# Games with Second-Order Expected Utility*

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**Abstract**

This paper studies games when agents have second-order expected utility. It examines the theoretical predictions of the model and compares its performance in explaining behavior in experimental data on games with that of quantal response equilibrium.

**Keywords**: second-order Expected utility, game theory, probability matching, mixed strategies

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# 1 Introduction

This paper studies games when players have second-order expected utility, that is they do not regard compound lotteries as equivalent to simple lotteries with the same probability distribution over final outcomes. In particular players may display different attitudes to risk depending on the source of the randomness. Models with second-order expected utility have been widely studied in decision theory but rather little in game theory. This paper studies two-person Normal form games when players have second-order expected utility and shows that this model yields more reasonable predictions on the comparative statics of mixed strategy equilibria than standard game theory and can explain phenomena such as probability matching. It also shows that it fits data from studies of generalized matching pennies.

In more detail, the current paper considers games where players display different attitudes to risk in considering gambles involving their own decisions and randomization by other players. These may be because they feel more confident in their beliefs about their own actions and prefer bets in which they feel more expertise. A considerable literature, see for example Tversky and Fox (1995), suggests that agents do indeed prefer to bet on topics in which they feel more expertise.

As a result agents have preferences of the form

$$\sum_j q_j U\left(\sum_i p_i a_{ij}\right)$$

where $p_i$ is the agent's own mixed strategy, $q_j$ the other agent's mixed strategy and $a_{ij}$ the usual game payoffs, and $U$ the player's utility function with respect to gambles induced by the other agent's choices. Under these preferences the compound gamble induced by $q$ and $p$ in not equivalent to a simple gamble over outcomes. More importantly they imply, that if agents are risk averse with respect to the other's strategy, $U$ is concave, then they have an incentive to play a mixed strategy to hedge against the uncertainty caused by the other players' randomization. This gives more reasonable comparative statics for mixed strategy equilibria than standard expected-utility preferences and can explain phenomena such as probability matching.

The paper develops simple theoretical properties of games under second-order expected utility. In particular it shows how these preferences can yield more reasonable comparative statics for mixed strategy equilibria than standard ones and can explain probability matching.

The paper also examines the empirical performance of the model. It shows that in games where players face risk in the payoff they will receive as a result of the actions of the other player, the model performs well. In particular it performs well in comparison to quantal response equilibrium, which is the workhorse model in modeling deviations from Nash equilibrium.

Segal (1987) suggested that the Ellsberg paradox could be explained by a model in which agents conceive of the themselves as a facing a two-stage lottery: at the first stage a probability distribution is chosen, at the second a draw is made from the distribution. If agents preferences do not satisfy the reduction axiom, then the Ellsberg paradox can be explained. Segal (1990) develops this model further in the context of risk. In these papers, expected utility is not assumed.

Neilson (1993) and Neilson (2010) provide an axiomatization for a model when agents use expected utility at both stages. Other axiomatizations of somewhat more general versions of the model are provided by Nau (2001), Nau (2006), Klibanoff et al. (2005) and Ergin and Gul (2009). Seo (2009) and Grant et al. (2009) also provide axiomatizations. Chew and Sagi (2008) provides a general analysis of source dependent attitudes to risk.

Tversky and Fox (1995) provide experimental evidence that baseball fans prefer to bet on the results of baseball games rather than chance events with similar probabilities. Abdellaoui et al. (2011) provide some further experimental evidence of source preference.

It has often been noted that procedures to induce risk neutrality in games, such as Roth and Malouf (1979)'s binary lottery procedure, have mixed results — see for example Camerer (2003) Chapter 1.A.1.2.7. The preferences here are consistent with this observation: risk attitudes over objective lotteries may be different to those felt over gambles induced by other players. In particular the reduction of compound lotteries to simple lotteries assumed in such procedures need not hold.

The motivation of the work above is that agents may have differing attitudes to gambles involving subjective uncertainty (ambiguity) than to gambles involving objective uncertainty (risk). Kreps and Porteus (1978) propose a similar representation to second-order expected utility in the context of dynamic choices under risk. Similar preferences arise in the context of risk-sensitive control, sometimes utilized in macroeconomics (see Backus et al. (2005)). In this paper the interpretation is that the player feels a different attitude to risk in gambles where she determines the probabilities, and so these are objectively known, and gambles induced by the other players' choices, in which case the probabilities are not known objectively, even though in equilibrium her beliefs are correct.

There has been little study of second-order expected utility in the context of game theory. Crawford (1990) discusses the general question of the existence of Nash equilibrium with non-expected utility preferences. Dekel and

Segal (1991) discuss the existence of Nash equilibrium when agents' preferences may violate the reduction axiom.

Hanany et al. (2020) study sequential games in which there is ambiguity about players' types when agents have the smooth ambiguity preferences of Klibanoff et al. (2005), which can be thought of as a generalization of second-order expected utility. Battigalli et al. (2019) study issues of dynamic consistency in recurring games when agents have smooth ambiguity preferences. Ellis (2018) studies dynamic consistency for a range of models of ambiguity. Li et al. (2019) measure ambiguity attitudes in a sequential trust game.

Other papers have studied models of games with ambiguity but have assumed different kinds of preferences. Marinacci (2000) is an early example.

The paper most closely related to the current one is probably Eichberger and Kelsey (2011). They study a model of normal-form games under which player may view each others' strategies as ambiguous but use a model where players maximize a weighed average of the expected utility and the minimum and the maximum utilities of a strategy rather than second-order expected utility. Eichberger et al. (2009) and Eichberger and Kelsey (2014) also explore this model. This formulation allows players to be concerned about the possibility that their opponents might play actions which should never be played in equilibrium and so focuses on a different concern to this paper. In Eichberger and Kelsey (2011)'s model, unlike the current one, players do not have an incentive to play a mixed strategy to hedge against the randomization of the other player as payoffs are affine in own probabilities and it is shown in Section 3 that this leads to less satisfactory comparative statics for mixed strategy equilibria.

The paper proceeds as follows. Section 2 outlines the model. Section 3 shows that the model gives more intuitive predictions about the comparative statics of mixed strategy equilibria than does Nash Equilibrium. Section 4 that the model can explain probability matching.

Sections 5 considers the application of the model to data from experimental games. The games studied all have long runs of play, so it is reasonable to see whether equilibrium notions can explain behavior. It looks at data from studies of generalized matching pennies games and shows that second-order expected utility compares favorably with quantal response equilibrium, even when the latter is augmented to include risk aversion (Goeree et al. (2003)). It also discusses data from games studied by Selten and Chmura (2008) where second-order expected utility fits less well as risk is not an important consideration for players. Section 6 concludes.

## 2 The Model

There are two players, 1 and 2, playing a Normal-form game with finitely many (pure) actions. Player 1 has actions $1, \ldots, n_1$ and player 2 actions $1, \ldots, n_2$. A mixed strategy $p$ for player 1 is a probability distribution $p = (p_1, \ldots, p_{n_1})$ over his pure actions. Similarly a pure strategy $q$ for player 2 is a probability distribution over her pure actions, $q = (q_1, \ldots, q_{n_2})$. The framework can easily be generalized to the case of more than two players but for simplicity attention is restricted to the two player case.

If player 1 takes action $i$ and player 2 takes action $j$, player 1 receives a first-order utility payoff of $a_{ij}$ and player 2 receives a payoff of $b_{ij}$. Each player has a second-order utility function $U^i$ and if player 1 chooses a mixed strategy $p$ and player 2 a mixed strategy $q$ then the expected overall payoff of player 1 is

$$\Pi^1(p,q) = \sum_j q_j U^1 \left( \sum_i p_i a_{ij} \right) \tag{1}$$

and of player 2

$$\Pi^2(p,q) = \sum_i p_i U^2 \left( \sum_j q_j b_{ij} \right) \tag{2}$$

In this formulation, $a_{ij}$ and $b_{ij}$ are the utilities relevant for gambles involving a player's own random choices. $U^1$ and $U^2$ represent utility for gambles generated by the other player's choices. In the simplest case, $a_{ij}$ and $b_{ij}$ are monetary payoffs, so players are risk neutral over their own gambles but risk averse over gambles created by the other player. In general, this formulation allows for the degree of risk aversion to depend on the source of the uncertainty.

One could adopt more symmetric notation for second and first-order utility and write $a_{ij}$ as $a(\{ij\})$ so that (1) and (2) become:

$$\Pi^1(p,q) = \sum_j q_j U^i \left( \sum_i p_i a(\{ij\}) \right) \tag{3}$$

and

$$\Pi^2(p,q) = \sum_i p_i U^2 \left( \sum_j q_j b(\{ij\}) \right) \tag{4}$$

This makes it clear that both $U^1$ and $a$, for example, are utility functions but the earlier notation is used for simplicity.

One can interpret these preferences in several ways. One interpretation is simply that agents have non-expected utility preferences under risk as they do not regard the lottery induced by $p$ and $q$ as equivalent to the compound lottery $r = (p_i q_j)$ over consequences $ij$.

Another interpretation is in terms of ambiguity and risk. In this interpretation $a_{ij}$ represent utilities over gambles with probabilities known objectively to player 1, that is involving risk. Gambles induced by the choices of player 2 are not seen as given objectively but reflect subjective beliefs by player 1 and so involve ambiguity. Under this interpretation these preferences are those introduced by Neilson (1993).

It will be assumed throughout the paper that the following mild assumption holds

**Assumption 1.** *$U^1$ and $U^2$ are strictly increasing and continuous.*

It will also often be assumed that players are risk averse, so that

**Assumption 2.** *In addition to Assumption 1, $U^1$ and $U^2$ are concave.*

A standard argument (see Appendix A.1) shows that under Assumption 2 equilibrium always exists.

Finally to give some characterizations, particularly of the examples, it will be assumed, with primes denoting derivatives, that

**Assumption 3.** *In addition to Assumption 2, $U^1$ and $U^2$ are twice continuously differentiable with $U^{1'} > 0$ and $U^{2'} > 0$.*

It is straightforward to see that the assumption of Second Order Expected Utility does not affect the pure strategy equilibria of a game as players face no risk from the choices of the other player and in fact it does not affect the set of rationalizable strategies. A detailed discussion of these results can be found in Appendices A.1 and A.2. The assumption of Second Order Expected Utility does, however, have a substantial effect on mixed strategy equilibria of games and the next section focuses on these.

