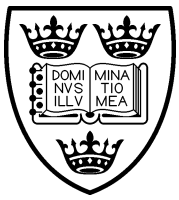
Pricing Basket Temperature Derivatives



Sarah Carter
St. Hugh's College
University of Oxford

Dissertation submitted in partial fulfilment of MSc in Mathematical and Computational Finance

Trinity Term 2008

Acknowledgements

I would like to thank my supervisor, Dr Ben Hambly, for his time, patience and direction.

Contents

1 Introduction **1**

2 The Market **3**

2.1 The Chicago Mercantile Exchange 3

2.1.1 Definitions 3

3 Modelling Temperature **6**

3.1 Properties of Temperature Data 6

3.2 Stochastic Models for Daily Average Temperature 8

3.2.1 Model 1: An Ornstein-Uhlenbeck Process driven by Fractional Brownian Motion 9

3.2.2 Model 2: An Ornstein-Uhlenbeck Model driven by a Lévy Process . 11

3.2.3 Model 3: An Ornstein-Uhlenbeck Process driven by Brownian Motion 12

4 Temperature Baskets **14**

4.1 The Model 15

4.1.1 Basket Indices 17

4.1.2 Girsanov's Theorem in $\mathbb{R}^n$ 19

4.2 Pricing Futures and Options on Basket Indices 21

4.2.1 Basket CAT and PRIM Futures 21

4.2.2 Call Options on Basket CAT and PRIM Futures 25

4.2.3 Basket CDD and HDD Futures 28

5 Hedging Spatial Temperature Risk **35**

5.1 A Spatial-Temporal Model for Temperature 35

5.2 Hedging Spatial Temperature Risk using Market Traded Futures 36

6 Conclusions **39**

Bibliography **41**

Chapter 1

Introduction

Weather derivatives are financial contracts whose payoffs depend on certain weather conditions. Since the first 'over the counter' weather derivative trade in 1997, followed in 1999 by the Chicago Mercantile Exchange introducing derivative contracts on temperature indices for several US cities, the weather derivatives market has exploded in volume. Both exchange traded and OTC contracts are now written on a range of weather indices, including temperature, hurricane, frost and precipitation for cities across the US, Canada, Europe and Asia [14]. According to the Weather Risk Management Association, the twelve months 2003-2004 saw $4.7 billion of weather transactions executed, this number increasing to $45.2 billion from 2005-2006. This marked increase in popularity is due to the need of many institutions, including energy, utility and insurance companies, and the agriculture, construction, retail, leisure and tourism industries, to hedge their exposure to weather risk and protect their revenues. Energy prices are strongly correlated with the weather, with demand for energy highly dependent upon weather fluctuations. Hence energy providers are one of the key players in the weather derivatives market. As their industry is highly competitive (and deregulated), the more effectively they hedge their weather risk, the less affected their prices will be by adverse conditions and the better they will perform. Specifically for electricity and natural gas sectors, temperature is the key factor influencing consumer demand, and temperature-based derivatives constitute the majority of trading volume in this young and expanding market.

Weather derivatives differ from insurance in several ways. Insurance compensates policyholders for low probability extreme events, for example hurricane damage, whereas weather derivatives are traded to 'cover' lower risk high probability events, such as a wet summer season. Holders of weather derivative contracts will receive their payoff irrespective of whether or not they have sustained significant economic loss as a consequence of the weather conditions. In contrast, the holder of an insurance policy must prove damages to receive

their compensation. Indeed, insurance will not compensate for reduced demand resulting from slight variations in weather conditions.

The weather derivatives market also provides the opportunity for two market participants, with opposite attitudes to certain weather conditions, to hedge each others risk. A warm winter may be beneficial to one institution but not to another, and a contract between the two parties may reduce both risk exposures. In particular, the expanding OTC weather derivatives market provides an increasing range of contracts to hedge more bespoke weather risk. Investors may also purchase derivatives contracts to speculate on the weather.

Weather derivatives are a typical example of an incomplete market. The underlying, the weather itself, has no direct value. It is not a tradeable or storable asset, and has dynamics determined by nature, largely independent of human activity. Consequently the standard Black Scholes hedging arguments cannot be applied, and pricing of contracts is more complex. There is no method to construct a portfolio of tradeable financial assets whose payoff exactly replicates that of a weather derivative. As such, no-arbitrage pricing cannot be applied in the conventional way. Furthermore the weather derivatives market is relatively illiquid. A key factor that will continue to contribute to this illiquidity is that weather is location-specific and non-standardised [11], resulting in a highly bespoke market.

In this dissertation I will focus on temperature derivative contracts. This will involve reviewing stochastic models proposed for daily average temperature. Since many market participants are affected by weather risk at more than one location, I develop the idea of a basket index on temperature at several cities, and will investigate pricing futures and options written on such basket indices. The basket weightings, to be determined by the investor, will reflect the relative risk at each location. I will conclude by looking at a spatial-temporal model for temperature, with spatial dependency modelled through a Gaussian random field. Using this model I will briefly discuss how to construct optimal replicating portfolios to hedge temperature risk over a region, or at a location where futures contracts do not trade.

Chapter 2

The Market

2.1 The Chicago Mercantile Exchange

The Chicago Mercantile Exchange (CME) organises trade in temperature derivatives for eighteen cities in the US, six in Canada, nine in Europe and two in Japan, [14]. Futures contracts written on monthly and seasonal temperature indices are traded, along with vanilla European options written on these temperature index futures. Seasonal Strip futures and options enable customers to select the months within a season that they would like to trade. The contracts available are continually expanding, both in terms of locations available and their specification. This increased availability allows customers to more effectively hedge their weather risk.

The underlying indices on which temperature derivatives are written, are Heating Degree Days, Cooling Degree Days, Cumulative Average Temperature and Pacific Rim.

2.1.1 Definitions

- **Daily Average Temperature**
  For a given location let T_τ^{max} and T_τ^{min} denote the maximum and minimum temperatures observed on day τ (over a 24-hour time period). We define the daily average temperature (DAT) on day τ by
  $$T(\tau) := \frac{T_\tau^{max} + T_\tau^{min}}{2}. \tag{2.1}$$

- **Heating Degree Days (HDD) and Cooling Degree Days (CDD)**
  For a given spatial location we define Heating Degree Days generated on day τ, $HDD(\tau)$, by
  $$HDD(\tau) := \max(T_{ref} - T(\tau), 0), \tag{2.2}$$
  and Cooling Degree Days generated on day τ, $CDD(\tau)$, by
  $$CDD(\tau) := \max(T(\tau) - T_{ref}, 0), \tag{2.3}$$

where T_{ref} is a reference temperature. For both HDD and CDD the reference temperature is typically set at $T_{ref} = 18^o$C (equivalently 65^oF in the US). This is motivated by the US energy markets, where 65^oF is regarded as the critical temperature below which users turn on their heating, and above which users turn on their air-conditioning. Cooling Degree Days only accumulate if DAT is above this threshold, whilst Heating Degree Days accumulate only if DAT is below it.

- **Heating Degree Day Index and Cooling Degree Day Index**
  We define the HDD index over the time interval $[\tau_1, \tau_2]$ by

$$HDD[\tau_1, \tau_2] := \sum_{\tau=\tau_1}^{\tau_2} \max\left(T_{ref} - T(\tau), 0\right). \tag{2.4}$$

  We analogously define the CDD index over the same time interval by

$$CDD[\tau_1, \tau_2] := \sum_{\tau=\tau_1}^{\tau_2} \max\left(T(\tau) - T_{ref}, 0\right). \tag{2.5}$$

  $HDD[\tau_1, \tau_2]$ represents the accumulated degrees (from τ_1 to τ_2) where the DAT is below T_{ref}. $CDD[\tau_1, \tau_2]$ represents the accumulated degrees where the DAT is above T_{ref}. These HDD/CDD indices reflect demand for energy needed to heat or cool over the period $[\tau_1, \tau_2]$.

- **Cumulative Average Temperature Index and Pacific Rim Index**
  We define the Cumulative Average Temperature (CAT) index over time interval $[\tau_1, \tau_2]$ by

$$\text{CAT}[\tau_1, \tau_2] := \sum_{\tau=\tau_1}^{\tau_2} T(\tau), \tag{2.6}$$

  and the Pacific Rim (PRIM) index over the same interval by

$$\text{PRIM}[\tau_1, \tau_2] := \sum_{\tau=\tau_1}^{\tau_2} \frac{1}{\tau_2 - \tau_1} T(\tau). \tag{2.7}$$

  For the US and Canadian cities, futures trade on HDD and CDD indices. Temperature is measured in degrees Fahrenheit, and contracts are denominated in US or Canadian dollars. However in Europe futures contracts have as their underlying the CAT or HDD indices. Temperature is measured in degrees Celsius, and contracts are denominated in British pounds. The absence of trade in CDD index futures for European cities reflects the substantially lower use of air conditioning, in comparison to the US. For the two Japanese cities futures are written on the PRIM index (the average DAT across a month or season) and the currency of denomination is the Japanese yen.

Trading in futures contracts continues up to the beginning of the index measurement period, at τ_1. The contracts are agreements to buy or sell twenty times the value of the underlying index at a future date, for a price determined at the outset. They are cash settled, and marked-to-market daily based upon the value of the index. Futures contracts may be bought for a month or a season, with their availability throughout the year depending upon the nature of the index. For example, monthly Heating Degree Day futures are only available in the US for the months October to April inclusive, when heating demands are high.

The CME also organises trade in European vanilla call and put options written on the above futures contracts. The expiry date τ of an option (written on a futures contract) will always be before the start of the index measurement period $[\tau_1, \tau_2]$. The options are cash settled, so if they are exercised the holder receives the difference between the pre-determined strike and the futures price at expiry.

We can also define the temperature indices in a continuous-time setting, often more convenient for pricing than the above market definitions.

In the continuous-time setting, we define the HDD, CDD, CAT and PRIM indices over the time interval $[\tau_1, \tau_2]$ by

$$HDD[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} \max\left(T_{ref} - T(\tau), 0\right) d\tau, \tag{2.8}$$

$$CDD[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} \max\left(T(\tau) - T_{ref}, 0\right) d\tau, \tag{2.9}$$

$$CAT[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} T(\tau) d\tau, \tag{2.10}$$

and

$$PRIM[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} \frac{1}{\tau_2 - \tau_1} T(\tau) d\tau. \tag{2.11}$$

Chapter 3

Modelling Temperature

In order to price options and futures on temperature indices we will need a model for the daily average temperature. In this Chapter I will discuss some of the models proposed in the literature, and their motivation based on properties of temperature data. The aim is to find a stochastic process describing temperature, that both encapsulates the properties of temperature fluctuations and is yet simple enough to allow for analytical pricing of contracts.

3.1 Properties of Temperature Data

Temperature is, by its nature, location specific. Any model proposed must have sufficient flexibility in its description to be fitted to a particular geographical region or city, whilst maintaining the 'global' properties of temperature. We model the daily average temperature, rather than the underlying temperature itself. This is because, as is highlighted in [11], the high frequency data is not available, and taking into account intra-day effects in a temperature model would make it particularly complicated. Since the majority of derivatives contracts are written on the daily average temperature itself, we restrict our attention to modelling this quantity.

As evidenced in the papers discussed below, a model for daily average temperature must capture the following properties:

- *The seasonality of temperature.* Daily average temperature varies throughout the year, following a strong seasonal pattern resulting from calendar effects. Temperature data for cities in the US (see [11]) and Europe (see [3]) have a bimodal distribution, with peaks corresponding to the cooler temperatures in winter, and the warmer summer temperatures.

- *Mean reversion.* Temperature should not deviate from its mean level (which varies temporally) for significant periods of time. Variations are around some 'normal' level.

- *Temperature has 'long memory'.* Daily average temperature is persistent, with pronounced cyclical dynamics and strong autocorrelation. A warm day is more likely to be followed by another warm day than a cool day. Large fluctuations in temperature tend to be persistent.

- *Seasonal volatility.* Not only does the average temperature follow a seasonal pattern, but the magnitude and pattern of variation about this level also varies seasonally. This is, in particular, discussed in [3].

- *A linear trend.* Where data is observed over long periods of time, as in [11], [12], [1] and [3], there is a statistically significant deterministic linear trend in the data. This is, in part, due to global warming. However, as explained in [11], the temperature measuring stations are typically located in cities and airports, where urbanisation and air pollution accelerate the increase in air temperature beyond the global average. This is described as the 'urban heating effect' in [1].

  Most of the above properties are discussed in [13]. This paper studies a daily air temperature index (STI) based on data from four weather stations across Spain, and over a twenty-nine year period. Changes in temperature are observed to have long memory (strong autocorrelation), and exhibit mean reversion and strong seasonal behaviour. They deduce that the conditioned volatility behaviour fits well in GARCH models, and suggest using stochastic interest rate models "incorporating mean reversion, heteroskedasticity and high autocorrelation".

