ENTAILMENT

The Formalisation of Inference

by

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I should like to express my gratitude to Professor Dana Scott, who supervised the preparation of this thesis. During the research stages of the work he allowed me to inflict my often barbarous and grotesque ideas upon him at regular intervals, and he was an infinite source of correction and inspiration. He was also kind enough to read through the whole of the final manuscript.

I should also like to thank Lloyd Humberstone for pointing out two not unimportant errors in the formal work in Chapters IV and IX.

But perhaps a logician's greatest debt is always to his typist, in my case Joyce Arnott, whose patience and skill have often seemed to exceed the bounds of logical possibility.

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The proofs in section 9.3 presuppose theorem 7.2 of chapter III, which in fact cannot be proved for ramified modal logic as it has been presented here. This point is discussed in the note added between pages 55 and 56.

Abstract

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Entailment is the relation which holds between the premisses and conclusion of a valid argument. Various standards of argument validity are prominent in the current literature: the classical, the relevant, and the intuitionistic. Correspondingly there exist various entailment relations. Classical entailment is studied in Chapters II - V, and relevant entailment in Chapters VI - IX.

Entailment is treated syntactically as a sentence-predicate, and sentences asserting entailment are axiomatised hierarchically, one axiom system for each level of language. The axiomatisations used are based on Genzen's sequent calculus. $\Box$ and $\Diamond$ are defined in terms of entailment, and thus are treated also as sentence-predicates. Accordingly the hierarchy of entailment systems constitutes modal logic at what Quine has termed the first grade of modal involvement. It is proved that the hierarchy of systems is of S5 strength. Thereby the influential conjectures of Montague (1963) are rebutted.

The semantics of entailment is studied. In particular semantics giving a substitutional interpretation to the quantifiers are investigated. Such semantics answer Quine's charge that quantification into first grade modal contexts is incoherent. The problem of finding a substitutional semantics which invalidates the Barcan Formula is tackled. Finally, with a view to

avoiding the standard objections to substitutional quantification whilst retaining the facility of quantifying into metalinguistic contexts, semantics which mix the substitutional and objectual interpretations are introduced.

Reasons given in the literature for rejecting classical entailment theory in favour of relevance logic are surveyed and dismissed. However, the question of the relevance of premisses to conclusion is obviously a bona fide part of the whole business of appraising the worth of an argument. It behooves us as formal logicians to attempt to formalise the relevance appraisal. Given a formal theory of premiss-conclusion relevance one may, if one wishes, include the relevance appraisal alongside the truth-preservation appraisal in the assessment of formal validity. This procedure generates a relevant entailment relation. Anderson and Belnap's system E of entailment is investigated and dismissed as being merely ad hoc, answering to no account of premiss-conclusion relevance. A formal semantical analysis of premiss-conclusion relevance is given. This analysis is used in the construction of a semantical characterisation of a system of relevant entailment. A Genzen-style axiomatisation of the system is then given.

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CHAPTER I

Groundwork

1. The Connection Between Entailment and Inference

1.1 The traditional use of the word "entails"

It seems that the term "entails" was first introduced into philosophy by G.E. Moore (1922), when he chose it to express "the converse of that relation which we assert to hold between a particular proposition q and a particular proposition p when we assert that q follows from or is deducible from p." (p.291) That is the sense in which the term "entails" will be used in this thesis: the relation of entailment is the converse of the relation of deducibility. Moreover, the relation will be taken to be the converse of logical deducibility. There is a customary sense of "follows" in which, for example, "Socrates is mortal" follows from "Socrates is a man". The admissibility of deductions of this sort turn on matters of fact rather than matters of logic: things might have been such that the premiss of the deduction was true whilst the conclusion false. Such deductions will not be counted as converse entailments. Moore does not discuss this point explicitly, but I think it is clear from the text that by "deducible" he intended to mean "logically deducible".

Lewis and Langford (1932) chose the word "implies" where Moore chose "entails". They write: "The chief business of a canon of deduction is to delineate correctly the properties of that relation which holds between any premiss, or set of premisses, and a conclusion which can validly be inferred. Commonly, this relation is called 'implication'." (p.235) We shall use the terms "entails" and "implies" interchangeably.

1.2 Entailment and argument

The term "argument" will be used so that an argument is a sequence of (English) sentences; and so that two different sequences constitute the same argument just in case they consist of the same sentence types in the same order. There is, of course, a different and widely used sense of "argument", such that the same argument can be expressed by two sequences which do not consist of the same sentence types. The former sense, however, is of greater interest to the formal logician, who requires an argument to be a fairly concrete object capable of being written down in exactly one way.

Since B is deducible from A just in case the argument ⌜A Therefore B⌝ is valid, entailment in Moore's sense is the relation which holds between the premiss (or, more generally, premisses) and the conclusion of a valid argument. Accordingly an assertion of entailment is true just in case the corresponding argument is valid.

In discussing arguments, the logician customarily attains greater generality by using dummy variables of various categories rather than actual expressions of English. In standard predicate logic dummy propositional and predicate variables are used. The former are eliminable by "real" sentences and the latter by "real" predicate-expressions. The result of replacing expressions occurring in an actual argument by variables (of appropriate categories) is usually called an argument form. Because our interest is in general forms of inference rather than inferences between particular English sentences, we will widen Moore's sense of "entails" to cater for argument forms as well as actual arguments. Entailment is taken to be a relation between the premisses and conclusion of an argument or argument form; and correspondingly formulae of formal logic will be permitted to figure as antecedents and consequents of entailment

sentences. Because of our concern with generality, it is entailments between formulae rather than actual sentences which will chiefly hold our attention.

1.3 Inference figures and entailment sentences

Typically the sequence of sentences comprising an argument will contain members which are neither premisses nor conclusion of the argument: in the course of drawing the conclusion from the premisses a host of modes of inference will be used, the conclusions of earlier inferences figuring as the premisses of later ones. In representing an argument in a fashion ripe for the formulation of the corresponding entailment sentence this "middle structure" may be ignored, for our concern is with saying merely that the original premisses entail the final conclusion. The format we adopt in this respect for representing arguments and argument forms is that of the inference figure, due to Gerhard Genzen (1935). An inference figure is a display of the form $\frac{A_1, \ldots, A_n}{B}$, the schematic letters representing sentences, or formulae in the case of an argument form. The figure indicates that B is deducible from the A_i's, and will be appraised as valid just in case the original argument or argument form is valid. Any argument or argument form may be represented minus its middle structure by writing its premisses as the upper members of an inference figure, and its conclusion as the lower member.

Inference figures provide a canonical method for the representation of logically correct inferences, and so lend themselves to the systematic treatment of inference. Taking certain inference figures as primitive, it will be possible to derive from them by the use of appropriate transformations all the remaining valid inference figures associated with some given language. Thus the logic of a language may be reduced to a recursive formal theory.

It is natural to call a valid inference figure a principle (or rule) of inference. The set of valid inference figures associated with a given language represent all the permissible deductive moves in the language; and given a recursive characterisation of these inference figures, appeal to the validity of figures provides a systematic means of licensing particular inferences in the language.

An entailment sentence corresponding to a given argument or argument form asserts that the premisses of the argument (or form) entail the conclusion. The sign "$\rightarrow$" is used to carry the concept of entailment symbolically, and an entailment sentence is formed by writing the premisses of the argument, separated by commas, on the left of an arrow, and the conclusion of the argument on the right of the arrow. Thus entailment sentences "horizontalise" inference figures. Where $\frac{A_1, \ldots, A_n}{B}$ is the inference figure representing a given argument (or form), its horizontalisation $A_1, \ldots, A_n \rightarrow B$ is the entailment sentence corresponding to that argument (or form). An entailment sentence will be said to be valid just in case the figure it horizontalises is valid (or, equivalently, just in case the corresponding argument form is valid).

Valid entailment sentences, then, are horizontalisations of principles of inference. Being linear objects, entailment sentences are somewhat easier to handle than the figures they horizontalise, and their use facilitates the construction of a systematic characterisation of the principles of inference of a language.

It is worth noting that component formulae of entailment sentences may themselves be entailment sentences. Quine (1953, p.168) has suggested that sentences containing iterations of the symbol for entailment will be devoid of interest. But this is incorrect. Many principles of inference

will themselves concern entailment sentences, and thus their horizontalisations will contain iterations of the symbol →. Examples are the horizontalisations of the principle of contraposition and of the so-called principle of conditional proof: $(A \to B) \to (\sim B \to \sim A)$ and $(A_1, \ldots, A_n, A \to B) \to (A_1, \ldots, A_n \to A \supset B)$. Again, in order to horizontalise certain rules in natural deduction formulations of predicate logic it is necessary to use nested arrows. The rule of ∨-elimination in Genzen's system NK (1935) is an example:

$$\frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

This is horizontalised by the sentence $(A \to C), (B \to C) \to (A \vee B \to C)$.

1.4 Argument validity

In the above account, entailment has been tied to argument validity; and so which entailment sentences one regarded as valid will depend on which arguments are regarded as valid. To appraise an argument as valid is not nowadays univocal, for standards of argument correctness against which the appraisal is made proliferate. The classical standard of argument correctness is that of truth-preservation: an argument is classically valid just in case there is no possible state of affairs in which its premisses are true but its conclusion false. There are two non-classical standards of argument correctness of major importance: that of the intuitionists, and that of relevance logicians. All intuitionistically or relevantly valid arguments are classically valid, but not vice versa. In these two schools of thought, truth preservation is taken as a necessary condition for validity, but not as a sufficient condition.

At the present stage in this work the classical standard of argument correctness is espoused. There are three motives for this. Firstly, I am not convinced that it is an incorrect standard, although not because I am dissatisfied by the arguments of relevance logicians and intuitionists, but rather because I am unhappy with the view that there is just one right standard of correctness. This position is discussed further in Chapter VI; and the notion of relevant entailment is thoroughly discussed in the later parts of this thesis. Secondly, I feel that I have some fresh and important things to say concerning the entailment theory determined by the classical standard. And lastly, no discussion of the classical standard is time wasted, in view both of its historical importance and of the fact that the intuitionist and relevance accounts take the classical standard as part, at any rate, of the whole story.

An inference figure (or, equivalently, argument or argument form) will, then, be said to be valid just in case there is no possible state of affairs in which its premisses are true but its conclusion false (relative to any replacement by actual expressions of any dummy variables occurring in the figure). Or, to spell this out without reference to states of affairs, an argument is valid just in case it is an exemplification of a valid form, where an argument form is valid just in case it has no exemplification with true premisses and a false conclusion (Copi, 1967, pp.20-21).

It is now possible to state the truth conditions of an entailment sentence: $A_1, \ldots, A_n \rightarrow B$ is true just in case <u>whenever</u> all the A_i are true, B is <u>also</u> true (and the sentence is <u>valid</u> just in case it is true in all instances or models). (This statement of the truth conditions of

entailment sentences is given in Scott, 1974, p.149.) In Chapter IV, these truth conditions are incorporated into a formal recursive semantics for classical entailment. The present statement is intended to give some intuition into the meaning of the sign →.

1.5 Entailment systems

An entailment system is a formal axiomatic system all of whose theorems are entailment sentences. Much of this thesis is taken up with the construction of such systems. In the later chapters a system of relevant entailment is constructed, whilst in the earlier chapters two axiomatisations of classical entailment are given, the axiomatisations differing in their treatment of quantification into a formula itself containing an entailment. There are two motives for this concentration on entailment systems.

Firstly, an entailment system constitutes a systematic formalisation of principles of inference. We maintain that the production of such systemalisations is of cardinal importance in the science of inference, formal logic. Yet it is an activity which has been largely neglected. Logicians have, in general, been content to regard systems of material implication as fulfilling this function. This is, however, a fundamental conceptual mistake, as argued in section 3.1 below. Exceptions to this generalisation are Lewis and Langford, on the classical front, and Anderson and Belnap on the relevance front. However, we claim that neither of these pairs of logicians have succeeded in formulating satisfactory entailment systems. This claim is argued in Chapter I section 4 (Lewis and Langford) and Chapter VIII (Anderson and Belnap). The systems set out in this thesis are intended as corrective to the pioneer work of these authors in their fields.

An entailment system of particular interest will be that having as theorems all and only valid entailments of the form A → B, where A and B are formulae whose only logical symbols are quantifiers and truth functional

connectives (and so, in particular, A and B contain no occurrences of →). Such a system provides a formalisation of the principles of inference of classical predicate logic. Of similar interest is the analogous system for relevant entailment, which provides a formalisation of the principles of inference of relevant predicate logic. A system fulfilling the former role is set out in Chapter II, and the latter role, in Chapter IX.

There is a second motive for our concentration on entailment systems. Historically, the provision of axiomatisations has been important, in that an axiomatic theory provides a formal, systematic, and precise account of the truths involving the target concepts of the theory, and can in an important sense be said to provide an analysis of these concepts. Thus, for example, Russell and Whitehead (building on the work of Frege) constructed a systematic formal analysis of the previously informal concepts of quantification, conjunction, negation, and so on; and Anderson and Belnap (building on the work of Ackermann) have, in their system E, constructed a systematic formal analysis of the previously informal concept of relevant entailment.

The provision of a formal analysis of the concept of entailment is one of the chief aims of this thesis. However, we do not follow the historical tradition of relying solely on the construction of an axiomatic theory to supply the formal analysis. We believe that the construction of a formal semantics for a concept is a far more powerful tool of analysis than the construction of axiomatic systems (providing, of course, the semantics is not a mere model theory for a pre-existing axiomatisation: too much recent work on the semantics of entailment has taken this sterile form). Much time is given in this thesis to constructing what are claimed to be natural and illuminating formal semantics for entailment, both classical and relevant,

and we consider these semantics to provide clear and correct formal analyses of the concept(s) of entailment.

Nonetheless, the axiomatisation of the theory determined by the semantics is considered to be an important complement to a semantical analysis. An axiomatic analysis will often deepen the insight and understanding supplied by the semantical one; and to record a piece of autobiography, I feel I have become very clear on the concept of entailment as a result of combining these two methods of analysis.

The formal entailment systems constructed in this thesis, then, are always axiomatisations of the body of truths generated by a pre-existing semantics (although our first system is in fact presented before the semantics); and we maintain that the methodology of constructing semantical theories of entailment prior to axiomatic systems of entailment has, particularly in the case of relevance logic, increased greatly the clarity and appeal of the analysis of entailment provided in this work (as compared to that supplied by Anderson and Belnap (1975), who follow the reverse procedure).

1.6 Significant logical structure

We will be interested only in entailment systems which treat the quantifiers, truth functional connectives, the identity predicate, and $\rightarrow$ itself as the sole logical constants. (A logical constant is any expression which makes the same contribution to the truth conditions of every sentence in which it occurs.) The horizontalisations of many classically (and relevantly) valid inference figures will not be provable in such systems. To cite examples from number theory and set theory. $x > 1 \rightarrow x > 0$ and $x = \{\emptyset\} \rightarrow \exists y(y \in x)$ will not be provable, although

clearly they are horizontalisations of classically (and relevantly) valid inference figures. The truth preserving character of these figures arises from the semantical roles of the predicates ">" and "$\in$", and of the singular terms "1", "0", and "$\{\emptyset\}$". A theory of inference which fails to assign distinguished logical roles to these expressions will be insensitive to the validity of the inference figures in question.

There are two ways in which an entailment system for classical predicate logic with identity might be extended to include all the principles of inference associated with a given body of necessary truths such as set theory or number theory. Firstly, the Zermelo-Fraenkel axioms or the Peano axioms, say, may simply be added to the axioms of the entailment theory, and the following rule used to derive the required entailment sentences: if B is provable in the system on the assumption of $A_1, \ldots, A_n$, then $A_1, \ldots, A_n \rightarrow B$ is a theorem of the system, where each of the A_i's and B is a sentence of set theory (or number theory, as the case may be). The deduction will be carried out using predicate logic as formalised by the entailment system itself.

However, this method is unpleasantly ad hoc: entailment sentences ought to be derivable directly in an entailment system, rather than by means of a rather contrived rule of conditional proof relying on the availability in the system of axioms which are not entailment sentences.

The second method of extending an entailment system to deal with set theory or number theory avoids this ad hocness, and seems to be the correct way to set about things. The method is to take a natural deduction formulation of set theory (or number theory), horizontalise the natural deduction rules by means of entailment sentences, and add these entailments as axioms to the entailment system for predicate logic. However, this

method is hampered in practice by the fact that little work has been carried out on natural deduction formulations of mathematics. Fitch (1952) and Prawitz (1965) have carried out a certain amount of work on natural deduction formulations of set theory, but I know of no such work on number theory or other areas of mathematics.

The fact that the principles of inference associated with a given body of necessary truths may be formalised by extending an entailment theory for predicate logic points up two of the motives for our policy of omitting from our systemalisation of inference all principles whose validity turns on the semantical role of expressions other than the logical constants of standard predicate calculus with identity (and $\rightarrow$ itself). Firstly, predicate logic is thereby shown to occupy a fundamental position in the theory of inference, and well deserves the undivided attention of an entailment theorist. And secondly, the additional axioms used provide no additional information about the notion of entailment, but rather concern the properties of the new logical constants under consideration. A syntactical and semantical study of inference which treats the above mentioned as the sole logical constants is sufficient to get thoroughly clear on the concept of entailment.

A third reason for ignoring principles of inference of the sort under discussion is that, in any case, the vast majority of them will elude systematic treatment. There is no limit to the possibilities of inexplicit abbreviation and analyticity in natural language, and the principles of inference thrown up by these abbreviations and analyticities will not form a succinctly specifiable set. For example, the validity of the inference figure:

$$\frac{\text{Russell is a brother}}{\text{Russell is male}}$$

depends on the semantical roles of the two predicates contained in it. To accommodate this principle of inference in an entailment system requires treating these predicates as logical constants, whose relevant logical properties must, in some way or other, be axiomatised. The result of doing this for all such "analytic inferences" of English would be an entailment system having very, very many axioms (although presumably only finitely many), and even given the patience, time, and lexicographical skill to construct this system, the result would be too cumbersome to be of practical interest. Because of the phenomena of analyticity and inexplicit abbreviation, it is in practice delusive to aim for a systematisation of all the principles of inference of the English language. However, we hasten to add that one should not conclude from this "so much the worse for formal logic". An entailment system for predicate logic sets out the apparatus sufficing for the basic business of inference considered apart from those "surface features" of language which real speakers will frequently make use of in the course of expressing themselves clearly, briefly, and stylishly (and considered apart from mathematical reasoning, where attention to additional logical constants is essential).

It is worth stressing that we consider the argument

$$\frac{\text{Russell is a brother}}{\text{Russell is male}}$$

to be a fully fledged valid argument (unlike various other authors, for example George (1972), Anderson and Belnap (1961)). We do not consider it to be merely enthymematically valid; that is, an enthymeme itself invalid in the proper sense of the term, but appraised as valid by courtesy in view of the validity of the parent argument from which the enthymeme is obtained by the suppression of a necessarily true premiss. (In the present case

the envisaged parent argument would be:

$$\frac{\text{Russell is a brother. All brothers are male.}}{\text{Russell is male.}})$$

This is because we take seriously the classical standard of argument validity: an argument is classically valid just in case there is no possible state of affairs in which its premiss is true and conclusion is false.

2. How is the Concept of Entailment to be Carried Grammatically?

The concern of this section is to consider whether "→" is best treated syntactically as a binary semantical predicate, viz an operator attaching the pairs of names of sentences to form a sentence of the metalanguage, or as a binary connective, viz an operator attaching to pairs of sentences to form a further sentence of the same language. The metalinguistic approach will be favoured.

Quine has often stressed that entailments are sentences of the metalanguage. He writes: "Implication [is a] relation between statements, and hence [is] properly attributed by putting a verb 'implies' between names of the statements related. Such discourse belongs to metalogic." (1966a, p.16) "When we say that one statement or schema implies another, we are not to write 'implies' between the statements or schema concerned, but between their names. In this way, we mention the schemata or statements, we talk about them, but use their names." (1972, p.38)

It is, however, difficult to find in Quine's writings any definite arguments for the correctness of the metalinguistic approach. Quine's attitude seems to be that in defending the approach one can do no more than draw attention to the distinction between using an expression and

mentioning it: once the distinction has been grasped, the incorrectness of treating implication as a connective should become obvious, and anyone who persists with that treatment has simply remained confused about use and mention. Certainly many logicians do persist with the connective treatment, and Quine remarks at one point that "by now perhaps I should have concluded that I must be the confused one, were it not for people who do turn out to see the distinction my way". (1961a, p.177) But this is hardly a satisfactory state of polemical affairs. Outstanding logicians who evidently have grasped the distinction between use and mention but nonetheless favour the connective approach to implication cannot be attacked by merely citing once again the use/mention distinction. Definite arguments for the metalinguistic approach are required.

Quine does provide one definite argument in (1953). However, the argument continues to rely on Quine's basic tenet that the syntactical distinction between logic and metalogic is sharp and obvious, and since this is precisely what is rejected by those apprised of the use/mention distinction but satisfied with a connective treatment of implication, the argument is of little use. The argument hinges on the following analysis of implication given by Quine: "one statement logically implies another if the material conditional formed of the two statements is logically valid", where "a statement is logically valid if it and all statements like it in logical structure are true" (ibid, p.163). And since, the argument goes, "is valid" is clearly a semantical predicate, it follows that "implies" is also. (p.164) The trouble with the argument is that anyone who likes treating implication syntactically as a connective is quite liable to be prepared to like a similar syntactical treatment of validity. Certainly it is usual practice in mathematical logic to assert validity metalinguistically. But usual practice need not be the only practice, and it might,

for example, simplify things syntactically to use a one place connective to carry the concept of validity in the construction of a formal system axiomatising the valid validity-assertions associated with predicate logic. Indeed, given Quine's analysis of implication, an entailment system for predicate logic will constitute part of such a system.

What, then, are the considerations which lead us to favour a metalinguistic approach to entailment?

Firstly, it seems to me quite certain that, conceptually, entailment is a relation between sentences. The fact that the conclusion of a given inference figure is deducible from its premiss is a fact about a particular pair of inscriptions, namely the premiss and the conclusion. Conceptually, entailment belongs to metalogic, no matter how we treat it syntactically. Carrying entailment in our syntax by means of a semantical predicate attaching to pairs of sentences accurately reflects this conceptual fact, whilst carrying entailment syntactically by means of a connective obscures the conceptual fact. This does not show that it is incorrect to treat entailment as a connective; but it does indicate that if we are interested in using a formal study of entailment to get as clear as possible about the concept of entailment, then the predicate approach is the one we should favour in constructing our formal syntax.

In the end, we may come to favour the connective approach. A very good reason for eventually coming to adopt it is the immense simplicity it lends to the formal syntax compared to the predicate approach. (The complexities involved in the predicate approach will become painfully apparent in the following chapters.) But I think it is a mistake to build the connective approach primitively into our formal thought. Once enough study and experimentation have been made with entailment construed as a semantical

predicate, we can then self consciously take on the mix between logic and metalogic involved in the connective approach to entailment without jeopardising the conceptual clarity gained as a result of our first metalinguistic studies. But to have made the mix at the outset would, for me at any rate, have been quite unilluminating: the clarity which I feel I have achieved on the concept of entailment as a result of the formal experimentation made during the researching of this thesis is quite certainly a result of the metalinguistic approach adopted.

A further reason for preferring to initially avoid a syntax in which entailment is carried by a connective concerns the question of iteration. When → is treated as a semantical predicate it attaches to pairs of names of sentences to form a sentence of the metalanguage. → treated thus can iterate: it can attach to names of sentences of the metalanguage to form a sentence of the meta metalanguage, to names of sentences of the meta metalanguage to form a sentence of the meta meta metalanguage, and so on. Thus where p is a propositional variable, ("p" → "p") is a sentence of the metalanguage, ("("p" → "p")" → "("p → "p")") is a sentence of the meta metalanguage, and so on.[1] An important point about such iterations is that they are always balanced: the number of occurrences of arrow on the left side of the principal arrow of the formula always equals the number of occurrences of arrow on the right side of the principal arrow. Formulae not balanced in this way are not well formed. For example, if a formula

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1. Since p is a variable, not a schematic letter, "p" is a name of a formula, not a name of the sixteenth letter of the roman alphabet, and so there is no gaffe committed in writing: "p" → "p". (See Quine (1940, section 6) for a discussion of the use of quotation marks in formal logic.) Note that for visual clarity outermost pairs of quotes are omitted in mentioning the above formulae. This policy is followed generally in this thesis: no distinction is made in the thesis-language between a formula (or symbol) and its name in the thesis-language.

is constructed with one arrow on the left of the main arrow and two on the right, the main arrow has been attached to a name of a sentence of the metalanguage, and a name of a sentence of the meta metalanguage, and the result is not in any member of the progression of metalanguages.

It is important to keep entailment formulae balanced in this way if they are to remain clearly related to inference figures. A formula containing one arrow horizontalises an inference figure whose upper and lower members are formulae of the predicate calculus. A formula containing three arrows horizontalises an inference figure whose upper and lower members are single-arrowed formula, and so in fact asserts the derivability of one principle of inference from another. For example, the formula $(A \to \sim B) \to (B \to \sim A)$ corresponds to the display

$$\dfrac{\dfrac{A}{\sim B}}{\dfrac{\sim B}{A}}$$

which indicates that the lower inference figure is derivable from the upper.[2] Similarly a formula containing five arrows corresponds to a more complex but quite clear relationship between inference figures; and so on.

The trouble with a connective approach to entailment is that it leads to formulae which are not balanced in the way just discussed. Since a connective may be written between any two sentences of the language to form a further sentence of the language, formulae like $(A \to (A \to B)) \to (A \to B)$ and $(A \to \sim A) \to \sim A$ are well formed. (In fact both these formulae figure as axioms in the Anderson-Belnap system of entailment E. An exposition and discussion of E is given in Chapter VIII below.) But how are the formulae to be related to inference figures? Perhaps the first may be read as

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2. Our policy of omitting explicit occurrences of quotation marks from entailment sentences is discussed in Chapter II, section 1.1.

indicating that the soundness of the figure $\frac{A}{B}$ is deducible from the fact that the soundess of this figure is deducible from the sentence A; and the second as indicating that ~A is deducible from the soundness of the figure $\frac{A}{\sim A}$. The formulae, then, mix concepts of logic, metalogic, and meta metalogic in a slightly alarming way, and clearly express entailments of very different category to those expressed by balanced formulae. There is, I believe, nothing fundamentally wrong with this mixing of concepts, nor with a syntax that countenances it. But, again, it seems to me to keep things clearer if the mix is not made at the outset of a formal study of entailment. I prefer to avoid entailments like those expressed by the above two formulae until I feel that I have got a good grasp on the concept of entailment.

Of course, one way of avoiding these unbalanced formulae whilst retaining the connective approach is simply to stipulate that no unbalanced formulae are well formed. But it seems to me that anyone prepared to use the connective approach deserves to bite the bullet and deal at the outset with the entailments engendered by unbalanced formulae. After all, the motivation for initially avoiding these entailments is that they mix concepts of logic and metalogic, and it is precisely this mixing that the connective approach countenances, whether all formulae are balanced or not.

A lengthy justification of carrying entailment syntactically by means of a connective has been given by Belnap (1975), and the remainder of this section is devoted to a discussion of his arguments.

Belnap's main intention in the paper is to argue for the conclusion that "although 'entails' in English is unmistakably a predicate, this grammatical fact should not lead us to forego investigation of entailment

qua connective". (p.490) Since I have not drawn on considerations of English grammar in my suggestion that initially a formal study of entailment should favour the predicate approach, the overall argument of Belnap's paper is irrelevant to my position. However his views do conflict in a damaging way with mine in that he would deny what was taken to be obvious at the beginning of the above discussion, namely that, conceptually, entailment is a relation between sentences and so a concept of metalogic. He claims that logic is concerned with rather wide generic concepts which do not specifically take either one form or another, for example the form of either a relation between sentences or of a relation between propositions, but which may equally well come in a number of different forms. As an example, we have the concept of negation-falsity: "by 'negation-falsity' we mean to refer to that generalised concept of nay-saying (we stretch for a generic term) which has not yet decided what grammatical clothes to wear." (p.481) The concept of entailment is one of these concepts: "we recognise the existence of a genuine conditional-entailment-implication topic". (p.490)

In so far as Belnap has an argument for the existence of these generic concepts, it seems to be an inference from the fact that amongst the characteristically logical constructions of English we often find constructions which differ widely in grammatical form but which are closely related in point of logical import. Speaking of the concept he calls negation-falsity, he writes: "Logicians customarily use 'negation' for a connective, and 'falsity' for a predicate, but this should not blind us to the existence of a single topic of which these two are specialisations." (p.481) He continues: "Similarly, although a certain school of logicians

systematically was 'conditional' for the connective and 'implication' for the predicate, it is no confusion to suggest that there is a single topic here." (pp.481-482) The generic "conditional-entailment-implication" concept may be carried in English in at least eleven ways, claims Belnap. (p.481) In particular it may be carried by the predicate "—— entails ——", or by the connective "(that ——) entails that ——". (ibid)

It is this latter reading that Belnap favours for the entailment connective axiomatised by the system E: "When we raise the question of the applicability to English of the logic of entailment, we intend the application to be made via the "'that —— entails that ——' connective." (p.491) However, it seems to me that there are two fundamental objections to this course.

Firstly, the proposed interpretation of $\rightarrow$ raises problems in the vicinity of propositional identity. If the formula $A \rightarrow \sim(\sim A.\sim B)$ is to be understood as saying that that A entails that $\sim(\sim A.\sim B)$, is it also to be understood as saying that that A entails that $(A \vee B)$? I think it must. For as I understand the function of the "that-" operator in English, if T is a sentence embedded in a sentence S by means of the "that-" operator, the substitution of a sentence synonymous with T for T in S results in a sentence synonymous with S. (Belnap certainly offers no alternative construal of the "that-" operator.) Thus, since $\sim(\sim A.\sim B)$ and $(A \vee B)$ are synonymous, so are, on Belnap's interpretation of $\rightarrow$, $A \rightarrow \sim(\sim A.\sim B)$ and $A \rightarrow A \vee B$. (That is, synonymous in the sense of meaning the same under their intended interpretation.) But this is an unwanted result. The former formula corresponds to the principle from A, infer $\sim(\sim A.\sim B)$, whilst the latter corresponds to the principle from A, infer $(A \vee B)$. Clearly

these two principles are distinct. But on Belnap's interpretation of →, they are treated as one and the same rule, and thus any entailment system using Belnap's entailment connective will conflate distinct principles of inference.

The second problem associated with Belnap's proposed interpretation of → arises from our lack of philosophical understanding of that-clauses themselves. These clauses are notoriously unclear, seem to easily generate paradoxes, and in general stoutly resist attempts at philosophical analysis. Yet Belnap is inviting us to build the use of that-clauses into our study of entailment at the initial stages, which is just where we want things to be clearest. I prefer to construct my formal syntax in a way which does not make my understanding of what I am doing depend on an understanding of that-clauses.

In the paper under discussion Belnap presents an argument which he might possibly use to counter both of the above objections. (He in fact constructs the argument in an attempt to show that his interpretation of → does not commit him to the existence of propositions.) The argument runs: "We do not think the instances of 'that' in 'that —— entails that ——' commit us to propositions. We think it an accident that, in English, entailment is carried by a functor requiring terms instead of sentences and thus requires prenominalisation." (p.491) "There are important grammatical constraints on the reading of functors in formal languages into English; but one should be wary of drawing exaggerated metaphysical or conceptual implications from the rules of English grammar." (p.488)

There is an element of truth in these remarks. The fact that the only connective reading of entailment in English involves the use of that-

clauses does not show that we have to understand an entailment connective in a formal language in terms of that-clauses. When reading the formal language into English we will have to use that-clauses, but we should not conclude from the poverty of English alone that the notion of a that-clause has been built into the formal language. However, if we use the connective approach we ought to have some understanding of how → is functioning in our formal language. I always dislike the sensation of the symbolism running away with me, and I used to get this feeling in the presence of formulae like $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$. The feeling became somewhat abated when I thought of → in terms of that-clauses, and perhaps it is the case that anyone who initially favours the connective approach will have to rely on that-clauses to attain the necessary understanding of his formal language. Speaking personally, I am glad I started off with the predicate approach: now my studies have passed through the initial stages I have sufficient grasp of the concept of entailment to be able to understand formulae like the above in terms of a harmless mixing of levels of language, rather than in terms of philosophically obscure that-clauses.

A further important strand of this part of Belnap's argument is that the logician is entitled to regard the occurrences of "that" in "that —— entails that ——" as playing a merely auxiliary role. Thus the connective → of the formal language may be understood in terms of the construction "that —— entails that ——" without the espousal of full blooded that-clauses and accordingly without commitment to propositions. He writes: "It is a crucial part of the argument that, on our view --- the propositional subnector 'that ——' can be waived as sometimes --- playing in English [a] low grade and philosophically uninteresting role: --- that of converting a sentence into a term without semantic increment."

(p.488; Belnap's emphasis). "We take the immediate constituents of 'that Tom runs entails that Tom moves' to be 'Tom runs', 'Tom moves' and the above exhibited connective ['that —— entails that ——'], with occurrences of 'that' playing an only grammatical role." (p.491) I am doubtful about this idea, in that I am uncertain that it would be possible to establish that "that" played a low-grade, merely grammatical role in "that —— entails that ——". (Presumably Belnap would accept that it does not always play such a role: there are, for example, vital differences between the sentences "Carl denies that Everest is Chomolongma" and "Carl denies the sentence "Everest is Chomolongma.'".) However, the important point I want to make about this strand of Belnap's argument is that it will not marry up with (what I take to be) his argument for the existence of generic concepts like the "conditional-entailment-implication" concept. For if the immediate constituents of "that Tom runs entails that Tom moves" are, as Belnap claims, the sentences "Tom runs" and "Tom moves" and the connective "that —— entails —— that", with occurrences of "that" playing a merely auxiliary role and carrying no commitment to propositions, then the availability in English of this way of expressing entailment in no way tends to show that, conceptually, entailment is not always and unequivocally a relation between sentences.

To conclude, then, I find contentious Belnap's suggestion that, conceptually, entailment is not purely and simply a relation between sentences; and I cannot find that he has furnished any appealing arguments to sustain the suggestion. Moreover, such argument as he does give is inconsistent with a further claim of his, which he requires in order to keep his account of entailment-as-a-connective clear of the philosophical obscurities of that-clauses.

3. Need The Study of Entailment Exceed The Standard Resources of Predicate Logic?

3.1 Entailment is not material implication

Russell thought, or at any rate wrote as if he thought, that the notion of implication or entailment could be defined in the propositional calculus of the Principia. He defines the proposition $A \supset B$ as the proposition $\sim A \vee B$ i.e. the proposition that either not-A or B is true (sic); asserts that $\supset$ is the relation of implication; and reads $A \supset B$ as: A implies B (1910, p.7). All this is immensely misleading, and several writers have sharply condemned Russell's obvious confusion here over use and mention, for example Quine (1961a, p.175) Scott (1971, p.789) and Gödel (1963, p.126).[3] Of course, the fact that Russell himself was confused over use and mention is no argument that material implication is not entailment: a dislike of use/mention mixes in the study of entailment has to be justified. (Just such a justification is given in the preceding section.) But doubtless Russell's original unreflecting conflation of entailment with a conditional connective has over the years been a sustaining force of the belief that there is nothing problematic about treating entailment as a connective.

Our chief reason for denying that material implication is implication is its relative weakness: $A \supset B$ is true in a state of affairs ω just in case if A is true in ω, then so is B; whereas we require $A \rightarrow B$ to be true in ω just in case B is true in any state of affairs in which A is true (vide 1.4, 3.3, and Chapter IV). This relative weakness of $\supset$ gives rise to the so called paradoxes of material implication: $A \supset (B \supset A)$; $\sim A \supset (A \supset B)$; $(A \supset B) \vee (B \supset A)$. If $\supset$ is read as "implies", each of these three sentences is intuitively false, yet each is derivable with obvious correctness from sentences which are clearly true. Hence their paradoxicality. The resolution

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3. Although Scott (ibid, p.789) has suggested that the original sin may have been Whitehead's.

is to give up the idea that material implication is any sort of implication at all: it is the truth functional if-then, nothing more.

R.K. Meyer has drawn the useful distinction between "systems which attempt to express a formalised entailment, and those systems which are content to indicate entailment" (1968, p.473). "A connective → indicates entailment for a logic L when it is held metalogically true that A entails B if and only if A → B is a theorem of L. A connective → expresses entailment if and only if A → B means that B is a logical consequence of A - e.g. that A → B shall be true on an interpretation I of L if and only if A entails B on I." (1971, p.815) Using this distinction, we can say that ⊃ does not express entailment, but does at some occurrences indicate entailment: ⊃ indicates entailment whenever it occurs as the principal operator of a tautology. However, even taking ⊃ to indicate entailment may lead to confusion, since in the same formula there may be occurrences of ⊃ which do indicate entailment and occurrences which do not. For example, this happens in both of $A \supset (B \supset A)$ and $\sim A \supset (A \supset B)$.

A plausible adequacy condition for a formal version of entailment is that it should support modus ponens. That is, a formal concept ⇒ is not a satisfactory version of entailment unless B can be inferred from $A \Rightarrow B$ and A. A currently important reason for denying that material implication is implication has been provided by relevance logicians, who maintain that modus ponens on ⊃ commits a fallacy of relevance, and hence is an incorrect form of inference. (See for example Anderson and Belnap 1975, p.165 and p.296ff.) This claim (which I believe to be mistaken) is discussed in Chapter VII below.

3.2 Entailment is not truth functionally definable

Lewis and Langford (1932, pp.237-238) provide the following argument that entailment can be identified with no binary operator definable in any truth functional calculus. Let I be such an operator. Then I is such that for any pair of true sentences A and B, either A I B holds or not-(A I B) holds (since I is a relation between truth values, and each of A and B has the truth value true). Hence, if we read A I B as "A implies B" or "B is deducible from A", we have that either every true sentence implies every true sentence, or no true sentence implies any true sentence, depending on how I is defined. If I is defined so that the latter holds, then clearly I is not a relation of implication, since it could never give rise to inference: for either true sentences would imply only those which are not true, or no true sentence would imply anything. Conversely, if I is defined so that the former holds, then every pair of true propositions is such that each can validly be deduced from the other. "As a statement about deducibility, this is quite surely an absurdity". (p. 238) "It seems to us that the falsity of the assertion 'Every true proposition is deducible from every other' hardly requires proof". (p.239) If the assertion were true, then any formal system whose theorems are true under some interpretation, would be such that every theorem of the system is deducible from any one theorem of the system. (p.238) It is to be concluded, then, that however I is defined, it does not express implication or entailment.

3.3 Does standard proof theory or model theory for predicate logic supply an entailment relation?

We use $\vdash$ in the standard proof theoretic way for the notion of

derivability within a given axiomatisation of predicate logic, and $\models$ in the standard model theoretic way for the notion of semantic consequence (see, for example, Enderton 1972, pp.83 and 102). Extentionally, of course, the two relations coincide, as proved by Gödel (1930). Both are notions of metalogic, each being a relation between sentences of the predicate calculus.

$\vdash$ bears a close resemblance to an entailment relation for predicate logic, in that $A \vdash B$ holds just in case B is deducible from A in conjunction with the axioms of a systematisation of predicate logic, using only the rules of inference of that system to effect the deduction. Provided this axiomatisation is complete (viz, all the valid formulae of predicate logic are provable in it), we will clearly want $A \rightarrow B$ to hold just in case $A \vdash B$. However, the definition of $\vdash$ considered as an <u>analysis</u> of entailment is simply unilluminating. $\vdash$ is defined in terms of derivability within an axiomatisation of predicate logic, whereas an analysis of the concept of entailment should be able to explain <u>why</u> the formal derivations in a particular system correspond to correct entailments.

The definition of the semantical notion $\models$, on the other hand, is not unilluminating. It runs: $A \models B$ if and only if B is true in every Tarski-interpretation of the language of predicate logic in which A is true. (An exposition of the notion of a Tarski-interpretation is given in Chapter IV, section 1.1.) This form of definition is the paradigm of semantic analysis of a concept. Moreover, the definition certainly provides the most natural semantical interpretation of the $\rightarrow$ of classical entailment theory: it captures the fact that the inference figure $\frac{A}{B}$ is classically valid just in case there is no possibility of the combination A true, B false. In a Tarski-style formal recursive semantics for the

language of the theory, we should expect to have the following clause for formulae of the form A → B: A → B is true in an interpretation (or valuation, or model, depending on the exact nature of the semantics) just in case there exists no interpretation in which A is true and B is false.

In fact, for a technical reason this clause is modified slightly in the semantics developed in this thesis. The reason concerns substitution into contexts governed by →: we shall wish to perform uniform substitutions on propositional variables in entailment sentences. (Indeed, in the following chapters we will take a rule of uniform substitution as read and allow ourselves the use of schematic letters in the usual way.) Without the availability of uniform substitution on propositional variables in entailment formulae the systematic study of entailment would be impossible: the set of valid entailment sentences could not be recursively axiomatised. Now, given uniform substitution, the following happens if the above clause for formulae of the form A → B is incorporated into a formal recursive semantics. Consider the formula p → q (where p and q are propositional variables). Clearly there will exist an interpretation in which p is true and q is false, and so p → q is false in every interpretation. Assuming the usual semantics for negation, ∼(p → q) is then true in every interpretation, and accordingly is valid (where an arbitrary formula is valid just in case it is true in every interpretation). But then by uniform substitution ∼(p → p) is valid, which is absurd.

To overcome this problem the semantics of → is modified such that A → B holds in an interpretation i just in case for every interpretation j in some <u>fixed set</u> of interpretations associated with i, A is true in j only if B is. p → q will not now fail in every interpretation, since with some

interpretation i we may associate a fixed set of interpretations in which p is true only if q is. In that case, $p \rightarrow q$ is true in i, so $\sim(p \rightarrow q)$ is false in i, and accordingly this latter formula no longer comes out as valid. (The general restrictions on the associated fixed set of interpretations need not concern us here: these fixed sets are defined recursively in Chapter IV.)

The relationship between sentences of the forms $A \rightarrow B$ and $A \models B$ is now as follows: $A \models B$ holds if and only if $A \rightarrow B$ is valid in the above semantics.

4. C.I. Lewis: The Founder of The Systematic Study of Entailment

As we have seen (section 3.2, above), Lewis felt strongly the inadequacy of material implication as the converse of the relation of deducibility. He saw clearly that no calculus based on material implication could provide a systematisation of the principles of inference of propositional logic; and he undertook to construct a calculus which did. He writes: "We have here the further purpose to develop a calculus based upon a meaning of 'implies' such that 'p implies q' will be synonymous with 'q is deducible from p'. The relation of material implication, which figures in most logistic calculuses of propositions, does not accord with this usual meaning of 'implies'. --- As we shall see, it is entirely possible so to develop the calculus of propositions that it accords with the usual meaning of 'implies'. --- We shall call the relation of implication [of the developed calculus] 'strict implication' and the system 'the calculus of Strict Implication'." (1932, pp.122-123) He claims of strict implication that it "expresses precisely that relation which holds when valid deduction is possible, and fails to hold when valid deduction is not possible. In that sense, the

system of Strict Implication may be said to provide that canon and critique of deductive inference which is the desideratum of logical investigation." (ibid, p.247)

Rather than treating strict implication as primitive, Lewis defined it in terms of a primitive notion of logical possibility. He tells us that "p implies q" is to mean that it is logically impossible that p should be true and q false; in symbols $(p \dashv q) =_{df} \sim\Diamond(p \wedge \sim q)$, where $\dashv$ is Lewis' symbol for strict implication, $\Diamond$ his symbol for logical possibility (ibid p.124). But despite this fact, the axioms of his calculus of Strict Implication S1 are all stated in terms of $\dashv$: all axioms of S1 are entailments of the form $A \dashv B$ (ibid pp.124-125). The production of this calculus was a landmark in the systematic study of entailment, for it successfully systematises the principles of inference of propositional logic, in that where A and B are formulae of the propositional calculus and $\frac{A}{B}$ is a (classically) valid inference figure, then $A \dashv B$ is a theorem of S1. (That this is so is a corrollary of Kripke's completeness theorem for modal logic (1959).) The horizontalisations of many valid inference figures whose upper and lower members are themselves entailments are not, however, provable in S1, and nor are various other entailments between iterated modalities. (The modalities are, standardly, entailment or implication, logical possibility ($\Diamond$) and logical necessity ($\Box$), which Lewis defined in terms of $\Diamond$: A is necessary just in case $\sim A$ is not possible, viz $\sim\Diamond\sim A$ (ibid, p.160).)[4] Lewis proposes a further four successively stronger systems of strict implication, S2 through S5, each representing a different way of treating iterations of $\Box$ and $\Diamond$ (ibid, Appendix II). Lewis indicates that he believes S1 to be the preferable system for formalising entailment (ibid, p.502), whereas we ourselves believe a system of S5 strength to be

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4. Lewis himself did not use a special symbol for necessity. The first published use of $\Box$ is in Barcan (1946), but the symbol itself was devised by Fitch. (See Hughes and Cresswell (1968), footnote 425).

correct (as argued in Chapter III, section 10).

Despite the immense importance of the Lewis systems in the formal study of inference, I believe there to be enough wrong with these systems from a philosophical point of view to warrant an attempt at reformulating the theory of classical entailment for propositional logic. This reformulation will largely occupy us until the end of Chapter III. The remainder of the present chapter is devoted to an account of what is wrong, philosophically, with the Lewis systems. The basic complaint is that despite his painstaking formal work Lewis does not succeed in making the notion of implication or entailment at all clear.

4.1 Lewis' confusion of mention with use

Lewis grafted his modal operators directly onto the propositional language of the Principia, giving them the same syntactical status as the truth functional connectives (1932, pp.123-124). Thus he mixed concepts of logic and metalogic in the way discussed in section 2, above. However, Lewis did not make this mix self-consciously: he was simply confused over the difference between language and metalanguage. Signs of this confusion are present in many passages in "Symbolic Logic". (A typical passage is that beginning at the bottom of page 82, where Lewis struggles with the difference between "~p" and ""p" is false".) Quine tells us that despite lengthy discussion with Lewis, he was unable to make him aware of confusing use and mention. (1961a, p.176)

Did it matter that Lewis was confused over the distinction between language and metalanguage? I think so. As suggested in section 2, above, the mixing of concepts of logic and metalogic is inimical to clear understanding, and is best avoided in the first stages of the development of the

formal study of inference. (Certainly I myself found Lewis' treatment of entailment very unclear when first introduced to it as an undergraduate. I remember expending much thought wrestling with the S1 axiom $(p \wedge (p \dashrightarrow q) \dashrightarrow q)$: isn't "p entails q" <u>about</u> p and q, so how can it be conjoined with p itself?) Again, it is unfortunate that what was historically the first attempt to systematise the principles of inference of propositional logic should, without explanation or excuse, have traded on a mixing of concepts of logic and metalogic. Quine's belief that this mixing is simply incoherent will doubtless have influenced many to under-rate the importance of Lewis' achievement, if not of the whole field of study itself. It seems to me that Quine's complaints about propositional modal logic certainly deserve an answer, and the reconstruction of classical entailment theory given in the next two chapters is seen as providing one.

However worrying Lewis' confusion of language with metalanguage may be from the point of view of clarity, it is in the end provable that the confusion was <u>formally</u> harmless. It can be shown that whenever a "confused" sentence is provable in a Lewis system, an unconfused "copy" of it is provable in a system in which implication is carried by means of a predicate of the metalanguage. This result, established in Chapter III, is considered to be one of the main achievements of this thesis, and is taken to defend propositional modal logic against Quine's criticism of it. But the very fact that Lewis' syntax can be seen as standing in need of justification is a sharp criticism of his work viewed as an attempted <u>analysis</u> of entailment.

4.2 <u>Logical possibility is an obscure notion</u>

A further inclarity is imported into Lewis' treatment of entailment

by his procedure of defining strict implication in terms of logical possibility. Perhaps Lewis chose this notion as primitive because of the wide philosophical currency it enjoyed, believing that its use would help clarify his new logical relation of strict implication. However that may be, the reverse effect is surely produced: the notion of logical possibility is philosophically obscure, and it seems a bad idea to base an account of entailment on it. Lewis himself agrees that "the words 'possible' and 'necessary' are highly ambiguous in ordinary discourse". (1932, p.160) Intending to explain his own use of the terms, he continues: "The meaning here assigned to $\Diamond p$ is the wide meaning of 'possibility' - namely, logical conceivability or the absence of self contradiction. And the resultant meanings of $\sim\Diamond p$ and $\sim\Diamond\sim p$ are the correspondingly narrow meanings of 'impossibility' and 'necessity': $\sim\Diamond p$ means "p is logically inconceivable"; and $\sim\Diamond\sim p$ means "It is not logically conceivable that p should be false"." (ibid, pp.160-161) But has anything really been made clearer? The obscure notion of possibility has been traded in for two more equally obscure notions - logical conceivability, and the absence of self contradiction. Again, in interpretation of the important theorem 18.7 $(p \strictif q .=. \sim\Diamond\sim(p \supset q))$ Lewis writes: "A strict implication is equivalent to a material implication which holds of necessity, or by purely logical analysis." (ibid, p.165) Despite the appealing ring to the phrase "purely logical analysis", this remark is hardly informative in the absence of an account of what purely logical analysis consists in.

There is nothing to be said for Lewis' procedure of defining implication in terms of logical possibility, and it is avoided in the present reconstruction of the theory of classical entailment. In this chapter, $\rightarrow$ has been introduced as a primitive device for horizontalising

inference figures, and its significance has been explained in terms of the usual (classical) accounts of argument validity (section 1.4 above). Already things are looking clearer than Lewis made them. In Chapter III Lewis' procedure is reversed, his $\Diamond$ and $\Box$ being defined in terms of $\rightarrow$ in a well motivated and illuminating way. The makings of a clear reconstruction of the obsure notions of logical possibility and logical necessity is thereby provided. In Chapters IV and V the business of clarification is completed with the provision of a formal semantics for classical entailment.

(Of course, Lewis' own work looks a lot clearer in the light of Kripke's possible world semantics for modal logic, which provides clear interpretations of the notions of necessity and possibility as they appear in the stronger Lewis systems. Perhaps we can say that the availability of the Kripke semantics makes tolerable the obscurity of Lewis' account discussed in this subsection; but of course it would be preferable for the obscurity not to be present at all.)

CHAPTER II

Formal Systems for Classical Entailment

In this chapter we present and discuss a hierarchy of entailment systems in which entailment is treated as a predicate of the metalanguage.

1. The Hierarchical Approach to Entailment.

1.1 The hierarchy of languages.

Since → is to function as a semantical predicate, the result of writing it between formulae of the predicate calculus will be a formula of the metalanguage, the result of writing it between formulae of the metalanguage, a formula of the meta metalanguage and so on. Thus for the formal study of entailment we require not a single formal language, as in Lewis' case, but rather a hierarchy of formal languages, each of which is a metalanguage for its immediate predecessor, the first member of the hierarchy being a formal language for standard predicate logic. We proceed to give a recursive definition of a suitable language hierarchy.

(α) The language L_o.

L_o consists of individual, function and predicate variables, individual constants, and the following logical symbols: $\sim$, $\wedge$, $\vee$, $\supset$, $\forall$, $\exists$ (plus auxiliary parentheses). The set of terms of L_o is the least set containing all individual variables and constants and such that if F is any n-place function variable and $t_1, \ldots, t_n$ are members of the set, then, $F(t_1, \ldots, t_n)$ is also a member. If P is any n-place predicate variable and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is called an atomic formula of L_o. The set of well formed formulae of L_o, WFF_o, is the least set containing all atomic formulae of L_o and such that if A, B are members of the WFF_o, then so are $(\sim A)$, $(A \vee B)$, $(A \wedge B)$, $(A \supset B)$, $(\forall y\ A)$ and $(\exists y\ A)$ (where y is any individual variable).[1]

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1. We adopt the convention of always omitting quasi-quotes in presenting schemata.

(β) The language L_n.

L_n consists of the whole of L_{n-1}, plus metalinguistic counterparts to each logical symbol of L_n, plus the logical symbol $\rightarrow$ in case L_{n-1} is L_o, plus metalinguistic commas and parentheses. The set of well formed formulae of L_n, WFF_n, is the least set such that

(i) where $\{A_1, \ldots, A_j\}$, $\{B_1, \ldots, B_k\}$ are any subsets of WFF_{n-1}, $(A_1, \ldots, A_j \rightarrow B_1, \ldots, B_k) \in WFF_n$

(ii) if A and B are in WFF_n, then so are $(\sim A)$, $(A \vee B)$, $(A \wedge B)$, $(A \supset B)$, $(\forall y\ A)$, $(\exists y\ A)$ (where y is any individual variable of L_o).

Outmost parentheses will usually be omitted, and inner parentheses often will. In an unbracketed formula, $\rightarrow$ is always assigned maximum scope, and the other logical symbols are ranked in scope in the standard way.

The set of all well formed formulae, WFF, is $\bigcup_{\infty}^{i = o} WFF_i$. A member of WFF is called a proper entailment if its principal logical symbol is $\rightarrow$. WFE is written for the set of proper entailments.

A member of WFF will be said to be of degree n if it is a member of WFF_n. Thus the degree of a formula is a measure of the nestedness of the formula with respect to $\rightarrow$.

We have chosen a syntax such that the metalinguistic formula $A \rightarrow B$ uses the formulae A and B as names of themselves. It would be possible to construct the hierarchy of metalanguages such that a formula of L_{n+1} is obtained by writing $\rightarrow$ between non autonymous names of formulae of L_n, say quotation mark names or perhaps set theoretic structural designators. However, this would merely involve unnecessary prolixity. The metalinguistic character of $A \rightarrow B$ in the present setup arises simply from the fact that $A \rightarrow B$ is a formula of L_{n+1} just in case both A and B are formulae of L_n. (This practice is used by

Scott (1971).)

Formulae like $(A \rightarrow \sim A) \rightarrow \sim A$ and $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ (both of which are axioms of Anderson and Belnap's system E) are not in WFF. All formulae in WFE are balanced, in that the principle arrow is always flanked by formulae of equal degree. This is essential in a metalinguistic approach to entailment : given that $\rightarrow$ is a metalinguistic predicate, it is incoherent to write it between formulae of different languages (i.e. of different degree).

Clause $\beta(i)$ allows entailment sentences with multiple consequents as well formed, whereas we have previously considered only entailment sentences with a single consequent. This generalisation of the form of entailment sentences facilitates the systematisation of principles of inference. The meaning of an entailment sentence of this extended form is : whenever all the antecedent formulae are true, then at least one of the consequent formulae is true. In terms of inference figures, the extension reflects the point that a valid inference figure $\frac{A_1, \ldots, A_n}{B}$ may be <u>weakened</u> without loss of validity to a figure of the following form: $\frac{A_1, \ldots, A_n}{B, B_1, \ldots, B_n}$ where the lower portion of the figure is understood disjunctively, just as the upper portion is understood conjunctively.

Since either or both of the sets exhibited in clause $\beta(i)$ may be null, formulae of the following three forms are well formed : $\Gamma \rightarrow$, $\rightarrow \Gamma$, and $\rightarrow$.[2] Bearing in mind the intended interpretation of

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2. Henceforward, capital greek letters will be used to represent any sequence of formulae of a given language (possibly the unit sequence or null sequence), whilst capital roman letters, with or without subscripts, will be used to represent single formulae.

entailment sentences, $\Gamma \rightarrow$ asserts that whenever all of Γ are true, then at least one member of the empty sequence of formulae is true, i.e. Γ is inconsistent. Alternatively, $\Gamma \rightarrow$ may be regarded as saying that any formula (of the appropriate degree) is implied by Γ, for if $\Gamma \rightarrow$ is true on the intended interpretation, then $\Gamma \rightarrow A$ is also true (for arbitrary A such that $\Gamma \rightarrow A$ is well formed). $\rightarrow \Gamma$ asserts that whenever a member of the empty sequence is true, at least one of Γ is true. So since, vacuously, every member of the empty sequence is true, $\rightarrow \Gamma$ asserts that at least one member of Γ is always true, or, in other words, Γ is a logically valid sequence. Again, similarly to $\Gamma \rightarrow$, the context may also be regarded as saying that any formula (of the appropriate degree) implies Γ. By analogous considerations, $\rightarrow$ is the absurd sentence of entailment theory : it indicates that falsehood is implied by truth. (Notice that $\rightarrow$ is a well formed formula of every $L_{n > o}$. Thus $\rightarrow$ is of arbitrary degree, and so it is not possible to assign a degree to formulae built up out of logical symbols alone (for example $((\rightarrow) \wedge (\rightarrow) \rightarrow (\rightarrow))$. However, any well formed formula containing at least one atomic formula of L_o will be of a definite degree.)