# 3 Mixed Strategy Equilibria

This section considers the effect of the assumption of Second Order Expected Utility on the mixed strategy equilibria of a game. It shows that comparative statics of mixed strategy are more reasonable under second-order expected utility than with standard preferences.

It is well known that mixed strategy equilibria can have unintuitive comparative statics. The experiments by Ochs (1995) make this point nicely. In

|   | $L$ | $R$ |
|---|---|---|
| $T$ | $1, 0$ | $0, 1$ |
| $B$ | $0, 1$ | $1, 0$ |

Figure 1: Matching Pennies

|   | $L$ | $R$ |
|---|---|---|
| $T$ | $a, 0$ | $0, 1$ |
| $B$ | $0, 1$ | $1, 0$ |

Figure 2: Ochs (1995)'s Game

the version of Matching Pennies shown in Figure 1, the unique mixed strategy equilibrium with standard preferences is for each player to randomize with probability 1/2 on each of their pure strategies and the proportion of players choosing these actions is consistent with this prediction in the results reported by Ochs (1995).

Now suppose that the row player's payoff to playing $T$ when the column player plays $L$ is given by a parameter $a > 0$: With standard preferences, the row player will still randomize with equal probability between $T$ and $B$ but the column player will put weight $1/(1 + a)$ on $L$ and $a/(1 + a)$ on $R$ to keep the column player indifferent between her pure strategies. As is well known, a player's randomization probabilities are determined by the other player's payoffs and not their own. This is counter-intuitive and is violated in Ochs (1995)'s experiments. If $a = 4$ or $a = 9$ the frequency with which row players choose $T$ is significantly greater than 1/2 in his experiments.

This behavior can explained, at least qualitatively, by second-order expected utility. Players are still willing to mix even if actions to do not yield exactly the same expected payoff and this relaxes the indifference condition for player 2 that rigidly pins down's player 1's choices independently of own payoffs with standard preferences.

In more detail, let $p$ be the probability that player 1 chooses $T$ and $q$ the probability that player 2 choose $L$. With second-order expected utility player 1 maximizes in Ochs (1995)'s game

$$qU^1(pa) + (1-q)U^1(1-p) \tag{5}$$

and player 2 maximizes

$$pU^2(1-q) + (1-p)U^2(q) \tag{6}$$

In the game of Figure 1, it is easy to see that with preferences satisfying Assumption 3 the equilibrium is still to randomize equally over all actions (cf. Lemma 8 in Appendix A.2). When $a > 1$, however, player 1 will wish to put more weight on strategy $T$. Player 2 will, however, still randomize rather than switch entirely to $R$ as insurance against player 1 choosing $B$ if $U^2$ is concave. For a similar reason player 1 will not switch entirely to $T$ if $a$ is not too large if $U^1$ is concave.

As an illustration, assume that both players have utility functions displaying constant relative risk aversion:

$$U^1(x) = U^2(x) = \frac{x^{1-\epsilon}}{1-\epsilon} \qquad \epsilon \geq 0 \tag{7}$$

As usual $\epsilon = 1$ corresponds to $U^1(x) = \ln x$.

Player 1 then solves

$$\max_p q\frac{(pa)^{1-\epsilon}}{1-\epsilon} + (1-q)\frac{(1-p)^{1-\epsilon}}{1-\epsilon} \tag{8}$$

and player 2 solves

$$\max_q p\frac{(1-q)^{1-\epsilon}}{1-\epsilon} + (1-p)\frac{q^{\epsilon}}{1-\epsilon} \tag{9}$$

This yields first-order conditions

$$qa^{1-\epsilon}p^{-\epsilon} - (1-q)(1-p)^{1-\epsilon} = 0 \tag{10}$$

$$-p(1-q)^{-\epsilon} + (1-p)q^{-\epsilon} = 0 \tag{11}$$

Re-arranging yields

$$\frac{p}{1-p} = \left(\frac{q}{1-q}\right)^{\frac{1}{\epsilon}} a^{\frac{1-\epsilon}{\epsilon}} \tag{12}$$

$$\frac{q}{1-q} = \left(\frac{p}{1-p}\right)^{-\frac{1}{\epsilon}} \tag{13}$$

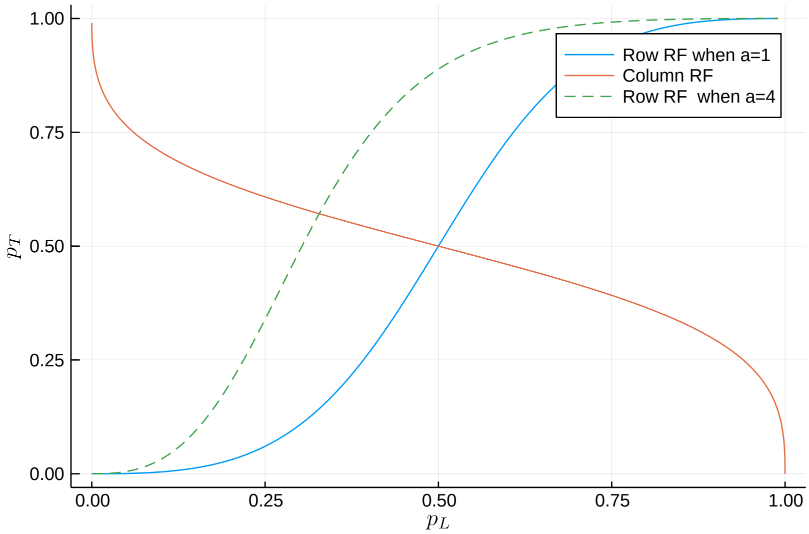


Figure 3: Reaction Functions in Ochs (1995) when $\epsilon = 0.4$

From (12) it follows that player 1's optimal choice of $p$ is increasing in $a$ for each value of $q$ provided $\epsilon < 1$: if player 1 is not too risk averse then the higher expected return of $T$ leads him to put more weight on it. If in contrast he is very risk averse, so that $\epsilon > 1$, then player 1 instead prefers to use the higher payoff of $T$ to reduce the weight he puts on it and increase the weight on $B$, thus improving insurance against player 2 choosing $R$.

Solving (12) and (13) for the equilibrium values of $p$ and $q$ confirms that the effect of $a$ persists in equilibrium. One finds that the equilibrium values of $p$ and $q$ are

$$p^* = \frac{a^{\frac{\epsilon(1-\epsilon)}{1+\epsilon^2}}}{1 + a^{\frac{\epsilon(1-\epsilon)}{1+\epsilon^2}}} \qquad \text{and} \qquad q^* = \frac{a^{\frac{\epsilon-1}{1+\epsilon^2}}}{1 + a^{\frac{\epsilon-1}{1+\epsilon^2}}} \tag{14}$$

In particular $p^*$ is increasing in $a$ and $q^*$ is decreasing in $a$ if $0 < \epsilon < 1$. The reverse conclusions hold if $\epsilon > 1$. That is, if players are not too risk averse then the comparative statics are the intuitive ones.

The result is illustrated graphically for $\epsilon = 0.4$. As can be seen from the Figure, when $a$ increases the row player's reaction function shifts upwards and the equilibrium value that the row player plays $T$ goes up:

Note that in a general game players may play a pure strategy in response to a mixed strategy under second-order expected utility. In the game here, however, players receive 0 under some realizations of an opponent's mixed strategy and with the utility functions here this leads them to avoid pure strategies unless the other player plays a pure strategy.

| | $L$ | $R$ |
|---|---|---|
| $T$ | $a, \alpha$ | $b, \beta$ |
| $B$ | $c, \gamma$ | $d, \delta$ |

Figure 4: General Case

Eichberger and Kelsey (2011) also consider matching pennies games. In their model players think that for any pure action they play with some probability the other player will play the best or worst action for them under it. This has the effect of adding constants to the payoffs of each action but expected payoffs are still affine in probabilities and so players must be exactly indifferent between their pure actions in order to randomize. This implies that, as under standard preferences, the row player must still randomize with probability 0.5 between $T$ and $B$ regardless of the value of $a$ if the column player is to randomize. This is inconsistent with the evidence in Ochs (1995)'s experiments. In Eichberger and Kelsey (2011)'s model if $a$ is large enough then the pure equilibrium $(T, R)$ is played if the row player is over-optimistic enough about the column player playing $L$ but the model cannot explain mixed equilibria where the probability of $T$ being played responds to $a$. A detailed analysis appears in Appendix A.3.

More generally, consider a general $2 \times 2$ game with a unique mixed-strategy equilibrium:

Let $p$ be the equilibrium probability that player 1 plays $T$ and $q$ the equilibrium probability that player 2 plays $L$.

**Lemma 1.** *Consider the game in Figure 4 with* $a > c \geq 0$, $0 \leq b < d$, $0 \leq \alpha < \beta$ *and* $\gamma > \delta \geq 0$. *If preferences satisfy Assumption 3, then there is a unique equilibrium, in which each player plays a mixed strategy.*

*(a)* $p$ *is strictly increasing in* $b$ *and strictly decreasing in* $c$.

*(b) If player 1's utility function,* $U^1$ *has coefficient of relative risk aversion lying between 0 and 1,* $0 < -xU^{1''}(x)/U^{1'}(x) < 1$, $T$ *then* $p$ *is strictly increasing in* $a$ *and strictly decreasing in* $d$.

*(c) Each of these parameter changes has the opposite effect on* $q$ *to its effect on* $p$.

A condition on risk aversion is needed for the comparative statics on $a$ as if player 1 is very risk averse he might decrease the probability with which he plays $T$ as $a$ increases to make sure his payoffs are not affected too much by whether column plays $L$ or $R$. Put otherwise, an increase in $a$ increases the expected return of playing $T$ but also increases its riskiness in the sense that the gap between the payoffs when $L$ and $R$ are played goes up, so one needs the player not to be too risk averse in order to be sure that he will put more weight on $T$. The condition is strong but is sufficient and not necessary, although in the illustration of Ochs (1995) with constant relative risk aversion above it is tight. It is analogous to well known conditions in portfolio for more of an asset to be bought if its return goes up — see for example Hadar and Seo (1990). The proof is in Appendix A.4.