  Both Cao and Wei [12] and Campbell and Diebold [11] study daily average temperature data recorded at US weather stations. In [12], the data is taken from five locations, (Atlanta, Chicago, Dallas, New York and Philadelphia) and over twenty years, whilst Campbell and Diebold study observations from ten cities (Atlanta, Chicago, Cincinnati, Dallas, Des Moines, Las Vegas, New York, Philadelphia, Portland and Tucson) and over a forty year period. Importantly, for all the locations studied futures and options contracts written on daily average temperature trade at the CME.

  A time series approach is taken in [11], to model the daily average temperature at the cities in question. They observe the strong seasonality in DAT. Importantly seasonal fluctuations in daily temperature vary noticeably with location of the city, in both the amplitude and pattern of the variation. This highlights the importance of flexibility of the model, to express seasonality specific to a certain location. [11] suggests using

low-order Fourier Series to produce a smooth seasonal component. The introduction of a Fourier approximation will substantially reduce the number of parameters that need to be estimated. The data analysis in [11] also highlights many of the properties of temperatures stated above. They observe a deterministic linear trend, and strong persistence of daily average temperature is captured using autoregressive lags.

Campbell and Diebold [11] propose the following model for temperature at the ten US cities. Daily average temperature is modelled as an ARMA time series, with a seasonal ARCH model for the volatility:

$$T(t) = \text{Trend}_t + \text{Seasonal}_t + \sum_{l=1}^{L} \rho_{t-l} T(t-l) + \sigma_t \epsilon_t, \tag{3.1}$$

where the terms have general form

$$\begin{aligned} \text{Trend}_t &= \beta_0 + \beta_1 t \\ \text{Seasonal}_t &= \sum_{p=1}^{P} \left(\delta_{c,p} cos\left(2\pi p \frac{d(t)}{365}\right) + \delta_{s,p} sin\left(2\pi p \frac{d(t)}{365}\right)\right) \\ \sigma_t^2 &= \sum_{q=1}^{Q} \left(\gamma_{c,q} cos\left(2\pi q \frac{d(t)}{365}\right) + \gamma_{s,q} sin\left(2\pi q \frac{d(t)}{365}\right)\right) + \sum_{r=1}^{R} \alpha_r \epsilon_{t-r}^2 \\ \epsilon_t &\sim (0,1) \ \text{ i.i.d,} \end{aligned} \tag{3.2}$$

and d(t) is a repeating step function that cycles through $1, ..., 365$.

The model incorporates a deterministic linear trend, and approximates seasonality in daily average temperature and its volatility by Fourier series. It accounts for the autoregressive effects of the variance (persistence) and the cyclical dynamics of temperature.

3.2 Stochastic Models for Daily Average Temperature

There are a number of mean-reverting models proposed in the literature for daily average temperature, varying primarily in their description of the random 'noise' of temperature variations. It is this unpredictable component of the daily average temperature that constitutes weather risk. We will consider three different prescriptions of this random noise term: a fractional Brownian Motion, a Lévy Process, and a standard Brownian Motion. For a particular location, let

- $T(t)$ denote the daily average temperature on day t.

- $\Lambda(t)$ be a deterministic function, describing the seasonal variation. It is the mean level to which the temperature reverts on day t.
- $\alpha(t)$ denote the rate of mean-reversion on day t.
- $\sigma(t)$ denote the volatility of daily temperature fluctuations. It is a seasonal function, taking into account the variation in volatility throughout the year.

We focus on models of the form

$$dT(t) = d\Lambda(t) - \alpha(t)(T(t) - \Lambda(t))dt + \sigma(t)dX(t),$$

where $X(t)$ is a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus daily average temperature follows an Ornstein-Uhlenbeck process, reverting to mean level $\Lambda(t)$, our seasonal component.

3.2.1 Model 1: An Ornstein-Uhlenbeck Process driven by Fractional Brownian Motion

Brody et al. [10] propose the following model for the time evolution of T(t):

$$dT(t) = -\alpha(t)\left(T(t) - \Lambda(t)\right)dt + \sigma(t)dB^H(t), \tag{3.3}$$

where $B^H(t)$ is a fractional Brownian Motion (FBM).

$B^H(t)$ is a continuous-time Gaussian process with mean 0, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^H)$ and characterised by its Hurst parameter $H \in (0, 1)$. It starts at zero ($B^H(0)$=0), and has correlation function

$$\mathbb{E}^H[B^H(t)B^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - \mid t - s \mid^{2H}\right) \quad \forall\, t, s \geq 0.$$

$\mathbb{E}^H$ is the expectation with respect to measure $\mathbb{P}^H$.

The value of the Hurst parameter determines what type of process the FBM is. For $H = \frac{1}{2}$ the process is in fact a standard Brownian Motion, as correlation between increments is zero. Increments are positively correlated for $H > \frac{1}{2}$, but negatively correlated for $H < \frac{1}{2}$.

It is the persistence in temperature data which motivates the choice of fractional Brownian Motion as the driving noise, as it is one of the simplest stochastic processes with this 'memory' property. In fact, it is when the Hurst parameter is greater than $\frac{1}{2}$ that the process exhibits long-range temporal dependence, since

$$\sum_{n=1}^{\infty} \mathbb{E}^H\left(B^H(1)(B^H(n+1) - B^H(n))\right) = \infty \quad \text{for H } > \frac{1}{2}.$$

In [10], Brody et al. propose FBM with Hurst exponent $H = 1+\gamma$. The parameter $\gamma \leq 0$ is determined from analysis of temperature fluctuations which remain after removing the annual cycle. For the data in question (the daily central England temperature record) they find $\gamma = -0.39$. Importantly this gives $H > \frac{1}{2}$, to account for the persistence of the data.

Assuming that $\alpha(t)$, $\Lambda(t)$ and $\sigma(t)$ are bounded and deterministic functions, it is deduced that $T(t)$ is a normally distributed random variable, with mean

$$\mathbb{E}^H(T(t)) = K(t)\left\{T(0) + \int_0^t \alpha(s)\Lambda(s)K(s)^{-1}ds\right\}, \tag{3.4}$$

and variance

$$\mathrm{Var}(T(t)) = K(t)^2 \int_0^t \int_0^t \sigma(u)\sigma(s)K(u)^{-1}K(s)^{-1}H(2H-1) \mid u-s \mid^{2H-2} duds, \tag{3.5}$$

where

$$K(t) = \exp\left(-\int_0^t \alpha(s)ds\right).$$

This model is therefore consistent with their empirical observations - that temperature fluctuations are normally distributed with long-range temporal dependencies, and revert to a mean level (that changes with time).

The paper goes on to investigate pricing derivatives written on cumulative HDD's and cumulative temperature, using the expected discounted value approach. Prices are expressed as solutions of certain partial differential equations. However these prices are only calculated at the time when the contract is entered, and no dynamics are given for the derivative value. As highlighted in [5], for a consistent mean-reversion model one should add $d\Lambda(t)$ to the right hand side of (3.3), giving

$$dT(t) = d\Lambda(t) - \alpha(t)\left(T(t) - \Lambda(t)\right)dt + \sigma(t)dB^H(t).$$

Although fractional Brownian Motion captures the persistence observed in temperature data, it is not a semimartingale and standard theory for incomplete markets breaks down. Arbitrage-free price dynamics based on this model are derived in [4], using quasi-conditional expectation, fractional stochastic calculus (based on chaos expansions) and the Wick integral. There is debate in the literature as to whether or not such fractional Brownian Motion models do indeed admit arbitrage [8].

3.2.2 Model 2: An Ornstein-Uhlenbeck Model driven by a Lévy Process

Benth's paper [5] proposes a non-Gaussian Ornstein-Uhlenbeck model for the evolution of temperature, with seasonal mean and volatility, and with residuals generated by a Lévy process L(t):

$$dT(t) = d\Lambda(t) - \alpha\left(T(t) - \Lambda(t)\right)dt + \sigma(t)dL(t). \tag{3.6}$$

The speed of mean reversion, α, is constant in this model. The paper analyses temperature data from seven Norwegian cities, using the following time series analogue of the continuous time model (3.6):

$$T_t - \Lambda_t = (1-\alpha)(T_{t-1} - \Lambda_{t-1}) + \sigma_t\epsilon_t, \quad t = 1, 2, \dots \ .$$

This has the form of the additive time series

$$T_t = \Lambda_t + c_t + \tilde{\epsilon}_t, \quad t = 0, 1, 2, \dots \ ,$$

where Λ_t is the seasonal component, c_t the cyclical component and $\tilde{\epsilon}_t$ the random noise.

Empirical studies of the data in [5] observe that the distributions have slight skew (i.e are not symmetrical). Examination of the residuals (obtained by dividing by the square root of the variance function) concludes they are not in fact normally distributed and have (semi-)heavy tails. At four of the seven measurement stations, a normal distribution to model these residuals is rejected at the 2% significance level.

It is these factors which motivate the proposition of Lévy process dynamics in the model for daily average temperature. The Lévy process is chosen to have marginals following the class of generalised hyperbolic distributions - a very flexible family of distributions, with known density and characteristic functions. Moreover this process will capture the skewness and heavy tails of residuals. Parameters of this family of distributions are fitted to the data using the maximum likelihood estimator.

Applying Itô's formula for semimartingales to (3.6), we obtain an explicit expression for the daily average temperature at time t:

$$T(t) = \Lambda(t) + \left(T(0) - \Lambda(0)\right)e^{-\alpha t} + \int_0^t \sigma(u)e^{-\alpha(t-u)}dL(u). \tag{3.7}$$

Once this model is formulated, the authors in [5] go on to derive prices for temperature derivatives, using the Esscher transform to specify probability measures. If the Lévy

process is in fact a standard Brownian Motion, the Esscher transform becomes the Girsanov transform, allowing for a greater range of explicit pricing formulae to be derived.

It is remarked in [5] that fractional Brownian Motion as a driving noise is not appropriate for the Norwegian temperature data. Fractionality is observed in the data, but it is believed that this can be better explained by combining a regression model for deseasonalized temperature with a temporally-dependent variance function.

3.2.3 Model 3: An Ornstein-Uhlenbeck Process driven by Brownian Motion

This daily average temperature model is proposed in [3] and [1], and has the form:

$$dT(t) = d\Lambda(t) - \alpha\left(T(t) - \Lambda(t)\right)dt + \sigma(t)dB(t), \tag{3.8}$$

where $B(t)$ is a standard Brownian Motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Applying Itô's Formula we obtain explicit dynamics for the daily average temperature:

$$T(t) = \Lambda(t) + (T(0) - \Lambda(0))e^{-\alpha t} + \int_0^t \sigma(u)e^{-\alpha(t-u)}dB(u).$$

[1] analyses forty years of temperature data from a Swedish airport, and proposes a monthly varying volatility function σ. Plotting daily temperature differences against a normal distribution, a close resemblance is observed. However, a normal approximation would underestimate very small or large differences in daily temperature from the seasonal average - the 'actual' distribution has heavier tails. Whilst recognising that a normal distribution would underestimate certain probabilities, [1] suggests a standard Brownian Motion to model the random noise term. There is, however, no statistical test to validate its use. It may prove to be dangerous to underestimate these very low probability extreme events. Certainly in considering temperature risk management, there is a larger probability of 'worst case' scenarios in realised temperature fluctuations, than would be suggested by modelling the noise by a standard Brownian Motion.

[3] analyses forty years of daily average temperature data from Stockholm. To fit their model to the data they reformulate the continuous-time dynamics of the SDE to a time series. They de-trend and de-seasonalize the daily average temperatures, and fit to an AR(1) model (to express the mean reverting property of temperature dynamics). Investigating the temporal dependency in daily residuals, they observe strong seasonality in daily volatility (with highest variation for Stockholm being in

the Winter period). In particular it is the autocorrelation of squared residuals that suggests there is a seasonally dependent variance. They then examine residuals obtained after removing this temporal contribution to the variance, leaving the noise term. The histogram of these residuals is indeed unimodal, and relatively symmetrical. However, the hypothesis of normality is rejected at the 1% significance level. This suggests that it would be more appropriate to model the noise by a Lévy process. However in [3] the assumption of independent identically distributed standard normal residuals is believed to be acceptable, for the purpose of calculating derivative prices. Similar observations as in [3] are made in [6], where daily average temperature data is analysed for twenty meteorological stations in Lithuania. The residuals obtained after removing all trends, applying AR(1), and removing effects of seasonal temperature fluctuations are close to being normally distributed. Persistence is evident for the first few lags, but the residuals become uncorrelated for increasing number of lags.

It is this final model, approximating the noise by a standard Brownian Motion, that we will adopt and extend to a n-dimensional setting. It will allow for analytical pricing of futures and options contracts. However there will be pricing errors, as a consequence of the underestimate of more extreme events by this model. More study into the size of mispricing errors should be undertaken, and adjustments incorporated into the prices obtained in the following section.

Chapter 4

Temperature Baskets

In this Chapter I consider a new aspect of temperature derivative pricing that, to the best of my knowledge, has not been explored in the literature. This will involve extending the temperature model in [3] to n-dimensions, to price basket temperature derivatives.

