

WELL-POSEDNESS OF AN INTEGRO-DIFFERENTIAL MODEL FOR ACTIVE BROWNIAN PARTICLES*

MARIA BRUNA[†], MARTIN BURGER[‡], ANTONIO ESPOSITO[§], AND SIMON SCHULZ[¶]

Abstract. We propose a general strategy for solving nonlinear integro-differential evolution problems with periodic boundary conditions, where no direct maximum/minimum principle is available. This is motivated by the study of recent macroscopic models for active Brownian particles with repulsive interactions, consisting of advection-diffusion processes in the space of particle position and orientation. We focus on one of such models, namely a semilinear parabolic equation with a nonlinear active drift term, whereby the velocity depends on the particle orientation and angle-independent overall particle density (leading to a nonlocal term by integrating out the angular variable). The main idea of the existence analysis is to exploit a priori estimates from (approximate) entropy dissipation. The global existence and uniqueness of weak solutions is shown using a two-step Galerkin approximation with appropriate cutoff in order to obtain nonnegativity, an upper bound on the overall density, and preserve a priori estimates. Our analysis naturally includes the case of finite systems, corresponding to the case of a finite number of directions. The Duhamel principle is then used to obtain additional regularity of the solution, namely continuity in time-space. Motivated by the class of initial data relevant for the application, which includes perfectly aligned particles (same orientation), we extend the well-posedness result to very weak solutions allowing distributional initial data with low regularity.

Key words. active particles, space-periodic problems, parabolic equations, Galerkin approximation, periodic heat kernel

MSC codes. 35K20, 35K58, 35Q70, 35Q92

DOI. 10.1137/21M1462039

1. Introduction. In this paper, we present a well-posedness theory (existence, uniqueness, and regularity) for an integro-differential equation describing a system of interacting active Brownian particles. Active Brownian particles are a model system for self-propelled particles, and have wide-ranging applications such as molecular motors, bacteria, animal swarms, and pedestrian flows (cf., e.g., [5, 8, 12, 13]). Denote by $\mathbf{X}_i \in \Omega \subset \mathbb{R}^2$ and $\Theta_i \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ the position and orientation of particle number i , respectively. The evolution of the system with N indistinguishable particles is given

*Received by the editors November 29, 2021; accepted for publication (in revised form) May 23, 2022; published electronically October 24, 2022.
<https://doi.org/10.1137/21M1462039>

Funding: The work of the first author was partially supported by the Royal Society University Research Fellowship grant URF/R1/180040 and a Humboldt Research Fellowship from the Alexander von Humboldt Foundation. The work of the second and third authors was supported by the German Science Foundation (DFG) through CRC TR 154, “Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks.” The work of the third author was also supported by the Advanced Grant Nonlocal-CPD of the ERC under the EU’s Horizon 2020 research and innovation programme (grant agreement 883363). The work of the fourth author was supported by the Royal Society Award RGF/EA/181043.

[†]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK (bruna@maths.cam.ac.uk).

[‡]Department Mathematik, Friedrich-Alexander Universität Erlangen-Nürnberg, Cauerstr. 11, D 91058 Erlangen, Germany (martin.burger@fau.de).

[§]Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK (antonio.esposito@maths.ox.ac.uk).

[¶]Department of Mathematics, University of Wisconsin-Madison, Van Vleck Hall, 480 Lincoln Dr., Madison, WI 53706, USA (smschulz2@wisc.edu).

by the following stochastic differential equations:

$$(1.1a) \quad d\mathbf{X}_i = \sqrt{2D_T} d\mathbf{W}_i + v_0 \mathbf{e}(\Theta_i) dt - \sum_{j \neq i} \nabla u((\mathbf{X}_i - \mathbf{X}_j)/\varepsilon) dt,$$

$$(1.1b) \quad d\Theta_i = \sqrt{2D_R} dW_i,$$

where D_T and D_R are the translational and rotational diffusion coefficients, respectively, \mathbf{W}_i and W_i are N independent Brownian motions in \mathbb{R}^2 and \mathbb{R} , respectively, v_0 is the self-propulsion speed, $\mathbf{e}(\theta) = (\cos \theta, \sin \theta)$, and u is the pairwise interaction potential, assumed to be radially symmetric, purely repulsive, and short-ranged ($\varepsilon \ll 1$). In contrast to passive Brownian particles (that is, $D_R = 0$ in (1.1b)), active Brownian particles perform a biased Brownian walk in the direction given by their orientation.

Starting from (1.1) and considering the associated Fokker–Planck equation for the joint probability density in $(\Omega \times \mathbb{T})^N$, one can derive the BBGKY hierarchy of coupled equations for the marginal probabilities. Depending on the assumptions on the interaction u , one can obtain different continuum models for the one-particle marginal density [4]. One of such models is studied by Speck et al in [16], where they obtain a closed equation by assuming that the system is homogeneous and neglecting the time-dependence of the pair correlation function. Following a rescaling of time and space, leading to a rescaled velocity or Péclet number $\text{Pe} = v_0/\sqrt{D_R D_T} \geq 0$, the resulting equation is (see [4] for details)

$$(1.2) \quad \partial_t f + \text{Pe} \nabla \cdot [f(1 - \rho) \mathbf{e}(\theta)] = D_e \Delta f + \partial_\theta^2 f,$$

where $D_e \in (0, 1]$ is an effective translational diffusion, the unknown $f = f(t, \mathbf{x}, \theta)$ is the phase space density, and $\rho = \rho(t, \mathbf{x})$ is the spatial density obtained through

$$\rho(t, \mathbf{x}) = \int_0^{2\pi} f(t, \mathbf{x}, \theta) d\theta.$$

Both densities are scaled such that they have mass $\phi \in [0, 1]$; this represents the effective occupied fraction of particles in Ω . By integrating (1.2) with respect to θ , we obtain the following equation for ρ :

$$(1.3) \quad \partial_t \rho + \text{Pe} \nabla \cdot [(1 - \rho) \mathbf{p}] = D_e \Delta \rho,$$

where $\mathbf{p}(t, \mathbf{x})$ is the polarization, defined by $\mathbf{p}(t, \mathbf{x}) = \int_0^{2\pi} \mathbf{e}(\theta) f(t, \mathbf{x}, \theta) d\theta$. As we shall see later, the equation for ρ , (1.3), plays an important role in the analysis of (1.2) in order to determine the sign of the nonlocality (in angle) $1 - \rho$, which is the reason why the usual maximum/minimum principle does not work in our case. It is also interesting to consider (1.2) in one spatial dimension, $\Omega \subset \mathbb{R}$. In this case, the set of possible orientations reduces to $\theta = 0, \pi$, or right- and left-moving particles, and the rotational diffusion reduces to discrete jumps between these two orientations. The one-dimensional version of (1.2) reads

$$(1.4) \quad \begin{aligned} \partial_t f_R + \text{Pe} \partial_x [f_R(1 - \rho)] &= \partial_{xx} f_R + f_L - f_R, \\ \partial_t f_L - \text{Pe} \partial_x [f_L(1 - \rho)] &= \partial_{xx} f_L + f_R - f_L, \end{aligned}$$

where f_R and f_L are the densities of right- and left-moving particles, respectively, and the space density is $\rho = f_R + f_L$. This model is derived in [4] from the one-dimensional version of (1.1), and in [14] as the hydrodynamic limit of an active lattice

gas. In fact, the two-dimensional model (1.2) can also be derived starting from an active lattice gas model, with a partial exclusion rule (only one particle is allowed by site, but particles can exchange sites in the diffusive step). This explains why models (1.2) and (1.4) present linear diffusion. If, instead, one considers a complete exclusion rule (resulting in an asymmetric simple exclusion process), nonlinearities of cross-diffusion type (involving terms $\rho \nabla f, f \nabla \rho$) appear in the continuum model [4]. Equation (1.4) coincides with the crowded version (with the additional $1 - \rho$ term in the advection) of the Goldstein–Taylor model [11, 17], and for this reason we refer to it as the *crowded Goldstein–Taylor model*. We shall see that well-posedness for the above system follows directly from that of (1.2), as consequence of our strategy based on a two-step approximation.

To the best of our knowledge, most of the results in the literature focus on time-periodic evolution equations or initial-value problems with Neumann or zero-flux boundary conditions (cf., e.g., [15]), while periodicity in space has not been studied in detail. Moreover, the mobility $f(1 - \rho)$ in the drift term does not yield a maximum principle due to the nonlocal dependence of ρ on f . In particular, it is not immediately clear whether one can use the theory of Ladyzhenskaya, Solonnikov, and Uralceva [15], or that established by Amann [1], based on fixed point theorems. Moreover, we note that the equation does not present a usual Wasserstein gradient flow structure, even in the absence of angular diffusion, again because of the mobility $f(1 - \rho)$ in the transport drift term; cf., for instance, [3]. In this regard, we also point out that this problem does not fall in the theory of generalized Wasserstein gradient flows for equations with nonlinear mobility—studied, for instance, in [9, 7] and the references therein—as the mobility in (1.2) is nonlocal in addition to being, effectively, nonlinear.

For these reasons, we provide a valuable strategy to tackle space-periodic nonlinear evolution PDEs with a nonlocal in angle drift term and periodic boundary conditions by means of a double space-discretization purely using periodicity. More precisely, in order to prove the existence and uniqueness of solutions in the sense of Definition 2.2, we first consider a Galerkin approximation with respect to the angular variable θ . We emphasize that, in what follows, we make the choice to expand in terms of the sine and cosine basis instead of the complex exponentials; this is so that all coefficients are real-valued, which is required to make sense of the positive part in the parabolic system that we study; cf. (3.7) in section 3. Existence of solutions to such a system is proven by means of a further Galerkin approximation in the space variable, again taking into account periodicity of the problem in the choice of the basis.

In this paper we provide the first rigorous result on a macroscopic model for Brownian active particles in phase space, thus starting with the possibly simplest model without cross-diffusion effects. However, it still carries two characteristic issues of these problems, namely the nonlinear drift and the nonlocal effects due to the overall density ρ . Note that here we work in the setting of a continuous angular variable, with discrete angles or directions as in the one-dimensional version of the model we directly see the relation to nonlinear parabolic systems. We stress that this strategy may be extended to the analysis of a larger class of space-periodic evolution equations, even some cross-diffusion systems; see [4]. This will indeed be the object of further investigation. Let us mention at this point the importance of the space-periodic setting also for pattern formation effects. A closed system with no-flux boundary conditions might lead to very different stationary flux behavior and may not lead to the phase separation phenomena observed in many microscopic simulations and discussed in [4].

This article is organized as follows. In section 2 we provide the general set of assumptions and the summary of the main results. Section 3 contains the proof of existence and uniqueness of weak solutions of (1.2) and (1.4). Here we provide a detailed explanation of the strategy used, which can be potentially adapted to a larger class of space-periodic nonlinear evolution problems. More precisely, well-posedness is proven via a two-step Galerkin discretization. This involves solving a semilinear parabolic system outside what is covered by standard parabolic theory, not due to particular complications, but rather the periodic nature of the problem. In section 4 we obtain higher regularity of solutions by applying the Duhamel principle. Section 5 is devoted to enlarging the class of initial data, among them those presenting a Dirac delta in the angular variable. This is relevant to the application of active Brownian particles, as it corresponds to all particles being originally perfectly aligned (same orientation). Finally, the appendices collect some technical results used in this paper.

2. Preliminaries and main results. In what follows, we refer to (1.2) and (1.4) as 2D and 1D problems, respectively, in reference to the dimension of the spatial domain Ω . Note that, mathematically, the 2D problem (1.2) lives in the three-dimensional space $\Upsilon := \Omega \times \mathbb{T}$. We choose the spatial domain Ω to be the d -dimensional torus \mathbb{T}^d , where $d \in \{1, 2\}$ and $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. This imposes periodic boundary conditions as considered in [4, 16]. The 2π -period is chosen by mathematical convenience, so that in the 2D problem $\Upsilon \equiv \mathbb{T}^3$. In what follows, we say that a function f is \mathbb{T}^M -periodic (for $M \in \{1, 2, 3\}$) if, for all $\xi \in \mathbb{R}^M$ and all $j \in \{1, \dots, M\}$, there holds $f(\xi) = f(\xi + 2\pi e_j)$, where $\{e_j\}_{j=1}^M$ are the standard unit vectors of \mathbb{R}^M .

DEFINITION 2.1 (function spaces). *Let $M \in \{1, 2, 3\}$. We define the following Banach spaces:*

$$\begin{aligned} L_{per}^p(\mathbb{T}^M) &:= \{f : \mathbb{R}^M \rightarrow \mathbb{R} \mid f \text{ is a.e. } \mathbb{T}^M\text{-periodic, and } f \in L_{loc}^p(\mathbb{R}^M)\}, \\ H_{per}^m(\mathbb{T}^M) &:= \{f : \mathbb{R}^M \rightarrow \mathbb{R} \mid f \text{ is a.e. } \mathbb{T}^M\text{-periodic, and } f \in H_{loc}^m(\mathbb{R}^M)\}, \end{aligned}$$

with $p \geq 1, m \in \mathbb{N} \cup \{0\}$ endowed with the norms $\|\cdot\|_{L^p((0, 2\pi)^M)}$ and $\|\cdot\|_{H^m((0, 2\pi)^M)}$, respectively. The topological dual of $H_{per}^m(\mathbb{T}^M)$ is denoted by $(H_{per}^m)'(\mathbb{T}^M)$. Similarly, we define the spaces

$$C_{per}^k(\bar{\mathbb{T}}^M) := \{f : \mathbb{R}^M \rightarrow \mathbb{R} \mid f \text{ is } \mathbb{T}^M\text{-periodic, and } f \in C^k(\mathbb{R}^M)\}$$

for $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, equipped with their respective norm $\|\cdot\|_{C^k([0, 2\pi]^M)}$.

For every $f, g : \Upsilon \rightarrow \mathbb{R}$, we use the notation $\langle \cdot, \cdot \rangle_{\Upsilon}$ to denote both the L^2 scalar product in Υ and duality between $(H^1)'$ and H^1 as a Gelfand triple with L^2 , while for any $a, b : \Omega \rightarrow \mathbb{R}$ all such pairings (in L^2 and H^1) are denoted by $\langle a, b \rangle_{\Omega}$. Moreover, let us specify that, by slight abuse of notation, any curve $f : [0, T] \rightarrow B$ will map $t \in [0, T] \mapsto f(t, \cdot) := f(t) \in B$, for any Banach space B we consider.

DEFINITION 2.2 (weak solution to the 2D model). *Let $f_0 \in L_{per}^2(\Upsilon)$ be such that*

$$(2.1) \quad \rho_0(\mathbf{x}) := \int_0^{2\pi} f_0(\mathbf{x}, \theta) d\theta \in [0, 1] \quad \text{for a.e. } \mathbf{x} \in \Omega.$$

A curve f is a weak solution to (1.2) if $f \in C([0, T]; L_{per}^2(\Upsilon)) \cap L^2([0, T]; H_{per}^1(\Upsilon))$, $f' \in L^2([0, T]; (H_{per}^1)'(\Upsilon))$, and for a.e. $t \in [0, T]$ and any $\varphi \in H_{per}^1(\Upsilon)$, there holds

$$(2.2) \quad \begin{aligned} \langle \partial_t f(t), \varphi \rangle_{\Upsilon} &= \langle Pe(1 - \rho(t))f(t)\mathbf{e}(\theta), \nabla \varphi \rangle_{\Upsilon} - \langle D_e \nabla f(t), \nabla \varphi \rangle_{\Upsilon} - \langle \partial_{\theta} f(t), \partial_{\theta} \varphi \rangle_{\Upsilon}, \\ f(0, \mathbf{x}, \theta) &= f_0(\mathbf{x}, \theta), \end{aligned}$$

with periodic boundary conditions on Υ , where $\rho(t, \mathbf{x}) = \int_0^{2\pi} f(t, \mathbf{x}, \theta) d\theta$, and the initial data is achieved in the sense $f(0) = f_0$ in $L^2_{per}(\Upsilon)$.

Our first main result is the following.

THEOREM 2.3 (existence and uniqueness for the 2D model). *Let $T > 0$ and $f_0 \in L^2_{per}(\Upsilon)$ be nonnegative and such that $\rho_0(\mathbf{x}) = \int_0^{2\pi} f_0(\mathbf{x}, \theta) d\theta \in [0, 1]$ for a.e. $\mathbf{x} \in \Omega$. Then, there exists a unique weak solution to (2.2) in the sense of Definition 2.2.*

Remark 2.4. In the proof of Theorem 2.3 we prove that $f \in L^\infty([0, T]; L^2_{per}(\Upsilon)) \cap L^2([0, T]; H^1_{per}(\Upsilon))$. Then, $f \in C([0, T]; L^2_{per}(\Omega))$ is deduced from [18, Chap. 3, sect. 1.4, Lem. 1.2], since $f \in L^2([0, T]; H^1_{per}(\Upsilon))$ and $f' \in L^2([0, T]; (H^1_{per})'(\Upsilon))$.

An analogous theorem is obtained for the 1D model (1.4); see Theorem 3.4 in subsection 3.5. We then consider the regularity of weak solutions. More precisely, we provide the following regularity result up to the initial time.

THEOREM 2.5 (regularity for the 2D model). *Let $f_0 \in L^\infty(\Upsilon)$. Then the unique weak solution of (2.2) provided by Theorem 2.3 satisfies $f \in C((0, T] \times \Upsilon)$. Furthermore, if $f_0 \in C(\bar{\Upsilon})$, the unique weak solution to (2.2) belongs to $C([0, T] \times \bar{\Upsilon})$.*

The previous statement also holds true for the 1D model (1.4); see subsection 4.1.

Finally, we extend the concept of solution in order to allow for initial data with a fixed orientation (see Definition 5.1). Note that initial data of this form are of significantly lower regularity than considered in the original Theorem 2.3; indeed, they are distributional in the angle variable, and hence ρ_0 must be defined by duality. Our existence result for initial data of this kind entails defining *very weak solutions*, and we refer the reader to section 5 for further details; for clarity of presentation, we do not include the definition of very weak solutions here. The main result proved therein is as follows.

THEOREM 2.6 (existence of very weak solutions for 2D model). *Let $T > 0$, f_0 be a nonnegative element of $L^2_{per}(\Omega; (H^1_{per})'(0, 2\pi))$ and $\rho_0(\mathbf{x}) := \langle f_0(\mathbf{x}, \cdot), 1 \rangle \in [0, 1]$ for a.e. $\mathbf{x} \in \Omega$. Then there exists a very weak solution of (1.2) in the sense of Definition 5.1, $f \in L^2([0, T]; H^1_{per}(\Omega; (H^1_{per})'(0, 2\pi))) \cap L^\infty([0, T]; L^2(\Omega; (H^1_{per})'(0, 2\pi))) \cap L^2((0, T) \times \Upsilon)$.*

Note that the result of Theorem 2.6 implies an instantaneous smoothing phenomenon, since the initial data is distributional in the angle variable, while the solution f belongs to $L^2((0, T) \times \Upsilon)$.

3. Well-posedness. In this section we consider the analysis of the 1D and 2D models. We tackle the 2D model first, and show in subsection 3.5 that the result for the 1D model follows from that of the 2D model.

A crucial ingredient for the well-posedness of (1.2) and (1.4) is that $\rho \in [0, 1]$ for all times. Equation (1.3) for the density ρ does not have an obvious maximum or minimum principle, since the drift term neither has a sign nor is zero. One can show that any solution to (1.3) (provided it exists) satisfies $\rho \leq 1$, starting with an initial datum $\rho_0 \in [0, 1]$. However, at this early stage of the analysis, the nonnegativity of the solution is a delicate issue. For this reason we consider a variant of (1.2), where we have $(1 - (\rho)_+)_+$ in the transport term. It will turn out that we are indeed solving the original equation, i.e., (1.2). This final step is shown in Step 5 of the Proof of Theorem 2.3; cf. subsection 3.3. The modified version of (1.2) we start with is

$$(3.1) \quad \begin{aligned} \partial_t f + \text{Pe} \nabla \cdot (f(1 - (\rho)_+)_+ \mathbf{e}(\theta)) &= D_e \Delta f + \partial_\theta^2 f, \\ f(0, \mathbf{x}, \theta) &= f_0(\mathbf{x}, \theta), \end{aligned}$$

in $\Upsilon = \mathbb{T}^3$, where $(a)_+ = \max\{a, 0\}$, for any $a \in \mathbb{R}$. Later, we show that we recover the solution to the original equation (1.2) (see subsection 3.3). We remind the reader that the nonlocality in the drift term means that there is no inherent maximum/minimum principle for (3.1). Our strategy for solving the equation is to construct an approximating sequence $(f^n)_{n \in \mathbb{N}}$ whose limit solves (3.1).

3.1. Approximated system in angle. We introduce the following Galerkin approximation in θ . Given the periodic boundary conditions and the term $\mathbf{e}(\theta)$ in (3.1), it is convenient to consider the basis $\{1, \cos(k(\cdot)), \sin(k(\cdot))\}_{k \in \mathbb{N}}$ of $L^2_{\text{per}}([0, 2\pi])$. Since $f_0(\mathbf{x}, \cdot) \in L^2_{\text{per}}([0, 2\pi])$ for a.e. $\mathbf{x} \in \Omega$, we can write

$$(3.2) \quad f_0(\mathbf{x}, \theta) = \frac{1}{2\pi} a_0(0, \mathbf{x}) + \frac{1}{\pi} \sum_{k \geq 1} (a_k(0, \mathbf{x}) \cos(k\theta) + b_k(0, \mathbf{x}) \sin(k\theta)),$$

where $a_k(0, \cdot), b_k(0, \cdot)$ are the Fourier coefficients of $f_0(\cdot, \theta)$. Note that the requirement $0 \leq \rho_0(\mathbf{x}) \leq 1$ (cf. Definition 2.2) implies that $0 \leq a_0(0, \mathbf{x}) \leq 1$ for a.e. $\mathbf{x} \in \Omega$.

