nature portfolio

Peer Review File

Characterising high-order interdependence via entropic conjugation

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This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
In this paper, Rosas et al., introduce the mathematical notion of "entropic conjugation" to explore the structure of information-theoretic measures of higher-order interactions in complex systems. Prior work on measures like the O-information, TSE-complexity, and PID has historically lacked a unifying foundation for understanding the relationships between measures. Here Rosas et al., provide compelling results pointing towards a way forward. They find that the standard battery of measures (O-information, S-information, TC, DTC, and TSE Complexity) can all be understood in the context of entropic conjugation.

The mathematics all appears to be robust, and I think that all of the results will be of great interest to the community of complexity and information theorists. My biggest critique is that the presentation is very dense. I understand that the authors are trying to cover a fair amount of mathematical ground, but I would like to see a little bit more effort paid to the problem of making the intuitions a little bit more accessible.

For exampe: Eq. 7 seems to come out of nowhere, but the measure u_k(X) is a core part of all of the subsequent results. Is there a parsimonious intuition pump that captures what u_k(X) really quantifies? There is a reference [24] to a somewhat obscure paper from the late 1970s which I can't imagine most readers will be familiar with. If it's possible to make u_k(X) more accessible, I think that will go a long way towards illuminating the underlying logic of the piece.

One possibility would be to describe how these measures (u_k(X), TC, DTC, etc) behave on a set of small, well-known distributions. For example, this paper goes through and details how a novel information theoretic measure behaves on three distributions: the logical XOR (pure synergy), the giant bit (pure redundancy), and the W-distribution (a mixture of synergy and redundancy).

https://www.nature.com/articles/s44260-024-00011-1

I know that many of these measures, such as the O-information, have already been applied to XOR distributions and the like in the past, but for some of these other measures, I think it would be helpful. Adding these to the already-existing "results" section (focusing on ferromagnetic spin systems) shouldn't be too hard, I don't think.

Minor comments:

- Given that all the explored measures can be written in terms of sums of u_k^n(X), it must be possible to define novel measures using different summations, correct? Are there new measures that you could propose that might have interesting properties?

- Have the authors thought at all about whether this framework also applies to local/pointwise instances? The signs of the expected values of measures like the O-information are straightforward, but if we go local, you can have instances where the local TC or local DTC are negative, which seems like it might complicate the interpretation of negative local O. But on the flip side, if localization is possible, this would provide interesting new insights into time-resolved dynamics.

- Typo: "minus sing" in the Conclusions.

- I am a little fuzzy on Eq. 25 - it seems like \phi is being defined in terms of functions on itself (\phi + \phi-star)? In \phi notation in general was kind of hard to follow. Perhaps this is my lack of math background, but I wasn't sure if it had any meaning beyond "an arbitary linear combination of all entropies."

Reviewer #2

(Remarks to the Author)
The authors propose a new conjugacy relation for Shannon entropy in multivariate systems. They then use it to analyze information-theoretic measures of higher-order dependence. They show that their approach clarifies the relation between previous measures.

In general, I found the manuscript interesting and clearly written, and the formalism is elegant. I have two minor suggestions before recommending for publication.

First, I am wondering about other conjugation relations. For instance, it seems that
H(X^a)* = H(X) - H(X^-a),
the negation of the authors' conjugation, may also be one. (BTW, this alternative conjugation satisfies a nice analogue of the rank-nullity theorem from linear algebra,
H(X) = H(X^a)* + H(X^-a) ,
which could be interesting to mention). In any case, I understand that the authors' conjugation is special from the point of view of the PID atoms. This is interesting but not entirely convincing to me, as the validity of the PID atoms has been called into question numerous times. I would appreciate a discussion of the possibility of other conjugation relations.

Second, the spin-glass example is a bit underwhelming, and I think could be expanded to better showcase the authors' approach. I leave it up to the authors, but it would be nice to get a clearer sense that we gain some new insight into the spin-glass system. For instance, could entropic conjugation suggest the appropriate dependency measure to use a given class of spin glasses (e.g., ferromagnetic vs. frustrated ones)? Could their approach suggest a set of order parameters for distinguishing different phases of spin glasses? Etc.

