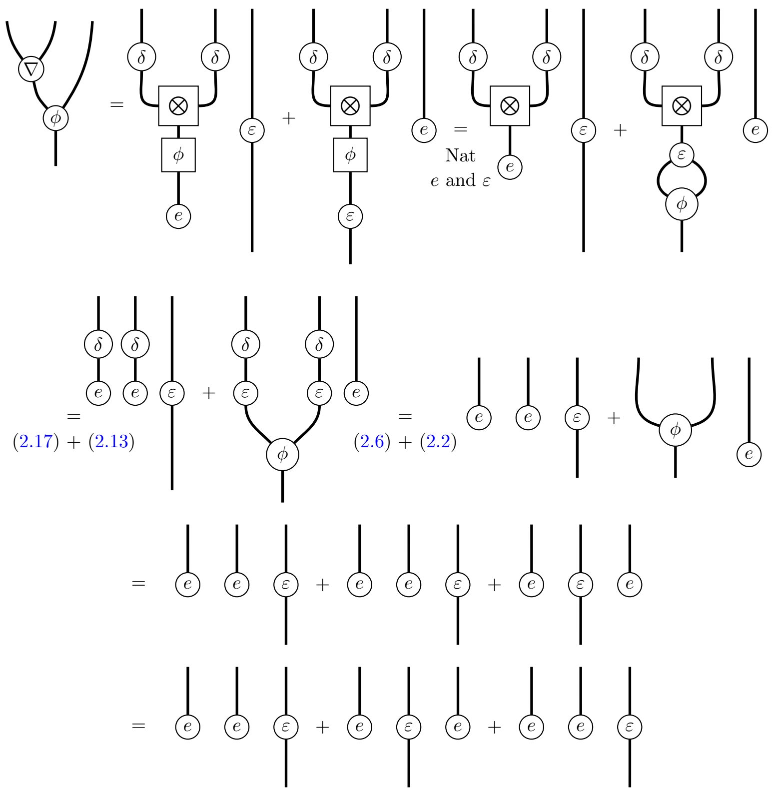
The following identities will be useful for proving the additive bialgebra modality coherences.

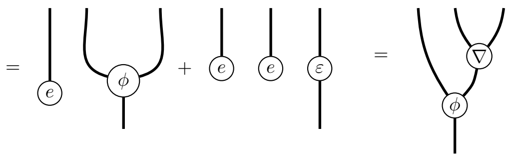
Lemma 2.3.13 *[7, Lemma 16] ϕ satisfies the following equalities:*



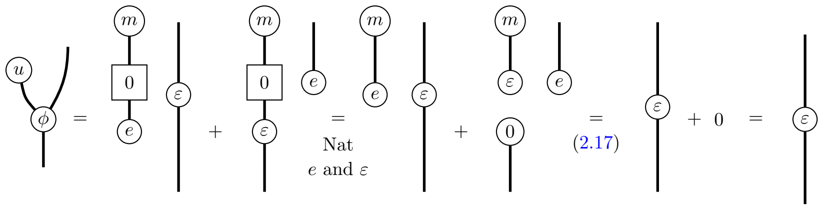
(2.32)

PROOF: For the first equality we have that:

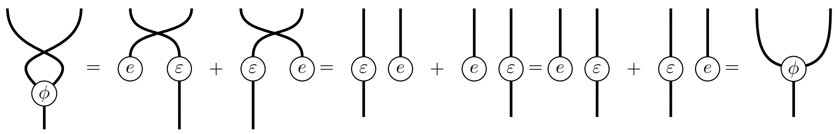




For left identity of the second equality we have that:



And similarly for the right identity. Lastly, for the third identity we have that:

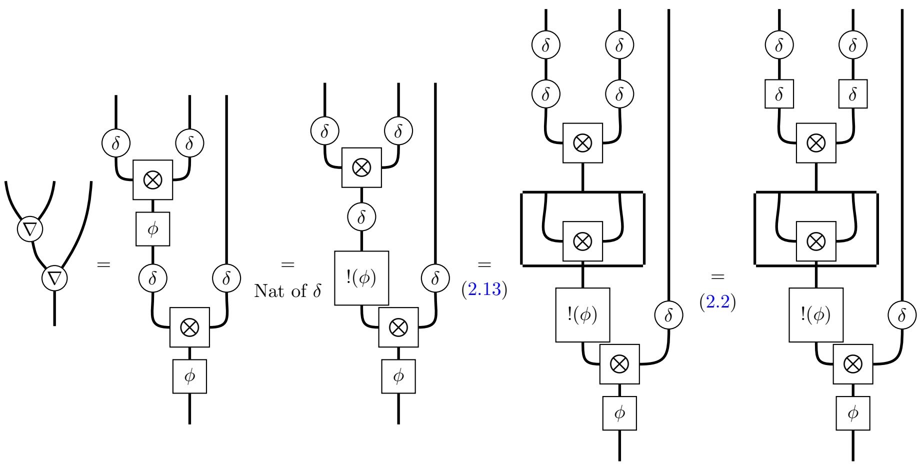


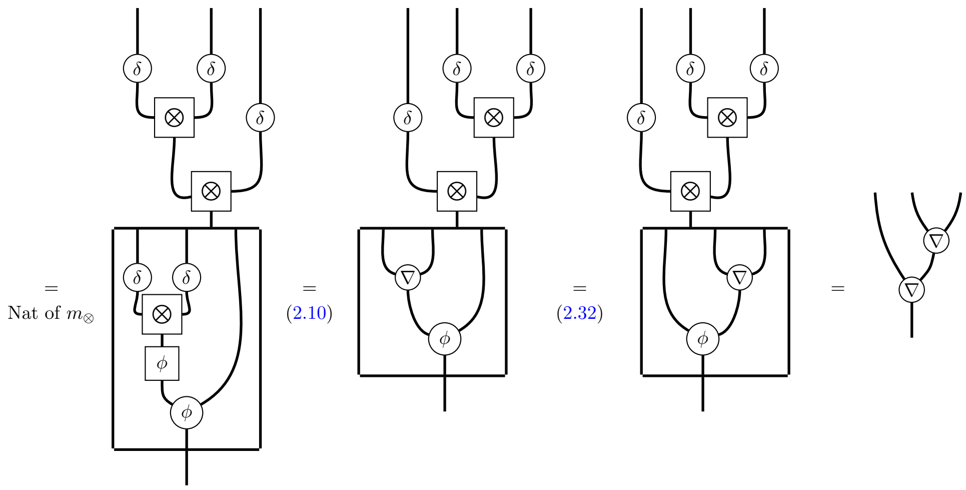
So we conclude that ϕ satisfies the desired equalities. □

We now show that $!A$ is a bialgebra.

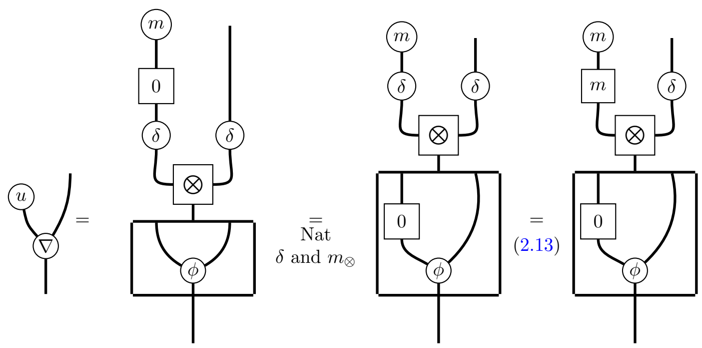
Lemma 2.3.14 *[7, Lemma 17] For each object A, $(!A, \nabla, u)$ is a commutative monoid.*

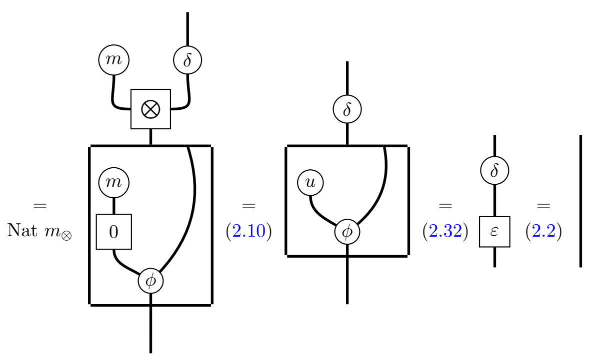
PROOF: This follows mostly from Lemma 2.3.13.
1. Associativity:



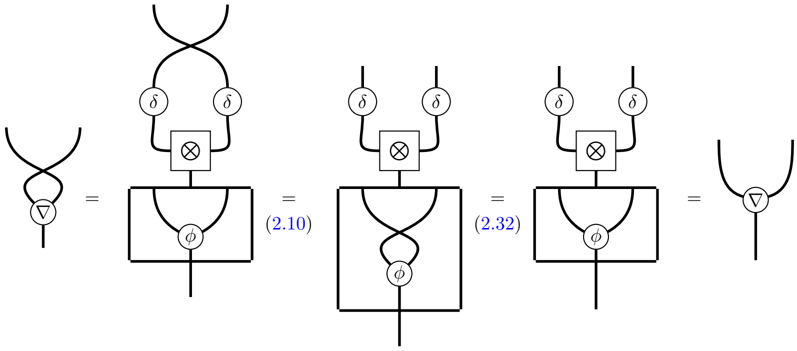


2. Unit Laws: (We only show one of them, since calculation for the other is similar)





3. Commutativity:

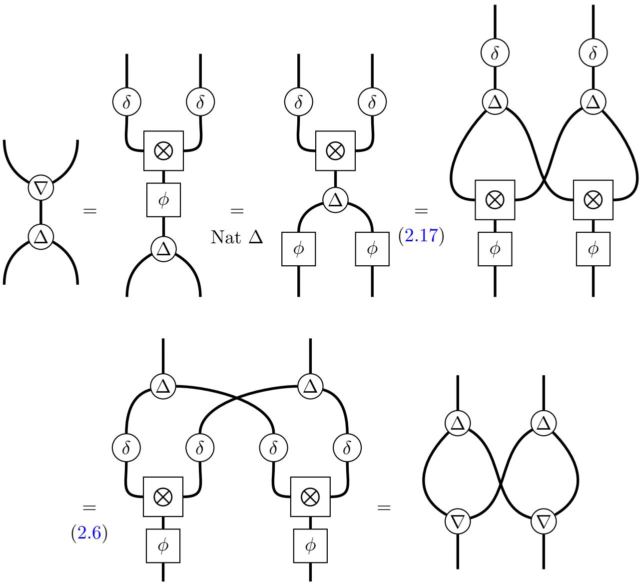


So we conclude that $(!A, \nabla, u)$ is a commutative monoid. □

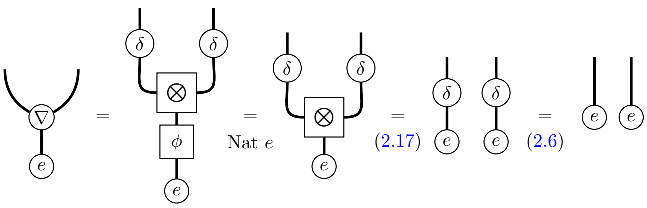
Lemma 2.3.15 *[7, Lemma 18] For each object A, $(!A, \nabla, u, \Delta, e)$ is a bialgebra.*

Proof: We need to check the four bialgebra compatibility relations:

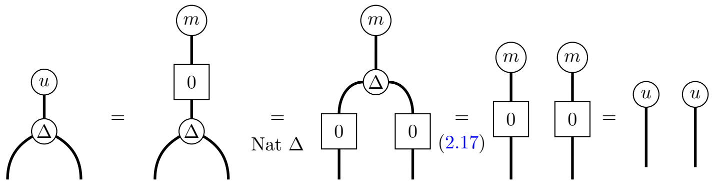
1. Multiplication and comultiplication compatibility:



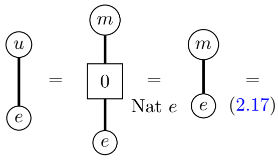
2. Multiplication and counit compatibility:



3. Comultiplication and unit compatibility:



4. Counit and unit compatibility:

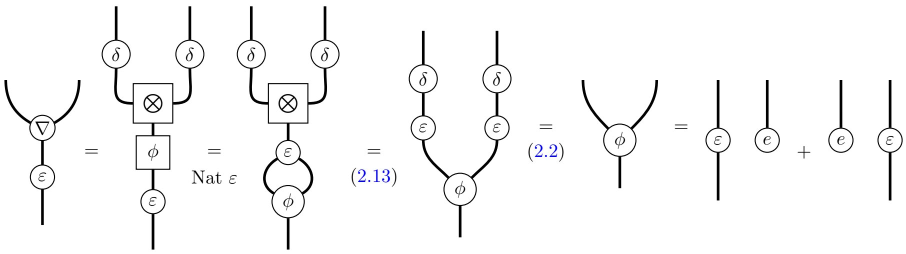


So we conclude that $(!A, \nabla, u, \Delta, e)$ is a bialgebra. □

Lastly, we show the remaining additive bialgebra modality identities.

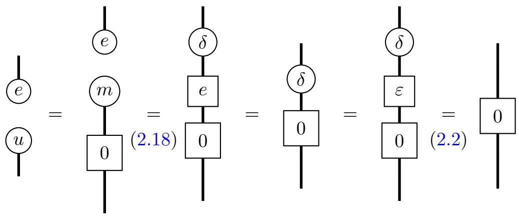
Proposition 2.3.16 *[7, Proposition 14] The monoidal coalgebra modality of an additive linear category is an additive bialgebra modality with* ∇ *and* u *defined as in (2.29).*

Proof: We first prove that we have a bialgebra modality by proving the compatibility of ε and ∇:

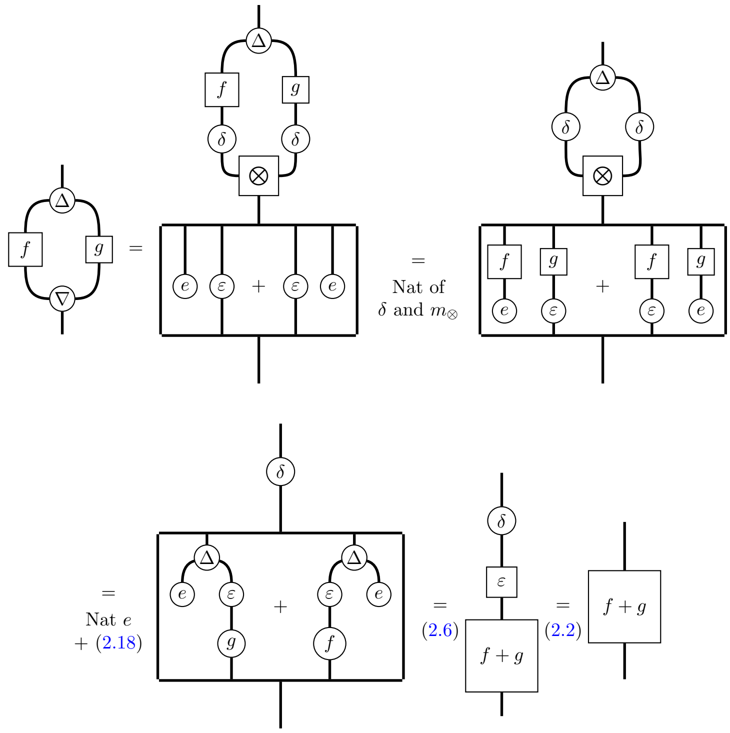


Finally, we show that the bialgebra modality is in fact additive by proving the compatibility with the additive structure:

1. $!(0) = eu$:



2. $!(f+g) = \Delta(!(f)\otimes !(g))\nabla$:



So we conclude that $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ is an additive bialgebra modality. □

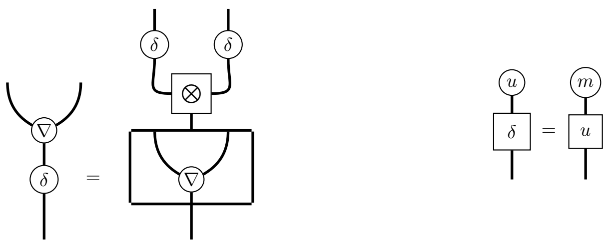
As first observed in [39, Theorem 3.1], the monoidal coalgebra modality structure and the bialgebra modality structure are compatible in the following sense:

Proposition 2.3.17 *[7, Proposition 2] In an additive linear category:*

(i) *u and ∇ are $!$-coalgebra morphisms, that is, the following diagram commutes:*

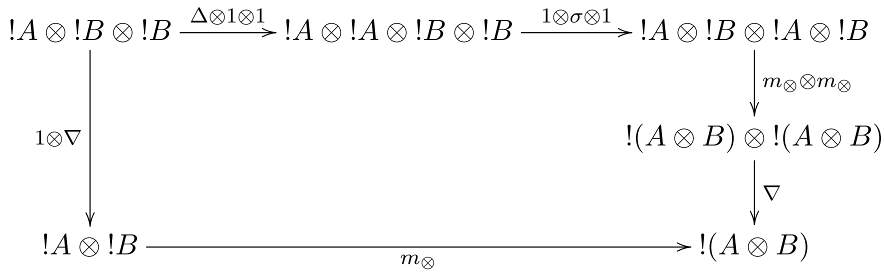
$$\begin{array}{ccccc} !A\otimes !A & \xrightarrow{\delta\otimes\delta} & !!A\otimes !!A & \xrightarrow{m_\otimes} & !(!A\otimes !A) \\ {\scriptstyle\nabla}\downarrow & & & & \downarrow{\scriptstyle !(\nabla)} \\ !A & & \xrightarrow[\delta]{} & & !!A \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{m_k} & !K \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle !(u)} \\ !A & \xrightarrow[\delta]{} & !!A \end{array}$$

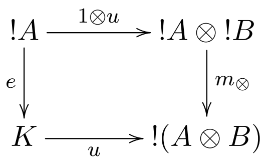
which in the graphical calculus is drawn as:



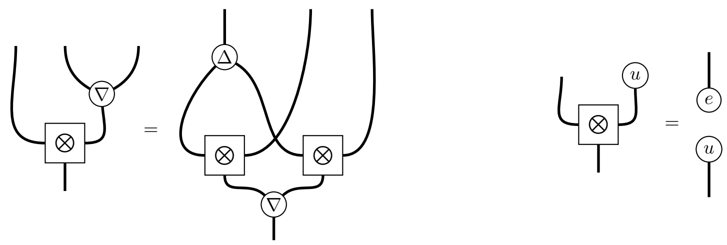
(2.33)

(ii) *The following diagrams commute:*



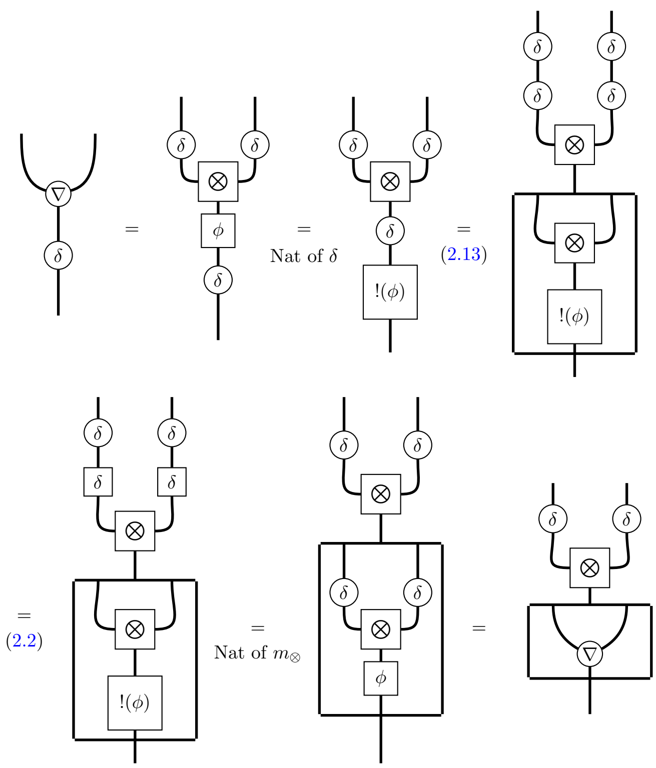


which are drawn as follows in the graphical calculus:

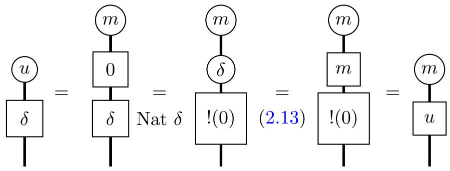


(2.34)

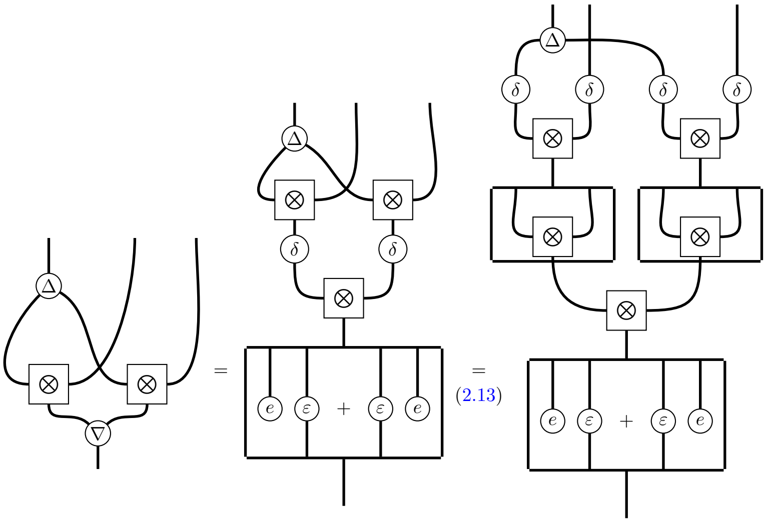
Proof: We first show that ∇ is a !-coalgebra morphism:

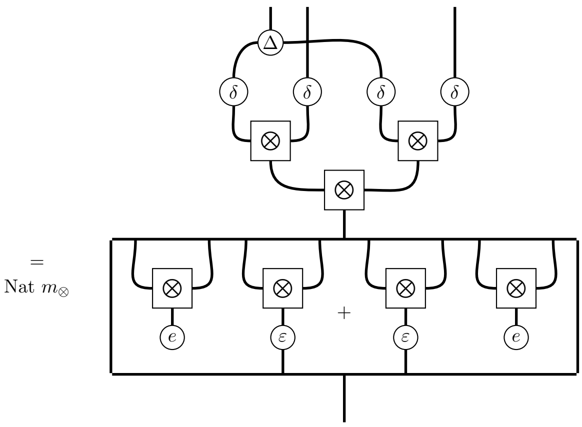


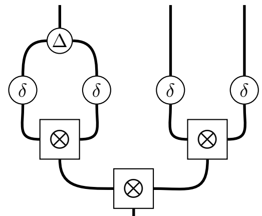
Next we show that u is also a !-coalgebra morphism:



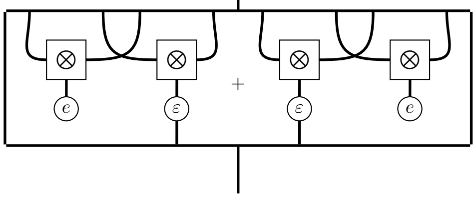
Next we show the compatibility between ∇ and $m_\otimes$:



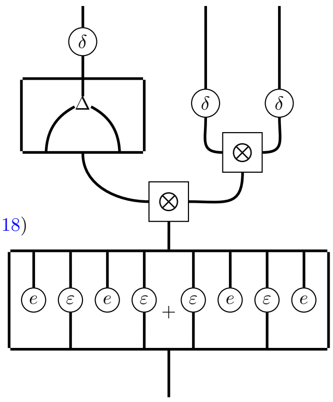




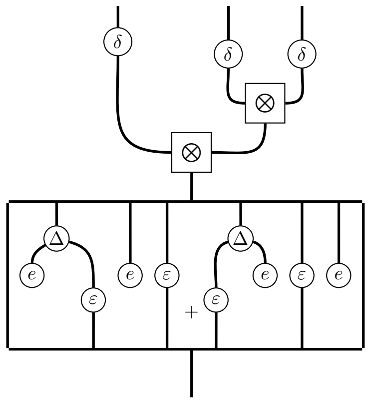
= (2.10)



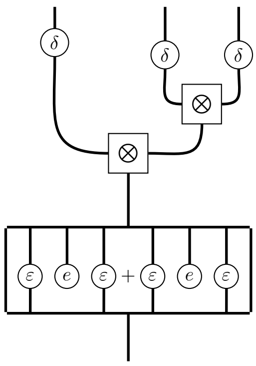
= (2.13)+(2.18)



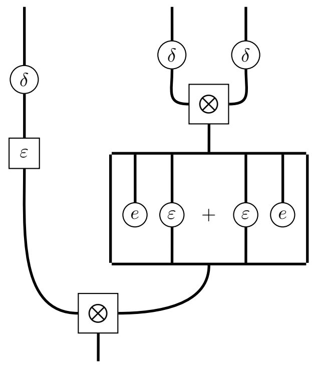
= Nat $m_\otimes$



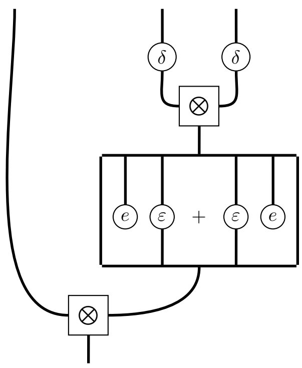
= (2.6)



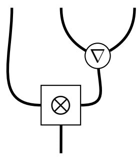
= Nat $m_\otimes$



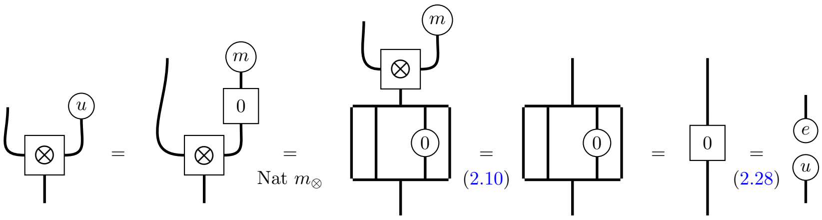
= (2.2)



=



Lastly we show the compatibility between u and $m_{\otimes}$:



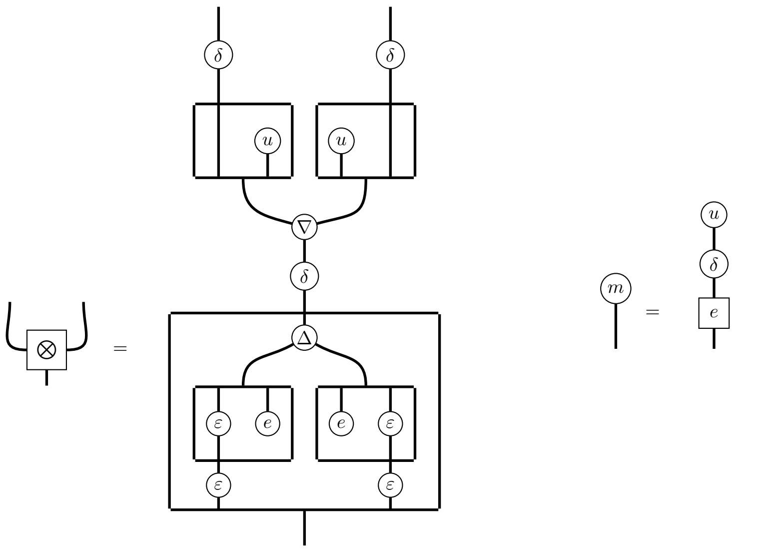
So we conclude that ∇ and u satisfy the desired identities. □

We will now prove the converse of Proposition 2.3.16, that is, we will now show how an additive bialgebra modality induces a monoidal coalgebra modality. The monoidal structure $m_{\otimes}$ and m_k are defined respectively as follows:

$$\begin{array}{ccccccc}
!A\otimes !B & \xrightarrow{\delta\otimes\delta} & !A\otimes !B & \xrightarrow{!(1\otimes u)\otimes !(u\otimes 1)} & !(!A\otimes !B)\otimes !(!A\otimes !B) & \xrightarrow{\nabla} & !(!A\otimes !B) \\
\downarrow m_{\otimes} \quad := & & & & & & \downarrow \delta \\
!(A\otimes B) & \xleftarrow{!(\varepsilon\otimes\varepsilon)} & !(!A\otimes !B) & \xleftarrow{!(!(\varepsilon\otimes e)\otimes !(e\otimes\varepsilon))} & !(!(!A\otimes !B)\otimes !(!A\otimes !B)) & \xleftarrow{!(\Delta)} & !!(!A\otimes !B)
\end{array} \tag{2.35}$$

$$\begin{array}{ccc}
K & \xrightarrow{u} & !K \\
\downarrow m_k \quad := & & \downarrow \delta \\
!(K) & \xleftarrow{!(e)} & !!K
\end{array}$$

which in the graphical calculus are drawn out as follows:



(2.36)

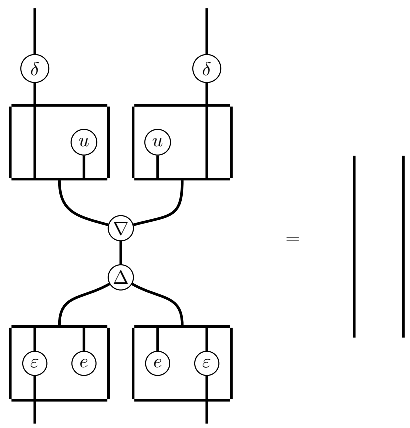
We will now carefully prove all the necessary coherences of a monoidal coalgebra modality.

Lemma 2.3.18 *[7, Lemma 21] $m_{\otimes}$ is a natural transformation.*

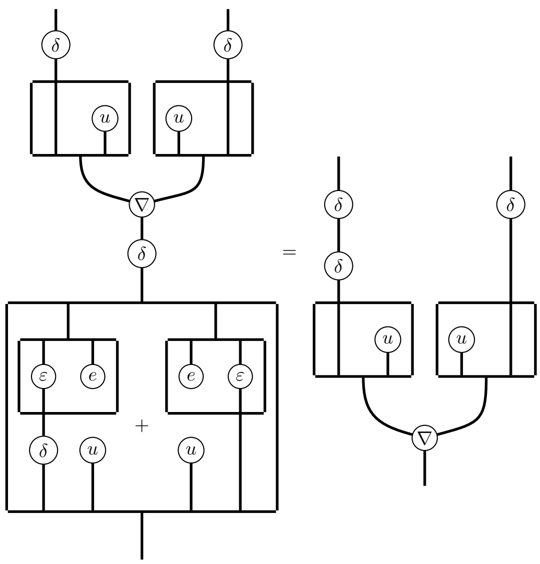
PROOF: By construction we have that $m_\otimes$ is natural. □

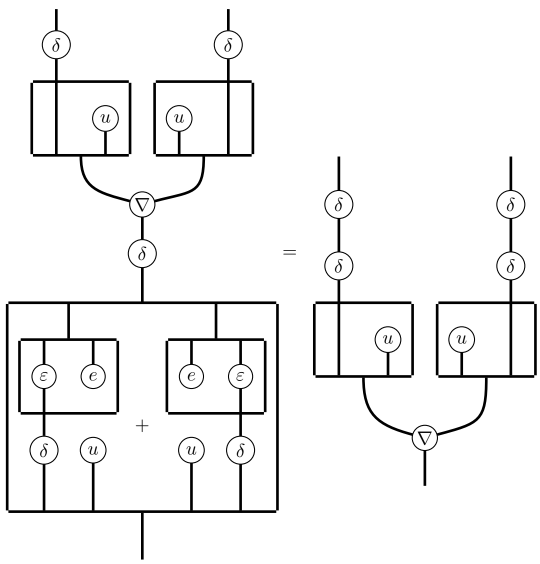
For multiple parts of the following proofs, we will need the following useful identities (for which we omit the proofs – they can be found in [7]).

Lemma 2.3.19 *[7, Lemma 22] The following equality holds:*

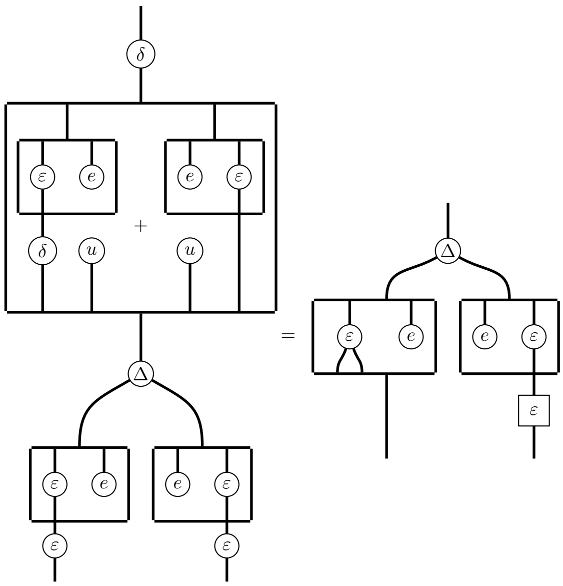


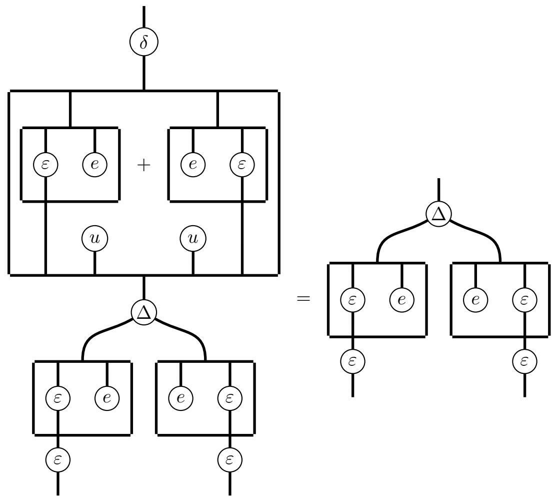
Lemma 2.3.20 *[7, Lemma 23] The following equalities hold:*

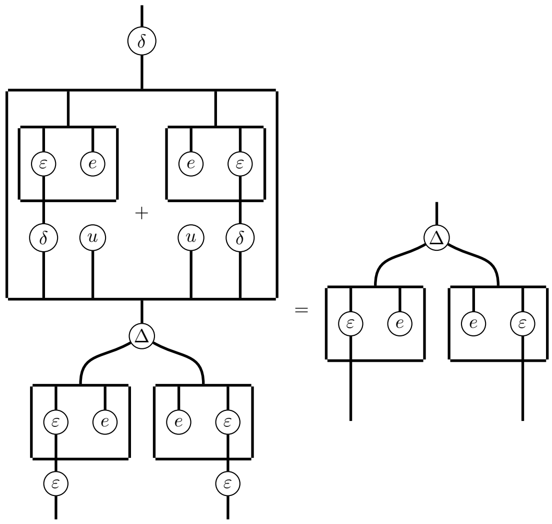




Lemma 2.3.21 *[7, Lemma 24] The following equalities holds:*





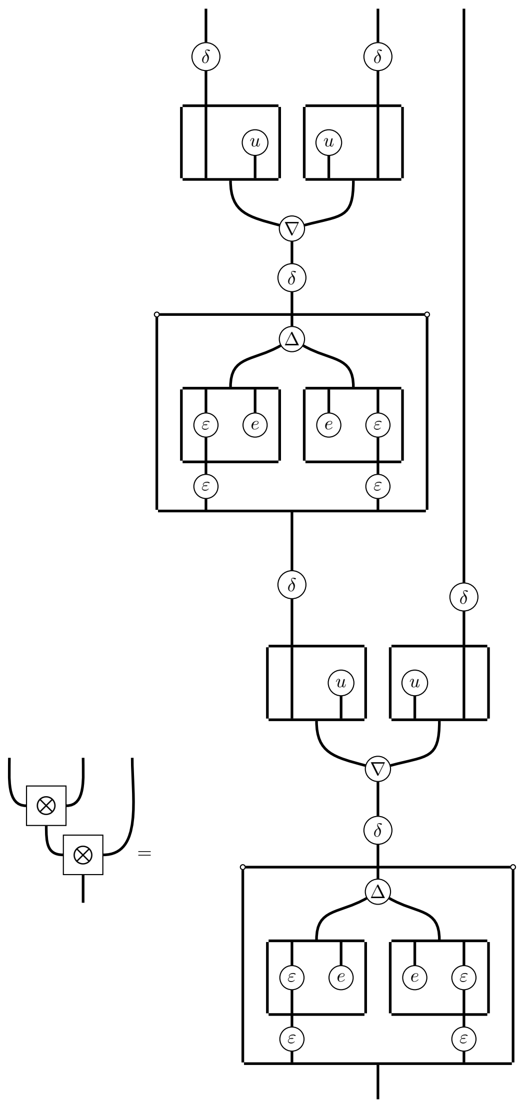


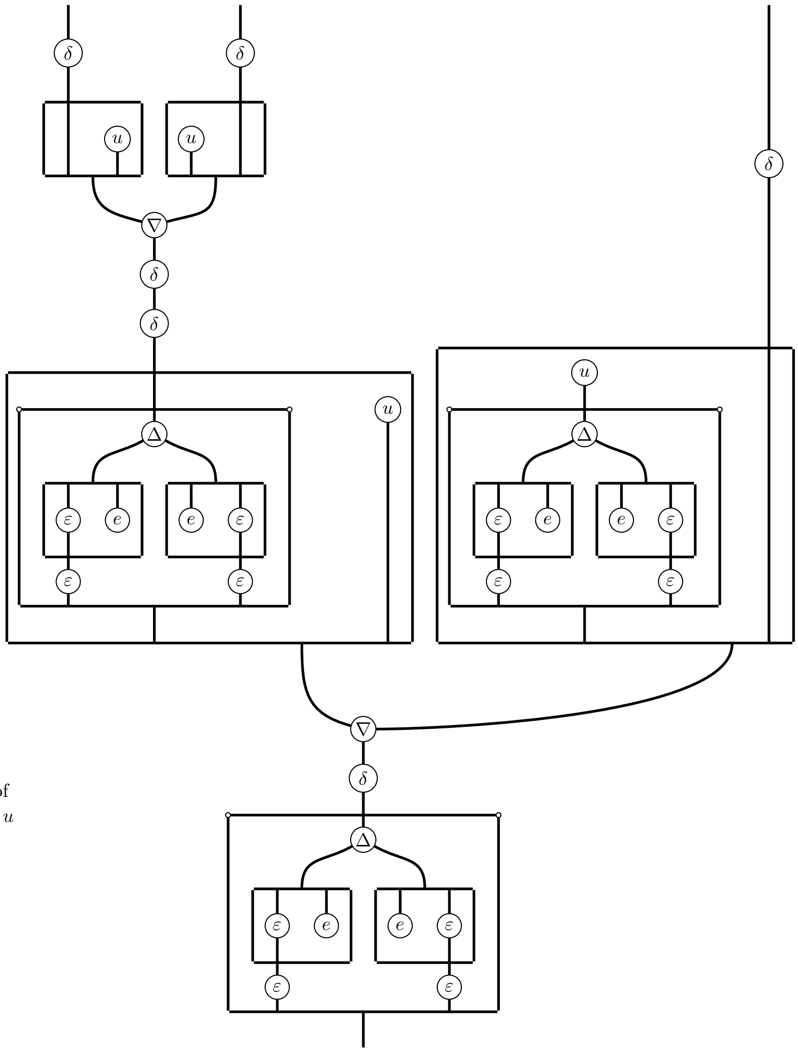
With the above identities, we can now show the symmetric monoidal comonad structure.

Lemma 2.3.22 *[7, Lemma 25]* $(!, m_\otimes, m_k)$ *is a symmetric monoidal functor.*

Proof: We must prove the associativity, unit, and symmetry coherences.

1. Associativity: Consider the following series of equalities:





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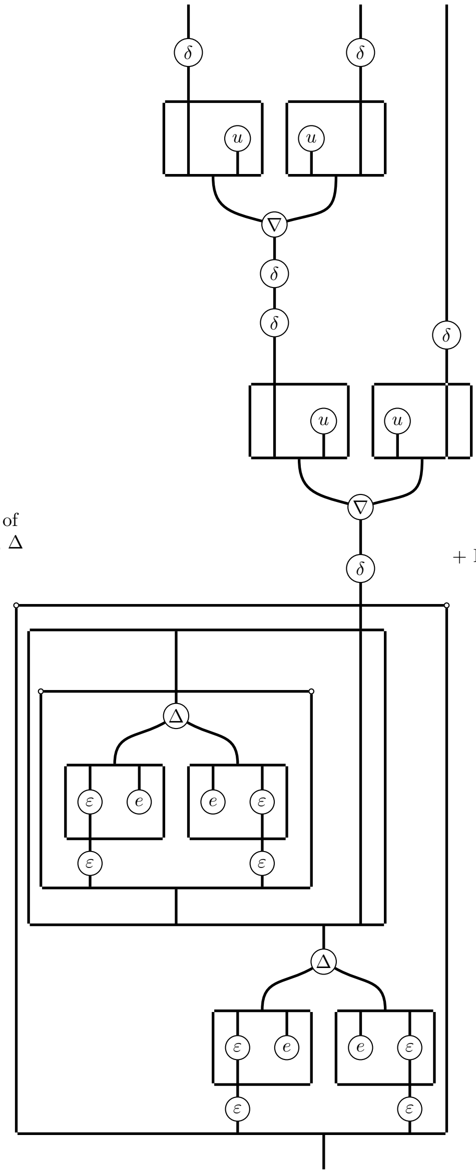

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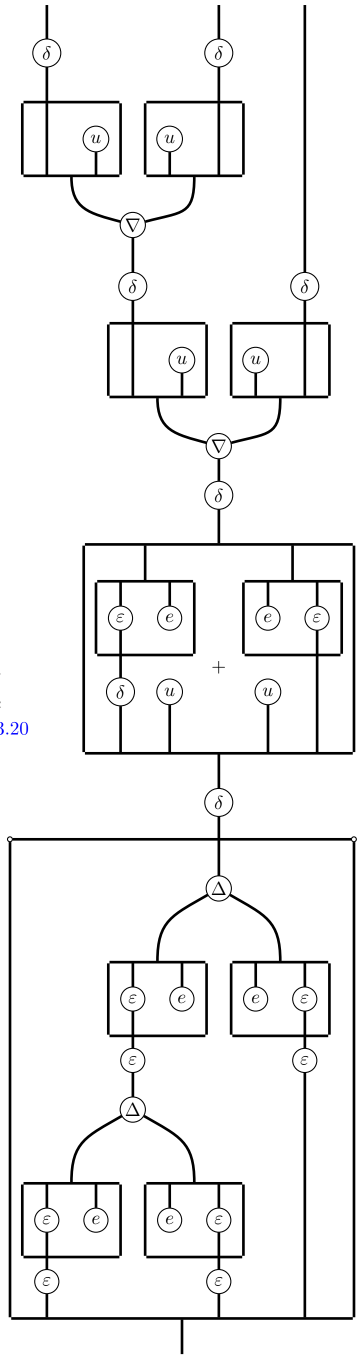
+ Lem 2.3.20



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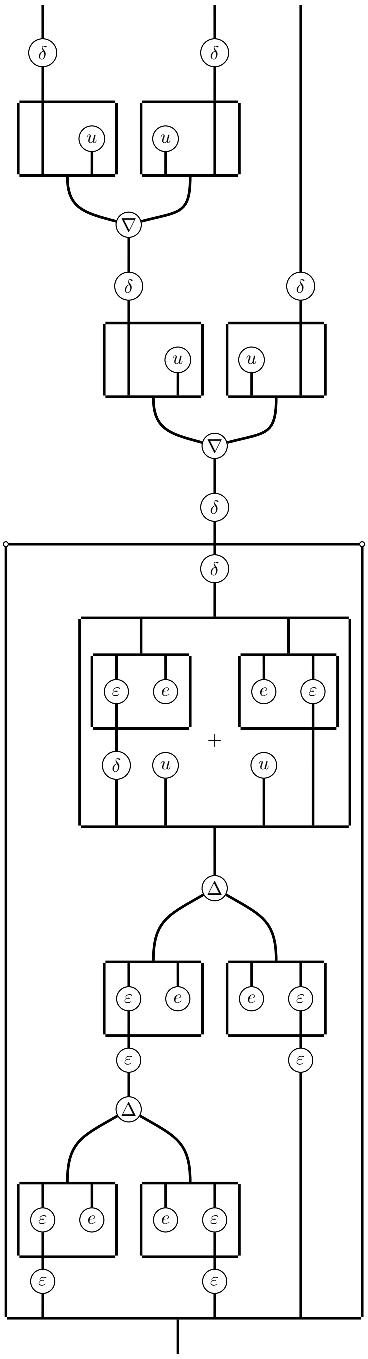
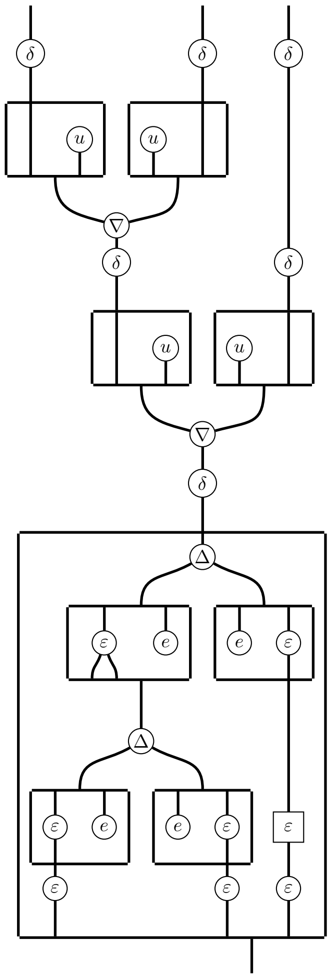
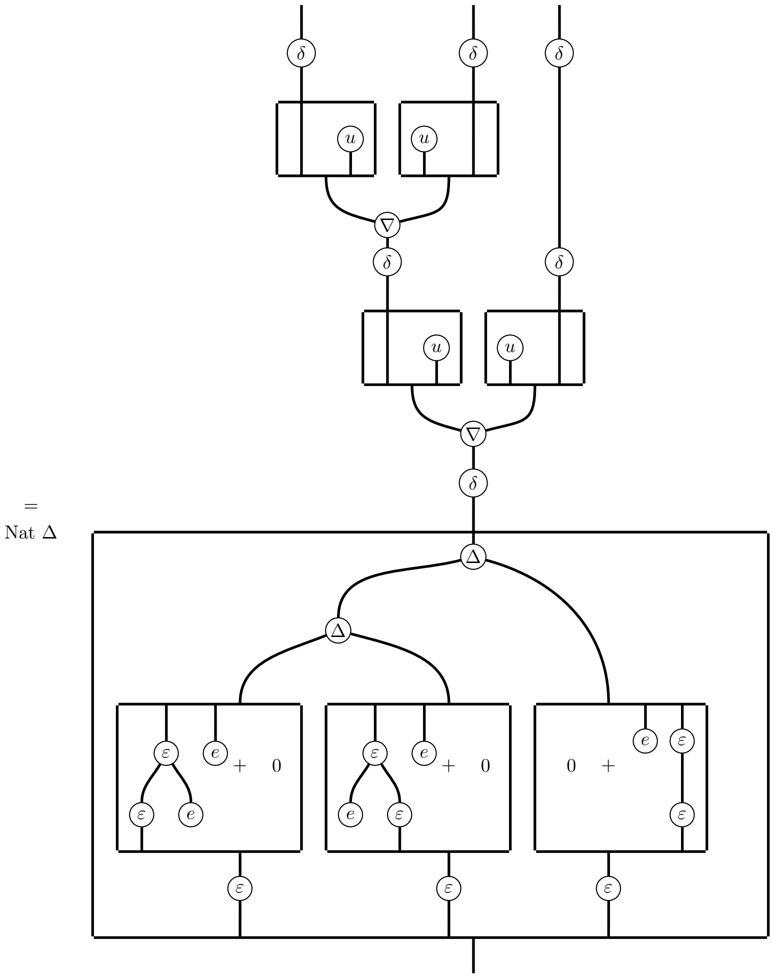
+ (2.2)



=
Lem 2.3.20

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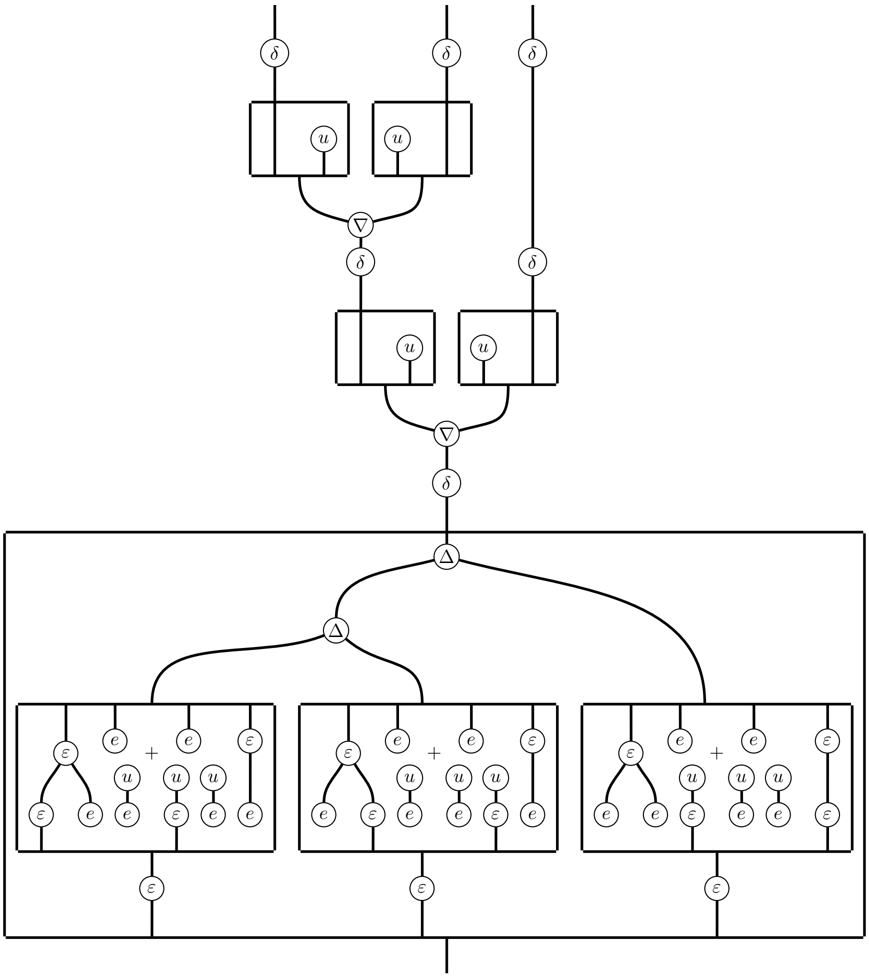
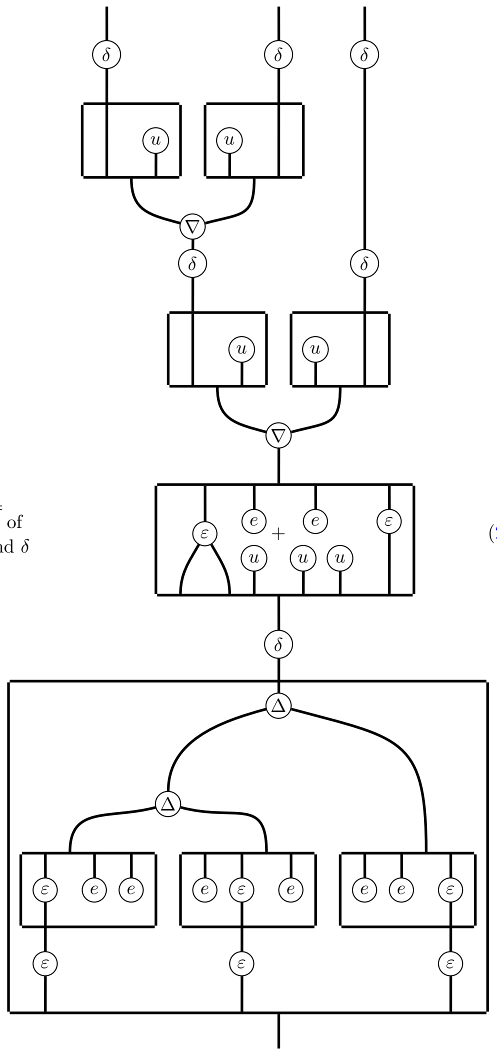


= Nat of Δ and δ



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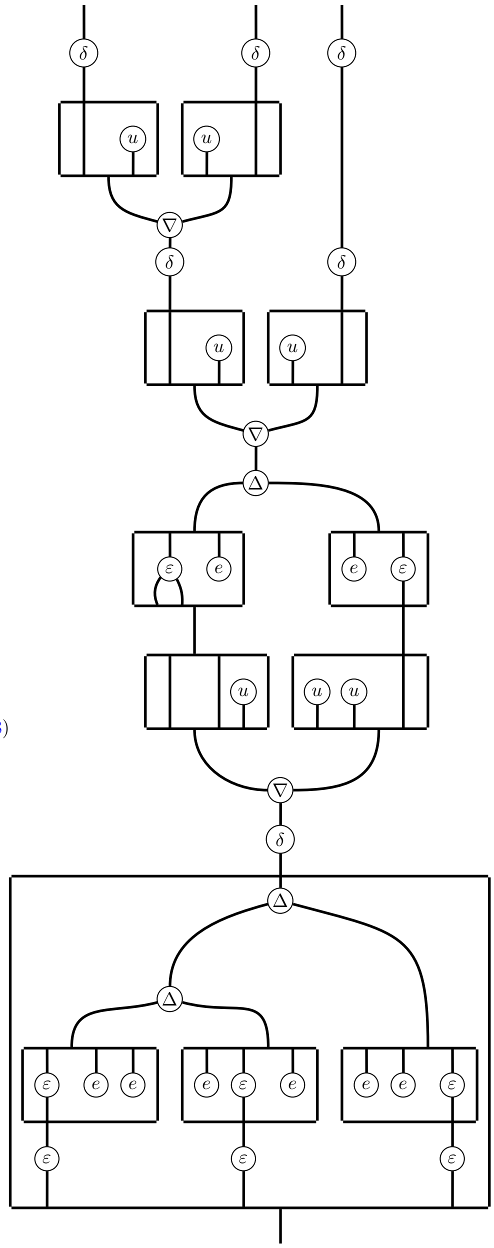
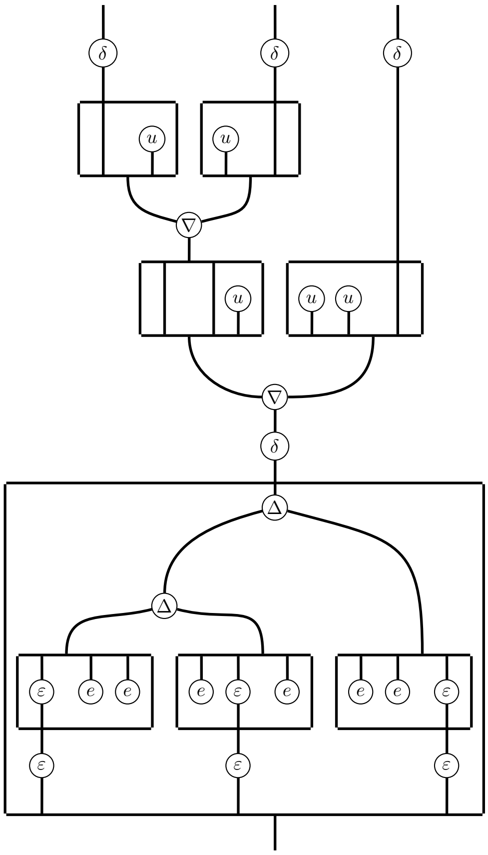


= Lem 2.3.19



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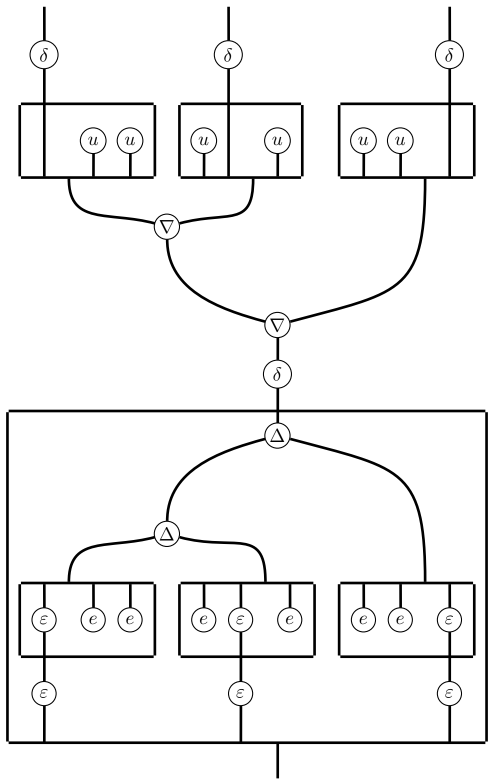
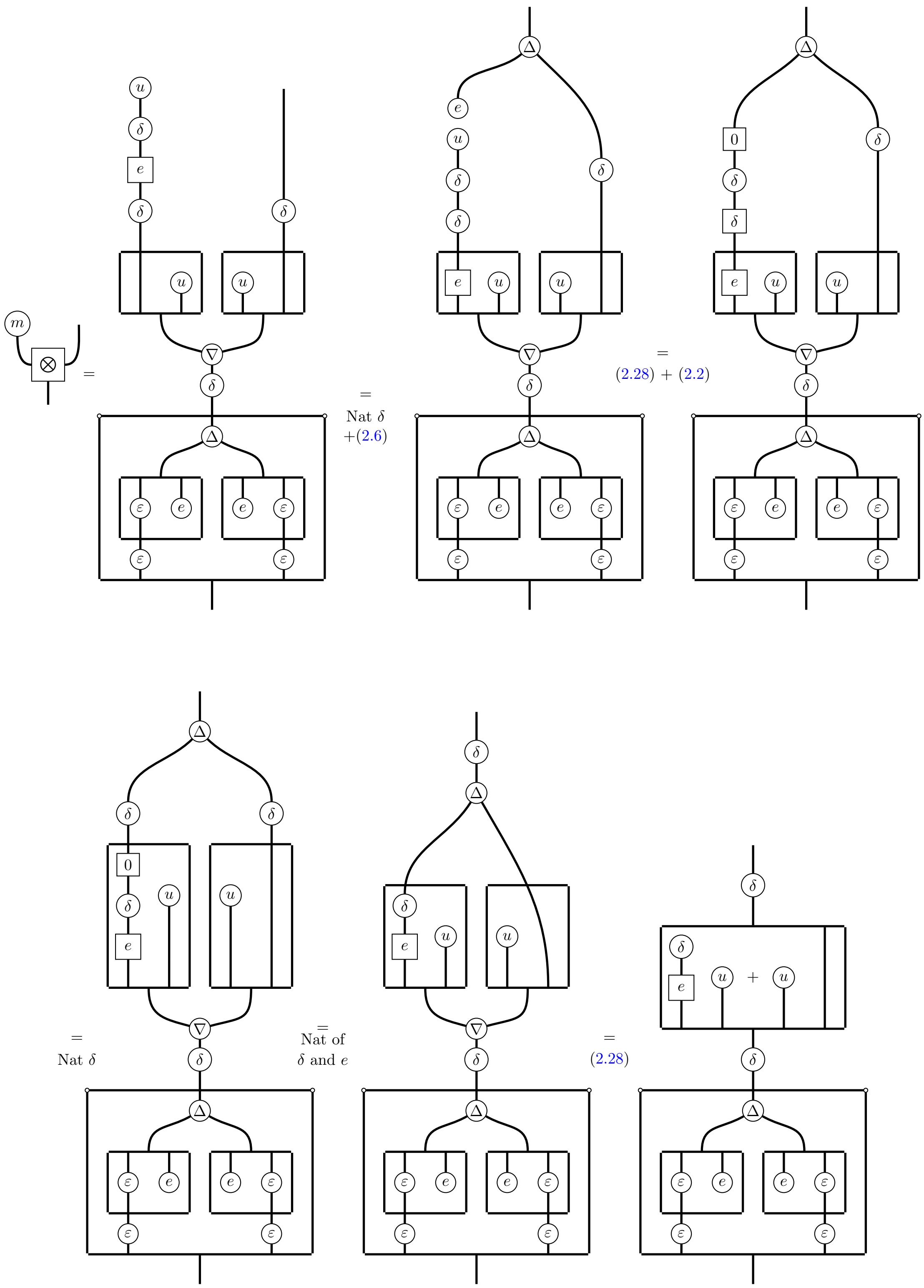


The last circuit is symmetric by associativity of the bialgebra, therefore reversing the sequence of equalities by symmetry gives the right associativity of $m_\otimes$. This proves the associativity coherence for $m_\otimes$.

2. Unit: We only prove the left unit identity, as the proof for the right unit identity is similar:



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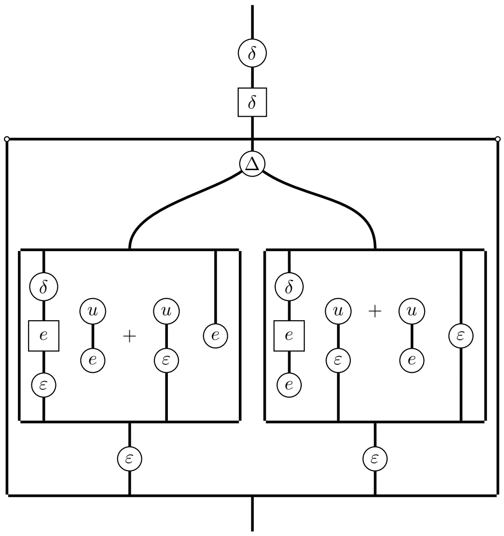

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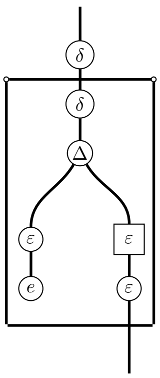

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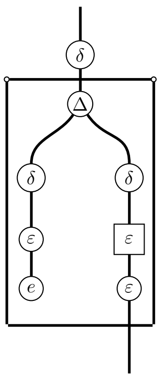

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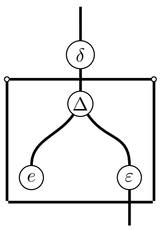

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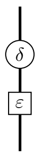

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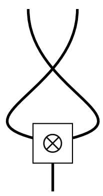
3. Symmetry:



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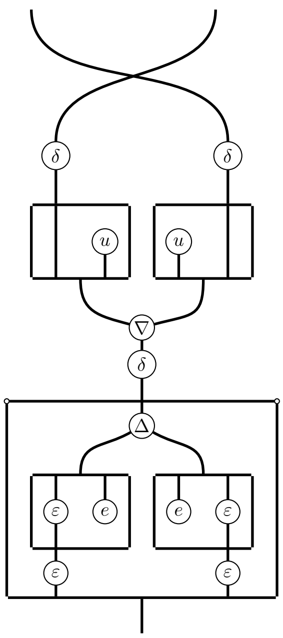

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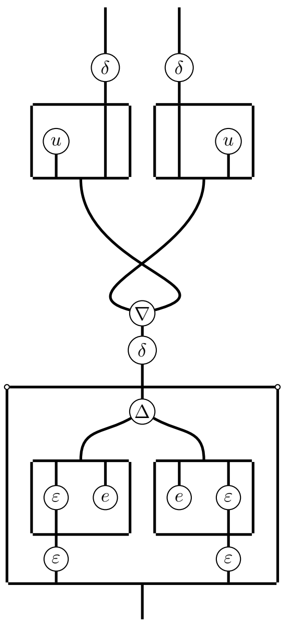

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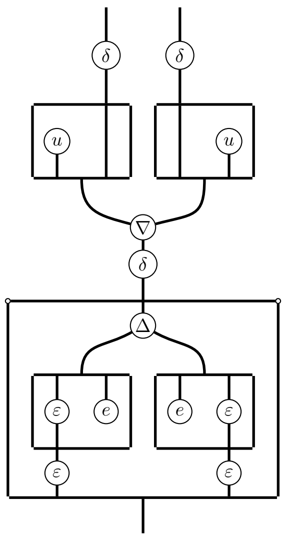
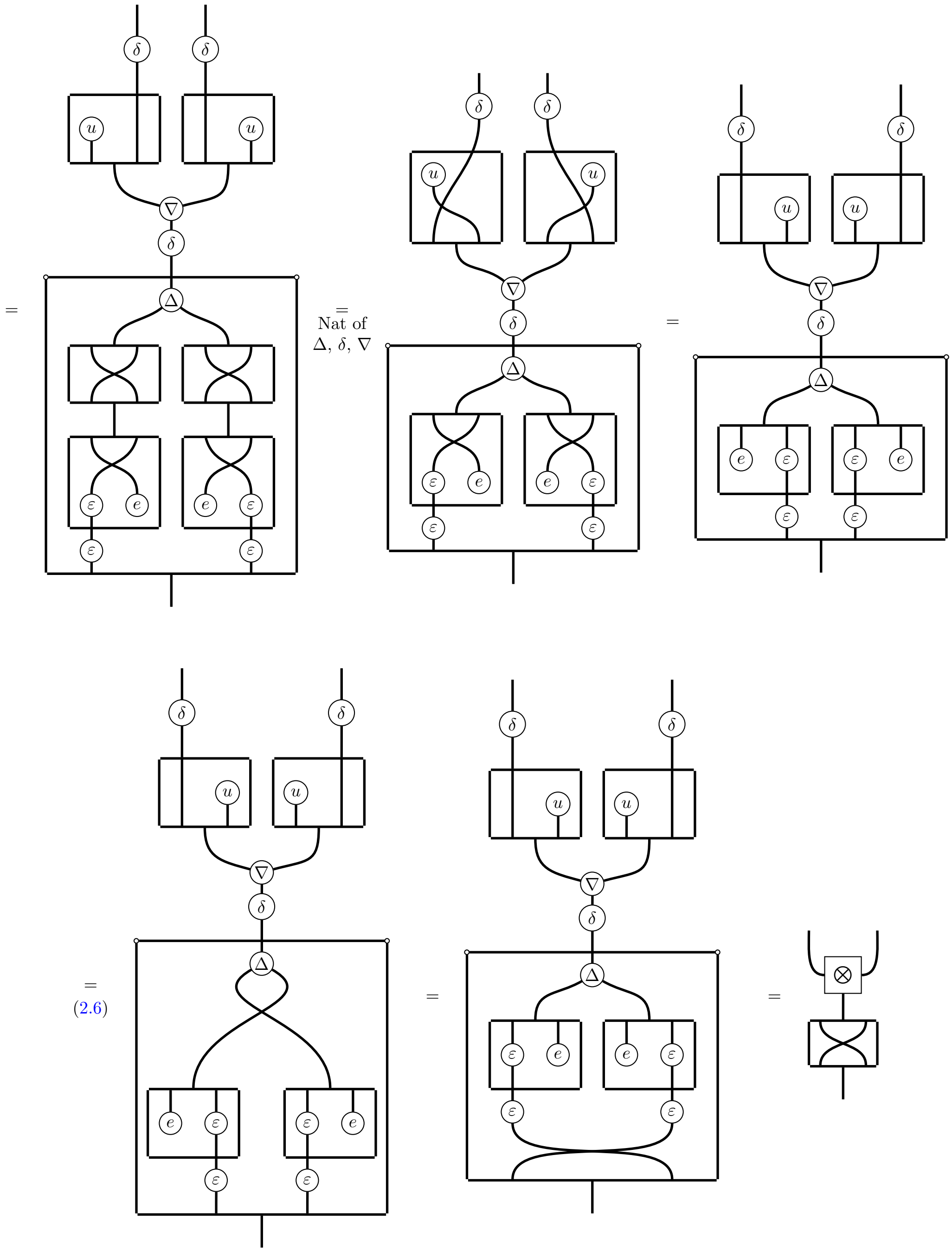


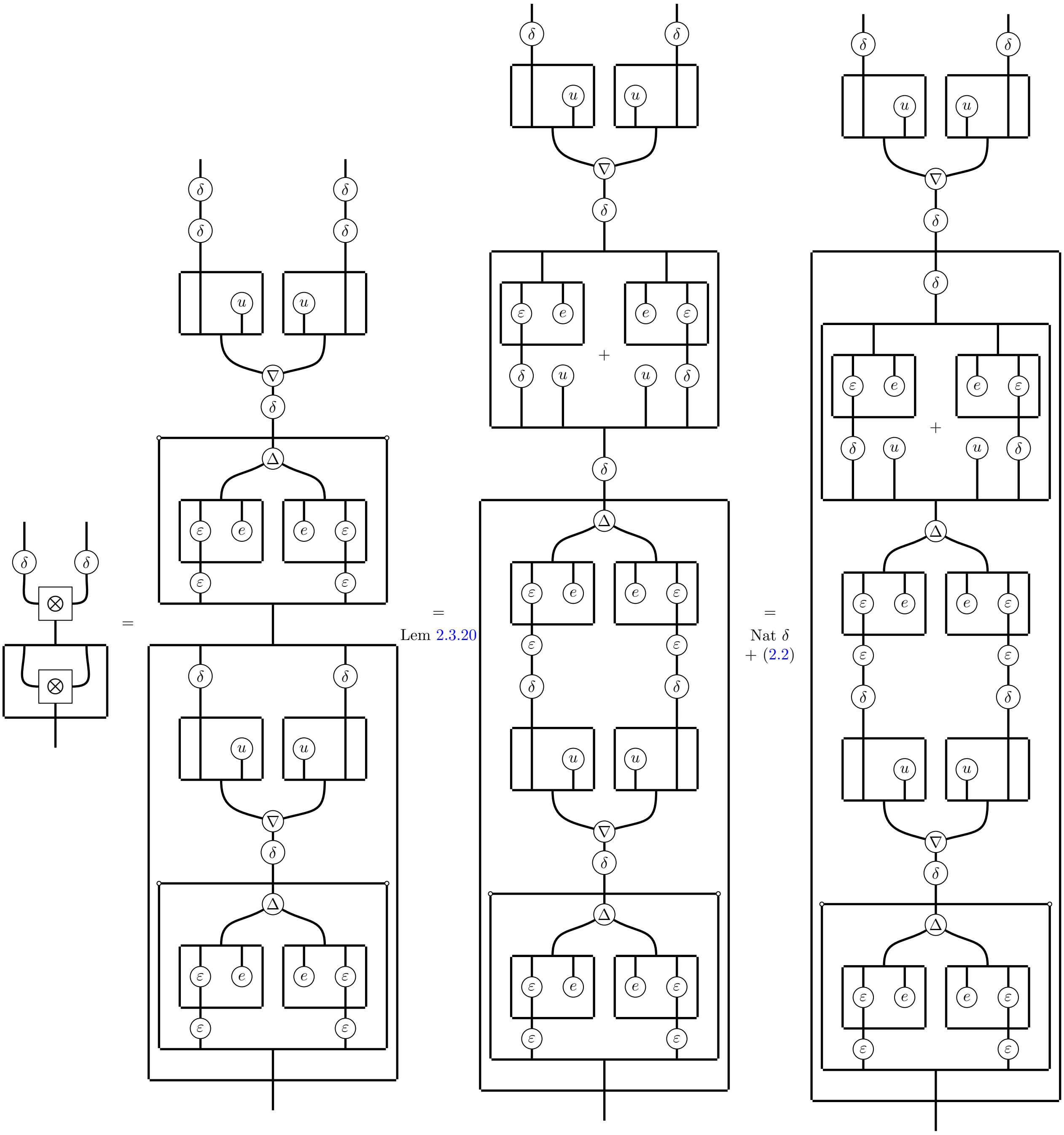


□

Lemma 2.3.23 *[7, Lemma 26]* $(!, \delta, \varepsilon, m_\otimes, m_k)$ *is a symmetric monoidal comonad.*

Proof: We need to show that δ and ε are monoidal natural transformation.