For some fixed $n \in \mathbb{N}$, we consider the finite-dimensional space

$$(3.3) \quad X_n = \text{span}\{\cos(k(\cdot)), \sin(k(\cdot))\}_{k=0, \dots, n}.$$

We try to find a solution f^n to the weak-formulation (2.2) adapted to (3.1) of the approximated system given by

$$(3.4) \quad \langle \partial_t f^n, \varphi \rangle_{\Upsilon} = \langle \text{Pe}(1 - (\rho^n)_+) + f^n(t) \mathbf{e}(\theta), \nabla \varphi \rangle_{\Upsilon} - \langle D_e \nabla f^n, \nabla \varphi \rangle_{\Upsilon} - \langle \partial_\theta f^n, \partial_\theta \varphi \rangle_{\Upsilon},$$

where $\rho^n = \int f^n d\theta$, for test functions $\varphi = \chi(\theta) \psi(\mathbf{x})$ with $\chi \in X_n$ and $\psi \in H^1_{\text{per}}(\Omega)$. We can express f^n as

$$(3.5) \quad f^n(t, \mathbf{x}, \theta) := \frac{1}{2\pi} a_0^n(t, \mathbf{x}) + \frac{1}{\pi} \sum_{k=1}^n (a_k^n(t, \mathbf{x}) \cos(k\theta) + b_k^n(t, \mathbf{x}) \sin(k\theta)),$$

where the Fourier coefficients $a_k^n(t, \mathbf{x}), b_k^n(t, \mathbf{x})$ need to be determined. We note that ρ^n above corresponds to the constant coefficient, that is,

$$(3.6) \quad \rho^n(t, \mathbf{x}) = \int_0^{2\pi} f^n(t, \mathbf{x}, \theta) d\theta \equiv a_0^n(t, \mathbf{x}).$$

From (3.4) we obtain a $(2n+1)$ -dimensional system of semilinear parabolic equations for $a_k^n(t, \mathbf{x}), b_k^n(t, \mathbf{x})$. In particular, testing (3.4) for each $\chi(\theta)$ in the basis of X_n gives the equation for the corresponding Fourier coefficient of f^n . More precisely, choosing $\chi = 1$ results in

$$(3.7a) \quad \partial_t a_0^n + \text{Pe} \nabla \cdot [(1 - (a_0^n)_+) \mathbf{J}_0^n] = D_e \Delta a_0^n, \quad \mathbf{J}_0^n = (a_1^n, b_1^n).$$

Repeating for $\chi = \cos(k\theta), \sin(k\theta)$ for $k = 1, \dots, n$ leads to

$$(3.7b) \quad \partial_t a_k^n + \frac{\text{Pe}}{2} \nabla \cdot [(1 - (a_0^n)_+) \mathbf{J}_k^n] = D_e \Delta a_k^n - k^2 a_k^n,$$

$$(3.7c) \quad \partial_t b_k^n + \frac{\text{Pe}}{2} \nabla \cdot [(1 - (a_0^n)_+) \mathbf{Q}_k^n] = D_e \Delta b_k^n - k^2 b_k^n,$$

in \mathbb{T}^2 , where the velocities \mathbf{J}_k^n and \mathbf{Q}_k^n are given by

$$(3.7d) \quad \begin{aligned} \mathbf{J}_1^n &= (2a_0^n + a_2^n, b_2^n), & \mathbf{Q}_1^n &= (b_2^n, 2a_0^n - a_2^n), \\ \mathbf{J}_l^n &= (a_{l+1}^n + a_{l-1}^n, b_{l+1}^n - b_{l-1}^n), & \mathbf{Q}_l^n &= (b_{l+1}^n + b_{l-1}^n, a_{l-1}^n - a_{l+1}^n), \\ \mathbf{J}_n^n &= (a_{n-1}^n, -b_{n-1}^n), & \mathbf{Q}_n^n &= (b_{n-1}^n, a_{n-1}^n) \end{aligned}$$

for $l = 2, \dots, n-1$, with periodic boundary conditions on Ω and initial conditions $a_k^n(0, \mathbf{x}), b_k^n(0, \mathbf{x})$ given by the corresponding Fourier coefficients of f_0 in (3.2).

Provided that (3.7) can be solved, we obtain an explicit description of each f^n in terms of its Galerkin coefficients a_k^n, b_k^n . In order to prove Theorem 2.3, we need to show the convergence (in a sense to be made precise) of the sequence $(f^n)_{n \in \mathbb{N}}$ towards some limit f , and we must show that f solves (3.1) in the weak sense prescribed by Definition 2.2. As previously mentioned, we will see that we are indeed solving (1.2).

3.2. Existence of solutions to the semilinear parabolic system with spatially periodic boundary conditions. In order to solve the semilinear parabolic system (3.7), we perform a further Galerkin approximation in space, with the aim of rewriting it as a system of ordinary differential equations (ODEs). While the well-posedness for systems similar to (3.7) is considered by Ladyzhenskaya, Solonnikov, and Uralceva [15] and Amann [2], it is not obvious how these results can be applied to space-periodic problems or to systems without a maximum principle (resp., L^∞ -bounds). This motivates our study. We define

$$\mathbf{c}^n(t, \mathbf{x}) = (a_0^n, a_1^n, \dots, a_n^n, b_1^n, \dots, b_n^n).$$

Since throughout this section we keep $n \in \mathbb{N}$ fixed, in order to avoid confusion we drop the superscript n .

THEOREM 3.1 (solution to (3.7)). *Let $\mathbf{c}(0, \mathbf{x}) \in (L_{per}^2(\Omega))^{2n+1}$ be the initial data. For any $T > 0$ there exist curves \mathbf{c} weak solutions to system (3.7) such that*

$$(3.8a) \quad \mathbf{c} \in (L^\infty([0, T]; L_{per}^2(\Omega))^{2n+1} \cap L^2([0, T]; H_{per}^1(\Omega)))^{2n+1},$$

$$(3.8b) \quad \mathbf{c}' \in (L^2([0, T]; (H_{per}^1(\Omega))'(\Omega)))^{2n+1}.$$

With $C_{0,n} = \sum_{k=0}^n (\|a_k(0)\|_{L^2(\Omega)}^2 + \|b_k(0)\|_{L^2(\Omega)}^2)$, there holds

$$(3.9a) \quad \sum_{k=0}^n \left(\|a_k(t)\|_{L^2(\Omega)}^2 + \|b_k(t)\|_{L^2(\Omega)}^2 \right) \leq C_{0,n} \exp \left(2CT \frac{Pe^2}{D_e} \right),$$

$$(3.9b) \quad \sum_{k=0}^n \int_0^T k^2 (\|a_k(t)\|_{L^2(\Omega)}^2 + \|b_k(t)\|_{L^2(\Omega)}^2) + D_e (\|\nabla a_k(t)\|_{L^2(\Omega)}^2 + \|\nabla b_k(t)\|_{L^2(\Omega)}^2) dt \\ \leq C(C_{0,n}, Pe, D_e, T),$$

$$(3.9c) \quad \|a_0'\|_{L^2([0, T]; (H_{per}^1(\Omega))'(\Omega))} \leq \bar{C}(Pe, \phi, T, C_{0,n}).$$

Proof. Step 1 (approximation of the system). We look for an approximate solution \mathbf{c}^m of (3.7) in X_m^2 (3.3),

$$X_m^2 = \text{span}\{\cos(px), \sin(px), \cos(qy), \sin(qy)\}_{p,q=0,\dots,m}.$$

From the $(2n+1)$ -dimensional system of PDEs (3.7), this further approximation leads to a $(2n+1)(2m+1)^2$ -dimensional system of ODEs for the Fourier coefficients of the elements of \mathbf{c}^m . Defining the vector of unknowns by $\Lambda^{n,m}(t)$ and its initial data by $\Lambda_0^{n,m}$ (which are given explicitly in Appendix A), we obtain an ODE of the following form:

$$(3.10) \quad \begin{cases} \frac{d\Lambda^{n,m}}{dt}(t) = F(t, \Lambda^{n,m}(t)) & \forall t \in [0, T], \\ \Lambda^{n,m}(0) = \Lambda_0^{n,m}, \end{cases}$$

where F is locally Lipschitz. Therefore the Cauchy–Lipschitz theorem implies that, for each $m \in \mathbb{N}$, there exists a solution to (3.10) in $C^1([0, T])$.

Step 2 (uniform estimates for the approximating solution). Testing the weak formulations for $\{a_k^{n,m}, b_k^{n,m}\}$ to perform the classical L^2 parabolic estimate, we obtain the following estimates for a.e. $t \in [0, T]$:

$$(3.11) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|a_0^{n,m}\|_{L^2(\Omega)}^2 &= \langle \partial_t a_0^{n,m}, a_0^{n,m} \rangle_\Omega \\ &\leq -\frac{D_e}{2} \|\nabla a_0^{n,m}\|_{L^2(\Omega)}^2 + \frac{\text{Pe}^2}{2D_e} \left(\|a_1^{n,m}\|_{L^2(\Omega)}^2 + \|b_1^{n,m}\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

$$(3.12) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|a_1^{n,m}\|_{L^2(\Omega)}^2 + \|b_1^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &= -D_e \left(\|\nabla a_1^{n,m}\|_{L^2(\Omega)}^2 + \|\nabla b_1^{n,m}\|_{L^2(\Omega)}^2 \right) - \left(\|a_1^{n,m}\|_{L^2(\Omega)}^2 + \|b_1^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\quad + \frac{\text{Pe}}{2} \langle (1 - (a_0^{n,m})_+) \mathbf{J}_1^{n,m}, \nabla a_1^{n,m} \rangle_\Omega + \frac{\text{Pe}}{2} \langle (1 - (a_0^{n,m})_+) \mathbf{Q}_1^{n,m}, \nabla b_1^{n,m} \rangle_\Omega \\ &\leq -\frac{3D_e}{4} \left(\|\nabla a_1^{n,m}\|_{L^2(\Omega)}^2 + \|\nabla b_1^{n,m}\|_{L^2(\Omega)}^2 \right) - \left(\|a_1^{n,m}\|_{L^2(\Omega)}^2 + \|b_1^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\quad + \frac{4\text{Pe}^2}{D_e} \|a_0^{n,m}\|_{L^2}^2 + \frac{\text{Pe}^2}{D_e} \left(\|a_2^{n,m}\|_{L^2(\Omega)}^2 + \|b_2^{n,m}\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

while, for $k \in \{2, \dots, n-1\}$,

$$(3.13) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\leq -\frac{3D_e}{4} \left(\|\nabla a_k^{n,m}\|_{L^2(\Omega)}^2 + \|\nabla b_k^{n,m}\|_{L^2(\Omega)}^2 \right) - k^2 \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\quad + \frac{\text{Pe}^2}{D_e} \left(\|a_{k+1}^{n,m}\|_{L^2(\Omega)}^2 + \|a_{k-1}^{n,m}\|_{L^2(\Omega)}^2 + \|b_{k+1}^{n,m}\|_{L^2(\Omega)}^2 + \|b_{k-1}^{n,m}\|_{L^2(\Omega)}^2 \right) \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|a_n^{n,m}\|_{L^2(\Omega)}^2 + \|b_n^{n,m}\|_{L^2(\Omega)}^2 \right) &\leq -\frac{3D_e}{4} \left(\|\nabla a_n^{n,m}\|_{L^2(\Omega)}^2 + \|\nabla b_n^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\quad - n^2 \left(\|a_n^{n,m}\|_{L^2(\Omega)}^2 + \|b_n^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\quad + \frac{\text{Pe}^2}{2D_e} \left(\|a_{n-1}^{n,m}\|_{L^2(\Omega)}^2 + \|b_{n-1}^{n,m}\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

By summing up eqs. (3.11)–(3.14), we obtain

$$(3.15) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \sum_{k=0}^n \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) + \sum_{k=0}^n k^2 \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\quad + D_e \sum_{k=0}^n \left(\|\nabla a_k^{n,m}\|_{L^2(\Omega)}^2 + \|\nabla b_k^{n,m}\|_{L^2(\Omega)}^2 \right) \\ &\quad \leq C \frac{\text{Pe}^2}{D_e} \sum_{k=0}^n \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) \end{aligned}$$

for some positive constant C independent of n, m , and Pe . Since on the left-hand side all the quantities but the time-derivative are positive, we also have

$$\frac{d}{dt} \sum_{k=0}^n \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) \leq 2C \frac{\text{Pe}^2}{D_e} \sum_{k=0}^n \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right),$$

whence, by Grönwall's inequality, using also (A.2) and the Plancherel theorem,

$$(3.16) \quad \sum_{k=0}^n \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) \leq C_{0,n} \exp \left(2CT \frac{\text{Pe}^2}{D_e} \right),$$

where $C_{0,n} = \sum_{k=0}^n \|a_k(0)\|_{L^2(\Omega)}^2 + \|b_k(0)\|_{L^2(\Omega)}^2$, which is bounded by $\|f_0\|_{L^2(\Upsilon)}^2$. From (3.16) we deduce that $\{a_k^{n,m}, b_k^{n,m}\}_{k=1}^n$ are bounded in $L^\infty([0, T]; L_{\text{per}}^2(\Omega))$ independently of m . As a direct consequence, by integrating (3.15) in time,

$$(3.17) \quad \begin{aligned} & \sum_{k=0}^n k^2 \int_0^T \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) dt \\ & + D_e \sum_{k=0}^n \int_0^T \left(\|\nabla a_k^{n,m}\|_{L^2(\Omega)}^2 + \|\nabla b_k^{n,m}\|_{L^2(\Omega)}^2 \right) dt \\ & \leq C \frac{\text{Pe}^2}{D_e} \sum_{k=0}^n \int_0^T \left(\|a_k^{n,m}\|_{L^2(\Omega)}^2 + \|b_k^{n,m}\|_{L^2(\Omega)}^2 \right) dt + C_{0,n} \\ & \leq CT \frac{\text{Pe}^2}{D_e} C_{0,n} \exp \left(2CT \frac{\text{Pe}^2}{D_e} \right) + C_{0,n} =: C(C_{0,n}, \text{Pe}, D_e, T), \end{aligned}$$

independent of m , and $\{a_k^{n,m}, b_k^{n,m}\}_{k=1}^n$ are uniformly bounded in $L^\infty([0, T]; L_{\text{per}}^2(\Omega)) \cap L^2([0, T]; H_{\text{per}}^1(\Omega))$. Next, we show $\|a_k^{m'}\|_{L^2([0, T]; (H_{\text{per}}^1(\Omega))')}$, $\|b_k^{m'}\|_{L^2([0, T]; (H_{\text{per}}^1(\Omega))')}$ for $0 \leq k \leq n$ are uniformly bounded in m .

Let us consider a test function $\psi \in H_{\text{per}}^1(\Omega)$ such that $\|\psi\|_{H^1(\Omega)} \leq 1$. Since a_0^m is a weak solution to (3.7a) for $k = 0$, we have for a.e. $t \in [0, T]$,

$$(3.18) \quad \begin{aligned} |\langle \partial_t a_0^m, \psi \rangle_\Omega| & \leq D_e |\langle \nabla a_0^m, \nabla \psi \rangle_\Omega| + |\langle \text{Pe}(1 - (a_0^m)_+) + (a_1^m, b_1^m), \nabla \psi \rangle_\Omega| \\ & \leq D_e \|\nabla a_0^m\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} + \text{Pe} \|a_1^m\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \\ & \quad + \text{Pe} \|b_1^m\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \\ & \leq \max \{ \text{Pe}, D_e \} \left(\|a_1^m\|_{L^2(\Omega)} + \|b_1^m\|_{L^2(\Omega)} + \|\nabla a_0^m\|_{L^2(\Omega)} \right). \end{aligned}$$

By taking the supremum over all $\psi \in H_{\text{per}}^1(\Omega)$ such that $\|\psi\|_{H^1(\Omega)} \leq 1$, squaring and integrating in time, we obtain

$$(3.19) \quad \int_0^T \|\partial_t a_0^m\|_{(H_{\text{per}}^1(\Omega))'(\Omega)}^2 dt \leq \bar{C}(\text{Pe}, \phi, T, C_0),$$

since a_1^m, b_1^m are uniformly bounded in $L^\infty([0, T]; L_{\text{per}}^2(\Omega)) \cap L^2([0, T]; H_{\text{per}}^1(\Omega))$, and the boundedness of $\|\nabla a_0^m\|_{L^2([0, T]; L_{\text{per}}^2(\Omega))}$ follows from (3.17). In turn, this gives that the sequence $\|a_0^{m'}\|_{L^2([0, T]; (H_{\text{per}}^1(\Omega))'(\Omega))}$ is uniformly bounded in m . With a similar computation we can prove a uniform bound in m for $\|a_k^{m'}\|_{L^2([0, T]; (H_{\text{per}}^1(\Omega))'(\Omega))}$, $\|b_k^{m'}\|_{L^2([0, T]; (H_{\text{per}}^1(\Omega))'(\Omega))}$, where $1 \leq k \leq n$.

Step 3 (convergence of the approximating solutions). For $0 \leq k \leq n$, the sequence $\{a_k^m, b_k^m\}_m$ is uniformly bounded in m in $L^\infty([0, T]; L_{per}^2(\Omega)) \cap L^2([0, T]; H_{per}^1(\Omega))$. The Banach–Alaoglu theorem yields that there exist subsequences $\{a_k^{m_l}, b_k^{m_l}\}_l$ and curves $a_k, b_k \in L^\infty([0, T]; L_{per}^2(\Omega)) \cap L^2([0, T]; H_{per}^1(\Omega))$ for $0 \leq k \leq n$, such that

$$(3.20a) \quad a_k^{m_l} \overset{*}{\rightharpoonup} a_k, \quad b_k^{m_l} \overset{*}{\rightharpoonup} b_k \quad \text{in } L^\infty([0, T]; L_{per}^2(\Omega)) \text{ as } m_l \rightarrow \infty,$$

$$(3.20b) \quad a_k^{m_l} \rightharpoonup a_k, \quad b_k^{m_l} \rightharpoonup b_k \quad \text{in } L^2([0, T]; H_{per}^1(\Omega)) \text{ as } m_l \rightarrow \infty.$$

The limits coincide since $H^1 \subset L^2 \equiv (L^2)' \subset (H^1)'$ and $\varphi \in L^2([0, T]; L_{per}^2(\Omega))$ is a common test function. Since $\|a_0^{m'}\|, \|a_k^{m'}\|, \|b_k^{m'}\|$ are bounded independently of m , for $1 \leq k \leq n$, in $L^2([0, T]; (H_{per}^1)'(\Omega))$, the Banach–Alaoglu theorem implies

$$(3.21) \quad a_k^{m_l'} \overset{*}{\rightharpoonup} g_{a_k}, \quad b_k^{m_l'} \overset{*}{\rightharpoonup} g_{b_k} \quad \text{in } L^2([0, T]; (H_{per}^1)'(\Omega))$$

for some $g_{a_k}, g_{b_k} \in L^2([0, T]; (H_{per}^1)'(\Omega))$, $0 \leq k \leq n$. By testing against a smooth compactly supported test function defined on $(0, T)$ and using the weak convergence $a_k^{m_l} \rightharpoonup a_k, b_k^{m_l} \rightharpoonup b_k$ in $L^2([0, T]; H_{per}^1(\Omega))$, we get $g_{a_k} = a_k', g_{b_k} = b_k'$. Applying the Aubin–Lions lemma to the sequence $\{a_0^m\}_{m \in \mathbb{N}}$, due to the uniform bound of $\|a_{0,m}'\|_{L^2([0, T]; (H_{per}^1)'(\Omega))}$ from (3.19), a_0^m strongly converges in $L^2([0, T]; L_{per}^2(\Omega))$.

The strong convergence of a_0^m , (3.20), and (3.21) allow us to pass to the limit in the weak form of system (3.7), thus to obtain the existence of weak solutions. We stress that strong convergence for a_0^m is needed in order to allow convergence through the positive part in the drift term. Indeed, the positive part function is continuous and it grows at most linearly, thus $(1 - (a_0^m)_+)_+ \rightarrow (1 - (a_0)_+)_+$ strongly in $L^2([0, T]; L_{per}^2(\Omega))$. As consequence of the lower semicontinuity of the norm, we obtain the regularity (3.8) and the uniform estimates (3.9). \square

3.3. Consistency of the approximating scheme. We now proceed with the proof of Theorem 2.3. We split the proof into two parts: existence and uniqueness. The proof of existence consists of five distinct steps.

Proof of Theorem 2.3—existence. First, let us remind the reader that we consider the variant (3.1) of the original problem, with periodic boundary conditions on Υ . We will see that we recover the solution to the original problem (1.2) in Step 5 of the proof.

Step 1 (approximating solution). As previously outlined, for each $n \in \mathbb{N}$, we consider the approximation of (3.1) given by (3.5) where the functions $\{a_k^n, b_k^n\}_{k=0}^n$ are a solution of the semilinear parabolic equations (3.7). Note that Theorem 3.1 provides existence for the coefficients $\{a_k^n, b_k^n\}_{k=0}^n$ satisfying (3.8) and (3.9). In the next step we will use that, for each $n \in \mathbb{N}$, there holds

$$(3.22a) \quad \sum_{k=0}^n \left(\|a_k^n(t)\|_{L^2(\Omega)}^2 + \|b_k^n(t)\|_{L^2(\Omega)}^2 \right) \leq C_0 \exp \left(2CT \frac{\text{Pe}^2}{D_e} \right),$$

$$(3.22b) \quad \sum_{k=0}^n k^2 \int_0^T \left(\|a_k^n(t)\|_{L^2(\Omega)}^2 + \|b_k^n(t)\|_{L^2(\Omega)}^2 \right) dt + D_e \sum_{k=0}^n \int_0^T \left(\|\nabla a_k^n(t)\|_{L^2(\Omega)}^2 + \|\nabla b_k^n(t)\|_{L^2(\Omega)}^2 \right) dt \leq C(C_0, \text{Pe}, D_e, T),$$

$$(3.22c) \quad \|a_0^{n'}\|_{L^2([0, T]; (H^1)'(\Omega))} \leq \bar{C}(\text{Pe}, D_e, T, C_0),$$

being $C_0 = \sum_{k=0}^{\infty} (\|a_k(0, \cdot)\|_{L^2(\Omega)}^2 + \|b_k(0, \cdot)\|_{L^2(\Omega)}^2) = \|f_0\|_{L^2(\Omega)}^2$ and C a constant independent of n (see eqs. (3.16) and (3.17) in the proof of Theorem 3.1).

