The Density of Rational Points on the Cubic Surface $X_0^3 = X_1X_2X_3$

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1 Introduction and Statement of Results

In this paper we are concerned with the problem of counting rational points of bounded height on rational cubic surfaces. For most such surfaces this question appears much too hard for current methods. We shall therefore examine a particular example, namely the surface

$$X_0^3 = X_1X_2X_3, \tag{1}$$

for which the problem is tractable, though by no means trivial. This surface contains the lines $X_0 = X_i = 0$, for $i = 1, 2, 3$, which we shall exclude from consideration. We therefore define

$$\mathcal{N}(H) = \#\{\mathbf{x} \in \mathbb{N}^4 : x_0^3 = x_1x_2x_3, \ \ \mathbf{x} \text{ primitive}, \ x_i \le H, \ (i = 0, 1, 2, 3)\},$$

where an integer vector is said to be primitive if its components have no common factor. The function $\mathcal{N}(H)$ counts points with the most naive height available. This suffices, however, to bring out the key features of the problem. We may observe at once that the condition $x_0 \le H$ is redundant, and that it suffices to assume that (x_1, x_2, x_3) is primitive.

Our principal result is the following.

Theorem *As H tends to infinity we have*

$$\mathcal{N}(H) = \frac{H(\log H)^6}{4 \times 6!} \prod_p \sigma_p + O(H(\log H)^5),$$

where

$$\sigma_p = (1 - \frac{1}{p})^7(1 + 7\frac{1}{p} + \frac{1}{p^2}).$$

It should be stressed that the entire difficulty of this question lies in the constraints imposed on the size of x_1, x_2 and x_3. If we merely require that $x_0 \le H$, the problem is far easier. Indeed if we let $F(x_0)$ denote the number of representations of x_0^3 as a product of three coprime factors, then it is apparent that $F(n)$ is a multiplicative function, and it is an easy exercise to show that $\sum_{n\le H} F(n)$ is asymptotically $cH(\log H)^8$, where c is an appropriate positive constant.

The estimate given by the theorem may be viewed as an instance of a very general conjecture of Manin (see Batyrev and Manin [1]), which predicts the occurence of such asymptotic formulae, and describes the exponents of H and $\log H$ which should appear. The particular surface (1) has attracted attention elsewhere, and versions of the asymptotic formula have been obtained by Batyrev and Tschinkel [2], by Fouvry [4], and, in work still in preparation, by Salberger and by de la Bretèche [3]. Batyrev and Tschinkel use an analytic method, based on a related non-singular toric variety. The approach employed by de la Bretèche is also analytic, but in the more traditional sense. Salberger's treatment is based on consideration of the 'universal torsor' of the surface (1). Like that of Batyrev and Tschinkel, it has the advantage of providing a full interpretation of the constants in the asymptotic formula. Technically it is 'elementary' though couched in highly geometric language. Fouvry's approach is elementary, in the traditional sense, though certainly not simple. In contrast to all these, we believe that our treatment, which is also elementary, has the advantage of being both straightforward and natural. Moreover there is no obvious obstacle to its generalization to other related counting problems.

2 Interpretation of the Theorem

The simplest way to interpret the theorem is to look at the density of solutions of (1) in each of the completions of $\mathbb{Q}$. In the case of a p-adic completion we define

$$\sigma_p' = \lim_{n\to\infty} \frac{1}{p^{3n}} \#\{\mathbf{x} \pmod{p^n} : x_0^3 \equiv x_1x_2x_3 \pmod{p^n},\ p \nmid \mathbf{x}\}.$$

It is not difficult to see that there are $\phi(p^n)^3$ solutions with $p \nmid x_1, x_2, x_3$, since then x_0, x_1 and x_2 determine x_3. Similarly there are $3\phi(p^n)^2p^{n-1}$ solutions in which p divides exactly one of x_1, x_2 or x_3, since in this case, if $p|x_3$ for example, x_1, x_2 and $p^{-1}x_0$ determine x_3 modulo p^n. Finally, if $k+j$ is a multiple of three, and is less than n, there are

$$\phi(p^{n-k})\phi(p^{n-j})\phi(p^{n-(k+j)/3})p^{j+k}$$

solutions in which $p^k||x_1$ and $p^j||x_2$. It follows that there are

$$3p^{3n}(1-\frac{1}{p})^3 \sum_{\substack{k,j=1 \\ 3|k+j}}^{\infty} p^{-(k+j)/3} + O(n^2p^{3n-[n/3]})$$