1. Compatibility between δ and $m_\otimes$:



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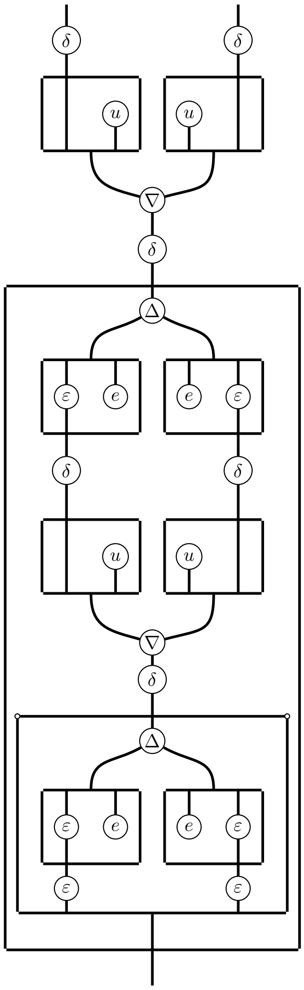
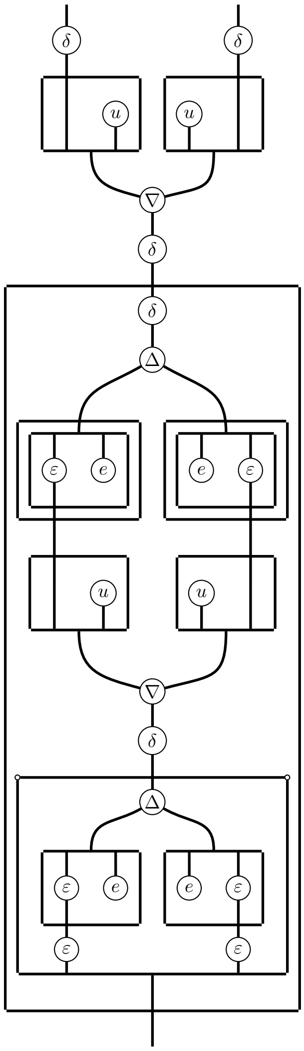
Lem 2.3.21



$=$
Nat δ
+ (2.6)



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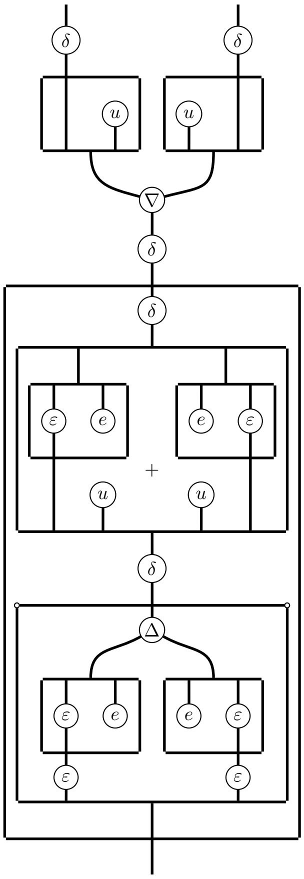
(2.28)



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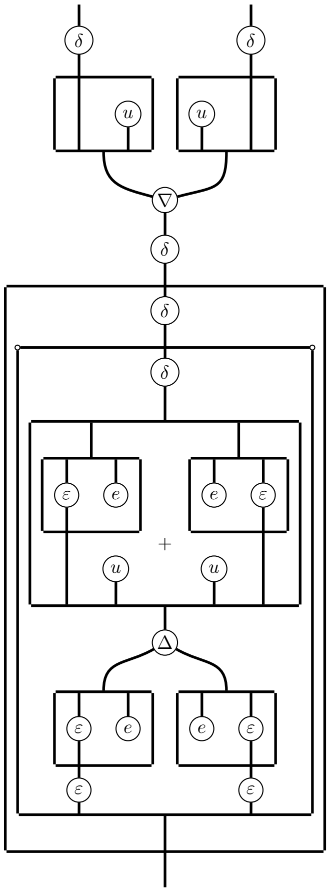
Nat δ
+ (2.2)



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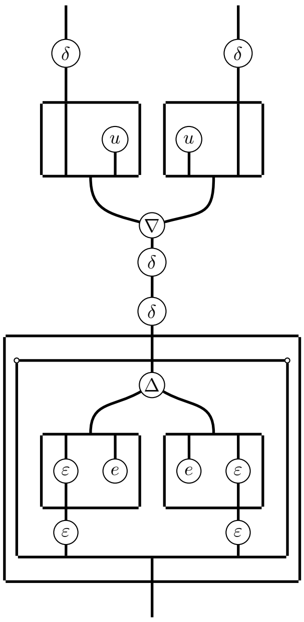
Lem 2.3.21
+ (2.2)



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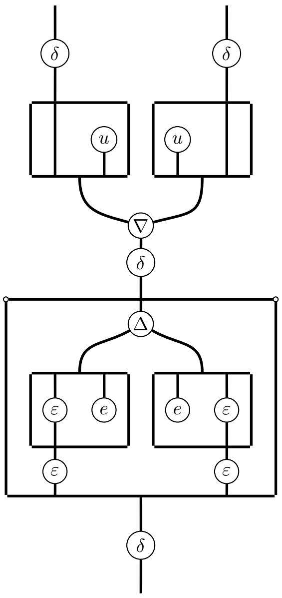
Nat δ



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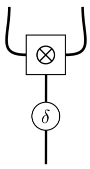
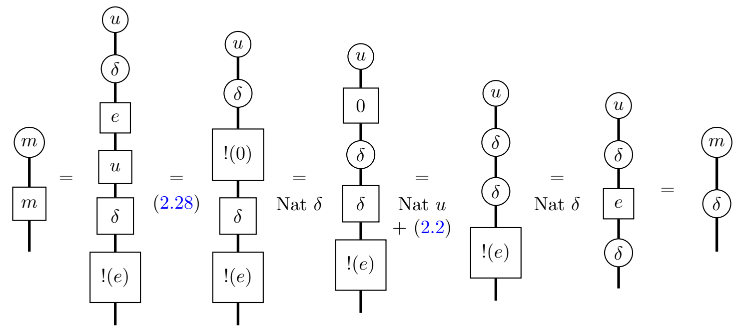
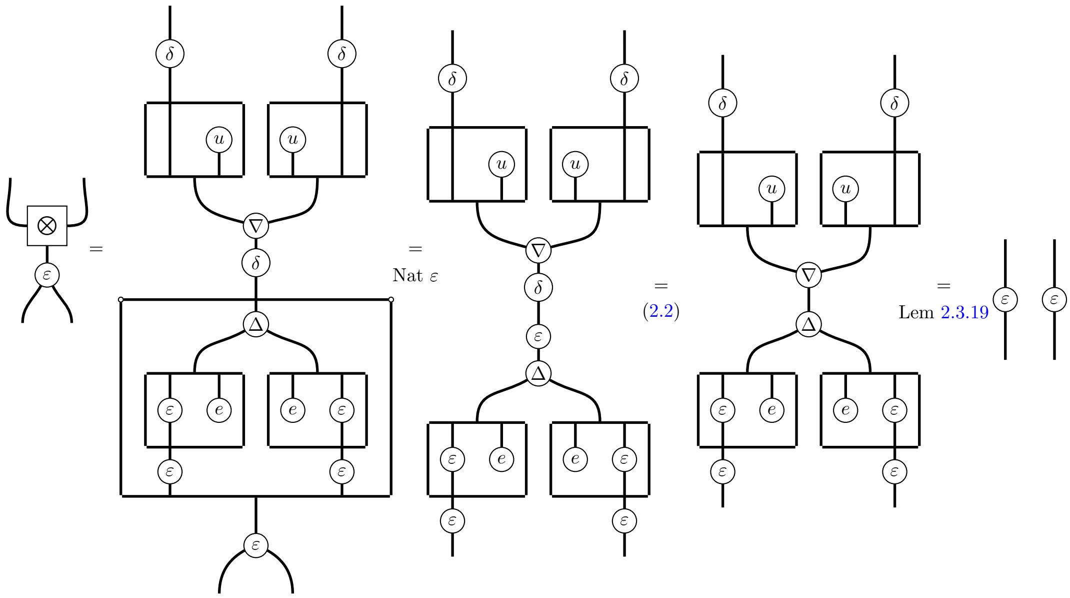


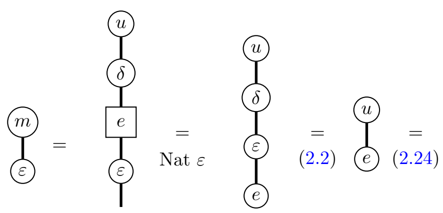
2. Compatibility between δ and m_k:



3. Compatibility between ε and $m_\otimes$:



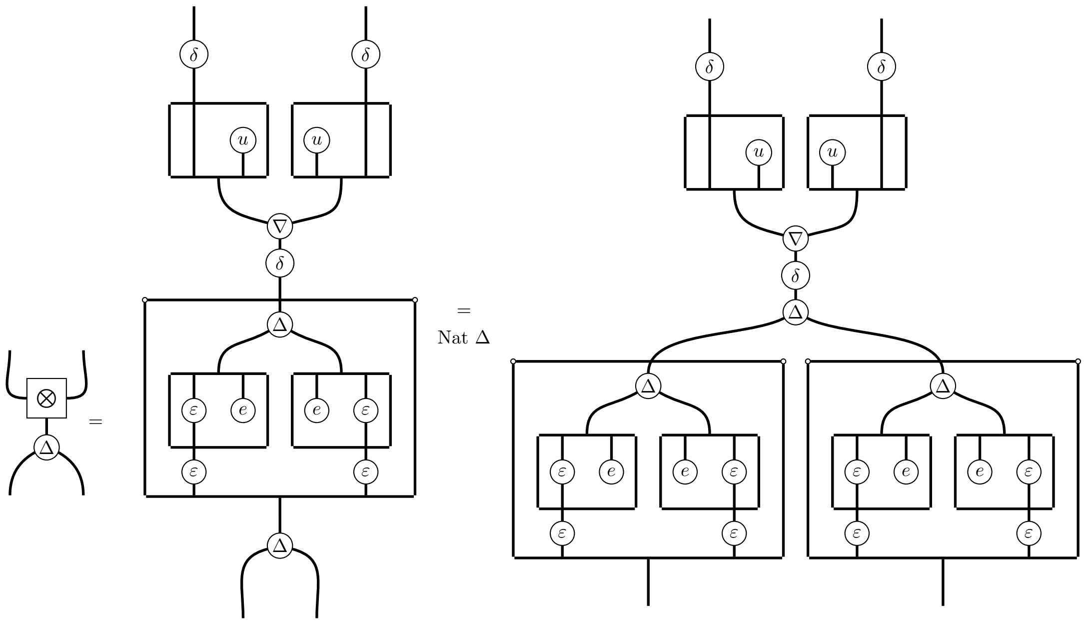
4. Compatibility between ε and m_k:

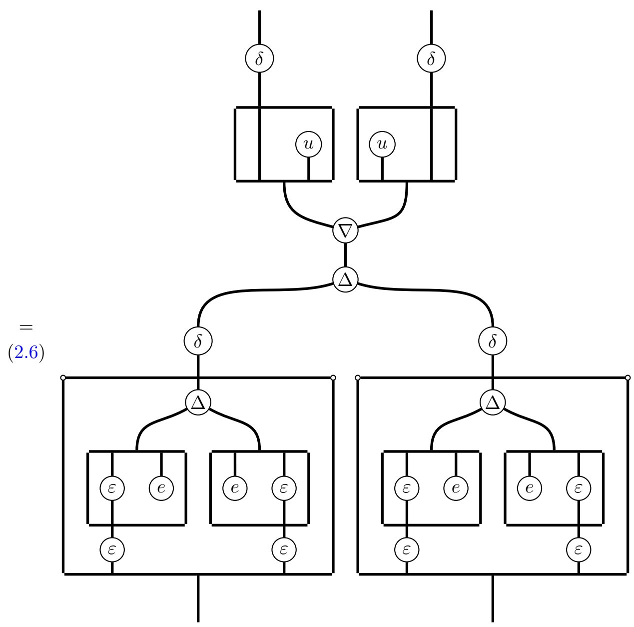


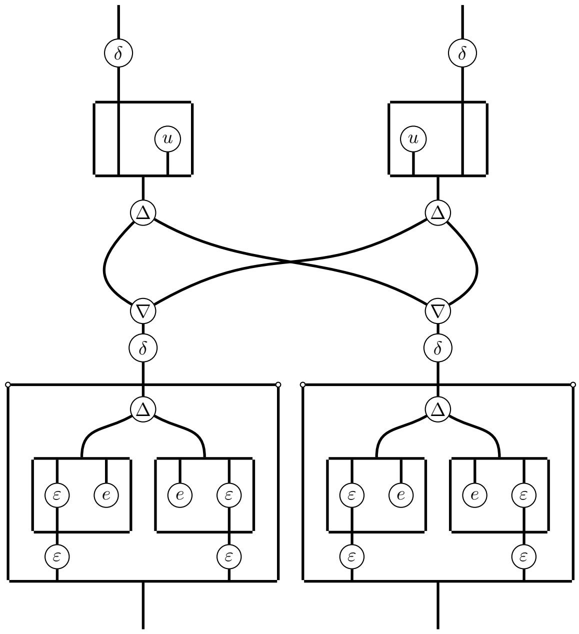
□

Proposition 2.3.24 *[7, Proposition 3] Every additive bialgebra modality is a monoidal coalgebra modality where $m_\otimes$ and m_k are defined as in (2.35).*

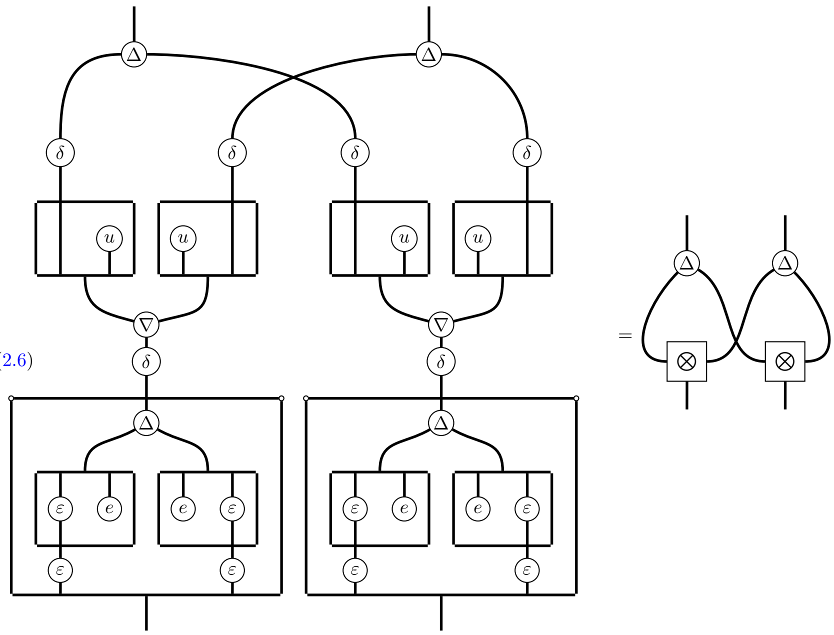
PROOF: We begin by proving that Δ and ε are monoidal transformations.
1. Compatibility between Δ and $m_\otimes$



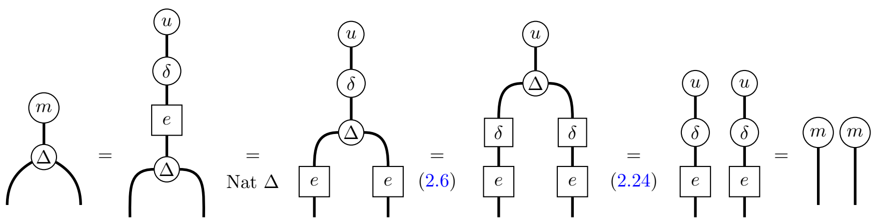




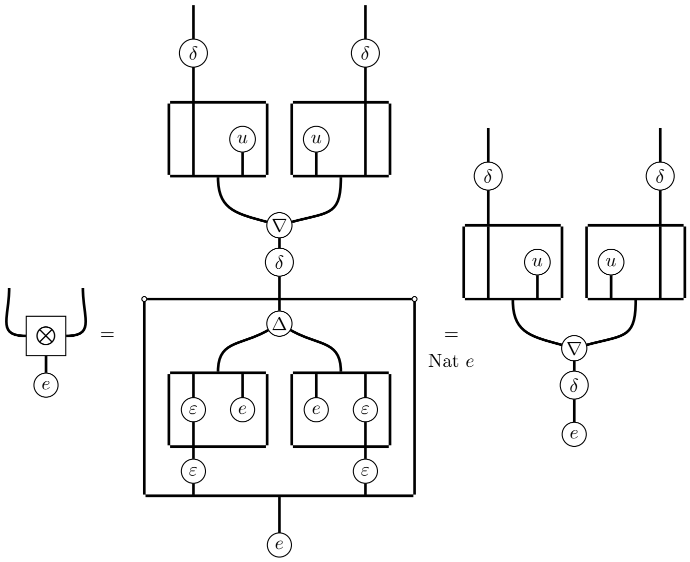
= (2.24)

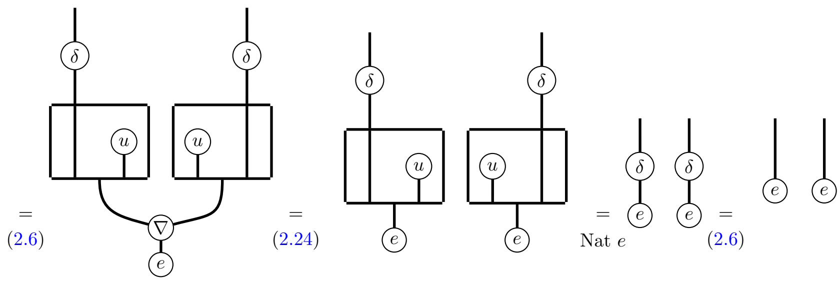


2. Compatibility between Δ and m_k:

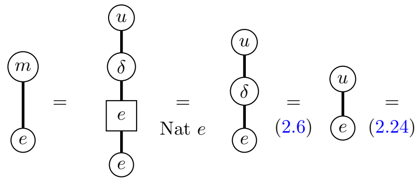


3. Compatibility between e and $m_\otimes$:

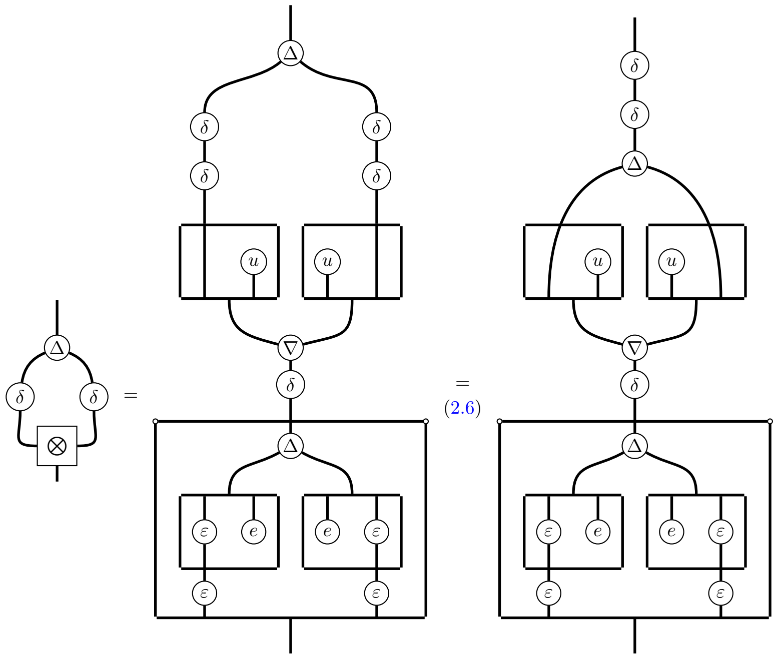


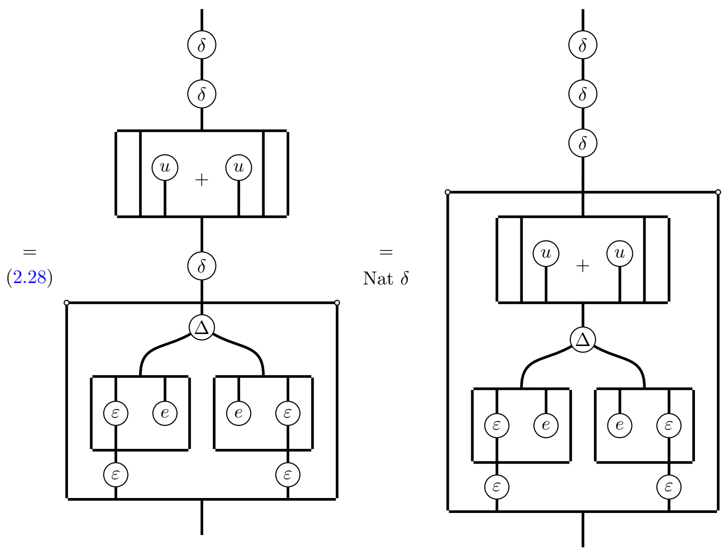


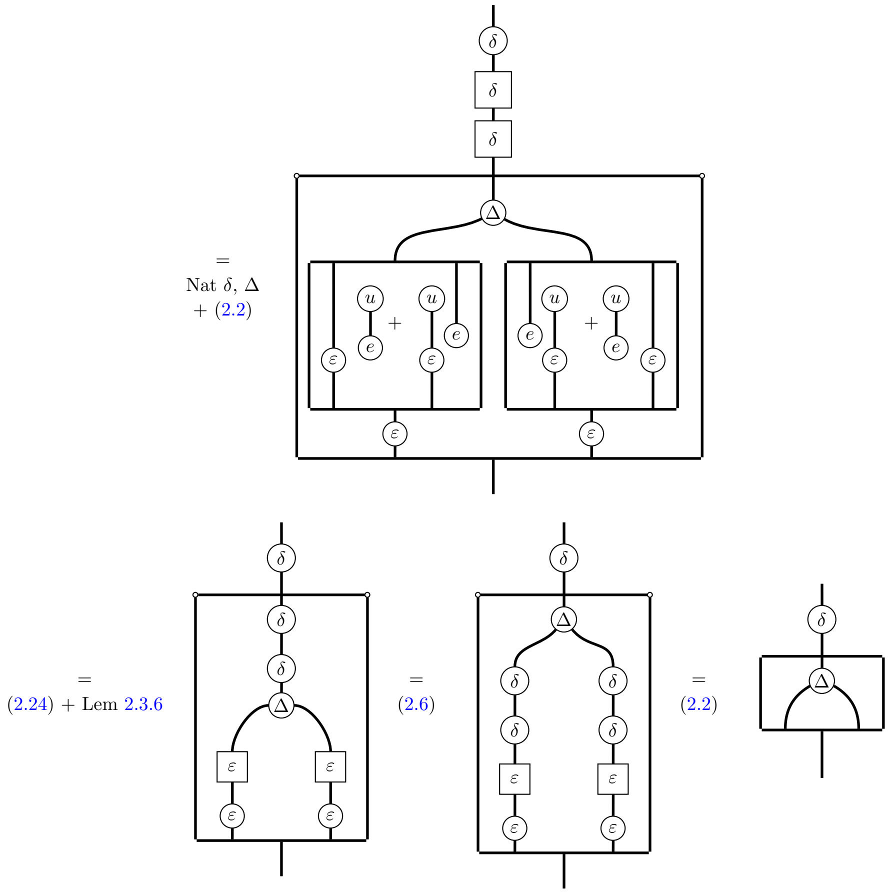
4. Compatibility between e and m_k:



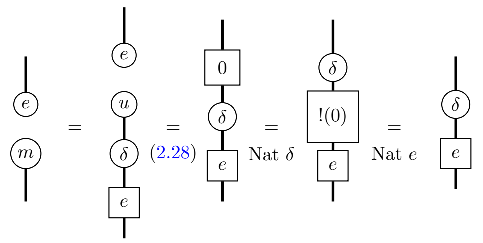
Now we prove that Δ is a !-coalgebra morphism:







And finally we prove that e is a !-coalgebra morphism:



So we conclude that $(!, \delta, \varepsilon, m_\otimes, m_k, \Delta, e)$ is a monoidal coalgebra modality. □

We can state the main result of this chapter, that the constructions between additive bialgebra modalities and monoidal coalgebra modalities are in fact inverses of each other.

Theorem 2.3.25 *[7, Theorem 1] For an additive symmetric monoidal category, the following are in bijective correspondence:*

(i) *Monoidal coalgebra modalities;*

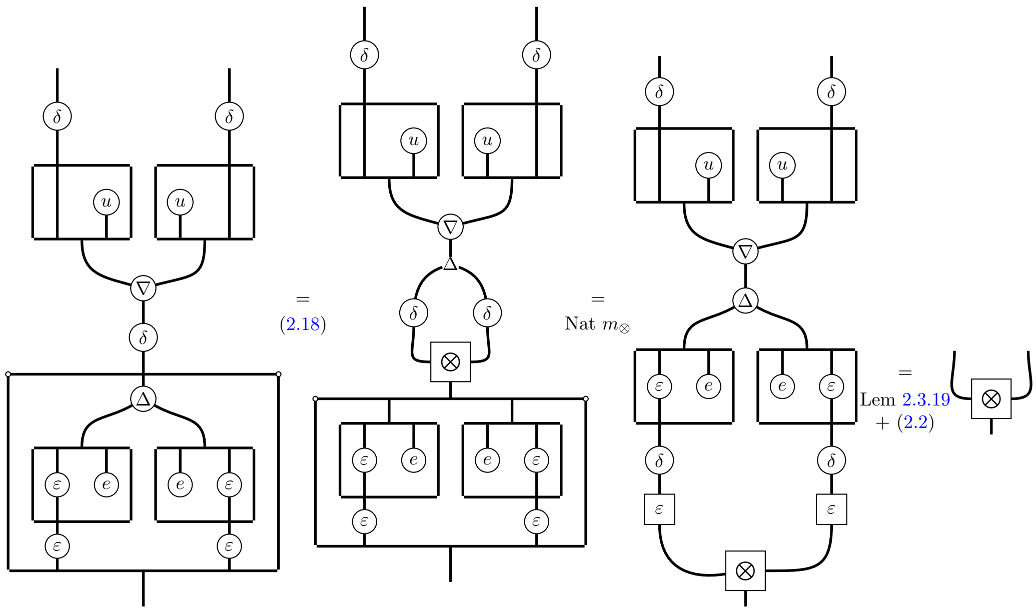
(ii) *Additive bialgebra modalities.*

Therefore, the following are equivalent:

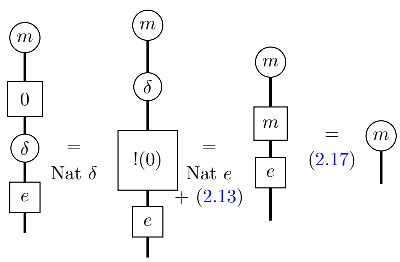
(i) *An additive linear category;*

(ii) *An additive symmetric monoidal category with an additive bialgebra modality.*

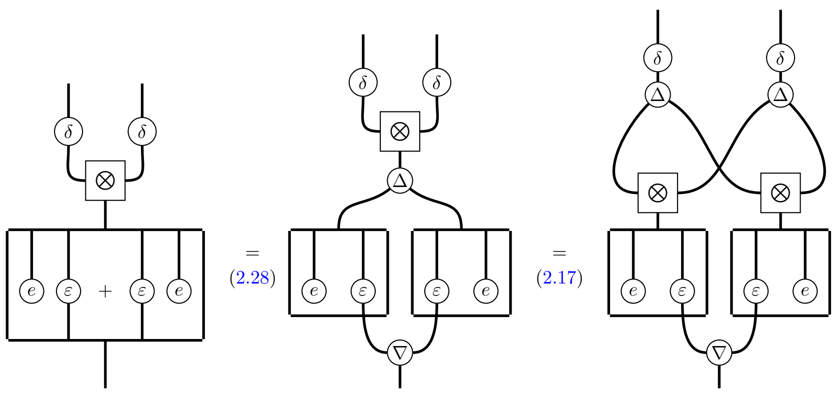
PROOF: Starting from a monoidal coalgebra modality, we first check that we re-obtain $m_\otimes$:

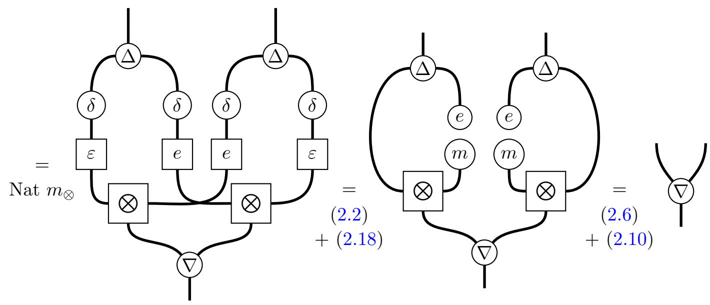


Next we check that we get back m_k:

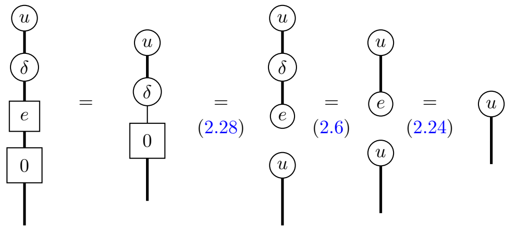


Starting with an additive bialgebra modality, we first check that we re-obtain the multiplication:





Finally we prove that we re-obtain the unit:



So we conclude that additive bialgebra modalities are in bijective correspondence with monoidal coalgebra modalities. □

We conclude this section with a discussion on additive bialgebra modalities in the presence of biproducts, which are equivalently described by the Seely isomorphisms and additive monoidal storage categories. We omit the proofs for the remainder of this section, and leave it to the reader to check these facts for themselves.

Definition 2.3.26 *An **additive monoidal storage category** [7, Definition 11] is a monoidal storage category which is also an additive symmetric monoidal category.*

Note that for an additive monoidal storage category, the product $\times$ is in fact a biproduct, and the the terminal object is in fact a zero object, which we will now denote as $\top = 0$. As noted in [8], the coalgebra modality of an additive monoidal storage category is an additive bialgebra modality where the multiplication and unit are defined using the Seely isomorphisms as follows:

$$\nabla := \ !A \otimes !A \xrightarrow{\chi^{-1}} !(A \times A) \xrightarrow{!(\nabla_\times)} !A$$

$$u = \ K \xrightarrow{\chi_0^{-1}} !0 \xrightarrow{!(0)} !A$$

where $\nabla_\times$ is the codiagonal map of the biproduct. Conversly, every additive bialgebra modality of an additive symmetric monoidal category with finite (bi)products satisfies the Seely isomorphisms where χ^{-1} and χ_0^{-1} are defined respectively as follows:

$$\chi^{-1} := \ !A \otimes !B \xrightarrow{!(\iota_0) \otimes !(\iota_1)} !(A \times B) \otimes !(A \times B) \xrightarrow{\nabla} !(A \times B)$$

$$\chi_0^{-1} := \ K \xrightarrow{u} !0$$

where ι_0 and ι_1 are the injection maps of the biproduct. Combining Theorem 2.2.8 and Theorem 2.3.25 together, we obtain the following:

Theorem 2.3.27 *[7, Theorem 6] For an additive symmetric monoidal category with finite (bi)products, the following are in bijective correspondence:*

(i) *Coalgebra Modalities with Seely isomorphisms;*

(ii) *Monoidal coalgebra modalities;*

(iii) *Additive bialgebra modalities.*

Therefore, the following are equivalent:

(i) *An additive monoidal storage category;*

(ii) *An additive linear category with finite (bi)products;*

(iii) *An additive symmetric monoidal category with finite (bi)products and an additive bialgebra modality.*

Every additive symmetric monoidal category with an additive bialgebra modality induces an additive monoidal storage category via the biproduct completion. We first recall the biproduct completion for an additive category [67]. Let $\mathbb{X}$ be an additive category. Define the biproduct completion of $\mathbb{X}$, $\mathsf{B}[\mathbb{X}]$, as the category whose objects are list of objects of $\mathbb{X}$: $(A_1, \ldots, A_n)$, including the empty list $()$, and whose maps are matrices of maps of $\mathbb{X}$, including the empty matrix:

$$(A_1, \ldots, A_n) \xrightarrow{[f_{i,j}]} (B_1, \ldots, B_m)$$

where $f_{i,j} : A_i \longrightarrow B_j$. The composition in $\mathsf{B}[\mathbb{X}]$ is the standard matrix multiplication:

$$[f_{i,j}][g_{l,k}] = [\sum f_{i,k} g_{k,j}]$$

while the identity is the standard identity matrix:

$$(A_1, \ldots, A_n) \xrightarrow{[\delta_{i,j}]} (A_1, \ldots, A_n)$$

where $\delta_{i,j} = 0$ if $i \neq j$, and $\delta_{i,i} = 1$. It is easy to see that $\mathsf{B}[\mathbb{X}]$ does in fact have biproducts:

Lemma 2.3.28 *[7, Lemma 10] $\mathsf{B}[\mathbb{X}]$ is a well-defined category with finite biproducts, and furthermore, the obvious inclusion functor $\mathcal{I} : \mathbb{X} \longrightarrow \mathsf{B}[\mathbb{X}]$ preserves the additive structure strictly.*

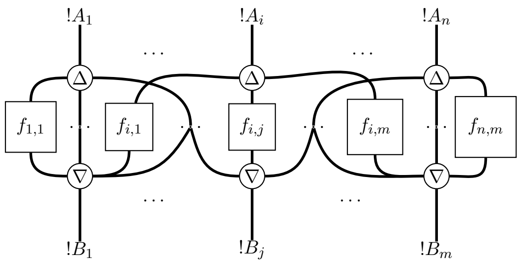
If $\mathbb{X}$ is an additive symmetric monoidal category, then so is $\mathsf{B}[\mathbb{X}]$. The monoidal unit is the same as in $\mathbb{X}$, the tensor product of objects is:

$$(A_1, \ldots, A_n) \otimes (B_1, \ldots, B_m) = (A_1 \otimes B_1, \ldots, A_1 \otimes B_m, \ldots, A_n \otimes B_n)$$

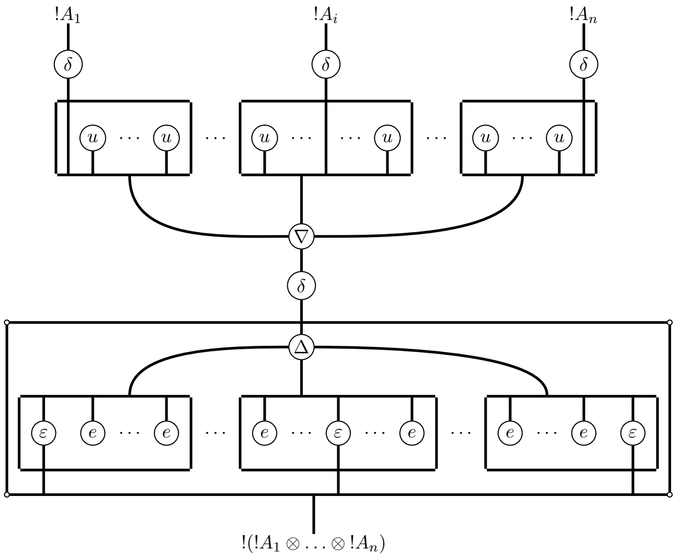
while the tensor product of maps is the standard Kronecker product of matrices.

Lemma 2.3.29 *[7, Lemma 11] If $\mathbb{X}$ is an additive symmetric monoidal category, then so is $\mathsf{B}[\mathbb{X}]$, and furthermore, the obvious inclusion functor $\mathcal{I} : \mathbb{X} \longrightarrow \mathsf{B}[\mathbb{X}]$ preserves the additive symmetric monoidal structure strictly.*

If $\mathbb{X}$ admits an additive bialgebra modality, then $\mathsf{B}[\mathbb{X}]$ is an additive monoidal storage category where the Seely isomorphisms are strict, i.e., equalities. As such, $\mathsf{B}[\mathbb{X}]$ an additive linear category, and therefore also has an additive bialgebra modality. The functor $! : \mathsf{B}[\mathbb{X}] \longrightarrow \mathsf{B}[\mathbb{X}]$ is defined on objects as $!(A_1, \ldots, A_n) = !A_1 \otimes \ldots \otimes !A_n$ and on a map $[f_{i,j}] : (A_1, \ldots, A_n) \longrightarrow (B_1, \ldots, B_m)$, $!([f_{i,j}]) : !A_1 \otimes \ldots \otimes !A_n \longrightarrow !B_1 \otimes \ldots \otimes !B_m$ is represented in the graphical calculus as:



The bialgebra structure is given by the standard tensor product of bialgebras, the comonad comultiplication $!(A_1, \ldots, A_n) \longrightarrow !!(A_1, \ldots, A_n)$ is represented in the graphical calculus as:



while the comonad counit is the following matrix:

$$\left[\varepsilon \otimes e \otimes \ldots \otimes e, \quad \ldots \quad , e \otimes \ldots \otimes \varepsilon \otimes \ldots \otimes e, \quad \ldots \quad , e \otimes e \otimes \ldots \otimes \varepsilon\right] : !A_1 \otimes \ldots \otimes !A_n \longrightarrow (A_1, \ldots, A_n)$$

Proposition 2.3.30 *[7, Proposition 8] If $\mathbb{X}$ is an additive symmetric monoidal category with an additive bialgebra modality, then $\mathsf{B}[\mathbb{X}]$ is an additive monoidal storage category where the obvious inclusion functor $\mathcal{I} : \mathbb{X} \longrightarrow \mathsf{B}[\mathbb{X}]$ preserves the additive bialgebra modality strictly.*

2.4 Constructing Non-Additive Bialgebra Modalities

In this final section of this chapter, we give a construction of a family of non-monoidal coalgebra modalities induced by a monoidal coalgebra modality. In the presence of additive structure, this construction also results in a family of non-additive bialgebra modalities induced by additive algebra modalities. These newly constructed coalgebra modalities should be thought of as the original coalgebra modality but in context. Indeed, these coalgebra modalities in context are a key ingredient for the notion of a *contextual* integral category [27], where the context is key in order to describe which variables are being integrated and which are kept in context (i.e. constant). In the next chapter, we will also mention how these constructed coalgebra modalities are also related to partial differentiation.

Let $\mathbb{X}$ be a linear category with monoidal coalgebra modality $(!, \delta, \varepsilon, m_\otimes, m_k, \Delta, e)$. for each object B consider the functor $!^B : \mathbb{X} \longrightarrow \mathbb{X}$ defined on objects as $!^B A = !B \otimes !A$, and on a map $f : A \longrightarrow C$ as $!^B(f) = 1 \otimes !(f) : !B \otimes !A \longrightarrow !B \otimes !C$. Consider the natural transformations:

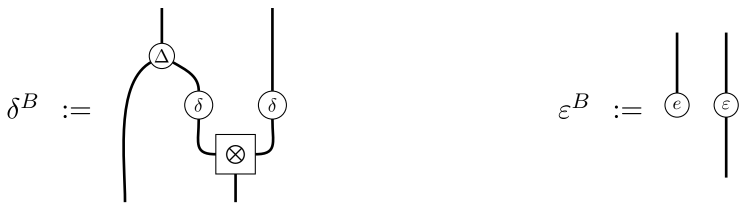
$$\delta^B : !^B(A) \longrightarrow !^B(!^B(A)) \qquad \qquad \varepsilon^B : !^B(A) \longrightarrow A$$

defined as follows:

$$\delta^B := \; !B \otimes !A \xrightarrow{\Delta \otimes 1} !B \otimes !B \otimes !A \xrightarrow{1 \otimes \delta \otimes \delta} !B \otimes !!B \otimes !!A \xrightarrow{1 \otimes m_\otimes} !B \otimes !(!B \otimes !A)$$

$$\varepsilon^B := \; !B \otimes !A \xrightarrow{e \otimes \varepsilon} A$$

which drawn in the graphical calculus are:

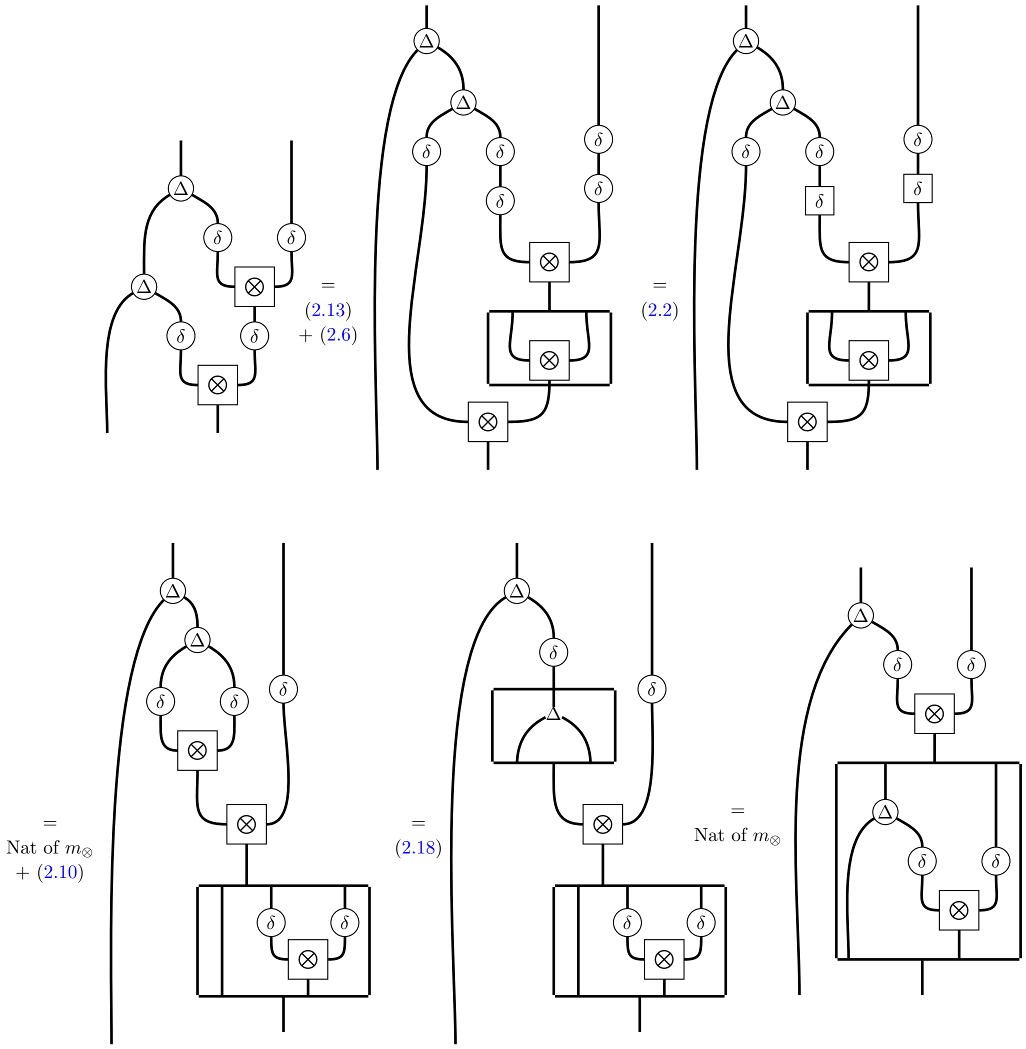


We first show that this gives a comonad:

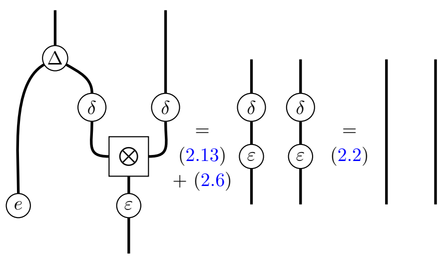
Lemma 2.4.1 *[7, Lemma 12] $(!^B, \delta^B, \varepsilon^B)$ is a comonad.*

Proof: We must show the following three identities:
1. $\delta^B \delta^B = \delta^B !^B(\delta^B)$: Here we use that δ is a monoidal transformation, the naturality of $m_\otimes$, the co-associativity of Δ, the associativity of $m_\otimes$, the co-associativity of the comonad, and that Δ is a

!-coalgebra morphism:

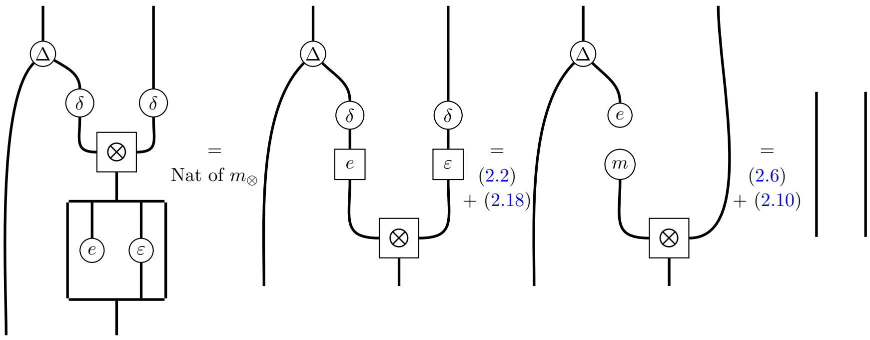


2. $\delta^B \varepsilon^B = 1$: Here we use the counit law of the comultiplication, that ε is a monoidal transformation, and the triangle identities of the comonad:



3. $\delta^B !^B(\varepsilon^B) = 1$: Here we use the naturality of $m_\otimes$, that e is a monoidal transformation, the

comonad triangle identities, the unit law of $m_\otimes$, and the counit law for the comultiplication:



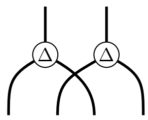
So we conclude that $(!^B, \delta^B, \varepsilon^B)$ is a comonad. □

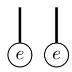
The comonoid structure of $!^B A$ is given by the standard tensor product of comonoids, that is, $\Delta^B : !^B A \longrightarrow !^B A \otimes !^B A$ and $e^B : !^B A \longrightarrow k$ are defined as follows:

$$\Delta^B := \ !B \otimes !A \xrightarrow{\Delta \otimes \Delta} !B \otimes !B \otimes !A \otimes !A \xrightarrow{1 \otimes \sigma \otimes 1} !B \otimes !A \otimes !B \otimes !A$$

$$e^B := \ !B \otimes !A \xrightarrow{e \otimes e} K$$

and which are drawn in the graphical calculus as follows:

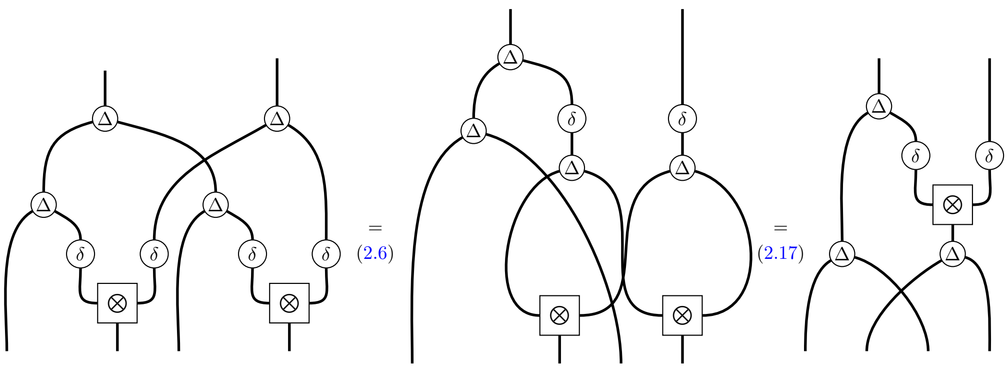
Δ^B = 

e^B = 

(2.37)

Lemma 2.4.2 *[7, Proposition 10]* $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B)$ *is a coalgebra modality.*

PROOF: By construction, $(!^B A, \Delta^B, e^B)$ is a cocommutative comonoid since it is the tensor product of cocommutative comonoids. Therefore, it remains to show that δ^B preserves the comultiplication, which follows from the fact that δ, Δ, and $m_\otimes$ are all comonoid morphisms:



So we conclude that $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B)$. □

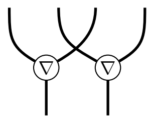
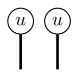
In general, this coalgebra modality is not monoidal, unless $!B \cong k$. Indeed, suppose that $\mathbb{X}$ was a monoidal storage category, and so had finite products $\times$ and a terminal object $\top$. Then by the Seely isomorphisms, $!^B A = !B \otimes !A \cong !(B \times A)$. Then for the terminal object, we would have that $!^B \top \cong !B$. Thus if $!B$ is not isomorphic to the monoidal unit k, it follows that $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B)$ does not have Seely isomorphisms, and is therefore not a monoidal coalgebra modality.

Now suppose that $\mathbb{X}$ was an additive linear category, with its induced additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$. The bialgebra structure of $!^B A$ is given by the standard tensor product of bialgebras, that is, the multiplication ∇^B and the unit u^B are defined respectively as follows:

$$\nabla^B := \; !B \otimes !A \otimes !B \otimes !A \xrightarrow{1 \otimes \sigma \otimes 1} !B \otimes !B \otimes !A \otimes !A \xrightarrow{\nabla \otimes \nabla} !B \otimes !A$$

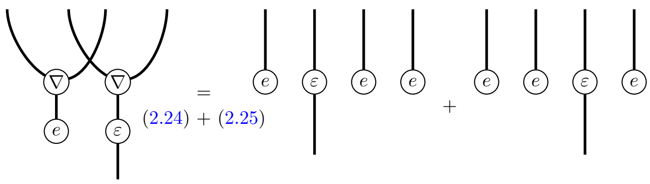
$$u^B := \; K \xrightarrow{u \otimes u} !B \otimes !A$$

which drawn in the graphical calculus are:

∇^B =  u^B =  (2.38)

Proposition 2.4.3 *[7, Proposition 10]* $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B, \nabla^B, u^B)$ *is a bialgebra modality.*

Proof: As before, that $(!^B A, \nabla^B, u^B, \Delta^B, e^B)$ is a bicommutative bialgebra follows from the fact that it is the tensor product of bicommutative bialgebras. So it remains only to show the compatibility relation between ε^B and the multiplication. Here we use the compatibility between the counit and the multiplication, and the compatibility between ε and the multiplication:



So we conclude that $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B, \nabla^B, u^B)$ is a bialgebra modality. □

As mentioned above, in general, this bialgebra modality is not additive, unless $!(B) \cong K$. In particular, for the zero map we have that:

$$!^B(0) = 1 \otimes !(0) = (1 \otimes e)(1 \otimes u) \neq (e \otimes e)(u \otimes u) = e^B u^B$$

In the next chapter, we will show that if the starting additive bialgebra modality came equipped with a codereliction, then these newly cosntructed bialgebra modalities also come equipped with a codereliction.

Chapter 3

Differential Categories Revisited

This chapter is based on [7, Sections 4,5,6,9], which in turn is based on the conference paper [26]. As such, the author would like to thank their coauthors Rick Blute, Robin Cockett, and Robert A. G. Seely, as well as the anonymous referee from "Applied Categorical Structures" for very helpful and constructive comments in their review. In particular, this chapter also introduces differential categories, codereliction, and the graphical calculus for differential categories.

Differential linear logic [36], as introduced by Ehrhard and Regnier [37, 38], is an extension of linear logic which includes a differentiation inference rule, as well as a cocontraction rule, coweaking rule, and codereliction rule for the exponential modality.

$$\frac{\Gamma,\ !A \vdash B}{\Gamma,\ !A,\ A \vdash B} \qquad \frac{\Gamma,\ !A \vdash B}{\Gamma,\ !A,\ !A, \vdash B} \qquad \frac{\Gamma,\ !A \vdash B}{\Gamma \vdash B} \qquad \frac{\Gamma,\ !A \vdash B}{\Gamma,\ A \vdash B}$$

As such, differential linear logic provides a syntactic proof-theoretic approach to differential calculus. Blute, Cockett, and Seely then introduced differential categories [8], which were the appropriate categorical structure for modelling differential linear logic. A differential category is an additive symmetric monoidal category with a coalgebra modality which comes equipped with a natural transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, called a **deriving transformation**, satisfying certain equations based on the properties of differentiation from calculus, such as the Leibniz rule (also known as the product rule) and the chain rule.

Those familiar with linear logic will quickly point to the fact that in order to properly capture differential linear logic, the coalgebra modality is required to be a monoidal coalgebra modality. Indeed, there are many familiar and important examples of differential categories with a monoidal coalgebra modality which include the opposite of the category vector spaces with the symmetric algebra monad [8], which captures polynomial differentiation, the category of finiteness spaces [35] and the category of vector spaces [17] with their respective free exponential modality, which captures power series differentiation, and also the category of convenient vector spaces with the smooth functional comonad [11], which captures differential calculus over convenient vector spaces. However, the full might of a monoidal coalgebra modality is in fact not required to axiomatize the basic properties of differentiation. Instead, as already mentioned, one only requires a coalgebra modality to express the product rule, chain rule, etc. Dropping the requirement of the need for a monoidal coalgebra modality opens the door to many interesting examples of differential categories with non-monoidal coalgebra modalities. A compelling example is the opposite category of real vector spaces equipped with the free $\mathcal{C}^\infty$-ring monad [33], which captures differentiation real smooth functions, a fundamental important example that any theory claiming to provide the foundations differentiation must capture. Furthermore, even with only a mere coalgebra modality, differential categories have been able to formalize numerous notions of differential calculus from classical commutative algebra such as, to list a few, Kähler differentials [12,13], differential algebras [60], antiderivatives [28,63], and De Rham cohomology [78].

For a bialgebra modality, another approach to differential structure is in terms of a natural transformation $\eta : A \to !A$ called a **codereliction** [8], which should be interpreted as a sort of linearization operator. Every codereliction induces a deriving transformation, and conversely, every deriving transformation which satisfies the ∇-rule, which is an extra compatibility with the multiplication of the bialgebra, induces a codereliction. In fact, these constructions are inverses of each other and thus, for a bialgebra modality, there is a bijective correspondence between coderelictions and deriving transformations that satisfy the ∇-rule [8, Theorem 4.12]. The issue, of course, is that not every deriving transformation satisfies the ∇-rule. Therefore, in general, coderelictions and deriving transformation need not be equivalent, and thus a deriving transformation is slightly weaker than a codereliction. In fact, from a differential linear logic point of view, coderelictions are the preferred choice for characterizing differential on a monoidal coalgebra modality. On the other hand, the notion of a deriving transformation is intuitively much closer to that of derivations from classical commutative algebra. This distinction between deriving transformation and coderelictions created a sort of rift between the Canadian approach to differential categories, which preferred deriving transformations, and the European approach to differential categories, which preferred coderelictions. Luckily, this story does have a happy ending of union and peace!

The main result of this chapter is that for a monoidal coalgebra modality, every deriving transformation satisfies the ∇-rule, and that therefore, in this case, deriving transformations and codereliction are equivalent! The key to this proof is using the main result from the previous chapter that in the presence of additive structure, a monoidal coalgebra modality is equivalent to an additive bialgebra modality. Using the extra coherences of an additive bialgebra modality, one can show that for a deriving transformation, the ∇-rule is equivalent to the product rule. This bridges the gap between deriving transformations and coderelictions, and reunites the Canadian and European schools of thought. Thus, there is only one notion of differentiation in linear logic.

As in the previous chapter, to help with the proofs of this chapter we will make use of the graphical calculus for differential categories using the same conventions as in [7,8]. We do our best in this chapter not to resize any of the string diagrams so that the reader can clearly read the proofs without having to constantly zoom in and out. As a result of this, the formatting of certain pages may seem quite bizarre, as the template does its best to accommodate these pictures of various sizes. We hope that this won't be too much of an inconvenience for the reader. If so, we once again refer the reader to the journal or conference paper version of this story [7, 26].

Chapter Outline: Section 3.1 reviews the definition of a differential category, provides a long list of examples and revisits the axioms of a deriving transformation. Section 3.2 reviews the notion of a codereliction, provides multiple examples and explains the bijective correspondence between coderelictions and deriving transformations that satisfy the ∇-rule. Section 3.3 is the main section of this chapter which studies differential structure for additive bialgebra modalities. In particular, we revisit the axioms of a codereliction and prove that a for an additive bialgebra modality, every deriving transformation that satisfies the ∇-rule. As such, we conclude that for an additive bialgebra modality, there is a bijective correspondence between coderelictions and deriving transformations. Furthermore, we also show that coderelictions for additive bialgebra modalities lift to the biproduct completion and induce new coderelictions for the constructed non-additive bialgebra modalities.

3.1 Differential Categories

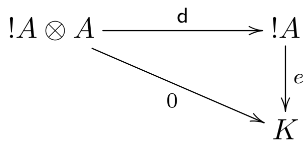
In this section we review differential categories, specifically the notion of a deriving transformation [8]. In particular we revisit the axioms of a deriving transformation for coalgebra modalities and bialgebra modalities. We also provide a long list of examples of differential categories, which highlight to wide variety of examples in the literature. For a full detailed introduction to differential categories, we refer the reader to [7, 8].

Definition 3.1.1 *A* ***differential category*** *[8, Definition 2.4] is an additive symmetric monoidal category (Definition 2.3.1) with a coalgebra modality* $(!, \delta, \varepsilon, \Delta, e)$ *(Definition 2.1.5) which comes equipped with a* ***deriving transformation*** *[7, Definition 7], that is, a natural transformation:*

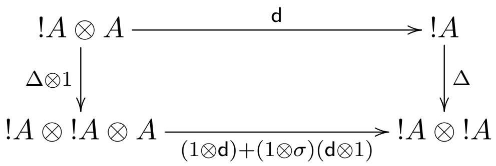
$$\mathsf{d} : !A \otimes A \longrightarrow !A$$

such that the following diagrams commute:

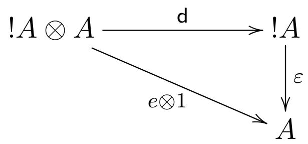
[d.1] *Constant Rule:*



[d.2] *Leibniz Rule (or Product Rule):*



[d.3] *Linear Rule:*



[d.4] *Chain Rule:*

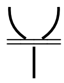
$$\begin{array}{ccccc}
!A \otimes A & & \xrightarrow{\quad \mathsf{d} \quad} & & !A \\
\downarrow{\scriptstyle \Delta\otimes 1} & & & & \downarrow{\scriptstyle \delta} \\
!A \otimes !A \otimes A & \xrightarrow[\delta\otimes\mathsf{d}]{} & !!A \otimes !A & \xrightarrow[\mathsf{d}]{} & !!A
\end{array}$$

[d.5] *Interchange Rule:*

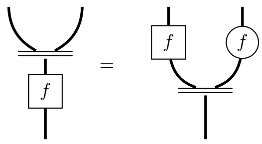
$$\begin{array}{ccccc}
!A \otimes A \otimes A & \xrightarrow{1\otimes\sigma} & !A \otimes A \otimes A & \xrightarrow{\mathsf{d}\otimes 1} & !A \otimes A \\
\downarrow{\scriptstyle \mathsf{d}\otimes 1} & & & & \downarrow{\scriptstyle \mathsf{d}} \\
!A \otimes A & & \xrightarrow[\quad \mathsf{d} \quad]{} & & !A
\end{array}$$

It should be noted that **[d.5]** was not originally a requirement in [8] but was later added to the definition to ensure that the coKleisli category of a differential category was a Cartesian differential category [9], which we discuss in the next chapter.

In the graphical calculus, the deriving transformation d is represented as:

$\mathsf{d} :=$ 

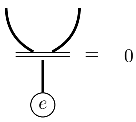
The naturality of d is drawn as follows:



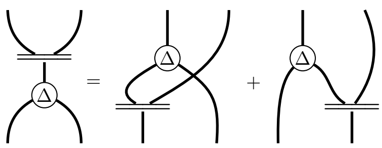
(3.1)

while the deriving transformation axioms **[d.1]**-**[d.5]** are drawn as follows:

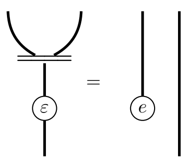
[d.1] Constant Rule:



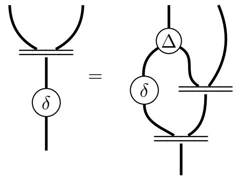
[d.2] Leibniz Rule (or Product Rule):



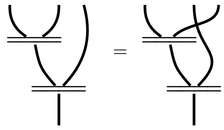
[d.3] Linear Rule:



[d.4] Chain Rule:



[d.5] Interchange Rule:



The deriving transformation axioms are probably best understood by studying Example 3.1.4 below, which arises from polynomial differentiation. Essentially, the deriving transformation induces a sort of external differential operator on the coKleisli category. This story is made precisely by the fact that the coKleisli category of a differential category is a Cartesian differential category [9, Proposition 3.2.1]. Indeed, the coKleisli maps, that is, maps of type $f : !A \longrightarrow B$, are important and are to be thought of as *smooth* maps from A to B as they are, in a certain sense, infinitely differentiable. The differential of a smooth map $f : !A \longrightarrow B$ is the map $\mathsf{D}[f] : !A \otimes A \longrightarrow B$ defined by precomposing with the deriving transformation, $\mathsf{D}[f] = \mathsf{d}f$. The first axiom **[d.1]** states that the derivative of a constant map is zero. The second axiom **[d.2]** is the Leibniz rule or the product rule for differentiation. For the third axiom, a subclass of smooth maps are the *linear* maps, which are coKleisli maps of the form $\varepsilon g : !A \longrightarrow B$ for some map $g : A \longrightarrow B$. Then the linear rule **[d.3]** says that the derivative of a linear map is constant with respect to the point at which it is taken. The fourth axiom **[d.4]** is the chain rule. The last axiom **[d.5]** is the interchange law, which naively states that differentiating with respect to x then y is the same as differentiation with respect to y then x.

On the other hand, what can we say about the coEilenberg-Moore category of a differential category? As discussed in [30], the answer is that it is a tangent category [19]. Indeed, for a differential category with finite biproducts, then the dual of the coEilenberg-Moore category is always a tangent category [30, Theorem 22], whose tangent structure is a generalization of that of the category of commutative rings where the tangent bundle functor is given by associating a commutative ring to its ring of dual number. If the the coEilenberg-Moore category admits coreflexive equalizers then it is a tangent category whose tangent structure is obtain via the adjoint construction from [19, Proposition 5.17]. In the special case that the coalgebra modality has Seely isomorphisms, it follows that the coEilenberg-Moore category is also a *representable* tangent category [30, Theorem 27], that is, where the tangent functor is representable functor, and where the associated exponent object, which is called the infinitesimal object, is $k \oplus k$. In summary, the coalgebras of the coalgebra modality of a differential category should be thought of as generalized smooth manifolds.

We now provide our main examples of differential categories, based on the examples from the previous chapter. A Venn diagram of separating examples of the different kinds of differential categories can be found in [7, Sections 9].

Example 3.1.2 REL is a differential category with the additive symmetric monoidal structure from Example 2.3.2 and coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ from Example 2.1.7, whose deriving transformation $\mathsf{d} : !X \times X \longrightarrow !X$ is the relation which relates a pair of a finite multiset and an element to the sum of said finite multiset and the element's associated characteristic function:

$$\mathsf{d} = \{\left((f, x), f + \eta_x\right) \mid f \in !X, x \in X\} \subseteq (!X \times X) \times !X$$

For more details on this differential category, see [8, Section 2.5.1].

Example 3.1.3 The dual notion of a differential category is that of a **codifferential category** [12, Definition 3.9], which is an additive symmetric monoidal category $\mathbb{X}$ with an algebra modality $(\mathsf{S}, \mu, \eta, \nabla, \mathsf{u})$ that comes equipped with a **deriving transformation**[1], that is, a natural transformation:

$$\mathsf{d} : \mathsf{S}A \longrightarrow \mathsf{S}A \otimes A$$

such that the dual diagrams of **[d.1]**-**[d.5]** commute. Therefore, $\mathbb{X}^{op}$ is a differential category. The intuition here is that $\mathsf{S}A$ can be thought of as a space of smooth scalar-valued maps on A and so d is a derivation on $\mathsf{S}A$ which sends a smooth function on A to its derivative. However, unlike derivations from classical algebra, d not only satisfies a generalization of the Leibniz rule but also satisfies a generalization of the chain rule. Also, as mentioned above, the Eilenberg-Moore category of a codifferential category with finite biproducts is a tangent category [30, Theorem 22].

Example 3.1.4 For any field $\mathbb{K}$, $\mathsf{VEC}_{\mathbb{K}}$ is a codifferential category with additive symmetric monoidal structure from Example 2.3.3 and algebra modality $(\mathsf{Sym}, \mu, \eta, \nabla, u)$ from Example 2.1.9 whose deriving transformation $\mathsf{d} : \mathsf{Sym}(V) \longrightarrow \mathsf{Sym}(V) \otimes V$ is defined on pure symmetric tensors as follows:

$$\mathsf{d}\left(v_1 \otimes_s \ldots \otimes_s v_n\right) = \sum_{i=0}^{n} \left(v_1 \otimes_s v_2 \otimes_s \ldots v_{i-1} \otimes_s v_{i+1} \otimes_s \ldots \otimes v_n\right) \otimes v_i$$

which we then extend by linearity. It turns out that this deriving transformation is precisely polynomial differentiation. To see this, let X be a basis for V, so $\mathsf{Sym}(V) \cong \mathbb{K}[X]$. Then the deriving transformation can be described as a map $\mathsf{d} : \mathbb{K}[X] \longrightarrow \mathbb{K}[X] \otimes V$ which maps a polynomial to its sum of its partial derivatives:

$$\mathsf{d}(p(\vec{x}) = \sum_{i=1}^{n} \frac{\partial p(\vec{x})}{\partial x_i} \otimes x_i$$

Thus $\mathsf{VEC}^{op}_{\mathbb{K}}$ is a differential category, whose differential structure captures polynomial differentiation. We note that this example can be generalized to the category of modules over any commutative semiring. For more details on this (co)differential category, see [8, Section 2.5.3].

Here is now a list of other important and interesting examples of (co)differential categories, but where we do not provide the full construction of the underlying (co)algebra modality.

[1]As in the literature, we keep the same terminology and notation for a deriving transformation in both the context of a differential category and a codifferential category

Example 3.1.5 As mentioned in Example 2.1.4, REL is self-dual. Therefore, REL is also a co-differential category. The algebra modality structure of ! is precisely the dual relations of that of Example 2.1.7, while the deriving transformation is the dual of the one found in Example 3.1.2. Explicitly, the deriving transformation $\mathsf{d}^\circ : !X \longrightarrow !X \times X$ can be described as relating a finite multiset to said finite multiset minus the characteristic functions of elements in its support:

$$\mathsf{d}^\circ = \{\big(f, (f - \eta_x, x)\big) \mid x \in \mathsf{supp}(f)\} \subset \mathsf{M}X \times (\mathsf{M}X \times X)$$

Note that for $x \in \mathsf{supp}(f)$, $f - \eta_x$ is well defined since if $y \neq x$, $f(y) - \eta_x(y) = f(y)$, while for x, $f(x) \neq 0$ which implies that $f(x) = n + 1$ for some n, and therefore $f(x) - \eta_x(x) = n$. To better understand this example, note that every finite multiset $f \in !X$ can be seen as a monomial in variables $\mathsf{supp}(f) = \{x_1, \ldots, x_n\}$, specifically $x_1^{f(x_1)} x_2^{f(x_2)} \ldots x_n^{f(x_n)}$. Therefore, the deriving transformation is the relation which relates a monomial to its derivative:

$$x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \sim (x_1^{k_1} x_2^{k_2} \ldots x_i^{k_i - 1} \ldots x_n^{k_n}, x_i)$$

Keen-eyed readers may note that normally when differentiating a monomial, a factor k_j should appear in front of the derivative. In REL, however, addition is idempotent so $1 + 1 + \ldots + 1 = 1$. Therefore, multiplying by $n \in \mathbb{N}$, $n \neq 0$, is the same as multiplying by 1.

Example 3.1.6 Any categorical model of differential linear logic [36] is a differential category, where the deriving transformation captures the inference rule:

$$\frac{\Gamma,\ !A \vdash B}{\Gamma,\ !A,\ A \vdash B}$$

Important examples include the category of finiteness spaces [35] and the category of convenient vector spaces [11] (the latter of which we discuss in a later chapter).

Example 3.1.7 Let $\mathbb{R}$ be the field of real numbers. While the codifferential structure on $\mathsf{VEC}_\mathbb{R}$ from Example 3.1.4 captures polynomial differentiation, $\mathsf{VEC}_\mathbb{R}$ has another codifferential structure where this time the deriving transformation corresponds to differentiating (real) smooth functions. The key to this example is the notion of C^∞-rings [75, Chapter I]. Recall that C^∞-rings are defined as the algebras of the Lawvere theory whose morphisms are smooth maps between the Euclidean spaces $\mathbb{R}^n$. Equivalently, a C^∞-ring is a set A equipped with a family of functions $\Phi_f : A^n \longrightarrow A$ indexed by the smooth functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and which satisfies certain coherence equations. For example, if M is a smooth manifold, then the set $C^\infty(M)$ defined as:

$$C^\infty(M) = \{f : M \longrightarrow \mathbb{R} \mid f \text{ is smooth}\}$$

is a C^∞-ring where for a smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\Phi_f : C^\infty(M)^n \longrightarrow C^\infty(M)$ is defined by post-composition by f:

$$\Phi_f\left(M \xrightarrow{f_1} \mathbb{R}, \ldots, M \xrightarrow{f_n} \mathbb{R} \right) = \left(M \xrightarrow{\langle f_1, \ldots, f_n \rangle} \mathbb{R}^n \xrightarrow{\quad f \quad} \mathbb{R} \right)$$

Every C^∞-ring A is a commutative $\mathbb{R}$-algebra where the multiplication is given by $\Phi_m : A \times A \longrightarrow A$ where $m : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the multiplication function $m(x, y) = xy$, and the unit is given by $\Phi_u(*) \in A$ where $u : \mathbb{R}^0 = \{*\} \longrightarrow \mathbb{R}$ is the constant function $u(*) = 1$. For every $\mathbb{R}$-vector space V, there exists a free C^∞-ring over V [33, Section 4], which we denote as $\mathsf{S}^\infty(V)$, and which in turn induces an algebra modality on $\mathsf{VEC}_\mathbb{R}$. If V is finite dimensional of dimension n, then $\mathsf{S}^\infty(V) \cong C^\infty(\mathbb{R}^n)$ as C^∞-rings, and in particular, $\mathsf{S}^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. Then $\mathsf{VEC}_\mathbb{R}$ is a codifferential category with respect to the algebra modality S^∞ and whose deriving transformation $\mathsf{d} : \mathsf{S}^\infty(V) \longrightarrow \mathsf{S}^\infty(V) \otimes V$ is induced by differentiating smooth functions. In particular for $\mathbb{R}^n$, the deriving transformation $\mathsf{d} : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ maps a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ to the sum of its partial derivatives:

$$\mathsf{d}(f) \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes x_i$$

Hence $\mathsf{VEC}_\mathbb{R}^{op}$ is a differential category, whose differential structure captures smooth function differentiation. This example was studied in detail by Cruttwell, Lucyshyn-Wright, and the author in [33].

Example 3.1.8 For any field $\mathbb{K}$, $\mathsf{VEC}_\mathbb{K}$ is a differential category with coalgebra modality Q defined as follows on a $\mathbb{K}$-vector space V:

$$\mathsf{Q}(V) = \bigoplus_{v \in V} \mathsf{Sym}(V)$$

and whose deriving transformation $\mathsf{d} : \mathsf{Q}(V) \otimes V \longrightarrow \mathsf{Q}(V)$ is defined as follows on pure symmetric tensors:

$$\mathsf{d}\left([v_1 \otimes_s \otimes \ldots \otimes_s \otimes v_n]_w \otimes v\right) = [v_1 \otimes_s \otimes \ldots \otimes_s \otimes v_n \otimes_s v]_w$$

which we then extend by linearity, and where $[-]_w$ indicates that said pure symmetric tensor lies in the $w \in V$ component of $\mathsf{Q}(V)$. If X is a basis for V, then $\mathsf{Q}(V) \cong \bigoplus\limits_{v \in V} \mathbb{K}[X]$, and so the deriving transformation $\mathsf{d} : \bigoplus\limits_{v \in V} \mathbb{K}[X] \otimes V \longrightarrow \bigoplus\limits_{v \in V} \mathbb{K}[X]$ is map which multiplies a polynomial by basis elements, and without changing the component part:

$$\mathsf{d}([p(\vec{x})]_w \otimes x_i) = [p(\vec{x})x_i]_w$$

When $\mathbb{K}$ is an algebraically closed field of characteristic 0, it turns out that $\mathbb{Q}(V)$ is the cofree cocommutative $\mathbb{K}$-coalgebra over V [76], so $\mathbb{Q}$ would be a free exponential modality in this case, and this differential category was studied in detail by Clift and Murfet in [17]. This differential category example generalizes to the category of modules over any commutative semiring, which was studied in detail by Garner and the author in [41], though $\mathsf{Q}(V)$ may no longer be the cofree cocommutative coalgebra. That said, it turns out that $\mathbb{Q}$ is always the initial monoidal coalgebra modality with a deriving transformation.

