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RESEARCH ARTICLE

Ambiguity-Averse Aggregation under Heterogeneous Beliefs

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Abstract

If a complete-markets economy is composed of expected utility maximizers with heterogeneous beliefs and logarithmic utility, then it may be well represented via a single expected utility-maximizing agent with a "consensus belief". For more general preferences, an "aggregation bias" necessitates alteration of the economy's fundamentals if such a representation is to be constructed. But no such aggregation bias arises if the representative agent is allowed to be ambiguity averse, in the sense that he maximizes his Choquet expected utility under a convex "consensus capacity". The economy thus becomes ambiguity averse in the aggregate as a consequence of heterogeneity in beliefs.

Keywords Complete markets · Heterogeneous agents · Aggregation · Representative agent · Ambiguity aversion · Choquet expected utility

JEL Classification: D46 · D59 · D81 · G12

1 Introduction

Constructing a representative agent from a weighted average of individual characteristics is a widely used approach to aggregation (Negishi 1960; Breeden and Litzenberger 1978; Constantinidès 1982). Under heterogeneous beliefs, consumers with logarithmic utility even have a "consensus belief", in the sense that their representative agent has a well-defined belief that is a weighted average of individual beliefs (Rubinstein 1974, 1976).[1] Such a belief may be derived under more general preferences in com-

[1] In fact, in the single-period markets studied here, agents with CARA utility also have a consensus belief (see Back 2017, §21.2).

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plete markets, subject to correction for an "aggregation bias" via an adjusted aggregate endowment (Calvet et al. 2018) or discount factor (Jouini and Napp 2006a, 2007). In this way, the complexities of diverse opinions may be bypassed with the simplicity of a single Savage (1954) decision-maker.

Here I show that no aggregation bias arises if, instead of a Savage decision-maker, the representative agent is allowed to be ambiguity averse. In particular, given a competitive equilibrium of a heterogeneous-beliefs expected utility economy, there exists an "equivalent equilibrium" in the economy where each consumer maximizes his Choquet expected utility (Schmeidler 1989) under a "consensus capacity", reproducing the original equilibrium prices and trading volumes whilst respecting marginal asset valuations in the manner required by Calvet et al. (2018) and Jouini and Napp (2006a, 2007). From this equivalent equilibrium, one may straightforwardly construct a representative agent with this same capacity, which is a non-additive probability that moreover is "convex" and hence exhibits ambiguity aversion. The equivalent equilibrium is then rendered unique by the requirement that the degree of ambiguity aversion be minimized. Thus, even with ambiguity-neutral Savage consumers, if they have heterogeneous beliefs, there is in general an ambiguity-averse representative agent. Indeed, such a representation is possible using the popular maxmin expected utility model (Gilboa and Schmeidler 1989) with a general compact, convex set of priors.

Intuitively, this representation is made possible by the indeterminacy of equilibria that can arise under ambiguity aversion, first noted by Dow and Werlang (1992), and further explored by Epstein and Wang (1994), before being shown to be nongeneric by Dana (2004) and Rigotti and Shannon (2012).[2] This nongenericity initially necessitates a restriction to economies with no aggregate risk, which at first appears strong. However, whilst nongeneric in a topological sense, Mandler (2013) shows that such economies are robust in an economic sense, since they emerge as the equilibrium outcome of ambiguity-averse agents choosing a level of investment in productive assets prior to trading: "Ambiguity aversion introduces a discontinuity in the rate of return on investment at just the points where investment results in indeterminacy; the discontinuity ensures that these particular investment levels arise robustly" (Mandler 2013, p. 742).

With this idea in mind, I show that the above consensus capacity extends to economies with aggregate endowment risk, via a representative economy with an initial stage of productive investment prior to trading. With sufficient ambiguity aversion in the consensus capacity, equilibrium choices must fully insure against aggregate risk, resulting in a trading economy exhibiting the equilibrium indeterminacy that allows construction of a convex consensus capacity. Thus, by modeling consumers as determining their subsequent endowments through such a prior productive investment, they are representable via a single ambiguity-averse agent. Such a general aggregative technique provides a strong argument that representative agents should be modelled as ambiguity averse wherever heterogeneous beliefs are thought to be important. That disagreement should have this effect on the 'average' consumer is intuitive, and offers a potential link between representative-agent models and financial-market measures of

[2] This theoretical literature is reviewed by Billot et al. (2020). For an overview of the many resulting applications, see Mukerji and Tallon (2004).

disagreement and uncertainty such as analyst forecast dispersion (Barron et al. 1998), which has been shown to be an important factor in empirical asset-pricing (Diether et al. 2002; Johnson 2004).

The literature on estimation of ambiguity aversion is broad, but hampered by the confounding of beliefs with ambiguity attitudes in the data. Beliefs may be identified in financial markets satisfying certain conditions (see, e.g., Ross 2015, Echenique and Saito 2015, and Kubler and Polemarchakis 2017), though this is under the standard expected utility model. Many measures of ambiguity itself have been proposed, starting with Dow and Werlang's (1992) use of the sum of Schmeidler's capacities, but also taking in entropy (Henry 2002), moment-based measures (Ui 2011; Maccheroni et al. 2013; Izhakian 2020) and confidence intervals (Bewley 2011; Boyle et al. 2011). Baillon et al. (2018) offer two indices to capture both the perceived level of ambiguity and ambiguity aversion itself. In the related area of robust decision-making under macroeconomic model uncertainty, Anderson et al. (2003) seek to recover multiple prior beliefs from the cautious behavior that they induce.

Heterogeneous beliefs pose modeling problems in a discipline dominated by common priors and the implied difficulty of "agreeing to disagree" (Aumann 1976), but at the same time, differences in beliefs are by now a well-documented phenomenon (Anderson 2005; Patton and Timmermann 2010; Buraschi et al. 2014; Carlin et al. 2014; Brunnermeier et al. 2021; Meeuwis et al. 2022; Giglio et al. 2021). An obvious response is to allow heterogeneous priors (Harrison and Kreps 1978) in the form of a "difference of opinion" over the model's fundamentals (Varian 1985, 1989; Abel 1999; Harris and Raviv 1993; Morris 1996; Detemple and Murthy 1994; Jouini and Napp 2007).[3] Models with such belief heterogeneity can account for various asset-pricing puzzles (Calvet et al. 2018; Jouini and Napp 2006a, 2007; Ottaviani and Sørensen 2015; Pohl et al. 2021), whilst a parallel literature has found that ambiguity aversion has similar explanatory power (Mukerji and Tallon 2001; Caballero and Krishnamurthy 2008; Chen et al. 2014; Ju and Miao 2012; Gollier 2011; Hansen and Sargent 2010; Uhlig 2010; Ilut and Schneider 2014; Collard et al. 2018). The approach to aggregation taken here offers a link between the two.