Step 2 (uniform estimates for the approximating solution). The uniform bound in $L^\infty([0, T]; L^2_{per}(\Upsilon))$ for the approximation f^n of (3.5) follows directly from (3.22). Indeed, for any $t \in [0, T]$ there holds

$$\|f^n(t)\|_{L^2(\Upsilon)}^2 \leq \frac{1}{\pi} \|a_0^n(t)\|_{L^2(\Omega)}^2 + \frac{4}{\pi} \sum_{k=1}^n \left(\|a_k^n(t)\|_{L^2(\Omega)}^2 + \|b_k^n(t)\|_{L^2(\Omega)}^2 \right) \leq \frac{C_0}{\pi} e^{2CT \frac{Pe^2}{D_e}}.$$

Note that in the penultimate inequality we used $|\sin(k\theta)|, |\cos(k\theta)| \leq 1$ for any $k \in \mathbb{N}$, Fubini theorem, and the orthogonality conditions for the trigonometric functions. Similarly, we obtain a uniform bound for f^n in $L^2([0, T]; H^1_{per}(\Upsilon))$. Indeed, $\int_0^T \|\nabla_{\xi} f^n(t)\|_{L^2(\Upsilon)}^2 dt = \int_0^T \int_{\Upsilon} |\nabla_{\mathbf{x}} f^n(t, \mathbf{x}, \theta)|^2 d\mathbf{x} d\theta dt + \int_0^T \int_{\Upsilon} |\partial_{\theta} f^n(t, \mathbf{x}, \theta)|^2 d\mathbf{x} d\theta dt$, from which it follows that, with $\xi = (\mathbf{x}, \theta)$,

$$\begin{aligned} \int_0^T \|\nabla_{\xi} f^n(t)\|_{L^2(\Upsilon)}^2 dt &\leq \frac{1}{\pi} \int_0^T \|\nabla a_0^n(t)\|_{L^2(\Omega)}^2 dt \\ &\quad + \frac{4}{\pi} \sum_{k=1}^n \int_0^T \left(\|\nabla a_k^n(t)\|_{L^2(\Omega)}^2 + \|\nabla b_k^n(t)\|_{L^2(\Omega)}^2 \right) dt \\ &\quad + \frac{2}{\pi} \sum_{k=1}^n k^2 \int_0^T \left(\|a_k^n(t)\|_{L^2(\Omega)}^2 + \|b_k^n(t)\|_{L^2(\Omega)}^2 \right) dt \\ &\stackrel{(3.22b)}{\leq} C(C_0, Pe, D_e, T), \end{aligned}$$

where this final constant is independent of n .

Finally, we prove that $\|(f^n)'\|_{L^2([0, T]; (H^1_{per})'(\Upsilon))}$ is uniformly bounded. This will be a crucial observation in order to pass to the limit in n in the equation, by applying the Aubin–Lions lemma. Let $\varphi \in H^1_{per}(\Upsilon)$ be a test function such that $\|\varphi\|_{H^1(\Upsilon)} \leq 1$. Since f^n is a weak solution to (3.1), we obtain for a.e. $t \in [0, T]$

$$\begin{aligned} (3.23) \quad |\langle \partial_t f^n(t), \varphi \rangle_{\Upsilon}| &\leq D_e |\langle \nabla f^n(t), \nabla \varphi \rangle_{\Upsilon}| + |\langle \partial_{\theta} f^n(t), \partial_{\theta} \varphi \rangle_{\Upsilon}| \\ &\quad + |\langle Pe(1 - (a_0^n)_+) + f^n(t) \mathbf{e}(\theta), \nabla \varphi \rangle_{\Upsilon}| \\ &\leq D_e \|\nabla f^n(t)\|_{L^2(\Upsilon)} \|\nabla \varphi\|_{L^2(\Upsilon)} + \|\partial_{\theta} f^n(t)\|_{L^2(\Upsilon)} \|\partial_{\theta} \varphi\|_{L^2(\Upsilon)} \\ &\quad + Pe \|f^n(t)\|_{L^2(\Upsilon)} \|\nabla \varphi\|_{L^2(\Upsilon)} \\ &\leq \max \{Pe, D_e\} (\|f^n(t)\|_{L^2(\Upsilon)} + \|\nabla f^n(t)\|_{L^2(\Upsilon)} + \|\partial_{\theta} f^n(t)\|_{L^2(\Upsilon)}). \end{aligned}$$

Taking the supremum over all $\varphi \in H^1_{per}(\Upsilon)$ with $\|\varphi\|_{H^1(\Upsilon)} \leq 1$, $\|\partial_t f^n(t)\|_{(H^1_{per})'(\Upsilon)} \leq C(Pe, D_e) (\|f^n(t)\|_{L^2(\Upsilon)} + \|\nabla f^n(t)\|_{L^2(\Upsilon)} + \|\partial_{\theta} f^n(t)\|_{L^2(\Upsilon)})$, whence

$$(3.24) \quad \int_0^T \|\partial_t f^n(t)\|_{(H^1_{per})'(\Upsilon)}^2 dt \leq \tilde{C}(Pe, D_e) \int_0^T \|f^n(t)\|_{H^1(\Upsilon)}^2 dt \leq \bar{C}(Pe, \phi, T, C_0),$$

due to the uniform bound on $\|f^n\|_{L^2([0, T]; H^1_{per}(\Upsilon))}$. Therefore, we have the desired bound for $(f^n)'$ in $L^2([0, T]; (H^1_{per})'(\Upsilon))$.

Step 3 (convergence of the approximation). Since the sequence $\{f^n\}_{n \in \mathbb{N}}$ of (3.5) is uniformly bounded in $L^\infty([0, T]; L^2_{per}(\Upsilon)) \cap L^2([0, T]; H^1_{per}(\Upsilon))$, we infer from the

Banach–Alaoglu theorem that there exist a subsequence $\{f^{n_k}\}_{k \in \mathbb{N}}$ and a curve $f \in L^\infty([0, T]; L^2_{per}(\Upsilon)) \cap L^2([0, T]; H^1_{per}(\Upsilon))$ such that

$$(3.25) \quad f^{n_k} \xrightarrow{*} f \quad \text{in } L^\infty([0, T]; L^2_{per}(\Upsilon)), \quad f^{n_k} \rightharpoonup f \quad \text{in } L^2([0, T]; H^1_{per}(\Upsilon)).$$

The limits coincide since $H^1 \subset L^2 \equiv (L^2)' \subset (H^1)'$ and any $\varphi \in L^2([0, T]; L^2_{per}(\Upsilon))$ is a common test function. The uniform boundedness of $\|(f^n)'\|_{L^2([0, T]; (H^1_{per})'(\Upsilon))}$ and the Banach–Alaoglu theorem also yields that

$$(3.26) \quad (f^{n_k})' \xrightarrow{*} f' \quad \text{in } L^2([0, T]; (H^1_{per})'(\Upsilon)),$$

where we verify that this latter limit is indeed f' by testing against a smooth compactly supported test function defined on $(0, T)$ and using the weak convergence $f^{n_k} \rightharpoonup f$ in $L^2([0, T]; H^1_{per}(\Upsilon))$. Henceforth, we do not relabel further subsequences.

Additionally, the aforementioned boundedness of $\|(f^n)'\|_{L^2([0, T]; (H^1_{per})'(\Upsilon))}$ yields, taking a further subsequence if necessary, by application of the Aubin–Lions lemma that $f^n \rightarrow f$ strongly in $L^2([0, T]; L^2_{per}(\Upsilon))$ as $n \rightarrow \infty$. We take a further subsequence such that, for a.e. $t \in [0, T]$, $f^n(t) \rightarrow f(t)$ strongly in $L^2_{per}(\Upsilon)$ as $n \rightarrow \infty$. Moreover, by further applying the Banach–Alaoglu theorem and the Aubin–Lions lemma to the sequence of curves $\{a_0^n\}_{n \in \mathbb{N}}$, due to the uniform bounds from (3.22), we also know that a_0^n strongly converges in $L^2([0, T]; L^2_{per}(\Omega))$. In other words, up to a subsequence, $\{\rho^n\}_{n \in \mathbb{N}}$ converges strongly in $L^2([0, T]; L^2_{per}(\Omega))$. Moreover, defining $\rho := \int_0^{2\pi} f \, d\theta$, where f is the aforementioned strong limit in $L^2([0, T]; L^2_{per}(\Omega))$, we have, for the convergent subsequence in question,

$$\begin{aligned} \|\rho - \rho^n\|_{L^2([0, T]; L^2(\Omega))}^2 &= \int_0^T \int_\Omega \left| \int_0^{2\pi} (f(t, \mathbf{x}, \theta) - f^n(t, \mathbf{x}, \theta)) \, d\theta \right|^2 \, dx \, dt \\ &\leq 2\pi \|f - f^n\|_{L^2([0, T]; L^2(\Upsilon))}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where we applied Jensen’s inequality, thus $\rho^n \rightarrow \rho$ strongly in $L^2([0, T]; L^2_{per}(\Omega))$.

This allows us to pass to the limit in the weak form of (3.1), thus to obtain that f is a weak solution to the revised (3.1). We stress that strong convergence for a_0^n , i.e., $\{\rho^n\}_{n \in \mathbb{N}}$, is needed in order to allow convergence through the positive part in the drift term. Indeed, the positive part function is continuous and it grows at most linearly, thus $(1 - (\rho^n)_+)_+ \rightarrow (1 - (\rho)_+)_+$ strongly in $L^2([0, T]; L^2_{per}(\Omega))$. More precisely, for a function $\psi \in C^1([0, T])$ such that $\psi(0) = \psi(T) = 0$, there holds (using, e.g., [10, App. E, Thm. 8]) $\int_0^T \langle \partial_t f^n(t), \varphi \rangle_\Upsilon \psi(t) \, dt = - \int_0^T \langle f^n(t), \varphi \rangle_\Upsilon \psi'(t) \, dt$, and hence

$$\begin{aligned} \int_0^T \langle f^n(t), \varphi \rangle_\Upsilon \psi'(t) \, dt &= - \int_0^T \langle \text{Pe}(1 - (a_0^n(t))_+) + f^n(t) \mathbf{e}(\theta), \nabla \varphi \rangle_\Upsilon \psi(t) \, dt \\ &\quad + D_e \int_0^T \langle \nabla f^n(t), \nabla \varphi \rangle_\Upsilon \psi(t) \, dt + \int_0^T \langle \partial_\theta f^n(t), \partial_\theta \varphi \rangle_\Upsilon \psi(t) \, dt. \end{aligned}$$

Using (3.25) and (3.26), the aforementioned strong convergence $(1 - (\rho^n)_+)_+ \rightarrow (1 - (\rho)_+)_+$ in $L^2(0, T; L^2_{per}(\Omega))$, we pass to the limit $n \rightarrow +\infty$ in the above and obtain, after another integration by parts in t ,

$$\begin{aligned} \int_0^T \langle \partial_t f(t), \varphi \rangle_\Upsilon \psi(t) \, dt &= \int_0^T \langle \text{Pe}(1 - (a_0(t))_+) + f(t) \mathbf{e}(\theta), \nabla \varphi \rangle_\Upsilon \psi(t) \, dt \\ &\quad - D_e \int_0^T \langle \nabla f(t), \nabla \varphi \rangle_\Upsilon \psi(t) \, dt - \int_0^T \langle \partial_\theta f(t), \partial_\theta \varphi \rangle_\Upsilon \psi(t) \, dt. \end{aligned}$$

Hence, since ψ was arbitrary, $f \in L^2([0, T]; H_{per}^1(\Upsilon))$ with $f' \in L^2([0, T]; (H_{per}^1(\Upsilon))')$ solves (3.1) weakly (i.e., duality with $H_{per}^1(\Upsilon)$ for a.e. $t \in [0, T]$).

Step 4 (initial data). Recall that f_0 has Fourier expansion (3.2) and $f^n(0, \cdot, \cdot) \rightarrow f_0$ strongly in $L_{per}^2(\Upsilon)$. Moreover, by extracting a subsequence after applying the Aubin–Lions lemma under (3.26), $f^n(t) \rightarrow f(t)$ strongly in $L_{per}^2(\Upsilon)$ for a.e. $t \in [0, T]$. By Remark 2.4, $f^n, f \in C([0, T]; L_{per}^2(\Upsilon))$. Hence, $t = 0$ is a Lebesgue point, and $f^n(0) \rightarrow f(0)$ strongly in $L_{per}^2(\Upsilon)$. By uniqueness of limits, $f(0) = f_0$ in $L_{per}^2(\Upsilon)$.

Step 5 (solution to the original equation and nonnegativity). As a direct consequence of the previous step, by integrating (3.1) in the angle variable, we infer that $\rho(t, \mathbf{x}) = \int_0^{2\pi} f(\mathbf{x}, t, \theta) d\theta$ is a curve belonging to $L^\infty([0, T]; L_{per}^2(\Omega)) \cap L^2([0, T]; H_{per}^1(\Omega))$, and is a weak solution of

$$(3.27) \quad \partial_t \rho + \text{Pe} \nabla \cdot ((1 - (\rho)_+)_{+} \mathbf{p}) = D_e \Delta \rho.$$

Moreover, by the weak-* lower semicontinuity of the norm, $\rho' \in L^2([0, T]; (H_{per}^1(\Omega))')$ as it is the weak-* limit curve of the sequence $\{a_0^{n'}\}_{n \in \mathbb{N}}$. Begin by noting that $(1 - (\rho)_+)_{-} = -(1 - \rho \mathbb{1}_{\rho > 0}) \mathbb{1}_{\rho > 1}$, which implies that, in the sense of distributions, $\partial_t (1 - (\rho)_+)_{-} = \partial_t \rho \mathbb{1}_{\rho > 1}$ and $\nabla (1 - (\rho)_+)_{-} = \nabla \rho \mathbb{1}_{\rho > 1}$. Testing (3.27) with $(1 - (\rho)_+)_{-}$,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (1 - (\rho)_+)_{-}^2 d\mathbf{x} = \text{Pe} \int_{\Omega} \nabla (1 - (\rho)_+)_{-} \cdot (1 - (\rho)_+)_{+} \mathbf{p} d\mathbf{x} - D_e \int_{\Omega} |\nabla \rho|^2 \mathbb{1}_{\rho > 1} d\mathbf{x}.$$

Observe that the first term on the right-hand side is null, since the supports of the two terms in the integrand are disjoint, while the final term is nonpositive. We thereby deduce $\frac{1}{2} \int_{\Omega} (1 - (\rho(t))_{+})_{-}^2 d\mathbf{x} \leq 0$, since we initially have $(1 - \rho_0(\mathbf{x}))_{-} = 0$ for a.e. $x \in \Omega$. As a result, $\rho(t, \mathbf{x}) \leq 1$ for a.e. $(t, \mathbf{x}) \in (0, T) \times \Omega$, as required.

Returning to (3.1) and testing with $(f)_{-}$, we get, after integrating by parts,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Upsilon} (f)_{-}^2 d\boldsymbol{\xi} &= \text{Pe} \int_{\Upsilon} \nabla (f)_{-} \cdot (f)_{-} (1 - (\rho)_+)_{+} \mathbf{e}(\theta) d\boldsymbol{\xi} - D_e \int_{\Upsilon} |\nabla (f)_{-}|^2 d\boldsymbol{\xi} - \int_{\Upsilon} |\partial_{\theta} (f)_{-}|^2 d\boldsymbol{\xi} \\ &\leq C(\text{Pe}, D_e) \int_{\Upsilon} \frac{1}{2} (f)_{-}^2 d\boldsymbol{\xi} - \frac{1}{2} D_e \int_{\Upsilon} |\nabla (f)_{-}|^2 d\boldsymbol{\xi} - \int_{\Upsilon} |\partial_{\theta} (f)_{-}|^2 d\boldsymbol{\xi}, \end{aligned}$$

where $\boldsymbol{\xi} = (\mathbf{x}, \theta)$. Dropping the two nonpositive terms, Grönwall's lemma implies

$$\int_{\Upsilon} (f(t, \mathbf{x}, \theta))_{-}^2 d\boldsymbol{\xi} \leq \exp(Ct) \int_{\Upsilon} (f_0(\mathbf{x}, \theta))_{-}^2 d\boldsymbol{\xi} = 0 \quad \text{a.e. } t \in (0, T),$$

since f_0 is nonnegative a.e. on Υ . Hence, $f(t, \mathbf{x}, \theta) \geq 0$ for a.e. $(t, \mathbf{x}, \theta) \in \Upsilon \times (0, T)$ and thus the space density $\rho(t, \mathbf{x}) \in [0, 1]$ for a.e. $(t, \mathbf{x}) \in \Upsilon \times (0, T)$. Hence the limiting curve f is a weak solution in the sense of Definition 2.2 to the original equation (1.2). We emphasize that $f \in C([0, T]; L_{per}^2(\Upsilon))$ is deduced from $f \in L^2([0, T]; H_{per}^1(\Upsilon))$ and $f' \in L^2([0, T]; (H_{per}^1(\Upsilon))')$; cf. Remark 2.4. \square

3.4. Uniqueness of solutions.

LEMMA 3.2. *Given $f_0 \in L_{per}^2(\Upsilon)$ nonnegative such that its corresponding ρ_0 satisfies $\rho_0(\mathbf{x}) = \int_0^{2\pi} f_0(\mathbf{x}, \theta) d\theta \in [0, 1]$ for a.e. $\mathbf{x} \in \Omega$, there exists at most one periodic weak solution, in the sense of Definition 2.2, of the problem (2.2).*

Proof. Let us consider two weak solutions to (1.2), f_1, f_2 , in the sense of Definition 2.2, and set $\tilde{f} := f_1 - f_2$ to be their difference. We refer to ρ_1, ρ_2 as the corresponding space densities, and to $\tilde{\rho} := \rho_1 - \rho_2$ as their difference. By means of a

direct computation, \bar{f} is a weak solution to

$$\partial_t \bar{f} + \text{Pe} \nabla \cdot ((1 - \rho_1) \bar{f} \mathbf{e}(\theta) - \bar{\rho} f_2 \mathbf{e}(\theta)) = D_e \Delta \bar{f} + \partial_\theta^2 \bar{f},$$

and $\bar{\rho}$ solves $\partial_t \bar{\rho} + \text{Pe} \nabla \cdot ((1 - \rho_1) \bar{\mathbf{p}} - \bar{\rho} \mathbf{p}_2) = D_e \Delta \bar{\rho}$, where we remind the reader $\bar{\mathbf{p}}(t, \mathbf{x}) = \int_0^{2\pi} \bar{f}(t, \mathbf{x}, \theta) d\theta$ and $\mathbf{p}_2(t, \mathbf{x}) = \int_0^{2\pi} f_2(t, \mathbf{x}, \theta) d\theta$, for any $(t, \mathbf{x}, \theta) \in [0, T] \times \Upsilon$. Note that, by directly integrating the previous equation for $\bar{\rho}$, $\int_\Omega \bar{\rho}(\mathbf{x}) d\mathbf{x} = 0$ and thus $\int_\Omega \int_0^{2\pi} \bar{f}(t, \mathbf{x}, \theta) d\theta d\mathbf{x} = 0$. Motivated by this latter “mean-zero condition”, we consider the space

$$H_z^1(\Upsilon) := \left\{ f \in H^1(\Upsilon) : \int_\Upsilon f(\xi) d\xi = 0 \right\},$$

and its dual $(H_z^1)'(\Upsilon)$, where subscript z refers to the mean-zero property in question. Let $L := (-\Delta)^{-1} : (H_z^1)'(\Upsilon) \rightarrow (H_z^1)(\Upsilon)$ be the inverse of the solution operator for the Poisson equation with periodic boundary condition, that is, $Lg = u$, where $\int_\Upsilon \nabla u \cdot \nabla \varphi = \langle g, \varphi \rangle$ for all $\varphi \in H_z^1(\Upsilon)$. In the following we shall use that L maps continuously $L_z^2(\Upsilon)$ into $H_z^2(\Upsilon)$. Moreover, we note the self-adjointness property

$$\langle \varphi, L\varphi \rangle = \langle (-\Delta) \circ (-\Delta)^{-1} \varphi, L\varphi \rangle = \|\nabla L\varphi\|_{L^2(\Upsilon)}^2 = \langle L\varphi, \varphi \rangle$$

for every $\varphi \in H_z^1(\Upsilon)$, where the duality product is understood in the sense of $L^2(\Upsilon)$. Similarly, using the commutativity of the partial derivatives, for any $\varphi \in H_z^1(\Upsilon)$,

$$\langle \partial_\theta \varphi, \partial_\theta L\varphi \rangle = \langle (-\Delta) \partial_\theta L\varphi, \partial_\theta L\varphi \rangle = \|\nabla \partial_\theta L\varphi\|_{L^2(\Upsilon)}^2.$$

Let $\lambda > 0$. For the next lines of computation, we use Definition 2.2, Young’s inequality, the bound $\rho_1 \leq 1$, and the self-adjointness property. We drop the time dependence for ease of presentation. We obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\nabla L\bar{f}\|_{L^2(\Upsilon)}^2 + \lambda \|\bar{\rho}\|_{L^2(\Omega)}^2 \right) &= \langle \partial_t \bar{f}, L\bar{f} \rangle + \lambda \langle \partial_t \bar{\rho}, \bar{\rho} \rangle \\ &= -D_e \|\bar{f}\|_{L^2(\Upsilon)}^2 - \|\nabla \partial_\theta L\bar{f}\|_{L^2(\Upsilon)}^2 - \lambda D_e \|\nabla \bar{\rho}\|_{L^2}^2 \\ &\quad + \text{Pe} \langle (1 - \rho_1) \bar{f} \mathbf{e}(\theta), \nabla L\bar{f} \rangle - \text{Pe} \lambda \langle \bar{\rho} \mathbf{p}_2, \nabla \bar{\rho} \rangle \\ &\quad + \text{Pe} \lambda \langle (1 - \rho_1) \bar{\mathbf{p}}, \nabla \bar{\rho} \rangle - \text{Pe} \langle \bar{\rho} f_2 \mathbf{e}(\theta), \nabla L\bar{f} \rangle \\ (3.28) \quad &\leq -D_e \left(1 - \frac{1}{2\varepsilon} - \frac{\lambda}{\sigma} \pi \right) \|\bar{f}\|_{L^2(\Upsilon)}^2 \\ &\quad - \lambda \left(D_e - \frac{\text{Pe}^2 \sigma}{D_e} \right) \|\nabla \bar{\rho}\|_{L^2(\Omega)}^2 + \frac{\lambda D_e}{2\sigma} \|\bar{\rho}\|_{L^2(\Omega)}^2 \\ &\quad + \frac{\text{Pe}^2 \varepsilon}{2D_e} \|\nabla L\bar{f}\|_{L^2(\Upsilon)}^2 - \text{Pe} \langle \bar{\rho} f_2 \mathbf{e}(\theta), \nabla L\bar{f} \rangle, \end{aligned}$$

where we also used $\|\bar{\mathbf{p}}\|_{L^2(\Omega)}^2 \leq 2\pi \|\bar{f}\|_{L^2(\Upsilon)}^2$, and noticed that for any $(t, \mathbf{x}) \in \Omega \times [0, T]$, $|\mathbf{p}_2(t, \mathbf{x})| \leq \int_0^{2\pi} f_2(t, \mathbf{x}, \theta) d\theta = \rho_2(t, \mathbf{x}) \leq 1$, which implies $\|\bar{\rho} \mathbf{p}_2\|_{L^2(\Omega)}^2 \leq \|\bar{\rho}\|_{L^2(\Omega)}^2$. In order to apply Grönwall’s inequality and conclude the argument, we must estimate the final term in (3.28). Since the operator L maps continuously $L_z^2(\Upsilon)$ into $H_z^2(\Upsilon)$, we have $L\bar{f} \in H_z^2(\Upsilon)$, thus $\nabla L\bar{f} \in H_z^1(\Upsilon)$. By means of the Sobolev embedding $H^1(\Upsilon) \hookrightarrow L^{p_*}(\Upsilon)$ for $p_* = 6$, and that $|\Upsilon| < +\infty$, we infer that $\nabla L\bar{f} \in L^{p'}(\Upsilon)$ for any $p' \leq p_* = 6$, and $f_2 \in L^q(\Upsilon)$ for any $q \leq p_* = 6$. In particular, for $\frac{6}{5} \leq p \leq \frac{3}{2}$ and