Example 3.1.9 Unlike symmetric algebras, exterior algebras [58, Section 8, Chapter XIX] only induce an algebra modality in a particular case. Let $\mathbb{K}$ be a field. For a $\mathbb{K}$-vector space V, let $E_n(V)$ be the subspace of $V^{\otimes^n}$ generated by the alternating tensor symmetries:

$$v_1 \otimes \ldots \otimes v_n - \mathsf{sign}(\tau)(v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n)})$$

for all $v_i \in V$ and all n-permutations τ, and where $\mathsf{sign}(\tau)$ is the sign of the permutation. Define the n-th exterior power of V as $\mathsf{Ext}_n(V) := V^{\otimes^n}/E_n(V)$ and let $v_1 \wedge \ldots \wedge v_n$ be the equivalence class of $v_1 \otimes \ldots \otimes v_n$ in $\mathsf{Ext}_n(V)$, which we refer to as pure wedge products. Note that if V is finite dimensional, then for all $n > \mathsf{dim}(V)$, $\mathsf{Ext}_n(V) = 0$ since in particular $v \wedge v = 0$. Therefore we can define an endofunctor $\mathsf{Ext} : \mathsf{FVEC}_{\mathbb{K}} \longrightarrow \mathsf{FVEC}_{\mathbb{K}}$, where $\mathsf{FVEC}_{\mathbb{K}}$ is the category of all finite dimensional $\mathbb{K}$-vector spaces, which maps a finite dimensional $\mathbb{K}$-vector space V to its exterior algebra $\mathsf{Ext}(V)$ defined as follows:

$$\mathsf{Ext}(V) := \bigoplus_{n=0}^{\mathsf{dim}(V)} \mathsf{Ext}_n(V) = \mathbb{K} \oplus V \oplus \mathsf{Ext}_2(V) \oplus \ldots \oplus \mathsf{Ext}_{\mathsf{dim}(V)}(V)$$

Unfortunately, in general, there are two problems with Ext being an algebra modality. The first is that $(\mathsf{Ext}(V), \mathsf{m}, \mathsf{u})$ is not a commutative $\mathbb{K}$-algebra but an *anticommutative* $\mathbb{K}$-(co)algebra since $v \wedge w = -w \wedge v$. The second is that due to this anticommutativity, it is not possible (in general) to construct a well defined monad multiplication map with the desired properties. Both of these problems are solved when $\mathbb{K} = \mathbb{Z}_2$, the field of integers modulo 2, since in this case $1 = -1$ and therefore $v \wedge w = w \wedge v$. In this case, $\mathsf{FVEC}_{\mathbb{Z}_2}$ is a codifferential category with algebra modality Ext and deriving transformation $\mathsf{d} : \mathsf{Ext}(V) \longrightarrow \mathsf{Ext}(V) \otimes V$ defined as follows on pure wedge products:

$$\mathsf{d}\left(v_1 \wedge \ldots \wedge v_n\right) = \sum_{i=1}^{n} \left(v_1 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_n\right) \otimes v_i$$

which we then extend by linearity. This example was studied by the author in [62].

Example 3.1.10 As mentioned, the previous example does not generalize to $\mathsf{FVEC}_{\mathbb{K}}$ for an arbitrary field $\mathbb{K}$. In fact, as it was shown by the author in [62, Corollary 3.6], if $\mathbb{K}$ is a field of characteristic zero, then $\mathsf{FVEC}_{\mathbb{K}}$ admits no non-trivial (co)differential category structure (i.e. where $!A \neq 0$).

Example 3.1.11 Another surprising non-example is that differential algebras [79] do not induce a differential category structure. For a field $\mathbb{K}$, recall that a $\mathbb{K}$-differential algebra is a commutative $\mathbb{K}$-algebra A equipped with a **derivation**, that is, an $\mathbb{K}$-linear map $\mathsf{D} : A \longrightarrow A$ which satisfies the **Leibniz rule**:

$$\mathsf{D}(ab) = a\mathsf{D}(b) + \mathsf{D}(a)b$$

For any $\mathbb{K}$-vector space V there exists a free $\mathbb{K}$-differential algebra $\mathsf{Diff}(V)$ over V [49], which induces an algebra modality on $\mathsf{VEC}_{\mathbb{K}}$. A construction of $\mathsf{Diff}(V)$ can be found in [7, Example 3]. In particular, if X is a basis for V, then $\mathsf{Diff}(V) \cong \mathbb{K}[X \times \mathbb{N}]$ with derivation D defined on monomials as follows:

$$\mathsf{D}\left((x_1, n_1)\ldots(x_m, n_m)\right) = \sum_{k=1}^{m} (x_1, n_1)\ldots(x_k, n_k + 1)\ldots(x_n, n_m)$$

As shown by Blute, Cockett, Seely, and the author in [7, Theorem 7], Diff does not have a deriving transformation (unless $\mathbb{K} = 0$). The reason for this amounts to the fact that differential algebras are simply axiomatized by the Leibniz rule, while deriving transformations also require the chain rule. On the other hand, if the theory of differential categories wishes to champion itself as the axiomatization of the fundamentals of differentiation: differential algebras should fit naturally in this story. In order to properly introduce differential algebras to the story of differential categories, the solution is that one must instead study differential algebras inside a codifferential category. This was done so by the author in [60], where these generalized versions of differential algebras are instead axiomatized by the chain rule.

Example 3.1.12 By way of contrast, the integration counterparts of differential algebras provide a surprising example of a differential category. The integration counterpart of differential algebras are known as Rota-Baxter algebras [48]. For a field $\mathbb{K}$, a $\mathbb{K}$-Rota-Baxter algebra algebra is a commutative $\mathbb{K}$-algebra A equipped with a **derivation**, that is, an $\mathbb{K}$-linear map $\mathsf{P} : A \longrightarrow A$ such that P satisfies the Rota-Baxter identity, that is, the following equality holds:

$$\mathsf{P}(a)\mathsf{P}(b) = \mathsf{P}(a\mathsf{P}(b)) + \mathsf{P}(\mathsf{P}(a)b) \quad \forall ab, \in A$$

The Rota-Baxter identity is integration by parts expressed only using integrals. For any $\mathbb{K}$-vector space V there exists a free $\mathbb{K}$-Rota-Baxter algebra $\mathsf{RB}(V)$ over V [48, Chapter 3], and is defined as follows:

$$\mathsf{RB}(V) := \mathsf{Sh}(\mathsf{Sym}(\mathsf{V})) \otimes \mathsf{Sym}(\mathsf{V})$$

where for Sh denotes the shuffle algebra. This induces an algebra modality RB on $\mathsf{VEC}_{\mathbb{K}}$, which comes equipped with deriving transformation defined simply as applying the deriving transformation for Sym from Example 3.1.4 and leaving the shuffle algebra component alone:

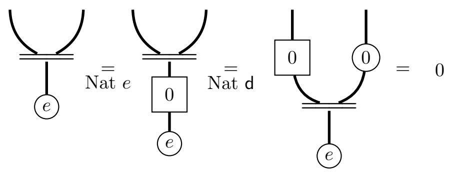
$$1 \otimes \mathsf{d} : \mathsf{Sh}(\mathsf{Sym}(M)) \otimes \mathsf{Sym}(M) \longrightarrow \mathsf{Sh}(\mathsf{Sym}(M)) \otimes \mathsf{Sym}(M) \otimes M$$

This example was studied by Blute, Cockett, Seely, and the author in [7, Example 4].

We now turn our attention back to revisiting the axioms of a deriving transformation. We first show that the constant rule **[d.1]** is in fact derivable:

Lemma 3.1.13 *[7, Lemma 3] For a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ on additive symmetric monoidal category, any natural transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$ satisfies the constant rule* **[d.1]***.*

Proof: By naturality of e and d, and the additive structure, we have the following equalities:



So we conclude that d satisfies **[d.1]**. □

Therefore, we may remove **[d.1]** from the axioms of a deriving transformation.

Corollary 3.1.14 *[7, Corollary 1]For a coalgebra modality* $(!, \delta, \varepsilon, \Delta, e)$ *on additive symmetric monoidal category* $\mathbb{X}$*, the following are equivalent for a natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow !A$*:*

(i) d *is a deriving transformation;*

(ii) d *satisfies the product rule* **[d.2]***, the linear rule* **[d.3]***, the chain rule* **[d.4]***, and the interchange rule* **[d.5]***.*

Next, we conclude this section by discussing the compatibility relation between deriving transformations and (not necessarily additive) bialgebra modalities (Definition 2.3.5). This is captured by the ∇-rule [8], which as we discussed above in the chapter's introduction, plays a crucial role in this chapter.

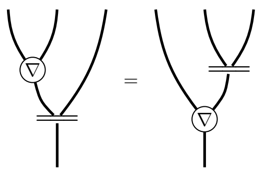
Definition 3.1.15 *[7, Definition 8] For a bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *on an additive symmetric monoidal category* $\mathbb{X}$*, a natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow !A$ *is said to satisfy the* ∇***-rule*** *if the following diagram commutes:*

[d.∇] ∇*-Rule:*

$$\begin{array}{ccc} !A \otimes !A \otimes A & \xrightarrow{\nabla \otimes 1} & !A \otimes A \\ {\scriptstyle 1 \otimes \mathsf{d}} \downarrow & & \downarrow {\scriptstyle \mathsf{d}} \\ !A \otimes !A & \xrightarrow[\nabla]{} & !A \end{array}$$

In the graphical calculus, the ∇-Rule is drawn as follows:

[d.∇] ∇-Rule:

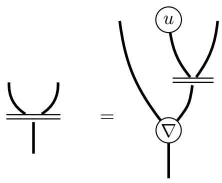


We first observe the following identity:

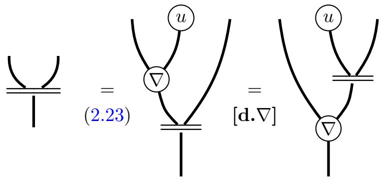
Lemma 3.1.16 *[7, Lemma 4] For a bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *on an additive symmetric monoidal category* $\mathbb{X}$*, for any natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow !A$ *which satisfies the* ∇*-rule,* **[d.∇]***, the following diagram commutes:*

$$\begin{array}{ccccc} !A \otimes A & \xrightarrow{1 \otimes u \otimes 1} & !A \otimes !A \otimes A & \xrightarrow{1 \otimes \mathsf{d}} & !A \otimes !A \\ & \searrow^{\mathsf{d}} & & & \downarrow {\scriptstyle \nabla} \\ & & & & !A \end{array}$$

which is drawn as follows in the graphical calculus:



Proof: Using [**d**.∇] and the monoid unit identity, we obtain the following:

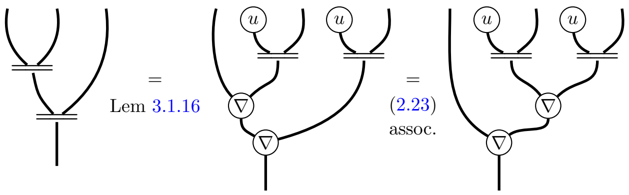


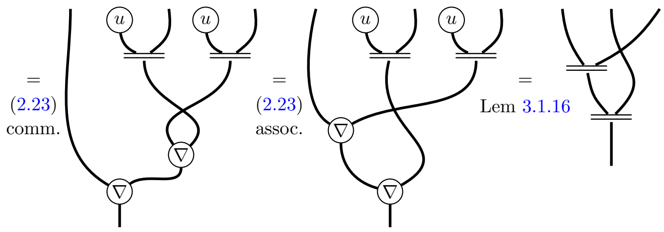
So the desired equality holds. □

Using the above identity, it follows that the ∇-rule implies the interchange rule:

Lemma 3.1.17 *[7, Lemma 5] For a bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *on an additive symmetric monoidal category* $\mathbb{X}$*, any natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow !A$ *which satisfies the* ∇*-rule,* [**d**.∇]*, also satisfies the interchange rule,* [**d.5**].

Proof: Using Lemma 3.1.16, and both associativity and commutativity of the multiplication, we have the following equality:





So we conclude that d satisfies [**d.5**]. □

Therefore, for bialgebra modalities, the ∇-rule can replace the interchange rule in the axiomatization of a deriving transformation.

Corollary 3.1.18 *[7, Corollary 2] For a bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *on an additive symmetric monoidal category* $\mathbb{X}$*, the following are equivalent for a natural transformation of type* $\mathsf{d} : !A \otimes A \longrightarrow !A$*:*

(i) d *is a deriving transformation which satisfies the* ∇*-rule* [**d**.∇]*;*

(ii) d *satisfies the product rule* [**d.2**]*, the linear rule* [**d.3**]*, the chain rule* [**d.4**]*, and the* ∇*-rule* [**d**.∇]*.*

We will soon see that for an additive bialgebra modality, every deriving transformation satisfies the ∇-rule. As such, most of the examples of deriving transformations above satisfy the ∇-rule.

3.2 Coderelictions

In this section we review the notion of a codereliction [8], which is a natural alternative way to characterize differentiation for bialgebra modalities. We also provide a list of examples of coderelictions, which has so far not often been done in differential category literature. We also revisit the axioms of a codereliciton and in particular show that every codereliction satisfies Fiore's proposed chain rule [39]. We conclude this section by explaining the bijective correspondence between coderelictions and deriving transformations which satisfy the ∇-rule.

Definition 3.2.1 *A **codereliction** [7, Definition 9] for a bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *on an additive symmetric monoidal category* $\mathbb{X}$*, is a natural transformation* $\eta : A \longrightarrow\ !A$*, such that the following diagrams commute:*

[dC.1] *Constant Rule:*

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & !A \\ & \searrow^{0} & \downarrow e \\ & & K \end{array}$$

[dC.2] *Product Rule:*

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & !A \\ & \searrow^{\eta \otimes u + u \otimes \eta} & \downarrow \Delta \\ & & !A \otimes !A \end{array}$$

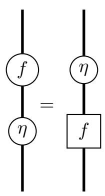
[dC.3] *Linear Rule:*

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & !A \\ & \searrow^{=} & \downarrow \varepsilon \\ & & A \end{array}$$

[dC.4] *Chain Rule:*

$$\begin{array}{ccccc} !A \otimes A & \xrightarrow{1 \otimes \eta} & !A \otimes !A & \xrightarrow{\nabla} & !A \\ \downarrow \Delta \otimes \eta & & & & \\ !A \otimes !A \otimes !A & & & & \downarrow \delta \\ \downarrow 1 \otimes \nabla & & & & \\ !A \otimes !A & \xrightarrow[\delta \otimes \eta]{} & !!A \otimes !!A & \xrightarrow[\nabla]{} & !!A \end{array}$$

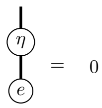
In the graphical calculus, the naturality of η is drawn as



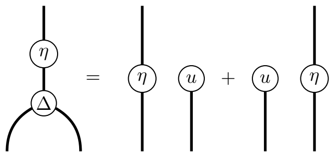
(3.2)

while the codereliction axioms **[dC.1]**-**[dC.4]** are drawn as follows:

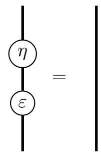
[dC.1] Constant Rule:



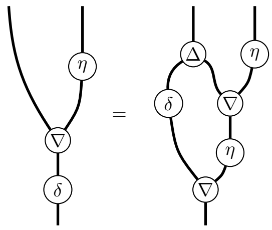
[dC.2] Product Rule:



[dC.3] Linear Rule:



[dC.4] Chain Rule:



The intuition for coderelictions is best understood as evaluating derivatives at zero in the point argument, which results in a linearization of smooth function. Recall that coKleisli maps $f : !A \longrightarrow B$ are thought of a smooth maps, while maps of the base category $g : A \longrightarrow B$ are thought of as linear maps. Therefore, the codereliction should be thought of a sort of linearizing operator. For every smooth map $f : !A \longrightarrow B$, we obtain a linear map $\eta f : A \longrightarrow B$, which we precompose by ε to reobtain a smooth map $\varepsilon\eta f : !A \longrightarrow B$. This idea is made precise in a following chapter when we discuss linearizing combinators [29]. As coderelictions are closely related to deriving transformations, the axioms of a codereliction are analogues of those of a deriving transformation. The first axiom **[dC.1]** says that linearization of a constant map is zero. The second axiom **[dC.2]** says that linearization of a product of smooth maps is the sum of the product of the linearization of one and the other evaluated at zero. The third axiom **[dC.3]** says that linearization of a smooth linear map is simply itself. And lastly the fourth axiom **[dC.4]** turns out to be precisely the chain rule for the deriving transformation, as we will soon see below.

Here is now our two main examples of coderelictions from our main examples of the previous chapter:

Example 3.2.2 The additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e)$ from Example 2.3.8 on REL comes equipped with a codereliction $\eta : X \longrightarrow !X$ which is precisely the dual relation of the comonad counit $\varepsilon : !X \longrightarrow X$. Explicitly, $\eta : X \longrightarrow !X$ is the relation which relates an element to its characteristic function:

$$\eta = \{(x, \eta_x) | \ \forall x \in X\} \subseteq X \times !X$$

Example 3.2.3 The dual notion of a codereliction for the dual notion of a bialgebra modality $(\mathsf{S}, \mu, \eta, \nabla, u, \Delta, e)$, for an additive symmetric monoidal category $\mathbb{X}$, would be a natural transformation:

$$\varepsilon : \mathsf{S}A \longrightarrow A$$

such that the dual diagrams of **[dC.1]**-**[dC.5]** commute. Therefore, ε is a codereliction for the dual of $(\mathsf{S}, \mu, \eta, \nabla, u, \Delta, e)$ in $\mathbb{X}^{op}$. As before with deriving transformations, the terminology is not ideal, but we use the term codereliction in both cases.

Example 3.2.4 The additive bialgebra modality $(\mathsf{Sym}, \mu, \eta, \nabla, u)$ from Example 2.3.10 on $\mathsf{VEC}_{\mathbb{K}}$ comes equipped with a codereliction $\varepsilon : \mathsf{Sym}(V) \longrightarrow V$ which is defined as projecting out the V component of the symmetric algebra. Explicitly, ε is defined as follows on pure symmetric tensors:

$$\varepsilon(1) = 0 \qquad \varepsilon(v) = v \qquad \varepsilon(v_1 \otimes_s \ldots \otimes_s v_n) = 0$$

which we then extend by linearity. In terms of polynomials, if X is a basis for V, then the codereliction $\varepsilon : \mathbb{K}[X] \longrightarrow V$ is defined as picking out the degree 1 terms of the polynomial, that is, its x_i terms. This can be described as follows:

$$\varepsilon(p(\vec{x})) = \sum_{i=1}^{n} \frac{\partial p(\vec{x})}{\partial x_i}(0) x_i$$

Note that evaluating a polynomial at zero extracts its constant term. The constant term of $\frac{\partial p(\vec{x})}{\partial x_i}$ is precisely the scalar factor of x_i. Therefore, $\frac{\partial p(\vec{x})}{\partial x_i}(0)x_i$ are precisely the degree 1 terms of $p(\vec{x})$.

Here are now some other examples of coderelictions but where, as before, we do not give the full structure of the bialgebra modality.

Example 3.2.5 Once again, since REL is self-dual, ! is also an additive bialgebra modality in the dual sense, where the structure is given by the dual relations of Example 2.3.8. Furthermore, the comonad counit $\varepsilon : !X \longrightarrow X$ is a codereliction in this dual sense.

Example 3.2.6 Any categorical model of differential linear logic has an additive bialgebra modality equipped with a codereliction which captures the inference rule:

$$\frac{\Gamma, \ !A \vdash B}{\Gamma, \ A \vdash B}$$

Example 3.2.7 The C^∞-ring algebra modality S^∞ from Example 3.1.7 on $\mathsf{VEC}_\mathbb{R}$ is not a bialgebra modality, as explained in [33, Section 5.2]. The main reason for this is that $C^\infty(\mathbb{R}^n)$ cannot be made into a bialgebra since if that were the case, then $C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m)$ and $C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m)$ would be isomorphic to one another. Famously, however, these two C^∞-rings are not isomorphic. That said, S^∞ still has a quasi-codereliction, Cruttwell, Lucyshyn-Wright, and the author in [33, Proposition 5.30], which is a natural transformation $\varepsilon : \mathsf{S}^\infty(V) \longrightarrow V$ which satisfies slightly weaker axioms. In particular, for $\mathbb{R}^n$, $\varepsilon : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is the linear map that evaluates the derivative of a smooth function at zero:

$$\varepsilon(f) = \left(\frac{\partial f}{\partial x_1}(\vec{0}), \ldots, \frac{\partial f}{\partial x_n}(\vec{0}) \right)$$

Note that when taking a polynomial, one reobtains previously the codereliction for Sym.

Example 3.2.8 The coalgebra modality Q from Example 3.1.8 on $\mathsf{VEC}_\mathbb{K}$ is additive bialgebra modality and which comes equipped with a codereliction $\eta : V \longrightarrow \mathsf{Q}(V)$ defined as inserting V into the $0 \in V$ component of $\mathsf{Q}(V)$:

$$\eta(v) = [v]_0$$

Example 3.2.9 The algebra modality Ext induced by exterior algebras from Example 3.1.9 on $\mathsf{FVEC}_{\mathbb{Z}_2}$ is additive bialgebra modality and which comes equipped with a codereliction $\varepsilon : \mathsf{Ext}(V) \longrightarrow V$ defined as projecting out the V component of the exterior algebra. Explicitly, ε is defined as follows on pure wedge products:

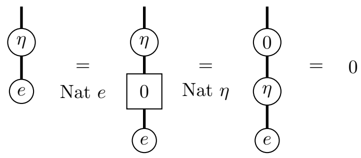
$$\varepsilon(1) = 0 \qquad \varepsilon(v) = v \qquad \varepsilon(v_1 \wedge \ldots \wedge v_n) = 0$$

which we then extend by linearity.

We now turn our attention to reviewing the axioms of a codereliction. As for the constant rule for the deriving transformation, the constant rule **[dC.1]** for a codereliction can be derived:

Lemma 3.2.10 *[7, Lemma 6] For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ on an additive symmetric monoidal category $\mathbb{X}$, any natural transformation $\eta : A \longrightarrow !A$ satisfies the constant rule* **[dC.1]**.

PROOF: By naturality of e and η, and the additive structure, we have the following equalities:



Therefore we conclude that η satisfies **[dC.1]**. □

Therefore, we may remove **[dC.1]** from the axiomatization of a codereliction.

Corollary 3.2.11 *[7, Corollary 3] For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ on an additive symmetric monoidal category $\mathbb{X}$, the following are equivalent for a natural transformation $\eta : A \longrightarrow !A$:*

(i) *η is a codereliction;*

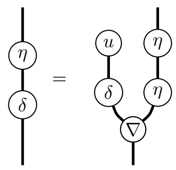
(ii) *η satisfies the product rule* **[dC.2]***, the linear rule* **[dC.3]***, and the chain rule* **[dC.4]***.*

In [39] an alternative axiom for the chain rule **[dC.4]** is used:

[dC.4′] Alternative Chain Rule:

$$\begin{array}{ccccc} A & & \xrightarrow{\quad\eta\quad} & & !A \\ \Big\downarrow{\scriptstyle u\otimes\eta} & & & & \Big\downarrow{\scriptstyle \delta} \\ !A\otimes !A & \xrightarrow[\delta\otimes\eta]{} & !!A\otimes !!A & \xrightarrow[\nabla]{} & !!A \end{array}$$

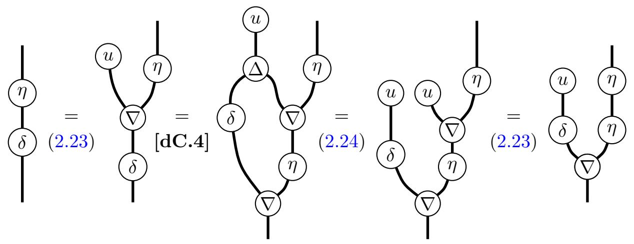
which is drawn in the graphical calculus as:



The alternative chain rule **[dC.4′]** tells us how to linearize the composition of smooth maps, i.e., the chain rule for linearization. As we will see in the next section, for a monoidal coalgebra modality, which was the setting assumed in [39], the two chain rules axioms **[dC.4]** and **[dC.4′]** are equivalent. However in the setting of a mere bialgebra modality, it is only the case that **[dC.4]** implies **[dC.4′]**. The reverse implication, however, does not appear to hold. Thus, at this stage we prove the implication in one direction:

Lemma 3.2.12 *For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ on an additive symmetric monoidal category $\mathbb{X}$, any natural transformation $\eta : A \longrightarrow\ !A$ which satisfies the chain rule* **[dC.4]** *also satisfies the alternative chain rule* **[dC.4′]***.*

Proof: The bialgebra structure gives the following chain of equalities:

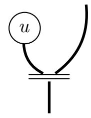


Therefore we conclude that η satisfies **[dC.4′]**. □

We conclude this section by discussing the equivalence between coderelictions and deriving transformations which satisfy the ∇-rule. Every deriving transformation d which satisfies the ∇-rule **[d.∇]** induces a codereliction defined as follows:

$$\eta := \ A \xrightarrow{\ u\otimes 1\ } !A\otimes A \xrightarrow{\ \mathsf{d}\ } !A$$

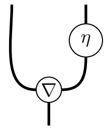
which drawn in the graphical calculus is:

$\eta \;=$ 

Conversely, every codereliction η induces a deriving transformation which satisfies the ∇-rule:

$$\mathsf{d} := \; !A \otimes A \xrightarrow{1 \otimes \eta} !A \otimes !A \xrightarrow{\nabla} !A$$

which is drawn as:

$\mathsf{d} \;=$ 

Furthermore, these constructions are inverses of each other and we obtain the following theorem:

Theorem 3.2.13 *[7, Theorem 2] For an additive symmetric monoidal category $\mathbb{X}$ with a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, deriving transformations which satisfy the ∇-rule* **[d.∇]** *are in bijective correspondence to coderelictions via the constructions given above.*

PROOF: Using the same computations as in [8, Theorem 4.12], which was for monoidal storage categories but does indeed still hold bialgebra modalities, the result follows. □

3.3 Differentiation for Additive Bialgebra Modalities

In this section we prove that for additive bialgebra modalities, there is only one notion of differentiation. In other words, will show that for an additive bialgebra modality, coderelictions and deriving transformations are equivalent. To prove this, we will first review the axioms of codereliction and show that, for an additive bialgebra modality, every codereliction satisfies the axioms proposed in [39]. Next, we will show that for deriving transformations of an additive bialgebra modality, the ∇-rule is equivalent to the product rule. Therefore, we can conclude that deriving transformations are indeed equivalent to codereliction in this case. We will conclude this section by discussing how coderelictions also lift to the biproduct completion and how they induce new coderelictions on the constructed non-additive bialgebra modalities.

As mentioned, we begin by examining coderelictions for additive bialgebra modalities. To do so, we will require the following useful "projection" and "injection" like maps.

Lemma 3.3.1 *For an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ on an additive symmetric monoidal category $\mathbb{X}$ and a natural transformation $\eta : A \longrightarrow !A$ which satisfies the linear rule* **[dC.3]**, *define the following four natural transformations:*

$$\mathsf{p}_0 = \; !A \otimes !B \xrightarrow{\varepsilon \otimes e} A \qquad \mathsf{p}_1 = \; !A \otimes !B \xrightarrow{e \otimes \varepsilon} B$$

$$\mathsf{i}_0 = A \xrightarrow{\eta \otimes u} !A \otimes !B \qquad \mathsf{i}_1 = B \xrightarrow{u \otimes \eta} !A \otimes !B$$

Then the following equalities holds:

$$\mathsf{i}_j\mathsf{p}_k = \begin{cases} 0 & \textit{if } j \neq k \\ 1 & \textit{if } j = k \end{cases} \tag{3.3}$$

$$!(\mathsf{i}_j)!(\mathsf{p}_k) = \begin{cases} eu & \textit{if } j \neq k \\ 1 & \textit{if } j = k \end{cases} \tag{3.4}$$

PROOF: For (3.3), η satisfies the constant rule **[dC.1]** and the linear rule **[dC.3]**, then from the properties of a bialgebra modality, we have:

$$\mathsf{i}_0\mathsf{p}_0 = (\varepsilon \otimes e)(\eta \otimes u) = 1_A \qquad \mathsf{i}_0\mathsf{p}_1 = (\varepsilon \otimes e)(u \otimes \eta) = 0$$
$$\mathsf{i}_1\mathsf{p}_0 = (e \otimes \varepsilon)(\eta \otimes u) = 0 \qquad \mathsf{i}_1\mathsf{p}_1 = (e \otimes \varepsilon)(u \otimes \eta) = 1_B$$

For (3.4), since $!(0) = eu$, from (3.3) we easily compute that:

$$!(\mathsf{i}_0)!(\mathsf{p}_0) = !(\mathsf{i}_0\mathsf{p}_0) = !(1_A) = 1_{!A} \qquad !(\mathsf{i}_0)!(\mathsf{p}_1) = !(\mathsf{i}_0\mathsf{p}_1) = !(0) = eu$$
$$!(\mathsf{i}_1)!(\mathsf{p}_0) = !(\mathsf{i}_1\mathsf{p}_0) = !(0) = eu \qquad !(\mathsf{i}_1)!(\mathsf{p}_1) = !(\mathsf{i}_1\mathsf{p}_1) = !(1_B) = 1_{!B}$$

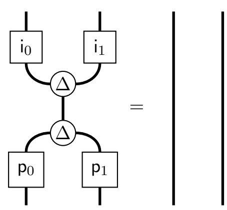
So we conclude that the desired equalities hold. □

Notice that (3.3) and (3.4) are reminiscent of the identities satisfied by the projection and injection maps of a biproduct. These maps will be key to the proof of Proposition 3.3.3 below. Next we compute the following useful identity:

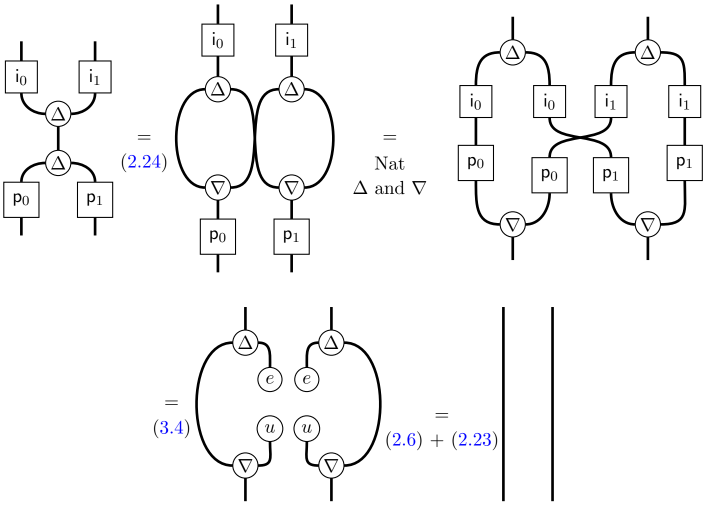
Lemma 3.3.2 *[7, Lemma 8] For an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ on an additive symmetric monoidal category $\mathbb{X}$ and a natural transformation $\eta : A \longrightarrow\ !A$ which satisfies the linear rule* **[dC.3]***, the following diagram commutes:*

$$\begin{array}{ccccccc} !A \otimes !B & \xrightarrow{!(\mathsf{i}_0)\otimes !(\mathsf{i}_1)} & !(!A \otimes !B) \otimes !(!A \otimes !B) & \xrightarrow{\nabla} & !(!A \otimes !B) & \xrightarrow{\Delta} & !(!A \otimes !B) \otimes !(!A \otimes !B) \\ & & & & & & \downarrow {\scriptstyle !(\mathsf{p}_0)\otimes !(\mathsf{p}_1)} \\ & & & = & & & !A \otimes !B \end{array}$$

which is drawn in the graphical calculus as follows:



PROOF: By the bialgebra identities, naturality of Δ and ∇, and the p_i and i_j identities, we have the following equality:



So we conclude the desired equality holds. □

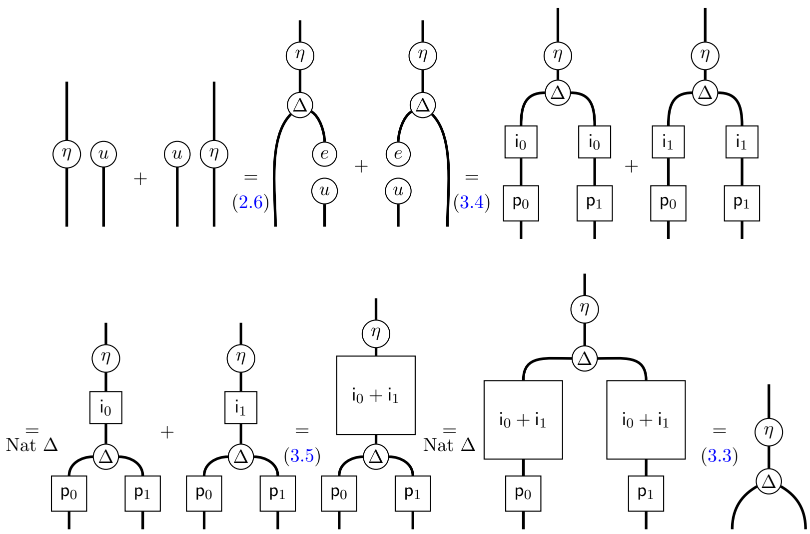
For an additive bialgebra modality, the linear rule **[dC.3]** implies the product rule **[dC.2]**:

Proposition 3.3.3 *[7, Proposition 4] For an additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *on an additive symmetric monoidal category* $\mathbb{X}$*, any natural transformation* $\eta : A \longrightarrow\ !A$ *which satisfies the linear rule* **[dC.3]***, also satisfies the product rule* **[dC.2]***.*

PROOF: Notice that by naturality of η, we have that:

$$\eta!(f+g) = (f+g)\eta = f\eta + g\eta = \eta!(f) + \eta!(g) \tag{3.5}$$

Then using the i_j and p_k identities, we obtain the following:



So we conclude that η satisfies [**dC.2**]. □

Therefore, for an additive bialgebra modality, a codereliction can equivalently be axiomatized by only two axioms:

Corollary 3.3.4 *[7, Corollary 4] For an additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$*, the following are equivalent for a natural transformation* $\eta : A \longrightarrow !A$*:*

(i) *η is a codereliction;*

(ii) *η satisfies the linear rule* [**dC.3**] *and the chain rule* [**dC.4**]*.*

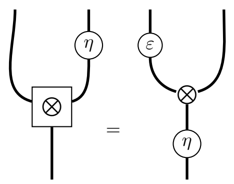
In [39], for a monoidal coalgebra modality, Fiore introduced another axiom relating η to the monoidal structure:

[**dC.m**] Monoidal Rule:

$$\begin{array}{ccc} !A \otimes B & \xrightarrow{1 \otimes \eta} & !A \otimes !B \\ {\scriptstyle \varepsilon \otimes 1} \big\downarrow & & \big\downarrow {\scriptstyle m_\otimes} \\ A \otimes B & \xrightarrow[\eta]{} & !(A \otimes B) \end{array}$$

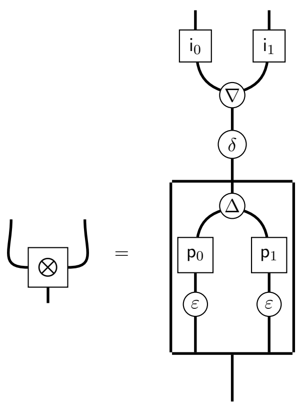
which drawn in graphical calculus is:

[**dC.m**] Monoidal Rule:



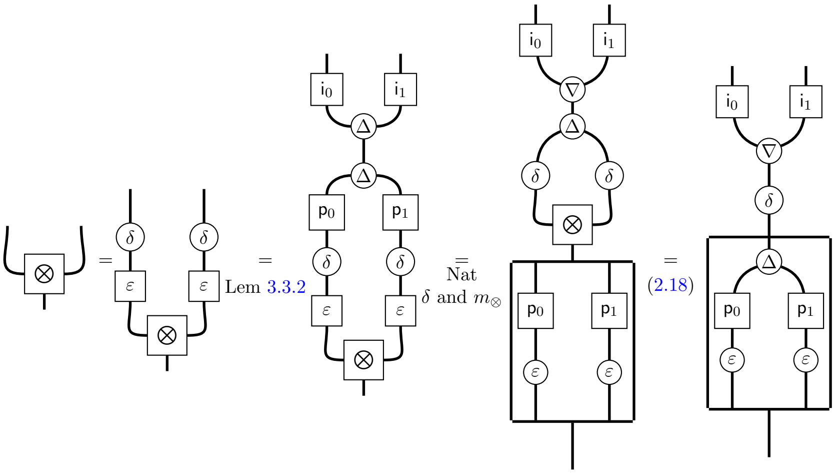
However, it turns out that coderelictions for the additive bialgebra modalities always satisfy the monoidal rule [**dC.m**]. We will first require the following identity:

Lemma 3.3.5 *The following equality holds:*



(3.6)

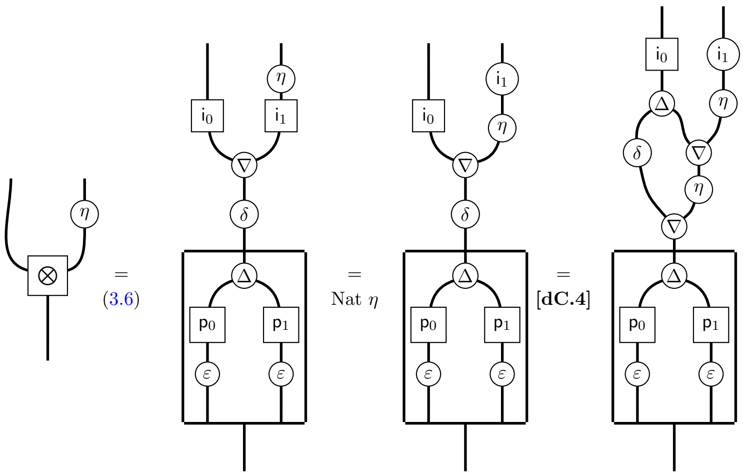
PROOF: By Lemma 3.3.2 and the fact that Δ is a !-coalgebra morphism, we first compute:

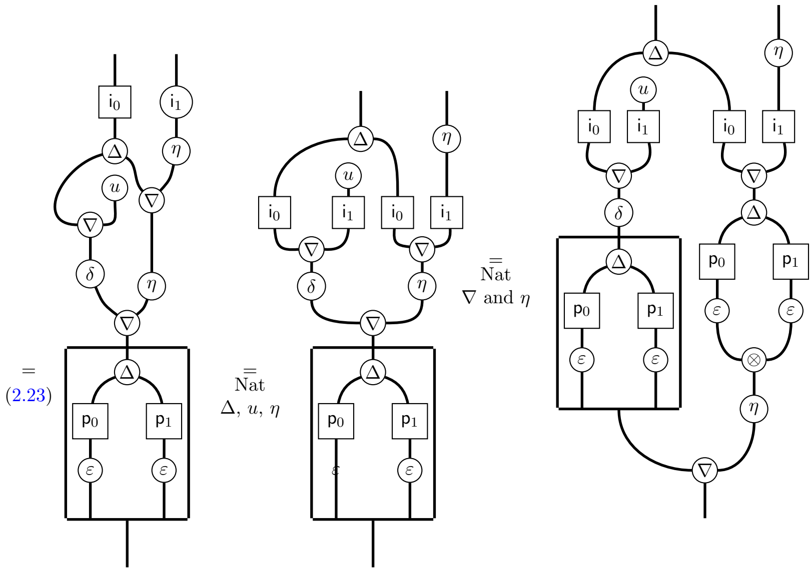


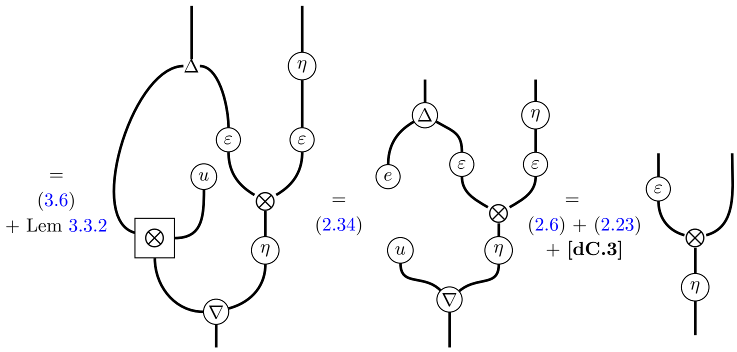
So the desired equality holds. □

Proposition 3.3.6 *[7, Proposition 5] For the induced additive bialgebra modality of an additive linear category: all coderelictions satisfy the monoidal rule* **[dC.m]**.

PROOF: Expressing $m_\otimes$ as in the above lemma, then by the linear rule **[dC.3]**, chain rule **[dC.4]**, the naturality of u and ∇, and Proposition 2.3.17, we have the following equality:





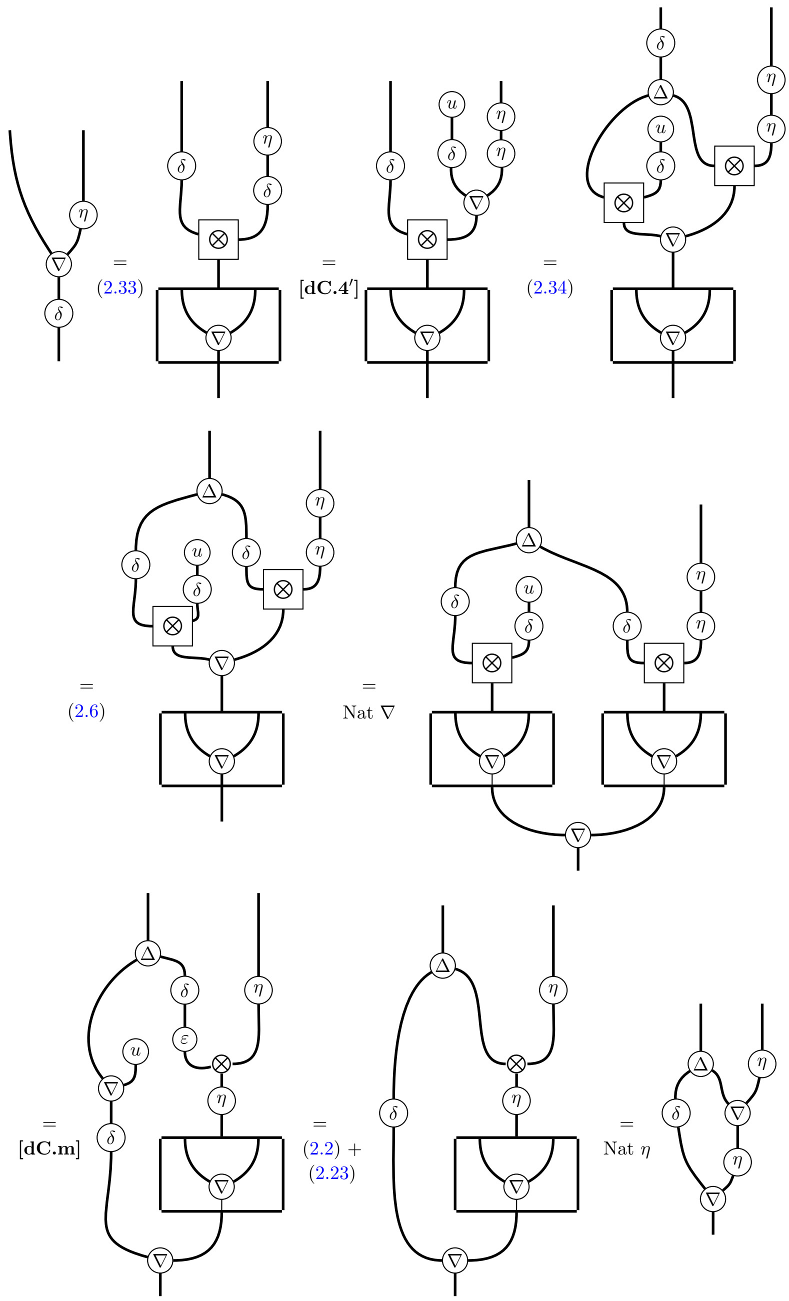


So we conclude that η satisfies **[dC.m]**. □

Conversly, the alternative chain rule **[dC.4′]** and the monoidal rule **[dC.m]** imply the chain rule **[dC.4]**.

Lemma 3.3.7 *[7, Lemma 9] For the induced additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ of an additive linear category $\mathbb{X}$, any natural transformation $\eta : A \longrightarrow !A$ which satisfies the alternative chain rule* **[dC.4′]** *and the monoidal rule* **[dC.m]** *also satisfies the chain rule* **[dC.4]**.

PROOF: Using Proposition 2.3.17, the alternative chain rule **[dC.4′]**, and the bialgebra modality identities we have:



So we conclude that η satisfies **[dC.4]**. □

In summary, we obtain the following regarding the axiomatization of a codereliction:

Corollary 3.3.8 *[7, Corollary 5] For the induced additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *of an additive linear category* $\mathbb{X}$*, the following are equivalent for a natural transformation* $\eta : A \longrightarrow !A$*:*

(i) *η is a codereliction;*

(ii) η *satisfies the linear rule* **[dC.3]** *and the chain rule* **[dC.4]***;*

(iii) η *satisfies the linear rule* **[dC.3]***, the alternative chain rule* **[dC.4′]** *and the monoidal rule* **[dC.m]***.*

Part *(iii)* of the above corollary is the definition of Fiore's creation map [39]. This shows that, for additive bialgebra modalities or equivalently monoidal coalgebra modalities, the original definition of a codereliction is equivalent to Fiore's creation map.

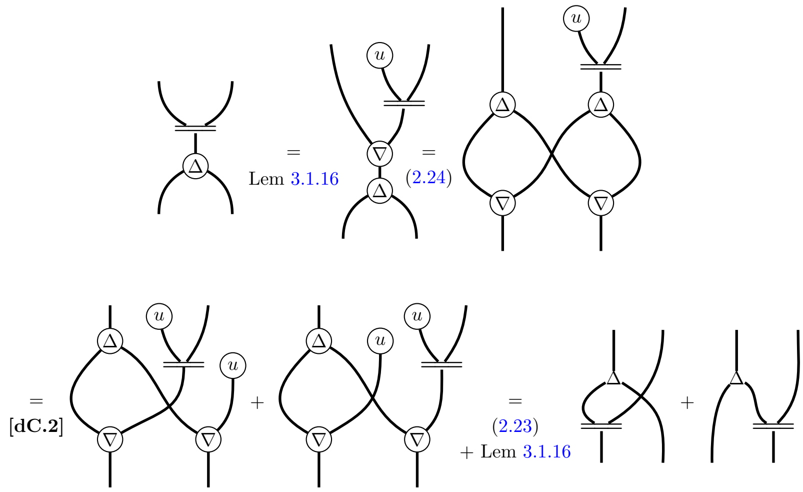
Turning our attention to deriving transformations for additive bialgebra modalities, we begin by noticing that satisfying the product rule is equivalent to satisfying the ∇-rule:

Proposition 3.3.9 *[7, Proposition 6] For an additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *on additive symmetric monoidal category* $\mathbb{X}$*, the following are equivalent for a natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow !A$ *which satisfies the linear rule* **[d.3]***:*

(i) d *satisfies the product rule* **[d.2]***;*

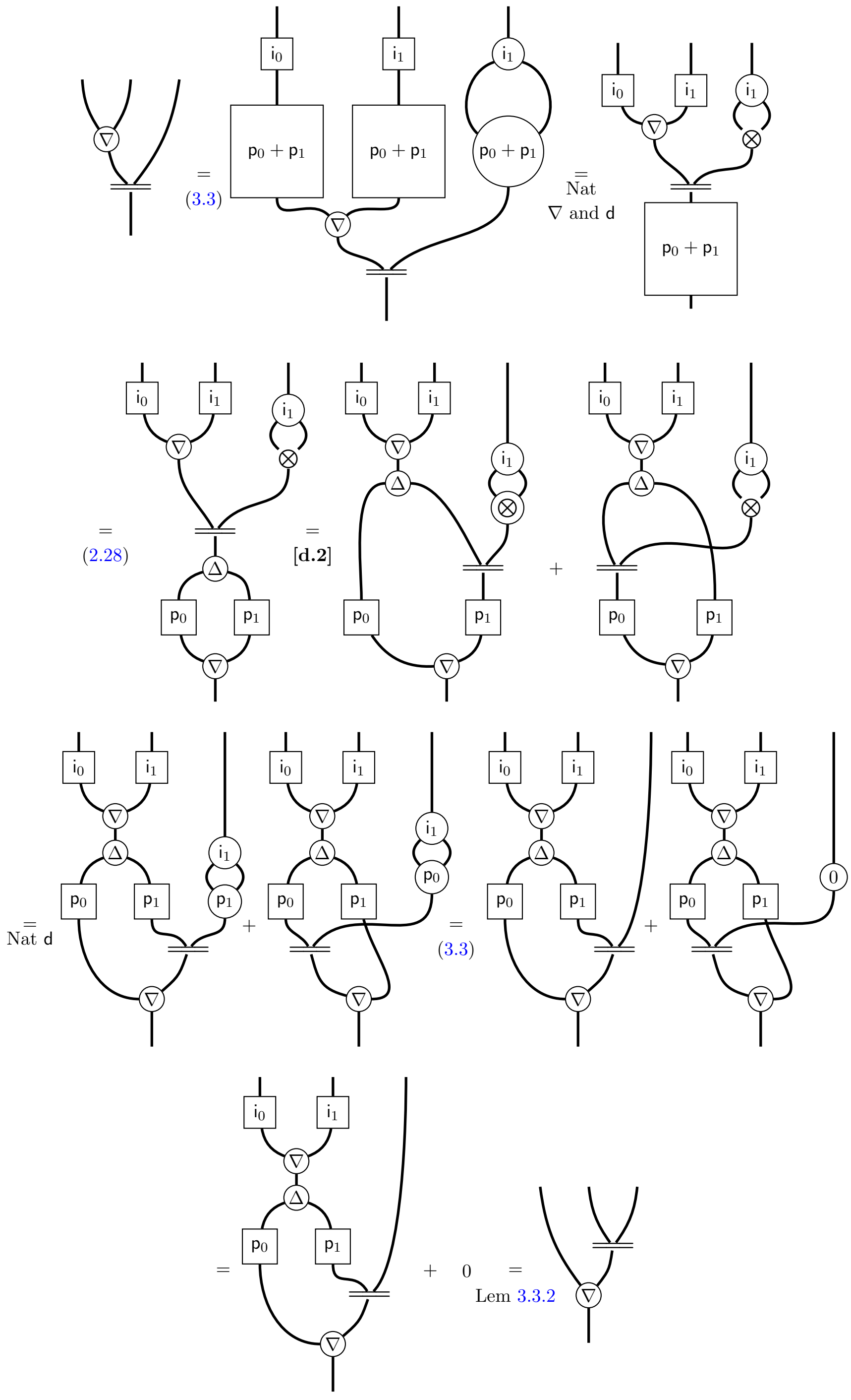
(ii) d *satisfies the* ∇*-rule* **[d.∇]***.*

Proof:
[d.∇] $\Rightarrow$ **[d.2]**: It is easy to see that since d satisfies **[d.3]** that $(u \otimes 1)\mathsf{d} : A \longrightarrow !A$ satisfies **[dC.3]**, the linear rule for coderelictions. However, by Lemma 3.3.3, this implies that $(u \otimes 1)\mathsf{d}$ satisfies **[dC.2]**, the product rule for coderelictions. Since d satisfies **[d.∇]**, then Lemma 3.1.16 holds. And so we have:



So we conclude that d satisfies **[d.2]**.

[d.2] $\Rightarrow$ **[d.∇]**: By the properties of i_j and p_k, Lemma 3.3.2, the additive bialgebra modality identities, and the additive structure, we have that:



So we conclude that d satisfies [**d.**∇]. $\square$

Therefore, for an additive bialgebra modality, we obtain the following axiomatization for deriving transformations:

Corollary 3.3.10 *[7, Corollary 6] For an additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *of an additive symmetric monoidal category* $\mathbb{X}$*, the following are equivalent for a natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow !A$*:*

(i) d *is a deriving transformation;*

(ii) d *satisfies the product rule* **[d.2]***, the linear rule* **[d.3]***, and the chain rule* **[d.4]***;*

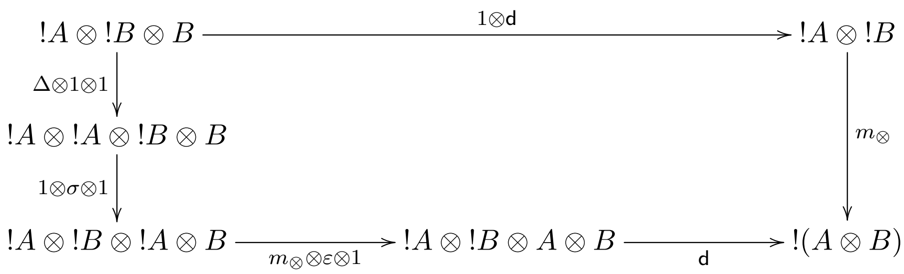
(iii) d *satisfies the linear rule* **[d.3]***, the chain rule* **[d.4]***, and the* ∇*-rule* **[d.∇]***.*

Therefore, as an immediate consequence of Theorem 3.2.13, we obtain the following theorem that for additive bialgebra modalities, deriving transformations and coderelictions are equivalent.

Theorem 3.3.11 *[7, Theorem 3] For an additive bialgebra modality, every deriving transformation satisfies the* ∇*-rule* **[d.∇]** *and thus is induced equivalently by a codereliction. Therefore, for an additive bialgebra modality on an additive symmetric monoidal category, there is a bijective correspondence between deriving transformations and coderelictions via the constructions of Theorem 3.2.13.*

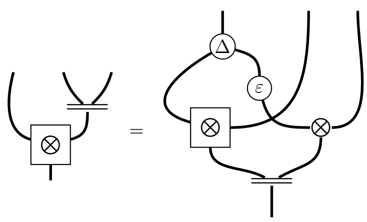
We turn our attention to the relation between the monoidal structure and the differential structure, that is, we explore deriving transformations of additive linear categories. The compatibility between a deriving transformation and the monoidal structure is described by the monoidal rule [39]– this is the strength rule which was the subject of Fiore's addendum:

[d.m] Monoidal Rule:



which in the graphical calculus is drawn as:

[d.m] Monoidal Rule:



Fiore's creation operator [39] was defined to satisfy the linear rule **[d.3]**, the chain rule **[d.4]**, the ∇-rule **[d.∇]**, and the monoidal rule **[d.m]**. Later, Fiore added an addendum and pointed out the latter was redundant. It turns out that when a natural transformation satisfies both the linear rule **[d.3]** and the chain rule **[d.4]**, then the monoidal rule is equivalent to both the ∇-rule and the Leibniz rule:

Proposition 3.3.12 *[7, Proposition 7] For the induced additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *of an additive linear category, the following are equivalent for a natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow A$ *which satisfies the linear rule* **[d.3]** *and the chain rule* **[d.4]***:*

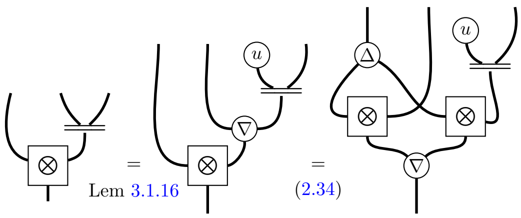
(i) d *satisfies the Leibniz rule* **[d.2]***;*

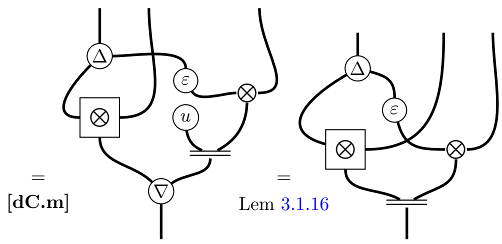
(ii) d *satisfies the* ∇*-rule* **[d.∇]***;*

(iii) d *satisfies the monoidal rule* **[d.m]**

Proof: Since this is an extension of Proposition 3.3.9, it suffices to show that the ∇-rule **[d.∇]** and the monoidal rule **[d.m]** are equivalent.

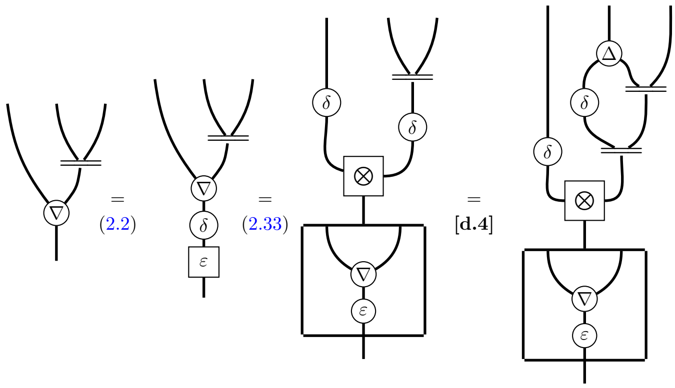
[d.∇] $\Rightarrow$ **[d.m]**: It is easy to see that since d satisfies the linear rule **[d.3]** and the chain rule **[d.4]**, $(u \otimes 1)\mathsf{d} : A \longrightarrow !A$ satisfies the codereliction linear rule **[dC.3]** and chain rule **[dC.4]**, and therefore by Corollary 3.3.8 is a codereliction and which by Proposition 3.3.6 satisfies the codereliction monoidal rule **[dC.m]**. Therefore, by Lemma 3.1.16 (since d satisfies **[d.∇]**) and one of the identities of Proposition 2.3.17, we have:

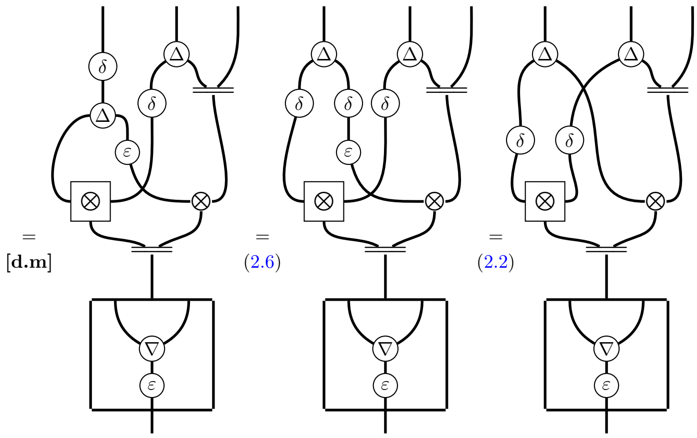


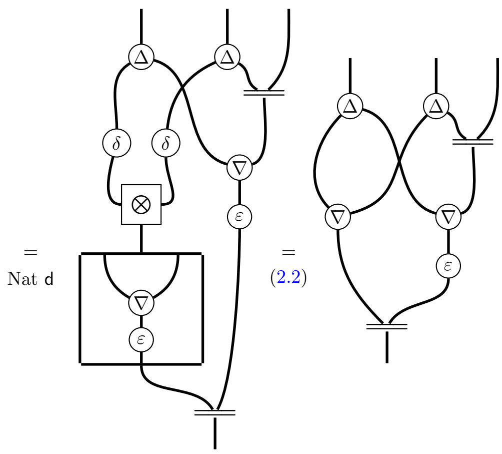


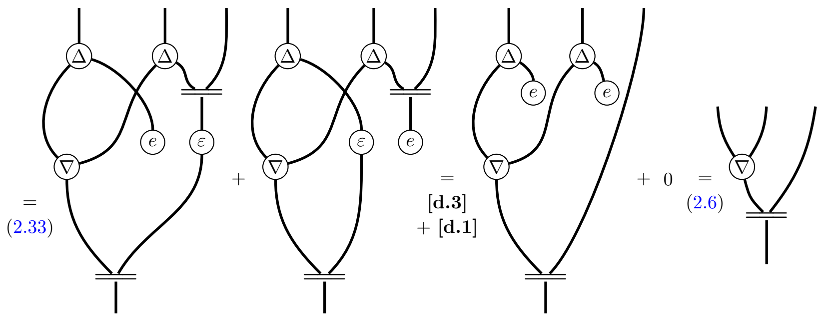
So we conclude that d satisfies **[d.∇]**.

[d.m] $\Rightarrow$ **[d.∇]**: Using the coalgebra modality identities, that ∇ is a !-coalgebra morphism, the monoidal rule **[d.m]**, the chain rule **[d.4]**, the bialgebra modality compatibility between ∇ and ε, the linear rule **[d.3]**, and the constant rule **[d.1]**, we have that:









So we conclude that d satisfies [**d.**∇]. ☐

Summarizing all this together, we obtain:

Corollary 3.3.13 *[7, Corollary 7] For the induced additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *of an additive linear category* $\mathbb{X}$*, the following are equivalent for a natural transformation* $\mathsf{d} : !A \otimes A \longrightarrow A$*:*

(i) d *is a deriving transformation;*

(ii) d *satisfies the product rule* [**d.2**]*, the linear rule* [**d.3**]*, and the chain rule* [**d.4**]*;*

(iii) d *satisfies the linear rule* [**d.3**]*, the chain rule* [**d.4**]*, and the* ∇*-rule* [**d.**∇]*;*

(iv) d *satisfies the linear rule* [**d.3**]*, the chain rule* [**d.4**]*, and the monoidal rule* [**d.m**]*.*

Finally, this gives the following theorem:

Theorem 3.3.14 *[7, Theorem 4] For the monoidal coalgebra modality of an additive linear category, all deriving transformations satisfy the monoidal rule* [**d.m**] *and are induced by a codereliction (for the induced additive bialgebra modality). Therefore, for a monoidal coalgebra modality on an additive symmetric monoidal category, there is a bijective correspondence between deriving transformations and coderelictions via the constructions of Theorem 3.2.13.*

It easily follows that coderelictions of monoidal coalgebra modalities (or equivalently additive bialgebra modalities) also lift to the biproduct completion. Therefore, the biproduct completion of a differential category with a monoidal coalgebra modality (or equivalently an additive bialgebra modality) is again a differential category.

Proposition 3.3.15 *[7, Proposition 9] Let* $\mathbb{X}$ *be a differential category with an additive bialgebra modality* $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ *and codereliction* $\eta : A \longrightarrow !A$*. Then the biproduct completion* $\mathsf{B}[\mathbb{X}]$ *is a differential category with an additive bialgebra modality as defined in Proposition 2.3.30 and codereliction defined as follows:*

$$\begin{bmatrix} \eta_{A_1} \otimes u \otimes \ldots \otimes u \\ \ldots \\ u \otimes \ldots \otimes \eta_{A_i} \otimes \ldots \otimes u \\ \ldots \\ u \otimes u \otimes \ldots \otimes \eta_{A_n} \end{bmatrix} : (A_1, \ldots, A_n) \longrightarrow !A_1 \otimes \ldots \otimes !A_n$$

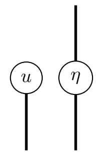
Furthermore, the obvious inclusion functor $\mathcal{I} : \mathbb{X} \longrightarrow \mathsf{B}[\mathbb{X}]$ preserves the differential category structure strictly.

We conclude this section with the observation that every codereliction of an additive bialgebra modality induces a codereliction on the non-additive bialgebra modalities from Section 2.4.

Proposition 3.3.16 *[7, Proposition 11] Let $\mathbb{X}$ be a differential category with an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ and codereliction $\eta : A \longrightarrow !A$. Define the natural transformation $\eta^B : A \longrightarrow !^B(A)$ as follows:*

$$\eta^B := A \xrightarrow{u \otimes \eta} !B \otimes !A$$

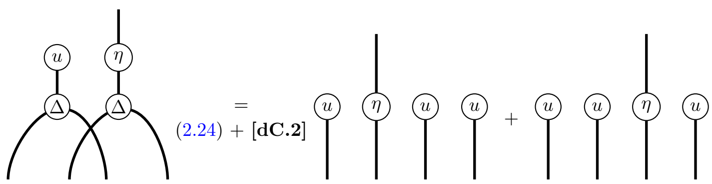
which is drawn as follows in the graphical calculus:

$\eta^B \quad :=$ 

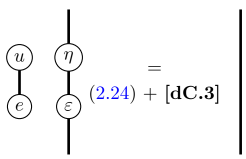
Then η^B is a codereliction for the bialgebra modality $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B, \nabla^B, u^B)$ as defined in Proposition 2.4.3.

Proof: We must show **[dC.2]**, **[dC.3]**, and **[dC.4]**:

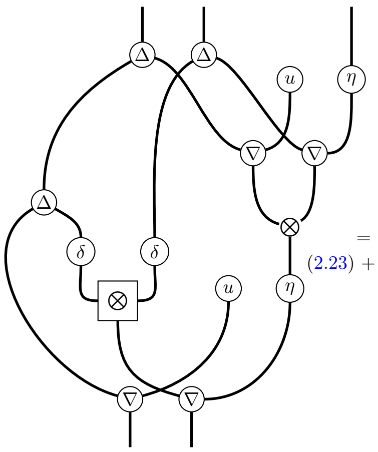
[dC.2]: Here we use the bialgebra identity between the unit and the comultiplication, and that η satisfies the product rule **[dC.2]**:



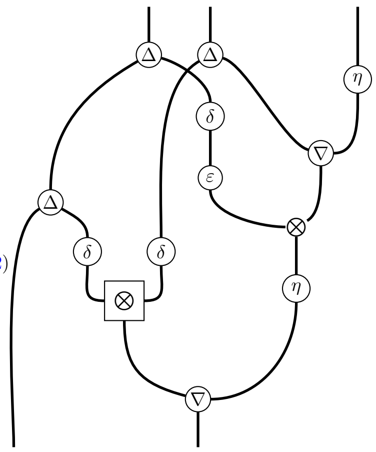
[dC.3]: Here we use the bialgebra identity between the unit and counit, and that η satisfies the linear rule **[dC.3]**:



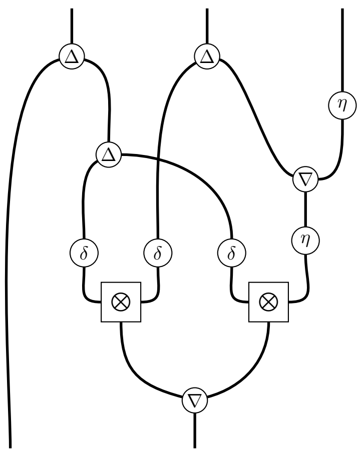
[dC.4]: Here we use coassociativity of the comultiplication, one of the identities of Proposition 2.3.17, that δ preserves the comultiplication, and that η satisfies the chain rule **[dC.4]** and the monoidal rule **[dC.m]**:



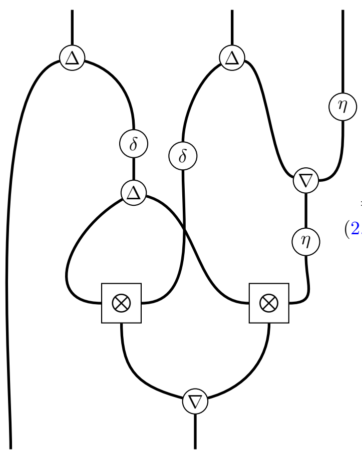
$=$
(2.23) + (2.2)



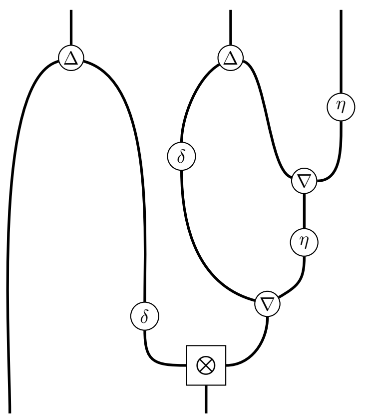
$=$
(2.6) + **[dC.m]**

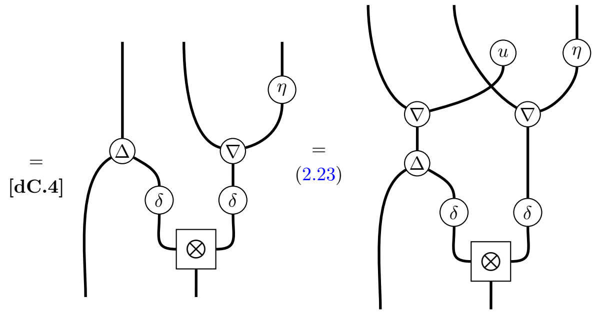


$=$
(2.6)



$=$
(2.34)





So we conclude that η^B is a codereliction. □

The induced deriving transformation $\mathsf{d}^B : !^B(A) \otimes A \to !^B(A)$ is easily computed out to be:

$$!B \otimes !A \otimes A \xrightarrow{1 \otimes \mathsf{d}} !B \otimes !A$$

Intuitively, this should be thought of as the partial derivative with respect to A and leaving B in context.

Chapter 4

Cartesian Differential Categories Revisited

This chapter is based on [29]. As such, the author would like to thank their coauthor Robin Cockett, and also Kristine Bauer for her help on this project, as well as Brenda Johnson and Sarah Yeakel for useful discussions at the 2018 Canadian Mathematical Society Summer Meeting which initiated this research project. In particular, this chapter also introduces the conventions and notations for Cartesian differential categories which we use throughout the remaining chapters.

Cartesian differential categories were introduced by Blute, Cockett, and Seely in [9] to provide the categorical semantics of Ehrhard and Regnier's differential λ-calculus [37]. Briefly, a Cartesian differential category comes equipped with a differential combinator D, which maps for every map $f : A \to B$ produces its derivatives $\mathsf{D}[f] : A \times A \to B$, and such that the axioms of D formalize the basic properties of the derivative from multivariable calculus over Euclidean spaces. The main example of a Cartesian differential category is the category of Euclidean spaces and real smooth functions between them. Other interesting examples include any category with finite biproducts, the Lawvere Theory of polynomials over a commutative semiring, and, of course, categorical models of the differential λ-calculus [14, 18, 69] (which are in fact called Cartesian differential *closed* categories). There are even also both free Cartesian differential categories, given by the term calculus as found in [9], and cofree Cartesian differential categories [31, 59], where in particular the composition is defined using the Faà di Bruno formula for the higher order chain rule. An important source of examples of Cartesian differential categories are the coKleisli categories of differential categories [7, 8], where the differential combinator is constructed using the deriving transformation.

Abelian functor calculus was developed by Johnson and McCarthy in [53], based on Goodwillie's functor calculus [45, 46, 47]. In [5], Bauer, Johnson, Osborne, Riehl, and Tebbe (BJORT) showed that, using the Abelian functor calculus, the homotopy category of the category of Abelian categories is a Cartesian differential category. The differential combinator $\nabla(-)$ (referred to as the *directional derivative* in [5, Section 6]) is defined as [5, Definition 6.1] $\nabla F(X, V) := D_1(F(X \oplus -))(V)$[1], where $D_1(G)$ is the *linearization* (or *linear approximation*) of a functor G [5, Section 5].

From the Cartesian differential category perspective, the BJORT construction is backwards. In

[1]Here the second argument is the linear argument.

any Cartesian differential category it is always possible to define the notion of a linear map. The notion of linearity in a Cartesian differential category is defined with respect to the differential combinator and often coincides with the classical notion from linear algebra. In particular, linearity in a Cartesian differential category always implies additivity. That said, there are examples of Cartesian differential categories where a map may be additive yet not linear. However, it always possible to linearize a map using the differential combinator. On the other hand, BJORT constructed their differential combinator using an already established notion of linear map and linearization. The goal of this chapter is to reverse engineer BJORT's construction by abstracting the notion of linear approximation D_1 from the (Abelian) functor calculus. To this end, we introduce the notion of a **linearizing combinator** and show that every Cartesian differential category comes equipped with a canonical system of linearizing combinators built from the differential combinator. Furthermore, we show that the differential combinator can be reconstructed à la BJORT using such a system of linearizing combinators. In this manner, we show that linearizing combinators do, in fact, provide an alternative axiomatization of Cartesian differential categories.

To better understand the BJORT construction, let us consider classical multivariable calculus. Given a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$, linearization $\mathsf{L}[f] : \mathbb{R} \longrightarrow \mathbb{R}$ is the best $\mathbb{R}$-linear function which is closest to f. This is given by the first degree term in its Maclaurin series expansion (i.e its Taylor series expansion at 0), that is, $\mathsf{L}[f](x) = f'(0)x$, which is indeed an $\mathbb{R}$-linear function. In terms of the differential combinator, its differential $\mathsf{D}[f] : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $\mathsf{D}[f](x, y) = f'(x)y$, and so $\mathsf{L}[f](x) = \mathsf{D}[f](0, x)$. Therefore, in an arbitrary Cartesian differential category, the linearizing combinator L is defined by first applying the differential combinator and then evaluating the derivative at zero in its first argument:

$$\dfrac{\dfrac{A \xrightarrow{\;f\;} B}{\text{Apply the differential combinator}\qquad A \times A \xrightarrow{\;\mathsf{D}[f]\;} B}}{\text{Evaluate at zero in the first argument}\qquad \mathsf{L}[f] := A \xrightarrow{\;\langle 0,1\rangle\;} A \times A \xrightarrow{\;\mathsf{D}[f]\;} B}$$

We can use this to derive an abstract notion of a linearizing combinator, L, for arbitrary Cartesian left additive categories, which satisfies axioms which parallel those of the differential combinator. These include a sort of chain rule for linearizing a composite and the fact that the linearization of a map is always additive. In particular, one can then show that D_1, from Abelian functor calculus, is an example of such an abstract linearizing combinator.

To define a differential combinator from linearization, the ability to perform linearization in context is required. We refer to linearization in context as *partial* linearization because differentiation in context is usually called partial differentiation. Consider the classical limit definition of the derivative of a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$:

$$\mathsf{D}[f](x, y) = \lim_{t \longrightarrow 0} \frac{f(x + ty) - f(x)}{t}$$

Note that if we evaluate at $x = 0$, then we obtain an expression of $\mathsf{L}[f]$ in terms of a limit:

$$\mathsf{L}[f](y) = \mathsf{D}[f](0, y) = \lim_{t \longrightarrow 0} \frac{f(ty) - f(0)}{t}$$

For a fixed x, define $g_x : \mathbb{R} \longrightarrow \mathbb{R}$ to be the smooth function defined as $g_x(y) = f(x + y)$. Then:

$$\mathsf{D}[f](x, y) = \lim_{t \longrightarrow 0} \frac{f(x + t \cdot y) - f(x)}{t} = \lim_{t \longrightarrow 0} \frac{g_x(ty) - g_x(0)}{t} = \mathsf{L}[g_x](y)$$

Therefore, the derivative of f is the linearization of the function $g_x(y) = f(x + y)$ in the variable y. In other words, if we let $g(x, y) = f(x + y)$, then $\mathsf{D}[f]$ is the partial linearization of $g(x, y)$ in its second argument while keeping the first argument constant. We may write this directly as:

$$\mathsf{D}[f](x, y) = \mathsf{L}[z \mapsto f(x + z)](y)$$

where we are viewing $z \mapsto f(x + z)$ as a function in the variable context x. This is precisely how BJORT define their differential combinator. In fact, every differential combinator in a Cartesian differential category can be defined in this fashion. However, there is a caveat: in an arbitrary Cartesian left additive category, it is not always possible to define partial linearization from total linearization. Indeed, for example, $\mathcal{C}^1$ functions have a total linearization combinator but do not have partial linearization since this would induce a differential combinator, which cannot be the case since the derivative of a $\mathcal{C}^1$ function is not necessarily a $\mathcal{C}^1$ function (see Example 4.4.15 below for more details). Thus, partial linearization, that is linearization in *context*, must be assumed.