For this reason, the range of empirical phenomena that may be represented within the framework of ambiguity-averse aggregation is broad. For example, belief heterogeneity offers a possible resolution of the equity premium puzzle and the risk free rate puzzle (Mehra and Prescott 1985; Weil 1989), and this can be captured via a more flexible consumption-based capital asset pricing model (CCAPM) under ambiguity-averse aggregation, which I derive in the Appendix, along with a detailed example of the construction of the consensus capacity. First though, I outline in Section 2 the standard expected utility economy with heterogeneous beliefs, before going on in Section 3 to prove the general possibility of using ambiguity aversion for aggregation in this economy, and in Section 4 to consider the case of aggregate risk.

[3] Alternatively, heterogeneous beliefs might result from disagreement over the informativeness of a signal (Scheinkman and Xiong 2003; Lam et al. 2000; Dumas et al. 2009; Xiong and Yan 2010), in which case convergence of beliefs need not result even from an infinite sequence of signals (Acemoglu et al. 2016).

2 The Model

An economy contains I consumers, each with consumption set $\mathbb{R}_+$. There is a finite space $S \equiv \{1, \ldots, K\}$ of *states of the world*. A *consumption plan* $c_i : S \to \mathbb{R}_+$ is a measurable $\mathbb{R}_+$-valued function. Consumer i's *endowment*—or equivalently a portfolio of Arrow–Debreu securities— is a particular known consumption plan denoted ω_i, his *belief* the probability measure p^i on $(S, 2^S)$, and his utility function the standard

$$U_i(c_i) = \mathrm{E}_{p^i}\left(u_i(c_i(s))\right), \tag{1}$$

where $u_i : \mathbb{R}_+ \to [-\infty, \infty)$ is a payoff function on consumption. As usual, this utility function relies on the consumer's satisfaction of the Savage (1954) axioms. The following assumptions are basic to the model:

Axiom 1 *The payoff functions $u_i : \mathbb{R}_+ \to [-\infty, \infty)$: (i) are C^1, strictly concave and strictly monotonic; and (ii) satisfy an Inada condition at 0—i.e. $u_i'(c) \to \infty$ as $c \downarrow 0$.*

Axiom 2 *We have $\infty > \overline{\omega} = \sup_{s\in S} \sum_{j=1}^{I} \omega_j(s) \geq \inf_{s\in S} \sum_{j=1}^{I} \omega_j(s) = \underline{\omega} > 0$.*

Axiom 3 *For all $s \in S$ and all i, $p^i(s) > 0$.*

Existence of competitive equilibrium is guaranteed under these (quite standard) axioms.

With complete markets, each consumer i then faces the problem

$$\begin{aligned} &\max_{c_i} \mathrm{E}_{p^i}\left(u_i(c_i(s))\right), \quad \text{s.t.}: \\ &\sum_{s\in S} q(s)c_i(s) \leq \sum_{s\in S} q(s)\omega_i(s); \\ &c_i(s) \geq 0, \quad \text{for all } s \in S; \end{aligned} \tag{2}$$

where $q : S \to \mathbb{R}_+ \cup \{\infty\}$ are the *state prices* of the consumption good. The first-order condition (FOC) for agent i's optimal choice of consumption specifies a number $\lambda_i > 0$ such that, for all $s \in S$:

$$u_i'(c_i(s))p^i(s) - \lambda_i q(s) = 0. \tag{3}$$

A *competitive equilibrium* of course solves (2) for each agent i under market clearing, $\sum_{j=1}^{I} c_j(s) = \sum_{j=1}^{I} \omega_j(s)$ for all s.

This static economy is chosen as the simplest possible setting for the aggregation exercise, matching Calvet et al. (2018), and could be extended to a dynamic economy along the lines of Jouini and Napp (2006b, a, 2007). The only further substantive restriction I make is that there is initially *no aggregate risk*, in the sense that $\omega(s) \equiv \sum_{j=1}^{I} \omega_j(s)$ is constant across all $s \in S$. This is not so restrictive an assumption as it might appear, as demonstrated by its endogenization (in the spirit of Mandler 2013) in Section 4.

3 Aggregation

Given an equilibrium of such a heterogeneous-beliefs economy, Calvet et al. (2018, based on a 2001 working paper) seek an "equivalent equilibrium" where those beliefs are aggregated into a common "consensus belief" that generates the same equilibrium prices and trading volumes as the original equilibrium. To satisfy this first "invariance condition" for aggregation, redistribution of the endowment is permitted; but in general an "aggregation bias" arises from the diversity of beliefs, which Calvet et al. correct via a scalar adjustment to the aggregate endowment. A fundamental of the original economy is thus altered to obtain a consensus belief, but in its place the authors argue for their second invariance condition that each asset should retain the same marginal valuation in the equivalent equilibrium.[4] Jouini and Napp (2006b) extend this approach to dynamic economies, albeit with the required aggregate endowment adjustment now nonscalar. With an equivalent equilibrium obtained in such a fashion, it is straightforward to construct a representative agent who maximizes expected utility under the consensus belief via the standard Negishi (1960) approach (see Jouini and Napp 2006a, §2.3). Zimper (2023) provides belief aggregation formulas that nest both approaches.

Of course, the aggregation bias is famously absent in the case of logarithmic utility (Rubinstein 1974, 1976; Zapatero 1998), where the consensus belief is a weighted arithmetic average of individual beliefs.[5] Under exponential (and more generally, CARA) utility, consensus beliefs are a weighted geometric average of individual beliefs (Wilson 1968), but aggregation bias necessitates a model adjustment to obtain an equivalent equilibrium in dynamic markets (Jouini and Napp 2006a, p. 755). The model adjustment need not, however, be to the aggregate endowment, as shown by Jouini and Napp's (2006a; 2007) representation via a discount factor adjustment in a fully dynamic discrete- or continuous-time setting.

In this section, I show that no adjustment to the economy's fundamentals is required for aggregation, as long as the representative agent is allowed to be ambiguity averse. In particular, there exists an equivalent equilibrium if the consensus characteristic is not a belief but a capacity, with respect to which the consumers maximize their Choquet expected utility in the manner advanced by Schmeidler (1986, 1989). Moreover, this equivalent equilibrium may still be used to construct an (ambiguity-averse) representative agent in the classic Negishi fashion (via the characterization of Pareto optima in Chateauneuf et al. 2000, Proposition 3.1, and Dana 2004, Proposition 3.1 and Corollary 3.2).

A *capacity* (Choquet 1953) on $(S, 2^S)$ is a set function $\nu : 2^S \to [0, 1]$ such that $\nu(\emptyset) = 0$, $\nu(S) = 1$ and $\nu(A) \leq \nu(B)$ for all $A, B \in 2^S$ such that $A \subseteq B$. This differs from a probability measure in that it is not required to be additive, but only to satisfy a weaker monotonicity property. A capacity ν is *convex* if, for all $A, B \in 2^S$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$. The *core* of ν is

[4] As Jouini and Napp (2006a, p. 764) note, this is equivalent to the requirement that each consumer's original demand be larger (resp., smaller) than his demand in the equivalent equilibrium if and only if his subjective probability on the relevant state is larger (resp., smaller) than the consensus probability.