$q = \frac{2p}{2-p} \leq 6$, using the Cauchy–Young inequality to get from the second line to the third, and the Gagliardo–Nirenberg inequality to get the final line, we see that the term $\text{Pe} |\langle \bar{\rho} f_2 \mathbf{e}(\theta), \nabla L \bar{f} \rangle|$ is bounded by

$$(3.29) \quad \begin{aligned} \text{Pe} \|\bar{\rho} f_2\|_{L^p(\Upsilon)} \|\nabla L \bar{f}\|_{L^{p'}(\Upsilon)} &\leq C(\text{Pe}, |\Upsilon|, p', p_*) \|\bar{f}\|_{L^2(\Upsilon)} \|\bar{\rho} f_2\|_{L^p(\Upsilon)} \\ &\leq \frac{\lambda D_e}{2} \|\bar{f}\|_{L^2(\Upsilon)}^2 + \frac{\tilde{c}}{\lambda D_e} \|f_2\|_{H^1(\Upsilon)}^2 \|\bar{\rho}\|_{L^2(\Omega)}^2, \end{aligned}$$

where the constants λ, \tilde{c} depend only on Υ, p, q via the embeddings. Using (3.29) in (3.28), we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\nabla L \bar{f}\|_{L^2(\Upsilon)}^2 + \lambda \|\bar{\rho}\|_{L^2(\Omega)}^2 \right) \\ &\leq -D_e \left(1 - \frac{\lambda}{2} - \frac{1}{2\varepsilon} - \frac{\lambda}{\sigma} \pi \right) \|\bar{f}\|_{L^2(\Upsilon)}^2 - \lambda \left(D_e - \frac{\text{Pe}^2 \sigma}{D_e} \right) \|\nabla \bar{\rho}\|_{L^2(\Omega)}^2 \\ &\quad + \frac{\text{Pe}^2 \varepsilon}{2D_e} \|\nabla L \bar{f}\|_{L^2(\Upsilon)}^2 + \frac{\lambda D_e}{2\sigma} \|\bar{\rho}\|_{L^2(\Omega)}^2 + \frac{\tilde{c}}{\lambda D_e} \|f_2\|_{H^1(\Upsilon)}^2 \|\bar{\rho}\|_{L^2(\Omega)}^2 \\ &\leq \frac{M(f_2)}{\lambda^2 \sigma} \left(\|\nabla L \bar{f}\|_{L^2(\Upsilon)}^2 + \lambda \|\bar{\rho}\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

where we set $\sigma = \frac{D_e^2}{\text{Pe}^2}$, $\lambda = (1 + \frac{\text{Pe}^2}{D_e^2} \pi)^{-1}$, $\varepsilon = \frac{1}{\lambda}$, and $M(f_2) := \max\{\frac{\text{Pe}^2 \lambda \sigma}{2D_e}, \frac{\lambda^2 D_e}{2} + \frac{\tilde{c} \sigma}{D_e} \|f_2\|_{H^1(\Upsilon)}^2\}$. Finally, since $f_2 \in L^2([0, T]; H_{\text{per}}^1(\Upsilon))$, by means of Grönwall's inequality we obtain uniqueness, as for any $t \in [0, T]$,

$$\|\nabla L \bar{f}(t)\|_{L^2(\Upsilon)}^2 + \lambda \|\bar{\rho}(t)\|_{L^2(\Omega)}^2 \leq \underbrace{\left(\|\nabla L \bar{f}(0)\|_{L^2(\Upsilon)}^2 + \lambda \|\bar{\rho}(0)\|_{L^2(\Omega)}^2 \right)}_{=0} e^{2 \int_0^t \frac{M(f_2(s))}{\lambda^2 \sigma} ds}. \quad \square$$

3.5. Existence and uniqueness of solutions to the 1D Model. The analysis carried out previously for (1.2) can be applied to obtain well-posedness for the 1D model (1.4). Therefore, we will not provide proofs but only point out the main differences.

Let $f_L(x, t)$ and $f_R(x, t)$ be the densities of the left- and right-moving particles, respectively. The (rescaled) one-dimensional version of (1.2) is

$$(3.30) \quad \begin{cases} \partial_t f_R + \text{Pe} \partial_x (f_R(1 - \rho)) = \partial_{xx} f_R + f_L - f_R, \\ \partial_t f_L - \text{Pe} \partial_x (f_L(1 - \rho)) = \partial_{xx} f_L + f_R - f_L. \end{cases}$$

The *space density* and the *polarization* are, respectively,

$$\rho := f_R + f_L, \quad p := f_R - f_L.$$

DEFINITION 3.3 (weak solution to the 1D model). *Let $f_{R,0}, f_{L,0} \in L_{\text{per}}^2([0, 2\pi])$ be nonnegative functions such that*

$$\rho_0(x) = f_{R,0}(x) + f_{L,0}(x) \in [0, 1] \quad \text{for a.e. } x \in [0, 2\pi].$$

A weak solution to system (3.30) is a pair of curves (f_R, f_L) such that $f_R, f_L \in L^2([0, T]; H_{\text{per}}^1([0, 2\pi]))$ and $f'_R, f'_L \in L^2([0, T]; (H_{\text{per}}^1)'([0, 2\pi]))$, and for a.e. $t \in [0, T]$

and any $\varphi, \phi \in H_{per}^1([0, 2\pi])$, it holds that

$$\begin{aligned} \langle \partial_t f_R(t), \varphi \rangle_\Omega &= Pe \langle f_R(t)(1 - \rho(t)), \partial_x \varphi \rangle_\Omega - \langle \partial_x f_R(t), \partial_x \varphi \rangle_\Omega + \langle f_L(t) - f_R(t), \varphi \rangle_\Omega, \\ \langle \partial_t f_L(t), \phi \rangle_\Omega &= -Pe \langle f_L(t)(1 - \rho(t)), \partial_x \phi \rangle_\Omega - \langle \partial_x f_L(t), \partial_x \phi \rangle_\Omega + \langle f_R(t) - f_L(t), \phi \rangle_\Omega, \\ (f_R, f_L)(x, 0) &= (f_{R,0}, f_{L,0}), \end{aligned}$$

with periodic boundary conditions on Ω , where $\rho(t, x) = f_R(t, x) + f_L(t, x)$, and the initial data is achieved in the sense $f_R(0) = f_{R,0}$ and $f_L(0) = f_{L,0}$ in $L_{per}^2([0, 2\pi])$.

Notice that adding/subtracting the equations in system (3.30) yields

$$(3.31) \quad \begin{cases} \partial_t \rho + Pe \partial_x (p(1 - \rho)) = \partial_{xx} \rho, \\ \partial_t p + Pe \partial_x (\rho(1 - \rho)) = \partial_{xx} p - 2p. \end{cases}$$

As already mentioned, existence and uniqueness of solutions to (3.30) can be proved with a strategy similar to that used for (1.2). The main difference is the absence of the angular variable, which simplifies the discretization of the system; indeed the angular variable is already discretized, as there are only two possible directions (left and right). It therefore suffices to write a Galerkin approximation only in space, as in subsection 3.2. Moreover, the solution (f_R, f_L) directly inherits the bound of the density ρ (unlike in the higher-dimensional case). This simplifies the uniqueness proof, since we need not employ the operator $(-\Delta)^{-1}$. The result is stated below.

THEOREM 3.4 (well-posedness for 1D model). *Let $(f_{R,0}, f_{L,0}) \in (L_{per}^2([0, 2\pi]))^2$ be a pair of nonnegative functions such that*

$$\rho_0(x) = f_{R,0}(x) + f_{L,0}(x) \in [0, 1] \quad \text{for a.e. } x \in [0, 2\pi].$$

For any $T > 0$, there exists a unique weak solution to (3.30) in the sense of Definition 3.3. Moreover, $f_R(t, x), f_L(t, x) \in [0, 1]$ for a.e. $t \in [0, T]$ and $x \in [0, 2\pi]$.

Remark 3.5. As per Remark 2.4, we deduce $f_R, f_L \in C([0, T]; L_{per}^2([0, 2\pi]))$.

4. Regularity. Having established well-posedness in section 3, we turn to the issue of regularity. More precisely, we provide here the proof of Theorem 2.5, which is performed in Lemmas 4.8 and 4.9. For ease of presentation, we start with the one-dimensional model.

4.1. Regularity for the one-dimensional model. Our strategy is to apply the Duhamel principle so as to treat (3.30) as a forced one-dimensional heat equation. Inspired by [6, Chap. 9], we provide the following definition.

DEFINITION 4.1. Define $\Phi : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ by the explicit formula

$$(4.1) \quad \Phi(t, x) := \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \cos nx \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}.$$

Note that the above may be rewritten in terms of the complex exponential basis as $\Phi(t, x) = \frac{1}{2\pi} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{-n^2 t}}{2\pi} e^{inx}$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$. In what follows it will be useful to rewrite Φ in terms of the Jacobi theta-3 function (denoted by θ_3), and in terms of the Jacobi theta function (denoted by ϑ , also sometimes by ϑ_{00}),

$$(4.2) \quad \Phi(t, x) = \frac{1}{2\pi} \theta_3\left(\frac{x}{2}, e^{-t}\right) = \frac{1}{2\pi} \vartheta\left(\frac{x}{2\pi}, \frac{it}{\pi}\right) \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}.$$

LEMMA 4.2. *The function Φ of Definition 4.1 is a smooth 2π -periodic function that satisfies $\partial_t \Phi - \partial_{xx} \Phi = 0$ as a pointwise equality in $(0, \infty) \times \mathbb{R}$, is nonnegative in $(0, \infty) \times \mathbb{R}$, and $\|\Phi(t, \cdot)\|_{L^1([0, 2\pi])} = 1$ for every $t \in (0, \infty)$. Additionally,*

$$(4.3) \quad \int_0^t \|\Phi(\tau, \cdot)\|_{L^2([0, 2\pi])}^2 d\tau = t + \sum_{n=1}^{\infty} \frac{1}{2n^2} (1 - e^{-2n^2 t}) \quad \forall t > 0,$$

and $\int_0^t \int_0^{2\pi} \Phi(t-s, x-y)^2 dy ds = \int_0^t \|\Phi(\tau, \cdot)\|_{L^2([0, 2\pi])}^2 d\tau$ for all $(t, x) \in (0, T] \times [0, 2\pi]$. Finally, for any 2π -periodic C^2 function ψ , $\lim_{t \rightarrow 0^+} \int_0^{2\pi} \Phi(t, x) \psi(x) dx = \psi(0)$.

The proof of Lemma 4.2 is in Appendix B. We use the function Φ as a periodic heat kernel, in the sense made precise by the following result. The initial data ψ is chosen in $L^1_{loc}(\mathbb{R})$ to justify the expansion in terms of the sine/cosine basis.

LEMMA 4.3. *Let $\psi \in L^1_{per}([0, 2\pi])$. Then, $\Psi(t, x) := \int_0^{2\pi} \Phi(t, x-y) \psi(y) dy$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$ is a smooth 2π -periodic function defined on the full-space $(0, \infty) \times \mathbb{R}$, which solves the Cauchy problem*

$$(4.4) \quad \begin{cases} \partial_t \Psi - \partial_{xx} \Psi = 0 & \text{in } (0, \infty) \times \mathbb{R}, \\ \Psi|_{t=0} = \psi & \text{on } \mathbb{R}, \end{cases}$$

where the initial data is achieved in L^1_{loc} , i.e., $\lim_{t \rightarrow 0^+} \|\Psi(t, \cdot) - \psi\|_{L^1([0, 2\pi])} = 0$.

The proof of this result is in Appendix B. An alternative way of generating periodic solutions of the heat equation is to convolve the classical one-dimensional heat kernel with 2π -periodic test functions. However, when performing integration by parts in the Duhamel formula (cf. (4.5) in the lemma below) the boundary terms do not vanish unless both the kernel and the “test function” are periodic. Hence, the kernel of Definition 4.1 is better suited to our analysis.

LEMMA 4.4 (regularity away from initial time). *Suppose that the periodic initial data for problem (3.30) is such that $f_{R,0}, f_{L,0} \in L^\infty(\Omega)$. Then, the unique weak solutions f_R and f_L of (3.30) belong to $C((0, T] \times \bar{\Omega})$.*

Proof. Recall from Theorem 3.4 that we have $f_R, f_L, \rho \in L^\infty([0, T] \times \Omega)$.

The key observation is that (3.30) can be interpreted as heat equations with a source on the right-hand side. Since we already derived the fundamental solution of the periodic heat equation in Lemmas 4.2 and 4.3, and we have already shown uniqueness in the relevant space of solutions to (3.30), we are in a position to apply the Duhamel principle and obtain implicit integral relations for f_R and f_L . In the first step of the proof, we justify that these Duhamel formulas are well-defined. In a second step, we prove the continuity of these functions away from the initial time.

Step 1 (Duhamel formula). We justify that the formula for f_R ,

$$(4.5) \quad \begin{aligned} f_R(t, x) = & \int_0^{2\pi} \Phi(t, x-y) f_{R,0}(y) dy \\ & - \int_0^t \int_0^{2\pi} \Phi(t-s, x-y) \left(\text{Pe} \partial_y \{f_R(s, y)[1 - \rho(s, y)]\} + f_R(s, y) - f_L(s, y) \right) dy ds, \end{aligned}$$

is well-defined for a.e. $(t, x) \in [0, T] \times \bar{\Omega}$. To begin with, whenever $t > 0$, the function $\Phi(t, \cdot)$ is smooth and is therefore square-integrable on the compact interval $[0, 2\pi]$. This shows that the integrand in the first integral in (4.5) is integrable. Meanwhile, the

equality (4.3) shows that $\Phi \in L^2([0, T]; L^2([0, 2\pi]))$, whence, using the boundedness of ρ ,

$$\begin{aligned} & \int_0^t \int_0^{2\pi} \Phi(t-s, x-y) |\partial_y \{f_R(s, y)[1-\rho(s, y)]\}| dy ds \\ & \leq 2 \int_0^t \left[\int_0^{2\pi} \Phi(t-s, x-y)^2 dy \right]^{1/2} \left[\int_0^{2\pi} |\partial_y f_R(s, y)|^2 + f_R(s, y)^2 |\partial_y \rho(s, y)|^2 dy \right]^{1/2} ds \\ & \leq \int_0^t \int_0^{2\pi} \Phi(t-s, x-y)^2 dy ds + \|f_R\|_{L^2(0, T; H^1(\Omega))}^2 + \int_0^t \int_0^{2\pi} f_R(s, y)^2 |\partial_y \rho(s, y)|^2 dy ds, \end{aligned}$$

where we have used the Young inequality. As before, we use a change of variable and the constraint $x \in [0, 2\pi]$ to bound the first term on the right-hand side of the above by $2\|\Phi\|_{L^2(0, T; L^2([0, 2\pi]))}^2$. Meanwhile,

$$\int_0^t \int_0^{2\pi} f_R(s, y)^2 |\partial_y \rho(s, y)|^2 dy ds \leq \|f_R\|_{L^\infty((0, T) \times \Omega)}^2 \|\rho\|_{L^2(0, T; H^1(\Omega))}^2,$$

using that $f_R \in L^\infty((0, T) \times \Omega)$ from Theorem 3.4. Similarly, using Lemma 4.2

$$\int_0^t \int_0^{2\pi} \Phi(t-s, x-y) |f_R(s, y) - f_L(s, y)| dy ds \leq 2\|\Phi\|_{L^2(0, T; L^2(\Omega))} \|f_R - f_L\|_{L^2(0, T; L^2(\Omega))}.$$

Therefore, the integrand in the second integral of (4.5) is also integrable. It is an exercise to verify that (4.5) and the equivalent formula for f_L satisfy (3.30) and are periodic on Ω . These formulas therefore coincide with the unique solutions of (3.30).

Step 2 (continuity away from initial time). Fix $(t, x) \in (0, T] \times \bar{\Omega}$, and let $(t_n, x_n) \in (0, T] \times \Omega$ be converging to (t, x) . Using the integrability from Step 1, the continuity of Φ for $t > 0$, and the Dominated Convergence theorem on (4.5) and the analogous formula for f_L , we directly obtain $|f_R(t_n, x_n) - f_R(t, x)| \rightarrow 0$ and $|f_L(t_n, x_n) - f_L(t, x)| \rightarrow 0$ as $n \rightarrow \infty$. It follows that $f_R, f_L \in C((0, T] \times \bar{\Omega})$. In turn, we also obtain $\rho \in C((0, T] \times \bar{\Omega})$. \square

LEMMA 4.5. *Suppose that $f_{R,0}, f_{L,0} \in C(\bar{\Omega})$ and are periodic. Then, the unique weak solutions f_R and f_L of (3.30) belong to $C([0, T] \times \bar{\Omega})$.*

Proof. We only write the proof for f_R , as the proof for f_L is identical. In view of the uniform continuity of f_R on the compact $\bar{\Omega}$, given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ for which if $x, y \in [0, 2\pi]$ are such that $|x - y| < \delta$, then $|f_{R,0}(x) - f_{R,0}(y)| < \varepsilon/4$. Now, given $t \in (0, \infty)$ and $x^* \in (0, 2\pi)$, there holds from (4.5)

$$\begin{aligned} |f_R(t, x) - f_{R,0}(x^*)| & \leq \left| \int_0^{2\pi} \Phi(t, x-y) f_{R,0}(y) dy - f_{R,0}(x^*) \right| \\ & \quad + \text{Pe} \int_0^t \int_0^{2\pi} \Phi(t-s, x-y) |\partial_y (f_R(s, y)(1-\rho(s, y)))| dy ds \\ & \quad + k \int_0^t \int_0^{2\pi} \Phi(t-s, x-y) |f_R(s, y) - f_L(s, y)| dy ds \\ & =: I_1(t, x) + I_2(t, x) + I_3(t, x), \end{aligned}$$

where we used the nonnegativity of Φ and $\int_0^{2\pi} \Phi(t, x) dx = 1$ for every $t > 0$ from Lemma 4.2. Using Hölder's inequality, the L^2 -boundedness of $\partial_x f_R, f_R, f_L$, and Lemma 4.2, we obtain $\lim_{t \rightarrow 0^+} \|I_2(t, \cdot)\|_{L^\infty([0, 2\pi])} = \lim_{t \rightarrow 0^+} \|I_3(t, \cdot)\|_{L^\infty([0, 2\pi])} = 0$.

It remains to control I_1 . Recall that since $f_{R,0} \in C(\bar{\Omega})$ is periodic, there exists a sequence $(\psi_m)_{m \in \mathbb{N}}$ of periodic elements of $C^2(\bar{\Omega})$ such that $\|f_{R,0} - \psi_m\|_{L^\infty(\bar{\Omega})} \rightarrow 0$ as $n \rightarrow \infty$. Then, since $\psi_m \in C^2(\bar{\Omega})$ for each $m \in \mathbb{N}$, we know that it has Fourier coefficients $a_{m,n}, b_{m,n}$ that decay as n^{-2} ; cf. (B.1) and (B.2). Then, using the convolution result for Fourier series,

$$\begin{aligned} \left| \psi_m(x) - \int_0^{2\pi} \Phi(t, x-y) \psi_m(y) dy \right| &= \frac{1}{\pi} \left| \sum_{n=1}^{\infty} (1 - e^{-n^2 t}) (a_{m,n} \cos nx + b_{m,n} \sin nx) \right| \\ &\leq C_m \sum_{n=1}^{\infty} \frac{1}{n^2} (1 - e^{-n^2 t}) \rightarrow 0 \quad \text{as } t \rightarrow 0^+, \end{aligned}$$

where we made use of the Dominated Convergence theorem applied to the atomic measure in the final line. Note that

$$\begin{aligned} I_1(t, x) &\leq |f_{R,0}(x^*) - f_{R,0}(x)| + |f_{R,0}(x) - \psi_m(x)| \\ &\quad + \left| \psi_m(x) - \int_0^{2\pi} \Phi(t, x-y) \psi_m(y) dy \right| + \left| \int_0^{2\pi} \Phi(t, x-y) (\psi_m(y) - f_{R,0}(y)) dy \right|, \end{aligned}$$

and elementary manipulations (cf. Step 3 of the proof of Lemma 4.2) relying also on the periodicity of ψ_m yield $\int_0^{2\pi} \Phi(t, x-y) \psi_m(y) dy = \int_0^{2\pi} \Phi(t, x-y) \psi_m(x+y) dy$. Thus, given $\varepsilon > 0$, by first picking x such that $|x - x^*| < \delta$ and then choosing m sufficiently large such that $\|f_{R,0} - \psi_m\|_{L^\infty(\bar{\Omega})} < \varepsilon/4$, and finally taking t sufficiently close to zero for this particular m (cf. convergence to initial data of Lemma 4.2), it follows that

$$I_1(t, x) < \frac{3}{4}\varepsilon + \|\psi_m - f_{R,0}\|_{L^\infty(\bar{\Omega})} \int_0^{2\pi} \Phi(t, x-y) dy < \varepsilon,$$

where we used the unit integral property of Φ . Hence, $\lim_{(t,x) \rightarrow (0^+, x^*)} I_1(t, x) = 0$, and we conclude that $\lim_{(t,x) \rightarrow (0^+, x^*)} |f_{R,0}(x^*) - f_R(t, x)| = 0$, which verifies the required continuity at the initial time. The same argument shows the result for f_L . \square

Remark 4.6. We showed that the weak solutions of the 1D model, which (cf. Remark 3.5), in principle, only belong to $C([0, T]; L^2_{per}([0, 2\pi])) \cap L^2([0, T]; H^1_{per}([0, 2\pi]))$, are in fact continuous under mild assumptions on the initial data. Although we suspect that it is possible to show higher regularity of the solutions, the method of proof required for such a result would be rather different to the current approach. For this reason, we leave this for further investigation.

