ESSAYS ON OPTIMAL INVESTMENT IN MATHEMATICAL FINANCE

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In memory of my father

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Abstract

This thesis comprises four essays on optimal investment in mathematical finance. The first two are concerned with optimal stock buying/selling w.r.t. its global maximum (minimum). We first aim to determine an optimal selling time so as to minimize the expectation of the square error between the selling price and the global maximum. Then, we formulate four stock buying/selling problems by minimizing/maximizing the expectation of the ratio of the buying/selling price to the global maximum (minimum) price. These are optimal stopping problems that can be formulated as variational inequality problems. We solve them by a partial differential equation approach. The latter two essays are related to optimal investment with behavioral preferences. In Essay III, we consider optimal asset liquidation for an investor with an S-shaped utility. We characterize the value function in the sense of viscosity solution due to the nonsmoothness of the payoff function and show that the optimal liquidation strategies are consistent with the disposition effect. In Essay IV, we consider a mean-semi-variance portfolio selection problem with probability distortion. Using a dual argument, we change the decision variable to a quantile function. We then apply the Lagrange method to solve the problem. It turns out that the distorted mean-semi-variance problem only admits an optimal solution with some proper distortion functions.

Contents

Chapter 1

Introduction

This chapter summarizes the work of this thesis. Firstly, I give overviews of four essays on optimal investment involved in the thesis. Essays I and II are related to stock selling/buying with respect to its global maximum (minimum) of the given period. These problems are called optimal prediction problems in probability literature. Essay III and IV are related to optimal investment with behavioral preferences. More precisely, Essay III considers optimal asset liquidation with an S-shaped utility, while Essay IV considers a mean-semi-variance portfolio selection problem with probability distortions.

1.1 Overviews

1.1.1 Stock selling/buying w.r.t. its global maximum/minimum

Assume that a discounted stock price, S_t, evolves according to

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where constants $\mu \in (-\infty, +\infty)$ and $\sigma > 0$ are the excess rate of return and the volatility rate, respectively, and $\{B_t; t > 0\}$ is a standard 1-dimensional Brownian motion on a filtered probability space $(\mathbb{S}, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Define by M_T and m_T respectively the global maximum and minimum of the stock price on $[0, T]$, namely,

$$\begin{cases} M_T = \max_{0\leq\nu\leq T} S_\nu, \\ m_T = \min_{0\leq\nu\leq T} S_\nu. \end{cases} \tag{1.1.1}$$

Consider an individual investor trading stock. It would be ideal if s/he could sell (buy) the stock exactly at the global maximum (minimum) of this given period. However, this mission is impossible since the global maximum (minimum) is not known yet when s/he decides to sell (buy) before the terminal date T. It is therefore crucial for the investor to predict the global maximum (minimum) price and choose the best selling (buying) time such that the transaction price is as *close* as possible to the ideal price, where the investor is assumed to be able to continuously monitor the stock price from time 0.

Essay I: Selling a Stock Close to Its Global Maximum

In Essay I, we are interested in

$$\inf_{\nu\in\mathcal{T}} \mathbb{E}[(S_\nu - M_T)^2], \tag{1.1.2}$$

where $\mathbb{E}$ stands for the expectation, $\mathcal{T}$ is the set of all $\mathscr{F}_t$-stopping time $\nu \in [0, T]$.

The earliest studies of the problem called optimal prediction problem in probabilistic literature can be traced back Graversen, Peskir and Shiryaev (2001) who measure the *closeness* by a square error distance between a standard Brownian motion and its global maximum. Indeed, Graversen, Peskir and Shiryaev (2001) consider

$$\inf_{\nu\in\mathcal{T}} \mathbb{E}[(B_\nu - \max_{0\leq s\leq 1} B_s)^2]. \tag{1.1.3}$$

By the time-change arguments, they reduce (1.1.3) to a time independent problem and show that the infimum is attained at the stopping time

$$\tau^* = \inf\left\{0 \leq t \leq 1 | \max_{0\leq s\leq t} B_s - B_t \geq C\sqrt{1-t}\right\},$$

for some constant C determined by an ODE problem. Their work is extended by Du Toit and Peskir (2007) to a drifted Brownian motion. Recently, Shiryaev, Xu and Zhou (2008b) further apply this idea to model optimal stock selling behavior, where they take into consideration the measure of relative error and use the geometric Brownian motion (GBM) assumption of stock price. A closely related topic will be addressed in Essay II.

Essay II: Buy Low and Sell High

In this Essay, we are interested in the following optimal decisions to buy or sell the stock over a given investment horizon $[0, T]$:

$$\text{Buying:} \qquad \min_{\nu\in\mathcal{T}} \mathbb{E}\left(\frac{S_\nu}{M_T}\right), \tag{1.1.4}$$

$$\min_{\nu\in\mathcal{T}} \mathbb{E}\left(\frac{S_\nu}{m_T}\right); \tag{1.1.5}$$

$$\text{Selling:} \qquad \max_{\nu\in\mathcal{T}} \mathbb{E}\left(\frac{S_\nu}{M_T}\right), \tag{1.1.6}$$

$$\max_{\nu\in\mathcal{T}} \mathbb{E}\left(\frac{S_\nu}{m_T}\right). \tag{1.1.7}$$

Some discussions on the motivations of the above problems are in order. When an investor trades a stock she naturally hopes to "buy low and sell high (BLSH)". Since one could never be able to "buy at the lowest and sell at the highest", there could be different interpretations on the maxim BLSH depending on the meanings of the "low" and the "high". Problems (1.1.4)–(1.1.7) make precise these in terms of optimal stopping (timing). Specifically, Problem (1.1.4) is equivalent to

$$\max_{\nu\in\mathcal{T}} \mathbb{E}\left(\frac{M_T - S_\nu}{M_T}\right),$$

i.e., the investor attempts to *maximize* the expected *relative* error between the buying price and the highest possible stock price by choosing a proper time to buy. This is motivated by a typical investment sentiment that an investor wants to stay away, as far as possible, from the highest price when she is buying. Similarly, Problem (1.1.5) is to *minimize* the expected relative error between the buying price and the lowest possible stock price. In the same spirit, Problem (1.1.6) (resp. (1.1.7)) is to minimize (resp. maximize) the expected relative error between the selling price and the highest (resp. lowest) possible stock price when selling. Therefore, Problems (1.1.4)–(1.1.7) capture BLSH from various perspectives.

Optimal stock trading involving relative error is first formulated and solved in Shiryaev, Xu and Zhou (2008b), where Problem (1.1.6) is investigated using a purely probabilistic approach. They find that it is optimal to hold the stock if $\mu/\sigma^2 \geq 1/2$, while it is optimal to sell the stock immediately if $\mu/\sigma^2 \leq 0$. There is, however, a case $\mu/\sigma^2 \in (0, 1/2)$ that remains unsolved in Shiryaev, Xu and Zhou (2008b)[1].

1.1.2 Behavioral preferences

In economic literature, people typically assume that investors derive utility only from consumption or total wealth and use expected utility theory (EUT) (dating back to von Neumann and Morgenstern 1944) to evaluate risky gambles. Initiated by Samuelson (1969) and Merton (1969), EUT has been applied in many portfolio choice models. In such models, it is often assumed the investors are risk-averse which is equivalent to assuming a concave utility function. However, such assumptions have been challenged by many puzzles and paradoxes. For example, Friedman and Savage (1948) find people often buy both insurance and lotteries, which contradicts the assumption of the concave utility function. The best example

[1]It is argued in Shiryaev, Xu and Zhou (2008a, 2008b) that this missing case is economically insignificant. The case is subsequently covered by du Toit and Peskir (2009) employing the same probabilistic approach and Dai, et al. (2010) employing the PDE approach.

against EUT is known as The Allais Paradox (Allais 1953), which plays an important role in stimulating the theoretical developments for non-EUT theories. The following example based on Allais (1953) is called *common ratio effect*. Consider the choice between $\mathbf{A}$ = (£3,000, 1) and $\mathbf{B}$ = (£4,000, 0.8; £0, 0.2), and the choice between $\mathbf{A}'$ = (£3,000, 0.25; £0, 0.75) and $\mathbf{B}'$ = (£4,000, 0.2; £0, 0.8), where $(\pounds x, y)$ denotes payoff $\pounds x$ with probability y. Because $\mathbf{A}' = 0.25 \times \mathbf{A} + 0.75 \times \mathbf{0}$ and $\mathbf{B}' = 0.25 \times \mathbf{B} + 0.75 \times \mathbf{0}$ where $\mathbf{0}$ = (£0, 1). If the individual follows EUT, one would conclude that he will either choose $\mathbf{A}$ in the first pair and $\mathbf{A}'$ in the second pair or $\mathbf{B}$ in the first pair and $\mathbf{B}'$ in the second pair. However, the survey results show that the majority choose $\mathbf{A}$ and $\mathbf{B}'$. This example shows people's preference is nonlinear with respect to the distribution function. In other words, people tend to distort the probability.

The failure of EUT motivates economists to develop other alternative models of choice. Recently, many alternative preference measures have been put forth. For example, Yaari's "dual theory of choice" (Yaari 1987) which attempts to resolve a number of puzzles and paradoxes associated with EUT. In this theory, the main feature is that investor distorts the probability decumulative function of the random payment, instead of distorting the payment directly by the utility function. As Yaari (1987) shows, the probability distortion function represents the risk preference in a different way. In particular, risk aversion is characterized by a convex distortion function. In psychological literature, the prospect theory of Kahneman and Tversky (1979), and Tversky and Kahneman (1992) later receives most attention along this line, which also contributes them the Nobel price in Economics. There are three key elements in the prospect theory.

- A reference point. The utility is defined over gains and losses relative to a reference point (or, break-even level/status quo), rather than over the absolute values[2];

- An S-shaped utility function and loss aversion. The utility function[3] is concave for gains and convex for losses, which is steeper for losses than for gains;

- A nonlinear probability distortion. People inflates a small probability and deflates a large probability.

Recently, more and more researchers have employed the prospect theory to study financial problems. For instance, Berkelaar, et al. (2004), Gomes (2005), Jin and Zhou (2008), He and Zhou (2010) treat portfolio selection; Benartzi and Thaler (1995), Barberis and Huang (2001), Barberis, et al. (2001) study asset pricing; De Giorgi, et al. (2008) study equilibrium problems; Xu and Zhou (2009) study optimal stopping problem.

In Essay III, we will incorporate the first two elements of the prospect theory into the investor's liquidation behavior and leave the probability distortion model for future research. It turns out that our results predict the disposition effect, which states that investors sell winners too early and ride losers too long. This effect has been extensively found in various financial markets. While in Essay IV, we introduce the probability distortion into mean-semi-variance portfolio selection problem. This work is motivated by Jin, Yan and Zhou (2005), where they show that continuous time mean-semi-variance portfolio selection problem admits no optimal solution. However, we show that distorted mean-semi-variance problem does exist optimal solutions with proper probability distortion, e.g. a convex power distortion function $w(z) = z^\gamma, \gamma > 1$.

[2] Markowitz (1952) probably first proposes the reference point, termed the customary wealth.
[3] This is called value function in Kahneman-Tversky's terminology.

Essay III: Behavioral Liquidation

The disposition effect has been supported by many empirical studies on individual investor trading activities. For instance, Odean (1998) document the disposition effect by analyzing the trading activity of U.S. individual customers of a discount brokerage house, while Dhar and Zhu (2006) analyze cross sectional difference in the disposition effect based on individual characteristics. Using the trading data from Chinese stock market, Feng and Seasholes (2005) find that experienced or sophisticated investors are 67% less prone to the disposition effect than the average investors in their sample, but can not eliminate this bias. Although these papers examine trading of stock, disposition effects have also been found in other settings. Genesove and Mayer (2001), Heath, Huddart and Lang (1999), and Frazzini (2006) uncover disposition effects in the real estate assets, in the exercise of executive stock options and in mutual funds, respectively. Recently, Crane and Hartzell (2008) take advantage of the transparency of real estate investment trusts (REITS) and find strong statistical evidence that REIT managers tend to sell winners and hold losers, where winners and losers are defined using changes in properties' prices since they were acquired. In addition, they find that REIT managers are significantly less likely to sell properties that have a loss relative to a reference point based on inflation or historical average returns, controlling for the properties' recent returns.

As we have seen, the disposition effect affects individual investors, stock/future traders, professional managers and financial institutions. However, the theoretical analysis of this effect is very informal. Sherin and Statman (1985) attribute this effect to four psychological factors: prospect theory preferences, mental accounting (Thaler 1985), regret aversion and self-control (Thaler and Sherin 1981). The first factor proposes that investors face an S-shaped utility function centered around some reference point. If the asset has gone up in value since purchase, the investor is in a gain region and behaves in a risk averse manner, which predicts that the investor may be inclined to sell the asset. Similarly, if the asset has gone down in value, the investor is in a loss region and thus risk seeking. So, he may be inclined to hold on to the asset.

Kyle, Ou-Yang and Xiong (2006) first propose an optimal stopping model to explain the disposition effect. Assuming the investor has exponential S-shaped preferences and the asset price follows the drifted Brownian motion, they find that the investor never (voluntarily) sells at a loss. In particular, selling at the break-even level is favorable for an asset with extremely poor Sharpe ratio. In addition to Kyle, Ou-Yang and Xiong (2006), Barberis and Xiong (2010) and Henderson (2009) also treat liquidation problems for investors with prospect preferences. Barberis and Xiong (2010) consider a piecewise linear (not real S-shaped) utility function over realized gains or losses relative to the break-even level together with a positive discount rate. Their model can explain a number of financial phenomenons. But, they predict that stock is only sold at gains never at losses, because of the positive discount rate and infinite time horizon. Thus, their model is consistent with a strong version of the disposition effect as they acknowledge. Henderson (2009) extend Kyle, Ou-Yang and Xiong (2006) to much more general setting and also take into account of partial liquidation. She finds that the behavior of "selling at break-even level" disappears when the utility function is piecewise power or partial liquidation is allowable. However, all of these papers are concerned liquidation in infinite time horizon and the dependence of the liquidation strategies on time is never revealed.

Essay IV: Mean-Semi-Variance Portfolio Selection with Probability Distortion

Risk is a central issue in modern financial investment. Markowitz (1952) first propose the single-period mean-variance portfolio selection model, where the variance is used to quantify the risk. From then on, it becomes a rather popular criterion to measure the investment performance in the financial market. However, there is also substantial amount of objection to the measurement of risk by variance. The main aspect of the mean-variance theory under criticism is the penalization on the upside return. Consequently, many alternative risk measures are proposed, such as downside risk measures, where only the return below its mean or a target level is counted as risk. One of the well-recognized downside risk measures is semi-variance. It is agreed by Markowitz (1959) himself that the semi-variance seems more plausible than variance as a measure of risk. Before 2000, most of work on mean-variance problem has been done in discrete time setting until the development of the stochastic linear quadratic (LQ) control. A series of papers (e.g. Zhou and Li 2000, Lim and Zhou 2002, etc.) investigate the continuous-time Markowitz models thoroughly with explicit solutions obtained in most cases. Jin, Yan and Zhou (2005) study the mean-semi-variance problem in continuous time. Their main result is that the mean-semi-variance problem admits *no* optimal solution and investors can construct a sequence of strategies to keep its mean level of the final wealth, but reduce its risk that measured by semi-variance to some infimum value asymptotically. This "negative" result also motivates them to study a general downside-risk model, which gives a very similar result under some mild conditions. The "negative" result implies that mean-semi-variance preference can not fully capture the investor's risk preference in certain sense.

Thus, it is natural to introduce the probability distortion into the conventional mean-semi-variance model and investigate this new risk preference. Essay IV aims to take this task.

1.2 The Scope of the Thesis

In Chapter 2, we will see that problem (1.1.2) can be reformulated as a standard optimal stopping problem. Its PDE formulation, described by a free boundary problem or an obstacle problem, seemingly resembles the pricing model of lookback options which has been extensively studied by numerous researchers (e.g., Shepp and Shiryaev 1994, Wilmott, Dewynne and Howison 1995, Yu, Kowk and Wu 2001, Lai and Lim 2004, Dai and Kwok 2005, Peskir 2005 and reference therein). However, the present problem involves a complicated obstacle function such that theoretical analysis of the free boundary is distinct from the previous ones.

Our main focus will be on the characterization of the free boundary, which is challenging due to the lack of analytical solutions. We will resort to a PDE approach and conduct a thorough qualitative analysis of the free boundary. The existence, boundness and smoothness of the free boundary are established. We will see that the case of $\mu \leq 0$ is relatively easy to handle since it is easy to show the value function has desired properties. However, it is not only challenging but also more important in practice to deal with the case of $\mu > 0$, which represents a good stock. It turns out that the behaviors of the free boundary heavily depend on the sign of μ. In the case of $\mu > 0$, the free boundary is shown to have two branches and exists only when time to maturity is shorter than a threshold value. This indicates that one should not sell a good stock too early if the investor has plenty of time to sell in the future. In the case of $\mu \leq 0$, we show that the free boundary has only one branch. It is worthwhile pointing out that these results are quite similar to those obtained by Du Toit and Peskir (2007), where the "stock price" is assumed to follow a drifted Brownian motion. They use a purely probabilistic approach and take advantage of the concise explicit expression of obstacle function to analyze the function H in (2.1.7) by a direct calculation. However, under the assumption of geometric Brownian motion(GBM) of stock price, explicit expression of function H is so complicated that it is intractable (at least for us) to extend their approach to tackle the present problem.

Chapter 3 is devoted to Essay II. Again, we take a PDE approach, which enables us to solve not only all the cases associated with Problem (1.1.6), but also all the other problems (1.1.4), (1.1.5) and (1.1.7) simultaneously. Moreover, we will derive completely the buying/selling regions for all the problems, leading to optimal *feedback* trading strategies, ones that would respond to all the scenarios in time and in (certain) well-defined states (rather than to the ones at $t = 0$ only).

The results derived from our models are quite intuitive and consistent with the common investment practice. For the two buying problems (1.1.4) and (1.1.5), our results dictate that one ought to buy immediately or never buy depending on whether the underlying stock is "good" or "bad" (which will be defined precisely). If, on the other hand, the stock is intermediate between good and bad, then one should buy if and only if either the current stock price is sufficiently cheap compared with the historical high (for Problem (1.1.4)) or it has sufficiently risen from the historical low (for Problem (1.1.5)). For the selling problems (1.1.6) and (1.1.7), one should never sell (i.e. hold until the final date) or immediately sell depending, again, on whether the stock is "good" or "bad" (which will however be defined differently from the buying problems). If the stock is in between good and bad, then one should sell if and only if either the current stock price is sufficiently close to the historical high (for Problem (1.1.6)) or it is sufficiently close to the historical low (for Problem (1.1.7)). In particular, at time $t = 0$ the stock price is trivially both historical high and low; hence the selling strategy would be to sell at $t = 0$, suggesting that this intermediate case is "bad" after all.

Chapter 4 studies the finite horizon liquidation behavior of the investor with prospect preferences. Following Kyle, Ou-Yang and Xiong (2006), we adopt the optimal stopping framework. Our primary interests will be the changes of the optimal liquidation strategies with respect to time t. Partial liquidation problem is not considered in this thesis. We assume the asset price follows GBM and the utility function is piecewise power function with index $\alpha \in (0, 1]$. In particular, we pay special attention to the case of $\alpha = 1$, since it has finite sensitivity at the origin (break-even level). We find that the liquidation strategies are dramatically different between $\alpha = 1$ and $\alpha < 1$ at the break-even level. When $\alpha = 1$, we find that for an asset with extremely good expected return, the only liquidation strategy is selling at the break-even level after certain time threshold depending on the asset return. Moreover, liquidation at break-even is very likely when the asset has a moderate return. If the asset has negative expected return, liquidation at a small loss is always possible. While $\alpha < 1$, we find the investor will never voluntarily liquidate at a small loss or break-even level no matter how bad the asset is. Because of the positive discount, the investors always prefer to defer large losses. As a consequence, even if the asset has extremely poor Sharpe ratio, the investor will wait to liquidate either at a small gain or some moderate loss. However, under both cases of $\alpha = 1$ and $\alpha < 1$, the optimal liquidation strategies show that the investors are more likely to liquidate the asset at a gain than at a loss. This result is consistent with the disposition effect.

Mathematically, the problem is challenging, since the utility function is not smooth and the smooth-fit principle does not hold on the whole free boundary that corresponds to the optimal liquidation strategy. The lack of desired properties on the payoff function results the *usual* variational approach failed. Thanks to the viscosity solution, we characterize the value function in the sense of viscosity solution and show that smooth fit principle holds partially on the free boundary.

The proof of smooth-fit at one single point enables us to see the role of the sensitivity of utility function at the break-even level. If the utility function has finite sensitivity, then loss aversion plays an important role on the liquidation strategies, which implies that more loss aversion more possibility to liquidate at break-even. However, if the utility function has infinite sensitivity at the break-even level, then the value function is smooth before maturity, regardless of the loss aversion at maturity. This helps us to understand that the liquidation at break-even is subject to the precise specification of prospect theory preference. Our approach can easily extend Kyle, Ou-Yang and Xiong (2006) to the finite horizon setting and the property of liquidation at the break-even level can be revealed in a similar way as the case of $\alpha = 1$.

In Chapter 5, we study the distorted mean-semi-variance portfolio selection problem. We model the traditional linear expectation by Choquet's integral. Choquet (1953) extend the probability measure $\mathbb{P}$ in $[0, 1]$ to a nonlinear probability measure V (also called the capacity) and define nonlinear expectation (also called the Choquet expectation) $C(\xi) := \int_{-\infty}^{0}[V(\xi \geq t) - 1]dt + \int_{0}^{+\infty} V(\xi \geq t)dt$. Many papers (e.g. Denneberg 1994 and the reference therein) have been devoted to the Choquet expectation and its applications. The main technical challenge is that such probability distortion (or nonlinear expectation) destroys the time consistency, which is necessary for the dynamic programming approach as well as the convexity, which is necessary for the convex duality approach. Fortunately, some pioneer work involving probability distortion into portfolio selection has been done in the literature (e.g. Jin and Zhou 2008, He and Zhou 2010). Jin and Zhou (2008) study a general continuous-time behavioral portfolio selection model under Kahneman and Tversky's (cumulative) prospect theory, featuring S-shaped utility (value) functions and probability distortions. They develop a quantile approach to overcome the difficulties arising from the probability distortion. He and Zhou

(2010) formulate a new portfolio choice model with probability distortion in continuous time for both complete and incomplete markets. Their formulation covers a wide body of existing and new models with law-invariant preference measures. The basic approach to solve their problems is the martingale approach (cf. Follmer and Leukert 1999, Pliska 1982), i.e. reducing the problem into two subproblems: one to solve a constrained static optimization problem on the terminal payment, and the other to replicate the optimal terminal payment. The second subproblem is normally straightforward to solve in a complete market. However, the first subproblem is very difficult, because it is lack of convexity in the decision variable. The key idea is to reformulate the first subproblem via quantile approach (e.g. Jin and Zhou 2008 for probability distortion problem, Schied 2004 for general risk measure problems), i.e. changing the decision variable from the random variable itself to its distribution or quantile. Basically, the distorted mean-semi-variance problem is solved by the same approach as Jin and Zhou (2008). But, we have an additional constraint on its distorted mean that complicates the problem. The main contribution of this work has three folds: firstly, we solve the feasibility problem, which is non trivial with probability distortion in the constraint; secondly, we give the necessary and sufficient conditions that characterize the existence of optimal solution, which solely depends on the probability distortion given the financial market; thirdly, we present the optimal solution whenever it exists and exploit the efficient frontier.

Finally, Chapter 6 concludes the thesis.

Chapter 2

Selling Stock Close to Its Global Maximum

In this chapter, we will make use of a PDE approach to study the optimal stock selling strategy which minimize the expected square error distance between the selling price and the ultimate maximum over the given period $[0, T]$. The rest of the chapter is organized as follows. In the subsequent section, we formulate problem (1.1.2) as a variational inequality problem. Section 2 is concerned with the existence and uniqueness of the solution to the variational inequality. The optimal selling strategy is analyzed in Section 3, where the monotonicity and smoothness of the free boundary are proved. We summarize the chapter in Section 4 while some auxiliary results and technical proofs are placed in an appendix.

2.1 PDE Formulation

In this section, we will reduce the nonstandard optimal stopping problem (1.1.2), because of the non-$\mathscr{F}_t$-adapted payoff, to a standard optimal stopping problem and then provide a PDE formulation for the latter one.

Denote by M_t the running maximum of the stock price over $[0,t]$, i.e., $M_t = \max_{0\le\nu\le t} S_\nu$. Then we have

$$\begin{aligned}\mathbb{E}(S_\nu - M_T)^2 &= \mathbb{E}\left[\mathbb{E}_\nu(S_\nu - M_T)^2\right] = \mathbb{E}\left[S_\nu^2 - 2S_\nu\mathbb{E}_\nu(M_T) + \mathbb{E}_\nu(M_T^2)\right] \\ &= \mathbb{E}\left[U(S_\nu, M_\nu, \nu)\right], \end{aligned} \tag{2.1.1}$$

where

$$\begin{aligned} U(S,M,t) &= S^2 - 2SU_1(S,M,t) + U_2(S,M,t), \\ U_i(S_t,M_t,t) &= \mathbb{E}_t(M_T^i),\ i=1,2,\ \ \mathbb{E}_t(\cdot) = \mathbb{E}(\cdot\,|\mathscr{F}_t). \end{aligned}$$

Equation (2.1.1) indicates that (1.1.2) is equivalent to a standard optimal stopping problem with an exercise payoff $U(S,M,t)$. In view of the dynamic programming approach and the Markov property of (S_t, M_t), we only need to consider the following problem

$$V(S,M,t) = \inf_{0\le\nu\le T-t} \mathbb{E}_t[U(S_{t+\nu}, M_{t+\nu}, t+\nu)], \tag{2.1.2}$$

where $(S_t, M_t) = (S,M)$ under $\mathbb{P}_{t,S,M}$ with $0 < S \le M < \infty$ given and fixed. Obviously, the original problem is

$$V(S_0,S_0,0) = \inf_{0\le\nu\le T} \mathbb{E}(S_\nu - M_T)^2.$$

Following Wilmott, Dewynne and Howison (1995), one can show that $U_i, i = 1,2$, and V satisfy[1]

$$\begin{cases} -\partial_t U_i - \mathcal{L}^0 U_i = 0, & (S,M,t)\in D, \\ \partial_M U_i(M,M,t) = 0, & U_i(S,M,T) = M^i, \end{cases} \tag{2.1.3}$$

[1] Equation (2.1.3) resembles the well-known pricing model of a European-style lookback option, whereas (2.1.4) resembles that of an American-style lookback option. However, problem (2.1.4) involves a complicated payoff function such that the theoretical analysis of free boundary is distinct from the standard literature.

and

$$\begin{cases} \max\{-\partial_t V - \mathcal{L}^0 V, V - U\} = 0, \qquad (S,M,t) \in D, \\ \partial_M V(M,M,t) = 0, \quad V(S,M,T) = (S-M)^2, \end{cases} \tag{2.1.4}$$

where $\mathcal{L}^0 = \frac{\sigma^2}{2} S^2 \partial_{SS} + \mu\, S \partial_S$ and $D = \{(S,M,t) : 0 < S < M,\ 0 \le t < T\}$.

Next, we will show that problems (2.1.3)-(2.1.4) can be reduced to one-dimensional time dependent problems. Indeed, by the transformation

$$\begin{aligned} \tau &= T - t, \quad x = \ln M - \ln S, \qquad V(S,M,t) = S^2 v^*(x,\tau), \\ U_1(S,M,t) &= S u_1(x,\tau),\ U_2(S,M,t) = S^2 u_2(x,\tau),\ U(S,M,t) = S^2 u(x,\tau). \end{aligned}$$

According to (2.1.3)-(2.1.4), u_1, u_2 and v^* satisfy

$$\begin{cases} \partial_\tau u_1 - \mathcal{L}^1_x u_1 = 0 \ \text{ in } \Omega, \\ \partial_x u_1(0,\tau) = 0,\ u_1(x,0) = e^x, \end{cases} \tag{2.1.5}$$

$$\begin{cases} \partial_\tau u_2 - \mathcal{L}^2_x u_2 = 0 \ \text{ in } \Omega, \\ \partial_x u_2(0,\tau) = 0,\ u_2(x,0) = e^{2x} \end{cases}$$

and

$$\begin{cases} \max\{\partial_\tau v^* - \mathcal{L}^2_x v^*, v^* - u\} = 0 \ \text{ in } \Omega, \\ \partial_x v^*(0,\tau) = 0,\ v^*(x,0) = (e^x - 1)^2, \end{cases}$$

where $\Omega = (0,\infty) \times (0,T]$ and

$$\mathcal{L}^1_x u_1 = \frac{\sigma^2}{2}\partial_{xx} u_1 - \left(\mu + \frac{\sigma^2}{2}\right)\partial_x u_1 + \mu u_1,\ \ \mathcal{L}^2_x u_2 = \frac{\sigma^2}{2}\partial_{xx} u_2 - \left(\mu + \frac{3\sigma^2}{2}\right)\partial_x u_2 + (2\mu + \sigma^2) u_2.$$

Denote $v = v^* - u$, then v satisfies

$$\begin{cases} \max\{\partial_\tau v - \mathcal{L}^2_x v - H, v\} = 0 \ \text{ in } \Omega, \\ \partial_x v(0,\tau) = 0,\ v(x,0) = 0, \end{cases} \tag{2.1.6}$$

where[2]

$$H \;=\; \mathcal{L}_x^2 u - \partial_\tau u = 2\mu + \sigma^2 + 2[\sigma^2 \partial_x u_1 - (\mu + \sigma^2) u_1]. \qquad (2.1.7)$$

In physics, problem (2.1.6) is known as an upper obstacle problem. We can define the selling region (stopping region) as follows:

$$SR = \{(x, \tau) \in [0, \infty) \times (0, T] : v(x, \tau) = 0\}.$$

2.2 Existence and Uniqueness

In the section, we will establish the existence and uniqueness of $W_p^{2,1}$ solution with at most exponential growth. Some general facts and notations about PDE theory will be required, which we briefly recall in the Appendix 2.6.1. Before we go through the existence theorem, let us first introduce two lemmas, which provide some prior estimates and characterize the desired properties of $H(x, \tau)$. Without loss of generality, we may assume $\sigma = 1$, as one could make a change of time $\bar{\tau} = \sigma^2 \tau$ otherwise. Thereby, we use $\alpha = \mu/\sigma^2$ instead of μ in the following.

Lemma 2.2.1. *Problem (2.1.5) has a unique solution $u_1(x, \tau)$ such that*

i) $$\max\{e^x, e^{\alpha\tau}\} \le u_1(x, \tau) \le \begin{cases} e^x + \frac{e^{2(|\alpha|+1)\tau}}{x+1} \;\; \textit{in } \Omega, & \textit{if } \alpha \ge 0; \\ \min\left\{e^x - \frac{e^{2\alpha x}}{2\alpha}, e^x + \frac{e^{2(|\alpha|+1)\tau}}{x+1}\right\} \;\; \textit{in } \Omega, & \textit{if } \alpha < 0. \end{cases}$$

In addition, $u_1(x, \tau) \ge e^x - x + \frac{\tau}{2}$ *if* $\alpha = 0$.

ii) $$e^x \ge \partial_x u_1(x, \tau) \ge \begin{cases} 0 \;\; \textit{in } \Omega, & \textit{if } \alpha > 0, \\ \max\{e^x - \frac{e^{8\tau}}{(x+1)^3}, 0\} \;\; \textit{in } \Omega, & \textit{if } \alpha = 0, \\ e^x - 1 \;\; \textit{in } \Omega, & \textit{if } \alpha < 0. \end{cases}$$

[2] If we consider the dynamics of payoff process $U(S_t, M_t, t)$, $H(x, t)$ is related to the drift. Thus, the sign of $H(x, t)$ decides whether $U(S_t, M_t, t)$ behaves as a supermartingale or submartingale locally.

Moreover, $\partial_x u_1 \le \partial_{xx} u_1$ *in* Ω.

iii) $\partial_\tau u_1 > 0$ *and* $\partial_{\tau x} u_1 \le 0$ *in* Ω.

The proof is placed in Appendix 2.6.2. This lemma is an auxiliary result for the next lemma.

Lemma 2.2.2. *The function* $H(x,\tau)$ *defined in (*2.1.7*) satisfies*

i) $\partial_\tau H > 0$ *if* $\alpha \ge -1$ *and* $\partial_x H > 0$ *if* $\alpha \le 0$, $\forall (x,\tau) \in \Omega$.

ii) $|H| \le 4(|\,\alpha| + 1)(e^x + e^{(|\,\alpha|+1)\tau})$ *in* $\overline{\Omega}$.

iii) If $\alpha \le 0$, *there exists a curve* $x = \gamma(\tau) \in C^\infty(0,T]$ *with* $\lim_{\tau\to 0^+} \gamma(\tau) = 0$ *such that*

$$H(\gamma(\tau),\tau) = 0, \quad H(x,\tau) > 0 \text{ in } \mathcal{M} \quad \text{and } H(x,\tau) < 0 \text{ in } \Omega \setminus \overline{\mathcal{M}}, \tag{2.2.1}$$

where $\mathcal{M} = \{(x,\tau) : x > \gamma(\tau), 0 < \tau \le T\}$.

iv) If $\alpha > 0$, *there exist a* $\tau_H \in (0, \frac{1}{\alpha}\ln(1+\frac{1}{2\alpha})]$ *independent of* T *and two strictly monotone curves* $x = \gamma_1(\tau) \in C^\infty(0,\tau_H]$ *(increasing),* $x = \gamma_2(\tau) \in C^\infty(0,\tau_H]$ *(decreasing) with* $\gamma_1(\tau_H) = \gamma_2(\tau_H)$, $\lim_{\tau\to 0^+}\gamma_1(\tau) = 0$ *and* $\lim_{\tau\to 0^+}\gamma_2(\tau) = \ln(\frac{1}{2\alpha}+1)$ *such that*

$$H(\gamma_i(\tau),\tau) = 0, i \in \{1,2\}, \; H(x,\tau) > 0 \text{ in } \mathcal{M} \text{ and } H(x,\tau) < 0 \text{ in } \Omega \setminus \overline{\mathcal{M}},$$

where $\mathcal{M} = \{(x,\tau) : \gamma_1(\tau) < x < \gamma_2(\tau), 0 < \tau \le \tau_H\}$.

The proof is placed in Appendix 2.6.3.

Remark 2.2.3. *One may find an explicit but tedious expression of H. Indeed, we can easily find $u_1(x,\tau)$ via computing $\mathbb{E}_t(M_T)$ or solving (2.1.5),*

$$u_1(x,\tau) = e^x \Phi(-d+\sigma\sqrt{\tau}) - \frac{\sigma^2}{2\mu} e^{2\mu/\sigma^2 x}\Phi(d - 2\mu/\sigma\sqrt{\tau}) + (1+\sigma^2/2\mu)\Phi(d)$$

and

$$\begin{aligned}\partial_x u_1(x,\tau) =& e^x\Phi(-d+\sigma\sqrt{\tau}) - e^{2\mu/\sigma^2 x}\Phi(d-2\mu/\sigma\sqrt{\tau}) \\ &+ \frac{1}{\sigma\sqrt{\tau}}\left(e^x\phi(-d+\sigma\sqrt{\tau}) + \frac{\sigma^2}{2\mu}e^{2\mu/\sigma^2 x}\phi(d-2\mu/\sigma\sqrt{\tau}) - (1+\sigma^2/2\mu)\phi(d)\right),\end{aligned}$$

where $d = \frac{-x+(\mu+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$, $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and $\Phi(\cdot) = \int_\infty^\cdot \phi(x)dx$. Combining with (2.1.7), we obtain a tedious expression of H, which seems intractable to obtain the above properties directly (at least for us). So, we employ a PDE approach to show those desired properties of H.

Now, we are ready to show the existence and uniqueness of solutions to problem (2.1.6). Since the domain Ω is unbounded, we first restrict attention to a bounded domain $\Omega_T^n \triangleq \Omega \cap \{(x,\tau) : 0 < x < n\}$, $\forall n \in \mathbb{Z}^+$ and consider

$$\begin{cases} \max\{\partial_\tau v_n - \mathcal{L}_x^2 v_n - H(x,\tau), v_n\} = 0 \quad \text{in } \Omega_T^n, \\ \partial_x v_n(0,\tau) = 0, \qquad v_n(n,\tau) = 0, \qquad v_n(x,0) = 0. \end{cases} \tag{2.2.2}$$

Proposition 2.2.4. *For any fixed $n \in \mathbb{Z}^+$, there exists a unique solution $v_n \in C(\overline{\Omega_T^n}) \cap W_p^{2,1}(\Omega_T^n)$ of problem (2.2.2) for any fixed $1 < p < +\infty$, $n \in \mathbb{Z}^+$. Moreover, if n is large enough, then we have the following estimates:*

$$-u \le v_n \le 0 \ \textit{in} \ \overline{\Omega_T^n}, \tag{2.2.3}$$

$$\begin{cases} \partial_x v_n \ge 0 \ \textit{in} \ \Omega_T^n, & \textit{if } \alpha \le 0, \\ \partial_\tau v_n \le 0 \ \textit{a.e. in} \ \Omega_T^n, & \textit{if } \alpha \ge -1. \end{cases} \tag{2.2.4}$$

Proof. Following Friedman(1982), we consider the following penalized problem to approximate problem (2.2.2):

$$\begin{cases} \partial_\tau v_{n,\varepsilon} - \mathcal{L}_x^2 v_{n,\varepsilon} - \beta_\varepsilon(-v_{n,\varepsilon}) = H(x,\tau) & \text{in } \Omega_T^n, \\ \partial_x v_{n,\varepsilon}(0,\tau) = 0, \quad v_{n,\varepsilon}(n,\tau) = 0, \quad v_{n,\varepsilon}(x,0) = 0, \end{cases} \tag{2.2.5}$$

where $\beta_\varepsilon(\xi)$ is the penalty function, satisfying

$$\begin{aligned} &\beta_\varepsilon(\xi) \in C^\infty(-\infty,+\infty),\ \forall\ 0 < \varepsilon << 1; \qquad \beta_\varepsilon(\xi) = 0,\ \text{ if } \xi \geq \varepsilon; \\ &\beta_\varepsilon(0) = -C_0 \triangleq -4(|\,\alpha| + 1)\left(e^n + e^{2(|\alpha|+1)T}\right); \\ &\beta_\varepsilon(\xi) \leq 0; \qquad 0 \leq \beta'_\varepsilon(\xi) \leq \frac{2}{\varepsilon}; \qquad \beta''_\varepsilon(\xi) \leq 0. \end{aligned} \tag{2.2.6}$$

The existence and uniqueness of solutions to nonlinear problem (2.2.5) are standard in the PDE theory. For completeness, we include a brief proof here. Denote $B = C(\overline{\Omega_T^n})$, and $D = \{w \in B\ :\ w \leq C_1 - u\}$, where $C_1 = \max_{\overline{\Omega_T^n}} u$. Then D is a closed convex set in B. For any $w \in D$, define a mapping $T : w \to v$, such that $v = Tw$ solve the following linear problem

$$\begin{cases} \partial_\tau v - \mathcal{L}_x^2 v = H(x,\tau) + \beta_\varepsilon(-w) & \text{in } \Omega_T^n, \\ \partial_x v(0,\tau) = 0, \quad v(n,\tau) = 0, \quad v(x,0) = 0. \end{cases} \tag{2.2.7}$$

The linear problem (2.2.7) has a unique solution $v \in W_p^{2,1}(\Omega_T^n)$, and

$$\|v\|_{W_p^{2,1}(\Omega_T^n)} \leq C\left(|H(x,\tau)|_{L^p(\Omega_T^n)} + |\beta_\varepsilon(-w)|_{L^p(\Omega_T^n)}\right), \tag{2.2.8}$$

where constant C (possibly different wherever it appears) is independent of ε. Actually, v is a smooth solution to (2.2.7) here.