From a categorical perspective, the notion of context is captured by simple slice categories [51], where a map $A \longrightarrow B$ in the simple slice is a map of type $C \times A \longrightarrow B$ in the base category. Maps in the simple slice category over an object C are said to be in "context C". Asking that a Cartesian left additive category has partial linearization is requiring that it comes equipped with a **system of linearizing combinators** and is the requirement that every simple slice category come equipped with a linearizing combinator L^C. Thus, for a map $f : C \times A \longrightarrow B$, $\mathsf{L}^C[f] : C \times A \longrightarrow B$ is its linearization in context C, and these linearizing combinators are compatible with one another. For example, given a map of type $C \times A \longrightarrow B$, we require that partially linearizing A then C is the same as partially linearizing C then A. For the Abelian functor calculus, BJORT's linearization of a multivariable functor at a single variable by holding all other inputs constant, D^1_1, is precisely a linearizing combinator in context. For a Cartesian differential category, every simple slice category is again a Cartesian differential category where the differential combinator in context is given by partial differentiation. As such, every Cartesian differential category comes equipped with a canonical system of linearizing combinators. Conversely, to define a differential combinator from partial linearization, one must first be able to precompose by a map which captures addition. In a Cartesian left additive category, for every object A, there is a map $\oplus_A := \pi_0 + \pi_1 : A \times A \longrightarrow A$ which makes A a commutative monoid. This allows the differential combinator D to be defined on a map by linearizing in context that map precomposed by $\oplus_A$, thus, generalizing the construction above.

$$A \xrightarrow{\quad f \quad} B$$

Precompose by addition $\qquad A \times A \xrightarrow{\quad \oplus_A \quad} A \xrightarrow{\quad f \quad} B$

Linearize in the second argument $\qquad \mathsf{D}[f] := \; A \times A \xrightarrow{\quad \mathsf{L}^A[\oplus_A f] \quad} B$

Furthermore, these constructions are inverses of each other, and so there is a bijective correspondence between differential combinators and systems of linearizing combinators. This shows that a Cartesian differential category is precisely a Cartesian left additive category with a system of linearizing combinators.

It is worth pointing out that the bijective correspondence between differential combinators and systems of linearizing combinators is analogous to the bijective correspondence between deriving transformations and coderelictions for differential categories [7, Theorem 4] (or as explained in the previous chapter). Linearizing combinators, thus, should also provide equivalent axiomatizations for generalizations of Cartesian differential categories including generalized Cartesian differential categories [32], differential restriction categories [22], and even tangent categories [19]. In each setting the precise form that linearization takes needs to be developed: hopefully this development, centred as it is on Cartesian differential categories, will be a useful guide.

$\otimes$-differential categories	Cartesian differential categories
Deriving transformations $\mathsf{d} : !A \otimes A \to !A$	Differential combinators D $\dfrac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$
Coderelictions $\eta : A \to !A$	Linearizing Combinators L $\dfrac{f : A \to B}{\mathsf{L}[f] : A \to B}$

Returning back to differential categories, recall that from a codereliction η, one defines a deriving transformation as $\mathsf{d} = (1 \otimes \eta)\nabla$. In the coKleisli category, the multiplication ∇ plays the role of pre-composing by addition $\oplus$, while $1 \otimes \eta$ plays the role of the linearizing the second argument, that is, the linearizing combinator in context L^C. The keen-eyed reader may note that the "partial" codereliction $1 \otimes \eta$ can easily be defined from the "total" codereliction. The reason for this is the presence of Seely isomorphisms $!(C \times A) \cong !C \otimes !A$ which allow us to split off the context part and then bring it back afterwards. Unfortunately, as previously mentioned, this does not work in arbitrary Cartesian differential categories. To do so, we require the base category to be Cartesian closed.

To show how partial linearization can arise from total linearization, we investigate linearization in Cartesian closed settings. For Cartesian *closed* left additive categories, we introduce the notion of an **exponentiable** linearizing combinator. We then show how such a total linearizing combinator gives rise to a **closed** systems of linearizing combinators: that is a system of linearizing combinators, which are compatible with the closed structure. To obtain a linearizing combinator in context, given a total exponentiable linearizing combinator, one employs the total linearization on the curry of the map and then one uncurries the result:

$$\cfrac{\cfrac{\cfrac{C \times A \xrightarrow{f} B}{\text{Curry} \quad A \xrightarrow{\lambda(f)} [C, A]}}{\text{Linearize} \quad A \xrightarrow{\mathsf{L}[\lambda(f)]} [C, A]}}{\text{Uncurry} \quad \mathsf{L}^C[f] := C \times A \xrightarrow{\lambda^{-1}(\mathsf{L}[\lambda(f)])} [C, A]}$$

Therefore, in the closed setting, partial linearization is equivalent to total linearization.

Chapter Outline: Section 4.1 is a background section which reviews the basic theory of Cartesian differential categories and Cartesian left additive categories, as well as the notion of linear maps and their basic properties. This section also provides a list of the main examples of Cartesian differential categories used in this thesis, including how the coKleisli category of a differential category is a Cartesian differential category. Section 4.2 introduces linearizing combinators, the main novel concept of study in this chapter. We show that every differential combinator induces a linearizing combinator, and afterwards we provide examples of these induced linearizing combinators in our main examples. Section 4.3 reviews the notion of partial differentiation and being linear in context. Section 4.4 discusses partial linearization by introducing systems of linearizing combinator. We then show how every system of linearizing combinators induces a differential combinator – following the BJORT construction. The first main result of this chapter is that there is a bijective correspondence between differential combinators and systems of linearizing combinators: thus, a Cartesian differential category is precisely a Cartesian left additive category with a system of linearizing combinators. We also provide an example of a linearizing combinator on a Cartesian left additive category which is not induced from a differential combinator or a system of linearizing combinators. Section 4.5 studies how to define partial linearization from total linearization in the closed setting by introducing exponentiable linearizing combinators and closed systems of linearizing combinators. We show that every closed system of linearizing combinators induces an exponentiable linearizing combinators, and conversely we also show how every exponentiable linearizing combinator induces a closed system of linearizing combinators. The second main result of this chapter states that a Cartesian closed differential category is precisely a Cartesian closed left additive category with a closed system of linearizing combinators, or equivalently an exponentiable linearizing combinator.

4.1 Cartesian Differential Categories

In this section, we review Cartesian left additive categories, Cartesian differential categories, and linear maps. We also provide the examples of Cartesian differential categories, which we will use throughout this thesis. For a more in-depth introduction to Cartesian differential categories, we refer the reader to the original paper [9].

The underlying structure of a Cartesian differential category is that of a Cartesian left additive category. A category is said to be *left* additive if it is *skew*-enriched [16] over the category of commutative monoids, which is made by precise by Garner and the author in [41, Section 2.1]. This allows one to have zero maps and sums of maps while allowing for maps which do not preserve the additive structure. Maps which do preserve the additive structure are called *additive* maps.

Definition 4.1.1 *A **left additive category** [9, Definition 1.1.1] is a category $\mathbb{X}$ such that each hom-set $\mathbb{X}(A,B)$ is a commutative monoid with addition $+ : \mathbb{X}(A,B) \times \mathbb{X}(A,B) \longrightarrow \mathbb{X}(A,B)$, $(f,g) \mapsto f+g$, and zero $0 \in \mathbb{X}(A,B)$, such that pre-composition preserves the additive structure, that is, $f(g+h) = fg + fh$ and $f0 = 0$. Furthermore, we say that:*

(i) *A map $f : A \longrightarrow B$ is **constant** if $0f = f$;*

(ii) *A map* $f : A \longrightarrow B$ *is* ***reduced*** *if* $0f = 0$*;*

(iii) *A map* $f : A \longrightarrow B$ *is* ***semi-additive*** *if* $(g + h)f = gf + hf$*;*

(iv) *A map* $f : A \longrightarrow B$ *is* ***additive*** *if it is both reduced and semi-additive.*

Next, we turn our attention to left additive categories with finite products. Recall that for a category with finite products we use $\times$ for the binary product, $\pi_0 : A \times B \longrightarrow A$ and $\pi_1 : A \times B \longrightarrow B$ for the projection maps, $\langle -, - \rangle$ for the pairing operation, so that $f \times g = \langle \pi_0 f, \pi_1 g \rangle$, and $\top$ for the chosen terminal object. Let $\tau_{A,B} : A \times B \longrightarrow B \times A$ denote the canonical natural *symmetry* isomorphism which is defined as follows:

$$\tau_{A,B} = \langle \pi_1, \pi_0 \rangle \tag{4.1}$$

We also denote the canonical natural *interchange* isomorphism by

$$c_{A,B,C,D} : (A \times B) \times (C \times D) \longrightarrow (A \times C) \times (B \times D)$$

which is defined as:

$$c_{A,B,C,D} := \langle \pi_0 \times \pi_0, \pi_1 \times \pi_1 \rangle \tag{4.2}$$

To simplify notation, we will often omit the subscripts of τ and c. Note that both τ and c are self-inverse, that is, $\tau\tau = 1$ and $cc = 1$.

Definition 4.1.2 *A* ***Cartesian left additive category*** *[59, Definition 2.3] is a left additive category* $\mathbb{X}$ *which has products for which all the projection maps* $\pi_0 : A \times B \longrightarrow A$ *and* $\pi_1 : A \times B \longrightarrow B$ *are additive.*

The definition of a Cartesian left additive category presented here is not precisely that given in [9, Definition 1.2.1], but was shown to be equivalent by the author in [59]:

Lemma 4.1.3 *[59, Lemma 2.4] In a Cartesian left additive category* $\mathbb{X}$ *(as defined in Definition 4.1.2):*

(i) $\langle f, g \rangle + \langle h, k \rangle = \langle f + h, g + k \rangle$ *and* $\langle 0, 0 \rangle = 0$*;*

(ii) *If* $f : C \longrightarrow A$ *and* $g : C \longrightarrow B$ *are additive then* $\langle f, g \rangle : C \longrightarrow A \times B$ *is additive;*

(iii) *The diagonal map* $\Delta_\times = \langle 1, 1 \rangle : A \longrightarrow A \times A$ *is additive;*

(iv) *If* $h : A \longrightarrow B$ *and* $k : C \longrightarrow D$ *are additive then* $h \times k$ *is additive;*

(v) *For any object* A*, for the unique map to the terminal object* $\mathtt{t}_A : A \longrightarrow \top$*,* $\mathtt{t}_A = 0$*.*

Proof: The proof of (i) is the same as the one found in [9, Lemma 1.2.3] and uses only that the projections π_i are additive. Then (ii) follows from (i), that is, assuming f and g are additive:

$$\begin{aligned}(h + k)\langle f, g \rangle &= \langle (h + k)f, (h + k)g \rangle \\ &= \langle hf + kf, hg + kg \rangle\end{aligned}$$

$$\begin{aligned} &= \langle hf, hg \rangle + \langle kf, kg \rangle \\ &= h\langle f, g \rangle + k\langle f, g \rangle \end{aligned}$$

$$\begin{aligned} 0\langle f, g \rangle &= \langle 0f, 0g \rangle \\ &= \langle 0, 0 \rangle \\ &= 0 \end{aligned}$$

and therefore $\langle f, g \rangle$ is additive. For (iii), [9, Proposition 1.1.2] tells us that all identity maps are additive, and therefore by (ii), $\Delta_\times = \langle 1, 1 \rangle$ is additive. For (iv), [9, Proposition 1.1.2] also tells us that additive maps are closed under composition, so if f and g are additive, then so is $\pi_0 f$ and $\pi_1 g$. Then again by (ii), $f \times g = \langle \pi_0 f, \pi_1 g \rangle$ is additive. Lastly for (v), since by definition, there is always a zero map of type $0 : A \longrightarrow \top$, by the universal property of the terminal object, the zero map must be the unique map to the terminal, and so $\mathsf{t}_A = 0$. □

As done by Garner and the author in [41, Section 2.1], it is possible to generalize (Cartesian) left additive category to instead be skew-enriched over the category of R-modules for some fixed commutative semiring R, which are called (Cartesian) left R-linear categories [41, Definition 2.1]. Taking $R = \mathbb{N}$, the standard semiring of natural numbers with addition, one reobtains the notion above, that is, a (Cartesian) left $\mathbb{N}$-linear category is precisely a (Cartesian) left additive category. That said, we've elected to work with (Cartesian) left additive categories in this thesis, though all the results of this chapter and the next can easily be generalized to the R-linear case.

In a Cartesian left additive category, define the *lifting* map $\ell_{A,B,C,D} : A \times D \longrightarrow (A \times B) \times (C \times D)$ as the map which inserts zeros in the middle two arguments, that is, define $\ell_{A,B,C,D}$ as follows:

$$\ell_{A,B,C,D} := \langle 1, 0 \rangle \times \langle 0, 1 \rangle \tag{4.3}$$

As before, to simplify notation, we will often omit the subscripts of ℓ when there is no confusion. It is important to note that in an arbitrary Cartesian left additive category, ℓ is *not* a natural transformation. However, ℓ is natural whenever g and h are reduced maps making $(f \times k)\ell = \ell\big((f \times g) \times (h \times k)\big)$. The lifting map ℓ is a crucial ingredient in constructing differential combinators and linearizing combinators in *context*, as will see in later sections.

Cartesian left additive categories can be equivalently axiomatized by equipping each object with a commutative monoid structure so all the projection maps, π_0 and π_1, are monoid morphisms. In this axiomatization of a Cartesian left additive category, the additive maps are precisely the monoid morphisms with respect to the canonical monoid structure. Here is how that monoid structure arises:

Lemma 4.1.4 *[9, Proposition 1.2.2, Lemma 1.2.3] In a Cartesian left additive category $\mathbb{X}$, for every object A define the map $\oplus_A : A \times A \longrightarrow A$ as $\oplus_A := \pi_0 + \pi_1$. Then:*

(i) *For every object A, $(A, \oplus_A, 0)$ is a commutative monoid, that is, the following equalities hold:*

$$\langle 0, 1 \rangle \oplus_A = 1 \qquad \langle 1, 0 \rangle \oplus_A = 1 \qquad \tau \oplus_A = \oplus_A \qquad c(\oplus_A \times \oplus_A)\oplus_A = (\oplus_A \times \oplus_A)\oplus_A$$

(ii) *For every pair of objects A and B, the following equalities hold:*

$$\oplus_{A\times B} = c(\oplus_A \times \oplus_B) \qquad \ell\oplus_{A\times B} = 1 \qquad \ell(\oplus_A \times \oplus_A) = 1$$

(iii) *A map $f : A \longrightarrow B$ is additive if and only if $\oplus_A f = \pi_0 f + \pi_1 f$ and $0f = 0$ (or equivalently if $\oplus_A f = (f \times f)\oplus_B$ and $0f = 0$).*

Cartesian differential categories are Cartesian left additive categories which come equipped with a differential combinator, which in turn is axiomatized by the basic properties of the directional derivative from multivariable differential calculus. It is important to note that unlike in [9, 19, 21], we use the convention used in the more recent work on Cartesian differential categories where the linear argument of $\mathsf{D}[f]$ is its second argument rather than its first argument. There are various equivalent ways of expressing the axioms of a Cartesian differential category. For this paper, we've chosen the one found in [59, Definition 2.6] (using the notation for Cartesian left additive categories introduced above).

Definition 4.1.5 *A* ***Cartesian differential category*** *[9, Definition 2.1.1] is a Cartesian left additive category $\mathbb{X}$ equipped with a* ***differential combinator*** *D, which is a family of operators $\mathsf{D} : \mathbb{X}(A, B) \longrightarrow \mathbb{X}(A \times A, B)$, $f \mapsto \mathsf{D}[f]$, where $\mathsf{D}[f]$ is called the derivative of f, such that the following seven axioms hold:*

[CD.1] $\mathsf{D}[f + g] = \mathsf{D}[f] + \mathsf{D}[g]$ *and* $\mathsf{D}[0] = 0$*;*

[CD.2] $(1 \times \oplus_A)\mathsf{D}[f] = (1 \times \pi_0)\mathsf{D}[f] + (1 \times \pi_1)\mathsf{D}[f]$ *and* $\langle 1, 0\rangle\mathsf{D}[f] = 0$*;*

[CD.3] $\mathsf{D}[1] = \pi_1$*,* $\mathsf{D}[\pi_0] = \pi_1\pi_0$ *and* $\mathsf{D}[\pi_1] = \pi_1\pi_1$*;*

[CD.4] $\mathsf{D}[\langle f, g\rangle] = \langle \mathsf{D}[f], \mathsf{D}[g]\rangle$*;*

[CD.5] $\mathsf{D}[fg] = \langle \pi_0 f, \mathsf{D}[f]\rangle\mathsf{D}[g]$ *(the chain rule);*

[CD.6] $\ell\ \mathsf{D}\big[\mathsf{D}[f]\big] = \mathsf{D}[f]$ *where ℓ is defined as in (4.3);*

[CD.7] $c\ \mathsf{D}\big[\mathsf{D}[f]\big] = \mathsf{D}\left[\mathsf{D}[f]\right]$ *where c is defined as in (4.2).*

To help with the intuition, it is useful to use the term logic of Cartesian differential categories as introduced in [9, Section 4], which expresses the differential combinator as:

$$\mathsf{D}[f](a, b) := \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b$$

Using this notation, the seven differential combinator axioms are expressed as:

[CD.1] $\frac{\mathsf{d}f(x)+g(x)}{\mathsf{d}x}(a) \cdot b = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b + \frac{\mathsf{d}g(x)}{\mathsf{d}x}(a) \cdot b$ and $\frac{\mathsf{d}0}{\mathsf{d}x}(a) \cdot b = 0$

[CD.2] $\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot (b + c) = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b + \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot c$ and $\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot 0 = 0$

[CD.3] $\frac{\mathsf{d}x}{\mathsf{d}x}(a) \cdot b = b$ and $\frac{\mathsf{d}\pi_i(x_0,x_1)}{\mathsf{d}(x_0,x_1)}(a_0, a_1) \cdot (b_0, b_1) = b_i$

[CD.4] $\frac{\mathsf{d}\langle f(x),g(x)\rangle}{\mathsf{d}x}(a) \cdot b = \left\langle \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b, \frac{\mathsf{d}g(x)}{\mathsf{d}x}(a) \cdot b \right\rangle$

[CD.5] $\frac{\mathsf{d}g\left(f(x)\right)}{\mathsf{d}x}(a) \cdot b = \frac{\mathsf{d}g(y)}{\mathsf{d}y}(f(a)) \cdot \left(\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b \right)$

[CD.6] $\frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(y)\cdot z}{\mathsf{d}(y,z)}(a, 0) \cdot (0, b) = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b$

[CD.7] $\frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(y)\cdot z}{\mathsf{d}(y,z)}(a, b) \cdot (c, d) = \frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(y)\cdot z}{\mathsf{d}(y,z)}(a, c) \cdot (b, d)$

[CD.1] says that the derivative of a sum is equal to the sum of the derivatives, and that the derivative of zero maps is zero. **[CD.2]** says that derivatives are additive in their second argument (which we make precise below in Definition 4.3.2). **[CD.3]** tells us what the derivatives of identity maps and projections maps are. **[CD.4]** says the derivative of a pairing of maps is equal to the pairing of the derivatives. **[CD.5]** is the chain rule which tells us what the derivative of a composition of functions is. Lastly, **[CD.6]** and **[CD.7]** may look somewhat mysterious for now but essentially they capture properties of partial differentiation, which we discuss below in Proposition 4.3.4. Briefly, **[CD.6]** says that derivatives are linear in the second argument (which we make precise below in Definition 4.3.5). While **[CD.7]** captures the symmetry of the partial derivatives. More discussion on the intuition for the differential combinator axioms can be found in [9, Remark 2.1.3].

Here are now our main examples of Cartesian differential categories which we will use throughout this chapter:

Example 4.1.6 Any category $\mathbb{X}$ with finite biproduct $\oplus$ is a Cartesian left additive category where the additive structure is given by the canonical one induced by the biproduct structure. As such, it follows that every map is additive in this case. In fact, a Cartesian left additive where every map is additive is precisely a category with finite biproducts. Furthermore, any category $\mathbb{X}$ with finite biproduct is a Cartesian differential category where the differential combinator is defined by precomposing with the second projection map:

$$\mathsf{D}[f] := A \times A \xrightarrow{\pi_1} A \xrightarrow{f} B$$

For example, if $\mathbb{K}$, then $\mathsf{VEC}_{\mathbb{K}}$ is a Cartesian differential category where for a $\mathbb{K}$-linear map $f : V \longrightarrow W$, its derivative $\mathsf{D}[f] : V \oplus V \longrightarrow W$ is defined as $\mathsf{D}[f](v \oplus w) = w$.

Example 4.1.7 Let $\mathbb{K}$ be a field. Define the category $\mathsf{POLY}_{\mathbb{K}}$ whose object are $n \in \mathbb{N}$, where a map $P : n \longrightarrow m$ is a m-tuple of polynomials in n variables, that is, $P = \langle p_1(\vec{x}), \ldots, p_m(\vec{)}\rangle$ with $p_i(\vec{x}) \in \mathbb{K}[x_1, \ldots, x_n]$. The identity maps $1_n : n \longrightarrow n$ are the tuples $1_n = \langle x_1, \ldots, x_n\rangle$ and where composition is given by standard polynomial substitution. $\mathsf{POLY}_{\mathbb{K}}$ is a Cartesian left additive category where the finite product structure is given by $n \times m = n + m$ with projection maps $\pi_0 : n \times m \longrightarrow n$ and $\pi_0 : n \times m \longrightarrow n$ defined as the tuples $\pi_0 = \langle x_1, \ldots, x_n\rangle$ and

$\pi_1 = \langle x_{n+1}, \ldots, x_{n+m}\rangle$, and where the additive structure is defined coordinate wise via the standard sum of polynomial. $\mathsf{POLY}_{\mathbb{K}}$ is also a Cartesian differential category where the differential combinator is given by the standard differentiation of polynomials, that is, for a map $P : n \longrightarrow m$, $P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x})\rangle$, its derivative $\mathsf{D}[P] : n \times n \longrightarrow m$ is defined as the tuple of the sum of the partial derivatives of the polynomials $p_i(\vec{x})$:

$$\mathsf{D}[P](\vec{x}, \vec{y}) := \left(\sum_{i=1}^{n} \frac{\partial p_1(\vec{x})}{\partial x_i} y_i, \ldots, \sum_{i=1}^{n} \frac{\partial p_n(\vec{x})}{\partial x_i} y_i \right)$$

where $\sum\limits_{i=1}^{n} \frac{\partial p_j(\vec{x})}{\partial x_i} y_i \in \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We note that this example generalize to the category of polynomials over an arbitrary commutative semiring.

Example 4.1.8 Let $\mathbb{R}$ be the set of real numbers. Define SMOOTH as the category whose objects are the Euclidean real vector spaces $\mathbb{R}^n$ (including the singleton $\mathbb{R}^0 = \{\top\}$) and whose maps are the real smooth functions $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ between them. SMOOTH is a Cartesian left additive category where the finite product structure is given by the Cartesian product of Euclidean spaces and where the additive structure is given by the standard sum of smooth functions. SMOOTH is also a Cartesian differential category, arguably the canonical example, where the differential combinator is defined as the directional derivative of a smooth function. Recall that a smooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is in fact a tuple $F = \langle f_1, \ldots, f_m\rangle$ of smooth functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. Then the Jacobian matrix of F at vector $\vec{x} \in \mathbb{R}^n$ is the matrix $\nabla(F)(\vec{x})$ of size $m \times n$ whose coordinates are the partial derivatives of the f_i:

$$\nabla(F)(\vec{x}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \frac{\partial f_1}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}) & \frac{\partial f_2}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_2}{\partial x_n}(\vec{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}) & \frac{\partial f_m}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_m}{\partial x_n}(\vec{x}) \end{bmatrix}$$

So for a smooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is then defined as:

$$\mathsf{D}[F](\vec{x}, \vec{y}) := \nabla(F)(\vec{x}) \cdot \vec{y} = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{x}) y_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i}(\vec{x}) y_i \right\rangle$$

where $\cdot$ is matrix multiplication and $\vec{y}$ is seen as a $n \times 1$ matrix. Note that $\mathsf{POLY}_{\mathbb{R}}$ is a sub-Cartesian differential category of SMOOTH.

Example 4.1.9 Every categorical model of the differential λ-calculus [37, 69] induces a Cartesian differential category [18, Theorem 4.3].

Example 4.1.10 We very briefly review the BJORT abelian functor calculus: for more complete details on this example of a Cartesian differential category, see [5]. Let $\mathbb{A}$ be an abelian category and let $\mathsf{Ch}(\mathbb{A})$ be its category of (non-negative) chain complexes. Define $\mathsf{HoAbCat}_{\mathsf{Ch}}$ as the category whose objects are abelian categories where a map from $\mathbb{A} \longrightarrow \mathbb{B}$ is a point-wise chain homotopy equivalence class of functors $\mathbb{A} \longrightarrow \mathsf{Ch}(\mathbb{B})$, and where composition and identity maps are defined as in [5, Definition 3.5]. By [5, Corollary 6.6], $\mathsf{HoAbCat}_{\mathsf{Ch}}$ is a Cartesian differential category where the differential combinator, which in this case is written as ∇, is defined for $F : \mathbb{A} \longrightarrow \mathsf{Ch}(\mathbb{B})$ as follows on objects:

$$\nabla F(X, V) := D_1 F(X \oplus -)(V)$$

where D_1 is the linearization operator as defined in [5, Definition 5.1] using cross effects of functors.

Example 4.1.11 Every Cartesian left additive category has a cofree Cartesian differential category over it which satisfies the obvious couniversal property. Cofree Cartesian differential categories were first constructed in [31] using the Faà di Bruno construction. In this chapter, we will use the alternative construction provided by the author as found in [59], as the differential combinator is simpler to express. For a Cartesian left additive category $\mathbb{X}$, let $\mathsf{P} : \mathbb{X} \longrightarrow \mathbb{X}$ be the product functor defined on objects as $\mathsf{P}(A) = A \times A$ and on maps as $\mathsf{P}(f) = f \times f$. Then define $\mathcal{D}(\mathbb{X})$ as the category whose objects are the same as $\mathbb{X}$ and where a map $A \longrightarrow B$ is a D-sequence which is a sequence of maps $(f_0, f_1, \ldots)$ where $f_n : \mathsf{P}^n(A) \longrightarrow B$ and satisfying the coherences found in [59, Definition 4.2]. Composition and identity maps are defined as in [59, Definition 3.6]. $\mathcal{D}(\mathbb{X})$ is a Cartesian differential category [59, Corollary 4.25] where the differential combinator is defined by shifting D-sequences to the left:

$$\mathsf{D}[(f_0, f_1, \ldots)] = (f_1, f_2, \ldots)$$

Maps in $\mathcal{D}(\mathbb{X})$ should be thought of a generalized sequences of the form $(f, \mathsf{D}[f], \mathsf{D}^2[f], \ldots)$.

We now turn our attention to a very important source of Cartesian differential categories. As previously mentioned numerous times throughout this thesis, the coKleisli category of a differential category (Definition 3.1.1) with finite (bi)products is a Cartesian left additive category. As we will be working with coKleisli categories, we will use the notation found in [10] and use interpretation brackets $[\![-]\!]$ to help distinguish between composition in the base category and coKleisli composition. So for a comonad $(!, \delta, \varepsilon)$ on a category $\mathbb{X}$, let $\mathbb{X}_!$ denote its coKleisli category, which is the category whose objects are the same as $\mathbb{X}$ and where $\mathbb{X}_!(A, B) = \mathbb{X}(!A, B)$ with composition and identity defined as:

$$[\![fg]\!] = \delta!([\![f]\!])[\![g]\!] \qquad\qquad [\![1]\!] = \varepsilon$$

If $\mathbb{X}$ if has finite products then so does $\mathbb{X}_!$ where on objects the product is defined as in $\mathbb{X}$ and where the remaining data is defined as follows:

$$[\![\pi_0]\!] = \varepsilon\pi_0 \qquad [\![\pi_1]\!] = \varepsilon\pi_1 \qquad [\![\langle f, g\rangle]\!] = \big\langle [\![f]\!], [\![g]\!] \big\rangle \qquad [\![f \times g]\!] = \big\langle !(\pi_0)[\![f]\!], !(\pi_1)[\![g]\!] \big\rangle$$

If $\mathbb{X}$ is a Cartesian left additive category then so is $\mathbb{X}_!$ where:

$$[\![f + g]\!] = [\![f]\!] + [\![g]\!] \qquad\qquad [\![0]\!] = 0 \qquad\qquad [\![\oplus_A]\!] = \varepsilon\oplus_A$$

where $\oplus_A$ is defined as in Lemma 4.1.4. As explained above in Example 4.1.6, since every category with finite biproducts is a Cartesian left additive category, it follows that every differential category with finite biproducts is a Cartesian left additive category. And therefore, the coKleisli category of a differential category with finite biproducts is a Cartesian left additive category also. Lastly, one then uses the deriving transformation to define the differential combinator of the coKleisli category.

Proposition 4.1.12 *[9, Proposition 3.2.1] Let $\mathbb{X}$ be a differential category with coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite (bi)products (which we denote here using the product notation). Then the coKleisli category $\mathbb{X}_!$ is a Cartesian differential category with Cartesian left additive structure defined above and differential combinator D defined as follows on a coKleisli map $[\![f]\!] : !A \longrightarrow B$:*

$$[\![\mathsf{D}[f]]\!] := \; !(A \times A) \xrightarrow{\chi} !A \otimes !A \xrightarrow{1 \otimes \varepsilon} !A \otimes A \xrightarrow{\mathsf{d}} !A \xrightarrow{[\![f]\!]} B$$

where $\chi : !(A \times B) \longrightarrow !A \otimes !B$ is defined as in Definition 2.2.7.

It is important to note that the above proposition does not require the coalgebra modality to be monoidal (Definition 2.2.3) or, equivalently, to have Seely isomorphisms (Definition 2.2.7). If that is the case, then the deriving transformation is induced by a codereliction $\eta : A \longrightarrow !A$ (Definition 3.2.1). As such, for a coKlesili map $f : !A \longrightarrow B$, its derivative $\mathsf{D}[f] : !(A \times A) \longrightarrow B$ could also be expressed as follows:

$$\mathsf{D}[f] := \; !(A \times A) \xrightarrow{\chi_{A,A}} !A \otimes !A \xrightarrow{1 \otimes \varepsilon_A} !A \otimes A \xrightarrow{1 \otimes \eta_A} !A \otimes !A \xrightarrow{\nabla_A} !A \xrightarrow{f} B$$

Here are now some examples of coKleisli categories of differential categories.

Example 4.1.13 The category of convenient vector spaces and smooth functions between them is an example of a coKleisli category of a differential category [11, 70]. For a detailed introduction to convenient vector spaces, see [55]. Briefly, recall that a locally convex space E is a topological $\mathbb{R}$-vector space which is Hausdorff and such that 0 has a neighbourhood basis of convex sets, and therefore we have a notion of converging limits. A curve of E is a function $\phi : \mathbb{R} \longrightarrow E$ and we say that a curve ϕ is differentiable if the limit:

$$\psi(x) := \lim_{t \longrightarrow 0} \frac{\phi(x+t) - \phi(x)}{t}$$

exists for all $x \in E$, and this defines a curve $\psi : \mathbb{R} \longrightarrow E$ which is called the derivative of ϕ. A curve ϕ is smooth if all its iterated derivatives exists, i.e, if it is infinitely differentiable. A convenient vector space [55, Theorem 2.14] is a locally convex space E such that for every smooth curve ϕ there exists a smooth curve $\tilde{\phi}$ such that $\tilde{\phi}' = \phi$. Alternatively, a convenient vector space is a locally convex vector space which c^∞-complete (which is called Mackey complete in [11, Definition 3.15]) If E and F are both convenient vector spaces, then a smooth function $f : E \longrightarrow F$ is a function f which preserves smooth curves, that is, if ϕ is a smooth curve of E then ϕf is a smooth curve of F. Let CON be the category of convenient vector spaces and smooth functions between them. By [11, Theorem 6.3], CON is isomorphic to the coKleisli category of a comonad on CON_{lin}, the category of convenient vector spaces and smooth *linear* functions (i.e. smooth functions which are also $\mathbb{R}$-linear). Furthermore, CON_{lin} is a differential category [11, Theorem 6.6] and therefore CON is a Cartesian differential category (see [70, Example 2.4.2] for full details). For a smooth function $f : E \longrightarrow F$, its derivative $\mathsf{D}[f] : E \times E \longrightarrow F$ is defined as follows:

$$\mathsf{D}[f](x, y) := \lim_{t \longrightarrow 0} \frac{f(x + t \cdot y) - f(x)}{t}$$

where $t \in \mathbb{R}$ and $\cdot$ is scalar multiplication. Lastly, note that SMOOTH is also a sub-Cartesian differential category of CON.

Example 4.1.14 For the differential category REL with the finite multiset coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ from Example 3.1.2, its coKleisli category $\mathsf{REL}_!$ is a Cartesian differential category which is studied in detail in [14, Section 5.1].

Example 4.1.15 Let $\mathbb{K}$ be a field. Then for the differential category $\mathsf{VEC}^{op}_{\mathbb{K}}$ with the symmetric algebra coalgebra modality $(\mathsf{Sym}, \mu, \eta, \nabla, u)$ from Example 3.1.4, its coKleisli category $\left(\mathsf{VEC}^{op}_{\mathbb{K}}\right)_{\mathsf{Sym}}$ is a Cartesian differential category where a coKleisli map $\mathsf{Sym}(V) \longrightarrow W$ should be thought of a polynomial map from V to W. In particular, $\mathsf{POLY}_{\mathbb{K}}$ is isomorphic to the full sub-Cartesian differential category of $\left(\mathsf{VEC}^{op}_{\mathbb{K}}\right)_{\mathsf{Sym}}$ where one only considers the $\mathbb{K}$-vector spaces of the form $\mathbb{K}^n$.

Example 4.1.16 For the differential category $\mathsf{VEC}^{op}_{\mathbb{R}}$ with the C^∞-ring coalgebra modality S^∞ from Example 3.1.7, its coKleisli category $\left(\mathsf{VEC}^{op}_{\mathbb{R}}\right)_{\mathsf{S}^\infty}$ is a Cartesian differential category where a coKleisli map $\mathsf{S}^\infty(V) \longrightarrow W$ should be thought of a smooth function from V to W. In particular, SMOOTH is isomorphic to the full sub-Cartesian differential category of $\left(\mathsf{VEC}^{op}_{\mathbb{R}}\right)_{\mathsf{S}^\infty}$ where one only considers the $\mathbb{R}$-vector spaces of the form $\mathbb{R}^n$.

Example 4.1.17 Let $\mathbb{K}$ be any arbitrary field. Then for the differential category $\mathsf{VEC}_{\mathbb{K}}$ with the coalgebra modality Q from Example 3.1.8, as was shown by Garner and the author, its coKleisli category $(\mathsf{VEC}_{\mathbb{K}})_{\mathsf{Q}}$ is a Cartesian differential category which is the cofree Cartesian differential $\mathbb{K}$-linear category over the Cartesian left additive category whose objects are $\mathbb{K}$-vector spaces and whose maps are arbitrary set functions between them [41, Proposition 4.14]. In the case that $\mathbb{K}$ is an algebraically closed field of characteristic 0, Clift and Murfet also studied this coKleisli category in [17].

It is worth mentioning that in [10], Blute, Cockett, and Seely characterized precisely the Cartesian differential categories which where the coKleisli category of differential categories with Seely isomorphisms. Such Cartesian differential categories are called **Cartesian differential storage categories**. In fact, as shown by Garner and the author, it is always possible to embed a Cartesian differential category into the coKleisli category of a differential category with Seely isomorphisms [41, Theorem 8.7].

We now turn our attention to discussing an important class of maps in a Cartesian differential category: linear maps.

Definition 4.1.18 *In a Cartesian differential category $\mathbb{X}$ with differential combinator D, a map f is said to be **linear** [9, Definition 2.2.1] if $\mathsf{D}[f] = \pi_1 f$.*

In term logic notation, a linear map is a map such that:

$$\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b = f(b)$$

When we need to emphasize the differential sense in which a map is linear we shall say that the map is D-linear.

Lemma 4.1.19 *[9, Lemma 2.2.2] In a Cartesian differential category $\mathbb{X}$ with differential combinator D,*

(i) *If f is linear then f is additive;*

(ii) *If f is linear then for every map g which is post-composable with f, $\mathsf{D}[fg] = (f \times f)\mathsf{D}[g]$;*

(iii) *If g is linear then for every map f which is pre-composable with g, $\mathsf{D}[fg] = \mathsf{D}[f]g$.*

(iv) *Identity maps are linear;*

(v) *Zero maps are linear;*

(vi) *Projection maps π_0 and π_1 are linear;*

(vii) *If f and g are linear and composable, then their composition fg is linear;*

(viii) *If f and g are linear and pairable, then their pairing $\langle f, g\rangle$ is linear;*

(ix) *If f and g are linear, then their product $f \times g$ is linear;*

(x) *If f and g are linear and summable, then their sum $f + g$ is linear;*

(xi) *If f is a retract and linear, and if for a map g which is post-composable with f their composite fg is linear, then g is linear;*

(xii) *If f is linear and an isomorphism, then its inverse f^{-1} is also linear.*

Corollary 4.1.20 *In a Cartesian differential category $\mathbb{X}$ with differential combinator D,*

(i) *The symmetry isomorphism $\tau : A \times B \longrightarrow B \times A$ is linear;*

(ii) *The interchange isomorphism $c : (A \times B) \times (C \times D) \longrightarrow (A \times C) \times (B \times D)$ is linear;*

(iii) *The lifting map $\ell : A \times D \longrightarrow (A \times B) \times (C \times D)$ is linear;*

(iv) *The sum map $\oplus_A : A \times A \longrightarrow A$ is linear.*

For a Cartesian differential category $\mathbb{X}$, define its subcategory of linear maps $\mathsf{LIN}[\mathbb{X}]$ to be the category whose objects are the same as $\mathbb{X}$ and whose maps are linear in $\mathbb{X}$. Lemma 4.1.19 tells us that $\mathsf{LIN}[\mathbb{X}]$ is a well-defined category and also that it has finite biproducts [9, Corollary 2.2.3], and thus is a Cartesian left additive category where every map is additive. $\mathsf{LIN}[\mathbb{X}]$ also inherits the differential combinator from $\mathbb{X}$ and so $\mathsf{LIN}[\mathbb{X}]$ is a Cartesian differential category where every map is linear. Therefore the obvious forgetful functor $\mathsf{U} : \mathsf{LIN}[\mathbb{X}] \longrightarrow \mathbb{X}$ preserves the Cartesian differential structure strictly. It is important to note that although additive and linear maps often coincide in the examples, it is important to recall that, in general, while every linear map is additive, not every additive map is necessarily linear.

A key observation for this paper is that f is linear if and only it $\langle 0, 1\rangle\mathsf{D}[f]$ is linear. We shall use this fact to construct the *linearizing combinator* of a Cartesian differential category (see Proposition 4.2.6).

Lemma 4.1.21 *[29, Lemma 2.8] In a Cartesian differential category,*

(i) *For any map f, $\langle 0,1\rangle \mathsf{D}[f]$ is linear.*

(ii) *f is linear if and only if $f = \langle 0,1\rangle \mathsf{D}[f]$.*

PROOF: For (i), we must show that $\mathsf{D}\left[\langle 0,1\rangle \mathsf{D}[f]\right] = \pi_1 \langle 0,1\rangle \mathsf{D}[f]$. First note that by Lemma 4.1.19.(iv), (v), and (viii) it follows that $\langle 0,1\rangle$ is linear. Therefore, we compute that:

$$\begin{aligned}
\mathsf{D}\left[\langle 0,1\rangle \mathsf{D}[f]\right] &= (\langle 0,1\rangle \times \langle 0,1\rangle)\mathsf{D}[\mathsf{D}[f]] && (\langle 0,1\rangle \text{ is linear + Lem.4.1.19.(ii)}) \\
&= \langle \langle 0,\pi_0\rangle, \langle 0,\pi_1\rangle\rangle \mathsf{D}[\mathsf{D}[f]] \\
&= \langle \langle 0,0\rangle, \langle \pi_0,\pi_1\rangle\rangle \mathsf{D}[\mathsf{D}[f]] && \textbf{[CD.7]} \\
&= \langle \langle 0,0\rangle, \langle \pi_0,0\rangle + \langle 0,\pi_1\rangle\rangle \mathsf{D}[\mathsf{D}[f]] \\
&= \langle \langle 0,0\rangle, \langle \pi_0,0\rangle\rangle \mathsf{D}[\mathsf{D}[f]] + \langle \langle 0,0\rangle, \langle 0,\pi_1\rangle\rangle \mathsf{D}[\mathsf{D}[f]] && \textbf{[CD.2]} \\
&= \langle \langle 0,\pi_0\rangle, \langle 0,0\rangle\rangle \mathsf{D}[\mathsf{D}[f]] + \langle 0,\pi_1\rangle \mathsf{D}[f] && \textbf{[CD.7]} + \textbf{[CD.6]} \\
&= \langle \langle 0,\pi_0\rangle, 0\rangle \mathsf{D}[\mathsf{D}[f]] + \langle 0,\pi_1\rangle \mathsf{D}[f] \\
&= 0 + \langle 0,\pi_1\rangle \mathsf{D}[f] && \textbf{[CD.2]} \\
&= \pi_1 \langle 0,1\rangle \mathsf{D}[f]
\end{aligned}$$

So we conclude that $\langle 0,1\rangle \mathsf{D}[f]$ is linear. Now suppose that f is linear, then we compute:

$$\begin{aligned}
\langle 0,1\rangle \mathsf{D}[f] &= \langle 0,1\rangle \pi_1 f && (f \text{ is linear}) \\
&= f
\end{aligned}$$

So $f = \langle 0,1\rangle \mathsf{D}[f]$. Conversely, suppose that $f = \langle 0,1\rangle \mathsf{D}[f]$. By (i), $\langle 0,1\rangle \mathsf{D}[f]$ is linear and so f is also linear. □

We conclude this section by proving some examples of linear maps in our examples of Cartesian differential categories from above.

Example 4.1.22 For a category with finite biproducts seen as a Cartesian differential category as in Example 4.1.6, every map is linear by definition. Conversely, a Cartesian differential category where every map is linear is precisely a category with finite biproducts.

Example 4.1.23 For the Cartesian differential category $\mathsf{POLY}_{\mathbb{K}}$ from Example 4.1.7, a map $p(\vec{x}) : n \longrightarrow 1$, which is a polynomial $p(\vec{x}) \in \mathbb{K}[x_1, \dots, x_n]$, is linear in the Cartesian differential sense if and only if it is a sum of monomials of degree 1, that is, $p(\vec{x})$ is of the form:

$$p(\vec{x}) = \sum_{i=1}^{n} a_i x_i$$

A map $P : n \longrightarrow m$, which is a tuple $P = \langle p_1(\vec{x}), \dots, p_m(\vec{)}\rangle$, is linear if and only if each $p_i(\vec{x})$ is linear.

Example 4.1.24 For the Cartesian differential category SMOOTH from Example 4.1.8, a smooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear in the Cartesian differential sense precisely when it is $\mathbb{R}$-linear in the classical sense, that is, $F(s\vec{x} + t\vec{y}) = sF(\vec{x}) + tF(\vec{y})$ for all $s, t \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

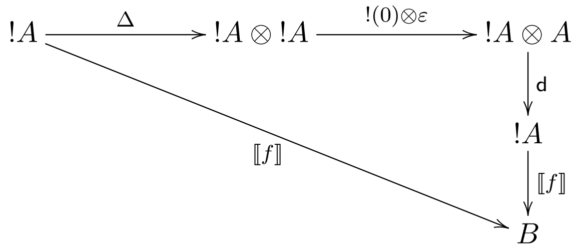
Example 4.1.25 For the Cartesian differential category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ from Example 4.1.10, a functor F is linear in the Cartesian differential sense if it linear in the abelian functor calculus sense, that is, if F preserves finite direct sums up to chain homotopy equivalence [5, Definition 5.5].

Example 4.1.26 For a Cartesian left additive category $\mathbb{X}$, in its cofree Cartesian differential category $\mathcal{D}(\mathbb{X})$ from Example 4.1.11, a D-sequence $(f_0, f_1, \ldots)$ is linear if and only if we have that:

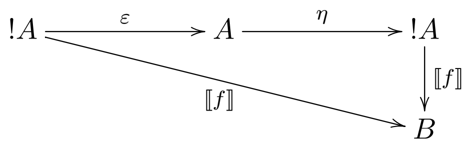
$$f_n = \underbrace{\pi_1 \ldots \pi_1}_{n-times} f_0$$

for all $n \in \mathbb{N}$ [59, Lemma 4.26].

Example 4.1.27 For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products, a coKleisli map $[\![f]\!] : !A \longrightarrow B$ is linear in the Cartesian differential sense if and only if the following diagram commutes:



In particular, for every map $g : A \longrightarrow B$ in $\mathbb{X}$, $\varepsilon_A g : !A \longrightarrow B$ is a linear map in the coKleisli category $\mathbb{X}_!$. If $(!, \delta, \varepsilon, \Delta, e)$ has the Seely isomorphisms, and so the deriving transformation is induced by a codereliction $\eta : A \longrightarrow !A$, a coKleisli map $[\![f]\!] : !A \longrightarrow B$ is linear if and only if the follow diagram commutes:



In other words, a coKleisli map is linear if and only if it is of the form εg for some map g of $\mathbb{X}$. As such, in the case of having Seely isomorphisms, $\mathsf{LIN}[\mathbb{X}_!] \cong \mathbb{X}$.

Example 4.1.28 For the Cartesian differential category CON from Example 4.1.13, a smooth function is linear in the Cartesian differential sense precisely when it is a (smooth) linear function in the ordinary sense of linear algebra.

4.2 Linearizing Combinators

In this section we introduce the notion of a *linearizing combinator* for a Cartesian left additive category. Linearizing combinators are generalizations of the linearization operation used in the abelian functor calculus [5, Definition 5.1]. In fact, we will show that every Cartesian differential category comes equipped with a canonical linearizing combinator. The basic idea is that a linearizing combinator produces the linear approximation of maps.

Definition 4.2.1 *A* ***linearizing combinator*** *[29, Definition 3.1]* L *on a Cartesian left additive category* $\mathbb{X}$ *is a family of operators, for each* $A, B \in \mathbb{X}$

$$\mathsf{L} : \mathbb{X}(A,B) \longrightarrow \mathbb{X}(A,B); f \mapsto \mathsf{L}[f]$$

such that the following six axioms hold:

[L.1] $\mathsf{L}[f+g] = \mathsf{L}[f] + \mathsf{L}[g]$ *and* $\mathsf{L}[0] = 0$

[L.2] $\mathsf{L}[f]$ *is additive, or equivalently by Lemma 4.1.4.(iii),* $\oplus_A \mathsf{L}[f] = \pi_0 \mathsf{L}[f] + \pi_1 \mathsf{L}[f]$ *and* $0\mathsf{L}[f] = 0$*;*

[L.3] $\mathsf{L}[1] = 1$*,* $\mathsf{L}[\pi_0] = \pi_0$*, and* $\mathsf{L}[\pi_1] = \pi_1$

[L.4] $\mathsf{L}\left[\langle f, g\rangle\right] = \left\langle \mathsf{L}[f], \mathsf{L}[g]\right\rangle$

[L.5] $\mathsf{L}[fg] = \mathsf{L}[f]\ \mathsf{L}\left[(1+0f)g\right]$

[L.6] $\mathsf{L}\left[\mathsf{L}[f]\right] = \mathsf{L}[f]$

The expression $\mathsf{L}[f]$ *is called the* ***linearization*** *of* f*.*

The basic intuition of a linearizing combinator L is that from an arbitrary map f, L produces a linear map $\mathsf{L}[f]$. Examples of linearizing combinators can be found at the end of this section. The motivating example of a linearizing combinator is the linearization operator from abelian functor calculus [5, Definition 5.1]. The main source of examples of linearizing combinators come from Cartesian differential categories, as we will see in Proposition 4.2.6 below, where the linearizing combinator is defined as the differential combinator evaluated at zero in the first argument. Indeed as explained in Lemma 4.1.21(i), for every map f, the composite $\langle 0, 1\rangle \mathsf{D}[f]$ is a linear map. As a simple example, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function, then $\mathsf{L}[f] : \mathbb{R} \longrightarrow \mathbb{R}$ is the $\mathbb{R}$-linear map defined as the degree 1 term of the Taylor expansion of f, that is, $\mathsf{L}[f](x) = f'(0)x$.

It may be useful for intuition to use notation similar to the term logic for Cartesian differential categories. Let us denote the linearizing combinator as follows:

$$\mathsf{L}[f](a) := \frac{\ell f(x)}{\ell x} \cdot a$$

Using this notation, the seven differential combinator axioms are expressed as:

[L.1] $\frac{\ell f(x)+g(x)}{\ell x} \cdot a = \frac{\ell f(x)}{\ell x} \cdot a + \frac{\ell g(x)}{\ell x} \cdot a$ and $\frac{\ell 0}{\ell x} \cdot a = 0$

[L.2] $\frac{\ell f(x)}{\ell x} \cdot (a+b) = \frac{\ell f(x)}{\ell x} \cdot a + \frac{\ell f(x)}{\ell x} \cdot b$ and $\frac{\ell f(x)}{\ell x} \cdot 0 = 0$

[L.3] $\frac{\ell x}{\ell x} \cdot a = a$ and $\frac{\ell \pi_i(x_0,x_1)}{\ell (x_0,x_1)} \cdot (a_0, a_1) = \pi_i(a_0, a_1) = a_i$

[L.4] $\frac{\ell \langle f(x), g(x)\rangle}{\ell x} \cdot a = \left\langle \frac{\ell f(x)}{\ell x} \cdot a, \frac{\ell g(x)}{\ell x} \cdot a \right\rangle$

[L.5] $\frac{\ell g\big(f(x)\big)}{\ell x} \cdot a = \frac{\ell g(f(0)+y)}{\ell y} \cdot \left(\frac{\ell f(x)}{\ell x} \cdot a\right)$

[L.6] $\frac{\ell \frac{\ell f(x)}{\ell x} \cdot y}{\ell y} \cdot a = \frac{\ell f(x)}{\ell x} \cdot a$

The axioms of a linearizing combinator are analogues of the first six axioms of a differential combinator. **[L.1]** says that the linearization of a sum of maps is equal to the sum of linearization of maps. **[L.2]** says the linearization of a map is additive. **[L.3]** tells us that identity maps and projection maps are already linearized. **[L.4]** says that the linearization of a pairing of maps is same as the pairing of the linearization of maps. **[L.5]** tells how to linearize a composite of maps, ie., the chain rule for linearization. And lastly, **[L.6]** says that the linearizing combinator is idempotent, that is, since a linearization of a map is already linearized, to apply the linearization combinator twice is the same as doing it once. These axioms can also be found throughout the BJORT paper [5]. Indeed, **[L.1]** is [5, Lemma 5.6.ii], **[L.2]** is [5, Lemma 5.6.i], **[L.3]** is [5, Lemma 5.16], **[L.4]** is [5, Lemma 5.18], and **[L.5]** is a generalization of [5, Propostion 5.10].

The keen-eyed reader may have noticed that on the right hand side of **[L.5]**, $\mathsf{L}\left[(1+0f)g\right]$ is a linearization of a composite of maps. In theory one could again apply **[L.5]** to $\mathsf{L}\left[(1+0f)g\right]$. However, the following calculation shows us that doing so does not result in any simplification:

$$\begin{aligned}
\mathsf{L}\left[(1+0f)g\right] &= \mathsf{L}[1+0f]\mathsf{L}\left[\big(1+0(1+0f)\big)\, g\right] && \textbf{[L.5]}\\
&= \mathsf{L}[1+0f]\mathsf{L}\left[(1+0+0f)\, g\right]\\
&= \mathsf{L}[1+0f]\mathsf{L}\left[(1+0f)g\right]\\
&= \big(\mathsf{L}[1]+\mathsf{L}[0f]\big)\,\mathsf{L}\left[(1+0f)g\right] && \textbf{[L.1]}\\
&= \Big(1+\mathsf{L}[0]\mathsf{L}\left[(1+0)f\right]\Big)\,\mathsf{L}\left[(1+0f)g\right] && \textbf{[L.3]} + \textbf{[L.5]}\\
&= \big(1+0\mathsf{L}\left[f\right]\big)\,\mathsf{L}\left[(1+0f)g\right] && \textbf{[L.1]}\\
&= (1+0)\,\mathsf{L}\left[(1+0f)g\right] && \textbf{[L.2]}\\
&= \mathsf{L}\left[(1+0f)g\right]
\end{aligned}$$

So **[L.5]** is indeed simplified as far as possible. That said, **[L.5]** does simplify when the maps are either reduced, semi-additive, or additive.

Lemma 4.2.2 *[29, Lemma 3.2] Let L be a linearizing combinator on a Cartesian left additive category $\mathbb{X}$.*

(i) *If $f : A \longrightarrow B$ is constant then $\mathsf{L}[f] = 0$.*

(ii) *For a reduced map f and any map g, $\mathsf{L}[fg] = \mathsf{L}[f]\ \mathsf{L}[g]$.*

(iii) *For a semi-additive map g and any map f, $\mathsf{L}[fg] = \mathsf{L}[f]\ \mathsf{L}[g]$.*

Proof: These are mostly straightforward calculations.

(i) Suppose that f is constant, that is, $0f = f$. Then we have that:

$$\begin{aligned} \mathsf{L}[f] = & \ \mathsf{L}[0f] && (f \text{ constant}) \\ = & \ \mathsf{L}[0]\ \mathsf{L}\left[(1+00)f\right] && \textbf{[L.5]} \\ = & \ 0\mathsf{L}\left[f\right] && \textbf{[L.1]} \\ = & \ 0 && \textbf{[L.2]} \end{aligned}$$

(ii) Suppose that f is reduced, that is, $0f = 0$. Then we have that:

$$\begin{aligned} \mathsf{L}[fg] = & \ \mathsf{L}[f]\ \mathsf{L}\left[(1+0f)g\right] && \textbf{[L.5]} \\ = & \ \mathsf{L}[f]\ \mathsf{L}\left[(1+0)g\right] && (f \text{ reduced}) \\ = & \ \mathsf{L}[f]\ \mathsf{L}[g] \end{aligned}$$

(iii) Suppose that g is semi-additive, that is, $(f+k)g = fg + kg$. Then we have that:

$$\begin{aligned} \mathsf{L}[fg] = & \ \mathsf{L}[f]\ \mathsf{L}\left[(1+0f)g\right] && \textbf{[L.5]} \\ = & \ \mathsf{L}[f]\ \mathsf{L}\left[g+0fg\right] && (g \text{ semi-additive}) \\ = & \ \mathsf{L}[f]\ \left(\mathsf{L}[g] + \mathsf{L}\left[0fg\right]\right) && \textbf{[L.1]} \\ = & \ \mathsf{L}[f]\ \Big(\mathsf{L}[g] + \mathsf{L}[0]\mathsf{L}\left[(1+0)fg\right]\Big) && \textbf{[L.5]} \\ = & \ \mathsf{L}[f]\ \left(\mathsf{L}[g] + 0\mathsf{L}\left[fg\right]\right) && \textbf{[L.1]} \\ = & \ \mathsf{L}[f]\ \left(\mathsf{L}[g] + 0\right) && \textbf{[L.2]} \\ = & \ \mathsf{L}[f]\ \mathsf{L}[g] \end{aligned}$$

□

For a linearizing combinator, the analogues of linear maps are the maps for which the linearizing combinator does nothing, that is, $\mathsf{L}[f] = f$, or in the term logic:

$$\frac{\ell f(x)}{\ell x} \cdot a = f(a)$$

Definition 4.2.3 *In a Cartesian left additive category* $\mathbb{X}$ *with a linearizing combinator* L*, a map* f *is said to be* ***L-linear*** *[29, Definition 3.3] if* $\mathsf{L}[f] = f$*.*

As we will see in Proposition 4.2.6, in a Cartesian differential category the L-linear maps are precisely the linear maps. As such, L-linear satisfy many of same basic properties as linear maps.

Lemma 4.2.4 *[29, Lemma 3.4] In a Cartesian left additive category* $\mathbb{X}$ *with a linearizing combinator* L*,*

(i) *For every map* f*,* $\mathsf{L}[f]$ *is* L*-linear;*

(ii) *If* f *is* L*-linear then* f *is additive;*

(iii) *If f is L-linear then for every map g which is post-composable with f, $\mathsf{L}[fg] = f\mathsf{L}[g]$;*

(iv) *If g is L-linear then for every map f which is pre-composable with g, $\mathsf{L}[fg] = \mathsf{L}[f]g$.*

(v) *Identity maps are L-linear;*

(vi) *Zero maps are L-linear;*

(vii) *Projection maps π_0 and π_1 are L-linear;*

(viii) *If f and g are L-linear and composable, then their composition fg is L-linear;*

(ix) *If f and g are L-linear and pairable, then their pairing $\langle f, g\rangle$ is L-linear;*

(x) *If f and g are L-linear, then their product $f \times g$ is L-linear;*

(xi) *If f and g are L-linear and summable, then their sum $f + g : A \longrightarrow B$ is L-linear;*

(xii) *If f is a retract and L-linear, and if for a map g which is post-composable with f their composite fg is L-linear, then g is L-linear.*

(xiii) *If f is L-linear and an isomorphism, then its inverse f^{-1} is also L-linear.*

Proof: Most of these follow directly from the axioms of a linearizing combinator. (i) follows from **[L.6]**, (*ii*) follows from **[L.2]**, (*v*) and (*vii*) follow from **[L.3]**, (*vi*) and (*xi*) follow from **[L.1]**, (vii) follows from **[L.4]**. For the rest, we mostly use **[L.5]** and Lemma 4.2.2.

(iii): Suppose that f is L-linear. By (ii), f is additive and therefore reduced. Then:

$$\begin{aligned} \mathsf{L}[fg] &= \mathsf{L}[f]\ \mathsf{L}[g] && \text{(Lemma 4.2.2.(ii))} \\ &= f\mathsf{L}[g] && (f \text{ is } \mathsf{L}\text{-linear}) \end{aligned}$$

(iv): Suppose that g is L-linear. By (ii), g is additive and therefore semi-additive. Then:

$$\begin{aligned} \mathsf{L}[fg] &= \mathsf{L}[f]\ \mathsf{L}[g] && \text{(Lemma 4.2.2.(iii))} \\ &= \mathsf{L}[f]g && (g \text{ is } \mathsf{L}\text{-linear}) \end{aligned}$$

(viii): Suppose that f and g are L-linear and composable. Then we have that:

$$\begin{aligned} \mathsf{L}[fg] &= f\mathsf{L}[g] && (f \text{ is } \mathsf{L}\text{-linear} + \text{(iii)}) \\ &= fg && (g \text{ is } \mathsf{L}\text{-linear}) \end{aligned}$$

So fg is L-linear.

(x): Suppose that f and g are L-linear. By (vii) and (viii), $\pi_0 f$ and $\pi_1 g$ are also L-linear. Then by (ix), the pairing of $\pi_0 f$ and $\pi_1 g$ is also L-linear, that is, $f \times g = \langle \pi_0 f, \pi_1 g\rangle$ is L-linear.

(xii): Suppose that f is a retract (with section f°) and L-linear, and that fg is L-linear. Then:

$$\mathsf{L}[g] = f^\circ f\mathsf{L}[g] \qquad (f \text{ is a retract of } f^\circ)$$

$$
\begin{aligned}
&= f^\circ\, \mathsf{L}[fg] && (f \text{ is } \mathsf{L}\text{-linear} + (\text{iii})) \\
&= f^\circ fg && (fg \text{ is } \mathsf{L}\text{-linear}) \\
&= g && (f \text{ is a retract of } f^\circ)
\end{aligned}
$$

So g is L-linear.

(xiii): Suppose that f is L-linear and an isomorphism. By (viii), the composite $ff^{-1} = 1$ is L-linear. Since f is a retract, by (xii) we have that f^{-1} is also L-linear. □

Once again, it is important to note that while every L-linear map is additive, not every additive map is necessarily L-linear. That said, since every L-linear map is additive, the subcategory of L-linear maps form a category with finite biproducts.

Lemma 4.2.5 *[29, Lemma 3.5] For a Cartesian left additive category $\mathbb{X}$ with a linearizing combinator L, define $\mathbb{X}^{\mathsf{L}}$ as the category of L-linear maps of $\mathbb{X}$. Then $\mathbb{X}^{\mathsf{L}}$ is a category with finite biproducts. Furthermore, for every map f in $\mathbb{X}$, $\mathsf{L}[f]$ is a map in $\mathbb{X}^{\mathsf{L}}$.*

Proof: That composition and identity maps in $\mathbb{X}^{\mathsf{L}}$ are well-defined follows from Lemma 4.2.4.(v) and (viii). That $\mathbb{X}^{\mathsf{L}}$ has finite products follows from Lemma 4.2.4.(vii) and (ix). That $\mathbb{X}^{\mathsf{L}}$ is a Cartesian left additive category follows from Lemma 4.2.4.(vi) and (xi). Note that a Cartesian left additive category where every map is additive is precisely a category with finite biproducts. By Lemma 4.2.4.(ii), it follows that every map in $\mathbb{X}^{\mathsf{L}}$ is additive. So $\mathbb{X}^{\mathsf{L}}$ is a category with finite biproducts. Lastly, by Lemma 4.2.4.(i), for every map f in $\mathbb{X}$, $\mathsf{L}[f]$ is a map in $\mathbb{X}^{\mathsf{L}}$. □

Note that in general, the linearizing combinator does not induce a functor from $\mathbb{X}$ to $\mathbb{X}^{\mathsf{L}}$. However by Lemma 4.2.2.(ii) and (iii), the linearizing combinator does induce a functor from the subcategories of reduced maps, semi-additive maps, and additive maps to $\mathbb{X}^{\mathsf{L}}$.

We now show that every Cartesian differential category comes equipped with a canonical linearizing combinator. Consider again the example of a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$, where $\mathsf{L}[f](x) = f'(0)x$. Recall that $\mathsf{D}[f](x, y) = f'(x)y$. Therefore, $\mathsf{L}[f](x) = \mathsf{D}[f](0, x)$. This construction generalizes to arbitrary Cartesian differential categories. In term logic notation,

$$\frac{\ell f(x)}{\ell x} \cdot a = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(0) \cdot b$$

Furthermore, maps which are linear in the Cartesian differential category sense, that is D-linear, are precisely those for which $\mathsf{L}[f] = f$, that is those which are L-linear.

Proposition 4.2.6 *[29, Proposition 3.6] Every Cartesian differential category, with differential combinator D, admits a linearizing combinator L_{D} defined as follows for every map $f : A \longrightarrow B$:*

$$\mathsf{L}_{\mathsf{D}}[f] := \langle 0, 1\rangle \mathsf{D}[f] \tag{4.4}$$

Furthermore,

(i) *For every map f, $\mathsf{L}_{\mathsf{D}}[f]$ is D-linear;*

(ii) *A map f is* D*-linear if and only if f is* L_{D}*-linear.*

Proof: First note that (i) and (ii) are precisely a reformulation of Lemma 4.1.21 in terms of L_{D}. Using this, we will now prove that L_{D} satisfies **[L.1]** to **[L.6]**. Each of the linearizing combinator axioms will follow mostly from the differential combinator axiom of the same number.

[L.1]: $\mathsf{L}_{\mathsf{D}}[f+g] = \mathsf{L}_{\mathsf{D}}[f] + \mathsf{L}_{\mathsf{D}}[g]$ and $\mathsf{L}_{\mathsf{D}}[0] = 0$

$$\begin{aligned} \mathsf{L}_{\mathsf{D}}[f+g] &= \langle 0,1\rangle \mathsf{D}[f+g] \\ &= \langle 0,1\rangle \big(\mathsf{D}[f] + \mathsf{D}[g]\big) && \textbf{[CD.1]} \\ &= \langle 0,1\rangle \mathsf{D}[f] + \langle 0,1\rangle \mathsf{D}[g] \\ &= \mathsf{L}_{\mathsf{D}}[f] + \mathsf{L}_{\mathsf{D}}[g] \end{aligned}$$

$$\begin{aligned} \mathsf{L}_{\mathsf{D}}[0] &= \langle 0,1\rangle \mathsf{D}[0] \\ &= \langle 0,1\rangle 0 && \textbf{[CD.1]} \\ &= 0 \end{aligned}$$

[L.2]: $\oplus_A \mathsf{L}_{\mathsf{D}}[f] = \pi_0 \mathsf{L}_{\mathsf{D}}[f] + \pi_1 \mathsf{L}_{\mathsf{D}}[f]$ and $0\mathsf{L}_{\mathsf{D}}[f] = 0$:

By Proposition 4.2.6.(i), $\mathsf{L}_{\mathsf{D}}[f]$ is D-linear and therefore by Lemma 4.1.19.(i), $\mathsf{L}_{\mathsf{D}}[f]$ is also additive, i.e., $\oplus_A \mathsf{L}_{\mathsf{D}}[f] = \pi_0 \mathsf{L}_{\mathsf{D}}[f] + \pi_1 \mathsf{L}_{\mathsf{D}}[f]$ and $0\mathsf{L}_{\mathsf{D}}[f] = 0$.

[L.3]: $\mathsf{L}_{\mathsf{D}}[1] = 1$ and $\mathsf{L}_{\mathsf{D}}[\pi_i] = \pi_i$:

By Lemma 4.1.19.(iv) and (vi), identity maps and projection maps are D-linear. Therefore by Proposition 4.2.6.(ii), identity maps and projection maps are also L_{D}-linear, i.e., $\mathsf{L}_{\mathsf{D}}[1] = 1$ and $\mathsf{L}_{\mathsf{D}}[\pi_i] = \pi_i$.

[L.4]: $\mathsf{L}_{\mathsf{D}}\big[\langle f,g\rangle\big] = \big\langle \mathsf{L}_{\mathsf{D}}[f], \mathsf{L}_{\mathsf{D}}[g]\big\rangle$

$$\begin{aligned} \mathsf{L}_{\mathsf{D}}\big[\langle f,g\rangle\big] &= \langle 0,1\rangle \mathsf{D}\big[\langle f,g\rangle\big] \\ &= \langle 0,1\rangle \big\langle \mathsf{D}[f], \mathsf{D}[g]\big\rangle && \textbf{[CD.4]} \\ &= \big\langle \langle 0,1\rangle \mathsf{D}[f], \langle 0,1\rangle \mathsf{D}[g]\big\rangle \\ &= \big\langle \mathsf{L}_{\mathsf{D}}[f], \mathsf{L}_{\mathsf{D}}[g]\big\rangle \end{aligned}$$

[L.5]: $\mathsf{L}_{\mathsf{D}}[fg] = \mathsf{L}_{\mathsf{D}}[f]\ \mathsf{L}_{\mathsf{D}}\big[(1+0f)g\big]$

$$\begin{aligned} \mathsf{L}_{\mathsf{D}}[f]\ \mathsf{L}_{\mathsf{D}}\big[(1+0f)g\big] &= \mathsf{L}_{\mathsf{D}}[f]\langle 0,1\rangle \mathsf{D}\big[(1+0f)g\big] \\ &= \mathsf{L}_{\mathsf{D}}[f]\langle 0,1\rangle \big\langle \pi_0(1+0f), \mathsf{D}[1+0f]\big\rangle \mathsf{D}[g] && \textbf{[CD.5]} \\ &= \mathsf{L}_{\mathsf{D}}[f]\langle 0,1\rangle \big\langle \pi_0 + \pi_0 0f, \mathsf{D}[1] + \mathsf{D}[0f]\big\rangle \mathsf{D}[g] && \textbf{[CD.1]} \\ &= \mathsf{L}_{\mathsf{D}}[f]\langle 0,1\rangle \big\langle \pi_0 + 0f, \pi_1 + \langle \pi_0 0, \mathsf{D}[0]\rangle \mathsf{D}[f]\big\rangle \mathsf{D}[g] && \textbf{[CD.3]} + \textbf{[CD.5]} \\ &= \mathsf{L}_{\mathsf{D}}[f]\langle 0,1\rangle \big\langle \pi_0 + 0f, \pi_1 + \langle 0,0\rangle \mathsf{D}[f]\big\rangle \mathsf{D}[g] && \textbf{[CD.1]} \end{aligned}$$

$$
\begin{aligned}
&= \mathsf{L_D}[f]\langle 0,1\rangle \langle \pi_0 + 0f, \pi_1 + 0\rangle\, \mathsf{D}[g] && \textbf{[CD.2]}\\
&= \mathsf{L_D}[f] \left\langle \langle 0,1\rangle(\pi_0 + 0f), \langle 0,1\rangle \pi_1 \right\rangle \mathsf{D}[g] \\
&= \mathsf{L_D}[f] \left\langle \langle 0,1\rangle\pi_0 + \langle\langle 0,1\rangle 0f, \langle 0,1\rangle \pi_1 \right\rangle \mathsf{D}[g] \\
&= \mathsf{L_D}[f]\, \langle 0 + 0f, 1\rangle\, \mathsf{D}[g] \\
&= \mathsf{L_D}[f]\, \langle 0f, 1\rangle\, \mathsf{D}[g] \\
&= \left\langle \mathsf{L_D}[f]0f, \mathsf{L_D}[f] \right\rangle \mathsf{D}[g] \\
&= \left\langle 0f, \langle 0,1\rangle \mathsf{D}[f] \right\rangle \mathsf{D}[g] \\
&= \left\langle \langle 0,1\rangle \pi_0 f, \langle 0,1\rangle \mathsf{D}[f] \right\rangle \mathsf{D}[g] \\
&= \langle 0,1\rangle \left\langle \pi_0 f, \mathsf{D}[f] \right\rangle \mathsf{D}[g] \\
&= \langle 0,1\rangle \mathsf{D}[fg] && \textbf{[CD.5]}\\
&= \mathsf{L_D}[fg]
\end{aligned}
$$

[L.6]: $\mathsf{L_D}\left[\mathsf{L_D}[f]\right] = \mathsf{L_D}[f]$:

By Proposition 4.2.6.(i), $\mathsf{L_D}[f]$ is D-linear. Therefore by Proposition 4.2.6.(ii), it follows that $\mathsf{L_D}[f]$ is also $\mathsf{L_D}$-linear which means that $\mathsf{L_D}\left[\mathsf{L_D}[f]\right] = \mathsf{L_D}[f]$.

So we conclude that $\mathsf{L_D}$ is a linearizing combinator. □

We conclude this section by providing examples of linearizing combinators by applying Proposition 4.2.6 to some of the examples of Cartesian differential categories from Section 4.1.

Example 4.2.7 For a category with finite biproducts seen as a Cartesian differential category as in Example 4.1.6, the linearizing combinator is simply the identity combinator:

$$\mathsf{L_D}[f] = \langle 0,1\rangle \mathsf{D}[f] = \langle 0,1\rangle \pi_1 f = f$$

This make sense since every map, in this example, is already D-linear by definition as explained in Example 4.1.22.

Example 4.2.8 For the Cartesian differential category $\mathsf{POLY}_{\mathbb{K}}$ from Example 4.1.7, let us first work out the the linearizing combinator for maps $n \longrightarrow 1$. Of course, a map $p(\vec{x}) : n \longrightarrow 1$ is a polynomial $p(\vec{x}) \in \mathbb{K}[x_1, \ldots, x_n]$, which can be written out as follows:

$$p(\vec{x}) = \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} a_{(k_1,\ldots,k_n)} x_1^{k_1} \ldots x_n^{k_n}$$

with only finitely many $a_{(k_1,\ldots,k_n)} \neq 0$. Then the linearizing combinator picks out the degree 1 terms of the polynomial, that is, $\mathsf{L}[p(\vec{x})] : n \longrightarrow 1$ is the polynomial:

$$\mathsf{L}[p(\vec{x})] = \sum a_{(0,0,\ldots,0,1,0,\ldots,0)} x_i$$

This can be described as follows:

$$\mathsf{L}[p(\vec{x})] = \sum_{i=1}^{n} \frac{\partial p(\vec{x})}{\partial x_i}(0) x_i$$

Note that this precisely the same as $\varepsilon(p(\vec{x}))$ from Example 3.2.4. More generally, for a map $P : n \to m$, which is a tuple $P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle$, $\mathsf{L}[P] = \langle \mathsf{L}[p_1(\vec{x})], \ldots, \mathsf{L}[p_m(\vec{x})] \rangle$. For example, consider the polynomial $p(x, y, z) = x^2y + 3x + z + 1$, then $\mathsf{L}[p(x, y, z)]$ picks out all monomials of degree 1 in $p(x, y, z)$, so $\mathsf{L}[p(x, y, z)] = 3x + z$.