Finally we note that clause β(ii) permits the binding of individual variables of L_o by metalinguistic quantifiers. For example, where P(y) is an atomic formula of L_o, the following is a formula of L_1 : $\forall y\ (\forall y\ P(y) \rightarrow P(y))$. That we allow such formulae as well formed may occasion some surprise, since one of Quine's main complaints about quantified modal logic is that quantification through a modal operator is a dubious business, and, moreover, something which would never occur in a system keeping use and mention distinct by treating "implies" as a metalinguistic predicate (1953, esp. p.170). Formulae of this sort receive a thorough discussion in Chapters IV and V, below, and Quine's objections to them are discussed in detail in Chapter IV, section 3.1.

It is one of the main contentions of this thesis that contra Quine, quantified modal logic is semantically well justified if the modalities are treated as metalinguistic predicates.

1.2 The hierarchy of entailment systems

Lewis' cumulative syntax allows him to derive all valid entailments within a single system. We, on the other hand, have to construct a hierarchy of entailment systems, one system for each language in the hierarchy. To allow the derivability of theorems having a different degree to the axioms of a system is to let in by the back door - as inference rules of the system - precisely those unbalanced entailments that we are concerned to avoid. Accordingly, to axiomatise all valid entailments, a separate set of axioms for the valid entailments of each degree has to be specified. The axioms of the first system in the hierarchy will be formulae of degree 1, and the system will axiomatise entailments whose antecedents and consequents are formulae of the predicate calculus; the axioms of the second system will be of degree 2, the system axiomatising entailments between first degree entailments, and so on. The inference rules of the system associated with L_n will be expressible as formulae of degree n + 1.

Entailment systems in the hierarchy will be called ramified entailment systems, in distinction to the cumulative entailment systems proposed by Lewis and Anderson-Belnap.

2. An Entailment System for Predicate Logic

By "an entailment system for predicate logic" is meant an axiomatisation of all and only those entailments horizontalising valid inference figures whose upper and lower members are formulae of the predicate calculus (i.e. of L_o). Such an entailment system will

constitute the first member of the hierarchy of systems just discussed.

When I first came to think about the construction of an entailment system for predicate logic, I rapidly realised that the work had already been carried out, although in a different context, by Gerhard Genzen. His system LK (logischer klassischer Kalkül) is such that $\Gamma \rightarrow \Delta$ is provable just in case $\frac{\Gamma}{\Delta}$ is a truth preserving inference figure, Γ and Δ representing sequences of formulae of the predicate calculus. (This claim is substantiated by the work done on semantics and completeness in Chapter IV, below.)

Genzen called a formula of the form $\Gamma \rightarrow \Delta$ a <u>sequent</u>, and LK is often called the classical sequent calculus. The formation rules for sequents are as for the proper entailments of L_1 (1.1, above, and Genzen 1935 pp.70-71). The desire to use LK as the first degree entailment system has of course conditioned our choice of syntax, although a further motive in our choice of a Genzen-style syntax was its extreme perspicuity for the study of entailment (as, hopefully, became apparent in section 1.3 of Chapter I). Our syntax is an extension of that presented by Genzen in (1935), in that he considers neither iterations of the symbol $\rightarrow$, nor truth functional compounds, or quantifications, of sequents.

Genzen's purpose in constructing LK was to produce a formalisation of predicate logic which reflected the structure of reasoning to be found in actual mathematical arguments, something not done by the so-called logistic formulations of predicate logic constructed by Russell and Hilbert-Ackermann, for example. It seems unlikely that Genzen himself thought of LK as an entailment system. Without a doubt the idea that a sequent horizontalises an inference figure is present in Genzen's explanation of his syntax (ibid., p.82). Nonetheless, there are two reasons for thinking that Genzen did not regard his $\rightarrow$ symbol

as expressing entailment. Firstly, he does not regard → either as a connective, or as a relation sign of the metalanguage : "The →, like commas, is an auxiliary symbol and not a logical symbol" (ibid., p.71). But entailment certainly is a logical concept, so presumably Genzen did not consider himself to be dealing with it. Secondly, in explaining the concept of a sequent, Genzen writes that the informal meaning of the sequent $A_1, \ldots, A_n \rightarrow B$ is no different to that of the formula $A_1 \wedge \ldots \wedge A_n \supset B$: "the expressions differ merely in their formal structure" (ibid., p.82). But as we have seen in Chapter I, sections 3.1 and 3.2, $\supset$ is too weak to express entailment. This would, again, tend to indicate that Genzen was not thinking of → as an entailment operator.

Nonetheless, whether or not Genzen himself thought of LK as an entailment system, the fact remains that LK is an ideal candidate for the first member of the hierarchy of systems described in 1.2, above.

3. An Exposition of Genzen's System LK.

Genzen's own exposition of LK may be found in (1935), pages 83-84.

3.1 Conventions for schematic letters.

Capital greek letters represent sequences of formulae of L_o, and capital roman letters represent single formulae of L_o. Lower case letters from the end of the roman alphabet represent individual variables of L_o, lower case letters from the beginning of the roman alphabet represent arbitrary terms of L_o. The schemata A(a), B(a), etc., represent formulae of L_o, the notation indicating that the term represented by a has any number of occurrences, possibly none, in the formula represented by A(a). Where A(a) and A(b) occur in the same formula or general context, it is to be understood that A(b)

is A(a) with occurrences of a (if any) replaced by occurrences of b. Where a is a variable all occurrences of it are replaced, but where a is an individual constant or complex term, some occurrences may be left unchanged. (This last convention is of importance in the statement of the quantificational rules of LK, below.)

A free variable which becomes bound by a quantifier is called by Genzen an eigenvariable (ibid., p.77). The variable x in the formulae $\Gamma \rightarrow \Delta$, A(x) and A(x), $\Gamma \rightarrow \Delta$ will be said to satisfy the eigenvariable condition just in case x does not occur free in Γ or Δ. Capitalisation of the variable x in these contexts will indicate that it satisfies the condition. (Genzen himself did not use this capitalisation convention.)

When a formula A is prefixed by a universal (existential) quantifier to form the formula $\forall y$ A ($\exists y$ A), y will be called the replacement variable. The replacement variable y will be said to satisfy the replacement condition just in case y does not occur bound in A. Capitalisation of a replacement variable indicates that the replacement condition is satisfied.

3.2 Axioms of LK.

All formulae of the form $A \rightarrow A$ are axioms of LK. (Genzen himself called formulae of this form initial sequents (ibid., p.83).)

3.3 The inference rules of LK.

(i) Structural rules[3]

Thinning: $\dfrac{\Gamma \to \Delta}{A, \Gamma \to \Delta}$ (Th L) $\qquad \dfrac{\Gamma \to \Delta}{\Gamma \to \Delta, A}$ (Th R)

Contraction: $\dfrac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}$ (Con L) $\qquad \dfrac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}$ (Con R)

Exchange: $\dfrac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta}$ (Ex L) $\qquad \dfrac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda}$ (Ex R)

Cut: $\dfrac{(\Gamma \to \Delta, A) \ (A, \Pi \to \Lambda)}{\Gamma, \Pi \to \Delta, \Lambda}$

(ii) Propositional rules

Negation: $\dfrac{\Gamma \to \Delta, A}{\sim A, \Gamma \to \Delta}$ ($\sim$ L) $\qquad \dfrac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \sim A}$ ($\sim$ R)

Conjunction, left: $\dfrac{A, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} \qquad \dfrac{B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}$ ($\wedge$ L)

Conjunction, right: $\dfrac{(\Gamma \to \Delta, A) \ (\Gamma \to \Delta, B)}{\Gamma \to \Delta, A \wedge B}$ ($\wedge$ R)

Disjunction, left: $\dfrac{(A, \Gamma \to \Delta) \ (B, \Gamma \to \Delta)}{A \vee B, \Gamma \to \Delta}$ ($\vee$ L)

Disjunction, right: $\dfrac{\Gamma \to \Delta, A}{\Gamma \to \Delta, A \vee B} \qquad \dfrac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \vee B}$ ($\vee$ R)

Material implication, left: $\dfrac{(\Gamma \to \Delta, A) \ (B, \Pi \to \Lambda)}{A \supset B, \Gamma, \Pi \to \Delta, \Lambda}$ ($\supset$ L)

Material implication, right: $\dfrac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}$ ($\supset$ R)

(iii) Quantificational rules

Universal quantifier: $\dfrac{A(a), \Gamma \to \Delta}{\forall Y \ A(Y), \Gamma \to \Delta}$ ($\forall$ L) $\qquad \dfrac{\Gamma \to \Delta, A(X)}{\Gamma \to \Delta, \forall Y \ A(Y)}$ ($\forall$ R)

Existential quantifier: $\dfrac{A(X), \Gamma \to \Delta}{\exists Y \ A(Y), \Gamma \to \Delta}$ ($\exists$ L) $\qquad \dfrac{\Gamma \to \Delta, A(a)}{\Gamma \to \Delta, \exists Y \ A(Y)}$ ($\exists$ R)

The stipulation that the eigenvariable in the upper sequent of ($\forall$R) and ($\exists$L) should satisfy the eigenvariable condition excludes obviously

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3. L and R in the abbreviated names for rules stand for 'left' and 'right'.

invalid inferences like the following: $\frac{A(x) \to A(x)}{A(x) \to \forall x\, A(x)}$ and $\frac{A(x) \to A(x)}{\exists x\, A(x) \to A(x)}$

3.4 Remarks on LK.

The number of non structural inference rules could be limited to three by defining $\wedge$, $\supset$, and $\forall$, say, in terms of $\vee$, $\sim$ and $\exists$ in the standard way. This procedure is avoided, however, the emphasis being placed on ease of proof construction rather than on formal parsimony.

From the motive of historical accuracy we have followed Genzen's procedure of stating the rules of LK by means of inference figures. In future, however, the rules of LK will be considered to be expressed by means of entailment sentences (which are easier to handle typographically than inference figures). Thus, for example, the cut rule is stated by the entailment sentence horizontalising the above cut figure: $(\Gamma \to \Delta, A), (A, \Pi \to \Lambda) \to (\Gamma, \Pi \to \Delta, \Lambda)$. The horizontalisations of the above inference figures will be called the meta axioms of LK; and to avoid confusion, axioms in the ordinary sense will be referred to as proper axioms. (Horizontalisations of rules of inference of deductive systems require a different semantical treatment to horizontalisations of principles of inference (in the sense of Chapter I, section 1.3). This matter is discussed in Chapter IV, section 2.).

Genzen intended that proofs in LK should have what he termed "tree form" (ibid., p.73). The statement proved in a proof tree (the endsequent, in Genzen's terminology) stands at the base of the "trunk". At the tip of each "branch" of the tree stands a proper axiom (or initial sequent). With the exception of the endsequent, each sequent in the proof is an upper sequent of exactly one inference figure; and, of course, the only inference figures which may legitimately occur in an LK proof tree are those expressing rules of LK. Distinct branches of a proof tree are merged at inference figures having two upper sequents. For

example, successive applications in a proof tree of Cut,

($\wedge$ R) and ($\sim$ L) may be represented like this:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \to \Delta,\ A} \quad \dfrac{\vdots}{A,\ \Pi \to B}}{\Gamma,\ \Pi \to \Delta,\ B} \qquad \dfrac{\vdots}{\Gamma,\ \Pi \to \Delta,\ C}}{\dfrac{\Gamma,\ \Pi \to \Delta,\ B \wedge C}{\sim(B \wedge C),\ \Gamma,\ \Pi \to \Delta}}$$

Dotted lines represent proof trees of the sequents ($\Gamma \to \Delta$, A), (A, $\Pi \to$ B) and (Γ, $\Pi \to \Delta$, C), respectively.

LK proofs in tree form copy in a perspicuous way the structure of the reasoning in actual mathematical arguments, and that they do so has been one of the main attractions of Genzen's work for those concerned with proof theory. However, the present concern is merely to use LK as an axiomatisation of valid first degree entailments, and accordingly we abandon Genzen's tree form for proofs in favour of the more tractable tabular form. A proof will be a tabular arrangement of sequents, each member of which is either a proper axiom, or follows from one or more preceding members in accordance with a rule of inference of the system. (Perhaps it is worth remarking that this obliteration of the tree form of LK proofs does not effect the provability of Genzen's Haupsatz 1935 , p.85) or its many important applications in proof theory (Takeuti (1975) provides an excellent survey of these.))

Finally we note that if no sequents having more than one formula in the consequent are permitted, LK provides a formalisation of intuitionistic predicate logic as formulated by Heyting (1966), in the sense that given this restriction $\to$ A is LK provable just in case A is provable in the Heyting system (Genzen 1935, pp.82, 117, 120). Thus LK may also be used to provide a formalisation of the intuitionistically valid argument forms of predicate logic. (Genzen used the name LJ for this variation of LK.)

4. Truth Functions and Quantifications of Proper Entailments.

An entailment system will be called a proper system if all its theorems are proper entailments i.e. formulae of the form $\Gamma \rightarrow \Delta$. Analogously, a system is improper if truth functional compounds, or quantifications, of proper entailments are provable in it. LK is a proper entailment system, whilst Lewis' S1 is improper. A proper entailment system suffices to provide a systematisation of principles of inference: all principles of inference can be horizontalised as proper entailment sentences, while truth functions and quantifications of proper entailment sentences do not correspond in any direct way to principles of inference (hence our choice of the word "proper"). However, many non-proper entailment sentences will, of course, be valid (for example $(A \rightarrow A) \vee \sim(A \rightarrow A)$), and so a proper entailment system will be inadequate if our interest is to axiomatise all the valid formulae of a given degree. Moreover, the use of improper systems is essential if a hierarchy of ramified entailment systems is to be used in a reconstruction of modal logic in which the modalities appear as metalinguistic predicates (as will be seen in the following Chapter). Accordingly we indicate how to convert LK into an axiomatisation of all valid first degree formulae.

4.1 Genzen's system LHK.

Genzen's logistic formulation of classical predicate logic LHK will be utilised. LHK is modelled on the logistic calculus presented in Hilbert-Ackermann (1938). Genzen's exposition of LHK may be found in (1935), pages 116-117.

The axioms of LHK are all formulae of L_o of the following forms:

1. $A \supset A$

2. $A \supset (B \supset A)$

3. $(A \supset (A \supset B)) \supset (A \supset B)$

4. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

5. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

6. $(A \wedge B) \supset A$

7. $(A \wedge B) \supset B$

8. $(A \supset B) \supset ((A \supset C) \supset (A \supset (B \wedge C)))$

9. $A \supset (A \vee B)$

10. $B \supset (A \vee B)$

11. $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

12. $(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$

13. $(\sim A) \supset (A \supset B)$

14. $A \vee \sim A$

15. $\forall x\, A(x) \supset A(a)$

16. $A(a) \supset \exists x\, A(x)$

The rules of inference of LHK are as follows:

(LHK1) $A, A \supset B \rightarrow B$

(LHK2) $A \supset B(X) \rightarrow A \supset \forall Y\, B(Y)$

(LHK3) $A(X) \supset B \rightarrow \exists Y\, A(Y) \supset B$

(In analogy to our previous practice, capitalization of the eigenvariable in $A(x) \supset B$ ($A \supset B(x)$) indicates that x does not occur free in B(A).)

Our reason for preferring LHK to a simpler logistic system of predicate logic is that since $\supset$, $\wedge$, $\vee$, $\sim$, $\forall$ and $\exists$ are all taken as primitive in each language of the hierarchy defined in 1.1, above, we require a logistic system containing axioms for each of these connectives.

4.2 The System EK.

LHK was intended by Genzen as an axiomatisation of the valid formulae of the predicate calculus, and accordingly he intended the schematic letters used in specifying the axioms and rules of the system to represent formulae of the language we call L_o. However, we intend to use LHK as an axiomatisation of those valid first degree formulae whose validity depends only on the logical properties of the truth functional connectives and the quantifiers (and not on the logical properties of →). Accordingly we intend the schematic letters used in the above axiom-and rule-schemata to represent formulae of L_1. Given this understanding, LK is extended to an axiomatisation of all valid first degree formulae as follows: the sixteen axioms of LHK are added to the proper axioms of LK, and the three inference rules of LHK are added to the rules of LK. Notice that it remains the case that all proper axioms of the system are of degree one, and all meta axioms, of degree two. The resultant system will be called EK. ("EK" stands for "classical entailment": it is intended to parallel "LK".)

5. Recursive Definition of the Hierarchy of Entailment Systems.

The proper entailment system for formulae of degree n will be called LK_n. LK_1 is LK. LK_{n+1} is defined in terms of LK_n as follows:

(i) Proper axioms of LK_{n+1}. These fall into three groups. Firstly, initial axioms: for every formula A of L_n, $A \rightarrow A$ is an axiom of LK_{n+1}. Secondly, reiteration axioms: if A is an axiom of LK_n, then $\rightarrow A$ is an axiom of LK_{n+1}. Thirdly, the axioms for → : every meta axiom of LK_n is a proper axiom of LK_{n+1}.

(ii) Inference rules of LK_{n+1}. These are exactly the same as for LK_n except that schematic letters must of course be understood as representing formulae of degree n. (In the case of LK_n they represent formulae of degree n - 1.)

The hierarchy of improper entailment systems, EK_1, EK_2, ... is defined quite analogously. EK_1 is EK, and EK_{n+1} is defined in terms of EK_n in exactly the same way that LK_{n+1} is defined in terms of LK_n.

The hierarchy of systems LK_n is seen as extending Genzen's work concerning first degree sequents to sequents of arbitrarily high degree; and the hierarchy of systems EK_n is seen as extending Genzen's work to truth functions and quantifications of sequents of arbitrarily high degree.

A formula will be said to be LK_n (EK_n) provable just in case the formula is provable in LK_n (EK_n).

6. Some Theorems.

The purpose of this section is twofold: firstly to illustrate our entailment systems by writing out some proofs, and secondly to establish some theorems which will be needed in the following Chapter.

Theorems are presented schematically, and the least n for which a theorem of that form is LK_n provable is written in brackets immediately after the display of the theorem. The proof following is then schematic for an $LK_{m \geq n}$ proof of a formula of L_m of the given form. Notice that any LK_n proof is also an EK_n proof, since EK_n extends LK_n.

6.1 $\sim\sim A \to A$ (1)

1.	$A \to A$	initial axiom
2.	$\to A, \sim A$	1, ($\sim$R)
3.	$\sim\sim A \to A$	2, ($\sim$L)

6.2 $\to (\sim\sim A \to A)$ (2)

1.	$(A \to A) \to (\to A, \sim A)$	proper axiom ($\sim$R)
2.	$\to (A \to A)$	reiteration proper axiom
3.	$\to (\to A, \sim A)$	2, 1, Cut
4.	$(\to A, \sim A) \to (\sim\sim A \to A)$	proper axiom ($\sim$L)
5.	$\to (\sim\sim A \to A)$	4, 3, Cut

6.3 $A \to \sim\sim A$ (1)

1.	$A \to A$	initial axiom
2.	$\sim A, A \to$	1, ($\sim$L)
3.	$A \to \sim\sim A$	2, ($\sim$R)

6.4 $\to (A \to \sim\sim A)$ (2)

1.	$(A \to A) \to (\sim A, A \to)$	proper axiom ($\sim$L)
2.	$\to (A \to A)$	reiteration axiom
3.	$\to (\sim A, A \to)$	2, 1, Cut
4.	$(\sim A, A \to) \to (A \to \sim\sim A)$	proper axiom ($\sim$ R)
5.	$\to (A \to \sim\sim A)$	4, 3, Cut

6.5 $(A \to \sim B) \to (B \to \sim A)$ (2)

1.	$(A \to \sim B) \to (\sim\sim B, A \to)$	proper axiom ($\sim$L)
2.	$(\sim\sim B, A \to) \to (A, \sim\sim B \to)$	proper axiom (Ex L)
3.	$(A \to \sim B) \to (A, \sim\sim B \to)$	2, 1, Cut
4.	$(A, \sim\sim B \to) \to (\sim\sim B \to \sim A)$	proper axiom ($\sim$R)
5.	$(A \to \sim B) \to (\sim\sim B \to \sim A)$	4, 3, Cut
6.	$(\sim\sim B \to \sim A), (B \to \sim\sim B) \to (B \to \sim A)$	proper axiom Cut
7.	$\to (B \to \sim\sim B)$	6.4
8.	$(\sim\sim B \to \sim A) \to (B \to \sim A)$	7, 6, (Ex L), Cut
9.	$(A \to \sim B) \to (B \to \sim A)$	8, 5, Cut.

6.6 $(\sim B \to \sim A) \to (A \to B)$ (2)

1.	$(A \to \sim\sim B),\ (\sim\sim B \to B) \to (A \to B)$	Cut proper axiom
2.	$\to (\sim\sim B \to B)$	6.2
3.	$(A \to \sim\sim B) \to (A \to B)$	2, 1 Cut
4.	$(\sim B \to \sim A) \to (A \to \sim\sim B)$	6.3
5.	$(\sim B \to \sim A) \to (A \to B)$	4, 3, Cut

This is the principle of contraposition.

6.7 $(\sim A, \Gamma \to) \to (\Gamma \to A)$ (2)

1.	$(\sim A, \Gamma \to) \to (\Gamma \to \sim\sim A)$	proper axiom ($\sim$R)
2.	$(\Gamma \to \sim\sim A),\ (\sim\sim A \to A) \to (\Gamma \to A)$	Cut proper axiom
3.	$\to (\sim\sim A \to A)$	6.2
4.	$(\Gamma \to \sim\sim A) \to (\Gamma \to A)$	3, 2, (Ex L), Cut
5.	$(\sim A, \Gamma \to) \to (\Gamma \to A)$	4, 1, Cut

This is the principle of reductio ad absurdum.

6.8 $(A, \Gamma \to B) \to (\Gamma \to A \supset B)$ (2)

This is a proper axiom ($\supset$R). We draw special attention to it here because it expresses the important principle of conditional proof. Notice that 6.8 is not a deduction theorem: it is not a metatheoretic result about a particular system. Our concern is to axiomatise principles of inference (in the sense of Chapter 1, section 1.3) rather than to axiomatise the permissible proof procedures associated with some particular logistic system. Theorem 6.8 asserts the derivability of the inference figure $\frac{\Gamma}{A \supset B}$ from the inference figure $\frac{A, \Gamma}{B}$: the theorem establishes that if the latter is a correct figure, then so is the former. 6.8 formalises the reasoning used by anyone who, having established that Γ, A entail B, concludes that Γ entail $A \supset B$.

6.9 $A \supset B, A \rightarrow B$ (1)

1.	$A \rightarrow A$	initial axiom
2.	$B \rightarrow B$	initial axiom
3.	$A \supset B, A \rightarrow B$	2, 1, ($\supset$L)

This is the principle of modus ponens for material implication. Again, it is not a metatheoretic result about some logistic system of propositional logic. Rather, it formalises the fact that anyone who argues in accordance with the pattern $\frac{A \supset B, A}{B}$ is arguing validly.

6.10 $\sim\forall x\, A(x) \rightarrow \exists x \sim A(x)$ (1)

1.	$A(x) \rightarrow A(x)$	initial axiom
2.	$\rightarrow A(x), \sim A(x)$	1, ($\sim$R)
3.	$\rightarrow A(x), \exists x \sim A(x)$	2, ($\exists$R)
4.	$\rightarrow \exists x \sim A(x), A(x)$	3, (Ex R)
5.	$\rightarrow \exists x \sim A(x), \forall x\, A(x)$	4, ($\forall$R)
6.	$\sim\forall x\, A(x) \rightarrow \exists x \sim A(x)$	5, ($\sim$L)

(Notice that x satisfies the eigenvariable condition in line 4.)

6.11 $\sim\exists x\, A(x) \rightarrow \forall x \sim A(x)$ (1)

1.	$A(x) \rightarrow A(x)$	initial axiom
2.	$\rightarrow A(x), \sim A(x)$	1, ($\sim$R)
3.	$\rightarrow \sim A(x), A(x)$	2, (Ex R)
4.	$\rightarrow \sim A(x), \exists x\, A(x)$	3, ($\exists$R)
5.	$\rightarrow \exists x\, A(x), \forall x \sim A(x)$	4, (Ex R), ($\forall$R)
6.	$\sim\exists x\, A(x) \rightarrow \forall x \sim A(x)$	5, (Ex R), ($\sim$L)

Theorems 6.10 and 6.11 are two of the principles of quantifier negation.

6.12 $A \wedge \sim A \rightarrow B$ (1)

1.	$A \rightarrow A$	initial axiom
2.	$\sim A, A \rightarrow$	1, ($\sim$L)
3.	$A \wedge \sim A, A \rightarrow$	2, ($\wedge$L)
4.	$A, A \wedge \sim A \rightarrow$	3, (Ex L)

	5.	$A \wedge \sim A, A \wedge \sim A \rightarrow$	4, ($\wedge$L)
	6.	$A \wedge \sim A \rightarrow$	5, (Con L)
	7.	$A \wedge \sim A \rightarrow B$	6, Th R
6.13	$A \rightarrow B \vee \sim B$		(1)
	1.	$B \rightarrow B$	initial axiom
	2.	$\rightarrow B, \sim B$	1, ($\sim$R)
	3.	$\rightarrow B, B \vee \sim B$	2, ($\vee$R)
	4.	$\rightarrow B \vee \sim B, B$	3, (Ex R)
	5.	$\rightarrow B \vee \sim B, B \vee \sim B$	4, ($\vee$R)
	6.	$\rightarrow B \vee \sim B$	5, (Con R)
	7.	$A \rightarrow B \vee \sim B$	6, (Th L)

Theorems 6.12 and 6.13 are two of the paradoxes of classical entailment, and are widely held to embody a fallacy of relevance. The paradoxes are discussed in Chapter VI, below.

6.14	$\rightarrow (\rightarrow \sim A, A)$		(2)
	1.	$(A \rightarrow A) \rightarrow (\rightarrow A, \sim A)$	proper axiom ($\sim$R)
	2.	$(\rightarrow A, \sim A) \rightarrow (\rightarrow \sim A, A)$	proper axiom (Ex R)
	3.	$(A \rightarrow A) \rightarrow (\rightarrow \sim A, A)$	2, 1, Cut
	4.	$\rightarrow (A \rightarrow A)$	reiteration axiom
	5.	$\rightarrow (\rightarrow \sim A, A)$	4, 3, Cut
6.15	$\rightarrow (B \rightarrow \sim A, B)$		(2)
	1.	$(B \rightarrow B) \rightarrow (B \rightarrow B, \sim A)$	proper axiom (Th R)
	2.	$(B \rightarrow B, \sim A) \rightarrow (B \rightarrow \sim A, B)$	proper axiom (Ex R)
	3.	$(B \rightarrow B) \rightarrow (B \rightarrow \sim A, B)$	2, 1, Cut
	4.	$\rightarrow (B \rightarrow B)$	reiteration axiom
	5.	$\rightarrow (B \rightarrow \sim A, B)$	4, 3, Cut.

6.16 $\rightarrow (A \supset B \rightarrow \sim A, B)$ (2)

1. $(B \rightarrow \sim A, B), (\rightarrow \sim A, A) \rightarrow (A \supset B \rightarrow \sim A, B)$ proper axiom ($\supset$L)
2. $\rightarrow (B \rightarrow \sim A, B)$ 6.15
3. $(\rightarrow \sim A, A) \rightarrow (A \supset B \rightarrow \sim A, B)$ 2, 1, Cut
4. $\rightarrow (\rightarrow \sim A, A)$ 6.14
5. $\rightarrow (A \supset B \rightarrow \sim A, B)$ 4, 3, Cut

6.17 $(\rightarrow A \supset B) \rightarrow (A \rightarrow B)$ (2)

1. $(A \supset B \rightarrow \sim A, B), (\rightarrow A \supset B) \rightarrow (\rightarrow \sim A, B)$ Cut proper axiom
2. $\rightarrow (A \supset B \rightarrow \sim A, B)$ 6.16
3. $(\rightarrow A \supset B) \rightarrow (\rightarrow \sim A, B)$ 2, 1, Cut
4. $(\rightarrow \sim A, B) \rightarrow (\rightarrow B, \sim A)$ proper axiom (Ex R)
5. $(\rightarrow A \supset B) \rightarrow (\rightarrow B, \sim A)$ 4, 3, Cut
6. $(\rightarrow B, \sim A) \rightarrow (\sim\sim A \rightarrow B)$ proper axiom ($\sim$L)
7. $(\rightarrow A \supset B) \rightarrow (\sim\sim A \rightarrow B)$ 6, 5, Cut
8. $(A \rightarrow \sim\sim A), (\sim\sim A \rightarrow B) \rightarrow (A \rightarrow B)$ Cut proper axiom
9. $\rightarrow (A \rightarrow \sim\sim A)$ 6.14
10. $(\sim\sim A \rightarrow B) \rightarrow (A \rightarrow B)$ 9, 8, Cut
11. $(\rightarrow A \supset B) \rightarrow (A \rightarrow B)$ 10, 7, Cut.

Since the converse of 6.17 is a proper axiom ($\supset$R), $A \rightarrow B$ and $\rightarrow (A \supset B)$ are equivalent. The significance of this fact emerges in the following Chapter, where modal logic is reconstructed in a ramified fashion.

7. The Interpretation of $\rightarrow$ Within the Hierarchy of Systems EK_n

In this section we establish two metatheorems concerning the hierarchy of systems EK_n which together licence the reading of $\rightarrow$ as "is provable".

7.1 If A is EK_n provable, then $\rightarrow A$ is EK_{n+1} provable.[4]

The theorem is proved by induction on the number of lines in an EK_n proof P of A. The number of lines in a proof P will be called the <u>length</u> of P, written Lg(P)

<u>Case I</u> Lg(P) = 1

In this case, A is a proper axiom of EK_n, so $\rightarrow A$ is a reiteration proper axiom of EK_{n+1}.

<u>Case II</u> Lg(P) = k + 1 and the theorem holds for the endsequent of any proof of length k or less. There are various instances to consider, according to the form of the final inference of P. Just two cases are exhibited here.

(i) A is obtained by ($\wedge$R). In this case P may be represented as follows:

(A_1) $\Gamma \rightarrow \Delta, B$

(A_2) $\Gamma \rightarrow \Delta, C$

(A) $\Gamma \rightarrow \Delta, B \wedge C$

Clearly there are EK_n proofs of A_1 and A_2 of length less than k, so by the induction hypothesis, $\rightarrow A_1$ and $\rightarrow A_2$ are EK_{n+1} provable.

Whence $\rightarrow A$ is EK_{n+1} provable, thus:

$(\Gamma \rightarrow \Delta, B), (\Gamma \rightarrow \Delta, C) \rightarrow (\Gamma \rightarrow \Delta, B \wedge C)$ proper axiom ($\wedge$R)

$\rightarrow (\Gamma \rightarrow \Delta, B)$

$(\Gamma \rightarrow \Delta, C) \rightarrow (\Gamma \rightarrow \Delta, B \wedge C)$ Cut

$\rightarrow (\Gamma \rightarrow \Delta, C)$

$\rightarrow (\Gamma \rightarrow \Delta, B \wedge C)$ Cut

(ii) A is obtained by modus ponens on $\supset$. In this case P may be represented:

$B \supset A$

B

A

---

4. From section 5, above, "EK_n provable" means "is a theorem of the system EK_n".

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It has been pointed out to me by Hans Kamp that an error occurs in subcase (ii) of the induction step of the following proof of theorem 7.2. I now believe that theorem 7.2 is in fact not provable. For example:

$(A \to B) \supset (\sim B \to \sim A)$ is provable at no level, although $\to((A \to B) \supset (\sim B \to \sim A))$ is provable at every level of the appropriate degree. Fundamentally the problem is that insufficient axioms exist at each level for formulae whose principle logical symbol is other than $\to$. Apart from simply adding more axioms, there are two responses one might make to this situation, although at this stage I have not fully investigated either of them. Firstly, rather than obtain EK_n from LK_n by adding LHK axioms, one may obtain EK_n by adding to the rules of LK_n Genzen's NK rules for the truth functional connectives. Then, for example, $(A \to B) \supset (\sim B \to \sim A)$ is provable at every level, thus: by applications of $(\sim L)$, $(\sim R)$, and (ExL) $\sim B \to \sim A$ is derivable on the assumption of $A \to B$, whence by $\supset$-introduction $(A \to B) \supset (\sim B \to \sim A)$. Secondly, with a view to simplifying the systems whilst maintaining their philosophical significance, one might abandon the study of entailment sentences whose principal symbol is other than $\to$, and restrict attention to the LK_n hierarchy. In this case the Key Metatheorem of the following chapter must be restated as: if C is a theorem of S5 of the form $\Box A$ or $A \to B$, then $\lceil C \rceil$ is ML provable.

Clearly there are EK_n proofs of $B \supset A$ and B of length less than k, so by induction $\rightarrow (B \supset A)$ and $\rightarrow B$ are both EK_{n+1} provable. Then $\rightarrow A$ is EK_{n+1} provable thus:

$B \supset A,\ B \rightarrow A$ proper axiom

$\rightarrow (B \supset A)$

$B \rightarrow A$ Cut

$\rightarrow B$

$\rightarrow A$ Cut

7.2 If $\Gamma \rightarrow \Delta$ is EK_{n+1} provable and all of Γ are EK_n provable, then $\overset{\vee}{\Delta}$ is EK_n provable, where $\overset{\vee}{\Delta}$ is any formula if Δ is empty, A if Δ is A, $A_1 \vee \ldots \vee A_k$ if Δ is $A_1, \ldots, A_k$.

The proof is an induction on the length of an EK_{n+1} proof P of $\Gamma \rightarrow \Delta$.

Case I $Lg(P) = 1$

In this case, $\Gamma \rightarrow \Delta$ is a proper axiom of EK_{n+1}. If $\Gamma \rightarrow \Delta$ is initial, the result holds trivially. If $\Gamma \rightarrow \Delta$ is a reiteration axiom (i.e. Γ is empty and Δ is a single formula), then by definition Δ is a proper axiom of EK_n. If $\Gamma \rightarrow \Delta$ is an axiom for $\rightarrow$, then $\Gamma \rightarrow \Delta$ is a meta axiom of EK_n, in which case the result is trivial.

Case II $Lg(P) = k + 1$ and the theorem holds for the endsequent of any EK_{n+1} proof of length k or less. There are various cases to consider, according to the form of the final inference in P. We take two.

(i) The final inference in P is a cut. There are two possibilities.

(i)a P is represented:

$$\begin{array}{l} \vdots \\ B, \Gamma \rightarrow \Delta \\ \vdots \\ \rightarrow B \\ \Gamma \rightarrow \Delta \text{ Cut} \end{array}$$

Clearly there are EK_{n+1} proofs of Γ, $B \rightarrow \Delta$ and $\rightarrow B$ of length less than k. By hypothesis, if all of Γ, B are EK_n provable, then so is $\check{\Delta}$; and B is EK_n provable. Whence, if all of Γ are EK_n provable, so is $\check{\Delta}$.

(i)b P is represented:

$$\begin{array}{l} \vdots \\ \Gamma \rightarrow \Delta, B \\ \vdots \\ B \rightarrow \\ \Gamma \rightarrow \Delta \quad \text{Cut} \end{array}$$

Clearly there exists an EK_{n+1} proof of $B \rightarrow$ of maximum length k - 1. Thus there is an EK_{n+1} proof of $\rightarrow \sim B$ of maximum length k, and so by the induction hypothesis $\sim B$ is EK_n provable. Also by hypothesis if all of Γ are EK_n provable so is $(\check{\Delta}) \vee B$. Thus if all of Γ are EK_n provable so are $(\check{\Delta}) \vee B$ and $\sim B$. But it is a derived rule of LHK that if $A \vee B$ and $\sim B$ are LHK provable, so is A. Whence if all of Γ are EK_n provable then $\check{\Delta}$ is EK_n provable (since EK_n contains LHK as its truth functional fragment).

(ii) The final inference in P is a (VL). In this case P may be represented:

$$\begin{array}{l} \vdots \\ B, \Gamma \rightarrow \Delta \\ \vdots \\ C, \Gamma \rightarrow \Delta \\ \hline B \vee C, \Gamma \rightarrow \Delta \end{array}$$

By induction, if all of B, Γ are EK_n provable, so is $\check{\Delta}$; and if all of C, Γ are EK_n provable, so is $\check{\Delta}$. It follows that $B \wedge D_1 \wedge \ldots \wedge D_n \supset \check{\Delta}$ and $C \wedge D_1 \wedge \ldots \wedge D_n \supset \check{\Delta}$ are EK_n provable, where Γ is $D_1, \ldots, D_n$ (since a usual form of deduction theorem holds for LHK). Now suppose $B \vee C$, Γ are EK_n provable. Then $(B \vee C) \wedge D_1 \wedge \ldots \wedge D_n$ is EK_n provable. The

following formula is a theorem of the LHK fragment of EK_n:

$(((B \wedge D_1 \wedge \ldots \wedge D_n \supset \check{\Delta}) \wedge (C \wedge D_1 \wedge \ldots \wedge D_n \supset \check{\Delta})) \wedge ((B \vee C) \wedge D_1 \wedge \ldots \wedge D_n)) \supset \check{\Delta}$

It follows that $\check{\Delta}$ is EK_n provable. Accordingly, if $B \vee C$, Γ are EK_n provable, then so is $\check{\Delta}$.

7.3 In the case where Γ is empty and Δ is A, theorem 7.2 reduces to the converse of theorem 7.1: if $\rightarrow$ A is EK_{n+1} provable, then A is EK_n provable (since it is vacuously true that every member of the empty sequence is EK_n provable). Combining theorem 7.1 with its converse yields: $\rightarrow$ A is EK_{n+1} provable if and only if A is EK_n provable. Thus each system in the hierarchy constitutes a syntactical metatheory for its predecessor; and a theorem $\rightarrow$ A of EK_{n+1} may be read as the metastatement: A is provable in EK_n.

In case A is of degree zero, $\rightarrow$ A may be read as "A is provable in LHK". Genzen himself established that $\rightarrow$ A is LK provable just in case A is LHK provable (1935, p.128).

CHAPTER III

Ramified Modal Logic

In this chapter it is shown that contrary to Montague (1963), it is possible to construct an S5 strength modal logic in which possibility, necessity, and implication function as metalinguistic predicates.

1. Quine's Three Grades of Modal Involvement.

In (1953) Quine distinguishes between what he terms the three grades of modal involvement. He writes:

"There are three different degrees to which we may allow our logic, or semantics, to embrace the idea of necessity. The first or least degree of acceptance is this: necessity is expressed by a semantical predicate attributable to statements as notational forms - hence attachable to names of statements. We write, e.g.: Nec '9 > 5', ... A second and more drastic degree in which the notion of necessity may be adopted is in the form of a statement operator. Here we have no longer a predicate, attaching to names of statements .... but a logical operator 'nec', which attaches to statements themselves, in the manner of the negation sign. Under this usage [the above example] would be rendered rather as : nec 9 > 5. ...Thus whereas 'Nec' is a predicate or verb, 'is necessary', which attaches to a noun to form a statement, 'nec' is rather an adverb, 'necessarily', which attaches to a statement to form a statement. Finally the third and gravest degree is expression of necessity by a sentence operator. This is an extension of the second degree, and goes beyond it in allowing the attachment of 'nec' not only to statements but also to open sentences, such as 'x > 5', preparatory to the ultimate attachment of quantifiers: ($\exists$x) nec (x > 5). "(pp.156-157)

Sentences of Lewis' modal logic exhibit the second grade of modal involvement. As discussed in Chapter 1, section 4.1, □, ◇ and ⊰ figure as connectives in Lewis' syntax, on all fours with the truth functional connectives. Lewis' modal systems have been extended by the addition of quantifiers, notably by Barcan (1946), Prior (1956) and Kripke (1963). The sentences of these quantified modal systems exhibit the third grade of modal involvement.

Quine's suggestion in the above paper is that the first grade of modal involvement is the only philosophically respectable one. His reasons for thinking this appear to be threefold. Two of the reasons concern problems he sees with quantification through a modal operator. These problems are discussed in the following Chapter, section 3.2. Quine's third reason for not wishing to exceed the first grade of modal involvement (which fairly clearly is his chief reason) is that at both the second and third grades of involvement the confusion of concepts of logic and metalogic occurs. Quine's thoughts on this topic have already been discussed in Chapter I, section 2. In that section I have indicated that I do not believe there to be any reason for <u>always</u> looking askance at a syntax which mixes concepts of logic and metalogic; but I have argued that it is bad practice to build this mixing primitively into our thought on entailment. For this reason I have suggested that Lewis' modal logic is not a good way to start on the systematic study of entailment (Chapter I, section 4.1). However once the study of entailment and the other modalities is progressing well on a metalinguistic approach, we may wish to return to a Lewis style syntax (perhaps in order to benefit from its greater formal simplicity); and it does not seem to me that there is anything wrong with this move, provided it is not made at so early a stage as to jeopardise

the conceptual clarity gained by use of the metalinguistic approach. However, in view of Quine's complaints over the confusion of use with mention in Lewis' modal logic, I think it deserves to be shown that there need be nothing formally or conceptually wrong with a Lewis style syntax. In this Chapter a formal result is established which I take to vindicate Lewis' syntax (although not Lewis' own use of it: his syntactical mixing of concepts of logic and metalogic was born of a genuine conceptual confusion (vide Chapter 1, section 4.1).)

1.1 Modal Logic at the first grade of involvement.

In this chapter we will construct a modal logic which does not mix concepts of logic and metalogic; that is, a modal logic in which the modalities are treated syntactically as metalinguistic predicates. The construction is based on the hierarchy of systems EK_n, in which the modality of implication is already treated as a predicate of the metalanguage. The result of the construction is a hierarchy of modal systems, each system containing all valid modal assertions about sentences of the language of the immediately preceding system. We refer to the hierarchy of systems as ramified modal logic. Ramified modal logic is modal logic at the first grade of modal involvement.

It is worth remarking that two assertions which Quine makes about modal logic at the first grade of involvement are incorrect. Firstly, he asserts that modal logic at the first grade will contain no "principles which, explicitly or implicitly (via '$\supset$', etc.) involve iteration of necessity". (ibid., p.169) But as said in Chapter I, section 1.3, there is no reason to suppose that $\rightarrow$ will not iterate. Secondly, Quine asserts that first grade modal logic will involve "if we are literal minded, a pair of quotation marks after each 'Nec'". (ibid., p.169) But in Chapter II, section 1.1, the hierarchy of

entailment metalanguages is constructed without the use of any functor forming names of expressions.

2. A Theorem Due to Montague Concerning First Grade Modal Logic.

Richard Montague (1963) has investigated the question of how much traditional modal logic can be retained if the modalities are treated as metalinguistic predicates. The answer he gives is discouraging. He establishes the following theorem: Suppose T is any theory such that

(i) T is an extension of Robinson's Arithmetic, and for all sentences A, B of T containing no free variables

(ii) $\vdash_T N[A] \supset A$

(iii) $\vdash_T N[N[A] \supset A]$

(iv) $\vdash_T N[A \supset B] \supset (N[A] \supset N[B])$

(v) $\vdash_T N[A]$, if A is a logical axiom

Then T is inconsistent. (1963, p.159)

N is a monadic predicate to be interpreted as the necessity predicate, and N[A] is the result of filling the gap in N by a name of A constructed by means of a Gödel numbering. Although N[A] captures formally Quine's metalinguistic ⌜Nec 'A'⌝, A and N[A] are both sentences of number theory: Montague utilises Gödel's (1931) method of mapping the metalanguage of number theory back into the language of number theory itself.

Robinson's Arithmetic is an extremely weak finitely axiomatisable fragment of number theory. As Montague remarks, condition (i) is not so strong as to rob the theorem of interest, since modal logic, like ordinary logic, ought to be applicable to arbitrary subject matter, including arithmetic. (ibid., p.160)

The assumptions of the theorem concern only those expressions N[A] in which A is without free variables. Thus, as Montague points out, the difficulties which the theorem exhibits for modal logic at the first grade of involvement are independent of the problems Quine raises concerning quantification into modal contexts (ibid, p.159: these problems are discussed in the following Chapter, section 3.2.)

The theorem tells us that if necessity is treated as a meta-linguistic predicate, then there is no consistent modal system extending Robinson's Arithmetic which has all of (ii) through (v) true of it. But these are all statements which we intuitively want to be true of any adequate modal system; and, indeed, they are all true of S1, Lewis' weakest system. Montague interprets the theorem as showing that the reconstruction of propositional modal logic at the first grade of modal involvement is a hopeless cause: "If necessity is to be treated --- as a predicate of sentences, as Carnap and Quine have urged, then virtually all of modal logic, even the weak system S1, must be sacrificed." (ibid, p.161)

Montague's interpretation of his theorem is not correct, however. It is certainly true on the basis of the theorem that there can be no single system providing an adequate reconstruction of propositional modal logic at the first grade of involvement. But the theorem does not show that there is no way of adequately reconstructing modal logic. The hierarchical approach to modal logic provides just such a way. On this approach we have not one system, but an infinite number, and although S1 is not contained in any one system of the hierarchy, it is contained in the hierarchy as a whole. (In fact S5 is contained in the hierarchy

constructed in this Chapter.)[1]

3. Dana Scott's Reconstruction of Lewis' Modal Logic

Scott (1971), like myself concerned with Lewis' mixing of language levels, has also given a reconstruction of propositional modal logic free of use/mention confusions. I propose to give a thumbnail sketch of Scott's method of reconstruction in order to establish the difference between his approach and that followed in this Chapter.

Scott proposes to give what he calls a "rational reconstruction" of Lewis' program (p.800), which depends on taking Lewis at his word when he sets out to construct a calculus comprising "only the most general principles of deductive inference" (1932, p.122). Scott contends that if we are concerned with producing a theory which formalises principles of deductive inference in their greatest generality, then we ought not

> to think of the domain of our theory as comprised of syntactical objects: the objects of the domain should be thought of quite generally. (p.799)

Lewis can then be viewed as having in effect injected a part of the

> metalanguage into the object "language" by means of his modal operators (p.800): since we are not thinking of the "language" as a set of syntactical objects, no use/mention confusion is committed by this procedure. (p.799)

The validity of the metalinguistic $A \rightarrow B$ ($A \vdash B$, in Scott's notation) can

> safely be "confused" with the assertion of the strict implication $A \dashv B$. (p.801)

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1. Montague's assessment of his formal result seems to have been widely accepted, and moreover applied in areas falling outside modal logic. For example Posy (1977) writes: "van Dantzig (1949) tried to legitimatise Brouwer's theory of the creative subject by interpreting it in terms of provability. However, Montague (1963) rules out any such reading if one takes provability in the usual syntactic way." (p.54) But Montague's result does not rule out any such reading if one is prepared to work with a hierarchy of systems, rather than pack everything into a single system.

Clearly Scott's method of sorting out the use/mention confusions of classical modal logic differs from my own method of successively formalising the metalanguage, meta-metalanguage, and so on, and constructing the modal system associated with each separate level of language. (I prefer this more literal-minded method, which avoids Scott's idea of "abstract propositional calculus" where " we do not care what the "statements" in S [the 'language'] actually are" (p.799).)

4. How To Construe Lewis' Modalities Metalinguistically

4.1 Necessity and possibility

In order to preserve continuity between ramified and classical modal logic, the symbols $\Box$ and $\Diamond$ are used for logical necessity and logical possibility, respectively. We introduce these symbols by definition into each member (except the first) of the hierarchy of languages contructed in the previous chapter. The following two definitions are used:

$$\Box A =_{df} \rightarrow A \qquad \text{(def. } \Box\text{)}$$

$$\Diamond A =_{df} \sim(\rightarrow \sim A) \qquad \text{(def. } \Diamond\text{)}$$

Derivatively, $\Diamond A$ and $\sim\Box\sim A$ are interchangeable, as are $\Box A$ and $\sim\Diamond\sim A$. Thus the inter-dependencies of $\Box$ and $\Diamond$ in the Lewis systems are preserved.

Necessity and possibility are, then, being construed metalinguistically: where A is a formula of L_n, $\Box A$ is a formula of L_{n+1}. The first grade of modal involvement has not been exceeded.

The above definition of $\Box$ is extremely natural. A sentence is defined as being logically necessary just in case it is entailed by the null sequence of premisses. The definition precises the philosophical view that a logically necessary sentence is a sentence which is true in itself:

a sentence whose truth does not "depend" on the truth of any other sentence of the language. For if it is true that A is deducible from the null sequence of sentences, then A is true no matter how truth values are assigned to other sentences of the language. Moreover, our definition of logical necessity is equivalent to Lewis' in the following sense: it is provable that $\Box A$ and $\sim A \dashv A$ are interchangeable at their every occurrence in theorems of S1 through S5; and it is provable that $\Box A$ and $\sim A \rightarrow A$ are interchangeable at their every occurrence in theorems of ramified modal logic. Thus we can say that both Lewis and ourselves see logical necessity and entailment as linked in the following way: a sentence is logically necessary just in case it is entailed by its own negation. Finally, we remark that if Kripke's (1959) possible world semantics for modal logic is taken as providing a correct semantical analysis of the notion of logical necessity, then the above definition does indeed capture this notion. Kripke counts $\Box A$ as true in a possible world just in case A is true in every possible world, and $A \dashv B$ as true in a possible world just in case if A is true in any world, B is also true in that world. The following extends this last truth clause to implications of the more general form $\Gamma \rightarrow A$: $\Gamma \rightarrow A$ is true in a world just in case if all of Γ are true in any world, then B is also true in that world. When Γ is empty, this clause reduces to the Kripke truth clause for $\Box A$.

4.2 Strict implication

The only inference figures that can be horizontalised in the stratified syntax we have so far developed are those whose upper and lower members are all of the same degree. Lewis, however, concerned himself with a wider range of inferences than can be expressed by such

figures. For example, he considered the inferences expressed by the following three figures to be valid, and said so by means of his theses $\Box A \rightarrowtail A$, $\Box A \rightarrowtail \Box\Box A$, and $\Box A \rightarrowtail \Box\Box\Box A$, respectively: $\frac{\Box A}{A}$ $\frac{\Box A}{\Box\Box A}$ $\frac{\Box A}{\Box\Box\Box A}$

Hitherto we have been concerned to avoid the horizontalisations of "confused" inference figures like these (see Chapter 1, section 2). However, if propositional modal logic is to be reconstructed at the first grade of modal involvement, such figures must be considered; and some way must be found of extending our syntax so that such figures can be horizontalised without discarding our policy of keeping language and metalanguage distinct.

What is required is some method of translating language into metalanguage. For example, writing "trans(A)" for the translation of A into the metalanguage, the first of the above three inference figures can be re-expressed without any mixing of language levels as $\frac{\Box A}{\text{trans}(A)}$. This figure may then be horizontalised by the sentence $\Box A \rightarrow \text{trans}(A)$, in which the metalinguistic predicate $\rightarrow$ attaches as it ought to a pair of sentences of the same language. Similarly, writing "trans($\Box$A)" for the translation of $\Box A$ into the metametalanguage, the second of the above two figures can be horizontalised without mixing levels as $\text{trans}(\Box A) \rightarrow \Box\Box A$.

This translation from language to metalanguage will be accomplished formally by the straightforward use of semantic ascent. The primitive * is introduced into each language $L_{n>o}$: for every A in WFF_{n-1}, *A is in WFF_n. *A may be interpreted as "A is true"; or, alternatively, * may be regarded as a merely syntactical device for accomplishing the desired translation. In either case, *A is the translation of A into the metalanguage, **A is the translation of A and *A into the meta metalanguage,

and so on. The informal point that *A, **A, etc., all have the same "meaning" as A is captured in a formal semantics by stipulating that for every A in WFF, the conditions under which *A is true are just those under which A is true (vide Chapter IV, section 1).

An alternative way of coping with this translation is to use a hierarchy of cumulative metalanguages. L_n is injected into L_{n+1} at each stage of the construction, so that for all $m \leq n$ WFF_m is a subset of WFF_n. Thus, for example, $\Box A \rightarrow A$ would be a formula of L_n, A at its first occurrence being a formula of L_{n-2} and at its second occurrence a formula of L_{n-1}. This method has the advantage of syntactical simplicity. However, simply grafting the object language onto the metalanguage seems to me to be a mere formal trick which does not really confront the problem of horizontalising unbalanced inference figures in such a way that $\rightarrow$ gets written only between formulae of the same language level. Certainly I would be surprised if Quine were to see a reconstruction of modal logic using this approach as constituting modal logic at the first grade of modal involvement. Thus this approach is avoided here.

Sentences like $\Box A \rightarrow *A$, $*\Box A \rightarrow \Box\Box A$ and $**\Box A \rightarrow \Box\Box\Box A$, then, are all sentences exhibiting the first grade of modal involvement. They say in a way which does not mix concepts of logic and metalogic what Lewis said in his "confused" syntax by means of $\Box A \dashv A$, $\Box A \dashv \Box\Box A$ and $\Box A \dashv \Box\Box\Box A$.

We now introduce the sign $\dashv$ as a way of abbreviating entailment sentences. A useful identity of form is thereby secured between Lewis' expressions of entailment and our own. $\dashv$ is introduced by the following definition.

$$A \rightarrowtriangle B =_{df} *^{i}A \rightarrow *^{j}B \qquad \text{(def. } \rightarrowtriangle\text{)}$$

$*^{i}$ is a notational device for abbreviating iterations of the symbol *, defined as follows:

$$*^{i}A \begin{cases} i = o : A \\ \\ i = n : **^{n-1}A \end{cases}$$

In (def. $\rightarrowtriangle$) the following restrictions are placed on i and j: at least one of i and j is zero and i - j is the degree of B minus the degree of A. The first restriction ensures that $A \rightarrowtriangle B$ abbreviates just one formula, whilst the second ensures that $A \rightarrowtriangle B$ is always well formed.

To give some examples, $A \rightarrowtriangle A \vee A$ is $A \rightarrow A \vee A$, $\Box A \rightarrowtriangle A$ is $\Box A \rightarrow *A$, $\Box A \rightarrowtriangle \Box\Box A$ is $*\Box A \rightarrow \Box\Box A$ and $(A \rightarrowtriangle B) \rightarrowtriangle (\Box\Box A \rightarrowtriangle \Box B)$ is $**(A \rightarrow B) \rightarrow (\Box\Box A \rightarrow *\Box B)$ (assuming A and B are of the same degree).

As established in theorem 6.17 of the previous chapter, $*^{i}A \rightarrow *^{j}B$ and $\Box(*^{i}A \supset *^{j}B)$ are equivalent. This parallels Lewis' result 18.7 : $p \rightarrowtriangle q \ . = . \sim\Diamond\sim(p \supset q)$ (1932, p.165). Thus in both classical and ramified modal logic, implication turns out to be the necessity of the material conditional.

5. Axioms for *

The following two LK style meta axioms govern the introduction of the symbol * into proofs.

$$((\rightarrow A), \Gamma \rightarrow \Delta) \rightarrow (*A, \Gamma \rightarrow \Delta) \qquad (*L)$$

$$(\Gamma \rightarrow \Delta, (\rightarrow A)) \rightarrow (\Gamma \rightarrow \Delta, *A) \qquad (*R)$$

In (*R), A may be any member of WFF, whilst in (*L) A must contain an arrow. (Of course, when (*L) or (*R) is added to a particular entailment system A has to be a formula of the appropriate degree.)

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Hans Kamp has pointed out to me that the restriction on the (*L) axiom should be that every occurrence in A of an atomic formula of L_o is in the scope of an →.

In Chapter II, section 7.3, it was established that where n is the degree of A, $\rightarrow$A may be read as "A is provable in EK_n" (or "A is provable in LHK" in case the degree of A is zero). In the light of this reading (*L) and (*R) are intuitively appealing entailments. (*R) says that if "A is provable" can be inferred from Γ, then "A is true" can be inferred from Γ; which is what we would expect, since if A is provable in a well behaved system, then A will be true (on the intended interpretation of the system). (*L) says that if a formula is inferable from "A is provable", then it is inferable from "A is true" provided that A is an entailment sentence (i.e. a sentence of the form $*^n(\Gamma \rightarrow \Delta)$, possibly with n = o, or a truth function or quantification of such). Again, this is what we would expect, since all true entailment sentences (of the appropriate degree) will be provable in a well behaved entailment system. At least, certainly all valid entailments will be provable in a well behaved system, and it seems to me doubtful that there could be any true entailments which are not valid (that is, always true). I cannot conceive of the relation of logical deducibility ever holding in virtue of some contingent feature of the actual world, whose alteration would result in the falsity of the corresponding entailment. (This point receives some related discussion in section 10, below.)