In the following, we prove T satisfying

i) $T(D) \subset D$,

ii) $\overline{T(D)}$ is compact set in B,

iii) T is continuous.

In fact, we have

$$\partial_\tau(C_1 - u) - \mathcal{L}_x^2(C_1 - u) = H(x,\tau) \geq \partial_\tau v - \mathcal{L}_x^2 v, \forall w \in D.$$

Moreover,

$$\partial_x(C_1 - u) = 0, (C_1 - u)(x,0) \geq 0 \text{ and } (C_1 - u)(n,\tau) \geq 0.$$

By the maximum principle, $v \leq (C_1 - u)$, then $T(D) \subset D$. From $w \leq C_1$ because of $u \geq 0$, we know that $\beta_\varepsilon(-w)$ is bounded, then $\|v\|_{W_p^{2,1}(\Omega_T^n)} \leq C_\varepsilon$, where C_ε is independent of w. Imbedding Theorem 2.6.6 implies that T is a compact mapping for p large enough. To show T is continuous, we only need to prove

$$\lim_{j\to\infty} v_j = v \text{ in } D,$$

where $\{w_j\} \subset D$, $w_j \to v$ in D, and $v_j = Tw_j$, $v = Tw$. Indeed, $v_j - v$ is the solution of

$$\begin{cases} \partial_\tau(v_j - v) - \mathcal{L}_x^2(v_j - v) = -\beta_\varepsilon^{'}(\cdot)(w_j - w) & \text{in } \Omega_T^n, \\ \partial_x(v_j - v)(0,\tau) = 0, \quad (v_j - v)(n,\tau) = 0, \quad (v_j - v)(x,0) = 0. \end{cases}$$

Therefore, by the estimate of maximum norm,

$$|v_j - v|_{L^\infty(\Omega_T^n)} \leq C|w_j - w|_{L^\infty(\Omega_T^n)} \to 0.$$

By the Schauder fixed point theorem, we show there exists a $v \in D$, such that $v = Tv$, while the uniqueness is implied by the comparison principle.

Denote $w := -u - \varepsilon e^{(2\alpha+1)\tau + \ |2\alpha+1|T} \leq -\varepsilon$, then

$$\partial_\tau w - \mathcal{L}_x^2 w - \beta_\varepsilon(-w) = H - (2\alpha+1)\varepsilon e^{(2\alpha+1)\tau + \ |2\alpha+1|T} \leq H.$$

In addition, by the definition of C_0 in (2.2.6) and the boundness of H, we have

$$\partial_\tau v - \mathcal{L}_x^2 v + \beta_\varepsilon^{'}(\cdot)v = \beta_\varepsilon(0) + H(x,\tau) \leq 0.$$

By the comparison principle, we can deduce that

$$-u - \varepsilon e^{(2\alpha+1)\tau + \,|2\alpha+1|T} \leq v_{n,\,\varepsilon} \leq 0, \tag{2.2.9}$$

According to the $W_p^{2,\,1}$ estimate, there exists a constant C independent of ε such that

$$\|v_{n,\,\varepsilon}\|_{\,W_p^{2,1}(\Omega_T^n)} + \|v_{n,\,\varepsilon}\|_{\,L^\infty(\overline{\Omega_T^n})} \leq C.$$

It follows that, possibly a subsequence,

$$v_{n,\,\varepsilon} \rightharpoonup v_n \ \text{ in } W_p^{2,1}(\Omega_T^n) \text{ weakly } \text{ and } \text{ in } C(\overline{\Omega_T^n}) \text{ uniformly } \quad \text{as } \varepsilon \to 0^+.$$

Thus, one can easily deduce that v_n is the $W_p^{2,1}$ solution of problem (2.2.2) and (2.2.3) simply follows by letting $\varepsilon \to 0^+$ in (2.2.9).

Next, we will show (2.2.4). Since v_n achieves its maximum on the line $x = n$, we have

$$\partial_x v_n(n, \tau) \geq 0 \ \ \forall\, \tau \in [\,0, T\,].$$

Moreover, the imbedding theorem leads to $\partial_x v_n \in C(\Omega_T^n)$, which implies that

$$\partial_x v_n(x, \tau) \geq 0 \ \text{ on } \partial_p\{(x, \tau) : v_n < 0\},$$

where $\partial_p A$ denotes the parabolic boundary of set A (cf. 2.6.4).

In the case of $\alpha \leq 0$, differentiating the equation in (2.2.2) with respect to x, we have

$$\begin{cases} \partial_\tau(\partial_x v_n) - \mathcal{L}_x^2(\partial_x v_n) = \partial_x H(x, \tau) \geq 0, & \text{in } \ \{(x, \tau) : v_n < 0\}, \\ \partial_x v_n \geq 0, & \text{on } \partial_p\{(x, \tau) : v_n < 0\}. \end{cases}$$

Then we obtain $\partial_x v_n \geq 0$ in $\overline{\Omega}_T^n$.

To show $\partial_\tau v_n \leq 0$ in Ω_T^n for the case of $\alpha \geq -1$, it suffices to show $w = v_n(x, \tau + \delta) - v_n(x, \tau) \leq 0$, for any $\delta > 0$. Suppose not, then

$$O = \{(x, \tau) : w > 0\} \neq \emptyset.$$

Since $\partial_\tau H \le 0$, it is obvious that w satisfies

$$\begin{cases} \partial_\tau w - \mathcal{L}_x^2 w \le H(x,\tau+\delta) - H(x,\tau) \le 0, \text{ in } O, \\ \partial_x w|_{\{x=0\}\cap\partial_p O} = 0, \ w|_{\partial_p O\setminus\{x=0\}} = 0. \end{cases}$$

Thus, $w \le 0$ in O, which contradicts the definition of O.

The proof of the uniqueness of the solution v_n follows by the comparison principle.

Theorem 2.2.5. *Problem (2.1.6) has a unique solution $v \in C(\overline{\Omega_T}) \cap W_p^{2,1}(\Omega_T^R)$ for any fixed $R > 0,\ 1 < p < +\infty$. Moreover,*

$$-u \le v \le 0 \ \ in \ \overline{\Omega}_T, \tag{2.2.10}$$

$$\begin{cases} \partial_x v \ge 0 \quad in \ \ \Omega_T, & if\, \alpha \le 0, \\ \partial_\tau v \le 0 \quad a.e. \ in \ \ \Omega_T, & if\, \alpha \ge -1. \end{cases} \tag{2.2.11}$$

Proof. We rewrite the problem (2.2.5) as

$$\begin{cases} \partial_\tau v_n - \mathcal{L}_x^2 v_n = f(x,\tau) & \text{in } \ \Omega_T^n, \\ \partial_x v_n(0,\tau) = 0, \quad v_n(n,\tau) = 0, \quad v_n(x,0) = 0, \end{cases}$$

where $f(x,\tau) = I_{\{v_n<0\}} H(x,\tau)$ and I_A is the indicator function of set A.

Notice that $v_n \in W_p^{2,1}(\Omega_T^n)$, then we see $f(x,\tau) \in L^p(\Omega_T^n)$ and

$$|f(x,\tau)| = |I_{\{v_n<0\}} H(x,\tau)| \le |H(x,\tau)| \qquad \text{in } \Omega_T^n.$$

Recalling (2.2.3), we infer

$$\|v_n\|_{W_p^{2.1}(\Omega_T^R)} \le C(\|v_n\|_{L^\infty(\Omega_T^R)} + \|H\|_{L^\infty(\Omega_T^R)}) \le C,$$

for any fixed $R \in (0,n)$ and some constant C depends on R but is independent of n. So, we see that there exists a function v^R and a subsequence of $\{v_n\}$, still denoted by itself, such that

$$v_n \rightharpoonup v^R \ \text{ in } \ W_p^{2.1}(\Omega_T^R) \text{ weakly}, \quad \text{as } n \to +\infty.$$

Furthermore, applying the imbedding theorem gives

$$v_n \to v^R \text{ in } C(\overline{\Omega_T^R}), \quad \text{as } n \to +\infty.$$

By the standard diagonal argument, extracting a subsequence $\{v_n^{(n)}\}$ such that for any $m \in \mathbb{Z}^+$, there holds

$$v_n^{(n)} \to v^m \text{ in } W_p^{2.1}(\Omega_T^m) \text{ weakly and } v_n^{(n)} \to v^m \text{ in } C(\overline{\Omega_T^m}) \text{ as } \quad n \to +\infty.$$

It is clear that $v^{m+1} = v^m$ in $[\,0, m\,] \times [\,0, T\,]$. So, we can define $v = v^m$ if $(x, \tau) \in [\,0, m\,] \times [\,0, T\,]$ and it is not difficult to deduce that v is the solution of the problem (2.1.6). Let $n \to +\infty$ in (2.2.3) and (2.2.4), (2.2.10) and (2.2.11) follow at once. The proof of the uniqueness follows by comparison principle.

Remark 2.2.6. *Though the solution $v \in W_p^{2,1}(\Omega_T^R)$, v is smooth in the domain Ω/SR by PDE regularity theory. Moreover, one can employ a truncation and regularization technique to show that $V(S, M, t) = S^2(v(\ln \frac{M}{S}, T-t)+u(\ln \frac{M}{S}, T-t))$ is the value function of our optimal stopping problem (1.1.2). This technique is standard in the PDE approach, so we omit the details and refer interested readers to Pascucci (2008).*

2.3 The Optimal Strategies

In this section we present the main results of this chapter.

Theorem 2.4.1. *(The optimal selling strategies)*

i) If $\alpha > 0$, there exists a critical point $\tau^* \in \left(0, \frac{1}{\alpha} \ln\left(\frac{1}{2\alpha} + 1\right)\right)$ independent of T. There exist a strictly increasing free boundary $x_{1s}^*(\tau)$ and a strictly decreasing boundary $x_{2s}^*(\tau)$ with $x_{1s}^*(\tau^*) = x_{2s}^*(\tau^*)$, such that

$$SR = \{(x, \tau) : x_{1s}^*(\tau) \le x \le x_{2s}^*(\tau), 0 < \tau \le \tau^*\}.$$

Moreover, $x_{1s}^*(\tau),\ x_{2s}^*(\tau) \in C[0,\tau^*] \cap C^\infty(0,\tau^*]$ *and*

$$\lim_{\tau\to 0^+} x_{1s}^*(\tau) = 0, \qquad \lim_{\tau\to 0^+} x_{2s}^*(\tau) = \ln\left(1+\frac{1}{2\alpha}\right).$$

ii) If $-1/2 \le \alpha \le 0$, *there exists a strictly increasing free boundary* $x_s^*(\tau) \in C[0,T] \cap C^\infty(0,T\,]$ *such that*

$$SR = \{(x,\tau) : x \ge x_s^*(\tau), 0 < \tau \le T\}, \quad \lim_{\tau\to 0^+} x_s^*(\tau) = 0.$$

Moreover, if $\alpha \in (-1/2, 0]$, *then* $\lim_{\tau\to\infty} x_s^*(\tau) = +\infty$.

iii) If $\alpha < -1/2$, *there exists a free boundary* $x_s^*(\tau) \in C[0,T] \cap C^\infty(0,T\,]$ *and a positive number* X, *independent of* T, *such that*

$$SR = \{(x,\tau) : x \ge x_s^*(\tau), 0 < \tau \le T\}, \quad \lim_{\tau\to 0^+} x_s^*(\tau) = 0, \quad x_s^*(\tau) \le X.$$

Moreover, $x_s^*(\tau)$ *is strictly increasing if* $-1 \le \alpha < -1/2$.

Proof. First we show part i). Since $\partial_\tau v \le 0$, it is easy to see

$$\left(\partial_\tau v - \mathcal{L}_x^2 v - H\right)|_{\tau=0} \le -(1 - 2\alpha(e^x - 1)) < 0, \quad \forall\ 0 < x < \ln\left(1+\frac{1}{2\alpha}\right).$$

which implies that SR is nonempty. On the other hand, $SR \subset \{H \ge 0\}$, since $\partial_\tau v - \mathcal{L}_x^2 v - H = -H \le 0$ if $v = 0$.

Define

$$\tau^* = \sup\{\tau : v(x,\tau) = 0, \exists\, x \in (0,\infty)\}, \text{ then } \tau^* \le \tau_H \le \frac{1}{\alpha}\ln\left(\frac{1}{2\alpha}+1\right).$$

For any $\tau \in (0,\tau^*]$, we can denote

$$x_{1s}^*(\tau) = \inf\{x \in [\,0,+\infty) : v(x,\tau) = 0\} \text{ and } x_{2s}^*(\tau) = \sup\{x \in [\,0,+\infty) : v(x,\tau) = 0\}.$$

Suppose $x_{1s}^*(\tau) < x_{2s}^*(\tau)$, the continuity of v implies that $(x_{1s}^*(\tau),\tau)$ and $(x_{2s}^*(\tau),\tau)$ belong to SR. We claim that the rectangle Q with corners at $(x_{1s}^*(\tau),0)$, $(x_{1s}^*(\tau),\tau)$, $(x_{2s}^*(\tau),0)$ and $(x_{2s}^*(\tau),\tau)$ is a subset of SR. In fact, due to $\partial_\tau v \le 0,\ v \le 0$, we see

$$v(x_{1s}^*(\tau),t) = v(x_{2s}^*(\tau),t) = 0,\ \forall\ 0 \le t \le \tau.$$

Suppose $\widehat{O} = Q \cap \{v < 0\} \neq \emptyset$, then

$$\begin{cases} \partial_\tau v - \mathcal{L}_x^2 v = H \geq 0 & \text{in } \widehat{O}, \\ v|_{\partial_p \widehat{O}} = 0. \end{cases}$$

According to the maximum principle, $v(x,\tau) \geq 0$ in $\widehat{O}$, which contradicts the definition of $\widehat{O}$.

The monotonicity of $x_{1s}^*(\tau)$ and $x_{2s}^*(\tau)$ easily follows by $\partial_\tau v \leq 0$. Now, we show $x_{1s}^*(\tau)$ and $x_{2s}^*(\tau)$ are continuous. Without loss of generality, we only show the continuity of $x_{1s}^*(\tau)$. Suppose not, then there exists a $\tau_0 \leq \tau^*$ such that

$$x_{1s}^*(\tau_0^+) = \lim_{t \to \tau_0^+} x_{1s}^*(t) > \lim_{t \to \tau_0^-} x_{1s}^*(t) = x_{1s}^*(\tau_0^-).$$

It follows

$$\begin{cases} \partial_\tau v - \mathcal{L}_x^2 v = H, \text{ in } (x_{1s}^*(\tau_0^-),\ x_{1s}^*(\tau_0^+)) \times [\,\tau_0, T\,], \\ v(x,\tau_0) = 0, \qquad\qquad \forall x \in [\,x_{1s}^*(\tau_0^-),\ x_{1s}^*(\tau_0^+)\,]. \end{cases}$$

Since $SR \subset \{(x,\tau) : H \geq 0\}$ and $H > 0$ in the interior of $\{(x,\tau) : H \geq 0\}$, $H(x,\tau_0) > 0$ for any $x \in (\,x_{1s}^*(\tau_0^-),\ x_{1s}^*(\tau_0^+)\,)$. It follows that $\partial_\tau v(x,\tau_0) = H(x,\tau_0) > 0$ on the line segment $(\,x_{1s}^*(\tau_0^-),\ x_{1s}^*(\tau_0^+)\,)$, which contradicts $\partial_\tau v \leq 0$ in Ω. By the same method, we can show

$$x_{1s}^*(\tau^*) = x_{2s}^*(\tau^*), \quad \lim_{\tau \to 0^+} x_{1s}^*(\tau) \leq 0, \quad \lim_{\tau \to 0^+} x_{2s}^*(\tau) \geq \ln\left(\frac{1}{2\alpha} + 1\right).$$

On the other hand, $\gamma_1(\tau) \leq x_{1s}^*(\tau) \leq x_{2s}^*(\tau) \leq \gamma_2(\tau) \leq \ln\left(1 + \frac{1}{2\alpha}\right)$, because $SR \subset \{H \geq 0\}$. So, $\lim\limits_{\tau \to 0^+} x_{1s}^*(\tau) = 0$ and $\lim\limits_{\tau \to 0^+} x_{2s}^*(\tau) = \ln\left(\frac{1}{2\alpha} + 1\right)$.

Let us show that $x_{1s}^*(\tau)$ is strictly increasing. Otherwise, there exists some $x_0 > 0$ and $0 \leq \tau_1 < \tau_2 \leq \tau^*$ such that $x_{1s}^*(\tau_1) = x_{1s}^*(\tau_2) = x_0$. Then,

$$\begin{cases} \partial_\tau v - \mathcal{L}_x^2 v = H & \text{in } (\,0, x_0\,] \times (\tau_1, \tau_2\,], \\ v(x_0,\tau) = 0, & \forall \tau \in [\,\tau_1, \tau_2\,]. \end{cases}$$

So, $\partial_\tau v$ satisfies

$$\begin{cases} \partial_\tau(v_\tau) - \mathcal{L}_x^2(v_\tau) = H_\tau \le 0 & \text{in } (\,0, x_0\,] \times (\tau_1\ \tau_2\,], \\ v_\tau(x_0,\tau) = 0,\ \ \forall\, \tau \in [\,\tau_1, \tau_2\,], \qquad v_\tau(x,\tau) \le 0, & \text{in } [\,0, x_0\,] \times [\,\tau_1, \tau_2\,]. \end{cases}$$

Hence, applying the strong maximum principle yields

$$\partial_{x\tau} v(x_0, \tau) > 0, \forall\, \tau \in [\,\tau_1, \tau_2\,]. \tag{2.4.1}$$

On the other hand, the continuity of $\partial_x v$ implies that $\partial_x v(x_0,\tau) = 0$ and $\partial_{x\tau} v(x_0,\tau) = 0$ for any $\tau \in (\tau_1, \tau_2\,]$. So, we get a contradiction. Similarly, we can prove $x^*_{2s}(\tau)$ is strictly decreasing.

The C^∞ smoothness of $x^*_{1s}(\tau)$ and $x^*_{2s}(\tau)$ follows by the standard method in Friedman (1975), since $\partial_\tau v \le 0$ and 0 is the upper obstacle of v.

Now, we prove part ii). Similar to part i), by $\partial_x v \ge 0$, we can define

$$x^*_s(\tau) = \inf\left\{x : v(x,\tau) = 0, 0 < \tau \le T\right\}.$$

First we show $x^*_s(\tau) < \infty$, for any $\tau \in [0, T]$. Thanks to (2.6.11), we can use the idea in Brezis and Friedman (1976). Indeed, we construct an auxiliary function

$$w(x,\tau) = \begin{cases} -\frac{\varepsilon}{R} e^{(2\alpha+1)\tau}(x-R)^2, & \text{if } x_0 \le x \le R, \\ 0, & \text{if } x > R, \end{cases}$$

where x_0 is a sufficient large number.

By the comparison principle, it is easy to verify $w(x,\tau) \le v(x,\tau)$ in $[x_0, +\infty) \times [0, T]$, for sufficiently small ε and large R. Hence, we deduce that $x^*_s(\tau) \le R$ for any $0 \le \tau \le T$.

Thanks to $\partial_\tau v \le 0$, we use a similar argument as in the proof of part i) to show that $x^*_s(\tau)$ is strictly increasing, $\lim\limits_{\tau \to 0^+} x^*_s(\tau) = 0$, $x^*_s(\tau) \in C[\,0, T\,] \cap C^\infty(0, T]$.

Now, let us show $\lim_{\tau\to\infty} x_s^*(\tau) = +\infty$. We need two different methods to tackle the cases of $\alpha = 0$ and $-1/2 < \alpha < 0$.

In the case of $\alpha = 0$, if $\lim_{\tau\to\infty} x_s^*(\tau) = x_\infty < \infty$, then

$$(x_\infty, \infty) \times (0, T] \subset SR \subset \{(x,\tau) : H(x,\tau) \geq 0\}, \forall\, T > 0.$$

However, from Lemma 2.2.1 and (2.1.7),

$$H(x,\tau) = 1+2\partial_x u_1 - 2u_1 \leq 1+2e^x-2\left(e^x - x + \frac{\tau}{2}\right) = 1+2x-\tau < 0, \text{ if } \tau > 1+2x,$$

which results in a contradiction.

In the case of $-1/2 < \alpha < 0$, we construct a supersolution of v as follows.

$$W = 2u_1 - e^{(1+2\alpha)\tau} - 1.$$

It is clear that

$$\begin{cases} \partial_\tau W - \mathcal{L}_x^2 W = H \geq \partial_\tau v - \mathcal{L}_x^2 v \quad \text{in } \Omega, \\ \partial_x W(0,\tau) = 0 = \partial_x v(0,\tau), \quad W(x,0) = 2e^x - 2 \geq 0 = v(x,0). \end{cases}$$

Applying the comparison principle and Lemma 2.2.1 part i), we deduce

$$v(x,\tau) \leq W(x,\tau) \leq 2e^x - \frac{e^{2\alpha x}}{\alpha} - e^{(1+2\alpha)\tau}. \tag{2.4.2}$$

So, for any fixed x^*, we can find a sufficiently large τ^* such that $v(x^*,\tau) < 0$ for any $\tau \geq \tau^*$. Hence it follows from $\partial_x v \geq 0$ that $x_s^*(\tau) > x^*$ for any $\tau \geq \tau^*$. Since the free boundary $x_s^*(\tau)$ is increasing, $\lim_{\tau\to\infty} x_s^*(\tau) = +\infty$ follows.

Next, we show part iii). It suffices to show the boundness of $x_s^*(\tau)$ as τ goes to ∞, and the smoothness of $x_s^*(\tau)$ in the case of $\alpha < -1$ because of the lack of $\partial_\tau v \leq 0$, while the other properties can be derived in the same line of part ii).

Let us first deal with the upper bound of $x_s^*(\tau)$. Denote

$$w = (2u_1 - 1) - 2e^{2x} + \frac{4}{2\alpha+1}\, e^{(2\alpha+1)x}.$$

It is easy to verify that

$$\partial_\tau w - \mathcal{L}_x^2 w = H, \qquad \partial_x w(0,\tau) = 0.$$

Moreover, part i) in Lemma 2.2.1 implies that

$$w < 2e^x - \frac{e^{2\alpha x}}{\alpha} - 2e^{2x} + \frac{4}{2\alpha+1}e^{(2\alpha+1)x} \le e^{2\alpha x}\left(\frac{4}{2\alpha+1} - \frac{1}{\alpha}\right) < 0.$$

So, $v \ge w$ follows by the comparison principle. Since $\partial_x v \ge 0$ for $\alpha \le 0$, we obtain

$$v(x,\tau) \ge v(0,\tau) \ge w(0,\tau) \ge -1 + \frac{4}{2\alpha+1}. \tag{2.4.3}$$

Define

$$w^* = \begin{cases} -X^{-3/2}(x-X)^2, & \text{if } X/2 \le x \le X, \\ 0, & \text{if } x > X. \end{cases}$$

Recalling (2.6.11), we have

$$\partial_\tau w^* - \mathcal{L}_x^2 w^* \le X^{-3/2} - (2\alpha+3)X^{-3/2}(x-X) \le X^{-3/2} + X^{-1/2} < H, \ \ \forall\, x \in [\,X/2, X\,],$$

provided X is large enough. According to (2.4.3), we see that $w^*(X/2,\tau) \le v(X/2,\tau)$ for any $\tau \in [\,0,T\,]$ if X is chosen to be large enough, but independent with T. Again by the comparison principle, we obtain $v \ge w^*$ for any $x \ge X/2$, and then $x_s^*(\tau) \le X$ follows.

Finally, it remains to show the smoothness of $x_s^*(\tau)$ in the case of $\alpha < -1$. The idea of the proof stems from Yi, Yang and Wang (2008). Denote $y = x - \alpha\tau$, $\overline{H}(y,\tau) = H(x,\tau)$ and $\overline{v}(y,\tau) = v(x,\tau)$. It is easy to see that $\overline{v}(y,\tau)$ satisfies

$$\begin{cases} \max\left\{\partial_\tau \overline{v} - \mathcal{L}_y^2 \overline{v} - \overline{H}(y,\tau), \overline{v}\right\} = 0 \text{ in } \Omega_y, \\ \partial_y \overline{v}(-\alpha\tau,\tau) = 0, \qquad \overline{v}(y,0) = 0, \end{cases}$$

where $\mathcal{L}_y^2 \overline{v} = \frac{1}{2}\,\partial_{yy}\overline{v} - \frac{3}{2}\partial_y \overline{v} + (2\alpha+1)\overline{v}$ and $\Omega_y = \{(y,\tau) : y > \alpha\tau, 0 < \tau \le T\ \}$.

In addition, $\partial_y \overline{H} = \partial_x H \geq 0$ and

$$\begin{aligned}\partial_\tau \overline{H} &= \alpha \partial_x H + \partial_\tau H = 2\partial_{x\tau} u_1 + 2\alpha \partial_{xx} u_1 - 2(\alpha+1)(\partial_\tau u_1 + \alpha \partial_x u_1) \\ &\leq 4\alpha \left[\partial_\tau u_1 + \left(\alpha + \frac{1}{2} \right) \partial_x u_1 - \alpha u_1 \right] - 2(\alpha+1)(\partial_\tau u_1 + \alpha \partial_x u_1) \\ &= 2(\alpha - 1)\partial_\tau u_1 + 2\alpha^2(\partial_x u_1 - u_1) - 2\alpha^2 u_1 < 0,\end{aligned}$$

where $\partial_{x\tau} u_1 \leq 0$ and eq (2.1.5) are used in the first inequality, and $\partial_x u_1 - u_1 \leq 0$, $\partial_\tau u_1 > 0$ and $u_1 > 0$ are used in the last inequality.

Thanks to $\partial_y \overline{H} \geq 0$ and $\partial_\tau \overline{H} \leq 0$, it is straightforward to show the new free boundary $y^*(\tau)$ is monotonically increasing and $y^*(\tau) \in C[0,T] \cap C^\infty(0,T]$. Since $x_s^*(\tau) = y^*(\tau) + \alpha\tau$, we obtain $x_s^*(\tau) \in C[0,T] \cap C^\infty(0,T]$, which completes the proof.

The extensive numerical results confirm all the theoretical results in Theorem 2.4.1. Figure 2.1 and 2.2 show the optimal selling boundary in the case of $\alpha > 0$ and $-1 \leq \alpha \leq 0$, respectively. Figure 2.3 shows the optimal selling boundary in the case of $\alpha < -1$ is first increasing and then decreasing to 0 as time goes to maturity T. Moreover, we also numerically examine the dependence of selling boundaries on the volatility σ. Intuitively, larger volatility will imply larger selling region, since the stock has more risk. In Figure 2.4, when σ increases from 0.2 to 0.3, there is a upward shift of both branches in the case of $\alpha > 0$. So, the impact of σ on selling region is non-monotonic, which is a bit counterintuitive.

2.5 Summary

The main results of the chapter are as follows.

- We formulate the optimal stock selling problem as a variational inequality

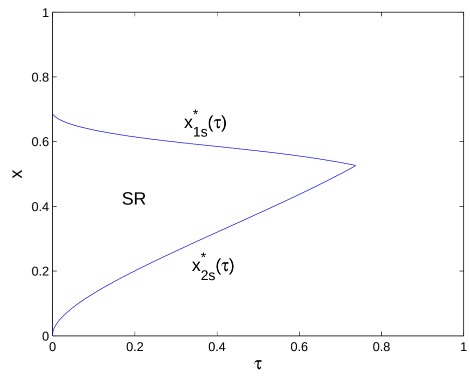

Figure 2.1: Two optimal selling boundaries. Parameter values used: $\mu = 0.045,\ \sigma = 0.3,\ T = 1$.

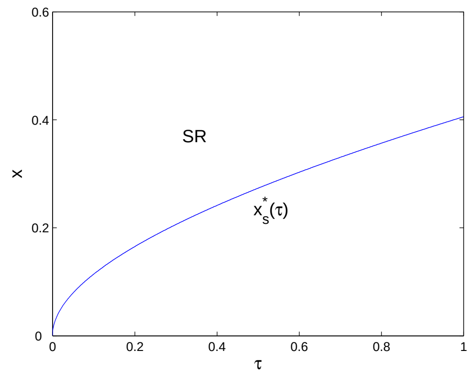

Figure 2.2: The monotonically increasing optimal selling boundary. Parameter values used: $\mu = -0.010,\ \sigma = 0.3,\ T = 1$.

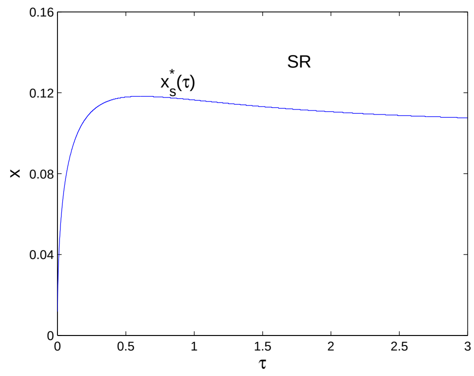

Figure 2.3: The non-monotonic optimal selling boundary. Parameter values used: $\mu = -0.032,\ \sigma = 0.4,\ T = 3.$

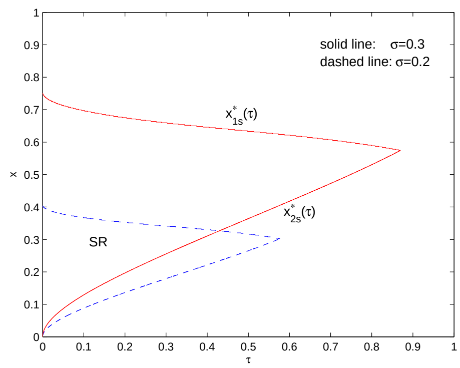

Figure 2.4: The effect of volatility on optimal selling boundaries. Parameter values used: $\mu = 0.040,\ T = 1.$

problem under the assumption of GBM[3].

- The existence and uniqueness of the variational inequality has been established.

- We carefully examine the optimal selling strategies, which heavily depend on sign of the stock return.

2.6 Appendix

2.6.1 Some primary results of PDEs

In this subsection, we will briefly recall some primary results from PDE theory, which can be found in the textbook of PDEs, e.g. Friedman (1964, 1982), Evans (1997), Gilbarg and Trudinger (1998) and Lieberman (2005).

Definition 2.6.1. *The Hölder Space $C^{\alpha}(\overline{\Omega})$ ($0 < \alpha < 1$, Ω open in R^n) consists of all the functions $u(x)$ which are Hölder continuous (exponent α), that is,*

$$H_{\alpha}(u) = \sup_{x,\ y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} < \infty.$$

$C^{\alpha}(\overline{\Omega})$ is a Banach space, with the norm

$$||u||_{\alpha} = ||u||_0 + H_{\alpha}(u),$$

where

$$||u||_0 = \sup_{x \in \Omega} |u(x)|.$$

[3]The PDE formulation would hold for a more general diffusion process, e.g. a mean reverting process with time dependent coefficeints, but the analysis of free boundaries (our main interests), may be much more difficult and thus no conclusion could be made.

Similarly, we say that $u \in C^{m+\alpha}(\overline{\Omega})$ *(m positive integer) if*

$$||u||_{m+\alpha} = \sum_{|\beta| \leq m} ||D^\beta u||_0 + \sum_{|\beta|=m} H_\alpha(D^\beta u) < \infty,$$

where β *is a multi-index and* $D^\beta u$ *is* β*-th derivative w.r.t* x.

In the above definition, let $Q_T = \Omega \times \{0 < t < T\}$ and denote by $C^{\alpha,\alpha/2}(Q_T)$ the space of functions $u(x,t)$ such that

$$H_\alpha(u) = \sup_{(x,t),\ (x',t') \in Q_T} \frac{|u(x,t) - u(x',t')|}{d^\alpha((x,t),(x',t'))} < \infty,$$

where

$$d((x,t),(x',t')) = |x - x'| + |t - t'|^{1/2}$$

is the parabolic distance.

Remark 2.6.2. *If* $\alpha = 0$ *and* 1, *we denote the continuous function space* $C^0(\overline{\Omega})$ *(or* $C(\overline{\Omega})$*) with the maximum normal* $L^\infty(\overline{\Omega})$ *and Lipschitz continuous function space* $C^{0,1}(\overline{\Omega})$ *(or* $C^{0+1}(\overline{\Omega})$*), respectively.*

Definition 2.6.3. *The Sobolev space* $W^{k,p}(\Omega)$ *(k positive integer,* $p \in [1,\infty)$*) consists of all locally integrable function* $u(x)$ *such that for each multi-index* β *with* $|\beta| \leq k$, $D^\beta u \in L^p(\Omega)$.

It can be shown that $W^{k,p}(\Omega)$ is a Banach space with the norm

$$||u||_{W^{k,p}(\Omega)} = \sum_{|\beta| \leq k} ||D^\beta u||_{L^p(\Omega)}.$$

Definition 2.6.4. *The Sobolev space with time variable* t, $W_p^{2k,k}(Q_T)$, *(k positive integer,* $p \in [1,\infty)$*) consists of all locally integrable function* $u(x,t)$ *such that*

$$D^\beta D_t^r u \in L^p(Q_T),\ \textit{for all}\ |\beta| + 2r \leq 2k,$$

where $D^\beta D_t^r u$ *is* β*-th derivative w.r.t.* x *and* r*-th derivative w.r.t.* t.

It can be shown that $W_p^{2k,k}(Q_T)$ is a Banach space with the norm

$$||u||_{W_p^{2k,k}(Q_T)} = \sum_{|\beta|+2r\leq 2k} ||D^\beta D_t^r u||_{L^p(Q_T)}.$$

A Banach space B_1 is said to be continuously imbedded in a Banach space B_2 if there exists a bounded, linear, one-to-one mapping: $B_1 \to B_2$.

Theorem 2.6.5. *(Imbedding Theorem I) Let Ω be a $C^{0,1}$ domain in R^n, $1 \leq p < \infty$. Then,*

i) if $kp < n$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $L^q(\Omega)$, $1 \leq q \leq \frac{np}{n-kp}$, and compactly imbedded in $L^q(\Omega)$, $1 \leq q < \frac{np}{n-kp}$.

ii) if $0 \leq m < k - n/p < m+1$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $C^{m,\alpha}(\overline{\Omega})$, $\alpha \leq k - n/p - m$, and compactly imbedded in $C^{m,\alpha}(\overline{\Omega})$, $\alpha < k - n/p - m$.

A similar result holds for the Sobolev space $W_p^{2k,k}(Q_T)$.

Theorem 2.6.6. *(Imbedding Theorem II) Let Ω be a $C^{0,1}$ domain in R^n, $1 \leq p < \infty$. Then,*

i) if $kp < (n+2)/2$, the space $W_p^{2k,k}(Q_T)$ is continuously imbedded in $L^q(Q_T)$, $1 \leq q \leq \frac{(n+2)p}{n+2-2kp}$, and compactly imbedded in $L^q(Q_T)$, $1 \leq q < \frac{(n+2)p}{n+2-2kp}$.

ii) if $kp > (n+2)/2$, the space $W_p^{2k,k}(Q_T)$ is continuously imbedded in $C^{\alpha,\alpha/2}(\overline{Q_T})$, $0 < \alpha \leq 2 - \frac{n+2}{kp}$, and compactly imbedded in $C^{\alpha,\alpha/2}(\overline{Q_T})$, $\alpha < 2 - \frac{n+2}{kp}$.

In the following, we recall the maximum principle, which plays a critical role in the PDE theory. They have various versions with different assumptions. So, we are only recall the simplest version of the maximum principle. Let us define a linear operators L by

$$Lu = u_t - a^{ij}(X)D_{ij}u + b^i(X)D_iu + c(X)u, \text{ in } Q_T.$$

We assume that L is uniformly parabolic, i.e. $\exists\ \Delta > \delta > 0$ such that $\delta|\xi|^2 \leq a^{ij}(X)\xi_i\xi_j \leq \Delta|\xi|^2$, for all $\xi \in R^n$ and $X \in Q_T$. All the coefficients are continuous and $c(X)$ is lower bounded. Denote by $\partial_p Q_T$ the parabolic boundary, i.e.

$$\partial_p Q_T = \partial\Omega \times (0, T) \cup \overline{\Omega} \times \{0\}. \tag{2.6.4}$$

Lemma 2.6.7. *(Weak maximum principle) Suppose $u \in C^{2,1}(Q_T) \cap C^0(\overline{Q_T})$. If $Lu \geq 0$ and $u \geq 0$ on $\partial_p Q_T$, then $u \geq 0$ on $\overline{Q_T}$.*

A crucial and immediate corollary from the lemma is the following comparison principle, which implies the uniqueness result.