$$= 3(2+p^{-1})(1-p^{-1})p^{3n-1} + O(n^2p^{3n-[n/3]})$$

solutions in which p divides exactly two of x_1, x_2 or x_3. We therefore see that

$$\sigma'_p = (1-p^{-1})(1+7p^{-1}+p^{-2}).$$

We may also calculate the real density of solutions as

$$\sigma_\infty = \int_0^H \int_0^H \int_0^H \frac{dx_1dx_2dx_3}{3(x_1x_2x_3)^{2/3}} = 9H.$$

The product of the p-adic densities is divergent, and the standard procedure is to introduce a 'renormalising factor' of $(1-p^{-1})^6$, compensated for by a factor $(\log H)^6$. This leads us to compare $\mathcal{N}(H)$ with

$$9H(\log H)^6 \prod_p (1-p^{-1})^7(1+7p^{-1}+p^{-2}),$$

and we see that the two differ by a factor which tends to $36 \times 6!$.

The renormalization procedure above has a distinct air of arbitrariness about it, and a more sophisticated interpretation of the theorem comes from consideration of the 'main term' in the analysis of the Hardy-Littlewood circle method. We shall not go into this in detail. However we point out that the 'main term' involves a partial sum of the 'singular series', taking the form

$$\sum_{q \le Q} a_q.$$

If this sum is evaluated via Perron's formula, one needs to examine the residue of $Z(s)Q^s s^{-1}$ at $s=0$, where $Z(s) = \sum_1^\infty a_q q^{-s}$. Since the function $Z(s)$ turns out to have a pole of order 7 at the origin, this process produces a 'main term' containing a factor $(\log Q)^6/6!$. This goes some way towards explaining the presence of the factor 6! in our theorem. However there will inevitably be other terms arising out of the circle method, in addition to what has been called here the 'main term'.

3 Proof of the Theorem

As has already been remarked, $\mathcal{N}(H)$ is the number of primitive integer vectors $\mathbf{x} = (x_1, x_2, x_3)$ with $1 \le x_i \le H$ for $i = 1, 2, 3$, and such that $x_1x_2x_3$ is a cube. For such a vector we write $x_i = y_i z_i^3$ where y_i is cube-free. Then $y_1y_2y_3$ will be a cube, y^3, say, and the vectors $\mathbf{y} = (y_1, y_2, y_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$ will be primitive. Moreover if we set $w_1 = (y_2, y_3)$, and similarly $w_2 = (y_1, y_3)$, $w_3 = (y_1, y_2)$, we must have $(z_i, w_i) = 1$ for $i = 1, 2, 3$. One may readily check, conversely, that these conditions imply that the original vector $\mathbf{x}$ is primitive. Notice that the integers w_1, w_2 and w_3 must be coprime in pairs.

We proceed to count the number of possible vectors $\mathbf{z}$ corresponding to each vector $\mathbf{y}$. This will be achieved by means of the following estimate.

Lemma 1 *Let w_1, w_2 and w_3 be positive integers, which are coprime in pairs. Let Z_1, Z_2 and Z_3 be any positive real numbers. Then there are*

$$\frac{Z_1Z_2Z_3}{\zeta(3)} f(w_1w_2w_3) + O(g(w_1w_2w_3)Z_1Z_2Z_3\{\max Z_i^{-1/2}\})$$

primitive integer vectors (z_1, z_2, z_3) such that $1 \le z_i \le Z_i$ and $(z_i, w_i) = 1$ for $i = 1, 2, 3$.

Here $f(d)$ and $g(d)$ are multiplicative functions defined by

$$f(p^e) = \frac{1 - p^{-1}}{1 - p^{-3}}, \quad g(p^e) = 1 + p^{-1/2}, \tag{2}$$

for every prime power $p^e > 1$.

We shall prove this in the next section.

It is now apparent, on taking $Z_i = (H/y_i)^{1/3}$, that

$$\mathcal{N}(H) = \mathcal{M}(H) + \mathcal{R}(H)$$

where

$$\mathcal{M}(H) = H\zeta(3)^{-1} \sum_{\mathbf{y}} \frac{f(w_1w_2w_3)}{y}, \tag{3}$$

and

$$\mathcal{R}(H) \ll H^{5/6} \sum_{\mathbf{y}} g(w_1w_2w_3)(y_1y_2y_3)^{-1/3} \sum_i y_i^{1/6}.$$

Here $\mathbf{y} = (y_1, y_2, y_3)$ runs over primitive integer vectors with $1 \le y_i \le H$, subject to the restriction that $y_1y_2y_3$ is a cube.