An increase in $b$ improves both the return of playing $T$ and also reduces its risk, since $b$ becomes closer to $a$, and so one only needs that player 1 be risk averse to sign the effects of this change and not the extra condition on relative risk aversion. The comparative statics for $d$ and $c$ follow for similar reasons to those for $a$ and $b$ respectively. $p$ and $q$ move in opposite directions as the column player prefers the off-diagonal elements of the matrix..

The assumption on payoffs being positive is used as the assumption on relative risk aversion is otherwise not sensible. Constant relative risk aversion, as used in the illustration above, is also only meaningful if payoffs are positive. In applications where payoffs may be negative, it could be applied by measuring payoffs relative to the lowest possible payoff for each player or some lower payoff level.

**Example 1.** *As a further application of Lemma 1, consider the interpretation of Figure 4 with the row player as the public deciding whether to speed (T speed, B not) and the column player as the police deciding whether to enforce the law (L not, R enforce) considered by Tsebelis (1989). Tsebelis (1989) points out that if the penalty for speeding increases, so that b falls, then under standard preferences this has no effect on the probability that the motorist speeds. This result has generated considerable controversy as to whether this game is an appropriate model of law enforcement (see for example Hirshleifer and Rasmusen (1992)). Note however that under Second Order Expected Utility one finds the intuitively reasonably result: if the penalty for speeding goes up then the probability of speeding falls.*

Lemma 1 could be extended to general games: if a player is not too risk then an increase in the potential payoffs received playing a strategy will increase the increase the incentive to play it and will in general affect the equilibrium probability with which that strategy is played. To sign the effect one needs information about the nature of the game: for example at an unstable equilibrium, such as the mixed equilibrium in a $2 \times 2$ coordination game, the other player's reaction might be such as to lead to perverse comparative statics.

| | $L$ | $R$ |
|---|---|---|
| $L$ | 1 | 0 |
| $R$ | 0 | 1 |

Figure 5: Probability Matching

The interpretation of mixed strategies is controversial. This is discussed in the context of the current model in Appendix A.5.

# 4 Probability Matching

Before proceeding to empirical evidence, this section gives another example of how the incentive to play mixed strategies under Second Order Expected Utility can be used to explain phenomena difficult to rationalize with standard preferences. It shows it can be used to explain probability matching, a phenomenon which has often puzzled investigators.

In experiments subjects are repeatedly asked to predict the outcome of a chance event, for example whether a light will be blue or green or whether it will light up on their left or right. The outcomes are chosen randomly by a device with fixed probabilities independent of past history and of the subject's responses. Small rewards are paid for correct answers. If agents are expected utility maximizers they should always pick the most likely outcome as their prediction. By contrast many subjects randomize, matching the probabilities of the outcomes. So for example if a blue light will show 60% of the time and a green 40%, they predict blue 60% of the time.

There is some evidence that subjects eventually learn to play as expected utility maximizers but any convergence is slow. Shanks et al. (2002) for example report evidence of convergence when subjects received extensive feedback and large monetary rewards but only when the number of repetitions is large. A survey for economists can be found in Vulkan (2000).

The situation can be represented as follows. For simplicity consider the case of two outcomes. Nature chooses $L$ with probability $p$, $R$ with probability $1 - p$. The agent is an expected-utility maximizer and so payoffs are measure in utils. The agent predicts $L$ or $R$ before nature's choice is revealed and receives a payoff of 1 util for a correct prediction, 0 otherwise. The game is illustrated in Figure 5 with the agent as the row player and only her payoffs shown:

Suppose that the subject predicts $L$ with probability $\pi$. The expected payoff to an expected-utility maximizer is

$$p\pi 1 + (1-p)(1-\pi)1 \tag{15}$$

Hence an expected utility maximizer will choose $L$ with probability 1 when

$$p > 1/2$$

and $R$ with probability 1 when $p < 1/2$. Note that since payoffs are measured in utils, the possibility of conventional risk aversion has already been taken into account and so cannot help to explain the phenomenon.

Now suppose that the decision-maker regards the choices of nature as ambiguous or is simply less comfortable in betting on them than on her own choices. Her utility from choosing $L$ with probability $\pi$ is then

$$pU(\pi) + (1-p)U(1-\pi) \tag{16}$$

Randomization now acts as a hedge against uncertainty and non-extreme choices will be optimal.

In particular suppose the agent has second-order utility function

$$U(x) = \ln x \tag{17}$$

Her payoff function is then

$$p \ln \pi + (1-p)\ln(1-\pi) \tag{18}$$

and it is straightforward to check that the optimal choice of $\pi$ is

$$\pi = p \tag{19}$$

That is probability matching is optimal.

Precise probability matching of course depends on the utility function being logarithmic. If her second-order utility function belongs to the power family:

$$U(x) = \frac{x^{1-\epsilon}}{1-\epsilon} \tag{20}$$

then it is easy to check that the optimal solution for $\pi$ satisfies:

$$\frac{\pi}{1-\pi} = \left(\frac{p}{1-p}\right)^{1/\epsilon} \tag{21}$$

Logarithmic utility is of course a limiting case of power utility with $\epsilon = 1$. If $\epsilon$ is close to 1 then behavior close to probability matching is optimal, while as $\epsilon$ approaches zero behavior approaches that under standard expected utility.

Second-order expected utility is by no means the only explanation for these phenomena. For example probability matching could be explained by the Luce form of quantal response equilibrium (see Goeree et al. (2016)), which assumes that an agent chooses actions with probability proportional to their expected payoffs. Unlike this explanation and many others, however, the explanation given by second-order expected utility is based on an explicit model of optimizing behavior.

This example also illustrates two key differences with standard preferences: a player need not be indifferent between her pure strategies in order to be willing to randomize and, if her second-order utility function is strictly concave, will have a unique best response even if she is indifferent.

# 5 Empirical Results

This section examines the empirical fit of second-order expected utility. The first subsection looks at the comparative fit of the models in variants of Ochs (1995)'s game studied in Section 3. These are games which have often be used to study the performance of quantal response equilibria and other models. The second subsection looks at a game introduced by Goeree et al. (2003). The latter introduced this as an example where quantal response equilibrium fits badly and risk aversion seems an important consideration. The third subsection examines the games studied by Selten and Chmura (2008). Second-order expected utility compares favorably in its fit of the data in the first two subsections with quantal response equilibria, even when QRE is extended to incorporate risk aversion as in Goeree et al. (2003). It fits the third set of data less well and reasons for this are discussed. In essence in the latter games, players do not face much risk from the choices of the other players and risk aversion is unimportant, whereas it is important in the first two sets of games.

Under quantal response equilibria, each player takes a random response based on the expected payoff of each action. In the popular logit form, the probability that player 1 chooses action $i$ is given by

$$p_i = \frac{e^{\lambda \Pi_i(q)}}{\sum_k e^{\lambda \Pi_k(q)}} \tag{22}$$

where $\Pi_k(q)$ denotes the expected payoff to action $k$ given that player 2 plays strategy $q$ and $\lambda$ is a parameter. This can be interpreted in various ways, for example that players make errors or have randomly disturbed payoffs.

|  | $L$ | $R$ |
|---|---|---|
| $T$ | $a, 0$ | $0, 1$ |
| $B$ | $0, 1$ | $1, 0$ |

Figure 6: Ochs (1995)'s Game

Goeree et al. (2003) incorporate risk aversion by assuming that players have constant relative risk aversion: the utility of payoff $x$ is

$$U(x) = Ax^{1-r} \tag{23}$$

$A$ is a scaling parameter. Here $A$ is set to 1.[1] The model is otherwise unchanged. This class of models will be abbreviated as RQRE.

Standard quantal response equilibrium and quantal response equilibrium augmented to include risk aversion are compared with second order-expected utility with second-order utility functions

$$U^1(x) = U^2(x) = \frac{x^{1-\epsilon}}{1-\epsilon} \tag{24}$$

Second-order expected utility will be abbreviated as SOEU.

Even if quantal response equilibrium is extended to incorporate risk aversion, it differs in some ways with second-order expected utility. In particular quantal response models, at least in the logit form, predict that players will always randomize and so, for example, may play a strictly dominated strategy, whereas in second-order expected utility they will never do so (see Appendix A).

## 5.1 Ochs (1995)'s games

This section looks at Ochs (1995)'s game and variants studied by McKelvey et al. (2000). Ochs (1995) studies games of the form shown in Figure 6.

In his experiments the parameter $a$ takes the value 1,9 or 4. These are labeled games 1 to 3. As discussed in the previous section, under expected utility the choice probabilities of the row player should be unaffected by the value of $a$ but this is strongly contradicted by his data.

[1] Goeree et al. (2003) chose $A$ to ensure that the best outcome has utility 1 and the worst payoff 0. This affects the reported value of $\lambda$ but not $r$ and does not otherwise affect the results.

McKelvey et al. (2000) consider four related games. In game A, $a = 9$ and in game D $a = 4$. In game B, payoffs are the same as in game A except that the column player's payoffs are multiplied by 4. In game C, the payoffs are the same as in game A except that all payoffs are multiplied by 4. It is straightforward to check that the re-scalings in B and C should not affect behavior under second-order expected utility with constant relative risk aversion but will affect it with risk-neutral quantal response equilibria. McKelvey et al. (2000) find that behavior is indeed not much affected.

Ochs (1995) Game 1 is omitted as the predicted choice probabilities for each player are 0.5 for all parameter values in all of the models considered and so game 1 does not affect the estimated parameters.[2]

The model with second-order expected utility is estimated by maximum likelihood using the equilibrium probabilities given in (14). QRE and RQRE are re-estimated here using the method of Goeree et al. (2016) chapter 6. The results are in close agreement with the earlier papers. Calculations were performed in `Julia`. Parameter estimates are given in Table 1.

The fitted probabilities, with $p_T$ and $p_L$ denoting the probability or proportion of times that $T$ and $L$ respectively are chosen, in Table 2 together with the likelihoods for each game implied by the estimated parameters as a simple measure of fit.