Four further meta axioms permit the commutation of * and $\sim$ in proofs:

$$\left.\begin{array}{l}(\sim *^i A,\ \Gamma \rightarrow \Delta) \rightarrow (*^i \sim A,\ \Gamma \rightarrow \Delta) \\ (*^i \sim A,\ \Gamma \rightarrow \Delta) \rightarrow (\sim *^i A,\ \Gamma \rightarrow \Delta)\end{array}\right\} (*/\sim L)$$

$$\left.\begin{array}{l}(\Gamma \rightarrow \Delta\ ,\ \sim *^i A) \rightarrow (\Gamma \rightarrow \Delta,\ *^i \sim A) \\ (\Gamma \rightarrow \Delta\ ,\ *^i \sim A) \rightarrow (\Gamma \rightarrow \Delta,\ \sim *^i A)\end{array}\right\} (*/\sim R)$$

These meta axioms reflect the point that *A has the same truth conditions as A: $\sim *^i A$ is true if and only if $*^i \sim A$ is true if and only if $\sim A$ is true. Thus exchanging positions of $\sim$ and * in a formula occurring in the antecedent or consequent of a sequent will not alter the truth conditions of the sequent as a whole. By the same general consideration, * commutes with the quantifiers and distributes over $\rightarrow$, $\vee$, $\wedge$ and $\supset$. The meta axioms licencing the distribution of * over $\rightarrow$ in proofs are as follows:

$$\left.\begin{array}{l}(*^i(A \rightarrow B),\ \Gamma \rightarrow \Delta) \rightarrow (*^j(*^k A \rightarrow *^k B),\ \Gamma \rightarrow \Delta) \\ (*^j(*^k A \rightarrow *^k B),\ \Gamma \rightarrow \Delta) \rightarrow (*^i(A \rightarrow B),\ \Gamma \rightarrow \Delta)\end{array}\right\} (*/\rightarrow L)$$

$$\left.\begin{array}{l}(\Gamma \rightarrow \Delta,\ *^i(A \rightarrow B)) \rightarrow (\Gamma \rightarrow \Delta,\ *^j(*^k A \rightarrow *^k B)) \\ (\Gamma \rightarrow \Delta,\ *^j(*^k A \rightarrow *^k B)) \rightarrow (\Gamma \rightarrow \Delta,\ *^i(A \rightarrow B))\end{array}\right\} (*/\rightarrow R)$$

The restriction on i, j and k is that $i = j+k$.

The distribution axioms for $\vee$, $\wedge$ and $\supset$ and the commutation axioms for $\forall$ and $\exists$ are similar, and we omit the statements of them. They are annotated (*/ $\vee$ L), (*/ $\vee$ R), (*/ $\wedge$ L), (*/ $\wedge$ R), (*/ $\supset$ L), (*/ $\supset$ R), (*/ $\forall$ L), (*/ $\forall$ R), (*/ $\exists$ L) and (*/ $\exists$ R).

We also require axioms permitting * to distribute over, or commute with, the principal logical sign of a line of a proof when this sign is other than $\rightarrow$. For example:

$$\left.\begin{array}{l}\sim *^i A \supset *^i \sim A \\ *^i \sim A \supset \sim *^i A\end{array}\right\} (*/\sim)$$

$$\left.\begin{array}{l}*^i \forall x A \supset \forall x *^i A \\ \forall x *^i A \supset *^i \forall x A\end{array}\right\} (*/\forall)$$

$$\left.\begin{array}{l} *^{i}(A \vee B) \supset *^{j}(*^{k}A \vee *^{k}B) \\ *^{j}(*^{k}A \vee *^{k}B) \supset *^{i}(A \vee B) \end{array}\right\} (*/\vee)$$

$(i = j+k)$

Axioms for $\wedge$, $\supset$ and $\exists$ are also required. They are similar, and we omit the statement of them.

To illustrate the use of these axioms in a proof, suppose $\sim *^{i}A$ is proved and $*^{i}\sim A$ is required: the latter formula is derivable by the rule of modus ponens for $\supset$ applied to the former formula and the first $(*/\sim)$ proper axiom.

Finally we state a pair of axioms permitting the derivation in a proof of $\rightarrow(\Gamma \rightarrow \Delta)$ from $*(\Gamma \rightarrow \Delta)$, and $*A$ from $\rightarrow A$ (where A is any formula of the appropriate degree):

$$(\rightarrow A) \supset *A \qquad (\rightarrow:*)$$

$$*(\Gamma \rightarrow \Delta) \supset \rightarrow(\Gamma \rightarrow \Delta) \qquad (*:\rightarrow)$$

Thus, for example, $*A$ is derivable in a proof by modus ponens on $\supset$ applied to $\rightarrow A$ and $(\rightarrow:*)$. Reading $\rightarrow A$ as "A is provable" these two conditionals are intuitively acceptable. $(\rightarrow:*)$ says that if A is provable, then A is true, and $(*:\rightarrow)$ says that if $\Gamma \rightarrow \Delta$ is true, then it is provable (in EK_n, where n is the degree of $\Gamma \rightarrow \Delta$).

6. The Hierarchy of Ramified Modal Systems

The hierarchy of ramified modal systems, ML_1, ---, ML_n, --- is based on the hierarchy of entailment systems EK_1, ---, EK_n, --- constructed in Chapter II, section 5. The new hierarchy is defined by the following recursion:

(1) ML_1 is EK_1 extended by the addition of the following proper axioms: $(*/\sim)$, $(*/\vee)$, $(*/\wedge)$, $(*/\supset)$, $(*/\forall)$, $(*/\exists)$ and $(\rightarrow:*)$. (Notice that

the remaining new axioms discussed in the preceding section have no application in ML_1, all of whose theorems are of degree one.)

(2) ML_2 is EK_2 extended in the following way:

(i) all the axioms mentioned in (1) are added to the proper axioms of EK_2 together with the * axiom (*:→).

(ii) if A is a * axiom of ML_1, →A is added to the proper axioms of EK_2.

(iii) the entailments (*L), (*R), (*/∼L), (*/∼R), (*/→L), (*/→R), (*/∨L), (*/∨R), (*/∧L), (*/∧R), (*/⊃L), (*/⊃R), (*/∀L), (*/∀R), (*/∃L) and (*/∃R) are added to the meta axioms of EK_2.

(3) ML_{n+1} is defined in terms of ML_n in exactly the same way that EK_{n+1} is defined in terms of EK_n where $n > 1$.

Since classical modal logic contains the whole of the predicate calculus (or propositional calculus in the case of the quantifier - free Lewis systems) we add a predicate calculus to the hierarchy as the zeroth member. Genzen's predicate calculus LHK is chosen. Thus ML_o = LHK.

A formula will be called ML_n provable if it is a theorem of ML_n, and ML provable if it is ML_n provable for some n.

7. Some Theorems of Ramified Modal Logic

We illustrate our modal systems by writing out some proofs. Theorems are presented schematically, and the least n for which a theorem of that form is ML_n provable will be written in brackets immediately after the display of schema. The proof following is schematic for an $ML_m \geq n$ proof of a formula of L_m of the given form.

7.1 $(*^{k}(A \rightarrowtriangle B) \wedge (B \rightarrowtriangle C)) \rightarrowtriangle (A \rightarrowtriangle C)$ (2)

1. $(*^{i+k}A \rightarrow *^{k}B), (*^{k}B \rightarrow C) \rightarrow (*^{i+k}A \rightarrow C)$ Cut proper axiom
2. $*^{k}(*^{i}A \rightarrow B), (*^{k}B \rightarrow C) \rightarrow (*^{i+k}A \rightarrow C)$ 1, (*/ → L)
3. $*^{k}(*^{i}A \rightarrow B) \wedge (*^{k}B \rightarrow C) \rightarrow (*^{i+k}A \rightarrow C)$ 2, (∧L), (ExL), (Con L)
4. $(*^{k}(A \rightarrowtriangle B) \wedge (B \rightarrowtriangle C)) \rightarrowtriangle (A \rightarrowtriangle C)$ 3, (def.⇢)

The proof assumes that degree A < degree B < degree C, and in particular that degree B-degree A = i, degree C - degree B = k. The theorem is a ramified version of Lewis S1 axiom 11.6: $((A \rightarrowtriangle B) \wedge (B \rightarrowtriangle C)) \rightarrowtriangle (A \rightarrowtriangle C)$. The other ramified versions of this axiom are $((A \rightarrowtriangle B) \wedge *^{k}(B \rightarrowtriangle C)) \rightarrowtriangle (A \rightarrowtriangle C)$, $((A \rightarrowtriangle B) \wedge (B \rightarrowtriangle C)) \rightarrowtriangle *^{k}(A \rightarrowtriangle C)$ and $((A \rightarrowtriangle B) \wedge (B \rightarrowtriangle C)) \rightarrowtriangle (A \rightarrowtriangle C)$. This last version is well formed only in case A, B and C are of equal degree. All versions are proved analogously to 7.1.

7.2 $\Box A \rightarrowtriangle A$ (2)

1. $(\rightarrow A) \rightarrow (\rightarrow A)$ initial axiom
2. $(\rightarrow A) \rightarrow *A$ 1, (*R)
3. $\Box A \rightarrowtriangle A$ 2, (def. □), (def. ⇢)

This is Lewis' S1 theorem 18.42

7.3 $(A \rightarrowtriangle B) \rightarrowtriangle (\Box A \supset \Box B)$ (2)

1. $(A \rightarrow B), (\rightarrow A) \rightarrow (\rightarrow B)$ Cut proper axiom
2. $(A \rightarrow B) \rightarrow (\rightarrow A \supset \rightarrow B)$ 1, (⊃R)
3. $(A \rightarrowtriangle B) \rightarrowtriangle (\Box A \supset \Box B)$ 2, (def. □), (def. ⇢)

7.4 $(*A \wedge (A \rightarrowtriangle B)) \rightarrowtriangle B$ (2)

1.	$(A \rightarrow B) \rightarrow (A \rightarrow B)$	initial axiom
2.	$*A \wedge (A \rightarrow B) \rightarrow (A \rightarrow B)$	1, ($\wedge$L)
3.	$(A \rightarrow B) \rightarrow (\rightarrow A \supset B)$	proper axiom ($\supset$R)
4.	$*A \wedge (A \rightarrow B) \rightarrow (\rightarrow A \supset B)$	3, 2, Cut
5.	$*A \wedge (A \rightarrow B) \rightarrow (*(A \supset B))$	4, (*R)
6.	$*A \wedge (A \rightarrow B) \rightarrow (*A \supset *B)$	5, (*/$\supset$R)
7.	$*A \wedge (A \rightarrow B) \rightarrow *A$	[similar 1-2]
8.	$*A \wedge (A \rightarrow B) \rightarrow *A \wedge (*A \supset *B)$	7, 6, ($\wedge$R)
9.	$*A \rightarrow *A$	initial axiom
10.	$*B \rightarrow *B$	initial axiom
11.	$*A \supset *B, *A \rightarrow *B$	10, 9, ($\supset$L)
12.	$*A \wedge (*A \supset *B) \rightarrow *B$	11, ($\wedge$L), (ExL), (Con L)
13.	$*A \wedge (A \rightarrow B) \rightarrow *B$	12, 8, Cut
14.	$*A \wedge (A \rightarrowtriangle B) \rightarrowtriangle B$	12, (def. $\rightarrowtriangle$)

7.4 is a ramified version of Lewis' S1 axiom 11.7: $(A \wedge (A \rightarrowtriangle B)) \rightarrowtriangle B$. Being unbalanced, 11.7 is not a member of WFF. 7.4 is itself in WFF only in case A and B are of equal degree. In case degree B-degree A is k, 7.4 becomes: $(*^{k+1}A \wedge (A \rightarrowtriangle B)) \rightarrowtriangle B$; and in case degree A-degree B is k, 7.4 becomes: $(*A \wedge (A \rightarrowtriangle B)) \rightarrowtriangle *^{k+1}B$. (These two are proved in the same manner as 7.4.)

7.5 $\Diamond (A \wedge B) \rightarrowtriangle \Diamond A$ (2)

1.	$(A \rightarrow A) \rightarrow (A \wedge B \rightarrow A)$	proper axiom ($\wedge$L)
2.	$\rightarrow(A \rightarrow A)$	reiteration axiom
3.	$\rightarrow(A \wedge B \rightarrow A)$	2, 1, Cut
4.	$(A \wedge B \rightarrow A) \rightarrow (\sim A, A \wedge B \rightarrow)$	proper axiom ($\sim$L)
5.	$\rightarrow(\sim A, A \wedge B \rightarrow)$	4, 3, Cut
6.	$(\sim A, A \wedge B \rightarrow) \rightarrow (A \wedge B, \sim A \rightarrow)$	proper axiom (ExL)
7.	$\rightarrow(A \wedge B, \sim A \rightarrow)$	6, 5, Cut
8.	$(A \wedge B, \sim A \rightarrow) \rightarrow (\sim A \rightarrow \sim(A \wedge B))$	proper axiom ($\sim$R)
9.	$\rightarrow(\sim A \rightarrow \sim(A \wedge B))$	8, 7, Cut
10.	$(\sim A \rightarrow \sim(A \wedge B)), (\rightarrow \sim A) \rightarrow (\rightarrow \sim(A \wedge B))$	Cut proper axiom
11.	$(\rightarrow \sim A) \rightarrow (\rightarrow \sim(A \wedge B))$	10, 9, Cut
12.	$\sim(\rightarrow \sim(A \wedge B)) \rightarrow \sim(\rightarrow \sim A)$	11, ($\sim$L), (ExL), ($\sim$R)
13.	$\Diamond(A \wedge B) \rightarrowtriangle \Diamond A$	12, (def. $\Diamond$), (def. $\rightarrowtriangle$)

This is Lewis' Consistency Postulate.

7.6 $(A \rightarrowtriangle B) \rightarrowtriangle (\sim \Diamond B \rightarrowtriangle \sim \Diamond A)$ (3)

The proof of this theorem is simplified by the use of the following two derived rules of inference:

$$\frac{(\Gamma \rightarrow \Delta) \rightarrow (A, \Pi \rightarrow \Lambda)}{(\Gamma \rightarrow \Delta) \rightarrow (B, \Pi \rightarrow \Lambda)} \quad (DR1) \qquad \frac{(\Gamma \rightarrow \Delta) \rightarrow (\Pi \rightarrow \Lambda, A)}{(\Gamma \rightarrow \Delta) \rightarrow (\Pi \rightarrow \Lambda, B)} \quad (DR2)$$

$$\rightarrow (B \rightarrow A) \qquad\qquad \rightarrow (A \rightarrow B)$$

That (DR1) and (DR2) are derived rules of any $ML_{n \geq 2}$ is established by metatheorems 8.8 and 8.9 of the following section.

1. $(\rightarrow(A \rightarrow *^{i}B)) \rightarrow (\rightarrow(A \rightarrow *^{i}B))$ initial axiom

2. $\rightarrow((A \rightarrow *^{i}B) \rightarrow \sim\sim(\rightarrow \sim A) \vee \sim(\rightarrow \sim *^{i}B))$ (theorem)

3. $(\rightarrow(A \rightarrow *^{i}B)) \rightarrow (\rightarrow(\sim\sim(\rightarrow \sim A) \vee \sim(\rightarrow \sim *^{i}B)))$ 2, 1, (DR2)

4. $(\rightarrow \sim\sim(\rightarrow \sim A \vee \sim(\rightarrow *^{i}B)) \rightarrow (\sim\sim(\rightarrow \sim *^{i}B) \rightarrow \sim\sim(\rightarrow \sim A))$ (theorem)

5. $(\rightarrow(A \rightarrow *^{i}B)) \rightarrow (\sim\sim(\rightarrow \sim *^{i}B) \rightarrow \sim\sim(\rightarrow \sim A))$ 4, 3, Cut

6. $*(A \rightarrow *^{i}B) \rightarrow (\sim\sim(\rightarrow \sim *^{i}B) \rightarrow \sim\sim(\rightarrow \sim A))$ 5, (*L)

7. $\rightarrow((\sim\sim(\rightarrow *^{i}\sim B)) \rightarrow (\sim\sim(\rightarrow \sim *^{i}B)))$ (theorem)

8. $*(A \rightarrow *^{i}B) \rightarrow (\sim\sim(\rightarrow *^{i}\sim B) \rightarrow \sim\sim(\rightarrow \sim A))$ 7, 6, (DR1)

9. $\rightarrow(\sim\sim *^{i}(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow *^{i}\sim B))$ (theorem)

10. $*(A \rightarrow *^{i}B) \rightarrow (\sim\sim *^{i}(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow \sim A))$ 9, 8, (DR1)

11. $(\sim\sim *^{i}(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow \sim A)) \rightarrow (\sim *^{i}\sim(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow \sim A))$ proper axiom (*/~L)

12. $*(A \rightarrow *^{i}B) \rightarrow (\sim *^{i}\sim(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow \sim A))$ 11, 10, Cut

13. $(\sim *^{i}\sim(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow \sim A)) \rightarrow (*^{i}\sim\sim(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow \sim A))$ proper axiom (*/~L)

14. $*(A \rightarrow *^{i}B) \rightarrow (*^{i}\sim\sim(\rightarrow \sim B) \rightarrow \sim\sim(\rightarrow \sim A))$ 13, 12, Cut

15. $(A \backsim B) \backsim (\sim\Diamond B \backsim \sim\Diamond A)$ 14, (def. $\Diamond$), def. $\backsim$)

This is the characteristic axiom of S3. The proof assumes that degree A $\geq$ degree B. The reverse case is treated analogously.

7.7 $(\Gamma \rightarrow \Delta) \backsim \Box(\Gamma \rightarrow \Delta)$ (3)

1. $(\rightarrow(\Gamma \rightarrow \Delta)) \rightarrow (\rightarrow(\Gamma \rightarrow \Delta))$ initial axiom

2. $\bar{*}(\Gamma \rightarrow \Delta) \rightarrow (\rightarrow(\Gamma \rightarrow \Delta))$ 1, (*L)

3. $(\Gamma \rightarrow \Delta) \backsim \Box(\Gamma \rightarrow \Delta)$ 2, (def. $\Box$), (def. $\backsim$)

Where Γ is empty and Δ is A we have the special case:

$\Box A \backsim \Box\Box A$

This is the characteristic axiom of S4. Notice that 7.7 is provable in $LK_{n\geq 3}$ supplemented by (*L) alone. In particular (*:$\rightarrow$) is not used.

(A special case of (*:→) is *□A ⊃ □□A, which is the ramified analogue to □A ⊃ □□A, a formula equivalent in classical modal logic to the characteristic S4 axiom.) I find it immensely pleasing that the characteristic S4 axiom is provable in a Genzen-style entailment system supplemented just with a well motivated rule for *-introduction.

7.8 $\Box A \rightarrowtriangle \Box\Box\Box A$ (4)

1.	$\Box\Box\Box A \rightarrow \Box\Box\Box A$	initial axiom
2.	$*\Box\Box A \rightarrow \Box\Box\Box A$	1, (def. □), (*L)
3.	$\Box*\Box A \rightarrow \Box\Box\Box A$	2, (def. □), (* /→ L)
4.	$**\Box A \rightarrow \Box\Box\Box A$	3, (def. □), (*L)
5.	$\Box A \rightarrowtriangle \Box\Box\Box A$	4, (def. ⇾)

7.9 $\Diamond A \rightarrowtriangle \Box\Diamond A$ (3)

1.	$(\rightarrow \sim(\rightarrow \sim A)) \rightarrow (\rightarrow \sim(\rightarrow \sim A))$	initial axiom
2.	$*\sim(\rightarrow \sim A) \rightarrow (\nrightarrow\sim(\rightarrow \sim A))$	1, (*L)
3.	$\Diamond A \rightarrowtriangle \Box\Diamond A$	2, (def. ◇), (def. □), (def. ⇾)

This is the characteristic axiom of S5.

7.10 $\sim\Diamond A \rightarrowtriangle (A \rightarrowtriangle B)$ (2)

1.	$(\rightarrow \sim*^{i}A) \rightarrow (\sim B \rightarrow \sim*^{i}A)$	proper axiom (ThL)
2.	$(\sim B \rightarrow \sim*^{i}A) \rightarrow (*^{i}A \rightarrow B)$	(theorem)
3.	$(\rightarrow \sim*^{i}A) \rightarrow (*^{i}A \rightarrow B)$	2, 1, Cut
4.	$(\rightarrow *^{i}\sim A) \rightarrow (\rightarrow \sim*^{i}A)$	proper axiom (*/∼R)
5.	$(\rightarrow *^{i}\sim A) \rightarrow (*^{i}A \rightarrow B)$	4, 3, Cut
6.	$*^{i}(\rightarrow \sim A) \rightarrow (*^{i}A \rightarrow B)$	5, (*/→L)

7.	$\sim\!\!\sim *^i(\rightarrow \sim A) \rightarrow (*^i A \rightarrow B)$	6, (~R), (~L)
8.	$*^i\!\sim\!\!\sim(\rightarrow \sim A) \rightarrow (*^i A \rightarrow B)$	7, (*/~L)
9.	$\sim\!\Diamond A \supset (A \supset B)$	8, (def. $\Diamond$), (def. $\supset$)

7.11 $\Box A \supset (B \supset A)$ (2)

1.	$(\rightarrow *^i A) \rightarrow (B \rightarrow *^i A)$	proper axiom (Th L)
2.	$*^i(\rightarrow A) \rightarrow (B \rightarrow *^i A)$	1, (*/→L)
3.	$\Box A \supset (B \supset A)$	2, (def. $\Box$), (def. $\supset$)

7.10 and 7.11 are the paradoxes of strict implication.

7.12 $(A\rightarrow) \rightarrow (\sim\!\Diamond A)$ (2)

1.	$(A\rightarrow) \rightarrow (\rightarrow \sim A)$	proper axiom (~R)
2.	$(A\rightarrow) \rightarrow \sim\!\sim(\rightarrow \sim A)$	1, (~L), (~R)
3.	$(A\rightarrow) \rightarrow \sim\!\Diamond A$	2, (def. $\Diamond$)

7.13 $(\sim\!\Diamond A) \rightarrow (A\rightarrow)$ (2)

1.	$\sim\!\sim(\rightarrow \sim A) \rightarrow (\rightarrow \sim A)$	(theorem)
2.	$(\rightarrow \sim A) \rightarrow (\sim\!\sim A\rightarrow)$	proper axiom (~L)
3.	$\rightarrow(A \rightarrow \sim\!\sim A)$	(theorem)
4.	$(\rightarrow \sim A) \rightarrow (A\rightarrow)$	3, 2, (DR1)
5.	$\sim\!\sim(\rightarrow \sim A) \rightarrow (A\rightarrow)$	4, 1, Cut
6.	$(\sim\!\Diamond A) \rightarrow (A\rightarrow)$	5, (def. $\Diamond$)

Theorems 7.12 and 7.13 establish that the sequent A→ may be read as asserting the logical impossibility of A.

8. Some Metatheorems

We present a list of metatheorems concerning the hierarchy of ramified modal systems. These metatheorems are essentially lemmata for the results of the next section, and their proofs are relegated to Appendix I.

8.1 If A is ML_n provable, $\rightarrow A$ is ML_{n+1} provable.

8.2 If $\Gamma \rightarrow A$ is ML_{n+1} provable and every member of Γ is ML_n provable, then A is ML_n provable.

8.3 If $\rightarrow^{*i+1} A$ is ML_{n+1} provable, then $\rightarrow^{*i} A$ is ML_n provable.

8.4 If $\rightarrow^{*i} A$ is ML provable, then $\rightarrow A$ is ML provable.

8.5 If $\rightarrow^{*i} A$ is ML provable, then A is ML provable.

8.6 If $\rightarrow^{*i} A$ is ML_n provable, then $\rightarrow^{*i+1} A$ is ML_{n+1} provable.

8.7 If A is ML provable, then $\rightarrow^{*i} A$ is ML provable.

8.8 If $(\Gamma \rightarrow \Delta) \rightarrow (A, \Pi \rightarrow \Lambda)$ and $\rightarrow(B \rightarrow A)$ are ML_n provable, then $(\Gamma \rightarrow \Delta) \rightarrow (B, \Pi \rightarrow \Lambda)$ is ML_n provable. (This metatheorem establishes that (DR1) is a derived rule.)

8.9 If $(\Gamma \rightarrow \Delta) \rightarrow (\Pi \rightarrow \Lambda, A)$ and $\rightarrow(A \rightarrow B)$ are ML_n provable, then $(\Gamma \rightarrow \Delta) \rightarrow (\Pi \rightarrow \Lambda, B)$ is ML_n provable. (This metatheorem establishes that (DR2) is a derived rule.)

9. The Comparison of Classical and Ramified Propositional Modal Logic

Rather than use Lewis' axiomatisation of classical propositional modal logic we utilise a variation T' of Feys' system T (1937). T is aimed at axiomatising modal logic by extending an axiomatisation of standard propositional logic. The idea of doing this originated with Gödel (1933), and Feys obtained T by dropping an axiom from Gödel's system. T is

deductively equivalent to the system M independently constructed by von Wright (1951). It is known that T contains S2 (and therefore S1); although T neither contains nor is contained by S3 (Prior 1957, pp.123-124). Further, the system obtained by adding $\Box A \supset \Box\Box A$ to T is deductively equivalent to S4; and the result of adding $\Diamond A \supset \Box\Diamond A$ is deductively equivalent to S5. T does not use Lewis' rule of substitution of strict equivalents (1932, p.125). The treatment of this rule in ramified modal logic is complicated, and this is our reason for preferring T to a Lewis system.

T' is obtained from T by writing $\rightarrowtriangle$ in place of the principal occurrence of $\supset$ in each axiom of T, $\rightarrowtriangle$ being defined in the following classical fashion: $A \rightarrowtriangle B =_{df} \Box(A \supset B)$. The fact that the primary operator of each axiom of T' is $\rightarrowtriangle$ rather than $\supset$ somewhat simplifies the proof of the key metatheorem of section 9.3, below. Since T' and T are deductively equivalent, they are identically related to the Lewis systems. We present T and T' side by side.

	The System T	The System T'
A1	$A \vee A \supset A$	$A \vee A \rightarrowtriangle A$
A2	$B \supset A \vee B$	$B \rightarrowtriangle A \vee B$
A3	$A \vee B \supset B \vee A$	$A \vee B \rightarrowtriangle B \vee A$
A4	$(B \supset C) \supset (A \vee B \supset A \vee C)$	$(B \supset C) \rightarrowtriangle (A \vee B \supset A \vee C)$
A5	$\Box A \supset A$	$\Box A \rightarrowtriangle A$
A6	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$\Box(A \supset B) \rightarrowtriangle (\Box A \supset \Box B)$
R1	Modus ponens on $\supset$	Modus ponens on $\supset$
R2	If A is a thesis, $\Box A$ is a thesis (N)	N
R3		Modus ponens on $\rightarrowtriangle$.

That $T' \subseteq T$ is a consequence of N and the definition of $\rightarrowtriangle$. That $T \subseteq T'$ is a consequence of the following instance of A5 of T': $(A \rightarrowtriangle B) \rightarrowtriangle (A \supset B)$.

$T' + (\Box A \rightarrowtriangle \Box\Box A)$ will be called T4. $T' + (\Diamond A \rightarrowtriangle \Box\Diamond A)$ will be called T5. T4 is deductively equivalent to $T + (\Box A \supset \Box\Box A)$; and T5 is deductively equivalent to $T + (\Diamond A \supset \Box\Diamond A)$. Thus T4 is deductively equivalent to S4 and T5 is deductively equivalent to S5.

9.1 "Confused" classical theses

Since the modalities are treated as connectives in Lewis' syntax, very many classically well formed formulae will not be well formed on our ramified approach. This will occur whenever, from the ramified viewpoint, language and metalanguage are confused within a formula. In section 4.2 we discussed a particular kind of such formulae: those of the form $A \rightarrowtriangle B$, where A and B are themselves well formed on our approach, but are of unequal degree. In that section a method was introduced for translating classical formulae of this form into the language of ramified modal logic. If, for example, the degree of A minus the degree of B is k, $A \rightarrow *^{k}B$ is the translation into the language of ramified modal logic of the classical formula $A \rightarrowtriangle B$. A ramified version of $\rightarrowtriangle$ was then defined so that the ramified formula $A \rightarrowtriangle B$ is an abbreviation of the ramified formula $A \rightarrow *^{k}B$.

However, this method does not suffice for the translation of all "confused" classical theses into the language of ramified modal logic. The method must be generalised to cover formulae whose principal logical sign is other than $\rightarrowtriangle$, and whose subformulae are not necessarily themselves already in WFF. The translation of an arbitrary classical formula A, written [A], is defined by recursion on the complexity of A. (A suggestive reading of [A] is "the levelling of A".)

(0) A is an atomic formula : [A] = A

(1) A is $\Box B$: $[A] = \Box[B]$

(2) A is $B \rightarrowtriangle C$: $[A] = [B] \rightarrowtriangle [C]$

(3) A is $\sim B$: $[A] = \sim[B]$

(4) A is $B \wedge C$: $[A] = *^i[B] \wedge *^j[C]$

(5) A is $\forall xB$: $[A] = \forall x[B]$

The restrictions on i and j in clause (4) are as follows: at least one of i, j is zero, and where b is the degree of [B], c is the degree of [C], either $b \leq c$ and $i = c - b$, or $b > c$ and $j = b - c$.

$[B \supset C]$, $[B \vee C]$, $[\exists xB]$ and $[\Diamond A]$ are computed with the aid of the standard definitions of these connectives.

To illustrate the method of translation, $[(A \wedge (A \rightarrowtriangle \Box B)) \rightarrowtriangle \Box B]$ is computed where A and B represent atomic formulae. (The formula to be translated is an instance of the "confused" S1 axiom $(A \wedge (A \rightarrowtriangle B)) \rightarrowtriangle B$.) Applying clause (2) yields $[A \wedge (A \rightarrowtriangle \Box B)] \rightarrowtriangle [\Box B]$. Clause (4) applied to the antecedent of this yields $*^2A \wedge [A \rightarrowtriangle \Box B] \rightarrowtriangle [\Box B]$. A further application of clause (2) yields $*^2A \wedge (A \rightarrowtriangle [\Box B]) \rightarrowtriangle [\Box B]$. Two applications of clause (1) yield $*^2A \wedge (A \rightarrowtriangle \Box B) \rightarrowtriangle \Box B$. This latter is a member of WFF_3, abbreviating the formula: $(*^2A \wedge (*A \rightarrow \Box B)) \rightarrow *\Box B$.

We are now in a position to state formally the relationship between classical and ramified modal logic: if A is a theorem of T5 (or S5), then [A] is ML provable. This key metatheorem is established in section 9.3 below.

9.2 The inference rules of classical modal logic

Neither the rule of Necessitation, nor the rule of Modus Ponens on $\rightarrowtriangle$ (both of which are rules of T') may coherently figure as an inference

rule of a ramified modal system. Both state that a formula may be derived from others of a different degree, and accordingly they have no application in a ramified modal system, all of whose theorems are of the same degree. It is, however, possible to reconstrue each of these rules such that they are applicable to ramified modal logic. The former is reconstrued as: if A is ML provable, infer that $\Box A$ is ML provable; and the latter as: if A and $A \rightarrow B$ are ML provable, infer that B is ML provable. Thus the rules no longer concern a single system, but the hierarchy of systems. In this subsection we state and establish two metatheorems concerning the hierarchy of ramified modal systems which correspond to these reconstruals of Necessitation and Modus Ponens. These metatheorems are of the first importance in establishing the Key Metatheorem of the next subsection.

The Necessitation Metatheorem: if A is ML provable, then $\Box A$ is ML provable.

This theorem is an immediate consequence of metatheorem 8.1 and (def. $\Box$).

The Modus Ponens Metatheorem: if A and $A \rightarrow B$ are ML provable, then B is ML provable.

There are three cases to consider in the proof:

(i) degree A = degree B. In this case the result is immediate from metatheorem 8.2 and (def. $\rightarrow$).

(ii) degree A < degree B. Then for some i, for some n, $*^{i+1} A \rightarrow B$ is ML_n provable. Since $(\rightarrow *^i A) \rightarrow *^{i+1} A$ is ML_n provable (from $(\rightarrow *^i A) \rightarrow (\rightarrow *^i A)$ by (*R)), it follows that $(\rightarrow *^i A) \rightarrow B$ is ML_n provable. So by metatheorem 8.2, if $\rightarrow *^i A$ is ML provable, so is B. But by metatheorem 8.7, if A is ML provable, so is $\rightarrow *^i A$. Thus B is ML provable.

(iii) degree A > degree B. Then for some i, for some n, $A \rightarrow *^{i}B$ is ML_{n+1} provable. If A is ML_n provable, then by 8.1, $\rightarrow A$ is ML_{n+1} provable, so $\rightarrow *^{i}B$ is ML_{n+1} provable. Whence by 8.5, if A is ML provable, then so is B.

9.3 The Key Metatheorem

The Key Metatheorem states the relationship between classical propositional modal logic and ramified modal logic. The metatheorem is: if A is a theorem of T5, then [A] is ML provable.

Lemma: if A is an axiom of T5, then [A] is ML provable. We consider just two cases. The others are treated similarly.

(i) A is of the form $\Box(B \supset C) \supset (\Box B \supset \Box C)$ (i.e. is an instance of axiom schema A6). Suppose degree B minus degree C is i. Then [A] is $\Box([B] \supset *^{i}[C]) \supset (\Box[B] \supset *^{i}\Box[C])$. By 9.3.2 of Appendix I, [A] is ML provable.

The case where degree C exceeds degree B is treated analogously.

(ii) A is of the form $(C \supset D) \supset (B \vee C \supset B \vee D)$ (i.e. is an instance of axiom schema A4). Suppose that degree C minus degree B is i, degree D minus degree B is j, and degree D exceeds degree C. Thus degree D minus degree C is (j - i). [A] is $(*^{j-i}[C] \supset [D]) \supset (*^{j-i}(*^{i}[B] \vee [C]) \supset (*^{j}[B] \vee [D]))$. By 9.3.1 of Appendix I, [A] is ML provable.

Other possible orderings of the degrees of B, C and D are treated analogously.

Proof of the Key Metatheorem

The proof is an induction on the length of a T5 proof P of A.

Case I Lg(P) = 1. In this case A is an axiom of T5, and so by the preceding lemma [A] is ML provable.

Case II Lg(P) = k+1 and the metatheorem holds for the final formula of any T5 proof of length k or less.

(i) Suppose A is obtained by Modus Ponens on $\rightarrow$. Then for some classical formula B there are T5 proofs of lengths k or less of B $\rightarrow$ A and B. By the induction hypothesis, [B $\rightarrow$ A] and [B] are ML provable. By definition [B $\rightarrow$ A] is [B] $\rightarrow$ [A]. Thus by the Modus Ponens Metatheorem, [A] is ML provable.

(ii) Suppose A is obtained by Necessitation. Then for some classical formula B, A is $\Box$B and there exists a T5 proof of length k of B. By induction, [B] is ML provable. Applying the Necessitation Metatheorem, $\Box$[B] is ML provable. But by definition $\Box$[B] is [$\Box$B]. Hence [A] is ML provable.

(iii) Suppose A is obtained by Modus Ponens on $\supset$. Then for some classical formula B there are T5 proofs of length k or less of B and B $\supset$ A. By induction, [B] and [B $\supset$ A] are ML provable. [B $\supset$ A] is $*^i$[B] $\supset$ [A] or [B] $\supset$ $*^i$[A], according to whether degree A > degree B or vice versa. In either case [A] is ML provable, since modus ponens on $\supset$ is a rule of LHK. (Notice that by metatheorems 8.4, 8.7 and 8.2, A is ML provable if and only if $*^i$A is ML provable.)

Since (i) through (iii) exhaust the possibilities for the final inference of P, [A] is ML provable.

QED

9.4 Significance of the Key Metatheorem

The Metatheorem establishes two facts. Firstly, it is formally established that Montague's interpretation of his theorem is incorrect: it is possible to construct an S5 strength modal logic that does not exceed

the first grade of modal involvement. Secondly, it is formally established that whenever a "confused" formula is provable in S5, an unconfused translation of it is provable in the ramified system of the appropriate degree. This shows that one can pursue classical modal logic without confusing concepts of logic and metalogic. Since any theorem of classical modal logic can always be translated into a provable ramified formula, one can use a classical syntax without compromising the position that, conceptually, entailment and necessity belong to metalogic. Thus the Key Metatheorem vindicates classical propositional modal logic against Quine's complaints.

10. Remarks on Ramified Modal Logic and S4

It is possible to weaken the hierarchy of ramified modal systems to S4 strength by imposing the following restriction on the rule for *-introduction on the left:

$$((\rightarrow A), \Gamma \rightarrow \Delta) \rightarrow (*A, \Gamma \rightarrow \Delta)$$

where A is a formula of the form $*^n(\Pi \rightarrow \wedge)$ (possibly with n = o). Hitherto the restriction on the rule was simply that A should contain an →.

In motivating (*L) in its original form (section 5, above), it was remarked that the axiom said that if a formula is inferable from "A is provable", then it is inferable from "A is true", provided that A contains an arrow; and it was maintained that this is plausible, since any true sentence containing an arrow will be provable in a well behaved entailment system (of the appropriate degree). In its new form, (*L) says that if a formula is inferable from "A is provable", then it is inferable from "A is true", provided that A is of the form $*^n(\Pi \rightarrow \wedge)$. Thus if (*L) in

its original form is an intuitively correct entailment, so is (*L) in its new form: the new form is a special case of the old.

However, (*L) in its new form has no intuitive appeal as the rule for *-introduction. No justification for the extra restriction can be given, save the ad hoc one that imposing that restriction is the way to weaken the hierarchy of systems to S4 strength. The original restriction on (*L), on the other hand, is well justified. It states the most general conditions under which the rule is validity preserving: unless A contains an arrow, it will not be the case that if A is true, then A is provable, and so it will not be the case that if a formula is inferable from "A is provable", it is inferable from "A is true".

Notice that the restriction on (*L) in its new form would be well motivated if the entailment systems in the hierarchy were reworked so that only formulae of the form $*^n(\Pi \rightarrow \Lambda)$ were provable in them: it would then be the new restriction on (*L) which stated the most general conditions under which the rule was validity preserving. However, if the hierarchy were reworked in this way, it would no longer contain S4. In particular, the Modus Ponens Metatheorem (section 9.2, above) would no longer hold: for any formula B whose principal logical operator is other than $\rightarrow$ or *, A and $A \multimap B$ might be provable in the hierarchy without B also being provable.

Lewis was of the belief that "the interests of logical study" are best served by his weaker systems, although he adduces no arguments in support of this opinion, and, indeed, remarks that "prevailing good use in logical inference --- is not sufficiently precise and self conscious to determine clearly which of [S1 through S5] expresses the acceptable principles of deduction". (1932, pp.501-502) However, the fact that the

hierarchy of entailment systems EK_n becomes a hierarchy of modal systems of S5 strength when supplemented with a pair of well motivated rules for introducing the truth predicate *, provides a powerful argument that out of all the Lewis systems, it is S5 which gives the best axiomatisation of entailment. Moreover, work done on the semantics of classical entailment favours S5. The natural semantics for entailment turns out to be of S5 strength (as will be seen in the following chapter), and it is only by means of ad hoc restrictions that the semantics can be weakened to S4 strength or less. (Vide Kripke (1963) and Prior (1962) for an account of these restrictions.) Finally, we remark that where $\Box$ is defined in accordance with (def. $\Box$) (section 4.1, above), the characteristic S5 axiom does horizontalise an intuitively correct principle of inference, namely:

$$\frac{*\sim(\rightarrow \sim A)}{\rightarrow(\sim(\rightarrow \sim A))}$$

If $\sim A$ fails to follow from the null set of premisses, then surely this fact itself follows from the null set of premisses: the failure of $\sim A$ to follow from the null set can hardly depend upon some contingent matter. (We are, of course, arguing that S5 is the correct modal system for alethic modalities only, and not that it is necessarily the correct system when $\Box$ and $\Diamond$ are interpreted as epistemic modalities or tense operators, for example (cf von Wright (1951) and Prior (1957)).

11. Ramified Modal Logic and Quantified S5

Barcan Marcus (1946) pioneered the addition of quantifiers to modal logic. She chose Lewis' system S2 as her modal basis and supplemented it with the following definition, axioms, and rule.

D1: $\forall xA = \sim\exists x\sim A$

A1: $\forall xA \rightarrow B$ where B results from substituting any variable y for all free occurrences of x in A, provided A does not contain an occurrence of free x within the scope of a universal quantifier binding y.

A2: $\forall x(A \supset B) \rightarrow (\forall xA \supset \forall xB)$

A3: $A \rightarrow \forall xA$ provided x does not occur free in A

A4: $\Diamond\exists xA(x) \rightarrow \exists x\Diamond A(x)$

Generalisation: if $\vdash A(x)$, then $\vdash \forall yA(y)$

We will designate as QS5 the system resulting from the addition to Lewis' S5 of A1 through A3, D1, and Generalisation. (Prior (1956) has shown that A4 is derivable in QS5.) There are various alternative ways of axiomatising QS5. Prior (ibid) himself supplements Feys' system T with the axiom $\Diamond A \supset \Box\Diamond A$, two rules for introducing the existential quantifier due to Łukasiewicz (1951), and the definition $\forall xA = \sim\exists x\sim A$. We vary Prior's axiomatisation by extending the system T5 of section 9, above, with the following four rules for quantifier introduction:

E1: if $\vdash A \rightarrow B$, then $\vdash \exists xA \rightarrow B$, provided x does not occur free in B

E2: if $\vdash A \rightarrow B$, then $\vdash A \rightarrow \exists xB$

U1: if $\vdash A \rightarrow B$, then $\vdash A \rightarrow \forall xB$, provided x does not occur free in A

U2: if $\vdash A \rightarrow B$, then $\vdash \forall xA \rightarrow B$

These are Łukasiewicz' four rules for quantifier introduction with $\rightarrow$ as the sign of implication (1951, pp.62, 86).

A non equivalent version of quantified S5 is presented by Kripke (1963). This system is discussed in Chapter V. The Kripke system is included in QS5, but not vice versa.

In the following subsection, the Key Metatheorem of section 9, above, is extended to QS5.

11.1 If A is a theorem of QS5, then [A] is ML provable.

This theorem is established by extending in the obvious way the proof of the Key Metatheorem, and the details are omitted. We merely establish the following lemma:

If $A \rightarrowtriangle B$ is ML provable, then so are

(i) $\exists xA \rightarrowtriangle B$ (provided x does not occur free in B)

(ii) $A \rightarrowtriangle \exists xB$

(iii) $A \rightarrowtriangle \forall xB$ (provided x does not occur free in A)

(iv) $\forall xA \rightarrowtriangle B$

All four cases are proved similarly, and we cite the proof just for (iii).

Suppose $A \rightarrowtriangle B$ is ML provable. $A \rightarrowtriangle B$ is either (1) $*^{i}A \rightarrow B$, or (2) $A \rightarrow *^{i}B$.

Case (1) Applying ($\forall$R), $*^{i}A \rightarrow \forall xB$ is ML provable. (That the eigenvariable condition is satisfied is a hypothesis of the theorem.)

Case (2) Applying ($\forall$R), $A \rightarrow \forall x*^{i}B$ is ML provable. Whence by (*/$\forall$R), $A \rightarrow *^{i}\forall xB$ is ML provable.

This extension of the Key Metatheorem lacks some of the philosophical interest of the original. Quine has condemned classical propositional modal logic on the grounds that it exceeds the first grade of modal involvement. The Key Metatheorem vindicates classical propositional modal logic by showing that it can be mapped into a modal logic which does not exceed the first grade of involvement. However, quantified modal logic cannot be analogously vindicated against Quine by showing that it, too, can be mapped into ramified modal logic. For Quine's position is that quantified modal logic at the

first grade of involvement is, if anything, worse off than quantified modal logic at the third grade of involvement (1953, p.170). This position is discussed in section 3.1 of the following chapter.

Chapter IV

The Semantics of Classical Entailment

In this chapter we have a chief and a subsidiary aim. Firstly, to discuss the semantics of classical entailment; and secondly, to argue that, contra Quine, quantified modal logic at the first grade of modal involvement is semantically well justified.

1. The Semantical Theory *S1*

A valuation is a function from members of WFF to truth values. Writing 1 for the truth value True and 0 for the truth value False, a valuation is a function $v:WFF \rightarrow \{0, 1\}$. For any formula A, $v(A) = 1$ may be read as "A is true", and $v(A) = 0$ as "A is false".

Not all valuations will respect the intended meanings of the logical constants: some valuations will make both A and $\sim A$ true, some will make $A \wedge B$ true without making both of A and B true, and so on. We are interested in those sets of valuations whose members respect the intended meanings of the logical constants $\sim$, $\wedge$, $\vee$, $\supset$, $\forall$, $\exists$, $\rightarrow$ and *: sets of intended valuations, for short. A set of intended valuations is defined by recursion on the complexity of formulae.

The following abbreviations are used in the definition. $\hat{\Gamma}$ is $A_1 \wedge \text{---} \wedge A_n$, where Γ is A_1, ---, A_n. $\check{\Gamma}$ is $A_1 \vee \text{---} \vee A_n$. $op(1) = 0$ and $op(0) = 1$. $max(i, j) = 1$ if either of i, j is 1; otherwise $max(i, j) = 0$. $min(i, j) = 0$ if either of i, j is 0; otherwise $min(i, j) = 1$. $(A)t/u$ is defined as follows:

$(A)t/u$

- t is a closed term: the result of rewriting A with t replacing every free occurrence, if any, of the individual variable u
- t is an open term: the result of rewriting A with t replacing every free occurrence of u which is not within the scope of a quantifier binding a variable occurring in t

We now define recursively a set V of intended valuations. Members of V evaluate atomic formulae arbitrarily, subject to the restriction that <u>every</u> atomic formula is assigned a truth value. Then for every u in V, for all A, B ε WFF, for all Γ, $\Delta \subseteq$ WFF:

(i) $u(\sim A) = op(u(A))$

(ii) $u(A \wedge B) = \min(u(A), u(B))$

(iii) $u(A \vee B) = \max(u(A), u(B))$

(iv) $u(A \supset B) = \max(op(u(A)), u(B))$

(v) $u(\forall xA) = 1$ iff $u((A)t/x) = 1$ for every term t of L_o

(vi) $u(\exists xA) = 1$ iff $u((A)t/x) = 1$ for some term t of L_o

(vii) $u(*A) = u(A)$

(viii) $u(\Gamma \rightarrow \Delta) = u(\hat{\Gamma} \rightarrow \check{\Delta})$

(ix) $u(\rightarrow A) = u((A \supset A) \rightarrow A)$

(x) $u(A \rightarrow) = u(A \rightarrow \sim(A \supset A))$

(xi) $u(A \rightarrow B) = 1$ only if $u((A)t/y \rightarrow (B)t/y) = 1$ where y is any free variable, t is any term

(xii) $u(A \rightarrow B) = 1$ if and only if for all w ε V, if $w(A) = 1$, then $w(B) = 1$

(xiii) $u(\rightarrow) = 0$

This recursion enables us to define countably many sets of intended valuations. All intended valuations will occur in the union of all these sets. This union is written V_I.

A formula is <u>valid</u> just in case it is true in every valuation in V_I.

Clauses (i) through (iv) are taken in essence from Tarski's (1936) semantics for predicate logic.

Clauses (v) and (vi) give a substitutional interpretation to the quantifiers. Substitutional quantification is discussed in 1.1, below.

The restriction in clauses (v) and (vi) to substituting terms which do not occur bound in A (made via the definition of the substitution function) ensures that, for example, $\forall x \exists y A(x, y)$ is not made false by the falsity of $\exists y A(y, y)$. ("Everyone talked to someone" does not imply "Someone talked to himself".)

Clause (ix) counts $\rightarrow A$ as true in a valuation just in case it is true in that valuation that A is entailed by a formula which is true in every intended valuation, $A \supset A$; and clause (x) counts $A \rightarrow$ as true in a valuation just in case it is true in that valuation that A entails a formula which is false in every intended valuation, $\sim(A \supset A)$. This semantical treatment of $\rightarrow A$ and $A \rightarrow$ accords well with the intuitive readings suggested for these contexts in Chapter II, section 1.1, namely "A is logically valid" and "A is inconsistent", respectively. Clauses (ix) and (x) may be replaced by the following more literal clauses: $u(\rightarrow A) = u(\emptyset \rightarrow A)$ and $u(A \rightarrow) = u(A \rightarrow \emptyset)$, where $\emptyset$ is the null sequence of formulae and $u(\emptyset) = 0$ for every u in V. Within classical logic the two approaches are equivalent. In relevance logic, however, the latter approach is not available, for clearly no formula is relevant to the null sequence of formulae, yet we will want many instances of $\rightarrow A$ (i.e. $\Box A$) to be valid in a semantics for relevance logic. It is in order to preserve a maximum of continuity between classical and relevance semantics that the former approach is used here.

Clause (xi) ensures that intended valuations respect the uniform substitution of suitable terms for variables free in arrow formulae.

The combined import of clauses (viii) and (xii) is that $\Gamma \rightarrow \Delta$ is true in $u \in V$ just in case every valuation in V which makes all of Γ true also makes at least one of Δ true. (It should be stressed that the

if --- then --- occurring in clause (xii) is to be taken truth functionally.) This approach to the semantics of entailment is that of Scott (1974, p.149).

1.1 The substitutional interpretation of the quantifiers

The standard Tarskian interpretation of quantification involves the use of a set of objects over which the quantifiers of the language are taken as ranging in a particular instance or model. This set is usually called the domain of the model, and different domains are assigned to the quantifiers of the language by different models. Each model assigns a member of its domain to each individual variable and individual constant of the language, and assigns a set of n-tuples of members of its domain to each n-place predicate variable of the language. The notion of truth in a model is defined by recursion on the complexity of formulae. An atomic formula $P(t_1, ---, t_n)$ is counted as true in a model just in case the n-tuple of objects assigned by the model to $t_1, ---, t_n$ is a member of the set of n-tuples assigned by the model to the n-place predicate variable P. A formula of the form $\forall xA(x)$ is counted as true in a model M with domain D just in case A(x) is true in M for every assignment of a member of D to the free variable x; and a formula of the form $\exists xA(x)$ is true in M just in case A(x) is true in M for some assignment of a member of D to the free variable x. (In detail, Tarski's original formulation uses the notion of the satisfaction of a formula in a model by a sequence of objects. We have presented a simplified version of his account.)

This sort of interpretation of the quantifiers has been dubbed the objectual interpretation by Quine. (1966, p.105) Whether or not a quantified formula is true in a model is made to depend upon how things

are with objects. The substitutional interpretation of the quantifiers, in contrast, abstains from all reference to objects. The notion of a domain of quantification is not used. A universal quantification $\forall xA(x)$ is counted as true in a valuation (which might in parallel to the objectual account also be called a model) just in case every suitable substitution instance of A(x) is true in that valuation; and an existential quantification $\exists xA(x)$ is counted as true in a valuation just in case some suitable substitution instance of A(x) is true in that valuation. (Unsuitable substitution instances are those in which variables become accidentally bound as a result of substitution.) At the top end of the recursion, atomic formulae are made true or false arbitrarily by a valuation. This is also the case in the objectual account, although the arbitrary assignment of a truth value is made less directly: the truth or falsity of an atomic formula depends upon arbitrary assignments of objects to individual variables and n-tuples of objects to predicate variables.

The substitutional interpretation of quantification was first suggested by Barcan Marcus (1961, p.314). (Although Quine reports that he has learned in conversation with Leśniewski that this logician of the Warsaw School has independently constructed a substitutional semantics for the quantifiers. (1968a, note 15)) Barcan Marcus has herself credited the substitutional reading of the quantifiers to Russell and Whitehead (in the recorded informal discussion between Barcan Marcus, Quine, Kripke, and Føllesdal appearing in Wartofsky (1962)). However, Quine has maintained that Russell and Whitehead's apparent departure from Frege's objectual reading of the quantifiers is an illusion engendered by their confusion over the difference between use and mention. (ibid.) In (1961),

Barcan Marcus presents a formal recursive substitutional semantics for an extremely limited language. (p.319) As far as I am aware, the first published presentation of a fully general formal recursive semantics for the predicate calculus giving a substitutional interpretation to the quantifiers appears in Dunn and Belnap (1968). Dunn and Belnap do not extend their semantics beyond predicate logic: no attention is given to modal or entailment logic.

The present substitutional semantics differs from that of Dunn and Belnap in that they allow only closed atomic formulae to take truth values, and count a universal quantification $\forall xA(x)$ (existential quantification $\exists xA(x)$) as true in a valuation just in case $A(t)$ is true in the valuation for all closed terms t (for some closed term t), where a formula or term is closed just in case it contains no free variables. (1968, p.179) (Perhaps Dunn and Belnap make this restriction in deference to the idea that a formula containing a free variable is neither true nor false, but rather is true of some things, false of others.) Barcan Marcus similarly allows $\exists xA(x)$ to be true just in case $A(t)$ is true for some closed term t. (1961, p.319) Given that open sentences are neither true nor false, then it is certainly the case that the truth of a quantification can be unpacked substitutionally only in terms of the truth of closed substitution instances. However, there seems to be no good reason for prohibiting the assignment of truth values to open sentences in a formal semantics. Perhaps it is counterintuitive to say that an open sentence is true or false, but the notion of truth in a valuation or model need not be regarded as a formal reconstruction of our intuitive concept of truth. In *S*1 all open sentences receive truth values in all

valuations, and the truth conditions of quantifications are stated in terms of the truth conditions of both closed and open substitution instances.

This policy renders *S*1 immune to the objection often made to the substitutional interpretation of the quantifiers that the interpretation yields the wrong truth conditions for quantified sentences in case not all the objects talked about in the language are specifiable by quantifier-free closed terms of the language (for example Quine (1961a), Wallace (1970)). The rider "quantifier-free" rules out definite descriptions as suitable closed terms. Definite descriptions are, usually, treated as abbreviations for quantificational contexts in the Russellian way, and thus if they are allowed to figure as substituting terms in the specification of the truth conditions of quantifications, the semantics is no longer a proper recursion on the logical complexity of formulae. (Kripke (1976) has made this point.) To give an example of the kind of case envisaged by the objection, consider a meteorite whose spatial coordinates are C. The sentence $\exists x$(x is at C) ought to be true. But in the likely event that the language possesses no quantifier-free closed term specifying the meteorite, a substitutional semantics giving the truth conditions of the above sentence in terms of closed substitution instances will assess the sentence as false. However, in *S*1 this sentence will be true in any valuation making an atomic sentence "x is at C" true. Moreover, $\exists!x$(x is at C) will be true in any *S*1 valuation making an atomic sentence "x is at C" true but every (quantifier-free) substitution instance of "x is at C" false. (This approach may be extended

by enlarging the formal language with a stock of uncountably many individual variables in order to cover the case where an individual is unspecifiable by a closed term of the language for reason of being a member of an uncountable totality, for example, the real numbers.)

We use the substitutional interpretation of the quantifiers in order to show, contra Quine, that there is nothing wrong with quantifying into formulae containing an → (section 3.1, below). However, in section 4 below, it is argued that there is something fundamentally wrong with substitutional quantification; and in section 5 it is shown how to blend substitutional and objectual quantification so as to get the best of both worlds.

1.2 The comparison of *S*1 with Kripke's possible world semantics

Expositions of the possible world semantics may be found in Kripke (1959) and Kripke (1963).

Kripke (1959) defines a complete assignment for a formula A in a non-empty domain D as a function which to every free variable of A assigns an element of D, and to every n-place predicate variable of A assigns a set of ordered n-tuples of members of D. A model of A in D is an ordered pair (G, K), where G ε K and K is a set of complete assignments for A in D such that every member of K agrees with G in its assignments to free individual variables of A. Where H ε K and B is a subformula of A, the truth value assignment made to B by H (written H(B)) is defined recursively. Where B is an atomic formula $P(x_1, ---, x_n)$, ψ is the set of ordered n-tuples assigned by H to P, and $\alpha_1, ---, \alpha_n$ are the elements of D assigned by H to $x_1, ---, x_n$, H(B) = 1 if and only if $(\alpha_1, ---, \alpha_n) \varepsilon \psi$. $H(\sim B) = 1$ if and only if H(B) = 0. $H(B \wedge C) = 1$ if and only if H(B) = 1 and H(C) = 1.