[5] "Consensus beliefs" should be distinguished from the "composite beliefs" of Rubinstein (1974), which are the special case of an unweighted arithmetic average of individual beliefs.

$$\text{core}(\nu) = \left\{ \pi \in \mathbb{R}_+^K \mid \sum_{j=1}^{K} \pi^j = 1 \text{ and } \pi(A) \geq \nu(A),\ \forall A \in 2^S \right\}.$$

Given a random variable $c = (c(1), \ldots, c(K))$ on $(S, 2^S)$, the Choquet integral of c is

$$\mathrm{E}_\nu(c) = \int_{-\infty}^{0} (\nu(c \geq t) - 1)dt + \int_{0}^{\infty} \nu(c \geq t)dt,$$

and if $c(1) \leq c(2) \leq \cdots \leq c(K)$,

$$\mathrm{E}_\nu(c) = \sum_{j=1}^{K-1} [\nu(\{j, \ldots, K\}) - \nu(\{j+1, \ldots, K\})]c(j) + \nu(\{K\})c(K).$$

If ν is convex, then $\text{core}(\nu) \neq \emptyset$ and

$$\mathrm{E}_\nu(c) = \min_{\pi \in \text{core}(\nu)} \mathrm{E}_\pi(c), \quad \forall c \in \mathbb{R}^K.$$

Hence, an agent who maximizes his *Choquet expected utility* under a convex capacity ν,

$$V_i(c) = \min_{\pi \in \text{core}(\nu)} \mathrm{E}_\pi(u_i(c)), \quad \forall c \in \mathbb{R}_+^K,$$

is a special case of the maxmin expected utility model (Gilboa and Schmeidler 1989). The function $V_i : \mathbb{R}_+^K \to \mathbb{R}$ is strictly concave and superdifferentiable. Let $\partial V_i(c) = \{a \in \mathbb{R}^K \mid V_i(c) - V_i(c') \geq a(c - c'),\ \forall c' \in \mathbb{R}_+^K\}$ denote V_i's superderivatives at c. Note that, if $c(1) = c(2) = \cdots = c(K)$, then the set of superderivatives of $V_i(c)$ is proportional to the core of ν (see Dana 2004, p. 573), specifically $\partial V_i(c) = u_i'(c(1))\,\text{core}(\nu)$.

Suppose that, instead of maximizing expected utility under their heterogeneous beliefs $(p^1, \ldots, p^I)$, the economy's consumers maximize Choquet expected utility under a common convex capacity $\tilde{\nu}$.[6] In this case, V_i is "continuously superdifferentiable" in the sense that the set of superderivatives varies continuously with s (indeed, it is constant), yielding existence of competitive equilibrium (see Epstein and Wang 1994, p. 300). For $j = 1, \ldots, K$, let δ_j be the vector with $\delta_l^j = 0$ if $l \neq j$ and $\delta_l^l = 1$. Then we also know the following:

Lemma 1 *(Dana 2004) If $\omega(1) = \omega(2) = \cdots = \omega(K)$, a price $\tilde{q}$ is an equilibrium price if and only if it is proportional to a probability $\tilde{\pi}$ in core$(\tilde{\nu})$, the associated equilibrium consumptions being constant over states and equal to the expected value*

[6] In a general setting allowing for aggregate risk, Xia and Zhou (2016) analyze the rank-dependent case of a common, possibly nonconvex capacity, whilst Jin et al. (2019) and Boonen and Ghossoub (2020) examine heterogeneous rank-dependent capacities. Rigotti and Shannon (2012) analyze the more general class of variational preferences.

of their initial endowments with respect to that probability. The maximal (resp., the minimal) price of consumption in state j equals $-\mathrm{E}_\nu(\delta_j)$ (resp., $\mathrm{E}_\nu(\delta_j)$). Consumer i's maximal (resp., minimal) consumption equals $-\mathrm{E}_\nu(\omega_i)$.

I now define the central notion of ambiguity-averse aggregation.

Definition 1 Given equilibrium state prices q^* and consumption plans c^* for the original economy with heterogeneous beliefs, I will call a capacity $\tilde{\nu}$ a *consensus capacity* if there exist adjusted endowments $(\tilde{\omega}_1, \ldots, \tilde{\omega}_I) \in \mathbb{R}^{SI}$ with $\sum_{i=1}^{I} \tilde{\omega}_i = \omega$ under which $\tilde{\nu}$ generates an *equivalent equilibrium* satisfying the following *invariance conditions*:

1. the equivalent equilibrium prices $(\tilde{q}(s))_{s\in S} \propto \tilde{\pi} \in \mathrm{core}(\tilde{\nu})$ and trading volumes $(\tilde{c}_i(s) - \tilde{\omega}_i(s))_{s\in S,\, i=1,\ldots,I}$ are the same as the prices $(q^*(s))_{s\in S}$ and trading volumes $(c_i^*(s) - \omega_i(s))_{s\in S,\, i=1,\ldots,I}$ in the original equilibrium;
2. each consumer's marginal valuation of each asset in the original equilibrium must belong to the "spread" of marginal valuations forming the set of superderivatives of V_i in the equivalent equilibrium,

$$\forall i, s,\ \exists \pi \in \mathrm{core}(\tilde{\nu}): \quad p^i(s)u_i'(c_i^*(s)) = u_i'(\tilde{c}_i(1))\pi(s); \qquad \text{and}$$

3. there exists no capacity ν satisfying conditions 1 and 2 with $\|\nu - \tilde{\pi}\| < \|\tilde{\nu} - \tilde{\pi}\|$.

Compared to those of Calvet et al. (2018) and Jouini and Napp (2006a, 2007), the first of these invariance conditions is unchanged, whilst the second is a natural adaptation to the ambiguity-averse setting, where there are "spreads" in equilibrium marginal asset valuations bounded by the marginal valuations prevailing under each possible state ordering. The third condition is of course specific to the current setting, and designed to achieve a minimal departure from the expected utility case. Note also that individual adjusted endowments may have negative entries here (whilst still summing to the original aggregate endowment), which is essential if we are to reproduce all possible trading volumes with final allocations constrained to lie on the full-insurance line. This is of course immaterial for the construction of the representative agent. In Appendix A, I work through an extended two-state, two-consumer example of this approach to aggregation.

Theorem 1 *If there is no aggregate risk ($\omega(1) = \omega(2) = \cdots = \omega(K)$), then there exists a unique convex consensus capacity.*

Proof Fix the equivalent equilibrium prices $(\tilde{q}(s))_{s\in S} \propto \tilde{\pi}$ to be equal to those in the original equilibrium. Now, under Choquet expected utility the full-insurance line is still the set of Pareto optima (Chateauneuf et al. 2000), of which the equilibrium set is a subset by the First Welfare Theorem. Hence, offsetting the adjusted endowments $(\tilde{\omega}_1, \ldots, \tilde{\omega}_I)$ from the full-insurance line by the vector capturing the original trading volumes, we obtain a hyperplane of adjusted endowment points consistent with matching the original prices and trading volumes.