Corollary 2.6.8. *(Comparison principle) Suppose $u, v \in C^{2,1}(Q_T) \cap C^0(\overline{Q_T})$. If $Lu \geq Lv$ and $u \geq v$ on $\partial_p Q_T$, then $u \geq v$ on $\overline{Q_T}$.*

Lemma 2.6.9. *(Strong maximum principle) Suppose $u \in C^{2,1}(Q_T) \cap C^0(\overline{Q_T})$ and Ω is a connected domain. If $Lu \geq 0$ and u attains its non-positive minimum over $\overline{Q_T}$ at an interior point, then u is constant within Q_T.*

2.6.2 The proof of Lemma 2.2.1

Proof. It is easy to see that $\partial_x u_1$ in (2.1.5) is not continuous at $(0, 0)$. So, we need to smooth the initial value of $u_1(x, 0)$. Let us consider the following approximated problem to (2.1.5):

$$\begin{cases} \partial_\tau u_\varepsilon^1 - \mathcal{L}_x^1 u_\varepsilon^1 = 0 \text{ in } \ \Omega, \\ \partial_x u_\varepsilon^1(0, \tau) = 0, \ u_\varepsilon^1(x, 0) = \pi_\varepsilon(x), \end{cases} \tag{2.6.1}$$

where $\pi_\varepsilon(x) = \begin{cases} e^x + \frac{\varepsilon}{4}\left[\left(\frac{x}{\varepsilon}-1\right)^4 - 1\right], & \forall\ 0 \le x \le \varepsilon, \\ e^x - \frac{\varepsilon}{4}, & \forall\ x > \varepsilon. \end{cases}$

It is not hard to check that $\pi_\varepsilon(x) \in C^{3+1}[0,+\infty)$ and $\lim_{\varepsilon\to 0^+}\pi_\varepsilon(x) = e^x$, for any $x \ge 0$. Moreover, for any $x \in [0,\varepsilon]$, $\pi_\varepsilon(x)$ satisfies

$$\begin{cases} e^x - \frac{\varepsilon}{4} \le \pi_\varepsilon(x) \le e^x, \\ \pi'_\varepsilon(x) = e^x + (\frac{x}{\varepsilon}-1)^3, \quad \pi'_\varepsilon(0) = 0, \quad e^x - 1 \le \pi'_\varepsilon(x) \le e^x, \\ \pi''_\varepsilon(x) = e^x + \frac{3}{\varepsilon}(\frac{x}{\varepsilon}-1)^2, \quad \pi'''_\varepsilon(x) = e^x + \frac{6}{\varepsilon^2}(\frac{x}{\varepsilon}-1). \end{cases}$$

Next, we estimate the uniform bound of u^1_ε with respect to ε and prove that u^1_ε converges to u_1.

Denote $w = e^x - \varepsilon e^{\alpha\tau}$ (w is some generic function and possibly different wherever it appears) and it is easy to see

$$\begin{cases} \partial_\tau w - \mathcal{L}^1_x w = 0 \ \text{ in } \Omega, \\ \partial_x w(0,\tau) = 1,\ w(x,0) = e^x - \varepsilon. \end{cases}$$

Then the comparison principle implies

$$u^1_\varepsilon \ge w = e^x - \varepsilon e^{\alpha\tau} \qquad \text{in } \overline{\Omega}. \tag{2.6.2}$$

By the same method, it is easy to check

$$u^1_\varepsilon \ge (1-\varepsilon)e^{\alpha\tau} \qquad \text{in } \overline{\Omega}. \tag{2.6.3}$$

Furthermore, denote $w = e^x - x + \frac{\tau}{2} - \varepsilon$, if $\alpha = 0$. It is also easy to show

$$u^1_\varepsilon \ge w \quad \text{in } \overline{\Omega}. \tag{2.6.4}$$

To achieve a precise upper bound of u^1_ε, we construct different auxiliary functions with respect to α. Firstly, we denote $W = e^x + \frac{e^{2(|\alpha|+1)\tau}}{x+1}$. It is easy to see

$$\begin{cases} \partial_\tau W - \mathcal{L}^1_x W = e^{2(|\alpha|+1)\tau}\left(\frac{2|\alpha|+2}{x+1} - \frac{1}{(x+1)^3} - \frac{2\alpha+1}{2(x+1)^2} - \frac{\alpha}{x+1}\right) > 0 \text{ in } \Omega, \\ \partial_x W(0,\tau) = 1 - e^{2(|\alpha|+1)\tau} \le 0, \quad W(x,0) \ge e^x. \end{cases}$$

So, we conclude

$$u_\varepsilon^1 \le e^x + \frac{e^{2(|\alpha|+1)\tau}}{x+1} \quad \text{in } \overline{\Omega}. \tag{2.6.5}$$

Particularly, we denote $W = e^x - \frac{1}{2\alpha}\, e^{2\alpha x}$, if $\alpha < 0$. Then, by comparison principle, we deduce

$$u_\varepsilon^1 \le W \quad \text{in } \overline{\Omega}. \tag{2.6.6}$$

Thanks to (2.6.2)–(2.6.6), one can easily show that

$$u_\varepsilon^1 \to u_1 \ \text{ in } \ C^{\alpha,\alpha/2}(\overline{\Omega}), \ \ u_\varepsilon^1 \to u_1 \ \text{ in } \ C^k(\overline{\Omega \cap \{0 \le x \le R\} \setminus B_\delta(0,0)}), \ \text{as } \varepsilon \to 0^+,$$

for any $\alpha \in (0,1)$, $k \in Z^+$, $R > \delta > 0$ and $B_\delta(0,0) = \{(x,\tau) : x^2 + \tau^2 \le \delta^2\}$. So, part i) follows by letting $\varepsilon \to 0^+$ in (2.6.2)–(2.6.6).

Now we show part ii). Differentiating (2.6.1) with respect to x and denoting $g_1 = \partial_x u_\varepsilon^1$, we have

$$\begin{cases} \partial_\tau g_1 - \mathcal{L}_x^1 g_1 = 0 \quad \text{in } \Omega, \\ g_1(0,\tau) = 0, \qquad g_1(x,0) = \pi_\varepsilon'(x) \ge e^x - 1 \ge 0. \end{cases} \tag{2.6.7}$$

So, we obtain

$$\partial_x u_\varepsilon^1 = g_1 \ge 0 \quad \text{in } \overline{\Omega}. \tag{2.6.8}$$

Additionally, if $\alpha = 0$, denote $w = e^x - \frac{e^{8\tau}}{(x+1)^3}$, then

$$\begin{cases} \partial_\tau w - \mathcal{L}_x^1 w = -\frac{16(x+1)^2 - 12 - 3(x+1)}{2(x+1)^5}\, e^{8\tau} < 0 \qquad \text{in } \Omega, \\ w(0,\tau) \le 0, \qquad w(x,0) \le \pi_\varepsilon'(x); \end{cases}$$

if $\alpha < 0$, denote $w = e^x - 1$, then

$$\begin{cases} \partial_\tau w - \mathcal{L}_x^1 w = \alpha < 0 \qquad \text{in } \Omega, \\ w(0,\tau) = 0, \qquad w(x,0) \le \pi_\varepsilon'(x). \end{cases}$$

Hence, we deduce

$$\partial_x u_\varepsilon^1 = g_1 \ge w = \begin{cases} e^x - \frac{e^{8\tau}}{(x+1)^3} & \text{if } \alpha = 0, \\ e^x - 1 & \text{if } \alpha \le 0, \end{cases} \qquad \text{in } \overline{\Omega}. \tag{2.6.9}$$

Taking $\varepsilon \to 0^+$ in (2.6.8) and (2.6.9), it follows that

$$\partial_x u_1 \geq \begin{cases} 0 & \text{if } \alpha > 0, \\ \max\{e^x - \frac{e^{8\tau}}{(x+1)^3}, 0\} & \text{if } \alpha = 0, \\ e^x - 1 & \text{if } \alpha \leq 0, \end{cases} \quad \text{in } \overline{\Omega}.$$

Denote $g_2 = \partial_x u_\varepsilon^1 - e^x$, then we have

$$\begin{cases} \partial_\tau g_2 - \mathcal{L}_x^1 g_2 = 0 & \text{in } \Omega, \\ g_2(0,\tau) = -e^x, \quad g_2(x,0) = \pi_\varepsilon'(x) - e^x \leq 0. \end{cases}$$

Similarly, we can infer

$$\partial_x u_\varepsilon^1 - e^x = g_2 \leq 0 \ \text{ in } \overline{\Omega}, \qquad \text{then } \ \partial_x u_1 \leq e^x \ \text{ in } \Omega.$$

Differentiating (2.6.7) with respect to x and denoting $g_3 = \partial_x g_1 = \partial_{xx} u_\varepsilon^1$, we have

$$\begin{cases} \partial_\tau g_3 - \mathcal{L}_x^1 g_3 = 0 & \text{in } \ \Omega, \\ g_3(0,\tau) \geq 0, \qquad g_3(x,0) = \pi_\varepsilon''(x) \geq e^x \geq \pi_\varepsilon'(x). \end{cases}$$

In view of the comparison principle, we get

$$\partial_{xx} u_\varepsilon^1 = g_3 \geq \partial_x u_\varepsilon^1 \quad \text{in } \overline{\Omega}, \text{ then } \ \partial_{xx} u_1 \geq \partial_x u^1 \geq 0 \text{ in } \Omega.$$

Let us move to part iii). Denote $g_4 = \partial_\tau u_\varepsilon^1$. It is easy to see that g_4 satisfies

$$\begin{cases} \partial_\tau g_4 - \mathcal{L}_x^1 g_4 = 0 & \text{in } \ \Omega, \\ \partial_x g_4(0,\tau) = 0, \qquad g_4(x,0) = \mathcal{L}_x^1 \pi_\varepsilon(e^x) = \frac{1}{2}\pi_\varepsilon''(x) - \left(\alpha + \frac{1}{2}\right)\pi_\varepsilon'(x) + \alpha \pi_\varepsilon(x). \end{cases}$$

A close examination shows

$$\frac{1}{2}\pi_\varepsilon''(x) - \left(\alpha + \frac{1}{2}\right)\pi_\varepsilon'(x) + \alpha\pi_\varepsilon(x) \geq -|\,\alpha|\,\varepsilon,$$

provided small enough ε.

Thus, we obtain

$$\partial_\tau u^1_\varepsilon = g_4 \geq -|\,\alpha|\,\varepsilon e^{\alpha\tau} \text{ in } \overline{\Omega}. \tag{2.6.10}$$

Letting $\varepsilon \to 0^+$, we get $\partial_\tau u_1 \geq 0$ in $\overline{\Omega}$. Then, $\partial_\tau u_1 > 0$ in Ω follows by the strong maximum principle.

Differentiating (2.6.7) with respect to τ and denoting $g_5 = \partial_\tau g_1 = \partial_{x\tau} u^1_\varepsilon$, then we have

$$\begin{cases} \partial_\tau g_5 - \mathcal{L}^1_x g_5 = 0 \ \text{ in } \Omega, \\ g_5(0,\tau) = 0, \\ g_5(x,0) = \partial_x(\mathcal{L}^1_x \pi_\varepsilon(e^x)) = \frac{1}{2}\,\pi'''_\varepsilon(x) - \left(\alpha + \frac{1}{2}\right)\pi''_\varepsilon(x) + \alpha\pi'_\varepsilon(x). \end{cases}$$

A close examination shows

$$\frac{1}{2}\,\pi'''_\varepsilon(x) - \left(\alpha + \frac{1}{2}\right)\pi''_\varepsilon(x) + \alpha\pi'_\varepsilon(x) \leq 0,$$

provided small enough ε. Thus, we show

$$\partial_{x\tau} u^1_\varepsilon = g_5 \leq 0 \text{ in } \overline{\Omega} \text{ and } \ \partial_{x\tau} u^1 \leq 0 \text{ in } \Omega.$$

The proof is complete.

2.6.3 The proof of Lemma 2.2.2

Proof. Part i) directly follows by Lemma 2.2.1. Indeed,

$$\partial_\tau H(x,\tau) = 2\partial_{x\tau} u_1 - 2(\alpha+1)\partial_\tau u_1 < 0, \ \text{ if } \alpha \geq -1,$$

and

$$\partial_x H(x,\tau) = 2(\partial_{xx} u_1 - \partial_x u_1) - 2\alpha\partial_x u_1 > 0, \ \text{ if } \alpha \leq 0,$$

where the strictly inequalities follow from the strong maximum principle.

According to Lemma 2.2.1 part i) and part ii), we have

$$H=(2\alpha+1)+2(\partial_x u_1-u_1)-2\alpha u_1\leq 2\alpha+1+2|\,\alpha|\left(e^x+e^{2(|\alpha\,|+1)\tau}\right)\leq 4(|\,\alpha|+1)\left(e^x+e^{2(|\alpha\,|+1)\tau}\right)$$

and

$$H\geq(2\alpha+1)-2(\alpha+1)u_1\geq -4(|\,\alpha|+1)\left(e^x+e^{2(|\alpha\,|+1)\tau}\right).$$

So, part ii) follows.

Next, we prove part iii). We first claim $H(0,\tau)<0$ for any $\tau\in[0,T]$ and $H(x,\tau)>0$ for x large enough. Then combining $\partial_x H>0$ in Ω, we conclude that there exists a curve $\gamma(\tau)<\infty$, such that (2.2.1) holds. Indeed,

$$H(0,\tau)=2\alpha+1-2(\alpha+1)u_1(0,\tau)\leq\begin{cases}2\alpha+1-2(\alpha+1)=-1<0, & \text{if } \alpha\geq -1,\\ 2\alpha+1-2(\alpha+1)(1-\frac{1}{2\alpha})=\frac{1}{\alpha}<0, & \text{if } \alpha<-1.\end{cases}$$

Additionally,

$$\begin{cases}\text{if } \alpha<-1, & H=2\alpha+1+2\partial_x u_1-2(\alpha+1)u_1\geq 2\alpha+1+2(e^x-1)\geq 0,\\ & \forall\, x\geq \ln\left(\frac{1}{2}-\alpha\right);\\ \text{if } -1\leq\alpha<0, & H=2\alpha+1+2\partial_x u_1-2(\alpha+1)u_1\geq 2\alpha+1+2(e^x-1)\\ & -2(\alpha+1)\left(e^x-\frac{1}{2\alpha}\right)\geq -2\alpha e^x+\left(2\alpha+\frac{1}{\alpha}\right)\geq 0,\ \ \forall\, x\geq\ln\left(1+\frac{1}{2\alpha^2}\right);\\ \text{if } \alpha=0, & H=1+2\partial_x u_1-2u_1\geq 1+2\left(\left(e^x-\frac{e^{8\tau}}{(x+1)^3}\right)-\left(e^x+\frac{e^{2\tau}}{x+1}\right)\right)\\ & \geq 1-\frac{2\left(e^{8T}+e^{2T}\right)}{x+1}\geq\frac{1}{2},\ \ \forall\, x\geq 8e^{8T}.\end{cases}\tag{2.6.11}$$

Note that if $\alpha\leq 0$,

$$H(x,0)=2\alpha+1+2e^x-2(\alpha+1)e^x=1-2\alpha(e^x-1)>1,\ \forall x>0.$$

We conclude $\lim_{\tau\to 0^+}\gamma(\tau)=0$. The smoothness of $\gamma(\tau)$ follows from the smoothness of H in Ω. Moreover, $\gamma(\tau)$ is strictly increasing if $-1\leq\alpha\leq 0$, since $\partial_\tau H<0$ in Ω in this case.

Finally, we show part iv). Since H is no longer monotone in x, the proof will be different. Notice that

$$H(x,0) = 1 - 2\alpha(e^x - 1) \begin{cases} > 0 & \text{if } 0 \le x < \ln\left(\frac{1}{2\alpha} + 1\right), \\ = 0 & \text{if } x = \ln\left(\frac{1}{2\alpha} + 1\right), \\ < 0 & \text{if } x > \ln\left(\frac{1}{2\alpha} + 1\right), \end{cases}$$

and

$$H(x,\tau) = 1 + 2(\partial_x u_1 - u_1) - 2\alpha(u_1 - 1) \le 1 - 2\alpha(\max\{e^x,\, e^{\alpha\tau}\} - 1) < 0,$$

provided $x > \ln\left(\frac{1}{2\alpha} + 1\right)$ or $\tau > \frac{1}{\alpha}\ln(\frac{1}{2\alpha}+1)$. We can define

$$\tau_H = \sup\left\{\tau : H(x,\tau) = 0,\ \exists\, x > 0\right\}, \quad \text{then } \ \tau_H \le \frac{1}{\alpha}\ln(\frac{1}{2\alpha}+1).$$

Furthermore, $\partial_\tau H < 0$ in Ω implies that there exists a curve $\Gamma = \tau(x)$, $0 < x \le \ln\left(\frac{1}{2\alpha} + 1\right)$ with $\lim_{x\to 0^+} \tau(x) = 0$ and $\tau\left(\ln\left(\frac{1}{2\alpha} + 1\right)\right) = 0$, such that $H(x,\tau(x)) = 0$, by the implicit function theorem. The smoothness of Γ follows by the smoothness of H in Ω.

Since Γ is enclosed by the rectangle with corners at $(0,0)$, $(\ln\left(\frac{1}{2\alpha} + 1\right), 0)$, $(0,\tau_H)$ and $(\ln\left(\frac{1}{2\alpha} + 1\right), \tau_H)$, we claim that there exists a $x_0 \in \left(0, \ln\left(\frac{1}{2\alpha} + 1\right)\right)$ such that $\tau(x)$ is increasing in $(0, x_0]$ and decreasing in $\left[x_0, \ln\left(\frac{1}{2\alpha} + 1\right)\right]$. Suppose not, then there exist $0 < x_1 < x_2 < \infty$ such that $H_x(x_i,\tau) = 0$, $i = 1,2$, for some $\tau > 0$. However, $H_x(x,\tau)$ satisfies

$$\begin{cases} \partial_\tau(H_x) - \mathcal{L}_x^1(H_x) = 0 & \text{in } \Omega, \\ H_x(0,\tau) = 2\partial_{xx} u_1(0,\tau) > 0,\ H_x(x,0) = -2\alpha e^x < 0. \end{cases}$$

Thus, we conclude that there exists at most one root of $\partial_x H(x,\tau) = 0$ for any $\tau \in (0,T]$, since the number of roots of $\partial_x H(x,\tau) = 0$, for $x \in [0,+\infty)$, is non-increasing in τ, which leads to a contradiction. Therefore, this curve Γ can be rewritten as $x = \gamma_1(\tau)$ and $x = \gamma_2(\tau)$, for any $\tau \in (0,\tau_H]$, which are increasing and decreasing in τ, respectively. Moreover $\gamma_1(\tau_H) = \gamma_2(\tau_H)$. The proof is complete.

Chapter 3

Buy Low and Sell High

In this chapter, four stock buying/selling problems are formulated as optimal stopping problems so as to capture the investment motto "buy low and sell high". The remainder of the chapter is organized as follows. In Section 1 we give mathematical preliminaries needed in solving the four problems, and in Section 2 we present the main results. Some concluding remarks are given in Section 3, while the proofs are relegated to an appendix.

3.1 Preliminaries

As they stand Problems (1.1.4)–(1.1.7) are not standard optimal stopping time problems since they all involve the global maximum and minimum of a stochastic process which are not adapted. In this section we first turn these problems into standard ones, and then present the corresponding free-boundary PDEs for solving them. We do these in two sub-sections for buying and selling respectively.

3.1.1 Buying problems

We start with the buying problem (1.1.4). Denote by M_t the running maximum stock price over $[0,t]$, i.e., $M_t = \max_{0\le\nu\le t} S_\nu$. Then, we have

$$\begin{aligned}
\mathbb{E}\left(\frac{S_\tau}{M_T}\right) &= \mathbb{E}\left(\frac{S_\tau}{\max\{M_\tau, \max_{\tau\le s\le T} S_s\}}\right) = \mathbb{E}\left(\max\left\{\frac{M_\tau}{S_\tau}, \max_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1} \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\max\left\{\frac{M_\tau}{S_\tau}, \max_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1}\Big|\mathscr{F}_\tau\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\min\left\{e^{-x}, e^{-\max_{t\le s\le T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma(B_s-B_t)\right\}}\right\}\right]\Big|_{x=\log\frac{M_\tau}{S_\tau},\ t=\tau}\right] \\
&= \mathbb{E}\left[\Psi\left(\log\frac{M_\tau}{S_\tau}, \tau\right)\right] \qquad (3.1.1)
\end{aligned}$$

where

$$\Psi(x,t) = \mathbb{E}\left[\min\left\{e^{-x}, e^{-\max_{t\le s\le T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma(B_s-B_t)\right\}}\right\}\right], \forall (x,t)\in\Omega^+,$$

and $\Omega^+ = (0,+\infty)\times[0,T)$.

The expression of $\Psi(x,t)$ is as follows: [cf. Shiryaev, Xu and Zhou (2008b)]

$$\Psi(x,t) = \begin{cases} \frac{3\sigma^2-2\mu}{2(\sigma^2-\mu)}e^{(\sigma^2-\mu)(T-t)}\Phi(d_1) + e^{-x}\Phi(d_2) + \frac{\sigma^2}{2(\mu-\sigma^2)}e^{\frac{2(\mu-\sigma^2)x}{\sigma^2}}\Phi(d_3) & \text{if } \mu\ne\sigma^2, \\ e^{-x}\Phi(d_2) + (1+x+\frac{\sigma^2(T-t)}{2})\Phi(d_3) - \sigma\sqrt{\frac{T-t}{2\pi}}e^{-\frac{d_3^2}{2}} & \text{if } \mu=\sigma^2, \end{cases} \qquad (3.1.2)$$

where $d_1 = \frac{-x+(\mu-\frac{3}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = \frac{x-(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_3 = \frac{-x-(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $\Phi(\cdot) = \int_{-\infty}^{\cdot}\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}ds$.

Equation (3.1.1) implies that (1.1.4) is equivalent to a standard optimal stopping problem with a terminal payoff Ψ and an underlying adapted state process

$$X_t = \log\frac{M_t}{S_t},\ X_0 = 0.$$

In view of the dynamic programming approach, we need to consider the following problem

$$V(x,t) = \min_{0\le\tau\le T-t}\mathbb{E}_{t,x}\left(\Psi(X_{\tau+t}, \tau+t)\right), \qquad (3.1.3)$$

where $X_t = x$ under $\mathbb{P}_{t,x}$ with $(x,t) \in \Omega^+$ given and fixed. Obviously, the original problem is $V(0,0) = \min_{0\le\tau\le T} \mathbb{E}\left(\frac{S_\tau}{M_T}\right)$.

It is a standard exercise to show that $V(\cdot,\cdot)$, the value function, satisfies the following free-boundary PDE (also known as the variational inequalities)

$$\begin{cases} \max\{\mathscr{L}V,\, V-\Psi\} = 0, \text{ in } \Omega^+, \\ V_x(0,t) = 0, \; V(x,T) = \Psi(x,T), \end{cases} \tag{3.1.4}$$

where the operator $\mathscr{L}$ is defined by

$$\mathscr{L} = -\partial_t - \frac{\sigma^2}{2}\partial_{xx} - (\frac{1}{2}\sigma^2 - \mu)\partial_x. \tag{3.1.5}$$

Therefore, the buying region for Model (1.1.4) is

$$BR = \left\{(x,t) \in \widetilde{\Omega^+} : V(x,t) = \Psi(x,t)\right\}, \tag{3.1.6}$$

where $\widetilde{\Omega^+} = \Omega^+ \cup \{x = 0\}$.

Let us now turn to the alternative buying model (1.1.5). Denote by m_t the running minimum stock price over $[0,t]$, i.e., $m_t = \min_{0\le\nu\le t} S_\nu$. Then, we have

$$\begin{aligned} \mathbb{E}\left(\frac{S_\tau}{m_T}\right) &= \mathbb{E}\left(\frac{S_\tau}{\min\{m_\tau, \min_{\tau\le s\le T} S_s\}}\right) = \mathbb{E}\left(\min\left\{\frac{m_\tau}{S_\tau}, \min_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1} \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(\min\left\{\frac{m_\tau}{S_\tau}, \min_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1}\Big|\mathscr{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\max\left\{e^{-x}, e^{-\min_{t\le s\le T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma(B_s-B_t)\right\}}\right\}\right]\Big|_{x=\log\frac{m_\tau}{S_\tau},\; t=\tau}\right] \\ &= \mathbb{E}\left[\psi\left(\log\frac{m_\tau}{S_\tau}, \tau\right)\right] \end{aligned}$$

where

$$\psi(x,t) = \mathbb{E}\left[\max\left\{e^{-x}, e^{-\min_{t\le s\le T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma(B_s-B_t)\right\}}\right\}\right], \forall (x,t) \in \Omega^-,$$

and $\Omega^- = (-\infty, 0) \times [0,T)$.

Similar to (3.1.2), we can find the expression of $\psi(x,t)$, $\forall(x,t)\in\Omega^-$, as follows

$$\psi(x,t)=\begin{cases}\frac{3\sigma^2-2\mu}{2(\sigma^2-\mu)}e^{(\sigma^2-\mu)(T-t)}\Phi(d_1'^{-x}\Phi(d_2')+\frac{\sigma^2}{2(\mu-\sigma^2)}e^{\frac{2(\mu-\sigma^2)x}{\sigma^2}}\Phi(d_3') & \text{if } \mu\neq\sigma^2,\\ e^{-x}\Phi(d_2')+(1+x+\frac{\sigma^2(T-t)}{2})\Phi(d_3')-\sigma\sqrt{\frac{T-t}{2\pi}}e^{-\frac{d_3'^2}{2}} & \text{if } \mu=\sigma^2,\end{cases} \tag{3.1.7}$$

where $d_1'=\frac{x-(\mu-\frac{3}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2'=\frac{-x+(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_3'=\frac{x+(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

Thus, we define the associated value function as

$$v(x,t)=\min_{0\leq\tau\leq T-t}\mathbb{E}_{t,x}\left(\psi(X_{\tau+t},\tau+t)\right),$$

where $X_t=\log\frac{m_t}{S_t}=x$ under $\mathbb{P}_{t,x}$ with $(x,t)\in\Omega^-$ given and fixed. The variational inequalities that $v(x,t)$ satisfies are given as follows

$$\begin{cases}\max\{\mathscr{L}v,\ v-\psi\}=0,\ \text{in } \Omega^-,\\ v_x(0,t)=0,\ v(x,T)=\psi(x,T),\end{cases} \tag{3.1.8}$$

where $\mathscr{L}$ is defined by (3.1.5).

The buying region for Model (1.1.5) is, therefore

$$BR=\left\{(x,t)\in\widetilde{\Omega^-}:\ v(x,t)=\psi(x,t)\ \right\}, \tag{3.1.9}$$

where $\widetilde{\Omega^-}=\Omega^-\cup\{x=0\}$.

3.1.2 Selling problems

We now consider the selling problems. In a similar manner, we introduce the value function associated with problem (1.1.6) as follows:

$$U(x,t)=\max_{0\leq\tau\leq T-t}\mathbb{E}\left(\Psi(X_{\tau+t},\tau+t)\right),\forall(x,t)\in\Omega^+.$$

It is also easy to see that U satisfies

$$\begin{cases}\min\{\mathscr{L}U,U-\Psi\}=0,\ \text{in } \Omega^+,\\ U_x(0,t)=0,\ U(x,T)=\Psi(x,T),\end{cases} \tag{3.1.10}$$

where $\mathscr{L}$ and Ψ are as given in (3.1.5) and (3.1.2) respectively. The corresponding selling region is

$$SR = \left\{(x,t) \in \widetilde{\Omega^+} : U(x,t) = \Psi(x,t)\right\}. \tag{3.1.11}$$

For Problem (1.1.7), we introduce the value function

$$u(x,t) = \max_{0\le\tau\le T-t} \mathbb{E}\left(\psi(X_{\tau+t}, \tau+t)\right), \forall (x,t) \in \Omega^-.$$

It is easy to see that u satisfies

$$\begin{cases} \min\{\mathscr{L}u, u-\psi\} = 0, \text{ in } \Omega^-, \\ u_x(0,t) = 0, \ u(x,T) = \psi(x,T), \end{cases} \tag{3.1.12}$$

where $\mathscr{L}$ and ψ are as given in (3.1.5) and (3.1.7) respectively. The corresponding selling region is as follows:

$$SR = \left\{(x,t) \in \widetilde{\Omega^-} : u(x,t) = \psi(x,t)\right\}. \tag{3.1.13}$$

3.2 Optimal Buying and Selling Strategies

In this section we present the main results of this chapter. Again, we divide the section into two sub-sections dealing with the buying and selling decisions respectively.

The following "goodness index" of the stock is crucial in defining whether the stock is "good", "bad" or "intermediate" for the four problems under consideration:

$$\alpha = \frac{\mu}{\sigma^2}.$$

3.2.1 Buying strategies

The following result characterizes the optimal buying strategy for Problem (1.1.4).

Theorem 3.2.1. *(Optimal Buying Strategies against the Highest Price) Let BR be the buying region as defined in (3.1.6).*

i) If $\alpha \leq 0$, then $BR = \emptyset$;

ii) If $0 < \alpha < 1$, then there is a monotonically decreasing boundary $x_b^*(t)$: $[0, T) \to (0, +\infty)$ such that

$$BR = \{(x, t) \in \Omega^+ : x \geq x_b^*(t), 0 \leq t < T\}. \tag{3.2.1}$$

Moreover, $\lim_{t \to T^-} x_b^*(t) = 0$, and

$$\lim_{T-t \to \infty} x_b^*(t) = \begin{cases} \frac{8(1-\alpha)}{(3-2\alpha)(2\alpha-1)} & \text{if } \frac{1}{2} < \alpha < 1, \\ +\infty & \text{if } 0 < \alpha \leq \frac{1}{2}; \end{cases} \tag{3.2.2}$$

iii) If $\alpha \geq 1$, then $BR = \widetilde{\Omega^+}$.

The PDE technique adopted to show the main results is quite similar to Chapter 2. However, some elegant transformations have to be introduced to derive those desired properties. For easy presentation, we place the proof in Appendix 3.4.2.

So, if the buying criterion is to stay away as much as possible from the highest price, then one should never buy if the stock is "bad" ($\alpha \leq 0$), and immediately buy if the stock is "good" ($\alpha \geq 1$). If the stock is somewhat between good and bad ($0 < \alpha < 1$), then one should buy as soon as the ratio between the historical high and the current stock price, M_t/S_t, exceeds certain time-dependent level (or equivalently the current stock price is sufficiently – depending on when the time is – away in proportion from the historical high). Moreover, in this case when the terminal date is sufficiently near one should always buy.

The other buying problem (1.1.5) is solved in the following theorem.

Theorem 3.2.2. *(Optimal Buying Strategies against the Lowest Price) Let BR be the buying region as defined in (3.1.9).*

i) If $\alpha \le 0$, *then* $BR = \emptyset$*;*

ii) If $0 < \alpha < 1$, *then there is a monotonically increasing boundary* $x_b^*(t)$: $[0, T) \to (-\infty, 0)$ *such that*

$$BR = \{(x, t) \in \Omega^- : x \le x_b^*(t), 0 \le t < T\}. \tag{3.2.3}$$

Moreover, $\lim_{t \to T^-} x_b^*(t) = 0$, *and*

$$\lim_{T-t \to \infty} x_b^*(t) = -\infty.$$

iii) If $\alpha \ge 1$, *then* $BR = \widetilde{\Omega^-}$.

We place the proof in Appendix 3.4.3.

The above results suggest that, if the buying criterion is to buy at a price as close to the lowest price as possible, then one should never buy if the stock is "bad" ($\alpha \le 0$), and immediately buy if the stock is "good" ($\alpha \ge 1$). If the stock is intermediate between good and bad ($0 < \alpha < 1$), then one should buy as soon as the ratio between the current price and the historical low, S_t/m_t, exceeds certain time-dependent level (or the current stock price is sufficiently away in proportion from the historical low). Moreover, in this case when the terminal date is sufficiently near one should always buy. On the other hand, one tend not to buy if the duration of the investment horizon is exceedingly long.

Comparing Theorems 3.2.1 and 3.2.2, we find that the two buying models (1.1.4) and (1.1.5), albeit different in formulation, produce quite similar trading behaviors. The only difference lies in the (endogenous) criteria (those in terms of M_t/S_t and S_t/m_t) to be used to trigger buying for the intermediate case $0 < \alpha < 1$.

3.2.2 Selling strategies

The first selling problem (1.1.6) is solved in the following theorem.

Theorem 3.2.3. *(Optimal Selling Strategies against the Highest Price) Let SR be the optimal selling region as defined in (3.1.11).*

i) If $\alpha > \frac{1}{2}$, then $SR = \emptyset$.

ii) If $\alpha = \frac{1}{2}$, then $SR = \{(x,t) : x = 0\}$.

iii) If $0 < \alpha < \frac{1}{2}$, then $\{(x,t) : x = 0\} \subset SR$. Moreover, there exists a boundary $x_s^*(t) : [0,T) \to [0,+\infty)$ such that

$$SR = \left\{(x,t) \in \widetilde{\Omega^+} : x \le x_s^*(t)\right\}, \tag{3.2.4}$$

and $x_s^*(t) \le x_b^*(t) < \infty$, where $x_b^*(t)$ is defined in (3.2.1).

iv) If $\alpha \le 0$, then $SR = \widetilde{\Omega^+}$.

The proof is placed in Appendix 3.4.4.

The above results indicate that, apart from the definitely holding case $\alpha > \frac{1}{2}$ and the immediately selling case $\alpha \le 0$, there is an intermediate case $0 < \alpha \le \frac{1}{2}$ where one should sell only when M_t/S_t is sufficiently small. At $t = 0$ this quantity is automatically the smallest; hence one should also sell at $t = 0$ if $0 < \alpha \le \frac{1}{2}$. This therefore fills the gap case $0 < \alpha \le \frac{1}{2}$ missing in Shiryaev, Xu and Zhou (2008b). Thus, one may conclude the selling strategy is bang-bang here, i.e. either hold the stock to time T or sell at time 0. However, our results are much stronger than that. Suppose the investor define the benchmark to be $\max\{1.3S_0, M_T\}$, then we are still able to illustrate the optimal strategy. Figure 3.1 shows that the red point is the new starting point and the optimal selling strategy depends on the ratio between the current price and the benchmark $\max\{1.3S_0, M_t\}$.

On the other hand, Figure 3.1 also shows that $x_s^*(t)$ is monotonically decreasing and $x_s^*(t) > 0$ when $0 < \alpha < \frac{1}{2}$, although we are yet to be able to establish these analytically. Moveover, $\lim_{T-t\to\infty} x_s^*(t) = \frac{1}{2\alpha-1}\log\left(1-(2\alpha-1)^2\right)$, which can be shown as (3.2.2), is confirmed by our numerical results.

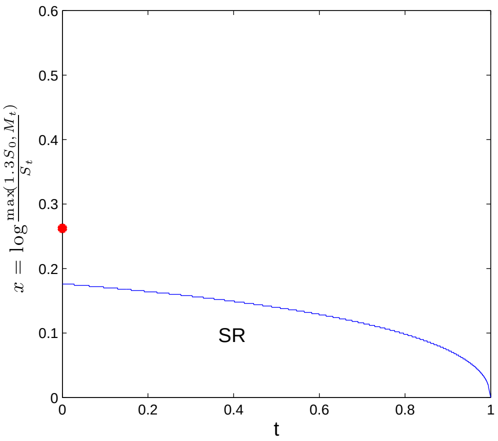

Figure 3.1: Optimal Selling Strategy When $0 < \alpha < 0.5$.

The following is on Problem (1.1.7).

Theorem 3.2.4. *(Optimal Selling Strategies against the Lowest Price) Let SR be the optimal selling region as defined in (3.1.13).*

i) If $\alpha > \frac{1}{2}$, then $SR = \emptyset$.

ii) If $\alpha = \frac{1}{2}$, then $SR = \{x = 0\}$.

iii) If $0 < \alpha < \frac{1}{2}$, then $\{x = 0\} \subset SR$. Moreover, there exists a boundary $x_s^*(t) : [0, T) \to (-\infty, 0]$ such that

$$SR = \left\{(x,t) \in \widetilde{\Omega^-} : x \geq x_s^*(t)\right\}, \tag{3.2.5}$$

and $x_s^*(t) \geq x_b^*(t) > -\infty$, where $x_b^*(t)$ is defined in (3.2.3).

iv) If $\alpha \leq 0$, then $SR = \widetilde{\Omega^-}$.

The proof is the same as that for Theorem 3.2.3. We omit it here.

The trading behavior derived from this model is virtually the same as the other selling model at time $t = 0$.

Incidentally, numerical results show that $x_s^*(t)$ is monotonically increasing. However, it is an open problem to establish $\lim_{T-t\to\infty} x_s^*(t)$ since the solution to the stationary problem is not unique.

3.3 Summary

The main results of this chapter are as follows.

- We formulate four stock buying/selling problems as optimal stopping problems so as to capture the investment motto "buy low and sell high".
- The free boundary PDE approach, as opposed to the probabilistic approach taken by Shiryaev, Xu and Zhou (2008b), is employed to solve all the problems thoroughly.
- We derive the optimal trading strategies in all cases for the four problems which are simple and consistent with the normal investment behaviors.

3.4 Appendix

3.4.1 Some transformations

Before proving our main results, let us present some transformations, which play a critical role in the analysis and are also used by Dai and Zhong (2011):

$$F(x,t) = \log\left(\Psi(x,t)\right), \ \overline{V}(x,t) = \log\left(\frac{V(x,t)}{\Psi(x,t)}\right), \text{ and } \overline{U}(x,t) = \log\left(\frac{U(x,t)}{\Psi(x,t)}\right), \tag{3.4.1}$$

and introduce two lemmas which are useful for both buying and selling cases. Without loss of generality, we assume that $\sigma = 1$, as one could make a change of time $\bar{t} = \sigma^2 t$, otherwise. Thus, we use α instead of μ in the following.

A direct calculation ([cf. Shiryaev, Xu and Zhou 2008b]) shows that $\Psi(x,t)$ satisfies

$$\begin{cases} \mathscr{L}\Psi = \Psi_x + (1-\alpha)\,\Psi, \text{ in } \Omega^+, \\ \Psi_x(0,t) = 0, \Psi(x,T) = e^{-x}. \end{cases} \tag{3.4.2}$$

Then, $F(x,t)$ satisfies

$$\begin{cases} -F_t - \frac{1}{2}(F_{xx} + F_x^2) + (\alpha - \frac{3}{2})F_x + (\alpha - 1) = 0, \text{ in } \Omega^+, \\ F_x(0,t) = 0, \ F(x,T) = -x. \end{cases} \tag{3.4.3}$$

Accordingly, (3.1.4) and (3.1.10) reduce to

$$\begin{cases} \max\{\mathscr{L}_0\,\overline{V} + F_x - (\alpha - 1), \ \overline{V}\} = 0, \ \text{ in } \Omega^+, \\ \overline{V}_x(0,t) = 0, \ \overline{V}(x,T) = 0, \end{cases} \tag{3.4.4}$$

and

$$\begin{cases} \min\{\mathscr{L}_0\,\overline{U} + F_x - (\alpha - 1), \ \overline{U}\} = 0, \ \text{ in } \Omega^+, \\ \overline{U}_x(0,t) = 0, \ \overline{U}(x,T) = 0, \end{cases} \tag{3.4.5}$$

respectively, where $\mathscr{L}_0 = -\partial_t - \frac{1}{2}\left[\partial_{xx} + (\partial_x)^2 + 2F_x\partial_x\right] + (\alpha - \frac{1}{2})\partial_x$.