$H(\forall xB(x)) = 1$ if and only if $H(B(x))' = 1$ for every assignment of an element of D to x. $H(\Box B) = 1$ if and only if $F(B) = 1$ for every $F \varepsilon K$. A is said to be valid in a model (G, K) of A in D if and only if $G(A) = 1$; and A is said to be valid in a domain D if and only if A is valid in every model of A in D. Finally, A is said to be <u>universally valid</u> if and only if A is valid in every non-empty domain. (1959, p.2)

Kripke's semantics, then, gives an objectual interpretation to quantification. Because of this the semantics is unsuitable for ramified modal logic: on Kripke's objectual interpretation, quantification into the ramified formula $\Box A(x)$, where $\Box$ is a metalinguistic predicate,is not coherent. (Vide section 3.1, below, for an explanation of this point.) Kripke himself, of course, views $\Box$ and $\rightarrowtail$ as connectives of the object language, not as predicates of the metalanguage.

Leaving the treatment of quantification aside, *S*1 and the Kripke semantics are essentially similar. Each gives the same treatment to quantifier free non modal formulae. Complete assignments assign truth values arbitrarily to atomic formulae, and the truth functions are dealt with in a Tarskian fashion. Further, the treatments of entailment and the modalities provided by the two semantics are essentially the same. Defining $A \rightarrowtail B$ as $\Box(A \supset B)$, it is readily established that $A \rightarrowtail B$ is true in a complete assignment G (with respect to the model (G, K)) just in case for every $F \varepsilon K$, if $F(A) = 1$, then $F(B) = 1$. This is exactly parallel to the treatment of strict implications in *S*1. Defining $A \rightarrowtail B$ as $*^{i}A \rightarrow *^{j}B$ in accordance with (def. $\rightarrowtail$), $A \rightarrowtail B$ is true in an intended valuation $u \varepsilon V$ just in case for every $w \varepsilon V$, if $w(A) = 1$, then $w(B) = 1$ (for note that $w(*A) = w(A)$).

A Kripke model (G, K) corresponds to a pair (w, V), where w ε V and V is a set of intended valuations. Paralleling Kripke's treatment of validity, we can say that a formula is valid in the model (w,V) just in case it is true in the valuation w, and that a formula is universally valid just in case it is valid in every model. The set of universally valid formulae of *S*1 is then identical with the set of formulae valid in *S*1 on the original definition of validity as truth in all intended valuations. The Kripke-style treatment of validity has, as it were, built into the fact that no formula of the form $\Gamma \rightarrow \Delta$ can be evaluated in an intended valuation considered in isolation: a set of intended valuations containing the valuation in question has to be considered in evaluating the formula. However, the original treatment of validity is simpler, and we shall continue to use it. (Just as validity in *S*1 can be reworked in Kripke's fashion, universal validity in the Kripke semantics can be reworked in the fashion of *S*1 as truth in all complete assignments.)

Kripke speaks of complete assignments as representing possible worlds. (ibid, p.3) The idea is that the result of assignment a truth value to every sentence of the language - which is what a complete assignment does - is a complete specification of a possible state of affairs, or possible world. (In Carnap's (1947) terminology, the result of such an assignment of truth values is a state description.) One such complete specification will be a specification of the actual world. Other complete assignments will result in specifications of "worlds which are conceivable but not actual". (ibid, p.3) The actual world is singled out as the first member of the ordered pair (G, K) constituting a Kripke model (although the actual world is semantically undistinguished

and for formal purposes any member of K may be chosen as G). K represents the set of worlds possible relative to the actual world. Membership of K is arbitrary, save as restricted by the recursive definition of a complete assignment: "it seems plausible to assume --- that any combination of possible worlds may be associated with the real world." (ibid) Thus the semantics may be seen as counting a formula of the form □A as true in the actual world just in case A is true in every world possible relative to the actual world. (ibid)

In Kripke's terms, an intended valuation may be thought of as specifying a possible world, and a set of intended valuations V may be thought of as representing the set of worlds possible with respect to any member of V. Since the present semantics counts □A as true in any valuation u ε V just in case A is true in every w ε V, we can say that our semantics gives the same intuitive interpretation to □ as the Kripke semantics: □A is true in a possible world w just in case A is true in every world possible relative to w.

2. The Completeness of the Hierarchy of Ramified Modal Systems

A logic will be called sound with respect to a semantics just in case all its axioms are valid in the semantics and all its rules of inference are validity preserving. (Thus the meta axioms of a sound ramified system, which constitute the rules of inference of the system, are required to be validity preserving entailments, rather than valid entailments. It is readily shown that all valid entailments are validity preserving. The converse, however, does not hold. *A → □A, for example, is validity preserving in *S*1 but not valid.)

A logic will be called inclusive with respect to a semantics just in case all formulae of the language valid in the semantics are theorems of the logic.

A logic will be called complete with respect to a semantics just in case it is both sound and inclusive with respect to the semantics.

Ramified modal logic is complete with respect to the semantics *S*1. In the interest of keeping the text as readable as possible, the proof of this fact is relegated to Appendix II, sections 1 and 2.

3. Quantification into Modal Contexts

A variable x will be said to occur freely in A ε WFF_n just in case x occu freely in some zero degree subformula of A (possibly A itself), and A contains no quantifier binding that occurrence of x. (Recall that all atomic formulae are formulae of L_o only; vide Chapter II, section 1.1.) x will be said to occur bound in A ε WFF_n just in case A contains a quantifier of L_n binding x, or x occurs bound in some subformula of A.

A formula of $L_{n>o}$ containing a quantifier will be said to exhibit quantification within a modal context provided it contains no quantifier with an occurrence of → or * within its scope; otherwise it will be said to exhibit quantification into a modal context. So, for example, $A(t) \rightarrow \exists xA(x)$ and $*\forall x(A(x) \vee \sim A(x))$ exhibit quantification within a modal context if $A(x) \varepsilon WFF_o$, and into a modal context if $A(x) \varepsilon WFF_{n>o}$, whilst $\forall x(A(x) \rightarrow \exists xA(x))$ exhibits quantification into a modal context no matter what the degree of A(x) is.

Formulae exhibiting quantification into a modal context differ in a radical respect from formulae exhibiting quantification within a modal context: the quantifier which accomplishes quantification into a modal

context is metalinguistic (or meta-metalinguistic, and so on) whilst it binds a variable of the object language, L_o.

3.1 Quine's views on quantification into modal contexts at the first grade of modal involvement

Quine's position is that quantification into modal contexts at the first grade of modal involvement is simply incoherent; and indeed, one of his chief motives in developing the distinction between the three grades of modal involvement is to make the point that whilst it is tempting to quantify into open modal sentences at the third grade of involvement, there is no such temptation if we limit ourselves to the first grade of involvement, where such quantification is, he would say, clearly out of place. (1953, esp. p.170) The first grade modal logic developed in the preceding Chapter countenances quantification into modal contexts, and in order to justify this I want to examine and criticise Quine's arguments for his position.

The argument that quantification into first grade modal sentences is out of place goes as follows. The first grade sentence Nec 'x > 5', for example, is not an open sentence with free x, but a sentence of the metalanguage asserting the necessity of a particular open sentence of the object language. (Quine's first grade modal operator Nec is explained in Chapter III, section 1, above.) Nec 'x > 5' mentions, and does not use, the sentence x > 5, and only by confusing mention with use can Nec 'x > 5' be seen as containing an occurrence of the variable x. Properly understood, Nec 'x > 5' contains the letter x not as a variable, but as a constituent symbol of a complex name, and so to write $\exists x$ Nec 'x > 5' is like writing $\exists x$(Socrates is mortal): in neither case is the quantifier followed by a germane occurrence of its variable. (1953, p.170)

This argument of Quine's is not convincing, however. Certainly x occurs in Nec 'x > 5' as a constituent symbol in the name of a formula; but why should x at that same occurrence not also be a variable accessible to a quantifier of the metalanguage? To be sure, if, as Quine does, we have in mind an objectual interpretation of quantification, x does not occur as a variable of quantification in Nec 'x > 5'. Being simply a constituent symbol of a name, it no more serves to refer to objects in the domain over which quantification occurs than the occurrence of "rice" in "Beatrice" serves to refer to rice. But this shows just that the quantifier in $\exists x$ Nec 'x > 5' cannot be interpreted objectually; it ought not to persuade us that no good account can be given of quantification as it occurs here. Indeed, if as Barcan Marcus encourages (1961, p.88ff; 1962, p.252) the modal sentence $\exists x \Box A(x)$ is read as "some substitution instance of $\Box A(x)$ is true", Quine's problem does not arise: this, unlike the objectual reading "there exists at least one object, x, such that A(x)", does not presuppose for its intelligibility that x occurs referentially in $\Box A(x)$. Barcan Marcus' substitutional reading of quantification is, of course, captured formally in the semantics *S*1; and the availability of this semantics (together with the development of it *S*2 presented in section 5, below) is taken to fully rebut Quine's charge that it is incoherent to quantify into modal contexts at the first grade of involvement.

As a postscript to this discussion, it is worth saying that Quine would see me as having oversimplified things in the preceding paragraph. I claim that the mere fact that x occurs in a quotation mark name in Nec 'x > 5' makes x non-referential at that occurrence. However, in (1961) Quine writes: "It would not be quite accurate to conclude that an occurrence of a name within single quotes is <u>never</u> referential. --- The point about

quotation is not that it must destroy referential occurrence, but that it can (and ordinarily does) destroy referential occurrence." (p.141)

Quine gives two examples of sentences containing what he sees as a referential occurrence of a proper name occurring within a quotation mark name (p.141):

(1) 'Giorgione played chess' is true

(2) 'Giorgione' named a chess player

Quine's criterion for referential occurrence is as follows: an occurrence of a name "n" in a sentence A is referential just in case A remains true when the occurrence of "n" is supplanted by any other name of the object n. (pp.139-140) On this criterion, the occurrences of "Giorgione" in (1) and (2) are referential, whilst any singular term t in the context Nec 't > 5' is non-referential (substitute "the number of the planets" for "9" in Nec '9 > 5'). (p.144)

So, then, Quine agrees with me that the occurrence of x in Nec 'x > 5' is non-referential, but he would say that my reason for thinking it so is inadequate. However, I do not see Quine's sentences (1) and (2) as counterexamples to the general thesis that if a singular term of the object language occurs in a sentence of the metalanguage within a meta-linguistic designator of an expression of the object language, then the singular term is non referential at that occurrence (and so, for example, x occurs non referentially in Quine's first grade modal sentence Nec 'x > 5' and my first grade modal sentence →(x > 5)). For the only reference made in (1) is to the actual sentence: Giorgione played chess. To see the occurrence of "Giorgione" in (1) as referential is to mistakenly view the singular term

(3) "Giorgione played chess"

as somehow making a reference to Giorgione himself. The object, Giorgione, does not enter into the fact asserted by (1), namely the circumstance of the referent of (3) being a sentence of a certain sort, a true one. Rather, Giorgione enters into the fact which makes the referent of (3) such a sentence, namely the circumstance of Giorgione being a chess player. Similarly Giorgione does not enter into the fact asserted by (2), namely the circumstance of the particular inscription, "Giorgione", naming a chess player.

It would seem, then, that Quine's criterion of referential occurrence stated above is too broad: it counts as referential, occurrences of singular terms which are in fact non referential, as witnessed by (1) and (2).

3.2 Quine's views on quantification into modal contexts at the third grade of modal involvement

Quine has three main arguments against quantifying into third grade modal sentences.

The first argument runs as follows. Sentences in which the modalities function as connectives (third grade modal sentences) receive "their more fundamental expression" in paraphrases of them in which the modalities function as semantical predicates (first grade modal sentences). (1961, p.144; 1953, p.166) But quantified third grade sentences cannot be paraphrased as first grade modal sentences (as Quine would incorrectly say). Thus the legitimacy of such quantifications is dubious. However, Quine indicates that this is intended as a subsidiary argument only. (1961, p.21) Whether or not first grade modal sentences are viewed as more basic than third grade modal sentences is irrelevant to his chief objection to quantifying into third grade modal contexts, namely that such contexts are referentially opaque. (ibid)

A context is referentially opaque "when by putting a statement ∅ into that context, we can cause a purely referential occurrence [of a term] in ∅ to be not purely referential in the whole context" (1953, p.158), where an occurrence of a term t in a sentence is purely referential just in case the truth value of the sentence is invariant under any substitution for t of a co-referential term (ibid). Quine's argument that it is incorrect to quantify into referentially opaque contexts may be expressed as follows. Let t be a term, A(t) be a context in which t has an occurrence which is not purely referential, and let t' be a term co-referential with t but such that A(t) and A(t') have opposite truth values. Consider the existential quantification $\exists x A(x)$. This asserts, Quine would say, that there is some object satisfying the condition A. But it is unclear what this object is. It is not the object t, for if t satisfies the condition A, then so will t', t and t' being the same object. But by hypothesis if A(t) is true, A(t') is false, and vice versa. But if the object is not t, then clearly $\exists x A(x)$ cannot rightly be inferred from A(t), and nor can A(t) instantiate $\exists x A(x)$. In short, quantification into an opaque context is not coherent. (cf 1953, pp.179-171; 1961, p.148)

This argument depends on the incorrect assumption that the only interpretation of quantification is the objectual one. The argument fails to carry through if quantification into opaque contexts is understood substitutionally: the substitutional interpretation of existential quantification has no implications concerning the behaviour under substitution of a term occupying a position accessible to a quantifier.

Barcan Marcus has proposed a substitutional interpretation of the quantifiers of her Lewis-style language for quantified modal logic in order to avoid Quine's argument that quantification into referentially opaque contexts is incoherent. (1961, p.88ff) In section 4, below, it is argued, however, that it is not correct to interpret all the quantifiers of a language substitutionally. If the semantics is to be adequate, some quantifications have to be treated objectually. Thus Barcan Marcus' proposal for avoiding Quine's argument is not satisfactory; and for the same reason the semantics *S*1 is not a complete answer to Quine's charge that quantification into first grade modal contexts is incoherent. In section 5, below, *S*1 is reworked such that the quantifiers of the basis language L_o are interpreted objectually, whilst the substitutional interpretation is retained for all metalinguistics quantifiers. (Such a semantics is termed semi substitutional.) But it seems that this sort of approach is not rightly available to those who view modal operators as connectives and do not use a stratified syntax. On that approach, it would be purely ad hoc to insist that a quantifier prefixed to an open sentence containing a modal operator should be treated substitutionally, whilst a quantifier prefixed to a non-modal open sentence should be treated objectually. No such ad hocness attaches to our approach, where the use of different interpretations of the quantifiers coincides with the distinction between language and metalanguage. Here, then, is a further reason for preferring to do quantified modal logic at the first grade of modal involvement. The semantics of quantification into first grade modal contexts is straightforward, whereas the semantics of quantification into third grade modal contexts is not.

There are two ways in which one may seek to retain an objectual semantics for classical modal logic without running foul of Quine's argument that objectual quantification into referentially opaque contexts is incoherent. Firstly, one may deny that modal contexts are referentially opaque; or, secondly, one may exclude from the ontology of modal logic "all objects which --- are nameable by names which fail of interchangeability in modal contexts. To do so would be to sweep away all examples indicative of the opacity of modal contexts." (Quine, 1961, p.150)

This latter course was originally suggested by Church (1943) in response to Quine's first presentation of his argument against quantification into modal contexts (Quine, 1943). The problem with the approach is that it tends to lead to a collapse of modal distinctions, as Quine has pointed out (1960, pp.197-198). The approach amounts to retaining in the ontology of modal logic only those objects such that any two conditions uniquely determining one of them are necessary equivalent, i.e. only those objects x such that $\forall y(A(y) \equiv (y = x)) \wedge \forall y(B(y) \equiv (y = x)) \supset \Box\forall y(A(y) \equiv B(y))$ (where $\equiv$ symbolises material equivalence). But an easy proof (ibid p.198) establishes that a consequence of this condition is $A \supset \Box A$, for every formula A.

Carnap (1947) has presented a modal logic with its ontology circumscribed in the manner under discussion, the variables of the logic ranging over what Carnap calls "individual concepts" (ibid, p.47). $A \supset \Box A$ is not in fact derivable in Carnap's system. Føllesdal (1968) tells us that the reason for this is that Carnap restricts his syntax so that no definite description may contain a modal operator. He gives

no justification for this restriction, merely remarking that it is imposed "in order to avoid certain complications, which cannot be explained here" (1947, p.184). As Føllesdal remarks, if one lifts the restriction, one sees what the complications are: modal distinctions collapse. (1968, p.178) In the absence of a semantical justification of this merely ad hoc restriction then, Carnap has failed to show how to quantify objectually into referentially opaque modal contexts.

The remaining possibility for retaining an objectual semantics for quantification into modal contexts without running foul of Quine's charge of incoherence is simply to deny, contra Quine, that modal contexts are referentially opaque. This approach was first suggested by Smullyan (1948), and has been followed by a large number of modal logicians.

Quine asserts the referential opacity of modal contexts on the basis of the truth of

(1) $\Box(9 > 7)$

and (2) 9 = the number of the planets

and the falsity of (3) □(the number of the planets > 7) (1961, pp.143-144). That is, the occurrence of 9 in (1) is not purely referential, since substitution of the co-referential term "the number of the planets" produces the falsehood (3); and so prefixing □ to a sentence produces a referentially opaque context, since the occurrence of 9 in 9 > 7 is purely referential. However (to slightly vary Smullyan's argument against Quine's example: cf (1948), page 35), if the falsity of (3) is to show that the open sentence $\Box(x > 7)$ cannot coherently be construed as imposing a condition on objects, then (3) must be read as:

(4) The number of the planets satisfies the condition $\Box(x > 7)$

Yet (4) is true: the number 9, however it is specified, satisfies the condition that □(x > 7). (3) is false only if it is read as:

(5) It is necessary that the number of the planets satisfies the condition (x > 7).

Clearly it is the combination (1) true, (4) false, rather than (1) true, (5) false, which would show that the occurrence of x in □(x > 7) is not coherently accessible to an objectually interpreted quantifier. But the former combination does not obtain, so no objection to quantifying into modal contexts has been adduced.

The important point to which Smullyan has drawn attention is that (3) is ambiguous, according as the descriptive phrase occurring in it is assigned wider scope, to yield (4), or narrower scope, to yield (5). In Russell's theory of descriptions, the truth value of a sentence is indifferent to the scope of any description occurring it except when the description lacks denotation. In (1948), Smullyan presents a generalisation of Russell's theory such that the scope of a non empty description is relevant to the truth value of a modal sentence such as (3).

An alternative way of dealing formally with the ambiguity of (3) is presented by Thomason and Stalnaker (1968), who assign scope to the modal operator □ itself. When □ is assigned wider scope, (3) is rendered by (5), and when □ is assigned narrower scope, (3) is rendered by (4). Thus Quine's argument that prefixing □ to a sentence produces a referentially opaque context is blocked in a general fashion: the blocking does not depend on the fact that (3) contains a definite description. Indeed the approach can, so far as I can see, be applied straightforwardly to modal sentences containing no singular terms other than variables of quantification - an important generality in view of Quine's thesis that

singular terms other than variables of quantification are eliminable (1960, pp.176-185).

It seems, then, that Quine's assertion that all modal contexts are referentially opaque is incorrect. The context $\Box A(x)$ is opaque only if $\Box$ is assigned wide scope. Thus where $\Box$ is assigned narrow scope in $\Box A(x)$ the objectually interpreted quantification $\exists x \Box A(x)$ is not incoherent. However, the modal logician with a preference for a Lewis-style syntax and an objectual interpretation of quantification is not yet out of the wood. As Quine says "a reversion to Aristotelian essentialism is required if quantification into modal contexts is to be insisted on. An object, of itself and by whatever name or none, must be seen as having some of its traits necessarily and others contingently." (1961, p.155) "More formally, what Aristotelian essentialism says is that you can have open sentences - 'Fx' and 'Gx' - such that $\exists x((\text{nec } Fx) \wedge Gx \wedge \sim \text{nec } Gx)$." (1953, p.174) As a pertinent example, we have $\exists x(\text{nec}(x > 5) \wedge$ there are just x planets $\wedge \sim$nec (there are just x planets)). (ibid) The number nine, on the essentialist view, possesses the trait of being greater than five necessarily, or essentially, and the trait of being the number of the planets contingently, or accidentally. "We can see pretty directly that any quantified modal logic is bound to show such favouritism among the traits of an object; for surely it will be held, for each thing x, on the one hand that necessarily $(x = x)$, and on the other hand that $\sim$ necessarily $(p \wedge (x = x))$, where 'p' stands for an arbitrary contingent truth." (1961, p.155)

Quine finds the doctrine of Aristotelian essentialism odious, and suggests that Carnap, and Lewis himself, would also repudiate the doctrine.

However many recent philosophers of modal logic have found the doctrine acceptable, for example Barcan Marcus (1971), Brody (1973), Plantinga (1970). I myself have been unable to find any clear reasons for either accepting or rejecting the doctrine, and have come to suspect that the doctrine is a purely metaphysical one, which cannot be either established or rejected by means of linguistic analysis. If this is so, then an objectual interpretation of quantification into modal contexts makes the truth of sentences of quantified modal logic depend on the truth of a purely metaphysical doctrine. And this seems to me to be a reductio ad absurdum of the objectual interpretation. Quantified modal logic simply is not such a mysterious business as the objectual interpretation makes it out to be.

Our discussion of Quine's objections to quantified modal logic at the third grade of modal involvement has, then, produced a case for the metalinguistic approach to quantified modal logic developed in the preceding chapter. A purely substitutional interpretation of the quantifiers of third grade modal logic is unsatisfactory (such a semantics being unsatisfactory in general, as argued in the following section); the use of a semi substitutional interpretation is merely ad hoc, there being no proper justification for interpreting some occurrences of an object language quantifier objectually, and some substitutionally; and an objectual interpretation encumbers quantified modal logic with too heavy a load of metaphysical baggage. In the case of ramified modal logic, on the other hand, a semi substitutional interpretation of quantification is well motivated, and since all metalinguistic quantifiers are interpreted substitutionally, the semantics carries no unwelcome metaphysical commitments.

The metalinguistic context □A(x) is referentially opaque. □A(x) is not a condition on objects: rather, (□A(x))t/x is true for some specifications t of an object, false for others, and the substitutional interpretation of existential quantification into such a context reflects that fact. We operate with what Quine calls a non-Aristotelian view of necessity: in ramified modal logic "necessity resides in the way in which we say things, and not in the things we talk about" (1953, p.174).

4. Objectual Quantification Reconsidered

Quine's chief objection to a substitutional interpretation of quantification is that such a semantics severs truth from reference. For example, in (1961a) he writes: substitutional quantification "abstracts from reference altogether. Quantification ordinarily so called is purely and simply the logical idiom of objective reference. When we reconstrue it in terms of substituted expressions rather than real values [of variables], we waive reference. We preserve distinctions between true and false, as in truth-function logic itself, but we cease to depict the referential dimension." (pp.102-103) For which elements of a substitutionally interpreted language might carry the burden of reference? Not the existential quantifier: "substitutional quantification makes good sense, explicable in terms of truth and substitution, no matter what substitution class we take - even that whose sole member is the left hand parenthesis. To conclude that entities are being assumed that trivially, and that far out, is simply to drop ontological questions." (1968, p.106) Nor is existential import carried by the names of the language, in terms of which quantifications will most naturally be cashed out. Any name N may occur coherently in the context "N does not exist",

a sentence which would be absurd if mere namehood were enough to indicate existential import. (1968, p.93) To be sure, if we have objectually interpreted quantifiers to hand we can spot the constructions within which singular terms do occur as referring expressions: N occurs in purely referential position in the context A(N) just in case N can coherently be replaced by a variable bound to a quantifier prefixing the whole context. (1960, p.142ff, p.166; 1961, p.144ff) However, substitutional quantification itself brings no way of distinguishing between referential and non-referential occurrences of names, for any item is always accessible to variables of quantification. Finally, what of demonstratives: are they to be the sole reference-carrying devices of a substitutionally interpreted language? Without wanting to put words into Quine's mouth, perhaps the line to be taken here is that demonstratives can be regarded as unanalysable only on pain of circumscribing our understanding of the apparatus of language, and once analysed as quantificational structures as Quine has suggested (1960, pp.163,184), they carry no existential import unless a substitutionally interpreted quantifier can be shown to do so. Moreover, are we not entitled to leave demonstratives out of account? For substitutional quantification is supposed to make sense for a language containing no singular terms other than grammatical names and variables of quantification.

These comments are of course intended to apply to a natural language (albeit one partly regimented), and perhaps Quine would agree that existential import <u>is</u> carried by the individual constants of a formal language countenancing in principle no empty singular terms, and, consequently, that a substitutional semantics for this language does not divorce truth from reference. Each member of our hierarchy of languages

is intended to be such a language. But even so, a problem remains. Taking the individual constants of the formal language as the substitution class may well produce a semantics which correctly marries truth and reference, but except in the special case where every individual to be talked about is designated by a constant of the language, this semantics will generate the wrong truth conditions for quantified sentences (as remarked in section 1.1, above). This problem has been overcome in the formal semantics *S*1 by placing individual variables of the language alongside individual constants in the substitution class. But then once again the semantics "ceases to depict the referential dimension", or at least does so only partly. A free variable does not serve to pick out a particular individual in the domain of discourse.

Our hierarchy of entailment systems is formulated within a hierarchy of metalanguages. Each of these languages has a distinct set of logical symbols, and, in particular, the quantifiers of each language are peculiar to it. Hitherto the quantifiers of each language in the hierarchy have been given the same interpretation. However, there is no incoherency involved in giving different interpretations to the quantifiers at different levels, and this is the procedure we adopt in order to solve our present problems with reference. The quantifiers of the basis language L_o will be interpreted objectually, the whole language being given a classical Tarski-style semantics. No problem about objectual quantification into first grade modal contexts arises, since L_o is devoid of modal predicates. Quantification in the higher order languages is treated substitutionally, as before. Every sentence of any language of the hierarchy is evaluated ultimately in terms of its zero degree subformulae,

and so reference is transferred by osmosis to the modal sentences occupying the upper levels of the language hierarchy.

A semantics which interprets L_o quantifiers objectually and metalinguistic quantifiers substitutionally will be called semi-substitutional. Such a semantics will contain two pairs of clauses for the quantifiers, the first for quantifiers occurring within formulae of L_o, and the second for higher order quantification. In the next section a semi-substitutional reworking of the semantics *S*1 is presented.

5. The Semantical Theory *S*2

A model m is an ordered quadtuple <D, s, f, F>, where D is a non empty domain of individuals, s is a denumerable sequence of members of D (each member of D occurring at least once in s), F is a set of functions each of which assigns a set of n-tuples of members of D to every n-place predicate variable of L_o, and f is a member of F. The set of all such ordered quadtuples is designated Mod.

The individual assigned in m to the i^{th} individual variable of L_o, x, is designated m(x). m(x) is set as the i^{th} member of s (which will often be written s(x) when it is understood that x is the i^{th} variable). m(P), the assignment in m to the predicate variable P, is the set f(P).

The pair <D, s> is called the basis of m. The cardinality of a model basis <D, s> is the cardinality of D, and the cardinality of a model is the cardinality of its basis.

A predicate variant of a model <D, s, f, F> is any model <D, s, g, F>. (Note that g is a member of F, by the definition of a model). Mod_m is written for the set of predicate variants of a model m.

An i-variant of s is any sequence of members of D differing from s in at most the i^{th} place. An i-variant of a model <D, s, f, F> is any quadtuple <D, s', f, F>, where s' is an i-variant of s. Notice that an i-variant of a model will not in general itself be a model, since s' may not exhaust the members of D. An i-variant of a model which is not itself a model will be called a quasi model.

m(A) = 1 (m(A) = 0) is written for: the formula A is true (false) in the model or quasi model m. Truth in a model or quasi model is defined by recursion on the complexity of formulae, as follows:

(i) $m(P(x_i, ---, x_j)) = 1$ iff $\langle m(x_i), ---, m(x_j)\rangle \varepsilon\ m(P)$, where $P(x_i, ---, x_j)$ is an atomic formula, P is a predicate variable, x_i is the i^{th} variable of L_o, ---, x_j is the j^{th} variable of L_o

(ii) $m(\sim A) = op(m(A))$

(iii) $m(A \vee B) = max(m(A), m(B))$

(iv) $m(A \wedge B) = min(m(A), m(B))$

(v) $m(A \supset B) = max(op(m(A)), m(B))$

(vi) $m(\forall xA) = 1$ iff $m'(A) = 1$ for every i-variant m' of m, where x is the i^{th} variable and $\forall xA\ \varepsilon\ WFF_o$

(vii) $m(\exists xA) = 1$ iff $m'(A) = 1$ for some i-variant m' of m, where x is the i^{th} variable and $\exists xA\ \varepsilon\ WFF_o$

(viii) $m(*A) = m(A)$

(ix) $m(\Gamma \rightarrow \Delta) = m(\hat{\Gamma} \rightarrow \check{\Delta})$

(x) $m(\rightarrow A) = m\ ((A \supset A) \rightarrow A)$

(xi) $m(A \rightarrow) = m(A \rightarrow \sim(A \supset A))$

(xii) $m(\rightarrow) = 0$

(xiii) $m(A \rightarrow B) = 1$ only if $m((A)t/u \rightarrow (B)t/u) = 1$,

where t, u are any variables.

(xiv) $m(A \rightarrow B) = 1$ iff for every predicate variant of m, w, if $w(A) = 1$, then $w(B) = 1$.

(xv) $m(\forall xA) = 1$ iff $m((A)t/x)$ for every variable t,

where $\forall xA \in WFF_{n>o}$

(xvi) $m(\exists xA) = 1$ iff $m((A)t/x) = 1$ for some variable t,

where $\exists xA \in WFF_{n>o}$.

S2 assumes for simplicity that the language is devoid of singular constants. Dealing with a language which does contain individual constants involves a merely routine complication of the definition of a model.

Apart from its semi-substitutional nature, *S2* is essentially similar to the Kripke semantics described in section 1.2, above. A model represents a possible world, and the set of predicate variants of a model m, Mod_m, represents the set of worlds possible relative to m. A pair (m, Mod_m) corresponds to a Kripke model (G, K). Clause (xiv) gives to $\Box$ essentially the same treatment as it receives in the Kripke semantics.

A complete assignment in the Kripke semantics is a union of three functions, one assigning elements of the domain to individual variables of the language, a second assigning n-tuples of elements of the domain to n-adic predicate variables, and a third assigning truth values to formulae. We have preferred to keep these three functions separate, incorporating the first two into the model itself (via s and f, respectively), and defining the third with respect to particular models in the Tarskian fashion.

5.1 Rigid designators

A singular term is a rigid designator, in Kripke's phrase, just in case it denotes the same object in every possible world in which that object exists (Kripke, 1971, pp.145-146).

In the semantical theory *S*1 it has to be assumed that all singular terms of the language, open and closed, are rigid designators, lest the theory yield the wrong truth conditions for sentences exhibiting quantification into modal contexts. For example, *S*1 counts $\exists x \Box A(x)$ as true in some member of a set of intended valuations V just in case there exists a term t such that A(t) is true in every member of V. But clearly if t is not a rigid designator the truth of A(t) in every valuation will provide no inducement to assent to $\exists x \Box A(x)$. Since a purely substitutional semantics makes no use of the notion of a domain of quantification, it is impossible to state formally within such a semantics that a term is a rigid designator. This constitutes a further shortcoming of purely substitutional modal semantics. If the availability of rigid designators constitutes an important semantical fact, it ought to be possible to say so in the semantical theory.

In the present semi-substitutional semantics it is not required that terms should be rigid designators in the sense given above, i.e. denote the same individual in every model with that individual in its domain. This requirement has to be satisfied only if all models are considered in the evaluation in a given model of a formula exhibiting quantification into a modal context. The correct interplay between quantifiers and modalities is achieved in the present set up provided merely that all terms designate rigidly in every model in the set of predicate

variants of a model m, for all models m. That is, it must be the case that for every model m, every term t, every m_1, m_2 in Mod_m, $m_1(t) = m_2(t)$. Since every model in Mod_m has the same basis, this requirement is met; and it was, of course, the desire to meet this condition which motivated the present treatment of the assignment to a free variable in a model.

5.2 The equivalence of *S*1 and *S*2

*S*1 and *S*2 are equivalent in the sense that a formula is valid in *S*1 just in case it is valid in *S*2. This fact is established in Appendix II, section 3.

Since ramified modal logic is complete with respect to *S*1 (section 2, above), and *S*1 and *S*2 are equivalent, it follows that ramified modal logic is also complete with respect to *S*2.

6. Epistemological Problems with Possible Worlds

Kripke interprets the complete assignments of his (1959) semantics as complete descriptions of worlds, possible or actual (vide section 1.2, above). The models of *S*2 may be similarly interpreted. The objects in the domain of a model represent the individuals existing in the possible world described by the model, and the definition of the truth predicate recursively specifies all the sentences true in that possible world. Intuitively, *S*2 counts a sentence A(t) as necessary in a world w just in case A(t) is true in every world possible relative to w. If this account of necessity is to have any appeal, the same individual must be assigned to t in each world possible

relative to w.[1] Thus, it seems, our semantics requires that we should be able to recognise the individuals existing in one possible world (in the domain of one model) as the same individuals when they appear in another possible world (in the domain of another model). The notion of an individual existing in two or more distinct possible worlds is often called the notion of transworld identity. A number of philosophers have maintained that the notion is a dubious one, and so, by implication, that any semantics using it is unsatisfactory. We proceed to review some of the problems attaching to transworld identity.

Chisholm (1967) has raised some epistemological problems engendered by the notion of transworld identity. Call our present world w_1. A world possible relative to w_1 may be specified as follows: "consider some one of the entities of this world, alter its description slightly, adjust the descriptions of the other entities in the world to fit this alteration" (p.1). Suppose we start with Adam and allow him to live for 931 years instead of only 930, and then alter the descriptions of the other entities in the world to fit the change made to Adam (so, for example, Eve will now have the property of being married to a man who lives for 931 years). Call this possible world w_2. The question is: is the Adam of w_1 the same person as the Adam of w_2? And how are we to decide? But, claims Chisholm, surely we can never know that just one individual is involved here. How could we ever know that the man having

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1. David Lewis (1968) has suggested that this requirement is too strong. In his semantics he requires just that in every world w' possible relative to w, t should denote that individual in w' which is the counterpart of the denotation of t in w. However, this sort of approach seems to me to block the inference from $\Box A(t)$ to $\exists x \Box A(x)$: if the truth of $\Box A(t)$ in w is made to depend on the truth of statements about the counterparts of t in the worlds possible relative to w, then surely $\Box A(t)$ can be true in w, whilst it is false in w that there is some object (in w) satisfying the condition $\Box A(x)$. (Note that Lewis adopts an objectual interpretation of quantification in his semantics: vide page 114.) Further cogent criticisms of Lewis' approach are presented in Feldman (1971).

properties P_1, ---, P_n and living for 930 years in w_1 is the same man as the man having properties P_1, ---, P_n and living for 931 years in w_2? (pp.2-3)

Again, suppose we specify another possible world w_3 by giving Noah some of the properties possessed by Adam in w_1, and giving Adam some of the properties possessed by Noah in w_1 (and then accommodate the rest of the world to what we have done). Suppose we continue this through a sequence of worlds until we arrive finally at a possible world w_n which would seem to be exactly like our present world w_1 except for the fact that the Adam of w_n may be traced back to the Noah of w_1 and the Noah of w_n may be traced back to the Adam of w_1. (p.3) "Should we say of the Adam of w_n that he is identical with the Noah of w_1 and should we say of the Noah of w_n that he is identical with the Adam of w_1? In other words, is there an x such that x is Adam in w_1 and x is Noah in w_n, and is there a y such that y is Noah in w_1 and y is Adam in w_n? And how are we to decide?" (pp.3-4)

Suppose that somehow we arrive at an affirmative answer to this question. The object that is Adam in w_1 is Noah in w_n, and the object that is Noah in w_1 is Adam in w_n. Thus w_1 and w_n are distinct. But how could one ever know that they are distinct? "How is one to tell the difference between the two worlds w_1 and w_n? Shall we say that, though they are diverse, they are yet indiscernable from each other?" (p.4)

These three puzzles, then, have to be solved by anyone who would maintain that we can recognise the individuals existing in one possible world as the same individuals when they appear in another possible world.

A further epistemological problem with transworld identity is as follows. Consider the real world w_1, an object o existing in the real world, and a possible alternative to w_1, w_2. Suppose that w_2 contains objects b_1, ---, b_n each of which resembles o in various ways and to a greater or lesser degree. Each b_i has some properties that o possesses, but not all of them, and each b_i has some properties that o does not possess. How are we to decide which b_i is the object o of w_1 as it appears in w_2? How many properties of o must b_j possess before we are prepared to say that b_j is the transworld identical of o? Will some sorts of properties count more than others in selecting the transworld identical of o? If so, which? Are there any properties not possessed by o whose possession by b_j disqualifies b_j in our search for the transworld identical of o? If so, which? Clearly the recognition of an object as the same object in different possible worlds is a formidable task.

Brody (1973) has suggested that the problem with transworld identity is not an epistemological problem, but rather a problem of what meaning can be assigned to the notion of identity in this context. "Even if one could not tell in many (or even all) cases --- which, if any, object in the actual world is identical with [some object in a possible world] it would not follow, unless one was using some very suspicious verificationist assumption, that there is something wrong with the concept of identity through possible worlds. The real problem is that it is unclear as to what it means for an actual entity e_1 to be identical with an entity e_2 that exists in a possible world." (p.357) Brody argues that it is unclear what a claim of transworld identity amounts

to as follows. "The identity of an actual physical object with a possible physical object does not --- involve any requirement of spatio-temporal continuity, so we must be employing a different conception of identity [to our usual conception of identity of physical objects] when we talk about their identity. And similar arguments can be constructed to show that the identity of actual entities that are not physical objects cannot be the same type of identity as the identity of one of those actual entities with a corresponding possible entity." (ibid.)

I agree with Brody that it is difficult to see what might be meant by a claim of transworld identity. However, I do not agree that there is no epistemological aspect to the problem of transworld identity. For a semantics which uses the notion of transworld identity not only presupposes that the notion makes sense, but also presupposes that we can tell which object in a possible world is identical with some object in the actual world. If it is epistemologically impossible to do this, we will be unable to assign a truth value in any world to a formula of the form $\exists x \Box A(x)$.

Quine has taken exception to the very notion of a non-actual entity. "Take, for instance, the possible fat man in that doorway; and, again, the possible bald man in that doorway. Are they the same possible man, or two possible men? How do we decide? How many possible men are there in that doorway? Are there more possible thin ones than fat ones? How many of them are alike? Or would their being alike make them one? Are no two possible things alike? Is this the same as saying that it is impossible for two things to be alike? Or, finally, is the concept of

identity simply inapplicable to unactualised possibles? But what sense can be found in talking of entities which cannot meaningfully be said to be identical with themselves and distinct from one another? These elements are well-nigh incorrigible." (1948, p.4)

These problems with transworld identity infect not only possible world semantics for modal logic, but also the possible world semantics proposed by Hintikka (1962) for epistemic and doxastic logics. In (1975) Hintikka sets out a solution of these problems. Individuals are to be replaced in a possible world semantics by <u>individuating functions</u>. "What becomes of an individual is now an atemporal and non-spatial 'world line' through several possible worlds. In equivalent terms each individual in the full sense of the word is now essentially a function which picks out from several possible worlds a member of their domains as the 'embodiment' of that individual in this possible world or perhaps rather as the role which that individual plays under a given course of events. These functions I have called individuating functions." (p.30) Thus the whole notion of transworld identity is excised from a possible world semantics. The semantics no longer operates with the notion of the same individual existing in distinct possible worlds w_1, ---, w_n. In its place is the notion of a particular individuating function having a value at each of w_1, ---, w_n. Individuating functions are partial functions: they may fail to take a value at some worlds. This corresponds in the old terminology to an individual existing in some possible worlds and failing to exist in others.

Hintikka is unhappy with, in the semantics, "envisaging in purely abstract terms some unspecified system of world lines" (i.e. some unspecified

system of individuating functions). (p.36) Rather if the semantics is to show an "infinitesimal degree of realism" the semanticist should envisage a set of individuating functions as actually constructed, relative to a given set of possible worlds. (pp.33-34, 36) How, then, are the individuating functions to be constructed, given a set of possible worlds? "The world lines cannot be drawn so as to connect blank worlds. They depend on the properties and relations that obtain in the several possible worlds we are considering. They come about, as it were, only through comparisons between these possible worlds." (p.33) That is, the set of ordered pairs $\langle w_i, a_k \rangle$ constituting an individuating function (where w_i is a world and a_k is an element of the domain of w_i) must be constructed by comparing the given possible worlds and determining which member of the domain of w_1 is the manifestation in w_1 of the individual I, which member of the domain of w_2 is the manifestation in w_2 of the individual I, and so on. In this way the individuating function which, as it were, codifies the individual I, is built up.

In a restricted sense, then, we have returned to one aspect of the problem of transworld identity: we require criteria for determining when a member of the domain of w_1 is the embodiment of the same individual embodied by a member of the domain of w_2. On this matter Hintikka writes: "one might --- think of one's identificationary task as being comparable to identifying the originals of the characters of a roman à clef.[2] --- It is clear that our actual criteria of identifying a character in a roman à clef are multiple and not without some vagueness. --- It is nevertheless also clear what the main criteria of cross-identification are. They might be labelled 'continuity plus similarity', for the considerations that go beyond continuity [i.e. spatio-temporal continuity]

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2. The Concise Oxford Dictionary informs me that a roman à clef is a novel in which real persons or events appear in disguise.

seem to rely largely on the sharing of properties and relations." (pp.29-30)

Perhaps this proposed criterion of cross-world identification seems a little thin. However, Hintikka seems to think the criterion will suffice for the construction of the individuating functions associated with the set of worlds alternative to the actual world required by his semantics for epistemic and doxastic logic, viz all the worlds (epistemic alternatives) compatible with b's knowing what he knows in the actual world, or all the worlds (doxastic alternatives) compatible with what b believes in the actual world. (p.28) Clearly these sets of worlds are very limited compared to the set of logically possible alternatives to the actual world, and the problem of cross-identification of objects is much less severe in the former case than in the latter. Hintikka adopts the position that in the latter case the problem is sufficiently severe to result in the failure of a possible world sematics for the alethic modalities. He writes: "If world lines are to be drawn on the basis of similarities between worlds and of their intrinsic regularities, then this whole enterprise will break down when very dissimilar and irregular worlds have to be considered. And this is the case with the logical modalities." (p.36)

Hintikka's conclusion from this is that "Quine is thus seen to be right in his suspicion of quantified logical modalities. Quantifiers and logical modalities simply do not mix." (p.37) (Although Hintikka would say that a possible world semantics for the epistemic and doxastic modalities is satisfactory, and so these modalities will coherently mix with quantifiers.) However, it seems to me that Hintikka's conclusion is unwarranted. All of the problems we have been considering concerning possible individuals

and transworld identity arise only on a rather dubious interpretation of the possible world semantics. In the following subsection we explain what this dubious attitude is, and suggest a way of viewing the semantics *S2* such that the problem of transworld identity does not arise.

6.1 Epistemological respectability regained

In the foregoing discussion of transworld identity it has been presupposed that the entities occurring in the domains of models are, from the point of view of the semanticist occupying the real world, non-actual entities. The semanticist is then seen as faced with the epistemologically formidable task of identifying an actual object from his own real world with some non actual object in the domain of each of the possible worlds in question. However, this picture of things seems to me to be completely wrongheaded. The semanticist is not, as it were, presented gratis with a set of possible worlds, which he then uses in his semantics. Rather, he constructs for himself a set of models, and he is at liberty to select whatever sets of entities he likes as the domains of the models. Sets of real entities from the real world is the selection I would advocate. Thus the same object will appear in the domains of various models, in the straightforward and ordinary sense of "same". In such a context the notion of transworld identity is not in the least perplexing.

If all models have domains drawn from the actual world any object in any model will be exactly the same as that object as it is in the real world. So how, it may be asked, can any model be a possible alternative

to the actual world? This kind of question is based on a wrong conception of the nature of a formal semantics. A semanticist engages merely in the construction of an extensional model theory, and not in the somewhat grander task of creating a universe of worlds which are possible alternatives to the real world. In extensional model theory, a set of objects is chosen as the domain of a model, a function is defined which assigns members of the domain to singular terms of language, a second function is defined which assigns n-tuples of members of the domain to predicate constants of the language, and a third function is defined which assigns truth values to formulae of the language. Thus the sentence "John is fat" may be true in a model. But clearly it does not follow that the object o assigned to the term "John" in the model is in actual fact fat. For example, my friend, John, whom I assign in the model to the term "John", may be a thin man. In the counterfactual situation represented by the model John is a fat man, but the object 'John' used in the model to represent John as he appears in this counterfactual situation, possesses only properties which John actually possesses. Or again, the object o assigned in the model to the term "John" might be a number, if a situation is being modelled numerically - as, for example, surveyors preparing data for use by a computer model sections of the earth's surface in terms of numbers.

Properly viewed, 'worlds' of a possible world semantics are mere representations or modellings of possible worlds or counterfactual situations. The objects used in the modelling need have none of the properties of the objects which they represent. Even the model designated in the semantics as the actual world may be a mere representation of the actual world, analogous to the surveyor's model of the earth in terms of numbers.

An objection which might be made at this stage is that this way of thinking trivialises possible world semantics. The semantics remains useful for merely formal purposes, completeness and independence proofs and the like, but its philosophical importance evaporates. Kripke's insight was that modal logic could be made sense of as talk about possible worlds, and it is that feature of the possible world semantics which has captivated philosophers: to use a vague idiom, the possible world semantics is seen as providing a theory of meaning for modal sentences. But write full-blooded possible worlds out of the semantics, the objection goes, and its value as a sense-making instrument is lost.

An objection along these lines, however, misses the point. Full-blooded possible worlds have not been "written out" of the semantics. Full-blooded possible worlds are precisely what the models of the semantics model. I feel that much of the criticism which has been made of the possible world semantics has been engendered by the temptation to equate the model with the thing modelled. The whole problem of transworld identity arises only if we wrongly picture ourselves as in some sense physically presented with the 'populations' of possible worlds, in the same way in which we are in a sense presented with the domains of the models we work with. Given this picture it is difficult to avoid asking how a member of the population of one possible

world is to be re-identified amongst the population of another possible world. Here is a classic example of a philosophical problem arising as a result of a philosopher's being held captive by a false picture (cf Wittgenstein (1953), sections 109, 115); and in this case the false picture arises, I think, through a tendency to equate the model with what is modelled. We <u>are</u> physically presented, so to speak, with the domains of the models we work with, and if the semantics is to be able to deal with a sentence like $\exists x \Diamond F(x)$ we must be able to re-identify an object occurring in one domain when it occurs in another domain. But we are not physically presented with the things modelled, possible worlds or counterfactual situations: in their case it is out of place to think in terms of the re-identification of objects. Viewed aright, counterfactual situations do not promote questions about transworld identity. As Kripke writes: "If we say 'If Nixon had bribed such and such a Senator, Nixon would have gotten Carswell through', what is *given* in the very description of that situation is that it is a situation in which we are speaking of Nixon". (1971, p.148)

False pictures corrected, then, the problem of transworld identity reduces to the problem of how to construct models with overlapping domains such that given any two models m_1 and m_2, any object o_1 in the domain of m_1, and any object o_2 in the domain of m_2, the semanticist can tell whether or not o_1 is the same object as o_2.

In the specification of the semantical theory *S*2, above, it was left open how the domains of the models used in the theory might be built up. The following theorem shows that in establishing the validity or invalidity of a formula in the semantics we may choose any denumerable set and consider only models whose domains are subsets of this set. In particular, we may choose any denumerable set of objects occurring in the real world. In that case our semantical theory will not commit us to a disreputable ontology of non-actual objects. The set of natural numbers may prove a useful choice for this denumerable set. The actual state of affairs and every possible state of affairs will then be modelled in the semantics in terms of numbers and their relations. This choice provides a particularly easy criterion of transworld identity within the semantics: an object o_1 in the domain of a model m_1 is the same object as an object o_2 in the domain of a model m_2 just in case o_1 and o_2 are the same number.

Theorem Let μ be any denumerable set. Then a formula A is valid in *S*2 just in case A is true in every model whose domain is a subset of μ.

The proof of this theorem is given in Appendix II, section 4.

Chapter V

Non Denoting Singular Terms

In this Chapter informal semantical arguments are presented to show that certain formulae provable in ramified modal logic, and validated by the formal semantics of the preceding Chapter, are in fact unacceptable entailments. By allowing that some singular terms of L_o should fail to denote we produce intuitively satisfying reworkings of the semantical theories *S*1 and *S*2 which invalidate the unacceptable formulae. The set of sentences validated by the new semantics is then axiomatised in a ramified way. The axiomatisation is obtained by modifying the hierarchy of systems constructed in Chapter III in such a way as to countenance the presence in the language of non denoting singular terms.

1. The Barcan Formula and its Relatives

The following three formulae are provable in any system $ML_{n\geq 2}$:

(1) $(\Diamond\exists xA(x)) \rightarrow (\exists x\Diamond A(x))$

(2) $(\forall x\Box A(x)) \rightarrow (\Box\forall xA(x))$

(3) $(\exists x\Box A(x)) \rightarrow (\Box\exists xA(x))$

(1) and (2), which are equivalent, are known indiscriminately in the literature as the Barcan Formula. (In the first instance the name was coined by Prior (1956) for (1), which as noted in Chapter III section 11, figures as an axiom in Barcan Marcus' (1946) addition of quantifiers to S2.) (3) is the existential analogue of (2). The converses of (1) and (2) are aso ML provable, and the following discussion applies mutatis mutandis to them also. The converse of (3) is not ML provable. This formula does not express a correct entailment, as can be seen intuitively by taking A(x) as: x is the last word of sentence S. Clearly $\Box\exists xA(x)$ is then true, whilst $\exists x\Box A(x)$ is false.

1.1 Informal semantical arguments that formulae (1) through (3) are invalid.

Various authors have rightly maintained that (1) through (3) do not express correct entailments.[1] We cite three typical arguments. Prior (1957, p.26) questions (1) on the following ground: suppose it is true that someone will some day fly to the moon, but not anyone who now exists; in this case it is possible that someone will fly to the moon (i.e. the antecedent of (1) holds), but nonetheless it does not follow that there is now someone who will possibly fly to the moon (i.e. the consequent of (1) does not follow). Hughes and Cresswell (1968, p.170) advance a similar consideration against (2): even if everything that actually exists is necessarily A (i.e. the antecedent of (2) holds), this does not preclude the possibility that there might be (or might have been) some things which are not A at all - and in that case it would not be a necessary truth that everything is A (i.e. the consequent of (2) does not follow). Finally, Hintikka (1961, p.124) writes in criticism of (3): we may admit for the sake of argument that every wheel is necessarily round, yet we would not therefrom conclude that since there happen to be wheels in existence, the existence of round objects is a necessary and unavoidable feature of the world.

Each of these arguments invites us to construe quantification into modal contexts objectually. $\exists x \Diamond A(x)$ is read: there is (now) something, x, such that $\Diamond A(x)$; $\forall x \Box A(x)$ as: everything, x, that actually exists is such that $\Box A(x)$; $\exists x \Box A(x)$ as there is something, x, such that $\Box A(x)$. If $\Box$ and $\Diamond$ are construed as connectives these readings are, perhaps, acceptable. If,

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1. Although curiously Belnap (1967) lists (2) as an axiom of EQ, the quantified extension of the Anderson-Belnap system E of entailment. However, (2) does not re-appear in an axiomatisation of EQ given in a 1973 draft of the as yet unpublished volume II of Anderson-Belnap's "Entailment: The Logic Relevance and Necessity".

however, □ and ◊ are construed as semantical predicates, these objectual readings are incoherent (as argued in section 3.1 of the preceding Chapter); and accordingly the above arguments do not apply as they stand to formulae (1) through (3) as they appear in ramified modal logic. Can the arguments be reworked to invalidate (1) through (3) when quantification into modal contexts is interpreted substitutionally? In (1962, pp.257-258) Barcan Marcus seems to suggest that they cannot. However, Barcan Marcus is incorrect if she does intend to imply that any well motivated semantics in which quantification is treated substitutionally will validate the Barcan Formula; and we demonstrate the falsity of that thesis by producing two purely substitutional semantics in which the Barcan Formula fails.

1.2 Substitutional semantics and non denoting singular terms

The same semantical point underlies each of the above three arguments, namely, that each of (1) through (3) may be evaluated in such a way that - to speak in the objectual idiom - the quantifier in the antecedent is assigned a different domain from the quantifier in the consequent. For example, (2) is valid if $\Box\forall xA(x)$ is true in every state of affairs in which $\forall x\Box A(x)$ is true. But $\Box\forall xA(x)$ is true in a state of affairs just in case $\forall xA(x)$ is true in every state of affairs; and so the consequent of (2) may be falsified by the circumstance of $A(x)$ failing of some object not occurring in the domain of quantification assigned to the antecedent. Roughly, we may say that the possibility of falsifying (1) through (3) arises from the fact that objects existing in one state of affairs may fail to exist in another, and we want to construct a semantics capturing this fact in which the quantifiers are interpreted substitutionally.

Prima facie, there is a difficulty in doing this, since unlike classical Tarski-style semantics, a purely substitutional model, or valuation, does not carry with it a domain of objects over which the quantifiers of the language are interpreted as ranging. Thus there is no immediate feature of a substitutional model which corresponds to the set of objects existing in the state of affairs represented by the model. To solve this problem we associate with each substitutional model or valuation a subset of the singular terms of the language under consideration, and count $\forall xA(x)$ true if $A(t)$ is true for every term t in the set, false if $A(t)$ is false for some term t in the set. The idea is that these designated singular terms are all the terms which denote in that valuation, all other singular terms being treated as empty.

An extremely natural formal semantical counterpart to the idea of more or less objects existing in different possible states of affairs can now be found: a valuation v countenances more objects than a valuation v' just in case (i) the designated terms of v' are strictly included in those of v, (ii) if $v'(t_1 = t_2) = 0$ where t_1 and t_2 are designated in v', then $v(t_1 = t_2) = 0$, and (iii) there exists at least one term t designated in v but not in v' such that $v(t = t') = 0$ for every term t' designated in v'. (Notice that we are assuming that designated terms are rigid designators, in the terminology of Chapter IV, section 5.1.) A pair of valuations satisfying these three conditions may be described in the objectual idiom as having different domains associated with them. Such a pair of valuations will be called ontologically distinct.

As Strawson (1950) first canvassed, the question of the truth or falsity of a sentence containing an empty name does not naturally arise. (Strawson himself would say "statement" rather than "sentence", as it is

these former which he treats as the bearers of truth and falsity.) In the formal semantics *S*3, below, we cater for this fact of linguistic practice, making a sentence A truth valueless in a valuation where any singular term in A is undesignated. *S*3 thus provides a semantical characterisation of the logical theory of English prior to any artifical meddling designed to close truth value gaps. In the formal semantics *S*4, the truth value gaps countenanced by *S*3 are closed by the artifice of making any atomic sentence containing a non designated term false, and allowing the truth values of complex sentences to depend on the truth values of their constituents in the usual way. (This is the procedure for eliminating truth value gaps set out by Quine (1950, p.221.)

2. The Semantical Theory *S*3

A valuation v is a function from a subset (possibly empty) F_v of WFF onto {0, 1}. F_v is subject to the restriction that if an atomic formula A does not occur in F_v, then nor does any truth function of A. (Quantifications of A and arrow formulae containing A may occur in F_v.)When a formula A does not occur in F_v, v(A) is said to be undefined and A is said to be truth-valueless in v. D_v is written for the set of terms, open and closed, occurring in the atomic members of F_v (the v-designated terms). When F_v is empty, D_v is empty, and when this occurs we have, in the objectual idiom, quantification over the empty domain. If D_v is empty, any universal quantification is trivially true, and any existential quantification is trivially false.

A set V of intended valuations is defined by recursion on the complexity of formulae.

(0) atomic formulae in the domain of an intended valuation are evaluated arbitrarily (each receiving a truth value).

Clauses (i), (ii), (iii) and (iv), for ~, ∧, ∨ and ⊃ respectively, are as for the sematics *S*1 of the preceding Chapter.