I claim that there is a nonempty set N, such that for each $\nu \in N$: $\tilde{\pi}$ belongs to $\mathrm{core}(\nu)$ (i.e. the first invariance condition for a consensus capacity is satisfied); and there exist (market-clearing) equivalent equilibrium consumptions $(\tilde{c}_i(s))_{s\in S, i\in I}$ at which the spread of each consumer's marginal asset valuations contains his marginal asset valuations in the original equilibrium (i.e. the second invariance condition is satisfied). To see this, let ν^0 be the capacity with $\nu^0(E) = 0$ for every $E \in 2^S \backslash S$. Since core(ν^0) is the set of all probability measures on $(S, 2^S)$, the first claim is immediate, with $N \supset \{\nu^0\}$ by strict positivity of the equilibrium prices.

Turning to the second claim, note that Axioms 1–3 ensure that each agent in the original economy demands the entire endowment of any given state-contingent good when it has a zero price (a property sometimes assumed directly under the name "desirability"). Hence, for each state $s \in S$, $\pi(s) = 0$ for some $\pi \in$ core(ν^0), under which the demand of each agent $i = 1, \ldots, I$ is the aggregate endowment, $c_i(s) = \omega(s)$; strict concavity of u_i then implies that i's marginal valuation of s-consumption is strictly less than that in the original equilibrium (where i's demand was strictly less than this by Axiom 2's bounded endowments, strict positivity of equilibrium prices, Axiom 1's Inada condition, and Axiom 3's possibility assumption). Meanwhile, for any convex ν, core(ν) contains the Dirac measure on s, under which $c_i(s) = 0$ (by desirability of the other state-contingent goods) and i's marginal valuation is infinite. It follows by continuity of u_i' that core(ν) contains elements satisfying the second invariance condition for each ν in some $N \supset \{\nu^0\}$ by strict positivity of the equilibrium prices (and hence the marginal valuations).

Now, in general, core(ν) is the convex hull of the (at most $K!$) extremal points $(\pi_\sigma(j))_{j=1}^K$ with $\pi_\sigma(j) = \nu\{\sigma(j), \sigma(j+1), \ldots, \sigma(K)\} - \nu\{\sigma(j+1), \ldots, \sigma(K)\}$ and σ a permutation of S. It follows that the set N of capacities satisfying the first and second invariance conditions is nonempty and compact, and the minimization problem in the third invariance condition has a solution by Weierstrass' extreme value theorem. Since the Euclidean distance is monotone on the set of convex capacities, this solution is in fact unique. □

Multiple priors

Theorem 1 establishes the existence *a fortiori* of a representation of the original economy using the maxmin expected utility model of Gilboa and Schmeidler (1989). Indeed, it is not necessary to assume that the set of priors in such a representation is the core of a convex capacity, since I have used only the convexity and compactness of core(ν), along with the Pareto optimality of full insurance and the indeterminacy of equilibrium in the absence of aggregate risk; Dana's (2004) Propositions 4.1 and 4.2 thus establish that the analysis is essentially unchanged under a maxmin EU representative agent with a general compact, convex set of priors. Indeed, Gilboa and Schmeidler's (1993) Proposition 2.1 establishes that convexity of a non-additive measure is sufficient for a multiple-priors representation when utility is affine. By contrast, since indeterminacy would fail under the smooth indifference curves allowed by the "smooth ambiguity" model of Klibanoff et al. (2005), any mechanism for ambiguity-averse aggregation within that framework would be quite different to here.

4 Aggregate Risk

As previously mentioned, the assumption of no aggregate risk is much less restrictive if we allow the aggregate economy to assume a dynamic structure where static trade is preceded and determined by productive investment decisions. Mandler (2013) shows that indeterminacy arises as a robust feature of trade in such a setting, and it is this phenomenon that allows the previous subsection's construction of a convex consensus capacity to apply under more general aggregate endowments. In this subsection, I flesh out the details of this observation.

Suppose now that we have a *dynamic (expected utility) economy* with two periods, and with one event S arising in period 1 and two states L and H arising in period 2. There are hence three consumption goods, and I agents are endowed with some amount of each; the aggregate endowment in state H exceeds that in state L. In addition to being able (during an intermediate period) to trade period-2 consumption in the manner of Section 2's economy, the agents may also choose to forego period-1 consumption in favor of investment in a productive asset that generates period-2 output. Specifically, if agent i invests k_i of the period-1 good in the productive asset, then he receives a return of γk_i units of consumption in state L (in addition to his existing state-L endowment), where $\gamma > 0$.[7] With complete markets, he thus faces the problem

$$\max_{c_i} \; p^i(L) f_i(c_i(S), c_i(L)) + p^i(H) f_i(c_i(S), c_i(H)), \quad \text{s.t.}:$$
$$q(L)c_i(L) + q(H)c_i(H) \leq q(L)\omega_i(L) + q(H)\omega_i(H) + \gamma(\omega_i(S) - c_i(S));$$
$$c_i(s) \geq 0, \quad \text{for all } s \in S;$$

where $f_i : \mathbb{R}^2_+ \to [-\infty, \infty)$ is a payoff function on two-period consumption pairs (subject to analogous axioms to u_i), and the price of period-1 consumption is normalized to 1. If each f_i is additively separable with no discounting, this is clearly just the special case of problem (2) with $q(S) = \gamma$.

In this dynamic economy, the endowments in period 2 are endogenously determined by the agents' period-1 investment decisions; call the static trading economy prevailing after these investment decisions have been made the *intermediate economy*. I will say that there exists an *intermediate consensus capacity* if there exists a capacity $\tilde{\nu}$ that satisfies the first invariance condition in Definition 1 (i.e. replicates prices and trading volumes) for the full dynamic economy, and that moreover satisfies the second and third invariance conditions for the intermediate economy. Since Choquet expected utility under $\tilde{\nu}$ is the special case of maxmin expected utility (Gilboa and Schmeidler 1989) with the interval $[\tilde{\nu}(L), 1-\tilde{\nu}(H)]$ of probabilities placed on state L, the dynamic economy defined by the parameters $((f_i, \tilde{\nu}(L), 1-\tilde{\nu}(H), \omega_i(S), \omega_i(L), \omega_i(H))_{i=1}^I, \gamma)$ fits within the Mandler (2013) model.

Proposition 1 *In such a dynamic economy, with or without period-2 aggregate risk, there exists a unique convex intermediate consensus capacity.*

[7] The restriction of the investment's effect to state L is for simplicity alone.

Proof Given equilibrium state prices q^* and consumption plans c^* for the dynamic economy with heterogeneous beliefs, there exists a compact set of capacities that, when held by all agents, satisfy the first invariance condition of Definition 1 for the dynamic economy and imply $\tilde{c}_i(L) = \tilde{c}_i(H)$ for all i. To see this, note that L-consumption and H-consumption become perfect complements as $\tilde{\nu}$ approaches a capacity ν^0 with $\nu^0(L) = \nu^0(H) = 0$; in that limit, the equivalent equilibrium must have $\tilde{c}_i(L) = \tilde{c}_i(H)$ for all i. Since equilibrium exists in the ambiguity-averse economy, it follows under sufficient ambiguity aversion that the equilibrium period-1 investments $k_i^* \equiv \omega(S) - c_i^*(S)$ must remove period-2 aggregate risk, $\omega(L) + \sum_{i=1}^{I} k_i^* = \omega(H)$, in order that the agents' period-2 full-insurance lines coincide. Moreover, any $q^* = (1, q^*(L), q^*(H))$ is proportional to some $\tilde{\pi}$ belonging to core(ν^0), as then required by Lemma 1. Finally, with the adjusted endowments $(\tilde{\omega}_1, \ldots, \tilde{\omega}_I)$ offset from the full-insurance line by the vector of the original trading volumes, the first invariance condition is satisfied. If the removal of period-2 aggregate risk and the first invariance condition are satisfied by a capacity $\tilde{\nu}$, then they are also satisfied by the compact set of capacities whose core contains core$(\tilde{\nu})$; from this set, a unique convex capacity can be found to satisfy Definition 1's second and third invariance conditions for the intermediate economy by Theorem 1. □

Even with aggregate risk then, an appealing dynamic production economy allows ambiguity-averse aggregation: sufficient ambiguity aversion warrants the removal of aggregate risk in the intermediate economy, where equilibrium trade occurs exactly as in the previous subsection. Whether or not this is a feature of the original economy, it allows a general construction for a representative agent.