The OTC weather derivatives market is booming. Many institutions, including those in the energy, construction and tourism sectors, need to hedge temperature risk in more than one location. Rather than purchasing futures and options on temperature indices at each city, it would be more cost efficient and practically sensible to purchase these contracts on a basket index of temperature for the locations in question. Further, the weightings in the basket will allow greater emphasis to be placed on those locations where the institution has more exposure to temperature risk.

Suppose we have a finite number of spatial locations x_i, $i = 1, ..., n$, at which daily average temperature is recorded. We investigate future and option prices on baskets of daily average temperature indices at these locations. The temperature model proposed in [3] is extended to a n-dimensional setting, taking into account correlation in temperature between spatial locations. Both [5] and [12] observe that temperature variations are positively correlated for all cities investigated. The closer together two locations are geographically, the greater the correlation between their daily average temperatures, whereas locations sufficiently far apart will have low correlation in their temperature variations. In [6] correlation analysis is performed on the time series for each pair of temperature stations investigated in Lithuania. Since these locations are close, all correlations are close to 1. Understanding temperature correlations between cities is of crucial importance, as risk is increased rather than diversified by issuing temperature derivatives at a number of locations.

We will model the random noise in temperature evolution to be Brownian Motion, and propose that spatial correlations between locations are through this random noise term. We investigate dynamics driven by standard Brownian Motion, instead of by a Lévy Process or fractional Brownian Motion, ensuring that our model will allow analytical pricing of derivatives contracts.

4.1 The Model

Let $W_i(t)$, $i = 1, ..., n$, be standard 1-dimensional Brownian Motions on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We interpret $\mathbb{P}$ to be the actual probability measure, as observed from empirical studies of data. Let $T_i(t)$ denote the daily average temperature at location x_i. At each spatial location we propose the following Ornstein-Uhlenbeck model for the evolution of temperature with time:

$$dT_i(t) = d\Lambda_i(t) - \alpha_i \left(T_i(t) - \Lambda_i(t)\right) dt + \sigma_i(t) dW_i(t). \tag{4.1}$$

$\Lambda_i(t)$ is a deterministic function, describing the trend and seasonality of temperature at location x_i, whilst $\sigma_i^2(t)$ is a bounded, continuously differentiable function, modelling daily volatility of temperature variations. Both functions take the form of truncated Fourier series at each location, and are given by

$$\Lambda_i(t) = a_i + b_i t + a_{i,0} + \sum_{j=1}^{J_1} a_{i,j} \sin\left(\frac{2j\pi(t - f_j)}{365}\right) + \sum_{k=1}^{K_1} b_{i,k} \cos\left(\frac{2k\pi(t - g_k)}{365}\right), \tag{4.2}$$

and

$$\sigma_i^2(t) = c_i + \sum_{j=1}^{J_2} c_{i,j} \sin\left(\frac{2j\pi t}{365}\right) + \sum_{k=1}^{K_1} d_{i,k} \cos\left(\frac{2k\pi t}{365}\right). \tag{4.3}$$

The model proposed at each location is mean reverting at speed α_i, around a function that varies temporally depending on seasonality or any linear trends, and with seasonal volatility in temperature fluctuations.

Consider $W_i(t)$ and $W_j(s)$, the random noise components of daily average temperature at locations x_i and x_j and times t, s respectively. We propose that they are correlated spatially, with covariance

$$\operatorname{cov}\left(W_i(t), W_j(s)\right) = \min(t, s) f\left(|x_i - x_j|\right), \tag{4.4}$$

where f is a function of the distance between the two spatial locations in question. The correlation ρ_{ij} between Brownian Motions $W_i(t)$ and $W_j(t)$ therefore satisfies

$$dW_i(t) dW_j(t) = \rho_{ij} dt = f(|x_i - x_j|) dt. \tag{4.5}$$

This function f must have certain properties. Firstly we stipulate that

$$f\left(|x_i - x_i|\right) = f(0) = \rho_{ii} = 1, \tag{4.6}$$

as the same random noise components must be perfectly correlated. Further we require that $f\left(|x_i - x_j|\right)$ should monotonically decrease towards zero as x_i and x_j get further and further apart. Importantly, since $f\left(|x_i - x_j|\right) = f\left(|x_j - x_i|\right)$ we have $\rho_{ij} = \rho_{ji}$. Detailed analysis of correlation in temperature residuals for a number of cities would give a more detailed prescription of the function f. However a sensible suggestion could take the form $f(r) = e^{-\gamma r}$, for $r > 0$ and $\gamma > 0$ a constant.

We express the stochastic differential equation (4.1) for locations $i = 1, ..., n$ as the n-dimensional system

$$d\mathbf{T}(t) = d\mathbf{\Lambda}(t) - \boldsymbol{\alpha}\left(\mathbf{T}(t) - \mathbf{\Lambda}(t)\right)dt + \boldsymbol{\sigma}(t)d\mathbf{W}(t), \tag{4.7}$$

where

$$\mathbf{T}(t) = \begin{pmatrix} T_1(t) \\ T_2(t) \\ \vdots \\ T_n(t) \end{pmatrix}, \mathbf{\Lambda}(t) = \begin{pmatrix} \Lambda_1(t) \\ \Lambda_2(t) \\ \vdots \\ \Lambda_n(t) \end{pmatrix}, \mathbf{W}(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_n(t) \end{pmatrix},$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}, \boldsymbol{\sigma}(t) = \begin{pmatrix} \sigma_1(t) & 0 & \cdots & 0 \\ 0 & \sigma_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n(t) \end{pmatrix}.$$

We know that $\mathbf{W}(t)$ is normally distributed with zero mean and covariance matrix $\mathbf{\Omega}t$, i.e $\mathbf{W}(t) \sim N(\mathbf{0}, \mathbf{\Omega}t)$, where

$$\mathbf{\Omega} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{pmatrix}. \tag{4.8}$$

The linear transformation property of multivariate normal distributions tells us that

$$X \sim N(\mu, \Sigma) \;\Rightarrow\; AX \sim N(A\mu, A\Sigma A^T).$$

Hence if $Z \sim N(0, \mathbf{I}t)$ and $X = AZ$, then $X \sim N(0, AA^Tt)$. Applying Cholesky factorization to $\mathbf{\Sigma}$ we obtain a lower triangular form for $\mathbf{A}$, which is convenient for calculations.

Using this theory we can express $\mathbf{W}(t)$ in terms of an n-dimensional Brownian Motion $\mathbf{B}(t)$, by

$$\mathbf{W}(t) = \mathbf{L}\mathbf{B}(t), \tag{4.9}$$

where $\mathbf{L}\mathbf{L}^T = \mathbf{\Omega}$, $\mathbf{L}$ is lower triangular, and

$$\mathbf{B}(t) = \begin{pmatrix} B_1(t) \\ B_2(t) \\ \vdots \\ B_n(t) \end{pmatrix}, \text{ with } dB_i(t)dB_j(t) = \delta_{ij}dt.$$

Then we rewrite (4.7) in terms of $\mathbf{B}(t)$, as

$$d\mathbf{T}(t) = d\mathbf{\Lambda}(t) - \boldsymbol{\alpha}\left(\mathbf{T}(t) - \mathbf{\Lambda}(t)\right)dt + \boldsymbol{\sigma}(t)\mathbf{L}d\mathbf{B}(t). \tag{4.10}$$

4.1.1 Basket Indices

Our aim is to price derivatives on temperature indices HDD, CDD, CAT and PRIM written on baskets of temperatures, rather than on temperature at a single location. We introduce the basket

$$A(\tau) := \sum_{i=1}^{n} w_i T_i(\tau), \tag{4.11}$$

where w_i is the weight allocated to spatial location x_i, and $\sum_{i=1}^{n} w_i = 1$. So baskets are a weighted sum of the daily average temperatures at the n spatial locations in question. I use this to propose the concept of a basket index.

In the continuous-time setting, the basket HDD index over $[\tau_1, \tau_2]$ is given by:

$$HDD_A[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} \max(T_{ref} - A(\tau), 0)d\tau, \tag{4.12}$$

and similarly for the basket CDD index,

$$CDD_A[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} \max(A(\tau) - T_{ref}, 0)d\tau, \tag{4.13}$$

the basket CAT index,

$$CAT_A[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} A(\tau)d\tau, \tag{4.14}$$

and basket PRIM index,

$$PRIM_A[\tau_1, \tau_2] := \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} A(\tau)d\tau. \tag{4.15}$$

There is a CDD/HDD/CAT parity, since

$$\max(x - c, 0) - \max(c - x, 0) \equiv x - c$$

$$\Rightarrow CDD_A[\tau_1, \tau_2] - HDD_A[\tau_1, \tau_2] = CAT_A[\tau_1, \tau_2] - T_{ref}(\tau_2 - \tau_1). \tag{4.16}$$

In conventional basket option pricing, each member of the basket is modelled as a geometric Brownian Motion in the standard Black and Scholes framework. Asset prices

are driven by the instantaneous correlation between the Brownian Motions in the basket, and each asset has a lognormal risk-neutral distribution. Hence to determine the distribution of the basket, one must determine the distribution of a linear combination of lognormal random variables. However, this distribution is unknown. Pricing even simple basket options in this set-up is thus a complicated task. A common method is to determine an approximate distribution for the basket, using a moment-matching principle. This technique ensures that as many as possible of the terminal moments of the 'true' basket are reproduced by its approximation, see Brigo and Mercurio [9].

However in this set-up, daily average temperature at each location is modelled as an Ornstein-Uhlenbeck process, a Gaussian process. Temperatures are correlated spatially through the random noise term. The risk-neutral distribution of each temperature is normal, and hence the basket, as a weighted sum of normal random variables, is itself normally distributed. This is much simpler than the conventional case above, and allows explicit prices and dynamics to be calculated for options and futures on baskets of temperature.

First consider pricing a generic futures contract. Recall that we have an incomplete market - our underlying, temperature, is not a storable entity and is not tradeable. Hence futures contracts cannot be hedged, and so to price them we will apply the 'expected discounted value approach'. Let Y_A denote the random payoff at expiry for the owner of a futures contract on our basket index. The holder enters into the contract at time $t < \tau_1$ for the measurement period $[\tau_1, \tau_2]$. They pay at τ_2 the price $F(t, \tau_1, \tau_2)$ against receiving the random payment Y_A. By risk-neutral pricing, the futures price $F(t, \tau_1, \tau_2)$ must satisfy

$$0 = e^{-r(\tau_2 - t)} \mathbb{E}_{\mathbb{Q}} \left[Y_A - F(t, \tau_1, \tau_2) \mid \mathcal{F}_t \right] \tag{4.17}$$

where $\mathbb{Q}$ is a risk-neutral probability measure, and $r > 0$ is the constant (continuously compounding) rate of interest, earned on a risk-free bond. $F(t, \tau_1, \tau_2)$ is then given by

$$F(t, \tau_1, \tau_2) = \mathbb{E}_{\mathbb{Q}} \left[Y_A \mid \mathcal{F}_t \right]. \tag{4.18}$$

We know that this futures price must be $\mathcal{F}_t$ adapted, since it is decided at time t based on information available up to the current time. Importantly we observe that $F(t, \tau_1, \tau_2)$ is a $\mathbb{Q}$ -martingale.

We must now specify our risk-neutral measure $\mathbb{Q}$, and address the issues of existence and uniqueness of a risk-neutral probability measure. Recall that probability measures

$\mathbb{P}$ and $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if they agree which sets in $\mathcal{F}$ have probability zero.

Our futures have as their underlying a weighted basket of temperatures, which is patently not tradeable. One can't buy or sell temperature in the market, indeed the only tradeable in the temperature derivatives market is the risk-free bond, which is trivially a martingale when discounting with respect to any equivalent measure $\mathbb{Q}$. By the First Fundamental Theorem of Asset Pricing, since there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted prices of tradeable assets are $\mathbb{Q}$ - martingales, our financial market is arbitrage-free, and $\mathbb{Q}$ is a risk-neutral measure. So in this market all equivalent measures $\mathbb{Q}$ are in fact risk-neutral measures. For a complete market we would require every contingent claim to be attainable, so that we could hedge all of our risk. But with only one tradeable, the risk-free bond is trivially the only attainable claim, and so the market is incomplete. This is consistent with the risk-neutral measure $\mathbb{Q}$ not being unique.

Currently trading on the CME are futures for CAT, CDD, HDD and PRIM indices at each spatial location. We assume that these contracts are traded with sufficient liquidity so that we can determine the market price of risk from them, and calibrate our prices to the market. Since these futures are tradeable assets their price dynamics should be arbitrage-free, under our assumption of sufficient liquidity. Introduce equivalent martingale measure $\mathbb{Q}^{\boldsymbol{\theta}}$, where $\boldsymbol{\theta}$ is a n-dimensional vector on $[0, \tau_{max}]$ representing the (time-dependent) market price of risk. τ_{max} is a fixed horizon time, including measurement periods for all futures contracts. We then calibrate to the market for temperature trading at each x_i, to determine $\boldsymbol{\theta}$ and the risk-neutral measure used by the market. Since options are written on tradeable futures, once we have determined our risk-neutral measure by calibrating to the futures market, we use the same measure to derive options prices. These issues are discussed in detail for a single spatial location in [7].