4.2. Regularity for the 2D model. In accordance with the expression for the one-dimensional periodic kernel in terms of complex exponentials (cf. subsection 4.1), we find that the periodic heat kernel in three spatial dimensions (i.e., when $\Upsilon = [0, 2\pi]^3$) is given by the explicit formula

$$(4.6) \quad \tilde{\Phi}(t, \mathbf{x}) = \frac{1}{(2\pi)^3} + \sum_{\mathbf{n} \in \mathbb{Z}^3 \setminus \{0\}} \frac{e^{-|\mathbf{n}|^2 t}}{(2\pi)^3} e^{i\mathbf{n} \cdot \mathbf{x}} \quad \forall (t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^3.$$

Note that the above may be rewritten in terms of the Jacobi theta-3 function as $\tilde{\Phi}(t, \mathbf{x}) = (2\pi)^{-3} \theta_3(x_1/2, e^{-t}) \theta_3(x_2/2, e^{-t}) \theta_3(x_3/2, e^{-t})$ for every $(t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^3$.

As per the proof of Lemma 4.2, we obtain that $\tilde{\Phi}$ is nonnegative in $(0, \infty) \times \mathbb{R}^3$ with (4.7)

$$\|\tilde{\Phi}(t, \cdot)\|_{L^1(\Upsilon)} = 1 \quad \text{and} \quad \int_0^t \|\tilde{\Phi}(\tau, \cdot)\|_{L^2(\Upsilon)}^2 d\tau = t + \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{1}{2|\mathbf{n}|^2} (1 - e^{-|\mathbf{n}|^2 t}) \quad \forall t > 0,$$

and that $\tilde{\Phi}$ is a smooth Υ -periodic solution of the heat equation in $(0, \infty) \times \mathbb{R}^3$. In turn, an identical proof to that of Lemma 4.3 yields the following result.

LEMMA 4.7. *Let $\psi \in L^1_{per}(\Upsilon)$. Then, $\tilde{\Psi}(t, \xi) := \int_{\Upsilon} \tilde{\Phi}(t, \xi - \mathbf{z}) \psi(\mathbf{z}) d\mathbf{z}$ for all $(t, \xi) \in (0, \infty) \times \mathbb{R}^3$ is a smooth Υ -periodic function defined on the full-space $(0, \infty) \times \mathbb{R}^3$, and solves the Cauchy problem*

$$(4.8) \quad \begin{cases} \partial_t \tilde{\Psi} - \Delta_{\xi} \tilde{\Psi} = 0 & \text{in } (0, \infty) \times \mathbb{R}^3, \\ \tilde{\Psi}|_{t=0} = \psi & \text{on } \mathbb{R}^3, \end{cases}$$

where the initial data is achieved in L^1_{loc} , i.e., $\lim_{t \rightarrow 0^+} \|\tilde{\Psi}(t, \cdot) - \psi\|_{L^1(\Upsilon)} = 0$.

In turn, we have the following regularity result.

LEMMA 4.8. *Suppose the initial data for problem (1.2) is such that $f_0 \in L^{\infty}_{per}(\Upsilon)$. Then the unique weak solution f of (1.2) supplied by Theorem 2.3 belongs to $C((0, T] \times \Upsilon)$.*

Proof. There are two steps to the proof: the first is to show the boundedness of f (which requires building an appropriate test function); the second is to argue via a Duhamel formula (cf. Proof of Lemma 4.4).

Step 1 (boundedness of f). Recall from Theorem 2.3 that the weak solution of (1.2), f , a priori only belongs to the space $L^{\infty}([0, T]; L^2_{per}(\Upsilon)) \cap L^2([0, T]; H^1_{per}(\Upsilon))$. We show that $f \in L^{\infty}((0, T) \times \Upsilon)$.

For any $k > 0$, one verifies that $(f - k)_+ \in L^{\infty}([0, T]; L^2_{per}(\Omega)) \cap L^2([0, T]; H^1_{per}(\Upsilon))$ and $\partial_t(f - k)_+ \in L^2([0, T]; (H^1_{per}(\Upsilon))')$. Indeed, there holds $(f - k)_+ = \mathbf{1}_{\{f > k\}}(f - k)$, from which it follows that, in the sense of distributions, $\nabla_{t, \mathbf{x}, \theta}(f - k)_+ = \mathbf{1}_{\{f > k\}} \nabla_{t, \mathbf{x}, \theta} f$ a.e. in $(0, T) \times \Upsilon$, where $\nabla_{t, \mathbf{x}, \theta}$ is the vector of derivatives $(\partial_t, \nabla, \partial_{\theta})$, and we directly deduce the boundedness of the relevant norms. Thus, $(f(t) - k)_+$ is an admissible test function for the weak formulation of Definition 2.2. Note, additionally, that, by [18, Chap. 2, Lem. 1.2], there holds in the sense of distributions, for a.e. $t \in (0, T)$, $\frac{d}{dt} \frac{1}{2} \int_{\Upsilon} (f(t) - k)_+^2 d\mathbf{x} = \langle \partial_t f(t), (f(t) - k)_+ \rangle_{\Upsilon}$. Using $(f(t) - k)_+$ as the test function in the weak formulation of Definition 2.2, integrating in \mathbf{x} , using the boundedness of ρ , Hölder's inequality, and the Young inequality, we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|(f(t) - k)_+\|_{L^2(\Upsilon)}^2 \right) + \int_{\Upsilon} \left(\frac{D_e}{2} |\nabla(f(t) - k)_+|^2 + |\partial_{\theta}(f(t) - k)_+|^2 \right) d\mathbf{x} d\theta \\ \leq \frac{\text{Pe}^2}{2D_e} \|(f(t) - k)_+\|_{L^2(\Upsilon)}^2 \end{aligned}$$

for a.e. $t \in (0, T)$. It follows from Grönwall's lemma that there holds

$$\frac{1}{2} \|(f(t) - k)_+\|_{L^2(\Upsilon)}^2 \leq \frac{1}{2} \|(f_0 - k)_+\|_{L^2(\Upsilon)}^2 \exp \left(\frac{\text{Pe}^2}{D_e} t \right) \quad \text{a.e. } t \in (0, T).$$

By selecting $k = \|f_0\|_{L^{\infty}(\Upsilon)}$, the right-hand side of the above is null. It follows that $f(t) \leq \|f_0\|_{L^{\infty}(\Upsilon)}$ for a.e. $t \in (0, T)$, i.e., $\|f\|_{L^{\infty}((0, T) \times \Upsilon)} \leq \|f_0\|_{L^{\infty}(\Upsilon)}$.

Step 2 (Duhamel formula for f). Without loss of generality for this final step, we only consider the case where we have normalized (1.2) by the positive constant D_e , so that $\partial_t - D_e \Delta - \partial_\theta^2$ may be rewritten as $\partial_t - \Delta_\xi$ with ξ the usual 3-dimensional vector.

As per the proof of Lemma 4.4, in view of the uniqueness provided by Theorem 2.3, it follows from (1.2) and the Duhamel principle that f satisfies the following implicit equation, where the kernel $\tilde{\Phi}$ was defined in (4.6) (cf. Lemma 4.7):

$$(4.9) \quad f(t, \xi) = \int_{\Upsilon} \tilde{\Phi}(t, \xi - \mathbf{z}) f_0(\mathbf{z}) d\mathbf{z} + \text{Pe} \int_0^t \int_{\Upsilon} \tilde{\Phi}(t-s, \xi - \mathbf{z}) \nabla_{\mathbf{z}} \cdot (f(s, \mathbf{z}) (1 - \rho(s, \mathbf{z})) \mathbf{e}(\theta)) d\mathbf{z} ds$$

for a.e. $(t, \xi) \in [0, T] \times \tilde{\Upsilon}$. Note that the above is well-defined: using the boundedness of f from Step 1, we proceed to show the integrability of the integrand in (4.9) as per Step 1 of the proof of Lemma 4.4. We now deduce that $f \in C([0, T] \times \tilde{\Omega})$ using the continuity of $\tilde{\Phi}$ for $t > 0$ and the Dominated Convergence theorem, exactly as we did in Step 2 of the proof of Lemma 4.4. \square

Finally, by following the strategy of the proof of Lemma 4.5 to the letter, we obtain the following result concerning continuity up to the initial time.

LEMMA 4.9. *Suppose that the periodic initial data f_0 for problem (1.2) belongs to $C(\tilde{\Upsilon})$. Then, the unique weak solution f of (1.2) belongs to $C([0, T] \times \tilde{\Upsilon})$.*

We note, as per Remark 4.6, that it is likely possible to improve Lemma 4.9. However, the method for proving such a result is different from the current approach, which uses the Duhamel principle, and so we leave this for further investigation.

5. Extension to very weak solutions for the 2D model. The assumption on the initial data in $L_{per}^2(\Upsilon)$ rules out initial configurations with a fixed orientation, that is, $f_0(x, \theta) = h^0(x) \delta_{\theta=\theta^*}$ for some $\theta^* \in [0, 2\pi)$ and $h^0 \in L_{per}^2(\Omega)$, where the latter is the usual Dirac delta in the angular variable only. Since such initial data belongs to $L^2(\Omega; H^s(0, 2\pi))$ for $s < -1/2$, we turn our attention to initial data $f_0 \in L_{per}^2(\Omega; (H_{per}^1)'(0, 2\pi))$. For the reader's convenience, we clarify that, for any normed vector space V ,

$$(5.1) \quad f \in L_{per}^2(\Omega; V) \iff \begin{cases} f: \mathbb{R}^2 \rightarrow V, \\ \int_{\Omega} \|f(\mathbf{x}, \cdot)\|_V^2 d\mathbf{x} < +\infty, \text{ and} \\ f(\mathbf{x} - \mathbf{w}, \cdot) = f(\mathbf{x}, \cdot) \quad \text{a.e. } \mathbf{x} \in \Omega \quad \forall \mathbf{w} \in (2\pi\mathbb{Z})^2, \end{cases}$$

where the final equality on the right-hand side holds in V .

In view of the previous discussion, we define the following Hilbert spaces to give meaning to a weaker formulation of the equation (cf. Definition 5.1):

$$(5.2) \quad \begin{aligned} X &:= L^2([0, T]; H_{per}^1(\Upsilon)) \cap L^2([0, T]; L_{per}^2(\Omega; H_{per}^2(0, 2\pi))), \\ Y &:= H_{per}^1(\Upsilon) \cap L_{per}^2(\Omega; H_{per}^2(0, 2\pi)). \end{aligned}$$

Note that $\|u\|_{X'}^2 = \int_0^T \|u(t)\|_{Y'}^2 dt$ and $\|u(t)\|_{Y'} = \sup_{\substack{\varphi \in C_{per}^\infty(\tilde{\Upsilon}) \\ \|\varphi\|_Y \leq 1}} |\langle u(t), \varphi \rangle_{\Upsilon}|$.

The initial density must now be defined in duality, i.e.,

$$(5.3) \quad \rho_0(\mathbf{x}) := \langle f_0(\mathbf{x}, \cdot), 1 \rangle_{(H_{per}^1)' \times H_{per}^1} \quad \text{a.e. } x \in \Omega,$$

where $\langle \cdot, \cdot \rangle_{(H_{per}^1)' \times H_{per}^1}$ is the duality product on $(0, 2\pi)$. Note that for $f_0(\mathbf{x}, \theta) = h^0(\mathbf{x})\delta_{\theta=\theta^*}$, the initial space density is $\rho_0(\mathbf{x}) = h^0(\mathbf{x})$.

We now extend the concept of solution established in Definition 2.2 to admit a larger class of functions (i.e., smaller class of test functions). With slight abuse of terminology, we name these solutions *very weak solutions*, even though only the regularity in the angular variable is affected.

DEFINITION 5.1. *A very weak solution of (1.2) is a curve f belonging to $L^2([0, T]; H_{per}^1(\Omega; (H_{per}^1)'(0, 2\pi))) \cap L^\infty([0, T]; L_{per}^2(\Omega; (H_{per}^1)'(0, 2\pi))) \cap L^2((0, T) \times \Upsilon)$, with $\partial_t f \in X'$ such that, for every test function $\varphi \in Y$, there holds for a.e. $t \in [0, T]$*

$$(5.4) \quad \begin{aligned} \langle \partial_t f(t), \varphi \rangle_\Upsilon &= \langle Pe(1 - \rho(t))f(t)\mathbf{e}(\theta), \nabla \varphi \rangle_\Upsilon - \langle D_e \nabla f(t), \nabla \varphi \rangle_\Upsilon + \langle f(t), \partial_\theta^2 \varphi \rangle_\Upsilon, \\ f(0, \mathbf{x}, \theta) &= f_0(\mathbf{x}, \theta) \quad \text{on } \Upsilon \times \{0\}, \end{aligned}$$

with periodic boundary conditions on Υ , where $\rho(t, \mathbf{x}) = \int_0^{2\pi} f(t, \mathbf{x}, \theta) d\theta$, and where the initial data is achieved in the sense $f(0) = f_0$ in Y' .

Since $Y \subset L_{per}^2(\Omega; H_{per}^2(0, 2\pi))$, we have the inclusions $L_{per}^2(\Omega; (H_{per}^1)'(0, 2\pi)) \subset L_{per}^2(\Omega; (H_{per}^2)'(0, 2\pi)) \subset Y'$. It follows that $f \in L^2([0, T]; L_{per}^2(\Omega; (H_{per}^1)'(0, 2\pi)))$ and $\partial_t f \in X'$ imply $f \in H^1([0, T]; Y')$. Hence, by [10, sect. 5.9.2, Thm. 2], $f \in C([0, T]; Y')$. Therefore, $f(0) \in Y'$ is well-defined in the previous definition.

We now give the proof of Theorem 2.6; the main theorem of this section. Since a very weak solution is an element of $L^2((0, T) \times \Upsilon)$, this result implies that the solutions never concentrate in angle to a Dirac mass, even when the initial data does. This illustrates an instantaneous smoothing property of the equation.

Proof of Theorem 2.6. The idea of the proof is to construct a sequence of regularized initial data that approaches f_0 in the required norm, and to show that the corresponding regularized solutions converge to a very weak solution.

Step 1 (regularized sequence of initial data and solutions). Due to Lemma C.1 in Appendix C, there exists $(f_0^m)_{m \in \mathbb{N}}$ nonnegative elements of $C_{per}^\infty(\tilde{\Upsilon})$ such that

$$(5.5) \quad \|f_0 - f_0^m\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))} \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

with $\rho_0^m(\mathbf{x}) := \int_0^{2\pi} f_0^m(\mathbf{x}, \theta) d\theta \in [0, 1]$ for a.e. $\mathbf{x} \in \Upsilon$. Note that (5.5) of course implies that there exists a positive constant M independent of m such that

$$(5.6) \quad \sup_m \|f_0^m\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))} \leq M < +\infty.$$

Since each f_0^m is an admissible initial datum for Theorem 2.3, we deduce that there exists a unique $f_m \in L^\infty([0, T]; L_{per}^2(\Upsilon)) \cap L^2([0, T]; H_{per}^1(\Upsilon))$ with $\partial_t f_m \in L^2([0, T]; (H_{per}^1)'(\Upsilon))$ solving (1.2) in the weak sense given by Definition 2.2. Moreover, we have the uniform bound $0 \leq \rho_m(t, \mathbf{x}) \leq 1$ for a.e. $(t, \mathbf{x}) \in (0, T) \times \Omega$. Note, however, that we do not have a uniform estimate on $\|f_m\|_{L^2((0, T); H^1(\Upsilon))}$ or even on $\|f_m\|_{L^2((0, T) \times \Upsilon)}$, since these latter estimates depend on $\|f_0^m\|_{L^2(\Upsilon)}$, which explodes in the limit as $m \rightarrow \infty$. We therefore require a different estimate on $(f_m)_{m \in \mathbb{N}}$.

Step 2 (an auxiliary Dirichlet problem). In contrast to the inverse Laplacian introduced in subsection 3.4, we introduce $L_\theta := (-\partial_\theta^2)^{-1} : (H_{per}^1)'(0, 2\pi) \rightarrow H_{per}^1(0, 2\pi)$, which is understood to be the unique solution of the Dirichlet problem:

$$(5.7) \quad \begin{cases} (-\partial_\theta^2)L_\theta f = f & \text{in } (0, 2\pi), \\ L_\theta f(0) = L_\theta f(2\pi) = 0. \end{cases}$$

Let us note that for any function $f \in (H_{per}^1)'(0, 2\pi)$, there holds

$$\|f\|_{(H_{per}^1)'(0, 2\pi)} = \sup_{\substack{\varphi \in H_{per}^1(0, 2\pi) \\ \|\varphi\|_{H^1(0, 2\pi)} \leq 1}} \langle f, \varphi \rangle_{(H_{per}^1)' \times H_{per}^1} \leq \|\partial_\theta L_\theta f\|_{L_{per}^2(0, 2\pi)}.$$

Indeed, using the property $(-\partial_\theta^2)L_\theta f = f$, we have, where the duality product is between H_{per}^1 and its dual,

$$|\langle f, \varphi \rangle| = |\langle (-\partial_\theta^2) \circ (-\partial_\theta^2)^{-1} f, \varphi \rangle| = |\langle \partial_\theta L_\theta f, \partial_\theta \varphi \rangle_{L^2 \times L^2}| \leq \|\partial_\theta L_\theta f\|_{L^2} \|\partial_\theta \varphi\|_{L^2}.$$

Note that the integration by parts in the above involves no additional boundary term due to the periodicity of f and $L_\theta f$. By choosing the specific test function $\varphi = L_\theta f \in H_{per}^1(0, 2\pi)$ in the previous estimate, we obtain the maximum value in the duality product defining the $(H_{per}^1)'$ norm, hence

$$(5.8) \quad \|f\|_{(H_{per}^1)'(0, 2\pi)} = \|\partial_\theta L_\theta f\|_{L^2(0, 2\pi)}.$$

Since $L_\theta f \in H_0^1(0, 2\pi)$, the Poincaré inequality in one dimension yields

$$(5.9) \quad \|L_\theta f\|_{L^2(0, 2\pi)}^2 \leq C_P \|\partial_\theta L_\theta f\|_{L^2(0, 2\pi)}^2 = C_P \|f\|_{(H_{per}^1)'(0, 2\pi)}^2 \leq C_P \|f\|_{L_{per}^2(0, 2\pi)}^2,$$

where the constant C_P depends only on the domain $(0, 2\pi)$, and (5.8) was used to obtain the equality. In turn, given a function f such that $\nabla f \in L_{per}^2(0, 2\pi) \subset (H_{per}^1)'(0, 2\pi)$, there holds the estimate

$$(5.10) \quad \|L_\theta \nabla f\|_{L^2(0, 2\pi)}^2 \leq C_P \|\nabla f\|_{(H_{per}^1)'(0, 2\pi)}^2 \leq C_P \|\nabla f\|_{L^2(0, 2\pi)}^2,$$

and, directly from (5.8),

$$(5.11) \quad \|\partial_\theta L_\theta \nabla f\|_{L^2(0, 2\pi)}^2 = \|\nabla f\|_{(H_{per}^1)'(0, 2\pi)}^2 \leq \|\nabla f\|_{L^2(0, 2\pi)}^2.$$

Step 3 (energy estimate on first derivative independent of m). We now apply the operator L_θ to the sequence $(f_m)_{m \in \mathbb{N}}$. Note that for a.e. $t \in (0, T)$,

$$\|L_\theta f_m(t)\|_{H^1(\Upsilon)}^2 = \|L_\theta f_m(t)\|_{L^2(\Upsilon)}^2 + \|\nabla L_\theta f_m(t)\|_{L^2(\Upsilon)}^2 + \|\partial_\theta L_\theta f_m(t)\|_{L^2(\Upsilon)}^2.$$

It then follows from (5.9) and (5.10) that

$$\begin{aligned} \|L_\theta f_m(t)\|_{H^1(\Upsilon)}^2 &\leq \int_\Omega \|L_\theta f_m(t, \mathbf{x}, \cdot)\|_{L^2(0, 2\pi)}^2 d\mathbf{x} + \int_\Omega \|\nabla L_\theta f_m(t, \mathbf{x}, \cdot)\|_{L^2(0, 2\pi)}^2 d\mathbf{x} \\ &\quad + \int_\Omega \|\partial_\theta L_\theta f_m(t, \mathbf{x}, \cdot)\|_{L^2(0, 2\pi)}^2 d\mathbf{x} \\ &\leq 2C_P \int_\Omega (\|f_m(t, \mathbf{x}, \cdot)\|_{L^2(0, 2\pi)}^2 + \|\nabla f_m(t, \mathbf{x}, \cdot)\|_{L^2(0, 2\pi)}^2) d\mathbf{x}. \end{aligned}$$

Thus, $\|L_\theta f_m(t)\|_{H^1(\Upsilon)}^2 \leq 2C_P \|f_m\|_{H^1(\Upsilon)}^2 < +\infty$. Therefore, $L_\theta f_m(t)$ is an admissible test function for the weak formulation of Definition 2.2. This latter observation enables us to test the equation with $L_\theta f_m$ and to obtain an estimate independent of m . We get, for a.e. $(t, \mathbf{x}) \in (0, T) \times \Omega$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f_m(t, \mathbf{x}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)}^2 &= \frac{1}{2} \frac{d}{dt} \|\partial_\theta L_\theta f_m(t, \mathbf{x}, \cdot)\|_{L^2(0, 2\pi)}^2 \\ &= \langle \partial_t f_m(t, \mathbf{x}, \cdot), L_\theta f_m(t, \mathbf{x}, \cdot) \rangle_{(H^1)' \times H^1}, \end{aligned}$$

where we used that $\partial_t f \in L^2([0, T]; (H_{per}^1)'(\Upsilon))$ and $L_\theta f_m \in L^2([0, T]; H_{per}^1(\Upsilon))$. Hence, by integrating the previous line with respect to the space variable and using the weak formulation, also noting that $\langle \nabla f_m, \nabla L_\theta f_m \rangle_\Upsilon = \|\nabla \partial_\theta L_\theta f_m\|_{L^2(\Upsilon)}^2$ and $\langle \partial_\theta f_m, \partial_\theta L_\theta f_m \rangle_\Upsilon = \|f_m\|_{L^2(\Upsilon)}^2$,

$$(5.12) \quad \frac{1}{2} \frac{d}{dt} \|f_m(t)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 = \text{Pe} \langle (1 - \rho_m) f_m \mathbf{e}(\theta), \nabla L_\theta f_m \rangle_\Upsilon - \|\nabla \partial_\theta L_\theta f_m\|_{L^2(\Upsilon)}^2 - \|f_m\|_{L^2(\Upsilon)}^2,$$

where we used the equality $(-\partial_\theta^2) L_\theta f_m = f_m$, integrated by parts in θ in the last two duality products, and used the Dirichlet problem (5.7). Next, observe that $f_m \mathbf{e}(\theta) = (-\partial_\theta^2) L_\theta f_m \mathbf{e}(\theta) = -\partial_\theta(\partial_\theta L_\theta f_m \mathbf{e}(\theta)) - \partial_\theta L_\theta f_m \mathbf{e}^\perp(\theta)$, where $\mathbf{e}^\perp(\theta) = (-\sin \theta, \cos \theta)$. Using the fact that ρ_m is independent of the angle variable, it follows from an integration by parts that

$$\begin{aligned} \langle (1 - \rho_m) f_m \mathbf{e}(\theta), \nabla L_\theta f_m \rangle_\Upsilon &= \langle (1 - \rho_m) \partial_\theta L_\theta f_m \mathbf{e}(\theta), \nabla \partial_\theta L_\theta f_m \rangle_\Upsilon \\ &\quad - \langle (1 - \rho_m) \partial_\theta L_\theta f_m \mathbf{e}^\perp(\theta), \nabla L_\theta f_m \rangle_\Upsilon, \end{aligned}$$

whence the Hölder inequality and the bound for ρ_m give $|\langle (1 - \rho_m) f_m \mathbf{e}(\theta), \nabla L_\theta f_m \rangle_\Upsilon| \leq \|\partial_\theta L_\theta f_m\|_{L^2(\Upsilon)} \|\nabla \partial_\theta L_\theta f_m\|_{L^2(\Upsilon)} + \|\partial_\theta L_\theta f_m\|_{L^2(\Upsilon)} \|\nabla L_\theta f_m\|_{L^2(\Upsilon)}$. Thus, by applying the Poincaré inequality (5.9) to bound $\nabla L_\theta f_m$ by $\nabla \partial_\theta L_\theta f_m$ and the Young inequality,

$$|\langle (1 - \rho_m) f_m \mathbf{e}(\theta), \nabla L_\theta f_m \rangle_\Upsilon| \leq \text{Pe}(1 + C_P) \|\partial_\theta L_\theta f_m\|_{L^2(\Upsilon)}^2 + \frac{1}{2\text{Pe}} \|\partial_\theta \nabla L_\theta f_m\|_{L^2(\Upsilon)}^2.$$