A Two-State Example

To motivate the details of ambiguity-averse aggregation, I will outline an extended example with two states of the world $\{L, H\}$ and two consumers $\{A, B\}$. This example illustrates the "aggregation bias" arising in general from heterogeneous beliefs, and how it may be corrected with an ambiguity-averse representative agent. I employ the Choquet expected utility approach to ambiguity aversion (Schmeidler 1989), the essence of which is to relax the assumption that the "consensus belief" of the representative agent need sum to 1. Instead, his perceived likelihood of the states L and H is captured by a "consensus capacity" $\tilde{\nu}$ that is strictly convex in the sense that $\tilde{\nu}(L) + \tilde{\nu}(H) < 1 = \tilde{\nu}(L \cup H)$, because there is "ambiguous probability" in $\tilde{\nu}(L \cup H)$ that he is not sure how to allocate between L and H. In this sense, the representative consumer has less "confidence" in each state occurring than he would have in the standard case of ambiguity neutrality. This generates "spreads" in the price at which he would be willing to buy and sell consumption in each state, suggesting a natural modification to the "invariance conditions" for aggregation employed by Calvet et al. (2018) and Jouini and Napp (2006a, 2007), the general statement of which is then given in Definition 1. It is these spreads that allow the aggregation bias to be overcome.

Turning to the example then, each consumer has the (non-CARA) payoff function $u(c) = \sqrt{c}$, but consumer A is endowed with a unit of consumption in state L, whilst consumer B is endowed with a unit of consumption in state H. Consumer A believes that state H has probability $p^A(H) = 1/2$, whilst consumer B believes that it has probability $p^B(H) = 3/4$. Then competitive equilibrium with $q(L) = 1$ and $q(H) = q$ has

$$\begin{aligned} q^* &\approx 1.931 \\ \Rightarrow \quad c_A^*(L) &\approx 0.659,\ c_A^*(H) \approx 0.177,\ c_B^*(L) \approx 0.341,\ c_B^*(H) \approx 0.823 \\ c_A^*(L) - \omega_A(L) &\approx -0.341,\ c_A^*(H) - \omega_A(H) \approx 0.177, \\ c_B^*(L) - \omega_B(L) &\approx 0.341,\ c_B^*(H) - \omega_B(H) \approx -0.177 \\ \lambda_A^* &\approx 0.308, \lambda_B^* \approx 0.214, \end{aligned}$$

where the approximation is to three decimal places.

Calvet et al. (2018) and Jouini and Napp (2006a, 2007) authorize a redistribution of endowments to match this equilibrium price (and trading volumes) under a consensus belief. Here, under such a redistribution, this *first invariance condition* for aggregation is satisfied by the belief $\tilde{p} = (0.341, 0.659)$—with a tilde in general denoting the value of a variable in the equivalent equilibrium. However, this violates the *second invariance condition* on marginal asset valuations,

$$\begin{aligned} p^A(H)u'(c_A^*(H)) = \tilde{p}(H)u'(\tilde{c}_A(H)) &\qquad p^B(H)u'(c_B^*(H)) = \tilde{p}(H)u'(\tilde{c}_B(H)) \\ \Leftrightarrow \qquad \tilde{c}_A(H) \approx 0.307 &\qquad \tilde{c}_B(H) \approx 0.635, \end{aligned}$$

since no redistribution will yield an equilibrium that fails to exhaust the endowment. Calvet et al. (2018) respond to this by reducing the market endowment; Jouini and Napp (2006a, 2007) increase the discount factor above 1. Instead, I will make the adjustment by introducing ambiguity aversion into the consensus characteristic.

But how should I go about this? Are the invariance conditions employed by Calvet et al. (2018) and Jouini and Napp (2006a, 2007) still appropriate for ambiguity-averse aggregation? We know from Lemma 1 that, with all consumers sharing a convex capacity $\tilde{\nu}$ in the absence of aggregate risk, there is an interval of equilibrium price ratios $\tilde{q} \in [\tilde{\nu}(H)/(1 - \tilde{\nu}(H)), (1 - \tilde{\nu}(L))/\tilde{\nu}(L)]$. This can be seen in the (partial) Edgeworth box in Figure 1 as resulting from the nondifferentiability of the utility functions along the bisectrix, where the ordering of states under the Choquet integral changes. Chateauneuf et al. (2000) establish that the set PO of Pareto optima in such a setting is identical to that of an expected utility economy under a consensus belief (i.e. the bisectrix). The limits of the equilibrium set EQ are then determined by the extreme superderivatives of V_A and V_B that separate the consumers' weakly preferred sets: the upper limit of the set has the shallowest separating superderivative, with consumer A's standard FOC holding under the Choquet integral with $c_A(H) \geq c_A(L)$ and that of consumer B holding under the Choquet integral with $c_B(H) \leq c_B(L)$; the lower limit of the set has the steepest separating superderivative, with each consumer's standard FOC holding under the converse orderings.

The resulting equilibria all lie at a point of nondifferentiable utility, and are associated with "spreads" in the price at which the consumers would be willing to buy and sell each asset. For instance, at point E (which would not be an equilibrium under