4.1.2 Girsanov's Theorem in $\mathbb{R}^n$

We state Girsanov's Theorem, before applying it to our daily average temperature model.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with n-dimensional Brownian Motion $\mathbf{B}(t) = (B_1(t), ..., B_n(t))$. Note that the component processes of $\mathbf{B}(t)$ are independent under $\mathbb{P}$, by the definition of multidimensional Brownian Motion.

Let $\boldsymbol{\theta} = (\theta_1(t), ..., \theta_n(t))$ be an n-dimensional adapted process on $[0, \tau_{max}]$.

Define

$$Z^{\boldsymbol{\theta}}(t) := \exp\left(-\int_0^t \boldsymbol{\theta}(u)\cdot d\mathbf{B}(u) - \frac{1}{2}\int_0^t \|\boldsymbol{\theta}(u)\|^2 du\right), \tag{4.19}$$

where

$$\|\boldsymbol{\theta}(u)\|^2 = \sum_{i=1}^{n} \theta^i(u)^2.$$

Let

$$\tilde{\mathbf{B}}(t) = \mathbf{B}(t) + \int_0^t \boldsymbol{\theta}(u)du, \tag{4.20}$$

and assume that

$$\mathbb{E}\int_0^t \|\boldsymbol{\theta}(u)\|^2 Z(u)^2 du < \infty.$$

Then the process $\tilde{\mathbf{B}}(t)$ is a n-dimensional Brownian Motion under the probability measure $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(D) = \int_D Z_{\tau_{max}}(w)\mathbb{P}(dw) \quad \forall D \in \mathcal{F}_{\tau_{max}}. \tag{4.21}$$

Note that for each j we have

$$\tilde{B}_j(t) = B_j(t) + \int_0^t \theta_j(u)du, \quad j = 1, ..., n,$$

and importantly the component processes of $\tilde{\mathbf{B}}(t)$ are independent under $\tilde{\mathbb{P}}$.

We apply this theory to our stochastic differential equation (4.10),

$$d\mathbf{T}(t) = d\boldsymbol{\Lambda}(t) - \boldsymbol{\alpha}\left(\mathbf{T}(t) - \boldsymbol{\Lambda}(t)\right)dt + \boldsymbol{\sigma}(t)\mathbf{L}d\mathbf{B}(t),$$

recalling that the $B_i(t)$ are independent.

Putting

$$Z^{\boldsymbol{\theta}}(t) := \exp\left(\int_0^t (\boldsymbol{\sigma}(u)\mathbf{L})^{-1}\boldsymbol{\theta}(u)\cdot d\mathbf{B}(u) - \frac{1}{2}\int_0^t \|\boldsymbol{\sigma}(u)\mathbf{L}\|^{-2}\|\boldsymbol{\theta}(u)\|^2 du\right), \tag{4.22}$$

we deduce that

$$\mathbf{B}^{\theta}(t) = \mathbf{B}(t) - \int_0^t (\boldsymbol{\sigma}(u)\mathbf{L})^{-1}\boldsymbol{\theta}(u)du$$

is a standard Brownian Motion under equivalent martingale measure $\mathbb{Q}^{\boldsymbol{\theta}}$, defined as in (4.21). Then $d\mathbf{B}^{\theta}(t) = d\mathbf{B}(t) - (\boldsymbol{\sigma}(t)\mathbf{L})^{-1}\boldsymbol{\theta}(t)dt$, and

$$d\mathbf{T}(t) = d\boldsymbol{\Lambda}(t) + \left(\boldsymbol{\theta}(t) - \boldsymbol{\alpha}\left(\mathbf{T}(t) - \boldsymbol{\Lambda}(t)\right)\right)dt + \boldsymbol{\sigma}(t)\mathbf{L}d\mathbf{B}^{\theta}(t) \tag{4.23}$$

is our stochastic differential equation under the equivalent measure $\mathbb{Q}^{\boldsymbol{\theta}}$.

We can solve this SDE using the standard method for Ornstein-Uhlenbeck processes. Let

$$\boldsymbol{\beta}(t) = \begin{pmatrix} e^{\alpha_1 t} & 0 & \cdots & 0 \\ 0 & e^{\alpha_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\alpha_n t} \end{pmatrix},$$

and note that $\boldsymbol{\beta}(0) = \mathbf{I}$, the n-dimensional identity matrix.

Then, multiplying (4.23) through by $\boldsymbol{\beta}(t)$ we obtain

$$d\left(\boldsymbol{\beta}(t)\mathbf{T}(t)\right) = d\left(\boldsymbol{\beta}(t)\boldsymbol{\Lambda}(t)\right) + \boldsymbol{\beta}(t)\boldsymbol{\theta}(t)dt + \boldsymbol{\beta}(t)\boldsymbol{\sigma}(t)\mathbf{L}d\mathbf{B}^{\boldsymbol{\theta}}(t). \tag{4.24}$$

Integrating this SDE from 0 to t:

$$\begin{aligned}\boldsymbol{\beta}(t)\mathbf{T}(t) - \boldsymbol{\beta}(0)\mathbf{T}(0) = \boldsymbol{\beta}(t)\boldsymbol{\Lambda}(t) - \boldsymbol{\beta}(0)\boldsymbol{\Lambda}(0) + \int_0^t \boldsymbol{\beta}(u)\boldsymbol{\theta}(u)du \\ + \int_0^t \boldsymbol{\beta}(u)\boldsymbol{\sigma}(u)\mathbf{L}d\mathbf{B}^{\boldsymbol{\theta}}(u),\end{aligned} \tag{4.25}$$

and so

$$\begin{aligned}\mathbf{T}(t) = \boldsymbol{\Lambda}(t) + \boldsymbol{\beta}(t)^{-1}\left(\mathbf{T}(0) - \boldsymbol{\Lambda}(0)\right) + \int_0^t \boldsymbol{\beta}(t)^{-1}\boldsymbol{\beta}(u)\boldsymbol{\theta}(u)du \\ + \int_0^t \boldsymbol{\beta}(t)^{-1}\boldsymbol{\beta}(u)\boldsymbol{\sigma}(u)\mathbf{L}d\mathbf{B}^{\boldsymbol{\theta}}(u).\end{aligned} \tag{4.26}$$

Observing the i^{th} row we can express the temperature at location x_i as

$$\begin{aligned}T_i(t) = \Lambda_i(t) + e^{-\alpha_i t}\left(T_i(0) - \Lambda_i(0)\right) + \int_0^t e^{\alpha_i(u-t)}\theta_i(u)du \\ + \int_0^t e^{\alpha_i(u-t)}\sigma_i(u)\sum_{j=1}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(u).\end{aligned} \tag{4.27}$$

4.2 Pricing Futures and Options on Basket Indices

4.2.1 Basket CAT and PRIM Futures

We use the above theory to calculate prices for CAT futures on temperature baskets, and options on these futures.

Recall the basket CAT index given by equation (4.14), and the definition of our basket A in (4.11). Putting these together

$$CAT_A[\tau_1, \tau_2] = \int_{\tau_1}^{\tau_2}\left(\sum_{i=1}^{n} w_i T_i(\tau)\right)d\tau = \sum_{i=1}^{n} w_i\left(\int_{\tau_1}^{\tau_2} T_i(\tau)d\tau\right). \tag{4.28}$$

Thus our futures price $F_{CAT}(t, \tau_1, \tau_2; A)$ satisfies

$$0 = e^{-r(\tau_2 - t)} \mathbb{E}_{\boldsymbol{\theta}} \left[\sum_{i=1}^{n} w_i \left(\int_{\tau_1}^{\tau_2} T_i(\tau) d\tau \right) - F_{CAT}(t, \tau_1, \tau_2; A) \mid \mathcal{F}_t \right],$$

and is given by

$$F_{CAT}(t, \tau_1, \tau_2; A) = \mathbb{E}_{\boldsymbol{\theta}} \left[\sum_{i=1}^{n} w_i \left(\int_{\tau_1}^{\tau_2} T_i(\tau) d\tau \right) \mid \mathcal{F}_t \right]. \tag{4.29}$$

We therefore need to determine $\int_{\tau_1}^{\tau_2} T_i(\tau) d\tau$. Let

$$\tilde{T}_i(t) := T_i(t) - \Lambda_i(t). \tag{4.30}$$

Then, from (4.27)

$$\tilde{T}_i(t) = e^{-\alpha_i t} \tilde{T}_i(0) + \int_0^t e^{\alpha_i (u-t)} \theta_i(u) du + \int_0^t e^{\alpha_i (u-t)} \sigma_i(u) \sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(u). \tag{4.31}$$

But by (4.23) we have

$$d\tilde{\mathbf{T}}(t) = \left(\boldsymbol{\theta}(t) - \boldsymbol{\alpha} \tilde{\mathbf{T}}(t) \right) dt + \boldsymbol{\sigma}(t) \mathbf{L} d\mathbf{B}^{\boldsymbol{\theta}}(t). \tag{4.32}$$

Integrating from τ_1 to τ_2 and observing the i^{th} row:

$$\tilde{T}_i(\tau_2) = \tilde{T}_i(\tau_1) - \alpha_i \int_{\tau_1}^{\tau_2} \tilde{T}_i(\tau) d\tau + \int_{\tau_1}^{\tau_2} \theta_i(\tau) d\tau + \int_{\tau_1}^{\tau_2} \sigma_i(\tau) \sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(\tau). \tag{4.33}$$

Rearranging we obtain

$$\begin{aligned} \int_{\tau_1}^{\tau_2} T_i(\tau) d\tau = \int_{\tau_1}^{\tau_2} \Lambda_i(\tau) d\tau + \alpha_i^{-1} \int_{\tau_1}^{\tau_2} \theta_i(\tau) d\tau + \alpha_i^{-1} \int_{\tau_1}^{\tau_2} \sigma_i(\tau) \sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(\tau) \\ + \alpha_i^{-1} \left(\tilde{T}_i(\tau_1) - \tilde{T}_i(\tau_2) \right). \end{aligned} \tag{4.34}$$

To determine the futures price we will take the expectation conditioned on information available up to time t, $\mathcal{F}_t$. So it is now sensible to express $\tilde{T}_i(\tau_1)$ and $\tilde{T}_i(\tau_2)$ in terms of $\tilde{T}_i(t)$. Returning to the stochastic differential equation (4.24), we integrate from t up to τ giving

$$\begin{aligned} \mathbf{T}(\tau) = \boldsymbol{\Lambda}(\tau) + \boldsymbol{\beta}(\tau)^{-1} \boldsymbol{\beta}(t) \left(\mathbf{T}(t) - \boldsymbol{\Lambda}(t) \right) + \int_t^{\tau} \boldsymbol{\beta}(\tau)^{-1} \boldsymbol{\beta}(u) \boldsymbol{\theta}(u) du \\ + \int_t^{\tau} \boldsymbol{\beta}(\tau)^{-1} \boldsymbol{\beta}(u) \boldsymbol{\sigma}(u) \mathbf{L} d\mathbf{B}^{\theta}(u). \end{aligned} \tag{4.35}$$

The i^{th} row of this SDE is

$$T_i(\tau) = \Lambda_i(\tau) + e^{-\alpha_i(\tau-t)}\left(T_i(t) - \Lambda_i(t)\right) + \int_t^\tau e^{-\alpha_i(\tau-u)}\theta_i(u)du + \int_t^\tau e^{-\alpha_i(\tau-u)}\sigma_i(u)\sum_{j=1}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(u), \tag{4.36}$$

and so substituting back into equation (4.34) we obtain

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} T_i(\tau)d\tau \\
&= \int_{\tau_1}^{\tau_2} \Lambda_i(\tau)d\tau - {\alpha_i}^{-1}\left[\left(T_i(t) - \Lambda_i(t)\right)\left(e^{-\alpha_i(\tau_2-t)} - e^{-\alpha_i(\tau_1-t)}\right)\right] \\
&- {\alpha_i}^{-1}\int_t^{\tau_2} \theta_i(u)\left(e^{-\alpha_i(\tau_2-u)} - \mathbb{1}_{[t,\tau_1]}(u)e^{-\alpha_i(\tau_1-u)} - \mathbb{1}_{[\tau_1,\tau_2]}(u)\right)du \\
&- {\alpha_i}^{-1}\int_t^{\tau_2} \sigma_i(u)\left(e^{-\alpha_i(\tau_2-u)} - \mathbb{1}_{[t,\tau_1]}(u)e^{-\alpha_i(\tau_1-u)} - \mathbb{1}_{[\tau_1,\tau_2]}(u)\right)\sum_{j=1}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(u).
\end{aligned} \tag{4.37}$$