As noted above, the results for QRE and RQRE are in close agreement with reported earlier papers.[3]

Games A–C are re-scalings of each other, so the estimated probabilities for SOEU are identical as it is scale invariant when utility has the CRRA form (24). The estimated probabilities for RQRE are slightly closer to the data for Games 2–3 than for SOEU but the difference in the likelihoods of

[2]When fitting quantal response equilibria to Ochs (1995)'s games, McKelvey and Palfrey (1995) normalize payoffs so that each player's maximum payoff is always 1. This has no effect on the estimates for second-order expected utility with constant relative risk aversion but does affect the fit slightly with Quantal Response Equilibria as the scalings are different for each player (so not equivalent to a simple change in $\lambda$). The rationale for the rescaling is that in Ochs (1995), players are paid in lotteries. For consistency with earlier papers, this scaling is used in fitting QRE and RQRE.

[3]Goeree et al. (2003) normalize utility, which affects the reported value of $\lambda$ but not of $r$. The estimated probabilities are very close but Goeree et al. (2003) report a value of -5898.8 for the likelihood of RQRE on the McKelvey et al. (2000) data, which is higher than the likelihood would be if the model fitted perfectly (-5911.4). The number given here is close to that in the working paper version of Goeree et al. (2003) (-5929.0), Goeree et al. (2002). I am grateful to the Advisory Editor for drawing my attention to this paper. Friedman (2021), Appendix K, also re-estimates RQRE and reports a number of the likelihood (-5925.7) very close to that given here.

| Experiment | Model | $\epsilon$ | $\lambda$ | $r$ | Log Likelihood | # Obs |
|---|---|---|---|---|---|---|
| Games 2–3 | SOEU | 0.43 | | | -1239.3 | 960 |
| (Ochs 1995) | | (0.04) | | | | |
| | QRE | | 3.04 | | -1247.6 | |
| | | | (0.32) | | | |
| | RQRE | | 3.38 | 0.42 | -1238.6 | |
| | | | (0.68) | (0.06) | | |
| Games A–D | SOEU | 0.41 | | | -5918.1 | 4800 |
| (McKelvey et | | (0.01) | | | | |
| al. (2000)) | QRE | | 4.43 | | -6287.6 | |
| | | | (0.27) | | | |
| | RQRE | | 1.98 | 0.43 | -5926.8 | |
| | | | (0.15) | (0.02) | | |

Table 1: Parameter Estimates for Ochs (1995) and McKelvey et al. (2000) (Asymptotic standard errors in brackets)

the observed data is small. For games A–D SOEU fits the data better than RQRE overall and also in the individual games B–D.

Since second-order expected utility is not nested within the QRE models, the likelihoods cannot be compared formally by using likelihood ratio rests. Appendix B.2 presents formal tests. The principal criterion used is the Bayesian Information Criterion, but the Akaike Information Criterion and Shi (2015)'s version of Vuong (1989)'s test are also reported. Details of these statistics are given in Appendix B.1. Under the Bayesian Information Criterion SOEU is preferred to both QRE and RQRE in both Ochs (1995) and McKelvey et al. (2000)'s data.

Parameter constancy tests are reported in Appendix C.1. For Ochs (1995)'s experiments all three models seem to satisfy parameter constancy but for McKelvey et al. (2000) only second-order expected utility does. Lack of parameter constancy is a well known problem with QRE. Brunner et al. (2011) argue that it is unreasonable to expect constancy across a range of different games and that variations in $\lambda$ may reflect how subjects view different games. Lack of constancy across rather similar games with the same groups of subjects does, however, seem unreasonable. Goeree et al. (2003) note that the estimated value of the risk parameter $r$ seems roughly constant across games in the RQRE model. The estimated $\lambda$ parameters do, however, still vary significantly across the games. SOEU therefore seems to perform better

| Game | Data | | SOEU | | | QRE | | | RQRE | | | No. of |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $p_T$ | $p_L$ | $p_T$ | $p_L$ | LLik | $p_T$ | $p_L$ | LLik | $p_T$ | $p_L$ | LLik | Obs. |
| 2 | 0.595 | 0.258 | 0.612 | 0.258 | -558.4 | 0.651 | 0.265 | -561.2 | 0.585 | 0.261 | -558.3 | 448 |
| 3 | 0.542 | 0.336 | 0.571 | 0.339 | -680.8 | 0.617 | 0.312 | -686.5 | 0.556 | 0.334 | -680.1 | 512 |
| | | | | | | | | | | | | |
| A | 0.643 | 0.241 | 0.612 | 0.246 | -2170.9 | 0.723 | 0.122 | -2291.4 | 0.648 | 0.262 | -2169.3 | 1800 |
| B | 0.630 | 0.244 | 0.612 | 0.246 | -1458.4 | 0.560 | 0.105 | -1562.5 | 0.574 | 0.242 | -1465.2 | 1200 |
| C | 0.594 | 0.257 | 0.612 | 0.246 | -1495.5 | 0.562 | 0.101 | -1614.2 | 0.578 | 0.232 | -1496.9 | 1200 |
| D | 0.550 | 0.328 | 0.571 | 0.331 | -793.1 | 0.639 | 0.226 | -819.0 | 0.591 | 0.346 | -795.1 | 600 |

Table 2: Predicted Data Probabilities and Log Likelihoods
(Asymptotic standard errors in brackets)

| | $L$ | $R$ |
|---|---|---|
| $T$ | $200, 160$ | $160, 10$ |
| $B$ | $370, 200$ | $10, 370$ |

Figure 7: Goeree et al. (2003)'s Game

than either the QRE models as far as parameter constancy is concerned.

## 5.2 Goeree et al. (2003)'s game

Goeree et al. (2003) introduce the following game:

They note that in any quantal response equilibrium without risk aversion the probability of $L$ is at most 0.5, which is contradicted by the data.

The equilibrium of this game with second-order expected utility can be found numerically for each $\epsilon$ by using the first-order conditions for each player and solving the resulting mixed complementarity problem[4] This is then used to estimate $\epsilon$ by maximum likelihood.

The estimated model parameters and predicted probabilities are given in the Table 3.

SOEU fits the data considerably better than QRE. It fits the data slightly worse than RQRE but the difference is marginal. With two parameters

[4]See for example Facchinei and Pang (2003) for an extended treatment of complementarity problems.

| Model | $\epsilon$ | $\lambda$ | $r$ | $p_T$ | $p_L$ | Log Likelihood | # Obs |
|---|---|---|---|---|---|---|---|
| Data | | | | 0.47 | 0.65 | | 340 |
| SOEU | 0.48 | | | 0.42 | 0.63 | -457.0 | |
| | (0.07) | | | | | | |
| QRE | | 0.68 | | 0.53 | 0.47 | -480.4 | |
| | | (0.01) | | | | | |
| RQRE | | 0.16 | 0.45 | 0.47 | 0.65 | -455.2 | |
| | | (0.06) | (0.07) | | | | |

Table 3: Parameter Estimates and Predicted Data Probabilities for Goeree et al. (2003)
(Asymptotic standard errors in brackets)

and two choice probabilities to fit, RQRE is able to fit the data almost perfectly. With only one parameter, SOEU is able to fit the data very well but not perfectly. Whether the extra parameter is worthwhile depends on one's objective — explanation or prediction. As discussed in Appendix B.2, The Bayesian Information Criterion which emphasizes the former prefers SOEU but the Akaike Information Criterion which emphasizes the latter prefers RQRE.

## 5.3 Selten and Chmura (2008)'s Games

Selten and Chmura (2008) analyze a set of 12 games. These form pairs with game 1 paired with game 7, game 2 with game 8 and so on. Each of games 1 to 6 is a zero-sum game. Each of games 7 to 12 is obtained from the corresponding game in games 1 to 6 by adding a constant to the column player's payoffs when row plays $T$ and to the row player's payoffs when the column player pays $R$. Both Nash equilibrium and QRE in the logit form predict that these transformations should not affect behavior. They do, however, affect risk so behavior in pairs of games may differ under second-order expected utility and RQRE.

In Selten and Chmura (2008)'s experiments these games are played for 200 rounds. Each of games 1–6 was played by 12 subject groups but games 7–12 were played only by 6 subject groups. Selten and Chmura (2008) and Brunner et al. (2011) report results for the first and last 100 periods of play as well as the whole 200 periods of play. All solution concepts fit better in the last 100 periods. The same is true here but the results are qualitatively similar to the results for the whole sample, so only the latter are reported.

| | L | R |
|---|---|---|
| T | 10, 8 | 0, 18 |
| B | 9, 9 | 10, 8 |

**Game 1**

| | L | R |
|---|---|---|
| T | 9, 4 | 0, 13 |
| B | 6, 7 | 8, 5 |

**Game 2**

| | L | R |
|---|---|---|
| T | 8, 6 | 0, 14 |
| B | 7, 7 | 10, 4 |

**Game 3**

| | L | R |
|---|---|---|
| T | 7, 4 | 0, 11 |
| B | 5, 6 | 9, 2 |

**Game 4**

| | L | R |
|---|---|---|
| T | 7, 2 | 0, 9 |
| B | 4, 5 | 8, 1 |

**Game 5**

| | L | R |
|---|---|---|
| T | 7, 1 | 1, 7 |
| B | 3, 5 | 8, 0 |

**Game 6**

| | L | R |
|---|---|---|
| T | 10, 12 | 4, 22 |
| B | 9, 9 | 14, 8 |

**Game 7**

| | L | R |
|---|---|---|
| T | 9, 7 | 3, 16 |
| B | 6, 7 | 11, 5 |

**Game 8**

| | L | R |
|---|---|---|
| T | 8, 9 | 3, 17 |
| B | 7, 7 | 13, 4 |

**Game 9**

| | L | R |
|---|---|---|
| T | 7, 6 | 2, 13 |
| B | 5, 6 | 11, 2 |

**Game 10**

| | L | R |
|---|---|---|
| T | 7, 4 | 2, 11 |
| B | 4, 5 | 10, 1 |

**Game 11**

| | L | R |
|---|---|---|
| T | 7, 3 | 3, 9 |
| B | 3, 5 | 10, 0 |

**Game 12**

Figure 8: Selten and Chmura (2008)'s Games

The sample size is much larger here than in the previous games. If however there is correlation in play then the difference may be exaggerated.