We now analyse further the conditions on $\mathbf{y}$. Any prime factor of $y_1y_2y_3$ must divide exactly two of y_1, y_2 and y_3. Moreover it will divide one of these with exponent 1 and the other with exponent 2. We therefore define y_{ij} (for $i \ne j$) to be the product of those primes p for which $p||y_i$ and $p^2||y_j$. With this notation we find that the numbers y_{ij} are square-free and coprime in pairs. Moreover we have

$$y_1 = y_{12}y_{13}(y_{21}y_{31})^2, \quad y_2 = y_{21}y_{23}(y_{12}y_{32})^2, \quad \text{and} \quad y_3 = y_{31}y_{32}(y_{13}y_{23})^2.$$

It follows that $w_1 = y_{23}y_{32}$ and similarly for w_2 and w_3, whence $\prod_i w_i = \prod_{i,j} y_{ij}$. We note also that $\prod_{i,j} y_{ij} = y$, where $\prod_i y_i = y^3$.

We are now ready to estimate $\mathcal{R}(H)$. We have

$$\begin{aligned}\mathcal{R}(H) &\ll H^{5/6} \sum_{y_{ij}:\, y_i \le H}{}^{*}\, g(w_1w_2w_3)(y_1y_2y_3)^{-1/3} \sum_i y_i^{1/6} \\ &\ll H^{5/6} \sum_{y_{ij}:\, y_i \le H}{}^{*}\, g(w_1w_2w_3) y_1^{-1/3} y_2^{-1/3} y_3^{-1/6},\end{aligned}$$

by symmetry. Here we have used Σ^* to denote restriction to variables y_{ij} which are square-free and coprime in pairs. It follows that

$$\mathcal{R}(H) \ll H^{5/6} \sum_{y_{ij}}{}^{*} \frac{\prod_{i,j} g(y_{ij})}{y_{12}y_{21}(y_{13}y_{23})^{2/3}(y_{31}y_{32})^{5/6}},$$

where the summation is subject to the conditions

$$y_{12}y_{13}(y_{21}y_{31})^2 \le H, \quad y_{21}y_{23}(y_{12}y_{32})^2 \le H, \quad \text{and} \quad y_{31}y_{32}(y_{13}y_{23})^2 \le H. \tag{4}$$

If we substitute $y_{31}y_{32} = u$, this leads to the estimate

$$\mathcal{R}(H) \ll H^{5/6} \sum_{y_{ij} \le H,\ (i \ne 3)} \frac{\prod_{i,j:\ (i\ne 3)} g(y_{ij})}{y_{12}y_{21}(y_{13}y_{23})^{2/3}} \sum_{u \le U} G(u) u^{-5/6},$$

in which $U = H(y_{13}y_{23})^{-2}$ and

$$G(u) = \sum_{u = u_1u_2} g(u_1u_2).$$

Since the Dirichlet series generating function for $G(u)$ has a double pole at $s = 1$, a standard Tauberian argument shows that

$$\sum_{u \le U} G(u)u^{-5/6} \ll U^{1/6} \log U \ll H^{1/6}(\log H)(y_{13}y_{23})^{-1/3},$$

whence

$$\begin{aligned}\mathcal{R}(H) &\ll H(\log H) \sum_{y_{ij} \le H,\ (i\ne 3)} \frac{\prod_{i,j:\ (i\ne 3)} g(y_{ij})}{y_{12}y_{21}y_{13}y_{23}} \\ &\ll H(\log H)\{\sum_{h \le H} g(h)/h\}^4.\end{aligned}$$