Example 4.2.9 For the Cartesian differential category SMOOTH from Example 4.1.8, the linearizing combinator is defined as evaluating the directional derivative at zero in the first argument. Explicitly, for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, which recall is a tuple of smooth functions $F = \langle f_1, \ldots, f_n \rangle$:

$$\mathsf{L}[F](\vec{x}) = \nabla(F)(\vec{0}) \cdot \vec{x} = \left\langle \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0})x_i, \ldots, \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0})x_i \right\rangle$$

where recall that $\nabla(F)$ is the Jacobian matrix. In particular, for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathsf{L}[f](\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0})x_i$$

which is precisely $\varepsilon(f)$ from Example 3.2.7. For example, consider $g(x, y) = e^x \cos(y)$. Its derivative is worked out to be $\mathsf{D}[g](x, y, z, w) = e^x \cos(y)z - e^x \sin(y)w$. Then evaluating at 0 in the first two arguments, we obtain that $\mathsf{L}[g](x, y) = e^0 \cos(0)x - e^0 \sin(0)y = x$.

Example 4.2.10 For the Cartesian differential category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ from Example 4.1.10, the linearizing combinator is precisely the linearization operator D_1 as defined in [5, Definition 5.1], which in turn is defined using cross effects [5, Definition 2.1].

Example 4.2.11 For a Cartesian left additive category $\mathbb{X}$, in its cofree Cartesian differential category $\mathcal{D}(\mathbb{X})$ from Example 4.1.11, its linearizing combinator is defined as follows for a D-sequence $(f_0, f_1, f_2, \ldots)$:

$$\mathsf{L}\left[(f_0, f_1, f_2, \ldots)\right] = (\langle 0, 1 \rangle f_1, \mathsf{P}(\langle 0, 1 \rangle)f_2, \mathsf{P}^2(\langle 0, 1 \rangle)f_3, \ldots)$$

where recall that P is the product functor $\mathsf{P}(-) = - \times -$. However by [59, Lemma 4.26], or by the axioms of a D-sequence [59, Definition 4.2], it follows that $\mathsf{L}\left[(f_0, f_1, f_2, \ldots)\right]$ can be simplified to:

$$\mathsf{L}\left[(f_0, f_1, f_2, \ldots)\right] = (\langle 0, 1 \rangle f_1, \pi_1\langle 0, 1 \rangle f_1, \pi_1\pi_1\langle 0, 1 \rangle f_1, \ldots) = i_\bullet \cdot (\langle 0, 1 \rangle f_1)$$

where the notation $i_\bullet \cdot -$ was introduced in [59].

Example 4.2.12 For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ and deriving transformation $\mathsf{d} : !A \otimes A \to !A$, and finite products, the linearizing combinator for the coKleisli category $\mathbb{X}_!$ is defined as follows for a coKleisli map $[\![f]\!] : !A \to B$:

$$[\![\mathsf{L}[f]]\!] := \ !A \xrightarrow{\Delta_A} !A \otimes !A \xrightarrow{!(0) \otimes \varepsilon_A} !A \otimes A \xrightarrow{\mathsf{d}_A} !A \xrightarrow{[\![f]\!]} B$$

If $(!, \delta, \varepsilon, \Delta, e)$ has Seely isomorphisms, then the linearizing combinator can alternatively be expressed using the codereliction map:

$$[\![\mathsf{L}[f]]\!] := \ !A \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} !A \xrightarrow{[\![f]\!]} B$$

This makes precisely the interpretation that coderelictions should be thought of as linearizing operators, as explained in Section 3.2.

Example 4.2.13 For the Cartesian differential category CON from Example 4.1.13, the linearizing combinator is defined as follows on a smooth function f:

$$\mathsf{L}[f](x) := \lim_{t \longrightarrow 0} \frac{f(t \cdot x) - f(0)}{t}$$

4.3 Differentiation in Context

We would like to prove the converse of Proposition 4.2.6, that is, from a linearizing combinator we would like to construct a differential combinator following the same construction as in [5]. However, to do so requires the ability to partially linearize maps, that is, to linearize on certain variables while keeping others constant. This, equivalently, means we would like to able to linearize with respect to a fixed "context". From a categorical perspective, a map in a fixed "context" C is interpreted as a map in the **simple slice category** over C. Simple slice categories for a given category organize themselves into a fibration, called the simple fibration [51, Chapter 1].

Definition 4.3.1 *Let $\mathbb{X}$ be a category with finite products. For each object C, the* ***simple slice category*** *[51, Definition 1.3.1] over C is the category $\mathbb{X}[C]$ where:*

(i) *The objects are the objects of $\mathbb{X}$, $ob\left(\mathbb{X}[C]\right) := ob\left(\mathbb{X}\right)$;*

(ii) *The hom-sets are defined as $\mathbb{X}[C](A, B) := \mathbb{X}(C \times A, B)$;*

(iii) *The identity maps are the projection maps $\pi_1 : C \times A \longrightarrow A$;*

(iv) *The composition of maps $f : C \times A \longrightarrow B$ and $g : C \times B \longrightarrow D$ is the map $\langle \pi_0, f \rangle g : C \times A \longrightarrow D$.*

For each map $h : C' \longrightarrow C$ in $\mathbb{X}$, define the ***substitution functor*** *$h^* : \mathbb{X}[C] \longrightarrow \mathbb{X}[C']$ on objects as $h^*(A) := A$ and on maps as $h^*(f) := (h \times 1)f$.*

Alternatively, the simple slice category over C can be described as the coKleisli category for the comonad $C \times -$. Note that for the terminal object $\top$ there is an isomorphism of categories $\mathbb{X}[\top] \cong \mathbb{X}$. Every simple slice $\mathbb{X}[C]$ admits finite products where on objects the product is the same as in $\mathbb{X}$, the projection maps are respectively $\pi_1\pi_0 : C \times (A \times B) \longrightarrow A$ and $\pi_1\pi_1 : C \times (A \times B) \longrightarrow B$, and the pairing of maps is the same as in $\mathbb{X}$. If $\mathbb{X}$ is a Cartesian left additive category, then so is every simple slice $\mathbb{X}[C]$ [9, Corollary 1.3.5] where the sum and zero maps are defined again as in $\mathbb{X}$. As such, we can easily define what it means for a map to be additive in context.

Definition 4.3.2 *[29, Definition 4.2] In a Cartesian left additive category $\mathbb{X}$, we say that a map $f : C \times A \longrightarrow B$ is:*

(i) ***Constant in its second argument*** *if it is constant in $\mathbb{X}[C]$, that is, if $\langle \pi_0, 0 \rangle f = f$;*

(ii) ***Reduced in its second argument*** *if it is reduced in $\mathbb{X}[C]$, that is, if $\langle 1, 0 \rangle f = 0$;*

(iii) ***Semi-additive in its second argument*** *if it is semi-additive in* $\mathbb{X}[C]$*, that is, if* $\langle \pi_0, g+h\rangle f = \langle \pi_0, g\rangle f + \langle \pi_0, h\rangle f$*;*

(iv) ***Additive in its second argument*** *if it is additive in* $\mathbb{X}[C]$*, that is, if* $\langle 1, 0\rangle f = 0$ *and* $\langle \pi_0, g+h\rangle f = \langle \pi_0, g\rangle f + \langle \pi_0, h\rangle f$*.*

Lemma 4.3.3 *[29, Lemma 4.3] In a Cartesian left additive category* $\mathbb{X}$*,*

(i) *A map* $f : C \times A \longrightarrow B$ *is constant in its second argument if and only if* $f = \pi_0 g$ *for some map* $g : C \longrightarrow B$*;*

(ii) *A map* $f : C \times A \longrightarrow B$ *is additive in its second argument if and only if* $\langle \pi_0, 0\rangle f = 0$ *and* $(1 \times \oplus_A) f = (1 \times \pi_0) f + (1 \times \pi_1) f$*.*

Proof: For (i), it is immediate that $\langle \pi_0, 0\rangle \pi_0 g = \pi_0 g$, so $g = \pi_0 f$ is constant in its second argument. Conversely, if f is constant in its second argument then set $g = \langle 1, 0\rangle f$ then:

$$\begin{aligned} \pi_0 g &= \pi_0 \langle 1, 0\rangle f \\ &= \langle \pi_0, 0\rangle f \\ &= f && (f \text{ is constant in its second argument}) \end{aligned}$$

So, $f = \pi_0 g$. For (ii), since being additive in its second argument is the same as being additive in the simple slice, (ii) is simply re-expressing Lemma 4.1.4.(iii) using simple slice composition. □

In classical multivariable differential calculus, the standard way of defining partial differentiation, or in other words differentiation in context, is by evaluating at zero certain terms of the total derivative. This is also how one obtains partial derivatives in a Cartesian differential category. In fact, for a Cartesian differential category, every simple slice is also a Cartesian differential category whose differential combinator is given by partial differentiation, which amounts to evaluating at zero in the context arguments of the total derivative.

Proposition 4.3.4 *[9, Corollary 4.5.2] Let* $\mathbb{X}$ *be a Cartesian differential category with differential combinator* D*. Then each simple slice* $\mathbb{X}[C]$ *is a Cartesian differential category with differential combinator* D^C *defined as follows for a map* $f : C \times A \longrightarrow B$ *in* $\mathbb{X}$*:*

$$\mathsf{D}^C[f] = C \times (A \times A) \xrightarrow{\langle 1 \times \pi_0, 0 \times \pi_1\rangle} (C \times A) \times (C \times A) \xrightarrow{\mathsf{D}[f]} B \tag{4.5}$$

Furthermore, for every map $h : C' \longrightarrow C$ *in* $\mathbb{X}$*, the substitution functor* h^* *preserves the differential combinator in context [10, Proposition 4.1.3], that is,* $(h \times 1)\mathsf{D}^{C'}[f] = \mathsf{D}^C\left[(h \times 1) f\right]$*.*

In term logic notation, the differential combinator in context is written as:

$$\mathsf{D}^C[f](c, a, b) = \frac{\mathsf{d} f(c, x)}{\mathsf{d} x}(a) \cdot b := \frac{\mathsf{d} f(z, x)}{\mathsf{d}\langle z, x\rangle}(c, a) \cdot (0, b)$$

Since D^C is a differential combinator it satisfies, in context, **[CD.1]** to **[CD.7]**. In particular, we may re-express **[CD.6]** and **[CD.7]** as follows for the non-context differential combinator:

[CD.6] $\frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(c)\cdot y}{\mathsf{d}y}(a)\cdot(b) = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(c)\cdot b$

[CD.7] $\frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(x)\cdot a}{\mathsf{d}y}(c)\cdot b = \frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(x)\cdot b}{\mathsf{d}y}(c)\cdot a$

As with additivity, we can also define what it means for a map to be linear in context.

Definition 4.3.5 *In a Cartesian differential category $\mathbb{X}$, a map $f : C \times A \longrightarrow B$ is* ***linear in its second argument*** *[29, Definition 4.5] if it is linear in $\mathbb{X}[C]$, that is, if $\mathsf{D}^C[f] = \langle \pi_0, \pi_1\pi_1\rangle f = (1 \times \pi_1)f$.*

Being linear in context can also be expressed using the lifting map ℓ, and as such by **[CD.6]**, it follows that derivatives of maps are also linear in their second argument.

Lemma 4.3.6 *[29, Lemma 4.6] In a Cartesian differential category $\mathbb{X}$ with differential combinator D:*

(i) *A map $f : C \times A \longrightarrow B$ is linear in its second argument if and only if $\ell\mathsf{D}[f] = f$.*

(ii) *For every map $f : A \longrightarrow B$, $\mathsf{D}[f] : A \times A \longrightarrow B$ is linear in its second argument.*

PROOF: For (i), we first observe that for an arbitrary map $f : C \times A \longrightarrow B$, we compute:

$$\begin{aligned}
\langle \pi_0, \langle 0, \pi_1\rangle\rangle D^C[f] =&\ \langle \pi_0, \langle 0, \pi_1\rangle\rangle\langle 1 \times \pi_0, 0 \times \pi_1\rangle D[f]\\
&= \langle\langle \pi_0, \langle 0, \pi_1\rangle\rangle(1 \times \pi_0), \langle \pi_0, \langle 0, \pi_1\rangle\rangle(0 \times 1)\rangle D[f]\\
&= \langle\langle \pi_0, \langle 0, \pi_1\rangle\pi_0\rangle, \langle 0, \langle 0, \pi_1\rangle\pi_1\rangle\rangle D[f]\\
&= \langle\langle \pi_0, 0\rangle, \langle 0, \pi_1\rangle\rangle D[f]\\
&= \langle \pi_0\langle 1, 0\rangle, \pi_1\langle 0, 1\rangle\rangle D[f]\\
&= (\langle 1, 0\rangle \times \langle 0, 1\rangle)D[f]\\
&= \ell D[f]
\end{aligned}$$

So $\langle \pi_0, \langle 0, \pi_1\rangle\rangle D^C[f] = \ell D[f]$. Now recall that by Lemma 4.1.21.(ii), in a Cartesian differential category a map g is linear if and only if $\langle 0, 1\rangle D[g] = g$. Since every simple slice category is again a Cartesian differential category, then putting Lemma 4.1.21.(ii) into a context C means that a map $f : C \times A \longrightarrow B$ is linear in its second argument if and only if $\langle \pi_0, \langle 0, \pi_1\rangle\rangle D^C[f] = f$. However by the above calculations, we may re-express by saying that a map $f : C \times A \longrightarrow B$ is linear in its second argument if and only if $\ell\mathsf{D}[f] = f$. For (ii), for any map $f : A \longrightarrow B$, by **[CD.6]** we have that $\ell\ \mathsf{D}\big[\mathsf{D}[f]\big] = \mathsf{D}[f]$. Then by (i), it follows that $\mathsf{D}[f]$ is linear in its second argument. □

We conclude this section by taking a look at the differential combinators in context and maps which are linear in context in the examples of Cartesian differential categories from Section 4.1.

Example 4.3.7 For a category with finite biproducts seen as a Cartesian differential category as in Example 4.1.6, the differential combinator in context C is defined on a map $f : C \times A \longrightarrow B$ as follows:

$$\mathsf{D}^C[f] = (1 \times \pi_1)f$$

A map $f : C \times A \longrightarrow B$ is linear in its second argument if and only if $f = (0 \times 1)f$, or in other words, f does not depend on its first argument. For example, for $\mathsf{VEC}_{\mathbb{K}}$, for a $\mathbb{K}$-linear map $f : U \times V \longrightarrow W$:

$$\mathsf{D}^U[f](u, v, w) = f(u, w)$$

and $f : U \times V \longrightarrow W$ is linear in its second argument in the Cartesian differential sense if and only if $f(u, v) = f(0, v)$.

Example 4.3.8 For the Cartesian differential category $\mathsf{POLY}_{\mathbb{K}}$ from Example 4.1.7, for a map $P : k \times n \longrightarrow m$, which is a tuple $P = \langle p_1(\vec{z}, \vec{x}), \ldots, p_m(\vec{z}, \vec{x})\rangle$ with $p_i(\vec{z}, \vec{x}) \in \mathbb{K}[z_1, \ldots, z_k, x_1, \ldots, x_n]$, its partial derivative is $\mathsf{D}^k[P] : k \times n \times n \longrightarrow m$ defined as the following tuple of polynomials:

$$\mathsf{D}^k[P](\vec{z}, \vec{x}, \vec{y}) = \left\langle \sum_{i=1}^{m} \frac{\partial p(\vec{z}, \vec{x})}{\partial x_i} y_i, \ldots, \sum_{i=1}^{n} \frac{\partial p_m(\vec{z}, \vec{x})}{\partial x_i} y_i \right\rangle$$

A map $p(\vec{z}, \vec{x}) : k \times n \longrightarrow m$, which is a polynomial $p(\vec{z}, \vec{x}) \in \mathbb{K}[z_1, \ldots, z_k, x_1, \ldots, x_n]$, is linear in its second argument if it is of the form:

$$p(\vec{z}, \vec{x}) = \sum_{i=1}^{n} q_i(\vec{z}) x_i$$

for some polynomials $q_i(\vec{z}) \in \mathbb{K}[z_1, \ldots, z_k]$. Then $P : k \times n \longrightarrow m$, $P = \langle p_1(\vec{z}, \vec{x}), \ldots, p_m(\vec{z}, \vec{x})\rangle$, is linear in its second argument if each of the $p_i(\vec{z}, \vec{x})$ are.

Example 4.3.9 For the Cartesian differential category SMOOTH from Example 4.1.8, for a smooth function $F : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $F = \langle f_1, \ldots, f_m\rangle$, its partial derivative is the smooth function $\mathsf{D}^{\mathbb{R}^k}[F] : \mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow \mathbb{R}^m$ defined as follows:

$$\mathsf{D}^{\mathbb{R}^k}[F](\vec{z}, \vec{x}, \vec{y}) = \nabla(F)(\vec{z}, \vec{x}) \cdot (\vec{0}, \vec{y}) = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{z}, \vec{x}) y_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i}(\vec{z}, \vec{x}) y_i \right\rangle$$

A smooth function $F : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear in its second argument if it is $\mathbb{R}$-linear in its second argument, that is, $F(\vec{z}, s\vec{x} + t\vec{y}) = sF(\vec{z}, \vec{x}) + tF(\vec{z}, \vec{y})$ for all $s, t \in \mathbb{R}$, $\vec{z} \in \mathbb{R}^k$, and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Example 4.3.10 For the Cartesian differential category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ from Example 4.1.10, the differential combinator in context C is defined on a functor $F : C \times A \longrightarrow \mathsf{Ch}(B)$ as follows:

$$\nabla^C F(Z, X, V) := D_1^2 F(Z, X \oplus -)(V)$$

where D_1^i is the partial linearization operator as defined in [5, Convention 5.11]. A functor $F : C \times A \longrightarrow \mathsf{Ch}(B)$ is linear in its second argument if F preserves finite direct sums up to chain homotopy equivalence in its second argument.

Example 4.3.11 For a Cartesian left additive category $\mathbb{X}$, in its cofree Cartesian differential category $\mathcal{D}(\mathbb{X})$ from Example 4.1.11, the differential combinator in context C is worked out to be as follows for a D-sequence $(f_0, f_1, f_2, \ldots) : C \times A \longrightarrow B$ (so $f_n : \mathsf{P}^n(C \times A) \longrightarrow B$):

$$\mathsf{D}^C\left[(f_0, f_1, f_2, \ldots)\right] = \Big((\langle 1 \times \pi_0, 0 \times \pi_1\rangle) f_1, \mathsf{P}\left(\langle 1 \times \pi_0, 0 \times \pi_1\rangle\right) f_2, \mathsf{P}^2\left(\langle 1 \times \pi_0, 0 \times \pi_1\rangle\right) f_3, \ldots\Big)$$

A D-sequence $(f_0, f_1, f_2, \ldots) : C \times A \longrightarrow B$ is linear in its second argument if and only if we have that:

$$f_n = \underbrace{\ell \ldots \ell}_{n-times} f_0$$

for all $n \in \mathbb{N}$.

Example 4.3.12 For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products, the differential combinator in context C for the coKleisli category $\mathbb{X}_!$ is worked out to be as follows for a coKleisli map $[\![f]\!] : !(C \times A) \longrightarrow B$:

$$[\![\mathsf{D}^C[f]]\!] \quad := \quad \begin{array}{l} !\left(C \times (A \times A)\right) \xrightarrow{\langle 1\times\pi_0, \pi_1\pi_1\rangle} !\left((C \times A) \times A\right) \xrightarrow{\quad\chi\quad} !(C \times A) \times !A \xrightarrow{1\otimes\varepsilon} \\ !(C \times A) \otimes A \xrightarrow{1\otimes\langle 0,1\rangle} !(C \times A) \otimes (C \times A) \xrightarrow{\quad\mathsf{d}\quad} !(C \times A) \xrightarrow{\quad f\quad} B \end{array}$$

If $(!, \delta, \varepsilon, \Delta, e)$ has the Seely isomorphisms, then the partial derivative can alternatively be expressed using the codereliction map as follows:

$$[\![\mathsf{D}^C[f]]\!] \quad := \quad \begin{array}{l} !\left(C \times (A \times A)\right) \xrightarrow{\quad\chi\quad} !C \otimes !(A \times A) \xrightarrow{1\otimes\chi} !C \otimes (!A \otimes !A) \xrightarrow{1\otimes(1\otimes\varepsilon)} \\ !C \otimes (!A \otimes A) \xrightarrow{1\otimes(1\otimes\eta)} !C \otimes (!A \otimes !A) \xrightarrow{1\otimes\nabla} !C \otimes !A \xrightarrow{\chi^{-1}} !(C \times A) \xrightarrow{f} B \end{array}$$

In either case, a coKleisli map $[\![f]\!] : !(C \times A) \longrightarrow B$ is linear in its second argument if and only if it is of the form $f = \chi(1 \otimes \varepsilon)g$ for some map $g : !C \otimes A \longrightarrow B$ in $\mathbb{X}$.

Example 4.3.13 For the Cartesian differential category CON from Example 4.1.13, the differential combinator in context C is defined as follows on a smooth function $f : C \times E \longrightarrow F$:

$$\mathsf{D}^C[f](z, x, y) := \lim_{t \longrightarrow 0} \frac{f(z, x + t \cdot y) - f(z, x)}{t}$$

A smooth function is linear in its second argument in the Cartesian differential sense precisely when it is $\mathbb{R}$-linear in its second argument.

4.4 System of Linearizing Combinators

Partial linearization is a key operation in [5] as it used in the construction of the differential combinator. However, while it is always possible to define partial differentiation from total differentiation, in general it is not necessarily possible to define partial linearization from total linearization (see Example 4.4.15 at the end of this section). As such, we need to separately define the notion of linearizing combinators in contexts, which we call a *system* of linearizing combinators, which amounts to requiring that each simple slice admits a linearizing combinator. In fact, we will show that systems of linearizing combinators are in bijective correspondence with differential combinators. Therefore, systems of linearizing combinators provide an alternative axiomatization of Cartesian differential categories.

Definition 4.4.1 *A* ***system of linearizing combinators*** *[29, Definition 5.1] on a Cartesian left additive category $\mathbb{X}$ is a family of linearizing combinators L^C, where L^C is a linearizing combinator for the simple slice category $\mathbb{X}[C]$, that is, the following axioms hold:*

[L.1] $\mathsf{L}^C[f+g] = \mathsf{L}^C[f] + \mathsf{L}^C[g]$ *and* $\mathsf{L}^C[0] = 0$

[L.2] $\mathsf{L}^C[f]$ *is additive in its second argument, or equivalently by Lemma 4.3.3.(ii):*

$$(1 \times \oplus_A)\mathsf{L}^C[f] = (1 \times \pi_0)\mathsf{L}^C[f] + (1 \times \pi_1)\mathsf{L}^C[f] \qquad \langle 1, 0\rangle \mathsf{L}^C[f] = 0$$

[L.3] $\mathsf{L}^C[\pi_1] = \pi_1$, $\mathsf{L}^C[\pi_1\pi_0] = \pi_1\pi_0$, *and* $\mathsf{L}^C[\pi_1\pi_1] = \pi_1\pi_1$

[L.4] $\mathsf{L}^C\left[\langle f, g\rangle\right] = \left\langle \mathsf{L}^C[f], \mathsf{L}^C[g] \right\rangle$

[L.5] $\mathsf{L}^C[\langle \pi_0, f\rangle g] = \langle \pi_0, \mathsf{L}^C[f]\rangle\ \mathsf{L}^C\left[\left\langle \pi_0, \pi_1 + \langle \pi_0, 0\rangle f\right\rangle g\right]$

[L.6] $\mathsf{L}^C\left[\mathsf{L}^C[f]\right] = \mathsf{L}^C[f]$

and such that the following two extra axioms hold:

[L.7] *Let $\alpha : C \times (A \times B) \longrightarrow (C \times A) \times B$ and $\beta : C \times (A \times B) \longrightarrow (C \times B) \times A$ be the canonical natural isomorphisms respectively defined as follows:*

$$\alpha = \langle 1 \times \pi_0, \pi_1\pi_1\rangle \qquad \beta = \langle 1 \times \pi_1, \pi_1\pi_0\rangle$$

and for a map $f : C \times (A \times B) \longrightarrow D$, define the maps $\mathsf{L}^C_0[f] : C \times (A \times B) \longrightarrow D$ and $\mathsf{L}^C_1[f] : C \times (A \times B) \longrightarrow D$ respectively defined as follows:

$$\mathsf{L}^C_0[f] := \beta \mathsf{L}^{C \times B}[\beta^{-1} f] \qquad \mathsf{L}^C_1[f] := \alpha \mathsf{L}^{C \times A}[\alpha^{-1} f]$$

Then for any map $f : C \times (A \times B) \longrightarrow D$, $\mathsf{L}^C_1[\mathsf{L}^C_0[f]] = \mathsf{L}^C_0[\mathsf{L}^C_1[f]]$.

[L.8] *For any map $h : C \longrightarrow C'$ in $\mathbb{X}$, the substitution functor h^* (as defined in Definition 4.3.1) preserves the linearizing combinator in context, that is, $(h \times 1)\mathsf{L}^{C'}[f] = \mathsf{L}^C[(h \times 1)f]$*

In term logic notation, we write partial linearization as follows:

$$\mathsf{L}^C[f](c, a) := \frac{\ell f(c, x)}{\ell x} \cdot a$$

then the extra two axioms **[L.7]** and **[L.8]** are expressed as:

[L.7] $\frac{\frac{\ell f(c,x,y)}{\ell x} \cdot a}{\ell y} \cdot b = \frac{\frac{\ell f(c,x,y)}{\ell y} \cdot b}{\ell x} \cdot a$

[L.8] $h^*\left(\frac{\ell f(c,x)}{\ell x} \cdot a\right) = \frac{\ell f(h(c),x)}{\ell x} \cdot a$

and we leave it to the reader to work out how to write **[L.1]** to **[L.6]** in context. **[L.8]** simply says that partial linearization is unaffected by changes in the context argument. On the other hand, **[L.7]** is admittedly slightly complex at first glance without using the term logic. However, as made clear with the term logic, **[L.7]** amounts to the linearizing combinator analogue of **[CD.7]** and states the symmetry of partial linearization. Indeed, $\mathsf{L}_0^C[f]$ is the linearization of A while keeping C and B in context, while $\mathsf{L}_1^C[f]$ is the linearization of B while keeping C and A in context. In particular, **[L.7]** is also a generalization of [5, Lemma 5.18] when $C = \top$ (the terminal object), which in this case amounts to saying that linearizing A first then linearizing B (while keeping the other variables in context) is the same as linearizing B first then A. As an example, consider the polynomial function $f(x,y) = xy + 2xy^3 + 3x + 4y$. The total linearization of f, that is, linearizing f jointly in x and y is the polynomial $\mathsf{L}[f](x,y) = 3x+4y$. Linearizing f in terms of x while keeping y in context picks out the terms where x is of degree 1, and therefore results in the polynomial $\mathsf{L}_0[f] = xy + 2xy^3 + 3x$, which is now linear in x. On the other hand, linearizing f in terms of y while keeping x in context results in the polynomial $\mathsf{L}_1[f] = xy + 4y$, which this time is linear in y. Linearizing $xy + 2xy^3 + 3x$ in terms of y or linearizing $xy + 4y$ in terms of x both results in $\mathsf{L}_1[\mathsf{L}_0[f]] = \mathsf{L}_0[\mathsf{L}_1[f]] = xy$, which is an example of **[L.7]**. In Proposition 4.4.4, we will provide an equivalent alternative version of **[L.7]** which requires less setup.

Our first observation is that since there is an isomorphism between the base category and the simple slice category over the terminal object, it follows that a system of linearizing combinators also induces a linearizing combinator on the base category.

Proposition 4.4.2 *[29, Proposition 5.2] Let $\mathbb{X}$ be a Cartesian left additive category with a system of linearizing combinators L^C. Then $\mathbb{X}$ has a linearizing combinator L defined as follows for a map $f : A \longrightarrow B$:*

$$\mathsf{L}[f] = A \xrightarrow{\langle 0,1\rangle} \top \times A \xrightarrow{\mathsf{L}^\top[\pi_1 f]} B \tag{4.6}$$

where $\top$ is the terminal object. Furthermore:

(i) *For every map $f : A \longrightarrow B$ and every object C, $\mathsf{L}^C[\pi_1 f] = \pi_1 \mathsf{L}[f]$;*

(ii) *If f is L-linear then for every object C, $\pi_1 f$ is L^C-linear;*

(iii) *For every map L-linear map f, $(h \times f)\mathsf{L}^{C'}[g] = \mathsf{L}^C[(h \times f)g]$;*

(iv) *For a map $f : A \times B \longrightarrow C$, define $\mathsf{L}_0[f] : A \times B \longrightarrow C$ and $\mathsf{L}_1[f] : A \times B \longrightarrow C$ respectively as:*

$$\mathsf{L}_0[f] := \tau \mathsf{L}^B[\tau f] \qquad\qquad \mathsf{L}_1[f] := \mathsf{L}^A[f]$$

where τ is the canonical symmetry isomorphism as defined in (4.1). Then for every map $f : A \times B \longrightarrow C$, $\mathsf{L}_0[\mathsf{L}_1[f]] = \mathsf{L}_1[\mathsf{L}_0[f]]$.

PROOF: First note that for the terminal object, $\pi_1 : \top \times A \longrightarrow A$ and $\langle 0,1\rangle : A \longrightarrow \top \times A$ are inverses of each other. We now show that L satisfies **[L.1]** to **[L.6]**, which are proved using their simple slice version.

[L.1]: $\mathsf{L}[f+g] = \mathsf{L}[f] + \mathsf{L}[g]$ and $\mathsf{L}[0] = 0$

$$\begin{aligned}
\mathsf{L}[f+g] &= \langle 0,1\rangle \mathsf{L}^\top[\pi_1(f+g)] \\
&= \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f + \pi_1 g] \\
&= \langle 0,1\rangle \left(\mathsf{L}^\top[\pi_1 f] + \mathsf{L}^\top[\pi_1 g]\right) && \textbf{[L.1]} \\
&= \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f] + \langle 0,1\rangle \mathsf{L}^\top[\pi_1 g] \\
&= \mathsf{L}[f] + \mathsf{L}[g]
\end{aligned}$$

$$\begin{aligned}
\mathsf{L}[0] &= \langle 0,1\rangle \mathsf{L}^\top[\pi_1 0] \\
&= \langle 0,1\rangle \mathsf{L}^\top[0] \\
&= \langle 0,1\rangle 0 && \textbf{[L.1]} \\
&= 0
\end{aligned}$$

[L.2]: $\oplus_A \mathsf{L}[f] = \pi_0 \mathsf{L}[f] + \pi_1 \mathsf{L}[f]$ and $0\mathsf{L}[f] = 0$

$$\begin{aligned}
\oplus_A \mathsf{L}[f] &= \oplus_A \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f] \\
&= \langle 0,\oplus_A\rangle \mathsf{L}^\top[\pi_1 f] \\
&= \langle 0,1\rangle (1\times \oplus_A) \mathsf{L}^\top[\pi_1 f] \\
&= \langle 0,1\rangle (1\times \pi_0) \mathsf{L}^\top[\pi_1 f] + \langle 0,1\rangle (1\times \pi_1) \mathsf{L}^\top[\pi_1 f] && \textbf{[L.2]} \\
&= \langle 0,\pi_0\rangle \mathsf{L}^\top[\pi_1 f] + \langle 0,\pi_1\rangle \mathsf{L}^\top[\pi_1 f] \\
&= \pi_0 \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f] + \pi_1 \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f] \\
&= \pi_0 \mathsf{L}[f] + \pi_1 \mathsf{L}[f]
\end{aligned}$$

$$\begin{aligned}
0\mathsf{L}[f] &= 0\langle 0,1\rangle \mathsf{L}^\top[\pi_1 f] \\
&= \langle 0,0\rangle \mathsf{L}^\top[\pi_1 f] \\
&= \langle 0,1\rangle (1\times 0) \mathsf{L}^\top[\pi_1 f] \\
&= \langle 0,1\rangle 0 && \textbf{[L.2]} \\
&= 0
\end{aligned}$$

[L.3]: $\mathsf{L}[1] = 1$ and $\mathsf{L}[\pi_i] = \pi_i$

$$\begin{aligned}
\mathsf{L}[1] &= \langle 0,1\rangle \mathsf{L}^\top[\pi_1] \\
&= \langle 0,1\rangle \pi_1 && \textbf{[L.3]} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\mathsf{L}[\pi_i] &= \langle 0,1\rangle \mathsf{L}^\top[\pi_1 \pi_i] \\
&= \langle 0,1\rangle \pi_1 \pi_i && \textbf{[L.3]}
\end{aligned}$$

$$= \pi_i$$

[L.4]: $\mathsf{L}\left[\langle f, g\rangle\right] = \left\langle \mathsf{L}[f], \mathsf{L}[g]\right\rangle$

$$\begin{aligned}
\mathsf{L}\left[\langle f, g\rangle\right] &= \langle 0,1\rangle \mathsf{L}^\top[\pi_1\langle f,g\rangle] \\
&= \langle 0,1\rangle \mathsf{L}^\top[\langle \pi_1 f, \pi_1 g\rangle] \\
&= \langle 0,1\rangle \left\langle \mathsf{L}^\top[\pi_1 f], \mathsf{L}^\top[\pi_1 g]\right\rangle && \textbf{[L.4]} \\
&= \left\langle \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f], \langle 0,1\rangle \mathsf{L}^\top[\pi_1 g]\right\rangle \\
&= \left\langle \mathsf{L}[f], \mathsf{L}[g]\right\rangle
\end{aligned}$$

[L.5]: $\mathsf{L}[fg] = \mathsf{L}[f]\ \mathsf{L}\left[(1+0f)g\right]$

$$\begin{aligned}
\mathsf{L}[fg] &= \langle 0,1\rangle \mathsf{L}^\top[\pi_1 fg] \\
&= \langle 0,1\rangle \mathsf{L}^\top[\langle \pi_0, \pi_1 f\rangle \pi_1 g] \\
&= \langle 0,1\rangle \langle \pi_0, \mathsf{L}^\top[\pi_1 f]\rangle\ \mathsf{L}^\top\left[\left\langle \pi_0, \pi_1 + \langle \pi_0, 0\rangle \pi_1 f\right\rangle \pi_1 g\right] && \textbf{[L.5]} \\
&= \langle 0,1\rangle \langle \pi_0, \mathsf{L}^\top[\pi_1 f]\rangle\ \mathsf{L}^\top\left[\langle \pi_0, \pi_1 + 0f\rangle \pi_1 g\right] \\
&= \langle 0,1\rangle \langle \pi_0, \mathsf{L}^\top[\pi_1 f]\rangle\ \mathsf{L}^\top\left[(\pi_1 + 0f)\, g\right] \\
&= \left\langle \langle 0,1\rangle \pi_0, \mathsf{L}^\top[\pi_1 f]\right\rangle\ \mathsf{L}^\top\left[(\pi_1 + 0f)\, g\right] \\
&= \left\langle 0, \langle\langle 0,1\rangle \mathsf{L}^\top[\pi_1 f]\right\rangle\ \mathsf{L}^\top\left[(\pi_1 + 0f)\, g\right] \\
&= \left\langle 0, \mathsf{L}[f]\right\rangle\ \mathsf{L}^\top\left[(\pi_1 + 0f)\, g\right] \\
&= \mathsf{L}[f]\langle 0,1\rangle\ \mathsf{L}^\top\left[(\pi_1 + 0f)\, g\right] \\
&= \mathsf{L}[f]\langle 0,1\rangle\ \mathsf{L}^\top\left[\pi_1\,(1 + 0f)\, g\right] \\
&= \mathsf{L}[f]\ \mathsf{L}\left[(1+0f)g\right]
\end{aligned}$$

[L.6]: $\mathsf{L}\left[\mathsf{L}[f]\right] = \mathsf{L}[f]$

$$\begin{aligned}
\mathsf{L}\left[\mathsf{L}[f]\right] &= \langle 0,1\rangle \mathsf{L}^\top\left[\pi_1 \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f]\right] \\
&= \langle 0,1\rangle \mathsf{L}^\top\left[\mathsf{L}^\top[\pi_1 f]\right] && (\pi_1 \text{ and } \langle 0,1\rangle \text{ are inverses}) \\
&= \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f] && \textbf{[L.6]} \\
&= \mathsf{L}[f]
\end{aligned}$$

So we conclude that L is a linearizing combinator. For (i), for every object C, we compute:

$$\begin{aligned}
\mathsf{L}^C[\pi_1 f] &= \mathsf{L}^C[(0\times 1)\pi_1 f] \\
&= (0\times 1)\mathsf{L}^\top[\pi_1 f] && \textbf{[L.8]} \\
&= \langle 0, \pi_1\rangle \mathsf{L}^\top[\pi_1 f] \\
&= \pi_1 \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f]
\end{aligned}$$

$$= \pi_1 \mathsf{L}[f]$$

So $\mathsf{L}^C[\pi_1 f] = \pi_1 \mathsf{L}[f]$. For (ii), suppose that f is L-linear, that is, $\mathsf{L}[f] = f$. Then it follows that $\mathsf{L}^C[\pi_1 f] = \pi_1 f$ and so $\pi_1 f$ is L^C-linear. For (iii), suppose again that f is L-linear, and so $\pi_1 f$ is $\mathsf{L}^{C'}$-linear. By Lemma 4.2.4.(ii), $\pi_1 f$ is also additive (and so reduced) in the simple slice category. Then using Lemma 4.2.2 with respect to simple slice composition, we have that:

$$\begin{aligned}
\mathsf{L}^C[(h \times f)g] &= \mathsf{L}^C[(h \times 1)(1 \times f)g] \\
&= (h \times 1)\mathsf{L}^{C'}[(1 \times f)g] && \textbf{[L.8]} \\
&= (h \times 1)\mathsf{L}^{C'}[\langle \pi_0, \pi_1 f\rangle g] \\
&= (h \times 1)\langle \pi_0, \mathsf{L}^{C'}[\pi_1 f]\rangle \mathsf{L}^{C'}[g] && \text{(Lemma 4.2.2.(ii))} \\
&= (h \times 1)\langle \pi_0, \pi_1 f\rangle \mathsf{L}^{C'}[g] && (f \text{ is } \mathsf{L}\text{-linear, so } \pi_1 f \text{ is } \mathsf{L}^{C'}\text{-linear}) \\
&= (h \times 1)(1 \times f)\mathsf{L}^{C'}[g] \\
&= (h \times f)\mathsf{L}^{C'}[g]
\end{aligned}$$

So we have that $(h \times f)\mathsf{L}^{C'}[g] = \mathsf{L}^C[(h \times f)g]$, when f is L-linear. Lastly (iv) is a special case of **[L.7]** when $C = \top$. First observe that $\beta = (1 \times \tau)\alpha$, so we compute:

$$\begin{aligned}
\mathsf{L}_0[\mathsf{L}_1[f]] &= \tau \mathsf{L}^B[\tau \mathsf{L}^A[f]] \\
&= \tau \mathsf{L}^B[\tau \mathsf{L}^A[\langle 0, 1 \times 1\rangle \pi_1 f]] \\
&= \tau \mathsf{L}^B[\tau \mathsf{L}^A[(\langle 0, 1\rangle \times 1)\alpha^{-1}\pi_1 f]] \\
&= \tau \mathsf{L}^B[\tau(\langle 0, 1\rangle \times 1)\mathsf{L}^{\top \times A}[\alpha^{-1}\pi_1 f]] && \textbf{[L.8]} \\
&= \tau \mathsf{L}^B[(\langle 0, 1\rangle \times 1)\beta^{-1}\alpha \mathsf{L}^{\top \times A}[\alpha^{-1}\pi_1 f]] \\
&= \tau(\langle 0, 1\rangle \times 1)\mathsf{L}^{\top \times B}[\beta^{-1}\alpha \mathsf{L}^{\top \times A}[\alpha^{-1}\pi_1 f]] && \textbf{[L.8]} \\
&= (\langle 0, 1\rangle \times 1)\alpha^{-1}\beta \mathsf{L}^{\top \times B}[\beta^{-1}\alpha \mathsf{L}^{\top \times A}[\alpha^{-1}\pi_1 f]] \\
&= (\langle 0, 1\rangle \times 1)\alpha^{-1}\mathsf{L}^{\top}_0[\mathsf{L}^{\top}_1[\pi_1 f]] \\
&= (\langle 0, 1\rangle \times 1)\alpha^{-1}\mathsf{L}^{\top}_1[\mathsf{L}^{\top}_0[\pi_1 f]] && \textbf{[L.7]} \\
&= (\langle 0, 1\rangle \times 1)\alpha^{-1}\alpha \mathsf{L}^{\top \times A}[\alpha^{-1}\beta \mathsf{L}^{\top \times B}[\beta^{-1}\pi_1 f]] \\
&= (\langle 0, 1\rangle \times 1)\mathsf{L}^{\top \times A}[\alpha^{-1}\beta \mathsf{L}^{\top \times B}[\beta^{-1}\pi_1 f]] \\
&= \mathsf{L}^A[(\langle 0, 1\rangle \times 1)\alpha^{-1}\beta \mathsf{L}^{\top \times B}[\beta^{-1}\pi_1 f]] && \textbf{[L.8]} \\
&= \mathsf{L}^A[\tau(\langle 0, 1\rangle \times 1)\mathsf{L}^{\top \times B}[\beta^{-1}\pi_1 f]] \\
&= \mathsf{L}^A[\tau \mathsf{L}^B[(\langle 0, 1\rangle \times 1)\beta^{-1}\pi_1 f]] && \textbf{[L.8]} \\
&= \mathsf{L}^A[\tau \mathsf{L}^B[\tau\langle 0, 1 \times 1\rangle \pi_1 f]] \\
&= \mathsf{L}^A[\tau \mathsf{L}^B[\tau \pi_1 f]] \\
&= \mathsf{L}_1[\mathsf{L}_0[f]]
\end{aligned}$$

So we conclude that $\mathsf{L}_0[\mathsf{L}_1[f]] = \mathsf{L}_1[\mathsf{L}_0[f]]$. □

The following lemma will be useful in the proofs of Proposition 4.4.4 and Proposition 4.4.13:

Lemma 4.4.3 *[29, Lemma 5.3] In a Cartesian left additive category $\mathbb{X}$ with a system of linearizing combinators L^C,*

(i) *For every map $h : C \longrightarrow C'$, $\mathsf{L}^C[\pi_0 h] = 0$;*

(ii) *For every map $f : (C \times A) \times (B \times D) \longrightarrow E$, $\ell\ \mathsf{L}^{C\times A}[f] = \mathsf{L}^C[\ell f]$;*

(iii) *For every map $f : C \times A \longrightarrow B$, $\oplus_{C\times A}\mathsf{L}^C[f] = c\ \mathsf{L}^{C\times A}\left[(\oplus_C \times \oplus_A)f\right]$*

Proof: For (i), first note that by Lemma 4.3.3.(i), $\pi_0 h$ is constant in the simple slice category. Then by Proposition 4.4.2.(i), it follows that $\mathsf{L}^C[\pi_0 h] = 0$. For (ii), recall that $\ell = \langle 1, 0\rangle \times \langle 0, 1\rangle$. By Lemma 4.2.4.(v), (vi) and (ix), $\langle 0, 1\rangle$ is L-linear, and so (ii) is simply an application of Proposition 4.4.2.(iii). For (iii), recall that $\oplus_A = \pi_0 + \pi_1$ and so by Lemma 4.2.4, $\oplus_A$ is L-linear. Note that by Lemma 4.1.4.(ii) that $\oplus_{C\times A} = c(\oplus_C \times \oplus_A)$, and so (iii) is simply an application of Proposition 4.4.2.(iii). □

As previously discussed, it may be tempting to assume that from a linearizing combinator L on the base category, one should be able to define the linearizing combinator in context L^C by doing the same evaluate at zero trick as for differential combinators. This however does not work. Instead, in order to prove the converse of Proposition 4.4.2, we will require the extra assumption that our Cartesian left additive category be Cartesian closed, which we discuss in Section 4.5.

Our next observation is that **[L.7]** can equivalently be stated in a more compact way as **[L.7.a]** below, using the canonical interchange isomorphism. This equivalent version will be more useful in the proofs of Proposition 4.4.5 and Proposition 4.4.13, while on the other hand, **[L.7]** is somewhat more intuitive and will be more useful in Section 4.5. In term logic, **[L.7.a]** is expressed as follows:

[L.7.a] $\dfrac{\frac{\ell f(c,x,y,z)}{\ell(y,z)}\cdot(x,b,z)}{\ell(x,z)}\cdot(a,d) = \dfrac{\frac{\ell f(c,x,y,z)}{\ell(x,z)}\cdot(a,y,z)}{\ell(y,z)}\cdot(b,d)$

The proof that **[L.7]** and **[L.7.a]** are equivalent include probably the "nastiest" calculations in this thesis.

Proposition 4.4.4 *[29, Proposition 5.4] In the presence of the other axioms **[L.1]**-**[L.6]** and **[L.8]**, **[L.7]** is equivalent to the following:*

[L.7.a] *For a map $f : (C \times A) \times (B \times D) \longrightarrow E$, $c\ \mathsf{L}^{C\times B}\left[c\ \mathsf{L}^{C\times A}[f]\right] = \mathsf{L}^{C\times A}\left[c\ \mathsf{L}^{C\times B}[c\ f]\right]$*

where recall that c is the canonical natural interchange isomorphism as defined in (4.2).

Proof: Suppose that **[L.7]** holds. Then for any $f : (C \times A) \times (B \times D) \longrightarrow E$, we compute that:

$$\begin{aligned}
&c\ \mathsf{L}^{C\times B}\left[c\ \mathsf{L}^{C\times A}[f]\right] = \\
&= c\left((1\times 1)\times(1\times 1)\right)\mathsf{L}^{C\times B}\left[c\ \mathsf{L}^{C\times A}[f]\right] \\
&= c\Big(\left((1\times 1)\times(1\times 0)\right) + \left((1\times 1)\times(0\times 1)\right)\Big)\mathsf{L}^{C\times B}\left[c\ \mathsf{L}^{C\times A}[f]\right] \\
&= c\left(\left((1\times 1)\times(1\times 0)\right)\mathsf{L}^{C\times B}\left[c\ \mathsf{L}^{C\times A}[f]\right] + \left((1\times 1)\times(0\times 1)\right)\mathsf{L}^{C\times B}\left[c\ \mathsf{L}^{C\times A}[f]\right]\right) && \textbf{[L.2]}
\end{aligned}$$

$$
\begin{aligned}
&= c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\,\mathsf{L}^{C\times A}[f]\Big] + c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[c\,\mathsf{L}^{C\times A}[f]\Big]\\
&= c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&+ c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&= c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\Big(\big((1\times 1)\times(1\times 0)\big) + \big((1\times 1)\times(0\times 1)\big)\Big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&+ c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[c\Big(\big((1\times 1)\times(1\times 0)\big) + \big((1\times 1)\times(0\times 1)\big)\Big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&= c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\Big(\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f] + \big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big)\Big]\\
&+ c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[c\Big(\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f] + \big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big)\Big]\\
&= c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f] + c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&+ c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f] + c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&= c\big((1\times 1)\times(1\times 0)\big)\Big(\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f]\Big] + \mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\Big) && \textbf{[L.2]}\\
&+ c\big((1\times 1)\times(0\times 1)\big)\Big(\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f]\Big] + \mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\Big) && \textbf{[L.2]}\\
&= c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&+ c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&+ c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&+ c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
&= c\big((1\times 1)\times(1\times 0)\big)\,\mathsf{L}^{C\times B}\Big[c\,\mathsf{L}^{C\times A}[\big((1\times 1)\times(1\times 0)\big)\,f]\Big]\\
& && \text{(Lem.4.2.4.(v)+(vi)+(x) + Prop.4.4.2.(iii))}\\
&+ c\,\mathsf{L}^{C\times B}\Big[\big((1\times 1)\times(1\times 0)\big)\,c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
& && \text{(Lem.4.2.4.(v)+(vi)+(x) + Prop.4.4.2.(iii))}\\
&+ c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times B}\Big[\big((1\times 1)\times(1\times 0)\big)\,c\,\mathsf{L}^{C\times A}[f]\Big] && \text{(Nat. of } c\text{)}\\
&+ c\,\mathsf{L}^{C\times B}\Big[\big((1\times 1)\times(0\times 1)\big)\,c\big((1\times 1)\times(0\times 1)\big)\,\mathsf{L}^{C\times A}[f]\Big]\\
& && \text{(Lem.4.2.4.(v)+(vi)+(x) + Prop.4.4.2.(iii))}\\
&= c\,\mathsf{L}^{C\times B}\Big[\big((1\times 1)\times(1\times 0)\big)\,c\,\mathsf{L}^{C\times A}[\big((1\times 1)\times(1\times 0)\big)\,f]\Big]\\
& && \text{(Lem.4.2.4.(v)+(vi)+(x) + Prop.4.4.2.(iii))}
\end{aligned}
$$

$$+\ c\,\mathsf{L}^{C\times B}\left[c\left((1\times 1)\times(1\times 0)\right)\left((1\times 1)\times(0\times 1)\right)\mathsf{L}^{C\times A}[f]\right] \qquad \text{(Nat. of } c\text{)}$$
$$+\ c\left((1\times 1)\times(0\times 1)\right)\left((1\times 1)\times(1\times 0)\right)\mathsf{L}^{C\times B}\left[c\,\mathsf{L}^{C\times A}[f]\right]$$
(Lem.4.2.4.(v)+(vi)+(x) + Prop.4.4.2.(iii))
$$+\ c\,\mathsf{L}^{C\times B}\left[c\left((1\times 1)\times(1\times 0)\right)\left((1\times 1)\times(0\times 1)\right)\mathsf{L}^{C\times A}[f]\right] \qquad \text{(Nat. of } c\text{)}$$
$$=\ c\,\mathsf{L}^{C\times B}\left[c\left((1\times 1)\times(1\times 0)\right)\mathsf{L}^{C\times A}[\left((1\times 1)\times(1\times 0)\right)f]\right] \qquad \text{(Nat. of } c\text{)}$$
$$+\ c\,\mathsf{L}^{C\times B}\left[c\left((1\times 1)\times(0\times 0)\right)\mathsf{L}^{C\times A}[f]\right]$$
$$+\ c\left((1\times 1)\times(0\times 0)\right)\mathsf{L}^{C\times B}\left[c\,\mathsf{L}^{C\times A}[f]\right]$$
$$+\ c\,\mathsf{L}^{C\times B}\left[c\left((1\times 0)\times(0\times 1)\right)\mathsf{L}^{C\times A}[f]\right]$$
$$=\ c\,\mathsf{L}^{C\times B}\left[c\,\mathsf{L}^{C\times A}[\left((1\times 1)\times(1\times 0)\right)\left((1\times 1)\times(1\times 0)\right)f]\right]$$
(Lem.4.2.4.(v)+(vi)+(x) + Prop.4.4.2.(iii))
$$+\ c\,\mathsf{L}^{C\times B}[c0] \qquad \textbf{[L.2]}$$
$$+\ c0 \qquad \textbf{[L.2]}$$
$$+\ c\,\mathsf{L}^{C\times B}\left[(\pi_0\times\pi_1)\ell\mathsf{L}^{C\times A}[f]\right]$$
$$=\ c\,\mathsf{L}^{C\times B}\left[c\,\mathsf{L}^{C\times A}[\left((1\times 1)\times(1\times 0)\right)f]\right]+c\,\mathsf{L}^{C\times B}[0]+0+c\,(\pi_0\times\pi_1)\mathsf{L}^{C}\left[\mathsf{L}^{C}[\ell f]\right]$$
(Lem.4.2.4.(vii) + Prop.4.4.2.(iii) + Lem.4.4.3.(iii))
$$=\ c\,\mathsf{L}^{C\times B}\left[c\,\mathsf{L}^{C\times A}[\left((1\times 1)\times\pi_0\right)\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,0+c\,(\pi_0\times\pi_1)\mathsf{L}^{C}\left[\mathsf{L}^{C}[\ell f]\right] \qquad \textbf{[L.1]}$$
$$=\ c\,\mathsf{L}^{C\times B}\left[c\left((1\times 1)\times\pi_0\right)\mathsf{L}^{C\times A}[\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+0+\mathsf{L}^{C}\left[\mathsf{L}^{C}[\ell f]\right]$$
(Lem.4.2.4.(v)+(vii)+(x) + Prop.4.4.2.(iii))
$$=\ c\,\mathsf{L}^{C\times B}\left[c\left((1\times 1)\times\pi_0\right)\mathsf{L}^{C\times A}[\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^{C}[\ell f] \qquad \textbf{[L.6]}$$
$$=\ c\,\mathsf{L}^{C\times B}\left[\left((1\times 1)\times\pi_0\right)\beta^{-1}\alpha\mathsf{L}^{C\times A}[\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^{C}[\ell f]$$
$$=\ c\left((1\times 1)\times\pi_0\right)\mathsf{L}^{C\times B}\left[\beta^{-1}\alpha\mathsf{L}^{C\times A}[\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^{C}[\ell f]$$
(Lem.4.2.4.(vii) + Prop.4.4.2.(iii))
$$=\ \left((1\times 1)\times\pi_0\right)\alpha^{-1}\beta\mathsf{L}^{C\times B}\left[\beta^{-1}\alpha\mathsf{L}^{C\times A}[\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^{C}[\ell f]$$
$$=\ \left((1\times 1)\times\pi_0\right)\alpha^{-1}\beta\mathsf{L}^{C\times B}\left[\beta^{-1}\alpha\mathsf{L}^{C\times A}[\alpha^{-1}\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^{C}[\ell f]$$
$$=\ \left((1\times 1)\times\pi_0\right)\alpha^{-1}\mathsf{L}^{C}_0\left[\mathsf{L}^{C}_1[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^{C}[\ell f]$$

So we have the following equality:

$$c\,\mathsf{L}^{C\times B}\left[c\,\mathsf{L}^{C\times A}[f]\right]=\left((1\times 1)\times\pi_0\right)\alpha^{-1}\mathsf{L}^{C}_0\left[\mathsf{L}^{C}_1[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^{C}[\ell f] \qquad (4.7)$$

On the other hand, using the above equality, we compute that:

$$\mathsf{L}^{C\times A}\left[c\,\mathsf{L}^{C\times B}[c\,f]\right]=\ cc\,\mathsf{L}^{C\times A}\left[c\,\mathsf{L}^{C\times B}[c\,f]\right] \qquad (c \text{ is self-inverse})$$

$$\begin{aligned}
&= c\left(\left((1\times 1)\times \pi_0\right)\alpha^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)c\ f]\right]+\mathsf{L}^C[\ell c\ f]\right) && ((4.7)\text{ for } cf)\\
&= c\left((1\times 1)\times \pi_0\right)\alpha^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)c\ f]\right]+c\,\mathsf{L}^C[\ell c\ f]\\
&= \left((1\times 1)\times \pi_0\right)\beta^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\beta\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\mathsf{L}^C[\ell c\ f]\\
&= \left((1\times 1)\times \pi_0\right)\beta^{-1}\mathsf{L}_1^C\left[\mathsf{L}_0^C[\beta\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\mathsf{L}^C[\ell c\ f] && \textbf{[L.7]}\\
&= \left((1\times 1)\times \pi_0\right)\beta^{-1}\alpha\mathsf{L}^{C\times B}\left[\alpha^{-1}\beta\mathsf{L}^{C\times A}[\beta^{-1}\beta\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\mathsf{L}^C[\ell c\ f]\\
&= \left((1\times 1)\times \pi_0\right)\alpha^{-1}\beta\mathsf{L}^{C\times B}\left[\beta^{-1}\alpha\mathsf{L}^{C\times A}[\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\mathsf{L}^C[\ell c\ f]\\
&= \left((1\times 1)\times \pi_0\right)\alpha^{-1}\beta\mathsf{L}^{C\times B}\left[\beta^{-1}\alpha\mathsf{L}^{C\times A}[\alpha^{-1}\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\mathsf{L}^C[\ell c\ f]\\
&= \left((1\times 1)\times \pi_0\right)\alpha^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\mathsf{L}^C[\ell c\ f]\\
&= \left((1\times 1)\times \pi_0\right)\alpha^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\mathsf{L}^C[\ell f]\\
&= \left((1\times 1)\times \pi_0\right)\alpha^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+c\,\ell\mathsf{L}^C[f] && (\text{Lem.4.4.3.(iii)})\\
&= \left((1\times 1)\times \pi_0\right)\alpha^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\ell\,\mathsf{L}^C[f]\\
&= \left((1\times 1)\times \pi_0\right)\alpha^{-1}\mathsf{L}_0^C\left[\mathsf{L}_1^C[\alpha\left((1\times 1)\times\langle 1,0\rangle\right)f]\right]+\mathsf{L}^C[\ell f] && (\text{Lem.4.4.3.(iii)})\\
&= c\,\mathsf{L}^{C\times B}\left[c\,\mathsf{L}^{C\times A}[f]\right] && (4.7)
\end{aligned}$$

So we conclude that **[L.7.a]** holds.

Conversly, suppose that **[L.7.a]** holds. For a map $f : C\times(A\times B)\longrightarrow D$, define the map $f^\circ : (C\times A)\times(B\times\top)\longrightarrow D$ as the following composite:

$$f^\circ := \ (C\times A)\times(B\times\top)\xrightarrow{(1\times 1)\times\pi_0}(C\times A)\times B\xrightarrow{\alpha^{-1}}C\times(A\times B)\xrightarrow{f}D$$

First recall that for the terminal object $\top$, $\pi_0 : X\times\top\longrightarrow X$ is an isomorphism with inverse $\langle 1,0\rangle : X\longrightarrow X\times\top$. Therefore, we have the following equality:

$$f = \alpha\left((1\times 1)\times\langle 1,0\rangle\right)f^\circ \tag{4.8}$$

We also have the following equalities (which we leave to the reader to check for themselves):

$$\beta^{-1}\alpha\left((1\times 1)\times\langle 1,0\rangle\right) = \left((1\times 1)\times\langle 1,0\rangle\right)c = \alpha^{-1}\beta\left((1\times 1)\times\langle 1,0\rangle\right) \tag{4.9}$$

Lastly, by similar calculations near the end of the proof of Proposition 4.4.2 (which did not require **[L.7]**), one can show that for any map $g : (C\times X)\times(Y\times\top)\longrightarrow D$, the following equality holds:

$$\mathsf{L}^{C\times X}\left[\left((1\times 1)\times\langle 1,0\rangle\right)g\right] = \left((1\times 1)\times\langle 1,0\rangle\right)\mathsf{L}^{C\times X}[g] \tag{4.10}$$

Therefore, we compute that:

$$\mathsf{L}_0^C[\mathsf{L}_1^C[f]] = \ \beta\mathsf{L}^{C\times B}\left[\beta^{-1}\alpha\mathsf{L}^{C\times A}[\alpha^{-1}f]\right]$$

$$= \beta \mathsf{L}^{C\times B}\left[\beta^{-1}\alpha \mathsf{L}^{C\times A}\left[\left((1\times 1)\times\langle 1,0\rangle\right) f^\circ\right]\right] \quad (4.8)$$
$$= \beta \mathsf{L}^{C\times B}\left[\beta^{-1}\alpha \left((1\times 1)\times\langle 1,0\rangle\right) \mathsf{L}^{C\times A}[f^\circ]\right] \quad (4.10)$$
$$= \beta \mathsf{L}^{C\times B}\left[\left((1\times 1)\times\langle 1,0\rangle\right) c\, \mathsf{L}^{C\times A}[f^\circ]\right] \quad (4.9)$$
$$= \beta \left((1\times 1)\times\langle 1,0\rangle\right) \mathsf{L}^{C\times B}\left[c\, \mathsf{L}^{C\times A}[f^\circ]\right] \quad (4.10)$$
$$= \alpha \left((1\times 1)\times\langle 1,0\rangle\right) c\, \mathsf{L}^{C\times B}\left[c\, \mathsf{L}^{C\times A}[f^\circ]\right] \quad (4.9)$$
$$= \alpha \left((1\times 1)\times\langle 1,0\rangle\right) \mathsf{L}^{C\times A}\left[c\, \mathsf{L}^{C\times B}[cf^\circ]\right] \quad \textbf{[L.7.a]}$$
$$= \alpha \mathsf{L}^{C\times A}\left[\left((1\times 1)\times\langle 1,0\rangle\right) c\, \mathsf{L}^{C\times B}[cf^\circ]\right] \quad (4.10)$$
$$= \alpha \mathsf{L}^{C\times A}\left[\alpha^{-1}\beta \left((1\times 1)\times\langle 1,0\rangle\right) \mathsf{L}^{C\times B}[cf^\circ]\right] \quad (4.9)$$
$$= \alpha \mathsf{L}^{C\times A}\left[\alpha^{-1}\beta \mathsf{L}^{C\times B}\left[\left((1\times 1)\times\langle 1,0\rangle\right) cf^\circ\right]\right] \quad (4.10)$$
$$= \alpha \mathsf{L}^{C\times A}\left[\alpha^{-1}\beta \mathsf{L}^{C\times B}\left[\beta^{-1}\alpha\left((1\times 1)\times\langle 1,0\rangle\right) f^\circ\right]\right] \quad (4.9)$$
$$= \alpha \mathsf{L}^{C\times A}\left[\alpha^{-1}\beta \mathsf{L}^{C\times B}\left[\beta^{-1} f\right]\right] \quad (4.8)$$
$$= \mathsf{L}^C_1[\mathsf{L}^C_0[f]]$$

So we conclude that **[L.7]** holds. □

We now turn our attention to the relationship between differential combinators and systems of linearizing combinators. We first show that every differential combinator induces a system of linearizing combinators. Indeed, since every simple slice category of a Cartesian differential category is again Cartesian differential category, and every Cartesian differential category comes equipped with a canonical linearizing combinator, it follows that every Cartesian differential category admits a system of linearizing combinators. In term logic notation, the partial linearization is given as follows:

$$\frac{\ell f(c,x)}{\ell x}\cdot a = \frac{\mathsf{d} f(c,x)}{\mathsf{d} x}(0)\cdot b$$

Proposition 4.4.5 *[29, Proposition 5.5] Every Cartesian differential category $\mathbb{X}$, with differential combinator D, admits a system of linearizing combinators where the linearizing combinators $\mathsf{L}_{\mathsf{D}^C}$ for the simple slice categories are defined as in Proposition 4.2.6. As to not overload the subscripts, we denote this linearizing combinator instead by $\mathsf{L}^C_{\mathsf{D}} := \mathsf{L}_{\mathsf{D}^C}$. Equivalently, $\mathsf{L}^C_{\mathsf{D}}$ is defined as follows on a map $f : C\times A \longrightarrow B$:*

$$\mathsf{L}^C_{\mathsf{D}}[f] = C\times A \xrightarrow{\ \ell\ } (C\times A)\times(C\times A) \xrightarrow{\ \mathsf{D}[f]\ } B \qquad (4.11)$$

where ℓ is the lifting map as defined as in (4.3). Furthermore,

(i) *For every map $f : C\times A \longrightarrow B$, $\mathsf{L}^C_{\mathsf{D}}[f]$ is linear in its second argument;*

(ii) *A map* $f : C \times A \longrightarrow B$ *is linear in its second argument if and only if* f *is* L^C_D*-linear.*

(iii) $\mathsf{L} = \mathsf{L}_\mathsf{D}$*, where* L *is the induced linearizing combinator from Proposition 4.4.2 and* L_D *is the induced linearizing combinator from Proposition 4.2.6.*

PROOF: By Proposition 4.3.4, every simple slice category of a Cartesian differential category is again a Cartesian differential category with differential combinator $\mathsf{D}^C[f]$. Then by applying Proposition 4.2.6 to the simple slice categories, we obtain a linearizing combinator L^C_D for each simple slice category. So L^C_D satisfies **[L.1]** and **[L.6]** as in Definition 4.4.1. Furthermore, since a map of type $C \times A \longrightarrow B$ is linear in its second argument if it linear in the simple slice category, it follows from Proposition 4.2.6.(i) that for every map $f : C \times A \longrightarrow B$, $\mathsf{L}^C_\mathsf{D}[f]$ is linear in its second argument. Similarly, by Proposition 4.2.6.(ii), a map $f : C \times A \longrightarrow B$ is linear in its second argument (i.e. $\mathsf{D}^C[f] = (1 \times \pi_1)f$) if and only if f is L^C_D-linear (i.e. $\mathsf{L}^C_\mathsf{D}[f] = f$).

For a map $f : C \times A \longrightarrow B$, by Proposition 4.3.4 and by definition of composition in the simple slice category, $\mathsf{L}^C_\mathsf{D}[f] : C \times A \longrightarrow B$ is easily worked out to be:

$$\mathsf{L}^C_\mathsf{D}[f] = \big\langle \pi_0, \langle 0, \pi_1 \rangle \big\rangle\, \mathsf{D}^C[f] \tag{4.12}$$

Which can equivalently be rewritten as:

$$\mathsf{L}^C_\mathsf{D}[f] = (1 \times \langle 0, 1 \rangle)\mathsf{D}^C[f] \tag{4.13}$$

Expanding out the definition of $\mathsf{D}^C[f]$, we obtain:

$$\begin{aligned}
\mathsf{L}^C_\mathsf{D}[f] &= (1 \times \langle 0, 1 \rangle)\mathsf{D}^C[f] && (4.13)\\
&= (1 \times \langle 0, 1 \rangle)\langle 1 \times \pi_0, 0 \times \pi_1 \rangle \mathsf{D}[f] && (4.5)\\
&= \big\langle (1 \times \langle 0, 1 \rangle)(1 \times \pi_0), (1 \times \langle 0, 1 \rangle)(0 \times \pi_1) \big\rangle\, \mathsf{D}[f] \\
&= \langle 1 \times 0, 0 \times 1 \rangle\, \mathsf{D}[f] \\
&= \big(\langle 1, 0 \rangle \times \langle 0, 1 \rangle \big)\, \mathsf{D}[f] \\
&= \ell\, \mathsf{D}[f]
\end{aligned}$$

So we have that $\mathsf{L}^C_\mathsf{D}[f] = \ell \mathsf{D}[f]$. We now show that **[L.8]** and **[L.7.a]** also hold (which recall by Proposition 4.4.4 is equivalent to showing that **[L.7]** holds) :

[L.8]: $(h \times 1)\mathsf{L}^{C'}_\mathsf{D}[f] = \mathsf{L}^C_\mathsf{D}[(h \times 1)f]$

$$\begin{aligned}
(h \times 1)\mathsf{L}^{C'}_\mathsf{D}[f] &= (h \times 1)(1 \times \langle 0, 1 \rangle)\mathsf{D}^{C'}[f] && (4.13)\\
&= (1 \times \langle 0, 1 \rangle)(h \times 1)\mathsf{D}^{C'}[f] \\
&= (1 \times \langle 0, 1 \rangle)\mathsf{D}^{C}[(h \times 1)f] && (\text{Proposition } 4.3.4)\\
&= \mathsf{L}^C_\mathsf{D}[(h \times 1)f]
\end{aligned}$$

[L.7.a]: $c\ \mathsf{L}^{C \times B}_\mathsf{D}\left[c\ \mathsf{L}^{C \times A}_\mathsf{D}[f]\right] = \mathsf{L}^{C \times A}_\mathsf{D}\left[c\ \mathsf{L}^{C \times B}_\mathsf{D}[c\ f]\right]$

We leave it to the reader to check for themselves that following equality holds (which can be checked by a straightforward but tedious calculation):

$$c\ell(c \times c)(\ell \times \ell)c = \ell(c \times c)(\ell \times \ell)\left((c \times c) \times (c \times c)\right) \tag{4.14}$$

Then we have that:

$$\begin{aligned}
c\,\mathsf{L}_{\mathsf{D}}^{C\times B}\left[c\,\mathsf{L}_{\mathsf{D}}^{C\times A}[f]\right] &= c\ell\;\mathsf{D}\left[c\ell\mathsf{D}[f]\right] && (4.11)\\
&= c\ell(c\times c)(\ell\times\ell)\;\mathsf{D}\left[\mathsf{D}[f]\right] && \text{(Cor. 4.1.20.(ii)+(iii) + Lem. 4.1.19.(ii))}\\
&= c\ell(c\times c)(\ell\times\ell)c\;\mathsf{D}\left[\mathsf{D}[f]\right] && \textbf{[CD.7]}\\
&= \ell(c\times c)(\ell\times\ell)\left((c\times c)\times(c\times c)\right)\mathsf{D}\left[\mathsf{D}[f]\right] && (4.14)\\
&= \ell\mathsf{D}\left[c\ell(c\times c)\mathsf{D}[f]\right] && \text{(Cor. 4.1.20.(ii)+(iii) + Lem. 4.1.19.(ii))}\\
&= \ell\mathsf{D}\left[c\ell\;\mathsf{D}[cf]\right] && \text{(Cor. 4.1.20(ii) + Lem. 4.1.19.(ii))}\\
&= \mathsf{L}_{\mathsf{D}}^{C\times A}\left[c\,\mathsf{L}_{\mathsf{D}}^{C\times B}[c\,f]\right]
\end{aligned}$$

We conclude that every Cartesian differential category has a system of linearizing combinators. We now show that the linearizing combinators from Proposition 4.4.2 and Proposition 4.2.6 are the same:

$$\begin{aligned}
\mathsf{L}[f] &= \langle 0,1\rangle\pi_1\mathsf{L}[f]\\
&= \langle 0,1\rangle\mathsf{L}^{C}[\pi_1 f] && \text{(Proposition 4.4.2.(i))}\\
&= \langle 0,1\rangle\ell\mathsf{D}[\pi_1 f] && (4.11)\\
&= \langle 0,1\rangle\ell(\pi_1\times\pi_1)\mathsf{D}[f] && \text{(Lem 4.1.19.(ii)+(vi))}\\
&= \langle 0,1\rangle(\langle 1,0\rangle\times\langle 0,1\rangle)(\pi_1\times\pi_1)\mathsf{D}[f]\\
&= \langle 0,1\rangle(0\times 1)\mathsf{D}[f]\\
&= \langle 0,1\rangle\mathsf{D}[f]\\
&= \mathsf{L}_{\mathsf{D}}[f]
\end{aligned}$$

So we conclude that $\mathsf{L} = \mathsf{L}_{\mathsf{D}}$. □

We now apply Proposition 4.4.5 to some of the examples of Cartesian differential categories from Section 4.1 to obtain examples of systems of linearizing combinators in context, specifically using the construction given in (4.11).

Example 4.4.6 For a category with finite biproducts seen as a Cartesian differential category as in Example 4.1.6, for a map $f : C \times A \longrightarrow B$, the linearizing combinator in context C is defined as evaluating f at zero in its first argument:

$$\mathsf{L}^{C}[f] = (0 \times 1)f$$

For example, in $\mathsf{VEC}_{\mathbb{K}}$, for a $\mathbb{K}$-linear map $f : U \times V \longrightarrow W$, $\mathsf{L}^{U}[f](u,v) = f(0,v)$.

Example 4.4.7 For the Cartesian differential category $\mathsf{POLY}_{\mathbb{K}}$ from Example 4.1.7, a $p(\vec{z}, \vec{x}) : k \times n \longrightarrow m$, which is a polynomial $p(\vec{z}, \vec{x}) \in \mathbb{K}[z_1, \ldots, z_k, x_1, \ldots, x_n]$, can be written as follows:

$$p(\vec{z}, \vec{x}) = \sum_{(k_1,\ldots,k_n) \in \mathbb{N}^n} q_{(k_1,\ldots,k_n)}(\vec{z}) x_1^{k_1} \ldots x_n^{k_n}$$

with $q_{(k_1,\ldots,k_n)}(\vec{z}) \in \mathbb{K}[z_1, \ldots, z_k]$ and with only finitely many $q_{(k_1,\ldots,k_n)}(\vec{z}) \neq 0$. Then the partial linearization picks out the degree 1 terms in variable x_i, that is, $\mathsf{L}^k[p(\vec{z}, \vec{x})] : k \times n \longrightarrow 1$ is the polynomial:

$$\mathsf{L}^k[p(\vec{x})] = \sum q_{(0,0,\ldots,0,1,0,\ldots,0)}(\vec{z}) x_i$$

More generally, for a map $P : n \longrightarrow m$, which is a tuple $P = \langle p_1(\vec{z}, \vec{x}), \ldots, p_m(\vec{z}, \vec{x}) \rangle$, its partial linearization is $\mathsf{L}^k[P] = \langle \mathsf{L}^k[p_1(\vec{x})], \ldots, \mathsf{L}^k[p_m(\vec{x})] \rangle$. For example consider the polynomial function $p(z, x) = z^3 x + z^2 x^3 + x + 1$. The partial linearization of $p(z, x)$ is defined by picking out the terms which are linear in x, that is, $\mathsf{L}^1[p(z, x)] = z^3 x + x$.