Parameter estimates for this set of games are given in Table 4:

| Model | $\epsilon$ | $\lambda$ | $r$ | Log Likelihood | # Obs |
|---|---|---|---|---|---|
| SOEU | 0.69 | | | -107294.1 | 86400 |
| | (0.02) | | | | |
| QRE | | 1.00 | | -102116.1 | |
| | | (0.12) | | | |
| RQRE | | 1.10 | 0.16 | -101997.2 | |
| | | (0.01) | (0.01) | | |

Table 4: Parameter Estimates for Selten and Chmura (2008)
(Asymptotic standard errors in brackets)

and the fitted probabilities in Table 5:

| Game | Data | | SOEU | | QRE | | RQRE | |
|---|---|---|---|---|---|---|---|---|
| | $p_T$ | $p_L$ | $p_T$ | $p_L$ | $p_T$ | $p_L$ | $p_T$ | $p_L$ |
| Game 1 | 0.079 | 0.690 | 0.091 | 0.909 | 0.044 | 0.627 | 0.043 | 0.606 |
| Game 2 | 0.217 | 0.527 | 0.182 | 0.727 | 0.155 | 0.573 | 0.150 | 0.583 |
| Game 3 | 0.163 | 0.793 | 0.273 | 0.909 | 0.166 | 0.762 | 0.163 | 0.753 |
| Game 4 | 0.286 | 0.736 | 0.364 | 0.818 | 0.273 | 0.729 | 0.275 | 0.736 |
| Game 5 | 0.327 | 0.664 | 0.364 | 0.727 | 0.306 | 0.653 | 0.313 | 0.671 |
| Game 6 | 0.445 | 0.596 | 0.456 | 0.636 | 0.416 | 0.605 | 0.437 | 0.618 |
| Game 7 | 0.141 | 0.564 | 0.116 | 0.887 | 0.044 | 0.627 | 0.057 | 0.584 |
| Game 8 | 0.250 | 0.587 | 0.231 | 0.682 | 0.155 | 0.573 | 0.171 | 0.554 |
| Game 9 | 0.254 | 0.827 | 0.330 | 0.892 | 0.166 | 0.762 | 0.178 | 0.736 |
| Game 10 | 0.366 | 0.700 | 0.423 | 0.796 | 0.273 | 0.729 | 0.291 | 0.721 |
| Game 11 | 0.331 | 0.652 | 0.442 | 0.700 | 0.306 | 0.653 | 0.333 | 0.652 |
| Game 12 | 0.439 | 0.604 | 0.552 | 0.610 | 0.416 | 0.605 | 0.459 | 0.610 |

Table 5: Predicted Data Probabilities for Selten and Chmura (2008)

Second-order expected utility fits this data relatively poorly compared to QRE. It predicts that in games 1–6 plays will play according to the Nash equilibrium because players face no risk — the equilibrium strategy guarantees the same payoff regardless of the the choice of the other player (see

Lemma 8 in Appendix A.2). The fact that attitude towards risk plays little role in explaining play in games 1–6 is reinforced by the estimates for $r$ obtained for RQRE: significant statistically but very small. Formal model comparison tests are reported in Appendix B.

Second-order expected utility fits the behavior of the row player relatively well, indeed better than ordinary QRE. It has however more difficulty in explaining the behavior of the column player. As the row player is close to playing their equilibrium strategy the column player loses little by not playing a best response and QRE is able to capture this well, whereas because players face little risk here they are relatively sensitive to their opponents' strategies under SOEU. Note however that although QRE fits better overall, its prediction that play should be the same in each of the pairs of games is rejected at any reasonable level of significance in each of the pairs from games 1–4 and games 7–10, though not the remaining pairs.

All three theories, however, fail parameter constancy tests — see Appendix C.2. While QRE and RQRE perform better overall on this data none of the theories therefore seem entirely satisfactory.

# 6 Conclusions

This paper has considered second-order expected utility as an explanation for play in games. It showed that it could explain probability matching and possesses reasonable comparative statics for mixed strategy equilibria.

Overall the empirical results give a positive picture of second-order expected utility. It performs better than quantal response equilibrium in Ochs (1995)'s and McKelvey et al. (2000)'s data as well as in Goeree et al. (2003)'s experiments. It is not a complete explanation for behavior, however, and in Selten and Chmura (2008)'s data where risk is less important it performs less well than QRE.

When quantal response equilibrium is augmented to include risk, its performance in Ochs (1995), McKelvey et al. (2000) and Goeree et al. (2003)'s games improves but second-order expected utility still has comparable or slightly better performance in these games. Overall it seems that risk is an important factor in explaining behavior in these games, whether modeled by augmenting QRE or by second-order expected utility. RQRE may seem a reasonable compromise theory with good overall performance but is still suffers from the parameter constancy problems of QRE, so it also is not a complete theory.

Second Order Expected Utility is not a complete explanation for behavior but it seems worthy of further exploration.

## Appendices

# A General Results

This appendix gives some general results on the properties of games with second-order expected utility. The first subsection discusses the relationship between elimination of dominated strategies and rationalizability. It shows that if utility functions are concave the two notions coincide. It also discuses the existence of equilibrium and notes that second-order expected utility only affects the mixed equilibria of a game.

The second subsection gives a simple characterization of equilibrium when payoff functions are differentiable. It uses this to show that any rationalizable combination of strategies, and in particular any equilibrium, is also rationalizable when players have standard expected utility preferences. It also studies zero-sum games and shows that equilibria in this class of games coincide with those under standard expected utility preferences.

The remaining subsections deal with particular theoretical details arising in the main text and the interpretation of mixed strategies.

## A.1 Dominance and Equilibrium

If $U^1$ and $U^2$ are concave, then payoff functions are concave, so it is immediate that equilibrium exists:

**Lemma 2.** *Under Assumption 2, a Nash equilibrium exists.*

Equilibrium need not exist if the $U^i$ are convex. For example it is a consequence of the results below that in matching pennies, any equilibrium must be mixed. If however the $U^i$ are convex players will always prefer a pure strategy to a mixed one.

Below the case when the $U^i$ are affine, so that preferences reduce to expected utility, will be referred as the case of standard preferences.

**Lemma 3.** *Under Assumption 1, a strategy $p$ for player 1 is best reply to a pure strategy of player 2 if and only if it is best reply when player 1 has standard preferences. A similar statement holds for player 2.*

This is clear since if player 2 plays strategy $j$, say, then maximizing player 1's payoffs requires maximizing $U^1\left(\sum_i p_i a_{ij}\right)$, which is equivalent to maximizing $\sum_i p_i a_{ij}$ since $U^1$ is strictly increasing.

An immediate consequence is

**Lemma 4.** *Under Assumption 1* $(p^*, q^*)$ *is a pure strategy equilibrium if and only if it is a pure strategy equilibrium when both players have standard preferences.*

That is the assumption of second-order expected utility only has a potential affect on the mixed equilibria. This is clear since failure of reduction is only relevant if there is uncertainty.

A strategy $p$ for player 1 is said to *strictly dominate* a strategy $p'$ if it obtains a strictly higher payoff against any strategy $q$ of player 2. Dominance is defined similarly for player 2.

**Lemma 5.** *Under Assumption 1,* $p$ *strictly dominates* $p'$ *for player 1 if and only if it does so when player 1 has standard preferences. A similar statement holds for player 2.*

This is clear since $U^i$ is simply a monotone transformation of standard preferences. Since payoffs are linear in $q$ it is enough to check dominance against pure strategies of player 2. Then note:

$$\begin{aligned} U^1\left(\sum_i p_i a_{ij}\right) > U^1\left(\sum_i p'_i a_{ij}\right) \quad \text{for all } j \\ \iff \sum_i p_i a_{ij} > \sum_i p'_i a_{ij} \quad \text{for all } j \end{aligned}$$

If Assumption 2 holds then, as in standard games, iterated deletion of strictly dominated strategies and rationalizability produce the same answer. The sets of strategies surviving iterated deletion of dominated strategies and the set of rationalizable strategies are defined in the usual way. More precisely, let $S^i$ be the set of all strategies (pure and mixed) for player $i$. Set $\Sigma_0^i = \tilde{\Sigma}_0^i = S^i$ for $1, 2$. For $n \geq 1$, let

$$\Sigma_n^i = \{\sigma \in \Sigma_{n-1}^i : \sigma \text{ is a best reply to some } \tau \in \Sigma_{n-1}^j,\ j \neq i\}$$

and

$$\tilde{\Sigma}_n^i = \{\sigma \in \tilde{\Sigma}_{n-1}^i : \nexists \sigma' \in \tilde{\Sigma}_{n-1}^i \text{ with } \Pi^i(\sigma', \tau) > \Pi^i(\sigma, \tau)\, \forall \tau \in \tilde{\Sigma}_{n-1}^j,\ j \neq i\}$$

The set of rationalizable strategies for player $i$ is $\bigcap_{n=0}^{\infty} \Sigma_n^i$ and the set of strategies for $i$ surviving iterated deletion of strictly dominated strategies is $\bigcap_{n=0}^{\infty} \tilde{\Sigma}_n^i$.

**Lemma 6.** *Under Assumption 2, p is not strictly dominated for player 1 if and only if it is a best reply to some strategy of player 2. A similar statement holds for player 2.*

*Proof.* It is clear that if a strategy is strictly dominated it is never a best reply. It is only necessary to prove that if a strategy of player 1 is undominated then it is a best reply to some strategy of player 2. For a strategy $p$ of player 1, let

$$Z(p) = \left\{ U^1 \left( \sum_i p_i a_{i1} \right), \ldots, U^1 \left( \sum_i p_i a_{in_2} \right) \right\} \tag{25}$$

be the vector of payoffs to $p$ determined by the pure actions of player 2. Let

$$Z = \{Z(p') : p' \text{ a strategy of player 1}\} \tag{26}$$

Since $U^1$ is continuous $Z$ is compact.

Now let

$$Z_- = \{z : \exists\, z' \in Z \text{ s.t. } z \leq z'\} \tag{27}$$

where $\leq$ denotes component-wise inequality. Since $Z$ is compact, $Z_-$ is closed. Moreover since $U^1$ is concave, $Z_-$ is convex.