As a result, BR (3.1.6) and SR (3.1.11) can be rewritten as

$$BR = \{(x,t) \in \widetilde{\Omega^+} : \overline{V} = 0\} \text{ and } SR = \{(x,t) \in \widetilde{\Omega^+} : \overline{U} = 0\}.$$

Lemma 3.4.1. *Let $\overline{V}(x,t)$ and $\overline{U}(x,t)$ be the solutions to Problems (3.4.4) and (3.4.5), respectively. Then,*

$$\overline{V}(x,t) < \overline{U}(x,t) \ \textit{in} \ \Omega^+.$$

Proof. It is easy to see that $\mathscr{L}_0\overline{V} \le \mathscr{L}_0\overline{U}$ in Ω^+. Applying the strong maximum principle gives the result. □

Lemma 3.4.2. *Suppose $F(x,t)$ is the solution to (3.4.3). Then $F(x,t)$ has the following properties:*

i) $-1 < F_x(x,t) < 0, \forall \ (x,t) \in \Omega^+$;

ii) $F_{xx}(x,t) \leq 0, \forall\ (x,t) \in \Omega^+$;

iii) $F_{xt}(x,t) \leq 0, \forall\ (x,t) \in \Omega^+$;

iv) $F_x(x,t;\alpha+\delta) \leq F_x(x,t;\alpha)+\delta, \forall\ \delta > 0,\ (x,t) \in \Omega^+$;

v) $\lim_{x\to\infty} F_x(x,t) = -1, \forall\ t \in [0,T)$.

Proof. Denote $\widetilde{F}(x,t) = F_x(x,t)$, and $F^{xx}(x,t) = F_{xx}(x,t)$. It is easy to verify that $\widetilde{F}$ and F^{xx} satisfy

$$\begin{cases} -\widetilde{F}_t - \frac{1}{2}(\widetilde{F}_{xx} + 2\widetilde{F}\widetilde{F}_x) + (\alpha - \frac{3}{2})\widetilde{F}_x = 0, \text{ in } \Omega^+, \\ \widetilde{F}(0,t) = 0,\ \widetilde{F}(x,T) = -1, \end{cases}$$

and

$$\begin{cases} -F^{xx}_t - \frac{1}{2}(F^{xx}_{xx} + 2F_x F^{xx}_x + 2(F^{xx})^2) + (\alpha - \frac{3}{2})F^{xx}_x = 0,\ \text{in } \Omega^+, \\ F^{xx}(0,t) \leq 0,\ F^{xx}(x,T) = 0, \end{cases}$$

respectively. By virtue of the (strong) maximum principle, we obtain parts i) and ii).

To show part iii), let us define $F_x^{-\delta}(x,t) = F_x(x,t-\delta)$. It suffices to show $G(x,t) = F_x^{-\delta}(x,t) - F_x(x,t) \geq 0$, $\forall \delta \geq 0$. It is easy to verify that $G(x,t)$ satisfies

$$\begin{cases} -G_t - \frac{1}{2}\left(G_{xx} + 2F_x^{-\delta}G_x + 2F_{xx}G\right) + (\alpha - \frac{3}{2})G_x = 0,\ \text{in } (-\infty,0) \times [\delta,T), \\ G(0,t) = 0,\ G(x,T) = F_x(x,T-\delta) + 1 \geq 0. \end{cases}$$

Thanks to the maximum principle, we see $G(x,t) \geq 0$, in $(-\infty,0] \times [\delta,T)$, which results in $F_{xt}(x,t) \leq 0$.

Next we prove part iv). Denote $\widetilde{F}^\delta(x,t) = F_x(x,t;\alpha+\delta)$ and $\widehat{F}(x,t) = F_x(x,t;\alpha)+\delta$. Let $P = \widetilde{F}^\delta - \widehat{F}$. Then, it suffices to show $P < 0$ in Ω^+. It is easy to check that $\widetilde{F}^\delta(x,t)$ and $\widehat{F}(x,t)$ satisfy

$$\begin{cases} -\widetilde{F}^\delta_t - \frac{1}{2}(\widetilde{F}^\delta_{xx} + 2\widetilde{F}^\delta\widetilde{F}^\delta_x) + (\alpha + \delta - \frac{3}{2})\widetilde{F}^\delta_x = 0, \text{ in } \Omega^+, \\ \widetilde{F}^\delta(0,t) = 0,\ \widetilde{F}^\delta(x,T) = -1 \end{cases} \tag{3.4.6}$$

and

$$\begin{cases} -\widehat{F}_t - \frac{1}{2}(\widehat{F}_{xx} + 2\widehat{F}\widehat{F}_x) + (\alpha + \delta - \frac{3}{2})\widehat{F}_x = 0, \text{ in } \Omega^+, \\ \widehat{F}(0,t) = \delta, \ \widehat{F}(x,T) = -1 + \delta, \end{cases} \tag{3.4.7}$$

respectively. Subtracting (3.4.7) from (3.4.6), we obtain

$$\begin{cases} -\mathbb{P}_t - \frac{1}{2}(P_{xx} + \widehat{F}P_x + \widetilde{F}_x^{\delta}P) + (\alpha + \delta - \frac{3}{2})P_x = 0, \text{ in } \Omega^+, \\ P(0,t) = -\delta, \ P(x,T) = -\delta. \end{cases}$$

Applying the maximum principle gives the desired result.

To show part v), note that

$$\Phi(-\frac{x}{a}) \sim O\left(\frac{1}{x}e^{-\frac{x^2}{2a^2}}\right), \text{ as } x \to +\infty, \forall a > 0.$$

It follows

$$\lim_{x\to+\infty} F_x(x,t) = \lim_{x\to+\infty} \frac{\Psi_x(x,t)}{\Psi(x,t)} = \lim_{x\to+\infty} \frac{-e^{-x}\Phi(d_2) + e^{2(\alpha-1)x}\Phi(d_3)}{\Psi(x,t)} = -1,$$

which completes the proof. □

Due to Lemma 3.4.2 part ii) and iii), we can have the following proposition. We prove it by penalty method which has been applied in Chapter 2.

Proposition 3.4.3. *The variational inequality problem (3.4.4) has a unique solution $\overline{V}(x,t) \in W_p^{2,1}(\Omega_N^+)$, $1 < p < +\infty$, where Ω_N^+ is any bounded set in Ω^+. Moreover, $\forall (x,t) \in \Omega^+$, we have*

i) $0 \le \overline{V}_x \le 1$;

ii) $\overline{V}_t \ge 0$;

iii) $\overline{V}(x,t;\alpha) \le \overline{V}(x,t;\alpha+\delta)$ *for* $\delta > 0$;

Proof. Using the penalization approach [cf. Friedman (1982)], it is not hard to show that (3.4.4) has a unique solution $\overline{V}(x,t) \in W_p^{2,1}(\Omega_N^+), 1 < p < +\infty$, where

Ω_N^+ is any bounded set in Ω^+. To prove part i), we only need to confine to the non-coincidence set $\Lambda = \{(x,t) \in \Omega^+ : \overline{V} < 0\}$. Denote $w = \overline{V}_x$ and $\omega = \overline{V}_x - 1$, then w and ω satisfy

$$\begin{cases} -w_t - \frac{1}{2}(w_{xx} + 2ww_x + 2F_{xx}w + 2F_x w_x) + (\alpha - \frac{1}{2})w_x = -F_{xx}, \quad \text{in } \Lambda \\ w|_{\partial\Lambda} = 0, \end{cases}$$

and

$$\begin{cases} -\omega_t - \frac{1}{2}(\omega_{xx} + 2\omega\omega_x + 2F_{xx}\omega + 2F_x\omega_x) + (\alpha - \frac{3}{2})\omega_x = 0, \text{ in } \Lambda \\ \omega|_{\partial\Lambda} = -1, \end{cases}$$

respectively. Since $-F_{xx} \geq 0$, one can deduce $w \geq 0$ and $\omega \leq 0$ in Λ by the maximum principle, which is desired.

To show part ii), we denote $\widetilde{V}(x,t) = \overline{V}(x, t-\delta)$. It suffices to show $Q(x,t) = \widetilde{V}(x,t) - \overline{V}(x,t) \leq 0$ in Ω^+, $\forall\ \delta \geq 0$. If it is false, then

$$\Delta = \{(x,\tau) \in \Omega : Q(x,t) > 0\} \neq \emptyset.$$

It is easy to verify that $Q(x,t)$ satisfies

$$\begin{cases} -Q_t - \frac{1}{2}\left(Q_{xx} + (\widetilde{V}_x + \overline{V}_x)Q_x + 2F_x Q_x\right) + (\alpha - \frac{1}{2})Q_x \\ \qquad\qquad\qquad\qquad \leq -\delta F_{xt}(\cdot,\cdot)(\widetilde{V}_x - 1) \leq 0, \qquad \text{in } \Delta, \\ Q|_{\partial\Delta} = 0, \end{cases}$$

where we have used part iii) of Lemma 3.4.2 and $\widetilde{V}_x \leq 1$. Applying the maximum principle, we get $Q \leq 0$ in Δ, which contradicts the definition of Δ.

At last, let us show part iii). If it is not true, then

$$\mathscr{O} = \{(x,t) \in \Omega : H(x,t) < 0\} \neq \emptyset,$$

where $H(x,t) = \overline{V}(x,t;\alpha+\delta) - \overline{V}(x,t)$. Denote $F_x^\delta(x,t) = F_x(x,t;\alpha+\delta)$. It can be verified that

$$\begin{cases} -H_t - \frac{1}{2}(H_{xx} + H_x^2 + 2\overline{V}_x H_x + 2F_x^\delta H_x) + (\alpha + \delta - \frac{1}{2})H_x \\ \qquad\qquad\qquad\qquad \geq -(F_x^\delta - F_x - \delta)(1 - \overline{V}_x) \quad \text{in } \mathscr{O}, \\ H|_{\partial\mathscr{O}} = 0. \end{cases}$$

By part iv) in Lemma 3.4.2, $F_x^\delta < F_x + \delta$, which, together with $\overline{V}_x \le 1$, gives

$$-(F_x^\delta - F_x - \delta)(1 - \overline{V}_x) \ge 0.$$

Again applying the maximum principle, we get $H \ge 0$ in $\mathscr{O}$, which is a contradiction with the definition of $\mathscr{O}$. The proof is complete. $\square$

3.4.2 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1. According to part i) in Lemma 3.4.2 and (3.4.4),

$$\mathscr{L}_0\overline{V} \le -(F_x + 1) + \alpha < 0, \text{ for } \alpha \le 0.$$

Applying the strong maximum principle, we infer $\overline{V} < 0$ in Ω for $\alpha \le 0$. Part i) then follows.

If $\alpha \ge 1$, part i) in Lemma 3.4.2 leads to

$$F_x - (\alpha - 1) \le 0.$$

So, $\overline{V} = 0$ is a solution to (3.4.4), which implies part iii).

It remains to show part ii). Since $\overline{V}_x \ge 0$, we can define a boundary

$$x_b^*(t) = \inf\{x \in (0, \infty) : \overline{V}(x,t) = 0\}, \text{ for any } t \in [0, T).$$

Due to $\overline{V}_t \ge 0$, we infer that $x_b^*(t)$ is monotonically decreasing in t. Let us prove $x_b^*(t) > 0$ for all t. If not, then there exists a $t_0 < T$, such that $x_b^*(t) = 0$ for all $t \in [t_0, T)$. This leads to $\overline{V}(x,t) = 0$, in $(0, +\infty) \times [t_0, T)$. By (3.4.4), we have $\mathscr{L}_0\overline{V} + F_x - (\alpha - 1) \le 0$ in $(0, +\infty) \times [t_0, T)$, namely,

$$F_x \le \alpha - 1 \text{ in } (0, +\infty) \times [t_0, T),$$

which is a contradiction with $F_x(0,t) = 0, \forall t > 0$.

Further, it can be shown that $x_b^*(t) < \infty$ in terms of the standard argument of Brezis and Friedman (1976), where $\lim_{x\to+\infty} F_x(x,t) = -1$ will be used. Indeed, due to $\lim_{x\to\infty} F_x(x,\tau) = -1$, we see there exists an $x^* > 0$, such that $\sigma^2 F_x - (\alpha - \sigma^2) < -\frac{\alpha}{2}$ uniformly in τ, if $x > x^*$, where $F_{x\tau}(x,\tau) \le 0$ is used. We now construct an auxiliary function $W(x) = \begin{cases} -\frac{\varepsilon(R-x)^2}{R} & \text{if } x^* < x < R, \\ 0 & \text{if } x \ge R, \end{cases}$ and aim to show $W(x) \le \overline{V}(x,\tau)$, $\forall x > x^*$, where ε and R will be determined later.

Note that if $x^* < x < R$,

$$\begin{aligned} \mathscr{L}_0 W - \frac{\alpha}{2} &\le -\frac{\sigma^2\varepsilon}{R} - \frac{2\sigma^2 F_x \varepsilon(R-x)}{R} + (\alpha - \frac{\sigma^2}{2})\frac{2\varepsilon(R-x)}{R} - \frac{\alpha}{2} \\ &\le -\frac{\sigma^2\varepsilon}{R} + 2\sigma^2|F_x|\varepsilon + 2\Big|\alpha - \frac{\sigma^2}{2}\Big|\varepsilon - \frac{\alpha}{2} < 0, \end{aligned}$$

by choosing ε sufficiently small.

It is easy to see $\mathscr{L}_0 W - \frac{\alpha}{2} = -\frac{\alpha}{2} < 0$, if $x \ge R$. So that

$$\mathscr{L}_0 W + \sigma^2 F_x - (\alpha - \sigma^2) < 0, \ \forall x > x^*.$$

Since we can always choose R sufficiently large such that

$$W(x^*) < \overline{V}(x^*,\tau), \forall \tau \in [0,T],$$

we obtain $W(x) \le \overline{V}(x,\tau)$ in $[x^*,\infty) \times (0,T]$, by the comparison principle, which implies $x_b^*(\tau) \le R < +\infty$.

In addition, due to the monotonicity of $x_b^*(t)$, we deduce that $\{x = 0\} \notin BR$. So, (3.2.1) follows.

To show $\lim_{t\to T^-} x_b^*(t) = 0$, let us assume the contrary, i.e., $\lim_{t\to T^-} x_b^*(t) = x_0 > 0$. Then, we have

$$\mathscr{L}_0 \overline{V} + F_x - (\alpha - 1) = 0, \ \forall\ 0 < \ x < x_0, 0 \le t < T,$$

which, combined with $\overline{V}(x,T) = 0$ for all x, gives

$$\overline{V}_t|_{t=T} = F_x|_{t=0} - (\alpha - 1) = -\alpha < 0, \ \forall\ 0 < x < x_0.$$

This conflicts with $\overline{V}_t \geq 0$.

At last, we need to prove (3.2.2), which leads us to consider a stationary problem of (3.1.4). Here, we prove the case of $\frac{1}{2} < \alpha < 1$, while the case of $0 < \alpha \leq \frac{1}{2}$ can be done similarly.

Noting that $\lim_{T-t\to+\infty} \Psi(x,t) = 0$, if $\frac{1}{2} < \alpha < 1$, it follows that

$$\lim_{T-t\to+\infty} V(x,t) = 0,$$

which is not desired. Thus, we define

$$\begin{aligned}
\Psi^{\infty}(x) &= \lim_{t\to\infty} \sqrt{2\pi}(T-t)^{\frac{3}{2}} e^{\frac{(\alpha-\frac{1}{2})^2}{2}(T-t)} \Psi(x,t), \\
V^{\infty}(x) &= \lim_{t\to\infty} \sqrt{2\pi}(T-t)^{\frac{3}{2}} e^{\frac{(\alpha-\frac{1}{2})^2}{2}(T-t)} V(x,t).
\end{aligned}$$

A direct calculation shows

$$\Psi^{\infty}(x) = \frac{2\left(x + \frac{2}{3-2\alpha}\right)}{\left(\alpha - \frac{1}{2}\right)^2 \left(\frac{3}{2} - \alpha\right)} e^{-(\frac{3}{2}-\alpha)x}. \tag{3.4.8}$$

According to (3.1.4), we see $V^{\infty}(x)$ satisfies,

$$\begin{cases}
-\frac{1}{2}V^{\infty}_{xx} + (\alpha - \frac{1}{2})V^{\infty}_x - \frac{1}{2}(\alpha - \frac{1}{2})^2 V^{\infty} = 0, \ 0 < x < x_{\infty}, \\
V^{\infty}_x(0) = 0, \ V^{\infty}(x_{\infty}) = \Psi^{\infty}(x_{\infty}), \quad V^{\infty}_x(x_{\infty}) = \Psi^{\infty}_x(x_{\infty}),
\end{cases} \tag{3.4.9}$$

where x_{∞} is the free boundary.

It is easy to check that

$$V^{\infty}(x) = \begin{cases}
\frac{2}{(\alpha-\frac{1}{2})^3}(\frac{2}{2\alpha-1} - x)e^{\frac{2\alpha-1}{2}x - x_{\infty}}, & \text{if } 0 \leq x < x_{\infty}, \\
\Psi^{\infty}(x), & \text{if } x \geq x_{\infty},
\end{cases}$$

satisfies (3.4.9), where $x_{\infty} = \frac{8(1-\alpha)}{(3-2\alpha)(2\alpha-1)}$. The proof is complete. □

3.4.3 Proof of Theorem 3.2.2

Proof of Theorem 3.2.2. The proof is similar to Theorem 3.2.1. The only difference is that now we have $\overline{v}_x \leq 0$ instead of $0 \leq \overline{V}_x \leq 1$, where $\overline{v}(x,t) = \log \frac{v(x,t)}{\psi(x,t)}$. Thus, we can define the free boundary as

$$x_b^*(t) = \sup\{x \in (-\infty, 0) : \overline{v}(x,t) = 0\}, \text{ for any } t \in [0,T). \tag{3.4.10}$$

The existence follows immediately. Now, we turn to the asymptotic behavior of $x_b^*(t)$.

Note that $\lim_{t\to+\infty} \psi(x,t) = +\infty$, if $0 < \alpha < 1$. So, we denote

$$\psi^\infty(x) = \lim_{T-t\to\infty} e^{(\alpha-1)(T-t)}\psi(x,t) = \frac{3-2\alpha}{2(1-\alpha)}, \text{ and } v^\infty(x) = \lim_{T-t\to\infty} e^{(\alpha-1)(T-t)}v(x,t).$$

According to (3.1.4), it is easy to see $v^\infty(x)$ satisfies

$$\begin{cases} -\frac{1}{2}v^\infty_{xx} + (\alpha - \frac{1}{2})v^\infty_x - (\alpha-1)v^\infty = 0, \ x_\infty < x < 0, \\ v^\infty_x(0) = 0, \ v^\infty_x(x_\infty) = 0, \quad v^\infty(x_\infty) = \frac{3-2\alpha}{2(1-\alpha)}, \end{cases} \tag{3.4.11}$$

where x_∞ is the free boundary.

The general solution to (3.4.11) is

$$v^\infty(x) = Ae^x + Be^{2(\alpha-1)x}, \ x_\infty \le x \le 0. \tag{3.4.12}$$

Due to $v^\infty_x(0) = 0$, (3.4.12) can be rewritten as

$$v^\infty(x) = A\left(e^x - \frac{1}{2(\alpha-1)}e^{2(\alpha-1)x}\right), \ x_\infty \le x \le 0, \tag{3.4.13}$$

which results in

$$v^\infty_x(x) = A(e^x - e^{2(\alpha-1)x}).$$

Due to $v^\infty_x(x_\infty) = 0$, we obtain $A = 0$ and $v^\infty(x) = 0$ for all x, which contradicts $v^\infty(x_\infty) = \frac{3-2\alpha}{2(1-\alpha)}$. Thus, there is no finite free boundary x_∞, i.e., $x_\infty = -\infty$. Since $\alpha < 1$, and $0 \le \lim_{x\to-\infty} v^\infty(x) \le \frac{3-2\alpha}{2(1-\alpha)}$, we deduce $A = 0$ by (3.4.13), i.e., $v^\infty(x) \equiv 0 < \psi^\infty(x)$. The proof is complete. □

3.4.4 Proof of Theorem 3.2.3

Similar to Theorem 3.2.1, we can deal with the cases of $\alpha \leq 0$ and $\alpha \geq 1$ easily. However, the case of $0 < \alpha < 1$ is more challenging because we no longer have the monotonicity of $\overline{U}$ w.r.t. t. To overcome the difficulty, we introduce an auxiliary problem:

$$\begin{cases} \mathscr{L}_0\overline{U}^* + F_x - (\alpha - 1) = 0, \text{ in } \Omega^+, \\ \overline{U}^*_x(0,t) = 0, \overline{U}^*(x,T) = 0. \end{cases} \tag{3.4.14}$$

Lemma 3.4.4. *Let $\overline{U}^*(x,t)$ be the solution to (3.4.14). Then for any $t \in [0,T)$,*

$$\begin{aligned} \overline{U}^*(0,t) &> 0 \text{ if } \alpha > \frac{1}{2}, \\ \overline{U}^*(0,t) &= 0 \text{ if } \alpha = \frac{1}{2}, \\ \overline{U}^*(0,t) &< 0 \text{ if } \alpha < \frac{1}{2}. \end{aligned}$$

Proof. $\overline{U}^*(x,t) = \log\left(\frac{U^*(x,t)}{\Psi(x,t)}\right)$ is the solution to (3.4.14), where $U^*(x,t)$ is the value function associated with a simple strategy: holding the stock until expiry T, i.e. $U^*(x,t) = \mathbb{E}\left(\frac{S_T}{M_T} \mid \log\frac{M_t}{S_t} = x\right)$. According to Shiryaev, Xu and Zhou (2008b), Lemma 3.4.4 automatically follows at $x = 0$. □

Proposition 3.4.5. *Problem (3.4.5) has a unique solution $\overline{U}(x,t) \in W_p^{2,1}(\Omega_N^+)$, $1 < p < +\infty$, where Ω_N^+ is any bounded set in Ω^+. Moreover, for any $(x,t) \in \Omega^+$,*

i) $0 \leq \overline{U}_x \leq 1$;

ii) $\overline{U}(x,t;\alpha) \leq \overline{U}(x,t;\alpha+\delta)$ *for* $\delta > 0$;

iii) $\overline{U}(x,t) = \overline{U}^*(x,t) > 0$ *for* $\alpha \geq \frac{1}{2}$. *And, for any* $t \in [0,T)$,

$$\overline{U}(0,t) > 0, \text{ if } \alpha > \frac{1}{2}, \tag{3.4.15}$$

$$\overline{U}(0,t) = 0, \text{ if } \alpha = \frac{1}{2}, \tag{3.4.16}$$

$$\overline{U}(0,t) = 0, \text{ if } \alpha < \frac{1}{2}. \tag{3.4.17}$$

Proof. The proofs of part i) and ii) are the same as that of Proposition 3.4.3. Now let us prove part iii).

It is easy to see $\overline{U}^*_x(x,t) > 0$ in Ω^+ by virtue of $F_{xx} \le 0$ and the strong maximum principle. Combining with Lemma 3.4.4, we infer $\overline{U}^*(x,t) > 0$ in Ω^+ when $\alpha \ge \frac{1}{2}$. So, $\overline{U}^*(x,t)$ must be the solution to (3.4.5), which yields $\overline{U}(x,t) = \overline{U}^*(x,t)$ for $\alpha \ge \frac{1}{2}$. Then (3.4.15) and (3.4.16) follow. To show (3.4.17), clearly we have $\overline{U}(0,t) \ge 0$. Thanks to part ii) and (3.4.16), we infer $\overline{U}(0,t) \le 0$ for $\alpha < \frac{1}{2}$, which leads to (3.4.17). This completes the proof. $\square$

Now, we are going to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. Part i) and ii) follow by part iii) of Proposition 3.4.5. The proof of part iv) is similar to that of part i) in Theorem 3.2.1. Now let us prove part iii). Thanks to (3.4.17), we immediately get $\{x = 0\} \subset SR$. Combining with $\overline{U}_x \ge 0$, we can define

$$x^*_s(t) = \sup\{x \in [0,+\infty) : \overline{U}(x,t) = 0\}, \text{ for any } t \in [0,T).$$

We only need to show that $x^*_s(t) < \infty$. Let $x^*_b(t)$ be the free boundary as given in part ii) of Theorem 3.2.1. Due to Lemma 3.4.1, we infer $x^*_s(t) \le x^*_b(t)$, which, combined with $x^*_b(t) < \infty$, yields the desired result.

At last, let us prove $\lim_{T-t\to\infty} x^*_s(t) = \frac{1}{2\alpha-1}\log\left(1-(2\alpha-1)^2\right)$. Again, consider the stationary problem of (3.1.10). Denote

$$\Psi^\infty(x) = \lim_{T-t\to\infty} \Psi(x,t) \text{ and } U^\infty(x) = \lim_{T-t\to\infty} U(x,t).$$

It is easy to see that $\Psi^\infty(x) = e^{-x} + \frac{1}{2(\alpha-1)}e^{2(\alpha-1)x}$, if $0 < \alpha < \frac{1}{2}$, and $U^\infty(x)$ satisfies

$$-\frac{1}{2}U^\infty_{xx} + (\alpha - \frac{1}{2})U^\infty_x = 0, \quad \forall x > x_\infty, \tag{3.4.18}$$

$$U^\infty_x(x_\infty) = \Psi^\infty_x(x_\infty), \qquad U^\infty(x_\infty) = \Psi^\infty(x_\infty), \tag{3.4.19}$$

where x_∞ is the free boundary.

The general solution to (3.4.18) is $U^\infty(x) = A + Be^{(2\alpha-1)x}$. Since $\lim_{x\to\infty} U^\infty(x) = 0$, we see $A = 0$. Thus, it is easy to get $U^\infty(x) = (e^{-x_\infty} + \frac{1}{2(\alpha-1)} e^{2(\alpha-1)x_\infty}) e^{(2\alpha-1)(x-x_\infty)}$ from (3.4.19), where $x_\infty = \frac{1}{2\alpha-1} \log\left(1-(2\alpha-1)^2\right)$. The proof is complete. $\square$

Chapter 4

Behavioral Liquidation

In this chapter, we propose a simple optimal liquidation model with prospect theory preferences to study the disposition effect. The organization of the chapter is as follows. The problem formulation is presented in Section 1. Section 2 and 3 are devoted to the piecewise linear utility and real S-shaped utility, respectively. Due to the lack of explicit solution, we present numerical results in Section 4. We summarize the chapter in Section 5. Some proofs are placed in an appendix.

4.1 Formulation

We consider the liquidation decision of an investor who owns an asset that costs him K initially. The asset could be house, stock or some ongoing project. In order to meet some financial constraints, e.g. liquidity constraints, the investor plans to liquidate this asset in the period $[0, T]$. Let S_t denote the asset price at time t for $t \in [0, T]$, and assume S_t evolves as GBM, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

One may alternatively write

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}.$$

Let τ be the liquidation time chosen by the investor, then the profit or loss Y_τ, which is defined by the difference between the liquidation price S_τ and the cost K, i.e. $Y_\tau = S_\tau - K$, is realized at time $t = \tau$. The reason for the inclusion of the initial cost has clear financial interpretation. For example, one may think the asset is a house and the investor borrowed money K from bank to buy it. This cost K can also be considered as the reference point in the prospect theory, which is often chosen to be the break-even point. We assume the investor has a prospect preference, which reflects in his utility function $u(\cdot)$. For tractability, we assume

$$u(x) = \begin{cases} x^\alpha, & \text{if } x \geq 0, \\ -\lambda\left(-x\right)^\alpha, & \text{if } x < 0, \end{cases}$$

where $\alpha \in (0, 1]$ and the parameter $\lambda > 1$ governs the loss aversion. Figure 4.1 plots the shape of $u(x)$. The dashed line represents the shape with the parameters of $\alpha = 0.88$ and $\lambda = 2.25$ in Tversky and Kahneman (1992), while the solid line shows the shape with the parameters of $\alpha = 1$ and $\lambda = 2.25$. Note that the inclusion of $\alpha = 1$ has its own interesting, since the sensitivity of the utility is finite at the break-even point.

Thus, the investor's objective can be summarized in the following optimal stopping problem

$$\sup_{0 \leq \tau \leq T} \mathbb{E}\left[u(Y_\tau)e^{-\beta\tau}\right] \tag{4.1.1}$$

where $\mathbb{E}$ stands for the expectation, τ is a stopping time, $\beta \geq 0$ is the investor's discount factor.

Remark 4.1.1. *It is worth clarifying some differences with Barberis and Xiong (2010), Kyle, Ou-Yang and Xiong (2006) and Hendersen (2009). They model*

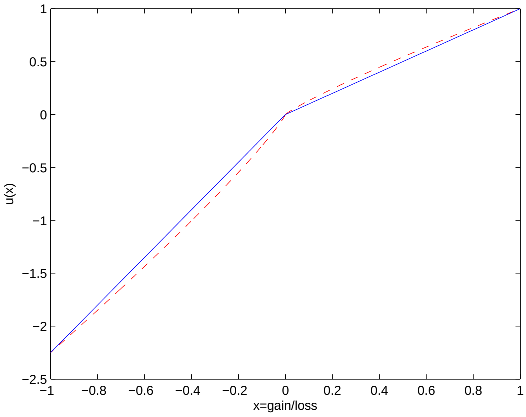

Figure 4.1: The utility function.

the problem with infinite horizons, and the latter two papers assume $\beta = 0$. *In their models, the positive discount rate, i.e.* $\beta > 0$, *will lead to a strong disposition effect, which means no loss realization. We introduce the discount rate in our finite horizon problem, since utility discounting has its own economical interpretation. We may also assume that the discount rate is greater than the interest rate, if the interest rate is non-zero without changing the problem essentially.*

4.2 Existence of Optimal Liquidation Time

In this section, we will study the existence of optimal liquidation time and characterize the value function in the sense of viscosity solution.

Thanks to the dynamic programming approach, we only need to consider the following parameterized problem

$$V(t,S) = \sup_{t \le \tau \le T} \mathbb{E}_{t,S}\left[u(S_\tau - K)e^{-\beta(\tau - t)}\right], \tag{4.2.2}$$

where $S_t = S$ under $\mathbb{P}_{t,S}$ with $(t,S) \in [0,T] \times (0,\infty)$ given and fixed. The original problem (4.1.1) is simply as

$$V(0,S_0) = \sup_{0\le\tau\le T} \mathbb{E}_{0,S_0}\left[u(S_\tau - K)e^{-\beta\tau}\right].$$

Note that one can freely choose $\tau = t$, which gives $V(t,S) \ge u(S-K)$. If an immediate liquidation is optimal, then the value function must equal the liquidation payoff. Thus, we can denote the holding region as

$$HR = \{(t,S) \in [0,T]\times(0,\infty) : V(t,S) > u(S-K)\}$$

and the selling region as

$$SR = \{(t,S) \in [0,T]\times(0,\infty) : V(t,S) = u(S-K)\}.$$

Theorem 4.2.1. *The value function $V(t,S)$ is continuous on $[0,T]\times(0,\infty)$ and there exists an optimal stopping time $\tau_{SR}(t,S) = \inf\{s \ge t : (s,S_s) \in SR\}$. Moreover, $V(t,S)$ is decreasing in t.*

Proof. Given any $(t,S) \in [0,T]\times(0,\infty)$, it is easy to see that

$$\begin{aligned} |V(t,S)| &= |\sup_{t\le\tau\le T} \mathbb{E}_{t,S} u(S_\tau - K)e^{-\beta(\tau-t)}| \le \mathbb{E}_{t,S} \sup_{t\le\tau\le T} |u(S_\tau - K)| \\ &\le \mathbb{E}_{t,S} \sup_{t\le\tau\le T} (S_\tau^\alpha + \lambda K^\alpha) = \lambda K^\alpha + S^\alpha \mathbb{E} \sup_{0\le\tau\le T} e^{(\mu-\frac{\sigma^2}{2})\tau + B_\tau} \\ &\le \lambda K^\alpha + CS^\alpha, \end{aligned} \tag{4.2.3}$$

where $C > 0$ is some constant, independent of t and S.

Now, let us prove the continuity of $V(t,S)$. For any (t,S) and (t_0,S_0) given, without loss of generality, we assume $t \geq t_0$. Then,

$$
\begin{aligned}
&|V(t_0,S_0) - V(t,S)| = |\sup_{t_0\le\tau\le T}\mathbb{E}_{t_0,S_0}u(S_\tau-K)e^{-\beta(\tau-t_0)} - \sup_{t\le\tau\le T}\mathbb{E}_{t,S}u(S_\tau-K)e^{-\beta(\tau-t)}| \\
&\le |\sup_{t_0\le\tau\le T}\mathbb{E}_{t_0,S_0}u(S_\tau-K)e^{-\beta(\tau-t_0)} - \sup_{t\le\tau\le T}\mathbb{E}_{t_0,S_0}u(S_\tau-K)e^{-\beta(\tau-t_0)}| \\
&\quad + |\sup_{t\le\tau\le T}\mathbb{E}_{t_0,S_0}u(S_\tau-K)e^{-\beta(\tau-t_0)} - \sup_{t\le\tau\le T}\mathbb{E}_{t,S}u(S_\tau-K)e^{-\beta(\tau-t)}| \\
&\le \mathbb{E}\sup_{t_0\le\tau\le t}|u(S_0X_\tau^{t_0}-K)e^{-\beta(\tau-t_0)} - u(S_0X_t^{t_0}-K)e^{-\beta(t-t_0)}| \\
&\quad + \mathbb{E}\sup_{t\le\tau\le T}|u\left(S_0e^{(\mu-\frac{\sigma^2}{2})(t-t_0)+\sigma(B_t-B_{t_0})}X_\tau^t-K\right)e^{-\beta(t-t_0)} - u(SX_\tau^t-K)|I_{\left\{|B_t-B_{t_0}|<(t-t_0)^{\frac14}\right\}} \\
&\quad + \mathbb{E}\sup_{t\le\tau\le T}|u\left(S_0e^{(\mu-\frac{\sigma^2}{2})(t-t_0)+\sigma(B_t-B_{t_0})}X_\tau^t-K\right)e^{-\beta(t-t_0)} - u(SX_\tau^t-K)|I_{\left\{|B_t-B_{t_0}|\ge(t-t_0)^{\frac14}\right\}},
\end{aligned}
$$

where $X_\tau^t = e^{(\mu-\frac{\sigma^2}{2})(\tau-t)+\sigma(B_\tau-B_t)}$.

Thus, the continuity of $V(t,S)$ follows from the continuity of $u(x)$ and a prior estimate (4.2.3).

Since both $V(t,S)$ and $u(S-K)$ are continuous, SR is closed and $\tau_{SR}(t,S)$ is indeed a stopping time. By the general results of optimal stopping theorem (cf. Peskir and Shiryaev 2006), we conclude that $\tau_{SR}(t,S)$ is an optimal stopping time for problem (4.2.2).

The monotonicity of $V(t,S)$ in t simply follows from (4.2.2). Indeed, for any $0 \le t_1 \le t_2 \le T$, we have

$$
\begin{aligned}
V(t_1,S) &= \sup_{0\le\tau\le T-t_1}\mathbb{E}\left[u(S_{t_1+\tau}-K)e^{-\beta\tau}|S_{t_1}=S\right] \\
&\le \sup_{0\le\tau\le T-t_2}\mathbb{E}\left[u(S_{t_1+\tau}-K)e^{-\beta\tau}|S_{t_1}=S\right] \\
&= \sup_{0\le\tau\le T-t_2}\mathbb{E}\left[u(S_{t_2+\tau}-K)e^{-\beta\tau}|S_{t_2}=S\right] \\
&= V(t_2,S).
\end{aligned}
$$

The proof is complete.

It is well-known that there is a connection between the optimal stopping problem (4.2.2) and variational inequalities (or obstacle problem) (4.2.4) as we have seen in Chapter 2 and 3.

$$\begin{cases} \min\{-\partial_t V - \mathcal{L}V, V - u(S-K)\} = 0, \text{ in } [0,T) \times (0,\infty) \\ V(T,S) = u(S-K), \end{cases} \tag{4.2.4}$$

where $\mathcal{L}V = \frac{1}{2}\sigma^2 S^2 \partial_{SS} V + \mu S \partial_S V - \beta V$.

However, we can not expect that $V(t,S)$ is smooth enough to satisfy the variational inequalities (4.2.4) in "normal" sense, since the obstacle $u(S-K)$ is not smooth enough. Indeed, we will later show that the value function $V(t,S)$ is not always differentiable at $S = K$ for any t, which implies $V(t,S)$ is not a $W_p^{2,1}$ solution of (4.2.4). Thanks to the linear growth of $V(t,S)$ in (4.2.3), we can fully characterize the value function $V(t,S)$ in the sense of viscosity solution[1]. Let us first introduce the definition of viscosity solution of (4.2.4).

Definition 4.2.2. *A function $V \in C([0,T] \times (0,\infty))$ is called a viscosity subsolution of (4.2.4) if*

$$V(T,S) \le u(S-K), \ \forall S \in (0,\infty), \tag{4.2.5}$$

and for any $\varphi \in C^{1,2}([0,T] \times (0,\infty))$ and each $(t_0, S_0) \in [0,T) \times (0,\infty)$ such that $V \le \varphi$ and $V(t_0,S_0) = \varphi(t_0,S_0)$, we have

$$\min\{-\partial_t \varphi(t_0,S_0) - \mathcal{L}\varphi(t_0,S_0), \varphi(t_0,S_0) - u(S_0-K)\} \le 0. \tag{4.2.6}$$

A function $V \in C([0,T] \times (0,\infty))$ is called a viscosity supersolution of (4.2.4) if $V \ge \varphi$ and the inequalities "$\le$" in (4.2.5)-(4.2.6) are changed to "$\ge$". Further, if

[1] Thank Professor M. Zervos for pointing me Lamberton (2009) during my viva. Lamberton (2009) proves the value function for a general optimal stopping problem with irregular reward functions can be characterized as the unique solution of a variational inequality in the sense of distributions under some mild assumptions.