(v) $v(\forall xA) = 1$ if $v((A)t/x = 1$ for every term t in D_v

$v(\forall xA) = 0$ if $v((A)t/x = 0$ for some term t in D_v

$v(\forall xA)$ is undefined otherwise (i.e. in case there is no t in D_v such that A(t) is in F_v, which will happen in case $\forall xA(x)$ contains singular terms, other than bound variables, which are not in D_v).

(vi) $v(\exists xA) = 1$ if $v((A)t/x) = 1$ for some term t in D_v

$v(\exists xA) = 0$ if $v((A)t/x = 0$ for every term t in D_v

$v(\exists xA)$ is undefined otherwise.

Clause (vii), (viii), (ix), (x) and (xi), for *A, $\Gamma \rightarrow \Delta$, $\rightarrow A$, $A \rightarrow$, and $(\rightarrow)$ respectively, are as for *S*1.

(xii) $v(A \rightarrow B) = 1$ only if $v((A)t/u \rightarrow (B)t/u) = 1$, where u is any free variable, t is any term.

(xiii) $v(A \rightarrow B) = 1$ if and only if for all $w \varepsilon V[A, B]$, if $w(A) = 1$, then $w(B) = 1$, where $V[A_1, ---, A_n]$ is written for the set of those members of V in which each of A_1 through A_n receives a truth value.

A formula is <u>valid</u> just in case it is true in every intended valuation.

Different intended valuations represent different possible states of affairs, and a set of intended valuations V represents the set of states of affairs possible relative to any member of V. Since the atomic formulae in any F_v, and in particular formulae of the form $t_i = t_j$, are evaluated arbitrarily by v, there exists at least one set of intended valuations containing at least one pair of ontologically distinct valuations. This fact results in the invalidation of the Barcan Formula in *S*3.

Open as well as closed singular terms may be undesignated in a valuation. Thus, a formula containing no singular terms other than free variables may

be truthvalueless. This feature of the semantics is not essential: it is possible to arrange things so that only closed terms are undesignated and accordingly truth value gaps manifest themselves only in conjunction with closed formulae. However, the present treatment is slightly simpler, and, we claim, not wholly unintuitive, since English sentences containing unbound pronouns - the natural language counterparts of free variables - plausibly lack truth value when the referent of the pronoun is not sufficiently distinguished by context. Moreover, the present treatment allows us the greater generality of quantification over the empty domain.

The restriction on the occurrence of truth functional formulae in F_V ensures that if a constituent of a truth function is truthvalueless, then so is the whole formula. This treatment is not an essential feature of the semantics. It is easy to cater for an intuition that, for example, $\sim A$ holds just in case A is not true, in which case $\sim A$ should be assigned Truth when A is truthvalueless.

A quantified formula is truthvalueless in a valuation in case it contains closed terms or free variables which are not designated in that valuation. $\forall x A(x)$ may be true in a valuation without every substitution instance of $A(x)$ being true in that valuation: it is required just that substitution instances having a designated term as substituens be true. This ties quantification to existence: $\forall x A(x)$ is not to be counted untrue on the ground that $A(t)$ is untrue where t is a non designated ("empty") term.

Clause (xiii) counts $\Box A$ as true just in case A is true in every valuation $w \in V$ such that $w(A)$ is defined; or, in other words, just in case there is no valuation in V in which A is false. *S*3 can be strengthened to *S*3.1 by replacing clause (xiii) by the clause for $\rightarrow$ in *S*1, namely $v(A \rightarrow B) = 1$ iff for all $w \in V$, if $w(A) = 1$, then $w(B) = 1$. In this case $\Box A$ is counted as

true in v just in case A is true in every valuation in V. Prior (1957) in effect allows both kinds of necessity in his system Q, the latter as L and the former as ~M~.

Formula (3) of section 1 is valid in *S*3.1 but invalid in *S*3. We establish this. Let u be an intended valuation, let V be a set of intended valuations containing u, let w be a member of V, and assume $w(\exists x \Box A(x)) = 1$. So $w(\Box A(t)) = 1$ for some $t \varepsilon D_w$. Using *S*3.1, A(t) is true in every valuation in V, so $\exists x A(x)$ is true in every valuation in V, and accordingly $w(\Box \exists x A(x)) = 1$, whence formula (3) is valid. Now let u be an intended valuation, let V be a set of intended valuations containing u, let w be a member of $V[\exists x \Box A(x), \Box \exists x A(x)]$, and assume $w(\exists x \Box A(x)) = 1$. So $w(\Box A(t)) = 1$ for some $t \varepsilon D_w$. Using *S*3, A(t) is true in every valuation in V[A(t)]. But consider a $v \varepsilon V$ such that A(t) is truthvalueless in v and $v(\exists x A(x)) = 0$. In this case $\exists x A(x)$ is not true in every member of $V[\exists x A(x)]$ and so $w(\Box \exists x A(x)) = 0$. Thus formula (3) is invalid.

Formulae (1) and (2) are invalid in both *S*3 and *S*3.1. We demonstrate this in the case of (2) and *S*3. Let u be an intended valuation, let V be a set of intended valuations containing u, let w be a member of $V[\forall x \Box A(x), \Box \forall x A(x)]$, and assume $w(\forall x \Box A(x)) = 1$. Thus for every term t in D_w, for every $v \varepsilon V[A(t)]$, $v(A(t)) = 1$. But consider a $v' \varepsilon V$ such that $v'(\forall x A(x)) = 0$ (i.e. for some term t' in D_v, but not D_w, $v'(A(t')) = 0$). In this case $\forall x A(x)$ is not true in every valuation in $V[\forall x A(x)]$, and so $w(\Box \forall x A(x)) = 0$. Thus formula (2) is invalid in *S*3.

Not all the inference rules of LK and its higher order neighbours remain validity preserving in *S*3 and *S*3.1. Of the quantifier rules, ($\forall$R) and ($\exists$L) are validity preserving in both *S*3 and *S*3.1, whilst ($\forall$L) is validity

preserving in neither and (∃R) is validity preserving in *S*3.1 but not in *S*3. This is to be expected, since it is presupposed in LK that all the singular terms of the language denote. Similarly, the assumption of truth value gaps invalidates certain of the propositional rules of LK. For example, if a disjunction is counted as truthvalueless if one of its disjuncts is so, the following instance of (∨R) is not validity preserving when C is truthvalueless: $(A \rightarrow B) \rightarrow (A \rightarrow B \vee C)$.

3. The Semantical Theory *S*4

We now supply a theory in which the truth value gaps countenanced by *S*3 become closed: any atomic sentence counted as truthvalueless in an *S*3 valuation is counted as false in the corresponding *S*4 valuation. Of the two, *S*4 will be our preferred theory, and when we axiomatise ramified modal logic in such a way that formulae (1) through (3) are not theses, we will axiomatise the logic characterised by *S*4 rather than that characterised by *S*3. This is because we believe with Quine (1960, p.177) that no benefit is to be gained by ccuntenancing truth value gaps in our formal logic. Truth value gaps are "an irksome complication, as complications are that promise no gain in understanding" (ibid).

A valuation is defined as a function onto {0, 1} from the Cartesian product of WFF with any set Υ of sets of terms, open and closed, of L_o. Every pair in WFF × Υ appears in the domain of the valuation, so all members of WFF are evaluated by an *S*4 valuation (unlike an *S*3 valuation, which evaluates a selected subset of WFF only). The idea is that where v is a valuation on WFF × Υ and T ε Υ, v(A, T) is an evaluation of the formula A treating T as all the denoting terms of the language, other singular terms being treated as empty.

A set V of intended valuations is as usual defined by recursion on the complexity of formulae.

Atomic clauses

(i) where A is atomic pairs <A, T> are evaluated arbitrarily provided all terms occurring in A are members of T and A is not of the form (x = y).

(ii) v(A, T) = 0 where A is an atomic formula at least one of whose singular terms is not in T.

At this stage of our investigation it becomes useful to introduce a binary predicate constant "=" into L_o, to be interpreted as the identity predicate. Formulae of the form = (x, y) will be written x = y. Three further clauses are required for atomic formulae of this form.

(iii) $v(t = t, T) = 1$ for every $t \varepsilon T$

(iv) if $v(t_1 = t_2, T) = 1$ and $v(t_2 = t_3, T) = 1$, then $v(t_1 = t_3, T) = 1$

(v) if $v(t_1 = t_2, T) = 1$, then $v(t_2 = t_1, T) = 1$

Note that $v(t_1 = t_2, T) = 0$ in case either or both of t_1, t_2 is not in T.

The clauses for $\sim$, $\vee$, $\wedge$ and $\supset$ are as usual.

Quantifiers

$v(\forall xA(x), T) = 1$ if and only if $v(A(t), T) = 1$ for every $t \varepsilon T$ not occurring bound in A(x)

$v(\exists xA(x), T) = 1$ if and only if $v(A(t), T) = 1$ for some $t \varepsilon T$ not occurring bound in A(x).

If provision is made for T to be empty, every universal quantification is trivially true, and every existential quantification is trivially false. This corresponds to objectual quantification over the empty domain.

The clauses for substitution, $\Gamma \rightarrow \Delta$, $A \rightarrow$, $\rightarrow A$, and $(\rightarrow)$ are as usual.

Entailment

Firstly we introduce some notation. The expression $T[A_1, ---, A_n]$ indicates that all terms other than bound variables occurring in the formulae A_1 through A_n are in the set of terms T. $\Upsilon[A_1, ---, A_n]$ denotes any set of sets of terms each of whose members, T, is such that $T[A_1, ---, A_n]$. Where V is a set of intended valuations, $V[A_1, ---, A_n]$ is written for the set of members of V with domain WFF $\times$ $\Upsilon[A_1, ---, A_n]$. The clause for entailment is then as follows:

$v(A \rightarrow B, T) = 1$ if and only if, if $w(A, S) = 1$, then $w(B, S) = 1$ for every $w \varepsilon V[A, B]$ and every set of terms S occurring in the domain of w.

As in the case of *S*3.1 a stronger clause for $\rightarrow$ may be used, yielding *S*4.1, namely:

$v(A \rightarrow B, T) = 1$ if and only if, if $w(A, S) = 1$, then $w(B, S) = 1$ for every $w \varepsilon V$ and every set of terms S occurring in the domain of w.

$v(A) = 1$ will be written in case $v(A, T) = 1$ for every set of terms T occurring in the domain of v. A formula A is valid just in case $v(A) = 1$ for every intended valuation v.

Formula (3) is invalid in *S*4, but valid in *S*4.1 in the special case where A(x) is atomic. We establish these facts.

Firstly we show that (3) is invalid in *S*4. Let u be an intended valuation, let V be a set of intended valuations containing u, let T be a set of terms in the domain of u, let w be a valuation in $V[\exists x \Box A(x), \Box \exists x A(x)]$, let S be a set of terms in the domain of w, and assume $w(\exists x \Box A(x), S) = 1$. Thus $w(\Box A(t), S) = 1$ for some $S \varepsilon T$, and so $v(A(t)) = 1$ for every $v \varepsilon V[A(t)]$.

Now consider a v' in V[∃xA(x)] such that v'(∃xA(x), U) = 0 for some set of terms U in the domain of v' (i.e. v' ∉ V[A(t)]). In this case it is false that v(∃xA(x)) = 1 for every v ε V[∃xA(x)], and so w(□∃xA(x), S) = 0. Thus u(∃x□A(x) → □∃xA(x), T) = 0, and accordingly formula (3) is invalid in *S*4.

We now show that (3) is valid in *S*4.1 in the special case where A(x) is atomic. Let u be an intended valuation, let V be a set of intended valuations containing u, let T be a set of terms in the domain of u, let w be a valuation in V, let S be a set of terms in the domain of w, and assume w(∃x□A(x), S) = 1. Thus for some t ε S, for every v ε V, v(A(t)) = 1. Since A(t) is by supposition atomic, t must occur in every set of terms in the domain of any valuation in V. Thus for every v ε V, v(∃xA(x)) = 1, and so w(□∃xA(x), S) = 1. Accordingly u(∃x□A(x) → □∃xA(x), T) = 1, whence formula (3) with atomic A(x) is valid in *S*4.1.

Formulae (1) and (2) are invalid in both of *S*4 and *S*4.1. (∀R) and (∃L) are validity preserving in *S*4 and *S*4.1, (∀L) is validity preserving in neither *S*4 nor *S*4.1, and (∃R) is not validity preserving in *S*4, but is validity preserving in *S*4.1 in the special case where A(t) is atomic. All the propositional inference rules of LK are validity preserving in both *S*4 and *S*4.1.

4. The Semantical Theory *S*5

In the preceding Chapter it was argued that no purely substitutional semantics is an adequate semantics for the theory of entailment, since such a semantics (a) severs truth from reference, and (b) lacks the resources to enable the assumptions of the theory concerning rigid designation to be expressed formally. We now proceed to construct the semi substitutional analogue of *S*4, *S*5. As in the case of the semi substitutional semantics

*S*2, we pretend for simplicity that the language contains no singular terms other than individual variables.

A model m is an ordered septuple $<\mu, s, \Upsilon, G, T, g, e>$. μ is a denumerable set of entities. s is a function assigning members of μ to individual variables of L_o. Υ is a set of sets of individual variables of L_o, and $T \varepsilon \Upsilon$. T will be called the set of designated terms of m. All and only members of T are treated in m as the non empty singular terms of the language. G is a set of functions each of which assigns a set of n-tuples of members of μ to every n-place predicate variable of L_o, and $g \varepsilon G$. e is the assignment to the predicate constant = (the identity predicate). e is the set of pairs <a, a>, for every $a \varepsilon \mu$.

The set of all such ordered septuples will be designated Mod.

The assignment in m to the variable x, m(x) is s(x) if $x \varepsilon T$. Otherwise m(x) is μ itself. Since $\mu \notin \mu$, any atomic formula containing a variable not in T will be false in m. The domain of m, D_m, is defined as the range of s restricted to T. Thus D_m is the set of those objects which are denoted by those terms of L_o which are treated as non-empty in m. D_m is the set of objects over which the quantifiers of L_o are interpreted as ranging in m.

The set μ will be called the Meinong universe of m after Alexius Meinong, who was prepared to admit non actual, or subsistent, objects into his theory of reference (1904). The idea is that whilst D_m is the set of "actual" objects countenanced in m - the quantificational domain of m - μ is the set of all objects, actual and possible, countenanced in m, in the sense that $\Diamond\exists xA(x)$ is true in m only if $\exists xA(x)$ is true in a model whose domain is included in μ. It is provable that given any denumerable set $\mathcal{H}$, a formula A is true in every model in Mod (i.e. is valid) just in case A is true in

every model with Meinong universe $\mathcal{H}$. (The proof is a routine adaptation of the proof of the analogous theorem for $S2$ established in Appendix II, section 4, and we omit it.) This theorem establishes that our semantical theory need deal only with models having a common Meinong universe, which may be chosen arbitrarily. In particular, $\mathcal{H}$ may be chosen as some set of actual objects existing in the "meta world" of the semanticist. (As in the case of $S2$, the set of natural numbers is likely to prove a useful choice.) In that case the semantical theory will not commit the semanticist to an ontology of possible objects. Moreover, although objects in the Meinong universe of a model but not in the domain of the model are merely possible, and not actual, from the point of view of the state of affairs represented by the model, it will not follow, if the Meinong universe is chosen as suggested, that such objects are non-actual from the point of view of the semanticist. Thus $S5$ need engender no problems of trans-world identity. (These problems and their origin are discussed in Chapter IV, section 6.)

An i-variant of the function s is any function s' differing from s at most in its assignment to the i^{th} variable of L_o, v_i, and such that $s'(v_i)$ is a member of the domain of m. An i-variant of a model $<\mu, s, \Upsilon, G, T, g, e>$ is any septuple $<\mu, s', \Upsilon, G, T, g, e>$, where s' is an i-variant of s. An i-variant of a model is always itself a model.

An <u>alternative</u> to a model $<\mu, s, \Upsilon, G, T, g, e>$ is any model $<\mu, s, \Upsilon, G, S, f, e>$. The notion of an alternative to a model is an extension to $S5$ of the notion of a predicate variant of a model, used in $S2$. The set of alternatives to a model m will be designated Mod_m. Since the variable assignment function s is invariant throughout Mod_m, the terms of L_o are rigid designators throughout Mod_m; that is, a term

has the same denotation in any model in Mod_m in which it is treated as non empty.

The notation $m[A_1, ---, A_n]$ indicates that all free variables occurring in A_1 through A_n are in the set of designated terms of m. $Mod_m\ [A_1, ---, A_n]$ is $\{w \in Mod_m | w[A_1, ---, A_n]\}$.

The recursion defining the truth predicate in *S*5 differs from that used in *S*2 just in the following respects.

Quantification in L_o

$m(\forall xA(x)) = 1$ if and only if $m'(A(y)) = 1$ for every i-variant m' of m, where y, the i^{th} variable, is designated in m and does not occur bound in A(x).

$m(\exists xA(x)) = 1$ if and only if $m'(A(y)) = 1$ for some i-variant m' of m, with y as above.

In *S*2 the clause for a universally quantified formula of L_o read as follows: $m(\forall xA(x)) = 1$ if and only if $m'(A(x)) = 1$ for every i-variant m' of m, where x is the i^{th} variable. This clause cannot be used in *S*5 since it yields the wrong truth conditions in case the bound variable x does not occur in the designated terms of m. The present departure from the *S*2 clause for existentially quantified formulae of L_o is similarly motivated.

Where no terms are designated in m, and so the domain of m is empty, every existential quantification of L_o will be trivially false. The following more prolix but equivalent formulation of the clause for universal quantification may be used: $m(\forall xA(x)) = 1$ if and only if for every designated variable y not occurring bound in A(x), for every i-variant m' of m (where y is the i^{th} variable) $m'(A(y)) = 1$. Thus where the domain of m is empty (i.e. m has no designated terms) every universal quantification of L_o will be trivially true.

Higher order quantification

$m(\forall xA(x)) = 1$ if and only if $m(A(y)) = 1$ for every variable y designated in m which does not occur bound in A(x).

$m(\exists xA(x)) = 1$ if and only if $m(A(y)) = 1$ for some variable y designated in m which does not occur bound in A(x).

Entailment

$m(A \rightarrow B) = 1$ if and only if for all $w \in Mod_m$ [A, B], if $w(A) = 1$, then $w(B) = 1$.

As usual, *S*5 may be strengthened to *S*5.1 by using the following clause for entailment:

$m(A \rightarrow B) = 1$ if and only if for all $w \in Mod_m$, if $w(A) = 1$, then $w(B) = 1$.

*S*5 counts $\Box A$ as true in m just in case A is true in every model which (i) has the same Meinong universe as m but not necessarily the same designated terms, and so not necessarily the same domain (ii) contains all variables occurring free in A amongst its designated terms, and (iii) has the same assignment function for individual variables as m. It is the first of these considerations which renders the Barcan Formula invalid in *S*5, and the combination of the first with the second which renders invalid in *S*5 the formula $\exists x\Box A(x) \rightarrow \Box\exists xA(x)$. (In *S*5.1, which lacks restriction (ii), this formula is valid in the special case where A(x) is atomic.) The third consideration ensures that all variables designate rigidly in all models invoked in assessing the truth in some model of $\Box A$.

4.1 The equivalence of *S*4 and *S*5

The semantics *S*4 and *S*5 are equivalent in the sense that any formula valid in the former is valid in the latter, and vice versa. This fact is established in Appendix III, section 1.

5. The Reformulation of Ramified Modal Logic

In this section we axiomatise in a ramified way the formulae valid in *S*4 and *S*5. The result is a ramified modal logic in which the incorrect entailments (1) through (3) of section 1 are not provable. The axiomatisation is obtained by modifying our previous hierarchy of modal systems ML_o, ---, ML_n, --- in such a way as to countenance the presence in L_o of non-denoting singular terms. The resulting systems will be named QML_o, ---, QML_n, ---. "QML" stands for "quantified modal logic", and the nomenclature indicates that it is this hierarchy of systems, and not our previous hierarchy, which provides the correct axiomatisation of an entailment theory admitting the binding of variables occurring within the scope of an arrow by a quantifier falling outside the scope of the arrow (although the old hierarchy is adequate for a theory of entailment countenancing quantification in formulae of L_o only).

All structural, propositional, and * meta axioms of any system $ML_{n\geq 1}$ are both valid and validity preserving in *S*5. All non quantificational proper axioms of any system $ML_{n\geq 1}$ whose principal logical sign is other than $\rightarrow$ are valid in *S*5. Of the quantificational meta axioms of $ML_{n\geq 1}$, ($\forall$L) and ($\exists$R) are neither valid nor validity preserving in *S*5, whilst ($\forall$R) and ($\exists$L) are validity preserving but not valid.

The first problem to be faced in modifying the hierarchy of ramified modal systems, then, is to find replacements for ($\forall$L) and ($\exists$R). The reason for their failure is that in each case the term which gets replaced by a bound variable might be non denoting. Thus the obvious way to amend each axiom is to somehow write in the condition that the replaced term must not be empty. Consider the introduction of $\forall$ into the antecedent of a sequent. The meta axiom which licences this move must say that if B is

deducible from $(A(t),\Gamma)$ (i.e. if $A(t),\Gamma \rightarrow B$), then B is deducible from $\forall xA(x),\Gamma$ provided the term t denotes. Following a recommendation of Quine, the metalinguistic assertion that the term t denotes may be rendered into the object language as $\exists x(x = t)$. The assertion that B is deducible from $\forall xA(x),\Gamma$ provided the term t denotes may then be written as: $\exists y(y = t), \forall xA(x),\Gamma \rightarrow B$. Thus our new $(\forall L)$ meta axiom may be written: $(A(t), \Gamma \rightarrow \Delta) \rightarrow (\exists y(y = t), \forall xA(x), \Gamma \rightarrow \Delta)$ (replacing B by Δ for full generality). Finally, since we want to be able to use the meta axiom in systems where the degree of the formulae in Γ, Δ is greater than 1, we prefix n occurrences of the symbol * to the formula $\exists y(y = t)$, where n is the degree of the formulae in Γ,Δ (and thus of A(t) also). In its final form, then, the replacement for $(\forall L)$, which will be named $[\forall L]$ is:

$(A(t), \Gamma \rightarrow \Delta) \rightarrow (*^{n}\exists y(y = t), \forall xA(x), \Gamma \rightarrow \Delta)$, where n is the degree of A(t). Similar considerations lead us to state the meta axiom governing $\exists$ introduction on the right, $[\exists R]$, as:

$(\Gamma \rightarrow \Delta, A(t)) \rightarrow (*^{n}\exists y(y = t), \Gamma \rightarrow \Delta, \exists xA(x))$, where n is the degree of A(t). $[\forall L]$ and $[\exists R]$ are both valid and validity preserving in *S*5. (The idea of carrying sentences of the form $\exists x(x = t)$ in the antecedent of an entailment in order to dispense with the existential presuppositions of classical quantification theory appears in Hintikka (1959). However, Hintikka's axioms are not used here; and in fact the idea occurred to me independently.)

In certain cases we want to be able to eliminate a denoting pre-supposition $*^{n}\exists x(x = t)$ from a theorem. For example, $\forall x(A(x) \wedge B(x)) \rightarrow \forall x(*^{n}\exists y(y = x) \supset A(x))$ is provable from $A(x) \wedge B(x) \rightarrow A(x)$ by $[\forall L]$, $(\supset R)$

and ($\forall$R), and we want to be able to eliminate the denoting presupposition from it to finally obtain $\forall x(A(x) \wedge B(x)) \rightarrow \forall xA(x)$. Similarly, we want to be able to eliminate the presupposition from $\exists x(A(x) \wedge B(x) \wedge {*^n}\exists y(y = x)) \rightarrow \exists xA(x)$, to obtain $\exists x(A(x) \wedge B(x)) \rightarrow \exists xA(x)$. Notice, however, that elimination should not be possible, for instance, in $\forall xA(x)$, ${*^n}\exists y(y = t) \rightarrow \exists xA(x)$, which says that $\exists xA(x)$ is deducible from $\forall xA(x)$ provided that some term denotes. The adoption of the following two proper axioms provides the most economical means of accomplishing the desired eliminations:

$$\forall x({*^n}\exists y(y = x) \supset A(x)) \rightarrow \forall xA(x) \qquad \text{(Id 1)}$$

$$\exists xA(x) \rightarrow \exists x(A(x) \wedge {*^n}\exists y(y = x)) \qquad \text{(Id 2)}$$

Thus, for example, $\forall x(A(x) \wedge B(x)) \rightarrow \forall xA(x)$ is derivable by Cut from (Id 1) and $\forall x(A(x) \wedge B(x)) \rightarrow \forall x({*^n}\exists y(y = x) \supset A(x))$.

Each system ML_n contains an axiomatisation of those logical truths of degree n whose principal logic sign is other than $\rightarrow$. In particular, $\forall xA(x) \supset A(t)$ and $A(t) \supset \exists xA(x)$ figure amongst these axioms. Both formulae are invalid in *S*5, and we replace them by the valid:

$$(\forall xA(x) \wedge {*^n}\exists y(y = t)) \supset A(t) \qquad [\forall \supset L]$$

$$(A(t) \wedge {*^n}\exists y(y = t)) \supset \exists xA(x) \qquad [\exists \supset R]$$

As in the case of arrow-formulae, denoting presuppositions will sometimes be eliminable, and we accomplish the desired eliminations by means of the following analogues to the two identity axioms of the preceding paragraph:

$$\forall x({*^n}\exists y(y = x) \supset A(x)) \supset \forall xA(x) \qquad \text{(Id 3)}$$

$$\exists xA(x) \supset \exists x(A(x) \wedge {*^n}\exists y(y = x)) \qquad \text{(Id 4)}$$

The next problem arises from the fact that although ($\forall$R) and ($\exists$L) are validity preserving in *S*5, they are no longer themselves valid entailments. In this respect they are like the horizontalisation of the rule of necessitation, $A \nrightarrow \Box A$, which in a semantics of S5 strength is validity

preserving without itself being valid. It is of course acceptable practice to use validity preserving but invalid entailments as the meta axioms - inference rules - of a system: it is required of an inference rule of a system merely that the validities of the system be closed under it. However, although the invalidity of (∀R) and (∃L) does not prohibit their use as meta axioms, it does of course mean that they must not be allowed to figure as proper axioms; and therein lies the present problem, since each system ML_{n+1} contains every meta axiom of ML_n amongst its proper axioms. The reason for having arranged things in this way is that we wanted to be able to mimic ML_n proofs in ML_{n+1} in such a way that if A is ML_n provable, □A is ML_{n+1} provable. (That this can be done is the content of the Necessitation Metatheorem of Chapter III: if A is ML provable, so is □A.) Thus, for example, consider the following ML_n proof P:

$$\vdots$$

$$\Gamma \rightarrow \Delta, A(X)$$

$$\Gamma \rightarrow \Delta, \forall y A(y) \qquad (\forall R)$$

Assuming that it is possible to mimic in ML_{n+1} the inferences which produce $\Gamma \rightarrow \Delta, A(X)$, so that $\Box(\Gamma \rightarrow \Delta, A(X))$ is ML_{n+1} provable, the use of (∀R) as a proper axiom enables the final quantificational inference of P to be mimicked, thus:

$$\vdots$$

$$\Box(\Gamma \rightarrow \Delta, A(X))$$

$$(\Gamma \rightarrow \Delta, A(X)) \rightarrow (\Gamma \rightarrow \Delta, \forall y A(y))$$

$$\Box(\Gamma \rightarrow \Delta, \forall y A(y)) \qquad \text{Cut, (def. } \Box)$$

We obviate the need for assuming the meta axioms (∀R) and (∃L) as proper axioms by reformulating them in the following way:

$$*^{i}(\Gamma \rightarrow \Delta,\ A(X)) \rightarrow *^{i}(\Gamma \rightarrow \Delta,\ \forall yA(y)) \qquad [\forall R]$$

$$*^{i}(A(X),\ \Gamma \rightarrow \Delta) \rightarrow *^{i}(\exists yA(y),\ \Gamma \rightarrow \Delta) \qquad [\exists L]$$

When these meta axioms are used in the ramified system for formulae of degree n, $A(X)$, Γ, Δ may have any degree less than n, say m, and $i = n-m$. Where we want to apply the meta axioms in their old form, i is set at 0. In order to mimic the above proof P, we set i at 1 and apply the meta axiom $[\forall R]$ in the following way:

$$\vdots$$

$\Box(\Gamma \rightarrow \Delta,\ A(X))$

$\Box B \supset *B$ (proper axiom schema, as before)

$*(\Gamma \rightarrow \Delta,\ A(X))$ modus ponens on $\supset$

$*(\Gamma \rightarrow \Delta,\ \forall xA(x))$ $[\forall R]$

$*B \supset \Box B$ (as before a proper axiom schema provided B contains an $\rightarrow$)

$\Box(\Gamma \rightarrow \Delta,\ \forall xA(x))$ modus ponens on $\supset$

Similarly the meta axioms $(A \supset B(X)) \rightarrow (A \supset \forall yB(y))$ and $(A(X) \supset B) \rightarrow (\exists yA(y)) \supset B)$ are validity preserving but are not themselves valid entailments, and we replace them with the following meta axioms, which obviate the need for ever assuming formulae of that form as proper axioms:

$$*^{i}(A \supset B(X)) \rightarrow *^{i}(A \supset \forall yB(y)) \qquad [\forall \supset R]$$

$$*^{i}(A(X) \supset B) \rightarrow *^{i}(\exists yA(y) \supset B) \qquad [\exists \supset L]$$

In order to completely axiomatise the theory of *S*5 some axioms of a form not previously found in ramified modal logic are required. Firstly, some axioms for the identity predicate are needed.

(Id 5) $\exists x(x = t) \supset (t = t)$, where t is an arbitrary term

(Id 6) $s_1 = t_1 \wedge \text{---} \wedge s_n = t_n \supset f(s_1, \text{---}, s_n) = f(t_1, \text{---}, t_n)$, where f is any n place function variable and s_1, --- s_n, t_1, --- t_n are arbitrary terms

(Id 7) $s_1 = t_1 \wedge \text{---} \wedge s_n = t_n \wedge P(s_1, \text{---}, s_n) \supset P(t_1, \text{---}, t_n)$ where P is any n place predicate variable or constant (and may be "=" itself), and $s_1, \text{---}, s_n, t_1, \text{---}, t_n$ are arbitrary terms.

Secondly, the following pair of special axioms are needed, which allow the assertion of a denoting presupposition on the basis of the truth of an atomic formula:

(Atm 1) $*^i A(t) \rightarrow *^i \exists x(x = t)$, where A(t) is atomic

(Atm 2) $*^i A(t) \supset *^i \exists x(x = t)$, where A(t) is atomic.

These axioms reflect the point that we are counting an atomic formula A(t) as false if t is empty, i.e. if $\exists x(x = t)$ is false. Thus where A(t) is atomic $*^i A(t) \supset *^i \exists x(x = t)$ is logically true for all i, and (Atm 2) ensures that such logical truths are provable. Similarly $\dfrac{*^i A(t)}{*^i \exists x(x = t)}$ is a truth preserving inference figure for any i, and (Atm 1) ensures that the horizontalisations of these figures are provable.

5.1 A checklist of the new axioms

For convenience of reference we list the axioms just introduced.

[$\forall$L] $(A(t), \Gamma \rightarrow \Delta) \rightarrow (*^n \exists y(y = t), \forall x A(x), \Gamma \rightarrow \Delta)$

where n is the degree of A(t)

[$\forall$R] $*^n(\Gamma \rightarrow \Delta, A(X)) \rightarrow *^n(\Gamma \rightarrow \Delta, \forall y A(y))$

where if [$\forall$R] is a meta axiom of the system for degree m and the degree of $\forall y A(y)$ is k, $n = m - (k + 1)$.

[$\exists$L] $*^n(A(X), \Gamma \rightarrow \Delta) \rightarrow *^n(\exists y A(y), \Gamma \rightarrow \Delta)$

where n is as in [$\forall$R]

[$\exists$R] $(\Gamma \rightarrow \Delta, A(t)) \rightarrow (*^n \exists y(y = t), \Gamma \rightarrow \Delta, \exists x A(x))$

where n is the degree of A(t)

$[\forall \supset L]$ $(\forall xA(x) \wedge {}^{*n}\exists y(y = t)) \supset A(t)$

where n is the degree of A(t)

$[\forall \supset R]$ ${}^{*n}(A \supset B(X)) \rightarrow {}^{*n}(A \supset \forall yB(y))$

where if $[\forall \supset R]$ is a meta axiom of the system for degree m and the degree of A is k, n = m - k

$[\exists \supset L]$ ${}^{*n}(A(X) \supset B) \rightarrow {}^{*n}(\exists yA(y) \supset B)$

where n is as for $[\forall \supset R]$.

$[\exists \supset R]$ $(A(t) \wedge {}^{*n}\exists y(y = t)) \supset \exists xA(x)$

where n is the degree of A(t)

(Id 1) $\forall x({}^{*n}\exists y(y = x) \supset A(x)) \rightarrow \forall xA(x)$

where n is the degree of A(x)

(Id 2) $\exists xA(x) \rightarrow \exists x(A(x) \wedge {}^{*n}\exists y(y = x))$

where n is the degree of A(x)

(Id 3) $\forall x({}^{*n}\exists y(y = x) \supset A(x)) \supset \forall xA(x)$

where n is the degree of A(x)

(Id 4) $\exists xA(x) \supset \exists x(A(x) \wedge {}^{*n}\exists y(y = x))$

where n is the degree of A(x)

(Id 5) $\exists x(x = t) \supset (t = t)$

(Id 6) $s_1 = t_1 \wedge \text{---} \wedge s_n = t_n \supset f(s_1, \text{---}, s_n) = f(t_1, \text{---}, t_n)$

where f is any n place function variable

(Id 7) $s_1 = t_1 \wedge \text{---} \wedge s_n = t_n \wedge P(s_1, \text{---}, s_n) \supset P(t_1, \text{---}, t_n)$

where P is any n place predicate variable or constant

(Atm 1) ${}^{*n}A(t) \rightarrow {}^{*n}\exists x(x = t)$

where A(t) is atomic and n + 1 is the degree of the provable formulae of the system

(Atm 2) ${}^{*n}A(t) \supset {}^{*n}\exists x(x = t)$

where A(t) is atomic and n is the degree of the provable formulae of the system

5.2 The recursive definition of the hierarchy of systems QML_o, ---, QML_n, ---

We indicate how to modify the recursion defining the hierarchy ML_o, ---, ML_n, --- in order to obtain a definition of QML_o, ---, QML_n, ---.

$\underline{QML_o}$ QML_o is ML_o (i.e. Genzen's LHK) with proper axioms $[\forall \supset L]$ and $[\exists \supset R]$ in place of their counterparts in ML_o, with meta axioms $[\forall \supset R]$ and $[\exists \supset L]$ in place of their counterparts in ML_o, and with (Id 3), (Id 4), (Id 5), (Id 6), (Id 7) and (Atm 2) as additional proper axioms.

$\underline{QML_1}$ QML_1 is ML_1 with proper axioms $[\forall \supset L]$ and $[\exists \supset R]$ in place of their counterparts in ML_1, with meta axioms $[\forall L]$, $[\forall R]$, $[\exists L]$, $[\exists R]$, $[\forall \supset R]$ and $[\exists \supset L]$ in place of their counterparts in ML_1, with meta axioms $[\forall L]$, $[\forall R]$, $[\exists L]$, $[\exists R]$, $[\forall \supset R]$ and $[\exists \supset L]$ in place of their counterparts in ML_1, and with (Id 1) through (Id 7), (Atm 1) and (Atm 2) as additional proper axioms.

$\underline{QML_{n+1}}$ QML_{n+1} is defined in terms of QML_n in just the same way as ML_{n+1} is defined in terms of ML_n, except that

(i) the meta axioms $[\forall R]$, $[\exists L]$, $[\forall \supset R]$, $[\exists \supset L]$ of QML_n do not figure amongst the proper axioms of QML_{n+1} (and nor do the analogous meta axioms of ML_n)

(ii) QML_{n+1} has proper axioms $[\forall \supset L]$ and $[\exists \supset R]$ in place of their counterparts in ML_{n+1}

(iii) QML_{n+1} has meta axioms $[\forall L]$, $[\forall R]$, $[\exists L]$, $[\exists R]$, $[\forall \supset R]$ and $[\exists \supset L]$ in place of their counterparts in ML_{n+1}

(iv) QML_{n+1} has (Id 1) through (Id 7), (Atm 1) and (Atm 2) as proper axioms.

All proper axioms of any system QML_n are valid in *S*5, and all meta axioms of QML_n are validity preserving in *S*5. That is, every system QML_n is sound with respect to *S*5.

6. Kripke's Addition of Quantifiers to S5

Motivated by the belief that the Barcan Formula does not express a correct entailment, Kripke (1963) has added quantifiers to S5 in such a way that neither this formula nor its existential analogue are provable. Thus Kripke's quantified S5 is not equivalent to the Marcus-Prior version of quantified S5 discussed in Chapter III, above.

Any formula without free variables obtained by prefixing universal quantifiers and necessity signs, in any order, to a formula A will be called a Kripke closure of A. The axioms of Kripke's quantified S5 - which we will call KS5 - are the Kripke closures of all formulae of the following forms (1963, p.69):

(0) truth functional tautologies

(1) $A \supset \forall xA$, where x is not free in A

(2) $\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$

(3) $\forall y(\forall xA(x) \supset A(y))$

(4) $\Box A \supset A$

(5) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(6) $\Diamond A \supset \Box\Diamond A$

(The closures - in the usual sense - of (0) through (3) are Quine's (1940) axioms for a non modal predicate logic in which only closed formulae are provable.)

Modus ponens on $\supset$ is the only rule of inference of KS5. (Necessitation can be obtained as a derived rule.)

All the axioms of KS5 are derivable in Marcus-Prior QS5. However, the converse does not hold. KS5, unlike QS5, has no open sentences as theorems. It is this fact which enables the proof of the Barcan Formula to be blocked in KS5, its availability in QS5 depending on an application

of necessitation to the open sentence $\forall x \Box A(x) \supset \Box A(y)$ (as Kripke remarks on pp. 68-69).

The relationship between KS5 and ramified modal logic is familiar: if A is provable in KS5, then [A] is QML provable. ([A] is defined in Chapter III, section 9.1). We omit the proof of this theorem, which differs only in detail from the proof of the Key Metatheorem of Chapter III.

7. The Completeness of Reformulated Ramified Modal Logic

The hierarchy of systems QML_o, ---, QML_n, --- is complete with respect to *S*4 and *S*5. The general strategy of our proof of this fact resembles that of our proof of the completeness of the hierarchy ML_o, ---, ML_n, --- with respect to *S*1 and *S*2: we use the fact that KS5 is complete with respect to Kripke's (1963) possible world semantics, together with the version of the Key Metatheorem given in the preceding section. How the proof proceeds is described in Appendix III, section 2.

Chapter VI

The Paradoxes of Classical Entailment

At this point we turn our attention from classical logic to relevance logic. The classical account of entailment has during the last half century been persistently criticised on the ground that it countenances as correct various theses about entailment which are putatively incorrect. These theses are commonly known as the paradoxes of strict implication. In this chapter we discuss the paradoxes and evaluate the consequences for classical entailment theory of their theoremhood.

1. The Paradoxes of Strict Implication

In S2 and the stronger Lewis systems the following are provable:

$$\sim\Diamond A \rightarrow (A \rightarrow B) \qquad \text{(P1)}$$

$$\Box A \rightarrow (B \rightarrow A) \qquad \text{(P2)}$$

P1 asserts that a logically impossible sentence entails any sentence, and P2 asserts that a logically necessary sentence is entailed by any sentence.

Hallden (1948) has shown that (P1) and (P2) are not theses of Lewis' weakest system, S1. However, the following two analogues to (P1) and (P2) are provable in S1:

$$\sim\Diamond A \supset (A \rightarrow B) \qquad \text{(P1)}'$$

$$\Box A \supset (B \rightarrow A) \qquad \text{(P2)}'$$

The following two formulae are provable in any of Lewis' systems:

$$A \wedge \sim A \rightarrow B \qquad \text{(P3)}$$

$$A \rightarrow B \vee \sim B \qquad \text{(P4)}$$

(P1) and (P3) are indiscriminately known in the literature as the first paradox of strict implication, and (P2) and (P4) are indiscriminately known as the second paradox of strict implication. Where it becomes necessary to distinguish (P1) from (P3), (P2) from (P4), (P1) and (P2) will be referred to as the general forms of the paradoxes.

The ramified analogies to (P1) through (P4) (which are typographically indistinguishable from (P1) through (P4), respectively) are provable in each ramified entailment system $ML_{n\geq 1}$. (Vide Chapter III, theorems 7.10 and 7.11.)

In (1932) Lewis casually remarks that (P1) and (P2) "are paradoxical only in the sense of expressing logical truths which are easily overlooked". (p.248) However in an appendix to the second edition of the book, written in 1959, Lewis offers the following diagnosis of the counterintuitiveness of the paradoxes. "That to infer in such cases is affected with a sense of paradox reflects the futility of drawing any inference when the premise is not only known false but is not even rational to suppose; and the gratuitous character of inferring what could be known true without reference to any premise. --- None of the paradoxical paradigms, either those which indicate what is deducible from the logically impossible, or those which formulate sufficient conditions of the logically necessary, are capable of any use for the characteristic purpose of inferring." (pp.513-514 Few, however, agree with Lewis' diagnosis of the paradoxicality of (P1) and (P2). There seems to be almost complete agreement amongst those who object to the presence of (P1) and (P2) in a theory of entailment that

their parodoxicality is to be located in the fact that they countenance fallacies of relevance. Both licence as acceptable, inferences whose premisses and conclusion are completely diverse in subject matter.

2. General Attitudes to the Paradoxes

Bennett (1969, p.198) speculates that many philosophers "accept Lewis' analysis [of entailment] only reluctantly, seeing the paradoxes as something unpalatable which must be choked down" - because the remainder of Lewis' analysis seems smooth and satisfactory, and they see no way of preserving what is correct in the analysis whilst rejecting the paradoxes. On the other hand, a large number of philosophers see the paradoxes as stating clear and fundamental truths about the relation of entailment. For example, both W. Kneale (1945) and K.R. Popper (1947) use the first and second paradoxes as integral parts of their respective accounts of the nature of logic (as Bennett has observed (1954, p.451)).

A third attitude is that of those philosophers who maintain that a satisfactory systematic analysis of entailment should contain no analogues to the paradoxes; for they see the paradoxes as false if construed as statements about entailment. For example, Anderson (1972a) writes: "The rest of Strict Implication theory is interesting enough, but results like $(P \wedge \sim P) \rightarrow Q$ are simply excrescences. ... We want [an analysis of entailment] ... which excises the unsightly warts on [Lewis'] theory." (pp.363-364)

3. Two Common Defences of the Paradoxes

Hughes and Cresswell (1968) give the following expression to what is perhaps the commonest argument that (P1) through (P4) are true of the

relation of entailment: "What we require in a valid inference is the logical guarantee that we shall not have the premiss(es) true but the conclusion false. Now, by this account, the 'paradoxes' are sound principles of deducibility; and hence it is not their presence in but their absence from a system which would tell against its claim to be a correct logic of entailment." (p.336) This argument, however, simply ignores the fact that relevance logicians are proposing, for what they consider to be good reasons, a new standard of inference validity whereby truth preservation is not a sufficient condition for valid inference. A defence of the paradoxes must involve a defence of the classical standard of argument validity against the objections raised to it by relevance logicians. (These objections are set out in the following section.)

Hughes and Cresswell also give the following version of a commonly found defence of the paradoxes: "The principle that $(p \wedge \sim p)$ entails q is at worst an innocent one: it could never lead us astray in practice by taking us from a true premiss to a false conclusion, since no proposition of the form $(p \wedge \sim p)$ can every be true." (ibid, p.338) (Clearly this approach is also applicable to the second paradox.) The obvious reply to this sort of defence is made by Anderson and Belnap: "We grant that [$(p \wedge \sim p)$ entails q] is harmless in this sense, but still contend that it is harmful in another sense, namely, in being false." (1962a, p.17)

4. Reasons Given in the Literature for Rejecting the Paradoxes as Truths about Entailment

I think it is true to say that this area of philosophy has been hallmarked by reliance on dogma rather than on argumentation. This state

of affairs is illustrated by the survey provided in the first of the following four subsections. In the remaining three subsections, three serious arguments against the paradoxes are presented, one due to von Wright and two due to Anderson and Belnap. The second Anderson-Belnap arguments, which is currently extremely influential, is evaluated in section 5 below.

4.1 The appeal to intuition

It is, unhappily, very common to find in the literature an author rejecting the paradoxes on the basis of an intuitive feeling that they are outrageous or repugnant. In each of the following sample quotations the paradoxes are dogmatically asserted to be false of entailment on the basis alone of the author's psychological reaction to them. Nelson (1930): "These paradoxes convince me that Strict Implication is not what is ordinarily meant by implication. --- The paradoxes seem so utterly devoid of rationality that I consider them a reductio ad absurdum of any view which involves them." (p.446) Duncan-Jones (1935): "[$p \wedge \sim p \rightarrow q$, and $q \rightarrow p \vee \sim p$] have nothing to be said for them in a calculus of entailment": they are "outrageous". (p.71) G. von Wright (1957): "It appears to be the essence of entailment to be a relation between propositions which subsists quite independently of ... the modal status of the propositions." (p.177) It is interesting to notice that Lewis himself set the precedent for appeals to intuition: "For the validity of the trains of reasoning in these proofs we must, of course, appeal to intuition. A point of logic being in question, no other course is possible." (1932, p.252)

The approach of the "new wave" of Lewis' critics hardly differs from that of these older writings. In making out a case against the paradoxes, Hockney and Wilson (1965) assert that the paradoxes "clash violently with our pre-analytic intuitions". (p.214) And in the writings of Anderson and Belnap appeals are frequently made in the course of attacks on the paradoxes to our "preanalytic ideas" and our "untutored intuitions" (i.e. intuitions untutored by classical logic). For example, Anderson writes: "[the] central controversial problem [is] whether a certain move in formal logic [is] appropriate, in the sense of being true to our preanalytic intuitions." (1972a, p.360) Now, to be sure, in supplying a formal analysis of entailment, we are analysing a concept we possess prior to the commencement of analysis, the features of which we expect our analysis to capture formally. But the simple appeal to felt characteristics of this concept is useless in polemic against the Lewis analysis. For a philosophical position cannot be refuted by an intuition which is not generally shared; and, as noted above, the intuitions of those who oppose Lewis are certainly not generally shared. Furthermore an appeal to intuition has no "epistemological backing", and intuitions about entailment seem to be very much a personal affair. For example, Duncan-Jones felt $A \rightarrow A \vee B$ to be counter-intuitive (1935, p.78), whereas Anderson and Belnap take it to express a valid principle of inference.

The bankruptcy of the appeal-to-intuition approach to rejecting the paradoxes is neatly highlighted by the following. In (1962b) Anderson and Belnap write: "Modal fallacies arise when it is claimed that entailments ... are entailed by contingent propositions ... [The

elimination of modal fallacies] clearly accords with (untutored) intuitions." (p.44) In (1969) Routley and Routley set out a convincing case that entailments from contingent propositions to entailments are not at all fallacious. And in recognition of their arguments, Anderson admits that the above claim of (1962b) is "simply and flatly false". (1972a, p.367)

4.2 The first paradox and reductio ad absurdum

von Wright (1957) claims that Lewis' analysis of entailment "cannot account for inverse proof. And this, it would seem, is enough to wreck it." (p.174) His argument is that it is essential to inverse proof (i.e. reductio ad absurdum) that we be able to discriminate between propositions which are entailed, and propositions which are not entailed, by the negation of a necessary proposition. Yet the first paradox denies that we can so discriminate. (pp.173-174) However, it is not clear why von Wright thinks we must be able to so discriminate in order to carry out inverse proof. Bennett is surely right when he says: "To show by reductio that P is impossible, we have only to show ... that $P \rightarrow Q$, where Q is known to be impossible. If P entails everything else as well, tant mieux." (1969, p.209)

4.3 The first Anderson-Belnap argument

This argument concerns Lewis' theorem 18.61:

$\Box A \wedge (A \wedge B \dashv C) \dashv (B \dashv C)$ (1932, p.165)[1]

18.61 states that strict implication tolerates the suppression of necessarily true premisses. Equating strict implication with entailment, Lewis places the following interpretation on the theorem: "The omission of a premise

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1. It is worth noting that 18.61 is closely related to (P2). It is derivable from (P2) as follows: assume $\Box A$ and $A \wedge B \dashv C$; by (P2) $B \dashv A$; thus $B \dashv C$. Conversely, (P2) is a corollary of 18.61. (Take A as C in 18.61 and use the fact that $\Box(A \wedge B \dashv A)$.)

which is a priori or logically undeniable does not effect the validity of a deduction." (1932, p.165) But, demand Anderson and Belnap, "how could this be so? Can we make no distinction between valid and invalid arguments when, as may happen, all the propositions in the argument are necessary? We believe, rather, that the situation is as [the following] citation from Copi may be misinterpreted as saying: 'where a necessary premise is missing, without that premise the inference is invalid.'[2] Necessary premises are just as necessary as premises that are not necessary." (1961, p.717: their emphases)

The fundamental issue here between Lewis and Anderson-Belnap is that Lewis accepts, whilst Anderson-Belnap reject, the classical standard of argument validity whereby truth preservation is a necessary and sufficient condition for the validity of an inference. Having accepted the classical standard, Lewis accepts the consequence of it that necessary premisses may be suppressed without affecting validity. Anderson-Belnap, on the other hand, take this consequence as demonstrating the absurdity of a standard of argument validity which takes truth preservation to be sufficient for validity: if necessary premisses may be suppressed without affecting validity, there is no such thing as an invalid argument all of whose propositions are necessary - since it can hardly matter which necessary propositions figure as the premisses of such an argument.

This line of criticism of the classical standard of validity incorporates the normative claim that our standard of validity should lead us to assess as invalid some arguments all of whose propositions

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2. Copi (1953) p.205.

are necessary. "All bachelors are unmarried; therefore 2 + 2 = 4" is a likely example of the sort of argument that an advocate of this line of criticism would have in mind. In such an argument, Anderson-Belnap tell us, "a necessary premise is suppressed: to wit, the conclusion." (1961, p.717) The fact that the criticism is based on this normative proposal opens the following line of defence of the classical standard of validity (which is taken by Bennett (1969, p.232)). A distinction must be drawn between valid arguments and satisfactory arguments. An argument is valid just in case the truth of the premisses guarantees the truth of the conclusion, but is satisfactory only if the person to whom it is addressed is able to see how the premisses establish the conclusion. Given such a distinction, a proponent of the classical standard of validity can urge against Anderson-Belnap that the standard is not too low - one merely has to recognise the fact, which was not denied in the first place, that not all classically valid arguments are satisfactory. In particular, there is plenty of scope for constructing valid but grossly unsatisfactory arguments from necessary premisses to necessary conclusions.

In response to this manoeuver a relevance logician might maintain that the blanket appraisal of satisfactoriness obscures important differences. Some classically valid arguments from necessary premisses to necessary conclusions will be unsatisfactory for the relatively trivial reasons of being "too congested or unclear or tortuous or the like to be capable of helping anyone to get from the premise to the conclusion." (This is Bennett's description of an unsatisfactory argument (1969, p.232).) Other classically valid arguments from necessary premisses to necessary

conclusions will be unsatisfactory because the premisses are not relevant to the conclusion. (An example of such an argument was given in the preceding paragraph.) And the relevance of premisses to conclusion, the relevance logician will say, is a sufficiently important consideration to be built into the appraisal of validity itself, rather than merely incorporated into an appraisal of satisfactoriness made at the pragmatic stage, and not the logical stage, of the evaluation of an argument. The force of this response to a Bennett-like distinction between valid arguments and satisfactory arguments is discussed in section 5, below.

It should be remarked that suppressing a necessary truth in a classically valid argument will not always produce a fallacy of relevance. For example, any good account of relevance will assess $2 < 3$ and $2 < 4$ as mutually relevant; and no fallacy of relevance results from the suppression of the necessarily true $3 < 4$ in the argument: $2 < 3$ and $3 < 4$, therefore $2 < 4$. In the argument quoted above Anderson and Belnap appear to object to all suppressions of necessarily true premisses (although I am inclined to suspect that their eagerness to make a pun has resulted in a misleading presentation of their argument). This position supports the addition of not only a relevance component to the appraisal of validity, but of a further component invalidating arguments resulting from the suppression of a necessary truth not occasioning loss of relevance. In fact, Anderson and Belnap do not repeat the argument in their encyclopaedaic (1975), which perhaps indicates that they have become dissatisfied with it. Nevertheless it does seem to me that anyone looking for arguments against the classical account of entailment

may in the manner I have just discussed find grist to their mill in Anderson-Belnap's observation that if entailment is identified with strict implication, then we can never distinguish between valid and invalid arguments when all the propositions in the argument are necessary.

4.4 The second Anderson-Belnap argument

This argument first appears in (1962b) and is reproduced in (1975). Kielkopf, in his recent comprehensive survey of the field of relevance logic, tells us that the argument has had considerable influence (1977, p.18). We reproduce it verbatim from (1975), pages 17-18:

"Imagine, if you can, a situation as follows. A mathematician writes a paper on Banach spaces, and after proving a couple of theorems he concludes with a conjecture. As a footnote to the conjecture, he writes: "In addition to its intrinsic interest, this conjecture has connections with other parts of mathematics which might not immediately occur to the reader. For example, if the conjecture is true, then the first order functional calculus is complete; whereas if it is false, then it implies that Fermat's last conjecture is correct." The editor replies that the paper is obviously acceptable, but he finds the final footnote perplexing; he can see no connection whatever between the conjecture and the "other parts of mathematics", and none is indicated in the footnote. So the mathematician replies, "Well, I was using 'if ... then ——' and 'implies' in the way that logicians have claimed I was: the first order functional calculus *is* complete, and necessarily so, so anything implies that fact - and if the conjecture is false it is presumably impossible, and hence implies anything. And if you object to this usage, it is simply because you have not understood the technical sense of

'if ... then ——' worked out so nicely for us by logicians." And to this the editor counters: "I *understand* the technical bit all right, but it is simply not correct. In spite of what most logicians say about us, the standards maintained by this journal require that the antecedent of an 'if ... then ——' statement must be *relevant* to the conclusion drawn. And you have given no evidence that your conjecture about Banach spaces is relevant either to the completeness theorem or to Fermat's conjecture."

"Now it might be thought that our mathematician's footnote should be regarded as true, "if ... then ——" being taken materially or (more likely) strictly - but simply uninteresting because of its triviality. But notice that the editor's reaction was *not* "But heavens, that's trivial" (as the contention that the mathematical "if ... then ——" is the same as material "implication" would require); any such reaction on the part of an editor would properly be judged insane. His thought was rather, "I can't see any reason for thinking that this is *true*."

"No, the editor's point is that though the technical meaning is clear, it is simply not the same as the meaning ascribed to "if ... then ——" in the pages of his journal. Furthermore, he has put his finger precisely on the difficulty: to argue from the necessary truth of *A* to *if B then A* is simply to commit a fallacy of relevance. The fancy that relevance is irrelevant to validity strikes us as ludicrous."

(Although the classical "if —— then ——" figures in the exposition of the argument, Anderson and Belnap intend the argument to apply equally to classical entailment (vide (1975), p.491).)

5. Appraisal of the Situation

It seems to me that the second Anderson-Belnap argument establishes no more than that anyone who argues from premisses to a conclusion which is not relevant to them, has argued badly. The argument provides no grounds for the stronger claim that anyone who so argues has argued invalidly. After all, everyone is prepared to agree that someone who argues from obviously false premisses, or who argues in a tortuous or unclear fashion, is arguing badly but may nonetheless be arguing validly. Why, then, should an argument whose premisses and conclusion lack relevance to one another be appraised as invalid, rather than merely as a bad piece of argumentation? Similarly Nelson and Duncan Jones, in the papers cited in 4.1 above, have in their peevish complaints about strict implication drawn attention to the fact that arguments whose premisses and conclusion are not relevant to one another are bad arguments, but have done nothing to show that the premisses of such arguments do not entail their conclusions. What relevance logicians require, but have failed to provide, is an argument that an appraisal of the relevance of conclusion to premisses should be built into the assessment of the validity of an argument. Only then will they have made out a case that there is a need for a non classical account of entailment.