By the linearity of expectation

$$\begin{aligned}
&F_{CAT}(t,\tau_1,\tau_2;A) \\
&= \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i=1}^{n} w_i\left(\int_{\tau_1}^{\tau_2} T_i(\tau)d\tau\right) \mid \mathcal{F}_t\right] = \sum_{i=1}^{n} w_i\mathbb{E}_{\boldsymbol{\theta}}\left[\int_{\tau_1}^{\tau_2} T_i(\tau)d\tau \mid \mathcal{F}_t\right],
\end{aligned} \tag{4.38}$$

and so from (4.37) we deduce

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\theta}}\left[\int_{\tau_1}^{\tau_2} T_i(\tau)d\tau \mid \mathcal{F}_t\right] \\
&= \int_{\tau_1}^{\tau_2} \Lambda_i(u)du + {\alpha_i}^{-1}\left[e^{-\alpha_i(\tau_1-t)} - e^{-\alpha_i(\tau_2-t)}\right]\left(T_i(t) - \Lambda_i(t)\right) + \Theta_i(t,\tau_1,\tau_2),
\end{aligned} \tag{4.39}$$

where

$$\begin{aligned}
&\Theta_i(t,\tau_1,\tau_2) \\
&= \mathbb{E}_{\boldsymbol{\theta}}\left(-{\alpha_i}^{-1}\int_t^{\tau_2} \theta_i(u)\left(e^{-\alpha_i(\tau_2-u)} - \mathbb{1}_{[t,\tau_1]}(u)e^{-\alpha_i(\tau_1-u)} - \mathbb{1}_{[\tau_1,\tau_2]}(u)\right)du \mid \mathcal{F}_t\right) \\
&= {\alpha_i}^{-1}\left(\int_t^{\tau_2} \theta_i(u)\left(1 - e^{-\alpha_i(\tau_2-u)}\right)du - \int_t^{\tau_1} \theta_i(u)\left(1 - e^{-\alpha_i(\tau_1-u)}\right)du\right).
\end{aligned} \tag{4.40}$$

Observe that for each $j = 1, ..., i$, as a consequence of the martingale property of Itô integrals

$$\mathbb{E}_{\boldsymbol{\theta}}\left(\int_t^{\tau_2} \sigma_i(u)\left(e^{-\alpha_i(\tau_2-u)} - \mathbb{1}_{[t,\tau_1]}(u)e^{-\alpha_i(\tau_1-u)} - \mathbb{1}_{[\tau_1,\tau_2]}(u)\right)L_{ij}dB_j^{\boldsymbol{\theta}}(u) \mid \mathcal{F}_t\right) = 0,$$

and so the last term in equation (4.37) does not contribute to the expectation in (4.39).

Combining (4.38) and (4.39) we obtain the price of our futures contract on the basket CAT index:

$$F_{CAT}(t,\tau_1,\tau_2;A)$$
$$=\sum_{i=1}^{n} w_i \left\{ \int_{\tau_1}^{\tau_2} \Lambda_i(u)du + \Theta_i(t,\tau_1,\tau_2) + {\alpha_i}^{-1}\left[e^{-\alpha_i(\tau_1-t)} - e^{-\alpha_i(\tau_2-t)}\right](T_i(t)-\Lambda_i(t))\right\}. \tag{4.41}$$

Importantly we need to specify the market price of risk θ_i at each location, in order to 'pin down' the basket CAT futures price. We do this by calibrating to today's market-observed CAT futures curve.

For each location x_i we take $w_i = 1$, and $w_j = 0$ for $j \neq i$. This gives our theoretical CAT futures price at that spatial location alone, with no basket. But this contract trades in the market (on the CME) and has a price $F_{CAT}(0,\tau_1,\tau_2;x_i)$. By setting

$$F_{CAT}(0,\tau_1,\tau_2;x_i)$$
$$=\left\{ \int_{\tau_1}^{\tau_2} \Lambda_i(u)du + \Theta_i(0,\tau_1,\tau_2) + {\alpha_i}^{-1}\left[e^{-\alpha_i(\tau_1)} - e^{-\alpha_i(\tau_2)}\right](T_i(0)-\Lambda_i(0))\right\}, \tag{4.42}$$

we determine our θ_i through the function $\Theta_i(0,\tau_1,\tau_2)$.

Applying Itô's formula to (4.41) we determine the F_{CAT} dynamics:

$$dF_{CAT}(t,\tau_1,\tau_2;A)$$
$$=\frac{\partial F_{CAT}}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial F_{CAT}}{\partial \tilde{T}_i}d\tilde{T}_i(t) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 F_{CAT}}{\partial \tilde{T}_i \partial \tilde{T}_j} d<\tilde{T}_i(t),\tilde{T}_j(t)> \tag{4.43}$$
$$=\sum_{i=1}^{n} w_i\left[\left(e^{-\alpha_i(\tau_1-t)} - e^{-\alpha_i(\tau_2-t)}\right)\left(\left(T_i(t)-\Lambda_i(t)-{\alpha_i}^{-1}\theta_i(t)\right)dt + {\alpha_i}^{-1}d\tilde{T}_i(t)\right)\right]. \tag{4.44}$$

From equation (4.32) we note that

$$d\tilde{T}_i(t) = \left(\theta_i(t) - \alpha_i(t)(T_i(t)-\Lambda_i(t))\right)dt + \sigma_i(t)\sum_{j=1}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(t),$$

and so

$$dF_{CAT}(t,\tau_1,\tau_2;A) = \sum_{i=1}^{n}\left(w_i\alpha_i^{-1}\sigma_i(t)\left(e^{-\alpha_i(\tau_1-t)} - e^{-\alpha_i(\tau_2-t)}\right)\sum_{j=1}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(t)\right). \tag{4.45}$$

Putting

$$\psi_i(t,\tau_1,\tau_2) := {\alpha_i}^{-1}\sigma_i(t)\left(e^{-\alpha_i(\tau_1-t)} - e^{-\alpha_i(\tau_2-t)}\right), \tag{4.46}$$

we obtain

$$dF_{CAT}(t,\tau_1,\tau_2;A) = \sum_{i=1}^{n} w_i \psi_i(t,\tau_1,\tau_2) \left(\sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(t) \right). \tag{4.47}$$

Recall that $PRIM_A[\tau_1,\tau_2] = \frac{1}{\tau_2-\tau_1} CAT_A[\tau_1,\tau_2]$, and since

$$\mathbb{E}_{\boldsymbol{\theta}}\left(PRIM_A[\tau_1,\tau_2] \mid \mathcal{F}_t\right) = \frac{1}{\tau_2-\tau_1}\mathbb{E}_{\boldsymbol{\theta}}\left(CAT_A[\tau_1,\tau_2] \mid \mathcal{F}_t\right),$$

it follows trivially that

$$F_{PRIM}(t,\tau_1,\tau_2;A) = \frac{1}{\tau_2-\tau_1} F_{CAT}(t,\tau_1,\tau_2;A), \tag{4.48}$$

and

$$dF_{PRIM}(t,\tau_1,\tau_2;A) = \frac{1}{\tau_2-\tau_1} dF_{CAT}(t,\tau_1,\tau_2;A). \tag{4.49}$$

4.2.2 Call Options on Basket CAT and PRIM Futures

Now consider the time-t price of a European call option on the basket CAT future contract, with exercise date $\tau \le \tau_1 < \tau_2$ and strike price K.
Let $C\left(F_{CAT}(t,\tau_1,\tau_2;A), t : K, \tau\right)$ denote the time-t price of this call option. By risk-neutral pricing it must satisfy

$$C\left(F_{CAT}(t,\tau_1,\tau_2;A), t : K, \tau\right) = e^{-r(\tau-t)}\mathbb{E}_{\boldsymbol{\theta}}\left[\max(F_{CAT}(\tau,\tau_1,\tau_2;A) - K, 0) \mid \mathcal{F}_t\right]. \tag{4.50}$$

But

$$\int_t^\tau dF_{CAT}(u,\tau_1,\tau_2;A) = F_{CAT}(\tau,\tau_1,\tau_2;A) - F_{CAT}(t,\tau_1,\tau_2;A),$$

and so by (4.47),

$$F_{CAT}(\tau,\tau_1,\tau_2;A) = F_{CAT}(t,\tau_1,\tau_2;A) + \sum_{i=1}^{n} w_i \int_t^\tau \psi_i(u,\tau_1,\tau_2) \left(\sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(u) \right). \tag{4.51}$$

Then for $F_{CAT}(\tau,\tau_1,\tau_2;A) > K$ we require

$$\sum_{i=1}^{n} w_i \int_t^\tau \psi_i(u,\tau_1,\tau_2) \left(\sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(u) \right) > K - F_{CAT}(t,\tau_1,\tau_2;A). \tag{4.52}$$

To solve this problem we need to determine the distribution of

$$\sum_{i=1}^{n} w_i \int_t^\tau \psi_i(u,\tau_1,\tau_2) \left(\sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(u) \right) = \sum_{i=1}^{n} w_i \sum_{j=1}^{i} \left(\int_t^\tau \psi_i(u,\tau_1,\tau_2) L_{ij} dB_j^{\boldsymbol{\theta}}(u) \right)$$

at time t.

By standard properties of the Itô integral,

$$\int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u)$$

is normally distributed, with

$$\mathbb{E}_{\boldsymbol{\theta}}\left(\int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u) \mid \mathcal{F}_t\right) = 0.$$

By Itô's isometry

$$\mathbb{E}_{\boldsymbol{\theta}}{}^2\left(\int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u) \mid \mathcal{F}_t\right) = \int_t^\tau \psi_i^2(u,\tau_1,\tau_2)L_{ij}^2du.$$

Hence

$$\int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u) \sim N\left(0, \int_t^\tau \psi_i^2(u,\tau_1,\tau_2)L_{ij}^2du\right). \tag{4.53}$$

Thus

$$\sum_{i=1}^n w_i \sum_{j=1}^i \left(\int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u)\right),$$

as a weighted sum of Gaussian random variables, is itself normally distributed. Note that the $\int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u)$ are independent for each j, since $B_j^{\boldsymbol{\theta}}(u)$, $j = 1, ..., n$, are independent by Girsanov's Theorem. So, by standard properties of the normal distribution,

$$w_i \sum_{j=1}^i \int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u) \sim N\left(0, w_i^2 \sum_{j=1}^i \int_t^\tau \psi_i^2(u,\tau_1,\tau_2)L_{ij}^2du\right). \tag{4.54}$$

Let

$$Z_i := w_i \sum_{j=1}^i \int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u). \tag{4.55}$$

We need to determine the variance of

$$\sum_{i=1}^n w_i \sum_{j=1}^i \left(\int_t^\tau \psi_i(u,\tau_1,\tau_2)L_{ij}dB_j^{\boldsymbol{\theta}}(u)\right) = \sum_{i=1}^n Z_i,$$

taking into account the covariance between terms Z_i. Consider $\text{cov}(Z_1, Z_j)$ for $j > 1$. Both Z_1 and Z_j have in common an integral with respect to the standard Brownian Motion $B_1^{\boldsymbol{\theta}}(u)$. By the independence of $B_1^{\boldsymbol{\theta}}(u)$ and $B_j^{\boldsymbol{\theta}}(u)$ for $j > 1$, the covariance is only between these two integrals. Hence

$$\begin{aligned}\text{cov}(Z_1, Z_j) &= \text{cov}\left(w_1 \int_t^\tau \psi_1(u,\tau_1,\tau_2)L_{11}dB_1^{\boldsymbol{\theta}}(u), w_j \int_t^\tau \psi_j(u,\tau_1,\tau_2)L_{j1}dB_1^{\boldsymbol{\theta}}(u)\right) \\ &= w_1w_jL_{11}L_{j1}\int_t^\tau \psi_1(u,\tau_1,\tau_2)\psi_j(u,\tau_1,\tau_2)du, \text{ for } j > 1. \end{aligned} \tag{4.56}$$

Comparing terms Z_i and Z_j for $j > i$ in this way, we conclude that

$$\begin{aligned}\operatorname{cov}(Z_i, Z_j) &= \operatorname{cov}\left(Z_i, w_j \sum_{k=1}^{i} \int_t^\tau \psi_j(u, \tau_1, \tau_2) L_{jk} dB_k^{\boldsymbol{\theta}}(u)\right) \\ &= w_i w_j \left(\sum_{k=1}^{i} L_{ik} L_{jk}\right) \int_t^\tau \psi_i(u, \tau_1, \tau_2)\psi_j(u, \tau_1, \tau_2) du, \text{ for } j > i. \quad (4.57)\end{aligned}$$

Then

$$\begin{aligned}\operatorname{Var}\left(\sum_{i=1}^{n} Z_i\right) &= \sum_{i=1}^{n} \operatorname{Var}(Z_i) + 2 \sum_{i,j \ i<j} \operatorname{cov}(Z_i, Z_j) \\ &= \sum_{i=1}^{n} w_i^2 \sum_{j=1}^{i} \int_t^\tau \psi_i^2(u, \tau_1, \tau_2) L_{ij}^2 du \\ &\quad + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \left(\sum_{k=1}^{i} L_{ik} L_{jk}\right) \int_t^\tau \psi_i(u, \tau_1, \tau_2)\psi_j(u, \tau_1, \tau_2) du \\ &:= \Pi_{t,\tau}^2. \quad (4.58)\end{aligned}$$