$V \in C([0,T] \times (0,\infty))$ *is both a viscosity subsolution and supersolution of (4.2.4), then it is called a viscosity solution of (4.2.4).*

Theorem 4.2.3. *The value function $V(t,S)$ is the unique viscosity solution of (4.2.4).*

Proof. The proof is similar to Øksendal and Reikvam (1998). We first treat the viscosity subsolution. Since $V(T,S) = u(S-K)$, (4.2.5) holds. Let $\varphi \in C^{1,2}([0,T] \times (0,\infty))$ and $(t_0,S_0) \in [0,T) \times (0,\infty)$ such that $V \le \varphi$ and $V(t_0,S_0) = \varphi(t_0,S_0)$. If $(t_0,S_0) \in SR$, then $V(t_0,S_0) = u(S_0-K)$ and $\varphi(t_0,S_0) = u(S_0-K)$. So, (4.2.6) follows. If $(t_0,S_0) \notin SR$, then for any $t_0 < \tau \le \tau_{SR}$, we have

$$\begin{aligned} V(t_0,S_0) &= \mathbb{E}_{t_0,S_0}\left[V(\tau,S_\tau)e^{-\beta(\tau-t_0)}\right] \\ &\le \mathbb{E}_{t_0,S_0}\left[\varphi(\tau,S_\tau)e^{-\beta(\tau-t_0)}\right] \\ &= \varphi(t_0,S_0) + \mathbb{E}_{t_0,S_0}\int_{t_0}^{\tau} e^{-\beta(\tau-t_0)}\left[\partial_t\varphi(s,S_s) + \mathcal{L}\varphi(s,S_s)\right]ds. \end{aligned}$$

So, we have $\mathbb{E}_{t_0,S_0}\int_{t_0}^{\tau} e^{-\beta(\tau-t_0)}\left[\partial_t\varphi(s,S_s) + \mathcal{L}\varphi(s,S_s)\right]ds \ge 0$, for any $t_0 < \tau \le \tau_{SR}$. Dividing by $\mathbb{E}_{t_0,S_0}(\tau-t_0)$ and letting $\tau \to {t_0}^+$, we get

$$\partial_t\varphi(t_0,S_0) + \mathcal{L}\varphi(t_0,S_0) \ge 0,$$

which implies (4.2.6). Thus, V is a viscosity subsolution.

Now let us treat the viscosity supersolution. Let $\varphi \in C^{1,2}([0,T] \times (0,\infty))$ and $(t_0,S_0) \in [0,T) \times (0,\infty)$ such that $V \ge \varphi$ and $V(t_0,S_0) = \varphi(t_0,S_0)$. Then for any stopping time $\tau > t_0$, we have

$$\begin{aligned} V(t_0,S_0) &\ge \mathbb{E}_{t_0,S_0}\left[V(\tau,S_\tau)e^{-\beta(\tau-t_0)}\right] \ge \mathbb{E}_{t_0,S_0}\left[\varphi(\tau,S_\tau)e^{-\beta(\tau-t_0)}\right] \\ &= \varphi(t_0,S_0) + \mathbb{E}_{t_0,S_0}\int_{t_0}^{\tau} e^{-\beta(\tau-t_0)}\left[\partial_t\varphi(s,S_s) + \mathcal{L}\varphi(s,S_s)\right]ds, \end{aligned}$$

which implies $\mathbb{E}_{t_0,S_0}\int_{t_0}^{\tau} e^{-\beta(\tau-t_0)}\left[\partial_t\varphi(s,S_s) + \mathcal{L}\varphi(s,S_s)\right] \le 0$.

Let $\tau \to t_0^+$, then

$$\partial_t \varphi(t_0, S_0) + \mathcal{L}\varphi(t_0, S_0) \leq 0.$$

Combining with $V(t_0, S_0) \geq u(S_0 - K)$, we obtain

$$\min\{-\partial_t \varphi(t_0, S_0) - \mathcal{L}\varphi(t_0, S_0), \varphi(t_0, S_0) - u(S_0 - K)\} \geq 0,$$

which shows V is a viscosity supersolution.

Since V is at most linear growth, the uniqueness directly follows by the comparison principle in Crandall, Ishii and Lions (1992). The proof completes.

According to Theorem 4.2.3, we can numerically calculate the value function together with the liquidation region whenever the explicit solution of (4.2.4) is not available. To solve (4.2.4), one may apply the penalty method (cf. Forsyth and Vetzal 2002, Dai and Zhong 2010) or projected SOR method (cf. Kwok 2008).

4.3 Piecewise Linear Utility

In this section, we will study the optimal liquidation strategy with a piecewise linear utility, i.e. $\alpha = 1$.

4.3.1 Optimal liquidation strategies

In order to characterize the optimal liquidation strategies, we first study the possible liquidation region and then show that the liquidation region is connected if it exists.

Let us start from the dynamics of the payoff process $u(S_t - K)e^{-\beta t}$. Since $u(x)$

has a kink at 0, we need to apply Itô-Tanaka formula, which gives

$$\begin{aligned} d\left\{u(S_t-K)e^{-\beta t}\right\} &= e^{-\beta t}\Big[u'(S_t-K)I_{\{S_t\neq K\}}dS_t - \beta u(S_t-K)dt \\ &\quad + \frac{(u'(0^+)-u'(0^-))}{2}dL_K^S(t)\Big] \\ &= e^{-\beta t}(I_{\{S_t>K\}}+\lambda I_{\{S_t<K\}})\left[(\mu-\beta)S_t+\beta K)dt+\sigma S_t dB_t\right] \\ &\quad + \frac{(1-\lambda)e^{-\beta t}}{2}dL_K^S(t), \ \forall t\in[0,T], \end{aligned}$$

where $L_K^S(t)$ is the local time of S_t at the level K.

Proposition 4.3.1. *Given $\beta > 0$.*

i) If $\mu \geq \beta$, then $HR \supset \{(t,S) : S \neq K, 0 \leq t < T\}$.

ii) If $0 < \mu < \beta$, then $HR \supset \{(t,S) : 0 < S < \frac{\beta K}{\beta-\mu}$, and $S \neq K$, $0 \leq t < T\}$.

iii) If $\mu \leq 0$, then $HR \supset \{(t,S) : 0 < S < \frac{\beta K}{\beta-\mu}$, $0 \leq t < T\}$.

Proof. We only prove part i), and others are similar. It suffices to show

$$V(t,S) > u(S-K), \text{ for all } S \neq K.$$

Let us consider a small ball B_δ centered at (t,S), with radius $\delta > 0$, such that $B_\delta \cap \{S = K\} = \emptyset$.

Denote $\tau = \inf\{s \geq t : (s, S_s) \notin B_\delta\} \bigwedge T$, where $S_t = S$. Then,

$$\begin{aligned} V(t,S) &\geq \mathbb{E}_{t,S}u(S_\tau-1)e^{-\beta(\tau-t)} \\ &= u(S-K) + \mathbb{E}_{t,S}\int_t^\tau e^{-\beta(s-t)}(I_{\{S_s>K\}}+\lambda I_{\{S_s<K\}})\big((\mu-\beta)S_s+\beta K\big)ds \\ &> u(S-K), \end{aligned}$$

where the last strict inequality is due to the positiveness of the integrand and $\tau > t$, $P_{t,S} - a.s.$. The proof completes.

This proposition implies that if holding the asset at the current price is optimal, then it is optimal to hold the asset at a lower price. This is consistent with the disposition effect, since it will sell the winner before the loser.

Proposition 4.3.2. *(Shape of SR)*

i) If $(t_0, S) \in SR$, then $(t, S) \in SR$, for all $t > t_0$.

ii) Given $\mu < \beta$. If $(t_0, S_0) \in SR$ for $S_0 \geq \frac{\beta K}{\beta - \mu}$, then $(t_0, S) \in SR$, for all $S > S_0$.

Proof. Part i) follows by the monotonicity of $V(t, S)$ in t. Indeed, for all $t > t_0$, such that $(t_0, S) \in SR$, we have

$$0 \leq V(t, S) - u(S - K) \leq V(t_0, S) - u(S - K) = 0,$$

which implies $(t, S) \in SR$.

Now we show part ii). Since $(t_0, S_0) \in SR$, we see that $(t, S_0) \in SR$, $\forall t \in [t_0, T]$, by part i). Denote $\tau = \inf\{s \geq t_0 : S_s = S_0\}$, with $S_{t_0} = \widetilde{S} > S_0$. Thus, the optimal stopping time $\tau_{SR} = \tau_{SR}(t_0, \widetilde{S}) \leq \tau$ and

$$\begin{aligned}
V(t_0, \widetilde{S}) &= \mathbb{E}_{t_0, \widetilde{S}} u(S_{\tau_{SR}} - K) e^{-\beta(\tau_{SR} - t_0)} \\
&= u(\widetilde{S} - K) + \mathbb{E}_{t_0, \widetilde{S}} \Big[\int_{t_0}^{\tau_{SR}} e^{-\beta(t - t_0)} (I_{\{S_t > K\}} \\
&\quad + \lambda I_{\{S_t < K\}}) \big[\big((\mu - \beta) S_t + \beta K \big) dt + \sigma S_t dB_t \big] + \frac{(1 - \lambda) e^{-\beta(t - t_0)}}{2} dL_K^S(t) \Big] \\
&\leq u(\widetilde{S} - K) + \mathbb{E}_{t_0, \widetilde{S}} \int_{t_0}^{\tau_{SR}} e^{-\beta(t - t_0)} (I_{\{S_t > K\}} + \lambda I_{\{S_t < K\}}) \big((\mu - \beta) S_t + \beta K \big) dt \\
&\leq u(\widetilde{S} - K). \qquad (4.3.7)
\end{aligned}$$

The last inequality is because the integrand is non-positive. From (4.3.7), we conclude $V(t_0, \widetilde{S}) = u(\widetilde{S} - K)$, thus $(t_0, \widetilde{S}) \in SR$, which completes the proof.

This proposition shows that the liquidation region expands as time t approaches the maturity T. Moreover, by this proposition, we can almost figure out the optimal selling region SR except the break-even level $S = K$. Since the investor is loss aversion, one may expect that the investor will sell the asset at the break-even level if time remained is not much. This speciality is shown in the following proposition.

Proposition 4.3.3. *(The Break-even Level) Given $\beta > 0$ and any value of other parameters,*

i) $(t, K) \in SR$, *for all* $t \geq T - \delta$, *where* $\delta > 0$ *is small enough;*

ii) $(t, K) \notin SR$, *for* $T - t$ *large enough.*

Proof. We show part i) by contradiction. Let us firstly consider the scenario $\mu \geq \beta$. Suppose that part i) fails. Due to Proposition 4.3.1 and 4.3.2, we can conclude $SR = \{(t, S) : t = T\}$, which gives that

$$\begin{aligned} V(t,K) &= e^{-\beta(T-t)}\mathbb{E}_{t,K}u(S_T - K) \\ &= e^{-\beta(T-t)}\left[\mathbb{E}_{t,K}(S_T - K) - (\lambda - 1)\mathbb{E}_{t,K}(S_T - K)^-\right] \\ &= K\left[e^{(\mu-\beta)(T-t)} - e^{-\beta(T-t)}\right] - K(\lambda - 1)e^{-\beta(T-t)}\mathbb{E}\left(e^{(\mu-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}B_1} - 1\right)^- \\ &=: G(\sqrt{T-t}). \end{aligned}$$

It is easy to see that $G(\cdot)$ is smooth on $[0, +\infty)$ and $G(0) = 0$. Furthermore,

$$G'(0+) = -K(\lambda - 1)\sigma\mathbb{E}[-B_1 I_{\{B_1<0\}}] = -K(\lambda - 1)\sigma\mathbb{E}B_1^- < 0.$$

So, $V(t, K) = G(\sqrt{T-t}) < 0$ for $T - t$ is small enough. On the other hand, we have $V(t, K) \geq u(K - K) = 0$, which results in a contradiction.

For the case of $\mu < \beta$, denote the corresponding optimal value of the problem as $V_\mu(t, K)$, then $0 \leq V_\mu(t, K) \leq V_\beta(t, K) = 0$. Hence, $(t, K) \in SR$, for all $t \geq T - \delta$, where $\delta > 0$ is small enough.

To show part ii), it is equivalent to show that $V(0, K) > 0$, for T large enough. Denote $\tau = \inf\{t : S_t = K + 1\}$, where $S_0 = K$. Then

$$\begin{aligned} V(0,K) &\geq \mathbb{E}_{0,K}u(S_{\tau\wedge T} - K)e^{-\beta(\tau\wedge T)} \\ &= \mathbb{E}[e^{-\beta\tau}I_{\{\tau<T\}}] + \mathbb{E}_K[u(S_T - K)e^{-\beta T}I_{\{\tau\geq T\}}] \\ &\geq \mathbb{E}[e^{-\beta\tau}I_{\{\tau<T\}}] + u(-K)e^{-\beta T}P(\tau \geq T) \qquad (4.3.8) \end{aligned}$$

Notice that the first term in (4.3.8) is positive and increasing in T, while the second term goes to 0, as $T \to \infty$. Hence, we conclude that $V(0, K) > 0$, $\forall T > T_0$, for some large T_0. The proof is complete.

Proposition 4.3.3 shows that it is always optimal to sell the asset at its break-even level when time is close enough to the maturity. This sounds very reasonable by intuition. Since the investor is loss aversion, he is not willing to gamble on a favorable game if the game is not repeatable for enough times. However, combining part ii) and Proposition 4.3.2 part ii), one can see that it is never optimal to liquidate the asset at the loss situation or the break even level, if the investor has enough time to defer the liquidation. As we have said before, $\beta > 0$ plays a critical role to defer losses.

Now we are in a position to summarize the main result in this section.

Theorem 4.3.4. *(The optimal selling strategies) Given $\beta > 0$.*

i) If $\mu \geq \beta$, then there exists a $t_0 > 0$, such that the selling region is $SR = \{(t, S) : t_0 \leq t < T, S = K\} \cup \{(t, S) : t = T\}$.

ii) If $0 < \mu < \beta$, then there exist a $t_0 > 0$ and a decreasing function $b(t) : [0, T) \longmapsto R^+$, such that the selling region is $SR = \{(t, S) : t_0 \leq t \leq T, S = K\} \cup \{(t, S) : S \geq b(t)\} \cup \{(t, S) : t = T\}$. Furthermore, $b(t)$ is continuous and satisfies

$$b(T^-) = \frac{\beta K}{\beta - \mu}, \quad \lim_{T-t \to \infty} b(t) = \frac{\gamma K}{\gamma - 1}, \tag{4.3.9}$$

where $\gamma = \left(-(\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2\beta\sigma^2}\right) \Big/ \sigma^2$.

iii) If $\mu = 0$, then there exists a decreasing function $b(t) : [0, T) \longmapsto R^+$, such that the selling region is $SR = \{(t, S) : S \geq b(t)\} \cup \{(t, S) : t = T\}$. Furthermore, $b(t)$ satisfies (4.3.9) and there exists $t_0 > 0$, such that $b(t) = K$, $\forall t \in [t_0, T)$.

iv) If $\mu < 0$, then there exists a decreasing function $b(t) : [0, T) \longmapsto R^+$, such that the selling region is $SR = \{(t, S) : S \geq b(t)\} \cup \{(t, S) : t = T\}$. Furthermore, $b(t)$ satisfies (4.3.9).

v) The function $b(t)$ defined in part ii), iii) and iv) is strictly decreasing with respect to t in $\{t \in [0, T) : b(t) \neq K\}$.

Proof. Part i) follows by Proposition 4.3.1 and 4.3.3 part i). Thus, we only need to show part ii), since part iii) and iv) hold similarly. Using the same argument as Part i), we obtain the existence of t_0. Now let us focus on the function $b(t)$. Thanks to Proposition 4.3.1 and 4.3.2 part ii), we can define

$$b(t) = \inf\left\{S \geq \frac{\beta K}{\beta - \mu} : V(t, S) = u(S - K), \forall t \in [0, T)\right\}.$$

Due to the continuity of V and u, we see $V(t, b(t)) = u(b(t) - K)$. Since $V(t, S)$ is decreasing in t, then for all $s > t$,

$$0 \leq V(s, b(t)) - u(b(t) - K) \leq V(t, b(t)) - u(b(t) - K) = 0,$$

which implies $b(s) \leq b(t)$. Thus, we show $b(t)$ is decreasing.

Now let us derive the continuity of $b(t)$. It is easy to show that $b(t)$ is right-continuous. Indeed, $b(t^+) = \lim_{s \to t^+} b(s) \leq b(t)$ by the monotonicity of $b(t)$ and $b(t^+) \geq b(t)$ by $V(t, b(t^+)) = u(b(t^+) - K)$.

Suppose $b(t)$ is not left-continuous at $t_0 \in [0, T)$, i.e. $b(t_0^-) > b(t_0)$. Due to strong Markov property of S_t, $V(t, S)$ is $C^{1,2}$ in the holding region HR and satisfies the Kolmogorov backward equation:

$$\begin{cases} \partial_t V + \mathcal{L}V = 0, \ \forall (t, S) \in HR \\ V|_{\partial HR} = u(S - K), \ V(T, S) = u(S - K). \end{cases} \tag{4.3.10}$$

Especially, (4.3.10) holds in $[0, t_0) \times (b(t_0), b(t_0^-))$ and $V(t_0, S) = u(S - K), \forall S \in (b(t_0), b(t_0^-))$. Without loss of generality, we can assume $K \notin (b(t_0), b(t_0^-))$, otherwise we choose $(K, b(t_0^-))$ instead. Since $u(S - K)$ is smooth if $S \neq K$, we have, for any $S \in (b(t_0), b(t_0^-))$,

$$\partial_t V|_{t=t_0} = -\mathcal{L}V|_{t=t_0} = -\mathcal{L}u(S-K) = (\beta-\mu)S + \beta K > 0,$$

which contradicts with V decreasing in t.

So, we have shown the continuity of $b(t)$ on $[0, T)$. The same argument can be used to show $b(T^-) = \frac{\beta K}{\beta-\mu}$.

The proof of $\lim_{T-t\to\infty} b(t) = \frac{\gamma K}{\gamma-1}$ is placed in Appendix 4.7.1.

To complete the proof, we will apply a PDE approach to show $b(t)$ is strictly decreasing if $b(t) \neq K$. Let us first assume that smooth fit principle hold at $b(t) \neq K$, i.e.

$$S :\longmapsto V_S(t,S) \text{ is } C^1 \text{ at } (t, b(t)), \text{ and } V_S(t, b(t)) = u'(b(t) - K), \tag{4.3.11}$$

which will be shown later. Suppose there exists $0 \leq t_1 < t_2 < T$, such that $b(t_1) = b(t_2) = S_0$. Then, we have (4.3.10) holds in $(t_1, t_2) \times (S_0 - \epsilon, S_0)$ and $V(t, S_0) = u(S_0 - K)$ for all $t \in (t_1, t_2)$, where $\epsilon > 0$ is small enough such that $K \notin (S_0 - \epsilon, S_0)$.

Denote $v(t,S) = \partial_t V(t,S), \forall (t,S) \in [t_1, t_2] \times [S_0 - \epsilon, S_0]$. We have

$$\begin{cases} \partial_t v + \mathcal{L}v = 0, \text{ in } (t_1, t_2) \times (S_0 - \epsilon, S_0), \\ v(t, S_0) = 0, \forall t \in (t_1, t_2). \end{cases} \tag{4.3.12}$$

Since V is decreasing in t, i.e. $v \leq 0$ in $[t_1, t_2] \times [S_0 - \epsilon, S_0]$, v attends its maximum at $S = S_0$. By Hopf's Lemma, we have $v_S(t, S_0) > 0$ for $t_1 < t < t_2$. However, $v_S(t, S_0) = 0$, because of $V_S(t, S_0) = u'(S_0 - K)$ for all $t \in (t_1, t_2)$. Hence, we get a contradiction.

Now, let us prove (4.3.11). We first show the scenario of $b(t) > K$. Denote $S = b(t)$, then

$$\frac{V(t, S-\epsilon) - V(t,S)}{-\epsilon} \leq \frac{u(S-\epsilon-K) - u(S-K)}{-\epsilon} \tag{4.3.13}$$

for all small $\epsilon > 0$. Taking $\epsilon \to 0^+$ in (4.3.13) we find

$$\limsup_{\epsilon\to 0^+} \frac{V(t,S-\epsilon)-V(t,S)}{-\epsilon} \leq u'(S-K) = 1. \tag{4.3.14}$$

Denote τ_ϵ=$\tau_{SR}(t,S-\epsilon)$ and $\tau_K = \inf\{s > t : S_s^{x-\epsilon} = K\}$, where $S_t = S-\epsilon > K$. We have

$$\begin{aligned}
\frac{V(t,S-\epsilon)-V(t,S)}{-\epsilon} &\geq \frac{\mathbb{E}_{t,S-\epsilon}u(S_{\tau_\epsilon}-K)e^{-\beta(\tau_\epsilon-t)} - \mathbb{E}_{t,S}u(S_{\tau_\epsilon}-K)e^{-\beta(\tau_\epsilon-t)}}{-\epsilon}\\
&= \frac{\mathbb{E}\left(u(S^{S-\epsilon}_{\tau_\epsilon-t}-K)-u(S^{S}_{\tau_\epsilon-t}-K)\right)e^{-\beta(\tau_\epsilon-t)}}{-\epsilon}\\
&\geq \mathbb{E}\left(e^{(\mu-\frac{\sigma^2}{2}-\beta)(\tau_\epsilon-t)+\sigma(B_{\tau_\epsilon}-B_t)}I_{\{\tau_\epsilon<\tau_K\}} - \lambda e^{\mu(\tau_\epsilon-t)+\sigma(B_{\tau_\epsilon}-B_t)}I_{\{\tau_\epsilon\geq\tau_K\}}\right),
\end{aligned}$$

where we use the fact that $u(\cdot)$ is Lipschitz continuous in the last inequality.

Since $b(t)$ is decreasing in t, we see $\tau_\epsilon \to t$ as $\epsilon \to 0^+$. Moreover, τ_K is increasing in ϵ. Applying the dominated convergence theory, we get

$$\liminf_{\epsilon\to 0^+} \frac{V(t,x-\epsilon)-V(t,x)}{-\epsilon} \geq 1 = u'(S-K). \tag{4.3.15}$$

Combining (4.3.14) and (4.3.15), we see that $S :\longmapsto V_S(t,S)$ is differentiable at $(t,b(t))$ and $V_S(t,b(t)) = u(b(t)-K)$. A small modification of the proof above shows that $S :\longmapsto V_S(t,S)$ is C^1 at $(t,b(t))$. The case of $b(t) < K$ is similar, we omit it. The proof is complete.

From (4.3.11), we can see that the smooth-fit principle holds on the optimal liquidation boundary $b(t) \neq K$. However, no conclusion has been made on the boundary $b(t) = K$. We will address this in the next subsection.

4.3.2 Sensitivity of the parameters

Now, we aim to study how the liquidation region shifts under different parameters, and what roles the discount factor β and loss aversion λ play in the prediction of disposition effect and break-even effect. In general, the effects are very complicated, so we only focus on four critical quantities of the liquidation region SR. These quantities are as follows:

i) The starting point of the selling boundary: $b(T^-)$;

ii) The asymptotic behavior of the selling boundary: $\lim_{T-t\to\infty} b(t)$;

iii) The first liquidation time at the break-even level:

$$\underline{t} = \inf\left\{t < T : (t, K) \in SR\right\};$$

iv) The"last" liquidation time at the break-even level:

$$\bar{t} = \begin{cases} T & \text{if } \mu \geq 0; \\ \sup\{t : b(t) = K\} & \text{if } \mu < 0. \end{cases}$$

Smooth-fit at one single point

Before we study those effects of the parameters, we need to first show $\underline{t} < \bar{t}$ in the case of $\mu < 0$. In order to prove this, we require another characterization of $\underline{t}$.

Theorem 4.3.5. *(Smooth-fit at one single point)The value function $V(t, S)$ is differentiable at $(\underline{t}, K)$, i.e. $V_S(\underline{t}, K^+) = V_S(\underline{t}, K^-)$. Moreover, for any $t > \underline{t}$, $V_S(\underline{t}, K^+) < V_S(\underline{t}, K^-)$.*

The proof is quite technical and placed in the Appendix.

The effect of loss aversion

Thanks to the expressions of $b(T^-)$ and $\lim_{T-t\to\infty} b(t)$, we know both of them are independent of λ. In the following, we only focus on $\underline{t}$ and $\bar{t}$.

Proposition 4.3.6. *Denote $V^\lambda(t,S)$ be the corresponding value function $V(t,S)$ with loss aversion coefficient λ, so are other notations.*

i) If $1<\lambda_1<\lambda_2$, then $SR^{\lambda_1}\subset SR^{\lambda_2}$;

ii) The "last" liquidation time for the break-even level: $\bar{t}^\lambda=\bar{t}^1$, which is independent of λ;

iii) If $1<\lambda_1<\lambda_2$, then $\underline{t}^{\lambda_1}>\underline{t}^{\lambda_2}$.

Proof. We first show part i). Since $u^{\lambda_1}(x)=x^+-\lambda_1x^-=\max\{u^{\lambda_2}(x),\,\frac{\lambda_1}{\lambda_2}u^{\lambda_2}(x)\}$, we have $V^{\lambda_1}\geq V^{\lambda_2}$, and $V^{\lambda_1}\geq\frac{\lambda_1}{\lambda_2}V^{\lambda_2}$. Thus, for any $(t,S)\in SR^{\lambda_1}$,

$$\begin{aligned}V^{\lambda_1}(t,S) &= u^{\lambda_1}(S-K)=\max\left\{u^{\lambda_2}(S-K),\frac{\lambda_1}{\lambda_2}u^{\lambda_2}(S-K)\right\}\\ &\leq \max\left\{V^{\lambda_2}(t,S),\frac{\lambda_1}{\lambda_2}V^{\lambda_2}(t,S)\right\}\leq V^{\lambda_1},\end{aligned}$$

which implies $V^{\lambda_2}(t,S)=u^{\lambda_2}(S-K)$. So, $(t,S)\in SR^{\lambda_2}$, and $SR^{\lambda_1}\subset SR^{\lambda_2}$.

As to part ii), we only need to consider the case of $\mu<0$, otherwise, $\bar{t}=T$. Let us first consider the problem with $\lambda=1$, which can be solved in the same way as American option problem. It is well-known that smooth-fit principle holds on the strictly increasing free boundary for the case of $\lambda=1$. So, we have $\partial_SV^1(\bar{t}^1,K^-)$ $=1$.

Given any $S<K$, then $(\bar{t}^1,S)$ is in HR^1, and $0<\tau_{SR}(\bar{t}^1,S)\leq\tau_K^1$, where $\tau_K^1=\inf\left\{t\geq\bar{t}^1: S_t=K\right\}\wedge T$, with $S_{\bar{t}^1}=S$. Thus,

$$\begin{aligned}(S-K) &< V^1(\bar{t}^1,S)=\sup_{\bar{t}^1\leq\tau\leq\tau_K^1}\mathbb{E}(S_\tau-K)e^{-\beta\left(\tau-\bar{t}^1\right)}\\ &= \frac{1}{\lambda}\sup_{\bar{t}^1\leq\tau\leq\tau_K^1}\mathbb{E}\lambda(S_\tau-K)e^{-\beta\left(\tau-\bar{t}^1\right)}\\ &\leq \frac{1}{\lambda}V^\lambda(\bar{t}^1,S),\end{aligned}$$

which implies $\left(\bar{t}^1,S\right)\in HR^\lambda$. So, $\bar{t}^\lambda\geq\bar{t}^1$.

By the proof of part i), we have $\bar{t}^{\lambda} \leq \bar{t}^{1}$. So,

$$\bar{t}^{\lambda} = \bar{t}^{1} \text{ and } \partial_S V^{\lambda}(\bar{t}^{\lambda}, K^-) = \lambda \partial_S V^1(\bar{t}^{1}, K^-) = \lambda.$$

It remains to prove part iii). By part i), we see $\underline{t}^{\lambda_1} \geq \underline{t}^{\lambda_2}$. We only need to show $\underline{t}^{\lambda_1} \neq \underline{t}^{\lambda_2}$. Suppose not, then $\underline{t}^{\lambda_1} = \underline{t}^{\lambda_2} = \underline{t}$. By the proof of part i) and part ii), we have

$$V^{\lambda_1}(\underline{t}, S) = \frac{\lambda_1}{\lambda_2} V^{\lambda_2}(\underline{t}, S), \text{ for any } S < K.$$

On the other hand, it is easy to see $V^{\lambda_1}(\underline{t}, S) = V^{\lambda_2}(\underline{t}, S)$ for any $S > K$. Moreover, by Theorem 4.3.5,

$$\partial_S V^{\lambda_1}(\underline{t}, K^-) = \partial_S V^{\lambda_1}(\underline{t}, K^+) = \partial_S V^{\lambda_2}(\underline{t}, K^+) = \partial_S V^{\lambda_2}(\underline{t}, K^-),$$

which results in a contradiction. The proof completes.

Noted that a small modification of the proof can show that $V^1(t, S) = \frac{1}{\lambda} V^{\lambda}(t, S)$, for any $t \geq \bar{t}^{1}$ and $S < K$, which implies $SR^{\lambda} \cap \{(t, S) : S < K\}$ is independent of λ. Combining with part iii), larger loss aversion implies a stronger breakeven effect, since it is more likely to liquidate at the breakeven level for a larger λ. Moreover, part i) shows that the stronger loss aversion the larger selling region. Combining these two, we can see that larger λ will show a stronger disposition effect.

The effect of discount factor

From the expression of $b(T^-)$, it is easy to see that $b(T^-)$ is increasing in β if $\mu < 0$, and decreasing in β if $\mu > 0$. Moreover, from (4.3.9), a direct calculation shows that $\lim_{T-t\to\infty} b(t)$ is strictly decreasing in β. So, we only need to study the effect of β on $\underline{t}$ and $\bar{t}$.

Proposition 4.3.7. *Denote $V^{\beta}(t, S)$ be the corresponding value function $V(t, S)$ with the discount rate β, so are other notations.*

i) If $0 < \beta_1 < \beta_2$, *and* $\mu < 0$, *then* $\overline{t}^{\beta_1} \leq \overline{t}^{\beta_2}$;

ii) If $0 < \beta_1 < \beta_2$, *and* $\mu < \beta_2$, *then* $\underline{t}^{\beta_1} \leq \underline{t}^{\beta_2}$;

iii) If $\beta_1 < \beta_2 \leq \mu$, *then* $\underline{t}^{\beta_1} = \underline{t}^{\beta_2}$.

Proof. we only show part ii) and iii), while part i) is similar to part ii). Suppose part ii) fails, i.e. $\underline{t}^{\beta_1} > \underline{t}^{\beta_2}$. We consider $\mu \leq 0$ and $\mu > 0$ separately. Let us first treat the case of $\mu \leq 0$. For any $(t, S) \in [\underline{t}^{\beta_1}, T] \times (0, K]$, denote $\tau^{\beta_1} = \tau^{\beta_1}_{SR}(t, S)$ be the optimal stopping time for the investor with discount factor β_1. Denote $\tau_K = \inf\{s \geq t : S_s = K\} \wedge T$, then $\tau^{\beta_1} \leq \tau_K$. Thus,

$$V^{\beta_1}(t, S) = \mathbb{E}_{t,S} u(S_{\tau^{\beta_1}} - K) e^{-\beta_1\left(\tau^{\beta_1} - t\right)} \leq \mathbb{E}_{t,S} u(S_{\tau^{\beta_1}} - K) e^{-\beta_2\left(\tau^{\beta_1} - t\right)} \leq V^{\beta_2}(t, S) \leq 0,$$

where the first inequality follows by $u(S_{\tau^{\beta_1}} - K) \leq 0$.

In particular, we have $V^{\beta_1}(\underline{t}^{\beta_1}, S) \leq V^{\beta_2}(\underline{t}^{\beta_1}, S)$, which implies

$$\partial_S V^{\beta_1}(\underline{t}^{\beta_1}, K^-) \geq \partial_S V^{\beta_2}(\underline{t}^{\beta_1}, K^-).$$

By Theorem 4.3.5, we have

$$\partial_S V^{\beta_1}(\underline{t}^{\beta_1}, K^-) = \partial_S V^{\beta_1}(\underline{t}^{\beta_1}, K^+) = u'(0^+) = 1, \text{ then } \partial_S V^{\beta_2}(\underline{t}^{\beta_1}, K^-) \leq 1.$$

On the other hand,

$$\partial_S V^{\beta_2}(\underline{t}^{\beta_1}, K^-) > \partial_S V^{\beta_2}(\underline{t}^{\beta_2}, K^-) = \partial_S V^{\beta_2}(\underline{t}^{\beta_2}, K^+) = 1,$$

which results in a contradiction.

Now, we consider the case of $\mu > 0$. Then, $\tau^{\beta_1} = \tau^{\beta_2} = \tau_K$, for any $(t, S) \in [\underline{t}^{\beta_1}, T] \times (0, K]$. So,

$$\begin{aligned} V^{\beta_1}(t, S) &= \mathbb{E}_{t,S} u(S_{\tau_K} - K) e^{-\beta_1(\tau_K - t)} \\ &= e^{(\beta_2 - \beta_1)(T-t)} \mathbb{E}_{t,S} u(S_{\tau_K} - K) e^{-\beta_2(\tau_K - t)} = e^{(\beta_2 - \beta_1)(T-t)} V^{\beta_2}(t, S), \end{aligned}$$

which gives

$$\partial_S V^{\beta_1}(\underline{t}^{\beta_1}, K^-) = e^{(\beta_2-\beta_1)(T-\underline{t}^{\beta_1})}\partial_S V^{\beta_2}(\underline{t}^{\beta_1}, K^-).$$

Fixing $(t, S) \in [\underline{t}^{\beta_1}, T] \times (K, \infty)$, it can be shown similarly,

$$\begin{aligned} V^{\beta_1}(t,S) &= \mathbb{E}_{t,S} u(S_{\tau^{\beta_1}} - K) e^{-\beta_1\left(\tau^{\beta_1} - t\right)} \\ &\leq e^{(\beta_2-\beta_1)(T-t)} \mathbb{E} u(S_{\tau^{\beta_1}} - K) e^{-\beta_2\left(\tau^{\beta_1} - t\right)} \leq e^{(\beta_2-\beta_1)(T-t)} V^{\beta_2}(t,S), \end{aligned}$$

which implies $\partial_S V^{\beta_1}(\underline{t}^{\beta_1}, K^+) \leq e^{(\beta_2-\beta_1)(T-\underline{t}^{\beta_1})}\partial_S V^{\beta_2}(\underline{t}^{\beta_1}, K^+)$.

By Theorem 4.3.5,

$$\partial_S V^{\beta_2}(\underline{t}^{\beta_1}, K^-) > \partial_S V^{\beta_2}(\underline{t}^{\beta_2}, K^-) = \partial_S V^{\beta_2}(\underline{t}^{\beta_2}, K^+) \geq \partial_S V^{\beta_2}(\underline{t}^{\beta_1}, K^+),$$

which gives $\partial_S V^{\beta_1}(\underline{t}^{\beta_1}, K^-) > \partial_S V^{\beta_1}(\underline{t}^{\beta_1}, K^+)$.

Thus, we obtain a contradiction, then part ii) follows.

Now we show part iii). Thanks to part ii), it is easy to show

$$V^{\beta_1}(t,S) = e^{(\beta_2-\beta_1)(T-t)} V^{\beta_2}(t,S), \ \forall t \geq \underline{t}^{\beta_2},$$

which gives $\partial_S V^{\beta_1}(\underline{t}^{\beta_2}, K^-) = \partial_S V^{\beta_1}(\underline{t}^{\beta_2}, K^+)$.

So, $\underline{t}^{\beta_1} \geq \underline{t}^{\beta_2}$ by Theorem 4.3.5. By part ii), we have

$$\underline{t}^{\beta_1} = \underline{t}^{\beta_2},$$

which completes the proof.

Remark 4.3.8. *Numerical results show that both inequalities in part i) and ii) are strict, but currently we can not prove it. Moreover, when* $\mu = 0$, *one can easily show* $\underline{t}^{\beta} = T - \frac{\log\lambda}{\beta}$, *by the smooth-fit principle at* $(\underline{t}^{\beta}, K)$.

From the proof of part ii), it is easy to see the it is less likely to liquidate in a loss situation for a larger discount factor when $\mu < 0$. While $\lim_{T-t\to\infty} b(t)$ is strictly decreasing with respect to β, so it is more likely to liquidate in a small gain for a larger discount factor in the long term. In other words, larger β implies a "stronger" disposition effect.

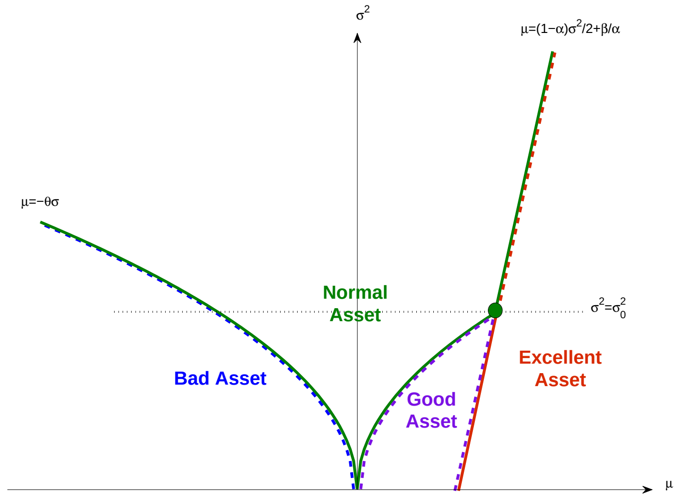

Figure 4.2: Four asset classes, where $\theta = \sqrt{2(1-\alpha)\beta/\alpha}$ and $\sigma_0^2 = \frac{2\beta}{\alpha(1-\alpha)}$.

4.4 Piecewise Power Utility

In this section, we will study the optimal selling strategies with a real S-shaped utility, i.e. $\alpha \in (0, 1)$ and examine how an S-shaped preference contributes in the disposition effect. Since the analysis will be similar to the case of $\alpha = 1$ but a bit tedious, we will omit most details or only notify the modifications. For easy presentation, we classify the parameters into four groups in Figure 4.2. One may consider the cases in $\alpha = 1$ as the limiting cases of $\alpha \to 1^-$.