Example 4.4.8 For the Cartesian differential category SMOOTH from Example 4.1.8, for a smooth function $F : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $F = \langle f_1, \ldots, f_m \rangle$, its partial linearization is the smooth function $\mathsf{L}^{\mathbb{R}^k}[F] : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined as follows:

$$\mathsf{L}^{\mathbb{R}^k}[F](\vec{z}, \vec{x}) = \nabla(F)(\vec{z}, \vec{0}) \cdot (\vec{0}, \vec{x}) = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{z}, \vec{0}) x_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(\vec{z}, \vec{0}) x_i \right\rangle$$

In particular, for a smooth function $f : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}$, its partial linearization is the sum of its partial derivatives in the $\mathbb{R}^n$ variables evaluated at zero in its first $\mathbb{R}^n$:

$$\mathsf{L}^{\mathbb{R}^k}[f](\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{z}, \vec{0}) x_i$$

Example 4.4.9 For the Cartesian differential category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ from Example 4.1.10, the partial linearizing combinator is precisely the partial linearization operator D_1^i as defined in [5, Convention 5.11]. Explicitly, for a functor $F : C \times A \longrightarrow \mathsf{Ch}(B)$, its partial linearization is $\mathsf{L}^C[F] = \mathsf{D}_1^1[F]$.

Example 4.4.10 For a Cartesian left additive category $\mathbb{X}$, in its cofree Cartesian differential category $\mathcal{D}(\mathbb{X})$ from Example 4.1.11, the linearizing combinator in context C is worked out to be as follows for a D-sequence $(f_0, f_1, f_2, \ldots) : C \times A \longrightarrow B$ (so $f_n : \mathsf{P}^n(C \times A) \longrightarrow B$):

$$\mathsf{L}^C\left[(f_0, f_1, f_2, \ldots)\right] = (\ell f_1, \mathsf{P}(\ell) f_2, \mathsf{P}^2(\ell) f_3, \ldots)$$

where recall that P is the product functor $\mathsf{P}(-) = - \times -$.

Example 4.4.11 For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products, the linearizing combinator in context C for the coKleisli category $\mathbb{X}_!$ is worked out to be as follows for a coKleisli map $[\![f]\!] : !(C \times A) \longrightarrow B$:

$$[\![\mathsf{L}^C[f]]\!] := \begin{array}{l} !(C \times A) \xrightarrow{\chi} !C \otimes !A \xrightarrow{1 \otimes \varepsilon} !C \otimes A \xrightarrow{!(\langle 1, 0 \rangle) \otimes \langle 0, 1 \rangle} \\ !(C \times A) \otimes (C \times A) \xrightarrow{\mathsf{d}} !(C \times A) \xrightarrow{[\![f]\!]} B \end{array}$$

If $(!, \delta, \varepsilon, \Delta, e)$ has Seely isomorphisms, then the linearizing combinator in context can alternatively be expressed using the codereliction map:

$$[\![\mathsf{L}^C[f]]\!] := !(C \times A) \xrightarrow{\chi} !C \otimes !A \xrightarrow{1 \otimes \varepsilon} !C \otimes A \xrightarrow{1 \otimes \eta} !C \otimes !A \xrightarrow{\chi^{-1}} !(C \times A) \xrightarrow{[\![f]\!]} B$$

Example 4.4.12 For the Cartesian differential category CON from Example 4.1.13, the partial linearizing combinator is defined as follows on a smooth function $f : C \times E \longrightarrow F$:

$$\mathsf{L}^C[f](z, x) := \lim_{t \longrightarrow 0} \frac{f(z, t \cdot x) - f(z, 0)}{t}$$

We now prove the converse of Proposition 4.4.5 and show that from a system of linearizing combinators one can build a differential combinator. The construction is a generalization of the differential combinator found in [5]. The derivative of f can be defined by linearizing in context f precompose by the addition map, which in the term logic is written as:

$$\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b = \frac{\ell f(x + a)}{\ell x} \cdot b$$

For example, consider the polynomial function $f(x) = x^3 + x$, then:

$$f(x + y) = (x + y)^3 + x + y = x^3 + 3x^2y + 3xy^2 + y^3 + x + y$$

The linearization of $f(x + y)$ in terms of y is $3x^2y + y$ which is precisely the directional derivative $\mathsf{D}[f](x, y)$.

Proposition 4.4.13 *[29, Proposition 5.13] Every Cartesian left additive category $\mathbb{X}$ with a system of linearizing combinators L^C is a Cartesian differential category with differential combinator D_L defined as follows on a map $f : A \longrightarrow B$:*

$$\mathsf{D}_\mathsf{L}[f] := \mathsf{L}^A[\oplus_A f] \tag{4.15}$$

where $\oplus_A$ is defined as in Lemma 4.1.4. Furthermore,

(i) *For every map $f : A \longrightarrow B$, $\mathsf{D}_\mathsf{L}[f]$ is L^A-linear;*

(ii) *A map $f : C \times A \longrightarrow B$ is linear in its second argument if and only if f is L^C-linear.*

(iii) *$\mathsf{L} = \mathsf{L}_{\mathsf{D}_\mathsf{L}}$, where $\mathsf{L}_{\mathsf{D}_\mathsf{L}}$ is the induced linearizing combinator from Proposition 4.2.6 and L is the induced linearizing combinator from Proposition 4.4.2.*

Proof: We must show that D_L satisfies **[CD.1]** to **[CD.7]**.

[CD.1] $\mathsf{D}_\mathsf{L}[f + g] = \mathsf{D}_\mathsf{L}[f] + \mathsf{D}_\mathsf{L}[g]$ and $\mathsf{D}_\mathsf{L}[0] = 0$

$$\begin{aligned} \mathsf{D}_\mathsf{L}[f + g] &= \mathsf{L}^A[\oplus_A(f + g)] \\ &= \mathsf{L}^A[\oplus_A f + \oplus_A g] \\ &= \mathsf{L}^A[\oplus_A f] + \mathsf{L}^A[\oplus_A g] && \textbf{[L.1]} \end{aligned}$$

$$= \mathsf{D}_{\mathsf{L}}[f] + \mathsf{D}_{\mathsf{L}}[g]$$

$$\begin{aligned} \mathsf{D}_{\mathsf{L}}[0] &= \mathsf{L}^A[\oplus_A 0] \\ &= \mathsf{L}^A[0] \\ &= 0 && \textbf{[L.1]} \end{aligned}$$

[CD.2] $(1 \times \oplus_A)\mathsf{D}_{\mathsf{L}}[f] = (1 \times \pi_0)\mathsf{D}_{\mathsf{L}}[f] + (1 \times \pi_1)\mathsf{D}_{\mathsf{L}}[f]$ and $\langle 1, 0\rangle \mathsf{D}_{\mathsf{L}}[f] = 0$

$$\begin{aligned} (1 \times \oplus_A)\mathsf{D}_{\mathsf{L}}[f] &= (1 \times \oplus_A)\mathsf{L}^A[\oplus_A f] \\ &= (1 \times \pi_0)\mathsf{L}^A[\oplus_A f] + (1 \times \pi_1)\mathsf{L}^A[\oplus_A f] && \textbf{[L.2]} \\ &= (1 \times \pi_0)\mathsf{D}_{\mathsf{L}}[f] + (1 \times \pi_1)\mathsf{D}_{\mathsf{L}}[f] \end{aligned}$$

$$\begin{aligned} \langle 1, 0\rangle \mathsf{D}_{\mathsf{L}}[f] &= \langle 1, 0\rangle \mathsf{L}^A[\oplus_A f] \\ &= 0 && \textbf{[L.2]} \end{aligned}$$

[CD.3] $\mathsf{D}_{\mathsf{L}}[1] = \pi_1$, $\mathsf{D}_{\mathsf{L}}[\pi_0] = \pi_1\pi_0$, and $\mathsf{D}_{\mathsf{L}}[\pi_1] = \pi_1\pi_1$

$$\begin{aligned} \mathsf{D}_{\mathsf{L}}[1] &= \mathsf{L}^A[\oplus_A] \\ &= \mathsf{L}^A[\pi_0 + \pi_1] \\ &= \mathsf{L}^A[\pi_0] + \mathsf{L}^A[\pi_1] && \textbf{[L.1]} \\ &= 0 + \mathsf{L}^A[\pi_1] && \text{(Lemma 4.4.3.(i))} \\ &= \pi_1 && \textbf{[L.3]} \end{aligned}$$

$$\begin{aligned} \mathsf{D}_{\mathsf{L}}[\pi_i] &= \mathsf{L}^{A\times B}[\oplus_{A\times B}\pi_i] \\ &= \mathsf{L}^{A\times B}\left[(\pi_0 + \pi_1)\pi_i\right] \\ &= \mathsf{L}^{A\times B}[\pi_0\pi_i + \pi_1\pi_i] && (\pi_i \text{ is additive}) \\ &= \mathsf{L}^{A\times B}[\pi_0\pi_i] + \mathsf{L}^{A\times B}[\pi_1\pi_i] && \textbf{[L.1]} \\ &= 0 + \mathsf{L}^{A\times B}[\pi_1\pi_i] && \text{(Lemma 4.4.3.(i))} \\ &= \pi_1\pi_i && \textbf{[L.3]} \end{aligned}$$

[CD.4] $\mathsf{D}_{\mathsf{L}}\left[\langle f, g\rangle\right] = \left\langle \mathsf{D}_{\mathsf{L}}[f], \mathsf{D}_{\mathsf{L}}[g]\right\rangle$

$$\begin{aligned} \mathsf{D}_{\mathsf{L}}\left[\langle f, g\rangle\right] &= \mathsf{L}^A[\oplus_A\langle f, g\rangle] \\ &= \mathsf{L}^A[\langle \oplus_A f, \oplus_A g\rangle] \\ &= \left\langle \mathsf{L}^A[\oplus_A f], \mathsf{L}^A[\oplus_A g]\right\rangle && \textbf{[L.4]} \\ &= \left\langle \mathsf{D}_{\mathsf{L}}[f], \mathsf{D}_{\mathsf{L}}[g]\right\rangle \end{aligned}$$

[CD.5] $\mathsf{D}_{\mathsf{L}}[fg] = \langle \pi_0 f, \mathsf{D}_{\mathsf{L}}[f]\rangle \mathsf{D}_{\mathsf{L}}[g]$

$$\mathsf{D}_{\mathsf{L}}[fg] = \mathsf{L}^A[\oplus_A fg]$$

$$
\begin{aligned}
&= \mathsf{L}^A\left[\langle \pi_0, \oplus_A f\rangle \pi_1 g\right] \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[\left\langle \pi_0, \pi_1 + \langle \pi_0, 0\rangle \oplus_A f\right\rangle \pi_1 g\right] && \textbf{[L.5]} \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[\left(\pi_1 + \langle \pi_0, 0\rangle \oplus_A f\right) g\right] \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[\left(\pi_1 + \pi_0\langle 1, 0\rangle \oplus_A f\right) g\right] \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[(\pi_1 + \pi_0 f)\, g\right] && \text{(Lemma 4.1.4.(i))} \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[\left((f\times 1)\pi_1 + (f \times 1)\pi_0\right) g\right] \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[(f\times 1)\,(\pi_1 + \pi_0)\, g\right] \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[(f\times 1)\,(\pi_0 + \pi_1)\, g\right] \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^A\left[(f\times 1) \oplus_B g\right] \\
&= \langle \pi_0, \mathsf{L}^A[\oplus_A f]\rangle (f \times 1)\mathsf{L}^B[\oplus_B g] && \textbf{[L.8]} \\
&= \langle \pi_0 f, \mathsf{L}^A[\oplus_A f]\rangle\, \mathsf{L}^B[\oplus_B g] \\
&= \langle \pi_0 f, \mathsf{D}_\mathsf{L}[f]\rangle \mathsf{D}_\mathsf{L}[g]
\end{aligned}
$$

[CD.6] $\ell \mathsf{D}_\mathsf{L}\left[\mathsf{D}_\mathsf{L}[f]\right] = \mathsf{D}_\mathsf{L}[f]$

$$
\begin{aligned}
\ell\, \mathsf{D}_\mathsf{L}\left[\mathsf{D}_\mathsf{L}[f]\right] &= \ell\, \mathsf{L}^{A\times A}\left[\oplus_{A\times A}\mathsf{D}_\mathsf{L}[f]\right] \\
&= \mathsf{L}^A\left[\ell \oplus_{A\times A} \mathsf{D}_\mathsf{L}[f]\right] && \text{(Lemma 4.4.3.(ii))} \\
&= \mathsf{L}^A\left[\mathsf{D}_\mathsf{L}[f]\right] && \text{(Lemma 4.1.4.(ii))} \\
&= \mathsf{L}^A\left[\mathsf{L}^A[\oplus_A f]\right] \\
&= \mathsf{L}^A[\oplus_A f] && \textbf{[L.6]} \\
&= \mathsf{D}_\mathsf{L}[f]
\end{aligned}
$$

[CD.7] $c\, \mathsf{D}_\mathsf{L}\left[\mathsf{D}_\mathsf{L}[f]\right] = \mathsf{D}_\mathsf{L}\left[\mathsf{D}_\mathsf{L}[f]\right]$

$$
\begin{aligned}
c\, \mathsf{D}_\mathsf{L}\left[\mathsf{D}_\mathsf{L}[f]\right] &= c\, \mathsf{L}^{A\times A}\left[\oplus_{A\times A}\mathsf{L}^A[\oplus_A f]\right] \\
&= c\, \mathsf{L}^{A\times A}\left[c(\oplus_A \times \oplus_A)\mathsf{L}^A[\oplus_A f]\right] && \text{(Lemma 4.1.4.(ii))} \\
&= c\, \mathsf{L}^{A\times A}\left[c\, \mathsf{L}^{A\times A}[(\oplus_A \times \oplus_A) \oplus_A f]\right] && \text{(Lemma 4.4.3.(iii))} \\
&= \mathsf{L}^{A\times A}\left[c\, \mathsf{L}^{A\times A}[c(\oplus_A \times \oplus_A) \oplus_A f]\right] && \textbf{[L.7.a]} \\
&= \mathsf{L}^{A\times A}\left[c\, \mathsf{L}^{A\times A}[(\oplus_A \times \oplus_A) \oplus_A f]\right] && \text{(Lemma 4.1.4.(i))} \\
&= \mathsf{L}^{A\times A}\left[c(\oplus_A \times \oplus_A)\mathsf{L}^A[\oplus_A f]\right] && \text{(Lemma 4.4.3.(iii))} \\
&= \mathsf{L}^{A\times A}\left[\oplus_{A\times A}\mathsf{L}^A[\oplus_A f]\right] && \text{(Lemma 4.1.4.(ii))} \\
&= \mathsf{D}_\mathsf{L}\left[\mathsf{D}_\mathsf{L}[f]\right]
\end{aligned}
$$

So we conclude that D_{L} is a differential combinator. Next, it follows immediately from **[L.6]** that:

$$\begin{aligned} \mathsf{L}^A[\mathsf{D}_{\mathsf{L}}[f]] &= \mathsf{L}^A[\mathsf{L}^A[\oplus_A f]] \\ &= \mathsf{L}^A[\oplus_A f] && \textbf{[L.6]} \\ &= \mathsf{D}_{\mathsf{L}}[f] \end{aligned}$$

Therefore, $\mathsf{D}_{\mathsf{L}}[f]$ is L^A-linear. Now suppose that a map $f : C \times A \longrightarrow B$ was L^C-linear, that is, $\mathsf{L}^C[f] = f$. Then we compute that:

$$\begin{aligned} \ell\mathsf{D}_{\mathsf{L}}[f] &= \ell\mathsf{L}^{C\times A}[\oplus_{C\times A} f] \\ &= \mathsf{L}^C[\ell \oplus_{C\times A} f] && \text{(Lemma 4.4.3.(ii))} \\ &= \mathsf{L}^C[f] && \text{(Lemma 4.1.4.(ii))} \\ &= f && (f \text{ is } \mathsf{L}^C\text{-linear}) \end{aligned}$$

Then by Lemma 4.3.6.(i), f is linear in its second argument. Conversly, suppose that $f : C \times A \longrightarrow B$ is linear in its second argument, that is, $\ell\mathsf{D}_{\mathsf{L}}[f] = f$. Then we have that:

$$\begin{aligned} \mathsf{L}^C[f] &= \mathsf{L}^C[\ell \oplus_{C\times A} f] && \text{(Lemma 4.1.4.(ii))} \\ &= \ell\mathsf{L}^{C\times A}[\oplus_{C\times A} f] && \text{(Lemma 4.4.3.(ii))} \\ &= \ell\mathsf{D}_{\mathsf{L}}[f] \\ &= f && (f \text{ is linear in its second argument}) \end{aligned}$$

Therefore, f is L^C-linear. Lastly, we show that, in this case, the constructions of the linearizing combinators from Proposition 4.4.2 and Proposition 4.2.6 are the same:

$$\begin{aligned} \mathsf{L}_{\mathsf{D}_{\mathsf{L}}}[f] &= \langle 0, 1\rangle\mathsf{D}_{\mathsf{L}}[f] \\ &= \langle 0, 1\rangle\mathsf{L}^A[\oplus_A f] \\ &= \langle 0, 1\rangle(0 \times 1)\mathsf{L}^A[\oplus_A f] \\ &= \langle 0, 1\rangle\mathsf{L}^{\top}[(0 \times 1) \oplus_A f] && \textbf{[L.8]} \\ &= \langle 0, 1\rangle\mathsf{L}^{\top}[(0 \times 1)(\pi_0 + \pi_1)f] \\ &= \langle 0, 1\rangle\mathsf{L}^{\top}[\big((0 \times 1)\pi_0 + (0 \times 1)\pi_1\big) f] \\ &= \langle 0, 1\rangle\mathsf{L}^{\top}[(0 + \pi_1) f] \\ &= \langle 0, 1\rangle\mathsf{L}^{\top}[\pi_1 f] \\ &= \mathsf{L}[f] \end{aligned}$$

So we conclude that $\mathsf{L} = \mathsf{L}_{\mathsf{D}_{\mathsf{L}}}$. □

We may now state the main result of this chapter.

Theorem 4.4.14 *[29, Theorem 5.14] For a Cartesian left additive category $\mathbb{X}$, there is a bijective correspondence between:*

(i) *Differential combinators;*

(ii) *Systems of linearizing combinators.*

Therefore, a Cartesian differential category is precisely a Cartesian left additive category equipped with a system of linearizing combinators.

Proof: It suffices to show that the constructions of Proposition 4.4.13 and Proposition 4.2.6 are inverses of each other. Starting with a differential combinator D, we first show that $\mathsf{D}_{\mathsf{L}_\mathsf{D}} = \mathsf{D}$:

$$\begin{aligned}
\mathsf{D}_{\mathsf{L}_\mathsf{D}}[f] &= \mathsf{L}^A_\mathsf{D}[\oplus_A f] \\
&= \ell\, \mathsf{D}\,[\oplus_A f] && (4.11) \\
&= \ell\,(\oplus_A \times \oplus_A)\,\mathsf{D}[f] && \text{(Cor. 4.1.20(iv) + Lem. 4.1.19.(ii))} \\
&= \mathsf{D}[f] && \text{(Lemma 4.1.4.(ii))}
\end{aligned}$$

Next, starting with a system of linearizing combinators L^C, we show that $\mathsf{L}^C_{\mathsf{D}_\mathsf{L}} = \mathsf{L}^C$:

$$\begin{aligned}
\mathsf{L}^C_{\mathsf{D}_\mathsf{L}}[f] &= \ell\, \mathsf{D}_\mathsf{L}[f] \\
&= \ell\, \mathsf{L}^{C\times A}[\oplus_{C\times A} f] && (4.11) \\
&= \mathsf{L}^C[\ell \oplus_{C\times A} f] && \text{(Lemma 4.4.3.(ii))} \\
&= \mathsf{L}^C\,[f] && \text{(Lemma 4.1.4.(ii))}
\end{aligned}$$

Thus, differential combinators and systems of linearizing combinators are in bijective correspondence. Therefore, we conclude that a Cartesian differential category is precisely a Cartesian left additive category equipped with a system of linearizing combinators. □

We conclude this section by providing an example of a Cartesian left additive category which has a total linearization but does not have partial linearization. This means that it is not possible, in general, to derive partial linearization from the presence of a total linearizing combinator.

Example 4.4.15 Recall that a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is a tuple $F = \langle f_1, \ldots, f_m \rangle$ of functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, is a $\mathcal{C}^1$ function if for each f_i, all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists and are continuous. Then define $\mathcal{C}^1$-DIFF be the category whose objects are the Euclidean real vector spaces $\mathbb{R}^n$ and whose maps are $\mathcal{C}^1$ functions $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ between them. $\mathcal{C}^1$-DIFF is a Cartesian left additive category in the obvious way, and note that SMOOTH is a sub-Cartesian left additive category of $\mathcal{C}^1$-DIFF. Notice that $\mathcal{C}^1$-DIFF has a (total) linearizing combinator L defined in the same way as the linearizing combinator in SMOOTH, that is, for a $\mathcal{C}^1$ function $F = \langle f_1, \ldots, f_m \rangle$:

$$\mathsf{L}[F](\vec{x}) = \left\langle \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0})x_i, \ldots, \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0})x_i \right\rangle$$

However, this category, while having a total linearizing combinator, does not have partial linearization. If $\mathcal{C}^1$-DIFF had partial linearization then $\mathcal{C}^1$-DIFF would also have a differential combinator, but this can't be since the derivative of $\mathcal{C}^1$ functions are not necessarily $\mathcal{C}^1$ functions. Explicitly, consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f = x^{\frac{3}{2}}$, which is a $\mathcal{C}^1$ function since its derivative $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

exists and is continuous. If partial linearization was possible, then we would be able to define $\mathsf{D}[f] : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$\mathsf{D}[f](x, y) = \mathsf{L}[z \mapsto f(x + z)](y) = \frac{3}{2}(x + y)^{\frac{1}{2}}$$

However, this linearization is not define when $x + y < 0$ and therefore is not a map in $\mathcal{C}^1$-DIFF. So we conclude that $\mathcal{C}^1$-DIFF has a total linearizing combinator, however, it is not induced by a differential combinator and, therefore, the category does not have partial linearization.

4.5 Linearizing Combinators in the Closed Setting

We would like to prove the converse of Proposition 4.4.2, that is, we would like to define partial linearization from total linearization. As previously discussed, in general this is not necessarily possible. However in the setting of a Cartesian closed category, it is possible to construct a system of linearizing combinators from a linearizing combinator on the base category. The key to this construction is the ability to curry and uncurry maps, which allows us to move the context of a map from its domain to its codomain. Indeed, given a map $f : C \times A \longrightarrow B$, to linearize A while keeping C in context, one takes the total linearization of its curry $\lambda(f) : A \longrightarrow [C, B]$ and then uncurry to obtain $\mathsf{L}^C[f] : C \times A \longrightarrow B$. For this to work, one must also require that the linearizing combinator be compatible with the closed structure, which we call an *exponential* linearizing combinator. Furthermore, we will also show that Cartesian *closed* differential categories are precisely Cartesian *closed* left additive categories equipped with an exponential linearizing combinators.

We begin this section by setting up notation for Cartesian closed categories and reviewing some basic, but very important, properties. For a Cartesian closed category $\mathbb{X}$, we denote the internal-hom by $[C, A]$, the evaluation map by $\epsilon_{C,A} : C \times [C, A] \longrightarrow A$ (from now on we will omit the subscripts and simply write ϵ when there is no confusion), and the curry of a map $f : C \times A \longrightarrow B$ as the map $\lambda(f) : A \longrightarrow [C, B]$, that is, $\lambda(f)$ is the unique map such that:

$$(1 \times \lambda(f))\epsilon = f$$

Conversly, define the un-curry of a map of type $g : A \longrightarrow [C, B]$ as the map $\lambda^{-1}(g) : C \times A \longrightarrow B$ which is defined as:

$$\lambda^{-1}(g) := (1 \times g)\epsilon$$

Therefore, $\lambda\left(\lambda^{-1}(g)\right) = g$ and $\lambda^{-1}\left(\lambda(f)\right) = f$.

Next we review the notion of Cartesian closed differential categories. As the name suggests, Cartesian closed differential categories are Cartesian differential categories whose underlying category is also Cartesian closed and such that the differential combinator is compatible with the curry operator. Furthermore, Cartesian closed differential categories provide suitable models to interpret *differential* λ-calculus [37]. Cartesian closed differential categories are also sometimes called differential λ categories. For a more in-depth introduction to Cartesian closed differential categories, we refer the reader to [14, 18, 69].

We must first discuss the notion of Cartesian closed left additive categories:

Definition 4.5.1 *A **Cartesian closed left additive category** [9, Section 1.4] is a Cartesian left additive category which is also a Cartesian closed category such that the currying operator preserves the additive structure, that is, $\lambda(f+g) = \lambda(f) + \lambda(g)$ and $\lambda(0) = 0$ (note that this implies that $\lambda^{-1}(f+g) = \lambda^{-1}(f) + \lambda^{-1}(g)$ and $\lambda^{-1}(0) = 0$).*

As shown in [18, Lemma 4.10], there are two equivalent ways of expressing compatibility between the closed structure and the differential combinator: one in terms of the curry operator and one in terms of the evaluation map.

Definition 4.5.2 *A **Cartesian closed differential category** [18, Section 4.6] (also known as a **differential** λ **category** [14, 69]) is a Cartesian differential category which is also a Cartesian closed left additive category such that one of the following additional axioms hold:*

[CD.λ] *For every map $f : C \times A \longrightarrow B$, $\mathsf{D}[\lambda(f)] = \lambda\left(\mathsf{D}^C[f]\right)$, where D^C is defined as in (4.5).*

or equivalently,

[CD.ev] *Evaluation maps $\epsilon : C \times [C, A] \longrightarrow A$ are linear in their second argument (Definition 4.3.5), that is, $\mathsf{D}^C[\epsilon] = (1 \times \pi_1)\epsilon$, or equivalently by Lemma 4.3.6.(i), $\ell\mathsf{D}[\epsilon] = \epsilon$.*

Here are now some examples of Cartesian closed differential categories.

Example 4.5.3 Every model of the differential λ-calculus [37] induces a Cartesian closed differential category [18, Theorem 4.3], and conversely every Cartesian closed differential category gives rise to a model of the differential λ-calculus [14, Theorem 4.12].

Example 4.5.4 Let $\mathbb{X}$ be a differential category with coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products. Suppose that $\mathbb{X}$ is also a symmetric monoidal closed category and that $(!, \delta, \varepsilon, \Delta, e)$ has Seely isomorphisms. Then the coKleisli category $\mathbb{X}_!$ is a Cartesian closed differential category [10, Theorem 4.4.2]. The internal-homs in the coKleisli category $\mathbb{X}_!$ are defined as $[A, B] = !A \multimap B$. Examples of such coKleisli categories are discussed in [14, Section 5], which include the relational model and the finiteness space model.

Example 4.5.5 CON_{lin} is a differential category with Seely isomorphisms such that CON_{lin} is symmetric monoidal closed [11, Theorem 4.2]. Therefore, since CON is isomorphic to the coKleisli category of the comonad $!$ on CON_{lin}, it follows that CON is also a Cartesian closed differential category (see [55, Theorem 3.12] for its Cartesian closed structure). In particular, for convenient vector spaces E and F, if we let $\mathcal{L}(E, F)$ denote the set of (smooth(linear function between E and F and $\mathcal{C}^\infty(E, F)$ the set of all smooth functions between E and F, then $\mathcal{C}^\infty(E, F) \cong L(!E, F)$ [11, Theorem 6.3].

We now turn our attention to the main objective of this section: on how to define partial linearization from total linearization in the setting of a Cartesian closed left additive category. To do so, we introduce the notions of linearizing combinators and systems of linearizing combinators which are compatible with the closed structure. We begin with *closed* systems of linearizing combinators, which are the Cartesian closed differential category version of systems of linearizing combinators.

Definition 4.5.6 *A* ***closed system of linearizing combinators*** *[29, Definition 6.6] on a Cartesian left additive category $\mathbb{X}$ is a system of linearizing combinators L^C on $\mathbb{X}$ such that the following extra axiom holds:*

[L.λ] *For every map $f : C \times A \longrightarrow B$, $\mathsf{L}[\lambda(f)] = \lambda\left(\mathsf{L}^C[f]\right)$, where L is defined as in (4.6).*

As we will see in Theorem 4.5.12, to give a Cartesian closed differential category is precisely to give a closed system of linearizing combinators. As such, **[L.λ]** is the linearizing combinator analogue of **[CD.λ]**. Therefore, the extra axiom of a closed system of linearizing combinators can equivalently be defined in terms of the evaluation map, **[L.ev]**, which is the linearizing combinator analgoue of **[CD.ev]**.

Lemma 4.5.7 *[29, Lemma 6.7]* **[L.λ]** *is equivalent to the following:*

[L.ev] *Evaluation maps $\epsilon : C \times [C, A] \longrightarrow A$ are L^C-linear, that is, $\mathsf{L}^C[\epsilon] = \epsilon$.*

Proof: Suppose that **[L.λ]** holds. Since $\epsilon = \lambda^{-1}(1)$, we have that:

$$\begin{aligned}
\mathsf{L}^C[\epsilon] &= \lambda^{-1}\left(\lambda\left(\mathsf{L}^C[\epsilon]\right)\right) \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda(\epsilon)]\right) && \textbf{[L.}\lambda\textbf{]} \\
&= \lambda^{-1}\left(\mathsf{L}[1]\right) \\
&= \lambda^{-1}(1) && \textbf{[L.3]} \\
&= \epsilon
\end{aligned}$$

So $\mathsf{L}^C[\epsilon] = \epsilon$, and so ϵ is L^C-linear. Conversly, suppose that **[L.ev]** holds. Then we compute:

$$\begin{aligned}
\lambda\left(\mathsf{L}^C[f]\right) &= \lambda\left(\mathsf{L}^C[\lambda^{-1}\left(\lambda(f)\right)]\right) \\
&= \lambda\left(\mathsf{L}^C[(1 \times \lambda(f))\epsilon]\right) \\
&= \lambda\left(\mathsf{L}^C[\langle \pi_0, \pi_1\lambda(f)\rangle\epsilon]\right) \\
&= \lambda\left(\langle \pi_0, \mathsf{L}^C[\pi_1\lambda(f)]\rangle\ \epsilon\right) && (\textbf{[L.ev]} + \text{Lem.4.2.4.(iv)}) \\
&= \lambda\left(\langle \pi_0, \pi_1\mathsf{L}[\lambda(f)]\rangle\ \epsilon\right) && (\text{Prop.4.4.2.(i)}) \\
&= \lambda\left((1 \times \mathsf{L}[\lambda(f)])\epsilon\right) \\
&= \lambda\left(\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right)\right) \\
&= \mathsf{L}[\lambda(f)]
\end{aligned}$$

So $\mathsf{L}[\lambda(f)] = \lambda\left(\mathsf{L}^C[f]\right)$. □

We now define *exponentiable* linearizing combinators, which from a system of linear maps perspective is the analogue of an exponentiable system of maps [10, Definition 2.2.1]. To do so, we must first review the canonical monads of the form $[C, -]$ in a Cartesian closed category. For a pair of maps $f : C \longrightarrow D$ and $g : A \longrightarrow B$, define the map $[f, g] : [D, A] \longrightarrow [C, B]$ as:

$$[f, g] := \lambda\left((f \times 1)\epsilon g\right)$$

Intuitively, $[f, g]$ is the map which pre-composes by f and post-composes by g. In particular, note that $[-, -]$ is contravariant in its first argument and covariant in its second argument, that is:

$$[fh, kg] = [h, k][f, g]$$

For each object C, define the functor $E^C : \mathbb{X} \longrightarrow \mathbb{X}$ on objects as $E^C(A) = [C, A]$ and on maps $E^C(f) = [1, f]$. E^C is monad where the monad unit $\eta^C_A : A \longrightarrow [C, A]$ and the monad multiplication $\mu^C_A : [C, [C, A]] \longrightarrow [C, A]$ are defined respectively as follows:

$$\eta^C_A := \lambda(\pi_1) \qquad \mu^C_A := \lambda\left(\langle \pi_0, \epsilon\rangle \epsilon\right) \tag{4.16}$$

Once again, as to not overload notation, we will omit the subscripts and superscripts and simply write η and μ when there is no confusion.

Definition 4.5.8 *An **exponentiable linearizing combinator** [29, Definition 6.8]* L *on a Cartesian left additive category* $\mathbb{X}$ *is a linearizing combinator* L *on* $\mathbb{X}$ *such that the following extra three axioms hold:*

[EL.1] $\mathsf{L}[\eta] = \eta$ *and* $\mathsf{L}[\mu] = \mu$

[EL.2] $\mathsf{L}\left[[f, g]\right] = \left[f, \mathsf{L}[g]\right]$

[EL.3] *For a map* $f : A \times B \longrightarrow C$*, define* $\mathsf{L}_0[f] : A \times B \longrightarrow C$ *and* $\mathsf{L}_1[f] : A \times B \longrightarrow C$ *respectively as follows:*

$$\mathsf{L}_0[f] := \tau\lambda^{-1}\left(\mathsf{L}\left[\lambda\left(\tau f\right)\right]\right) \qquad \mathsf{L}_1[f] := \lambda^{-1}\left(\mathsf{L}\left[\lambda\left(f\right)\right]\right)$$

where τ *was the canonical symmetry isomorphism defined in (4.1). Then for every map* $f : A \times B \longrightarrow C$, $\mathsf{L}_0[\mathsf{L}_1[f]] = \mathsf{L}_1[\mathsf{L}_0[f]]$.

As we will see in Proposition 4.5.11, from an exponentiable linearizing combinator we will be able to construct a closed system of linearizing combinators by uncurying the linearization of the curry. In other words, we will be able to define total linearization from partial linearization. We first show that, as expected, a Cartesian closed left additive category with a closed systems of linearizing combinators is in fact a Cartesian closed differential category, and its induced linearizing combinator is an exponentiable linearizing combinator. Alternatively, we could have instead shown that the induced linearizing combinator and system of linearizing combinators of a Cartesian closed differential category are respectively exponentiable and closed. Therefore, a Cartesian closed differential category is precisely a Cartesian closed left additive category with a closed system of linearizing combinators.

Proposition 4.5.9 *[29, Proposition 6.9] For Cartesian closed left additive category* $\mathbb{X}$ *with a closed systems of linearizing combinators* L^C*:*

(i) *The induced linearizing combinator* L *from Proposition 4.4.2 is an exponentiable linearizing combinator and* $\mathsf{L}^C[f] = \lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right)$.

(ii) *The induced differential combinator* D_L *from Proposition 4.4.13 satisfies* **[CD.**λ**]** *(or equivalently* **[CD.ev]***) and* $\mathsf{D}_\mathsf{L}[f] = \lambda^{-1}\left(\mathsf{L}[\lambda(\oplus_A f)]\right)$.

Therefore, a Cartesian closed left additive category with a closed systems of linearizing combinators is a Cartesian closed differential category.

Proof: First note that $\mathsf{L}^C[f] = \lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right)$ follows immediately from **[L.**λ**]**, and therefore we also have that $\mathsf{D}_\mathsf{L}[f] = \lambda^{-1}\left(\mathsf{L}[\lambda(\oplus_A f)]\right)$. Next we show that L satisfies **[EL.1]**, **[EL.2]**, and **[EL.3]**.

[EL.1]: $\mathsf{L}[\eta] = \eta$ and $\mathsf{L}[\mu] = \mu$

$$\begin{aligned} \mathsf{L}[\eta] &= \mathsf{L}[\lambda(\pi_1)] \\ &= \lambda(\mathsf{L}^C[\pi_1]) && \textbf{[L.}\lambda\textbf{]} \\ &= \lambda(\pi_1) && \textbf{[L.3]} \\ &= \eta \end{aligned}$$

$$\begin{aligned} \mathsf{L}[\mu] &= \mathsf{L}[\lambda\left(\langle \pi_0, \epsilon\rangle\epsilon\right)] \\ &= \lambda(\mathsf{L}^C[\left(\langle \pi_0, \epsilon\rangle\epsilon\right)]) && \textbf{[L.}\lambda\textbf{]} \\ &= \lambda\left(\langle \pi_0, \epsilon\rangle\epsilon\right) && (\textbf{[L.ev]} + \text{Lem.4.2.4.(viii)}) \\ &= \mu \end{aligned}$$

[EL.2]: $\mathsf{L}\left[[f,g]\right] = \left[f, \mathsf{L}[g]\right]$:

$$\begin{aligned} \mathsf{L}\left[[f,g]\right] &= \mathsf{L}[\lambda\left((f\times 1)\epsilon g\right)] \\ &= \lambda\left(\mathsf{L}^C[(f\times 1)\epsilon g]\right) \\ &= \lambda\left((f\times 1)\mathsf{L}^{C'}[\epsilon g]\right) && \textbf{[L.8]} \\ &= \lambda\left((f\times 1)\mathsf{L}^{C'}[\langle \pi_0, \epsilon\rangle \pi_1 g]\right) \\ &= \lambda\left((f\times 1)\langle \pi_0, \epsilon\rangle\mathsf{L}^{C'}[\pi_1 g]\right) && (\textbf{[L.ev]} + \text{Lem.4.2.4.(iii)}) \\ &= \lambda\left((f\times 1)\langle \pi_0, \epsilon\rangle\pi_1\mathsf{L}[g]\right) && (\text{Prop.4.4.2.(i)}) \\ &= \lambda\left((f\times 1)\epsilon\mathsf{L}[g]\right) \\ &= \left[f, \mathsf{L}[g]\right] \end{aligned}$$

[EL.3]: $\mathsf{L}_0[\mathsf{L}_1[f]] = \mathsf{L}_1[\mathsf{L}_0[f]]$:

Note that by **[L.**λ**]**, $\lambda^{-1}\left(\mathsf{L}[f]\right) = \mathsf{L}^C[\lambda^{-1}(f)]$. As such, it immediately follows that the L_0 and L_1 as defined in **[EL.3]** are precisely the same as L_0 and L_1 defined in Proposition 4.4.2.(iv). Therefore **[EL.3]** is precisely Proposition 4.4.2.(iv).

So we conclude that L is an exponentiable linearizing combinator. Next we must check that $\mathsf{D_L}$ satisfies **[CD.λ]** or equivalently **[CD.ev]**. By **[L.ev]** , ϵ is L^C-linear and so by Proposition 4.4.13.(ii), ϵ is linear in its second argument. Therefore, **[CD.ev]** holds and we conclude that a Cartesian closed left additive category with a closed systems of linearizing combinators is a Cartesian closed differential category. □

Corollary 4.5.10 *[29, Corollary 6.10] For a Cartesian closed differential category $\mathbb{X}$ with differential combinator D:*

(i) *The induced system of linearizing combinators L^C_D from Proposition 4.4.5 is a closed system of linearizing combinators.*

(ii) *The induced linearizing combinator $\mathsf{L_D}$ from Proposition 4.2.6 is an exponential linearizing combinator.*

Proof: We must show that L^C_D satisfies **[L.λ]** or equivalently **[L.ev]**. By **[CD.ev]** , ϵ is D-linear in its second argument, and so by Proposition 4.4.5.(ii), ϵ is L^C_D-linear. Therefore, **[L.ev]** holds and we conclude that L^C_D is a closed system of linearizing combinators. By Proposition 4.5.9.(i), the induced linearizing combinator from Proposition 4.4.2 is an exponentiable linearizing combinator. However by Proposition 4.4.5.(ii), the induced linearizing combinator from Proposition 4.2.6 is the precisely the same as the one from Proposition 4.4.2. Therefore, $\mathsf{L_D}$ is an exponentiable linearizing combinator.

□

We now prove the converse of Proposition 4.5.9, that in the closed setting we may define partial linearization from total linearization, that is, we will show that an exponentiable linearizing combinator induces a closed system of linearizing combinators. As a consequence, it follows that a Cartesian closed differential category is precisely a Cartesian left additive category with an exponentiable linearizing combinator.

Proposition 4.5.11 *[29, Proposition 6.11] For Cartesian closed left additive category $\mathbb{X}$ with an exponential linearizing combinator L:*

(i) *$\mathbb{X}$ comes equipped with a closed system of linearizing combinators L^C defined as follows for a map $f : C \times A \longrightarrow B$:*

$$\mathsf{L}^C[f] = \lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right)$$

and the resulting induced linearizing combinator from Proposition 4.4.2 is precisely L.

(ii) *$\mathbb{X}$ is a Cartesian closed differential category with differential combinator $\mathsf{D_L}$ defined as follows for a map $f : A \longrightarrow B$:*

$$\mathsf{D_L}[f] = \lambda^{-1}\left(\mathsf{L}[\lambda(\oplus_A f)]\right)$$

and furthermore, this differential combinator is precisely the induced differential combinator from Proposition 4.4.13.

Therefore, a Cartesian closed left additive category with an exponential linearizing combinator is a Cartesian closed differential category.

Proof: First, here are some useful identities which hold in any Cartesian closed category:

$$\lambda\left((f \times g)hk\right) = g\lambda(h)[f,k] \qquad (f \times g)\lambda^{-1}(h')k = \lambda^{-1}\left(gh'[f,k]\right) \tag{4.17}$$

Now we show that L^C satisfies **[L.1]**-**[L.8]** and **[L.ev]**:

[L.1]: $\mathsf{L}^C[f+g] = \mathsf{L}^C[f] + \mathsf{L}^C[g]$ and $\mathsf{L}^C[0] = 0$

$$\begin{aligned}
\mathsf{L}^C[f+g] &= \lambda^{-1}\left(\mathsf{L}[\lambda(f+g)]\right) \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda(f)+\lambda(g)]\right) \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda(f)]+\mathsf{L}[\lambda(g)]\right) && \textbf{[L.1]} \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right)+\lambda^{-1}\left(\mathsf{L}[\lambda(g)]\right) \\
&= \mathsf{L}^C[f]+\mathsf{L}^C[g]
\end{aligned}$$

$$\begin{aligned}
\mathsf{L}^C[0] &= \lambda^{-1}\left(\mathsf{L}[\lambda(0)]\right) \\
&= \lambda^{-1}\left(\mathsf{L}[0]\right) \\
&= \lambda^{-1}(0) && \textbf{[L.1]} \\
&= 0
\end{aligned}$$

[L.2]: $(1 \times \oplus_A)\mathsf{L}^C[f] = (1 \times \pi_0)\mathsf{L}^C[f] + (1 \times \pi_1)\mathsf{L}^C[f]$ and $\langle 1, 0\rangle \mathsf{L}^C[f] = 0$:

$$\begin{aligned}
(1 \times \oplus_A)\mathsf{L}^C[f] &= (1 \times \oplus_A)\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right) \\
&= \lambda^{-1}\left(\oplus_A \mathsf{L}[\lambda(f)]\right) && (4.17) \\
&= \lambda^{-1}\left(\pi_0\mathsf{L}[\lambda(f)] + \pi_1\mathsf{L}[\lambda(f)]\right) && \textbf{[L.2]} \\
&= \lambda^{-1}\left(\pi_0\mathsf{L}[\lambda(f)]\right) + \lambda^{-1}\left(\pi_1\mathsf{L}[\lambda(f)]\right) \\
&= (1 \times \pi_0)\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right) + (1 \times \pi_1)\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right) && (4.17) \\
&= (1 \times \pi_0)\mathsf{L}^C[f] + (1 \times \pi_1)\mathsf{L}^C[f]
\end{aligned}$$

$$\begin{aligned}
\langle 1, 0\rangle \mathsf{L}^C[f] &= \langle 1, 1\rangle (1 \times 0)\mathsf{L}^C[f] \\
&= \langle 1, 1\rangle (1 \times 0)\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right) \\
&= \langle 1, 1\rangle \lambda^{-1}\left(0\mathsf{L}[\lambda(f)]\right) && (4.17) \\
&= \langle 1, 1\rangle \lambda^{-1}(0) && \textbf{[L.2]} \\
&= \langle 1, 1\rangle 0 \\
&= 0
\end{aligned}$$

[L.3]: $\mathsf{L}^C[\pi_1] = \pi_1$ and $\mathsf{L}^C[\pi_1\pi_i] = \pi_1\pi_i$:

$$\begin{aligned}
\mathsf{L}^C[\pi_1] &= \lambda^{-1}\left(\mathsf{L}[\lambda(\pi_1)]\right) \\
&= \lambda^{-1}\left(\mathsf{L}[\eta]\right) \\
&= \lambda^{-1}(\eta) && \textbf{[EL.1]}
\end{aligned}$$

$$\begin{aligned}
&= \lambda^{-1}\left(\lambda(\pi_1)\right) \\
&= \pi_1
\end{aligned}$$

$$\begin{aligned}
\mathsf{L}^C[\pi_1\pi_i] &= \lambda^{-1}\left(\mathsf{L}[\lambda(\pi_1\pi_i)]\right) \\
&= \lambda^{-1}\Big(\mathsf{L}\left[\lambda(\pi_1)[1,\pi_i]\right]\Big) && (4.17) \\
&= \lambda^{-1}\Big(\mathsf{L}\left[\eta[1,\pi_i]\right]\Big) \\
&= \lambda^{-1}\Big(\eta\ \mathsf{L}\left[[1,\pi_i]\right]\Big) && (\textbf{[EL.1]} + \text{Lem.4.2.4.(iii)}) \\
&= \lambda^{-1}\Big(\eta\left[1,\mathsf{L}\left[\pi_i\right]\right]\Big) && \textbf{[EL.2]} \\
&= \lambda^{-1}\left(\eta\left[1,\pi_i\right]\right) && \textbf{[L.3]} \\
&= \lambda^{-1}\left(\lambda(\pi_1)\left[1,\pi_i\right]\right) \\
&= \lambda^{-1}\left(\lambda(\pi_1\pi_i)\right) && (4.17) \\
&= \pi_1\pi_i
\end{aligned}$$

[L.4]: $\mathsf{L}^C\left[\langle f,g\rangle\right] = \Big\langle \mathsf{L}^C[f], \mathsf{L}^C[g]\Big\rangle$

Recall that in any Cartesian closed category, we always have that $[C, A \times B] \cong [C, A] \times [C, B]$. So let $\theta_{C,A,B} : [C, A] \times [C, B] \longrightarrow [C, A \times B]$ be the natural isomorphism defined as follows:

$$\theta := \lambda\left(\langle(1\times\pi_0)\epsilon, (1\times\pi_1)\epsilon\rangle\right)$$

with inverse $\theta^{-1}_{C,A,B} : [C, A \times B] \longrightarrow [C, A] \times [C, B]$ defined as follows:

$$\theta^{-1}_{C,A,B} := \langle [1,\pi_0], [1,\pi_1]\rangle$$

To not overload notation, we will omit the subscripts of θ and θ^{-1}. We first compute that:

$$\begin{aligned}
\mathsf{L}[\theta^{-1}] &= \mathsf{L}\left[\langle[1,\pi_0],[1,\pi_1]\rangle\right] \\
&= \Big\langle \mathsf{L}\left[[1,\pi_0]\right], \mathsf{L}\left[[1,\pi_1]\right]\Big\rangle && \textbf{[L.4]} \\
&= \left\langle[1,\mathsf{L}\left[\pi_0\right]],[1,\mathsf{L}\left[\pi_1\right]]\right\rangle && \textbf{[EL.2]} \\
&= \langle[1,\pi_0],[1,\pi_1]\rangle && \textbf{[L.3]}
\end{aligned}$$

Therefore, θ^{-1} is L-linear. Since θ^{-1} is an isomorphism, by Lemma 4.2.4.(xiii), it follows that θ is also L-linear. Next observe that in any Cartesian closed category, the following equalities holds:

$$\lambda(\langle f,g\rangle) = \left\langle\lambda(f),\lambda(g)\right\rangle\theta \qquad \Big\langle\lambda^{-1}(h),\lambda^{-1}(k)\Big\rangle = \lambda^{-1}\left(\langle h,k\rangle\,\theta\right) \tag{4.18}$$

As such, we can compute that:

$$\begin{aligned}
\mathsf{L}^C\left[\langle f,g\rangle\right] &= \lambda^{-1}\left(\mathsf{L}[\lambda(\langle f,g\rangle)]\right) \\
&= \lambda^{-1}\Big(\mathsf{L}\left[\langle\lambda(f),\lambda(g)\rangle\theta\right]\Big) && (4.18)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^{-1}\left(\mathsf{L}\left[\langle \lambda(f), \lambda(g)\rangle\right] \theta\right) && (\theta \text{ is } \mathsf{L}\text{-linear + Lem.4.2.4.(iv)})\\
&= \lambda^{-1}\left(\left\langle \mathsf{L}\left[\lambda(f)\right], \mathsf{L}\left[\lambda(g)\right]\right\rangle \theta\right) && \textbf{[L.4]}\\
&= \left\langle \lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right), \lambda^{-1}\left(\mathsf{L}[\lambda(g)]\right)\right\rangle && (4.18)\\
&= \left\langle \mathsf{L}^C[f], \mathsf{L}^C[g]\right\rangle
\end{aligned}
$$

[L.5]: $\mathsf{L}^C[\langle \pi_0, f\rangle g] = \langle \pi_0, \mathsf{L}^C[f]\rangle\ \mathsf{L}^C\left[\left\langle \pi_0, \pi_1 + \langle \pi_0, 0\rangle f\right\rangle g\right]$

First note that in a Cartesian closed left additive category, since post-composition preserves the additive structure, it follows that we always have the following equalities:

$$[f, 0] = 0 \qquad\qquad [f, g+h] = [f, g] + [f, h] \tag{4.19}$$

Next, note that by **[EL.1]**, η and μ are L-linear. So in particular, by Lemma 4.2.4.(ii), η and μ are also additive. Also, recall that the monad identities are:

$$\mu\mu = [1, \mu]\mu \qquad\qquad \eta\mu = 1 = [1, \eta]\mu \tag{4.20}$$

Lastly, note that we have the following equality in any Cartesian closed category:

$$\lambda(\langle \pi_0, f\rangle g) = \lambda(f)[1, g]\mu \qquad\qquad \lambda^{-1}\left(h[1, k]\mu\right) = \langle \pi_0, \lambda^{-1}(h)\rangle \lambda^{-1}(k) \tag{4.21}$$

Therefore, we compute:

$$
\begin{aligned}
\mathsf{L}^C[\langle \pi_0, f\rangle g] &= \lambda^{-1}\left(\mathsf{L}[\lambda\left(\langle \pi_0, f\rangle g\right)]\right)\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)[1, \lambda(g)]\mu\right]\right) && (4.21)\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)[1, \lambda(g)]\right]\mu\right) && (\textbf{[EL.1]} + \text{Lem.4.2.4.(iv)})\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left(1 + 0\lambda(f)\right)[1, \lambda(g)]\right]\mu\right) && \textbf{[L.5]}\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left([1, \eta]\mu + 0\lambda\left(f\right)\eta\mu\right)[1, \lambda(g)]\right]\mu\right) && (4.20)\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left([1, \eta] + 0\lambda\left(f\right)\eta\right)\mu[1, \lambda(g)]\right]\mu\right) && (\mu \text{ is additive})\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left([1, \eta] + 0\eta[1, \lambda(f)]\right)\mu[1, \lambda(g)]\right]\mu\right) && (\text{Naturality of } \eta)\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left([1, \eta] + 0[1, \lambda(f)]\right)\mu[1, \lambda(g)]\right]\mu\right) && (\eta \text{ is additive})\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left([1, \eta] + [1, 0][1, \lambda(f)]\right)\mu[1, \lambda(g)]\right]\mu\right) && (4.19)\\
&= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left([1, \eta] + [1, 0\lambda(f)]\right)\mu[1, \lambda(g)]\right]\mu\right) && (\text{Functoriality of } [1, -])
\end{aligned}
$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\eta+0\lambda(f)\right]\mu[1,\lambda(g)]\right]\mu\right) \quad (4.19)$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\eta+\lambda\left((1\times 0)f\right)\right]\mu[1,\lambda(g)]\right]\mu\right) \quad (4.17)$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\eta+\lambda\left(\langle\pi_0,0\rangle f\right)\right]\mu[1,\lambda(g)]\right]\mu\right)$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\lambda(\pi_1)+\lambda\left(\langle\pi_0,0\rangle f\right)\right]\mu[1,\lambda(g)]\right]\mu\right)$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)\right]\mu\;[1,\lambda(g)]\right]\mu\right)$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)\right][1,[1,\lambda(g)]]\,\mu\right]\mu\right) \quad \text{(Naturality of } \mu)$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)\right][1,[1,\lambda(g)]]\right]\mu\mu\right)$$

$$(\textbf{[EL.1]} + \text{Lem.4.2.4.(iv)})$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\mathsf{L}\left[\left[1,\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)[1,\lambda(g)]\right]\right]\mu\mu\right) \quad \text{(Functoriality of } [1,-])$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\left[1,\mathsf{L}\left[\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)[1,\lambda(g)]\right]\right]\mu\mu\right) \quad \textbf{[EL.2]}$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\left[1,\mathsf{L}\left[\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)[1,\lambda(g)]\right]\right][1,\mu]\mu\right) \quad (4.20)$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\left[1,\mathsf{L}\left[\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)[1,\lambda(g)]\right]\mu\right]\mu\right) \quad \text{(Functoriality of } [1,-])$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\left[1,\mathsf{L}\left[\lambda\left(\pi_1+\langle\pi_0,0\rangle f\right)[1,\lambda(g)]\mu\right]\right]\mu\right) \quad (\textbf{[EL.1]} + \text{Lem.4.2.4.(iv)})$$

$$= \lambda^{-1}\left(\mathsf{L}\left[\lambda(f)\right]\left[1,\mathsf{L}\left[\lambda\left(\left\langle\pi_0,\pi_1+\langle\pi_0,0\rangle f\right\rangle g\right)\right]\right]\mu\right) \quad (4.21)$$

$$= \langle\pi_0,\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right)\rangle\,\lambda^{-1}\left(\mathsf{L}\left[\lambda\left(\left\langle\pi_0,\pi_1+\langle\pi_0,0\rangle f\right\rangle g\right)\right]\right) \quad (4.21)$$

$$= \langle\pi_0,\mathsf{L}^C[f]\rangle\,\mathsf{L}^C\left[\left\langle\pi_0,\pi_1+\langle\pi_0,0\rangle f\right\rangle g\right]$$

[L.6]: $\mathsf{L}^C\left[\mathsf{L}^C[f]\right]=\mathsf{L}^C[f]$

$$\mathsf{L}^C\left[\mathsf{L}^C[f]\right] = \lambda^{-1}\left(\mathsf{L}\left[\lambda\left(\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right)\right)\right]\right)$$

$$\begin{aligned}
&= \lambda^{-1}\Big(\mathsf{L}\left[\mathsf{L}[\lambda(f)]\right]\Big) \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right) && \textbf{[L.6]} \\
&= \mathsf{L}^C[f]
\end{aligned}$$

[L.7]: $\mathsf{L}_1^C[\mathsf{L}_0^C[f]] = \mathsf{L}_0^C[\mathsf{L}_1^C[f]]$

Recall that in any Cartesian closed category, we always have that $[A,[C,B]] \cong [C \times A, B]$. So let $\phi_{A,C,B} : [A,[C,B]] \longrightarrow [C \times A, B]$ be the natural isomorphism defined as follows:

$$\phi_{A,C,B} = \lambda\left(\alpha^{-1}(1 \times \epsilon)\epsilon\right)$$

As before, as to not overload notation, we will omit the subscripts of ϕ. We first note that the following equality holds:

$$\phi = \left[\pi_1, [\pi_0, 1]\right]\mu \tag{4.22}$$

Therefore, we can compute that:

$$\begin{aligned}
\mathsf{L}[\phi] &= \mathsf{L}\left[\left[\pi_1, [\pi_0, 1]\right]\mu\right] && (4.22) \\
&= \mathsf{L}\left[\left[\pi_1, [\pi_0, 1]\right]\right]\mu && (\textbf{[EL.1]} + \text{Lem.4.2.4.(iv)}) \\
&= \left[\pi_1, \mathsf{L}\left[[\pi_0, 1]\right]\right]\mu && \textbf{[EL.2]} \\
&= \left[\pi_1, [\pi_0, \mathsf{L}[1]]\right]\mu && \textbf{[EL.2]} \\
&= \left[\pi_1, [\pi_0, 1]\right]\mu && \textbf{[L.3]} \\
&= \phi && (4.22)
\end{aligned}$$

So ϕ is L-linear. Next observe that for a map $f : C \times (A \times B) \longrightarrow D$ (or a map $g : B \longrightarrow [A,[C,D]]$) we can apply the curry operator (or uncurry operator) twice and the following equalities hold in any Cartesian closed category (which we leave to the reader to check for themselves):

$$\lambda(\lambda(f))\phi = \lambda(\alpha^{-1}f) \qquad \lambda^{-1}\left(\lambda^{-1}(g)\right) = \alpha\lambda^{-1}(g\phi) \tag{4.23}$$

$$\lambda\left(\tau\lambda(f)\right)\phi = \lambda(\beta^{-1}f) \qquad \lambda^{-1}\left(\tau\lambda^{-1}(g)\right) = \beta\lambda^{-1}(g\phi) \tag{4.24}$$

As such, we can compute the following:

$$\begin{aligned}
\mathsf{L}_0^C[f] &= \beta\mathsf{L}^{C\times B}[\beta^{-1}f] \\
&= \beta\lambda^{-1}\left(\mathsf{L}[\lambda(\beta^{-1}f)]\right) \\
&= \beta\lambda^{-1}\left(\mathsf{L}[\lambda\left(\tau\lambda(f)\right)\phi]\right) && (4.24) \\
&= \beta\lambda^{-1}\left(\mathsf{L}[\lambda\left(\tau\lambda(f)\right)]\phi\right) && (\phi \text{ is } \mathsf{L}\text{-linear} + \text{Lem.4.2.4.(iv)}) \\
&= \lambda^{-1}\left(\tau\lambda^{-1}\left(\mathsf{L}[\lambda\left(\tau\lambda(f)\right)]\right)\right) && (4.24)
\end{aligned}$$

$$= \lambda^{-1}\left(\mathsf{L}_0[\lambda(f)]\right)$$

$$\begin{aligned}
\mathsf{L}_1^C[f] &= \alpha \mathsf{L}^{C\times A}[\alpha^{-1}f] \\
&= \alpha\lambda^{-1}\left(\mathsf{L}[\lambda(\alpha^{-1}f)]\right) \\
&= \alpha\lambda^{-1}\left(\mathsf{L}[\lambda(\lambda(f))\phi]\right) && (4.23) \\
&= \alpha\lambda^{-1}\left(\mathsf{L}[\lambda(\lambda(f))]\phi\right) && (\phi \text{ is } \mathsf{L}\text{-linear + Lem.4.2.4.(iv)}) \\
&= \lambda^{-1}\left(\lambda^{-1}\left(\mathsf{L}[\lambda(\lambda(f))]\right)\right) && (4.23) \\
&= \lambda^{-1}\left(\mathsf{L}_1[\lambda(f)]\right)
\end{aligned}$$

So we have the following equalities:

$$\mathsf{L}_1^C[f] = \lambda^{-1}\left(\mathsf{L}_0[\lambda(f)]\right) \qquad \mathsf{L}_1^C[f] = \lambda^{-1}\left(\mathsf{L}_1[\lambda(f)]\right) \tag{4.25}$$

Then we have that:

$$\begin{aligned}
\mathsf{L}_1^C[\mathsf{L}_0^C[f]] &= \lambda^{-1}\left(\mathsf{L}_1\left[\lambda\left(\lambda^{-1}\left(\mathsf{L}_0[\lambda(f)]\right)\right)\right]\right) && (4.25) \\
&= \lambda^{-1}\left(\mathsf{L}_1\left[\mathsf{L}_0[\lambda(f)]\right]\right) \\
&= \lambda^{-1}\left(\mathsf{L}_0\left[\mathsf{L}_1[\lambda(f)]\right]\right) && \textbf{[L.7]} \\
&= \lambda^{-1}\left(\mathsf{L}_0\left[\lambda\left(\lambda^{-1}\left(\mathsf{L}_1[\lambda(f)]\right)\right)\right]\right) \\
&= \mathsf{L}_0^C[\mathsf{L}_1^C[f]] && (4.25)
\end{aligned}$$

[L.8]: $(h\times 1)\mathsf{L}^{C'}[f] = \mathsf{L}^C[(h\times 1)f]$

We first observe that for any map h:

$$\begin{aligned}
\mathsf{L}[h,1] &= [h, \mathsf{L}[1]] && \textbf{[EL.2]} \\
&= [h,1] && \textbf{[L.3]}
\end{aligned}$$

Therefore, $[h,1]$ is L-linear.

$$\begin{aligned}
(h\times 1)\mathsf{L}^{C'}[f] &= (h\times 1)\lambda^{-1}\left(\mathsf{L}[\lambda(f)]\right) \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda(f)][h,1]\right) && (4.17) \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda(f)[h,1]]\right) && ([h,1] \text{ is } \mathsf{L}\text{-linear + Lem.4.2.4.(iv)}) \\
&= \lambda^{-1}\left(\mathsf{L}[\lambda\left((h\times 1)f\right)]\right) && (4.17) \\
&= \mathsf{L}^C[(h\times 1)f]
\end{aligned}$$

[L.ev]: $\mathsf{L}^C[\epsilon] = \epsilon$

$$\mathsf{L}^C[\epsilon] = \lambda^{-1}\left(\mathsf{L}[\lambda(\epsilon)]\right)$$

$$
\begin{aligned}
&= \lambda^{-1}\left(\mathsf{L}[1]\right) \\
&= \lambda^{-1}(1) && \textbf{[L.3]} \\
&= \epsilon
\end{aligned}
$$

So we conclude that L^C is a closed system of linearizing combinators. We also have that:

$$
\begin{aligned}
\langle 0,1\rangle \mathsf{L}^\top[\pi_1 f] &= \langle 0,1\rangle \lambda^{-1}\left(\mathsf{L}[\lambda(\pi_1 f)]\right) \\
&= \langle 0,1\rangle \lambda^{-1}\left(\mathsf{L}\left[\lambda(\pi_1)[1,f]\right]\right) && (4.17) \\
&= \langle 0,1\rangle \lambda^{-1}\left(\mathsf{L}\left[\eta[1,f]\right]\right) \\
&= \langle 0,1\rangle \lambda^{-1}\left(\eta\ \mathsf{L}\left[[1,f]\right]\right) && (\textbf{[EL.1]} + \text{Lem.4.2.4.(iii)}) \\
&= \langle 0,1\rangle \lambda^{-1}\left(\eta\ \left[1,\mathsf{L}[f]\right]\right) && \textbf{[EL.2]} \\
&= \langle 0,1\rangle \lambda^{-1}\left(\lambda(\pi_1)\left[1,\mathsf{L}[f]\right]\right) \\
&= \langle 0,1\rangle \lambda^{-1}\left(\lambda\left(\pi_1 \mathsf{L}[f]\right)\right) && (4.17) \\
&= \langle 0,1\rangle \pi_1 \mathsf{L}[f] \\
&= \mathsf{L}[f]
\end{aligned}
$$

Therefore, $\mathsf{L}[f] = \langle 0,1\rangle \mathsf{L}^\top[\pi_1 f]$ and so L is precisely the induced linearizing combinator from Proposition 4.4.2. Next we must show that D_L is a differential combinator which also satisfies **[CD.λ]** (or equivalently **[CD.ev]**). However, we have that:

$$
\begin{aligned}
\mathsf{D}_\mathsf{L}[f] &= \lambda^{-1}\left(\mathsf{L}[\lambda(\oplus_A f)]\right) \\
&= \mathsf{L}^A[\oplus_A f]
\end{aligned}
$$

Therefore, D_L is precisely the induced differential combinator from Proposition 4.4.13. Furthermore, by Proposition 4.5.9.(ii), D_L satisfies **[CD.λ]** (or equivalently **[CD.ev]**). So we conclude that a Cartesian left additive category with an exponential linearizing combinator is a Cartesian closed differential category. □

We conclude this section by stating the second main result of this chapter.

Theorem 4.5.12 *[29, Theorem 6.12] For a Cartesian closed left additive category $\mathbb{X}$, there are bijective correspondences between:*

(i) *Differential combinators D on $\mathbb{X}$ which satisfy* **[CD.λ]** *(or equivalently* **[CD.ev]***);*

(ii) *Closed systems of linearizing combinators L^C on $\mathbb{X}$;*

(iii) *Exponentiable linearizing combinators L on $\mathbb{X}$.*

Therefore, a Cartesian closed differential category is precisely a Cartesian closed left additive category equipped with a exponentiable linearizing combinator or equivalently a Cartesian closed left additive category equipped with a closed system of linearizing combinators.

PROOF: That (i) and (ii) are in bijective correspondence follows from Theorem 4.4.14 and Proposition 4.5.9.(ii). On the other hand, that (ii) and (iii) are in bijective correspondence follows form Proposition 4.5.9.(i) and Proposition 4.5.11.(i) (which when put together shows that their respective constructions are inverses of each other). □

In future work, a question which now should be answered is whether, in fact, for the Cartesian differential category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ from Example 4.1.10, its linearization combinator is exponentiable and if $\mathsf{HoAbCat}_{\mathsf{Ch}}$ is a Cartesian closed differential category.

Chapter 5

Exponential Functions for Cartesian Differential Categories

This chapter is based on [64], in which we generalize the notion of exponential functions for Cartesian differential categories. The author would first like to thank Geoff Cruttwell and Robin Cockett for their support in this project, their editorial comments, and very useful discussions which greatly helped the development of this research project. The author would also like to thank the anonymous referee from "Applied Categorical Structures" for their review and editorial comments.

Since their introduction, Cartesian differential categories have a rich literature and have been successful in generalizing many concepts from classical differential calculus. More recently, Cartesian differential categories have also started to find their way in applications. In particular, Cockett and Cruttwell have introduced the notion of dynamical systems and their solutions in tangent categories [20], which generalize ordinary differential equations in this context, specifically initial value problems. Since every Cartesian differential category is a tangent category, this implies that dynamical systems allow one to study differential equations in a Cartesian differential category. In classical differential calculus, one of the most important tools used for solving differential equations is the exponential function e^x. Therefore, it is desirable to generalize the exponential function for Cartesian differential categories.

The exponential function e^x admits numerous equivalent characterization. It can either be defined as the inverse of the natural logarithm function $ln(x)$, or as the limit:

$$e^x = \lim_{n \longrightarrow \infty} (1 + \frac{x}{n})^n$$

or as the convergent power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

or even as the solution to $f'(x) = f(x)$ with initial condition $f(0) = 1$. However in arbitrary Cartesian differential categories, functions need to be defined at zero (which excludes $ln(x)$) and one does not necessarily have a notion of convergence, infinite sums, or even (unique) solutions to initial value problems. Therefore one must look for a more algebraic characterization of the exponential function. In classical algebra, an exponential ring [86] is a ring equipped with an endomorphism e which is a monoid morphism from the additive structure to the multiplicative structure, that is:

$$e(x+y) = e(x)e(y) \qquad\qquad e(0) = 1$$

The canonical example of an exponential ring is the field of real numbers $\mathbb{R}$ with the exponential function e^x. While this seems promising, arbitrary objects in a Cartesian differential category do not necessarily come equipped with a multiplication. Rather than requiring this extra ring structure on objects, it turns out that the differential combinator D will allow us to bypass the need for a multiplication.

In the category of real smooth functions, which recall is the canonical example of a Cartesian differential category, the differential combinator D applied to the exponential function e^x is the smooth function $\mathsf{D}[e^x] : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined as:

$$\mathsf{D}[e^x](x, y) = e^x y$$

Thus the multiplication of $\mathbb{R}$ appears in $\mathsf{D}[e^x]$. Inspired by this observation, the generalization of the exponential function in a Cartesian differential category can be defined simply in terms of an endomorphism $e : A \longrightarrow A$ which is compatible with the differential combinator D in the sense that

$$\mathsf{D}[e](0, x) = x \qquad\qquad e(x+y) = \mathsf{D}[e](x, e(y))$$

We call such endomorphisms **differential exponential maps**, which is the main novel notion of study in this paper. Differential exponential maps generalize the exponential function for Cartesian differential categories. Indeed for e^x, the differential exponential maps axioms correspond precisely to $e^0 x = x$ and $e^x e^y = e^{x+y}$ respectively.

Previously, we mentioned that not every object in a Cartesian differential category has a multiplication. However, it turns out that every differential exponential map $e : A \longrightarrow A$ does induce a commutative rig structure on A, and thus A does come equipped with a multiplication. The construction is once again inspired by the classical exponential function e^x. Applying the differential combinator on e^x twice we obtain:

$$\mathsf{D}^2[e^x]((x, y), (z, w)) = e^x yz + e^x w$$

Setting $x = 0$ and $w = 0$, one re-obtains precisely the multiplication of $\mathbb{R}$:

$$\mathsf{D}^2[e^x]((0, y), (z, 0)) = yz$$

The unit for the multiplication is, of course, obtained by evaluating e^x at 0, $e^0 = 1$. This construction is easily generalized to an arbitrary Cartesian differential category and one can show that every differential exponential map induces a commutative rig. Commutativity of the multiplication follows from the symmetry of the partial derivatives axiom **[dC.7]** of the differential combinator. The unit identities for the multiplication will follow from both differential exponential map axioms. Proving associativity is a bit trickier but essentially follows from the observation that:

$$D^3[e^x]((0, x), (y, 0), (z, 0), (0, 0)) = xyz$$

and then using both the differential combinator axioms and differential exponential map axioms, one shows that both sides of the associativity law are equal to the third-order partial derivative.

Conversely, it is possible to obtain differential exponential maps from special kinds of commutative rigs. Indeed, one can alternatively axiomatize an object equipped with a differential exponential map instead as a commutative rig equipped with an endomorphism which satisfies the three fundamental properties of the exponential function that $e^0 = 1$, $e^{x+y} = e^x e^y$, and $\frac{\partial e^x}{\partial x} = e^x$. We call such rigs: **differential exponential rigs**. One of the main results of this paper it that there is a bijective correspondence between differential exponential maps and differential exponential rigs. In the category of real smooth functions, interesting examples of differential exponential rigs include $\mathbb{R}$ with the exponential function e^x, $\mathbb{R}^2$ equipped with the complex numbers multiplication and the complex exponential function $e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$, $\mathbb{R}^2$ equipped with the split complex numbers multiplication and the split complex exponential function $e^{x+jy} = e^x \cosh(y) + je^x \sinh(y)$ [77], and also $\mathbb{R}^2$ equipped with the dual numbers multiplication and the dual numbers exponential function $e^{x+y\varepsilon} = e^x + e^x y\varepsilon$ [80].

As one of the main motivations for their development, differential exponential maps do allow one to solve a certain class of linear dynamical systems in any Cartesian differential category. Specifically, one can solve the dynamical systems which generalize the initial value problems of the form $f'(x) = f(x)a$ with initial condition $f(0) = b$ (for some constants a and b), whose classical solution is $f(x) = e^{ax}b$. The types of differential equations are of particular interest in control systems theory [3, Chapter 5]. Furthermore, it turns out that a differential exponential map is indeed the a solution to the dynamical system which generalizes the initial value problem $f'(x) = f(x)$ with initial condition $f(0) = 1$. In future work on solving differential equations in a Cartesian differential category, differential exponential maps will hopefully play a key role. Such an application can be found in [24, Section 5], where it is shown by Cockett, Cruttwell, and the author that in a tangent category, a differential curve object admits a canonical differential exponential map, which induces solutions to many interesting dynamical systems including one which in turn induces an action on differential bundles.

Recall that an important example of a Cartesian differential category is the coKleisli category of the comonad ! of a differential category (with finite products). In this setting, a differential exponential map is an endomorphism in the coKleisli category and therefore a map of type $!A \longrightarrow A$ in the base category. When the coalgebra modality ! has Seely isomorphisms, the differential structure is captured by the codereliction map $\eta : A \longrightarrow !A$ and it turns out that a commutative rig in the coKleisli category is precisely a commutative monoid (over the tensor product) in the base category. As such, a differential exponential map in the coKleisli category can alternatively be given by a commutative monoid A in the base category equipped with a monoid morphism $!A \longrightarrow A$ which is a retract of the codereliction. We call such commutative monoids **!-differential exponential algebras**, and there is a bijective correspondence between !-differential exponential algebras and differential exponential maps in the coKleisli category.

Chapter Outline: In Section 5.1 we introduce differential exponential maps and in particular show that the category of differential exponential maps is a Cartesian tangent category. We provide examples of differential exponential maps in the category of smooth real functions, which include the classical exponential function, the complex exponential function, the split complex exponential function, and the dual number exponential function. In Section 5.2 we introduce differential exponential rigs. We show that every differential exponential rig induces a differential exponential map and conversely that every differential exponential map induces a differential exponential rig. We also show that these constructions are inverses of each other. As an immediate consequence, the category of differential exponential rigs is also a Cartesian tangent category. In Section 5.3 we explain the relationship between differential exponential maps and solutions to differential equations in arbitrary Cartesian differential categories. In particular, we show that every differential exponential map is a solution to the expected differential equation and that a certain class of dynamical systems admit a solution. We also show that in the presence of unique solutions to differential equations, one can build a differential exponential map. In Section 5.4 we study differential exponential maps in the coKleisli category of a differential category and give equivalent characterizations in these cases. As such, we introduce !-differential exponential algebras. We show that every !-differential exponential algebra induces a differential exponential map in the coKleisli category and conversely that every differential exponential map in the coKleisli category induces a !-differential exponential algebra, and that, once again, these constructions are inverses of each other . We conclude this paper with Section 5.5 which discusses some interesting potential future work to do with differential exponential maps.