Some typos:
"This leads to identify ..."
"axes of variability by via"
"results in a minus sing"

Reviewer #3

(Remarks to the Author)
This is an important contribution written by some of the world expert in high-order phenomena. The new insight provided by this work is that entropic conjugation may be seen as a principle to investigate the space of possible high-order quantities. In this way the authors shed light on the relations between existent metrics while opening new perspectives of research for the field. The manuscript is well written, in my opinion it deserves publication. Concerning the Ising systems the authors may cite and compare with the analysis of Ising models contained in "Gradients of O-information: Low-order descriptors of high-order dependencies", Scagliarini et al., Phys. Rev. Res. 5 013025, 2023.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
I am happy with the changes the authors have made - I appreciate the increased focus on pedagogy, and personally feel like I can better appreciate the material (which is very cool!)

Reviewer #2

(Remarks to the Author)
The authors' revision has addressed my concerns. I recommend the manuscript for publication.

Responses to comments from reviewers of paper

Manuscript ID COMMSPHYS-24-1346

"Characterising high-order interdependence via entropic conjugation"

RC *Reviewer's Comment*

AR Authors' Response

We would like to thank the editor and the reviewers for their thoughtful feedback on our paper. The received comments have helped us to substantially improve our manuscript. Our changes include a substantial rewrite of the conceptual and technical contributions of our paper, as well as new experimental results. In order to aid the review process, the modified text in the resubmitted manuscript has been highlighted using colour blue.

On behalf of all co-authors, thank you for your consideration.
Fernando Rosas

Responses to comments from Reviewer 1

RC *In this paper, Rosas et al., introduce the mathematical notion of "entropic conjugation" to explore the structure of information-theoretic measures of higher-order interactions in complex systems. Prior work on measures like the O-information, TSE-complexity, and PID has historically lacked a unifying foundation for understanding the relationships between measures. Here Rosas et al., provide compelling results pointing towards a way forward. They find that the standard battery of measures (O-information, S-information, TC, DTC, and TSE Complexity) can all be understood in the context of entropic conjugation.*

The mathematics all appears to be robust, and I think that all of the results will be of great interest to the community of complexity and information theorists. My biggest critique is that the presentation is very dense. I understand that the authors are trying to cover a fair amount of mathematical ground, but I would like to see a little bit more effort paid to the problem of making the intuitions a little bit more accessible.

AR We thank the reviewer for the very positive feedback. We also greatly appreciate the suggestions to make the text more accessible. Following this suggestion, our revised manuscript has significantly expanded the presentation of contents in order to provide more insights and better guide the introduction of ideas and motivations behind them.

RC *For exampe: Eq. 7 seems to come out of nowhere, but the measure $u_k(X)$ is a core part of all of the subsequent results. Is there a parsimonious intuition pump that captures what $u_k(X)$ really quantifies? There is a reference [24] to a somewhat obscure paper from the late 1970s which I can't imagine most readers will be familiar with. If it's possible to make $u_k(X)$ more accessible, I think that will go a long way towards illuminating the underlying logic of the piece.*

AR We thank the reviewer for bringing this up, we completely agree with this observation. To better motivate the introduction of the $u_k(X)$ quantities, our revised manuscript provides a more extensive rationale. The new presentation starts from the more accessible $r_k(X)$ quantities, and uses them to derive the $u_k(X)$ as the natural basis of the space of linear combinations of entropies satisfying labelling-symmetry and dependency.

RC *One possibility would be to describe how these measures ($u_k(X)$, TC, DTC, etc) behave on a set of small, well-known distributions. For example, this paper goes through and details how a novel information theoretic measure behaves on three distributions: the logical XOR (pure synergy), the giant bit (pure redundancy), and the W-distribution (a mixture of synergy and redundancy).*

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I know that many of these measures, such as the O-information, have already been applied to XOR distributions and the like in the past, but for some of these other measures, I think it would be helpful. Adding these to the already-existing "results" section (focusing on ferromagnetic spin systems) shouldn't be too hard, I don't think.

AR We thank the reviewer for the excellent suggestion. Following this advice, we have included an analysis of the W-distribution, which provides a nice confirmation that the u_k quantities follow our intuitions. We have added additional numerical analyses on a new collection of datasets, which we hope will provide further insights about the nature of these quantities and applicabiility of the proposed framework.

RC *Minor comments:*

- Given that all the explored measures can be written in terms of sums of $u_k^n(X)$, it must be possible to define novel measures using different summations, correct? Are there new measures that you could propose that might have interesting properties?

AR The reviewer is absolutely right. However, we are planning to leave that aspect of this work to be fully developed by an upcoming publication. Consequently, we added a remark in the discussion about this possibility as an interesting direction for future work.