4.4.1 Optimal liquidation strategies

We start with the dynamics of the payoff process. Since $u(x)$ is not smooth enough and has unbounded sensitivity at 0, we cannot apply Itô-Tanaka formula globally. Fortunately, some local behavior of $u(S_t - K)$ will be enough for our analysis. So, we may apply Itô formula on some stopped process.

Define $\tau = \inf\left\{t \geq 0 : S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t} = K\right\}$, then for any $S_0 = S > K$,

$$\begin{aligned} d\left\{u(S_t - K)e^{-\beta t}\right\} =& e^{-\beta t}\left(\frac{\sigma^2}{2}S_t^2 u''(S_t - K) + \mu S_t u'(S_t - K) - \beta u(S_t - K)\right) dt \\ &+ e^{-\beta t}\mu S_t u'(S_t - K) dB_t \\ =& e^{-\beta t}(S_t - K)^{\alpha - 2} f(S_t) dt + e^{-\beta t}\mu S_t u'(S_t - K) dB_t, \forall t \in [0, \tau], \end{aligned}$$

where $f(S) = \left(\mu - \frac{(1-\alpha)}{2}\sigma^2 - \frac{\beta}{\alpha}\right) S^2 + \left(\frac{2\beta}{\alpha} - \mu\right) SK - \frac{\beta}{\alpha}K^2$.

Similarly, for any $S_0 = S < K$, we have

$$d\left\{u(S_t - K)e^{-\beta t}\right\} = -\lambda e^{-\beta t}(K - S_t)^{\alpha - 2} f(S_t) dt + e^{-\beta t}\mu S_t u'(S_t - K) dB_t, \forall t \in [0, \tau].$$

In order to study whether the payoff process behaves as a submartingale or supmartingale locally, we only need to study the set

$$H = \{S > K : f(S) > 0\} \cup \{0 < S < K : f(S) < 0\},$$

which determines the sign of drift.

Lemma 4.4.1. *Given $\beta > 0$[2], denote $x_{\pm} = \frac{K\left(\mu - 2\beta/\alpha \pm \sqrt{\mu^2 - 2(1-\alpha)\sigma^2\beta/\alpha}\right)}{2(\mu - (1-\alpha)\sigma^2/2 - \beta/\alpha)}$ to be two different roots of $f(S) = 0$ if they exist. Then, for*

i) excellent asset, $H = (0, K) \cup (x_+, \infty)$, where $x_+ > K$;

ii) good asset, $H = (0, K) \cup (x_+, x_-)$, where $x_- > x_+ > K$;

iii) normal asset, $H = (0, K)$;

iv) bad asset, $H = (0, x_+) \cup (x_-, K)$, where $0 < x_+ < x_- < K$.

This lemma follows by a direct calculation.

[2] If $\beta = 0$, then the case of part ii) disappears and part iv) is replaced by (x_-, K). The following analysis still holds.

Proposition 4.4.2. *Given $\beta > 0$, the holding region*

$$HR \supset \{(t, S) : 0 \leq t < T, \text{ and } S \in H\}.$$

The proof is similar to Propsition4.3.1.

From this proposition, one can easily observe that it is never optimal to realize a small loss, but always possible to realize a small gain. This property also holds for the case of $\beta = 0$. But, with a positive discount factor, then it is always optimal to defer a large loss, which is very consistent with human behaviors in the financial market.

Proposition 4.4.3. *(Shape of SR) Given $\beta > 0$.*

i) If $(t_0, S) \in SR$, then $(t, S) \in SR$, for all $t > t_0$.

ii) For an excellent asset, if there exist (t_0,S_1) and (t_0,S_2), satisfying $K < S_1 \leq S_2 < x_+$, contained in SR, then $(S, t_0) \in SR$ for any $S \in (S_1, S_2)$.

iii) For a good asset, if there exist (t_0,S_1) and (t_0,S_2), satisfying $K < S_1 \leq S_2 < x_+$, (or (t_0, S_3), satisfying $S_3 > x_-$) contained in SR, then $(S, t_0) \in SR$ for any $S \in (S_1, S_2)$, (or $S \in (S_3, \infty)$).

iv) For a normal asset, if there exist (t_0, S_1) satisfying $S_1 > K$, contained in SR, then $(S, t_0) \in SR$ for any $S \in (S_1, \infty)$.

v) For a bad asset, if there exist (t_0,S_1) and (t_0,S_2), satisfying $x_+ < S_1 \leq S_2 < x_-$, (or (t_0, S_3), satisfying $S_3 > K$) contained in SR, then $(S, t_0) \in SR$ for any $S \in (S_1, S_2)$, (or $S \in (S_3, \infty)$).

Proof. Part i) follows by the monotonicity of $V(t, S)$ in t. The proof of part ii) to v) are similar. Without loss of generality, we only prove part iii).

Given $(t_{0,}S_1)$ and $(t_{0,}S_2)$ contained in SR and satisfying $K < S_1 \leq S_2 < x_+$, by part i), we see that $(t, S_i) \in SR$, for all $t \in [t_0, T]$, where $i = 1, 2$.

Denote $\tau = \inf\{s \geq t_0 : S_s \notin (S_1, S_2)\} \bigwedge T$, where $S_{t_0} = S \in (S_1, S_2)$, $\mathbb{P}_{t_0,S} - a.s.$. Thus, the optimal stopping time $\tau_{SR} \leq \tau$, $\mathbb{P}_{t_0,S} - a.s.$. Moreover,

$$\begin{aligned} u(S-K) &\leq V(t_0, S) = \mathbb{E}_{t_0,S}\ u(S_{\tau_{SR}} - K)e^{-\beta(\tau_{SR}-t_0)} \\ &= u(S-K) + \mathbb{E}_{t_0,S}\left(\int_{t_0}^{\tau_{SR}} e^{-\beta(t-t_0)}\left((S_t-K)^{\alpha-2}f(S_t)dt + \mu S u'(S_t-K)dB_t\right)\right) \\ &\leq u(S-K) + \mathbb{E}_{t_0,S}\left(\int_{t_0}^{\tau_{SR}} e^{-\beta(t-t_0)}(S_t-K)^{\alpha-2}f(S_t)dt\right) \\ &\leq u(S-K), \end{aligned} \tag{4.4.1}$$

where the last inequality is due to the negative integrand.

Thus, we conclude $V(t_0, S) = u(S-K)$, which implies $(t_0, S) \in SR$. The proof completes.

From Proposition 4.4.3, we can almost figure out the optimal selling region SR except on the break-even level. The following two propositions are going to character this.

Proposition 4.4.4. *Given $\beta > 0$, then $(t, S) \notin SR$, $\forall S \leq K$, if $T - t$ is large enough.*

The proof is the same as part ii) of Proposition 4.3.3. Thus, for the infinite problem, the investor will never realize a loss. This is also the essential reason that Barberis and Xiong (2010) derive a "strong" disposition effect, i.e. no loss realized unless liquidation shock arrives.

Proposition 4.4.5. *Given $t < T$, then $(t, K) \notin SR$.*

The proof is placed in the Appendix. As we have seen in part i) of Proposition 4.3.3, it is always optimal to liquidate the asset at break-even when time is close to maturity, because of the loss aversion. However, this proposition states that it is never optimal to sell at the break-even level no matter how close to the maturity and how loss aversion the investor is. This may be a bit counter-intuitive. Actually, the proof of Proposition 4.3.3 implicitly relies on the finite sensitivity on the break-even level of the utility function, which is violated here. The infinite sensitivity on the break-even level makes the investor no longer loss aversion around the breakeven level before maturity.

Now we are ready to present the optimal selling strategies in the following theorem.

Theorem 4.4.6. *(The optimal selling strategies) Given $\beta > 0$.*

i) For an excellent asset , there exist a critical point $t^* > 0$ independent of T and two free boundaries $b_{1s}(t)$ decreasing and $b_{2s}(t)$ increasing with $b_{1s}(t^*) = b_{2s}(t^*)$, such that

$$SR = \{(t,S) : b_{1s}(t) \le S \le b_{2s}(t), t^* \le t < T\} \cup \{(T,S) : S > 0\}.$$

Moreover, $\lim_{t\to T^-} b_{1s}(t) = K$, $\lim_{t\to T^-} b_{2s}(t) = x_+$.

ii) For a good asset, there exist a critical point $t^* > 0$ independent of T and three free boundaries $b_{1s}(t) : [t^*, T) \to (0,\infty)$, decreasing, $b_{2s}(t) : [t^*, T) \to (0,\infty)$, increasing, with $b_{1s}(t^*) = b_{2s}(t^*)$ and $b_{3s}(t) : [0,T) \to (x^+,\infty)$, decreasing, such that

$$\begin{aligned} SR &= \{(t,S) : b_{1s}(t) \le S \le b_{2s}(t),\ t^* \le t < T\} \cup \{(t,S) : S \ge b_{3s}(t), t < T\} \\ &\quad \cup \{(T,S) : S > 0\}. \end{aligned}$$

Moreover, $\lim_{t\to T^-} b_{1s}(t) = K$, $\lim_{t\to T^-} b_{2s}(t) = x_+$ *and* $\lim_{t\to T^-} b_{3s}(t) = x_-$, $\lim_{T-t\to\infty} b_{3s}(t) = \frac{\gamma K}{\gamma-\alpha}$.

iii) For a normal asset, there exists a free boundary $b_{1s}(t) : [0, T) \to (0, \infty)$, *decreasing, such that*

$$SR = \{(t, S) : S \geq b_{3s}(t), 0 \leq t < T\} \cup \{(T, S) : S > 0\}.$$

Moreover, $\lim_{t \to T^-} b_{1s}(t) = K$ *and* $\lim_{T-t \to \infty} b_{1s}(t) = \frac{\gamma K}{\gamma - \alpha}$.

iv) For a bad asset, there exist a critical point $t^* > 0$ *independent of* T *and three free boundaries* $b_{1s}(t) : [t^*, T) \to (0, K)$, *decreasing,* $b_{2s}(t) : [t^*, T) \to (0, K)$, *increasing, with* $b_{1s}(t^*) = b_{2s}(t^*)$ *and* $b_{3s}(t) : [0, T) \to [K, \infty)$, *decreasing, such that*

$$\begin{aligned} SR &= \{(t, S) : b_{1s}(t) \leq S \leq b_{2s}(t),\ t^* \leq t < T\} \cup \{(t, S) : S \geq b_{3s}(t), t < T\} \\ &\quad \cup \{(T, S) : S > 0\}. \end{aligned}$$

Moreover, $\lim_{t \to T^-} b_{1s}(t) = x_+$, $\lim_{t \to T^-} b_{2s}(t) = x_-$ *and* $\lim_{t \to T^-} b_{3s}(t) = K$, $\lim_{T-t \to \infty} b_{3s}(t) = \frac{\gamma K}{\gamma - \alpha}$.

The proof is similar to Theorem 4.3.4 and thus omitted.

4.5 Numerical Results

In this section, we apply the penalty method to numerically study the optimal liquidation strategies, the effect of loss aversion and discounting factor on the liquidation boundary, then move to Monte-Carlo simulations. By simulating the sample path of asset, we can study the probability of liquidation at a gain (or loss) and also the expected holding time of the asset.

4.5.1 Optimal liquidation boundaries

In Figure 4.3-4.6, we show the plots of the optimal liquidation strategies as a function of t in these four cases. The blue solid line and red dash line are the optimal liquidation boundaries for the investor with piecewise linear utility ($\alpha = 1$) and piecewise power utility ($\alpha < 1$), respectively. These plots verify the properties stated in Theorem 4.3.4 and 4.4.6. Figure 4.3 shows that for an excellent asset, it is only optimal to liquidate at the break-even level when time remained is not much for an investor with piecewise linear utility, while for an investor with S-shaped utility, there are two branches of the liquidation boundary and it is only optimal to liquidate the asset when the price falls in the region enclosed by these two branches, i.e. in a small gain region. Similarly, in Figure 4.4, the investor with piecewise linear utility will liquidate a good asset either at break-even when time remained is not much or at a large gain region; while the investor with S-shaped utility will liquidate the good asset either in a small gain region when time remained is not much or in a large gain region. Moreover, from Figure 4.3-4.6, we can summarize that liquidation at a loss is optimal only for a bad asset, i.e. $\mu < 0$, while liquidation at a (small) gain is optimal for most cases, except for an extremely good asset. This result shows that the investor is more readily to sell the asset at a (small) gain than at a loss, which predicts the disposition effect. Comparing with liquidation behaviors between $\alpha = 1$ and $\alpha < 1$, the main difference is at the break-even level. It can be seen that it is always optimal to liquidate the asset at the break-even level as long as the time remaining is short enough no matter how good the asset is for $\alpha = 1$, while it is always not optimal to liquidate the asset at the break-even level no matter how bad the asset is for $\alpha < 1$. The intuition is that for $\alpha = 1$, the investor has finite sensitivity at the break-even level and $\lambda > 1$ makes the investor loss aversion, which implies the investor not willing to take risk. This phenomenon is also called break-even effect.

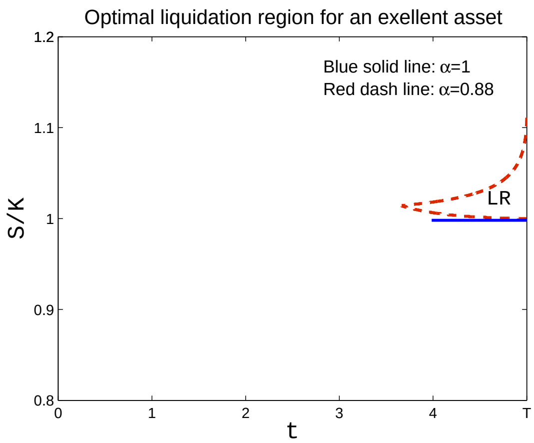

Figure 4.3: Excellent Asset

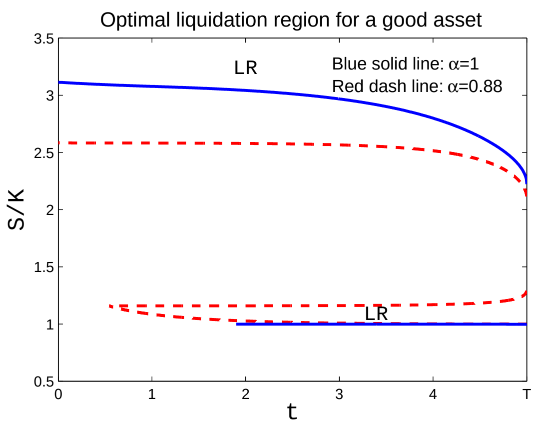

Figure 4.4: Good Asset

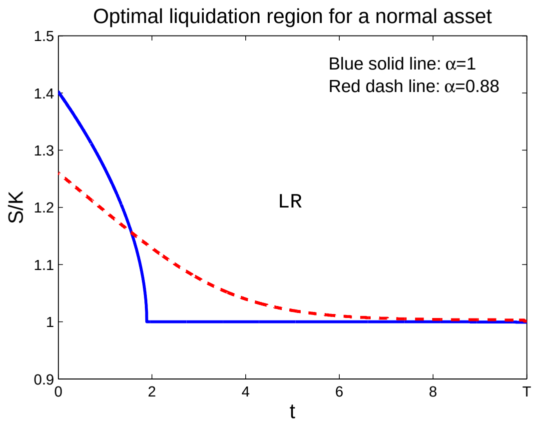

Figure 4.5: Normal Asset

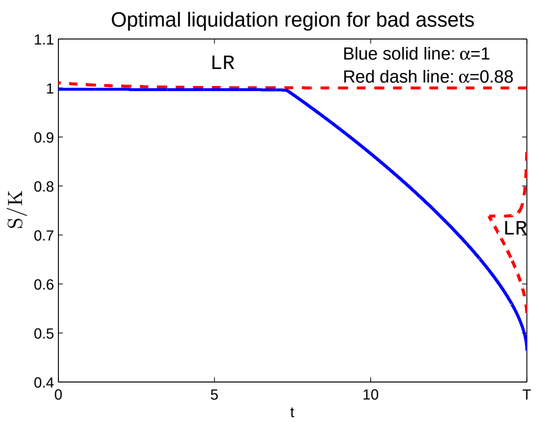

Figure 4.6: Bad Asset

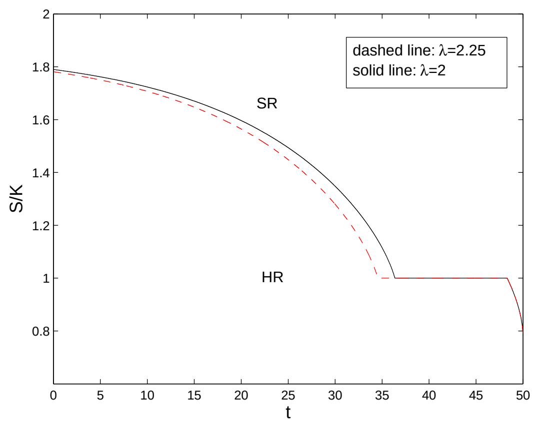

Figure 4.7: The effect of loss aversion.

4.5.2 Effect of loss aversion and discounting factor

We now investigate the impacts of loss aversion and discount factor on the optimal liquidation strategies. Figure 4.7 shows the effect of the loss aversion coefficient λ in the case of $\mu < 0$. It can be observed that the optimal selling region is independent of λ for $t > \bar{t}$. Moreover, the first time of liquidating at the break-even level is earlier for larger λ, which implies liquidating at the break-even level is more likely for an investor with stronger loss aversion. In Figure 4.8, we show the dependence of the optimal liquidation strategies on the discount factor β. Intuitively, the discount factor β plays a different role in the gain and loss, so the effect of β will be a bit complicate. Since future loss is less painful for the investor with larger β, while future gain is also less favorable, SR in loss situation should be smaller for larger β, and SR in gain situation should be larger for larger β. This intuition is almost figured out in Figure 4.8. In particular, in a long time (infinite) horizon, it is optimal to realize a smaller gain for larger β, and not optimal to realize a loss.

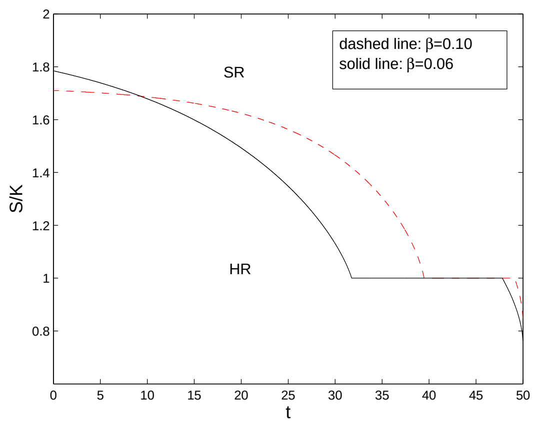

Figure 4.8: The effect of the discount factor.

4.5.3 Monte-Carlo simulations

We also simulate the sample path to find the probability of liquidation at a gain (or loss) and the average holding time (AHT) of the asset. Since the liquidation strategies change significantly against the parameters, we only vary the asset return μ, time horizon T and fix all the other parameters. In our numerical calculations, we choose the parameter values: $\sigma = 0.4, \beta = 0.07, \alpha = 0.88$, and $\lambda = 2.25$. The value of μ and T are depicted in each table. Table 4.1-4.4 clearly show that the probability of liquidation in gain is much higher than the probability in loss and the average holding time (AHT) is longer for the asset which is liquidated at a loss than one liquidated at a gain. These results are consistent with the disposition effect.

4.6 Summary

The main results of this chapter are as follows.

- We formulate a finite horizon optimal liquidation problem for an investor

Table 4.1: Excellent Asset.

Parameter values: $\mu = 0.10,\ T = 5.$

No. of Path	10^5		2×10^5		4×10^5	
	Prob(%)	AHT	Prob(%)	AHT	Prob(%)	AHT
Gain	70.3	4.551	70.3	4.549	70.3	4.549
Loss	29.7	5	29.7	5	29.7	5

Table 4.2: Good Asset.

Parameter values: $\mu = 0.06,\ T = 5.$

No. of Path	10^5		2×10^5		4×10^5	
	Prob(%)	AHT	Prob(%)	AHT	Prob(%)	AHT
Gain	84.0	1.246	84.1	1.246	84.1	1.248
Loss	16.0	5	15.9	5	15.9	5

Table 4.3: Normal Asset.

Parameter values: $\mu = 0.04,\ T = 5.$

No. of Path	10^5		2×10^5		4×10^5	
	Prob(%)	AHT	Prob(%)	AHT	Prob(%)	AHT
Gain	94.7	0.216	94.7	0.216	94.7	0.216
Loss	5.3	5	5.3	5	5.3	5

Table 4.4: Bad Asset.

Parameter values: $\mu = -0.08,\ T = 15.$

No. of Path	10^5		2×10^5		4×10^5	
	Prob(%)	AHT	Prob(%)	AHT	Prob(%)	AHT
Gain	97.7	0.061	97.7	0.063	97.7	0.063
Loss	2.3	14.987	2.3	14.989	2.3	14.988

with prospect theory preferences.

- We show the existence and uniqueness of the optimal liquidation strategies.
- The main application of our model is to predict the disposition effect and break-even effect.
- Extensive numerical results are provided.

4.7 Appendix

4.7.1 The proof of $\lim_{T-t\to\infty} b(t)$

To discern the limit behavior of $b(t)$ when $\mu < \beta$, we need to study the infinite horizon problem,

$$\overline{V}(S) = \sup_{\tau} \mathbb{E}_S \left(u(S_\tau - K)e^{-\beta\tau} \right). \tag{4.7.1}$$

Proposition 4.7.1. *Let $W(S)$ be the unique solution to (4.7.2)*

$$\min\{-\mathcal{L}W, W - u(S-K)\} = 0, \; in \; (0,\infty) \tag{4.7.2}$$

Then,

i) $\overline{V}(S) = W(S) = \begin{cases} \frac{S^\gamma}{\gamma b^{\frac{\gamma}{\gamma-1}}}, & \text{if } S < b, \\ S - K, & \text{if } S \geq b, \end{cases}$ *where* $\gamma = \frac{-(\mu - \frac{\sigma^2}{2}) + \sqrt{(\mu-\frac{\sigma^2}{2})^2 + 2\beta\sigma^2}}{\sigma^2}$, *and* $b = \frac{\gamma K}{\gamma - 1} > K$.

ii) $\lim_{T-t\to\infty} V(t,S) = \overline{V}(S)$, *where $V(t,S)$ is defined in (4.2.2).*

iii) $\lim_{T-t\to\infty} b(t) = b$.

Proof. Part i) is a well-known result, which can be proved directly by the general optimal stopping theory (cf. Peskir and Shiryaev 2006). As to part ii), it is obviously, $V(t,S) \leq \overline{V}(S)$. Conversely, denote $\tau = \inf\{t > 0 : S_t \geq b\}$, then $\overline{V}(S) = \mathbb{E}_S u(S_\tau - K)e^{-\beta\tau}$. And if we define $\tau_t = \tau \bigwedge (T-t)$, then for all $S < b$,

$$\begin{aligned} V(t,S) &\geq \mathbb{E}_{t,S} u(S_{t+\tau_t} - K)e^{-\beta\tau_t} \\ &\geq (b-K)\mathbb{E}_S e^{-\beta\tau} I_{\{\tau \leq T-t\}} + u(S_{T-t} - K)e^{-\beta(T-t)}\mathbb{E}_S I_{\{\tau > T-t\}} \\ &\geq (b-K)\mathbb{E}_S e^{-\beta\tau} I_{\{\tau \leq T-t\}} - \lambda K e^{-\beta(T-t)} \end{aligned}$$

Let $T-t \to \infty$, we have $\lim_{T-t\to\infty} V(t,S) \geq (b-K)\mathbb{E}_S e^{-\beta\tau} = \overline{V}(S)$, for all $S < b$.

Now, we show part iii). It is obvious that $b(t) \leq b$, since $V(t,S) \leq \overline{V}(S)$. Thus, $\lim_{T-t\to\infty} b(t) \leq b$. Conversely, fix any $S < b$ and denote $\epsilon = \overline{V}(S) - u(S-K) > 0$. So, there exists $T_\epsilon > 0$ such that $V(t,S) > \overline{V}(S) - \frac{\epsilon}{2} = u(S-K) + \frac{\epsilon}{2}$ for all $T-t > T_\epsilon$. Thus, $b(t) \geq S$, which results in $\lim_{T-t\to\infty} b(t) \geq b$. The proof completes.

4.7.2 The proof of Theorem 4.3.5

Proof. We show this result in two steps:

i) construct a function $\varphi(t,S)$ which is differentiable at (t^*, K) for some t^*;

ii) show $\varphi(t,S) = V(t,S)$ in $[t^*, T] \times (0,\infty)$ and $\underline{t} = t^*$.

We only show the case of $0 < \mu < \beta$, while the other cases can be done similarly. Since $u(S-K)$ has a kink at $S = K$, we separate (4.2.4) into two equations:

$$\begin{cases} -\partial_t W_1 - \mathcal{L} W_1 = 0, \text{ in } [0,T) \times (0,K) \\ W_1(T,S) = \lambda(S-K), \quad W_1(t,K) = 0, \end{cases} \tag{4.7.3}$$

and

$$\begin{cases} \min\{-\partial_t W_2 - \mathcal{L} W_2, W_2 - (S-K)\} = 0, \text{ in } [0,T) \times (K,\infty) \\ W_2(t,K) = 0, \quad W_2(T,S) = (S-K). \end{cases} \tag{4.7.4}$$

We claim that there exists a unique $t^* < T$, such that $\partial_S W_1(t^*, K^-) = \partial_S W_2(t^*, K^+)$ and $\partial_S W_1(t, K^-) > \partial_S W_2(t, K^+)$ for all $t > t^*$, where $W_1(t,S)$ and $W_2(t,S)$ are the unique solutions of (4.7.3) and (4.7.4) in proper sense. Indeed, it is easy to show that (4.7.3) admits a unique smooth solution. By the maximum principle, one also can easily show that $W_1 \geq \lambda e^{-\beta(T-t)}(S-K)$ and $\partial_t W_1(t,S) \leq 0$.Thus, $\partial_S W_1(t,K^-)$ increases in t and

$$\partial_S W_1(t,K^-) \leq \lambda e^{-\beta(T-t)}.$$

In addition, since $\partial_t W_1(t,K)=0$ achieves its maximum, we can obtain $\partial^2_{tS} W_1(t,K^-) > 0$, which implies $\partial_S W_1(t,K^-)$ strictly increasing in t.

On the other hand, by the standard penalty method (cf. Friedman 1982), we can easily show (4.7.4) admits a unique solution $W_2(t,S)$ in $W^{2,1}_p(\Omega_N)$, where $p>1$ and Ω_N is any bounded set in $[0,T)\times[K,\infty)$. Moreover, $W_2(t,S)$ is decreasing in t. Thus, $\partial_S W_2(t,K^+)$ is increasing in t. Since $\partial_S W_1(T,K^-)=\lambda>1=\partial_S W_2(T,K^+)$, there exists a unique t^* such that $\partial_S W_1(t^*,K^-)=\partial_S W_2(t^*,K^+)$ and $\partial_S W_1(t,K^-)>\partial_S W_2(t,K^+)$ for all $t>t^*$.

Now, let us denote

$$\varphi(t,S)=\begin{cases} W_1(t,S), & \text{if } S<K, \\ W_2(t,S), & \text{if } S\geq K, \end{cases}$$

and claim that $\varphi(t,S)=V(t,S)$ in $[t^*,T]\times(0,\infty)$. Indeed, we only need to show $\varphi(t,S)$ is a viscosity solution of (4.2.4) in $[t^*,T]\times(0,\infty)$, then by Theorem 4.2.3, we have $\varphi(t,S)=V(t,S)$.

It is obvious that $\varphi(t,S)$ is a viscosity subsolution of (4.2.4) since $\varphi(t,S)$ satisfies either $-\partial_t\varphi(t,S)-\mathcal{L}\varphi(t,S)=0$ or $\varphi(t,S)=u(S-K)$, in $(t^*,T]\times(0,\infty)$.

Now let us prove $\varphi(t,S)$ is a viscosity supersolution of (4.2.4) in $(t^*,T]\times(0,\infty)$. For any $(t,S)\in(t^*,T]\times\{K\}$, there is no $\phi(t,S)\in C^{1,2}\left((t^*,T]\times(0,\infty)\right)$ such that $\phi\leq\varphi$, because of $\partial_S\varphi(t,K^-)>\partial_S\varphi(t,K^+)$. So $\varphi(t,S)$ is a viscosity supersolution of (4.2.4) on $(t^*,T]\times\{K\}$. For any $(t,S)\in(t^*,T]\times(0,K)$, $\varphi(t,S)=W_1(t,S)$, which is a smooth solution of (4.7.3). Then, $-\partial_t\varphi-\mathcal{L}\varphi=0$, and $\varphi(t,S)\geq$

$\lambda e^{-\beta(T-t)}(S-K)$, which implies

$$\min\{-\partial_t\varphi - \mathcal{L}\varphi, \varphi - \lambda(S-K)\} \geq 0.$$

For any $(t,S) \in (t^*,T] \times (K,\infty)$, it is not hard to show $W_2(t,S)$ is a viscosity solution, thus also a viscosity supersolution, of (4.7.4). Thus, $\varphi(t,S) = W_2(t,S)$ satisfies

$$\min\{-\partial_t\varphi - \mathcal{L}\varphi, \varphi - (S-K)\} \geq 0.$$

Therefore, $\varphi(t,S)$ is indeed a viscosity solution of (4.2.4) in $(t^*,T] \times (0,\infty)$. So, $\varphi(t,S) = V(t,S)$ in $(t^*,T] \times (0,\infty)$. Since both V and φ are continuous on $[0,T] \times (0,\infty)$, we conclude that $V(t,S) = \varphi(t,S)$ in $[t^*,T] \times (0,\infty)$, especially, $V(t^*,K) = \varphi(t^*,K) = 0$ and $V_S(t^*,K^+) = V_S(t^*,K^-)$.

Thus, by the definition of $\underline{t}$, we have $\underline{t} \leq t^*$. Now, we are going to show $\underline{t} = t^*$.

Suppose not, then $\underline{t} < t^*$. By part ii) of Theorem 4.3.4, it is easy to see $\varphi(t,S) = V(t,S)$ in $[\underline{t},T] \times (0,\infty)$, then $\partial_S V(t,K^-) < \partial_S V(t,K^+)$ for any $t \in [\underline{t},t^*)$. Let us fix a $t_0 \in (\underline{t},t^*)$, then $\exists \theta > 0$, such that

$$V(t_0,S) > \begin{cases} k^-(S-K) & \text{if } S \in [K(1-\theta),K), \\ k^+(S-K) & \text{if } S \in [K,K(1+\theta)], \end{cases} \tag{4.7.5}$$

where $k^- = (2\partial_S V(t_0,K^-) + \partial_S V(t_0,K^+))/3$ and $k^+ = (\partial_S V(t_0,K^-) + 2\partial_S V(t_0,K^+))/3$.

Denote $f(\nu) = V(t_0 - \nu^2, K)$ and $\eta = \left(\mu - \frac{\sigma 2}{2}\right)\nu^2 + \sigma\nu B_1$, for $0 < \nu << 1$. It is easy to evaluate that

$$\mathbb{E}I_{\{|e^\eta - 1| > \theta\}} = o(v^4).$$

Then,

$$
\begin{aligned}
f(\nu) &= V(t_0-\nu^2,K) \geq \mathbb{E}_{t-\nu^2,K}\left[e^{-\beta\nu^2}V(t_0,S_{t_0})\right] \\
&= \mathbb{E}\left[e^{-\beta\nu^2}V\left(t_0, Ke^{\left(\mu-\frac{\sigma^2}{2}\right)\nu^2+\sigma B_{\nu^2}}\right)\right] = e^{-\beta\nu^2}\mathbb{E}\left[V\left(t_0,Ke^{\eta}\right)\right] \\
&\geq e^{-\beta\nu^2}\left\{-\lambda K\mathbb{E}I_{\{|e^\eta-1|>\theta\}} + k^-\mathbb{E}K(e^\eta-1)I_{\{-\theta\leq e^\eta-1\leq 0\}} + k^+\mathbb{E}K(e^\eta-1)I_{\{0\leq e^\eta-1\leq\theta\}}\right\} \\
&\geq e^{-\beta\nu^2}\left\{-\lambda K\mathbb{E}I_{\{|e^\eta-1|>\theta\}} + k^-K\mathbb{E}(e^\eta-1)I_{\{|e^\eta-1|\leq\theta\}}\right. \\
&\qquad \left. +(k^+-k^-)K\mathbb{E}(e^\eta-1)I_{\{0\leq e^\eta-1\leq\theta\}}\right\} \\
&\geq -\lambda K\mathbb{E}I_{\{|e^\eta-1|>\theta\}} + k^-K\mathbb{E}(e^\eta-1) - k^-K\mathbb{E}(e^\eta-1)I_{\{|e^\eta-1|>\theta\}} \\
&\qquad + (k^+-k^-)K\mathbb{E}\eta I_{\left\{0\leq\eta\leq\frac{\theta}{2}\right\}} + o(v) \\
&\geq -k^-K\sqrt{\mathbb{E}(e^\eta-1)^2}\sqrt{\mathbb{E}I_{\{|e^\eta-1|>\theta\}}} + (k^+-k^-)K\sigma\nu\mathbb{E}B_1 I_{\left\{-(\mu-\frac{\sigma^2}{2})\nu\leq B_1\leq\frac{\theta}{2\sigma\nu}-(\mu-\frac{\sigma^2}{2})\nu\right\}} \\
&\qquad + o(\nu) \\
&= (k^+-k^-)K\sigma\nu\mathbb{E}B_1 I_{\left\{-(\mu-\frac{\sigma^2}{2})\nu\leq B_1\leq\frac{\theta}{2\sigma\nu}-(\mu-\frac{\sigma^2}{2})\nu\right\}} + o(\nu).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f'(0) &= \lim_{\nu\downarrow 0}\frac{f(\nu)-f(0)}{\nu} = \lim_{\nu\downarrow 0}(k^+-k^-)K\sigma\mathbb{E}B_1 I_{\left\{-(\mu-\frac{\sigma^2}{2})\nu\leq B_1\leq\frac{\theta}{2\sigma\nu}-(\mu-\frac{\sigma^2}{2})\nu\right\}} \\
&= (k^+-k^-)K\sigma\mathbb{E}B_1^+ > 0.
\end{aligned}
$$

Combining with $f(0)=V(t_0,K)=0$, we get $f(v)>0$, for ν small enough.

Therefore, $V(t_0-\nu^2,K)>0$, which contradicts with $(t_0-\nu^2,K)\in SR$. Thus, we show $\underline{t}=t^*$, which completes the proof.

4.7.3 The proof of Proposition 4.4.5

Proof. We prove it by contradiction. Suppose not, then there exists a $t_0<T$, s.t. $(t_0,K)\in SR$. Thanks to Proposition 4.4.4 part i), we see $(t,K)\in SR$ for any $t\in[t_0,T]$. Moreover, Proposition 4.4.3 also shows that there always exists a $S_0<K$, such that $\{(t,S): S_0\leq S<K\}\subset HR$.

Now, let $W(t, S)$ be the solution to the following linear PDE,

$$\begin{cases} \partial_t W + \mathcal{L}W = 0, \text{ in } [t_0, T) \times (S_0, K), \\ W(T, S) = -(K - S)^\alpha, \forall S \in [S_0, K], \\ W(t, K) = 0, \ W(t, S_0) = V(t, S_0), \forall t \in [t_0, T]. \end{cases}$$

Applying the $W_p^{2,1}$-estimate in PDE, we have

$$W(t, S) \in W_p^{2,1}(\Omega_\delta), \text{ for any } p > 1,$$

where $\Omega_\delta = [t_0, T] \times [S_0, K] \backslash \{(t, S) : (t - T)^2 + (S - K)^2 \leq \delta^2\}$ and $0 < \delta < (T - t_0)/4$.

Choosing p large enough, we infer that $\partial_S W(t, S)$ is continuous at (t, K) for $t_0 \leq t < T - \delta$ by the imbedding Theorem, and thus bounded by some constant C_δ.

By the strong Markov property of S_t, it is easy to see that $V(t, S) = W(t, S)$. Thus, $\partial_S V(t, K^-) < \infty$ for $t_0 \leq t < T - \delta$. On the other hand, $V(t, S) \geq u(S - K)$ for $S > K$, and $V(t, K) = 0 = u(K - K)$ for $t \in [t_0, T]$. So, $\partial_S V(t, K^+) \geq \partial_S u(0^+) = \infty > \partial_S V(t, K^-)$.

Thus, a similar analysis in the proof of $\underline{t} = t^*$ in Theorem 4.3.5 will show that $V(t - \delta, K) > 0$, for any δ small enough, which contradicts $(t - \delta, K) \in SR$. The proof completes.

Chapter 5

Mean-Semi-Variance Portfolio Selection with Probability Distortion

In this chapter, we study the distorted mean-semi-variance portfolio selection problem. A quantile approach is adopted to tackle the main difficulty of probability distortion, which makes our problem lose the convexity in decision variable and also the time consistency. The remainder of this chapter is organized as follows. Section 1 formulates the problem. Section 2 is devoted to solve the optimal terminal wealth via quantile formulation. In Section 3, we give some specific examples, while Section 4 solves the efficient frontier and efficient strategy when the optimal solution exists. Section 5 summarizes this chapter.

5.1 Formulation

In this section, we will set up the continuous-time market and formulate our distorted mean-semi-variance portfolio selection problem.

5.1.1 The financial market

Let $T > 0$ is a fixed terminal time and $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed filtered complete probability space on which a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted m-dimensional Brownian motion $W(t) := (W_1(t), W_2(t), \cdots, W_m(t))^\top$ is defined, where the sign $\top$ here and after means the transpose of a matrix or a vector. We denote by $\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ the set of all $\mathbb{R}^m$-valued, measurable stochastic processes $f(\cdot) = \{f(t) : 0 \leq t \leq T\}$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $\mathbb{E}\int_0^T |f(t)|^2 dt < +\infty$ and by $\mathcal{L}^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^m)$ the set of all $\mathbb{R}^m$-valued, $\mathcal{F}_T$-measurable random variables X such that $\mathbb{E}|X|^2 < +\infty$.