Hence

$$\sum_{i=1}^{n} w_i \sum_{j=1}^{i} \left(\int_t^\tau \psi_i(u, \tau_1, \tau_2) L_{ij} dB_j^{\boldsymbol{\theta}}(u)\right) \sim N(0, \Pi_{t,\tau}^2).$$

Returning to (4.52), for $F_{CAT}(\tau, \tau_1, \tau_2; A) > K$ we require

$$\Pi_{t,\tau} X > K - F_{CAT}(t, \tau_1, \tau_2; A), \quad \text{where } X \sim N(0, 1).$$

This is equivalent to

$$X > \frac{K - F_{CAT}(t, \tau_1, \tau_2; A)}{\Pi_{t,\tau}} := d_1(t, \tau).$$

Denote $F_{CAT_t} := F_{CAT}(t, \tau_1, \tau_2; A)$. Then

$$\begin{aligned}C\left(F_{CAT_t}, t; K, \tau\right) &= e^{-r(\tau - t)} \int_{d_1(t,\tau)}^{\infty} \left(F_{CAT_t} + \Pi_{t,\tau} x - K\right) \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \\ &= e^{-r(\tau - t)} \left(F_{CAT_t} - K\right) \Phi\left(-d_1(t, \tau)\right) + \Pi_{t,\tau} e^{-r(\tau - t)} \int_{d_1(t,\tau)}^{\infty} \frac{x e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx. \quad (4.59)\end{aligned}$$

Hence

$$C\left(F_{CAT_t}, t : K, \tau\right) = e^{-r(\tau - t)} \left(F_{CAT_t} - K\right) \Phi\left(d(t, \tau)\right) + \frac{\Pi_{t,\tau}}{\sqrt{2\pi}} e^{-r(\tau - t)} e^{-\frac{1}{2}d(t,\tau)^2}, \quad (4.60)$$

where

$$d(t, \tau) = \frac{F_{CAT}(t, \tau_1, \tau_2; A) - K}{\Pi_{t,\tau}}. \quad (4.61)$$

We observe that this call price is in fact independent of the market price of risk $\boldsymbol{\theta}$. This is because we can hedge the option using the underlying CAT futures contract.

We need only modify this slightly to price a European call option on the basket PRIM futures contract, with exercise date $\tau \leq \tau_1 < \tau_2$ and strike K. The call price is given by

$$C\left(F_{PRIM}(t,\tau_1,\tau_2;A), t : K, \tau\right) = e^{-r(\tau-t)}\mathbb{E}_{\boldsymbol{\theta}}\left[\max(F_{PRIM}(\tau,\tau_1,\tau_2;A) - K, 0) \mid \mathcal{F}_t\right], \tag{4.62}$$

where

$$F_{PRIM}(t,\tau_1,\tau_2;A) = \frac{1}{\tau_2-\tau_1}F_{CAT}(t,\tau_1,\tau_2;A).$$

Dividing (4.51) by $(\tau_2 - \tau_1)$, we obtain

$$\begin{aligned} &F_{PRIM}(\tau,\tau_1,\tau_2;A) \\ &= F_{PRIM}(t,\tau_1,\tau_2;A) + \frac{1}{(\tau_2-\tau_1)}\sum_{i=1}^{n} w_i \int_t^{\tau} \psi_i(u,\tau_1,\tau_2)\left(\sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(u)\right), \end{aligned} \tag{4.63}$$

and therefore

$$F_{PRIM}(\tau,\tau_1,\tau_2;A) \sim N\left(F_{PRIM}(t,\tau_1,\tau_2;A), \frac{\Pi_{t,\tau}^2}{(\tau_2-\tau_1)^2}\right). \tag{4.64}$$

Following the same working as for the CAT case above, the call price on a basket PRIM future is given by

$$\begin{aligned} &C\left(F_{PRIM_t}, t : K, \tau\right) \\ &= e^{-r(\tau-t)}\left(F_{PRIM_t} - K\right)\Phi\left(d(t,\tau)(\tau_2-\tau_1)\right) + \frac{\Pi_{t,\tau}}{(\tau_2-\tau_1)}\frac{e^{-r(\tau-t)}}{\sqrt{2\pi}}e^{-\frac{1}{2}d(t,\tau)^2(\tau_2-\tau_1)^2}. \end{aligned} \tag{4.65}$$

4.2.3 Basket CDD and HDD Futures

We now determine prices for HDD futures on temperature baskets. Recall that

$$HDD_A[\tau_1,\tau_2] := \int_{\tau_1}^{\tau_2} \max(T_{ref} - A(\tau), 0)d\tau = \int_{\tau_1}^{\tau_2} \max\left(T_{ref} - \sum_{i=1}^{n} w_i T_i(\tau), 0\right) d\tau.$$

Thus our futures price $F_{HDD}(t,\tau_1,\tau_2;A)$ satisfies

$$0 = e^{-r(\tau_2-t)}\mathbb{E}_{\boldsymbol{\theta}}\left[\int_{\tau_1}^{\tau_2} \max\left(T_{ref} - \sum_{i=1}^{n} w_i T_i(\tau), 0\right) d\tau - F_{HDD}(t,\tau_1,\tau_2;A) \mid \mathcal{F}_t\right],$$

and is given by

$$F_{HDD}(t,\tau_1,\tau_2;A) = \mathbb{E}_{\boldsymbol{\theta}}\left(\int_{\tau_1}^{\tau_2} \max\left(T_{ref} - \sum_{i=1}^{n} w_i T_i(\tau), 0\right) d\tau \mid \mathcal{F}_t\right) \tag{4.66}$$

$$= \int_{\tau_1}^{\tau_2} \mathbb{E}_{\boldsymbol{\theta}}\left(\max\left(T_{ref} - \sum_{i=1}^{n} w_i T_i(\tau), 0\right) \mid \mathcal{F}_t\right) d\tau, \tag{4.67}$$

where we have used the Fubini-Tonelli Theorem to interchange expectation and integration. Let $c = T_{ref}$ be the reference temperature. We need to calculate

$$\mathbb{E}_{\boldsymbol{\theta}}\left(\max\left(c - \sum_{i=1}^{n} w_i T_i(\tau), 0\right) \mid \mathcal{F}_t\right).$$

By (4.36) we can write

$$A(\tau) = \sum_{i=1}^{n} w_i T_i(\tau) = \sum_{i=1}^{n} w_i \left(X_i(t,\tau) + Y_i(t,\tau)\right), \tag{4.68}$$

where

$$X_i(t,\tau) = \Lambda_i(\tau) + e^{-\alpha_i(\tau-t)}\left(T_i(t) - \Lambda_i(t)\right) + \int_t^{\tau} e^{-\alpha_i(\tau-u)}\theta_i(u)du, \tag{4.69}$$

and

$$Y_i(t,\tau) = \int_t^{\tau} e^{-\alpha_i(\tau-u)}\sigma_i(u)\sum_{j=1}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(u) = \sum_{j=1}^{i}\int_t^{\tau} e^{-\alpha_i(\tau-u)}\sigma_i(u)L_{ij}dB_j^{\boldsymbol{\theta}}(u). \tag{4.70}$$

We need to determine the distribution of our basket $A(\tau)$ at time t. Note $X_i(t,\tau)$ is deterministic. However

$$\int_t^{\tau} e^{-\alpha_i(\tau-u)}\sigma_i(u)L_{ij}dB_j^{\boldsymbol{\theta}}(u) \sim N\left(0, \int_t^{\tau} e^{-2\alpha_i(\tau-u)}\sigma_i^2(u)L_{ij}^2du\right)$$

at t, by Itô's isometry. Since the $B_j^{\boldsymbol{\theta}}(u)$ are independent for each j, we just sum the variances to obtain the variance of $Y_i(t,\tau)$, giving

$$Y_i(t,\tau) \sim N\left(0, \sum_{j=1}^{i}\int_t^{\tau} e^{-2\alpha_i(\tau-u)}\sigma_i^2(u)L_{ij}^2du\right). \tag{4.71}$$

Then $\sum_{i=1}^{n} w_i Y_i(t,\tau)$, as a sum of Gaussian random variables, is normally distributed with

$$\mathbb{E}\left(\sum_{i=1}^{n} w_i Y_i(t,\tau)\right) = \sum_{i=1}^{n} w_i \mathbb{E}\left(Y_i(t,\tau)\right) = 0.$$

But

$$\mathrm{Var}\left(\sum_{i=1}^{n} w_i Y_i(t,\tau)\right) = \sum_{i=1}^{n}\mathrm{Var}(w_i Y_i(t,\tau)) + 2\sum_{i,j\ i<j}\mathrm{cov}(w_i Y_i, w_j Y_j),$$

and so we need to determine the covariance between the Y_i for $i = 1, ..., n$. Put

$$\Gamma_i^2(t,\tau) := \sum_{j=1}^{i} \int_t^\tau e^{-2\alpha_i(\tau-u)} \sigma_i^2(u) L_{ij}^2 du. \tag{4.72}$$

Then

$$\text{Var}\left(\sum_{i=1}^{n} w_i Y_i(t,\tau)\right) = \sum_{i=1}^{n} w_i^2 \Gamma_i^2(t,\tau) + 2 \sum_{i,j \ i<j} w_i w_j \text{cov}(Y_i, Y_j).$$

Consider $\text{cov}(Y_1, Y_j)$ for $j > 1$. We follow the same arguments as for pricing the call on CAT futures. Both Y_1 and Y_j have in common an integral with respect to the standard Brownian Motion $B_1^{\boldsymbol{\theta}}(u)$, and by the independence of $B_1^{\boldsymbol{\theta}}(u)$ and $B_j^{\boldsymbol{\theta}}(u)$ for $j > 1$, the covariance is only between the these two integrals. Hence

$$\text{cov}(Y_1, Y_j) = L_{11} L_{j1} \int_t^\tau e^{-(\alpha_1+\alpha_j)(\tau-u)} \sigma_1(u) \sigma_j(u) du, \text{ for } j > 1.$$

Comparing terms Y_i and Y_j for $j > i$ in this way, we conclude that

$$\text{cov}(Y_i, Y_j) = \left(\sum_{k=1}^{i} L_{ik} L_{jk}\right) \int_t^\tau e^{-(\alpha_i+\alpha_j)(\tau-u)} \sigma_i(u) \sigma_j(u) du, \text{ for } j > i. \tag{4.73}$$

Let

$$\Delta_{i,j}(t,\tau) := \int_t^\tau e^{-(\alpha_i+\alpha_j)(\tau-u)} \sigma_i(u) \sigma_j(u) du. \tag{4.74}$$

Then

$$\text{Var}\left(\sum_{i=1}^{n} w_i Y_i(t,\tau)\right) = \sum_{i=1}^{n} w_i \left[w_i \Gamma_i^2(t,\tau) + 2 \sum_{j=i+1}^{n} w_j \left(\sum_{k=1}^{i} L_{ik} L_{jk}\right) \Delta_{i,j}(t,\tau) \right], \tag{4.75}$$

and so from (4.68) we deduce that

$$A(\tau) \sim N\left(\sum_{i=1}^{n} w_i X_i(t,\tau), \sum_{i=1}^{n} w_i \left[w_i \Gamma_i^2(t,\tau) + 2 \sum_{j=i+1}^{n} w_j \left(\sum_{k=1}^{i} L_{ik} L_{jk}\right) \Delta_{i,j}(t,\tau) \right]\right). \tag{4.76}$$

Let

$$U(t,\tau) := \sum_{i=1}^{n} w_i X_i(t,\tau), \qquad E(t,\tau) := \sum_{i=1}^{n} w_i^2 \Gamma_i^2(t,\tau), \tag{4.77}$$

and

$$G(t,\tau) := \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \left(\sum_{k=1}^{i} L_{ik} L_{jk}\right) \Delta_{i,j}(t,\tau). \tag{4.78}$$

We can then express $A(\tau)$ in terms of the standard normal random variable $X \sim N(0,1)$, as

$$A(\tau) = U(t,\tau) + (E(t,\tau) + 2G(t,\tau))^{\frac{1}{2}} X.$$

We need to determine

$$\mathbb{E}_{\boldsymbol{\theta}}\left(\max\left(c-\sum_{i=1}^{n}w_iT_i(\tau),0\right)\mid\mathcal{F}_t\right),$$

and so consider

$$c>A(\tau)=\sum_{i=1}^{n}w_iT_i(\tau).$$

This requires

$$\begin{aligned}&c-U(t,\tau)>\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}X\\ \Longrightarrow\quad &X<\frac{c-U(t,\tau)}{\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}}:=f(t,\tau).\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{\theta}}\left(\max\left(c-\sum_{i=1}^{n}w_iT_i(\tau),0\right)\mid\mathcal{F}_t\right)=\int_{-\infty}^{f(t,\tau)}\left(c-A(\tau,x)\right)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx\\ &=\int_{-\infty}^{f(t,\tau)}\left(f(t,\tau)\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}-\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}x\right)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx\\ &=\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}\left\{f(t,\tau)\Phi\left(f(t,\tau)\right)-\int_{-\infty}^{f(t,\tau)}\frac{xe^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx\right\}\\ &=\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}\left\{f(t,\tau)\Phi\left(f(t,\tau)\right)+\phi\left(f(t,\tau)\right)\right\},\end{aligned}\tag{4.79}$$

where ϕ is the standard normal density. Then by (4.67)

$$F_{HDD}(t,\tau_1,\tau_2;A)=\int_{\tau_1}^{\tau_2}\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}\left\{f(t,\tau)\Phi\left(f(t,\tau)\right)+\phi\left(f(t,\tau)\right)\right\}d\tau,\tag{4.80}$$

where

$$\begin{aligned}f(t,\tau)&=\frac{c-U(t,\tau)}{\left(E(t,\tau)+2G(t,\tau)\right)^{\frac{1}{2}}}\\ &=\frac{c-\sum_{i=1}^{n}w_iX_i(t,\tau)}{\left(\sum_{i=1}^{n}w_i^2\Gamma_i^2(t,\tau)+2\sum_{i=1}^{n}\sum_{j=i+1}^{n}w_iw_j\left(\sum_{k=1}^{i}L_{ik}L_{jk}\right)\Delta_{i,j}(t,\tau)\right)^{\frac{1}{2}}}.\end{aligned}\tag{4.81}$$