RC *- Have the authors thought at all about whether this framework also applies to local/pointwise instances? The signs of the expected values of measures like the O-information are straightforward, but if we go local, you can have instances where the local TC or local DTC are negative, which seems like it might complicate the interpretation of negative local O. But on the flip side, if localization is possible, this would provide interesting new insights into time-resolved dynamics.*

AR This is another very interesting direction for future work. While it is true that the non-negativity of the TC, DTC, and more generally of the $u_k(X)$ quantities may not hold, that would not disrupt much in the framework. Hence, we included another comment in the discussion regarding the possibility of using this framework to analyse and create new pointwise metrics.

RC - *Typo: "minus sing" in the Conclusions.*

AR The typo has been fixed, thanks.

RC - *I am a little fuzzy on Eq. 25 — it seems like ϕ is being defined in terms of functions on itself $(\phi + \phi^\star)$? In ϕ notation in general was kind of hard to follow. Perhaps this is my lack of math background, but I wasn't sure if it had any meaning beyond "an arbitary linear combination of all entropies."*

AR We thank the reviewer for bringing this up, we agree that the usage of the ϕ notation in those passages could have been clearer. To improve this, we have included additional clarifications regarding the ϕ notation, both in Definition 2 and in Theorem 2. We have also included more context and motivation before the definition and theorem are stated. Furthermore, we included a brief example after Theorem 2, which we hope may help to clarify the implications of the result and avoid potential ambiguities as the ones noted by the reviewer.

Responses to comments from Reviewer 2

RC *The authors propose a new conjugacy relation for Shannon entropy in multivariate systems. They then use it to analyze information-theoretic measures of higher-order dependence. They show that their approach clarifies the relation between previous measures. In general, I found the manuscript interesting and clearly written, and the formalism is elegant. I have two minor suggestions before recommending for publication.*

AR We thank the reviewer for the very positive comments regarding our work.

RC *First, I am wondering about other conjugation relations. For instance, it seems that $(H(X^a))^* = H(X) - H(X^-a)$, the negation of the authors' conjugation, may also be one. (BTW, this alternative conjugation satisfies a nice analogue of the rank-nullity theorem from linear algebra, $H(X) = (H(X^a))^* + H(X^-a)$, which could be interesting to mention). In any case, I understand that the authors' conjugation is special from the point of view of the PID atoms. This is interesting but not entirely convincing to me, as the validity of the PID atoms has been called into question numerous times. I would appreciate a discussion of the possibility of other conjugation relations.*

AR We thank the reviewer for bringing this interesting and very relevant suggestion, which helps to better frame our contribution. Following this suggestion, the revised manuscript provides more extensive motivation related to our choice for the conjugation, and also includes a new Appendix B discussing the space of alternative linear involutions.

RC *Second, the spin-glass example is a bit underwhelming, and I think could be expanded to better showcase the authors' approach. I leave it up to the authors, but it would be nice to get a clearer sense that we gain some new insight into the spin-glass system. For instance, could entropic conjugation suggest the appropriate dependency measure to use a given class of spin glasses (e.g., ferromagnetic vs. frustrated ones)? Could their approach suggest a set of order parameters for distinguishing different phases of spin glasses? Etc.*

AR We are thankful to the reviewer for raising this observation, which triggered productive discussions between co-authors regarding what could be a useful extension for the presented numerical analyses. We decided to include a relatively similar set of analyses on a collection of datasets related to choral music. Despite how different these systems are, analyses show a remarkable similarity with the results found for spin models. We hope that these additional analyses may serve to further illustrate the value of the proposed framework to better understand the nature and relationship between existent high-order information metrics.

RC *Some typos: "This leads to identify ..." "axes of variability by via" "results in a minus sing"*

AR These typos have been fixed, thanks for noticing them.

Responses to comments from Reviewer 3

RC *This is an important contribution written by some of the world expert in high-order phenomena. The new insight provided by this work is that entropic conjugation may be seen as a principle to investigate the space of possible high-order quantities. In this way the authors shed light on the relations between existent metrics while opening new perspectives of research for the field. The manuscript is well written, in my opinion it deserves publication.*

AR Many thanks for the very positive feedback.

RC *Concerning the Ising systems the authors may cite and compare with the analysis of Ising models contained in "Gradients of O-information: Low-order descriptors of high-order dependencies", Scagliarini et al., Phys. Rev. Res. 5 013025, 2023.*

AR Many thanks for the suggestion. We have mentioned this paper as a motivation for the current analysis. Additionally, the discussion section in our revised manuscript now mentions that the extension of entropy conjugation to study directed quantities such as the gradients of O-information is an important direction for future work.