The business of evaluating an argument may be divided into two parts. Firstly comes the formal appraisal of validity, and secondly comes an informal appraisal of worthiness as a piece of argumentation. This second, or posterior, appraisal typically takes into account such factors as the truth of premisses, the relevance of conclusion to the premisses, the clarity with which the argument presents the bearing the premisses have

on the conclusion, and so on. A possible line of argument open to a relevance logician, with which I am in complete sympathy, is as follows. Formal logic is fundamentally the science of evaluating arguments. The formal logician should seek to construct precise and rigorous accounts of as many of the components of argument appraisal as possible. But in contravention of this, attention has been focussed exclusively on the appraisal of truth preservation. No doubt many of the dimensions on which arguments are appraised are not amenable to formal treatment. But it is a scandal that formal logicians have been content to leave the appraisal of the relevance of conclusion to premisses at the informal, intuitive level. It is by no means prima facie impossible that a formal syntactical account of premiss/conclusion relevance should be found, and every attempt should be made to construct one.

Given a formal account of premiss/conclusion relevance (if, indeed, it does prove possible to construct one) the formal logician is faced with two alternatives. He may keep the appraisal of premiss/conclusion relevance as part of the posterior appraisal of an argument, although now a fully formal part. The classical standard of validity is retained, and with it classical entailment theory. Alternatively he may prefer to build the appraisal of premiss/conclusion relevance into the validity assessment itself, resulting in a relevant entailment theory. Perhaps the former approach might be preferred on the grounds that it keeps separate issues separate: on the one hand we have the appraisal of truth preservation, dealt with formally by our theory of entailment, and on the other hand we have the appraisal of premiss/conclusion relevance,

dealt with formally by our theory of relevance. Perhaps the latter approach might be preferred on the ground that in building the relevance appraisal alongside the truth preservation appraisal in the assessment of validity, due homage is paid to the importance, felt by some, of the relevance appraisal.

It must be stressed that these two choices are not rivals in any deep sense. They are merely alternative formal handlings of the same situation. To opt for the first alternative need imply no belittlement of the importance of the appraisal of premiss/conclusion relevance, whilst to opt for the second alternative need not imply the attitude that classical logicians were wrong all along. However, the mere fact that the second sort of approach might be viable does show that the classical way of treating the concept of entailment is not necessarily the only way: classical entailment is not the god some of its defenders make it out to be.

The method of procedure of system-building relevance logicians has not in fact been to first construct a general, formal account of premiss/conclusion relevance and then construct an entailment system incorporating this account. Rather, the trend has been to construct straight-off entailment systems which exclude the paradoxes, no consideration being given to the sort of account of relevance contained by the system. (Examples are the systems of Emch (1930), Parry (1933), Vrenduin (1939), Ackermann (1956), Anderson and Belnap (1975). The lack of a unified account of relevance in the Anderson-Belnap system is further discussed in Chapter VIII, below.) This practice is unfortunate, in two ways. Firstly, an entailment system presented without reference

to an account of relevance is extremely hard to evaluate. How is it to be decided whether or not A relevantly implies B without an account of when A and B are mutually relevant? (Indeed, one would think it is next to impossible to construct a system of relevant entailment without such an account. I have long harboured the suspicion that entailment systems often have been constructed in a more or less ad hoc manner, exclusion of the classical paradoxes being the chief concern. This procedure of course necessitates the cooking up at a later date of philosophical arguments to justify the exclusion of whatever else gets excluded; and one does occasionally get the hopefully false impression that exactly this has happened.) Secondly, the practice seems over the years to have engendered the inexplicit and incorrect belief that the only way to deal with the appraisal of premiss/conclusion relevance is via a theory of relevant entailment. And this belief has led to the current unfortunate feeling amongst relevance logicians that classical entailment theory and relevant entailment theory are as opponents on a battlefield. As indicated in the preceding paragraph, I do not share this feeling.

In Chapter IX a formal account of premiss/conclusion relevance is developed for sentences whose only significant logical operators are the truth functional connectives. A theory of relevant entailment incorporating this account is developed both semantically and syntactically. The theory is not equivalent to the Anderson-Belnap system E of entailment. Hence if our account of relevance is correct, E is not a satisfactory axiomatisation of relevant entailment.

Attempts have failed to extend this account of relevance to sentences analysed as subject-predicate and quantificational constructions. Perhaps this is due to a lack of skill and persistence. However, I have come to suspect that premiss/conclusion relevance in general is an irreducibly pragmatic notion not amenable to formal characterisation. If so there is no hope, in the general case, of shifting the relevance component of the assessment of the worthiness of a piece of argumentation into the actual assessment of formal validity. Thus, in the general case, the classical theory of entailment cannot be replaced by a relevance theory of entailment, for there can be no completely general theory of relevant entailment. In particular, there is no theory of relevant entailment to be had once one proceeds beyond the logic of truth functions.

6. A Note on an Argument Given by Bennett

A theme frequently harped on in the anti-Lewis writings is that the paradoxes fail miserably to square up with what is ordinarily meant by entailment or implication, since no sane person would ever seriously put forward an argument from inconsistent premisses to an irrelevant conclusion, or from an arbitrary premiss to a necessary conclusion. For example Anderson, writing of the Anderson-Belnap system E of entailment, remarks that E " is paradox free, while still allowing as valid anything that any sane man would regard as valid." (1963, p.7) The paradoxes, the credo goes, simply do not express what would be regarded as correct principles of inference by anyone whose logical intuitions have not "been numbed through hearing and repeating the logicians' fairy tales of the past half century" (Anderson and Belnap, 1962a, p.20).

This line of objection to the paradoxes has been thoroughly scotched by Bennett (1969), who presents an argument from inconsistent premisses to a conclusion completely diverse from them in subject matter, which is nonetheless an argument which a sane man might seriously put forward in the context Bennett describes. We reproduce verbatim the appropriate passage from Bennett's paper.

"A certain Oracle has said, "There will be rain this month or the King will die," and a tribesman is interested in this prediction, seriously wonders whether it is true, and expresses it to himself in the form: "The Oracle spoke truly." He accepts the sound agricultural maxim that

(1) *If there is rain this month, the harvest will be ruined,*

and the unchallenged tribal dogma that

(2) *If the harvest is ruined, the sky god will be angry.*

It occurs to him that these two beliefs imply that

If the Oracle spoke truly, then either the sky god will be angry or the King will die.

"If the Oracle spoke truly," he mutters, "it's going to be a bad month for someone; and the thought of the sky god's anger is terrifying. Perhaps the Oracle didn't speak truly but I daren't depend on that, so I'll take precautions. I know that

(3) *If I sacrifice a goat, the sky god won't be angry,*

and although that is expensive it will be worth it, if only for my peace of mind." He sacrifices a goat, and continues: "Now that

(4) *I have sacrificed a goat*

I can rest assured that

The sky god won't be angry,

and what a relief *that* is!" A little later, rain comes spattering down, and the tribesman notes:

(5) *"It is raining*

and so, after all,

The Oracle did speak truly!

I am glad that I sacrificed that goat to placate the sky god! Still, wasn't I satisfied that some disaster would occur if it turned out that the Oracle spoke truly? Yes, indeed! Since the Oracle *did* speak truly, it is settled that

Either the sky god will be angry or the King will die;

and I - fool that I am - have placated the sky god!" Whereupon he bitterly accuses himself of regicide.

"The tribesman's premises are numbered (1) to (5). The formal presentation of this argument, with the other indented propositions as lemmas, is left to the reader. The crucial point about the argument is that the tribesman reaches the conclusion that *The King will die* although it is not one of the premises nor even a constituent of any compound premise. The argument is aided by the availability of *The Oracle spoke truly;* but that is not a premise, or a constituent in any compound premise, either." (pp.227-229)

There is, then, no hope of motivating a departure from classical entailment theory by a claim that the paradoxes could never be exemplified by arguments which could be sensibly put forward in some real life situation. But does the argument not also undercut the case we ourselves have built for an investigation of relevant entailment, namely that relevance between premisses and conclusion is a requisite of good

argumentation? For although the tribesman's argument is certainly bad, its badness arises from the fact that its premisses are inconsistent, rather than from the fact that its conclusion is completely disparate in subject matter from its premisses: the lack of premiss/conclusion relevance is of no consequence for an audience acquainted with the oracle's pronouncement. However, one cannot conclude from this "so much the worse for relevant entailment" unless one is prepared for parallel reasons to conclude "so much the worse for classical entailment". A piece of argumentation which is perfectly satisfactory in its context may be classically invalid. When presenting an argument we will often omit to state contingently true propositions with which we know an audience to be acquainted, resulting in the classical invalidity of the argument. Similarly, the tribesman in Bennett's argument has quite legitimately omitted to use as a premiss the proposition "The oracle said 'There will be rain this month or the king will die.'", with resultant lack of premiss/conclusion relevance. But this latter case militates against relevance logic no more than a former case militates against classical logic. Both classical logic and relevance logic deal with general cases, abstract settings. The relevance logician is no more committed to the view that premiss/conclusion relevance is a feature of every satisfactory piece of real life argumentation than the classical logician is committed to the view that truth preservation is a feature of every satisfactory piece of real life argumentation.

Chapter VII

The Lewis Independent Proofs:
A Survey of Attempts to Fault Them

In (1932) Chapter VIII, Lewis sets out what have become known as his independent proofs of the first and second paradoxes. The strategy of the proofs is to assume the antecedent of a paradox and deduce from it the consequent of the paradox using apparently obvious modes of inference. It is concluded that the "paradox" - the statement asserting that the consequent is deducible from the antecedent - is true. For Lewis, the availability of the proofs placed it beyond doubt that his identification of converse deducibility, or entailment, with strict implication was correct. (vide pp.250-251) Anyone who accepts the apparently obvious modes of inference used in the proofs must, consistently, accept the paradoxes. Conversely, anyone who maintains the paradoxes to be false of entailment must accept the onus of faulting the proofs.

In this chapter I shall discuss only the first independent proof. The following statement of the proof differs from Lewis' original version (p.250) only in that it does not use his procedure of registering deductions by means of a strict implication. That feature of Lewis' version promotes the objection that the proof shows nothing about entailment: since the proof is presented in terms of strict implication, Lewis proves merely that the first paradox is true of strict implication. However, the proof is readily reformulated in a non-question-begging way:

(1)	$A \wedge \sim A$	assumption
(2)	A	(2) is deducible from (1) by simplification

(3)	$\sim A$	(3) is deducible from (1) by simplification
(4)	$A \vee B$	(4) is deducible from (2) by addition
(5)	B	(5) is deducible from (4) and (3) by disjunctive syllogism

Hence B is deducible from $A \wedge \sim A$; which is to say $A \wedge \sim A$ entails B.

Apart from the noted principles of inference, the proof tacitly relies on the principle of the transitivity of entailment: (A entails B and B entails C) entails (A entails C). This principle is used twice: firstly, in passing from

$A \wedge \sim A$ entails A and A entails $A \vee B$ to

$A \wedge \sim A$ entails $A \vee B$; and, secondly, in passing from

$A \wedge \sim A$ entails $\sim A \wedge (A \vee B)$ and $\sim A \wedge (A \vee B)$ entails B to

$A \wedge \sim A$ entails B. Also, the proof uses a principle of adjunction, by which $A \wedge \sim A$ entails $\sim A \wedge (A \vee B)$ is available from $A \wedge \sim A$ entails $\sim A$ and $A \wedge \sim A$ entails $A \vee B$.

The first independent proof is ostensibly a proof of $A \wedge \sim A \rightarrow B$. However, Lewis claims that the proof establishes the first paradox in its general form: "Thus any proposition one chooses may be deduced from the denial of a --- necessary truth: the theorem $\sim\Diamond p . \supset . p \supset q$ states a fact about deducibility." (p.250-251) Lewis' arguments for this claim begin on page 249, and the independent proof is itself just a step in this extended argument (a fact which is generally unremarked). As I understand it, the argument runs as follows. It is to be established that an arbitrary proposition q is deducible from any proposition r such that $\sim\Diamond r$. Let r be any proposition such that $\sim\Diamond r$. Thus it is the case that $\sim\Diamond\sim\sim r$. Now, the subformula q of any true proposition $\sim\Diamond\sim q$ is always "expressible" as a formula of the form $p \vee \sim p$ (as Lewis puts it: "Every

tautology is expressible as some proposition of the general form p ∨ ~p" - "remembering that ~◇~p means 'p is a tautology'" (p.249)). Thus for some p, ~r is "expressible" as p ∨ ~p. But "the negation of any proposition which is of the form ~p ∨ p is a corresponding proposition of the form p ∧ ~p." (p.250) Thus the demonstration that q is deducible from p ∧ ~p is also a demonstration that q is deducible from ~~r, and so from r itself: "This demonstration is a paradigm in which p may be any proposition so chosen that ~p ∨ p will express the tautology which is in question [viz, ~r]; and q may be any proposition whatever. Thus any proposition one chooses may be deduced from the denial of a tautological or necessary truth." (pp.250-251)

The key question in evaluating this extended argument is what Lewis means by "expressible" when he says that every tautology is expressible as some proposition of the general form p ∨ ~p. I think his meaning becomes clear in the course of some worked examples which he presents on page 249. He means that every tautology strictly implies and is strictly implied by some proposition of the form p ∨ ~p. But in this case Lewis' extended argument yields its conclusion only on the question begging presupposition that strict implication is entailment. Given that the independent proof establishes that p ∧ ~p entails an arbitrary proposition q, and given that r, the denial of a necessary truth, entails p ∧ ~p, there is a logical obligation to accept that r entails q (provided it is accepted that entailment is transitive), and thus that the general form of the first paradox does indeed express a truth about entailment. But given only that r strictly implies p ∧ ~p, there is no such obligation. What Lewis requires, if his extended argument is not to beg the question, is an

argument that any necessary truth entails, and is entailed by, a proposition of the form p ∨ ~p. It is an open question whether such an argument could be found to the satisfaction of all those who oppose the identification of entailment with strict implication.

Whether or not the first independent proof may be used to establish the first paradox in its general form, it is certainly the case that the proof establishes the paradox A ∧ ~A → B. Anyone who wishes to deny the paradox must fault the proof, and so must reject, and provide a reason for rejecting, at least one of the five principles of inference in the proof. I propose to consider various attempts in the literature to fault the proof.[1]

1. E.J. Nelson (1930)

In this article Nelson constructs a calculus of what he terms intensional relations. The intensional relations which he takes as primitive are consistency, conjunction, and negation, symbolised respectively as poq, pq, and -p. (p.444) He tells us that "intensional relations are not truth functions. --- The meanings of --- propositions are to determine the relation" (p.443). In particular "'consistency' is not a truth function; rather it holds between the meanings of the propositions in relation to one another." (p.443, Nelson's emphasis). It seems, then, that the propositional variables in the formula poq do

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1. This chapter is taken almost unchanged from my Bachelor of Philosophy thesis "Entailment Without Paradoxes" (1976). It was my original intention to present here an extended version of this earlier survey, covering proposals made by Bayliss (1931), Cleave (1974), Emch (1936), Geach (1970), Körner (1946), Parry (1933), Strawson (1948), and Vredenduin (1939). However, in the wake of the publication of the comprehensive and thorough Kielkopf (1977) this work would have been mere repetition. Accordingly the project was abandoned.

not range over truth values, but rather over the meanings, or Sinne, of propositions (sentences). Other remarks of Nelson's provide further indication that this is how his use of propositional variables is to be interpreted. For example: "-p symbolises the propositional function which is the proper contradictory of the propositional function symbolised by p. -p does not mean 'p is false'." (ibid) "I do not take pq to mean 'p is true and q is true', but simply 'p and q', which is a unit or whole, not simply an aggregate, and expresses the joint force of p and q." (ibid) It seems, then, that Nelson is to be interpreted as having constructed an intensional propositional calculus, unlike the extensional propositional calculus of the Principia in that its propositional variables range over meanings, or Sinne, rather than over truth values. (Such variables are termed intensional propositional variables by Church (1956, p.28).)

In terms of his primitives Nelson defines various other intensional relations, in particular intensional disjunction, pVq, and entailment, pEq. (pp.444-446) The definition of entailment is as follows: $pEq =_{df} -(po-q)$

On the basis of his definitions and explanations of his primitive notions Nelson denies the entailments (i) pqEp and (ii) pEpVq. (pp.447-448) It seems to me, however, that Nelson is mistaken in denying the first of these entailments. Unpacking his definitions, (i) is equivalent to -(pqo-p). Now, surely the conjoined Sinne of any two sentences will never be intensionally consistent with the contradiction of the Sinn of the first. Since -(pqo-p) asserts that this will never be the case, it seems that it, and accordingly (i), are true. To be sure, though, Nelson's explications of his primitive intensional connectives are not precise enough to enable a

definite decision to be made. If we unpack (ii), we arrive at a formula which asserts something like: the Sinn of any sentence S will never be intensionally consistent with the Sinn of a sentence which says that the contradiction of the Sinn of S is consistent with the contradiction of the Sinn of any other sentence T. To say the least, it is hard to assess the truth of this. (The sentence's opaqueness stems at least partly from Nelson's manoeuvre of carrying consistency, which is a concept of metalogic, by means of a connective of his propositional calculus.) So, then, it is doubtful whether Nelson is in a position to deny (i) and (ii).

Nelson says that it is to the entailments (i) and (ii) that "the so-called Paradoxes of Implication are due". (p.448) He goes on to claim that the absence of (i) and (ii) from his intensional propositional calculus renders the entailment calculus contained in it free of the paradoxes which beset the Lewis systems. So far as it goes that is true. However, Nelson has provided no argument at all that the <u>extensional</u> analogues of formulae like p-pEq and pEqV-q are incorrect entailments; and, after all, it is such extensional formulae which are provable in the Lewis systems and which provide the chief objection to the identification of entailment with strict implication. It seems fair to say, then, that Nelson has largely missed the point. Certainly his discussion of the entailments pqEp and pEpVq has no bearing on the correctness of the deduction of p from p$\wedge\sim$p, and pVq from p, in the Lewis independent proof. (Bennett has made this point. (1954, p.454)) Furthermore, entailments between truth functions cannot even be expressed within Nelson's calculus, the vocabulary of which does not contain the truth functional connectives,

and which in any case lacks propositional variables ranging over truth values, as are required in a truth function calculus. His calculus, therefore, is hardly an adequate entailment system.

2. A.E. Duncan-Jones (1935)

In this article, Duncan-Jones constructs a system of entailment which is often taken to be very similar to that of Nelson (e.g. by Bennett (1954, p.455) and Pollock (1966, p.188)). Duncan-Jones himself considers his system to differ from Nelson's only slightly. (vide page 70) However, the two systems are radically different, in that Nelson's comprises an intensional propositional calculus whereas Duncan-Jones obtains his system by augmenting standard truth functional propositional calculus with the relation of entailment, which he introduces as primitive. Non-truth functional negation, conjunction and disjunction do not occur in Duncan-Jones' system, and he does not suppose that they do. (vide page 74)

Duncan-Jones is not directly concerned with attacking Lewis' first independent proof. Rather, he is concerned with another independent proof for the first paradox, discovered by Nelson (1930, pp.447-448). The proof derives from the principle of Antilogism, and is stated by Duncan-Jones as:

(1) $(p \wedge qEr)E(p \wedge \sim rE \sim q)$ [2] (Antilogism)

(2) $p \wedge qEp$ (Simplification)

(3) $p \wedge \sim pE \sim q$

Nelson thinks that his denial of pqEq will fault this proof, but is mistaken since the proof still carries through for the truth functional connectives. Duncan-Jones himself is unwilling to deny the truth functional entailment $p \wedge qEp$. (p.76) Instead, he restricts Antilogism in such a way

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2. Like Nelson, Duncan-Jones writes "E" for "entails".

that the above proof fails. The restriction is effected by writing N for E in Antilogism, where by definition pNq just in case pEq and p is not a conjunction which would still entail q if one of its conjuncts were dropped. (pp.76-77)

The following casts doubt on the utility of Duncan-Jones' manoeuvre. Suppose pEp∨q. Clearly pNp∨q. Applying Contraposition (which Duncan-Jones accepts in the form (pNq)E(~qN~p)), we get ~(p∨q)N~p, whence, truth functionally, ~p∧~qN~p. Using this in conjunction with restricted Antilogism, we have the first paradox in the form ~p∧pNq, which, by definition of N, gives ~p∧pEq.

Thus, if Duncan-Jones' method of faulting Nelson's independent proof is to work he must deny the entailment pEp∨q. He does in fact say "it seems to me that [pEp∨q] is false" (p.78); but he gives no reasons for denying the entailment, and nor does he point out that he must deny it if his treatment of the Nelson independent proof is to be satisfactory. Moreover, this denial is not at all comparable to Nelson's denial of pEpVq. Whilst it may be possible to make out a case for the falsity of Addition for intensional disjunction[3], it is extremely difficult to see how a denial of Addition for extensional disjunction could be supported.

3. von Wright (1957)

von Wright suggests the following analysis of entailment: "p entails q if, and only if, by means of logic, it is possible to come to know the truth of p ⊃ q without coming to know the falsehood of p or the truth of q." (p.181). (A similar account is given by Geach (1958)). He restates this in terms of the notion of logical demonstration, or proof:

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3. And, indeed, Anderson and Belnap do present a plausible case for this in (1962a), using a notion of intensional disjunction less vague than that of Nelson.

"p entails q if, and only if, it is possible that p ⊃ q is demonstrated, but neither ~p nor q is demonstrated". (p.182) von Wright claims that on this analysis of entailment the first paradox is false. "For, consider how p∧~p ⊃ q may be proved. It can be obtained neither by substitution nor by detachment from any formula which is provable in a truth table without also disproving p∧~p. And if a truth table is used directly to prove the formula in question, this truth table would also prove the falsehood of p∧~p." (p.187)

von Wright's analysis blocks the above Lewis independent proof. For, in the course of the proof, both p and ~p are deduced from p∧~p: and to deduce a proposition and its negation from some premiss is a way of demonstrating the falsity of the premiss. Hence, the proof does not show that there is some way of demonstrating p∧~p ⊃ q without also demonstrating ~(p∧~p). Accordingly the proof fails to show that p∧~p entails q. However, the fact that von Wright's analysis blocks the independent proof is not sufficient to show that the proof is faulty. To do that, some argument is required to show that the analysis is <u>correct</u>. And von Wright offers absolutely no argument to this effect. Moreover, von Wright's analysis fails to block an analogue to Nelson's independent proof of the first paradox, noted above. (Pollock (1966) makes this point.) The analogue runs:

(1) (p∧~q ⊃ r) ⊃ (p∧~r ⊃ q)

(2) p∧~q ⊃ p

Therefore (3) p∧~p ⊃ q

The truth table demonstration of (1) does not also demonstrate either the falsity of p∧~p or the truth of q; and similarly for the truth table demonstration of (2). So, on von Wright's analysis of entailment, the

above proof of $p \wedge \sim p \supset q$ by substitution and detachment suffices to show that $p \wedge \sim p$ entails q.

4. Inconsistent Premisses

It has sometimes been claimed that inconsistent premisses entail nothing; and, in particular, that nothing is deducible from the premiss $A \wedge \sim A$ (For example, Strawson (1948, p.186), Körner (1946, p.158)). This doctrine, if true, clearly faults the Lewis independent proof. But the doctrine does not have much to be said for it. The inference from $A \wedge \sim A$ to A in the first independent proof is both truth preserving and relevant. What more can reasonably be required of a valid inference? Perhaps the feeling that inconsistent premisses entail nothing can be traced to a belief that no-one would ever seriously argue from inconsistent premisses. However, this belief was shown to be false in the final section of the preceding chapter.

It is standard practice to establish the inconsistency of a set of statements by inferring an explicit contradiction from them. Are we to take the doctrine that inconsistent premisses entail nothing as showing that this much used practice of reductio ad absurdum is somehow illegitimate - a consequence of the doctrine in which Quine senses "a reductio ad absurdum of the doctrine itself" (1953, p.5). Again, what are we to make of the case where the inconsistency of a theory is discovered fairly late in its development? Were the theorems previously derived merely apparently entailed by the axioms? But then how is the notion of apparent entailment to be explained? Does Γ apparently entail A just in case A is deducible from Γ and Γ is inconsistent?

5. Anderson and Belnap

It was noted above that one of the principles of inference assumed by the Lewis independent proof is that of Disjunctive Syllogism; viz. "from $(A \vee B) \wedge \sim A$ infer B" (henceforward DS). In (1962a), Anderson and Belnap challenge the proof by denying that DS is a valid principle of inference. (p.18) In other words, they deny the truth of $(A \vee B) \wedge \sim A$ entails B (henceforward DSE). The denial is repeated in (1975), where the paper (1962a) is reprinted virtually unchanged as the first part of Chapter III.

In (1963), Anderson remarks "Candour compels me to admit, again in the interests of historical accuracy, that one of the principle reasons for dropping this rule [Disjunctive Syllogism] was that in the presence of this ... rule, almost none of the arguments ... cited above, can be carried through." (p.10) The arguments referred to are to the effect that their system E of entailment "is paradox free, while still allowing as valid anything that any sane man would regard as valid". (ibid, p.7) Anderson goes on to claim that "once one entertains the possibility that [Disjunctive Syllogism is not a valid rule of inference] a host of strong philosophical arguments designed to support that thesis springs to mind". (p.10) But one may well have strong a priori doubts over the validity of any such arguments. For anyone who rejects DS is logically committed to rejecting various other principles of inference. Amongst those which must go are modus ponens for material implication (from $A \supset B$ and A infer B), and the principle: from $\sim A | \sim B$ and $\sim A$ infer B (where $|$ is Scheffer's stroke). Yet what could be more obviously correct than this latter, which says that from the assertions that not both of A and B are false, and that A is false, it can be inferred that B is true?

Anderson and Belnap propose the following necessary and sufficient condition for the truth of first degree entailments in (1962a). They call a conjunction (disjunction) primitive just in case each conjunct (disjunct) of it is either a propositional variable or the negate of a propositional variable (which is to say, is an atom). (p.11) They say an entailment is in normal form just in case its antecedent is of the form $A_1 \vee \ldots \vee A_n$, and its consequent is of the form $B_1 \wedge \ldots \wedge B_m$, where each A_i is a primitive conjunction, and each B_j is a primitive disjunction. (p.12) (Presumably Anderson and Belnap count any atom as trivially both a primitive conjunction and a primitive disjunction - otherwise DSE and many other entailments have no normal form.) An entailment F has a normal form G just in case G is in normal form and is derivable from F by applying various specified truth functional manipulations to the antecedent and consequent of F. (p.13) Anderson and Belnap then claim that a first degree entailment is valid if and only if it has a normal form $A_1 \vee \ldots \vee A_n \rightarrow B_1 \wedge \ldots \wedge B_m$ such that each A_i shares an atom with every B_j. (p.13) They call first degree entailments satisfying this condition tautological entailments. (ibid)

DSE has a normal form $(\sim A \wedge A) \vee (\sim A \wedge B) \rightarrow B$ and thus fails the above necessary and sufficient condition for tautological entailmenthood, $(\sim A \wedge A)$ failing in general to share a variable with B. (p.20) (Anderson and Belnap point out (p.23) that if one normal form of an entailment fails the condition, then all will.) However, this failure does not, in itself, tend to show the incorrectness of DSE. For Anderson and Belnap give no arguments that the set of tautological entailments does coincide with the set of correct first degree entailments. Rather, their strategy

in (1962a) (and the reprint of it in (1975)) is to advocate the condition of tautological entailmenthood as capturing what they consider to be good in various previous unsatisfactory attempts to construct paradox excluding criteria for the validity of first degree entailments (in particular those of von Wright (1957), Geach (1958), and Smiley (1959)). Then they defend their condition against the criticism that it judges DSE as incorrect by providing two arguments that DSE indeed is false. But at the same time they engage in a critique of the first Lewis independent proof, urging that it uses a faulty principle of inference, namely DS. The net result is that the overall strategy of their paper is obscured. Thus, for example, we have Bennett, in criticising the treatment of Lewis' proof in (1962a), writing that the denial of DSE "is prima facie extremely implausible, yet I cannot find that they [Anderson and Belnap] offer any arguments for it." (1969, p.205)

I shall state and criticise Anderson and Belnap's two arguments for the rejection of DSE.

The first is (what they call) an independent proof of the incorrectness of DSE (p.21) It runs:

(1) $\sim A \wedge (A \vee B)$ entails B

(1) entails (2) $(\sim A \wedge A) \vee (\sim A \wedge B)$ entails B

(2) entails (3) $\sim A \wedge A$ entails B

But (3) is absurd. QED

The strategy of this proof is to deduce an obvious falsehood from DSE using only intuitively acceptable principles of inference and therefrom conclude that DSE itself is false. And this, of course, bears some resemblance to the strategy of the Lewis independent proofs. However, there is an

important disanalogy between a proof using this sort of strategy, and a Lewis-style proof. A Lewis-style proof shows that if we accept certain entailments, then we must (logically) accept one of the paradoxes. Whereas the above argument shows that if we reject the first paradox, then we must reject certain entailments. What the above proof shows is not that the rejection of the first paradox commits us to rejecting DSE, but rather that it commits us to rejecting either DSE or one of the other entailments relied on in conducting the proof. The argument itself gives no indication which entailment must be rejected.

For purposes of comparison, I present the following "independent proof" of the falsity of A entails A ∨ B.

(1) A entails A ∨ B

(1) entails (2) A ∧ ~A entails (A ∨ B) ∧ ~A

(2) entails (3) A ∧ ~A entails B

But (3) is absurd. QED

Clearly, this argument does not tend to show the falsity of (1) any more than Anderson and Belnap's tends to show the falsity of DSE. All it shows is that the rejection of (3) demands, logically, the rejection either of (1) or of one of the principles of inference used in the proof. And similarly for Anderson and Belnap's argument.

I shall now consider Anderson and Belnap's second argument for the rejection of DS. Their approach is to attempt to show that an argument "A or B; ~A; therefore B" is not generally valid when "A or B" is a truth functional disjunction. The argument is valid, trivially, whenever either ~A alone entails B, or A alone entails B. (p.19) The argument is non-trivially valid when "there is relevance between the disjuncts" A, B; and is non-trivially valid in no other case. (ibid)

Anderson and Belnap do not say explicitly what they mean by "relevance" here. However, it emerges fairly plainly from their discussion (especially page 22) that they take it to be a necessary and sufficient condition for the relevance of the disjuncts of "A or B" that the disjunction be a sufficient condition for the truth of the (possibly counter-factual) subjunctive conditional: "If it were not the case that A, then it would be the case that B". They term a disjunction intensional just in case there is relevance between the disjuncts.[4] (p.20)

It is worth stressing that the intensionality of a disjunction is, as the matter is presented by Anderson and Belnap, a logical, and not an epistemological, question. The disjunction "A or B" either is, or is not, a logically sufficient condition for the subjunctive conditional "if it were not the case that A, then it would be the case that B". If it is, the disjunction is intensional: if it is not, the disjunction is not intensional (in the Anderson-Belnap sense). Both Bennett (1969, p.205) and Woods (1965, p.317) have, in different ways, criticised Anderson and Belnap's rejection of DS on the basis of the mistaken premiss that whether or not a disjunction is intensional depends on the grounds one may have for accepting or asserting it.

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4. In case the reader feels any puzzlement over whether a disjunction which is intensional in Anderson-Belnap's sense, could be false, I offer the following example. Smith, on seeing someone across a room, declares to himself: "If Jones were not dead, I would swear that man is Jones", Thus the disjunction "Jones is dead or Smith swears that man is Jones" is intensional. But unbeknownst to Smith, Jones is not dead, and the man in question is indeed he. Thus the disjunction is false (for Smith, believing Jones to be dead does not swear that the man is he).

Anderson and Belnap argue, quite correctly, that a truth functional disjunction will not in general be intensional. For consider the truth functional "Napoleon was born in Corsica or 666 equals the sum of its factors". This disjunction is true but it is clearly false that if it were not the case that Napoleon was born in Corsica, then it would be the case that 666 equals the sum of its factors. Hence the above disjunction is not intensional. (p.22)

It follows from these considerations that DSE is not in general a correct entailment.

Anderson and Belnap are now able to advance the following case against the first independent proof. Since the move of Addition by which (4) is obtained from (2) consists in disjoining an arbitrary B to A, A and B will not be relevant to one another. Accordingly (5) may not be legitimately inferred by DS from (3) and (4). (p.19) (Numerals refer to the lines in my presentation of the independent proof at the beginning of this chapter.) "Hence there is a sense in which the real flaw in Lewis' argument is not a fallacy of relevance but rather a fallacy of ambiguity: the passage from (2) to (4) is valid only if the "∨" is read truth functionally, while the passage from (3) and (4) to (5) is valid only if the "∨" is taken intensionally." (p.19)

The crux of Anderson and Belnap's criticism of DSE is the claim that the argument "A or B; ~A; therefore B" is non-trivially valid only when "A or B" is an intensional disjunction. In view of the importance of this claim, one might expect the sustained argument of a case for it. However, Anderson and Belnap content themselves with a case consisting of the following two moves. Firstly, they provide an example of a valid

argument having the form in question, and point out that the disjunction in the premisses (let it be "A or B") may not be taken truth functionally, since the arguer would "naturally feel" that if A or B, then if it were not the case that A, it would be the case that B. (p.22) Secondly, they issue a challenge to anyone who disagrees with the claim, to produce an example of a "serious" argument having the form in question and such that there is no relevance between the disjuncts in the premiss. (p.22) Nothing else is said in support of the claim.

I offer the following as a counter example to Anderson and Belnap's claim:

Suppose I am sitting an intelligence test. The instructions for one of the questions read: "Below are a list of sentences of the form "either P or Q". Each of these sentences is true. Agree with or deny each of P and Q in each case." One of the sentences is: "Either Columbus was the first man to set foot on American soil (A), or N'Djemina is the capital of Chad (B).". I perceive that it is certainly false that Columbus was the first man to set foot on American soil, the continent having had an indigenous population prior to its discovery by European navigators. I conclude that N'Djemina is the capital of Chad.

Thus I have argued from A or B, and not-A, to B. I would make the following remarks.

(i) Clearly I have argued satisfactorily.

(ii) "A or B" is not an intensional disjunction, since "A or B" is true, whilst "If it were not the case that Columbus was the first man to set foot on American soil, then it would be the case that N'Djemina is the capital of Chad" is clearly false.

(iii) Whoever set the paper takes "A or B" to be true, knowing that he obtained "A or B" by disjoining a false sentence to a true one.

(iv) From (ii) and (iii) it is clear that "A or B" is 'A ∨ B". Hence my argument has the form "p ∨ q;~p; therefore q".

(v) In solving the question, I nowhere rely on the supposition that if A or B then if it were not the case that A, it would be the case that B. Indeed, before commencing to solve the question, I can see - if I even bother to think about it - that "If it were not the case that A, then it would be the case that B" is obviously false. Yet according to Anderson and Belnap, it is only in virtue of a knowledge of the truth of this conditional that I can solve the question in the way I am supposing myself to have done i.e. by arguing in accordance with DS.

Another counterexample to the claim that DS is a correct principle of inference only if the disjunction involved in the premisses is intensional, has been provided by Curley (1972). I reproduce the appropriate passage from page 202 of Curley's paper.

"There are --- cases which look like unequivocal counter-examples, as -

A: "Who said: 'Give me liberty or give me death?'"

B: "It was either Nathan Hale or Patrick Henry - I'm not sure which."

C. "It wasn't Nathan Hale. *He* said 'I only regret that I have but one life to give for my country.'"

A. "Then I guess it must have been Patrick Henry."

I think it clear that B will take his disjunctive statement to be confirmed if he later learns that it was Patrick Henry; that he would have accepted 'If it wasn't Nathan Hale, then it was Patrick Henry' as a paraphrase of

that disjunctive statement; and that he would find faintly absurd the suggestion that he should be prepared to assent to "If Patrick Henry hadn't said 'Give me liberty or give me death', Nathan Hale would have." So while I think it is true that anyone who uses an 'or' statement as a premise in a disjunctive syllogism will be equally prepared to assent to the corresponding indicative conditional (simply because this is a general feature of our use of disjunctive propositions), I do not think it is true that anyone who uses an 'or' statement as a premise in a disjunctive syllogism will necessarily be prepared to assert the corresponding subjunctive conditional."

Anderson (1972b) discusses Curley's counterexample. He agrees in the light of the example that the Anderson-Belnap claim of (1962a) was too strong: there are satisfactory arguments of the form in question, Anderson agrees, whose disjunctive premiss fails to support a subjunctive conditional of the stipulated type. (p.9) However, Anderson goes on to argue that the "or" in Curley's "It was either Nathan Hale or Patrick Henry" is nonetheless not truth functional. He writes: "I have doubts about whether B would "take his disjunctive statement to be confirmed if he later learns that it was Patrick Henry". --- For suppose that the conversation goes on with

D: "It was Patrick Henry; I just looked it up in Bartlett."

It would seem again "faintly absurd" if we were to hear:

B: "Aha! I was right all along."" (p.9)

Anderson's argument seems weak to me, however. Certainly the response he suggests for B would sound faintly absurd. But this is not the only response that would show B to be taking the truth of "It was Patrick Henry" to establish the truth of his original disjunction "It was

either Nathan Hale or Patrick Henry". Another such response is "I was right that it was one of the two, then." And surely it would not seem at all absurd if B were to respond in this way.

Despite Anderson's concession to Curley in (1972b), Anderson and Belnap repeat the claim of (1962a) unmodified in (1975). However, they do there add the rider "we do not claim that our argument on this point, such as it is, is conclusive." (p.177, their emphasis) One is then left wondering how they do see themselves as having supported their rejection of DSE.

Chapter VIII

The System E of Entailment

In view of the amount of discussion and approval which the work of Anderson and Belnap currently enjoys, we devote this chapter to an exposition and criticism of their system E of entailment. A critique of the Routley-Meyer semantics is included.

In (1956), Ackermann set out a paradox free entailment system which is extremely natural in that all of its fifteen axioms are entailments horizontalising simple and fundamental principles of inference of propositional logic. Anderson and Belnap have taken over (with due credit) Ackermann's system and amended it to the system of entailment they call E. (The chief change is the omission of modus ponens for material implication from the inference rules of the system.) E is a powerful, natural and technically simple paradox free entailment system. The metatheory and semantics of E have been intensively studied in the past decade. Further, Anderson, Belnap, and their colleagues (in particular Meyer and Routley) have thoroughly discussed the philosophical motivation of E in an impressive array of papers. It is widely accepted amongst those who reject the paradoxes that E provides, or very nearly provides, the correct formal analysis of entailment.

1. Exposition of E

The following axiomatisation of E is taken from Anderson and Belnap (1975), Chapter IV.

The primitives of E are $\&$, $\vee$, $^{-}$, and $\rightarrow$ (conjunction, disjunction, negation, entailment). A rule of substitution is dispensed with in favour of schemata. Punctuation dots are to be understood in accordance

with the conventions of Church (1956). Parentheses are restored by association from the right.

The axioms of E are:

E 1 $A \to A \to B \to B$

E 2 $A \to B \to . B \to C \to . A \to C$

E 3 $(A \to . A \to B) \to . A \to B$

E 4 $(A \,\&\, B) \to A$

E 5 $(A \,\&\, B) \to B$

E 6 $(A \to B) \,\&\, (A \to C) \to . A \to (B \,\&\, C)$

E 7 $\Box A \,\&\, \Box B \to \Box(A \,\&\, B)$ $[\Box A =_{df} A \to A \to A]$

E 8 $A \to A \vee B$

E 9 $B \to A \vee B$

E10 $(A \to C) \,\&\, (B \to C) \to . (A \vee B) \to C$

E11 $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee C$

E12 $A \to \bar{A} \to \bar{A}$

E13 $A \to \bar{B} \to . B \to \bar{A}$

E14 $\bar{\bar{A}} \to A$

The inference rules of E are:

Modus ponens or ($\to$E): If $A \to B$ is asserted, then from A to infer B.

Adjunction or ($\&$I): From A and B to infer $A \,\&\, B$.

Belnap (1960) has proved that $A \to B$ is a theorem of E only if A and B share a propositional variable. It follows immediately that neither $A \,\&\, \bar{A} \to B$ nor $A \to B \vee \bar{B}$ is provable in E. Further, since (i) at least one formula of each of the forms $\Box A$ and $\overline{\Diamond A}$ is provable in E, and (ii) modus ponens on $\to$ is an inference rule of E, it follows that neither $\Box A \to B \to A$ nor $\overline{\Diamond A} \to A \to B$ is a theorem of E.

Belnap has also proved that an entailment A → B where A and B contain no occurrences of → (a first degree entailment in Anderson and Belnap's terminology) is provable in E if and only if A → B is a tautological entailment. (Belnap, 1959) (Anderson and Belnap's necessary and sufficient condition of tautological entailmenthood is stated in section 5 of the preceding chapter.) Thus if Anderson and Belnap are right in their claim that the set of correct first degree entailments is the set of tautological entailments, then the first degree fragment of E is complete. However, as previously remarked, Anderson and Belnap have presented no arguments for the correctness of this claim, and in section 2 below, we indicate that the claim is in fact false.

1.1 Use and mention

It will be apparent from the axioms of E that → is a connective, and not a semantical predicate. Belnap (1975) has argued that such a treatment of entailment is satisfactory. In our review of this paper in Chapter I, section 2, we conclude that Belnap has provided no good arguments that this procedure, which involves a mixing of concepts of logic and metalogic, is satisfactory. However, we do take such results as theorems 9.3 and 11.1 of Chapter III to show that the procedure is formally acceptable. Nonetheless we have repeatedly stressed that the procedure is likely to be conceptually harmful in the early stages of an investigation into entailment. At what point in the study the benefit of the formal simplicity of the connective approach to entailment begins to outweigh the benefit of the conceptual clarity of the metalinguistic approach, is a matter for the decision of the individual researcher or

pedagogue. Certainly we would not wish to criticise Belnap and his colleagues for using the above axiomatisation of E in their ongoing researches. However, it is unfortunate, in our opinion, that the connective approach to entailment is used from the outset in books like Kielkopf (1977) and Anderson and Belnap (1975), which are intended partly, perhaps chiefly, as textbooks on entailment addressed to students innocent of the subject.

2. First Criticism of E: the non-theoremhood of DSE

Meyer and Dunn (1969) have proved that the following rule of inference is admissible in E: $\dfrac{\vdash_E A \vee B,\ \vdash_E \bar{A}}{\vdash_E B}$.

(A rule r is admissible in a system S just in case the theorems of S with r are exactly those of S.) This rule is often known as (γ), Ackermann having named it thus in his (1956) system Π', from which E is drawn.

The admissibility in E of (γ) must not be confused with the theoremhood in E of DSE: $(A \vee B) \,\&\, \bar{A} \rightarrow B$. That DSE is not a theorem of E is a consequence of Belnap's (1959) metatheorem stated in the last section. Moreover, if DSE is added to the axioms of E, both $A \,\&\, \bar{A} \rightarrow B$ and $A \rightarrow B \vee \bar{B}$ become theorems of the system (as Anderson has shown (1960, p.210)).

In the preceding chapter it emerged that Anderson and Belnap have provided no arguments to support their claim that DSE is an incorrect principle of inference. It also emerged that it is possible to exemplify DSE with "real life" arguments which appear to be perfectly satisfactory. In our discussion of the Bennett argument (Chapter VI, section 6) it was remarked that one should be wary of concluding that an argument exemplifies

a correct principle of inference from the fact alone that it is satisfactory in its context. An argument may be satisfactory in its context yet neither be truth preserving nor exhibit premiss/conclusion relevance. However, the satisfactoriness of the two exemplifications of DSE given above, unlike the Bennett argument, does not seem to depend upon special features of their contexts: they are satisfactory tout court. Thus the availability of such exemplifications counts heavily in favour of the formal correctness of DSE as a principle of inference. Furthermore, DSE prima facie exhibits premiss/conclusion relevance in that the conclusion is a constituent sentence of the disjunctive premiss, and not, moreover, a merely "idle" constituent as in the case of $A \wedge \sim A \wedge (B \vee C) \rightarrow B$. On the whole, then, it seems that there is very little leverage to be obtained for a denial of DSE.

There are two ways in which an advocate of the Anderson-Belnap approach to relevance logic can attack our case for DSE. Firstly, he can attempt to establish the non-satisfactoriness of the given exemplifications of DSE. But, as we have seen, the best criticism Anderson could find of Curley's sample argument has little thrust (vide Chapter VII, section 5). Secondly, moving away from disputed examples to general theory, he can produce a well motivated analysis of premiss/conclusion relevance which shows DSE to commit a fallacy of relevance. However, neither Anderson, Belnap, nor any of their colleagues, has attempted to state such an analysis. In the following chapter we supply an analysis of relevance; and it turns out that on this analysis the premisses and conclusion of DSE are indeed relevant.

Our position, then, is that DSE is a correct principle of inference of relevance logic. Thus since DSE is not a theorem of E, the system is incomplete. Moreover, this incompleteness cannot be remedied by adding DSE to the axioms of E, for the resulting system will contain the first and second paradoxes.

3. Second Criticism of E: Anderson and Belnap have no good account of relevance with which to motivate E

3.1 The proof-theoretic account of relevance

Ackermann, in his original (1956) presentation of Π', makes no attempt to explicate the theory of premiss/conclusion relevance ("logischer Zusammerhang") associated with the system. In (1958), Anderson and Belnap emend Π' to E, by dropping the rule (γ) and emending an axiom to render redundant a further rule (δ). In this paper they indicate that they have an account of relevance with which to motivate E. The account is presented in Anderson (1960). "We first formulate a system E* for which we define the notion in question [premiss/conclusion relevance], and then prove the equivalence of E* with E. E* is formulated with the help of the subordinate proof format of Fitch (1952). Our aim, roughly speaking, is to alter the subordinate proof technique in such a way as to keep track of all the hypotheses which are actually used as premisses in the application of rules leading to a proposition B, and then to restrict the introduction rule for implication in such a way that $A \rightarrow B$ will be said to follow from a proof of B on the hypothesis A, only if A is used in arriving at B." (Anderson, 1960, pp.203-204) Anderson then introduces a method of numerical subscripting to keep track of which hypotheses have been used in obtaining a line of a proof. (ibid p.204)

Anderson and Belnap (1975) use Anderson's proof theoretic account of relevance to motivate E. "It is crashingly obvious that we did not prove it [$(A \rightarrow A)$] <u>from</u> the hypothesis B. --- This fact suggests a solution to the problem [of characterising the notion of premiss/conclusion relevance]: we should devise a technique for keeping track of the steps used, and then allow application of the introduction rule [for $\rightarrow$] only when A is relevant to B in the sense that A is <u>used</u> in arriving at B." (1975, p.18) "The subscripting technique --- may be construed as a formal analysis of the intuitive idea that for A to be relevant to B it must be possible to <u>use</u> A in a deduction of B from A." (ibid, p.30)

At first sight Anderson's proof theoretic account of relevance - the "use account" - seems obviously unsatisfactory. For the first Lewis independent proof is a derivation of B from the hypothesis $A \wedge \sim A$ in which the hypothesis certainly gets used in arriving at the conclusion. Thus one is naturally inclined to say that the availability of the first independent proof shows that on the use account of relevance, B is relevant to $A \wedge \sim A$, which demonstrates the absurdity of the account. A detailed examination of Anderson's system E* does nothing to temper this reaction.

E* is rechristened FE in Anderson and Belnap (1975), where the system is presented as follows (p.276):

Hyp. A step may be introduced as the hypothesis of a new subproof, and each new hypothesis receives a unit class $\{k\}$ of numerical subscripts, where $A_{\{k\}}$ is the k'th hypothesis of the proof.

Rep. A_a may be repeated as a new line of the same subproof, retaining the relevance indices a.

Reit. $A \to B_a$ may be reiterated, retaining a, as a line of any subsequent subproof.

$\to$I. From a proof of B_a on hypothesis $A_{\{k\}}$ to infer $A \to B_{a-\{k\}}$ provided k is in a.

$\to$E. From A_a and $A \to B_b$ to infer $B_{a \cup b}$.

&I. From A_a and B_a to infer $A \,\&\, B_a$.

&E. From $A \,\&\, B_a$ to infer A_a. From $A \,\&\, B_a$ to infer B_a.

$\vee$I. From A_a to infer $A \vee B_a$. From B_a to infer $A \vee B_a$.

$\vee$E. From $A \vee B_a$, $A \to C_b$, and $B \to C_b$, to infer $C_{a \cup b}$.

$\sim$I. From a proof of $\bar{A}_a$ on the hypothesis $A_{\{k\}}$, to infer $\bar{A}_{a-\{k\}}$, provided k is in a.

$\sim$E. From B_a, and a proof of $\bar{B}_b$ on the hypothesis $A_{\{k\}}$, where k is in b, to infer $\bar{A}_{(a \cup b)-\{k\}}$.[1]

$\sim\sim$E. From $\bar{\bar{A}}_a$ to infer A_a.

This is a harmonious system of rules for introducing and eliminating the logical constants in a way which pays due attention to the hypotheses on which each item of the proof depends. The methods of computing the subscripts of the conclusion of each rule from the subscripts of the premisses are extremely natural. The restriction on $\to$I is also very natural, ensuring that the hypothesis $A_{\{k\}}$ has actually been used in the proof of B_a.

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1. Anderson and Belnap in fact call $\sim$E "Contrap" - presumably short for contraposition. $\sim$E is the name given to the rule in Anderson (1960).

Unfortunately, however, this system of rules is not equivalent to E. Axiom E11 of E, $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee C$, is not provable from the rules. In view of this Anderson simply adds a distribution rule to those listed above in order to obtain the full system FE: from $A \,\&\, (B \vee C)_a$ to infer $(A \,\&\, B) \vee C_a$. But if a distribution rule which is not an introduction or elimination rule can be tacked on at will, why not others? In particular, why may not DS be added in the form: from $\bar{A} \,\&\, (A \vee B)_a$ to infer B_a? In motivating the adoption of the distribution rule Anderson and Belnap write that the move from $A \,\&\, (B \vee C)$ to $(A \,\&\, B) \vee C$ "is valid in the sense that it meets every criterion of relevance and necessity satisfied by the other theorems of E". (1975, p.274) But the only criterion of relevance which Anderson and Belnap have stated is the necessary condition that the antecedent and consequent of an entailment share a variable (ibid, pp.32-33), and DS also satisfies this condition. So, since no one is supposing DS to commit a fallacy of necessity, the argument which Anderson and Belnap use to motivate the adoption of the distribution rule also suffices to motivate the adoption of DS. In fact Anderson (1960) rejects the adoption of DS for the simple reason that its addition leads to the provability of $A \,\&\, \bar{A} \rightarrow B$. (p.210)

It seems, then, that the paradox freedom of FE is not a consequence of its embodiment of the principle that $A \rightarrow B$ shall follow from a proof of B on hypothesis A only if A is used in arriving at B, but rather is a consequence of the fact that DS does not figure amongst the rules by which one is permitted to arrive at B. Thus, since the use account itself plainly does not show DS to commit a fallacy of relevance, it cannot correctly be claimed that E is motivated by the

use account, or that E is a system of relevant entailment incorporating the analysis of relevance provided by the use account.

The use account is not even part of the story of what account of relevance E answers to. For it is possible to reformulate classical logic without loss in such a way that we "keep track of all the hypotheses which are actually used as premisses in the application of rules leading to a proposition B, and then restrict the introduction rule for implication in such a way that A → B will be said to follow from a proof of B on the hypothesis A, only if A is used in arriving at B." (The quotation is from Anderson (1960).) To establish this, we present the following formulation of classical entailment theory. The rules Hyp, Rep, Reit, →I, →E, &I, &E, ∨I, ∨E, and ~~E are taken from FE. To these are added:

~I: from $A \rightarrow \Lambda_a$ to infer $\bar{A}_a$

~E: from A_a, $\bar{A}_b$ to infer $\Lambda_{a \cup b}$

ΛE: from Λ_a to infer A_a

&I, &E, ∨I, ∨E, ~I, ~E, ΛE and ~~E are the rules of Genzen's natural deduction system NK for classical propositional logic (1935, pp.77-81) augmented with Anderson's subscripting device for tracking hypotheses through a proof. Hyp, Rep and Reit modify NK to a Fitch-style natural deduction system. →I and →E extend NK to a classical entailment system. Λ is a sentential constant intended to denote Falsehood. 0 = 1 is often chosen for Λ. The three rules ~I, ~E and ΛE give a classical treatment to negation, and of course each of these rules has a strong kinship with the first paradox. Nonetheless the rules do not block the tracking of hypotheses through the proof, and their presence in the system does not alter the fact that A → B is asserted on the basis of a proof of B on hypothesis A, only if A is used in arriving at B.

The use account of relevance, then, is no account of relevance at all once there are truth functional connectives in the picture; and no motivation for a system of relevant entailment between truth functions is provided by the idea that a hypothesis A should be said to entail a conclusion B only if A is used in arriving at B. (Although it should be added that the use account does suffice to motivate a system of relevant entailment in the restricted case where → is the only logical symbol.)

The comparison of FE with the above classical entailment system illustrates the fact that it is in its treatment of negation that FE differs essentially from classical logic. It is to its non classical treatment of negation that E owes its paradox freedom. So, to be convinced that E has any claim to being a system of relevant entailment, as opposed to a merely paradox excluding system, one would need to see an account of relevance which motivates the treatment given to negation in E. No such account has been attempted, and it seems to me very doubtful that one could be found. (Just how tortured negation is in E is illustrated by the negation introduction and elimination rules in Anderson and Belnap's sequent calculus formulation of the first degree fragment of E (1975, p.178).) It seems likely, then, that E answers to no account of relevance, and the system owes its paradox freedom to ad hoc manoeuvres.

3.2 The semantical account of relevance

In (1960) Belnap discusses an account of relevance which he explicitly contrasts with Anderson's proof theoretic account of relevance. (p.144) This alternative approach to relevance is discussed again in Anderson

and Belnap (1975), where it is once more explicitly contrasted with Anderson's account. (p.32-33) (It is here that the terms "semantical" and "proof theoretical" are used to characterise the two approaches to relevance.)

This alternative approach to premiss/conclusion relevance may be described informally as the idea that a pair of sentences are mutually relevant when they have "common meaning content". (1975, p.33) "We would wish to reject A → B if A and B have nothing to do with one another, if, that is, A and B are totally disparate in meaning". (1960, p.144) We will call this approach to premiss/conclusion relevance the <u>semantic overlap</u> approach.

It should be stressed that whether or not sentences have "semantic overlap" is a logical, not a psychological, matter. Bennett (1969, p.222ff) has suggested an account of meaning connection which he calls "topic linkage", which is at least partly psychological: topic linkage involves "a question about what the arguer believes and about what he wants to know or achieve" (p.223). Semantic overlap, on the other hand, is solely a function of the semantics of the component terms of sentences.

As Belnap remarks, the problem is to characterise this notion of relevance formally. He continues: "Confining our attention to propositional logic, a partial solution becomes almost obvious once we notice that in propositional logic, commonality of meaning is carried by identity of propositional variables. Thus, for A to be relevant to B in the required sense, a necessary condition is that A and B have some propositional variable in common." (1960, p.144)

The solution is "partial" only: Belnap has stated a necessary, but not a sufficient, condition for two sentences of propositional logic to

be relevant in the required sense. This point is stressed again in (1975): "we propose as a necessary, but by no means sufficient, condition for the relevance of A to B --- that A and B must share a variable." (p.33) However, in neither place is the denial of sufficiency explained. A consequence of taking variable sharing as sufficient for relevance is that relevance becomes a non-transitive relation. Thus if an inference figure is to be taken as exhibiting an entailment just in case it is truth preserving and its premisses are relevant, in this sense, to its conclusion, then entailment also is non-transitive relation. A rather unclear passage on page 11 of (1962a) perhaps indicates that this is Anderson and Belnap's reason for denying the sufficiency of the variable sharing condition. However it seems to me obvious that relevance in the sense of semantic overlap <u>is</u> a non-transitive relation: A is relevant to A ∧ B and A ∧ B is relevant to B, but it need not be the case that A is relevant to B. Thus no argument against the sufficiency of the variable sharing condition is provided by the fact that if the condition is taken as sufficient, then relevance is made non-transitive.

It must be stressed that Belnap, having given no sufficient condition for relevance in the sense of semantic overlap, has provided no formal account of when a premiss and a conclusion are relevant in this sense. He has indicated informally a possible notion of relevance, and has stated a necessary condition for the presence of relevance in this sense. He has proved that all theorems of E satisfy this necessary condition. (Belnap, 1960b) But in no way has he established E is answerable to a semantic overlap account of relevance.