The Dirichlet series generating function for $g(h)$ has a simple pole at $s = 1$ and the usual Tauberian argument therefore shows that

$$\sum_{h \le H} g(h)/h \ll \log H.$$

Hence we conclude that

$$\mathcal{R}(H) \ll H(\log H)^5. \tag{5}$$

We turn now to the leading term $\mathcal{M}(H)$. We shall write (3) in the form

$$\mathcal{M}(H) = H\zeta(3)^{-1} \sum_{y_{ij}}{}^{**} y^{-1} \prod_{i \neq j} \mu^2(y_{ij}) f(y_{ij}),$$

where Σ^{**} denotes restriction to variables y_{ij} which are coprime in pairs, and subject to the inequalities (4), but are not necessarily square-free. We introduce the multiplicative function $h(d)$ defined by

$$h(p^e) = \begin{cases} f(p) - 1, & e = 1, \\ -f(p), & e = 2, \\ 0, & e \geq 3, \end{cases} \tag{6}$$

so that

$$\mu^2(n) f(n) = \sum_{d|n} h(d).$$

We may now write

$$\mathcal{M}(H) = H\zeta(3)^{-1} \sum_{d_{ij}} \prod_{i,j} h(d_{ij}) \sum_{y_{ij}:\, d_{ij}|y_{ij}}{}^{**} y^{-1},$$

where the variables d_{ij} (for $i \neq j$) run over all positive integers, without restriction.

In order to remove the constraint that the y_{ij} are coprime in pairs we introduce a factor

$$\prod_{i,j,k,l} \{ \sum_{\delta_{ijkl}|y_{ij}, y_{kl}} \mu(\delta_{ijkl}) \},$$

where (i, j, k, l) runs over quadruples of integers in the range $1 \leq i, j, k, l \leq 3$, subject to the conditions that $i \neq j$, $k \neq l$ and that the ordered pairs (i, j) and (k, l) are distinct. It may be observed that the above product contains factors corresponding to both the conditions $(y_{ij}, y_{kl}) = 1$ and $(y_{kl}, y_{ij}) = 1$. This however has no material effect on the argument.

We may now write

$$\mathcal{M}(H) = H\zeta(3)^{-1} \sum_{d_{ij}} \prod_{i,j} h(d_{ij}) \sum_{\delta_{ijkl}} \prod_{i,j,k,l} \mu(\delta_{ijkl}) \sum_{y_{ij}:\, r_{ij}|y_{ij}} y^{-1},$$

where r_{ij} is the lowest common multiple of d_{ij} and the various δ_{ijkl} for $(i, j) \neq (k, l)$. The variables y_{ij} are still subject to the inequalities (4). We now call on the following estimate, which will be proved in the next section.

Lemma 2 *Let r_{ij} be arbitrary positive integers, and write $R = \prod r_{ij}$. Then*

$$\sum_{y_{ij}:\, r_{ij}|y_{ij}} y^{-1} = \frac{1}{4}\frac{(\log H)^6}{6!R} + O(R^{-3/4}(\log H)^5),$$

the summation being over integers y_{ij} subject to the inequalities (4).

This result enables us to write

$$\mathcal{M}(H) = \kappa H(\log H)^6 + O(\lambda H(\log H)^5), \tag{7}$$

where

$$\kappa = (4\zeta(3)6!)^{-1} \sum_{d_{ij}} \prod_{i,j} h(d_{ij}) \sum_{\delta_{ijkl}} \prod_{i,j,k,l} \mu(\delta_{ijkl}) \prod_{i,j} r_{ij}^{-1},$$

and

$$\lambda = \sum_{d_{ij}} \prod_{i,j} |h(d_{ij})| \sum_{\delta_{ijkl}} \prod_{i,j,k,l} \mu^2(\delta_{ijkl}) \prod_{i,j} r_{ij}^{-3/4}.$$

It remains to evaluate κ, and to prove that the infinite sum λ is convergent. We tackle the latter problem first. Since the terms involved are all multiplicative functions, it suffices to consider the corresponding Euler factors, in which we choose a prime p, and restrict the variables to be powers of p. The summand will then vanish unless d_{ij} is 1, p or p^2, and each δ_{ijkl} is 1 or p. It follows that there are at most 3×2^{30} terms, since there are 30 admissable quadruples (i, j, k, l). Moreover, we see from (2) and (6) that $|h(p)| \le p^{-1}$ and $|h(p^2)| \le 1$. Thus, if $\delta_{ijkl} = p$ for any index (i, j, k, l), then $p|r_{ij}, r_{kl}$, whence

$$\prod_{i,j} |h(d_{ij})| r_{ij}^{-3/4} \le p^{-3/2}. \tag{8}$$

The above estimate holds indeed whenever $p^2 | \prod r_{ij}$. On the other hand, if $\prod r_{ij} = p$, so that each variable δ_{ijkl} is 1, there must be some index ij such that $d_{ij} = p$. In the latter case $|h(d_{ij})| \le p^{-1}$, whence (8) still holds. We therefore see that (8) holds except when each of the variables d_{ij} and δ_{ijkl} is 1. We may now conclude that the Euler factor for the prime p is $1+O(p^{-3/2})$. The corresponding product is convergent, and hence the sum λ is also convergent.