Suppose that $p$ is undominated. Let

$$W(p) = Z_- - Z(p) \tag{28}$$

$W(p)$ is closed and convex.

As $p$ is undominated

$$W(p) \cap \mathcal{R}^{n_2}_{++} = \{0\} \tag{29}$$

Applying the separating hyperplane theorem to $W(p)$ and $\mathcal{R}^{n_2}_{++}$, there exists a probability vector $q'$ such that (with . denoting inner product)

$$q'.w \leq 0 \quad \forall w \in W(p) \tag{30}$$

or equivalently

$$q'.z \leq q'.Z(p) \quad \forall z \in Z_- \tag{31}$$

In particular, this holds for $z$ in $Z$, so

$$\sum_j q'_j U^1 \left( \sum_i p'_i a_{ij} \right) \leq \sum_j q'_j U^1 \left( \sum_i p_i a_{ij} \right) \quad \forall p' \tag{32}$$

That is $p$ is a best response to $q'$, which concludes the proof.

| | 1 | 2 |
|---|---|---|
| 1 | $2, 0$ | $0, 1$ |
| 2 | $0, 1$ | $2, 0$ |

Figure 9: Counter-Example

**Corollary 1.** *Under Assumption 2, the set of rationalizable strategies for each player and the set of strategies surviving iterated deletion of strictly dominated strategies coincide.*

The proof of Lemma 6 is on the same lines as the proof for standard games but one needs to use Assumption 2 as payoffs are no longer linear in both $p$ and $q$. The corollary follows immediately from the lemma.

If Assumption 2 does not hold then the result need not hold. In particular a strategy may be undominated but never a best reply. The reason is similar to that in equilibrium theory where with non-convex preferences, Pareto-efficient points may not be supportable by prices. Consider the following example:

**Example 2.** *Consider the game in example in Figure 9 with* $U^1(x) = x^2$*:*

*In utility terms strategy 1 for player 1, the row player, earns 4 against strategy 1 of player 2 and 0 against strategy 2, while for strategy 2 payoffs are reversed. It is easy to check that the mixed strategy* $p = (1/2, 1/2)$ *is undominated and earns* $1 = U^1(0.5 \times 2 + 0.5 \times 0)$ *against both strategy 1 and strategy 2 of player 2 (or any mixture).* $p = 1/2$ *is, however, never a best reply for player 1. Against any mixed strategy* $(q, 1 - q)$ *of player 2, strategy 1 earns* $4q$ *and strategy 2 earns* $4(1 - q)$ *and at least one of these payoffs exceeds 1.*

It follows from Corollary 1, Lemma 3 and the usual equivalence between rationalizability and iterated deletion of strictly dominated strategies under standard preferences that

**Corollary 2.** *Under Assumption 2, the set of rationalizable strategies for each player is the same as under standard preferences.*

The set of equilibria need not of course be the same. This result is discussed further in the next subsection.

Note that in Lemma 6, players are assumed to choose randomized strategies explicitly. This is a controversial issue in the literature. Klibanoff (1996) follows this path but Epstein (1997) does not not. The interpretation of mixed strategies is discussed further in Appendix A.5.

## A.2 Differentiable Case

For the applications in the paper, utility functions are assumed to be twice continuously differentiable, so that Assumption 3 is satisfied. This allows a simple characterization of equilibrium.

Player 1 maximizes $\sum_j q_j U^i(\sum_i p_i a_{ij})$ subject to $p_i \geq 0$ and $\sum_i p_i = 1$. Since $U^1$ is concave and the constraints linear it follows that $p^*$ is a best response to $q^*$ if and only if there exists $\lambda^*$ such that

$$\sum_j q_j^* a_{ij} U^{1'} \left( \sum_i p_i^* a_{ij} \right) = \lambda^* \qquad p_i^* > 0 \tag{33a}$$

$$\sum_j q_j^* a_{ij} U^{1'} \left( \sum_i p_i^* a_{ij} \right) \leq \lambda^* \qquad p_i^* = 0 \tag{33b}$$

Primes denote derivatives.

If $U^1$ were linear, then, as $\sum_j q_j^* a_{ij}$ is the expected payoff to strategy $i$, (33) would reduce to the condition that each action played with positive probability has the same expected payoff. Here each action played has the same expected marginal utility. Actions with low expected payoffs may be played with some probability if they give good insurance. The first-order conditions are analogous to those in a portfolio problem: the $p_i$ can be thought of as the holding of asset $i$, which has payoff $a_{ij}$ in state $j$, and $q_j$ as the probability of state $j$.

The left-hand side of (33) can also be written as

$$\sum_j q_j^* s_j a_{ij} \tag{34}$$

where $s_j = U^{1'} \left( \sum_i p_i^* a_{ij} \right)$ can be thought of as distortion factors to beliefs. This is a well known interpretation of second-order expected utility (see for example Gollier (2011)). These factors are, however, endogenous and determined by player $i$'s choices. The portfolio interpretation seems more useful. One can, however, use this characterization to make a link to play under standard preferences.

The conditions in (33) imply that $p^*$ is a best reply to the beliefs $\tilde{q}_j = q^*_j s_j / \left(\sum_k q^*_k s_k\right)$ under standard preferences. It follows that:

**Lemma 7.** *(a) Under Assumption 3 a strategy $p^*$ for player 1 is best reply to some some strategy $q^*$ of player 2 if and only if it is a best reply to some strategy of player 2 under standard preferences. A similar statement holds for player 2.*

*(b) It follows that any strategy combination $(p^*, q^*)$ which is rationalizable, in particular any equilibrium, is also rationalizable under standard preferences. Conversely any strategy combination rationalizable under standard preferences is rationalizable with second-order expected utility preferences.*

*Proof.* The only part of the Lemma which requires proof is the if direction in part (a). If $p^*$ is a best reply to $q^*$ under standard preferences, there exists $\lambda^*$ such that

$$\sum_j q^*_j a_{ij} = \lambda^* \qquad p^*_i > 0 \tag{35a}$$

$$\sum_j q^*_j a_{ij} \leq \lambda^* \qquad p^*_i = 0 \tag{35b}$$

Let $\tilde{q}^*_j = q^*_j / U^{1'}\left(\sum_i p^*_i a_{ij}\right)$ for each $j$. (35) is then equivalent to

$$\sum_j \tilde{q}^*_j a_{ij} U^{1'}\left(\sum_i p^*_i a_{ij}\right) = \lambda^* \qquad p^*_i > 0 \tag{36a}$$

$$\sum_j \tilde{q}^*_j a_{ij} U^{1'}\left(\sum_i p^*_i a_{ij}\right) \leq \lambda^* \qquad p^*_i = 0 \tag{36b}$$

Normalizing $\tilde{q}^*_j$ to form beliefs $q^{**}_j = \tilde{q}^*_j / \sum_k \tilde{q}^*_k$, yields

$$\sum_j q^{**}_j a_{ij} U^{1'}\left(\sum_i p^*_i a_{ij}\right) = \lambda^{**} \qquad p^*_i > 0 \tag{37a}$$

$$\sum_j q^{**}_j a_{ij} U^{1'}\left(\sum_i p^*_i a_{ij}\right) \leq \lambda^{**} \qquad p^*_i = 0 \tag{37b}$$

Since the first-order conditions are sufficient this implies that $p^*$ is a best reply to beliefs $q^{**}$.

The only if direction in (a) follows from the discussion above.

Part (b) follows immediately from the definition of rationalizability. This result appears already in the previous sub-section as Corollary 2 but the proof here is more direct. Equilibria need not be equilibria under standard preferences but they are at least rationalizable under standard preferences. In this sense equilibrium under second-order expected utility is a refinement of rationalizability with standard preferences.

For future reference, it is useful to record the following simple observation:

**Lemma 8.** *Consider a constant-sum game, that is $a_{ij} + b_{ij} = C$ for some $C$ for all $i$ and $j$. If $(p^*, q^*)$ is a Nash equilibrium with standard preferences then it is one for any utility functions of the two players satisfying Assumption 3.*

*Proof.* Let $(p^*, q^*)$ be a Nash equilibrium with standard preferences. It needs to be shown that it satisfies the equilibrium conditions if the agents have second-order expected utility. Recall the first-order conditions for player 1 with second-order expected utility

$$\sum_j q_j^* a_{ij} U^{1'}\left(\sum_i p_i^* a_{ij}\right) = \lambda^* \qquad p_i^* > 0 \tag{38a}$$

$$\sum_j q_j^* a_{ij} U^{1'}\left(\sum_i p_i^* a_{ij}\right) \leq \lambda^* \qquad p_i^* = 0 \tag{38b}$$

for some $\lambda^*$.

Since the game is constant-sum $\sum_i p_i^* a_{ij}$ equals $C - \sum_i p_i^* b_{ij}$. $\sum_i p_i^* b_{ij}$ is player 2's expected payoff from playing action $j$ with standard preferences and must be the same for all $j$ with $q_j^* > 0$. Hence $\sum_i p_i^* a_{ij}$, and so also $U^{1'}\left(\sum_i p_i^* a_{ij}\right)$, is independent of $j$ for those $j$ with $q_j^* > 0$. It follows that (38) holds if and only if

$$\sum_j q_j^* a_{ij} = \tilde{\lambda}^* \qquad p_i^* > 0 \tag{39a}$$

$$\sum_j q_j^* a_{ij} \leq \tilde{\lambda}^* \qquad p_i^* = 0 \tag{39b}$$

holds for some $\tilde{\lambda}^*$. That is all actions played with positive probability by player $i$ have the same payoff and are better than any actions not played. This indeed is the case since $(p^*, q^*)$ is an equilibrium with standard preferences. It follows that (38) holds.

Since the first-order conditions are sufficient it follows that $p^*$ is a best reply to $q^*$. A similar argument shows that $q^*$ is best reply to $p^*$ and so $(p^*, q^*)$ is an equilibrium when agents have second-order expected utility preferences.