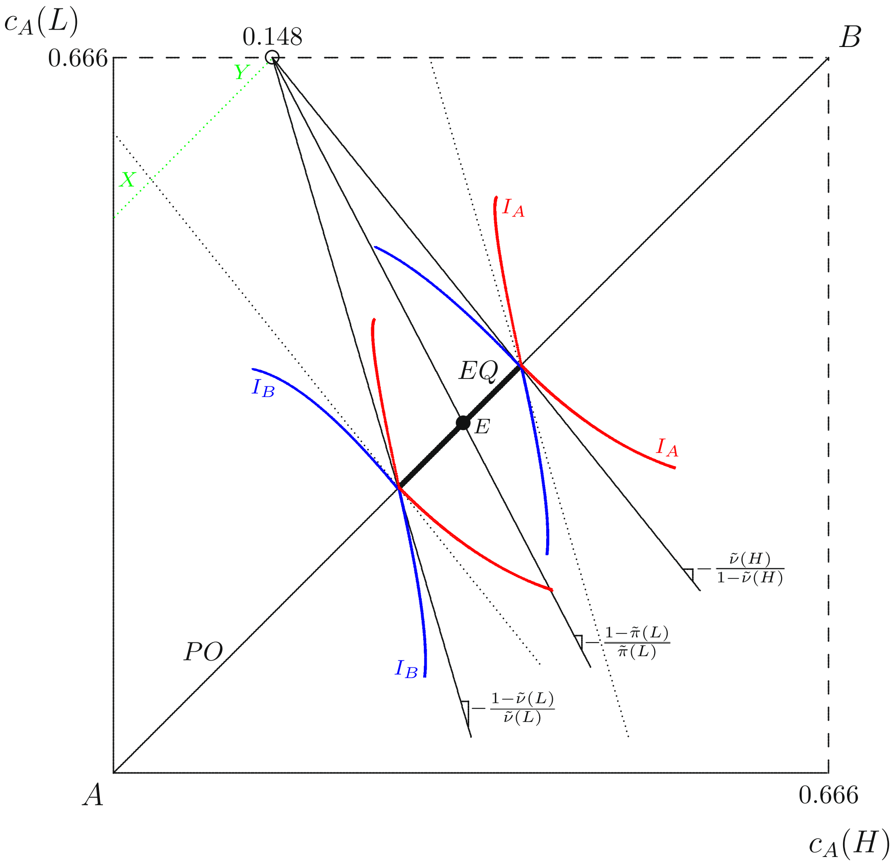

Fig. 1 A partial Edgeworth box illustrating the construction of the equivalent equilibrium E belonging to the indeterminate set EQ under ambiguity-averse aggregation

expected utility), the consensus consumer would be willing to trade L-consumption for H-consumption if H had probability $1-\tilde{\nu}(L)$, but his ambiguity aversion means that it has the lower measure $\tilde{\nu}(H)$. It is these spreads that give rise to the equilibrium indeterminacy that allows ambiguity-averse aggregation without altering any of the economy's fundamentals. At one limit of the equilibrium set,

$$\begin{aligned} \frac{\tilde{\nu}(L)}{2\sqrt{\hat{c}_A(L)}} &= \hat{\lambda}_A \qquad \frac{\tilde{\nu}(L)}{2\sqrt{\hat{c}_B(L)}} = \hat{\lambda}_B \\ \frac{1-\tilde{\nu}(L)}{2\sqrt{\hat{c}_A(H)}} &= \hat{\lambda}_A\hat{q} \qquad \frac{1-\tilde{\nu}(L)}{2\sqrt{\hat{c}_B(H)}} = \hat{\lambda}_B\hat{q} \\ \Rightarrow \quad \hat{c}_A(H) &= \frac{\tilde{\omega}_A(L)+\hat{q}\tilde{\omega}_A(H)}{\hat{q}+\left(\hat{q}\frac{\tilde{\nu}(L)}{1-\tilde{\nu}(L)}\right)^2} \qquad \hat{c}_B(H) = \frac{\tilde{\omega}_B(L)+\hat{q}\tilde{\omega}_B(H)}{\hat{q}+\left(\hat{q}\frac{\tilde{\nu}(L)}{1-\tilde{\nu}(L)}\right)^2} \end{aligned}$$

$$\Rightarrow \quad \hat{q} = \frac{1-\tilde{\nu}(L)}{\tilde{\nu}(L)}. \tag{4}$$

Meanwhile, at the other limit of the equilibrium set,

$$\begin{aligned} \frac{\tilde{\nu}(H)}{2\sqrt{\check{c}_A(H)}} &= \check{\lambda}_A \check{q} & \frac{\tilde{\nu}(H)}{2\sqrt{\check{c}_B(H)}} &= \check{\lambda}_B \check{q} \\ \frac{1-\tilde{\nu}(H)}{2\sqrt{\check{c}_A(L)}} &= \check{\lambda}_A & \frac{1-\tilde{\nu}(H)}{2\sqrt{\check{c}_B(L)}} &= \check{\lambda}_B \\ \Rightarrow \quad \check{c}_A(H) &= \frac{\tilde{\omega}_A(L)+\check{q}\tilde{\omega}_A(H)}{\check{q}+\left(\check{q}\frac{1-\tilde{\nu}(H)}{\tilde{\nu}(H)}\right)^2} & \check{c}_B(H) &= \frac{\tilde{\omega}_B(L)+\check{q}\tilde{\omega}_B(H)}{\check{q}+\left(\check{q}\frac{1-\tilde{\nu}(H)}{\tilde{\nu}(H)}\right)^2} \\ &\Rightarrow \quad \check{q} = \frac{\tilde{\nu}(H)}{1-\tilde{\nu}(H)}. \end{aligned} \tag{5}$$

Since there is an interval of equilibrium price ratios between these two limits, it follows that $\tilde{q} = (1-\tilde{\pi}(L))/\tilde{\pi}(L) = \tilde{\pi}(H)/(1-\tilde{\pi}(H))$ is an equilibrium price for any $\tilde{\pi}$ in the core of $\tilde{\nu}$. Moreover, since $\tilde{c}_A(L) = \tilde{c}_A(H)$ in equilibrium,

$$\begin{aligned} & \tilde{c}_A(L) + \tilde{q}\tilde{c}_A(L) = \tilde{\omega}_A(L) + \tilde{q}\tilde{\omega}_A(H) \\ \Leftrightarrow \quad & \tilde{c}_A(L) = \tilde{c}_A(H) = \tilde{\pi}(L)\tilde{\omega}_A(L) + (1-\tilde{\pi}(L))\tilde{\omega}_A(H) = \mathrm{E}_{\tilde{\pi}}\, \tilde{\omega}_A. \end{aligned}$$

For equality of trading volumes with the original equilibrium, we require that

$$\begin{aligned} 0.177 &\approx \mathrm{E}_{\tilde{\pi}}(\tilde{\omega}_A) - \tilde{\omega}_A(H) & -0.177 &\approx \mathrm{E}_{\tilde{\pi}}(\tilde{\omega}_B) - \tilde{\omega}_B(H) \\ \Rightarrow \quad \tilde{\omega}_A(L) - \tilde{\omega}_A(H) &\approx 0.518 & \tilde{\omega}_B(H) - \tilde{\omega}_B(L) &\approx 0.518 \\ &\Rightarrow \quad \tilde{\pi}(L) \approx 0.341. \end{aligned} \tag{6}$$

This delivers the first invariance condition of Calvet et al. (2018) and Jouini and Napp (2006a, 2007), satisfied by a budget constraint with slope $0.659/0.341 \approx q^*$ rooted at any point along the line XY in Figure 1.

But what of the second invariance condition? This requires that every consumer should be indifferent between marginal asset investments in the original equilibrium and the equivalent equilibrium, and hence that each asset should receive the same marginal (utility) valuation by each consumer in both equilibria. In the expected utility case, this is equivalent to imposing that each consumer's Lagrange multiplier be the same in both equilibria, and hence that the representative agent's utility function—which averages over individual consumer utilities with weights determined by their Lagrange multipliers—be the same in both equilibria. It is also equivalent to the condition that each consumer's original state-contingent demand be larger than his demand in the "equivalent equilibrium" if and only if his subjective probability on that state is larger than the aggregate common probability. Under ambiguity aversion, consumer FOCs need not hold with equality in equilibrium, where there is a spread of marginal asset valuations consistent with consumer optimization, as we have seen above. What then should the second invariance condition look like under ambiguity-averse aggregation?