Since the sum λ is absolutely convergent, the sum in the definition of κ is also absolutely convergent. We may therefore evaluate it by considering its Euler factors A_p, say. This will produce

$$\kappa = (4\zeta(3)6!)^{-1} \prod_p A_p. \tag{9}$$

To evaluate A_p we note that

$$A_p = \sum_{d_{ij}} \prod_{i,j} h(d_{ij}) \sum_{\delta_{ijkl}} \prod_{i,j,k,l} \mu(\delta_{ijkl}) \prod_{i,j} r_{ij}^{-1},$$

where the variables are restricted to run over powers of p. As observed above, this is, in effect, a finite sum. Since r_{ij} must divide p^2, we may write

$$p^{12}\prod_{i,j} r_{ij}^{-1} = \#\{\mathbf{n} \,(\mathrm{mod}\; p^2) : \, r_{ij}|n_{ij}\},$$

where $\mathbf{n}$ runs over 6-tuples of integers n_{ij}. It follows that

$$\begin{aligned} A_p &= p^{-12}\sum_{d_{ij}}\prod_{i,j} h(d_{ij})\sum_{\delta_{ijkl}}\prod_{i,j,k,l}\mu(\delta_{ijkl})\#\{\mathbf{n}\,(\mathrm{mod}\; p^2) : \, r_{ij}|n_{ij}\} \\ &= p^{-12}\sum_{\mathbf{n}(\mathrm{mod}\; p^2)}\sum_{d_{ij}|n_{ij}}\prod_{i,j} h(d_{ij})\sum_{\delta_{ijkl}|n_{ij},n_{kl}}\prod_{i,j,k,l}\mu(\delta_{ijkl}), \end{aligned}$$

since $r_{ij}|n_{ij}$ if and only if d_{ij} and each δ_{ijkl} divides n_{ij}. Since the variables δ_{ijkl} run over powers of p only, we find that

$$\sum_{\delta_{ijkl}|n_{ij},n_{kl}}\prod_{i,j,k,l}\mu(\delta_{ijkl})$$

vanishes unless at most one n_{ij} is a multiple of p. Otherwise the sum is 1. This observation shows that

$$A_p = p^{-12}\sum_{\mathbf{n}(\mathrm{mod}\; p^2)}\sum_{d_{ij}|n_{ij}}\prod_{i,j} h(d_{ij}),$$

the sum over $\mathbf{n}$ being restricted to sets of values for which at most one n_{ij} is a multiple of p. In the same way, we find that

$$\sum_{d_{ij}|n_{ij}}\prod_{i,j} h(d_{ij})$$

is $\prod f((p^2, n_{ij}))$ unless some n_{ij} is divisible by p^2, in which case the sum vanishes. The contribution to A_p arising from terms in which none of the n_{ij} is a multiple of p is therefore $p^{-12}(p^2-p)^6$. Similarly the contibution from terms in which exactly one of the n_{ij} is a multiple of p is $6p^{-12}(p^2-p)^5(p-1)f(p)$. In view of (2) we therefore find that

$$A_p = \frac{(1-p^{-1})^7}{1-p^{-3}}(1+7p^{-1}+p^{-2}).$$

The theorem now follows from (5), (7) and (9).

4 Proof of Lemmas 1 and 2

To establish Lemma 1 we begin with the observation that

$$\#\{v \in \mathbb{N} : v \le V, \;\; (v,w) = 1\} = \sum_{d|w}\mu(d)[\frac{V}{d}]$$

for $w \in \mathbb{N}$ and $V \geq 0$. We now employ the estimate

$$[\theta] = \theta + O(\theta^{1/2}),$$

which is valid for all $\theta \geq 0$. The astute reader will realize that, had the alternative error $O(1)$ been used at this point, we would be unable to handle the later errors purely in terms of convergent infinite sums. The approach used here considerably reduces the complications involved in dealing with these errors.