We also determine the dynamics of the HDD futures price, $dF_{HDD}(t,\tau_1,\tau_2;A)$, by applying Itô's formula:

$$\begin{aligned}&dF_{HDD}(t,\tau_1,\tau_2;A)\\ &=\frac{\partial F_{HDD}}{\partial t}dt+\sum_{i=1}^{n}\frac{\partial F_{HDD}}{\partial T_i}dT_i(t)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2F_{HDD}}{\partial T_i\partial T_j}d<T_i(t),T_j(t)>,\end{aligned}\tag{4.82}$$

where $dT_i(t)$ is given by (4.32) as

$$dT_i(t) = d\Lambda_i(t) + (\theta_i(t) - \alpha_i(t)(T_i(t) - \Lambda_i(t)))\,dt + \sigma_i(t)\sum_{j=i}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(t).$$

Since $F_{HDD}(t,\tau_1,\tau_2;A)$ is a $\mathbb{Q}^{\boldsymbol{\theta}}$-martingale, we need only focus on the 'risky' term in Itô's formula, and so

$$dF_{HDD}(t,\tau_1,\tau_2;A) = \sum_{i=1}^{n}\frac{\partial F_{HDD}}{\partial T_i}\sigma_i(t)\sum_{j=1}^{i} L_{ij}dB_j^{\boldsymbol{\theta}}(t).$$

Differentiating (4.80),

$$\begin{aligned}
&\frac{\partial F_{HDD}}{\partial T_i}\\
&= \int_{\tau_1}^{\tau_2}(E(t,\tau)+2G(t,\tau))^{\frac{1}{2}}\left(\frac{-w_i e^{-\alpha_i(\tau-t)}}{(E(t,\tau)+2G(t,\tau))^{\frac{1}{2}}}\right)\Phi\left(f(t,\tau)\right)d\tau\\
&+ \int_{\tau_1}^{\tau_2}(E(t,\tau)+2G(t,\tau))^{\frac{1}{2}}\,f(t,\tau)\phi\left(f(t,\tau)\right)\left(\frac{-w_i e^{-\alpha_i(\tau-t)}}{(E(t,\tau)+2G(t,\tau))^{\frac{1}{2}}}\right)d\tau\\
&+ \int_{\tau_1}^{\tau_2}(E(t,\tau)+2G(t,\tau))^{\frac{1}{2}}\,\phi\left(f(t,\tau)\right)\left(\frac{-w_i e^{-\alpha_i(\tau-t)}}{(E(t,\tau)+2G(t,\tau))^{\frac{1}{2}}}\right)(-f(t,\tau))\,d\tau\\
&= \int_{\tau_1}^{\tau_2} -w_i e^{-\alpha_i(\tau-t)}\Phi\left(f(t,\tau)\right)d\tau,
\end{aligned} \tag{4.83}$$

and substituting in we obtain

$$dF_{HDD}(t,\tau_1,\tau_2;A) = -\sum_{i=1}^{n}\left\{\int_{\tau_1}^{\tau_2} w_i\sigma_i(t)e^{-\alpha_i(\tau-t)}\Phi\left(f(t,\tau)\right)d\tau\sum_{j=1}^{i}L_{ij}dB_j^{\boldsymbol{\theta}}(t)\right\}. \tag{4.84}$$

These dynamics are particularly complicated, and thus call and put options written on basket HDD futures are likely to be priced numerically, using Monte Carlo methods. As an example, consider a European call option on the basket HDD futures contract, with exercise date $\tau \le \tau_1 < \tau_2$ and strike price K. By risk-neutral pricing, its time-t price $C\left(F_{HDD}(t,\tau_1,\tau_2;A), t : K, \tau\right)$ must satisfy

$$C\left(F_{HDD}(t,\tau_1,\tau_2;A), t : K, \tau\right) = e^{-r(\tau-t)}\mathbb{E}_{\boldsymbol{\theta}}\left[max(F_{HDD}(\tau,\tau_1,\tau_2;A) - K, 0) \mid \mathcal{F}_t\right]. \tag{4.85}$$

We can express $F_{HDD}(\tau,\tau_1,\tau_2;A)$ in terms of $F_{HDD}(t,\tau_1,\tau_2;A)$, by integrating our dynamics in (4.84) from t to τ. Then

$$\begin{aligned}
F_{HDD}(\tau,\tau_1,\tau_2;A) &= F_{HDD}(t,\tau_1,\tau_2;A)\\
&\quad - \sum_{i=1}^{n} w_i\int_t^{\tau}\left\{\int_{\tau_1}^{\tau_2}\sigma_i(u)e^{-\alpha_i(\tau-u)}\Phi\left(f(u,\tau)\right)d\tau\right\}\sum_{j=1}^{i}L_{ij}dB_j^{\boldsymbol{\theta}}(u).
\end{aligned} \tag{4.86}$$

Expressing $F_{HDD}(\tau, \tau_1, \tau_2; A)$ in terms of a normally distributed random variable would involve determining the variance of

$$\sum_{i=1}^{n} w_i \int_t^{\tau} \left\{ \int_{\tau_1}^{\tau_2} \sigma_i(u) e^{-\alpha_i(\tau-u)} \Phi\left(f(u,\tau)\right) d\tau \right\} \sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(u).$$

Although an expression similar to $\Pi_{t,\tau}^2$ (given in (4.58)) could be determined for this variance, explicit evaluation of the resulting integral would be more complex, as a consequence of the $\Phi\left(f(u,\tau)\right)$ term above. Since the variance would require numerical computation, a more sensible way to proceed would be to use Monte Carlo methods from the outset, simulating the risk-neutral temperature vector $\mathbf{T}(\tau)$ at the exercise time τ, and then deriving $F_{HDD}(\tau, \tau_1, \tau_2; A)$ by numerical integration. This is indeed suggested in [3], for pricing a call on HDD futures at just one spatial location.

Recall the parity relation given in equation (4.16) by

$$CDD_A[\tau_1, \tau_2] - HDD_A[\tau_1, \tau_2] = CAT_A[\tau_1, \tau_2] - T_{ref}(\tau_2 - \tau_1).$$

Since

$$F_{CAT}(t, \tau_1, \tau_2; A) = \mathbb{E}_{\boldsymbol{\theta}}\left[CAT_A[\tau_1, \tau_2] \mid \mathcal{F}_t\right],$$

and

$$F_{HDD}(t, \tau_1, \tau_2; A) = \mathbb{E}_{\boldsymbol{\theta}}\left[HDD_A[\tau_1, \tau_2] \mid \mathcal{F}_t\right],$$

we deduce that

$$\begin{aligned} F_{CDD}(t, \tau_1, \tau_2; A) &= \mathbb{E}_{\boldsymbol{\theta}}\left[CDD_A[\tau_1, \tau_2] \mid \mathcal{F}_t\right] \\ &= \mathbb{E}_{\boldsymbol{\theta}}\left[HDD_A[\tau_1, \tau_2] + CAT_A[\tau_1, \tau_2] - T_{ref}(\tau_2 - \tau_1)\right] \\ &= F_{CAT}(t, \tau_1, \tau_2; A) + F_{HDD}(t, \tau_1, \tau_2; A) - T_{ref}(\tau_2 - \tau_1). \end{aligned} \quad (4.87)$$

The F_{CDD} dynamics are given by

$$\begin{aligned} &dF_{CDD}(t, \tau_1, \tau_2; A) = dF_{CAT}(t, \tau_1, \tau_2; A) + dF_{HDD}(t, \tau_1, \tau_2; A), \\ &= \sum_{i=1}^{n} w_i \psi_i(t, \tau_1, \tau_2) \left(\sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(t) \right) \\ &- \sum_{i=1}^{n} \left\{ \int_{\tau_1}^{\tau_2} w_i \sigma_i(t) e^{-\alpha_i(\tau-t)} \Phi\left(f(t,\tau)\right) d\tau \sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(t) \right\} \\ &= \sum_{i=1}^{n} w_i \left\{ \psi_i(t, \tau_1, \tau_2) - \int_{\tau_1}^{\tau_2} \sigma_i(t) e^{-\alpha_i(\tau-t)} \Phi\left(f(t,\tau)\right) d\tau \right\} \left(\sum_{j=1}^{i} L_{ij} dB_j^{\boldsymbol{\theta}}(t) \right). \end{aligned} \quad (4.88)$$

In Benth and Šaltytė-Benth [3], expressions are derived for

$$F_{CAT}(t),\ dF_{CAT}(t),\ C\left(F_{CAT_t}, t\right),\ F_{HDD}(t) \text{ and } \ dF_{HDD}(t)$$

for indices at a single spatial location. We can use these expressions to 'check' our extension. Taking $w_1 = 1$, and $w_j = 0$ for $j > 1$ in (4.41), (4.47), (4.60), (4.80) and (4.84) respectively, our formulae agree exactly with those in [3] in this most simple 'basket' scenario, where all the weight is concentrated at one location.

Chapter 5

Hedging Spatial Temperature Risk

Many participants in the temperature derivatives market buy or issue contracts at a number of locations. It is important to understand the spatial correlations between these locations, to manage risk. Further many institutions are exposed to temperature risk at locations where derivatives do not trade in the market. Certainly the exchange traded contracts are all written on temperature indices at cities, and an investor may wish to understand the dependency between their location of risk, and a city that has contracts written on its temperature. Furthermore market participants may have temperature risk over a geographical region. An example is an electricity retailer, with demand depending on temperature over the whole area to which they provide electricity, see [2]. In order to manage this risk, they need to know how to best utilise available futures contracts to construct an optimal portfolio that best hedges their exposure.

In [2] optimal futures contracts are selected by minimising the variance in the portfolio with a particular temperature index. Synthetic futures contracts are constructed using what is traded in the market, to best replicate the investor's needs. This all requires a spatial-temporal model to describe daily average temperature.

5.1 A Spatial-Temporal Model for Temperature

A spatial-temporal model for daily average temperature is formulated in [6]. A Gaussian random field, which is mean reverting with seasonal variance, is used to encapsulate spatial dependency. This model is consistent with that proposed in [3], and extended in the previous Chapter.

Let $\mathcal{D}$ be a domain in $\mathbb{R}^2$, with piecewise smooth boundary $\partial\mathcal{D}$. Let $\mathbf{s}$ denote the spatial coordinates, and t the temporal coordinate. The random field

$$\left\{Z(\mathbf{s},t) \ : \ \mathbf{s} \in D \subset \mathbb{R}^2, \ t \in [0,\infty)\right\} \tag{5.1}$$

is the sum of a deterministic component $\mu(\mathbf{s},t)$ and a residual component $\eta(\mathbf{s},t)$,

$$Z(\mathbf{s};t) = \mu(\mathbf{s};t) + \eta(\mathbf{s};t). \tag{5.2}$$

The deterministic component is mean reverting at speed $\alpha(\mathbf{s})$, to a seasonal function $\Lambda(\mathbf{s},t)$. It describes the trends in daily average temperature, and is given by

$$\mu(\mathbf{s},t) = \Lambda(\mathbf{s},t) + \alpha(\mathbf{s})\left(Z(\mathbf{s};t-1) - \Lambda(\mathbf{s},t-1)\right). \tag{5.3}$$

Note the speed of mean reversion $\alpha(\mathbf{s})$ is assumed to be independent of time, as is suggested by empirical observations, see [6]. The residual term models fluctuations in daily average temperature. Due to seasonality evident in the autocorrelation of squared residuals, $\eta(\mathbf{s};t)$ is factorised to give

$$\eta(\mathbf{s};t) = \sigma(\mathbf{s};t)\epsilon(\mathbf{s};t), \tag{5.4}$$

where $\sigma(\mathbf{s};t)$ describes the seasonality observed in volatility. It is a deterministic periodic function, satisfying $\sigma(\mathbf{s};t) = \sigma(\mathbf{s};t+365) \ \forall \ t$.