Combining the above with (5.12), using (5.11) to rewrite $\|\nabla \partial_\theta L_\theta f_m\|_{L^2(\Upsilon)}^2$ and (5.8) to rewrite $\|\partial_\theta L_\theta f_m\|_{L^2(\Upsilon)}^2$, and integrating in time, we get, for a.e. $t \in (0, T)$,

$$\begin{aligned} \|f_m(t)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 &+ \int_0^t \|\nabla f_m(\tau)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 d\tau + 2 \int_0^t \|f_m(\tau)\|_{L^2(\Upsilon)}^2 d\tau \\ &\leq \|f_0^m\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 + \text{Pe}^2(1 + C_P) \int_0^t \|f_m(\tau)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 d\tau. \end{aligned}$$

Grönwall's lemma yields $\|f_m(t)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 \leq M^2 \exp(\text{Pe}^2(1 + C_P)T)$ a.e. $t \in (0, T)$, and we recall from (5.6) that the right-hand side is independent of m . Hence, we obtain for some positive constant M_1 independent of m ,

$$(5.13) \quad \begin{aligned} \text{esssup}_{t \in [0, T]} \|f_m(t)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 &+ \int_0^T \|\nabla f_m(t)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}^2 dt \\ &+ \int_0^T \|f_m(t)\|_{L^2(\Upsilon)}^2 dt \leq M_1. \end{aligned}$$

Thus, $\|f_m\|_{L^\infty([0, T]; L^2(\Omega; (H_{per}^1)'(0, 2\pi)))}, \|f_m\|_{L^2([0, T]; H^1(\Omega; (H_{per}^1)'(0, 2\pi)))}, \|f_m\|_{L^2((0, T) \times \Upsilon)}$ are bounded independently of m .

Step 4 (corresponding estimate on the time derivatives). We now return to (1.2) in order to obtain a similar uniform estimate on the sequence of time derivatives $(\partial_t f_m)_{m \in \mathbb{N}}$. Fix φ to be smooth and periodic on $\tilde{\Upsilon}$. Such a function is an admissible

test function for the weak formulation of the equation, and we get

$$\begin{aligned} |\langle \partial_t f_m(t), \varphi \rangle_{\Upsilon}| &\leq |\text{Pe} \langle (1 - \rho_m) f_m \mathbf{e}(\theta), \nabla \varphi \rangle_{\Upsilon}| + |\langle \nabla f_m, \nabla \varphi \rangle_{\Upsilon}| + |\langle \partial_\theta f_m, \partial_\theta \varphi \rangle_{\Upsilon}| \\ &\leq \text{Pe} \|f_m\|_{L^2(\Upsilon)} \|\nabla \varphi\|_{L^2(\Upsilon)} + \|f_m\|_{L^2(\Upsilon)} \|\partial_\theta^2 \varphi\|_{L^2(\Upsilon)} \\ &\quad + \|\nabla f_m\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))} \|\nabla \varphi\|_{L^2(\Omega; H^1(0, 2\pi))} \\ &\leq ((1 + \text{Pe}) \|f_m(t)\|_{L^2(\Upsilon)} + \|\nabla f_m(t)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))}) (\|\varphi\|_{H^1(\Upsilon)} \\ &\quad + \|\varphi\|_{L^2(\Omega; H^2(0, 2\pi))}) \end{aligned}$$

for a.e. $t \in (0, T)$. We therefore obtain, using (5.2), $\|\partial_t f_m(t)\|_{Y'} \leq ((1 + \text{Pe}) \|f_m(t)\|_{L^2(\Upsilon)} + \|\nabla f_m(t)\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))})$, whence, using (5.13) to bound the right-hand side of the above, we get, with M_2 a positive constant independent of m ,

$$(5.14) \quad \|\partial_t f_m\|_{X'} \leq M_2.$$

Step 5 (estimates on ρ independent of m). We infer uniform estimates on the sequence $(\rho_m)_{m \in \mathbb{N}}$ from those on $(f_m)_{m \in \mathbb{N}}$. First, $\rho_m(t, \mathbf{x}) = \int_0^{2\pi} f_m(t, \mathbf{x}, \theta) d\theta$ for a.e. $(t, \mathbf{x}) \in (0, T) \times \Omega$, from which it is easy to see, since the constant function $1/2\pi$ belongs to $H_{per}^1(0, 2\pi)$ and has norm equal to 1 therein,

$$(5.15) \quad |\rho_m(t, \mathbf{x})| \leq 2\pi \sup_{\|\varphi\|_{H_{per}^1(0, 2\pi)} \leq 1} \left| \int_0^{2\pi} f_m(t, \mathbf{x}, \theta) \varphi(\theta) d\theta \right| = 2\pi \|f_m(t, \mathbf{x}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)}$$

for a.e. $(t, \mathbf{x}) \in (0, T) \times \Omega$. The same estimate on the derivative yields $|\nabla \rho_m(t, \mathbf{x})| \leq 2\pi \|\nabla f_m(t, \mathbf{x}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)}$ a.e. in $(0, T) \times \Omega$. It follows directly from the comment under the uniform estimate (5.13) that there holds

$$(5.16) \quad \|\rho_m\|_{L^\infty([0, T]; L_{per}^2(\Omega))} + \|\rho_m\|_{L^2([0, T]; H_{per}^1(\Omega))} \leq M_3$$

for some M_3 independent of m . Finally, for the time derivatives, by returning to the weak formulation of (1.3) and arguing as we did to obtain (5.14), we obtain

$$|\langle \partial_t \rho_m(t), \varphi \rangle_\Omega| \leq \text{Pe} |\Omega|^{1/2} \|\nabla \varphi\|_{L^2(\Omega)} + \|\nabla \rho_m(t)\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)},$$

whence $\|\partial_t \rho_m(t)\|_{(H_{per}^1)'(\Omega)} \leq \text{Pe} |\Omega|^{1/2} + \|\nabla \rho_m(t)\|_{L^2(\Omega)}$ for a.e. $t \in (0, T)$. As a result, we deduce from (5.16) the uniform estimate

$$(5.17) \quad \|\partial_t \rho_m\|_{L^2([0, T]; (H_{per}^1)'(\Omega))} \leq M_4,$$

for some positive constant M_4 also independent of m .

Step 6 (compactness and passage to the limit in m). Due to the uniform estimate $\sup_m \|f_m\|_{L^2((0, T) \times \Upsilon)} < +\infty$ deduced under (5.13), we have from the Banach–Alaoglu theorem that there exists $f \in L^2((0, T) \times \Upsilon)$ such that

$$f_m \rightharpoonup f \quad \text{weakly in } L^2((0, T) \times \Upsilon),$$

up to a subsequence, which, with slight abuse of notation, we still label as $(f_m)_{m \in \mathbb{N}}$. In turn, there holds $\rho_m \rightharpoonup \rho$ weakly in $L^2((0, T) \times \Omega)$, where $\rho(t, \mathbf{x}) = \int_0^{2\pi} f(t, \mathbf{x}, \theta) d\theta$ for a.e. $(t, \mathbf{x}) \in (0, T) \times \Omega$. Meanwhile, given the estimates (5.16) and (5.17), the Aubin–Lions lemma implies the strong convergence

$$\rho_m \rightarrow \rho \quad \text{strongly in } L^2((0, T) \times \Omega).$$

The term $(1 - \rho_m)f_m$ is therefore the product of a strongly converging sequence and a weakly converging sequence in $L^2((0, T) \times \Omega)$. We are therefore able to pass to the limit weakly in $L^2((0, T) \times \Omega)$ in the nonlinear drift term.

Additionally, taking further subsequences if necessary, the remaining uniform estimates from (5.13) and (5.14), as well as the Banach–Alaoglu theorem, imply the convergences $\partial_t f_m \rightharpoonup^* \partial_t f$ in X' , $\nabla f_m \rightharpoonup^* \nabla f$ in $L^2([0, T]; L^2(\Omega; (H_{per}^1)'(0, 2\pi)))$, which proves the convergence of the relevant duality products to the limit equation. For the last duality product, we use an integration by parts to write $\langle \partial_\theta f_m, \partial_\theta \varphi \rangle_\Upsilon = -\langle f_m, \partial_\theta^2 \varphi \rangle_\Upsilon \rightarrow \langle f, \partial_\theta^2 \varphi \rangle_\Upsilon$ as $m \rightarrow \infty$, where we used the weak convergence $f_m \rightharpoonup f$ in $L^2((0, T) \times \Upsilon)$. Since the test function φ is chosen to belong to the space X , we obtain the limit equation as $m \rightarrow \infty$, as required.

Step 7 (initial data). Also recalling the discussion under Definition 5.1, the estimates (5.13) and (5.14) imply that $f_m, f \in H^1([0, T]; Y')$. By [10, sect. 5.9.2, Thm. 2], $f_m, f \in C([0, T]; Y')$. Recall from (5.5) $\|f_0 - f_0^m\|_{L^2(\Omega; (H_{per}^1)'(0, 2\pi))} \rightarrow 0$, from which $\|f_0 - f_0^m\|_{Y'} \rightarrow 0$; since $L^2(\Omega; (H_{per}^1)'(0, 2\pi)) \hookrightarrow Y'$ continuously. From (5.13), $(f_m)_{m \in \mathbb{N}}$ is bounded in $L^2([0, T]; L_{per}^2(\Upsilon))$. Using the Rellich–Kondrachov compactness embedding and properties of the adjoint map, $L_{per}^2(\Upsilon)$ embeds compactly into Y' ; since $Y \hookrightarrow H_{per}^1(\Upsilon)$ continuously, and $H_{per}^1(\Upsilon)$ is compactly embedded in $L_{per}^2(\Upsilon)$. Meanwhile, the sequence $(\partial_t f_m)_{m \in \mathbb{N}}$ is uniformly bounded in $X' = L^2([0, T]; Y')$. Thus the Aubin–Lions lemma implies $f_m \rightarrow f$ strongly in X' , and hence, up to taking a further subsequence, $f_m(t) \rightarrow f(t)$ strongly in Y' a.e. $t \in [0, T]$. Uniqueness of limits in Y' at the Lebesgue point $t = 0$ implies $f(0) = f_0$ in Y' . \square

Appendix A. The system of ODEs for the Galerkin approximation in space. Throughout this section, $m \in \mathbb{N}$ is fixed. For $k \in \{0, \dots, n\}$, we define

$$\begin{aligned} (A.1) \quad a_k^{n,m}(t, x, y) &:= \sum_{q=0}^m \sum_{p=0}^m \alpha_{1,p,q}^{k,m}(t) \cos px \cos qy + \sum_{q=1}^m \sum_{p=0}^m \alpha_{2,p,q}^{k,m}(t) \cos px \sin qy \\ &\quad + \sum_{q=0}^m \sum_{p=1}^m \alpha_{3,p,q}^{k,m}(t) \sin px \cos qy + \sum_{q=1}^m \sum_{p=1}^m \alpha_{4,p,q}^{k,m}(t) \sin px \sin qy, \\ b_k^{n,m}(t, x, y) &:= \sum_{q=0}^m \sum_{p=0}^m \beta_{1,p,q}^{k,m}(t) \cos px \cos qy + \sum_{q=1}^m \sum_{p=0}^m \beta_{2,p,q}^{k,m}(t) \cos px \sin qy \\ &\quad + \sum_{q=0}^m \sum_{p=1}^m \beta_{3,p,q}^{k,m}(t) \sin px \cos qy + \sum_{q=1}^m \sum_{p=1}^m \beta_{4,p,q}^{k,m}(t) \sin px \sin qy, \end{aligned}$$

where $\{\alpha_{j,p,q}^{k,m}, \beta_{j,p,q}^{k,m}\}$ are chosen to be the solutions of a specific ODE system.

Recall that the products of sines and cosines form an orthogonal basis of $L_{per}^2(\Omega)$, which motivates the approximations (A.1). Our aim is to recover the coefficients $\{a_k, b_k\}_{k=0}^n$ as the limits as $m \rightarrow \infty$ of the coefficients defined in (A.1). To do so, we impose that $\{a_k^{n,m}, b_k^{n,m}\}$ solve (3.7) weakly (i.e., in duality with $H_{per}^1(\Omega)$) and with initial data $\{a_k^{n,m}(0), b_k^{n,m}(0)\}$ computed by testing $\{a_k^0, b_k^0\}$ with products of sines and cosines up to frequencies m inclusive and adding these contributions, i.e.,

$$\begin{aligned} (A.2) \quad a_{k,m}^0(\mathbf{x}) &= \sum_{q=0}^m \sum_{p=0}^m C_{1,p,q} \cos px \cos qy + \sum_{q=1}^m \sum_{p=0}^m C_{2,p,q} \cos px \sin qy \\ &\quad + \sum_{q=0}^m \sum_{p=1}^m C_{3,p,q} \sin px \cos qy + \sum_{q=1}^m \sum_{p=1}^m C_{4,p,q} \sin px \sin qy, \end{aligned}$$

where $C_{1,p,q} = \int_{\Omega} \cos px \cos qy a_k^0(\mathbf{x}) d\mathbf{x}$, $C_{2,p,q} = \int_{\Omega} \cos px \sin qy a_k^0(\mathbf{x}) d\mathbf{x}$, $C_{3,p,q} = \int_{\Omega} \sin px \cos qy a_k^0(\mathbf{x}) d\mathbf{x}$, and $C_{4,p,q} = \int_{\Omega} \sin px \sin qy a_k^0(\mathbf{x}) d\mathbf{x}$ (up to multiples of 2π). The expression for $b_{k,m}^0$ is similar. By construction, $(a_{k,m}^0, b_{k,m}^0) \rightarrow (a_k^0, b_k^0)$ strongly in $L^2(\Omega)$ as $m \rightarrow \infty$.

By substituting appropriate test functions into the aforementioned weak formulations for $a_k^{n,m}$ and $b_k^{n,m}$, we obtain a system of ODEs with $\alpha_{j,p,q}^{k,m}, \beta_{j,p,q}^{k,m}$ as the unknowns.

Testing against $\varphi = 1$ in the equation for a_0^m in (3.7), we find $4\pi^2 \frac{d\alpha_{1,0,0}^{0,m}}{dt}(t) = 0$, whence $\alpha_{1,0,0}^{0,m}(t) = \alpha_{1,0,0}^{0,m}(0)$ for all $t \in [0, T]$. Moreover, testing against $\varphi = 1$ in the equations for a_1^m and b_1^m in (3.7), we obtain $4\pi^2 \frac{d\alpha_{1,0,0}^{k,m}}{dt}(t) = -4\pi^2 k^2 \alpha_{1,0,0}^{k,m}(t)$, $4\pi^2 \frac{d\beta_{1,0,0}^{k,m}}{dt}(t) = -4\pi^2 k^2 \beta_{1,0,0}^{k,m}(t)$ for $k \in \{1, \dots, n\}$. All coefficients $\{\alpha_{1,0,0}^{k,m}, \beta_{1,0,0}^{k,m}\}_{k=0}^n$ are therefore determined.

Then, testing the equations for $a_k^{n,m}, b_k^{n,m}$ in (3.7) with $\varphi = \cos px$ for $p \in \{1, \dots, m\}$, we get

$$\begin{aligned} 2\pi^2 \frac{d\alpha_{1,p,0}^{0,m}}{dt}(t) - \text{Pe} \langle (1 - (a_0^m(t))_+) + a_1^m(t), (-p \sin px) \rangle &= -D_e p^2 2\pi^2 \alpha_{1,p,0}^{0,m}(t), \\ 2\pi^2 \frac{d\alpha_{1,p,0}^{1,m}}{dt}(t) - \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + (2a_0^m(t) + a_{2,m}(t)), (-p \sin px) \rangle \\ &= -D_e p^2 2\pi^2 \alpha_{1,p,0}^{1,m}(t) - 2\pi^2 \alpha_{1,p,0}^{1,m}(t), \\ 2\pi^2 \frac{d\beta_{1,p,0}^{1,m}}{dt}(t) - \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + b_{2,m}(t), (-p \sin px) \rangle &= -D_e p^2 2\pi^2 \beta_{1,p,0}^{1,m}(t) \\ &\quad - 2\pi^2 \beta_{1,p,0}^{1,m}(t), \end{aligned}$$

while, for $1 < k < n$,

$$\begin{aligned} 2\pi^2 \frac{d\alpha_{1,p,0}^{k,m}}{dt}(t) - \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + (a_{k-1,m}(t) + a_{k+1,m}(t)), (-p \sin px) \rangle \\ &= -D_e p^2 2\pi^2 \alpha_{1,p,0}^{k,m}(t) - 2\pi^2 k^2 \alpha_{1,p,0}^{k,m}(t), \\ 2\pi^2 \frac{d\beta_{1,p,0}^{k,m}}{dt}(t) - \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + (b_{k-1,m}(t) + b_{k+1,m}(t)), (-p \sin px) \rangle \\ &= -D_e p^2 2\pi^2 \beta_{1,p,0}^{k,m}(t) - 2\pi^2 k^2 \beta_{1,p,0}^{k,m}(t), \end{aligned}$$

and

$$\begin{aligned} 2\pi^2 \frac{d\alpha_{1,p,0}^{n,m}}{dt}(t) - \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + a_{n-1,m}(t), (-p \sin px) \rangle &= -D_e p^2 2\pi^2 \alpha_{1,p,0}^{n,m}(t) \\ &\quad - 2\pi^2 n^2 \alpha_{1,p,0}^{n,m}(t), \\ 2\pi^2 \frac{d\beta_{1,p,0}^{n,m}}{dt}(t) - \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + b_{n-1,m}(t), (-p \sin px) \rangle &= -D_e p^2 2\pi^2 \beta_{1,p,0}^{n,m}(t) \\ &\quad - 2\pi^2 n^2 \beta_{1,p,0}^{n,m}(t). \end{aligned}$$

The ODEs for the coefficients $\{\alpha_{3,p,0}^{k,m}, \beta_{3,p,0}^{k,m}\}_{k=0}^n$ are obtained similarly by testing the equations with $\varphi = \sin px$ for $p \in \{1, \dots, m\}$ and have a similar structure to those for $\{\alpha_{1,p,0}^{k,m}, \beta_{1,p,0}^{k,m}\}_{k=0}^n$. The same is true for the ODEs for the coefficients

$\{\alpha_{1,0,q}^{k,m}, \beta_{1,0,q}^{k,m}\}_{k=0}^n$ (obtained by testing (3.7) with $\varphi = \cos qy$ for $q \in \{1, \dots, m\}$) and those for the coefficients $\{\alpha_{2,0,q}^{k,m}, \beta_{2,0,q}^{k,m}\}_{k=0}^n$ (obtained by testing (3.7) $\varphi = \sin qy$ for $q \in \{1, \dots, m\}$).