We now find that

$$\#\{v \in \mathbb{N} : v \leq V, \ \ (v,w)=1\} = V\frac{\phi(w)}{w} + O(V^{1/2}g(w)),$$

where $g(w) = \sum_{d|w} |\mu(d)|d^{-1/2}$, whence

$$\#\{\mathbf{v} \in \mathbb{N}^3 : v_i \leq V_i, \ \ (v_i,w_i)=1, \ (i=1,2,3)\} = \prod_i \{V_i\frac{\phi(w_i)}{w_i} + O(V_i^{1/2}g(w_i))\}.$$

At this point we observe that if $0 \leq a_i, b_i \leq c_i$ for $i = 1,2,3$ then

$$\prod_i \{a_i + O(b_i)\} = \prod_i a_i + O(c_1c_2c_3\{\max \frac{b_i}{c_i}\}).$$

If each of the V_i is at least 1 we may apply this with

$$a_i = V_i\frac{\phi(w_i)}{w_i}, \quad b_i = V_i^{1/2}g(w_i), \quad c_i = V_ig(w_i),$$

whence

$$\begin{aligned} &\#\{\mathbf{v} \in \mathbb{N}^3 : v_i \leq V_i, \ \ (v_i,w_i)=1 \ \ (i=1,2,3)\} \\ &\quad = \ V_1V_2V_3\frac{\phi(w_1w_2w_3)}{w_1w_2w_3} + O(V_1V_2V_3g(w_1w_2w_3)\{\max V_i^{-1/2}\}). \qquad (10) \end{aligned}$$

If V_i, say, is less than 1, the left hand side will vanish, and the estimate remains valid, since $V_1V_2V_3 \ll V_1V_2V_3V_i^{-1/2}$.

The number of primitive integer vectors (z_1,z_2,z_3) such that $1 \leq z_i \leq Z_i$ and $(z_i,w_i)=1$ for $i=1,2,3$, may now be found as

$$\sum_{d=1}^{\infty} \mu(d)\#\{\mathbf{z} \in \mathbb{N}^3 : z_i \leq Z_i, \ (z_i,w_i)=1, \ d|z_i, \ (i=1,2,3)\}.$$

Using (10) we see that this is

$$\sum_{d:\,(d,w_1w_2w_3)=1} \mu(d)\#\{\mathbf{v} \in \mathbb{N}^3 : v_i \leq Z_i/d, \ (v_i,w_i)=1, \ (i=1,2,3)\}$$

$$
\begin{aligned}
&= \sum_{d\colon (d,w_1w_2w_3)=1} \mu(d)\frac{Z_1Z_2Z_3}{d^3}\frac{\phi(w_1w_2w_3)}{w_1w_2w_3} \\
&\qquad +O\left(Z_1Z_2Z_3g(w_1w_2w_3)\{\max Z_i^{-1/2}\}\sum_{d=1}^{\infty} d^{-5/2}\right) \\
&= \frac{Z_1Z_2Z_3}{\zeta(3)}\frac{\phi(w_1w_2w_3)}{w_1w_2w_3}\prod_{p|w_1w_2w_3}(1-p^{-3})^{-1} \\
&\qquad +O(Z_1Z_2Z_3g(w_1w_2w_3)\{\max Z_i^{-1/2}\}) \\
&= \frac{Z_1Z_2Z_3}{\zeta(3)}f(w_1w_2w_3)+O(Z_1Z_2Z_3g(w_1w_2w_3)\{\max Z_i^{-1/2}\}),
\end{aligned}
$$

as required.

We turn now to Lemma 2. We begin by observing that it suffices to establish the result for the case in which the divisors r_{ij} are all 1. To see this we set

$$\Sigma(H)=\sum_{y_{ij}} y^{-1},$$

the variables being subject to the inequalities (4). In general, if we put $y_{ij}=r_{ij}z_{ij}$, we then see that the new variables z_{ij} lie in a region which is included in the set

$$z_{12}z_{13}(z_{21}z_{31})^2\le H,\quad z_{21}z_{23}(z_{12}z_{32})^2\le H,\quad \text{and}\quad z_{31}z_{32}(z_{13}z_{23})^2\le H.$$