Suppose there exists a financial market where $m + 1$ assets are traded continuously over a finite horizon $[0, T]$. One of the assets is a riskfree bond, whose price process $P_0(t)$ is subject to the following differential equation

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, t \in [0, T], \\ P_0(0) = p_0, \end{cases}$$

where the interest rate $r(\cdot)$ is a scalar-valued, $\mathcal{F}_t$-progressively measurable stochastic process with $\int_0^T |r(t)|dt < +\infty$, *a.s.*. The other m assets are stocks whose price processes $P_i(t), i = 1, 2, \cdots, m$, satisfy the following stochastic differential equations

$$\begin{cases} dP_i(t) = P_i(t)[b_i(t)dt + \displaystyle\sum_{j=1}^m \sigma_{ij}(t)dW_j(t)], & t \in [0, T], \\ P_i(0) = p_i, & i = 1, 2, \cdots, m, \end{cases}$$

where $b_i(t)$ and $\sigma_{ij}(t)$, the appreciation and volatility rates respectively, are scalar-valued, $\mathcal{F}_t$-progressively measurable stochastic processes with

$$\int_0^T \left[\sum_{i=1}^m |b_i(t)| + \sum_{i=1}^m \sum_{j=1}^m \sigma^2_{ij}(t)\right] dt < +\infty, \ a.s..$$

Set the excess rate of return process

$$B(t) := (b_1(t) - r(t), ..., b_m(t) - r(t))^\top,$$

and define the volatility matrix process $\sigma(t) := (\sigma_{ij}(t))_{m\times m}$. Basic assumptions imposed on the market parameters throughout this chapter are summarized as follows.

Assumption 5.1.1. *i) There exists $\delta > 0$ such that $\int_0^T |r(t)|dt \geq \delta$, a.s..*

ii) $Rank(\sigma(t)) = m$, a.e. $t \in [0, T]$, a.s..

iii) There exists an R^m-valued, uniformly bounded, $\mathcal{F}_t$-progressively measurable process $\theta(\cdot)$ such that $\sigma(t)\theta(t) = B(t)$, a.e. $t \in [0, T]$, a.s..

It is well known that under Assumption 5.1.1 there exists a unique risk-neutral probability measure Q defined by $\frac{dQ}{dP}|_{\mathcal{F}_t} = \rho(t)P_0(t)$, where

$$\rho(t) = \exp\left\{-\int_0^t [r(s) + |\theta(s)|^2/2]ds - \int_0^t \theta(s)^\top dW(s)\right\}$$

is the pricing kernel or state density price. Denote $\rho := \rho(T)$ with its cumulative distribution function $F_\rho(\cdot)$. It is clear that $0 < \rho < +\infty$, a.s. and $0 < E\rho < +\infty$.

Assumption 5.1.2. *ρ admits no atom.*

This assumption will be in force hereafter. This is a technical assumption to ensure the uniqueness of the optimal terminal wealth. It is satisfied, in particular, when $r(\cdot)$ and $\theta(\cdot)$ are deterministic with $\int_0^T ||\theta||^2(t)dt \neq 0$ (in which case ρ is a non-degenerate log-normal random variable).

Consider an agent, with an initial wealth $x_0 > 0$, whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Assume that the trading of shares takes place continuously in a self-financing fashion and there is no transaction cost, then $x(\cdot)$ satisfies (cf. Karatzas and Shreve 1998)

$$\begin{cases} dx(t) = [r(t)x(t) + B(t)^\top \pi(t)]dt + \pi(t)^\top \sigma(t)dW(t), t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{5.1.1}$$

where $\pi(t) := (\pi_1(t), \pi_2(t), \cdots, \pi_m(t))^\top$ with $\pi_i(t), i = 1, 2, \cdots, m$, denoting the total market value of the agent's wealth in the i-th stock at time t. The process $\pi(\cdot)$ is said to be a portfolio if it is $\mathcal{F}_t$-progressively measurable with

$$\int_0^T |\sigma(t)^\top \pi(t)|^2 dt < +\infty \text{ and } \int_0^T |B(t)^\top \pi(t)| dt < +\infty,\ a.s., \tag{5.1.2}$$

and it is admissible if $\pi(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$.

5.1.2 Problem formulation

Next we will introduce the distorted mean-semi-variance problem. Let $w(\cdot) : [0, 1] \to [0, 1]$ be the probability distortion function (or decumulative weighting function), representing a subjective inflation/deflation of the true probability. We assume that $w(\cdot)$ is generally non-linear, strictly increasing and continuously differentiable on $(0, 1)$ with $w(0) = 0$ and $w(1) = 1$. So the distortion at least preserves the order of the probabilities and does not distort any sure or null events.

Recall that the mathematical expectation of a random payoff X with cumulative distribution function $F_X(x) = \mathbb{P}(X \leq x)$ is

$$\mathbb{E}(X) := \int_0^{+\infty} \mathbb{P}(X \geq x) dx + \int_{-\infty}^0 [\mathbb{P}(X \geq x) - 1] dx.$$

With probability distortion, the distorted mean is defined by

$$\widetilde{\mathbb{E}}(X) := \int_0^{+\infty} w(\mathbb{P}(X \geq x)) dx + \int_{-\infty}^0 [w(\mathbb{P}(X \geq x)) - 1] dx$$

which is the Choquet expectation of X with respect to capacity $w \circ \mathbb{P}$ (a non-linear probability measure) and the distorted semi-variance (DSV)[1] is defined by

$$DSV(X) := -\widetilde{\mathbb{E}}[-(X - \widetilde{\mathbb{E}}(X))^2_-],$$

where x_+ and x_- denote respectively the positive part and negative part of a real number throughout this chapter.

[1]Due to no linearity of Choquet expectation, we choose a way to define the semi-variance that is consistent with the spirit of loss accounted as negative value.

Remark 5.1.1. *In prospect theory, for a random variable X (taking the origin as the reference point), its nonlinear expectation is defined by*

$$\widetilde{\widetilde{\mathbb{E}}}(X) := \widetilde{\mathbb{E}}(X_+) - \widetilde{\mathbb{E}}(X_-) = \int_0^{+\infty} w_+(\mathbb{P}(X_+ \geq x))dx - \int_0^{\infty} w_-(\mathbb{P}(X_- \geq x))dx.$$

Notice that there is a difference between the Choquet expectation and prospect expectation. But, both of them have their own interpretations. In the Choquet expectation, the investor only distorts the right tail events, while in prospect theory, the investor has different distortions for the right tail events and left tail events. In addition, if we choose $w_-(z) = 1 - w_+(1-z)$ and $w_+(z) = w(z)$, then the two distorted expectations are the same. To simplify the computation, we only consider the Choquet expectation or equivalently, we assume $w_-(z) = 1 - w_+(1-z)$ when we work in the prospect theory framework.

Note that if we put $w(z) = z$, the distorted mean and semi-variance of X are reduced the conventional mean and semi-variance of X.

The problem we will consider thereafter is

$$\begin{aligned} &\min \quad DSV(x(T)) \\ &\text{subject} \quad \text{to} \quad \begin{cases} \widetilde{\mathbb{E}}[x(T)] = k, \\ (x(\cdot), \pi(\cdot)) \text{ satisfies (5.1.1) and (5.1.2)}, \\ \pi(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m). \end{cases} \end{aligned} \tag{5.1.3}$$

Assumption 5.1.3. $k\mathbb{E}(\rho) > x_0$.

This assumption states that the investor expects higher terminal wealth k by investing in the stock market than the terminal wealth $\frac{x_0}{\mathbb{E}(\rho)}$ by only investing in the riskless zero coupon bond market. This assumption also implies that the investor has to take some risk to meet his investment target.

To solve problem (5.1.3), we employ a martingale method. In other words, we need solve the following two subproblems. The first subproblem is to find the optimal terminal wealth X^*. That is

$$\begin{aligned} \min \quad & DSV(X) \\ \text{subject} \quad \text{to} \quad & \begin{cases} \widetilde{\mathbb{E}}X = k \\ \mathbb{E}[\rho X] = x_0 \\ X \in \mathcal{L}^2_{\mathcal{F}_T}(\Omega; \mathbb{R}). \end{cases} \end{aligned} \tag{5.1.4}$$

The second subproblem is to find the portfolio $\pi(\cdot)$ that replicates X^*, where X^* is the solution of the first subproblem if it exists, i.e. $(x(\cdot), \pi(\cdot))$ satisfies

$$\begin{cases} dx(t) = [r(t)x(t) + B(t)^\top \pi(t)]dt + \pi(t)^\top \sigma(t) dW(t), \ t \in [0, T], \\ x(T) = X^*. \end{cases} \tag{5.1.5}$$

Equation (5.1.5) is called backward stochastic differential equation that has been well studied in the literature. The following proposition is taken from N. El Karoui, et al. (1997), which consider the existence of the second subproblem. Thus, we will mainly focus on the first subproblem.

Proposition 5.1.2. *For any ξ such that $\xi \in \mathcal{L}^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$, the following backward stochastic differential equation*

$$\begin{cases} dx(t) = [r(t)x(t) + \theta(t)^\top Z(t)]dt + Z(t)^\top dW(t), t \in [0, T], \\ x(T) = \xi, \end{cases} \tag{5.1.6}$$

admits a unique solution $(x(\cdot), Z(\cdot))$ and $x(t)$ is given by

$$x(t) = \rho(t)\mathbb{E}\left[\rho(T)\xi | \mathcal{F}_t\right], \quad \forall t \in [0, T], \ a.s.. \tag{5.1.7}$$

5.2 The Solution to the First Subproblem

This section is devoted to solve problem (5.1.4). Notice that problem (5.1.4) is not a convex optimization problem. Then, the usual Lagrange multiplier method can not be applied directly. Thanks to Jin and Zhou (2008), we can reformulate this problem via quantile approach and then recover the convexity.

5.2.1 Quantile formulation

According to the property of Choquet expectation, we have

$$\widetilde{\mathbb{E}}(X-k)=\widetilde{\mathbb{E}}X-k.$$

Then, denote $Y=X-k$, the first subproblem reduces to

$$\begin{aligned}
&\min \quad -\widetilde{\mathbb{E}}[-Y_-^2]\\
&\text{subject to} \quad \begin{cases}\widetilde{\mathbb{E}}Y=0\\ \mathbb{E}[\rho Y]=x_0-k\mathbb{E}(\rho)<0\\ Y\in\mathcal{L}^2_{\mathcal{F}_T}(\Omega;\mathbb{R}).\end{cases}
\end{aligned}\tag{5.2.1}$$

According to Jin and Zhou (2008), we are going to change the decision variable from a random variable Y to its quantile function and turn the problem into a convex optimization problem through some transformations. A quantile function is defined by the cumulative distribution function. For any cumulative distribution function $F(\cdot)$, denote by $F^{-1}(\cdot)$ its left-inverse function, i.e.,

$$F^{-1}(t)=\inf\{x\in\mathbb{R}:F(x)\geq t\}=\sup\{x\in\mathbb{R}:F(x)<t\},t\in[0,T].$$

Let $\mathbb{G}$ be the set of all these quantile functions $F^{-1}(\cdot)$, i.e.,

$$\mathbb{G}=\{G(\cdot):[0,1]\to\mathbb{R},\text{ non-decreasing, left-continuous}, G(0)=-\infty,\ G(1)=G(1^-)\}.$$

A direct calculation shows

$$\begin{aligned}
\widetilde{\mathbb{E}}(Y)&=\int_0^{+\infty}w(\mathbb{P}(Y\geq x))dx+\int_{-\infty}^0[w(\mathbb{P}(Y\geq x))-1]dx\\
&=\int_0^{+\infty}\int_x^{+\infty}d[-w(1-F_Y(y))]dx+\int_{-\infty}^0\int_{-\infty}^x dw(1-F_Y(y))dx\\
&=\int_{-\infty}^{+\infty}yd[-w(1-F_Y(y))]
\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathbb{E}}[-Y_-^2] &= 0 + \int_{-\infty}^{0} [w(\mathbb{P}(-Y_-^2 \geq x)) - 1]dx = \int_{-\infty}^{0} [w(\mathbb{P}(Y \geq -\sqrt{-x})) - 1]dx \\ &= -\int_{-\infty}^{0} y^2 d[-w(1 - F_Y(y))].\end{aligned}$$

Then, problem (5.2.1) can be rewritten as

$$\begin{aligned}&\min \quad \int_{-\infty}^{0} x^2 d[-w(1 - F_Y(x))] \\ &\text{subject to} \quad \begin{cases} \widetilde{\mathbb{E}}Y = \displaystyle\int_{-\infty}^{+\infty} xd[-w(1 - F_Y(x))] = 0 \\ \mathbb{E}[\rho Y] = x_0 - k\mathbb{E}(\rho) := y_0 < 0 \\ Y \in \mathcal{L}^2_{\mathcal{F}_T}(\Omega; \mathbb{R}). \end{cases}\end{aligned} \tag{5.2.2}$$

Now, we have expressed our objective in term of Y's distribution except for the initial budget constraint. Base on a crucial lemma in Jin and Zhou (2008, Theorem B.1), we can further reformulate the initial budget constraint via the quantile function.

Lemma 5.2.1. *Under Assumption 5.1.2, we have* $\mathbb{E}[\rho F_Y^{-1}(1 - F_\rho(\rho))] \leq \mathbb{E}[\rho Y]$, *where* $F_Y(\cdot)$ *is the cumulative distribution function of* Y. *Furthermore, if* $\mathbb{E}[\rho F_Y^{-1}(1 - F_\rho(\rho))] < \infty$, *then the inequality becomes equality if and only if* $Y = F_Y^{-1}(1 - F_\rho(\rho))$.

The economic interpretation of this lemma is that one can always replace a random payoff Y by $\overline{Y} := F_Y^{-1}(1 - F_\rho(\rho))$ which has the same probability law, but with no more (possibly strictly smaller) cost. So, if one's objective is law-invariance, then a dual argument yields that the optimal solution (if exists) Y^* must satisfy $\mathbb{E}(\rho Y^*) = \mathbb{E}\left[\rho F_{Y^*}^{-1}(1 - F_\rho(\rho))\right]$. Indeed, if $\mathbb{E}\left[\rho F_{Y^*}^{-1}(1 - F_\rho(\rho))\right] < \mathbb{E}(\rho Y^*)$, then $F_{Y^*}^{-1}(1 - F_\rho(\rho))$ would achieve the same risk value with a strictly smaller budget. Then, the investor can construct another $\overline{Y}$, which yields better performance. This result is summarized in the following lemma.

Lemma 5.2.2. *If problem (5.2.2) admits an optimal solution Y^* whose distribution function is $F_Y(\cdot)$, then $Y^* = F_Y^{-1}(1 - F_\rho(\rho))$, a.s..*

Proof. Denote $\overline{Y} := F_Y^{-1}(1-F_\rho(\rho))$, then $\overline{Y}$ has the same distribution as Y^*. Thus, we have $\widetilde{\mathbb{E}}(\overline{Y}) = \widetilde{\mathbb{E}}(Y^*)$.

Suppose $Y^* \neq \overline{Y}$, it follows from the uniqueness result in Lemma 5.2.1 that $y_1 := \mathbb{E}(\rho\overline{Y}) < \mathbb{E}(\rho Y^*) = y_0 < 0$. Define $Y_1 := \frac{y_0}{y_1}\overline{Y}$, then $\mathbb{E}(\rho Y_1) = y_0$.

Note that $\frac{y_0}{y_1} \in (0, 1)$, we have

$$\begin{aligned}
\widetilde{\mathbb{E}}(Y_1) = \widetilde{\mathbb{E}}\left(\frac{y_0}{y_1}\overline{Y}\right) &= \int_{-\infty}^{+\infty} yd[-w(1 - F_{\frac{y_0}{y_1}\overline{Y}}(y))] \\
&= \int_{-\infty}^{+\infty} \left(\frac{y_0}{y_1}x\right) d[-w(1 - F_{\overline{Y}}(x))] = \frac{y_0}{y_1}\widetilde{\mathbb{E}}(\overline{X}) = 0,
\end{aligned}$$

and

$$\begin{aligned}
-\widetilde{\mathbb{E}}[-(Y_1)_-^2] &= \int_{-\infty}^{0} y^2 d[-w(1 - F_{\frac{y_0}{y_1}\overline{Y}}(y))] \\
&= \left(\frac{y_0}{y_1}\right)^2 \int_{-\infty}^{0} x^2 d[-w(1 - F_{\overline{Y}}(x))] < -\widetilde{\mathbb{E}}[-(\overline{Y})_-^2].
\end{aligned}$$

Thus, Y_1 is also feasible for (5.2.2) and $-\widetilde{\mathbb{E}}[-(Y_1)_-^2] < -\widetilde{\mathbb{E}}[-(\overline{Y})_-^2] = -\widetilde{\mathbb{E}}[-(Y^*)_-^2]$, which contradicts with the optimality of Y^*.

Lemma 5.2.2 suggests that we only need to solve the problem (5.2.2) on the set of random variables $Y = G(Z_\rho)$, where $G(\cdot) \in \mathbb{G}$ and $Z_\rho := 1 - F_\rho(\rho)$, a particular uniform random variable.

Denote $G(\cdot) = F_Y^{-1}(\cdot)$, then

$$\begin{aligned}
\widetilde{\mathbb{E}}(Y) &= \int_0^1 G(z)w'(1 - z)dz = \mathbb{E}[G(Z_\rho)w'(1 - Z_\rho)], \\
-\widetilde{\mathbb{E}}[-Y_-^2] &= \int_0^1 G(z)_-^2 w'(1 - z)dz = \mathbb{E}[G(Z_\rho)_-^2 w'(1 - Z_\rho)]
\end{aligned}$$

and the budget constraint is

$$\mathbb{E}[F_\rho^{-1}(1 - Z_\rho)G(Z_\rho)] = y_0.$$

Thus, we reformulate our distorted mean-semi-variance problem as follows:

$$\begin{aligned} \min \quad & \mathbb{E}[G(Z_\rho)_-^2 w'(1-Z_\rho)] \\ \text{subject to} \quad & \begin{cases} \mathbb{E}[G(Z_\rho)w'(1-Z_\rho)] = 0 \\ \mathbb{E}[F_\rho^{-1}(1-Z_\rho)G(Z_\rho)] = y_0 \\ G(\cdot) \in \mathbb{G}. \end{cases} \end{aligned}$$

The integral version of this problem is

$$\begin{aligned} \min \quad & \int_0^{z_1} G(z)^2 w'(1-z)dz \\ \text{subject to} \quad & \begin{cases} \int_0^1 G(z)w'(1-z)dz = 0 \\ \int_0^1 F_\rho^{-1}(1-z)G(z)dz = y_0 \\ G(\cdot) \in \mathbb{G}, z_1 = \sup\{z : G(z) < 0\}. \end{cases} \end{aligned} \tag{5.2.3}$$

Once problem (5.2.3) is solved with optimal solution $G^*(\cdot)$, the optimal solution of (5.2.2) can be recovered by $Y^* = G^*(1 - F_\rho(\rho))$. Then, our optimal portfolio can be found by Proposition 5.1.2.

5.2.2 Feasibility and optimal solution

Due to the probability distortion involved in the constraints, the feasibility of problem (5.2.3) is no longer trivial. We need to address this question first. For simplicity, we define $M(z) := \frac{F_\rho^{-1}(1-z)}{w'(1-z)}$ and assume $M(z)$ is continuous on $[0,1]$.

Proposition 5.2.3. *Problem (5.2.3) has a feasible solution, if and only if there exists $z \in (0,1)$ such that $\int_z^1 [M(x) - \mathbb{E}(\rho)]d[-w(1-x)] < 0$.*

Proof. We first consider the "if" part. Assume that there exists a $z \in (0,1)$ such that $\int_z^1 [M(x) - \mathbb{E}(\rho)]d[-w(1-x)] < 0$. Set

$$G(y) = -\frac{y_0 \int_z^1 d[-w(1-x)]}{\int_z^1 [M(x) - \mathbb{E}(\rho)]d[-w(1-x)]}\mathbf{1}_{0\le y\le z} + \frac{y_0 \int_0^z d[-w(1-x)]}{\int_z^1 [M(x) - \mathbb{E}(\rho)]d[-w(1-x)]}\mathbf{1}_{z<y\le 1}.$$

It is easy to verify that $G(\cdot)$ is increasing and satisfies the constraints in (5.2.3). Thus, $G(\cdot)$ is feasible for problem (5.2.3).

Next we show the "only if" part. Assume that for any $z \in (0,1)$, $\int_z^1 [M(x) - \mathbb{E}(\rho)]d[-w(1-x)] \geq 0$. Notice that $\int_0^1 [M(z) - \mathbb{E}(\rho)]d[-w(1-z)] = 0$. Then, for any piece-wise constant function $G(\cdot) \in \mathbb{G}$, i.e.

$$G(z) = a_0 + \sum_{i=1}^{n} a_i \mathbf{1}_{z>z_i},$$

where $a_i > 0, i = 1, 2, \cdots, n$, and $0 < z_1 < ... < z_n < 1$, we have

$$\begin{aligned}\int_0^1 G(z)[M(z) - \mathbb{E}(\rho)]d[-w(1-z)] =& a_0 \int_0^1 [M(z) - \mathbb{E}(\rho)]d[-w(1-z)] \\ &+ \sum_{i=1}^{n} a_i \int_{z_i}^1 [M(z) - \mathbb{E}(\rho)]d[-w(1-z)] \geq 0,\end{aligned}$$

Hence, one can conclude that

$$\int_0^1 G(z)[M(z) - \mathbb{E}(\rho)]d[-w(1-z)] \geq 0,$$

for any $G(\cdot) \in \mathbb{G}$ by approximation and taking limits. Thereby, any $G(\cdot) \in \mathbb{G}$ satisfying $\int_0^1 G(z)d[-w(1-z)] = 0$ will result in

$$\int_0^1 G(z)M(z)d[-w(1-z)] = \int_0^1 G(z)F_\rho^{-1}(1-z) = y_0 \geq 0,$$

which contradicts with (5.2.2).

Consequently, there is no feasible solution for problem (5.2.3). The proof is complete.

Remark 5.2.4. *From now on, we call "there exists $z \in (0,1)$ such that $\int_z^1 [M(x) - \mathbb{E}(\rho)]d[-w(1-x)] < 0$" the* "feasibility condition"*. Note that the feasibility condition just depends on function $M(z)$ for any given probability distortion. We will later show what kind of $M(z)$ does not satisfy the feasibility condition, thereby there exists no feasible solution for problem (5.2.3).*

Note that problem (5.2.3) is convex in $G(\cdot)$, we can remove the constraints in problem (5.2.3) by Lagrange multipliers $\lambda > 0$, $\mu > 0$[2] (cf. Luenberger 1968). Then, we define

$$U(G(\cdot), \lambda, \mu) := \int_0^1 [G(z)_-]^2 w'(1-z)dz - 2\lambda \int_0^1 G(z)w'(1-z)dz + 2\mu \int_0^1 G(z)F_\rho^{-1}(1-z)dz - 2\mu y_0. \tag{5.2.4}$$

For ease presentation, we further denote $\tilde{\mu} = \frac{\lambda}{\mu} > 0$ and consider the following problem first.

$$\begin{aligned} v(\mu, \widetilde{\mu}) &:= \min_{G \in \mathbb{G}} U(G(\cdot), \mu\widetilde{\mu}, \mu) \\ &= \min_{G \in \mathbb{G}} \bigg\{ \int_0^{z_1} [G(z)]^2 w'(1-z)dz - 2\mu\widetilde{\mu} \int_0^{z_1} G(z)w'(1-z)dz \\ &\quad + 2\mu \int_0^{z_1} G(z)F_\rho^{-1}(1-z)dz \\ &\quad -2\mu\widetilde{\mu} \int_{z_1}^1 G(z)w'(1-z)dz + 2\mu \int_{z_1}^1 G(z)F_\rho^{-1}(1-z)dz \bigg\} - 2\mu y_0 \\ &= \min_{G \in \mathbb{G}} \bigg\{ \int_0^{z_1} [G(z) - \mu(\tilde{\mu} - M(z))]^2 w'(1-z)dz \\ &\quad - \int_0^{z_1} [\mu(\tilde{\mu} - M(z))]^2 w'(1-z)dz \\ &\quad -2\mu \int_{z_1}^1 G(z)[\tilde{\mu} - M(z)]w'(1-z)dz \bigg\} - 2\mu y_0, \end{aligned} \tag{5.2.5}$$

where $z_1 = \sup\{z : G(z) < 0\}$.

Theorem 5.2.5. *Define $J(\bar{z}, \tilde{\mu}) := \int_{\bar{z}}^1 [\tilde{\mu} - M(z)]d[-w(1-z)]$. Given $M(z)$ such that the feasibility condition holds, then problem (5.2.3) admits an optimal solution if and only if there exists $\tilde{\mu}^* > 0$ such that*

(1) $\forall \bar{z} \in (0, 1)$, $J(\bar{z}, \tilde{\mu}^*) \le 0$;

[2] This is because the first (second) equality constraint in (5.2.3) could be revised to the less-or-equal (greater-or-equal) inequality constraint without essentially changing the model.

(2) $\exists \bar{z}^* \in (0,1)$*, such that* $J(\bar{z}^*, \tilde{\mu}^*) = 0$.

Meanwhile, such $\tilde{\mu}^*$ *if exists, is unique.*

Proof. It is easy to see that if there exists $\tilde{\mu} > 0$ such that (1) and (2) are satisfied, problem (5.2.5) admits an optimal solution. Then problem (5.2.3) also has an optimal solution according to the Lagrange duality theory.

Conversely, assume that for any $\tilde{\mu} > 0$, there exists a $\bar{z} \in (0,1)$, such that $J(\bar{z}, \tilde{\mu}) > 0$. Set $z_1 = \bar{z}$. Note that bigger $G(z)_+$ leads to lower $U(G(\cdot), \mu\widetilde{\mu}, \mu)$. Letting $G(z)_+ \to +\infty$, we have $U(G(\cdot), \mu\widetilde{\mu}, \mu) \to -\infty$. So, $\inf_{G\in\mathbb{G}} U(G(\cdot), \mu\widetilde{\mu}, \mu) = -\infty$, for any $\tilde{\mu} > 0$. Thus, problem (5.2.3) has no optimal solution.

On the other hand, if there exists $\tilde{\mu}^* > 0$ such that for $\forall \bar{z} \in (0,1)$, we have $J(\bar{z}, \tilde{\mu}^*) \le 0$, but there exists no $\bar{z}^* \in (0,1)$ satisfying $J(\bar{z}^*, \tilde{\mu}^*) = 0$. That is, $J(\bar{z}, \tilde{\mu}^*) < 0$ for any $\bar{z} \in (0,1)$. Then, the optimal $G(\cdot)$ for (5.2.5) must satisfy $G(z) \le 0$, which is not feasible for problem (5.2.3). Again, problem (5.2.3) admits no optimal solution.

Now we will prove the uniqueness of $\tilde{\mu}^*$. Assume that there exist $\tilde{\mu}_i^*$ and $\bar{z}_i^*$, $i = 1, 2$, which satisfies (1) and (2), and $J(\bar{z}_i^*, \tilde{\mu}_i^*) = 0$. Without loss of generality, we assume $\tilde{\mu}_1^* - \tilde{\mu}_2^* > 0$. Then

$$\begin{aligned} J(\bar{z}_2^*, \tilde{\mu}_1^*) &= \int_{\bar{z}_2^*}^1 [\tilde{\mu}_2^* - M(z)] d[-w(1-z)] + (\tilde{\mu}_1^* - \tilde{\mu}_2^*) \int_{\bar{z}_2^*}^1 d[-w(1-z)] \\ &= (\tilde{\mu}_1^* - \tilde{\mu}_2^*) \int_{\bar{z}^*}^1 d[-w(1-z)] > 0, \end{aligned}$$

which results in a contradiction.

Therefore, problem (5.2.3) may not admit any optimal solution even if it has feasible solutions. From now on, we call “there exists $\widetilde{\mu} > 0$ such that (1) and (2) are satisfied” the “optimality condition”, which again depends on function $M(\cdot)$.

Remark 5.2.6. *In the standard mean-semi-variance problem,* $w(z) = z$ *and* $M(z)$ *is decreasing in* $(0,1)$*. Note that* $J(0, \tilde{\mu}) = \tilde{\mu} - \mathbb{E}(\rho)$*,* $J(1, \tilde{\mu}) = 0$*,* $\frac{\partial J(z,\tilde{\mu})}{\partial z} =$

$M(z) - \tilde{\mu}$, and $J(z, \tilde{\mu})$ is continuous of z and $\tilde{\mu}$, respectively. It is easy to see that for a fixed $\tilde{\mu} \geq \mathbb{E}(\rho)$, $J(0, \tilde{\mu}) \geq 0$, $J(z, \tilde{\mu})$ is increasing in $z \in (0, M^{-1}(\tilde{\mu}))$, and decreasing in $z \in (M^{-1}(\tilde{\mu}), 1)$. Thus $J(z, \tilde{\mu}) > 0$ for some $z \in (0,1)$. For fixed $\tilde{\mu} \in (0, \mathbb{E}(\rho))$, $J(0, \tilde{\mu}) < 0$, $J(z, \tilde{\mu})$ is increasing in $z \in (0, M^{-1}(\tilde{\mu}))$, and decreasing in $z \in (M^{-1}(\tilde{\mu}), 1)$. Thus, there exists some $\bar{z}$ such that $J(z, \tilde{\mu}) > 0$ for all $z \in (\bar{z}, 1)$. So, the standard mean-semi-variance problem violates the optimality condition, which implies no optimal solution exists. This result has been established by Jin, Yan and Zhou (2005) using a different approach.

Next we will give the optimal solution of problem (5.2.5) for some general $M(\cdot)$.

Theorem 5.2.7. *Given $M(\cdot)$ such that the optimality condition is satisfied, that is, there exist $\tilde{\mu}^*$ and $0 < \bar{z}_1 < \bar{z}_2 < \cdots < \bar{z}_n < 1$, such that $J(\bar{z}, \tilde{\mu}^*) \leq 0 \ \forall \bar{z} \in (0,1)$, $J(\bar{z}, \tilde{\mu}^*) < 0 \ \forall \bar{z} \in (0,1) \backslash \{\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n\}$ and $J(\bar{z}_i, \tilde{\mu}^*) = 0, i = 1, \cdots, n$. Then the optimal solution of problem (5.2.5) has the form $G(z) = G(z)\mathbf{1}_{0 \leq z \leq \bar{z}_1} + \sum_{i=1}^{n} k_i \mathbf{1}_{\bar{z}_i < z \leq \bar{z}_{i+1}}$, where $k_i \geq 0, i = 1, \cdots, n$, are constants and $\bar{z}_{n+1} = 1$ for convention.*

Proof. Note that (5.2.4) can be rewritten in the following form

$$\begin{aligned} U(G(\cdot), \mu\widetilde{\mu}, \mu) &= \int_0^1 [G(z)_-^2 - 2\mu(\tilde{\mu}^* - M(z))G(z)]w'(1-z)dz - 2\mu y_0 \\ &= \int_0^{\bar{z}_1} [G(z)_-^2 - 2\mu(\tilde{\mu}^* - M(z))G(z)]w'(1-z)dz \\ &\quad + \sum_{i=1}^{n} \int_{\bar{z}_i}^{\bar{z}_{i+1}} [G(z)_-^2 - 2\mu(\tilde{\mu}^* - M(z))G(z)]w'(1-z)dz - 2\mu y_0. \end{aligned} \tag{5.2.6}$$

Define $h(z) = \tilde{\mu}^* - M(z)$ and $H(z) = \int_z^1 [\tilde{\mu}^* - M(x)]w'(1-x)dx = J(z, \tilde{\mu}^*)$, then

$H'(z) = -h(z)w'(1-z)$, and

$$
\begin{aligned}
&\int_{\bar{z}_i}^{\bar{z}_{i+1}} [G(z)_-^2 - 2\mu(\tilde{\mu}^* - M(z))G(z)]w'(1-z)dz \\
&= \int_{\bar{z}_i}^{\bar{z}_{i+1}} G(z)_-^2 w'(1-z)dz + 2\mu \int_{\bar{z}_i}^{\bar{z}_{i+1}} G(z)dH(z) \\
&= \int_{\bar{z}_i}^{\bar{z}_{i+1}} G(z)_-^2 w'(1-z)dz + 2\mu(HG|_{\bar{z}_i}^{\bar{z}_{i+1}} - \int_{\bar{z}_i}^{\bar{z}_{i+1}} H(z)dG(z)) \\
&= \int_{\bar{z}_i}^{\bar{z}_{i+1}} G(z)_-^2 w'(1-z)dz - 2\mu \int_{\bar{z}_i}^{\bar{z}_{i+1}} H(z)dG(z), \quad i = 1, \cdots, n.
\end{aligned}
$$

The last equality follows by the fact that $H(\bar{z}_i) = H(\bar{z}_{i+1}) = 0$. Because of $H(z) \leq 0$, $z \in (\bar{z}_i, \bar{z}_{i+1}), i = 1, \cdots, n$, the optimal solution of problem (5.2.5) must satisfy $G(z)_- = 0$ and $dG(z) = 0, z \in (\bar{z}_i, \bar{z}_{i+1}), i = 1, \cdots, n$. The proof is complete.

5.2.3 Various cases of $M(z)$

In this section, we will study four typical but general enough cases of $M(\cdot)$ and present their explicit solutions if these exist. Notice that the case $M(z)$ with a finite number of monotonic pieces only incurs notational complexity in the approach below, rather than any essential difference. In addition, for many commonly used and interesting distortion functions, the resulting $M(z)$ has been included in the our four cases.

Case 1: There exists $z_0 \in [0, 1]$, such that $M(z)$ is strictly increasing on $(0, z_0)$, and strictly decreasing on $(z_0, 1)$, and $M(0) < M(1)$.

Case 2: There exists $z_0 \in [0, 1]$, such that $M(z)$ is strictly increasing on $(0, z_0)$, and strictly decreasing on $(z_0, 1)$, and $M(0) \geq M(1)$.

Case 3: There exists $z_0 \in [0, 1]$, such that $M(z)$ is strictly decreasing on $(0, z_0)$, and strictly increasing on $(z_0, 1)$, and $M(0) \geq M(1)$.

Case 4: There exists $z_0 \in [0,1]$, such that $M(z)$ is strictly decreasing on $(0, z_0)$, and strictly increasing on $(z_0, 1)$, and $M(0) < M(1)$.

Proposition 5.2.8. *In Case 1 with $M(1) > \mathbb{E}(\rho)$ and Case 4 with $M(0) < \mathbb{E}(\rho)$, there is no feasible solution for problem (5.2.3). In all other cases, there exist feasible solutions for problem (5.2.3).*

Proof. We only show the scenario of $M(\cdot)$ in Case 1 with $M(1) > \mathbb{E}(\rho)$, while the others follow similarly. Define $g(\tilde{z}) = \int_{\tilde{z}}^1 [M(z) - \mathbb{E}(\rho)] d[-w(1-z)] = -J(\tilde{z}, \mathbb{E}(\rho))$, then $g(0) = g(1) = 0$. Thanks to the shape of $M(z)$ and $M(0) < \mathbb{E}(\rho) < M(1)$, $g(\tilde{z})$ is increasing in $(0, M^{-1}(\mathbb{E}(\rho)))$ and decreasing in $(M^{-1}(\mathbb{E}(\rho)), 1)$. So, $g(\tilde{z}) \geq 0$ for $\tilde{z} \in [0,1]$. According to Proposition 5.2.3, there exists no feasible solution for problem (5.2.3). The proof is complete.

Theorem 5.2.9. *In Case 1 with $M(1) \leq \mathbb{E}(\rho)$ and Case 2, the infimum value of problem (5.2.3), which can not be achieved, is*

$$\frac{y_0^2}{\int_0^1 [M(1) - M(z)]^2 w'(1-z) dz - \int_0^{z_2^*} [M(z_2^*) - M(z)]^2 w'(1-z) dz}, \tag{5.2.7}$$

where z_2^* is determined by

$$\int_0^{z_2^*} [M(z_2^*) - M(z)] w'(1-z) dz = 0. \tag{5.2.8}$$

Moreover, $z_2^* \in [z_0, 1]$ in Case 1 with $M(1) \leq \mathbb{E}(\rho)$ and $z_2^* \in [z_0, \tilde{z}_0]$ in Case 2, where $\tilde{z}_0 \in [z_0, 1]$ such that $M(\tilde{z}_0) = M(0)$.

Before proving this theorem, we need to first introduce an auxiliary problem

$$\min_{G \in \mathbb{G}} \int_0^1 w'(1-z)[G(z) - \mu(\tilde{\mu} - M(z))]^2 dz. \tag{5.2.9}$$

The solution is summarized in the Lemma 5.2.10.

Lemma 5.2.10. *i) Assume $M(z)$ is as in Case 1 with $\mathbb{E}(\rho) \geq M(1)$, the optimal solution of (5.2.9) is $G^*(z) = \mu(\tilde{\mu} - M(z_2^*)) \mathbf{1}_{0 \leq z \leq z_2^*} + \mu(\tilde{\mu} - M(z)) \mathbf{1}_{z_2^* < z \leq 1}$ where $z_2^* \in [z_0, 1]$ is determined by (5.2.8).*

ii) Assume $M(z)$ is as in Case 2, the optimal solution of (5.2.9) is $G^*(z) = \mu(\tilde{\mu} - M(z_2^*))\mathbf{1}_{0\le z\le z_2^*} + \mu(\tilde{\mu} - M(z))\mathbf{1}_{z_2^*<z\le 1}$, where $z_2^* \in [z_0, \tilde{z}_0]$ is determined by (5.2.8).

The proof is placed in Appendix 5.6.1.

Proof of Theorem 5.2.9. Let's deal with $M(z)$ in Case 1 with $M(1) \le \mathbb{E}(\rho)$, while part ii) follows similarly. We first claim that such $M(z)$ violates the optimal condition. Recall that $J(z, \widetilde{\mu}) = \int_z^1 [\tilde{\mu} - M(x)] d[-w(1-x)]$, then if $\widetilde{\mu} > M(1)$, one can easily find a z^* such that $J(z^*, \widetilde{\mu}) > 0$.

While for $\widetilde{\mu} \in (0, M(1)]$, it is not difficult to show $J(z, \widetilde{\mu}) < 0$ for all $z \in (0, 1)$. Indeed, $J(0, \widetilde{\mu}) = \tilde{\mu} - \mathbb{E}(\rho) \le M(1) - \mathbb{E}(\rho) \le 0$, $J(1, \widetilde{\mu}) = 0$ and

$$\frac{\partial J(z, \tilde{\mu})}{\partial z} = \begin{cases} M(z) - \tilde{\mu} < 0, & \text{if } 0 < z < M^{-1}(\tilde{\mu}), \\ M(z) - \tilde{\mu} > 0, & \text{if } M^{-1}(\tilde{\mu}) < z < 1. \end{cases}$$

So, problem (5.2.3) has no optimal solution by Theorem 5.2.5.