The natural answer is that the marginal utility of an asset in the original equilibrium should lie between the equivalent equilibrium marginal utilities under the two possible Choquet-integral consumption orderings, i.e. it should belong to the "spread" of marginal asset valuations created by ambiguity aversion. Then the second invariance condition— together with the equality of $\tilde{c}_A(L)$ and $\tilde{c}_A(H)$ in equilibrium—implies the following inequalities:[8]

$$p^A(L)u'(c_A^*(L)) \geq \tilde{\nu}(L)u'(\tilde{c}_A(L)) \qquad p^A(H)u'(c_A^*(H)) \leq (1-\tilde{\nu}(L))u'(\tilde{c}_A(H)) \tag{7}$$

$$\begin{aligned} \tilde{c}_A(L) \gtrapprox \left(\frac{\tilde{\nu}(L)\times\sqrt{0.659}}{0.5}\right)^2 \qquad & \tilde{c}_A(H) \lessapprox \left(\frac{(1-\tilde{\nu}(L))\times\sqrt{0.177}}{0.5}\right)^2 \\ \Rightarrow \quad \left(\frac{\tilde{\nu}(L)\times\sqrt{0.659}}{0.5}\right)^2 & \lessapprox \left(\frac{(1-\tilde{\nu}(L))\times\sqrt{0.177}}{0.5}\right)^2 \\ \tilde{\nu}(L) & \lessapprox 0.341. \end{aligned} \tag{8}$$

$$\begin{aligned} p^A(H)u'(c_A^*(H)) \geq \tilde{\nu}(H)u'(\tilde{c}_A(H)) \qquad & p^A(L)u'(c_A^*(L)) \leq (1-\tilde{\nu}(H))u'(\tilde{c}_A(L)) \\ \tilde{c}_A(H) \gtrapprox \left(\frac{\tilde{\nu}(H)\times\sqrt{0.177}}{0.5}\right)^2 \qquad & \tilde{c}_A(L) \lessapprox \left(\frac{(1-\tilde{\nu}(H))\times\sqrt{0.659}}{0.5}\right)^2 \\ \Rightarrow \quad \left(\frac{\tilde{\nu}(H)\times\sqrt{0.177}}{0.5}\right)^2 & \lessapprox \left(\frac{(1-\tilde{\nu}(H))\times\sqrt{0.659}}{0.5}\right)^2 \\ \tilde{\nu}(H) & \lessapprox 0.659. \end{aligned} \tag{9}$$

For B, similarly,

$$\begin{aligned} p^B(L)u'(c_B^*(L)) \geq \tilde{\nu}(L)u'(\tilde{c}_B(L)) \qquad & p^B(H)u'(c_B^*(H)) \leq (1-\tilde{\nu}(L))u'(\tilde{c}_B(H)) \\ \tilde{c}_B(L) \gtrapprox \left(\frac{\tilde{\nu}(L)\times\sqrt{0.341}}{0.25}\right)^2 \qquad & \tilde{c}_B(H) \lessapprox \left(\frac{(1-\tilde{\nu}(L))\times\sqrt{0.823}}{0.75}\right)^2 \\ \Rightarrow \quad \left(\frac{\tilde{\nu}(L)\times\sqrt{0.341}}{0.25}\right)^2 & \lessapprox \left(\frac{(1-\tilde{\nu}(L))\times\sqrt{0.823}}{0.75}\right)^2 \\ \tilde{\nu}(L) & \lessapprox 0.341. \end{aligned} \tag{10}$$

[8] Note that consumer A maximizing his Choquet expected utility while maintaining the consumption ordering $c_A(L) \geq c_A(H)$ may be constrained from reducing L-consumption (and from increasing H-consumption). It follows that the first-order derivative of his Lagrangian with respect to L-consumption may be negative, and that with respect to H-consumption may be positive.

$$p^B(H)u'(c_B^*(H)) \geq \tilde{\nu}(H)u'(\tilde{c}_B(H)) \qquad p^B(L)u'(c_B^*(L)) \leq (1-\tilde{\nu}(H))u'(\tilde{c}_B(L))$$

$$\tilde{c}_B(H) \gtrapprox \left(\frac{\tilde{\nu}(H)\times\sqrt{0.823}}{0.75}\right)^2 \qquad \tilde{c}_B(L) \lessapprox \left(\frac{(1-\tilde{\nu}(H))\times\sqrt{0.341}}{0.25}\right)^2$$

$$\Rightarrow \quad \left(\frac{\tilde{\nu}(H)\times\sqrt{0.823}}{0.75}\right)^2 \lessapprox \left(\frac{(1-\tilde{\nu}(H))\times\sqrt{0.341}}{0.25}\right)^2$$

$$\tilde{\nu}(H) \lessapprox 0.659. \tag{11}$$

Now, if $\tilde{\nu}(L) \approx 0.341$, then $\tilde{c}_A(L) = \tilde{c}_A(H) \approx 0.307$ (from (8)) and $\tilde{c}_B(L) = \tilde{c}_B(H) \approx 0.635$ (from (10)), which cannot be an equilibrium since the endowment is not exhausted. The same applies (from (9) and (11)) if $\tilde{\nu}(H) \approx 0.659$. Since the slope of the budget constraint must be $0.659/0.341 \approx q^*$ in the equivalent equilibrium, this shows the need for strict ambiguity aversion (i.e. a strictly convex $\tilde{\nu}$) and for an equivalent equilibrium strictly in the interior of the resulting equilibrium set (i.e. ambiguity in both L and H). So let us introduce ambiguity aversion by determining $\tilde{\nu}(L) \leq 0.341$ and $\tilde{\nu}(H) \leq 0.659$, and looking for an equivalent equilibrium lying somewhere in the resulting equilibrium set EQ.

Note from (8), (9), (10) and (11) that, as $\tilde{\nu}$ approaches the capacity ν^0 with $\nu^0(s) = 0$ for each state $s \in \{L, H\}$, $\tilde{c}_A(L) + \tilde{c}_B(L)$ is constrained only to lie between 0 and an amount (roughly 2.171) in excess of the endowment. Hence, for sufficient ambiguity aversion, it is possible to choose equivalent equilibrium consumptions that both clear the market and satisfy the second invariance condition, in this case by allowing the upper bounds in (8) and (10) to bind. But whilst it is thus clear that I can find an equivalent equilibrium in this way, I would like to know the minimal degree of ambiguity aversion required for aggregation, as the smallest departure from a standard expected utility-maximizing representative agent; this will be my third "invariance condition".