5.1 Differential Exponential Maps

In this section, we introduce *differential exponential maps*, which generalizes the notion of the classical exponential function e^x for arbitrary Cartesian differential categories. See the previous chapter for the notation used for Cartesian differential category.

Definition 5.1.1 *A **differential exponential map** [64, Definition 4] in a Cartesian differential category $\mathbb{X}$ with differential combinator D (Definition 4.1.5) is a map $e : A \longrightarrow A$, such that the following diagrams commute:*

$$\begin{array}{ccc} A & \xrightarrow{\langle 0,1\rangle} & A \times A \\ & \searrow^{=} & \downarrow{\scriptstyle \mathsf{D}[e]} \\ & & A \end{array} \qquad \begin{array}{ccc} A \times A & \xrightarrow{1\times e} & A \times A \\ {\scriptstyle \oplus_A}\downarrow & & \downarrow{\scriptstyle \mathsf{D}[e]} \\ A & \xrightarrow[e]{} & A \end{array} \tag{5.1}$$

where recall that $\oplus_A = \pi_0 + \pi_1$, as defined in Lemma 4.1.4.

The intuition for a differential exponential map is best explained in Example 5.1.9.i, which shows that the classical exponential function e^x (which is, of course, the main motivating example) is a differential exponential map. Briefly, since e^x is its own derivative, applying the differential combinator on e^x results in the smooth function $\mathsf{D}[e^x] : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined as:

$$\mathsf{D}[e^x](x, y) = e^x y$$

The left diagram of (5.1) generalizes that $e^0 y = y$ (since $e^0 = 1$), while the right diagram generalizes that $e^x e^y = e^{x+y}$. The differential combinator is the key piece that allows one to bypass the need for a multiplication operation and a multiplicative unit in the definition of a differential exponential map. That said, in Section 5.2 we will see that every differential exponential map does induce a multiplication and that analogues of the three essential properties of the classical exponential function are satisfied (Proposition 5.2.5). And conversely, we will also see how one can also axiomatize differential exponential maps in terms of rig structure and analogues of the three essential properties of the classical exponential function (Proposition 5.2.4). And, as mentioned in the introduction, we also highlight that the definition of a differential exponential map does not require any added structure or property on the Cartesian differential category such as a notion of converging limits or infinite sums. Before giving examples of differential exponential maps, which can be found in Example 5.1.9, let us first consider the category of differential exponential maps and constructions of differential exponential maps.

For a Cartesian differential category $\mathbb{X}$, define its category of differential exponential maps as the category $\mathsf{DEM}[\mathbb{X}]$ whose objects are pairs (A, e) consisting of an object $A \in \mathbb{X}$ and a differential exponential map $e : A \longrightarrow A$, and where a map $f : (A, e) \longrightarrow (B, e')$ is a *linear* map (Definition 4.1.18) $f : A \longrightarrow B$ in $\mathbb{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ A & \xrightarrow[\;f\;]{} & B \end{array} \qquad (5.2)$$

and where composition and identity maps are as in $\mathbb{X}$. The reason for why maps in $\mathsf{DEM}[\mathbb{X}]$ are restricted to being linear will become apparent in the proof of Theorem 5.2.9. There is the obvious forgetful functor $\mathsf{U} : \mathsf{DEM}[\mathbb{X}] \longrightarrow \mathsf{LIN}[\mathbb{X}]$ which maps $\mathsf{U}(A, e) = A$ and $\mathsf{U}(f) = f$.

Lemma 5.1.2 *[64, Lemma 4] For a Cartesian differential category $\mathbb{X}$ with differential combinator D, the forgetful functor $\mathsf{U} : \mathsf{DEM}[\mathbb{X}] \longrightarrow \mathsf{LIN}[\mathbb{X}]$ creates all limits.*

PROOF: Let $\mathsf{F} : \mathbb{D} \longrightarrow \mathsf{DEM}[\mathbb{X}]$ be a functor such that the limit of the composite $\mathsf{U} \circ \mathsf{F} : \mathbb{D} \longrightarrow \mathsf{LIN}[\mathbb{X}]$ exists in $\mathsf{LIN}[\mathbb{X}]$ which we denote: $\lim\limits_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right)$ with projections $\pi_x : \lim\limits_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right) \longrightarrow \mathsf{U}\left(\mathsf{F}(X)\right)$. Note that $\mathsf{U}\left(\mathsf{F}(X)\right)$ is the underlying object of $\mathsf{F}(X)$, and so it comes equipped with a differential exponential map $e_X : \mathsf{U}\left(\mathsf{F}(X)\right) \longrightarrow \mathsf{U}\left(\mathsf{F}(X)\right)$. Therefore $\mathsf{F}(X) = \Big(\mathsf{U}\left(\mathsf{F}(X)\right), e_X\Big)$. Now for every map $f : X \longrightarrow Y$ in $\mathbb{D}$, since $\mathsf{F}(f)$ is a map in $\mathsf{DEM}[\mathbb{X}]$, the following diagram commutes:

$$\begin{array}{ccccc} & & \lim\limits_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right) & & \\ & {\scriptstyle \pi_X}\swarrow & & \searrow{\scriptstyle \pi_Y} & \\ \mathsf{U}\left(\mathsf{F}(X)\right) & & \xrightarrow{\;\mathsf{U}(\mathsf{F}(f))\;} & & \mathsf{U}\left(\mathsf{F}(Y)\right) \\ {\scriptstyle e_X}\downarrow & & & & \downarrow{\scriptstyle e_Y} \\ \mathsf{U}\left(\mathsf{F}(X)\right) & & \xrightarrow[\;\mathsf{U}(\mathsf{F}(f))\;]{} & & \mathsf{U}\left(\mathsf{F}(Y)\right) \end{array}$$

By the universal property of $\lim\limits_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right)$, there is a unique map:

$$\lim_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right) \xrightarrow{\lim\limits_{X \in \mathbb{D}} e_X} \lim_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right) \tag{5.3}$$

which makes the following diagram commute:

$$\begin{array}{ccc}
\lim\limits_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right) & \xrightarrow{\pi_X} & \mathsf{U}\left(\mathsf{F}(X)\right) \\
{\scriptstyle \exists! \ \lim\limits_{X \in \mathbb{D}} e_X} \Big\downarrow & & \Big\downarrow {\scriptstyle e_X} \\
\lim\limits_{X \in \mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right) & \xrightarrow[\pi_X]{} & \mathsf{U}\left(\mathsf{F}(X)\right)
\end{array}$$

We wish to show that $\lim\limits_{X \in \mathbb{D}} e_X$ is a differential exponential map. To do so, first note that for each $X \in \mathbb{D}$, π_X is linear and so by Lemma 4.1.19 it follows that:

$$\mathsf{D}\left[\lim_{X \in \mathbb{D}} e_X\right] \pi_X = (\pi_X \times \pi_X)\mathsf{D}\left[e_X\right] \tag{5.4}$$

Now since π_X is linear, it is also additive (Lemma 4.1.19) and therefore we obtain the following (where we omit the subscript on $\oplus$ so to not overload notation):

$$\begin{aligned}
\langle 0, 1\rangle \mathsf{D}\left[\lim_{X \in \mathbb{D}} e_X\right] \pi_X &= \langle 0, 1\rangle(\pi_X \times \pi_X)\mathsf{D}\left[e_X\right] && (5.4) \\
&= \langle 0\pi_X, \pi_X\rangle \mathsf{D}\left[e_X\right] \\
&= \langle 0, \pi_X\rangle \mathsf{D}\left[e_X\right] && (\pi_X \text{ is additive}) \\
&= \pi_X \langle 0, 1\rangle \mathsf{D}\left[e_X\right] \\
&= \pi_X && (5.1)
\end{aligned}$$

$$\begin{aligned}
\oplus(\lim_{X \in \mathbb{D}} e_X)\pi_X &= \oplus \pi_X e_X && (5.3) \\
&= (\pi_X \times \pi_X) \oplus e_X && (\pi_X \text{ is additive + Lemma 4.1.4}) \\
&= (\pi_X \times \pi_X)(1 \times e_X)\mathsf{D}[e_X] && (5.1) \\
&= \left(1 \times \lim_{X \in \mathbb{D}} e_X\right)(\pi_X \times \pi_X)\mathsf{D}\left[e_X\right] && (5.3) \\
&= \left(1 \times \lim_{X \in \mathbb{D}} e_X\right)\mathsf{D}\left[\lim_{X \in \mathbb{D}} e_X\right]\pi_X && (5.4)
\end{aligned}$$

Therefore for each $X \in \mathbb{D}$, we have that:

$$\langle 0, 1\rangle\, \mathsf{D}\left[\lim_{X \in \mathbb{D}} e_X\right] \pi_X = \pi_X \qquad \oplus(\lim_{X \in \mathbb{D}} e_X)\pi_X = \left(1 \times \lim_{X \in \mathbb{D}} e_X\right)\mathsf{D}\left[\lim_{X \in \mathbb{D}} e_X\right]\pi_X$$

Then by the universal property of the limit, it follows that:

$$\langle 0, 1\rangle \mathsf{D}\left[\lim_{X\in\mathbb{D}} e_X\right] = 1 \qquad\qquad \oplus(\lim_{X\in\mathbb{D}} e_X) = \left(1 \times \lim_{X\in\mathbb{D}} e_X\right)\mathsf{D}\left[\lim_{X\in\mathbb{D}} e_X\right]$$

and so we conclude that $\lim\limits_{X\in\mathbb{D}} e_X$ is a differential exponential map. From here, it is straightforward to conclude that the limit of $\mathsf{F} : \mathbb{D} \longrightarrow \mathsf{DEM}[\mathbb{X}]$ is the pair:

$$(\lim_{X\in\mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right), \lim_{X\in\mathbb{D}} e_X)$$

with projections $\pi_X : (\lim\limits_{X\in\mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right), \lim\limits_{X\in\mathbb{D}} e_X) \to \left(\mathsf{U}\left(\mathsf{F}(X)\right), e_X\right)$. Furthermore:

$$\mathsf{U}(\lim_{X\in\mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right), \lim_{X\in\mathbb{D}} e_X) = \lim_{X\in\mathbb{D}} \mathsf{U}\left(\mathsf{F}(X)\right) \qquad\qquad \mathsf{U}(\pi_X) = \pi_X$$

and it follows from the definition of $\lim\limits_{X\in\mathbb{D}} e_X$ that this is the unique cone over F with this property. Therefore, we conclude that $\mathsf{U} : \mathsf{DEM}[\mathbb{X}] \longrightarrow \mathsf{LIN}[\mathbb{X}]$ creates all limits. □

By Lemma 4.1.19, the projection maps of the product are linear. As such, an immediate consequence of Lemma 5.1.2 is that the product of differential exponential maps is again a differential exponential map. Therefore, the category of differential exponentials maps has finite products.

Corollary 5.1.3 *[64, Corollary 2] In a Cartesian differential category $\mathbb{X}$ with differential combinator D:*

(i) *For the terminal object $\top$, $1 : \top \longrightarrow \top$ is a differential exponential map;*

(ii) *If $e : A \longrightarrow A$ and $e' : B \longrightarrow B$ are differential exponential maps, then their product $e \times e' : A \times B \longrightarrow A \times B$ is a differential exponential map.*

Furthermore, for a Cartesian differential category $\mathbb{X}$, $\mathsf{DEM}[\mathbb{X}]$ has finite products where the terminal object is $(\top, 1)$, and where the product of (A, e) and (B, e') is $(A \times B, e \times e')$ with projection maps $\pi_0 : (A \times B, e \times e') \longrightarrow (A, e)$ and $\pi_1 : (A \times B, e \times e') \longrightarrow (B, e')$.

It is important to note that a differential exponential map for $A \times B$ is not necessarily the product of differential exponential maps, that is, of the form $e \times e'$. See Example 5.1.9 for three examples of differential exponential maps which are not the products of differential exponential maps.

Our next observation is that the category of differential exponential maps is also a Cartesian tangent category. We will not review the full definition of a tangent category here but we will highlight certain properties that will be important for this chapter. We invite the reader to read the full definition of a tangent category in [19, 21]. In particular, the differential combinator of a Cartesian differential category induces an endofunctor and this endofunctor makes a Cartesian differential category a Cartesian tangent category.

Proposition 5.1.4 *[19, Proposition 4.7] Every Cartesian differential category $\mathbb{X}$ with differential combinator D is a Cartesian tangent category where the **tangent functor** $\mathsf{T} : \mathbb{X} \longrightarrow \mathbb{X}$ is defined on objects as $\mathsf{T}(A) := A \times A$ and on morphisms as $\mathsf{T}(f) := \langle \pi_0 f, \mathsf{D}[f]\rangle$.*

Corollary 5.1.5 *For a Cartesian differential category* $\mathbb{X}$ *with differential combinator* D, $\mathsf{LIN}[\mathbb{X}]$ *is a Cartesian tangent category where the tangent functor* $\mathsf{T} : \mathsf{LIN}[\mathbb{X}] \longrightarrow \mathsf{LIN}[\mathbb{X}]$ *is defined on objects as* $\mathsf{T}(A) := A \times A$ *and on morphisms as* $\mathsf{T}(f) := f \times f$. *Furthermore, the forgetful functor* $\mathsf{U} : \mathsf{LIN}[\mathbb{X}] \longrightarrow \mathbb{X}$ *preserves the Cartesian tangent structure strictly.*

Here are now some useful properties involving the tangent functor (which we leave to the reader to check for themselves):

Lemma 5.1.6 *In a Cartesian differential category:*

(i) $\mathsf{D}[fg] = \mathsf{T}(f)\mathsf{D}[g]$;

(ii) $\langle 1, 0\rangle \mathsf{T}(f) = f\langle 1, 0\rangle$;

(iii) *If* f *is linear then* $\mathsf{T}(f) = f \times f$;

(iv) $\mathsf{T}(\langle f, g\rangle) = \langle \mathsf{T}(f), \mathsf{T}(g)\rangle c$;

(v) $\mathsf{D}[f \times g] = c\left(\mathsf{D}[f] \times \mathsf{D}[g]\right)$ *and* $\mathsf{T}(f \times g)c = c(\mathsf{T}(f) \times \mathsf{T}(g))$;

(vi) $\mathsf{T}(f + g) = \mathsf{T}(f) + \mathsf{T}(g)$ *and* $\mathsf{T}(0) = 0$;

(vii) $\mathsf{D}\left[\mathsf{T}(f)\right] = c\mathsf{T}(\mathsf{D}[f])$;

(viii) $\mathsf{T}(\oplus_A) = \oplus_A \times \oplus_A$;

(ix) $c\mathsf{T}(\oplus_A) = \oplus_{A\times A}$.

Next we show that the tangent functor maps differential exponential maps to differential exponential maps.

Lemma 5.1.7 *[64, Lemma 5] If* $e : A \longrightarrow A$ *is a differential exponential map, then its tangent* $\mathsf{T}(e) : A \times A \longrightarrow A \times A$ *is a differential exponential map.*

PROOF: We first show that $\langle 0, 1\rangle \mathsf{D}\left[\mathsf{T}(e)\right] = 1$:

$$\begin{aligned}
\langle 0, 1\rangle \mathsf{D}\left[\mathsf{T}(e)\right] &= \langle 0, 1\rangle c\mathsf{T}(\mathsf{D}[e]) && \text{(Lemma 5.1.6)}\\
&= \mathsf{T}(\langle 0, 1\rangle)\mathsf{T}(\mathsf{D}[e]) && \text{(Lemma 5.1.6)}\\
&= \mathsf{T}\left(\langle 0, 1\rangle \mathsf{D}[e]\right) && (\mathsf{T}\text{ is a functor})\\
&= \mathsf{T}(1) && (5.1)\\
&= 1 && (\mathsf{T}\text{ is a functor})
\end{aligned}$$

Next we show that $(1 \times \mathsf{T}(e))\mathsf{D}\left[\mathsf{T}(e)\right] = \oplus \mathsf{T}(e)$:

$$\begin{aligned}
(1 \times \mathsf{T}(e))\mathsf{D}\left[\mathsf{T}(e)\right] &= (1 \times \mathsf{T}(e))c\mathsf{T}(\mathsf{D}[e]) && \text{(Lemma 5.1.6)}\\
&= c\mathsf{T}(1 \times e)\mathsf{T}(\mathsf{D}[e]) && \text{(Lemma 5.1.6)}\\
&= c\mathsf{T}\left((1 \times e)\mathsf{D}[e]\right) && (\mathsf{T}\text{ is a functor})
\end{aligned}$$

$$
\begin{aligned}
&= c\mathsf{T}\left(\oplus_A e\right) && (5.1)\\
&= c\mathsf{T}(\oplus_A)\mathsf{T}(e) && (\mathsf{T} \text{ is a functor})\\
&= \oplus_{A\times A}\mathsf{T}(e) && (\text{Lemma } 5.1.6)
\end{aligned}
$$

So we conclude that $\mathsf{T}(e)$ is a differential exponential map. □

Proposition 5.1.8 *[64, Proposition 2] For a Cartesian differential category $\mathbb{X}$, $\mathsf{DEM}[\mathbb{X}]$ is a Cartesian tangent category where the tangent functor $\mathsf{T} : \mathsf{DEM}[\mathbb{X}] \longrightarrow \mathsf{DEM}[\mathbb{X}]$ is defined on objects as $\mathsf{T}(A, e) := (A \times A, \mathsf{T}(e))$ and on maps as $\mathsf{T}(f) = f \times f$, and where the remaining tangent structure is the same as for $\mathsf{LIN}[\mathbb{X}]$ (which is the same as for $\mathbb{X}$ and can be found in [19, Proposition 4.7]).*

Proof: The tangent functor T is well defined by Lemma 5.1.7. Since all the maps of the tangent structure of $\mathbb{X}$ are linear, it follows that by their respective naturality with the tangent functor of $\mathbb{X}$, they are also maps in $\mathsf{DEM}[\mathbb{X}]$ which are natural for its tangent functor. The existence of the necessary limits for tangent structure in $\mathsf{DEM}[\mathbb{X}]$ will follow from Corollary 5.1.5 and Lemma 5.1.2. And lastly, the required equalities for tangent structure will hold since they hold in $\mathsf{LIN}[\mathbb{X}]$. So we conclude that $\mathsf{DEM}[\mathbb{X}]$ is a tangent category. Furthermore, since $\mathbb{X}$ is a Cartesian tangent category, it follows that $\mathsf{DEM}[\mathbb{X}]$ is also a Cartesian tangent category. □

It may be tempting to think that $\mathsf{DEM}[\mathbb{X}]$ is also a Cartesian differential category, but this is not the case. Indeed note that even if $f : (A, e) \longrightarrow (B, e')$ and $g : (A, e) \longrightarrow (A, e')$ are maps in $\mathsf{DEM}[\mathbb{X}]$, their sum $f + g$ is not (in general) a map in $\mathsf{DEM}[\mathbb{X}]$ since it is not necessarily the case that $(f + g)e$ equals $e'(f + g)$. This implies that $\mathsf{DEM}[\mathbb{X}]$ is not a Cartesian left additive category, and so in particular not a Cartesian differential category.

Example 5.1.9 Here are now examples of differential exponential maps in the Cartesian differential category SMOOTH from Example 4.1.8.

(i) Consider the exponential function $e^x : \mathbb{R} \longrightarrow \mathbb{R}$, $x \mapsto e^x$. Since $\frac{\partial e^x}{\partial x}(a) = e^a$, we have that:

$$\mathsf{D}[e](a, b) = \frac{\partial e^x}{\partial x}(a)b = e^a b$$

The exponential function satisfies the left diagram of (5.1) since $e^0 = 1$:

$$\mathsf{D}[e](0, a) = e^0 a = a$$

while the right diagram of (5.1) is also satisfied since $e^a e^b = e^{a+b}$:

$$\mathsf{D}[e^x](a, e^b) = e^a e^b = e^{a+b}$$

Then the exponential function $e^x : \mathbb{R} \longrightarrow \mathbb{R}$ is a differential exponential map.

(ii) Applying Corollary 5.1.3.ii to the exponential function $e^x : \mathbb{R} \longrightarrow \mathbb{R}$, the point-wise exponential functions $e^x \times \ldots \times e^x : \mathbb{R}^n \longrightarrow \mathbb{R}^n$:

$$(x_1, \ldots, x_n) \mapsto (e^{x_1}, \ldots, e^{x_n})$$

are differential exponential maps in SMOOTH.

(iii) Applying Lemma 5.1.7 to the exponential function $e^x : \mathbb{R} \to \mathbb{R}$, the its tangent, that is the smooth function $\mathsf{T}(e^x) : \mathbb{R}^2 \to \mathbb{R}^2$, which is worked out to be:

$$(x, y) \mapsto (e^x, e^x y)$$

is a differential exponential map. To better understand $\mathsf{T}(e^x)$, consider the ring of dual numbers $\mathbb{R}[\varepsilon]$ [80, Section 1.1.3]:

$$\mathbb{R}[\varepsilon] = \{x + y\varepsilon |\ x, y \in \mathbb{R}, \varepsilon^2 = 0\}$$

As explained in [80, Section 1.1.5], the dual number exponential function is $e^{x+y\varepsilon} = e^x + e^x y\varepsilon$. It may be useful to the reader to work out this example using the power series definition of the exponential function:

$$e^{x+y\varepsilon} = \sum_{n=0}^{\infty} \frac{(x + y\varepsilon)^n}{n!}$$

Note that by the binomial theorem and power series multiplication, one can still derive that $e^{x+y\varepsilon} = e^x e^{y\varepsilon}$. Therefore, it remains to compute $e^{y\varepsilon}$. Since $\varepsilon^n = 0$ for all $n \geq 2$, we obtain:

$$e^{y\varepsilon} = \sum_{n=0}^{\infty} \frac{(y\varepsilon)^n}{n!} = \sum_{n=0}^{\infty} \frac{y^n \varepsilon^n}{n!} = 1 + y\varepsilon$$

So $e^{y\varepsilon} = 1 + y\varepsilon$. Therefore,

$$e^{x+y\varepsilon} = e^x e^{y\varepsilon} = e^x(1 + y\varepsilon) = e^x + e^x y\varepsilon$$

Writing dual numbers $x + y\varepsilon$ instead as (x, y), it becomes clear that $\mathsf{T}(e^x)$ is the real smooth function associated to the dual numbers exponential function. This relation was to be expected since tangent categories are closely related to dual numbers and Weil algebras [65]. Furthermore, note that in dual number notation, $\mathsf{D}[\mathsf{T}(e^x)]\,(a + b\varepsilon, c + d\varepsilon) = e^{a+b\varepsilon}(c + d\varepsilon)$.

(iv) Let $\mathbb{C}$ be the field of complex numbers:

$$\mathbb{C} = \{x + iy |\ x, y \in \mathbb{R}, i^2 = -1\}$$

The complex exponential function is $e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$, where cos and sin are the trigonometric cosine and sine functions respectively. The complex exponential function is of course derived from the power series definition:

$$e^{x+iy} = \sum_{n=0}^{\infty} \frac{(x + iy)^n}{n!}$$

and then simplifying by using that $e^{x+iy} = e^x e^{iy}$, $i^2 = -1$, and the Taylor series expansions of $\sin(x)$ and $\cos(x)$. The complex exponential function can be expressed as the smooth real function $\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$,

$$(x, y) \mapsto (e^x \cos(y), e^x \sin(y))$$

By the Leibniz rule and the derivative identities for both the exponential and trigonometric functions, we have that:

$$\mathsf{D}[\epsilon]\left((a,b),(c,d)\right) = \left(e^a\cos(b)c - e^a\sin(b)d, e^a\sin(b)c + e^a\cos(b)d\right)$$

Or using complex number notation: $\mathsf{D}[\epsilon](a+ib, c+id) = e^{a+ib}(c+id)$. It is well known that the complex exponential function satisfies the same basic properties as the real exponential function, that is, $e^0 = 1$ and $e^{z+w} = e^z e^w$ for $z, w \in \mathbb{C}$. As such, we can easily compute that (using complex number notation):

$$\mathsf{D}[\epsilon](0, a+ib) = e^0(a+ib) = a+ib$$
$$\mathsf{D}[\epsilon]\left(a+ib, e^{c+id}\right) = e^{a+ib}e^{c+id} = e^{(a+ib)+(c+id)}$$

Therefore, it follows that $\epsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a differential exponential map.

(v) Let $\mathbb{C}'$ be the ring of split-complex numbers [80, Section 1.1.2] (also known as hyperbolic two-complex numbers [77]):

$$\mathbb{C}' = \{x + jy |\ x, y \in \mathbb{R}, j^2 = 1\}$$

The split complex exponential function [77, Section 1.3] is instead defined using the hyperbolic cosine and sine functions cosh and sinh, that is, $e^{x+jy} = e^x\cosh(y) + je^x\sinh(y)$. As in the previous example, the split complex exponential function is derived from its power series definition:

$$e^{x+jy} = \sum_{n=0}^{\infty} \frac{(x+jy)^n}{n!}$$

and then simplifying by using that $e^{x+jy} = e^x e^{jy}$, $j^2 = 1$, and the Taylor series expansions of $\sinh(x)$ and $\cosh(x)$, as done in [77, Section 1.3]. Similar to the complex exponential function, the split complex exponential function can be expressed as the smooth real function $\epsilon' : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$:

$$(x,y) \mapsto (e^x\cosh(y), e^x\sinh(y))$$

By the Leibniz rule and the derivative identities for both the exponential and hyperbolic functions, we have that:

$$\mathsf{D}[\epsilon']\left((a,b),(c,d)\right) = \left(e^a\cosh(b)c + e^a\sinh(b)d, e^a\sinh(b)c + e^a\cosh(b)d\right)$$

Or using split complex numbers: $\mathsf{D}[\epsilon](a+jb, c+jd) = e^{a+jb}(c+jd)$. As explained in [77], the split complex exponential function satisfies that $e^0 = 1$ and $e^{u+v} = e^u e^v$ for $u, v \in \mathbb{C}'$. Then we can compute that (using split complex number notation):

$$\mathsf{D}[\epsilon'](0, a+jb) = \mathsf{D}[\epsilon](0, a+jb) = e^0(a+jb) = a+jb$$
$$\mathsf{D}[\epsilon']\left(a+jb, e^{c+jd}\right) = e^{a+jb}e^{c+jd} = e^{(a+jb)+(c+jd)}$$

Therefore, $\epsilon' : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a differential exponential map.

Note that Example 5.1.9.(iii), (iv), and (v) are not the product of differential exponential maps in the sense of Corollary 5.1.3.ii. That said, one could take the product of any of these differential exponential maps to obtain a multitude of other examples.

Example 5.1.10 Another example of a differential exponential map can be found in [24, Definition 5.20]. Briefly, in a Cartesian tangent category, a differential curve object [24, Definition 5.14] (viewed as an object in the Cartesian differential category of differential objects [19]) admits a canonical differential exponential map which arises as the solution to the well known associated differential equations. As a particular example, in the tangent category of real smooth manifolds, the differential curve object is $\mathbb{R}$ and its induced differential exponential map is the canonical exponential function e^x. In Section 5.3 we will discuss the link between differential exponential maps and differential equations, and in particular, we will see in Proposition 5.3.18 how a differential exponential map arises as the solution of a certain initial value problem.

Corollary 5.1.3.i tells us that the identity map of the terminal object is a differential exponential map. So we conclude this section with the observation that a differential exponential map is linear if and only if it is the identity map of a terminal object.

Lemma 5.1.11 *[64, Example 6] A differential exponential map $e : A \longrightarrow A$ is reduced (Definition 4.1.1) if and only if A is a terminal object. Therefore, a differential exponential map $e : A \longrightarrow A$ is additive (Definition 4.1.1) or linear if and only if A is a terminal object.*

PROOF: Suppose that $e : A \longrightarrow A$ is a differential exponential map which is reduced, that is, $0e = 0$. Then we have that:

$$\begin{aligned} e &= \langle 1, 0\rangle \oplus e && \text{(Lemma 4.1.4)} \\ &= \langle 1, 0\rangle(1 \times e)\mathsf{D}[e] && (5.1) \\ &= \langle 1, 0e\rangle\mathsf{D}[e] && \\ &= \langle 1, 0\rangle\mathsf{D}[e] && (e \text{ reduced}) \\ &= 0 && \textbf{[dC.2]} \end{aligned}$$

So $e = 0$. Now note that in a Cartesian left additive category, A is a terminal object if and only if $1_A = 0$ (we leave this to the reader to check for themselves). Then we have that:

$$\begin{aligned} 1_A &= \langle 0, 1\rangle\mathsf{D}[e] && (5.1) \\ &= \langle 0, 1\rangle\mathsf{D}[0] && (e = 0) \\ &= 0 && \textbf{[dC.1]} \end{aligned}$$

So $1_A = 0$, and so A is a terminal object. Conversely, if A is a terminal object, then we must have that $e = 1_A = 0$, and so clearly e is reduced. For the second statement, note that by definition every additive map is reduced, and since linear maps are additive (Lemma 4.1.19), they are also reduced. So if e is additive or linear, it is reduced and therefore A is a terminal object. Conversely, if A is a terminal object, then e must be the identity, and identity maps are always linear and additive (Lemma 4.1.19). □

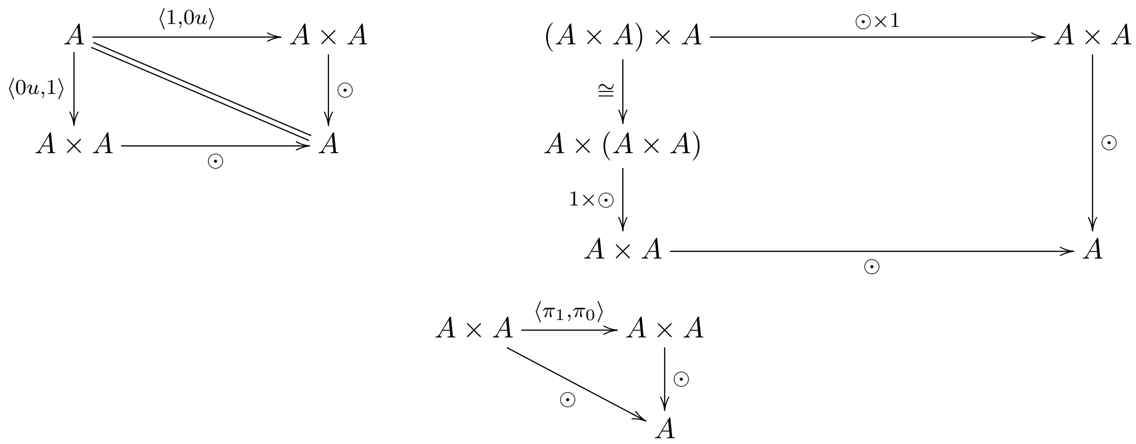
Example 5.1.12 For a category with finite biproducts seen as a Cartesian differential category as in Example 4.1.6, every map is linear by definition (Example 4.1.22).Therefore, in this case, the only differential exponential maps are the identity maps on terminal objects.

Note that for a Cartesian differential category $\mathbb{X}$, Lemma 5.1.11 also implies that the only differential object [19] in the Cartesian tangent category $\mathsf{DEM}[\mathbb{X}]$ is the terminal object.

5.2 Differential Exponential Rigs

In this section, we introduce *differential exponential rigs*, which provide an equivalent alternative characterization of differential exponential maps. We will show that every differential exponential map induces a differential exponential rig (Proposition 5.2.5) and that conversely, every differential exponential rig induces a differential exponential map (Proposition 5.2.4). We will also show that for a Cartesian differential category, its category of differential exponential maps is isomorphic to its category of differential exponential rigs (Theorem 5.2.9).

We begin by reviewing differential rigs, which are rigs in a Cartesian differential category whose multiplication is bilinear in the differential sense. Recall that in a category with finite products, a commutative monoid is a triple $(A, \odot, u)$ consisting of an object A, map $\odot : A \times A \longrightarrow A$, and a point $u : \top \longrightarrow A$ such that the following diagrams commute



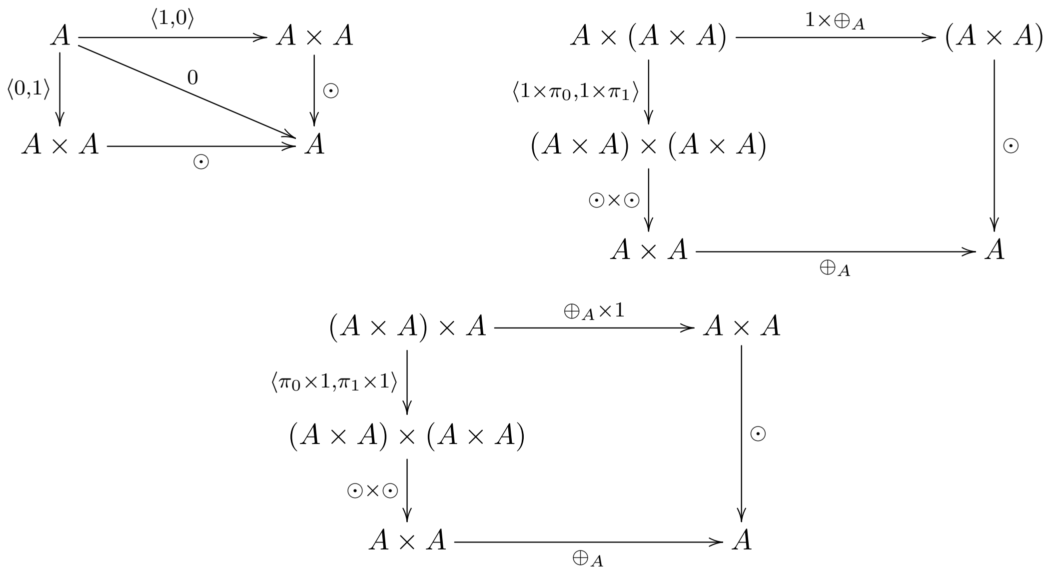
(5.5)

Definition 5.2.1 *A* ***differential rig*** *[64, Definition 5] in a Cartesian differential category $\mathbb{X}$, with differential combinator D is a triple $(A, \odot, u)$ consisting of an object A and two maps $\odot : A \times A \longrightarrow A$ and $u : \top \longrightarrow A$ such that $(A, \odot, u)$ is a commutative monoid and $\odot : A \times A \longrightarrow A$ is bilinear, that is, the following equality holds:*

$$\mathsf{D}[\odot] = (\pi_0 \times \pi_1) \odot + (\pi_1 \times \pi_0) \odot \tag{5.6}$$

We should justify the use of the term rig in differential rig. Indeed, the term (commutative) rig should imply that there are two (commutative) monoid structures that satisfy the expected distributivity axioms. But we know that in a Cartesian differential category, as discussed in Lemma 4.1.4, every object A already comes equipped with a commutative monoid structure $(A, \oplus, 0)$. So every differential rig $(A, \odot, u)$ does come with two commutative monoid structures. The required distributivity axioms are captured by the fact that $\odot$ is bilinear, and therefore additive in each argument – which is an equivalent way of saying that $\oplus$ and $\odot$ distribute over one another in the rig sense.

Lemma 5.2.2 *[64, Lemma 7] If $(A, \odot, u)$ is a differential rig, then $(A, \odot, u, \oplus, 0)$ is a commutative rig, that is, the following diagrams commute:*

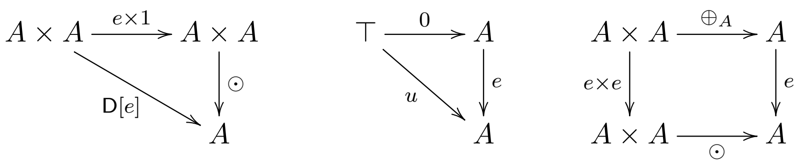


(5.7)

where $\oplus_A$ is defined as in Lemma 4.1.4.

A differential exponential rig is a differential rig equipped with an endomorphism which satisfies analogues of the three essential properties of the classical exponential function. This endomorphism will, of course, turn out to be a differential exponential map.

Definition 5.2.3 *A **differential exponential rig** [64, Definition 6] in a Cartesian differential category $\mathbb{X}$, with differential combinator D is a quadruple $(A, \odot, u, e)$ consisting of a differential rig $(A, \odot, u)$ and a map $e : A \longrightarrow A$, such that the following diagrams commute:*



(5.8)

where $\oplus$ is defined as in Lemma 4.1.4.

Examples of differential exponential rigs can be found below in Example 5.2.7 after we have proven Proposition 5.2.5 that every differential exponential map induces a differential exponential rig. If one keeps in mind the classical exponential function e^x, then the axioms of a differential exponential rig are straightforward to understand. The leftmost diagram of (5.8) generalizes that

$\mathsf{D}[e^x](x,y) = e^x y$, the middle diagram generalizes that $e^0 = 1$, and lastly the rightmost diagram generalizes that $e^{x+y} = e^x e^y$. Also note that the two right most diagrams of (5.8) says that e is a monoid morphism from $(A, \oplus, 0)$ to $(A, \odot, u)$. It is also worth discussing why the generalization of $e^{x+y} = e^x e^y$ is included in the axioms of a differential exponential rig, apart from being a desirable useful identity and one which is often included in algebraic generalizations of exponential functions [86]. Indeed in the classical case, the two leftmost diagrams of (5.8) are sufficient for characterizing the exponential function since e^x is the unique solution to the initial value problem $f'(x) = f(x)$ with $f(0) = 1$, and from that definition it is possible to prove that $e^{x+y} = e^x e^y$. In an arbitrary Cartesian differential category, however, it is not necessarily the case that $\mathsf{D}[e] = (e \times 1)\odot$ and $0e = u$ implies $\oplus e = (e \times e)\odot$. Furthermore, all three identities are necessary in proving that there is a bijective correspondence between differential exponential maps and differential exponential rigs. In turn, this allows us to drop the extra requirement for differential rig structure in characterizing these abstract exponential functions. That said, as we will see in Proposition 5.3.18, if one assumes uniqueness of solutions to certain initial value problems as in the classical case, then it is possible to derive $\oplus e = (e \times e)\odot$ from $\mathsf{D}[e] = (e \times 1)\odot$ and $0e = u$.

Proposition 5.2.4 *[64, Proposition 3] Let $(A, \odot, u, e)$ be a differential exponential rig. Then the underlying endomorphism $e : A \longrightarrow A$ is a differential exponential map.*

PROOF: We first show that $\langle 0, 1\rangle \mathsf{D}[e] = 1$:

$$\begin{aligned} \langle 0, 1\rangle \mathsf{D}[e] &= \langle 0, 1\rangle (e \times 1)\odot && (5.8) \\ &= \langle 0e, 1\rangle \odot \\ &= \langle 0u, 1\rangle \odot && (5.8) \\ &= 1 && (5.5) \end{aligned}$$

Next we show that $(1 \times e)\mathsf{D}[e] = \oplus_A e$:

$$\begin{aligned} (1 \times e)\mathsf{D}[e] &= (1 \times e)(e \times 1)\odot && (5.8) \\ &= (e \times e)\odot \\ &= \oplus_A e && (5.8) \end{aligned}$$

So we conclude that e is a differential exponential map. □

In order to show the converse of Proposition 5.2.4, consider the classical exponential function e^x and note that its second order derivative is:

$$\mathsf{D}^2[e^x]((x,y),(z,w)) = e^x yz + e^x w$$

Setting $x = 0$ and $w = 0$, one obtains precisely the multiplication of real numbers:

$$\mathsf{D}^2[e^x]((0,y),(z,0)) = yz$$

The unit for this multiplication is obtain from $e^0 = 1$. Generalizing this construction allows one to show how a differential exponential map induces a differential exponential rig.

Before doing the proof in an arbitrary Cartesian differential category, it may be worth mapping out how the proof will go using e^x. Commutativity of the multiplication, which is:

$$\mathsf{D}^2[e^x]((0,y),(z,0)) = \mathsf{D}^2[e^x]((0,z),(y,0))$$

follows from **[dC.7]**, which allows one to swap the middle two arguments of the second derivative. To show that e^0 is the unit, that is:

$$\mathsf{D}^2[e^x]((0,e^0),(x,0)) = x$$

we first observe that:

$$\mathsf{D}^2[e^x]((0,e^x),(y,0)) = e^x y = \mathsf{D}[e^x](x,y)$$

and then use the differential exponential map axiom to conclude that e^0 is indeed the unit. Associativity of the multiplication, which is that:

$$\mathsf{D}^2[e^x]\left((0,x),\left(\mathsf{D}^2[e^x]((0,y),(z,0)),0\right)\right) = \mathsf{D}^2[e^x]\left(\left(0,\mathsf{D}^2[e^x]((0,x),(y,0))\right),(z,0)\right)$$

is the trickiest part of the proof. Essentially, using the differential combinator axioms and the differential exponential map axioms, one can simplify both sides of the associativity law to the third order derivative of e^x (with 0 evaluated in the appropriate variables):

$$D^3[e^x]((0,x),(y,0),(z,0),(0,0)) = xyz$$

which itself turns out to be real numbers multiplication of three variables.

Proposition 5.2.5 *[64, Proposition 4] Let $e : A \longrightarrow A$ be a differential exponential map, and define the two maps $\odot_e : A \times A \longrightarrow A$ and $u_e : \top \longrightarrow A$ respectively as follows:*

$$\odot_e := \; A \times A \xrightarrow{\langle 0,1\rangle \times \langle 1,0\rangle} (A \times A) \times (A \times A) \xrightarrow{\mathsf{D}^2[e]} A$$

$$u_e := \; \top \xrightarrow{\;0\;} A \xrightarrow{\;e\;} A$$

Then $(A, \odot_e, u_e, e)$ is a differential exponential rig.

Proof: We will first prove that e satisfies the three identities of (5.8), as these will help simplify the proof that $(A, \odot_e, u_e)$ is a differential rig. We first prove that $\mathsf{D}[e] = (e \times 1)\odot_e$:

$$\begin{aligned}
\mathsf{D}[e] &= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,(\oplus_A \times \oplus_A)\mathsf{D}[e] && \text{(Lemma 4.1.4)}\\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}[\oplus_A e] && \text{($\oplus$ linear + Lemma 4.1.19)}\\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}\big[(1 \times e)\mathsf{D}[e]\big] && \text{(5.1)}\\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{T}(1 \times e)\mathsf{D}^2[e] && \text{(Lemma 5.1.6)}\\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,c\,\big(\mathsf{T}(1) \times \mathsf{T}(e)\big)\,\mathsf{D}^2[e] && \text{(Lemma 5.1.6)}\\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,c\,\big((1 \times 1) \times \mathsf{T}(e)\big)\,\mathsf{D}^2[e] && \text{(T is a functor)}\\
&= \langle \pi_1, \pi_0\rangle\,\big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\big((1 \times 1) \times \mathsf{T}(e)\big)\,\mathsf{D}^2[e]
\end{aligned}$$

$$\begin{aligned} &= \langle \pi_1, \pi_0 \rangle (1 \times e) \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big) \mathsf{D}^2[e] && \text{(Lemma 5.1.6)} \\ &= (e \times 1) \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big) \mathsf{D}^2[e] \\ &= (e \times 1)\odot_e \end{aligned}$$

Using the above equality, we can easily show that $\oplus_A e = (e \times e)\odot_e$:

$$\begin{aligned} \oplus_A e &= (1 \times e)\mathsf{D}[e] \\ &= (1 \times e)(e \times 1)\odot_e \\ &= (e \times e)\odot_e \end{aligned}$$

The remaining identity, $0e = u_e$ is automatic by construction. So e satisfies three identities of (5.8). Next we show that $(A, \odot_e, u_e)$ is a differential rig.

We first explain why $\odot_e$ is bilinear. In [31, Section 3] it was shown that for every map $f : A \longrightarrow B$, its second order partial derivative:

$$(A \times A) \times A \xrightarrow{(1\times 1)\times\langle 1,0\rangle} (A \times A) \times (A \times A) \xrightarrow{\mathsf{D}^2[f]} B$$

was bilinear in *context* A, so bilinear in its last two arguments A or equivalently bilinear with respect to the differential combinator of the simple slice category over A [9, Section 4.5]. By [10, Proposition 4.1.3], bilinear maps in context are preserved by pre-composition with maps which leave the bilinear arguments unaffected, that is, by pre-composition by maps of the form $(g \times 1) \times 1$. Therefore the composite:

$$(\top \times A) \times A \xrightarrow{(0\times 1)\times\langle 1,0\rangle} (A \times A) \times (A \times A) \xrightarrow{\mathsf{D}^2[f]} B$$

is bilinear in context $\top$. However maps which are bilinear in context $\top$ correspond precisely to bilinear maps without context. In this case, we obtain that the composite:

$$A \times A \xrightarrow{\langle 0,1\rangle\times\langle 1,0\rangle} (A \times A) \times (A \times A) \xrightarrow{\mathsf{D}^2[f]} B$$

is bilinear. Setting $f = e$, we conclude that $\odot_e$ is bilinear.

Now we show that $(A, \odot_e, u_e)$ is a commutative monoid by following the intuition provided above. First that $\odot_e$ is commutative, $\langle \pi_1, \pi_0 \rangle \odot_e = \odot_e$, follows from **[dC.7]**:

$$\begin{aligned} \langle \pi_1, \pi_0 \rangle \odot_e &= \langle \pi_1, \pi_0 \rangle \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big) \mathsf{D}^2[e] \\ &= \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big) c\mathsf{D}^2[e] \\ &= \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big) \mathsf{D}^2[e] && \textbf{[dC.7]} \\ &= \odot_e \end{aligned}$$

By commutativity, we need only show one of the unit identities, $\langle 0u_e, 1 \rangle \odot_e = 1$:

$$\begin{aligned} \langle 0u_e, 1 \rangle \odot_e &= \langle 0e, 1 \rangle \odot_e && (5.8) \\ &= \langle 0, 1 \rangle (e \times 1)\odot_e \\ &= \langle 0, 1 \rangle \mathsf{D}[e] && (5.8) \end{aligned}$$

$$= 1 \qquad (5.1)$$

Finally we prove associativity, which in the author's opinion is the most complex proof in this paper. Let $\alpha : (A \times A) \times A \longrightarrow A \times (A \times A)$ be the canonical associativity isomorphism $\alpha = \langle \pi_0, \pi_1 \times 1 \rangle$. By Lemma 4.1.19, α is linear and so is its inverse α^{-1}. As suggested above, our goal will be to show that the third-order derivative gives the three-fold multiplication, that is, we will show that we have the following equality:

$$\alpha(1 \times \odot_e)\odot_e = \Big((\langle 0,1 \rangle \times \langle 1,0 \rangle) \times \big\langle \langle 1,0 \rangle, 0 \big\rangle \Big) \mathsf{D}^3[e]$$

To do so, first note that by **[dC.5]** and **[dC.6]**, one can show that we have the following equality (which we leave as an exercise to the reader):

$$\alpha(1 \times \mathsf{D}[f])\mathsf{D}[g] = \big((1 \times 1) \times \langle 0,1 \rangle\big)\, \mathsf{D}\left[(1 \times f)\mathsf{D}[g]\right] \qquad (5.9)$$

Using the above identity, we compute that:

$$\begin{aligned}
\alpha(1 \times \odot_e)\mathsf{D}[e] =\ & \alpha \Big(1 \times \big(\langle 0,1 \rangle \times \langle 1,0 \rangle \big) \Big) (1 \times \mathsf{D}^2[e])\mathsf{D}[e] \\
=\ & \Big(\big(1 \times \langle 0,1 \rangle\big) \times \langle 1,0 \rangle \Big) \alpha(1 \times \mathsf{D}^2[e])\mathsf{D}[e] \\
=\ & \Big(1 \times \big(\langle 0,1 \rangle \times \langle 1,0 \rangle \big) \Big) \big((1 \times 1) \times \langle 0,1 \rangle\big)\, \mathsf{D}\left[(1 \times \mathsf{D}[e])\mathsf{D}[e]\right] && (5.9) \\
=\ & \big((1 \times \langle 0,1 \rangle) \times \langle 0, \langle 0,1 \rangle \rangle\big)\, \mathsf{D}\left[(1 \times \mathsf{D}[e])\mathsf{D}[e]\right] \\
=\ & \big((1 \times \langle 0,1 \rangle) \times \langle 0, \langle 0,1 \rangle \rangle\big)\, \mathsf{D}\left[\alpha^{-1} \big((1 \times 1) \times \langle 0,1 \rangle\big)\, \mathsf{D}\left[(1 \times e)\mathsf{D}[e]\right]\right] && (5.9) \\
=\ & \big((1 \times \langle 0,1 \rangle) \times \langle 0, \langle 0,1 \rangle \rangle\big)\, \mathsf{D}\left[\alpha^{-1} \big((1 \times 1) \times \langle 0,1 \rangle\big)\, \mathsf{D}\left[\oplus_A e\right]\right] && (5.1) \\
=\ & \big((1 \times \langle 0,1 \rangle) \times \langle 0, \langle 0,1 \rangle \rangle\big)\, \mathsf{D}\left[\alpha^{-1} \big((1 \times 1) \times \langle 0,1 \rangle\big) (\oplus_A \times \oplus_A)\mathsf{D}\left[e\right]\right] \\
& && \text{(Lemma 4.1.4 + Lemma 4.1.19)} \\
=\ & \big((1 \times \langle 0,1 \rangle) \times \langle 0, \langle 0,1 \rangle \rangle\big)\, \mathsf{D}\left[\alpha^{-1}(\oplus_A \times 1)\mathsf{D}\left[e\right]\right] && \text{(Lemma 4.1.4)} \\
=\ & \big((1 \times \langle 0,1 \rangle) \times \langle 0, \langle 0,1 \rangle \rangle\big) (\alpha^{-1} \times \alpha^{-1})\mathsf{D}\left[(\oplus_A \times 1)\mathsf{D}\left[e\right]\right] && \text{(Lemma 4.1.19)} \\
=\ & \big((1 \times \langle 0,1 \rangle) \times \langle 0, \langle 0,1 \rangle \rangle\big) (\alpha^{-1} \times \alpha^{-1}) \big((\oplus_A \times 1) \times (\oplus_A \times 1)\big)\, \mathsf{D}^2\left[e\right] \\
& && \text{(Lemma 4.1.4 + Lemma 4.1.19)} \\
=\ & \big((\langle 1,0 \rangle \times 1) \times \langle \langle 0,1 \rangle, 0 \rangle\big) \big((\oplus_A \times 1) \times (\oplus_A \times 1)\big)\, \mathsf{D}^2\left[e\right] \\
=\ & \big((1 \times 1) \times \langle 1,0 \rangle\big)\, \mathsf{D}^2\left[e\right] && \text{(Lemma 4.1.4)}
\end{aligned}$$

So we have that:

$$\alpha(1 \times \odot_e)\mathsf{D}[e] = \big((1 \times 1) \times \langle 1,0 \rangle\big)\, \mathsf{D}^2\left[e\right] \qquad (5.10)$$

Using this identity, we can simplify $\alpha(1 \times \odot_e)\odot_e$:

$$\begin{aligned}
\alpha(1 \times \odot_e)\odot_e =\ & \alpha(1 \times \odot_e) \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big)\, \mathsf{D}^2[e] \\
=\ & \alpha \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big) (1 \times \mathsf{T}(\odot_e))\mathsf{D}^2[e] && \text{(Lemma 5.1.6)} \\
=\ & \alpha \big(\langle 0,1 \rangle \times \langle 1,0 \rangle\big) (1 \times \mathsf{T}(\odot_e))c\mathsf{D}^2[e] && \textbf{[dC.7]}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)(\mathsf{T}(1) \times \mathsf{T}(\odot_e))c\mathsf{D}^2[e] && (\mathsf{T} \text{ is a functor})\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c\mathsf{T}(1 \times \odot_e)\mathsf{D}^2[e] && (\text{Lemma 5.1.6})\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c\mathsf{D}\left[(1 \times \odot_e)\mathsf{D}[e]\right] && (\text{Lemma 5.1.6})\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c\mathsf{D}\left[\alpha^{-1}\left((1 \times 1) \times \langle 1,0\rangle\right)\mathsf{D}^2[e]\right] && (5.10)\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c(\alpha^{-1} \times \alpha^{-1})\mathsf{D}\left[\left((1 \times 1) \times \langle 1,0\rangle\right)\mathsf{D}^2[e]\right] && (\text{Lemma 4.1.19})\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c(\alpha^{-1} \times \alpha^{-1})\\
&\quad \left(\left((1 \times 1) \times \langle 1,0\rangle\right) \times \left((1 \times 1) \times \langle 1,0\rangle\right)\right)\mathsf{D}^3[e] && (\text{Lemma 4.1.19})\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c(\alpha^{-1} \times \alpha^{-1})\\
&\quad \left(\left((1 \times 1) \times \langle 1,0\rangle\right) \times \left((1 \times 1) \times \langle 1,0\rangle\right)\right)c\mathsf{D}^3[e] && \textbf{[dC.7]}\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c(\alpha^{-1} \times \alpha^{-1})\\
&\quad \left(\left((1 \times 1) \times \langle 1,0\rangle\right) \times \left((1 \times 1) \times \langle 1,0\rangle\right)\right)c\mathsf{D}\left[c\mathsf{D}^2[e]\right] && \textbf{[dC.7]}\\
&= \alpha\left(\langle 0,1\rangle \times \langle 1,0\rangle\right)c(\alpha^{-1} \times \alpha^{-1})\\
&\quad \left(\left((1 \times 1) \times \langle 1,0\rangle\right) \times \left((1 \times 1) \times \langle 1,0\rangle\right)\right)c(c \times c)\mathsf{D}^3[e] && (\text{Lemma 4.1.19})\\
&= \left(\left(\langle 0,1\rangle \times \langle 1,0\rangle\right) \times \left\langle \langle 1,0\rangle, 0\right\rangle\right)\mathsf{D}^3[e]
\end{aligned}
$$

So we have that:

$$
\alpha(1 \times \odot_e)\odot_e = \left(\left(\langle 0,1\rangle \times \langle 1,0\rangle\right) \times \left\langle \langle 1,0\rangle, 0\right\rangle\right)\mathsf{D}^3[e] \tag{5.11}
$$

Now using the above identity and that we've already shown that $\odot_e$ is commutative, we finally can show that $\odot_e$ is associative, $(\odot_e \times 1)\odot_e = \alpha(1 \times \odot_e)\odot_e$:

$$
\begin{aligned}
(\odot_e \times 1)\odot_e &= (\odot_e \times 1)\langle \pi_1, \pi_0\rangle\odot_e && (5.5)\\
&= \langle \pi_1, \pi_0\rangle(1 \times \odot_e)\odot_e\\
&= \langle \pi_1, \pi_0\rangle\alpha^{-1}\left(\left(\langle 0,1\rangle \times \langle 1,0\rangle\right) \times \left\langle \langle 1,0\rangle, 0\right\rangle\right)\mathsf{D}^3[e] && (5.11)\\
&= \langle \pi_1, \pi_0\rangle\alpha^{-1}\left(\left(\langle 0,1\rangle \times \langle 1,0\rangle\right) \times \left\langle \langle 1,0\rangle, 0\right\rangle\right)\mathsf{D}\left[c\mathsf{D}^2[e]\right] && \textbf{[dC.7]}\\
&= \langle \pi_1, \pi_0\rangle\alpha^{-1}\left(\left(\langle 0,1\rangle \times \langle 1,0\rangle\right) \times \left\langle \langle 1,0\rangle, 0\right\rangle\right)(c \times c)\mathsf{D}^3[e] && (\text{Lemma 4.1.19})\\
&= \langle \pi_1, \pi_0\rangle\alpha^{-1}\left(\left(\langle 0,1\rangle \times \langle 1,0\rangle\right) \times \left\langle \langle 1,0\rangle, 0\right\rangle\right)(c \times c)c\mathsf{D}^3[e] && \textbf{[dC.7]}\\
&= \left(\left(\langle 0,1\rangle \times \langle 1,0\rangle\right) \times \left\langle \langle 1,0\rangle, 0\right\rangle\right)\mathsf{D}^3[e]\\
&= \alpha(1 \times \odot_e)\odot_e && (5.11)
\end{aligned}
$$

So $(A, \odot_e, u_e)$ is a differential rig, and therefore we conclude that $(A, \odot_e, u_e)$ is a differential exponential rig. □

In the proof of Theorem 5.2.9, we will show that the constructions of Proposition 5.2.4 and Proposition 5.2.5 are in fact inverse of each other. The construction from Proposition 5.2.5 is also compatible with some of the constructions of differential exponential maps in the following sense:

Lemma 5.2.6 *[64, Lemma 8] In a Cartesian differential category $\mathbb{X}$ with differential combinator D:*

(i) *For the terminal object $\top$, the following equality holds:*

$$\odot_{1_\top} = 0 \qquad\qquad u_{1_\top} = 1_\top$$

(ii) *If $e : A \longrightarrow A$ and $e' : B \longrightarrow B$ are differential exponential maps, then the following equality holds for the differential exponential map $e \times e' : A \times B \longrightarrow A \times B$:*

$$\odot_{e\times e'} = c(\odot_e \times \odot_{e'}) \qquad\qquad u_{e\times e'} = \langle u_e, u_{e'}\rangle$$

(iii) *If $e : A \longrightarrow A$ is a differential exponential map, then the following equality holds for the differential exponential map $\mathsf{T}(e) : A \times A \longrightarrow A \times A$:*

$$\odot_{\mathsf{T}(e)} = c\mathsf{T}(\odot_e) \qquad\qquad u_{\mathsf{T}(e)} = \langle u_e, 0\rangle$$

Proof:

(i) This is automatic by uniqueness of maps into the terminal object.

(ii) For the unit, this is mostly straightforward:

$$\begin{aligned} u_{e\times e'} &= 0(e \times e') \\ &= \langle 0e, 0e'\rangle \\ &= \langle u_e, u_{e'}\rangle \end{aligned}$$

For the multiplication, we have that:

$$\begin{aligned} \odot_{e\times e'} &= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}^2[e\times e'] \\ &= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}\Big[c\left(\mathsf{D}[e]\times \mathsf{D}[e']\right)\Big] && \text{(Lemma 5.1.6)} \\ &= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,(c\times c)\mathsf{D}\big[\mathsf{D}[e]\times \mathsf{D}[e']\big] && \text{(Lemma 4.1.19)} \\ &= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,(c\times c)c\left(\mathsf{D}^2[e]\times \mathsf{D}^2[e']\right) && \text{(Lemma 5.1.6)} \\ &= c\Big(\big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\times\big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\Big)\left(\mathsf{D}^2[e]\times \mathsf{D}^2[e']\right) \\ &= c(\odot_e \times \odot_{e'}) \end{aligned}$$

(iii) We first show the equality for the unit:

$$\begin{aligned} \langle u_e, 0\rangle &= \langle 0e, 0\rangle \\ &= 0e\langle 1,0\rangle \\ &= 0\langle 1,0\rangle\mathsf{T}(e) && \text{(Lemma 5.1.6)} \\ &= 0\mathsf{T}(e) && \text{($\langle 1,0\rangle$ is additive)} \end{aligned}$$

$$= u_{\mathsf{T}(e)}$$

For the multiplication, we have that:

$$\begin{aligned}
\odot_{\mathsf{T}(e)} &= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}^2[\mathsf{T}(e)] \\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}\big[c\mathsf{T}(\mathsf{D}[e])\big] && \text{(Lemma 5.1.6)} \\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,(c \times c)\mathsf{D}\big[\mathsf{T}(\mathsf{D}[e])\big] && \text{(Lemma 4.1.19)} \\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,(c \times c)c\mathsf{T}\left(\mathsf{D}^2[e]\right) && \text{(Lemma 5.1.6)} \\
&= c\left(\big(\langle 0,1\rangle \times \langle 1,0\rangle\big) \times \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\right)\mathsf{T}\left(\mathsf{D}^2[e]\right) \\
&= c\mathsf{T}\left(\big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}^2[e]\right) && \text{(Lemma 4.1.19)} \\
&= c\mathsf{T}(\odot_e)
\end{aligned}$$

□

Example 5.2.7 Here we apply Proposition 5.2.5 to the examples of differential exponential maps from the previous section to construct examples of differential exponential rigs in the Cartesian differential category SMOOTH from Example 4.1.8 (which are in fact rings, since SMOOTH has additive inverses).

(i) For the exponential function $e^x : \mathbb{R} \longrightarrow \mathbb{R}$, the induced multiplication is precisely given by the standard multiplication of real numbers, that is:

$$\odot_{e^x}(x, y) = xy$$

and $u_{e^x}(*) = 1$.

(ii) Applying Lemma 5.2.6.ii to the point-wise exponential $e^x \times \ldots \times e^x : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we obtain the point-wise multiplication of vectors, that is,

$$\odot_{e^x \times \ldots \times e^x}\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = (x_1y_1, \ldots, x_ny_n)$$

and $u_{e^x \times \ldots \times e^x}(*) = (1, \ldots, 1)$.

(iii) Applying Lemma 5.2.6.iii to the tangent exponential function $\mathsf{T}(e^x) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, we obtain the multiplication:

$$\odot_{\mathsf{T}(e)}\big((x_1, y_1), (x_2, y_2)\big) = (x_1x_2, x_1y_2 + y_1x_2)$$

with unit $u_{\mathsf{T}(e)}(*) = (1, 0)$. Observe that this ring structure on $\mathbb{R}^2$ is precisely that of the ring of dual numbers $\mathbb{R}[\varepsilon]$. Indeed, writing (x, y) as $x + y\varepsilon$ with $\varepsilon^2 = 0$, we see that $\odot_{\mathsf{T}(e)}$ is precisely the multiplication of dual numbers:

$$(x_1 + y_1\varepsilon)(x_2 + y_2\varepsilon) = x_1x_2 + (x_1y_2 + y_1x_2)\varepsilon$$

(iv) For the complex exponential function $\epsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, we obtain the multiplication:

$$\odot_\epsilon \big((x_1, y_1), (x_2, y_2)\big) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

with unit $u_\epsilon(*) = (1, 0)$. Unsurprisingly, this ring structure on $\mathbb{R}^2$ is that of complex numbers $\mathbb{C}$. Indeed, writing (x, y) as $x + iy$ with $i^2 = -1$, $\odot_\epsilon$ gives precisely complex number multiplication:

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

(v) For the split complex exponential function $\epsilon' : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, we obtain the multiplication:

$$\odot_{\epsilon'} \big((x_1, y_1), (x_2, y_2)\big) = (x_1x_2 + y_1y_2, x_1y_2 + x_2y_1)$$

with unit $u_{\epsilon'}(*) = (1, 0)$. This ring structure on $\mathbb{R}^2$ is that of split complex numbers $\mathbb{C}'$. Indeed, writing (x, y) as $x + jy$ with $j^2 = 1$, $\odot_{\epsilon'}$ gives split complex number multiplication:

$$(x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 + y_1y_2) + j(x_1y_2 + x_2y_1)$$

Example 5.2.8 As briefly discussed in Example 5.1.10, a differential curve object with solutions to linear systems admits a differential exponential map, and so by applying Proposition 5.2.5, a differential curve object is also a differential exponential rig [24, Corollary 5.26]. As an interesting application of this induced rig structure, in turns out that every differential bundle [21] is a module of the differential curve object [24, Proposition 5.4].

For a Cartesian differential category $\mathbb{X}$, define its category of differential exponential rigs as the category $\mathsf{DES}[\mathbb{X}]$ whose objects are differential exponential rigs $(A, \odot, u, e)$ and where a map $f : (A, \odot, u, e) \longrightarrow (B, \odot', u', e')$ is a *linear* map $f : A \longrightarrow B$ in $\mathbb{X}$ such that the following diagrams commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} \top & \xrightarrow{u} & A \\ & \searrow^{u'} & \downarrow{\scriptstyle u} \\ & & B \end{array} \qquad \begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ {\scriptstyle \odot}\downarrow & & \downarrow{\scriptstyle \odot'} \\ A & \xrightarrow[f]{} & B \end{array} \tag{5.12}$$

and where composition and identity maps are as in $\mathbb{X}$. Note that the two right most diagrams above imply that f is a monoid morphism.

Theorem 5.2.9 *[64, Theorem 1] For a Cartesian differential category $\mathbb{X}$, its category of differential exponential maps $\mathsf{DEM}[\mathbb{X}]$ is isomorphic to its category of differential exponential rigs $\mathsf{DES}[\mathbb{X}]$ via the inverse functors $\mathsf{E} : \mathsf{DEM}[\mathbb{X}] \longrightarrow \mathsf{DES}[\mathbb{X}]$ and $\mathsf{E}^{-1} : \mathsf{DES}[\mathbb{X}] \longrightarrow \mathsf{DEM}[\mathbb{X}]$ defined respectively as follows:*

$$\mathsf{E}(A, e) = (A, \odot_e, u_e, e) \qquad \mathsf{E}(f) = f \qquad \mathsf{E}^{-1}(A, \odot, u, e) = (A, e) \qquad \mathsf{E}^{-1}(f) = f$$

and therefore, there is a bijective correspondence between differential exponential maps and differential exponential rigs.

PROOF: We first need to show that E and E^{-1} are well-defined functors. By Proposition 5.2.4, E^{-1} is well-defined on objects, while E is well-defined on objects by Proposition 5.2.5. If f is a map in $\mathsf{DES}[\mathbb{X}]$ then by definition it is also a map in $\mathsf{DEM}[\mathbb{X}]$, so E^{-1} is well-defined on maps. Furthermore, E^{-1} clearly preserves composition and identity, and therefore E^{-1} is a well-defined functor. On the other hand, if $f : (A, e) \longrightarrow (B, e')$ is a map in $\mathsf{DEM}[\mathbb{X}]$, we must show that f also satisfies the three identities (5.12). By definition, one already has that $ef = fe'$, and so it remains to show that f is also a monoid morphism. Recall that since f is linear, f is also additive (Lemma 4.1.19). Now we first show that $u_e f = u_{e'}$:

$$\begin{aligned}
u_e f &= 0ef \\
&= 0fe' && (5.2)\\
&= 0e' && (f \text{ additive})\\
&= u_{e'}
\end{aligned}$$

Next we show that $\odot_e f = (f \times f)\odot_{e'}$:

$$\begin{aligned}
\odot_e f &= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}^2[e]f \\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}^2[ef] && (f \text{ linear + Lemma 4.1.19})\\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}^2[fe'] && (5.2)\\
&= \big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\big((f \times f) \times (f \times f)\big)\,\mathsf{D}^2[e'] && (f \text{ linear + Lemma 4.1.19})\\
&= \big(\langle 0f,f\rangle \times \langle f,0f\rangle\big)\,\mathsf{D}^2[e'] \\
&= \big(\langle 0,f\rangle \times \langle f,0\rangle\big)\,\mathsf{D}^2[e'] && (f \text{ additive})\\
&= (f \times f)\big(\langle 0,1\rangle \times \langle 1,0\rangle\big)\,\mathsf{D}^2[e'] \\
&= (f \times f)\odot_{e'}
\end{aligned}$$

Therefore f is a map in $\mathsf{DES}[\mathbb{X}]$, so E is well-defined on maps. Clearly E preserves composition and identity, and therefore E is also a well-defined functor.

Next we show that E and E^{-1} are inverses of each other. Clearly we have both that:

$$\mathsf{E}^{-1}\mathsf{E}(A, e) = (A, e) \qquad\qquad \mathsf{E}^{-1}\mathsf{E}(f) = f$$

For the other direction, clearly $\mathsf{E}\mathsf{E}^{-1}(f) = f$ and so it remains to show that we also have that $\mathsf{E}\mathsf{E}^{-1}(A, \odot, u, e) = (A, \odot, u, e)$, that is, we must show that $\odot = \odot_e$ and $u_e = u$. Starting with the unit:

$$\begin{aligned}
u_e &= 0e \\
&= u && (5.8)
\end{aligned}$$

Next for the multiplication, we first observe that:

$$\begin{aligned}
\mathsf{D}^2[e] &= \mathsf{D}\left[(e \times 1)\odot\right] && (5.8)\\
&= \mathsf{T}(e \times 1)\mathsf{D}\left[\odot\right] && (\text{Lemma 5.1.6})\\
&= c\big(\mathsf{T}(e) \times \mathsf{T}(1)\big)\,c\mathsf{D}\left[\odot\right] && (\text{Lemma 5.1.6})
\end{aligned}$$

$$
\begin{aligned}
&= c\left(\mathsf{T}(e)\times 1\right)c\mathsf{D}\left[\odot\right] && (\mathsf{T}\text{ is a functor})\\
&= c\left(\mathsf{T}(e)\times 1\right)c(\pi_0\times\pi_1)\odot + c\left(\mathsf{T}(e)\times 1\right)c(\pi_1\times\pi_0)\odot && (5.6)\\
&= c\left(\mathsf{T}(e)\times 1\right)(\pi_0\times\pi_1)\odot + c\left(\mathsf{T}(e)\times 1\right)(\pi_1\times\pi_0)\langle\pi_1,\pi_0\rangle\odot\\
&= c\left(\mathsf{T}(e)\times 1\right)(\pi_0\times\pi_1)\odot + c\left(\mathsf{T}(e)\times 1\right)(\pi_1\times\pi_0)\odot && (5.5)\\
&= c(\pi_0\times\pi_1)(e\times 1)\odot + c(\mathsf{D}[e]\times\pi_0)\odot && (\text{Definition of }\mathsf{T})\\
&= (\pi_0\times\pi_1)(e\times 1)\odot + c\left((e\times 1)\times(1\times 1)\right)(\odot\times\pi_0)\odot && (5.8)\\
&= (\pi_0\times\pi_1)(e\times 1)\odot + \left((e\times 1)\times(1\times 1)\right)c(\odot\times\pi_0)\odot\\
&= (\pi_0\times\pi_1)(e\times 1)\odot + \left((e\times 1)\times(1\times 1)\right)c\left((1\times 1)\times\pi_0\right)(\odot\times 1)\odot\\
&= (\pi_0\times\pi_1)(e\times 1)\odot\\
&\quad + \left((e\times 1)\times(1\times 1)\right)c\left((1\times 1)\times\pi_0\right)\alpha(1\times\odot)\odot && (5.5)\\
&= (\pi_0\times\pi_1)(e\times 1)\odot\\
&\quad + \left((e\times 1)\times(1\times 1)\right)\langle\pi_0,(\pi_1\times\pi_0)\rangle(1\times\langle\pi_1,\pi_0\rangle)(1\times\odot)\odot\\
&= (\pi_0\times\pi_1)(e\times 1)\odot\\
&\quad + \left((e\times 1)\times(1\times 1)\right)\langle\pi_0\pi_0,(\pi_1\times\pi_0)\rangle(1\times\odot)\odot && (5.5)\\
&= (\pi_0\times\pi_1)(e\times 1)\odot\\
&\quad + \left((e\times 1)\times(1\times 1)\right)\langle\pi_0\pi_0,(\pi_1\times\pi_0)\rangle\alpha^{-1}(\odot\times 1)\odot && (5.5)\\
&= (\pi_0\times\pi_1)(e\times 1)\odot + \left((e\times 1)\times(1\times 1)\right)\left((1\times 1)\times\pi_0\right)(\odot\times 1)\odot\\
&= (\pi_0\times\pi_1)(e\times 1)\odot + \left((e\times 1)\times\pi_0\right)(\odot\times 1)\odot\\
&= (\pi_0\times\pi_1)\mathsf{D}[e] + (\mathsf{D}[e]\times\pi_0)\odot
\end{aligned}
$$

So we have that:

$$
\mathsf{D}^2[e] = (\pi_0\times\pi_1)\mathsf{D}[e] + (\mathsf{D}[e]\times\pi_0)\odot \tag{5.13}
$$

Using the above identity, we can easily show that $\odot_e = \odot$:

$$
\begin{aligned}
\odot_e &= \left(\langle 0,1\rangle\times\langle 1,0\rangle\right)\mathsf{D}^2[e]\\
&= \left(\langle 0,1\rangle\times\langle 1,0\rangle\right)(\pi_0\times\pi_1)\mathsf{D}[e] + \left(\langle 0,1\rangle\times\langle 1,0\rangle\right)(\mathsf{D}[e]\times\pi_0)\odot && (5.13)\\
&= (0\times 0)\mathsf{D}[e] + (1\times 1)\odot && (5.1)\\
&= 0 + \odot && \textbf{[dC.2]}\\
&= \odot
\end{aligned}
$$

So $(A,\odot,u,e) = (A,\odot_e,u_e,e)$. Therefore we conclude that E and E^{-1} are inverse functors and that $\mathsf{DEM}[\mathbb{X}]$ is isomorphic to $\mathsf{DES}[\mathbb{X}]$. □

We conclude this section with the observation that as an immediate consequence of both Theorem 5.2.9 and Lemma 5.2.6: the category of differential exponential rigs is a Cartesian tangent category and that it is isomorphic as a Cartesian tangent category to the category of differential exponential maps.