Similarly, the ODEs for the coefficients $\{\alpha_{1,p,q}^{k,m}, \beta_{1,p,q}^{k,m}\}_{k=0}^n$ are obtained by testing (3.7) with $\varphi = \cos px \cos qy$ for $p, q \in \{1, \dots, m\}$. In detail, we obtain

$$\begin{aligned} \frac{d\alpha_{1,p,q}^{0,m}}{dt} &= \frac{\text{Pe}}{\pi^2} \langle (1 - (a_0^m(t))_+) + (a_1^m(t), b_1^m(t)), (-p \sin px \cos qy, -q \sin qy \cos px) \rangle_\Omega \\ &\quad - D_e(p^2 + q^2) \alpha_{1,p,q}^{0,m}(t), \end{aligned}$$

$$\begin{aligned} &\pi^2 \frac{d\alpha_{1,p,q}^{1,m}}{dt}(t) + D_e(p^2 + q^2) \pi^2 \alpha_{1,p,q}^{1,m}(t) + \pi^2 \alpha_{1,p,q}^{1,m}(t) \\ &= \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + (2a_0^m(t) + a_{2,m}(t), b_{2,m}(t)), (-p \sin px \cos qy, -q \sin qy \cos px) \rangle_\Omega, \\ &\pi^2 \frac{d\beta_{1,p,q}^{1,m}}{dt}(t) + D_e(p^2 + q^2) \pi^2 \beta_{1,p,q}^{1,m}(t) + \pi^2 \beta_{1,p,q}^{1,m}(t) \\ &= \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + (b_{2,m}(t), 2a_0^m(t) - a_{2,m}(t)), (-p \sin px \cos qy, -q \sin qy \cos px) \rangle_\Omega, \end{aligned}$$

while, for $1 < k < n$,

$$\begin{aligned} &\frac{2\pi^2}{\text{Pe}} \frac{d\alpha_{1,p,q}^{k,m}}{dt}(t) + \frac{2D_e}{\text{Pe}} (p^2 + q^2) \pi^2 \alpha_{1,p,q}^{k,m}(t) + \frac{2\pi^2}{\text{Pe}} k^2 \alpha_{1,p,q}^{k,m}(t) \\ &= \langle (1 - (a_0^m)_+) + (a_{k-1,m} + a_{k+1,m}, b_{k+1,m} - b_{k-1,m}), (-p \sin px \cos qy, -q \sin qy \cos px) \rangle_\Omega, \\ &\frac{2\pi^2}{\text{Pe}} \frac{d\beta_{1,p,q}^{k,m}}{dt}(t) + \frac{2D_e}{\text{Pe}} (p^2 + q^2) \pi^2 \beta_{1,p,q}^{k,m}(t) + \frac{2\pi^2}{\text{Pe}} k^2 \beta_{1,p,q}^{k,m}(t) \\ &= \langle (1 - (a_0^m)_+) + (b_{k-1,m} + b_{k+1,m}, a_{k-1,m} - a_{k+1,m}), (-p \sin px \cos qy, -q \sin qy \cos px) \rangle_\Omega, \end{aligned}$$

and

$$\begin{aligned} &\pi^2 \frac{d\alpha_{1,p,q}^{n,m}}{dt}(t) + D_e(p^2 + q^2) \pi^2 \alpha_{1,p,q}^{n,m}(t) + \pi^2 n^2 \alpha_{1,p,q}^{n,m}(t) \\ &= \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + (a_{n-1,m}(t), -b_{n-1,m}(t)), (-p \sin px \cos qy, -q \sin qy \cos px) \rangle_\Omega, \\ &\pi^2 \frac{d\beta_{1,p,q}^{n,m}}{dt}(t) + D_e(p^2 + q^2) \pi^2 \beta_{1,p,q}^{n,m}(t) + \pi^2 n^2 \beta_{1,p,q}^{n,m}(t) \\ &= \frac{\text{Pe}}{2} \langle (1 - (a_0^m(t))_+) + (b_{n-1,m}(t), a_{n-1,m}(t)), (-p \sin px \cos qy, -q \sin qy \cos px) \rangle_\Omega. \end{aligned}$$

The ODEs for the coefficients $\{\alpha_{3,p,q}^{k,m}, \beta_{3,p,q}^{k,m}\}_{k=0}^n$ are obtained similarly by testing the equations with $\varphi = \sin px \cos qy$ for $p, q \in \{1, \dots, m\}$, and have a similar structure to those of $\{\alpha_{1,p,q}^{n,m}, \beta_{1,p,q}^{n,m}\}_{k=0}^n$. The same is true for the ODEs for the coefficients $\{\alpha_{2,p,q}^{k,m}, \beta_{2,p,q}^{k,m}\}_{k=0}^n$ (obtained by testing the equations with $\varphi = \cos px \sin qy$ for $p, q \in \{1, \dots, m\}$) and those for the coefficients $\{\alpha_{4,p,q}^{k,m}, \beta_{4,p,q}^{k,m}\}_{k=0}^n$ (obtained by testing the equations with $\varphi = \sin px \sin qy$ for $p, q \in \{1, \dots, m\}$).

As was already mentioned in Step 1 of subsection 3.2, the previous system may be rewritten in the form of (3.10), i.e., $\Lambda'_m(t) = F(t, \Lambda_m(t))$ for $t \in [0, T]$, with initial condition $\Lambda_m(0) = \Lambda_{m,0}$, where $\Lambda_m = \{\alpha_{j,p,q}^{k,m}\}_{k,j,p,q}$ is the tensor of coefficients, with

initial value $\Lambda_{m,0}$, and F is locally Lipschitz since the function $x \mapsto x_+^2$ is locally Lipschitz. The initial data is given by the explicit formulas

$$(A.3) \quad 4\pi^2 \alpha_{1,0,0}^{0,m}(0) = \int_{\Omega} a_{0,m}^0 \, dx \, dy, \quad 4\pi^2 \alpha_{1,0,0}^{k,m}(0) = \int_{\Omega} a_{k,m}^0(x, y) \, dx \, dy,$$

where we recall the expression for $a_{k,m}^0$ in (A.2), and, for $p, q \in \{1, \dots, m\}$,

$$(A.4) \quad \begin{aligned} \alpha_{1,p,0}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{2\pi^2} \cos px \, dx \, dy, & \alpha_{3,p,0}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{2\pi^2} \sin px \, dx \, dy, \\ \alpha_{1,0,q}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{2\pi^2} \cos qy \, dx \, dy, & \alpha_{2,0,q}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{2\pi^2} \sin qy \, dx \, dy, \\ \alpha_{1,p,q}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{\pi^2} \cos px \cos qy \, dx \, dy, & \alpha_{3,p,q}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{\pi^2} \sin px \cos qy \, dx \, dy, \\ \alpha_{2,p,q}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{\pi^2} \cos px \sin qy \, dx \, dy, & \alpha_{4,p,q}^{k,m}(0) &= \int_{\Omega} \frac{a_{k,m}^0(x, y)}{\pi^2} \sin px \sin qy \, dx \, dy, \end{aligned}$$

and the formulas for the coefficients $\{\beta_{i,p,q}^{k,m}\}_{i,p,q,k,m}$ are identical, with the exception that $b_{k,m}^0$ replaces $a_{k,m}^0$ on the right-hand sides of the previous equations. Note that the above are well-defined real numbers due to the integrability of the initial data $a_{k,m}^0, b_{k,m}^0$ which is inherited from the integrability of $f_0 \in L_{per}^2(\Upsilon)$.

Appendix B. Properties of the periodic heat kernel. Below, we provide proofs of Lemma 4.2 and Lemma 4.3, respectively.

Proof of Lemma 4.2. Step 1 (smoothness away from initial time). Begin by noting that, by a standard argument using the Weierstraß M-test, the series representation (4.1) for Φ is well-defined, and moreover can repeatedly be differentiated term-by-term for positive times. As such, $\Phi \in C^\infty((0, \infty) \times \mathbb{R})$, and $\partial_t \Phi(t, x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} n^2 e^{-n^2 t} \cos nx = \partial_{xx} \Phi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$.

Step 2 (nonnegativity and L^1 -norm). Recall (cf. [19, section 21.3]) the infinite product representation for the Jacobi theta-3 function. For every $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$\Phi(t, x) = \frac{1}{2\pi} \theta_3\left(\frac{x}{2}, e^{-t}\right) = \frac{1}{2\pi} \prod_{n=1}^{\infty} (1 - e^{-2nt})(1 + 2e^{-t(2n-1)} \cos x + e^{-t(4n-2)}).$$

By the Young inequality, $1 + e^{-t(4n-2)} \geq 2e^{-t(2n-1)}$, hence every term in the above product is nonnegative. As such, Φ itself is nonnegative in $(0, \infty) \times \mathbb{R}$. Meanwhile, integrating the series term-by-term, we obtain $\int_0^{2\pi} \Phi(t, x) \, dx = 1$ for every $t \in (0, \infty)$. It therefore follows that $\|\Phi(t, \cdot)\|_{L^1([0, 2\pi])} = \int_0^{2\pi} \Phi(t, x) \, dx = 1$ for all $t \in (0, \infty)$. Also, the Plancherel theorem yields $\|\Phi(t, \cdot)\|_{L^2([0, 2\pi])}^2 = 1 + \sum_{n=1}^{\infty} e^{-2n^2 t}$. Integrating in time (using also $\sum n^{-2} < +\infty$) yields (4.3).

Step 3 (constant L^2 -norm). Furthermore, given any $(t, x) \in (0, T] \times [0, 2\pi]$, suc-

cessive changes of variables yield

$$\begin{aligned} \int_0^t \int_0^{2\pi} \Phi(t-s, x-y)^2 dy ds &= \int_0^t \int_0^{2\pi} \Phi(\tau, x-y)^2 dy d\tau \\ &= \int_0^t \int_0^x \Phi(\tau, w)^2 dw d\tau + \int_0^t \int_{x-2\pi}^0 \Phi(\tau, w+2\pi)^2 dw d\tau \\ &= \int_0^t \int_0^{2\pi} \Phi(\tau, w)^2 dw d\tau, \end{aligned}$$

as required, where we also used the periodicity of Φ to obtain the second equality.

Step 4 (convergence to initial data). It remains to verify the initial data. Fix any 2π -periodic C^2 function ψ . Then, the elementary theory of Fourier series shows that ψ admits the Fourier representation $\psi(x) = \frac{1}{2\pi}a_0 + \frac{1}{\pi} \sum_{n=1}^{\infty} a_n \cos nx + \frac{1}{\pi} \sum_{n=1}^{\infty} b_n \sin nx$ a.e. $x \in \mathbb{R}$, where, for $n \in \mathbb{N}$,

$$(B.1) \quad a_0 = \int_0^{2\pi} \psi(x) dx, \quad a_n = \int_0^{2\pi} \psi(x) \cos nx dx, \quad b_n = \int_0^{2\pi} \psi(x) \sin nx dx.$$

Note that due to the C^2 -regularity of ψ and its periodicity, two consecutive integrations by parts yield $a_n = -\frac{1}{n^2} \int_0^{2\pi} \psi''(x) \cos nx dx$ and $b_n = -\frac{1}{n^2} \int_0^{2\pi} \psi''(x) \sin nx dx$ so that there exists a positive constant C_ψ depending on ψ alone such that

$$(B.2) \quad |a_n| + |b_n| \leq C_\psi n^{-2} \quad \forall n \in \mathbb{N}.$$

Hence, $|\psi(0) - \int_0^{2\pi} \Phi(t, x) \psi(x) dx| = \frac{1}{\pi} |\sum_{n=1}^{\infty} (1 - e^{-n^2 t}) a_n| \leq \frac{C_\psi}{\pi} \sum_{n=1}^{\infty} n^{-2} (1 - e^{-n^2 t})$, and, by the Dominated Convergence theorem applied to the atomic measure (using that $\sum n^{-2} < +\infty$), the right-hand side vanishes in the limit as $t \rightarrow 0^+$. \square

Proof of Lemma 4.3. Recall from Lemma 4.2 that $\Phi \in C^\infty((0, \infty) \times \mathbb{R})$. Since $\psi \in L^1([0, 2\pi])$, an application of the Dominated Convergence theorem shows that $\Psi \in C^\infty((0, \infty) \times \mathbb{R})$, and that for any $m, n \in \mathbb{N} \cup \{0\}$, there holds $\partial_t^{(m)} \partial_x^{(n)} \Psi(t, x) = \int_0^{2\pi} \partial_t^{(m)} \partial_x^{(n)} \Phi(t, x-y) \psi(y) dy$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$. In turn, we directly obtain that $\partial_t \Psi - \partial_{xx} \Psi = 0$ pointwise in $(0, \infty) \times \mathbb{R}$.

It remains to verify the initial condition. Suppose for the time being that ψ is smooth enough, say $\psi \in C_{per}^2([0, 2\pi])$. From the Fourier series decomposition (B.1) for ψ we compute, using the Fubini–Tonelli theorem (or directly using the convolution result for Fourier series), the Fourier coefficients of Ψ :

$$(B.3) \quad \tilde{a}_0(t) = a_0, \quad \tilde{a}_n(t) = e^{-n^2 t} a_n, \quad \tilde{b}_n(t) = e^{-n^2 t} b_n \quad \text{for } n \in \mathbb{N}.$$

Using the Plancherel theorem for Fourier series of square integrable functions we get (up to a multiplicative constant)

$$(B.4) \quad \|\Psi(t, \cdot) - \psi\|_{L^2([0, 2\pi])}^2 = \sum_{n=1}^{\infty} (1 - e^{-n^2 t})^2 (a_n^2 + b_n^2) \rightarrow 0 \quad \text{as } t \rightarrow 0^+,$$

where we used the decay estimate (B.2) and the Dominated Convergence theorem applied to the atomic measure.

When ψ is only $L_{per}^1([0, 2\pi])$, we consider $\{\psi_m\}_{m \in \mathbb{N}}$ a sequence in $C_{per}^2([0, 2\pi])$

approximating ψ in $L^1([0, 2\pi])$. Then, given any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|\psi - \psi_m\|_{L^1([0, 2\pi])} < \varepsilon/3 \forall m \geq N(\varepsilon)$. We then have

$$\begin{aligned} \|\Psi(t, \cdot) - \psi\|_{L^1([0, 2\pi])} &\leq \|\Psi(t, \cdot) - \Psi_m(t, \cdot)\|_{L^1([0, 2\pi])} + \|\Psi_m(t, \cdot) - \psi_m\|_{L^1([0, 2\pi])} \\ &\quad + \|\psi_m - \psi\|_{L^1([0, 2\pi])}, \end{aligned}$$

where $\Psi_m = \int_0^{2\pi} \Phi(t, x - y) \psi_m(y) dy$. Note that the second term in the previous expression vanishes in the limit as $t \rightarrow 0^+$ by the argument in the previous paragraph. Next, using the Fubini–Tonelli theorem to exchange the order of the integrals, $\|\Psi(t, \cdot) - \Psi_m(t, \cdot)\|_{L^1([0, 2\pi])} \leq \|\Phi(t, \cdot)\|_{L^1([0, 2\pi])} \|\psi - \psi_m\|_{L^1([0, 2\pi])} = \|\psi - \psi_m\|_{L^1([0, 2\pi])}$, where we used from Lemma 4.2 that $\|\Phi(t, \cdot)\|_{L^1([0, 2\pi])} = 1$ for every $t \in (0, \infty)$. It follows that $\|\Psi(t, \cdot) - \psi\|_{L^1([0, 2\pi])} < 2\varepsilon/3 + \|\Psi_m - \psi_m\|_{L^1([0, 2\pi])}$ for all $m \geq N(\varepsilon)$. Thus, having fixed m large enough, by now letting $t \rightarrow 0^+$ and using the result from the previous paragraph, we get that $\|\Psi(t, \cdot) - \psi\|_{L^1([0, 2\pi])}$ vanishes in the limit as $t \rightarrow 0^+$. \square

Appendix C. Approximation lemma.

LEMMA C.1. *Let $f \in L^2_{per}(\Omega; (H^1_{per})'(0, 2\pi))$ be nonnegative and such that $\rho(\mathbf{x}) := \langle f(\mathbf{x}, \cdot), 1 \rangle_{(H^1_{per})' \times H^1_{per}} \in [0, 1]$ for a.e. $\mathbf{x} \in \Omega$. There exists a sequence $(f_\varepsilon)_{\varepsilon>0}$ of nonnegative elements of $C^\infty_{per}(\Upsilon)$ such that*

$$(C.1) \quad \|f - f_\varepsilon\|_{L^2_{per}(\Omega; (H^1_{per})'(0, 2\pi))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

with $\rho_\varepsilon(\mathbf{x}) = \int_0^{2\pi} f_\varepsilon(\mathbf{x}, \theta) d\theta \in [0, 1]$ for a.e. $\mathbf{x} \in \Omega$.

Proof. Step 1 (defining the approximation). Let $\beta \in C_c^\infty(\mathbb{R}^2)$ and $\gamma \in C_c^\infty(\mathbb{R})$ be nonnegative smooth bump functions, chosen such that β is compactly supported inside Ω and γ is compactly supported inside the interval $[0, 2\pi)$, and without loss of generality

$$(C.2) \quad \left(\int_\Omega \beta(\mathbf{y}) d\mathbf{y} \right) \left(\int_0^{2\pi} \gamma(\theta) d\theta \right) = 1.$$

Moreover, we impose that β is radially symmetric about the point (π, π) and that γ is symmetric about the point π . Consider the associated sequences $(\beta_\varepsilon)_{\varepsilon>0}$ and $(\gamma_\varepsilon)_{\varepsilon>0}$

$$\beta_\varepsilon(\mathbf{x}) := \varepsilon^{-2} \beta(\mathbf{x}/\varepsilon), \quad \gamma_\varepsilon(\theta) := \varepsilon^{-1} \gamma(\theta/\varepsilon)$$

for $\alpha > 2$. These functions are smooth and nonnegative, and β_ε is compactly supported in the square $\varepsilon\Omega$ while γ_ε is compactly supported in $[0, 2\pi\varepsilon]$. We define $\tilde{\gamma}_\varepsilon$ to be the 2π -periodic extension of γ_ε from $[0, 2\pi\varepsilon]$ to all of \mathbb{R} , so that $\tilde{\gamma}_\varepsilon \in H^1_{per}(0, 2\pi)$. Similarly, we define $\tilde{\beta}_\varepsilon$ to be the Ω -periodic extension of β_ε to all of \mathbb{R}^2 .

In turn, we define the sequence of approximations $(f_\varepsilon)_{\varepsilon>0}$ by the explicit formula

$$f_\varepsilon(\mathbf{x}, \theta) := \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) \langle f(\mathbf{y}, \cdot), \tilde{\gamma}_\varepsilon(\theta - \cdot) \rangle d\mathbf{y} \quad \text{a.e. } (\mathbf{x}, \theta) \in \Upsilon,$$

where, throughout the rest of this proof, the duality product is understood to be in the angle variable only, and denotes the duality product of $H^1_{per}(0, 2\pi)$.

Observe first that, due to the assumption that f is nonnegative, we immediately have f_ε nonnegative a.e. in Υ . Moreover, one may readily check (for instance, by

means of difference quotients) that f_ε is smooth. Similarly, it is straightforward to verify that f_ε is itself Υ -periodic. Thus, $f_\varepsilon \in C_{per}^\infty(\tilde{\Upsilon})$ for every $\varepsilon > 0$.

Step 2 (boundedness of the approximate density). Note that, in view of the aforementioned nonnegativity of f_ε , there holds the inequality $0 \leq \int_0^{2\pi} f_\varepsilon(\mathbf{x}, \theta) d\theta = \rho_\varepsilon(\mathbf{x})$. Additionally, we have $\rho_\varepsilon(\mathbf{x}) = \int_0^{2\pi} (\int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) \langle f(\mathbf{y}, \cdot), \tilde{\gamma}_\varepsilon(\theta - \cdot) \rangle d\mathbf{y}) d\theta$, and a standard argument involving Riemann sums and the Arzelà–Ascoli theorem (using also the Fubini–Tonelli theorem and the Dominated Convergence theorem) leads to being able to commute the duality bracket with the integral in the angle variable, i.e.,

$$(C.3) \quad \rho_\varepsilon(\mathbf{x}) = \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) \left\langle f(\mathbf{y}, \cdot), \int_0^{2\pi} \tilde{\gamma}_\varepsilon(\theta - \cdot) d\theta \right\rangle d\mathbf{y}.$$

Observe now that the integral inside the duality bracket is a constant. Indeed, since $\tilde{\gamma}_\varepsilon$ is smooth, and therefore bounded over any compact interval, the Dominated Convergence theorem yields $\frac{d}{ds} \int_0^{2\pi} \tilde{\gamma}_\varepsilon(\theta - s) d\theta = - \int_0^{2\pi} \tilde{\gamma}_\varepsilon'(\theta - s) d\theta = \tilde{\gamma}_\varepsilon(-s) - \tilde{\gamma}_\varepsilon(2\pi - s) = 0$, where the final equality follows from the periodicity of $\tilde{\gamma}_\varepsilon$; recall that it is the periodic extension to all of \mathbb{R} of the original mollifier γ_ε . It follows that

$$(C.4) \quad \int_0^{2\pi} \tilde{\gamma}_\varepsilon(\theta - s) d\theta = \int_0^{2\pi} \tilde{\gamma}_\varepsilon(\theta) d\theta = \int_0^{2\pi} \gamma_\varepsilon(\theta) d\theta = \int_0^{2\pi} \gamma(\theta) d\theta \quad \forall s \in \mathbb{R},$$

where we recall in passing that γ_ε is supported in $[0, 2\pi\varepsilon^\alpha]$, while γ is supported in $[0, 2\pi]$. Hence, we get $\rho_\varepsilon(\mathbf{x}) = (\int_0^{2\pi} \gamma(\theta) d\theta) (\int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) \langle f(\mathbf{y}, \cdot), 1 \rangle d\mathbf{y})$.

Also, applying the Dominated Convergence theorem and the Fubini–Tonelli theorem gives $\partial_{x_1} \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) d\mathbf{y} = \int_0^{2\pi} \int_0^{2\pi} \partial_1 \tilde{\beta}_\varepsilon(x_1 - y_1, x_2 - y_2) dy_1 dy_2 = - \int_0^{2\pi} (\tilde{\beta}_\varepsilon(x_1 - 2\pi, x_2 - y_2) - \tilde{\beta}_\varepsilon(x_1, x_2 - y_2)) dy_2 = 0$, where the final equality follows from the periodicity of $\tilde{\beta}_\varepsilon$; recall that this latter function is the Ω -periodic extension of β_ε . An identical argument shows that $\partial_{x_2} \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) d\mathbf{y} = 0$. Arguing as per (C.4) and making use of the symmetry property of β about the point (π, π) , we deduce that

$$(C.5) \quad \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) d\mathbf{y} = \int_\Omega \beta(\mathbf{y}) d\mathbf{y} \quad \forall \mathbf{x} \in \mathbb{R}^2.$$

We emphasize that the above property is not true for the function β_ε ; one must take its periodic extension in order to obtain this translation invariance, since the integral ranges only over the square Ω , and not over all of \mathbb{R}^2 .

Returning to (C.3) and using that $\langle f(\mathbf{y}, \cdot), 1 \rangle = \rho(\mathbf{y})$, which takes values in the interval $[0, 1]$, it follows that $0 \leq \rho_\varepsilon(\mathbf{x}) \leq (\int_0^{2\pi} \gamma(\theta) d\theta) (\int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) d\mathbf{y}) = (\int_0^{2\pi} \gamma(\theta) d\theta) (\int_\Omega \beta(\mathbf{y}) d\mathbf{y}) = 1$, as required, where we used the nonnegativity of $\tilde{\beta}_\varepsilon$ in the second inequality and (C.2) to obtain the final equality.