This follows since in a constant-sum game, player 2 is trying to minimize player 1's payoffs and so any action $j$ he plays is equally bad for player 1. Player 1's marginal utility, $U^{1'}$, is therefore independent of player 2's choice and so the conditions reduce to the usual ones. In essence, in a constant-sum game players face no risk with respect to the outcome of the other's choices and so risk aversion does not matter.

## A.3 Analysis of Ochs (1995) under Eichberger and Kelsey (2011) preferences

In Eichberger and Kelsey (2011)'s model each player believes that if they play any pure action then with probability $1-\delta$ the other player plays their equilibrium strategy but with probability $\delta$ does not and with probability $\alpha$ plays the best strategy for them given their action but with probability $1-\alpha$ the worst.

It follows that the expected payoffs to the row player playing $T$ and $B$ are

$$\pi_T = \delta(qa + (1-q)0) + (1-\delta)(\alpha a + (1-\alpha)0) \tag{40a}$$
$$\pi_B = \delta(q0 + (1-q)1) + (1-\delta)(\alpha 1 + (1-\alpha)0) \tag{40b}$$

respectively and to the column player playing $L$ and $R$ are

$$\pi_L = \delta(p0 + (1-p)1) + (1-\delta)(\alpha 1 + (1-\alpha)0) \tag{41a}$$
$$\pi_R = \delta(p1 + (1-q)0) + (1-\delta)(\alpha 1 + (1-\alpha)0) \tag{41b}$$

respectively.

In Eichberger and Kelsey (2011)'s model the expected payoff to any mixed strategy is simply the expected value of the payoffs of the component pure strategies, so $q\pi_L+(1-q)\pi_R$ for the column player. It follows that the column player will only randomize if she is indifferent between her pure strategies, which from (41) requires that $p = 1/2$ regardless of the value of $a$. That is in any mixed equilibrium $p$ does not depend on $a$. If $a$ is large enough then, provided $\alpha > 0$, the pure strategy equilibrium $(T, R)$ will be played but Eichberger and Kelsey (2011)'s model cannot explain the observed comparative statics when players still mix.

## A.4 Proof of Lemma 1

Under standard preferences the assumptions on payoffs imply that the game can have no pure-strategy equilibria and from Lemma 4 the same is true when agents have second-order expected utility preferences. If one agent plays a pure strategy, then it follows from the assumptions on payoffs that the other agent's unique best response is a pure strategy: if say agent 2 plays strategy $j$ with probability 1 then agent 1 maximizes $U^1(p_1 a_{1j} + p_2 a_{2j})$ and this has unique solution $p_1 = 0$ or $p_1 = 1$.

Equilibrium exists by Lemma 2, so it must be fully mixed. Let $p$ be the probability with which player plays $T$ and $q$ the probability with which player 2 plays $l$. Player 1's payoff as a function of $p$ and $q$ and $a$ (all the other parameters are held fixed) is

$$V^1(p,q;a) = qU^1(pa + (1-p)c) + (1-q)U^1(pb + (1-p)d) \tag{42}$$

and player 2's payoff function is

$$V^2(p,q) = pU^2(q\alpha + (1-q)\beta) + (1-p)U^2(q\gamma + (1-q)\delta) \tag{43}$$

So, letting subscripts denote partial derivatives,

$$V^1_p = q(a-c)U^{1'}(pa + (1-p)c) - (1-q)(d-b)U^{1'}(pb + (1-p)d) \tag{44}$$

Hence

$$V^1_{pp} = q(a-c)^2 U^{1''}|_{pa+(1-pc)c} + (1-q)(d-b)^2 U^{1''}|_{pb+(1-p)d} \quad \leq 0 \tag{45}$$

and

$$V^1_{pq} = (a-c)U^{1'}|_{pa+(1-p)c} + (d-b)U^{1'}|_{pb+(1-p)d} \quad > 0 \tag{46}$$

A similar calculation for player 2 shows that $V^2_{qq} \leq 0$ and $V^2_{qp} < 0$.

It follows from (46) that $V^1$ is strictly super-modular in $p$ and $q$ and so, applying, for example, Theorem 1 of Edlin and Shannon (1998), the best response correspondence of agent 1 is strictly increasing in the sense that if $p^* \in \text{argmax}_p V^1(p, q^*)$, $0 < p^* < 1$ and $p' \in \text{argmax}_p V^1(p, q')$ then $p' > p^*$ or $p' < p^*$ according as $q' > q^*$ or $q' < q^*$. A similar argument shows that player 2's best response correspondence is strictly decreasing in $p$ in the same sense. These two facts imply that equilibrium is unique: if $(p^*, q^*)$ and $(p', q')$ are distinct equilibria then if for example $p' > p^*$ then $q' < q^*$ as player 2 is optimizing but $q' < q^*$ implies $p' < p^*$, a contradiction.

Now consider comparative statics. The first-order conditions for the two-players are

$$V^1_p = 0, \qquad V^2_q = 0 \tag{47}$$

The Jacobian of this system is

$$J = \begin{pmatrix} V^1_{pp} & V^1_{pq} \\ V^2_{qp} & V^2_{qq} \end{pmatrix} \tag{48}$$

The determinant of $J$, $\Delta$, is strictly positive as under the assumptions on risk aversion $V^1_{pp} < 0$ and $V^2_{qq} < 0$ in addition to the already established results that $V^1_{pq} > 0$ and $V^2_{qp} < 0$. It follows that one can apply the implicit function to this system and find that

$$\frac{\partial p^*}{\partial \theta} = \frac{V^2_{qq}}{\Delta}\left(-V^1_{p\theta}\right) \tag{49a}$$

$$\frac{\partial q^*}{\partial \theta} = \frac{-V^2_{pq}}{\Delta}\left(-V^1_{p\theta}\right) \tag{49b}$$

where $\theta$ is one of $a$, $b$, $c$ or $d$.

It follows that $\frac{\partial p^*}{\partial \theta} > 0$ and $\frac{\partial q^*}{\partial \theta} < 0$ provided that $V^1_{p\theta} > 0$. Now

$$V^1_{pa} = qU^{1'}(pa + (1-p)c) + qp(a-c)U^{1''}(pa + (1-p)c) \tag{50}$$

Now $p(a-c) \leq pa + (1-p)c$ as $c \geq 0$, so $V^1_{pa} > 0$ holds if

$$U^{1'}(x) + xU^{1''}(x) > 0 \tag{51}$$

which is equivalent to the coefficient of relative risk aversion being less than one. It follows that $\frac{\partial p^*}{\partial a} > 0$ and $\frac{\partial q^*}{\partial a} < 0$ if $U^1$ has coefficient of relative risk aversion less than 1.

Now

$$V^1_{pb} = (1-q)U^{1'}(pb + (1-p)d) - (1-q)p(d-b)U^{1''}(pb + (1-p)d) \tag{52}$$

hence $V^1_{pb} > 0$ as $d > b$ and $U$ is increasing and concave. It follows that $\frac{\partial p^*}{\partial b} > 0$ and $\frac{\partial q^*}{\partial b} < 0$

Similar arguments prove the the results for the comparative statics with respect to $c$ and $d$.

## A.5 Interpretation of Mixed Strategies

The interpretation of mixed strategies is controversial. Here it is assumed that players can commit themselves to randomize. A common objection to non-expected utility preferences is lack of dynamic consistency. In particular if the $U^i$ are strictly concave players prefer randomization to playing pure strategies between which they are indifferent. In particular they prefer it to any pure strategy their randomization tells them to play, so why do they not randomize again? Here it is assumed that players can commit themselves to randomization. Alternatively, one might assume they do not have time to randomize again. For example in the experiments on probability matching discussed in Section 4, they may have limited time to submit an answer and so cannot randomize indefinitely.

Crawford (1990) considers equilibrium in belief to restore existence in games with non-expected utility, that is players do not actually randomize, but this is not pursued here.

Dekel and Segal (1991) suggest when preferences do not satisfy the reduction of compound lotteries then it is important whether players perceive themselves as moving first or second (even if they do not observe the choices made by the other player). Preferences of the form considered in (3)

$$\Pi^1(p, q) = \sum_j q_j U^1 \left( \sum_i p_i a\left(\{ij\}\right) \right) \tag{53}$$

would in Dekel and Segal (1991)'s interpretation mean that player 1 perceives himself as moving after player 2: player 2's choice determines the mixed lottery he faces. In contrast if preferences has the form

$$\tilde{\Pi}^1(p, q) = \sum_i p_i U^1 \left( \sum_j q_j a\left(\{ij\}\right) \right) \tag{54}$$

this would mean that player 1 perceives of himself as moving first. Dekel and Segal (1991) argue in favor of second interpretation but they assume that players cannot commit themselves to randomization. The fact that (54) is linear in own probabilities means, however, that the latter formulation is not useful in the applications in the paper. For example in probability matching, it will always be optimal to play the action with highest utility rather than randomize.

The current paper only considers normal-form games but the model could be applied to extensive form games. One might wonder about the assumption that a player always considers himself as moving first in that context. A player will, however, either observe the outcome of the other player's randomization, in which case he will be responding to a pure strategy, or not, in which case the situation is equivalent to choosing simultaneously as in a strategic-form and player is equally entitled to think of himself as moving first.

# B Formal Model Comparison

## B.1 Criteria Used

The models are estimated by Maximum Likelihood. Since second-order expected utility and QRE are non-nested, one cannot use standard likelihood-ratio tests to compare them. The principal criterion used is the Bayesian Information Criterion (BIC).

Suppose the modeler has a collection of $n$ observations, $y$, and candidate model $M_k$ has a $k$-dimensional parameter space $\Theta_k$ with likelihood function $L(\theta_k|y)$. Let $\widehat{\theta}_k$ be the maximum likelihood estimate obtained by maximizing $L$ over $\Theta_k$. The BIC for candidate model $M_k$ is given

$$\mathrm{BIC}_k = -2\ln L(\widehat{\theta}_k|y) + k\ln n \tag{55}$$

If the modeler has $L$ potential models $M_{k_1}, \ldots, M_{k_L}$ for the data then the BIC can be computed for each model and the one with lowest BIC is preferred. From a Bayesian point of view, choosing the model with lowest BIC is, in a large sample, equivalent to choosing the model with highest posterior probability. From a frequentist point of view, an appealing feature of the BIC is that if true model is contained in the set of candidate models then the BIC is consistent in the sense that it selects the true model with probability 1 as the sample size tends to infinity.