It follows that

$$\sum_{y_{ij}\colon r_{ij}|y_{ij}} y^{-1}\le R^{-1}\Sigma(H).$$

In the same way, the z_{ij} include all those which satisfy

$$z_{12}z_{13}(z_{21}z_{31})^2\le HR^{-2},\quad z_{21}z_{23}(z_{12}z_{32})^2\le HR^{-2},$$

and

$$z_{31}z_{32}(z_{13}z_{23})^2\le HR^{-2},$$

whence

$$\sum_{y_{ij}\colon r_{ij}|y_{ij}} y^{-1}\ge R^{-1}\Sigma(H/R^2).$$

However, if we know that

$$\Sigma(H)=\frac{1}{4}\frac{(\log H)^6}{6!}+O((\log H)^5), \tag{11}$$

the upper and lower bounds will both be

$$\frac{1}{4}\frac{(\log H)^6}{6!R}+O(R^{-3/4}(\log H)^5),$$

providing that $\log R \ll \log H$. On the other hand, if any of the r_{ij} exceeds H, then

$$\sum_{y_{ij}:\, r_{ij}|y_{ij}} y^{-1} = 0,$$

and Lemma 2 holds automatically, since we must now have $\log H \ll R^{1/4}$.

It therefore remains to establish (11). To do this, we observe that

$$\int_{y_{12}}^{y_{12}+1} \cdots \int_{y_{32}}^{y_{32}+1} \frac{dy_{12}}{y_{12}} \cdots \frac{dy_{32}}{y_{32}} \le \frac{1}{y} \le \int_{y_{12}-1}^{y_{12}} \cdots \int_{y_{32}-1}^{y_{32}} \frac{dy_{12}}{y_{12}} \cdots \frac{dy_{32}}{y_{32}},$$

since $y = \prod y_{ij}$. It follows that $\Sigma(H) \ge I(H)$, where

$$I(H) = \int \cdots \int \frac{dy_{12}}{y_{12}} \cdots \frac{dy_{32}}{y_{32}},$$

the integral being over the region (4), subject to $y_{ij} \ge 1$. Similarly we have

$$\sum_{y_{ij} \ge 2} (\prod y_{ij})^{-1} \le I(H),$$

the sum being subject to (4). However

$$\Sigma(H) \le \sum_{y_{ij} \ge 2} (\prod y_{ij})^{-1} + \sum_{y_{ij}=1 \text{ for some } ij} (\prod y_{ij})^{-1},$$

in which we may relax the conditions on the variables y_{ij} in the final sum to require merely that $y_{ij} \le H$. It then follows that

$$\Sigma(H) \le \sum_{y_{ij} \ge 2} (\prod y_{ij})^{-1} + 6\{\sum_{y \le H} y^{-1}\}^5 \le \sum_{y_{ij} \ge 2} (\prod y_{ij})^{-1} + O((\log H)^5).$$

We therefore deduce that $\Sigma(H) = I(H) + O((\log H)^5)$.

To evaluate the integral $I(H)$ we begin by substituting $y_{ij} = H^{t_{ij}}$, which shows that $I(H) = (\log H)^6 V$, where

$$V = \int \cdots \int dt_{12} \ldots dt_{32},$$

the final integral being over the 6-dimensional polyhedron P, given by

$$t_{12} + t_{13} + 2(t_{21} + t_{31}) \le 1, \quad t_{21} + t_{23} + 2(t_{12} + t_{32}) \le 1,$$

$$t_{31} + t_{32} + 2(t_{13} + t_{23}) \le 1,$$

with $t_{ij} \ge 0$.

We have failed to find a straightforward way of interpreting the volume V, which is clearly a rational number. The following method, suggested to us by

G. Mersmann, and carried out by Th. Kleinjung, does at least determine its value. Our original calculation was considerably more involved, and we are very grateful for their permission to describe their method.

We observe that the polyhedron $12P$ has coordinates in $\mathbb{Z}^6$. It follows that if

$$N(t) = \#\{\mathbf{n} \in \mathbb{Z}^6 : \mathbf{n} \in 12tP\}$$

for non-negative integers t, then $N(t)$ interpolates a polynomial, $\sum_{k=0}^{6} c_k t^k$, say (see Fulton [5; pages 110-114], for example). On taking $t \to \infty$ one sees that c_6 must be the volume of $12P$. A straightforward computer calculation to find $N(t)$ for $t = 0, 1, \ldots, 6$ then allows one to evaluate the coefficients c_k. In this way one obtains $c_6 = 5184/5$, and hence $V = (4 \times 6!)^{-1}$, as required. This completes the proof of Lemma 2.

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