Proposition 5.2.10 *[64, Proposition 5] For a Cartesian differential category $\mathbb{X}$, $\mathsf{DES}[\mathbb{X}]$ has finite products where the terminal object is $(\top, 0, 1_\top, 1_\top)$, and where the product of $(A, \odot, u, e)$ and $(B, \odot', u', e')$ is:*

$$(A \times B, c(\odot \times \odot'), \langle u, u' \rangle, e \times e')$$

with the obvious projection maps. $\mathsf{DES}[\mathbb{X}]$ is also a Cartesian tangent category where the tangent functor $\mathsf{T} : \mathsf{DES}[\mathbb{X}] \longrightarrow \mathsf{DES}[\mathbb{X}]$ is defined as follows:

$$\mathsf{T}(A, \odot, u, e) := (A \times A, c\mathsf{T}(\odot), \langle u, 0 \rangle, \mathsf{T}(e)) \qquad \mathsf{T}(f) = f \times f$$

and where the remaining tangent structure is the same as for $\mathbb{X}$ (which can be found in [19, Proposition 4.7]). Furthermore, both $\mathsf{E} : \mathsf{DEM}[\mathbb{X}] \longrightarrow \mathsf{DES}[\mathbb{X}]$ and $\mathsf{E}^{-1} : \mathsf{DEM}[\mathbb{X}] \longrightarrow \mathsf{DES}[\mathbb{X}]$ preserve the Cartesian tangent structure strictly.

5.3 Solutions to Dynamical Systems

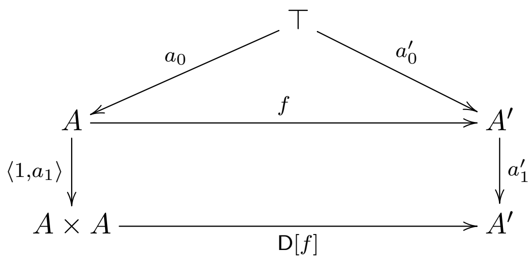
As introduced in [20, Section 5], ordinary differential equations in a Cartesian differential category are described as *dynamical systems*, while solutions for these differential equations are described as morphisms between these dynamical systems. In the classical case, the exponential function e^x can be defined as the unique solution to the initial value problem $f'(x) = f(x)$ with $f(0) = 1$. In this section, we explain how differential exponential maps provide solutions to certain (parametrized) dynamical systems and conversely how one can obtain a differential exponential map if one assumes that solutions are unique. See the work done by Cockett, Cruttwell, and the author in [24, Section 5] for more applications of differential exponential maps in regards to solving differential equations.

We note that in [20, 24], dynamical systems were defined for tangent categories and thus involve the tangent functor. Here we present the resulting definition specific to Cartesian differential categories, where dynamical systems can be described in terms of the differential combinator.

Definition 5.3.1 *[64, Definition 7] In a Cartesian differential category $\mathbb{X}$ with differential combinator D:*

(i) *A* ***dynamical system*** *[20, Definition 5.15] is a triple (A, a_0, a_1) consisting of an object A, a point $a_0 : \top \longrightarrow A$, and an endomorphism $a_1 : A \longrightarrow A$;*

(ii) *A* ***morphism of dynamical systems*** *$f : (A, a_0, a_1) \longrightarrow (A', a'_0, a'_1)$ is a map $f : A \longrightarrow A'$ such that the following diagram commutes:*



(5.14)

(iii) *If* $f : (A, a_0, a_1) \longrightarrow (A', a'_0, a'_1)$ *is a morphism of dynamical systems, we say that* f *is an* (A, a_0, a_1)***-solution*** *of* (A', a'_0, a'_1).

Example 5.3.2 A dynamical system in the Cartesian differential category SMOOTH from Example 4.1.8 can be seen as a triple $(\mathbb{R}^n, \vec{a}, F)$ where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth function and a point $\vec{a} \in \mathbb{R}^n$. If $(\mathbb{R}^n, \vec{a}, F)$ and $(\mathbb{R}^m, \vec{b}, G)$ are dynamical systems, then a $(\mathbb{R}^n, \vec{a}, F)$-solution of $(\mathbb{R}^m, \vec{b}, G)$ is a smooth function $H : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that:

$$H(\vec{a}) = \vec{b} \qquad\qquad \mathsf{D}[H](\vec{x}, F(\vec{x})) = G(H(\vec{x}))$$

which amounts to saying that H is a solution to a certain (large) system of differential equations. For a more explicit example, let $\overline{c} : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-zero constant function $\overline{c}(x) = c$, $c \neq 0$, and define the smooth function $g : \mathbb{R} \longrightarrow \mathbb{R}$ as $g(x) = -rx$ for some $r \in \mathbb{R}$. Then a $(\mathbb{R}, 0, \overline{c})$-solution of $(\mathbb{R}, a, g)$ is a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$f(0) = a \qquad\qquad \mathsf{D}[f](x, c) = -rf(x)$$

which is equivalent to saying that f is a solution to the following linear differential equation:

$$f(0) = b \qquad\qquad f'(x) + \lambda f(x) = 0$$

where $\lambda = \frac{r}{c}$. See [20, 24] for more details and intuition on dynamical systems.

For any differential rig $(A, \odot, u)$, there is a canonical dynamical system where the underlying endomorphism $\overline{u} : A \longrightarrow A$ is defined as follows:

$$\overline{u} := A \xrightarrow{\ 0\ } \top \xrightarrow{\ u\ } A \tag{5.15}$$

and we can ask that $(A, 0, \overline{u})$-solutions be compatible with the multiplication.

Definition 5.3.3 *[64, Definition 8] Let* $(A, \odot, u)$ *be a differential rig and* (A, a_0, a_1) *a dynamical system. An* $(A, \odot, u)$***-solution*** *of* (A, a_0, a_1) *is a map* $f : A \longrightarrow A$ *such that the following diagrams commute:*

$$\begin{array}{ccc} \top & \xrightarrow{\ 0\ } & A \\ & {\scriptstyle a_0} \searrow & \downarrow {\scriptstyle f} \\ & & A \end{array} \qquad\qquad \begin{array}{ccccc} A \times A & \xrightarrow{\ f \times 1\ } & A \times A & \xrightarrow{\ a_1 \times 1\ } & A \times A \\ & & {\scriptstyle \mathsf{D}[f]} \searrow & & \downarrow {\scriptstyle \odot} \\ & & & & A \end{array} \tag{5.16}$$

Lemma 5.3.4 *[64, Lemma 9] Let* $(A, \odot, u)$ *be a differential rig,* (A, a_0, a_1) *a dynamical system,* $f : A \longrightarrow A$ *an endomorphism. Then* f *is an* $(A, \odot, u)$*-solution of* (A, a_0, a_1) *if and only if* f *is an* $(A, 0, \overline{u})$*-solution of* (A, a_0, a_1) *such that the following diagram commutes:*

$$\begin{array}{ccccc} A \times A & \xrightarrow{\ \langle 1, \overline{u} \rangle \times 1\ } & (A \times A) \times A & \xrightarrow{\ \mathsf{D}[f] \times 1\ } & A \times A \\ & & {\scriptstyle \mathsf{D}[f]} \searrow & & \downarrow {\scriptstyle \odot} \\ & & & & A \end{array} \tag{5.17}$$

PROOF: Suppose that f is an $(A, 0, \overline{u})$-solution of (A, a_0, a_1). We first show that f is also a $(A, 0, \overline{u})$-solution of (A, a_0, a_1). The top triangle of (5.14) is precisely the left diagram of (5.16). So it remains to show that f also satisfies the bottom square of (5.14):

$$\begin{aligned} \langle 1, \overline{u}\rangle \mathsf{D}[f] &= \langle 1, \overline{u}\rangle (f \times 1)(a_1 \times 1)\odot && (5.16) \\ &= f a_1 \langle 1, \overline{u}\rangle \odot \\ &= f a_1 \langle 1, 0u\rangle \odot \\ &= f a_1 && (5.5) \end{aligned}$$

So f is an $(A, 0, \overline{u})$-solution of (A, a_0, a_1). As an immediate consequence, it follows that:

$$\begin{aligned} (\langle 1, \overline{u}\rangle \times 1)(\mathsf{D}[f] \times 1)\odot &= (f \times 1)(a_1 \times 1)\odot \\ &= \mathsf{D}[f] && (5.16) \end{aligned}$$

So f also satisfies (5.17). Conversely, suppose that f is an $(A, 0, \overline{u})$-solution of (A, a_0, a_1) which satisfies (5.17). We must show that f satisfies the two diagrams of (5.16). As before, the left diagram of (5.16) is the same as the top triangle of (5.14). So it remains to show that f also satisfies the right diagram of (5.16):

$$\begin{aligned} \mathsf{D}[f] &= (\langle 1, \overline{u}\rangle \times 1)(\mathsf{D}[f] \times 1)\odot && (5.17) \\ &= (f \times 1)(a_1 \times 1)\odot && (5.14) \end{aligned}$$

So we conclude that f is an $(A, \odot, u)$-solution of (A, a_0, a_1). □

Example 5.3.5 In the Cartesian differential category SMOOTH from Example 4.1.8, consider the differential rig induced from the exponential function e^x as defined in Example 5.2.7.i, that is, $\mathbb{R}$ with the standard multiplication of real numbers. Its canonical dynamical system as defined above is $(\mathbb{R}, 0, \overline{u_{e^x}})$ since $\overline{u_{e^x}}(x) = 1$. A $(\mathbb{R}, \odot_{e^x}, u_{e^x})$-solution of a dynamical system $(\mathbb{R}, a, g)$, where $a \in \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$, is a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$f(0) = a \qquad\qquad \mathsf{D}[f](x, y) = f'(x)y = g(f(x))y$$

By Lemma 5.3.4, setting $y = 1$, we see that f is also a solution to the differential equation $f'(x) = f(g(x))$ with initial value $f(0) = a$, or in other words, f is a $(\mathbb{R}, 0, \overline{u_{e^x}})$-solution of $(\mathbb{R}, a, g)$ such that f satisfies (5.17):

$$\mathsf{D}[f](x, y) = \mathsf{D}[f](x, 1)y$$

In fact, note that every arbitrary smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies (5.17) since:

$$\mathsf{D}[f](x, y) = f'(x)y = \mathsf{D}[f](x, 1)y$$

Then in this case, every $(\mathbb{R}, 0, \overline{u_{e^x}})$-solution of $(\mathbb{R}, a, g)$ is also a $(\mathbb{R}, \odot_{e^x}, u_{e^x})$-solution of $(\mathbb{R}, a, g)$.

For a differential exponential rig, its differential exponential map is the solution to the dynamical system which generalizes the initial value problem $f'(x) = f(x)$ with $f(0) = 1$.

Proposition 5.3.6 *[64, Proposition 6] Let $(A, \odot, u, e)$ be a differential exponential rig. Then the differential exponential map e is an $(A, \odot, u)$-solution of the dynamical system $(A, u, 1)$, and therefore e is also a $(A, 0, \overline{u})$-solution of $(A, u, 1)$.*

PROOF: The left diagram of (5.16) is precisely the middle diagram of (5.8) that $0e = u$. While the right diagram of (5.16) is precisely the left diagram of (5.8) that $\mathsf{D}[e] = (e \times 1)\odot$. Therefore, e is an $(A, \odot, u)$-solution of $(A, u, 1)$. By Lemma 5.3.4, it follows that e is also a $(A, 0, \overline{u})$-solution of $(A, u, 1)$. □

Example 5.3.7 In the Cartesian differential category SMOOTH from Example 4.1.8, $e^x : \mathbb{R} \longrightarrow \mathbb{R}$ is an $(\mathbb{R}, \odot_{e^x}, u_{e^x})$-solution of $(\mathbb{R}, u, 1)$. In other words, e^x is a solution to the following initial value problem:

$$\mathsf{D}[f](x, y) = f(x)y \qquad f(0) = 1$$

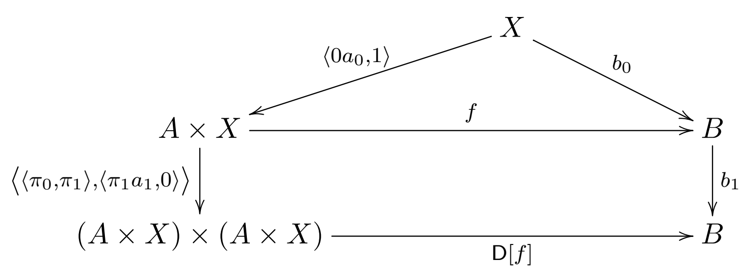
In fact, e^x is the unique $(\mathbb{R}, \odot_{e^x}, u_{e^x})$-solution of $(\mathbb{R}, u, 1)$. By Proposition 5.3.6, setting $y = 1$, e^x is also a $(\mathbb{R}, 0, \overline{u_{e^x}})$-solution of $(\mathbb{R}, u, 1)$, that is, e^x is the unique solution to the initial value problem $f'(x) = f(x)$ with $f(0) = 1$.

Note, however, that Proposition 5.3.6 does not say that a differential exponential map is a unique solution. Indeed, in an arbitrary Cartesian differential category, solutions of dynamical systems need not be unique, and so it is possible that for a differential rig $(A, \odot, u)$, there are multiple $(A, 0, \overline{u})$-solutions of $(A, u, 1)$. Furthermore, as discussed in Section 5.2, a $(A, 0, \overline{u})$-solution of $(A, u, 1)$ is not necessarily a differential exponential map since $\oplus_A e = (e \times e)\odot$ does not necessarily hold. That said, as we will see in Proposition 5.3.18, with the extra assumption that solutions be unique, then it follows that a $(A, 0, \overline{u})$-solution of $(A, u, 1)$ is, in this case, a differential exponential map. To do so, we first discuss the notion of solutions of *parametrized* dynamical systems.

Definition 5.3.8 *[64, Definition 9] In a Cartesian differential category $\mathbb{X}$ with differential combinator D:*

(i) *A* ***parametrized dynamical system*** *(over X or in context X) is a triple (B, b_0, b_1) consisting of an object B, a map $b_0 : X \longrightarrow B$, and an endomorphism $b_1 : B \longrightarrow B$;*

(ii) *If (A, a_0, a_1) is a dynamical system and (B, b_0, b_1) a parametrized dynamical system, then a* ***parametrized*** *(A, a_0, a_1)**-solution of*** *(B, b_0, b_1) is a map $f : A \times X \longrightarrow B$ such that the following diagram commutes:*



(5.18)

(iii) *For a dynamical system* (A, a_0, a_1)*, an endomorphism* $b_1 : B \longrightarrow B$ *is said to be* (A, a_0, a_1)***-complete*** *if for every map* $b_0 : X \longrightarrow B$*, there is an* (A, a_0, a_1)*-solution of the parametrized dynamical system* (B, b_0, b_1)*.*

Note that dynamical systems can be described as parametrized dynamical systems over the terminal object $\top$ and in this case (5.14) is the same as (5.18), modulo the isomorphism $A \cong A \times \top$.

Example 5.3.9 In the Cartesian differential category SMOOTH from Example 4.1.8, a parametrized dynamical system is simply a triple $(\mathbb{R}^m, K, G)$ with smooth functions $K : \mathbb{R}^k \longrightarrow \mathbb{R}^m$ and $G : \mathbb{R}^m \longrightarrow \mathbb{R}^m$. If $(\mathbb{R}^n, \vec{a}, H)$ is a dynamical system, then a parametrized $(\mathbb{R}^n, \vec{a}, H)$-solution of $(\mathbb{R}^m, K, G)$ is a smooth function $F : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^m$ such that:

$$F(\vec{a}, \vec{y}) = K(\vec{y}) \qquad\qquad \mathsf{D}[F]\left((\vec{x}, \vec{y}, K(\vec{y}), \vec{0}\right) = G(F(\vec{x}, \vec{y}))$$

As a particular example, let $m = 1$ with smooth functions $h : \mathbb{R}^k \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. A parametrized $(\mathbb{R}, 0, \overline{u_{e^x}})$-solution of $(\mathbb{R}, h, g)$ is a smooth function $f : \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ such that:

$$f(0, y_1, \ldots, y_k) = h(y_1, \ldots, y_k)$$
$$\frac{\partial f(t_0, t_1, \ldots, t_n)}{\partial t_0}(x, y_1, \ldots, y_k) = g\left(f(x, y_1, \ldots, y_k)\right)$$

As before, in the case of a differential rig, one can also ask for parametrized solutions to be compatible with the rig multiplication.

Definition 5.3.10 *[64, Definition 10] Let* $(A, \odot, u)$ *be a differential rig and* (A, a_0, a_1) *be a parametrized dynamical system over* X*.*

(i) *A* ***parametrized*** $(A, \odot, u)$***-solution*** *of* (A, a_0, a_1) *is a map* $f : A \times X \longrightarrow A$ *such that the following diagrams commute:*

$$\begin{array}{ccc} X & \xrightarrow{\langle 0,1 \rangle} & A \times X \\ & \searrow^{a_0} & \downarrow f \\ & & A \end{array} \qquad \begin{array}{ccccc} (A \times X) \times A & \xrightarrow{f \times 1} & A \times A & \xrightarrow{a_1 \times 1} & A \times A \\ \downarrow (1\times 1)\times\langle 1,0\rangle & & & & \downarrow \odot \\ (A \times X) \times (A \times X) & & \xrightarrow{\mathsf{D}[f]} & & A \end{array} \qquad (5.19)$$

(ii) *An endomorphism* $a_1 : A \longrightarrow A$ *is* $(A, \odot, u)$***-complete*** *if for every map* $a_0 : X \longrightarrow A$*, there is a parametrized* $(A, \odot, u)$*-solution of the parametrized dynamical system* (A, a_0, a_1)*.*

Lemma 5.3.11 *[64, Lemma 10] Let* $(A, \odot, u)$ *be a differential rig.*

(i) *Let* (A, a_0, a_1) *be a parametrized dynamical system over* X*. Then a map* $f : A \times X \longrightarrow A$ *is a parametrized* $(A, \odot, u)$*-solution of* (A, a_0, a_1) *if and only if* f *is a parametrized* $(A, 0, \overline{u})$*-solution of* (A, a_0, a_1) *such that the following diagram commutes:*

$$\begin{array}{ccc} (A \times X) \times A & \xrightarrow{\langle 1\times 1, \overline{u}\times 0\rangle \times 1} & \big((A \times X) \times (A \times X)\big) \times A \\ & & \downarrow{\scriptstyle \mathsf{D}[f]\times 1} \\ \downarrow{\scriptstyle (1\times 1)\times\langle 1,0\rangle} & & A \times A \\ & & \downarrow{\scriptstyle \odot} \\ (A \times X) \times (A \times X) & \xrightarrow[\mathsf{D}[f]]{} & A \end{array} \tag{5.20}$$

(ii) *If an endomorphism* $a_1 : A \longrightarrow A$ *is* $(A, \odot, u)$*-complete then* a_1 *is* $(A, 0, \overline{u})$*-complete.*

PROOF: Note that (i) is a generalization of Lemma 5.3.4 and it is proved by similar calculations. Now suppose that an endomorphism $a_1 : A \longrightarrow A$ is $(A, \odot, u)$-complete. Then for every map $a_0 : X \longrightarrow A$, there is a parametrized $(A, \odot, u)$-solution of (A, a_0, a_1), which is therefore also a parametrized $(A, 0, \overline{u})$-solution of (A, a_0, a_1). So we conclude that a_0 is also $(A, 0, \overline{u})$-complete. □

Example 5.3.12 In the Cartesian differential category SMOOTH from Example 4.1.8, let $(\mathbb{R}, h, g)$ be a parametrized dynamical system over $\mathbb{R}^k$ with smooth functions $h : \mathbb{R}^k \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. Then a parametrized $(\mathbb{R}, \odot_{e^x}, u_{e^x})$-solution of $(\mathbb{R}, h, g)$ is a smooth function $\mathbb{R} \times f : \mathbb{R}^k \longrightarrow \mathbb{R}$ such that:

$$f(0, y_1, \ldots, y_k) = h(y_1, \ldots, y_k)$$
$$\frac{\partial f(t_0, t_1, \ldots, t_n)}{\partial t_0}(x, y_1, \ldots, y_k)z = g\left(f(x, y_1, \ldots, y_k)\right) z$$

By Lemma 5.3.11, setting $z = 1$, we have that f is also a parametrized $(\mathbb{R}, 0, \overline{u_{e^x}})$-solution of $(\mathbb{R}, h, g)$.

We wish to show that for a differential exponential rig, a certain class of linear endomorphisms are complete, that is, always have a parametrized solution. This corresponds to the fact that in the classical case, e^x allows one to solve all initial value problems $f'(x) = af(x)$ with initial condition $f(0) = b$ for any constants a and b. In this case, the solution to this first order linear differential equation is $f(x) = e^{ax}b$. So the linear function which scalar multiplies by a is complete. This can be generalized to the multivariable case, which is of interest in control systems theory [3, Chapter 5].

Definition 5.3.13 *[64, Definition 11] Let* $(A, \odot, u)$ *be a differential rig. For a point* $a : \top \longrightarrow A$*, define the endomorphism* $\odot^a : A \longrightarrow A$ *as multiplication by* a*, that is:*

$$\odot^a := A \xrightarrow{\langle 1, 0a\rangle} A \times A \xrightarrow{\odot} A$$

As seen in the example below, the map $\odot^a$ is simply multiplication on the right by the point a, which by commutativity of $\odot$ is the same as multiplying on the left. In the case of a differential exponential rig, we will show that $\odot^a$ is complete.

Example 5.3.14 In the Cartesian differential category SMOOTH from Example 4.1.8, for $a \in \mathbb{R}$, $\odot^a_{e^x}(x) = xa$.

Lemma 5.3.15 *[64, Lemma 11] Let $(A, \odot, u)$ be a differential rig.*

(i) $\odot^0 = 0$ *and* $\odot^u = 1$*;*

(ii) *For every pair of points $a : \top \longrightarrow A$ and $b : \top \longrightarrow A$, $\odot^{a+b} = \odot^a + \odot^b$ and $\odot^a \odot^b = \odot^b \odot^a$;*

(iii) *For every point $a : \top \longrightarrow A$, $\odot^a$ is linear and the following diagrams commute:*

$$\begin{array}{ccc} A \times A & \xrightarrow{\ \odot\ } & A \\ {\scriptstyle 1\times\odot^a}\downarrow & & \downarrow{\scriptstyle \odot^a} \\ A \times A & \xrightarrow[\ \odot\ ]{} & A \end{array} \qquad\qquad \begin{array}{ccc} \top & \xrightarrow{\ u\ } & A \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle \odot^a} \\ & & A \end{array} \tag{5.21}$$

(iv) *For an endomorphism $f : A \longrightarrow A$, $\odot^{uf} = f$ if and only if the following diagram,*

$$\begin{array}{ccc} A \times A & \xrightarrow{\ \odot\ } & A \\ {\scriptstyle 1\times f}\downarrow & & \downarrow{\scriptstyle f} \\ A \times A & \xrightarrow[\ \odot\ ]{} & A \end{array} \tag{5.22}$$

(v) *For every point $a : \top \longrightarrow A$, $\odot^{u\odot^a} = \odot^a$.*

Proof: We leave these as an exercise for the reader to check for themselves. □

Proposition 5.3.16 *[64, Proposition 7] Let $(A, \odot, u, e)$ be a differential exponential rig. Then for every point $a : \top \longrightarrow A$, the endomorphism $\odot^a : A \longrightarrow A$ is $(A, \odot, u)$-complete.*

Proof: Let $a : \top \longrightarrow A$ be point and let $a_0 : X \longrightarrow A$ be an arbitrary map. Then $(A, a_0, \odot^a)$ is a parametrized dynamical system over X. Now consider the following composite:

$$A \times X \xrightarrow{\odot^a \times a_0} A \times A \xrightarrow{\mathsf{D}[e]} A$$

which by (5.8) is equal to:

$$(\odot^a \times a_0)\mathsf{D}[e] = (\odot^a \times a_0)(e \times 1)\odot$$

We need to show both equalities of (5.19). Starting with the left identity of (5.19), since by Lemma 5.3.15 $\odot^a$ is linear it is also additive, we have that:

$$\langle 0, 1\rangle(\odot^a \times a_0)\mathsf{D}[e] = \ \langle 0\odot^a, a_0\rangle\mathsf{D}[e]$$

$$
\begin{aligned}
&= \langle 0, a_0\rangle \mathsf{D}[e] && (\odot^a \text{ is additive})\\
&= a_0\langle 0, 1\rangle \mathsf{D}[e] &&\\
&= a_0 && (5.1)
\end{aligned}
$$

And for the right identity of (5.19):

$$
\begin{aligned}
&\big((1\times 1)\times\langle 1,0\rangle\big)\,\mathsf{D}\left[(\odot^a\times a_0)\mathsf{D}[e]\right] &&\\
&= \big((1\times 1)\times\langle 1,0\rangle\big)\,\mathsf{T}(\odot^a\times a_0)\mathsf{D}^2[e] && (\text{Lemma } 5.1.6)\\
&= \big((1\times 1)\times\langle 1,0\rangle\big)\,\mathsf{T}(\odot^a\times a_0)c\mathsf{D}^2[e] && \textbf{[dC.7]}\\
&= \big((1\times 1)\times\langle 1,0\rangle\big)\,c\left(\mathsf{T}(\odot^a)\times\mathsf{T}(a_0)\right)\mathsf{D}^2[e] && (\text{Lemma } 5.1.6)\\
&= \langle \pi_0\times 1, \pi_0\pi_1\rangle\,\big((1\times 1)\times\langle 1,0\rangle\big)\left(\mathsf{T}(\odot^a)\times\mathsf{T}(a_0)\right)\mathsf{D}^2[e] &&\\
&= \langle \pi_0\times 1, \pi_0\pi_1\rangle\left(\mathsf{T}(\odot^a)\times a_0\right)\big((1\times 1)\times\langle 1,0\rangle\big)\,\mathsf{D}^2[e] && (\text{Lemma } 5.1.6)\\
&= \langle \pi_0\times 1, \pi_0\pi_1\rangle\left(\mathsf{T}(\odot^a)\times a_0\right)\big((1\times 1)\times\langle 1,0\rangle\big)(\pi_0\times\pi_1)\mathsf{D}[e] &&\\
&\quad + \langle \pi_0\times 1, \pi_0\pi_1\rangle\left(\mathsf{T}(\odot^a)\times a_0\right)\big((1\times 1)\times\langle 1,0\rangle\big)(\mathsf{D}[e]\times\pi_0)\odot && (5.13)\\
&= \langle \pi_0\times 1, \pi_0\pi_1\rangle\left(\mathsf{T}(\odot^a)\times a_0\right)(\pi_0\times 0)\mathsf{D}[e] &&\\
&\quad + \langle \pi_0\times 1, \pi_0\pi_1\rangle\left(\mathsf{T}(\odot^a)\times a_0\right)(\mathsf{D}[e]\times 1)\odot &&\\
&= 0 + \langle \pi_0\times 1, \pi_0\pi_1\rangle\left(\mathsf{T}(\odot^a)\times a_0\right)(\mathsf{D}[e]\times 1)\odot && \textbf{[dC.2]}\\
&= \langle \pi_0\times 1, \pi_0\pi_1\rangle\left((\odot^a\times\odot^a)\times a_0\right)((e\times 1)\times 1)(\odot\times 1)\odot &&\\
& && (\text{Lemma } 5.3.15 + \text{Lemma } 5.1.6 + (5.8))\\
&= ((\odot^a\times a_0)\times 1)((e\times 1)\times 1)(\odot\times 1)(\odot^a\times 1)\odot && (\text{Lemma } 5.3.15)\\
&= ((\odot^a\times a_0)\times 1)(\mathsf{D}[e]\times 1)(\odot^a\times 1)\odot && (5.8)
\end{aligned}
$$

So we conclude that $(\odot^a\times a_0)\mathsf{D}[e]$ is an $(A,\odot,u)$-solution of $(A, a_0, \odot^a)$ and therefore that $\odot^a$ is $(A,\odot,u)$-complete. □

Example 5.3.17 In the Cartesian differential category SMOOTH from Example 4.1.8, for every point $a\in\mathbb{R}$ and a smooth function $h:\mathbb{R}^m\longrightarrow\mathbb{R}$, the smooth function $f:\mathbb{R}\times\mathbb{R}^m\longrightarrow\mathbb{R}$ defined as $f(x,\vec{y}) = e^{xa}h(\vec{y})$ is a parametrized $(\mathbb{R},\odot_{e^x},u_{e^x})$-solution of the parametrized dynamical system $(\mathbb{R},\odot^a_{e^x},h)$, that is:

$$
\frac{\partial f(t,\vec{u})}{\partial t}(x,\vec{y})z = af(x,\vec{y})z \qquad\qquad f(0,\vec{y}) = h(\vec{y})
$$

Setting $z=1$, it follows that f is also a solution to the differential equation:

$$
\frac{\partial f(t,\vec{u})}{\partial t}(x,\vec{y}) = af(x,\vec{y}) \qquad\qquad f(0,\vec{y}) = h(\vec{y})
$$

We would now like to prove the "converse" of Proposition 5.3.6, that is, we would like to obtain differential exponential maps as solutions to certain dynamical systems. To do so, we will require the extra assumption that solutions are unique, which is necessary to prove that $\oplus_A e = (e\times e)\odot$.

Proposition 5.3.18 *[64, Proposition 8] Let $(A,\odot,u)$ be a differential rig and suppose that:*

(i) *Parametrized $(A, \odot, u)$-solutions are unique if they exists, that is, if both f and g are parametrized $(A, \odot, u)$-solutions of a parametrized dynamical system (A, a_0, a_1), then $f = g$*

(ii) *The dynamical system $(A, u, 1)$ has an $(A, \odot, u)$-solution e.*

Then $(A, \odot, u, e)$ is a differential exponential rig.

Proof: We must show that e satisfies the three identities of (5.8). By definition of e being an $(A, \odot, u)$-solution of $(A, u, 1)$, $0e = u$ and $\mathsf{D}[e] = (e \times 1)\odot$. So it remains to show that $\oplus_A e = (e \times e)\odot$. To do so, we will show that $\oplus_A e$ and $(e \times e)\odot$ are both parametrized $(A, \odot, u)$-solutions of the parametrized dynamical system $(A, e, 1)$. Starting with $\oplus e$:

$$\langle 0, 1\rangle \oplus_A e = e \qquad (5.5)$$

$$\begin{aligned}
\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}[\oplus e] &= \big((1 \times 1) \times \langle 1, 0\rangle\big)\, (\oplus \times \oplus_A)\mathsf{D}[e] && \text{(Lemma 4.1.4 + Lemma 4.1.19)}\\
&= (\oplus_A \times 1)\mathsf{D}[e] && (5.5)\\
&= (\oplus_A \times 1)(e \times 1)\odot && \text{(Assumption (i) + (5.16))}
\end{aligned}$$

Next we work with $(e \times e)\odot$:

$$\begin{aligned}
\langle 0, 1\rangle(e \times e)\odot &= \langle 0e, e\rangle\odot\\
&= \langle u, e\rangle\odot && \text{(Assumption (ii) + (5.16))}\\
&= e && (5.5)
\end{aligned}$$

$$\begin{aligned}
&\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}[(e \times e)\odot] = \big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{T}(e \times e)\mathsf{D}[\odot] && \text{(Lemma 5.1.6)}\\
&= \big((1 \times 1) \times \langle 1, 0\rangle\big)\, c(\mathsf{T}(e) \times \mathsf{T}(e))c\mathsf{D}[\odot] && \text{(Lemma 5.1.6)}\\
&= \langle \pi_0 \times 1, \pi_0\pi_1\rangle\, \big((1 \times 1) \times \langle 1, 0\rangle\big)\, (\mathsf{T}(e) \times \mathsf{T}(e))c\mathsf{D}[\odot]\\
&= \langle \pi_0 \times 1, \pi_0\pi_1\rangle(\mathsf{T}(e) \times e)\, \big((1 \times 1) \times \langle 1, 0\rangle\big)\, c\mathsf{D}[\odot] && \text{(Lemma 5.1.6)}\\
&= \langle \pi_0 \times 1, \pi_0\pi_1\rangle(\mathsf{T}(e) \times e)\langle \pi_0 \times 1, \pi_0\pi_1\rangle\, \big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}[\odot]\\
&= \langle \pi_0 \times 1, \pi_0\pi_1\rangle\langle e \times e, \mathsf{D}[e]\rangle\, \big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}[\odot]\\
&= \langle \pi_0 \times 1, \pi_0\pi_1\rangle\langle e \times e, \mathsf{D}[e]\rangle\, \big((1 \times 1) \times \langle 1, 0\rangle\big)\, (\pi_0 \times \pi_1)\odot\\
&\quad + \langle \pi_0 \times 1, \pi_0\pi_1\rangle\langle e \times e, \mathsf{D}[e]\rangle\, \big((1 \times 1) \times \langle 1, 0\rangle\big)\, (\pi_1 \times \pi_0)\odot && (5.6)\\
&= \langle \pi_0 \times 1, \pi_0\pi_1\rangle\langle e \times e, \mathsf{D}[e]\rangle(\pi_0 \times 0) \odot + \langle \pi_0 \times 1, \pi_0\pi_1\rangle\langle e \times e, \mathsf{D}[e]\rangle(\pi_1 \times 1)\odot\\
&= 0 + \langle \pi_0 \times 1, \pi_0\pi_1\rangle\langle \pi_1 e, \mathsf{D}[e]\rangle\odot && (5.7)\\
&= \big\langle \pi_1 e, \langle \pi_0 \times 1, \pi_0\pi_1\rangle\mathsf{D}[e]\big\rangle \odot\\
&= \big\langle \pi_1 e, \langle \pi_0 \times 1, \pi_0\pi_1\rangle(e \times 1)\odot\big\rangle \odot && \text{(Assumption (ii) + (5.16))}\\
&= ((e \times e) \times 1)(\odot \times 1)\odot && (5.5)
\end{aligned}$$

So we have that $\oplus_A e$ and $(e \times e)\odot$ are both parametrized $(A, \odot, u)$-solutions of the parametrized dynamical system $(A, e, 1)$. However by assumption (ii), solutions are unique and therefore we have that $\oplus_A e = (e \times e)\odot$. And so we conclude that $(A, \odot, u, e)$ is a differential exponential rig. □

5.4 Differential Exponential Maps for Differential Categories

As we saw in Proposition 4.1.12, an interesting and important source of Cartesian differential categories are coKleisli categories of differential categories [8,9]. In this section, we study differential exponential maps in the coKleisli category of a differential category (with Seely isomorphisms). We also introduce !-differential exponential algebras and show that these are in bijective correspondence with differential exponential maps in the coKleisli category of a differential storage category.

Unfortunately we encounter a slight notational problem regarding differential exponential maps and coalgebra modality, since so far, both have been denoted by e. In this section, we use e for differential exponential maps and $\iota : !A \longrightarrow k$ for the counit of the coalgebra modality. Furthermore, unlike the first two chapters, we've elected to not use string diagrams in this chapter as we will need to work the coKleisli category and products.

A differential exponential map in the coKleisli category of a differential category would be a map of type $e : !A \longrightarrow A$ satisfying the required identities, which we can simplify slightly.

Proposition 5.4.1 *[64, Proposition 10] For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, \iota)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$ (Definition 3.1.1), and finite products, a map $e : !A \longrightarrow A$ is a differential exponential map in the coKleisli category $\mathbb{X}_!$ if and only if the following diagrams commute:*

$$\begin{array}{ccccc} !A & \xrightarrow{\Delta} & !A \otimes !A & \xrightarrow{!(0) \otimes \varepsilon} & !A \otimes A \\ & \searrow^{\varepsilon} & & & \downarrow \mathsf{d} \\ & & & & !A \\ & & & & \downarrow e \\ & & & & A \end{array} \qquad \begin{array}{ccccc} !(A \times A) & \xrightarrow{\chi} & !A \otimes !A & \xrightarrow{1 \otimes e} & !A \otimes A \\ \downarrow !(\oplus_A) & & & & \downarrow \mathsf{d} \\ & & & & !A \\ & & & & \downarrow e \\ !A & & \xrightarrow{e} & & A \end{array} \tag{5.23}$$

where $\chi : !(A \times B) \longrightarrow !A \otimes !B$ is defined as in Definition 2.2.7.

Proof: Let $e : !A \longrightarrow A$ be an arbitrary map. Then we leave it to the reader to check for themselves that we have the following three equalities (which are mostly straightforward calculations):

$$[\![\langle 0, 1\rangle \mathsf{D}[e]]\!] = \Delta(!(0) \otimes \varepsilon)\mathsf{d}e \qquad [\![\oplus_A e]\!] = !(\oplus_A)e \qquad [\![(1 \times e)\mathsf{D}[e]]\!] = \chi(1 \otimes e)\mathsf{d}e$$

Therefore, e is a differential exponential map in the coKleisli category if and only if $[\![\langle 0, 1\rangle \mathsf{D}[e]]\!] = [\![1]\!]$ and $[\![\oplus e]\!] = [\![(1 \times e)\mathsf{D}[e]]\!]$, which by the above equalities is precisely that both $\Delta(!(0) \otimes \varepsilon)\mathsf{d}e = \varepsilon$ and $!(\oplus)e = \chi(1 \otimes e)\mathsf{d}e$ hold. □

We now study differential exponential maps in the presence of the Seely isomorphisms. As before, we run into a slight notation problem for the unit. We will instead use $\nu : k \longrightarrow !A$ for the induced unit of the additive bialgebra modality. Explicitly, if $(!, \delta, \varepsilon, \Delta, \iota)$ is a coalgebra modality with Seely isomorphisms (Definition 2.2.7) and deriving transformation d, then $(!, \delta, \varepsilon, \Delta, \iota, \nabla, \nu)$ is the induced additive bialgebra modality (Definition 2.3.7) where note that in the notation for this chapter:

$$\nabla := \ !A \otimes !A \xrightarrow{\chi^{-1}} !(A \times A) \xrightarrow{!(\oplus_A)} !A \qquad \nu := \ k \xrightarrow{\chi_\top^{-1}} !\top \xrightarrow{!(0)} !A \tag{5.24}$$

Also, for the induced codereliction $\eta : A \longrightarrow !A$ (Definition 3.2.1) the note that in the notation for this chapter:

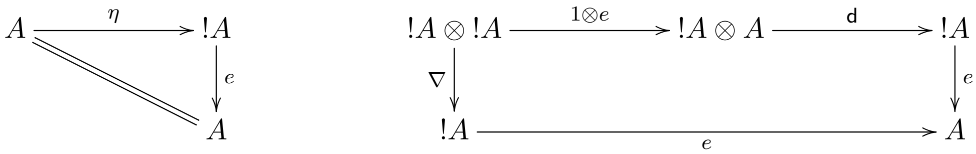
$$\eta := \ A \xrightarrow{\nu\otimes 1} !A \otimes A \xrightarrow{\mathsf{d}} !A \tag{5.25}$$

And so conversely, the deriving transformation is equal to the following composite:

$$\mathsf{d} := \ !A \otimes A \xrightarrow{1\otimes\eta} !A \otimes !A \xrightarrow{\nabla} !A \tag{5.26}$$

We may now review Proposition 5.4.1 in the presence of the Seely isomorphisms.

Proposition 5.4.2 *[64, Proposition 10] For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, \iota)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products such that $(!, \delta, \varepsilon, \Delta, \iota)$ has Seely isomorphisms, a map $e : !A \longrightarrow A$ is a differential exponential map in the coKleisli category $\mathbb{X}_!$ if and only if the following diagrams commute:*



(5.27)

Proof: Suppose that $e : !A \longrightarrow A$ is a differential exponential map in the coKleisli category. Using Proposition 5.4.1 we show that e satisfies (5.27):

$$\begin{aligned}
1 &= \eta\varepsilon && \textbf{[dC.3]}\\
&= \eta\Delta(!(0)\otimes\varepsilon)\mathsf{d}e && (5.23)\\
&= (\eta\otimes\nu)(!(0)\otimes\varepsilon)\mathsf{d}e + (\nu\otimes\eta)(!(0)\otimes\varepsilon)\mathsf{d}e && \textbf{[dC.2]}\\
&= 0 + \eta\varepsilon(\nu\otimes 1)\mathsf{d}e && (\text{Naturality of } \eta \text{ and } \nu)\\
&= (\nu\otimes 1)\mathsf{d}e && \textbf{[dC.3]}\\
&= \eta e && (5.25)
\end{aligned}$$

$$\begin{aligned}
\nabla e &= \chi^{-1}!(\oplus_A)e && (5.24)\\
&= \chi^{-1}\chi(1\otimes e)\mathsf{d}e && (5.23)\\
&= (1\otimes e)\mathsf{d}e
\end{aligned}$$

Conversely, suppose $e : !A \longrightarrow A$ satisfies (5.27). We show that e satisfies (5.23):

$$\begin{aligned}
\Delta(!(0)\otimes\varepsilon)\mathsf{d}e &= \Delta(!(0)\otimes\varepsilon)(1\otimes\eta)\nabla e && (5.26)\\
&= \Delta(\iota\otimes 1)(\nu\otimes 1)(1\otimes\varepsilon)(1\otimes\eta)\nabla e && (2.27)\\
&= \varepsilon\eta e && ((2.4) + (5.28))\\
&= \varepsilon && (5.27)
\end{aligned}$$

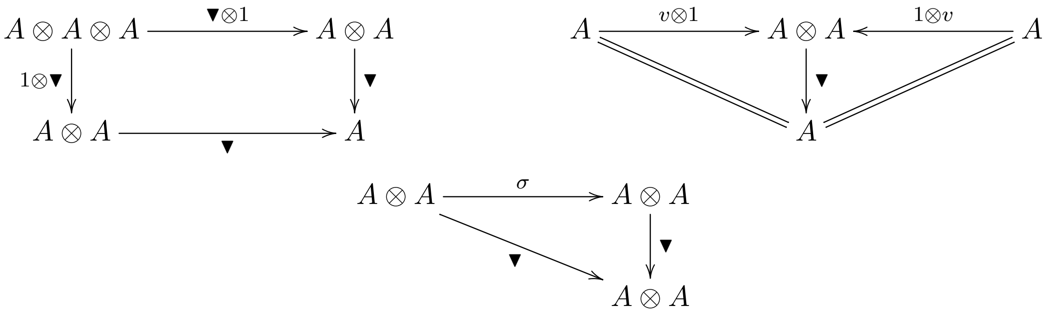
$$\chi(1\otimes e)\mathsf{d}e = \chi\nabla e \qquad (5.27)$$

$$= \chi\chi^{-1}!(\oplus_A)e \qquad (5.24)$$
$$= !(\oplus_A)e$$

Therefore, by Proposition 5.4.1, e is a differential exponential map. □

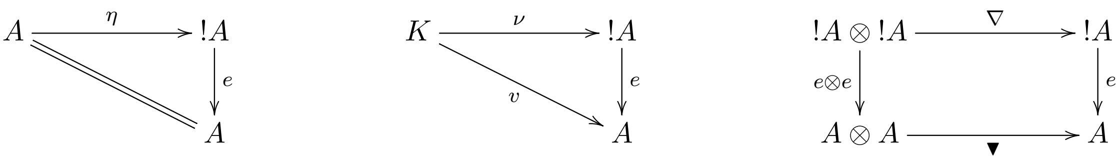
In the presence of the Seely isomorphisms, differential exponential maps in the coKleisli category can also be characterized by commutative monoids in the base category, which the analogue of how differential exponential maps are in bijective correspondence to differential exponential rigs.

Definition 5.4.3 *For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, \iota)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products such that $(!, \delta, \varepsilon, \Delta, \iota)$ has Seely isomorphisms, a* ***!-differential exponential algebra*** *[64, Definion 14] is a quadruple $(A, \blacktriangledown, v, e)$ consisting of an object A and maps $\blacktriangledown : A \otimes A \longrightarrow A$, $v : k \longrightarrow A$, and $e : !A \longrightarrow A$ such that $(A, \blacktriangledown, v)$ is a commutative monoid, that is, the following diagrams commute:*



(5.28)

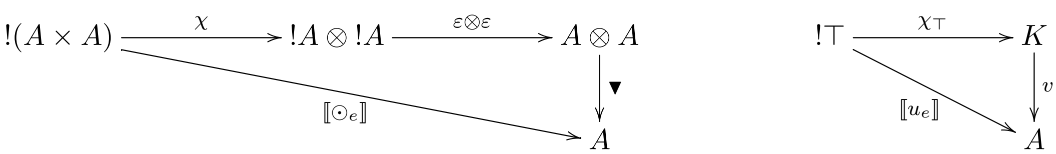
and also that the following diagrams commute:



(5.29)

Note in particular that for a !-differential exponential algebra, the two rightmost diagrams of (5.29) says that e is a monoid morphism. We now show that every !-differential exponential algebra induces a differential exponential map in the coKleisli category and vice-versa.

Proposition 5.4.4 *[64, Proposition 12] Let $(A, \blacktriangledown, v, e)$ be a !-differential exponential algebra. Then $e : !A \longrightarrow A$ is a differential exponential map in the coKleisli category and furthermore the following diagrams commute:*



where $\odot_e$ and u_e are defined as in Proposition 5.2.5.

PROOF: By Proposition 5.4.2, it suffices to show that e satisfies both diagrams of (5.27). However the left diagram of (5.27) is precisely the left most diagram of (5.29). So it remains to show that $\nabla e = (1 \otimes e)\mathsf{d}e$:

$$\begin{aligned}
(1 \otimes e)\mathsf{d}e &= (1 \otimes e)(1 \otimes \eta)\nabla e && (5.26)\\
&= (1 \otimes e)(1 \otimes \eta)(e \otimes e)\blacktriangledown && (5.29)\\
&= (e \otimes e)\blacktriangledown && (5.29)\\
&= \nabla e && (5.29)
\end{aligned}$$

So we conclude that e is a differential exponential map in the coKleisli category. Next we show that $[\![u_e]\!] = \chi_\top v$:

$$\begin{aligned}
[\![u_e]\!] &= [\![0e]\!]\\
&= \delta !(0)e\\
&= \delta\iota\nu e && (2.27)\\
&= \iota\nu e && \text{(Lemma 2.1.6)}\\
&= \iota v && (5.29)\\
&= \chi_\top v
\end{aligned}$$

To show the other equality, we observe that in the proof of [9, Proposition 3.2.1], it was computed out that we have the following equality for any $[\![f]\!] : !A \longrightarrow B$:

$$[\![\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[f]]\!] = \chi(\chi \otimes 1)(1 \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}f \qquad (5.30)$$

Using the above identity, we can show that:

$$\begin{aligned}
[\![\odot_e]\!] &= [\![\big(\langle 0, 1\rangle \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!]\\
&= [\![(\langle 0, 1\rangle \times 1)\, \big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!]\\
&= \delta!([\![\langle 0, 1\rangle \times 1]\!])[\![\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!]\\
&= \delta!(\langle !(\pi_0)[\![\langle 0, 1\rangle]\!], !(\pi_1)[\![1]\!]\rangle)[\![\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!]\\
&= \delta!(\langle !(\pi_0)\langle 0, \varepsilon\rangle, !(\pi_1)\varepsilon\rangle)[\![\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!]\\
&= \delta!(\langle \varepsilon\langle 0, \pi_0\rangle, \varepsilon\pi_1\rangle)[\![\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!] && \text{(Naturality of } \varepsilon)\\
&= \delta!(\varepsilon)!(\langle\langle 0, \pi_0\rangle, \varepsilon\pi_1\rangle)[\![\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!]\\
&= !(\langle\langle 0, \pi_0\rangle, \pi_1\rangle)[\![\big((1 \times 1) \times \langle 1, 0\rangle\big)\, \mathsf{D}^2[e]]\!] && (2.1)\\
&= !(\langle\langle 0, \pi_0\rangle, \pi_1\rangle)\chi(\chi \otimes 1)(1 \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}e && (5.30)\\
&= \Delta\, \big(!(\langle 0, \pi_0\rangle) \otimes !(\pi_1)\big)\, (\chi \otimes 1)(1 \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}e && \text{(Definition of } \chi)\\
&= \Delta(1 \otimes !(\pi_1))(\Delta \otimes 1)(!(0) \otimes !(\pi_0) \otimes 1)(1 \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}e && \text{(Definition of } \chi)\\
&= \Delta(\Delta \otimes 1)(!(0) \otimes !(\pi_0) \otimes !(\pi_1))(1 \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}e\\
&= \Delta(1 \otimes \Delta)(!(0) \otimes !(\pi_0) \otimes !(\pi_1))(1 \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}e && (2.4)\\
&= \Delta(1 \otimes \chi)(!(0) \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}e\\
&= \Delta(1 \otimes \chi)(\iota \otimes 1 \otimes 1)(\nu \otimes \varepsilon \otimes \varepsilon)(\mathsf{d} \otimes 1)\mathsf{d}e && (2.27)
\end{aligned}$$

$$
\begin{aligned}
&= \chi(\varepsilon \otimes \varepsilon)(\eta \otimes 1)\mathsf{d}e && ((2.4) + (5.25))\\
&= \chi(\varepsilon \otimes \varepsilon)(\eta \otimes 1)(1 \otimes \eta)\nabla e && (5.26)\\
&= \chi(\varepsilon \otimes \varepsilon)(\eta \otimes 1)(1 \otimes \eta)(e \otimes e)\blacktriangledown && (5.29)\\
&= \chi(\varepsilon \otimes \varepsilon)\blacktriangledown && (5.29)
\end{aligned}
$$

And so we have that $[\![\odot_e]\!] = \chi(\varepsilon \otimes \varepsilon)\blacktriangledown$. □

Proposition 5.4.5 *[64, Proposition 13] Let $e : !A \longrightarrow A$ be a differential exponential map in the coKleisli category of a differential storage category. Define the maps $\blacktriangledown_e : A \otimes A \longrightarrow A$ and $v_e : k \longrightarrow A$ respectively as follows:*

$$\blacktriangledown_e := A \otimes A \xrightarrow{\eta \otimes \eta} !A \otimes !A \xrightarrow{\nabla} !A \xrightarrow{e} A$$

$$v_e := k \xrightarrow{\nu} !A \xrightarrow{e} A$$

Then $(A, \blacktriangledown_e, v_e, e)$ is a !-differential exponential algebra.

Proof: We first show that $(A, \blacktriangledown_e, v_e)$ is a commutative monoid. Starting with showing that $\blacktriangledown_e$ is commutative:

$$
\begin{aligned}
\sigma\blacktriangledown_e &= \sigma(\eta \otimes \eta)\nabla e\\
&= (\eta \otimes \eta)\sigma\nabla e && \text{(Naturality of } \sigma)\\
&= (\eta \otimes \eta)\nabla e && (5.28)
\end{aligned}
$$

Since we've shown commutativity, we need only show one of the unit identities:

$$
\begin{aligned}
(1 \otimes v_e)\blacktriangledown_e &= (1 \otimes \nu)(1 \otimes e)(\eta \otimes \eta)\nabla e\\
&= (1 \otimes \nu)(\eta \otimes 1)(1 \otimes e)(1 \otimes \eta)\nabla e\\
&= (1 \otimes \nu)(\eta \otimes 1)(1 \otimes e)\mathsf{d}e && (5.26)\\
&= (1 \otimes \nu)(\eta \otimes 1)\nabla e && (5.27)\\
&= \eta e && (5.28)\\
&= 1 && (5.27)
\end{aligned}
$$

Lastly, we show that $\blacktriangledown_e$ is also associative:

$$
\begin{aligned}
(1 \otimes \blacktriangledown_e)\blacktriangledown_e &= (1 \otimes \eta \otimes \eta)(1 \otimes \nabla)(1 \otimes e)(\eta \otimes \eta)\nabla e\\
&= (1 \otimes \eta \otimes \eta)(1 \otimes \nabla)(\eta \otimes 1)(1 \otimes e)(1 \otimes \eta)\nabla e\\
&= (1 \otimes \eta \otimes \eta)(1 \otimes \nabla)(\eta \otimes 1)(1 \otimes e)\mathsf{d}e && (5.26)\\
&= (1 \otimes \eta \otimes \eta)(1 \otimes \nabla)(\eta \otimes 1)\nabla e && (5.27)\\
&= (\eta \otimes \eta \otimes \eta)(1 \otimes \nabla)\nabla e\\
&= (\eta \otimes \eta \otimes \eta)(\nabla \otimes 1)\nabla e && (5.28)\\
&= (\eta \otimes \eta \otimes \eta)(\nabla \otimes 1)\sigma\nabla e && (5.28)
\end{aligned}
$$

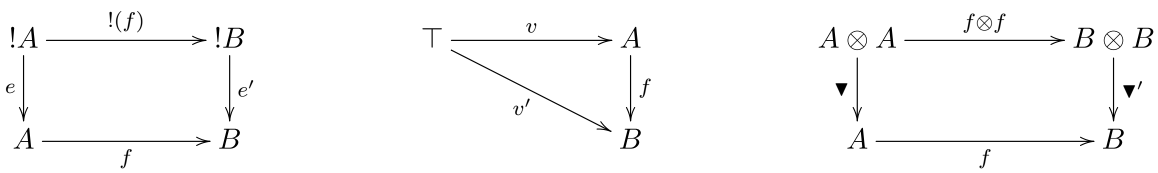
$$\begin{aligned}
&= (\eta \otimes \eta \otimes \eta)(\nabla \otimes 1)\sigma(1 \otimes e)\mathsf{d}e && (5.27)\\
&= (\eta \otimes \eta \otimes \eta)(\nabla \otimes 1)\sigma(1 \otimes e)(1 \otimes \eta)\nabla e && (5.26)\\
&= (\eta \otimes \eta \otimes \eta)(\nabla \otimes 1)(e \otimes 1)(\eta \otimes 1)\sigma\nabla e && (\text{Naturality of } \sigma)\\
&= (\eta \otimes \eta \otimes \eta)(\nabla \otimes 1)(e \otimes 1)(\eta \otimes 1)\nabla e && (5.28)\\
&= (\eta \otimes \eta \otimes 1)(\nabla \otimes 1)(e \otimes 1)(\eta \otimes \eta)\nabla e &&\\
&= (\blacktriangledown_e \otimes 1)\blacktriangledown_e &&
\end{aligned}$$

So we conclude that $(A, \blacktriangledown_e, v_e)$ is a commutative monoid. Next we show that e satisfies the three identities of (5.29). The left most diagram of (5.29) is precisely the left diagram of (5.27) and $v_e = \nu e$ by construction. So it remains only to show that $\nabla e = (e \otimes e)\blacktriangledown_e$:

$$\begin{aligned}
(e \otimes e)\blacktriangledown_e &= (e \otimes e)(\eta \otimes \eta)\nabla e &&\\
&= (e \otimes 1)(\eta \otimes 1)(1 \otimes e)(1 \otimes \eta)\nabla e &&\\
&= (e \otimes 1)(\eta \otimes 1)(1 \otimes e)\mathsf{d}e && (5.26)\\
&= (e \otimes 1)(\eta \otimes 1)\nabla e && (5.27)\\
&= (e \otimes 1)(\eta \otimes 1)\sigma\nabla e && (5.28)\\
&= \sigma(1 \otimes e)(1 \otimes \eta)\nabla e && (\text{Naturality of } \sigma)\\
&= \sigma(1 \otimes e)\mathsf{d}e && (5.26)\\
&= \sigma\nabla e && (5.27)\\
&= \nabla e && (5.28)
\end{aligned}$$

So we conclude that $(A, \blacktriangledown_e, v_e, e)$ is a !-differential exponential algebra. □

Finally we will show that the constructions of Proposition 5.4.4 and Proposition 5.4.5 are inverses of each other by showing that the category of !-differential exponential algebras is isomorphic to the category of differential exponential maps in the coKleisli category. For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, \iota)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products such that $(!, \delta, \varepsilon, \Delta, \iota)$ has Seely isomorphisms, define its category of !-differential exponential algebras $!\mathsf{DEA}[\mathbb{X}]$ as the category whose objects are !-differential exponential algebras $(A, \blacktriangledown, v, e)$ and where a map $f : (A, \blacktriangledown, v, e) \longrightarrow (B, \blacktriangledown', v', e')$ is a map $f : A \longrightarrow B$ such that the following diagrams commute:



(5.31)

and where composition and identity maps are as in $\mathbb{X}$. Note that the two right most diagrams above imply that f is a monoid morphism.

Theorem 5.4.6 *[64, Theorem 2] For a differential category $\mathbb{X}$ with coalgebra modality $(!, \delta, \varepsilon, \Delta, \iota)$ and deriving transformation $\mathsf{d} : !A \otimes A \longrightarrow !A$, and finite products such that $(!, \delta, \varepsilon, \Delta, \iota)$ has Seely* *isomorphisms, its category of* !*-differential exponential algebras* $\mathsf{!DEA}[\mathbb{X}]$ *is isomorphic to the category of differential exponential maps of the coKleisli category* $\mathsf{DEM}[\mathbb{X}_!]$ *via the inverse functors* $\mathsf{F} : \mathsf{!DEM}[\mathbb{X}] \longrightarrow \mathsf{DEM}[\mathbb{X}_!]$ *and* $\mathsf{F}^{-1} : \mathsf{DEM}[\mathbb{X}_!] \longrightarrow \mathsf{!DEA}[\mathbb{X}]$ *defined respectively as*

$$\mathsf{F}(A, \blacktriangledown, v, e) = (A, e) \qquad [\![\mathsf{F}(f)]\!] = \varepsilon f$$
$$\mathsf{F}^{-1}(A, e) = (A, \blacktriangledown_e, v_e, e) \qquad \mathsf{F}^{-1}([\![g]\!]) = \eta[\![g]\!]$$

Therefore, for differential categories with Seely isomorphisms, there is a bijective correspondence between differential exponential maps in the coKleisli category and !*-differential exponential algebras.*

PROOF: We first need to check that F and F^{-1} are well-defined. By Proposition 5.4.5, F is well-defined on objects and so it remains to check that it is also well-defined on maps. First note that by [9, Proposition 4.2.5], a map in the coKleisli category $\mathbb{X}_!$ is linear if and only if it is of the form $[\![g]\!] = \varepsilon g'$, and so every map in $\mathsf{DEM}[\mathbb{X}_!]$ is of this form. By definition $[\![\mathsf{F}(f)]\!]$ is linear and so it remains to show that $[\![e\mathsf{F}(f)]\!] = [\![\mathsf{F}(f)e']\!]$:

$$\begin{aligned}
[\![e\mathsf{F}(f)]\!] &= \delta !(e)[\![\mathsf{F}(f)]\!] \\
&= \delta !(e)\varepsilon f \\
&= \delta \varepsilon e f && \text{(Naturality of } \varepsilon\text{)} \\
&= ef && (2.1) \\
&= !(f)e' && (5.31) \\
&= \delta !(\varepsilon) !(f) e' && (2.1) \\
&= \delta !([\![\mathsf{F}(f)]\!])e' \\
&= [\![\mathsf{F}(f)e']\!]
\end{aligned}$$

So $\mathsf{F}(f)$ is a map in $\mathsf{DEM}[\mathbb{X}_!]$ and therefore F is well-defined. On the other hand, by Proposition 5.4.4, F^{-1} is well-defined on objects and so it again remains to check that it is also well-defined on maps. Note that since every map in $\mathsf{DEM}[\mathbb{X}_!]$ is of the form $[\![g]\!] = \varepsilon g'$, it follows from **[dC.3]** that we have that $\mathsf{F}^{-1}([\![g]\!]) = \eta \varepsilon g' = g'$. Since $[\![g]\!]$ is a map in $\mathsf{DEM}[\mathbb{X}_!]$, by Theorem 5.2.9, we also have that the following equalities hold:

$$[\![eg]\!] = [\![ge']\!] \qquad [\![\odot_e g]\!] = [\![(g \times g)\odot_{e'}]\!] \qquad [\![u_e g]\!] = [\![u_{e'}]\!]$$

Now since $[\![g]\!] = \varepsilon g'$, the above identities can easily be simplified out to be:

$$eg' = !(g')e' \qquad [\![\odot_e]\!]g' = !(g' \times g')[\![\odot_{e'}]\!] \qquad [\![u_e]\!]g' = [\![u_{e'}]\!] \tag{5.32}$$

Using these identities, we now show that $\mathsf{F}^{-1}([\![g]\!])$ satisfies (5.31):

$$\begin{aligned}
e\mathsf{F}^{-1}([\![g]\!]) &= eg' \\
&= !(g')e' && (5.32) \\
&= !(\mathsf{F}^{-1}([\![g]\!]))e'
\end{aligned}$$

$$\begin{aligned}
\blacktriangledown \mathsf{F}^{-1}(\llbracket g \rrbracket) &= (\eta \otimes \eta)(\varepsilon \otimes \varepsilon)\blacktriangledown \mathsf{F}^{-1}(\llbracket g \rrbracket) && \textbf{[dC.3]} \\
&= (\eta \otimes \eta)\chi^{-1}\chi(\varepsilon \otimes \varepsilon)\blacktriangledown \mathsf{F}^{-1}(\llbracket g \rrbracket) \\
&= (\eta \otimes \eta)\chi^{-1}\llbracket \odot_e \rrbracket \mathsf{F}^{-1}(\llbracket g \rrbracket) && \text{(Proposition 5.4.4)} \\
&= (\eta \otimes \eta)\chi^{-1}\llbracket \odot_e \rrbracket g' \\
&= (\eta \otimes \eta)\chi^{-1}!(g' \times g')\llbracket \odot_{e'} \rrbracket && (5.32) \\
&= (g' \otimes g')(\eta \otimes \eta)\chi^{-1}\llbracket \odot_{e'} \rrbracket && \text{(Naturality of } \chi \text{ and } \eta) \\
&= (g' \otimes g')(\eta \otimes \eta)\chi^{-1}\chi(\varepsilon \otimes \varepsilon)\blacktriangledown' && \text{(Proposition 5.4.4)} \\
&= (g' \otimes g')(\eta \otimes \eta)(\varepsilon \otimes \varepsilon)\blacktriangledown' \\
&= (g' \otimes g')\blacktriangledown' && \textbf{[dC.3]} \\
&= \left(\mathsf{F}^{-1}(\llbracket g \rrbracket) \otimes \mathsf{F}^{-1}(\llbracket g \rrbracket)\right)\blacktriangledown'
\end{aligned}$$

$$\begin{aligned}
\upsilon \mathsf{F}^{-1}(\llbracket g \rrbracket) &= \upsilon g' \\
&= \chi_\top^{-1}\chi_\top \upsilon g' \\
&= \chi_\top^{-1}\llbracket u_e \rrbracket g' && \text{(Proposition 5.4.4)} \\
&= \chi_\top^{-1}\llbracket u_{e'} \rrbracket && (5.32) \\
&= \upsilon' && \text{(Proposition 5.4.4)}
\end{aligned}$$

So $\mathsf{F}^{-1}(\llbracket g \rrbracket)$ is a map in $!\mathsf{DEA}[\mathbb{X}]$ and therefore F^{-1} is well-defined. We leave it to the reader to check for themselves that F and F^{-1} preserves both identities and composition, and are thus indeed functors. Lastly, we need to show that F and F^{-1} are inverses of each other. Clearly $\mathsf{F}\mathsf{F}^{-1}(A, e) = (A, e)$ and $\mathsf{F}\mathsf{F}^{-1}(\llbracket g \rrbracket) = \llbracket g \rrbracket$. In the other direction, we clearly have that $\mathsf{F}^{-1}\mathsf{F}(f) = f$ and so it remains to show that $(A, \blacktriangledown, \upsilon, e) = \mathsf{F}\mathsf{F}^{-1}(A, \blacktriangledown, \upsilon, e) = (A, \blacktriangledown_e, \upsilon_e, e)$, that is, we need to show that $\blacktriangledown = \blacktriangledown_e$ and $\upsilon = \upsilon_e$ – both of which follow immediately from (5.29):

$$\begin{aligned}
\blacktriangledown_e &= (\eta \otimes \eta)\nabla e \\
&= (\eta \otimes \eta)(e \otimes e)\blacktriangledown && (5.29) \\
&= \blacktriangledown && (5.29)
\end{aligned}$$

$$\begin{aligned}
\upsilon_e &= \nu e \\
&= \upsilon && (5.29)
\end{aligned}$$

So $(A, \blacktriangledown, \upsilon, e) = (A, \blacktriangledown_e, \upsilon_e, e)$. Therefore, we conclude that F and F^{-1} are inverse functors and that $!\mathsf{DEA}[\mathbb{X}]$ is isomorphic to $\mathsf{DEM}[\mathbb{X}_!]$. □

We conclude this section with some examples of !-differential exponential algebras.

Example 5.4.7 For the differential category REL with the finite multiset coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ from Example 3.1.2, recall that ! is the free exponential modality [73] on REL, that is, $!X$ is the cofree cocommutative cocomdiagrams over X in REL. Therefore !-coalgebras are precisely cocommutative comonoids in REL. By self-duality of REL, ! is also a monad such that $!X$

is the free commutative monoid over X in REL and !-algebras are precisely commutative monoids in REL. In fact, the codereliction η is the unit of the monad structure of !. It turns out that the !-differential exponential algebras are precisely the commutative monoids in REL (or equivalently the !-algebras). Indeed, every !-differential exponential algebra $(A, \blacktriangledown, v, e)$ is by definition a commutative monoid $(A, \blacktriangledown, v)$ and its associated !-algebra structure is precisely $e \subseteq !A \times A$. Conversely, given a commutative monoid $(A, \blacktriangledown, v)$, its associated !-algebra structure $e \subseteq !A \times A$ satisfies by definition that $\eta e = 1$ and is also a monoid morphism, and therefore $(A, \blacktriangledown, v, e)$ is a !-differential exponential algebra. In particular, since for every X, $(!X, \nabla, \nu)$ is a commutative monoid, there is a natural transformation $\mu \subseteq !!X \times !X$ such that $(!X, \nabla, \nu, \mu)$ is a !-differential exponential algebra. Explicitly, $\mu : !!X \longrightarrow !X$ is the dual relation of $\delta : !X \longrightarrow !!X$ as defined in Example 2.1.7, that is, μ is defined as follows:

$$\mu = \{(F, f)|\ f \in !X, F \in !!X \text{ s.t. } \sum_{g \in \mathsf{supp}(F)} g = f\} \subseteq !!X \times !X$$

Therefore, $\mu : !!X \longrightarrow !X$ is a differential exponential map in the coKleisli category $\mathsf{REL}_!$. Also, it turns out that every X comes equipped with a monoid structure in REL given by the dual of the copying relation, and so the induced !-algebra structure $e : !X \times X$ is given by:

$$e = \{(f, x)|\ x \in X, f \in !X \text{ s.t. } f(y) = 0 \text{ for all } y \neq x\} \subseteq !X \times X$$

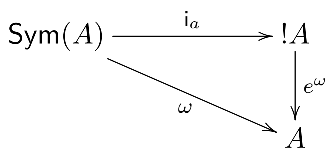
$$(f, x) \in e \Leftrightarrow f(y) = 0 \text{ for all } y \neq x$$

So for every X, $e : !X \times X$ is a differential exponential map in the coKleisli category $\mathsf{REL}_!$. For more examples, monoids in REL are studied in [52].

Example 5.4.8 Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. For the differential category $\mathsf{VEC}_{\mathbb{K}}$ with the coalgebra modality Q from Example 3.1.8, recall that:

$$\mathsf{Q}(V) = \bigoplus_{v \in V} \mathsf{Sym}(V)$$

and that in this case Q is the free exponential modality [73] on $\mathsf{VEC}_{\mathbb{K}}$, that is, $\mathbb{Q}(V)$ is the cofree cocommutative $\mathbb{K}$-coalgebra over V. In particular, Q-coalgebras are precisely cocommutative $\mathbb{K}$-coalgebras. It turns out that, once again, $\mathbb{Q}$-differential exponential algebras correspond precisely to commutative monoids in $\mathsf{VEC}_{\mathbb{K}}$ which are precisely the commutative $\mathbb{K}$-algebras or equivalently the Sym-algebras. By definition, every Q-differential exponential algebra $(A, \blacktriangledown, v, e)$ is a commutative $\mathbb{K}$-algebra and it turns out that its Sym-algebra structure is given by pre-composing $e : \mathsf{Q}(A) \longrightarrow A$ with the $0 \in V$ injection map $i_0 : \mathsf{Sym}(A) \longrightarrow !A$. Conversely, given a commutative $\mathbb{K}$-algebra $(A, \blacktriangledown, v)$, let $\omega : \mathsf{Sym}(A) \longrightarrow A$ be its induced Sym-algebra structure, and define $e^\omega : \mathsf{Q}(A) \longrightarrow A$ as the unique map which makes the following diagram commute for all injection maps $i_A : \mathsf{Sym}(A) \longrightarrow !A$, $a \in A$:



It follows that $(A, \blacktriangledown, \upsilon, e^{\omega})$ is a !-differential exponential algebra. In particular for every V, $(!V, \nabla, \nu)$ is a commutative $\mathbb{K}$-algebra. As such, there is a natural transformation $\mu : \mathsf{QQ}(V) \to \mathsf{Q}(V)$ such that $(\mathsf{Q}(V), \nabla, \nu, \mu)$ is a Q-differential exponential algebra, and thus μ is also a differential exponential map in the coKleisli category.

5.5 Future Work

As the exponential function e^x (and its generalizations) is so prominent and important throughout various fields and has numerous applications, the work in this chapter opens the door to numerous possibilities and applications for differential exponential maps. In particular, as the theory of differential equations in Cartesian differential categories develops, differential exponential maps should be a key component for this theory in the same way that the exponential function is a fundamental tool in solving classical differential equations. For such applications of differential exponential maps, see [24].

It is also of importance and of interest to find and study more examples of differential exponential maps in a variety of Cartesian differential categories. For example, one should consider studying differential exponential maps in cofree Cartesian differential categories [31, 59] and abelian functor calculus [5], as well as study !-differential exponential algebras in the differential categories of convenient vector spaces [11] and finiteness spaces [36]. Another possible source of examples is to construct differential exponential maps in the presence of infinite sums, which many categorical models of the differential λ-calculus [37, 69] have.

There are also certain interesting potential generalizations of differential exponential maps to consider. For example, the exponential function e^x can also be defined as the inverse of the natural logarithm function $ln(x)$. However the natural logarithm function is only a partial function of type $\mathbb{R} \to \mathbb{R}$, since $ln(x)$ is not defined at $x = 0$. As such, one must instead work in a differential restriction category [22], which allows one to consider partial functions and domains of definition. One could then generalize the natural logarithm function in a differential restriction category in such a way that differential exponential maps arise as their restriction inverse. On the other hand, one could also generalize differential exponential maps to tangent categories [19] and differential bundles [21], such that this notion should be a generalization of exponential maps for manifolds and Lie groups. That differential exponential maps can be axiomatized using only the basic structure of a Cartesian differential category (that is, without referencing extra requirements such as differential rig structure), will be particularly useful when generalizing exponential functions to tangent categories.

Regarding !-differential exponential algebras, it is interesting to point out that in both examples of differential categories studied in this chapter, there was a natural transformation $\mu : !!A \to !A$ which endows $!A$ with a !-differential exponential algebra structure. As such, it would be interesting to study differential storage categories with such a μ and understand what are the consequences from a differential linear logic [36] point of view. A natural question to ask is when does the codereliction $\eta : A \to !A$ and μ provide a monad structure on ! (with one of the monad identities already being a requirement for a !-differential exponential algebra), and conversely when does a monad structure on ! induce a natural !-differential exponential algebra structure.

Lastly, another possible direction would be to generalize the trigonometric functions and the hyperbolic functions in the same way for arbitrary Cartesian differential categories. Indeed, generalizations of the (hyperbolic) sine and cosine functions would be a pair of endomorphisms whose axioms are based on the fact that $\mathsf{D}[\sin(x)](x, y) = \cos(x)y$ and $\mathsf{D}[\cos(x)](x, y) = -\sin(x)y$ (resp. $\mathsf{D}[\sinh(x)](x, y) = \cosh(x)y$ and $\mathsf{D}[\cosh(x)](x, y) = \sinh(x)y$), as well as other algebraic properties of sin and cos (resp. sinh and cosh). It should also be expected that these generalized trigonometric or hyperbolic functions would also be in bijective correspondence with special sorts of differential rigs. Furthermore, since the (split) complex exponential function is constructed using e^x, $\cos(x)$ and $\sin(x)$ (resp. e^x, $\cosh(x)$, and $\sinh(x)$), combining a differential exponential map with these generalized trigonometric (or hyperbolic) functions should again result in a differential exponential map in a similar fashion. In fact, it would be desirable to find an overarching general notion that would encompass all of these varying concepts.

In conclusion, there are many potential interesting paths to take for future work.

Conclusion

We conclude this thesis with a very brief summary. The results in this thesis are all important results to the theory of differential categories. The main results of this thesis can be summarized as follows for each chapter:

(i) **Chapter 2**: additive bialgebra modalities are equivalent to monoidal coalgebra modalities (Theorem 2.3.25), and, in the presence of finite biproducts, to coalgebra modalities with Seely isomorphisms (Theorem 2.3.27).

(ii) **Chapter 3**: for an additive bialgebra modality (or equivalently a monoidal coalgebra modality), there is a bijective correspondence between coderelictions and deriving transformations (Theorem 3.3.11). Therefore, there is only one notion of differentiation in linear logic.

(iii) **Chapter 4**: for a Cartesian left additive category, there is a bijective correspondence between differential combinators and systems of linearizing combinators (Theorem 4.4.14). Therefore, a Cartesian differential category can equivalently be described as a Cartesian left additive category with a system of linearizing combinators.

(iv) **Chapter 5**: generalizing the notion of exponential functions for Cartesian differential categories, and showing that there is a bijective correspondence between differential exponential maps and differential exponential rigs (Theorem 5.2.9).

The theory of differential categories is an active field of research and continues to gain in popularity in a variety of research areas in both mathematics and computer science. There are many interesting on going and future possible research projects on differential categories. Hopefully in the near future, differential categories will reach their prime, become a fixture in the scientific community and find numerous applications in science and beyond.

“ *There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.* ”

-Nikolai Ivanovich Lobachevsky as quoted in [44]

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