Step 3 (convergence of approximations). We now study the convergence of the sequence $(f_\varepsilon)_{\varepsilon>0}$ in $L_{per}^2(\Omega; (H_{per}^1)'(0, 2\pi))$. Recall that

$$\|f(\mathbf{x}, \cdot) - f_\varepsilon(\mathbf{x}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)} = \sup_{\substack{\varphi \in C_{per}^\infty(0, 2\pi) \\ \|\varphi\|_{H^1(0, 2\pi)} \leq 1}} |\langle f(\mathbf{x}, \cdot) - f_\varepsilon(\mathbf{x}, \cdot), \varphi \rangle|$$

for a.e. $\mathbf{x} \in \Omega$, where the duality product above is that of $(H_{per}^1)'(0, 2\pi)$. Note that, due to the regularity of f_ε , one may write explicitly

$$(C.6) \quad \langle f(\mathbf{x}, \cdot) - f_\varepsilon(\mathbf{x}, \cdot), \varphi \rangle = \langle f(\mathbf{x}, \cdot), \varphi \rangle - \int_0^{2\pi} \left(\int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) \langle f(\mathbf{y}, \cdot), \tilde{\gamma}_\varepsilon(\theta - \cdot) \rangle d\mathbf{y} \right) \varphi(\theta) d\theta.$$

Using also the Fubini–Tonelli theorem and the Dominated Convergence theorem, a standard argument involving Riemann sums and the Arzelà–Ascoli theorem shows that this latter term is equal to $\int_{\Omega} \tilde{\beta}_{\varepsilon}(\mathbf{x} - \mathbf{y}) \langle f(\mathbf{y}, \cdot), \int_0^{2\pi} \tilde{\gamma}_{\varepsilon}(\theta - \cdot) \varphi(\theta) d\theta \rangle d\mathbf{y}$. Using the equalities in (C.2), (C.4), and (C.5), the first term on the right-hand side of (C.6) can be written as $\langle f(\mathbf{x}, \cdot), \varphi \rangle = \int_{\Omega} \tilde{\beta}_{\varepsilon}(\mathbf{x} - \mathbf{y}) \langle f(\mathbf{x}, \cdot), \varphi(\cdot) \int_0^{2\pi} \tilde{\gamma}_{\varepsilon}(\theta) d\theta \rangle d\mathbf{y}$ a.e. $\mathbf{x} \in \Omega$. It therefore follows from (C.6) that

(C.7)

$$\begin{aligned} \langle f(\mathbf{x}, \cdot) - f_{\varepsilon}(\mathbf{x}, \cdot), \varphi \rangle &= \int_{\Omega} \tilde{\beta}_{\varepsilon}(\mathbf{x} - \mathbf{y}) \left\langle f(\mathbf{x}, \cdot) - f(\mathbf{y}, \cdot), \varphi(\cdot) \int_0^{2\pi} \tilde{\gamma}_{\varepsilon}(\theta) d\theta \right\rangle d\mathbf{y} \\ &\quad + \int_{\Omega} \tilde{\beta}_{\varepsilon}(\mathbf{x} - \mathbf{y}) \left\langle f(\mathbf{y}, \cdot), \varphi(\cdot) \int_0^{2\pi} \tilde{\gamma}_{\varepsilon}(\theta) d\theta - \int_0^{2\pi} \tilde{\gamma}_{\varepsilon}(\theta - \cdot) \varphi(\theta) d\theta \right\rangle d\mathbf{y} \\ &=: I_{\varepsilon}(\mathbf{x}) + J_{\varepsilon}(\mathbf{x}). \end{aligned}$$

The term I_{ε} may be rewritten as $I_{\varepsilon}(\mathbf{x}) = c_{\gamma} \int_{\Omega} \tilde{\beta}_{\varepsilon}(\mathbf{x} - \mathbf{y}) \langle f(\mathbf{x}, \cdot) - f(\mathbf{y}, \cdot), \varphi \rangle d\mathbf{y}$, where $c_{\gamma} = \int_0^{2\pi} \gamma(\theta) d\theta$. Using that $\|\varphi\|_{H_{per}^1(0, 2\pi)} \leq 1$, we get

$$(C.8) \quad |I_{\varepsilon}(\mathbf{x})| \leq c_{\gamma} \int_{\Omega} \tilde{\beta}_{\varepsilon}(\mathbf{x} - \mathbf{y}) \|f(\mathbf{x}, \cdot) - f(\mathbf{y}, \cdot)\|_{(H_{per}^1(0, 2\pi))'} d\mathbf{y};$$

however, for the time being, we concentrate on the term J_{ε} . We have

(C.9)

$$|J_{\varepsilon}(\mathbf{x})| \leq \int_{\Omega} \tilde{\beta}_{\varepsilon}(\mathbf{x} - \mathbf{y}) \|f(\mathbf{y}, \cdot)\|_{(H_{per}^1(0, 2\pi))'} \left\| \int_0^{2\pi} (\tilde{\gamma}_{\varepsilon}(\theta) \varphi(\cdot) - \tilde{\gamma}_{\varepsilon}(\theta - \cdot) \varphi(\theta)) d\theta \right\|_{H^1(0, 2\pi)} d\mathbf{y},$$

and, in view of (C.4), we notice that for every $s \in \mathbb{R}$, $\int_0^{2\pi} (\tilde{\gamma}_{\varepsilon}(\theta) \varphi(s) - \tilde{\gamma}_{\varepsilon}(\theta - s) \varphi(\theta)) d\theta = - \int_{-s}^{2\pi-s} \tilde{\gamma}_{\varepsilon}(w) (\varphi(w+s) - \varphi(s)) dw$. Thus, using also the Dominated Convergence theorem to differentiate under the integral in the previous right-hand side, we get, for every $s \in \mathbb{R}$, $\frac{d}{ds} \int_0^{2\pi} (\tilde{\gamma}_{\varepsilon}(\theta) \varphi(s) - \tilde{\gamma}_{\varepsilon}(\theta - s) \varphi(\theta)) d\theta = - \int_{-s}^{2\pi-s} \tilde{\gamma}_{\varepsilon}(w) (\varphi'(w+s) - \varphi'(s)) dw$, where we remark that the derivative contribution due to the integral limits vanishes due to the fact that $\tilde{\gamma}_{\varepsilon}$ and φ are 2π -periodic. It follows that

$$\begin{aligned} &\left\| \int_0^{2\pi} \tilde{\gamma}_{\varepsilon}(\theta) \varphi(\cdot) - \tilde{\gamma}_{\varepsilon}(\theta - \cdot) \varphi(\theta) d\theta \right\|_{H_{per}^1(0, 2\pi)}^2 \\ (C.10) \quad &= \int_0^{2\pi} \left| \int_{-s}^{2\pi-s} \tilde{\gamma}_{\varepsilon}(w) (\varphi(w+s) - \varphi(s)) dw \right|^2 ds \\ &\quad + \int_0^{2\pi} \left| \int_{-s}^{2\pi-s} \tilde{\gamma}_{\varepsilon}(w) (\varphi'(w+s) - \varphi'(s)) dw \right|^2 ds =: K_{\varepsilon}^1 + K_{\varepsilon}^2. \end{aligned}$$

Using the Fundamental Theorem of Calculus along with the fact that $\|\varphi^{(m)}\|_{L^{\infty}(\mathbb{R})} < +\infty$ for any $m \in \mathbb{N} \cup \{0\}$ due to the smoothness and periodicity of φ , we get $|\int_{-s}^{2\pi-s} \tilde{\gamma}_{\varepsilon}(w) (\varphi(w+s) - \varphi(s)) dw| \leq \|\varphi'\|_{L^{\infty}(\mathbb{R})} \int_{-s}^{2\pi-s} \tilde{\gamma}_{\varepsilon}(w) |w| dw$, and (using the change of variables $\theta = w + s$)

$$K_{\varepsilon}^1 \leq \|\varphi'\|_{L^{\infty}(\mathbb{R})}^2 \int_0^{2\pi} \left| \int_0^{2\pi} \tilde{\gamma}_{\varepsilon}(\theta - s) |\theta - s| d\theta \right|^2 ds.$$

Our strategy is to split the previous right-hand side into three regions in order to bound it. While s is constrained to the interval $[0, 2\pi(1 - \varepsilon^{\alpha})]$, since $\theta \in [0, 2\pi]$, we

have that $-2\pi(1 - \varepsilon^\alpha) \leq \theta - s \leq 2\pi$, in which case $\tilde{\gamma}_\varepsilon(\theta - s)$ is nonzero only when $\theta - s \in [0, 2\pi\varepsilon^\alpha]$; we call this *Region 1*. On the other hand, when s is constrained to the interval $[2\pi(1 - \varepsilon^\alpha), 2\pi]$, since $\theta \in [0, 2\pi]$, we have that $-2\pi \leq \theta - s \leq 2\pi\varepsilon^\alpha$, in which case $\tilde{\gamma}_\varepsilon(\theta - s)$ is nonzero when $\theta - s \in [-2\pi, -2\pi(1 - \varepsilon^\alpha)] \cup [0, 2\pi\varepsilon^\alpha]$. We therefore have two more regions; *Region 2*, where $s \in [2\pi(1 - \varepsilon^\alpha), 2\pi]$ and $\theta - s \in [0, 2\pi\varepsilon^\alpha]$, and *Region 3*, where $s \in [2\pi(1 - \varepsilon^\alpha), 2\pi]$ and $\theta - s \in [-2\pi, -2\pi(1 - \varepsilon^\alpha)]$. It follows from the triangle inequality and the fact that the function $x \mapsto x^2$ is increasing on $[0, \infty)$ that K_ε^1 is bounded above by the sum of the following three terms, each corresponding to Regions 1, 2, and 3, respectively:

$$\begin{aligned} L_\varepsilon^1 &:= 4\pi^2 \varepsilon^{2\alpha} \|\varphi'\|_{L^\infty(\mathbb{R})}^2 \int_0^{2\pi(1-\varepsilon^\alpha)} \left(\int_0^{2\pi} \tilde{\gamma}_\varepsilon(\theta - s) d\theta \right)^2 ds, \\ L_\varepsilon^2 &:= 4\pi^2 \varepsilon^{2\alpha} \|\varphi'\|_{L^\infty(\mathbb{R})}^2 \int_{2\pi(1-\varepsilon^\alpha)}^{2\pi} \left(\int_{2\pi(1-\varepsilon^\alpha)}^{2\pi} \tilde{\gamma}_\varepsilon(\theta - s) d\theta \right)^2 ds, \\ L_\varepsilon^3 &:= \|\varphi'\|_{L^\infty(\mathbb{R})}^2 \int_{2\pi(1-\varepsilon^\alpha)}^{2\pi} \left(\int_0^{2\pi\varepsilon^\alpha} \tilde{\gamma}_\varepsilon(\theta - s)(s - \theta) d\theta \right)^2 ds, \end{aligned}$$

and it is straightforward to verify, using the nonnegativity of $\tilde{\gamma}_\varepsilon$ and (C.4), the estimates $|L_\varepsilon^1| + |L_\varepsilon^2| \leq C\varepsilon^{2\alpha}$ and $|L_\varepsilon^3| \leq C\varepsilon^\alpha$ for some positive constant C depending only on c_γ and $\|\varphi'\|_{L^\infty(\mathbb{R})}$. It follows that for a possibly larger constant C depending only on c_γ and $\|\varphi'\|_{L^\infty(\mathbb{R})}$, there holds $K_\varepsilon^1 \leq C\varepsilon^\alpha$ and, by the same argument, the same bound holds for K_ε^2 (with the only change being that the constant C also depends on $\|\varphi''\|_{L^\infty(\mathbb{R})}$). We therefore obtain from (C.10), for a possibly larger constant C depending only on c_γ , $\|\varphi'\|_{L^\infty(\mathbb{R})}$, and $\|\varphi''\|_{L^\infty(\mathbb{R})}$, that there holds

$$\left\| \int_0^{2\pi} \tilde{\gamma}_\varepsilon(\theta) \varphi(\cdot) - \tilde{\gamma}_\varepsilon(\theta - \cdot) \varphi(\theta) d\theta \right\|_{H_{per}^1(0, 2\pi)} \leq C\varepsilon^{\alpha/2}.$$

Thus, returning to the bound (C.9) on $|J_\varepsilon|$, taking supremum over test functions and integrating, followed by applying the Jensen inequality and using the monotonicity of the functions $x \mapsto x^2$ and $x \mapsto x^{1/2}$ on $[0, \infty)$, we get

$$(C.11) \quad \left\| \sup_{\substack{\varphi \in C_{per}^\infty(0, 2\pi) \\ \|\varphi\|_{H^1(0, 2\pi)} \leq 1}} J_\varepsilon \right\|_{L_{per}^2(\Omega)} \leq C_\Omega \varepsilon^{\alpha/2} \left(\int_\Omega \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y})^2 \|f(\mathbf{y}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)}^2 d\mathbf{y} d\mathbf{x} \right)^{1/2},$$

where C_Ω is a positive constant depending only on $|\Omega|$, c_γ , $\|\varphi'\|_{L^\infty(\mathbb{R})}$, and $\|\varphi''\|_{L^\infty(\mathbb{R})}$. Using the Fubini–Tonelli theorem to evaluate the double integral exactly, using also the same procedure as the one used for (C.5) to get $\int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y})^2 d\mathbf{x} = \varepsilon^{-2} \int_\Omega \beta(\mathbf{y})^2 d\mathbf{y}$,

$$(C.12) \quad \left\| \sup_{\substack{\varphi \in C_{per}^\infty(0, 2\pi) \\ \|\varphi\|_{H^1(0, 2\pi)} \leq 1}} J_\varepsilon \right\|_{L_{per}^2(\Omega)} \leq C_\Omega \varepsilon^{\frac{\alpha}{2}-1} \|f\|_{L_{per}^2(\Omega; (H_{per}^1)'(0, 2\pi))}$$

for some new positive constant C_Ω which again only depends on $|\Omega|$, c_γ , $\|\varphi'\|_{L^\infty(\mathbb{R})}$, and $\|\varphi''\|_{L^\infty(\mathbb{R})}$. Notice that since $\alpha > 2$, the above vanishes in the limit as $\varepsilon \rightarrow 0$.

Finally, we return to the term I_ε . By taking the supremum over test functions and integrating, we find that $\left\| \sup_{\substack{\varphi \in C_{per}^\infty(0, 2\pi) \\ \|\varphi\|_{H^1(0, 2\pi)} \leq 1}} I_\varepsilon \right\|_{L_{per}^2(\Omega)}$ is bounded above by

$$c_\gamma \left(\int_\Omega \left| \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{x} - \mathbf{y}) \|f(\mathbf{x}, \cdot) - f(\mathbf{y}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)} d\mathbf{y} \right|^2 d\mathbf{x} \right)^{1/2} =: c_\gamma L_\varepsilon.$$

We rewrite the innermost integral above using the change of variables $\mathbf{y} \mapsto \mathbf{z} = \mathbf{x} - \mathbf{y}$. Note that $\mathbf{z} \in [-2\pi, 2\pi] \times [-2\pi, 2\pi]$ and, moreover, $\tilde{\beta}_\varepsilon(\mathbf{z})$ is only nonzero when $\mathbf{z} \in \bigcup_{j=1}^4 A_\varepsilon^j$, where $A_\varepsilon^1 = [-2\pi, -2\pi(1-\varepsilon)] \times [0, 2\pi\varepsilon]$, $A_\varepsilon^2 = [0, 2\pi\varepsilon] \times [0, 2\pi\varepsilon]$, $A_\varepsilon^3 = [-2\pi, -2\pi(1-\varepsilon)] \times [-2\pi, -2\pi(1-\varepsilon)]$, and $A_\varepsilon^4 = [0, 2\pi\varepsilon] \times [-2\pi, -2\pi(1-\varepsilon)]$. It therefore follows that, using the triangle inequality for the norm of $L^2(\Omega)$ in the variable \mathbf{x} , there holds the inequality

$$L_\varepsilon \leq \sum_{j=1}^4 \left(\int_{\Omega} \left| \int_{A_\varepsilon^j} \tilde{\beta}_\varepsilon(\mathbf{z}) \|f(\mathbf{x}, \cdot) - f(\mathbf{x} - \mathbf{z}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)} d\mathbf{z} \right|^2 d\mathbf{x} \right)^{1/2} =: \sum_{j=1}^4 L_\varepsilon^j.$$

Using the Minkowski integral inequality for finite measure spaces, we get

$$(C.13) \quad L_\varepsilon^j \leq \int_{A_\varepsilon^j} \tilde{\beta}_\varepsilon(\mathbf{z}) \left(\int_{\Omega} \|f(\mathbf{x}, \cdot) - f(\mathbf{x} - \mathbf{z}, \cdot)\|_{(H_{per}^1)'(0, 2\pi)}^2 d\mathbf{x} \right)^{1/2} d\mathbf{z}.$$

We now have the following claim, the proof of which we omit as it is a standard exercise. The proof relies on observing that $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x} - \lim_{j \rightarrow \infty} \mathbf{z}_j)$ in $(H_{per}^1)'(0, 2\pi)$ for a.e. $\mathbf{x} \in \Omega$ and then integrating in \mathbf{x} . In order to pass to the limit in j , one needs to use the strong continuity property of the shift operator in $L_{per}^2(\Omega; X)$, i.e., $\lim_{\mathbf{w} \rightarrow 0} \|\tau_{\mathbf{w}} \mathbf{f} - \mathbf{f}\|_{L_{per}^2(\Omega; X)} = 0$, where $\tau_{\mathbf{w}} \mathbf{f} = \mathbf{f}(\cdot + \mathbf{w})$. The proof of this latter result is also straightforward (relying on identifying a convenient dense subset and applying the Dominated Convergence theorem) and so we omit it.

CLAIM C.2. *Let $f \in L_{per}^2(\Omega; (H_{per}^1)'(0, 2\pi))$. Suppose that $(\mathbf{z}_j)_{j \in \mathbb{N}}$ is a sequence of points in \mathbb{R}^2 converging towards the point $(2\pi m, 2\pi n) \in \mathbb{R}^2$ for some fixed $m, n \in \mathbb{Z}$. Then, $\lim_{j \rightarrow \infty} (\int_{\Omega} \|f(\mathbf{x}, \cdot) - f(\mathbf{x} - \mathbf{z}_j, \cdot)\|_{(H_{per}^1)'(0, 2\pi)}^2 d\mathbf{x})^{1/2} = 0$.*

By rewriting $\tilde{\beta}_\varepsilon(\mathbf{z})$ in each of the squares $\{A_\varepsilon^j\}_{j=1}^4$ in terms of the original bump function β , using the previous claim, and applying the Dominated Convergence theorem to the right-hand side of (C.13), we get $\lim_{\varepsilon \rightarrow 0} L_\varepsilon^j = 0$ for $j = 1, \dots, 4$. Hence $\|\sup_{\varphi \in C_{per}^\infty(0, 2\pi)} I_\varepsilon\|_{L_{per}^2(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. This, (C.12), and (C.7) prove (C.1). \square

Acknowledgments. This work was carried out while AE and SMS were post-doctoral researchers at FAU Erlangen-Nürnberg and the University of Cambridge, respectively.

REFERENCES

- [1] H. AMANN, *Existence and regularity for semilinear parabolic evolution equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 11 (1984), pp. 593–676, http://www.numdam.org/item?id=ASNSP_1984_4_11_4_593_0.
- [2] H. AMANN, *Global existence for semilinear parabolic systems*, J. Reine Angew. Math., 360 (1985), pp. 47–83, <https://doi.org/10.1515/crll.1985.360.47>.
- [3] L. AMBROSIO, N. GIGLI, AND G. SAVARÉ, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures Math. ETH Zürich, Birkhäuser Verlag, Basel, 2008.
- [4] M. BRUNA, M. BURGER, A. ESPOSITO, AND S. M. SCHULZ, *Phase separation in systems of interacting active Brownian particles*, SIAM J. Appl. Math., 82 (2022), pp. 1635–1660, <https://doi.org/10.1137/21M1452524>.
- [5] M. BRUNA, M. BURGER, J.-F. PIETSCHMANN, AND M.-T. WOLFRAM, *Active Crowds*, preprint, <https://arxiv.org/abs/2107.06392>, 2021.
- [6] R. CANNON, J., *The One-Dimensional Heat Equation*, Encyclopedia Math. Appl. 23, Addison-Wesley, Reading, MA, 1984.

- [7] J. A. CARRILLO, S. LISINI, G. SAVARÉ, AND D. SLEPČEV, *Nonlinear mobility continuity equations and generalized displacement convexity*, J. Funct. Anal., 258 (2010), pp. 1273–1309.
- [8] E. CRISTIANI, B. PICCOLI, AND A. TOSIN, *Multiscale Modeling of Pedestrian Dynamics*, MS&A. Model. Simul. Appl. 12, Springer, Cham, 2014.
- [9] J. DOLBEAULT, B. NAZARET, AND G. SAVARÉ, *A new class of transport distances between measures*, Calc. Var. Partial Differential Equations, 34 (2009), pp. 193–231.
- [10] L. EVANS, *Partial Differential Equations*, 2nd ed., Grad. Stud. Math. 19, American Mathematical Society, Providence, RI, 2010.
- [11] S. GOLDSTEIN, *On diffusion by discontinuous movements, and on the telegraph equation*, Quart. J. Mech. Appl. Math., 4 (1951), pp. 129–156.
- [12] G. GOMPPER, R. G. WINKLER, T. SPECK, A. SOLON, C. NARDINI, F. PERUANI, H. LÖWEN, R. GOLESTANIAN, U. B. KAUPP, L. ALVAREZ, T. KJØRBOE, E. LAUGA, W. C. K. POON, A. DESIMONE, S. MUIÑOS-LANDIN, A. FISCHER, N. A. SÖKER, F. CICHOS, R. KAPRAL, P. GASPARD, M. RIPOLL, F. SAGUES, A. DOOSTMOHAMMADI, J. M. YEOMANS, I. S. ARANSON, C. BECHINGER, H. STARK, C. K. HEMELRIJK, F. J. NEDELEC, T. SARKAR, T. ARYAKSAMA, M. LACROIX, G. DUCLOS, V. YASHUNSKY, P. SILBERZAN, M. ARROYO, AND S. KALE, *The 2020 motile active matter roadmap*, J. Phys.: Condens. Matter, 32 (2020), 193001.
- [13] D. HELBING AND P. MOLNAR, *Social force model for pedestrian dynamics*, Phys. Rev. E, 51 (1995), 4282.
- [14] M. KOURBANE-HOUSSENE, C. ERIGNOUX, T. BODINEAU, AND J. TAILLEUR, *Exact hydrodynamic description of active lattice gases*, Phys. Rev. Lett., 120 (2018), 268003.
- [15] O. LADYZHENSKAYA, V. A. SOLONNIKOV, AND N. N. URALCEVA, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs 23, American Mathematical Society, Providence, RI, 1968.
- [16] T. SPECK, A. M. MENZEL, J. BIALKÉ, AND H. LÖWEN, *Dynamical mean-field theory and weakly non-linear analysis for the phase separation of active Brownian particles*, J. Chem. Phys., 142 (2015), 224109.
- [17] G. I. TAYLOR, *Diffusion by continuous movements*, Proc. London Math. Soc. (2), 20 (1921), pp. 196–212.
- [18] R. TEMAM, *Navier-Stokes Equations. Theory and Numerical Analysis*, Studies in Mathematics and its Applications, Vol. 2, North-Holland, Amsterdam, New York, Oxford, 1977.
- [19] E. WHITTAKER AND N. WATSON, *A Course of Modern Analysis*, Cambridge Mathematical Library, 4th ed., Cambridge University Press, Cambridge, UK, 1927.