$\left\{\epsilon(\mathbf{s};t) : \mathbf{s} \in \mathcal{D} \subset \mathbb{R}^2 \ , \ t \in [0,\infty)\right\}$ is a zero-mean stationary Gaussian random field, with correlation function

$$\operatorname{corr}\left\{\epsilon(\mathbf{s};t), (\mathbf{s}+\mathbf{h_s};t)\right\} = \rho(\mathbf{h_s};\theta_\mathbf{s}). \tag{5.5}$$

Hence residuals are correlated in space but not in time, which is consistent with the clear spatial correlation structure observed in residuals for temperature data. [6] suggests the spherical spatial correlation function

$$\rho(\mid \mathbf{h_s} \mid;\theta_\mathbf{s}) = 1 - \frac{3}{2}\frac{\mid \mathbf{h_s} \mid}{\theta_\mathbf{s}} + \frac{1}{2}\left(\frac{\mid \mathbf{h_s} \mid}{\theta_\mathbf{s}}\right)^3. \tag{5.6}$$

5.2 Hedging Spatial Temperature Risk using Market Traded Futures

Let $\mathcal{D}$ represent a geographical region enclosing cities where temperature futures trade. [2] considers the above spatial-temporal model in a continuous-time setting. Let $T(t,x)$ denote the daily average temperature at time t and at location $x \in \mathcal{D}$. We

use the same notation as in Chapter 4 for the seasonal, mean-reverting and volatility components of daily average temperature, letting $\Lambda(t,x) \in C^{1,0}(\mathbb{R}^+ \times \mathcal{D})$ and $\sigma^2(t,x) \in C(\mathbb{R}^+ \times \mathcal{D})$ also depend on the location x. We let $\alpha(x)$ denote the speed of mean reversion to $\Lambda(t,x)$ at $x \in \mathcal{D}$. Let $q \in \mathrm{C}(\mathcal{D} \times \mathcal{D})$ be a continuous, symmetric and positive definite function modelling the spatial covariance in the random field (recall correlations between locations are strictly positive). Define a centred Gaussian random field $W(t,x)$, with covariance

$$\mathrm{cov}(W(t,x), W(s,y)) = \min(s,t)q(x,y) \quad s,t \in \mathbb{R}^+, \ x,y \in \mathcal{D}. \tag{5.7}$$

For details on how to construct $W(t;\cdot)$ "within the framework of Hilbert space valued random processes" see [2].

The evolution of daily average temperature at location $x \in \mathcal{D}$ and time t is given by the spatial-temporal model as

$$dT(t,x) = \left(\frac{\partial}{\partial t}\Lambda(t,x) - \alpha(x)(T(t,x) - \Lambda(t,x)) \right) dt + \sigma(t,x)dW(t,x). \tag{5.8}$$

Integrating this SDE from t up to τ, we obtain an explicit formula for $T(\tau,x)$ given in terms of daily average temperature at time t:

$$T(\tau,x) = \Lambda(\tau,x) + (T(t,x) - \Lambda(t,x))\, e^{-\alpha(x)(\tau-t)} + \int_t^\tau \sigma(u,x)e^{-\alpha(x)(\tau-u)}dW(u,x). \tag{5.9}$$

Suppose there are n locations $x_1, ..., x_n \ \in \mathcal{D}$ at which futures trade in the market. Let $I(\tau_1, \tau_2; x_i)$ denote the CAT, HDD, CDD or PRIM index written on daily average temperature at location x_i, and with measurement period $[\tau_1, \tau_2]$. Consider an agent who is exposed to temperature risk over an area $\mathcal{A} \in \mathcal{D}$, and enters the market at time $t < \tau_1$. In [2], the agent's temperature risk is expressed in the form

$$\int_{\mathcal{A}} I(\tau_1, \tau_2, y)\mu(dy), \tag{5.10}$$

with μ a measure on $(\mathcal{A}, \mathcal{B}(\mathcal{A}))$. In particular it could be a point mass at a location $\bar{y} \in \mathcal{A}$, where futures contracts do not trade. Note it is assumed that $I(\tau_1, \tau_2, y)$ is integrable with respect to the measure μ on $\mathcal{A}$.

The minimum variance problem is to find the optimal (adapted) strategy $\mathbf{a}(t) = (a_1(t), ..., a_n(t))$ which minimises the $\mathcal{L}^2$ distance between a synthetic futures contract, and the desired payoff in (5.10). Defining

$$R(t, \mathbf{a}(t)) := \mathbb{E}\left[\left(\int_{\mathcal{A}} I(\tau_1, \tau_2; y)d\mu(y) - \sum_{i=1}^n a_i(t) I(\tau_1, \tau_2; x_i) \right)^2 \mid \mathcal{F}_t \right], \tag{5.11}$$

our problem is to solve

$$\tilde{R}(t) = \min_{\mathbf{a}(t)} R(t, \mathbf{a}(t)), \tag{5.12}$$

to determine the optimal combination of market traded futures contracts in our replicating portfolio.

When the measure μ is a point mass at location $\bar{y}$, (5.11) simplifies to

$$R(t, \mathbf{a}(t)) = \mathbb{E}\left[\left(I(\tau_1, \tau_2; \bar{y}) - \sum_{i=1}^{n} a_i(t) I(\tau_1, \tau_2; x_i)\right)^2 \mid \mathcal{F}_t\right]. \tag{5.13}$$

Let $\tilde{\mathbf{a}}(t) = (\tilde{a}_1(t), ..., \tilde{a}_n(t))$ denote the minimiser of system (5.12). Then the synthetic futures contract $F^I_{\mathcal{A}}(t, \tau_1, \tau_2)$ is given by

$$F^I_{\mathcal{A}}(t, \tau_1, \tau_2) = \sum_{i=1}^{n} \tilde{a}_i(t) F^I(t, \tau_1, \tau_2; x_i), \tag{5.14}$$

where $F^I(t, \tau_1, \tau_2; x_i)$ denotes the time-t future price on index I at location x_i (this trades in the market).

In [2] the optimal solution $\tilde{\mathbf{a}}(t)$ is given as the solution of a linear system of equations. For optimality in (5.11), we require

$$\frac{\partial R}{\partial a_i} = 0 \text{ for } i = 1, ..., n.$$

Define

$$w(t, x, y) = \mathbb{E}\left[I(\tau_1, \tau_2; x) I(\tau_1, \tau_2; y)\right], \ x, y \in \mathcal{D}, \tag{5.15}$$

and $\mathbf{b}(t) := (b_1(t), ..., b_n(t))^T$, where

$$b_i(t) = \frac{\int_{\mathcal{A}} w(t, x_i, y) \mu(dy)}{w(t, x_i, x_i)}. \tag{5.16}$$

Putting

$$A_{ij}(t) = \frac{w(t, x_i, x_j)}{w(t, x_i, x_i)} \tag{5.17}$$

we define the matrix $A(t) := \{A_{ij}(t)\}_{i,j=1}^n$. Then, as proved in [2], $\tilde{\mathbf{a}}(t)$ solves the linear system

$$A(t)\mathbf{a}(t) = \mathbf{b}(t), \tag{5.18}$$

and so we can determine the weightings for our optimal synthetic portfolio. Although this portfolio is optimal, there is residual risk which cannot be fully hedged using the current futures market. $\tilde{R}(t) = R(t, \tilde{\mathbf{a}}(t))$ gives a measure of this outstanding exposure.

In [2] temporal risk is also considered, along with the spatial risk discussed above. An agent may wish to buy a synthetic futures contract over a time period not consistent with future measurement periods available in the market.

Chapter 6

Conclusions

The weather derivatives market is continually expanding. A range of contracts, issued on indices at a number of cities, trade on the CME. In this dissertation I have focused on the largest contributor to this market, temperature derivatives. Before pricing contracts, an underlying model for the daily average temperature is needed. I have reviewed stochastic models proposed in the literature, and their motivation based on properties of temperature data. The models are mean reverting, about a function that varies seasonally. Further analysis of temperature data in [3] and [6] proposes that the volatility of temperature fluctuations also varies seasonally, and is suitably modelled by a truncated Fourier series. Keeping in mind analytical pricing of derivatives contracts, we pursue standard Brownian Motion as the driving noise of the process. However, analysis of the noise residuals suggests that the 'actual' distribution has heavier tails, with extreme fluctuations in daily average temperature more likely than is suggested by a driving Brownian Motion. This area needs more investigation, and prices derived using Brownian Motion should be adjusted to compensate for its underestimate of the frequency of more extreme events.

I then moved on to pricing contracts written on basket indices of temperature at a number of locations. Many of the participants in this market have exposure to temperature risk at more than one location, or indeed over a region. A basket index allocates a weight to each city at which the investor has risk. Thus a basket option could be more attractive to the investor than buying futures or options on indices at each city. I extended the daily average temperature model in [3] to n-dimensions, taking into account correlation in noise between the locations in the basket. This correlation is a function $f(\mid x_i - x_j \mid)$ of the distance between the two cities, and decreases with increasing distance (as is suggested by empirical observations). Although I have described some general properties of this function, analysis of temperature data at a

range of cities would yield a more precise description of it. It may be that if locations are sufficiently close, we can model their daily average temperatures as being driven by the same Brownian Motion. Moreover, two locations sufficiently far apart may have temperature driven by independent Brownian Motions.

Pricing temperature derivatives needs some thought, as we have an incomplete market. Temperature has no direct value, as it is not a tradeable asset. In order to determine the market price of risk we must calibrate to the market, assuming that futures and options are traded with sufficient liquidity on the CME. We apply Girsanov's Theorem, and price under the risk-neutral measure $\mathbb{Q}^{\boldsymbol{\theta}}$. I have derived prices for futures on the CAT, PRIM, CDD and HDD basket indices, and calls on CAT and PRIM futures. Taking only one location in the basket, these prices agree with those determined in [3] at just one spatial location.

An interesting extension would be to price maximum/minimum type basket options on futures at a number of locations. Such contracts, also known as rainbow options, are typically calls or puts on the best or worst of n underlying assets. The holder of the contract will receive at maturity τ the maximum payoff using the realised value of the futures contracts at expiry. For example, for a CAT call option they would receive the payoff $\max(\tilde{F}_{CAT}(\tau) - K, 0)$ at τ, where $\tilde{F}_{CAT}(\tau) = \max_i(F_{CAT}(\tau; x_i))$ and K is the strike.

Finally I move on to discuss a spatial-temporal model for temperature. Such a model is particularly pertinent in this market, as many investors are exposed to temperature risk over a region, or at a location where derivatives contracts do not trade in the market. Since correlations in temperature between cities are all positive, investing in a portfolio of contracts does not necessarily diversify risk, and so an understanding of temperature dependency between locations is important. I discuss how [2] proposes to construct a synthetic futures contract using market traded futures, to best hedge bespoke temperature risk. This is very similar to the problem of determining the required basket weightings in Chapter 4.

Bibliography

[1] P. Alaton, B. Djehiche, and D. Stillberger. On modelling and pricing weather derivatives. *Applied Mathematical Finance*, 9(1):1–20, 2002.

[2] A. Barth, F.E. Benth, and J. Potthoff. Hedging of spatial temperature risk with market-traded futures. *http://folk.uio.no/fredb/*, 2008.

[3] F. E. Benth and J. Šaltytė-Benth. The volatility of temperature and pricing of weather derivatives. *Quantitative Finance*, 7(5):553–561, 2007.

[4] F.E. Benth. On arbitrage-free pricing of weather derivatives based on fractional Brownian Motion. *Applied Mathematical Finance*, 10(4):303–324, 2003.

[5] F.E. Benth and J. Šaltytė-Benth. Stochastic modelling of temperature variations with a view towards weather derivatives. *Applied Mathematical Finance*, 12(1):53–85, 2005.

[6] F.E. Benth, J. Šaltytė-Benth, and P. Jalinskas. A spatial-temporal model for temperature with seasonal variance. *Journal of Applied Statistics*, 34(7):823–841, 2007.

[7] F.E. Benth, J. Šaltytė-Benth, and S. Koekebakker. Putting a price on temperature. *Scandinavian Journal of Statistics*, 34(4):746–767, 2007.

[8] T. Björk and H. Hult. A note on Wick products and the fractional Black-Scholes model. *Finance and Stochastics*, 9(2):197–209.

[9] D. Brigo, F. Mercurio, F. Rapisarda, and R. Scotti. Approximated moment-mathcing dynamics for basket-options pricing. *Quantitative Finance*, 4(1):1–16, 2004.

[10] D.C. Brody, J. Syroka, and M. Zervos. Dynamical pricing of weather derivatives. *Quantitative Finance*, 2(3):189–198, 2002.

[11] S.D. Campbell and F.X. Diebold. Weather forecasting for weather derivatives. *Journal of the American Statistical Association*, 100(469):6–16, 2005.

[12] M. Cao and J. Wei. Weather derivatives evaluation and market price of weather risk. *Journal Of Futures Markets*, 24(11):1065–1089, 2004.

[13] H. Torro, V. Meneu, and E. Valor. Single factor stochastic models with seasonality applied to underlying weather derivatives variables. *Journal of Risk Finance*, 4(4):6–17, 2003.

[14] CME website. http://www.cme.com/trading/prd/weather/.