The Akaike Information Criterion (AIC) is also reported:

$$\mathrm{AIC}_k = -2\ln L(\widehat{\theta}_k|y) + 2k \tag{56}$$

This applies a lower penalty to the number of parameters than the BIC. Unlike the BIC, it is not consistent. The AIC can instead be thought of as appropriate where the main desire is predictive accuracy and an over-parameterized model is not considered important. The BIC is more appropriate when the goal is Bayesian selection of the most probable model or, as it is sometimes put, the goal is explanation rather than prediction. Shmueli (2010) discusses the difference between the aims of explanation and prediction in statistics. Note that, contrary to what is sometimes asserted, the BIC does not require models to be nested.

Cross-validation is advocated by Wright and Leyton-Brown (2017) and is is similar to the AIC in being primarily concerned with prediction. A good discussion of the differing objectives of the BIC compared to the AIC and cross-validation can be found in Efron and Hastie (2016).

The Bayesian Information Criterion is, as the name suggests, motivated by a Bayesian perspective. From a frequentist perspective it is more natural to look for a test to discriminate between competing models. Vuong (1989) suggests using the Likelihood ratio as a measure of fit. Even if neither model is correct this can be interpreted as an estimate of which model is closer to the true distribution in the sense of Kullback-Leibler distance. Vuong (1989)'s test has poor finite sample properties and the modification suggested by Shi (2015) is reported here.[5]

Vuong (1989) suggests using the statistic:

$$Z = \frac{n^{1/2}\widehat{\mathrm{LR}_n}}{\widehat{\omega}_n} \tag{57}$$

to compare two models, where $\widehat{LR}_n$ is sample mean of the log-likelihood ratio of the two models and $\widehat{\omega}$ is its sample variance.

Shi (2015) notes that both the numerator and denominator of (57) are biased in finite samples and suggests corrections to both to improve the performance of the test. Her procedure also has the advantage that one can apply the procedure whether or not the two models overlap, unlike in Vuong (1989)'s case where a first stage is required to test this. Shi (2015)'s test requires simulation to calculate critical values. The paper follows the procedure in Shi (2015) using 1000 draws in the simulation. In fact, Shi (2015)'s modifications make little difference to the value of the test statistic here. The test statistic can be modified to penalize the number of parameters by incorporating the AIC or BIC but again these make little difference and so are not reported.

In computing Shi (2015)'s test the use is made of the fact that the log-likelihood for an individual observation can be written as

$$l_{ij} = f_i + g_j \tag{58}$$

since players randomize independently. That is the probability that row plays $i$ and column plays $j$ is the product of the marginal probabilities that row plays $i$ and column plays $j$. It follows that

$$\operatorname{Var} l = \operatorname{Var} f + \operatorname{Var} g \tag{59}$$

This formulation has the advantage the right-hand side can be calculated only knowing the marginal frequencies of row and column play, whereas the left-hand side requires knowledge of the joint frequencies which are not always reported. The variance of the log-likelihoods, and so of the log-likelihood ratios, is therefore estimated as the sum of the variances of the row and column

[5] Schennach and Wilhelm (2017) also suggest a modification but Shi (2015)'s is slightly easier to apply here.

likelihoods. (59) need not hold exactly for estimated variances but it holds in expectation and asymptotically for all the models under consideration so using it should improve efficiency. Other non-linear moments are treated similarly.

## B.2 Model Comparison for Ochs (1995)

| Experiment | Model | Log Lik | BIC | AIC | S-V Test | # Obs |
|---|---|---|---|---|---|---|
| Games 2–3 (Ochs 1995) | SOEU | -1239.3 | 2485.5 | 2480.6 | — | 960 |
| | QRE | -1247.6 | 2502.0 | 2497.1 | 0.175 (2.06) | |
| | RQRE | -1238.6 | 2490.9 | 2481.1 | -0.057 (2.10) | |
| Games A–D (McKelvey et al. (2000)) | SOEU | -5918.1 | 11844.6 | 11838.2 | — | 4800 |
| | QRE | -6287.6 | 12583.6 | 12577.1 | 3.95 (2.08) | |
| | RQRE | -5926.8 | 11870.5 | 11857.6 | 0.117 (2.06) | |

Table 6: Model Comparison for Ochs (1995) and McKelvey et al. (2000) (5% Critical values for Shi-Vuong test in brackets)

For the assessment of differences in the Bayesian Information Criterion the scale suggested by Kass and Rafferty (1995) is commonly used

| $\Delta$BIC | Evidence against $H_0$ |
|---|---|
| 0 to 2 | Not worth more than a bare mention |
| 2 to 6 | Positive |
| 6 to 10 | Strong |
| $> 10$ | Very Strong |

Table 7: Kass and Rafferty (1995) scale

On this basis there is very strong evidence in favor of second-order expected utility over QRE in both experiments (recall the lower values of the BIC and AIC are better). There is positive evidence for SOEU over QRE augmented to include risk aversion in Ochs (1995)'s experiments and very strong in the McKelvey et al. (2000)'s. The Akaike information criterion is more favorable to the QRE-based models but the overall picture is similar. The Shi-Vuong test indicates that RQRE and SOEU cannot be separated - this contrasts with the BIC but from a frequentist perspective sample sizes, though large, are not large enough. Conflict between frequentist and bayesian criteria is not uncommon - see for example Efron and Hastie (2016) Chapter 13 for discussion. In addition the power in finite samples of the Shi-Vuong has not been thoroughly studied.

## B.3 Model Comparison for Goeree et al. (2003)

Goodness of fit statistics appear in Table 8: Second-order expected utility

| Model | Log Lik | BIC | AIC | S-V Test | # Obs |
|---|---|---|---|---|---|
| SOEU | -457.0 | 919.9 | 916.1 | — | 960 |
| QRE | -480.4 | 966.7 | 962.8 | 0.86 | |
| | | | | (2.06) | |
| RQRE | -455.2 | 922.1 | 914.4 | -.09 | |
| | | | | (2.05) | |

Table 8: Model Comparison for Goeree et al. (2003)
(5% Critical values for Shi-Vuong test in brackets)

outperforms QRE according to both BIC and AIC, although the Shi-Vuong test cannot separate them. It performs slightly better than RQRE according to the BIC but slightly worse according the AIC. With two parameters and two choice probabilities to fit, RQRE is able to fit the data almost perfectly. With only one parameter, SOEU is able to fit the data very well but not perfectly. Whether the extra parameter is worthwhile depends on one's objective — explanation or prediction, as discussed in Section B.1.

## B.4 Model Comparison for Selten and Chmura (2008)

| Model | Log Lik | BIC | AIC | S-V Test | # Obs |
|---|---|---|---|---|---|
| SOEU | -107294.1 | 214599.5 | 214590.2 | — | 86400 |
| QRE | -102116.1 | 204243.5 | 204234.1 | -13.6 | |
| | | | | (2.06) | |
| RQRE | -101461.5 | 204017.1 | 203998.4 | -13.9 | |
| | | | | (2.11) | |

Table 9: Model Comparison for Selten and Chmura (2008)
(5% Critical values for Shi-Vuong test in brackets)

# C Parameter Constancy Tests

## C.1 Parameter Constancy Tests for Ochs (1995)'s Games

The table below reports tests for the constancy of the estimated parameters within each group of experiments. Each experiment is allowed to have different parameter values. The reported BIC values take into account the resulting increase in the number of parameters.[6]

| Experiment | Model | Joint Log L | Sep Log L | p-value | Joint BIC | Sep BIC | # Obs |
|---|---|---|---|---|---|---|---|
| Games 2–3 (Ochs 1995) | SOEU | -1239.3 | -1239.3 | 0.92 | 2485.5 | 2492.4 | 960 |
| | QRE | -1247.6 | -1247.2 | 0.53 | 2502.0 | 2508.1 | |
| | RQRE | -1238.6 | -1238.3 | 0.58 | 2490.9 | 2504.0 | |
| Games A–D (McKelvey et al. (2000)) | SOEU | -5918.1 | -5917.6 | 0.93 | 11844.6 | 11869.1 | 4800 |
| | QRE | -6287.6 | -6187.3 | 0.00 | 12583.6 | 12408.6 | |
| | RQRE | -5926.8 | -5911.7 | 0.00 | 11870.5 | 11891.3 | |

Table 10: Parameter Constancy for Ochs (1995) and McKelvey et al. (2000)

For Ochs (1995)'s experiments all three models seem to satisfy parameter constancy but for McKelvey et al. (2000) only second-order expected utility

[6]For economy of space the AIC is not reported but the results are similar.

does. In the case of RQRE the BIC penalizes the extra parameter heavily enough that the improvement in likelihood, though statistically significant, is not worthwhile for RQRE in McKelvey et al. (2000), though the AIC would consider it so. The estimates for $\lambda$ under QRE vary substantially in actual size, so the difference is economically as well as statistically significant. For example in McKelvey et al. (2000) they range from 0.75 in game B to 7.33 in game D. Under RQRE the range for $\lambda$ in these experiments is reduced to 1.15 to 3.91 but the variation is still substantial.

## C.2 Parameter Constancy Test for Selten and Chmura (2008)

Second-order expected utility predicts that players will play the Nash equilibrium regardless of the value of $\epsilon$ in games 1–6, so this parameter is only identified from games 7–12. For this reason, the parameter constancy tests are restricted to games 7–12:

| Model | Joint Log L | Sep Log L | p-value | Joint BIC | Sep BIC | # Obs |
|---|---|---|---|---|---|---|
| SOEU | -37175.8 | -36488.3 | 0.00 | 74361.9 | 73038.2 | 28800 |
| QRE | -35677.5 | -35272.3 | 0.00 | 71365.3 | 70606.2 | |
| RQRE | -35393.2 | -34906.3 | 0.00 | 70806.9 | 69935.8 | |

Table 11: Parameter Constancy for Selten and Chmura (2008) Games 7–12

All three theories fail the parameter constancy tests. The BIC paints a similar story. While QRE and RQRE perform better overall on this data none of the theories therefore seem entirely satisfactory.

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