It seems to me that the semantic overlap approach to relevance is undoubtedly the approach to follow in the quest for an account of premiss/conclusion relevance. In the next chapter a formal analysis of semantic overlap is provided, and the analysis is used in the construction of a system of relevant entailment. This system is provably not equivalent to E.

3.3 The rejection of DSE

In both (1962a) and (1975) Anderson and Belnap claim that DSE [$\sim A \wedge (A \vee B) \rightarrow B$] commits "a fallacy of relevance" (1975, p.165). It emerges that the fallacy of relevance committed is that DSE licences the inference of B from $\sim A \wedge (A \vee B)$ when there is no "relevance between the disjuncts", A, B. (ibid) To draw the conclusion in the absence of such relevance "is a simple inferential mistake". (ibid) Anderson-Belnap's arguments for this latter assertion have been set out and criticised in Chapter VII section 5, above. For the purposes of the present discussion we remark that Anderson and Belnap are not rejecting DSE on the ground that its premisses are irrelevant to its conclusion, but rather that two constituents of the premisses are not relevant to one another. Thus their rejection of DSE has little to do with their demand for an analysis of entailment which takes into account the fact that relevance between premisses and conclusion is essential to sensible argumentation. Moreover, A and B are required to be relevant to one another in the sense that ⌜A or B⌝ must support the subjunctive conditional: if it were not the case that A, then it would be the case that B. (1975, p.176) Thus we have here yet a third sense of relevance.

Do Anderson and Belnap see DSE as committing a fallacy of premiss/conclusion relevance, as well as a fallacy of relevance of the kind just discussed? I think the answer is that they do, although the fact does not emerge with clarity from their awkward discussion of DSE in (1962a), nor from the repeat of it in (1975). The following argument can be pieced together, however. "$A \vee B \rightarrow C$ is valid if and only if $A \rightarrow C$ and $B \rightarrow C$ are both valid." (1975, p.156) Now, DSE has a normal form $(\sim A \wedge A) \vee (\sim A \wedge B) \rightarrow B$. (Anderson and Belnap define "normal form" on page 156: the definition is given above in Chapter VII, section 5.) Since an entailment and its normal form are equivalent (p.156), and $\sim A \wedge A \rightarrow B$ commits a fallacy of premiss/conclusion relevance, it follows that DSE commits a fallacy of premiss/conclusion relevance - or perhaps "involves a fallacy of premiss/conclusion relevance" is a better way of expressing the matter.

If this exegesis is correct, Anderson and Belnap emerge in the happy position of denying DSE on the same grounds on which they deny the first and second paradoxes: DSE commits a fallacy of premiss/conclusion relevance, albeit in a more circuitous way than the paradoxes. It remains to be seen, however, whether their argument that DSE commits such a fallacy is correct. The argument depends on the claim that $A \vee B \rightarrow C$ is valid if and only if $A \rightarrow C$ and $B \rightarrow C$ are both valid. Anderson and Belnap present no arguments for the claim: they simply state it. Certainly the claim is correct for classical entailment, but it seems to me unlikely to be true for relevant entailment. For surely C may be relevant to $A \vee B$ without being relevant to both A and B? The pair $(A \vee B, C)$ may satisfy Belnap's necessary condition for relevance whilst one of the pairs (A, C), (B, C) fails it.

4. The Routley-Meyer Semantics: Motivation for E?

Routley and Meyer (1972a, 1972b, 1973) set out model theoretic semantics for various relevance logics, in particular the systems R and NR. (NR is sometimes known as $R^{\Box}$.) The semantics develop earlier ideas of R and V Routley (1972). The essence of the approach is the use in the model theory of a group of structures broader than the set of all possible worlds. These structures are named "set ups" by the Routleys. The key feature of a set up is that, unlike a world, the set of sentences determined by it may be either or both inconsistent and negation incomplete: an arbitrary formula and its negation may both hold, or both fail to hold, in some set up. The set of worlds is strictly included in the set of set ups: a world is a consistent and negation complete set up. This key feature of set ups provides an easy way of falsifying the paradoxes of classical entailment

$A, \sim A \rightarrow B,\ B \rightarrow A, \sim A$: the first is falsified in an inconsistent set up in which both A and $\sim A$ hold but B fails, and the second in an incomplete set up in which B holds but both of A and $\sim A$ fail.

The requirement that a possible world determine a consistent and complete set of sentences is built into the usual semantical treatment of negation, whereby $\sim A$ holds in a structure just in case A fails to hold in that structure. It follows immediately that exactly one of every pair of sentences A, $\sim A$ must hold in every structure. The obvious way, then, to obtain inconsistent and incomplete structures is to modify the classical treatment of negation, and such a re-working of the truth conditions of negated sentences is the device which sets the Routley-Meyer semantics into motion. It is stipulated that $\sim A$ should hold in a

set up H just in case A fails to hold in H*, the complement of H, which is the image of H under a one-one function *. Thus H may be inconsistent, since A can hold in H and fail in H*, and H may be negation incomplete, since B may fail in H and hold in H*. When H = H* negation behaves normally, and such a set up is consistent and complete.

A striking feature of the Routley-Meyer semantics is that the same simple manouvre which enables the falsification of A, ~A → B and B → A, ~A also leads without further dressing to the falsification of DSE: ~A, A ∨ B → B. For consider an inconsistent set up H in which both A and ~A hold but B fails: using the usual interpretation of ∨, the premisses of DSE hold in H whilst the conclusion fails. It might be thought that this fact tends to provide a semantic justification of the claim that DSE embodies a fallacy of relevance, and thus supplies independent support for Anderson-Belnap's course of excluding DSE from their formalisation of relevant entailment. This thought is not correct, however, and two different arguments to that effect are provided below. It is also questioned whether the Routley-Meyer theory satisfies those natural requirements which distinguish an illuminating semantics from a merely formal model theory.

Firstly for the sake of definiteness we give a brief exposition of the Routley-Meyer semantical characterisation of the system R (Routley and Meyer 1973).

A relevant model structure (rms) is a quadtuple $<0, K, R, *>$, where K is a set, $0 \varepsilon K$, R is a ternary relation on K, and * is a unary operation on K.

definitions

$$a < b =_{df} R0ab$$

$$R^2abcd =_{df} \exists x(Rabx \,\&\, Rxcd \,\&\, x \in K)$$

postulates

$$R0aa$$

$$Raaa$$

$$R^2abcd \Rightarrow R^2acbd$$

$$R^2abcd \Rightarrow Rabc$$

$$Rabc \Rightarrow Rac^*b^*$$

$$a^{**} = a$$

Let $<0, K, R, *>$ be an rms, $\{T, F\}$ the set of classical truth values; L the language of the system R, WF the wff of L and Atm the atomic members of WF. A valuation v in $<0, K, R, *>$ is a function that assigns a truth value to each member of Atm for each set up in K; and the interpretation I associated with v is the extension of v to all members of WF. Formally:

(a) v is a valuation in $<0, K, R, *>$ provided that v is a function from $Atm \times K$ to $\{T, F\}$ satisfying the following condition, for all p in Atm and a, b in K:

$$a < b \,\&\, v(p, a) = T \Rightarrow v(p, b) = T$$

(b) I is the interpretation associated with v provided that I is a function from $WF \times K$ to $\{T, F\}$ that satisfies the following conditions, for all p in Atm, A, B in WF, and a in K:

(i) $I(p, a) = v(p, a)$

(ii) $I(A \wedge B, a) = T$ iff $I(A, a) = T$ and $I(B, a) = T$

(iii) $I(A \vee B, a) = T$ iff $I(A, a) = T$ or $I(B, a) = T$

(iv) $I(\sim A, a) = T$ iff $I(A, a^*) = F$

(v) $I(A \twoheadrightarrow B, a) = T$ iff for all b, c in K,

$$Rabc \text{ and } I(A, b) = T \Rightarrow I(B, c) = T$$

A formula A is said to be true at a point $a \varepsilon K$ on an interpretation I if $I(A, a) = T$.

It is provable that $0^* < 0$ and that for any formula A, any a, b in K

$$a < b \ \& \ I(A, a) = T \Rightarrow I(A, b) = T. \quad (1973, p.208)$$

It follows that 0 is a consistent and complete set up. This distinguished world plays a crucial rôle in the semantics: a formula is defined as valid in an rms just in case it is true at the point 0 on every interpretation in the structure. Finally a formula is R-valid if and only if valid in every rms. (1973, p.207) Thus if A contains only truth functional connectives as logical symbols, A is R-valid just in case A is classically valid.

$\twoheadrightarrow$ is the relevant conditional connective of the system R. (Routley and Meyer themselves use $\rightarrow$, which we reserve for entailment, relevant or otherwise.) $\Rightarrow$ abbreviates the verb "implies" (or some rough synonym) of the metalanguage, mathematical English. (1972a, p.72 n.1) Routley and Meyer tell us that they "have no story here about what 'implies' means in English", and suggest that no great harm is done if $\Rightarrow$ is read as "materially implies". (ibid) We may well wonder at this suggestion, since it is certainly required that $\Rightarrow$ support modus ponens.

Routley and Meyer (1973) suggest that the semantics given above may be extended to entailment formulae by the clause

$$I(A \rightarrow B, a) = T \text{ iff for all } b \varepsilon K, I(A, b) = T \Rightarrow I(B, b) = T$$

This clause is very natural, paralleling our earlier treatment of classical entailment. I do not know, however, what relation the extended semantics bears to E and NR.

The motivation of the system R was to formalise the theory of the relevant conditional underlying the entailment connective of E (vide Meyer 1968). Thus $\twoheadrightarrow$ underlies $\rightarrow$ in the same way that the classical conditional $\supset$ underlies the entailment connective $\multimap$ of Lewis' modal logic. R was in fact originally obtained simply by adding to E an axiom collapsing modalities, $A \rightarrow \Box A$ (Belnap 1967), although a more elegant systematisation has subsequently been devised (vide Routley and Meyer 1973). The system NR, devised by Meyer and set out in (1968), results from the addition to R of the usual axioms and rules which produce S4 from the set of classical tautologies. It was widely conjectured amongst relevance logicians that E and the nicely motivated NR were equivalent, the modal element removed from E to obtain R having been re-introduced into NR. In (1972a), Routley and Meyer extend the above semantics to NR with an S4-ish treatment of modality. They speculate that E is complete with respect to the new semantics, but remark that "if E and NR should happen to diverge on some formula, it will be E which turns out thereby to be semantically deficient; we do not think that will happen" (p.53). In point of fact, E and NR have been found not to coincide. G.E. Minc (1972) conjectured that the following theorem of NR is not provable in E:

$$(A \rightarrow (B \rightarrow C)) \wedge (B \rightarrow (A \vee C)) \rightarrow (B \rightarrow C)$$

and in (1973) L.L. Maksimova established the conjecture, constructing a semantics for E in which the formula fails. Thus the Routley-Meyer semantics for NR is not also a semantics for E.

I shall now attempt to evaluate the Routley-Meyer semantics.

It is crucial that the semantics preserve the classical meanings of the truth functional connectives. If this is not so, then at the level of first degree formulae the semantics loses contact with the enterprise of formalising the valid inference patterns between formulae of the classical propositional calculus. A specialised interest in just those patterns of inference which exhibit premiss/conclusion relevance will lead us to assign a stronger-than-classical meaning to →, but this interest does not licence, and indeed prohibits, a re-interpretation of any particles of the target language with whose inference patterns we are concerned. For example, the failure of the first degree formula $\sim A \wedge (A \vee B) \rightarrow B$ in a semantics which gives a non classical meaning to any of the connectives $\wedge$, $\vee$, $\sim$ will have no bearing on the key question of whether relevant entailment holds between the formula $\sim A \wedge (B \vee C)$ of classical logic, and the formula B. Routley-Meyer have a responsibility, then, to demonstrate that the innovatory treatment given to $\sim$ in their semantics nonetheless preserves its classical meaning.

Routley (1972, p.337-338) writes as follows: "The new negation is neither queer nor weak because it behaves exactly like classical negation over consistent and complete set ups. Since its behaviour diverges from that of classical negation only over a class of situations not admitted by modal logic, it is not so much the negation introduced by our new rule which is different as the class of situations that the rule entitles us to consider." These remarks do not convince me that negation has remained classical. For surely to understand the interpretation given to some connective by a semantics we need to consider the allowed behaviour of the

connective in all the structures admitted by the theory, and not merely in some subset of them? The fact that negation behaves classically in a truncation of the Routley-Meyer semantics countenancing only consistent and complete set ups is no indication at all that the semantics as it stands interprets negation classically. Routley's argument is simply a petitio principii for our purposes. He argues that since $\sim$ behaves classically in consistent and complete set ups, then $\sim$ is interpreted classically by the semantics, whereas what we are seeking is precisely some sort of argument to the conclusion that the classical meaning of $\sim$ is conserved when a semantics is extended by the inclusion of inconsistent and incomplete structures in the set of permitted models.

Prima facie it would seem that negation is not given a classical interpretation by Routley-Meyer. For doesn't it follow that $\sim$ is non-classical simply from the fact that $A \vee B$ and $\sim A$ can both be true in a model whilst B false in that model? Indeed, what better evidence could there be for the non-classical nature of $\sim$ than that both A and $\sim A$ can be true in the same model (a feature which seems to show that $\sim$ has, in fact, very little to do with negation at all, of which non contradiction is a fundamental property). A reply which Routley-Meyer might make here is that this criticism confuses in a facile way a classical Tarski-like notion of truth in a model with their notion of truth at a point in an interpretation. Having defined truth in a model for atomic sentences, Tarski states the truth conditions for complex sentences in terms of the truth of atomic sentences, and thereby recursively defines truth in a model for complex sentences also. But truth in a model is not the only semantical predicate that a model theoretic semantics can operate with,

and it may be that the predicate used by a particular theory is such as to coherently tolerate the holding in a structure of both a sentence and its classical negation. To take a naive example, a function V(A, x) to truth values, defined recursively in some semantical theory in the familiar way, might have properties which enable the interpretation: V(A, x) = T iff A is believed by x. Clearly the holding of V(A, x) = T and V(~A, x) = T would not by itself lead us to conclude that this semantics gives a non classical meaning to ~, for it is commonplace and unproblematic for an individual to hold (classically) contradictory beliefs.

I think it is fairly clear that Routley-Meyer do see their semantics as involving this kind of deviation from the Tarskian paradigm. They remark in (1973): "Negation requires --- the admission of theories that are inconsistent, incomplete, or both. Justification can be found in belief contexts, or in evaluative ones ---." (p.202) Similarly we have the Routleys writing: "we do encounter [inconsistent and incomplete] set ups in arguing from or about those sets of propositions which are picked out by people's beliefs, opinions, hypotheses and assertions. --- Such intensional functors select sets --- of propositions which may be inconsistent and are always incomplete; we may, for example, fear both P and ~P, and believe neither P nor ~P." (1972, p.344) It seems, then, that we are to regard the semantical function forming the core of the Routley-Meyer theory as some kind of "intensional functor" which parallels the belief predicate in the following respects: (i) I(A, a) = T and I(~A, a) = T may both hold (ii) I(A, a) = T and I(~A, a) = T are not contradictories (iii) these first two properties continue to hold when

~A is taken as the classical negation of A. Moreover this intensional functor must be powerful enough to succeed in assigning the classical meanings, as we have argued, to the truth functional connectives in the recursive semantics. The belief predicate itself, for example, is not powerful enough to fulfil this role. A clause of the form "A o B is believed by x iff A is believed by x or B is believed by x" succeeds - it seems to me - in assigning <u>some</u> definite meaning to o, but not the meaning of truth functional disjunction, since it will often be the case that x will believe A ∨ B but neither believe A nor believe B.

This kind of deviation from the Tarskian paradigm is quite legitimate. We find it, for example, in the Beth tree semantics for intuitionistic logic, where the semantical function used to define the logical constants certainly deviates in formal behaviour from the truth predicate of classical theory, and may be interpreted in terms of a statement's having been verified at a certain stage of knowledge. In particular, neither A nor its negation may have been verified at a particular stage, resulting in the failure in the semantics of the Law of Excluded Middle. This comparison between the Beth and Routley-Meyer semantics may, however, merely heighten our fears that Routley and Meyer have not succeeded in conserving the classical meanings of the truth functional connectives. It is widely accepted that the Beth semantics succeeds, to a certain extent anyway, in capturing the intuitive non classical meanings of the connectives of intuitionistic logic. It is, indeed, often for this reason that the Beth semantics is preferred for the study of intuitionistic systems to the earlier algebraic characterisations due to Jaśkowski,

Tarski, et al, which are merely formal, exhibiting no connection with the intended meanings of the logical constants.

How might we set about deciding the question of whether or not a semantical theory does succeed in assigning certain intended meanings to the logical constants? It would seem that the feature of the theory to consider is the natural interpretation of the semantical function used to recursively define the logical constants in the theory. Via this interpretation we can compare the recursive definition given of the constants with the actual or intended linguistic practices within which the constants receive their intended meanings, and so reach a decision as to whether the definition succeeds in capturing these meanings. For example, by this means we were able above to dismiss as failing to give the classical meaning to $\vee$ any theory containing an $\vee$-clause of the usual form and utilising a semantical function which admitted of interpretation in terms of belief, on the ground that "A $\vee$ B is believed by x" is not a sufficient condition for "A is believed by x or B is believed by x" when the intended meaning of $\vee$ is that of classical disjunction. Similarly interpretability of the semantical function in the Beth tree semantics in terms of verification at a stage of knowledge enables us to connect the formal semantics with our intuitions concerning the intended meanings of the intuitionistic logical constants.

What we should like to be able to do, then, is examine the interpretation Routley-Meyer propose for the semantical function used in their theory (truth at a point in an interpretation, as they read it), and use the proposed notion as the basis for an assessment of whether classical meanings are preserved by their recursive definition of the truth functional

connectives. Unfortunately, however, Routley-Meyer are completely mute on this point, save for drawing the previously noted analogies between the "intensional functor" of their theory and the notions of believing, fearing, and so on. The key semantical function of their theory receives no more than the bare formal treatment in their writings, and we are offered no account of how the formal explanation of the logical constants given in the semantics is to be related to linguistic practice. Perhaps Routley and Meyer would object here that they need give no account of the function, since it is defined recursively in the theory along with the logical constants themselves. But this would be to miss the point: unless we have an independent and intuitive grasp of the significance of the semantical function used in the theory, we are unable to relate the formal semantics to any body of linguistic practice, with the result that we cannot measure up the treatment given to the logical constants in the theory with the intended meanings we have in mind for them. The Routley-Meyer semantics is a merely formal model-theoretic characterisation of the set of sentences provable in NR: no connection is exhibited between the assessment of validity and the intended meanings of the logical constants. Thus no semantical argument for the incorrectness of the entailment DSE is yielded. Such an argument must show how the invalidity of DSE arises from a classical account of the meanings of the truth functional connectives together with a relevance account of the meaning of entailment. But, as we have seen, it is totally unclear what account of the meanings of any of the logical constants is given in the Routley-Meyer semantics.

To record a piece of mere autobiography, I have the feeling that the formal part of the Routley-Meyer theory was constructed first, and the

question of what it all meant was largely not thought about until afterwards. This is not a procedure likely to produce an illuminating semantics, and the result, I think it is true to say, has been an ad hoc and merely formal model theory. I derive little comfort from being able to see how the characterisation of the provable sentences of NR works formally without being able to see how the fact that it works arises from the meanings of the logical constants, as made clear by the theory itself.

We have seen the ad hoc nature of the Routley-Meyer theory exhibited in the construction of the semantical function used in the recursive definition of the logical constants without, apparently, a thought for its non-formal significance. A further good example of the ad hocness of the theory and its resultant lack of flavour is provided by the treatment of the function * figuring in the clause for negation:

$$I(\sim A, a) = T \text{ iff } I(A, a^*) = F$$

This unary operation on set ups carries the full weight of the novel treatment given to negation in the theory, and we might expect a full discussion of its significance and intuitive interpretation. On the contrary, though, in (1972), the paper in which the semantics receives its first and fullest yet motivational discussion, the Routleys restrict their remarks on * to the following: "the negation function [can be] retained by replacing (ii)' by

(iv) A is in H iff A is not in H*

where H* is the image of H under a one-one function *." (p.338) In the later paper (1973), Routley and Meyer do include a belated explanation of *. They write: "[we distinguish] a strong and a weak way of affirming

a sentence A in a given set up. The strong way is to assert A; the weak way is to omit the assertion of ~A. This yields for each set-up a the complementary set-up a*, where what is strongly affirmed in a is weakly affirmed in a* and vice versa. The wanted recursive clause then says that ~A holds in a just when A doesn't hold in a*; the reader will have noted that under normal circumstances, when we affirm just what we don't deny, a and a* coincide, whence the account of negation reduces to the usual one." (p.202) Perhaps this explanation does just about hang together - but some ears are likely to hear the creaking of unwilling ideas marshalled in support of a preconceived formal tactic.

Despite the failings of the Routley-Meyer semantics, might we not still be impressed to some extent by the fact that the same manoeuvre which enables the falsification of the paradoxes of classical entailment also leads to the falsification of DSE? The answer is that we should not be: the introduction of incomplete and inconsistent structures in conjunction with a re-working of the usual definitions of the truth functional connectives is a technique which may be used either to falsify the paradoxes and DSE whilst retaining the principles of Addition and Simplification, or to falsify the paradoxes and these last two principles whilst retaining DSE (recall that the first Lewis independent proof shows that if transitivity is to be retained and the paradoxes rejected, then at least one of DSE, Addition and Simplification must also be rejected). The following semantics is set out to substantiate this point.

A model structure is an ordered quadtuple $<0, K, 2, 3>$ where K is a set, $0 \in K$, and 2, 3 are monadic operations on K satisfying the following conditions (where a is any member of K and we write a^2 for 2(a), a^3 for 3(a)):

$$(a^3)^2 = a$$

$$(a^2)^3 = a$$

$$(a^3)^3 = (a^2)^2$$

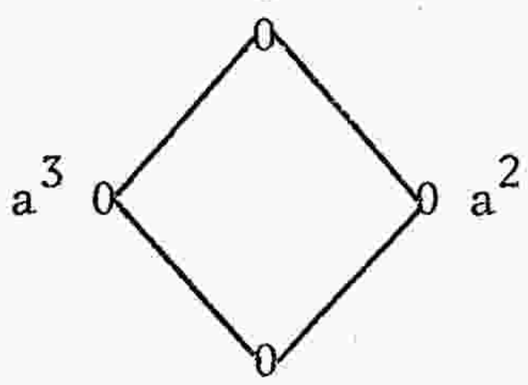
$$0^2 = 0$$

$$0^3 = 0$$

$$a \neq 0 \supset a \neq a^2 \neq a^3 \neq (a^3)^3$$

These conditions may be represented diagramatically as follows:

(i) $a \neq 0$



$(a^3)^3 : (a^2)^2$

(ii) where a = 0, all four points coincide

A valuation v is a function from Atm×K onto {T, F}, where Atm is the set of atomic well formed formulae and {T, F} is the set of classical truth values. I is the interpretation associated with v provided that I is a function from WF×K onto {T, F} which satisfies the following conditions for all A, B ε WF, a ε K (where WF is the set of all well formed formulae):

(i) $I(A, a) = v(A, a)$ if A is atomic

(ii) $I(A \wedge B, a) = T$ iff $I(A, a^2) = T$ and $I(B, a^3) = T$, or else
$I(A, a^3) = T$ and $I(B, a^2) = T$

(iii) $I(A \vee B, a) = T$ iff $(I(A, a^2) = T$ or $I(B, a^3) = T)$ and
$(I(A, a^3) = T$ or $I(B, a^2) = T)$

(iv) $I(\sim A, a) = T$ iff $I(A, a^2) = F$ and $I(A, a^3) = F$

(v) $I(A \rightarrow B, a) = T$ iff for all b ε K, if $I(A, b) = T$, then $I(B, b) = T$

A formula A holds in a model structure <0, K, 2, 3> just in case for every a ε K, every interpretation I, I(A, a) = T; otherwise A fails to hold in the structure. A is satisfiable in the model structure just in case for some a ε K, some interpretation I, I(A, a) = T. A is valid in the model structure just in case for every interpretation I, I(A, 0) = T.

This model theory is similar to that of Routley-Meyer in that tautologies will fail to hold and contradictions will be satisfiable. For example, it is easily checked that A ∧ ∼A is satisfiable in any model structure, and that A ∨ ∼A will fail to hold in any model structure.

As in the case of the Routley-Meyer theory, all and only truth table tautologies are the valid zero degree formulae of any model structure. It is worth remarking, however, that this circumstance is hardly an essential feature of the theory, having been secured with the utmost artificiality by means of the definition of validity and the special axioms for 0. This remark also applies to the Routley-Meyer theory, where essentially the same ploy is used. Thus the fact that the valid zero degree formulae of each theory are all and only truth table tautologies in no way tends to show that the truth functional connectives retain their classical meanings in either theory.

The present theory diverges from that of Routley-Meyer over higher degree validities. Disjunctive Syllogism is valid in either of the forms A ∧ (A ⊃ B) → B and ∼A ∧ (A ∨ B) → B. Disjunction and conjunction are now "twisted", none of the following being valid:

A ∧ B → A (Simplification), A → A ∨ B (Addition), A ∧ (B ∨ C) → (A ∧ B) ∨ C (Distribution), A, B → A ∧ B (Adjunction). Both ∧ and ∨ remain commutative

however. It seems likely that ~ corresponds in behaviour to classical negation to the extent that any formula having at most → and ~ as logical symbols is valid on the present approach if and only if classically valid (no counter-examples having been unearthed).

Our model theory agrees with that of Routley-Meyer in that both $A \wedge {\sim}A \rightarrow B$ and $A \rightarrow B \vee {\sim}B$ are invalid; and the same sort of trick is used to invalidate them in each theory. Routley-Meyer use an ad hoc deviation from the usual treatment of negation, whilst we use equally ad hoc deviations in the case of conjunction and disjunction. As in the case of the Routley-Meyer theory, it is completely obscure what account of the meanings of the logical constants has been given. There is no telling whether the model theory introduces non-classical versions of conjunction, disjunction and negation (and secures in a purely artificial way the validity of all classical truth table tautologies), or whether the theory succeeds in preserving the classical meanings of the truth functional connectives and introduces a non-classical (relevant) entailment relation. Our theory is mere model theory, and not a model theory which is also a semantics: the notions of conjunction, entailment, etc. have been "modelled", but the modelling brings no elucidation of the notions themselves. And this criticism seems to me to apply no less to the Routley-Meyer theory.

Chapter IX

A New Approach to Relevant Entailment

In this chapter we present a formal account of sentence relevance for sentences all of whose logical constants are truth functional. This account is then built into a formal semantics for relevant entailment. The semantics validates A, $A \supset B \rightarrow B$. Finally, a Genzen-style axiomatisation of the formulae validated by the semantics is given.

1. A Theory of Relevance

1.1 Pieces of information and their arithmetic

Declarative sentences give, or carry, information. Part of this information will be determined solely by the semantics of the sentence and a hearer's correct understanding of its semantics. Other parts of the information carried will be a function of this basic part of the information together with the hearer's knowledge, beliefs and aims, the context of utterance, and so on. We make the reasonable assumption that it is possible to isolate the basic, or semantically determined, information carried by a sentence. Each sentence gives a definite amount of information of this type, and we will term these amounts of information pieces of information. The notion of a piece of information is supposed to be intuitive rather than formal, and the precise analysis of the notion need not concern us here. A piece of information may be regarded as an English sentence, syntactically distinct sentences being distinct pieces of information; or a piece of information may be regarded as a list of sentences, distinct lists being distinct pieces of information. On the

other hand, pieces of information may be regarded as intensions. The sentence "John is tall" may be thought of as giving the piece of information that John is tall. Using Quine's intensional abstraction operator "[x]", which forms the name of a proposition from a sentence (1960, p.164), we may say that [A] is the piece of information carried by the sentence A, and that for the purposes of propositional logic, if A and B are atomic sentences, [A] and [B] are the same piece of information just in case A and B are the same sentence.

We make the intuitive assumption that there is a unary operation of contradiction applicable to any piece of information a to form the piece of information which contradicts a, or is exactly opposite to a. This piece of information will be designated $-a$. Thus, for example, if a is the piece of information that John is tall, $-a$ is the piece of information that John is not tall. We make the further intuitive assumption that there is a binary operation of addition applicable to any two pieces of information, a, b, to form the piece of information whch is exactly a and b taken together. This piece of information will be designated $a + b$.

It is natural to say that a sentence which is self-contradictory gives zero information, and so we assume that there is a null piece of information, 0_1. It is natural to define 0_1 as $a + -a$, where a is any piece of information. If we suppose that whenever we have $a + b$, we have a and b from the same source of information, then it is natural to postulate that for every piece of information a, $a + 0_1 = 0_1$: if a source issues both a piece of information and its contradiction we will treat the source as corrupt and yielding a sum total of zero

information. If, however, we suppose that we may have $a + b$ without having both a and b from a single source, then there is no reason why $a + 0_1 = 0_1$ should hold: rather it would seem that $a + 0_1 = a$. Thus there are different arithmetics of pieces of information depending on whether the first or second of these suppositions is made. The first supposition is made here.

It is also natural to say that tautologous sentences give zero information. However, this piece of zero information is not equal to 0_1. The fact that an information source issues a tautology will not lead us to regard the source as corrupt and yielding a sum total of zero information. Writing 0_2 for the piece of information carried by any tautology, it is natural to say that if a source issues 0_2 and a non zero piece of information a, then the sum total of information yielded by the source is a: that is, $a + 0_2 = a$. It follows immediately that $0_2 \neq 0_1$.

To sum up, then, we assume that there exists a set of pieces of information K which is closed under a unary operation - and a binary operation + and which contains two distinct zeros, 0_1 and 0_2.

Definition Let $Z \subset K$. Then the set of compounds of members of Z, $C(Z)$, is defined as the least set satisfying the following conditions:

(i) $Z \subseteq C(Z)$

(ii) if $a \in C(Z)$ then $-a \in C(Z)$

(iii) if $a \in C(Z)$ and $b \in C(Z)$, then $a + b \in C(Z)$

We assume that K satisfies the following condition:

(P1) Let X be any finite subset of K. Then there is a non empty subset Z of K such that no non zero compound of members of Z is a member of X (where an element of K is said to be zero just in case it it either 0_1 or 0_2).

(P1) says that given a finite set X of pieces of information, then it is possible to find a set Z of pieces of information which contains no member of X and such that no amount of compounding its members will produce a member of X: X and Z are completely diverse collections of information. Intuitively (P1) certainly ought to be the case.

We state the following postulates governing +, -, 0_1 and 0_2. For all $a \in K$, $b \in K$, $c \in K$:

(P2) $a + a = a$

(P3) $a + b = b + a$

(P4) $(a + b) + c = a + (b + c)$

(P5) if $a = b$ then $a + c = b + c$

(P6) $a \neq -a$

(P7) $--a = a$

(P8) $a + -(b + c) = -(-(a + -b) + -(a + - c))$

(P9) if $a = b$ then $-b = -a$

(P10) $0_1 = a + -a$

(P11) $0_2 = -0_1$

(P12) $a + 0_1 = 0_1$

(P13) $a + 0_2 = a$

(P14) if $a + b = 0_1$ then for some $c \in K$ $b = c + -a$

(note that c may be either 0_1 or 0_2)

(P15) if $b + c = 0_2$ then $b = 0_2$ and $c = 0_2$

Intuitively each of these postulates is true, given the intended intuitive meanings of the symbols it contains.

We now define an <u>information assignment</u>. An information assignment is a function $I : WFF_0 \rightarrow K$ satisfying the following conditions:

(i) I assigns members of K arbitrarily to propositional variables, except that no propositional variable is assigned 0_1 or 0_2

(ii) $I(A \wedge B) = I(A) + I(B)$

(iii) $I(\sim A) = -I(A)$

(Henceforward $\wedge$ and $\sim$ will be taken as the only primitive truth functional connectives of the language.)

The following theorem establishes that all formal contradictions and formal tautologies carry zero information:

<u>Theorem 1.1</u> If A is a truth table contradiction then $I(A) = 0_1$ for every information assignment I; and if A is a truth table tautology then $I(A) = 0_2$ for every information assignment I. (All theorems stated in this chapter are proved in Appendix IV. The proofs are omitted from the main text in the interests of readability.)

In general not only truth table contradictions are assigned 0_1 by an information assignment, for a sentence may carry contradictory information under an assignment without being a formal contradiction. For example, if I assigns α to the variable p and $-\alpha$ to the variable q, $I(p \wedge q) = 0_1$. Similarly, sentences other than truth table tautologies may receive 0_2 in an assignment. However, only formal contradictions are assigned 0_1 in <u>every</u> assignment, and only formal tautologies are

assigned 0_2 in every assignment. This is the content of the following theorem.

Theorem 1.2 If I(A) = 0_1 for every information assignment I, then A is a truth table contradiction; and if I(A) = 0_2 for every information assignment I, then A is a truth table tautology.

1.2 Relevance

An obvious sufficient condition for two sentences to be relevant, in the sense of having semantic overlap, is that they should give the same non zero piece of information. (The notion of semantic overlap was introduced in section 3.2 of the preceding chapter.) It seems clear that it is not a sufficient condition for the semantic overlap of two sentences that they should both give zero information, since two sentences may carry tautologous or self contradictory information under an information assignment whilst being semantically quite distinct; hence the restriction in the above sufficient condition to non zero pieces of information.

Although the above condition is sufficient for relevance it is not necessary. For suppose a and b are non zero pieces of information such that a, b and $a + b$ are all distinct (i.e. such that it is not the case that for some c, either $a = b + c$ or $b = a + c$), and suppose atomic sentences A and C both give the information a, and atomic sentence B gives the information b. A $\wedge$ B and C, then, do not give the same piece of information. However, we intuitively wish to say that A $\wedge$ B and C are relevant on the ground that the piece of information carrried by C is part of the information carried by A $\wedge$ B. Thus we state the following necessary and sufficient condition for two sentences to be relevant, in the sense of having semantic overlap: A is relevant to B just in case A,

or some constituent sentence of A, carries the same non zero piece of information as B, or some constituent sentence of B.

Again it may be suggested that this condition, whilst sufficient, is not necessary. For, it may be suggested, the atomic sentences "John is tall" and "John is handsome", whilst carrying distinct pieces of information, are nonetheless relevant, since each carries information about the same individual. However, we remark that the above condition is both necessary and sufficient for relevance in the case of propositional logic, where atomic sentences are treated as unstructured. Some of the problems associated with extending the account of relevance to atomic sentences analysed as subject-predicate constructions, and thus to quantificational logic, are discussed in the following subsection.

We now turn to the formalisation of the above account of relevance.

Definition Let $A \varepsilon WFF_0$. The set of subformulae of A, Sub A, is the least set satisfying the following conditions:

(i) $A \varepsilon$ Sub A

(ii) if $\sim B \varepsilon$ Sub A then $B \varepsilon$ Sub A

(iii) if $B \wedge C \varepsilon$ Sub A then $B \varepsilon$ Sub A and $C \varepsilon$ Sub A

Definition Let $A \varepsilon WFF_0$ and $B \varepsilon WFF_0$. Then A is relevant to B under an information assignment I - written $(A \text{ Rel } B)_I$ - just in case there is a C in Sub A, a D in Sub B, such that I(C) and I(D) are non zero and I(C) = I(D).

The notion of two formulae being relevant under an information assignment constitutes our formal account of relevance, in the sense of semantic overlap, for formulae all of whose logical constants are truth functional.

We remark that on our account of relevance, distinct propositional variables may be relevant under some information assignments: if I(p) = I(q) the atomic formulae p and q are relevant under I. Also a self contradictory or tautologous formula will be relevant under a given assignment to any formula carrying the same non zero information as some subformula of the contradiction or tautology. Thus under some information assignments the formulae $p \wedge \sim p$ and $p \vee \sim p$ will be relevant to the formula q, where p and q are distinct propositional variables. However, it will not of course be the case that this is so for every information assignment.

1.3 Relevance and structured atomic formulae

I have been unable to devise an extension of the above account of relevance to atomic formulae analysed as subject-predicate constructions, and thus have been unable to extend the account to quantificational logic. In this section the problems involved in the attempt to extend the account are reviewed. My reaction to these problems (as was stated in the final paragraph of section 5, Chapter VI) is that they most likely show relevance to be a pragmatic notion incapable in general of formal characterisation, and that there is no theory of relevant entailment to be had once one proceeds beyond the logic of truth functions.

Once atomic sentences are analysed as subject-predicate constructions, one intuitively wants to say that atomic sentences which carry distinct pieces of information may be relevant. For example, we might want to say that "John and Tom are tall" and "John and Tom are handsome" are relevant in that both sentences carry information about the same objects. But then we must consistently say that $\forall x F(x)$ and $\forall x G(x)$ are relevant for

all choices of F and G, since both sentences will carry information about the same objects. I have been unable to see how to cater for the intuition that atomic sentences giving information about the same objects are relevant without becoming committed to this absurd result. In any case, <u>is</u> it true that all atomic sentences giving information about the same object are relevant? Certainly some such sentences are. But are we to say that such diverse assertions as "John is a good Catholic" and "John likes cornflakes" are relevant?

Again, it seems intuitive to say that some sentences analysed as subject-predicate constructions are relevant to one another on the ground that each asserts the same property to hold of some individual or other. Thus, for example, "Tom is tall" and "John is tall" are relevant. But surely the compound predicates $F(x) \vee \sim F(x)$ and $G(x) \vee \sim G(x)$ denote the same property for every choice of F and G? Yet we would not want to say that the sentences $F(a) \vee \sim F(a)$ and $G(b) \vee \sim G(b)$ were relevant for all choices of a, b, F and G. Even apart from this, would we wish to say that all sentences asserting the same property to hold of some object or other are relevant? And if not, where and why is the line to be drawn? For example, surely the following sentences, both of which assert the same property to hold of some object, are not relevant to one another: the number of protons in a hydrogen atom equals one; the number of noses on an average human face equals one.

It is worth remarking that Anderson and Belnap make no attempt to confront these problems. In constructing the quantified extension of E, EQ, they simply augment E with a classical treatment of the quantifiers (vide Anderson (1960), Belnap (1967)). Accordingly Belnap's (1960b)

result for E extends to EQ: $A \rightarrow B$ is a theorem of EQ only if A and B share an atomic formula. Thus in extending their relevance logic to cover formulae analysed as subject-predicate and quantificational constructions, Anderson and Belnap take no account of the fact that sentences like "John is tall" and "John is handsome" appear to be relevant to one another, as do sentences like "Tom is tall" and "John is tall".

2. Semantical Characterisation of a System of Relevant Entailment

Let WF_n be the quantifier free fragment of WFF_n, with each distinct atomic formula replaced by a distinct propositional variable (for all n).

A first degree truth value assignment is a function from $WF_0 \cup WF_1$ onto the set {0, 1} of classical truth values. Not all such functions respect the intended meanings of the logical constants $\wedge$, $\sim$ and $\rightarrow$. We define a set V_I of intended first degree truth value assignments associated with an information assignment I by means of the following recursion. For all Γ, $\Delta \subseteq WF_0 \cup WF_1$, for all A, B ε $WF_0 \cup WF_1$, for all w ε V_I:

(o) w(A) is arbitrary if A is a propositional variable

(i) $w(\sim A) = op(w(A))$ [1]

(ii) $w(A \wedge B) = \min(w(A), w(B))$

(iii) $w(\Gamma \rightarrow \Delta) = w(\bigwedge\Gamma \rightarrow \bigvee\Delta)$

(iv) $w(\rightarrow A) = w(\sim(A \wedge \sim A) \rightarrow A)$

(v) $w(A \rightarrow) = w(A \rightarrow A \wedge \sim A)$

(vi) $w(A \rightarrow B) = 1$ iff $(A \text{ Rel } B)_I$ and for all u ε V_I, if u(A) = 1 then u(B) = 1.

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1. Notation used in this recursion is explained in Chapter IV, section 1.

There are denumerably many sets of intended truth value assignments associated with a given information assignment I. The set of all these sets will be designated U_I.

A <u>model</u> is a triple <u, V, I>, where I is an information assignment, V ε U_I and u ε V. A is true in a model <u, V, I> just in case u(A) = 1. A is <u>valid</u> just in case A is true in every model; viz just in case for every information assignment I, for every V ε U_I, for every u ε V, u(A) = 1.

This semantics is a modification of the semantical theory *S*1 for classical entailment (Chapter IV, section 1). Each of clauses (o) through (vi) figures in *S*1. The motivation of (iv) and (v) in connection with relevance logic is discussed in Chapter IV, section 1. The present semantics involves an extension of the classical notion of a possible world, discussed in Chapter IV, section 1.2. A possible world is now regarded as uniquely determined by an intended truth value assignment and an information assignment, rather than by a truth value assignment alone. Two truth value assignments may exactly agree in their assignments of truth values to formulae yet fail to determine the same possible world if identical formulae carry different information - have different "meanings" - relative to each assignment. Clause (vi) counts a first degree entailment A → B as true in a possible world just in case the entailment is truth preserving in that possible world and its antecedent and consequent are relevant in that possible world. Thus the semantics incorporates in a perspicuous way the claim, discussed in Chapter VI section 5, that the matter of the relevance of the premisses to the conclusion of an inference should be built into the assessment of the logical correctness of the inference.

The present notion of a model extends Kripke's (1959) notion of a model to relevance logic in the obvious way. (Kripke-models are described in Chapter IV, section 1.2.) The present definition of truth in a model parallels Kripke's definition of validity in a model, and the present definition of validity parallels Kripke's definition of universal validity. (ibid)

Theorem 2.1 A first degree entailment $A \rightarrow B$ is valid in the present semantics just in case $A \rightarrow B$ is classically valid and for every information assignment I, $(A \text{ Rel } B)_I$.

Theorem 2.2 $(A \text{ Rel } B)_I$ for all I if and only if Sub A $\cap$ Sub B $\neq \emptyset$.

It is an immediate corrollary of these two theorems that a first degree entailment $A \rightarrow B$ is valid in the present semantics just in case $A \rightarrow B$ is classically valid and A and B share a propositional variable. Thus the present semantics demonstrates that Belnap's necessary condition for the relevance of formulae of propositional logic is also a sufficient condition for the relevance of such formulae, provided relevance is understood in the sense of semantic overlap. (Belnap's necessary condition was stated in section 3.2 of the preceding chapter.)[2]

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2. The notion of a piece of information is used in a semantics for relevance logic by Urquhart (1972). However, Urquhart's approach differs considerably from that followed here. He defines a piece of information as "a set of basic sentences concerning a subject or subjects about which reasoning is being carried out". (p.159) He postulates an operation of addition on pieces of information, $\cup$, but does not work with the notion of the contradition of a piece of information. Also only a single zero piece of information is postulated, such that where X is any piece of information $X \cup 0 = X$. Urquhart asserts that the operation $\cup$ must satisfy the laws of set union. (p.160) It follows that "the set S of pieces of information is a semilattice under $\cup$ with 0 as the lattice zero". (p.160) Urquhart does not construct a theory of relevance. He proceeds immediately to

2.1 Relevant entailment is not transitive

Various philosophers have adopted the position that relevant entailment is not transitive: that is, that there exist formulae A, B and C such that A entails B and B entails C but A does not entail C (see, for example, Geach (1958), Lewy (1958), Maclachlan (1970)). It is readily

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recursively define a set of valuations from triples $\langle A, X, w_i \rangle$ to truth values, where A is a formula, X is a piece of information, and w_i is a member of a reflexive transitive relational structure W (a set of worlds with an S4 accessibility relation). (pp.161-162) 'v(A, X, w_i) = T' is to be read: it may be concluded that A on the basis of the piece of information X, given that the facts are as in possible world w_i. (p.161) The evaluation rule for entailments $A \rightarrow B$ is such that $v(A \rightarrow B, X, w_i) = T$ if and only if whenever A can be concluded on the basis of a piece of information Y then B can be concluded on the basis of $X \cup Y$, with respect to all relatively possible backgrounds of fact (i.e. with respect to the facts as they are in any alternative to w_i). (pp.160, 162) Finally, a formula A is valid iff $v(A, 0, w_i) = T$ for any semilattice S, for every w_i in W, where W is any reflexive transitive relational structure, 0 is the lattice zero, and v is any valuation defined with respect to S and W. (p.162) Where A is a pure entailment (i.e. a formula containing no logical constants other than $\rightarrow$) A is valid iff A is a theorem of $E\rightarrow$, the pure fragment of E. (p.162) However, the positive fragment of E is incomplete relative to the semantics. Moreover, Urquhart thinks that there is no acceptable way of accommodating within his semantics all the demands made on negation by the axioms of E. (p.166) He writes: "no explicit semantic justification was given in these papers [Ackermann (1956), Anderson and Belnap (1962a)] for the axioms postulated for negation. If the viewpoint of the present paper is correct, no such justification can be given." (p.166) This corroborates the conclusions drawn above in Chapter VIII, sections 3.1 and 4.

Urquhart remarks that negation must be treated non-classically in his semantics: if a classical treatment is given to negation the paradoxes are validated. (pp.164, 166) Thus Urquhart's work, in contrast to that of the present chapter, does not provide a semantical analysis of a relevant entailment relation holding between formulae of classical propositional logic. It seems likely that Urquhart has in fact produced a semantical characterisation of a relevant entailment relation holding between formulae of intuitionistic propositional logic (compare a remark of his on page 166).

shown that → is not transitive in the present semantics, and we are thus furnished with a decisive semantical argument for the correctness of this philosophical position. However, this position inevitably draws heavy flak, and it is likely to be argued that since any adequate entailment relation must be transitive, the availability of a well motivated semantics which shows relevant entailment to be non-transitive provides a reductio ad absurdum of the claim that an assessment of relevance should be built into an assessment of the logical correctness of an inference. On the other hand, there are likely to be those who seek to avoid the reductio by finding fault with the semantics itself. I shall comment on both these likely responses, taking the second first.

The availability of the first Lewis independent proof shows that if $A \wedge \sim A \rightarrow B$ is to be rejected, then either at least one of $A \wedge B \rightarrow B$, $A \rightarrow A \vee B$, $\sim A \wedge (A \vee B) \rightarrow B$ must be rejected, or it must be claimed that entailment does not support adjunction, or it must be claimed that entailment is not transitive. Our method of procedure has not been to select one of these courses at the outset and then tailor a formal system to fit, but rather has been to develop an account of relevance and use it to determine which course is correct; and we maintain that a paradox free system of entailment which is not answerable to a definite account of relevance is ipso facto <u>not</u> a theory of relevant entailment, but merely an ad hoc paradox-excluding system. We have adopted a semantic overlap account of relevance, and on this account of relevance it turns out that entailment is not transitive: obviously the classically

valid entailments $A \wedge \sim A \rightarrow (A \vee B) \wedge \sim A$ and $(A \vee B) \wedge \sim A \rightarrow B$, for example, have premisses and conclusions which are relevant in the sense of having semantic overlap, whilst the premiss and conclusion of $A \wedge \sim A \rightarrow B$ are not relevant in this sense. Given a semantic overlap approach to relevance there is nothing paradoxical about the fact that relevant entailment is not transitive: it is a natural consequence of the approach.

It is the case, then, that no relevant entailment relation which is transitive can be answerable to a semantic overlap account of relevance. But is there a satisfactory alternative account of relevance? This seems to me to be very unlikely - certainly the only proposed alternative, Anderson's "use account" of relevance, is demonstrably unsatisfactory (vide Chapter VIII, section 3.1). And if there is no alternative, the question of whether there is a philosophically acceptable relevant entailment relation becomes a question of whether a non-transitive entailment relation is philosophically acceptable. Moreover, even if there is a satisfactory alternative account of relevance which generates a transitive relevant entailment relation, this relation will fail to tolerate at least one of Simplification, Addition, Disjunctive Syllogism, and Adjunction. Thus this hypothetical entailment relation will be philosophically acceptable only if the rejection of one of these principles is philosophically acceptable. We have already seen that the rejection of Disjunctive Syllogism is not defensible, and it seems fairly clear that there is little to be said for the rejection of any of the other three principles. In short, then: either it is philosophically acceptable that relevant entailment is non transitive, or there is no philosophically

acceptable relation of relevant entailment.

It remains to be considered whether it is philosophically acceptable that an entailment relation should be non-transitive. The following statement by Anderson and Belnap is probably a fairly typical reaction to the suggestion that entailment is not transitive: "It seems in fact incredible that anyone should admit that B follows from A, and that C follows from B, but feel that some further evidence was required to establish that A entails C. What better evidence for A → C could one want?" (1975, p.154) This reaction seems to consist of two claims. Firstly, the claim that it is absolutely obvious that any acceptable entailment relation is transitive ("it seems in fact incredible ---"); and secondly the claim that given a valid argument from A to B, and given a valid argument from B to C, one automatically has a valid argument from A to C. Neither of these claims suffices to establish that a non-transitive entailment relation is philosophically unacceptable. The first claim carries little weight: it seems to me quite obvious that relevant entailment <u>is</u> philosophically acceptable and is <u>not</u> transitive. In any case, appeals to obviousness do not, or should not, get one very far in philosophical argumentation. Further, surely everyone would agree that "obviously entails" is a philosophically respectable non-transitive entailment relation? The second claim, whilst undoubtedly correct if truth preservation is taken as sufficient for argument validity, looks doubtful once it is decided to build an appraisal of premiss/conclusion relevance into the assessment of validity. Given a valid argument from A to B and a valid argument from B to C, in this

second sense of valid, we automatically have a valid argument from A to C only if the relevance of C to B and the relevance of B to A guarantees the relevance of C to A. Yet surely this will not generally be the case. For example, it seems undeniable that A is relevant to $A \wedge B$ and $A \wedge B$ is relevant to B, but it need not be the case that A is relevant to B.

The liking which logicians have for the transitivity of entailment is no doubt partly explained by utilitarian considerations. The principle that if we have a deduction of C from B and a deduction of B from A then we have a deduction of C from A is an essential tool for constructing proofs from hypotheses: loss of the principle would greatly hamper the formal logician. However, acknowledgement of the indispensibility of this principle commits one to the claim that entailment is transitive only on the assumption that if entailment is not transitive there can be no effective test to determine when an application of the principle is correct, and when it is not. But theorems 2.1 and 2.2, above, assure us that there is such a test in the present case: given a valid deduction from A to B and a valid deduction from B to C we have a valid deduction from A to C just in case A and C share a propositional variable.

It seems, then, that there is no reason, aside from plain prejudice, for thinking that a non-transitive entailment relation is ipso facto unacceptable.

2.2 The faulting of the first Lewis independent proof

As stated in Chapter VII, the first Lewis independent proof implicitly relies on the transitivity of entailment at two points, firstly

in passing from ⌜A ∧ ∼A entails A⌝ and ⌜A entails A ∨ B⌝ to ⌜A ∧ ∼A entails A ∨ B⌝, and secondly in passing from ⌜A ∧ ∼A entails ∼A ∧ (A ∨ B)⌝ and ⌜∼A ∧ (A ∨ B) entails B⌝ to ⌜A ∧ ∼A entails B⌝. This second inference is incorrect, however, on the present account of entailment: A ∧ ∼A → ∼A ∧ (A ∨ B) and ∼A ∧ (A ∨ B) → B are both valid entailments, but A ∧ ∼A → B is invalid. Therein lies the fault of the independent proof from the point of view of the relevance logic developed here.

We remark that all other inferences in the independent proof are drawn in accordance with entailments which are valid in the present semantics: the semantics validates each of Addition, Simplification, Disjunctive Syllogism, and Adjunction.

3. Axiomatisation of the System

In this section we modify Genzen's system LK so that Γ → Δ is provable in the resultant system if and only if Γ → Δ is valid in the semantics of section 2. Thus the system is an axiomatisation of all the principles of inference of propositional logic which are valid with respect to the relevance standard of argument validity. The system will be called LR, extending Genzen's nomenclature of LK for the logischer klassischer Kalkül and LJ for the logischer intuitionistischer Kalkül.[3]

The system LK is set out in Chapter II, section 3.

Since ∨ and ⊃ are now being treated as defined symbols, and we are concerned with propositional logic only, the left and right introduction rules for ∨, ⊃, ∀ and ∃ are elided from LK. The rules for conjunction introduction on the left and right remain validity preserving in our

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3. LR is equivalent to the logistic system RS presented in my Bachelor of Philosophy thesis "Entailment without Paradoxes" (1976).

relevance semantics, and are retained unchanged. The negation introduction rules of LK are $(\Gamma \to \Delta, A) \to (\sim A, \Gamma \to \Delta)$ and $(A, \Gamma \to \Delta) \to (\Gamma \to \Delta, \sim A)$. Since these rules allow the migration of formulae from one side of a sequent to another they are not validity preserving. For example, suppose $B \to C, D$ is valid, B and D sharing a propositional variable but neither B and C nor D and C sharing a variable. An application of ($\sim$L) yields the invalid $\sim D, B \to C$ (invalid since $\sim D \wedge B$ and C do not share a variable). Thus the negation rules must be amended. Genzen himself tells us that

> the effect of the above negation rules may alternatively be achieved by taking as proper axioms (initial sequents, in Genzen's own terminology) the sequents $\to A, \sim A$ and $\sim A, A \to$. (1935, p.86) Both of these sequents are valid in our relevance semantics, and so we adopt this procedure. As Genzen remarks, the Cut rule will not be eliminable if negation is treated in this way (ibid, p.85); but this fact is of no consequence for the current project.

In the presence of these new proper axioms for negation, the thinning rules lead immediately to the classical paradoxes: thinning the sequent $\to A, \sim A$ on the left yields $B \to A, \sim A$, which is a form of the second paradox, while thinning the sequent $\sim A, A \to$ on the right yields $\sim A, A \to B$, which is a form of the first paradox. However, thinning a valid sequent of the form $A_1, ---, A_n \to B_1, ---, B_m$ will never result in loss of validity, some A_i continuing to share a variable with some B_j no matter what additional formulae are added on the left or right of the arrow. This suggests that the thinning rules be restricted so that a sequent of the form $\to A_1, ---, A_n$ may be thinned

on the left only with a formula sharing a variable with some A_i, and similarly for thinning a sequent of the form $A_1, ---, A_n \rightarrow$ on the right. In order to state the restriction on the rules we introduce the following notation: Γ / Δ if and only if either (i) either Γ is empty or Δ is empty, or (ii) there is a formula A in Γ and a formula B in Δ such that A and B share a propositional variable. The thinning rules may now be stated as follows:

(ThL): $(\Gamma \rightarrow \Delta) \rightarrow (A, \Gamma \rightarrow \Delta)$ provided A, Γ / Δ

(ThR): $(\Gamma \rightarrow \Delta) \rightarrow (\Gamma \rightarrow \Delta, A)$ provided Γ / Δ, A

Imposing these conditions on the thinning rules is methodologically similar to Genzen's imposition of the eigenvariable condition on the rules ($\forall$R) and ($\exists$L).

The Cut rule of LK, $(\Gamma \rightarrow \Delta, A), (A, \Pi \rightarrow \Lambda) \rightarrow (\Gamma, \Pi \rightarrow \Delta, \Lambda)$, is not validity preserving in our relevance semantics, for clearly it may be the case that $\Gamma \rightarrow \Delta, A$ and $A, \Pi \rightarrow \Lambda$ are both valid without it being the case that there is a formula in Γ,Π and a formula in Δ, Λ which share a propositional variable with one another, in which case $\Gamma, \Pi \rightarrow \Delta, \Lambda$ is invalid. LR uses the following restricted Cut rule: $(\Gamma \rightarrow \Delta, A), (A, \Pi \rightarrow \Lambda) \rightarrow (\Gamma, \Pi \rightarrow \Delta, \Lambda)$ provided Γ, Π / Δ, Λ.

This completes the alteration of LK to LR. We now present the system as a whole.

3.1 The system LR

Capital roman letters are schematic for members of WF_0, capital greek letters for sequences of members of WF_0 (these sequences may be empty).