Hence, I now seek to minimize $\|\tilde{\nu} - \tilde{\pi}\|$ subject to (6), (8), (9), (10), (11) and market clearing. Looking for a solution where the upper bounds in (8) and (10) bind and exhaust the endowment,

$$\tilde{c}_A(H) + \tilde{c}_B(H) \approx \left(\frac{(1-\tilde{\nu}(L))\times\sqrt{0.177}}{0.5}\right)^2 + \left(\frac{(1-\tilde{\nu}(L))\times\sqrt{0.823}}{0.75}\right)^2 \approx 1,$$

we have $\tilde{\nu}(L) \approx 0.321$, and hence $\tilde{c}_A(H) = \tilde{c}_A(L) \approx 0.326$ and $\tilde{c}_B(H) = \tilde{c}_B(L) \approx 0.674$. The upper bounds in (9) and (11) then hold with equality for $\tilde{\nu}(H) \approx 0.649$, and the lower bounds in (8), (9), (10) and (11) then hold with strict inequality. It follows that

$$\tilde{c}_A(H) = \frac{1-\tilde{\pi}(L)-\tilde{\nu}(H)}{1-\tilde{\nu}(L)-\tilde{\nu}(H)}\left(\frac{\tilde{\omega}_A(L)+\hat{q}\tilde{\omega}_A(H)}{\hat{q}+\left(\hat{q}\frac{\tilde{\nu}(L)}{1-\tilde{\nu}(L)}\right)^2}\right)$$

$$
\begin{aligned}
&+\frac{\tilde{\pi}(L)-\tilde{\nu}(L)}{1-\tilde{\nu}(L)-\tilde{\nu}(H)}\left(\frac{\tilde{\omega}_A(L)+\check{q}\tilde{\omega}_A(H)}{\check{q}+\left(\check{q}\,\frac{1-\tilde{\nu}(H)}{\tilde{\nu}(H)}\right)^2}\right)\\
\Rightarrow\quad &\frac{1}{3}\left(\frac{\tilde{\omega}_A(H)+0.518+\frac{1-\tilde{\nu}(L)}{\tilde{\nu}(L)}\tilde{\omega}_A(H)}{\frac{1-\tilde{\nu}(L)}{\tilde{\nu}(L)}+1}\right)\\
&+\frac{2}{3}\left(\frac{\tilde{\omega}_A(H)+0.518+\frac{\tilde{\nu}(H)}{1-\tilde{\nu}(H)}\tilde{\omega}_A(H)}{\frac{\tilde{\nu}(H)}{1-\tilde{\nu}(H)}+1}\right)\approx 0.326\\
&\frac{\tilde{\omega}_A(H)+0.518+2.113\tilde{\omega}_A(H)}{3.113}+2\left(\frac{\tilde{\omega}_A(H)+0.518+1.849\tilde{\omega}_A(H)}{2.849}\right)\\
&\approx 0.978\tilde{\omega}_A(H)+0.166+2\tilde{\omega}_A(H)+0.367\approx 0.978\\
\Rightarrow\quad &\tilde{\omega}_A(H)\approx 0.148,\ \tilde{\omega}_A(L)\approx 0.666 .
\end{aligned}
$$

This completes the derivation of the equivalent equilibrium E in Figure 1.

B Asset Pricing

I now compare the standard CCAPM model with that in Section 4's heterogeneous-beliefs setting. For the standard case, I will set beliefs to be homogeneous and equal to some objective probability p, whilst the representative agent's utility from average consumption $(\bar{c}_1, \bar{c}_2)$ will be the average u_λ of the individual utilities $(u_1, \ldots, u_I)$ weighted by the inverse Lagrange multipliers $(\lambda_1^{-1}, \ldots, \lambda_I^{-1})$.

Suppose that there exists a riskless asset with prices $\overline{Q}_1 = 1$ and $\overline{Q}_2 = 1 + r^f$ for some predictable risk free rate r^f. Consider then a risky asset with positive prices Q_1 and Q_2, and resulting period-2 rate of return $R_2 = (Q_2/Q_1) - 1$. We must of course have

$$
q_1^* Q_1 = \mathrm{E}_p\left[q_2^* Q_2\right] \tag{12}
$$

for there to be no arbitrage opportunities. In the ambiguity-averse economy with intermediate consensus capacity $\tilde{\nu}$, since $q^* = \Lambda\tilde{\pi}u_\lambda'(\bar{c}_2)$ for any $\Lambda > 0$ and $\tilde{\pi} \in \mathrm{core}(\tilde{\nu})$ by Lemma 1, and letting $\tilde{\pi}_2 \equiv (\tilde{\pi}(L), \tilde{\pi}(H))$,

$$
\begin{aligned}
Q_1 &= \mathrm{E}_p\left[\frac{\Lambda\tilde{\pi}_2 u_\lambda'(\bar{c}_2)}{\Lambda\tilde{\pi}(S)u_\lambda'(\bar{c}_2)}Q_2\right]\\
\Leftrightarrow\qquad 1 &= \mathrm{E}_p\left[\tilde{\pi}_2(1+R_2)\right]\\
1 &= \mathrm{E}_p\left[\tilde{\pi}_2\right](1+\mathrm{E}_p[R_2]) + \mathrm{cov}_p\left(\tilde{\pi}_2, R_2\right).
\end{aligned}
$$

Then, applying (12) to the riskless asset yields $1/(1+r^f) = \mathrm{E}_p[q_2^*/q_1^*] = \mathrm{E}_p[\tilde{\pi}_2]$, so that

$$
1 + r^f = (1 + \mathrm{E}_p[R_2]) + (1 + r^f)\mathrm{cov}_p\left(\tilde{\pi}_2, R_2\right)
$$

$$\mathrm{E}_p[R_2] - r^f = -\frac{\mathrm{cov}_p(\tilde{\pi}_2, R_2)}{\mathrm{E}_p[\tilde{\pi}_2]}$$
$$\mathrm{E}_p[R_2] - r^f = -\mathrm{cov}_p\left[\frac{\tilde{\pi}_2}{\mathrm{E}_p[\tilde{\pi}_2]}, R_2\right].$$

This differs from the classic

$$\mathrm{E}_p[R_2] - r^f = -\mathrm{cov}_p\left[\frac{q_2^*}{\mathrm{E}_p[q_2^*]}, R_2\right]$$

through the replacement of q_2^* with the “price” $\tilde{\pi}_2 \in \mathrm{core}(\tilde{\nu}_2) \equiv \mathrm{core}((\tilde{\nu}(L), \tilde{\nu}(H)))$, raising or lowering the risk premium depending on the selection made from the indeterminate equilibrium prices.

The risk free rate r^f in particular is given by the solution to

$$1 + r^f = \frac{1}{\mathrm{E}_p[\tilde{\pi}_2]}, \qquad \tilde{\pi}_2 \in \mathrm{core}(\tilde{\nu}_2).$$

The risk free rate may thus be raised or lowered relative to the standard case (which would have the solution given by $\tilde{\pi} = q^*$). Thus, as Calvet et al. (2018) and Jouini and Napp (2006a, 2007) have observed for their models, belief heterogeneity offers a possible resolution of the equity premium puzzle (Mehra and Prescott 1985; Kocherlakota 1996) and the risk free rate puzzle (Weil 1989). Here, however, the interpretation is different; individual belief heterogeneity among expected utility maximizers translates into aggregate ambiguity aversion, with the resulting indeterminacy allowing an altered risk premium.

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