Next we consider the infimum value of problem (5.2.3). In the scenario of $\widetilde{\mu} > M(1)$, one can easily set $z_1 = z^*$, and let $G(z) \to \infty$, for $z > z_1$, then

$$v(\mu, \widetilde{\mu}) \to -\infty.$$

In the scenario of $\widetilde{\mu} \in (0, M(1)]$, it is easy to see that $z_1 = 1$ and problem (5.2.5) reduces to our auxiliary problem (5.2.9), in the sense that they admit the same optimal solution. According to Lemma 5.2.10, we see that

$$v(\mu, \widetilde{\mu}) = \int_0^{z_2^*} [\mu(M(z_2^*) - M(z))]^2 w'(1-z) dz - \int_0^1 [\mu(\tilde{\mu} - M(z))]^2 w'(1-z) dz - 2\mu y_0, \tag{5.2.10}$$

and the optimal solution of problem (5.2.5) is

$$G^*(z) = \mu(\tilde{\mu} - M(z_2^*))\mathbf{1}_{0\le z\le z_2^*} + \mu(\tilde{\mu} - M(z))\mathbf{1}_{z_2^*<z\le 1},$$

where $z_2^* \in [z_0, 1]$ is determined by (5.2.8).

To obtain the infimum value of problem (5.2.3), we need to consider problem

$$\max_{\mu>0} \max_{\tilde{\mu}>0} v(\mu, \tilde{\mu}).$$

Based on (5.2.10), a direct calculation shows that

$$\begin{aligned}
\max_{\tilde{\mu}>0} v(\mu, \tilde{\mu}) &= v(\mu, M(1)) \\
&= \int_0^{z_2^*} [\mu(M(z_2^*) - M(z))]^2 w'(1-z)dz \\
&\quad - \int_0^1 [\mu(M(1) - M(z))]^2 w'(1-z)dz - 2\mu y_0.
\end{aligned}$$

and $\max_{\mu>0} v(\mu, M(1))$ will achieve its maximum (5.2.7) at

$$\mu^* = \frac{y_0}{\int_0^{z_2^*}[M(z_2^*) - M(z)]^2 w'(1-z)dz - \int_0^1 [M(1) - M(z)]^2 w'(1-z)dz}.$$

One can easily check that $\mu^* > 0$, by showing

$$\int_0^{z_2^*} [M(z_2^*) - M(z)]^2 w'(1-z)dz - \int_0^1 [M(1) - M(z)]^2 w'(1-z)dz < 0.$$

Note that the corresponding value $G^*(z)$, giving by

$$G^*(z) = \mu^*[\tilde{\mu}^* - M(z_2^*)]\mathbf{1}_{0\le z\le z_2^*} + \mu^*[\tilde{\mu}^* - M(z)]\mathbf{1}_{z_2^*<z\le 1},$$

is not feasible for problem (5.2.3), due to $G^*(\cdot) \le 0$. According to the weak duality principle, we know (5.2.7) is a lower bound of the infimum value of problem (5.2.3). To complete our proof, we need to construct a sequences of $\{G_n(z)\}$, $n = 1, 2, \cdots$, which are feasible for (5.2.3) and $G_n(z) \to G^*(z)$, $n \to +\infty$, so (5.2.7) is also an upper bound. Set

$$G_n(z) = \mu_n\big\{[M(z_n) - M(z_2^*)]\mathbf{1}_{0\le z\le z_2^*} + [M(z_n) - M(z)]\mathbf{1}_{z_2^*<z\le z_n}\big\} + b_n \mathbf{1}_{z_n<z\le 1},$$

where μ_n, b_n, $n = 1, 2, \cdots$, are determined by the constraints in problem (5.2.3), that is,

$$\begin{cases} \mu_n = \dfrac{y_0}{A(z_n)} \\ b_n = -\dfrac{\int_0^{z_2^*}[M(z_n) - M(z_2^*)]w'(1-z)dz + \int_{z_2^*}^{z_n}[M(z_n) - M(z)]w'(1-z)dz}{\int_{z_n}^1 w'(1-z)dz}\mu_n, \end{cases}$$

where

$$\begin{aligned} A(z_n) := & \int_0^{z_2^*}[M(z_n) - M(z_2^*)][M(z) - \frac{\int_{z_n}^1 M(z)w'(1-z)dz}{\int_{z_n}^1 w'(1-z)dz}]w'(1-z)dz \\ & + \int_{z_2^*}^{z_n}[M(z_n) - M(z)][M(z) - \frac{\int_{z_n}^1 M(z)w'(1-z)dz}{\int_{z_n}^1 w'(1-z)dz}]w'(1-z)dz. \end{aligned}$$

It is easy to verify that

$$A(z_n) \to \int_0^{z_2^*}[M(z_2^*) - M(z)]^2 w'(1-z)dz - \int_0^1 [M(1) - M(z)]^2 w'(1-z)dz,$$

as $z_n \to 1$, $n \to +\infty$, then $\mu_n \to \mu^*$, $b_n \to +\infty$. Note that $A(z_n)$ is continuous for z_n and $A(z_2^*) = 0$, $A(1) < 0$. So we can select $\{z_n\} \subset (z_2^*, 1]$ such that $A(z_n) < 0$ and then $\mu_n > 0$, $b_n > 0$. Figure 5.1 illustrates $G^*(z)$ and $G_n(z)$. The proof is complete.

The following theorem summarizes the optimal solution in Case 3 and Case 4 with $M(0) \geq \mathbb{E}(\rho)$. Before we go through the details, we first show that such $M(z)$ in Case 3 and Case 4 with $M(0) \geq \mathbb{E}(\rho)$ satisfy the optimal condition in Lemma 5.2.11.

Lemma 5.2.11. *i) In Case 3, there exists a unique* $\tilde{\mu}^* \in [M(z_0), \min\{\mathbb{E}(\rho), M(1)\}]$ *such that* $J(z, \tilde{\mu}^*) \leq 0$, $\forall z \in (0, 1)$ *and there exists a unique*

$$z_{\tilde{\mu}^*} := \inf\{z \in (\bar{z}_0, z_0] : M(z) \leq \tilde{\mu}\}$$

satisfying $J(z_{\tilde{\mu}^*}, \tilde{\mu}^*) = 0$, *where* $\bar{z}_0 \in (0, z_0)$ *satisfies* $M(\bar{z}_0) = M(1)$.

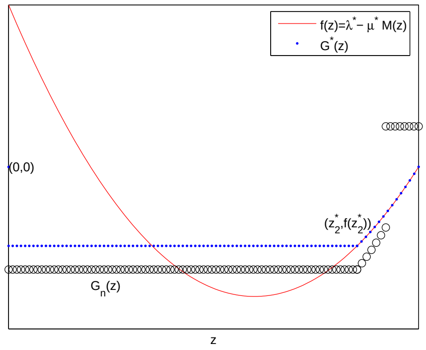

Figure 5.1: $G^*(z)$ and $G_n(z)$ in Case 1 with $M(1) \leq \mathbb{E}(\rho)$.

ii) In Case 4 with $M(0) \geq \mathbb{E}(\rho)$, there exists a unique $\tilde{\mu}^* \in [M(z_0), \mathbb{E}(\rho)]$ such that $J(z, \tilde{\mu}^*) \leq 0$, $\forall z \in (0,1)$ and there exists a unique $z_{\tilde{\mu}^*} \in (0, z_0)$ satisfying $J(z_{\tilde{\mu}^*}, \tilde{\mu}^*) = 0$.

The proof is placed in Appendix 5.6.2.

Theorem 5.2.12. *In Case 3 and Case 4 with $M(0) \geq \mathbb{E}(\rho)$, problem (5.2.3) admits an optimal solution*

$$G^*(z) = \mu^*(\tilde{\mu}^* - M(z))\mathbf{1}_{0\leq z\leq z^*} + b^*\mathbf{1}_{z^*\leq z\leq 1}, \tag{5.2.11}$$

and its corresponding optimal value is

$$\frac{y_0^2}{\int_0^{z^*}[M(z^*) - M(z)]^2 w'(1-z)dz}, \tag{5.2.12}$$

where

$$\begin{aligned} \tilde{\mu}^* =& \quad M(z^*), \quad \mu^* = \frac{-y_0}{\int_0^{z^*}[M(z^*) - M(z)]^2 w'(1-z)dz} > 0, \\ b^* =& \frac{-y_0}{\int_0^{z_2^*}[M(z^*) - M(z)]^2 w'(1-z)dz} \frac{\mathbb{E}(\rho) - M(z^*)}{\int_{z^*}^1 w'(1-z)dz}, \end{aligned} \tag{5.2.13}$$

and z^* is determined by

$$\int_{z^*}^{1}[M(z^*)-M(x)]d[-w(1-x)]=0. \tag{5.2.14}$$

Proof. It is easy to see that equation (5.2.14) admits a unique solution $z^* \in (0, z_0]$. By the definition of $z_{\tilde{\mu}^*}$ in Lemma 5.2.11, we know that $z_{\tilde{\mu}^*} \in (0, z_0]$ satisfies $M(z_{\tilde{\mu}^*}) = \tilde{\mu}^*$ and $\int_{z_{\tilde{\mu}^*}}^1 [\tilde{\mu}^* - M(x)]d[-w(1-x)] = 0$, so we have

$$\begin{aligned}0 &= \int_{z_{\tilde{\mu}^*}}^{1}\{[\tilde{\mu}^*-M(x)]-[\tilde{\mu}^*-M(z_{\tilde{\mu}^*})]\}d[-w(1-x)]\\ &= \int_{z_{\tilde{\mu}^*}}^{1}[M(z_{\tilde{\mu}^*})-M(x)]d[-w(1-x)].\end{aligned}$$

Then we have $z_{\tilde{\mu}^*} = z^*$ and $\tilde{\mu}^* = M(z^*)$ because both $z_{\tilde{\mu}^*} \in (0, z_0]$ and $z^* \in (0, z_0]$ are unique.

Now we are able to get the optimal solution of problem (5.2.3). According to Theorem 5.2.7, we know that the optimal solution of problem (5.2.5) must satisfy $G(z) \equiv b, z \in [z^*, 1]$, where b is some constant to be determined. On the other hand,

$$\begin{aligned}&\int_0^{z^*}[G(z)_-^2-2\mu(\tilde{\mu}^*-M(z))G(z)]w'(1-z)dz\\ &= \int_0^{z^*}[G(z)_-^2+2\mu(\tilde{\mu}^*-M(z))G(z)_-]w'(1-z)dz\\ &- \int_0^{z^*}2\mu(\tilde{\mu}^*-M(z))G(z)_+w'(1-z)dz.\end{aligned}$$

Because $\tilde{\mu}^* - M(z) < 0, z \in (0, z^*)$ and $\tilde{\mu}^* - M(z)$ is increasing on $z \in (0, z^*)$, the optimal solution of problem (5.2.5) must satisfy $G(z)_+ = 0$ and $G(z)_- = -\mu(\tilde{\mu}^* - M(z))$ on $z \in (0, z^*)$. So, the optimal solution of problem (5.2.5) is

$$G^*(z) = \mu(\tilde{\mu}^*-M(z))\mathbf{1}_{0\le z\le z^*} + b\mathbf{1}_{z^*\le z\le 1}, b > 0. \tag{5.2.15}$$

Inserting (5.2.15) into the two constraints in (5.2.3), we can obtain

$$\mu^* = \frac{-y_0}{\int_0^{z^*}[M(z^*)-M(z)]^2w'(1-z)dz} > 0,$$

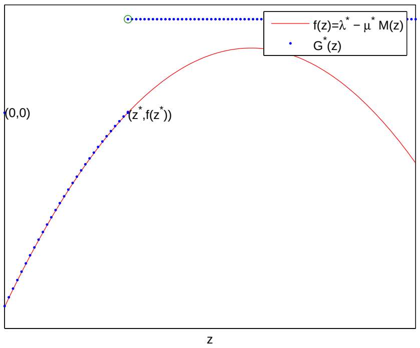

Figure 5.2: The optimal solution $G^*(z)$ to problem (5.2.3) in Case 3.

$$b^* = \frac{-y_0}{\int_0^{z^*}[M(z^*)-M(z)]^2 w'(1-z)dz} \frac{\mathbb{E}(\rho)-M(z^*)}{\int_{z^*}^1 w'(1-z)dz},$$

and then

$$\lambda^* = \mu^*\tilde{\mu}^* = \frac{-y_0 M(z^*)}{\int_0^{z^*}[M(z^*)-M(z)]^2 w'(1-z)dz} > 0. \tag{5.2.16}$$

Thanks to (5.2.11), one can easily deduce the minimum value of problem (5.2.3). Figure 5.2 shows the optimal solution $G^*(z)$ to problem (5.2.3).

5.3 Examples

In this section, we present some examples to illustrate our results. We show that some extensively used probability distortions are included in our four cases that we introduced before.

Example 5.3.1. *Take $w(z) = z^\gamma$ for some $\gamma > 1$ and assume ρ follows log-normal distribution, i.e., $F_\rho(x) = \Phi(\frac{\ln x-\tilde{\eta}}{\tilde{\sigma}})$, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution $N(0,1)$ and $\tilde{\eta}, \tilde{\sigma}$ are some constant parameters. Denote by $\phi(\cdot)$ the density function $N(0,1)$. It is easy to verify that $M(z) = \frac{F_\rho^{-1}(1-z)}{w'(1-z)}$ is strictly decreasing on $(0, z_0)$, and strictly increasing on $(z_0, 1)$*

for some $z_0 \in (0,1)$*, and* $M(0) = M(1) = +\infty$*. Then* $M(z)$ *is included in Case 3. Thus, the optimal solution of problem (5.2.3) is*

$$G^*(z) = \mu^*[M(z^*) - M(z)]\mathbf{1}_{0\le z\le z^*} + b^*\mathbf{1}_{z^*\le z\le 1},$$

where z^* *is determined by (5.2.14) and* μ^**,* b^* *are given by (5.2.13).*

Example 5.3.2. *For any concave probability distortion function* $w(\cdot)$*,* $w'(z)$ *is decreasing for* $z \in (0,1)$ *and* $w'(1-z)$ *is increasing for* $z \in (0,1)$*. since* $F_\rho^{-1}(1-z)$ *is decreasing in* $z \in (0,1)$ *for any* ρ*,* $M(z) = \frac{F_\rho^{-1}(1-z)}{w'(1-z)}$ *is decreasing in* $z \in (0,1)$*. In this case, one can easily check that such* $M(z)$ *satisfies the feasibility condition but violates the optimal condition, so problem (5.2.3) has no optimal solution but admits a finite infimum value. Notice that such* $M(z)$ *can also be considered as the limit version of Case 3 with* $z_0 \to 1$ *and* $\bar{z}_0 \to 1$*, then* $z^* \to 1$ *because of* $z^* \in [\bar{z}_0, z_0]$*. So the infimum value of (5.2.3) is*

$$\frac{y_0^2}{\int_0^1 [M(1) - M(z)]^2 w'(1-z)dz},$$

which cannot be attained. Moreover, one can construct a sequence of feasible solutions to problem (5.2.3)

$$G_n(z) = \mu_n[M(z_n) - M(z)]\mathbf{1}_{0\le z\le z_n} + b_n\mathbf{1}_{z_n< z\le 1},$$

where $n \to +\infty$ *and* $z_n \to 1$*, such that*

$$\int_0^1 [G_n(z)_-]^2 w'(1-z)dz \to \frac{y_0^2}{\int_0^1 [M(1) - M(z)]^2 w'(1-z)dz}, \quad z_n \to 1.$$

The constraints in problem (5.2.3) will determine μ_n*,* b_n*, that is*

$$\begin{cases} \mu_n = \dfrac{-y_0}{\int_0^{z_n} [M(z_n) - M(z)]^2 w'(1-z)dz} \\ b_n = \dfrac{-y_0}{\int_0^{z_n} [M(z_n) - M(z)]^2 w'(1-z)dz} \dfrac{\mathbb{E}(\rho) - M(z_n)}{\int_{z_n}^1 w'(1-z)dz}. \end{cases}$$

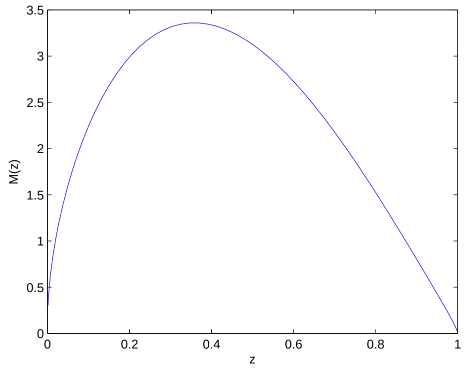

Figure 5.3: The $M(z)$ in inverse-S shaped probability distortion.

In particular, if one take $w(z) = z$, i.e. no distortion, then the infimum value of problem (5.2.3) is

$$\frac{y_0^2}{\int_0^1 [M(1) - M(z)]^2 w'(1-z)dz} = \frac{y_0^2}{\mathbb{E}[(\rho - \rho_0)^2]},$$

where $\rho_0 := \lim_{z\to 1} F_\rho^{-1}(1-z)$. This coincides with the result in Jin, Yan and Zhou (2005).

Example 5.3.3. *Take*

$$w(z) = \frac{z^\gamma}{[z^\gamma + (1-z)^\gamma]^{\frac{1}{\gamma}}},$$

for some $\gamma \in (0,1)$, which is the inverse-S shaped distortion, that is, $w(z)$ is concave initially and then convex. Assume ρ follows log-normal distribution, i.e., $F_\rho(x) = \Phi(\frac{\ln x - \tilde{\eta}}{\tilde{\sigma}})$. Letting $\gamma = 0.5$, $\tilde{\eta} = 0.1$, $\tilde{\sigma} = 0.4$, we plot $M(z)$ in Figure 5.3, which shows that $M(z)$ is first increasing, then decreasing and attains its maximum at some z_0. Moreover, $M(1) = lim_{z\to 1^-} \frac{F_\rho^{-1}(1-z)}{w'(1-z)} = 0 = lim_{z\to 0^+} \frac{F_\rho^{-1}(1-z)}{w'(1-z)} = M(0)$. Thus, such $M(z)$ has been included in Case 2. So, problem (5.2.3) admits no optimal solution for this inverse-S shaped distortion.

5.4 The Efficient Frontier and Efficient Strategy

5.4.1 The efficient frontier

In Case 3 (or Case 4 with $M(0) \le \mathbb{E}(\rho)$), the frontier of the distorted mean-semi-variance problem (5.1.4) is

$$\begin{aligned} -\widetilde{\mathbb{E}}[-(X-\widetilde{\mathbb{E}}(X))_-^2] &= -\widetilde{\mathbb{E}}[-Y_-^2] = \mathbb{E}[G^*(Z)_-^2 w'(1-Z)] \\ &= \frac{(x_0 - \mathbb{E}\rho\widetilde{\mathbb{E}}X)^2}{\int_0^{z^*}[M(z^*)-M(z)]^2 w'(1-z)dz}. \end{aligned}$$

The optimal solution to (5.2.3) is

$$G^*(z) = \left[\lambda^* - \mu^* \frac{F_\rho^{-1}(1-z)}{w'(1-z)}\right] \mathbf{1}_{0\le z\le z^*} + b^* \mathbf{1}_{z_2^*<z\le 1}.$$

Then the optimal solution to (5.2.1) is

$$Y^* = G^*(1-F_\rho(\rho)) = \left[\lambda^* - \mu^* \frac{F_\rho^{-1}(1-z)}{w'(1-z)}\right] \mathbf{1}_{\{1-F_\rho(\rho)\le z^*\}} + b^* \mathbf{1}_{\{z^*<1-F_\rho(\rho)\}},$$

where $z^* \in [\bar{z}_0, z_0]$ is determined by (5.2.14), $\lambda^*, \mu^*$and b^* is given by (5.2.16) and (5.2.11). The optimal terminal wealth is

$$X^* = Y^* + k = k + b^* \mathbf{1}_{\{\rho<F_\rho^{-1}(1-z^*)\}} + \left[\lambda^* - \mu^* \frac{\rho}{w'(F_\rho(\rho))}\right] \mathbf{1}_{\{\rho\ge F_\rho^{-1}(1-z^*)\}}.$$

Note that if we consider k as a "reference point", then $b^* \mathbf{1}_{\{\rho<F_\rho^{-1}(1-z^*)\}} > 0$ is gain part and $\left[\lambda^* - \mu^* \frac{\rho}{w'(F_\rho(\rho))}\right] \mathbf{1}_{\{\rho\ge F_\rho^{-1}(1-z^*)\}} < 0$ is loss part. The structure of the optimal terminal wealth is similar to that in Jin and Zhou (2008). The terminal wealth ending with a gain or loss is determined by the market being good or bad. In other words, the terminal state density price is lower or higher than one single threshold $F_\rho^{-1}(1-z^*)$, where z^* is determined by (5.2.14). Meanwhile, X^* is bounded from above or the gain is always some constant amount b^*.

5.4.2 The second subproblem

Now we aim to solve the second subproblem (5.1.5), which becomes

$$\begin{cases} dx(t) = [r(t)x(t) + B(t)^\top \pi(t)]dt + \pi(t)^\top \sigma(t) dW(t), \quad t \in [0,T], \\ x(T) = X^*. \end{cases}$$

We know that the optimal portfolio $\pi(\cdot)$ is the replicating portfolio for the claim

$$x(T) = Y^* + k = k + b^* \mathbf{1}_{\{\rho < F_\rho^{-1}(1-z^*)\}} + \left[\lambda^* - \mu^* \frac{\rho}{w'(F_\rho(\rho))}\right] \mathbf{1}_{\{\rho \geq F_\rho^{-1}(1-z^*)\}}.$$

Let $(x_1(\cdot), \pi_1(\cdot))$ replicate 1, $(x_2(\cdot), \pi_2(\cdot))$ replicate $\mathbf{1}_{\{\rho < F_\rho^{-1}(1-z^*)\}}$,and $(x_3(\cdot), \pi_3(\cdot))$ replicate

$$\left[\lambda^* - \mu^* \frac{\rho}{w'(F_\rho(\rho))}\right] \mathbf{1}_{\{\rho \geq F_\rho^{-1}(1-z^*)\}}.$$

Then, $x(t) = kx_1(t) + b^* x_2(t) + x_3(t)$, and $\pi(t) = k\pi_1(t) + b^* \pi_2(t) + \pi_3(t)$.

We assume all the market parameters are deterministic and time-invariant, i.e. $r(\cdot) \equiv r$, $B(\cdot) \equiv B$, $\sigma(\cdot) \equiv \sigma$ and then $\theta(\cdot) = \sigma^{-1} B =: \theta$. Given $\mathcal{F}_t$, $\rho(t,T) := \frac{\rho(T)}{\rho(t)}$ follows a log-normal distribution with parameter $(\tilde{\eta}_t, \tilde{\sigma}_t)$, where

$$\tilde{\eta}_t := -\left(r + \frac{|\theta|^2}{2}\right)(T-t) \quad \text{and} \quad \tilde{\sigma}_t^2 := |\theta|^2 (T-t). \tag{5.4.1}$$

In particular, $\rho(T) = \rho(0,T)$ follows a log-normal distribution with parameter $(\tilde{\eta}_0, \tilde{\sigma}_0)$.

Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the cumulative distribution function and the density function of the standard normal distribution, respectively. Define $c^* := F_\rho^{-1}(1-z^*)$.

By (5.1.7), we have

$$\begin{aligned} x_1(t) &= e^{r(t-T)}, \\ x_2(t) &= \mathbb{E}\left[\rho(t,T)\mathbf{1}_{\{\rho(T) < c^*\}} | \mathcal{F}_t\right] = \mathbb{E}\left[\rho(t,T)\mathbf{1}_{\{\rho(t,T) < \frac{c^*}{\rho(t)}\}} | \mathcal{F}_t\right] \\ &= \int_0^{\frac{c^*}{\rho(t)}} y d\Phi\left(\frac{\ln y - \tilde{\eta}_t}{\tilde{\sigma}_t}\right) = \frac{1}{\tilde{\sigma}_t}\int_0^{\frac{c^*}{\rho(t)}} \phi\left(\frac{\ln y - \tilde{\eta}_t}{\tilde{\sigma}_t}\right) dy \\ &=: g_2(t, \rho(t)), \end{aligned} \tag{5.4.2}$$

and

$$
\begin{aligned}
x_3(t) &= \mathbb{E}\left\{\rho(t,T)\left[\lambda^* - \mu^* \frac{\rho(T)}{w'(F_{\rho(T)}(\rho(T)))}\right]\mathbf{1}_{\{\rho(T)\geq c^*\}}|\mathcal{F}_t\right\} \\
&= \mathbb{E}\left\{\rho(t,T)\left[\lambda^* - \mu^*\rho(t,T)\rho(t)/\ w'(F_{\rho(T)}(\rho(t,T)\rho(t)))\right]\mathbf{1}_{\{\rho(t,T)\geq \frac{c^*}{\rho(t)}\}}|\mathcal{F}_t\right\} \\
&= \int_{\frac{c^*}{\rho(t)}}^{+\infty} y\left[\lambda^* - \mu^* y\rho(t)/w'\left(\Phi\left(\frac{\ln y + \ln\rho(t) - \tilde{\eta}_0}{\tilde{\sigma}_0}\right)\right)\right] d\Phi\left(\frac{\ln y - \tilde{\eta}_t}{\tilde{\sigma}_t}\right) \\
&=: g_3(t,\rho(t)).
\end{aligned}
\tag{5.4.3}
$$

It is well-known (e.g. Bielecki, et al. 2005) that the replicating portfolio of $x_2(t)$ is

$$\pi_2(t) = -B(\sigma\sigma^\top)^{-1}\rho(t)\frac{\partial g_2(t,\rho(t))}{\partial\rho}.$$

The replicating portfolio of $x_3(t)$ is

$$\pi_3(t) = -B(\sigma\sigma^\top)^{-1}\rho(t)\frac{\partial g_3(t,\rho(t))}{\partial\rho}.$$

5.5 Summary

The main results of the chapter are as follows.

- We formulate the “distorted” mean-semi-variance problem.
- The feasibility conditions are presented.
- We show the necessary and sufficient conditions of the existence of the optimal solution and present some examples.
- We recover the conventional mean-semi-variance results in Jin, Yan and Zhou (2005).

5.6 Appendix

5.6.1 The proof of Lemma 5.2.10

Proof. We only need prove part i), while part ii) follows similarly. For notation simplicity, define

$$f(z) = \mu(\tilde{\mu} - M(z)) = \lambda - \mu \frac{F_\rho^{-1}(1-z)}{w'(1-z)}, z \in [0,1].$$

Then we first claim that the optimal solution of problem (5.2.9) has two possible forms,

$$G_1(z) \equiv \lambda - \mu c, c \in [M(0), M(1)] \text{ or } G_2(z) = f(z_2)\mathbf{1}_{0\le z\le z_2} + f(z)\mathbf{1}_{z_2<z\le 1}, z_2 \in [z_0, 1],$$

where c and z_2 are constants to be determined.

Indeed, due to the non-decreasing property of $G(\cdot)$ and the decreasing property of $f(z)$ on $z \in (0, z_0)$, the optimal solution of problem (5.2.9) must satisfy $G(z) \equiv f(z_3)$ when $z \in [0, z_3]$ for some $z_3 \in (0, z_0)$. Otherwise, the value $\int_0^{z_3}[G(z) - f(z)]^2 w'(1-z)dz$ will be increased. If $f(z_3) \ge f(1)$, the optimal solution is $G(z) \equiv f(z_3)$, $z \in [0,1]$, otherwise the value $\int_{z_3}^1 [G(z) - f(z)]^2 w'(1-z)dz$ will be increased.

If $f(z_3) < f(1)$, there exists $z_2 \in [z_0, 1]$ such that $f(z_2) = f(z_3)$ because of the shape of $M(z)$. The optimal $G(\cdot)$ is $G(z) = f(z_2)\mathbf{1}_{0\le z\le z_2} + f(z)\mathbf{1}_{z_2<z\le 1}$, $z_2 \in [z_0, 1]$ and $f(z_2) = f(z_3)$, where $z_2 \in [z_0, 1]$ is to be determined. Otherwise the value $\int_{z_3}^1 [G(z) - f(z)]^2 w'(1-z)dz$ will be increased.

For a fixed μ, owing to $\mathbb{E}(\rho) \ge M(1)$, the first form $G_1(\cdot)$ reduces the problem (5.2.9) to

$$\min_{c\in[M(0),M(1)]} \mu^2 \int_0^1 [c - M(z)]^2 w'(1-z)dz = \mu^2 \int_0^1 [M(1) - M(z)]^2 w'(1-z)dz.$$

The second form $G_2(\cdot)$ reduces the problem (5.2.9) to

$$\min_{z_2\in[z_0,1]} \mu^2 \int_0^{z_2} w'(1-z)[M(z_2) - M(z)]^2 dz.$$

Denote $N(z_2) = \int_0^{z_2} w'(1-z)[M(z_2) - M(z)]^2 dz,\ z_2 \in [z_0, 1]$, we aim to find the minimizer of $N(z_2)$ over $[z_0, 1]$. Notice that

$$N'(z_2) = 2M'(z_2) \int_0^{z_2} w'(1-z)[M(z_2) - M(z)]dz = 2M'(z_2)n(z_2),$$

where $n(z_2) = \int_0^{z_2} w'(1-z)[M(z_2) - M(z)]dz$.

Thanks to the shape of $M(\cdot)$, we know that

$$\begin{aligned} n(z_0) &= \int_0^{z_0} w'(1-z)[M(z_0) - M(z)]dz \geq 0, \\ n(1) &= \int_0^{1} w'(1-z)[M(1) - M(z)]dz = M(1) - \mathbb{E}(\rho) \leq 0, \\ n'(z_2) &= \int_0^{z_2} w'(1-z)M'(z_2)dz < 0, z_2 \in (z_0, 1). \end{aligned}$$

So, there exists a unique $z_2^* \in [z_0, 1]$ which satisfies $n(z_2^*) = 0$. In addition, we have

$$N'(z_2) < 0,\ z_2 \in (z_0, z_2^*),\ N'(z_2^*) = 0 \text{ and } N'(z_2) > 0,\ z_2 \in (z_2^*, 1).$$

Then $N(z_2)$ attains its minimum at z_2^*. Moreover,

$$\int_0^{z_2^*} w'(1-z)[M(z_2^*) - M(z)]^2 dz < \int_0^1 [M(1) - M(z)]^2 w'(1-z)dz.$$

Thus, when $M(0) < M(1) \leq \mathbb{E}(\rho)$, the optimal solution of (5.2.9) is

$$G^*(z) = f(z_2^*)\mathbf{1}_{0 \leq z \leq z_2^*} + f(z)\mathbf{1}_{z_2^* < z \leq 1},$$

where z_2^* is determined by (5.2.8).

5.6.2 The proof of Lemma 5.2.11

Proof. We only need to prove part i), since part ii) follows similarly.

Note that $J(0, \tilde{\mu}) = \tilde{\mu} - \mathbb{E}(\rho)$, $J(1, \tilde{\mu}) = 0$, $\frac{\partial J(z,\tilde{\mu})}{\partial z} = M(z) - \tilde{\mu}$, $\frac{\partial J(z,\tilde{\mu})}{\partial \tilde{\mu}} \geq 0$ and $J(z, \tilde{\mu})$ is continuous of z and $\tilde{\mu}$, respectively. It is easy to see that if $\tilde{\mu} > \min\{\mathbb{E}(\rho), M(1)\}$, there exists $z \in (0, 1)$, such that $J(z, \tilde{\mu}) > 0$. If $\tilde{\mu} \leq M(z_0)$, then $J(z, \tilde{\mu}) < 0$, $\forall z \in (0, 1)$. If $\tilde{\mu} \in [M(z_0),\ \min\{\mathbb{E}(\rho), M(1)\}]$, then $J(0, \tilde{\mu}) = \tilde{\mu} - \mathbb{E}(\rho) \leq 0$. Fixing $\tilde{\mu}$, we define $z_{\tilde{\mu}} = M^{-1}(\tilde{\mu}) = \inf\{z \in (0, z_0] : M(z) \leq \tilde{\mu}\}$ and $\tilde{z}_{\tilde{\mu}} = \inf\{z \in [z_0, 1) : M(z) \geq \tilde{\mu}\}$, then $J(z, \tilde{\mu})$ is increasing in $(0, z_{\tilde{\mu}})$, decreasing in $(z_{\tilde{\mu}}, \tilde{z}_{\tilde{\mu}})$, and increasing in $(\tilde{z}_{\tilde{\mu}}, 1)$.

Next we show that there exists a unique $\tilde{\mu}^* \in [\ M(z_0), \min\{\mathbb{E}(\rho), M(1)\}\ ]$ such that $\forall z \in (0, 1)$, $J(z, \tilde{\mu}^*) \leq 0$ and $J(z, \tilde{\mu}^*) = 0$ if and only if $z = z_{\tilde{\mu}^*}$. If $\tilde{\mu} = M(z_0)$,

$$J(z_{\tilde{\mu}}, \tilde{\mu}) = \int_{z_{\tilde{\mu}}}^{1} [\tilde{\mu} - M(x)] d[-w(1-x)] = \int_{z_0}^{1} [M(z_0) - M(x)] d[-w(1-x)] < 0.$$

If $\tilde{\mu} = \min\{\mathbb{E}(\rho), M(1)\}$, then $J(z_{\tilde{\mu}}, \tilde{\mu}) > 0$ since

$$J(z_{\mathbb{E}(\rho)}, \mathbb{E}(\rho)) = \int_{z_{\mathbb{E}(\rho)}}^{1} [\mathbb{E}(\rho) - M(x)] d[-w(1-x)] > \int_{0}^{1} [\mathbb{E}(\rho) - M(x)] d[-w(1-x)] = 0$$

and

$$J(z_{M(1)}, M(1)) = \int_{z_{M(1)}}^{1} [M(1) - M(x)] d[-w(1-x)] > 0.$$

On the other hand, $z_{\tilde{\mu}^*}$ is decreasing and $\tilde{z}_{\tilde{\mu}}$ is increasing when $\tilde{\mu}^*$ is increasing. So

$$J(z_{\tilde{\mu}}, \tilde{\mu}) = \int_{z_{\tilde{\mu}}}^{\tilde{z}_{\tilde{\mu}}} [\tilde{\mu} - M(x)] d[-w(1-x)] + \int_{\tilde{z}_{\tilde{\mu}}}^{1} [\tilde{\mu} - M(x)] d[-w(1-x)]$$

is increasing for $\tilde{\mu}$, because of $\tilde{\mu} - M(x) > 0, x \in (z_{\tilde{\mu}}, \tilde{z}_{\tilde{\mu}})$ and $\tilde{\mu} - M(x) < 0, x \in (\tilde{z}_{\tilde{\mu}}, 1)$.

Therefore, there exists a unique $\tilde{\mu}^* \in [M(z_0), \min\{\mathbb{E}(\rho), M(1)\}]$ such that $J(z_{\tilde{\mu}^*}, \tilde{\mu}^*) = 0$, which completes the proof.

Chapter 6

Conclusions

This thesis comprises four essays on optimal investment in continuous financial markets. The first two are concerned with the optimal stock selling/buying with respect to its ultimate maximum (minimum) price, while the latter two consider the investment behavior of the investor with behavioral preferences.

Assuming the GBM of stock price, we formulate the optimal stock selling/buying problem as an optimal stopping problem in Chapter 2 and 3. More precisely, these problems are called optimal prediction problem in probability literature. Chapter 2 considers the optimal stock selling problem by minimizing the distance between its selling price and its ultimate maximum price, where the distance is measured by the expected square error. We make use of a PDE approach to study the optimal selling strategy. It turns out that the optimal selling strategy, heavily depending on the sign of the excess return of stock μ, is determined by the ratio of the running maximum to stock price as well as time to maturity. More precisely, in the case of $\mu > 0$, the optimal selling boundary has two branches between which is the selling region. Moreover, the two branches exist only when time to maturity is shorter than a threshold value, which indicates that selling is optimal only when time is sufficiently close to the maturity and the ratio of the running maximum to stock price lies between certain time-varying levels. However, in the case of $\mu \leq 0$, the optimal selling boundary has only one branch, and selling is optimal as time to maturity tends to zero.

In Chapter 3, we formulate four stock buying/selling problems so as to capture the investment motto “buy low and sell high”. The free boundary PDE approach, as opposed to the probabilistic approach taken by Shiryaev, Xu and Zhou (2008b), is employed to solve all the problems thoroughly. The optimal trading strategies derived are simple and consistent with the normal investment behaviors. For buying problems, apart from the straightforward extreme cases (depending on the quality of the underlying stock) where one should buy immediately or never buy, one ought to buy so long as the stock price has declined sufficiently from the historical high or risen sufficiently from the historical low. For selling problems, optimal strategies exhibit similar (or indeed opposite) patterns.

We then study finite horizon optimal liquidation for an investor with prospect theory preferences in Chapter 4. We solve the existence and uniqueness of the optimal liquidation strategies. Due to nonsmoothness of the payoff function, we characterize the value function in the sense of viscosity solution and show the smooth fit principle holds only on part of the free boundary. We also provide a thorough analysis on the optimal liquidation strategies. The main application of our model is to explain the disposition effect and break-even effect. Our results show the investor is much more likely to realize a small gain and defer a large loss. In particular, the investor with a piecewise power utility is never optimal to realize a small loss, but possible to realize a moderate loss. These properties are consistent with the disposition effect. As predicted by our model, loss aversion is the main driver of the break-even effect. The effect of loss aversion λ on the liquidation strategies implies that the more loss aversion the stronger break-even effect.

Finally, we consider the distorted mean-semi-variance portfolio selection problem in continuous time. This work is motivated by Jin, Yan and Zhou (2005), where they show there is no optimal solution for the conventual mean-semi-variance problem. According to Yaari's daul theory, probability distortion can reflect some of the investor's risk preference. We show that there do exist optimal solutions or the infimum risk is attained in our distorted mean-semi-variance model with some proper probability distortions, e.g. a convex probability distortion function $z^\gamma, \gamma > 1$, which reflects additional risk aversion of the investor. Via the quantile approach, we are able to solve the problem completely in the sense that we address the feasibility problem, show the necessary and sufficient conditions of the existence of the optimal solutions, present the close form solutions for the given examples. In addition, when there is no probability distortion, our model reduces to the conventional mean-semi-variance model, and our results recover the same case in Jin, Yan and Zhou (2005).

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