Axioms of LR

$A \rightarrow A$

$\rightarrow A, \sim A$

$\sim A, A \rightarrow$

Rules of LR

$(\Gamma \rightarrow \Delta) \rightarrow (A, \Gamma \rightarrow \Delta)$ provided $A, \Gamma / \Delta$ (Th L)

$(\Gamma \rightarrow \Delta) \rightarrow (\Gamma \rightarrow \Delta, A)$ provided $\Gamma / \Delta, A$ (Th R)

$(A, A, \Gamma \rightarrow \Delta) \rightarrow (A, \Gamma \rightarrow \Delta)$ (Con L)

$(\Gamma \rightarrow \Delta, A, A) \rightarrow (\Gamma \rightarrow \Delta, A)$ (Con R)

$(\Gamma, A, B, \Pi \rightarrow \Delta) \rightarrow (\Gamma, B, A, \Pi \rightarrow \Delta)$ (Ex L)

$(\Gamma \rightarrow \Delta, A, B, \Lambda) \rightarrow (\Gamma \rightarrow \Delta, B, A, \Lambda)$ (Ex R)

$(\Gamma \rightarrow \Delta, A), (A, \Pi \rightarrow \Lambda) \rightarrow (\Gamma, \Pi \rightarrow \Delta, \Lambda)$ provided $\Gamma, \Pi / \Delta, \Lambda$ Cut

$(A, \Gamma \rightarrow \Delta) \rightarrow (A \wedge B, \Gamma \rightarrow \Delta)$ ($\wedge$L)

$(B, \Gamma \rightarrow \Delta) \rightarrow (A \wedge B, \Gamma \rightarrow \Delta)$ ($\wedge$L)

$(\Gamma \rightarrow \Delta, A), (\Gamma \rightarrow \Delta, B) \rightarrow (\Gamma \rightarrow \Delta, A \wedge B)$ ($\wedge$R)

As Genzen remarks (1935, p.86), the three rules for $\wedge$ introduction may be replaced by the following three proper axioms: $A, B \rightarrow A \wedge B$, $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$. To do this would streamline LR, since all of its inference rules would then be what Genzen terms structural rules, the work done by Genzen's rules for introducing the logical constants being accomplished by the proper axioms of the system. On the other hand, the above presentation exhibits the least amount of alteration required to convert LK into a system of relevant entailment.

The four LK rules for introducing $\forall$ and $\exists$ may be added to LR. However, as argued in section 1.3 above, the result would not be an adequate

relevant entailment system for predicate logic, and this extension of LR is not considered to be worth investigation.

LR is a proper entailment system, in the terminology of Chapter II, section 4. That is, all its theorems are of the form $\Gamma \rightarrow \Delta$. It is possible to extend LR to an entailment system in which all valid truth functions of first degree entailments are provable, by augmenting it with a classical truth function calculus. This is because negation and conjunction are both treated classically in our relevance semantics. Thus LR may be extended to such a system by augmenting it with Genzen's system LHK in exactly the same way that the proper system LK was augmented with LHK to yield the system EK in Chapter II, section 4.2.

Theorem 3.1 If $\Gamma \rightarrow \Delta$ is a theorem of LR, then $\Gamma \rightarrow \Delta$ is classically valid and either Γ or Δ is empty or there is a formula A in Γ and a formula B in Δ such that A and B share a propositional variable.

Theorem 3.2 If $\Gamma \rightarrow \Delta$ is a theorem of LR, then $\Gamma \rightarrow \Delta$ is valid in the relevance semantics of section 2.

Theorem 3.3 Let $\Gamma \rightarrow \Delta$ be a member of WF_1. If $\Gamma \rightarrow \Delta$ is valid in our relevance semantics, then $\Gamma \rightarrow \Delta$ is a theorem of LR.

Appendix I

In this appendix we establish various theorems and metatheorems stated but not proved in Chapter III.

8.1 If A is ML_n provable, $\rightarrow A$ is ML_{n+1} provable.

The proof is a routine extension of the proof of metatheorem 7.1, Chapter II, which is the corresponding result for EK_n and EK_{n+1}.

8.2 If $\Gamma \rightarrow A$ is ML_{n+1} provable and every member of Γ is ML_n provable, then A is ML_n provable.

The proof is a routine extension of the proof of metatheorem 7.2, Chapter II, which is the corresponding result for EK_{n+1} and EK_n.

8.3 If $\rightarrow^{*i+1}A$ is ML_{n+1} provable, then $\rightarrow^{*i}A$ is ML_n provable.

Lemma: if A is ML_{n+1} provable, then $\bar{A}$ is ML_n provable, where $\bar{A}$ is defined as follows:

$A \varepsilon WFF_o : A$

A is $*B : B$

A is $C_1, ---, C_n \rightarrow D_1, ---, D_m : \sim(C_1 \wedge --- \wedge C_n) \vee (D_1 \vee --- \vee D_m)$

A is $C_1, ---, C_n \rightarrow : \sim(C_1 \wedge --- \wedge C_n)$

A is $\rightarrow D_1, ---, D_m : (D_1 \vee --- \vee D_m)$

A is $\sim B : \sim\bar{B}$

A is $B \vee C : \bar{B} \vee \bar{C}$

A is $B \wedge C : \bar{B} \wedge \bar{C}$

A is $B \supset C : \bar{B} \supset \bar{C}$

A is $\forall xB : \forall x\bar{B}$

A is $\exists xB : \exists x\bar{B}$

The lemma is proved by induction on the length of an ML_{n+1} proof P of A.

Case I Lg(P) = 1. In this case A is a proper axiom of ML_{n+1}. There are various cases to consider, all of which are treated identically. We take just one: A is $B \to B$. $\to \sim B \vee B$ is ML_{n+1} provable, so by 8.2, $\sim B \vee B$ is ML_n provable. Thus $\bar{A}$ is ML_n provable.

Case II Lg(P) = k+1 and the theorem holds for the endsequent of any ML_{n+1} proof of length k or less. For illustration we deal just with the case where A is obtained by Cut. P may be represented:

$$\begin{array}{l} \vdots \\ B \to C \\ \vdots \\ \to B \\ \\ \to C \quad \text{(i.e. A is } \to C\text{)}. \end{array}$$

By induction $(\sim B \vee C)$ and B are ML_n provable. $B \wedge (\sim B \vee C) \supset C$ is ML_n provable, so C is ML_n provable (modus ponens on $\supset$). Thus $\bar{A}$ is ML_n provable.

Proof of theorem: Suppose $\to *^{i+1} A$ is ML_{n+1} provable. By 8.2, $*^{i+1} A$ is ML_n provable, and by the lemma $*^i A$ is ML_{n-1} provable. Whence by 8.1, $\to *^i A$ is ML_n provable.

8.4 If $\to *^i A$ is ML provable, then $\to A$ is ML provable.

8.4 is a corrollary of 8.3. (For notice that $*^0 A$ is A.)

8.5 If $\to *^i A$ is ML provable, then A is ML provable.

8.5 is a corrollary of 8.4 and 8.2.

8.6 If $\to *^i A$ is ML_n provable, then $\to *^{i+1} A$ is ML_{n+1} provable.

Proof Suppose $\to *^i A$ is ML_n provable. By 8.1, $\to(\to *^i A)$ is ML_{n+1} provable. Applying (*R), $\to(*^{i+1} A)$ is ML_{n+1} provable.

8.7 If A is ML provable, then $\to *^i A$ is ML provable.

8.7 is a corrollary of 8.1 and 8.6.

8.8 If $(\Gamma \to \Delta) \to (A, \Pi \to \Lambda)$ and $\to(B \to A)$ are ML_n provable, then $(\Gamma \to \Delta) \to (B, \Pi \to \Lambda)$ is ML_n provable.

For consider the following ML_n proof

1.	$(A, \Pi \to \Lambda), (B \to A) \to (B, \Pi \to \Lambda)$	Cut proper axiom
2.	$\to(B \to A)$	(theorem)
3.	$(A, \Pi \to \Lambda) \to (B, \Pi \to \Lambda)$	2, 1, (ExL), Cut
4.	$(\Gamma \to \Delta) \to (A, \Pi \to \Lambda)$	(theorem)
5.	$(\Gamma \to \Delta) \to (B, \Pi \to \Lambda)$	4, 3, Cut.

8.9 If $(\Gamma \to \Delta) \to (\Pi \to \Lambda, A)$ and $\to(A \to B)$ are ML_n provable, then $(\Gamma \to \Delta) \to (\Pi \to \Lambda, B)$ is ML_n provable.

The proof is similar to the above.

Finally we establish the two first order theorems used in section 9.3.

9.3.1: $(*^{j-i}B \supset C) \rightarrowtriangle (*^{j-i}(*^{i}A \vee B) \supset (*^{j}A \vee C))$

1.	$*^{j}A \to *^{j}A \vee C, *^{j-i}B$	proper axiom (VR), (ThR)
2.	$*^{j-i}B \to *^{j}A \vee C, *^{j-i}B$	initial axiom, (ThR), (ExR)
3.	$*^{j}A \vee *^{j-i}B \to *^{j}A \vee C, *^{j-i}B$	2, 1, (VL)
4.	$*^{j-i}(*^{i}A \vee B) \to *^{j}A \vee C, *^{j-i}B$	3, (*/VL)
5.	$C \to *^{j}A \vee C$	proper axiom (VR)
6.	$*^{j-i}B \supset C, *^{j-i}(*^{i}A \vee B) \to *^{j}A \vee C$	5, 4, ($\supset$L)
7.	$(*^{j-i}B \supset C) \to (*^{j-i}(*^{i}A \vee B) \supset (*^{j}A \vee C))$	6, ($\supset$R)
8.	$(*^{j-i}B \supset C) \rightarrowtriangle (*^{j-i}(*^{i}A \vee B) \supset (*^{j}A \vee C))$	7, (def. $\rightarrowtriangle$)

9.3.2: $\Box(A \supset *^i B) \rightarrowtriangle (\Box A \supset *^i \Box B)$

1. $(A \to *^i B),\ (\to A) \to (\to *^i B)$	Cut proper axiom
2. $(\to *^i B) \to (\to *^i B)$	initial axiom
3. $(\to *^i B) \to *^i(\to B)$	2, $(*/\to R)$
4. $(A \to *^i B),\ (\to A) \to *^i(\to B)$	3, 1, Cut
5. $(A \to *^i B) \to ((\to A) \supset *^i(\to B))$	4, $(\supset R)$
6. $\to(A \supset *^i B) \to (A \to *^i B)$	(theorem 6.17, Chapter II)
7. $\to(A \supset *^i B) \to ((\to A) \supset *^i(\to B))$	6, 5, Cut
8. $\Box(A \supset *^i B) \overline{\rightarrowtriangle} (\Box A \supset *^i \Box B)$	7, (def. $\Box$), (def. $\rightarrowtriangle$)

Appendix II

1. Ramified Modal Logic is Sound with Respect to the Semantics *S*1.

1.1 The axioms of any system ML_n are valid in *S*1

For illustration we demonstrate the validity of (∀R) and (*L).

(i) (∀R): $(\Gamma \to \Delta, A(x)) \to (\Gamma \to \Delta, \forall y\, A(y))$, where x does not occur free in Γ or Δ and y does not occur bound in A(x). Let u be an intended valuation, let V be a set of intended valuations containing u, let w be any member of V, and assume $w(\Gamma \to \Delta, A(x)) = 1$. Thus $w(\hat{\Gamma} \to \check{\Delta} \vee A(x)) = 1$. By the eigenvariable condition and the substitution clause, $w(\hat{\Gamma} \to \check{\Delta} \vee A(t)) = 1$ for every term t not occurring bound in A(x). By clause (xii), if $v(\hat{\Gamma}) = 1$, then $v(\check{\Delta} \vee A(t)) = 1$ for every $v \in V$, with t as before. Assume $v \in V$ and $v(\hat{\Gamma}) = 1$. It follows that $v(\check{\Delta} \vee A(t)) = 1$, and so either $v(\check{\Delta}) = 1$ or $v(A(t)) = 1$ (for every term t not occurring bound in A(x)). In the former case, $v(\check{\Delta} \vee \forall y\, A(y)) = 1$. Since y does not occur bound in A(x) the same terms, if any, occur bound in A(x) and A(y). So in the latter of the above two cases, $v(A(t)) = 1$ for every term t not occurring bound in A(y), so $v(\forall y\, A(y)) = 1$; whence, again, $v(\check{\Delta} \vee \forall y\, A(y)) = 1$. Accordingly, if $v(\hat{\Gamma}) = 1$, then $v(\check{\Delta} \vee \forall y\, A(y)) = 1$ for arbitrary $v \in V$, whence $v(\hat{\Gamma} \to \check{\Delta} \vee \forall y\, A(y)) = 1$ for every $v \in V$; and, in particular, $w(\hat{\Gamma} \to \check{\Delta} \vee \forall y\, A(y)) = 1$. Applying clause (viii), $w(\Gamma \to \Delta, \forall y\, A(y)) = 1$. It follows that (∀R) is true in the arbitrary intended valuation u, and so is valid.

(ii) (*L): $((\to A), \Gamma \to \Delta) \to (*A, \Gamma \to \Delta)$, where A contains an →.

Lemma: for any formula A containing an arrow, for any set of intended valuations V, if A is true in some member of V, then A is true in all members of V.

The proof is an induction on the rank of A, (R(A)), defined as the number of subformulae of A containing an arrow (where A is a subformula of itself).

Let V be a set of intended valuations, u ε V, and u(A) = 1.

Case 1 R(A) = 1. In this case A is B → C, where B and C are arrow free. Since u(A) = 1, for all w ε V, if w(B) = 1, then w(C) = 1. Let v be an arbitrary member of V. Then v(B → C) = 1 (by clause (xii)). Accordingly v(A) = 1 for every v ε V.

Case II R(A) = k + 1 and the lemma holds for any formula A such that 0 < R(A) ≤ k.

(i) A is of the form B → C. Proceed as in I.

(ii) A is of the form *B. By induction w(B) = 1 for every w ε V. Thus w(*B) = 1 for every w ε V.

(iii) A is of the form ~B. By induction if w(B) = 1 for some w ε V, then w(B) = 1 for every w ε B. But u(B) ≠ 0, so not (for some w ε V, W(B) = 1). Thus for all w ε V, w(B) = 0; whence for all w ε V, w(A) ≠ 1.

The remaining possibilities for the composition of A are treated analogously.

We now use the lemma to show that (*L) is valid. Let u be an intended valuation, let V be a set of intended valuations containing u, let w be any member of V, and assume w((→A), Γ → Δ) = 1. Thus for every v ε V, if v((→A) ∧ $\hat{\Gamma}$) = 1, then v($\check{\Delta}$) = 1. Suppose further that v ε V and v(*A ∧ $\hat{\Gamma}$) = 1. Thus v(*A) = 1 and v($\hat{\Gamma}$) = 1; so v(A) = 1. By the lemma, w(A) = 1 for all w ε V, so v(→A) = 1. Accordingly v((→A) ∧ $\hat{\Gamma}$) = 1, so v($\check{\Delta}$) = 1. Thus if v(*A ∧ $\hat{\Gamma}$) = 1, then v($\check{\Delta}$) = 1 for arbitrary v ε V, whence s(*A ∧ $\hat{\Gamma}$ → $\check{\Delta}$) = 1 for all s ε V; and, in particular, w(*A ∧ $\hat{\Gamma}$ → $\check{\Delta}$) = 1. Applying clause (viii), w(*A, Γ → Δ) = 1. It follows that (*L) is true in the arbitrary intended valuation u, and so is valid.

1.2 The meta axioms of any system ML_n are validity preserving.

Lemma: if Γ → A is valid, then if each member of Γ is valid, A is valid.

Suppose $\Gamma \rightarrow \Delta$ is valid and each member of Γ is valid, and let u be an intended valuation and V be a set of intended valuations containing u. Then if $u(\hat{\Gamma}) = 1$, $u(A) = 1$; but $u(\hat{\Gamma}) = 1$, so A is true in the arbitrary intended valuation u. Thus A is valid.

Since all meta axioms of ML_n are proper axioms of ML_{n+1}, and by 1.1, all proper axioms of ML_{n+1} are valid, the lemma establishes that all meta axioms of ML_n are validity preserving.

2. Ramified Modal Logic is Inclusive With Respect to *S*1

(i.e. every member of WFF which is valid in *S*1 is ML provable.)

Rather than straightforwardly construct a completeness proof, we draw on the labours of Hughes and Cresswell (1968). They set out an objectual semantics for quantified S5 (p.147), which we will call HC, and they establish that QS5 is inclusive with respect to HC (pp.164-169). (QS5 is discussed and axiomatised in Chapter III,section 11, above.) Hughes and Cresswell axiomatise QS5 by adding Feys' system T and the S5 axiom $\Diamond A \supset \Box\Diamond A$ to their own axiomatisation of first order predicate logic. (pp.137-138, 141) Their axiomatisation of QS5 and our own are easily shown to be equivalent. We proceed to deduce that ramified modal logic is inclusive with respect to *S*1 from the fact that QS5 is inclusive with respect to HC.

Firstly we define the normalisation, $\bar{A}$, of $A \in WFF$:

(0) A is atomic: $\bar{A}$ is A

(1) A is $\sim B$: $\bar{A}$ is $\sim(\bar{B})$

(2) A is $B \wedge C$: $\bar{A}$ is $(\bar{B}) \wedge (\bar{C})$

(3) A is $\forall xB$: $\bar{A}$ is $\forall x(\bar{B})$

(4) A is $*B$: $\bar{A}$ is $\bar{B}$

(5) A is $B_1, \ldots, B_n \rightarrow C_1, \ldots, C_m$: $\bar{A}$ is $(\overline{B_1}) \wedge \ldots \wedge (\overline{B_n}) \rightarrow (\overline{C_1}) \vee \ldots \vee (\overline{C_m})$

The normalisations of formulae containing other logical constants may be computed with the aid of the usual definitions.

A routine induction on the lengths of proofs establishes the following lemma:

2.1 A is ML provable if and only if [Ā] is ML provable. ([A] is defined in Chapter III,section 9.1. Intuitively [Ā] is the result of normalising A and then re-inserting occurrences of *.)

The following lemma relates *S*1 and HC:

2.2 If A is valid in *S*1, then Ā is valid in HC. (In fact the converse also holds but is not required here.)

We proceed to establish lemma 2.2.

An HC model is an ordered triple <W, D, V>, where W is an index set of "worlds", D is a set of individuals (the domain), and V is a recursively defined binary function assigning a truth value to each pair <A, w_i>, where A is a formula and w_i a member of W. A formula A is valid in HC just in case for every HC model <W, D, V>, V(A, w_i) = 1 for every w_i in W. (pp.146-147) (In fact for reasons of generality Hughes and Cresswell define a model as a quadtuple <W, R, D, V>, where R is a binary relation on W. Varying the restrictions on R in the usual way yields semantics for T, S4 and S5, respectively. A semantics of S5 strength may be obtained either by requiring that R be reflexive, transitive and symmetrical, or by requiring that R holds between every two members of W. (p.74) This latter requirement is in fact equivalent to suppressing all mention of the relation R in the semantics (ibid.); and this is the procedure we adopt.)

Let m be an HC model <W, D, V>, let $w_i \varepsilon W$, and let u be a function from members of WFF to truth values (i.e. a valuation). Then u is the

valuation indexed by w_i, written $\underline{w_i}$, just in case $u(A) = V(\bar{A}, w_i)$ for every formula A. The set of valuations associated with m, V_m, is $\{\underline{w_i} \mid w_i \in W\}$. V_m is a set of intended valuations (i.e. V_m satisfies the recursive definition of a set of intended valuations given in Chapter IV, section 1.) Various cases have to be considered in establishing this fact. For purposes of illustration we consider just two.

(i) A is of the form $\Box B$. $\underline{w_i}(A) = V(\overline{\Box B}, w_i) = V(\Box(\bar{B}), w_i)$. $V(\Box(\bar{B}), w_i) = 1$ if and only if $V(\bar{B}, w_j) = 1$ for every $w_j \in W$. (p.147) But $V(\bar{B}, w_j) = 1$ for every $w_j \in W$ just in case $\underline{w_j}(B) = 1$ for every $\underline{w_j} \in V_m$. Thus $\underline{w_i}(\Box B) = 1$ just in case $\underline{w_j}(B) = 1$ for every $\underline{w_j} \in V_m$, as required.

(ii) A is of the form $\forall xB$. $\underline{w_i}(A) = V(\overline{\forall xB}, w_i) = V(\forall x(\bar{B}), w_i)$. $V(\forall x(\bar{B}), w_i) = 1$ if and only if for every value-assignment V' which makes the same assignment as V does to all variables other than x, $V'(\bar{B}, w_i) = 1$. (p.147) (HC value-assignments assign members of D to individual variables and n-tuples of members of D to n-place predicate variables, in the usual way.) Clearly $V'(\bar{B}, w_i) = 1$ just in case for some variable y, $V((\bar{B})y/x, w_i) = 1$. Thus $V(\forall x(\bar{B}), w_i) = 1$ just in case for every variable y, $V((\bar{B})y/x, w_i) = 1$. Since assignments of members of D to individual constants and complex terms will duplicate assignments made to individual variables, $\underline{w_i}(\forall xB) = 1$ just in case for every term t, $\underline{w_i}((B)t/x) = 1$, as required. (An easy induction establishes that $\overline{(B)t/x} = (\bar{B})t/x$.)

We now prove lemma 2.2. Suppose A is valid in *S*1. Let m be an HC model <W, D, V>. Since V_m is a set of intended valuations, $\underline{w_i}(A) = 1$ for every $\underline{w_i} \in V_m$. Accordingly $V(\bar{A}, w_i) = 1$ for every $w_i \in W$. It follows that $\bar{A}$ is valid in HC.

We are now in a position to prove that ramified modal logic is inclusive with respect to *S*1. Suppose A ε WFF and A is valid in *S*1. By lemma 2.2, $\bar{A}$ is valid in HC. Since QS5 is inclusive with respect to HC (Hughes and Cresswell, p.169), it follows that $\bar{A}$ is provable in QS5. We know from theorem 11.1 of Chapter III that for any formula A, if A is provable in QS5, then [A] is ML provable. It follows that [$\bar{A}$] is ML provable. Then by lemma 2.1, A is ML provable.

3. *S*1 and *S*2 are Equivalent

We aim to establish that a formula is valid in *S*1 if and only if it is valid in *S*2.

Firstly we give some definitions.

Let m be an *S*2 model. A set V of *S*1 valuations associated with m is defined by the following recursion: u ε V if and only if

(O) u(A) = 1 just in case m'(A) = 1, where A is atomic and m' is some predicate variant of m.

The remainder of the recursion consists of clauses (i) through (xiii) in the recursive definition of a set of intended valuations of *S*1.

Clearly V is a set of intended valuations.

The characteristic set C of intended valuations associated with m is that set of valuations associated with m satisfying the condition that for every atomic formula A, for every w ε Mod_m, there is a u ε C such that u(A) = w(A). (The proof that there is exactly one such set is trivial.)

The characteristic valuation associated with m, v_m, is the member of C such that $v_m(A) = m(A)$ for every atomic formula A.

Lemma 3.1

Let V be a set of intended valuations, and let u ε V. Then there exists an *S*2 model m such that u = v_m.

We proceed to construct such a model. The set of atomic formulae true in any w ε V will be written A(w). Let <D, s> be a model basis. Where P is any n-place predicate variable occurring in A(w), let f(P) be the set of n-tuples <s(x_1), ..., s(x_n)> such that P(x_1, ..., x_n) ε A(w). Where P does not occur in A(w), f(P) = ∅. Let F be the set of functions obtained in this way, one for each w ε V, and let g be the function obtained for u. The required model is <D, s, g, F>.

Lemma 3.2

Let m be a model, let C be the characteristic set of valuations associated with m, and let u ε C. Then there exists a model w in Mod_m such that u = v_w. For by the definition of C there is a predicate variant w of m such that A(u) are all the atomic formulae true in w; so u = v_w.

Lemma 3.3

Given any model m, any formula A, m(A) = 1 if and only if v_m(A) = 1.

Proof is by induction on the logical complexity of A. Suppose A is atomic. Then by definition v_m(A) = m(A). Now suppose the complexity of A is k + 1 and the lemma holds for any formula of complexity k or less. There are various cases to consider. We take just three.

(1) A ε WFF_o and is ∀xB(x)

(i) Assume m(∀x B(x)) = 1. Then m'(B(x)) = 1 for every i-variant m' of m (assuming x to be the i'th variable), and so m((B(x)) y/x) = 1, where y is any variable. By induction v_m((B(x)) y/x) = 1 for every variable y, and so v_m(∀x B(x)) = 1. (Recall that terms other than variables are being left out of count.)

(ii) Assume $v_m(\forall x\ B(x)) = 1$. Then $v_m((B(x))\ y/x) = 1$ for every variable y. By induction $m((B(x))\ y/x) = 1$ for every variable y. If B(x) contains no bound variables, $m(B(y)) = i$ for every variable y, so B(x) is true in every i-variant of m, in which case $m(\forall x\ B(x)) = 1$. Now suppose B(x) contains at least one bound variable, and let $v_1, \ldots, v_n$ be all the variables occurring bound in B(x). Let B_j be the result of substituting any variable new to B(x) for every occurrence of v_j in B(x). By similar reasoning to the above $m((B_j)\ y/x) = 1$ for every variable y. Let P_j be the set of i-variants of m in which B_j is true (x being the i'th variable). Then $\bigcup_{j=n}^{j=1} P_j$ is the set of all i-variants of m. Since B(x) is true in a particular model or quasi model just in case B_j is, B(x) is true in every i-variant of m. Accordingly $m(\forall x\ B(x)) = 1$.

(2) $A \in WFF_{n>o}$ and is $\forall x B(x)$. Assume $m(\forall x\ B(x)) = 1$. Then $m((B(x))\ y/x) = 1$ for every variable y. By induction $v_m((B(x))\ y/x) = 1$ for every variable y, so $v_m(\forall x\ B(x)) = 1$. The converse is proved similarly.

(3) A is $B \rightarrow D$

(i) Assume $m(B \rightarrow D) = 1$. Let C be the characteristic set of valuations associated with m, let $u \in C$, and assume $u(B) = 1$. By lemma 3.2, there is a predicate variant w of m such that $u = v_w$. By induction $w(B) = 1$. We know that for every predicate variant m' of m, if $m'(B) = 1$, then $m'(D) = 1$, so $w(D) = 1$. Accordingly for every $u \in C$, if $u(B) = 1$, then $u(D) = 1$, and so $v_m(B \rightarrow D) = 1$.

(ii) Assume $v_m(B \rightarrow D) = 1$. Let $w \in Mod_m$, let C be the characteristic set of valuations associated with w, and assume $w(B) = 1$. It follows from the definition of C that $v_m \in C$, so for every $u \in C$, if $u(B) = 1$, then $u(D) = 1$. By induction $v_w(B) = 1$, and so $v_w(D) = 1$. By induction again, $w(D) = 1$. Accordingly for every $w \in Mod_m$, if $w(B) = 1$, then $w(D) = 1$, and so $m(B \rightarrow D) = 1$.

Theorem 3.4

(i) If A is valid in *S*1, then A is valid in *S*2 (for arbitrary A).

(ii) If A is valid in *S*2, then A is valid in *S*1 (for arbitrary A).

Proof

(i) Assume A is true in every intended valuation. Let m be a model. Since v_m is an intended valuation, $v_m(A) = 1$. By lemma 3.3, $m(A) = 1$. Accordingly A is true in every model i.e. A is valid in *S*2.

(ii) Assume A is true in every model and let w be an intended valuation. By lemma 3.1 there is a model m such that $w = v_m$. Since $m(A) = 1$, $w(A) = 1$ by lemma 3.3. Accordingly A is true in every intended valuation i.e. A is valid in *S*1.

4. In this section we establish the following theorem:

4.1 Let μ be any denumerable set. Then a formula A is valid in *S*2 just in case A is true in every model whose domain is a subset of μ.

The proof of 4.1 is immediate from the following lemma:

4.2 Let μ be any denumerable set, let w be any model of cardinality k, let $\langle E, r\rangle$ be the basis of w, let D be a subset of μ of cardinality k, let δ be a bijection from E to D, and let s be the denumerable sequence $\langle\delta(r_1), \ldots, \delta(r_n), \ldots\rangle$ (writing r_n for the n'th member of r). Then there exists a model m with basis $\langle D, s\rangle$ such that for every formula A, $w(A) = m(A)$, and for every $w' \in Mod_w$ there exists an $m' \in Mod_m$ such that $w'(A) = m'(A)$.

Proof Let w be $\langle E, r, h, H\rangle$. Let $\Delta(r_i, \ldots, r_j)$ be $\langle\delta(r_i), \ldots, \delta(r_j)\rangle$ for every n-tuple $\langle r_i, \ldots, r_j\rangle$ of members of E. Since δ is a bijection so is Δ. For every member f of H, let f'(P) be the result of applying Δ to every member of f(P), where P is any predicate variable. Let G be $\{f' \mid f \in H\}$, and let m be $\langle D, s, h', G\rangle$. Clearly m is a model. We prove by induction on the logical complexity of A that $m(A) = w(A)$.

Case I Let A be $P(x_1, \ldots, x_n)$, where P is an n-place predicate variable. Suppose w(A) = 1. Then $\langle r_{x_1}, \ldots, r_{x_n} \rangle \varepsilon$ h(P) (writing r_v for the i'th member of r, where v is the i'th variable). Thus $\Delta(r_{x_1}, \ldots, r_{x_n}) \varepsilon$ h'(P). That is, $\langle s_{x_1}, \ldots, s_{x_n} \rangle \varepsilon$ h'(P), and so m(A) = 1. Conversely, suppose w(A) = 0. Then $\langle r_{x_1}, \ldots, r_{x_n} \rangle \notin$ h(P). Since Δ is a bijection, $\Delta(r_{x_1}, \ldots, r_{x_n}) \notin$ h(P), and so m(A) = 0. Now let w' be a predicate variant of w and suppose w'(A) = 1. In particular, let w' be $\langle E, r, g, H \rangle$. Consider the model m': $\langle D, s, g', G \rangle$. Since g ε H, g' ε G, so m' is a predicate variant of m. Further, since $\langle r_{x_1}, \ldots, r_{x_n} \rangle \varepsilon$ g(P), $\langle s_{x_1}, \ldots, s_{x_n} \rangle \varepsilon$ g'(P), and so m'(A) = 1. Accordingly if w'(A) = 1, then m'(A) = 1. The converse is established similarly.

Case II Suppose the complexity of A is k + 1 and the lemma holds for every formula of complexity k or less. There are various cases to consider. We take just two.

(i) A ε WFF_o and is $\forall x$ B(x). By induction, there exists a model u with the same basis as m such that for every formula C shorter than A, u(C) = w(C), and for every w' ε Mod_w there exists a u' ε Mod_u such that w'(C) = u'(C). In particular this is true for every atomic formula. But by step I an atomic formula is true in w just in case it is true in m, and true in a predicate variant w' of w just in case it is true in some predicate variant m' of m. Thus u and m are the same model. Now suppose w(A) = 1. We know from the considerations of part (ii) of the proof of lemma 3.3, above, that $w(\forall x\, B(x)) = 1$ if and only if for every variable y, every member of $\bigcup_{j=1}^{j=n} U(B_j)$ y/x is true in w. Thus for every variable y, every member of $\bigcup_{j=1}^{j=n} U(B_j)$ y/x is true in m (every member of this union being of length less than k + 1); and so $m(\forall x B(x)) = 1$. Accordingly if w(A) = 1, then m(A) = 1. The converse is established similarly. Analogous arguments establish that A is true in a predicate variant of w just in case A is true in a predicate variant of m.

(ii) A is $(B \rightarrow C)$. By induction there exists a model u with the same basis as m such that for every formula A' shorter than A, $u(A') = w(A')$, and for every $w' \varepsilon Mod_w$ there exists a $u' \varepsilon Mod_u$ such that $w'(A') = u'(A')$. By the argument in part (i) u and m are the same model. Now suppose $m(A) = 1$, let w' be a predicate variant of w, and suppose $w'(B) = 1$. By induction there exists a predicate variant m' of m such that $m'(B) = 1$. We know that for every predicate variant m' of m, if $m'(B) = 1$, then $m'(C) = 1$. Thus $m'(C) = 1$, and so $w'(C) = 1$. Accordingly for every predicate variant w' of w, if $w'(B) = 1$, then $w'(C) = 1$; whence $w(B \rightarrow C) = 1$. Moreover, since a formula of the form $B \rightarrow C$ is true in a model just in case it is true in every predicate variant of the model, if A is true in $w' \varepsilon Mod_w$, then A is true in $m' \varepsilon Mod_m$. The converses of these results are established similarly.

Appendix III

1. The Equivalence of *S*4 and *S*5

In this section we aim to establish that a formula is valid in *S*4 just in case it is valid in *S*5.

Firstly, we give some definitions. Let m be a model $<\mu, s, \Upsilon, G, T, g, e>$. A set V of valuations associated with m is defined by recursion on the complexity of formulae: $u \in V$ if and only if for some $\Upsilon_o \subseteq \Upsilon$, $u: WFF \times \Upsilon_o \rightarrow \{0, 1\}$ and for every $t \in \Upsilon_o$:

(i) where A is atomic, pairs $<A, t>$ are evaluated arbitrarily by u unless (1) A contains some term not occurring in t, in which case $u(A, t) = 0$, or (2) $t = T$, in which case $u(A, t) = m(A)$.

The remainder of the recursion is as for the definition of a set of intended valuations of *S*4.

Clearly V is a set of intended valuations.

The characteristic set C of intended valuations associated with m is that set of associated valuations such that for every w alternative to m there is a u in C satisfying the condition that for every atomic formula A, $u(A, S) = w(A)$, where S are the designated terms of w.

V_m is that subset of C such that $u \in V_m$ if and only if $u(A, \dot{T}) = m(A)$ for every atomic formula A.

Lemma 1.1

Let V be a set of intended valuations of domain $WFF \times \Upsilon$, let $u \in V$, and let $T \in \Upsilon$. Then there exists a model m such that for some μ, S, *G*, g and e, m is $<\mu, S, \Upsilon, G, T, g, e>$ and $u \in V_m$.

We construct such a model. Let μ be a Meinong universe and let s be a function from individual variables of L_o to members of μ. Where $w \in V$, A(w) will be written for the set of atomic sentences A such that

w(A, T) = 1. Where P is an n-place predicate variable occurring in some member of A(w), let g(P) be the set of n-tuples $\langle s(x_1), ---, s(x_n)\rangle$ such that $P(x_1, ---, x_n) \varepsilon A(w)$. Where P does not occur in A(w), $g(P) = \emptyset$. Let *G* be the set of functions obtained in this way, one for each $w \varepsilon V$, and let f be the funtion obtained for u. The required model is $\langle \mu, s, \Upsilon, G, T, f, e\rangle$.

Lemma 1.2

Let m be a model, let C be the characteristic set of intended valuations associated with m, let $u \varepsilon C$ and let S be a set of terms occurring in the domain of u. Then there is an alternative to m, w, whose designated terms are S and such that $u \varepsilon V_w$.

A(u) will be written for the set of atomic formulae A such that u(A, S) = 1. By the definition of C there is an alternative to m, w, whose designated terms are S and such that A(u) are all the atomic formulae true in w. Since w is an alternative to m, C is also the characteristic set of intended valuations associated with w. Thus $u \varepsilon V_w$.

Lemma 1.3

Let m be a model, let $w \varepsilon Mod_m [A_1, ---, A_n]$, and let C be the characteristic set of intended valuations associated with m. Then there is a u in $C[A_1, ---, A_n]$ such that $u \varepsilon V_w$.

We construct such a u. Let m be $\langle \mu, s, \Upsilon, G, T_o, f, e\rangle$, let T be the set of designated terms of w, and let Υ_o be $\{t \varepsilon \Upsilon \mid t[A_1, ---, A_n]\}$. We define the following function u: $WFF \times \Upsilon_o \rightarrow \{0, 1\}$ by recursion on the complexity of formulae.

For all $t \varepsilon \Upsilon_o$:

(i) where A is atomic, pairs <A, t> are evaluated arbitrarily by u, unless

(1) A contains some term not occurring in t, in which case u(A, t) = 0, or (2) t = T, in which case u(A, t) = w(A).

The remainder of the recursion is that used in the definition of a set V of intended valuations of *S*4, except for the clause for entailment, which reads:

u(A → B, t) = 1 if and only if for every v ε C[A, B], for every set of terms S occurring in the domain of v, if v(A, S) = 1, then v(B, S) = 1.

Clearly u ε C[A_1, ---, A_n]. Since w is an alternative to m, C is the characteristic set of intended valuations associated with w. Thus u ε V_w. Accordingly, u is the required valuation.

Lemma 1.4

For every formula A, every model m, every v ε V_m, if T are the designated terms of m, then m(A) = 1 if and only if v(A, T) = 1.

The lemma is proved by induction on the complexity of A. Let m be a model whose designated terms are T and let v ε V_m. In case A is atomic v(A, T) = m(A) by definition. Now suppose A is of complexity k + 1 and the lemma holds for all formulae of complexity k or less. There are various cases to consider. We take just the case where A is B→D. (The case where A is ∀xB(x) is treated exactly as in the proof of Lemma 3.3, Appendix II.)

(i) Assume m(B → D) = 1. Let C be the characteristic set of intended valuations associated with m, let u ε C[B, D], let S be a set of terms occurring in the domain of u, and assume u(B, S) = 1. By lemma 1.2, there is an alternative to m, w, whose designated terms are S and such that u ε V_w. Since u ε C[B, D], it is the case that w[B, D], and so w ε Mod_m[B, D]. By induction w(B) = 1. We know that for all w ε Mod_m [B, D], if w(B) = 1, then w(D) = 1; so w(D) = 1. By induction u(D, S) = 1. Thus v(B → D, T) = 1.

(ii) Assume $v(B \rightarrow D, T) = 1$. Let $w \varepsilon Mod_m [B, D]$, let S be the set of designated terms of w, let C be the characteristic set of intended valuations associated with m, and assume $w(B) = 1$. By lemma 1.3, there is a $u \varepsilon C[B, D]$ such that $u \varepsilon V_w$. By induction $u(B, S) = 1$. We know that for every $u \varepsilon C$, for every set of terms S occurring in the domain of u, if $u(B, S) = 1$, then $u(D, S) = 1$. Since $u \varepsilon V_w$ and w is an alternative to m, $V_w \subseteq C$. Thus $u(D, S) = 1$. By induction $w(D) = 1$. Thus $m(B \rightarrow D) = 1$.

It is immediate from lemma 1.4 that if a formula is valid in *S*4, then it is valid in *S*5; and it is immediate from lemma 1.4 and lemma 1.1 that if a formula is valid in *S*5, then it is valid in *S*4.

2. In this section we describe the general structure of the proof that the hierarchy QML_o, ---, QML_n, --- is inclusive with respect to *S*4 and *S*5 (i.e. if A is valid in *S*5, then A is QML provable.) As in the case of *S*2 and the hierarchy ML_o, ---, ML_n, --- we do not construct outright a proof of inclusiveness, but rather deduce inclusiveness from the fact that the classical modal system embedded in the hierarchy is complete (cf Appendix II, section 2).

In this case, the classical modal logic embedded in the hierarchy is KS5, as witnessed by the version of the Key Metatheorem stated in Chapter V, section 6:

2.1 If A is a theorem of KS5, [A] is QML provable.

We use the fact that KS5 is complete relative to Kripke's (1963) possible world semantics. That is:

2.2 A is a theorem of KS5 if and only if A is Kripke-valid.

As in our earlier proof of inclusiveness we require the notion of the normalisation of A ε WFF, $\bar{A}$. For the present purpose, $\bar{A}$ is as defined in Appendix II, section 2, except that the clause for atomic formulae in the definition is replaced by the following:

(0) A is atomic: (i) A contains no singular terms other than variables: $\bar{A}$ is A

(ii) A is $P(t_1, ---, t_n)$ where $t_i, ---, t_j$ are individual constants or singular terms (all other members of $t_1, ---, t_n$ being variables): $\bar{A}$ is A with each occurrence of t replaced by a variable new to A, for each t in $t_i, ---, t_j$, no distinct members of $t_i, ---, t_j$ being replaced by the same variable, and no non-distinct members of $t_i, ---, t_j$ being replaced by different variables.

The following lemmata permit the deduction from theorems 2.1 and 2.2 of the inclusiveness of ramified modal logic with respect to *S*5. (We omit the proofs of the lemmata.)

2.3 A is QML provable if and only if the universal closure of $[\bar{A}]$ (which will be written $\forall[\bar{A}]$) is QML provable.

2.4 $\forall[A] = [\forall A]$

2.5 If A is valid in *S*5, then $\forall\bar{A}$ is Kripke-valid.

Let A be a member of WFF and assume A is valid in *S*5. Applying 2.5 and 2.2, $\forall\bar{A}$ is KS5 provable. Applying 2.1, $[\forall\bar{A}]$ is QML provable, and so, by 2.4, $\forall[\bar{A}]$ is QML provable. Whence by 2.3, A is QML provable. Accordingly, the hierarchy $QML_o, ---, QML_n, ---$ is inclusive with respect to *S*5 (and so with respect to *S*4 also, by the result of the preceding section).

Appendix IV

In this Appendix the theorems stated in Chapter IX are proved.

Theorem 1.1 If A is a truth table contradiction then $I(A) = 0_1$ for every information assignment I; and if A is a truth table tautology then $I(A) = 0_2$ for every information assignment I.

We first establish a lemma.

Lemma 1 Let I be an information assignment. Then if A is a theorem of the system P_G, $I(A) = 0_2$.

P_G is the following complete axiomatisation of propositional logic set out by Church (1956, p.137):

Ax 1: $A \vee A \supset A$

Ax 2: $A \supset A \vee B$

Ax 3: $(B \supset C) \supset (A \vee B \supset C \vee A)$

Rule: from $A \supset B$ and A to infer B

To establish the lemma we show that each axiom of the system is assigned 0_2 by I, and that if A and $A \supset B$ are assigned 0_2 by I, so is B.

(i) Expanding Ax 1 to remove occurrences of the defined signs $\vee$ and $\supset$, we obtain $\sim(\sim\sim(\sim A \wedge \sim A) \wedge \sim A)$. Taking I(A) to be a, I(Ax 1) is $-(---(-a + -a) + -a)$. Applying (P2) and (P7), $I(Ax\ 1) = -(a + -a)$. Whence by (P10) and (P11), $I(Ax\ 1) = 0_2$.

(ii) Expanding Ax 2 we obtain $\sim(\sim\sim A \wedge \sim\sim(\sim A \wedge \sim B))$. Taking I(A) to be a and I(B) to be b, I(Ax 2) is $-(--a + --(-a + -b))$. Applying (P7), $I(Ax\ 2) = -(a + (-a + -b))$. Applying (P4) and (P3), $I(Ax\ 2) = -(-b + (a + -a))$. By (P10), $I(Ax\ 2) = -(-b + 0_1)$. Whence by (P12) and (P11), $I(Ax\ 2) = 0_2$.

(iii) Expanding Ax 3 we obtain:

$\sim(\sim\sim(\sim\sim B \wedge \sim C) \wedge \sim\sim(\sim\sim(\sim A \wedge \sim B) \wedge \sim\sim(\sim C \wedge \sim A)))$. Taking I(A) to be a, I(B) to be b and I(C) to be c, I(Ax 3) is:

$-(---(--b + -c) + --(---(-a + -b) + --(-c + -a)))$.

Applying (P7), I(Ax 3) is $-(-(b + -c) + (-(-a + -b) + (-c + -a)))$.

Applying (P3) and (P4), I(Ax 3) is $-(-(b + -c) + ((-a + -(-a + -b)) + -c))$.

Letting b be a in (P8) and applying (P10), (P11), (P3), (P13) and (P7), we get that $a + -(a + c) = a + -c$. Using this derived principle, I(Ax 3) is $-(-(b + -c) + ((-a + --b) + -c))$. Applications of (P3) and (P4) yield I(Ax 3) = $-((-c + -(-c + b)) + (-a + --b))$. Using the derived principle again, I(Ax 3) = $-((-c + -b) + (-a + --b))$. By (P3) and (P4), I(Ax 3) = $-((-a + -c) + (--b + -b))$. Applying (P7) and (P10), I(Ax 3) = $-((-a + -c) + 0_1)$. Whence by (P12) and (P11), I(Ax 3) = 0_2.

(iv) Expanding the rule we obtain: from $\sim(\sim\sim A \wedge \sim B)$ and A, to infer B. Suppose $I(\sim(\sim\sim A \wedge \sim B)) = 0_2$ and $I(A) = 0_2$. Then $-(0_2 + -I(B)) = 0_2$, whence $I(B) = 0_2$.

We now prove theorem 1.1. Suppose A is a tautology and let I be an information assignment. Since P_G is complete, A is a theorem of P_G, so $I(A) = 0_2$. Now suppose A is a contradiction. Since $\sim A$ is a tautology, $\sim A$ is a theorem of P_G, so $I(\sim A) = 0_2$. Thus $I(A) = -0_2$. From (P11), (P9) and (P7), $-0_2 = 0_1$. Accordingly $I(A) = 0_1$.

Theorem 1.2 If $I(A) = 0_1$ for every information assignment I, then A is a truth table contradiction; and if $I(A) = 0_2$ for every information assignment I, then A is a truth table tautology.

We first establish a lemma.

Lemma 2 There exists an information assignment I such that (i) every member of K is assigned by I to some formula, and (ii) if I(A) = I(B) then $A \equiv B$ (where $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$).

We model our theory of the arithmetic of pieces of information and show that such an assignment exists in the model. Let K be the set of

well formed formulae of the propositional calculus; let 0_1 be $(p \wedge \sim p)$ and let 0_2 be $(p \vee \sim p)$; let + be $\wedge$, let - be $\sim$ and let = be $\equiv$. Let I be the information assignment which assigns each propositional variable to itself. Clearly I satisfies (i) and (ii).

We now prove theorem 1.2. Proof is by induction on the logical complexity of A. If A is a propositional variable the theorem holds trivially, no propositional variable being assigned zero information. Let A be of complexity k + 1 and suppose the theorem holds for all formulae of complexity k or less. Suppose A is B $\wedge$ C and that $I(A) = 0_1$ for every information assignment I. In particular this is so for the information assignment given by lemma 2 - call it J. Since $J(B) + J(C) = 0_1$, by (P14) there exists a $d \in K$ such that $J(C) = d + -J(B)$. Since the range of J equals K, there is some formula D such that $J(D) = d$. Thus $J(C) = J(D) + -J(B)$. But $J(D) + -J(B) = J(D \wedge \sim B)$. So by lemma 2, $C \equiv D \wedge \sim B$, whence B $\wedge$ C is a contradiction. Now suppose $I(B \wedge C) = 0_2$ for every I. Thus $I(B) + I(C) = 0_2$. It follows from postulate (P15) that $I(B) = 0_2$ and $I(C) = 0_2$. By induction, then, B and C are both tautologies, and so B $\wedge$ C is a tautology. Now suppose A is $\sim$B and that $I(A) = 0_1$ for every I. Thus $I(B) = 0_2$ for every I. By induction B is a tautology, and so $\sim$B is a contradiction. Finally, suppose $I(\sim B) = 0_2$ for every I. Then $I(B) = 0_1$ for every I. By induction B is a contradiction, whence $\sim B$ is a tautology.

Theorem 2.1 A first degree entailment $A \rightarrow B$ is valid in our relevance semantics just in case $A \rightarrow B$ is classically valid and for every information assignment I, $(A \text{ Rel } B)_I$.

(i) Suppose $A \rightarrow B$ is valid. Then for every information assignment I, for every $V \in U_I$, for every $w \in V$, $w(A \rightarrow B) = 1$. Thus $(A \text{ Rel } B)_I$ for every

information assignment I. Secondly, it is clearly the case that if u is an intended valuation of $S1$, then $u \wedge WF_0 \cup WF_1$ is an intended truth value assignment of the present semantics. Thus A → B is true in every intended valuation of $S1$, and so A → B is classically valid.

(ii) Suppose A → B is classically valid and for every information assignment I, $(A\ Rel\ B)_I$. Let I be an information assignment, let $V \in U_I$, let $w \in V$, and suppose w(A) = 1. Since the present semantics and $S1$ coincide for formulae in WF_0, for every truth value assignment w of the present semantics there exists an $S1$ valuation v_w such that v_w and w coincide in their assignments to members of WF_0. Thus for some $S1$ valuation v_w, $v_w(A) = 1$, so $v_w(B) = 1$; whence w(B) = 1. Thus $(A\ Rel\ B)_I$ and for all $w \in V$, if w(A) = 1, then w(B) = 1. It follows that A → B is valid in the present semantics.

Theorem 2.2 $(A\ Rel\ B)_I$ for all I if and only if Sub A ∩ Sub B ≠ ∅.

We first establish a lemma.

Lemma 3 Let A and B be any formulae such that Sub A ∩ Sub B = ∅. Then there exists an information assignment I such that for every $x \in$ Sub A, for every $y \in$ Sub B, if I(x) and I(y) are non zero, then I(x) ≠ I(y).

Let J be any information assignment. By postulate (P1) there is a non empty subset of K, Z, such that no non-zero compound of members of Z is a member of the range of J ↾ Sub A. Let I: WF_0 → K be exactly like J, except that every propositional variable in Sub B is assigned a member of Z by I, complex formulae containing propositional variables in Sub B being assigned pieces of information in accordance with clauses (ii) and (iii) of the definition of an information assignment. Clearly I is an

information assignment. Since no propositional variable in Sub A occurs in Sub B, the range of I / Sub A equals the range of J / Sub A. Thus, since the range of I / Sub B is Z, I is the required assignment.

We now prove theorem 2.2.

(i) Suppose Sub A ∩ Sub B ≠ ∅. Then A and B have a propositional variable in common, and so for all I, $(A \text{ Rel } B)_I$

(ii) Suppose $(A \text{ Rel } B)_I$ for all I, and suppose for reductio Sub A ∩ Sub B = ∅. By the lemma, there exists an information assignment I such that for all x ε Sub A, for all y ε Sub B, either I(x) and I(y) are zero, or I(x) ≠ I(y). Thus it is not the case that $(A \text{ Rel } B)_I$, in contradiction to the original hypothesis.

Theorem 3.1 If Γ → Δ is a theorem of LR, then Γ → Δ is classically valid and either Γ or Δ is empty or there is a formula A in Γ and a formula B in Δ such that A and B share a propositional variable.

Suppose Γ → Δ is a theorem of LR. Clearly Γ → Δ will also be a theorem of LK. Thus Γ → Δ is classically valid, since LK is sound with respect to the semantics *S*1. An easy induction on the length of an LR proof of Γ → Δ establishes the second part of the theorem.

Theorem 3.2 If Γ → Δ is a theorem of LR, then Γ → Δ is valid in our relevance semantics.

This theorem is an immediate corrollary of theorems 2.1, 2.2 and 3.1.

Theorem 3.3 Let Γ → Δ be a member of WF_1. If Γ → Δ is valid in our relevance semantics, then Γ → Δ is a theorem of LR.

In the proof of this theorem we use the system LK^1, derived from LK by omitting the quantifier rules, omitting the rules for ⊃ and ∨ (which we

are treating as defined symbols), and omitting the rules for $\sim$ and $\wedge$, replacing them by the following initial sequents: $(\rightarrow A, \sim A)$, $(\sim A, A \rightarrow)$, $(A, B \rightarrow A \wedge B)$, $(A \wedge B \rightarrow A)$, $(A \wedge B \rightarrow B)$. If $\Gamma \rightarrow \Delta$ is quantifier-free, $\Gamma \rightarrow \Delta$ is provable in LK if and only if $\Gamma \rightarrow \Delta$ is provable in LK^1 (vide Genzen (1935) pp.85-86)).

We establish some lemmata.

Lemma 4 Let $\Gamma \rightarrow \Delta$ be a member of WF_1. If $\Gamma \rightarrow \Delta$ is LK provable, then $A, \Gamma \rightarrow \Delta$ is LR provable, where A is any formula sharing a variable with some formula in Δ, or any formula at all if Δ is empty.

If $\Gamma \rightarrow \Delta$ is LK provable, then $\Gamma \rightarrow \Delta$ is LK^1 provable, and vice versa. The lemma is proved by induction on the length of an LK^1 proof P of $\Gamma \rightarrow \Delta$.

Case I Suppose P is of length one. In this case $\Gamma \rightarrow \Delta$ is either $(B \rightarrow B)$, $(\sim B, B \rightarrow)$, $(\rightarrow B, \sim B)$, $(B, C \rightarrow B \wedge C)$, $(B \wedge C \rightarrow B)$ or $(B \wedge C \rightarrow C)$. Each of these is a proper axiom of LR. (We are taking the formulation of LR whereby the latter three sequents replace the $\wedge$-introduction rules.) In each case $A, \Gamma \rightarrow \Delta$, with A as specified in the lemma, is LR provable by thinning.

Case II Suppose P is of length k + 1 and the lemma holds for the endsequent of any LK^1 proof of length k or less. There are various cases to consider, according to the form of the final inference in P. We take two.

(i) The final inference in P is a left thin. P may be represented:

$$\begin{array}{c} \vdots \\ \Pi \rightarrow \Delta \\ \hline B, \Pi \rightarrow \Delta \end{array} \qquad \text{i.e. } \Gamma = B, \Pi$$

By induction $A \wedge B, \Pi \rightarrow \Delta$ is LR provable, where A is any formula sharing a variable with some formula in Δ, or any formula at all if Δ is empty. Whence $A, B, \Pi \rightarrow \Delta$ is LR provable from $A, B \rightarrow A \wedge B$ by Cut.

(ii) The final inference in P is a Cut. P may be represented:

$$\begin{array}{l} \vdots \\ \Pi_1 \rightarrow \wedge_1, B \\ \vdots \\ B, \Pi_2 \rightarrow \wedge_2 \\ \hline \Pi_1, \Pi_2 \rightarrow \wedge_1, \wedge_2 \end{array} \qquad \text{i.e. } \Gamma = \Pi_1, \Pi_2 \text{ and } \Delta = \wedge_1, \wedge_2$$

Suppose neither $\wedge_1$ nor $\wedge_2$ is empty. Let A be any formula sharing a variable with some formula in $\wedge_1$, and let C be any formula in $\wedge_2$. By induction $A, \Pi_1 \rightarrow \wedge_1, B$ and $C \vee \sim C, B, \Pi_2 \rightarrow \wedge_2$ are LR provable. Whence by exchange and Cut, $A, \Pi_1, C \vee \sim C, \Pi_2 \rightarrow \wedge_1, \wedge_2$ is LR provable. So, since $\rightarrow C \vee \sim C$ is LR provable, $A, \Pi_1, \Pi_2 \rightarrow \wedge_1, \wedge_2$ is LR provable. The cases where A shares a variable with some formula in $\wedge_2$, or one or both of $\wedge_1$ and $\wedge_2$ are empty, are treated analogously.

Lemma 5 Let $\Gamma \rightarrow \Delta$ be a member of WF_1. If $\Gamma \rightarrow \Delta$ is LK provable and Γ / Δ, then $\Gamma \rightarrow \Delta$ is LR provable.

(i) Suppose neither Γ nor Δ is empty. Then there is an A in Γ which shares a variable with some formula in Δ. By lemma 4, $A, \Gamma \rightarrow \Delta$ is LR provable, whence $\Gamma \rightarrow \Delta$ is LR provable by contraction.

(ii) Suppose Δ is empty. Let A be any formula in Γ. (Notice that since LK is consistent it will never be the case that both Γ and Δ are empty.) By lemma 4, $A, \Gamma \rightarrow \Delta$ is LR provable, whence $\Gamma \rightarrow \Delta$ is LR provable.

(iii) Suppose Γ is empty. Let A be any formula in Δ. By lemma 4, $\sim A, \Gamma \rightarrow \Delta$ is LR provable. Whence $\Gamma \rightarrow \Delta, A$ is LR provable from $\rightarrow A$, $\sim A$ and Cut, and so $\Gamma \rightarrow \Delta$ is LR provable by contraction.

We now prove theorem 3.3. Suppose $\Gamma \rightarrow \Delta \in WF_1$ is valid in the relevance semantics. By theorem 2.1, $\Gamma \rightarrow \Delta$ is classically valid. Thus $\Gamma \rightarrow \Delta$ is LK provable (by the theorem of section 2 of Chapter IV). From theorems 2.1 and 2.2 we know that $\Gamma \ / \ \Delta$. It follows from lemma 5 that $\Gamma \rightarrow \Delta$ is a theorem of LR.

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The following abbreviations are used:

JSL	The Journal of Symbolic Logic
J Phil	The Journal of Philosophy
Phil Stud	Philosophical Studies
Phil Rev	The Philosophical Review
PAS	Proceedings of the Aristotelian Society
PASS	Proceedings of the Aristotelian Society, supplementary volume
J Phil Log	Journal of Philosophical Logic

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