

# Integrable spin chains with random interactions

Fabian H.L. Essler<sup>1</sup>, Rianne van den Berg<sup>2</sup> and Vladimir Gritsev<sup>2,3</sup>

<sup>1</sup>*The Rudolf Peierls Centre for Theoretical Physics, Oxford University, Oxford, OX1 3NP, United Kingdom*

<sup>2</sup>*Institute for Theoretical Physics, Universiteit van Amsterdam,  
Science Park 904, Postbus 94485, 1098 XH Amsterdam, The Netherlands*

<sup>3</sup>*Russian Quantum Center, Skolkovo, Moscow 143025, Russia*

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We study a Yang-Baxter integrable quantum spin-1/2 chain with random interactions. The Hamiltonian is local and involves two and three-spin interactions with random parameters. We show that the energy eigenstates of the model are never localized and in fact exhibit perfect energy and spin transport at both zero and infinite temperatures. By considering the vicinity of a free fermion point in the model we demonstrate that this behavior persists under deformations that break Yang-Baxter integrability but preserve the free fermion nature of the Hamiltonian. In this case the ballistic behavior can be understood as arising from the correlated nature of the disorder in the model. We conjecture that the model belongs to a broad class of models avoiding localization in 1D.

## I. INTRODUCTION

After the fundamental paper by Anderson<sup>1</sup> it was believed for a long time that in one-dimensional random potentials all eigenstates are localized in the thermodynamic limit for arbitrarily weak disorder<sup>2–4</sup>. If Azbel resonances<sup>5,6</sup>, which form a set of measure zero, are neglected, the above statement is rigorously speaking valid only for white noise spatially uncorrelated randomness<sup>7</sup>. Later it was realized that the spatial correlations of the disorder potential can have profoundly influence Anderson localization<sup>8–10</sup>. In this case localization can be partially suppressed, at least for weak disorder<sup>11</sup>. In this context a delocalization-localization transition in 1D for *long-range* correlated disorder potentials has been intensively discussed in the literature<sup>12–17</sup>. On the other hand, it was found that models with specific short-range correlated disorder, so-called dimer models, exhibit conducting states<sup>18–21</sup>. In recent years considerable efforts have been made to understand the combined effects of disorder and interactions, which leads to the phenomenon of *many-body localization* (MBL)<sup>22–29</sup>, see Refs<sup>31–35</sup> for recent reviews. The MBL transition generally occurs at finite energy densities and is characterized by ergodicity breaking, the existence of an extensive number of quasi-local integrals of motion in the localized phase<sup>28–30</sup> and Poissonian level statistics. This is reminiscent of Yang-Baxter integrable many-body systems<sup>36,37</sup>, which also feature Poissonian level statistics and extensive numbers of conservation laws. In Yang-Baxter integrable systems the conserved charges are extensive but have (quasi) local densities. An interesting question is then whether there are any connections between Yang-Baxter integrability and MBL. An example of a Yang-Baxter integrable model that is localized is provided by disordered Richardson models<sup>38</sup>. However, this class of models is infinite-ranged whereas studies of MBL have focussed on models like the spin-1/2 Heisenberg chain with a random white-noise correlated magnetic field. Other forms of disorder such as random exchange interactions<sup>39,40</sup> have been

explored as well<sup>41–43</sup>, and there appears to be a wide spread belief that MBL behaviour is a rather generic phenomenon in the strong disorder regime. Non-MBL behaviour has been found in a disordered Hubbard chain<sup>44</sup>, but this could be related to the presence of non-abelian symmetries<sup>43,45</sup>. Another little explored issue is what effects correlations in the disorder have on MBL<sup>46–48</sup>.

In this work we study a Yang-Baxter integrable model of a Heisenberg-like spin chain with tuneable randomness and abelian symmetry. We employ a number of standard tools used to probe for (many-body) localized behaviour: inverse participation ratios, local quantum quench dynamics and transport properties in energy eigenstates. All methods point to the same conclusion: the model does not exhibit any traces of localization irrespective of the magnitudes of the interactions and disorder. On the contrary, we find that the model is an ideal conductor for both spin and energy. Moreover, we show in a non-interacting limit that by deforming the model by tuning the correlations between the random interaction parameters (the resulting model is no longer Yang-Baxter integrable) it is possible to induce localization. This suggests that the model we study here is a particular example of a broader class of strongly disordered models in one dimensions that do not exhibit MBL.

## II. THE MODEL

The Hamiltonian of our integrable chain contains nearest-neighbour, next-nearest neighbour and three-spin interactions with random couplings, *cf.* Fig. (1), and can be expressed in the form

$$H = \sum_{j=1}^{L/2} J_{2j}^{(1)} \left( [\vec{\sigma}_{2j-1} \cdot \vec{\sigma}_{2j}]_{\Delta_{2j}} + [\vec{\sigma}_{2j} \cdot \vec{\sigma}_{2j+1}]_{\Delta_{2j}} \right) + K_{2j} \left( [\vec{\sigma}_{2j} \cdot (\vec{\sigma}_{2j-1} \times \vec{\sigma}_{2j+1})]_{\Delta_{2j}^{-1}} + \Delta_{2j}^{-1} \right) + J_{2j}^{(2)} (\vec{\sigma}_{2j-1} \cdot \vec{\sigma}_{2j+1} - 1), \quad (1)$$

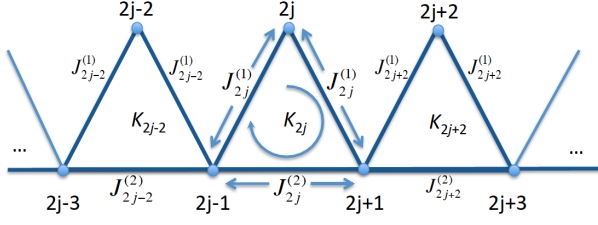


FIG. 1. Schematic representation of the disordered interacting spin chain studied here. Three types of position-dependent interactions with random parameters are present: a nearest-neighbour exchange  $J_{2j}^{(1)}$ , a next-nearest neighbour coupling  $J_{2j}^{(2)}$  and a three-spin interaction  $K_{2j}$ . Explicit expressions for the various terms in the Hamiltonian are given in the text. The ratios  $J_{2j}^{(1)}/J_{2j}^{(2)}$  and  $J_{2j}^{(1)}/K_{2j}$  are correlated.

where  $[\vec{\sigma}_j \cdot \vec{\sigma}_k]_{\Delta} = \sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \Delta(\sigma_j^z \sigma_k^z - 1)$ . The exchange couplings are parametrized as

$$J_{2j}^{(1)} = \frac{\sin^2 \eta \cosh \xi_{2j}}{\sin^2 \eta + \sinh^2 \xi_{2j}}, \quad J_{2j}^{(2)} = \frac{\cos \eta \sinh^2 \xi_{2j}}{\sin^2 \eta + \sinh^2 \xi_{2j}},$$

$$K_{2j} = \frac{\sin \eta \cos \eta \sinh \xi_{2j}}{\sin^2 \eta + \sinh^2 \xi_{2j}}, \quad \Delta_{2j} = \frac{\cos \eta}{\cosh \xi_{2j}}, \quad (2)$$

where  $\xi_{2j}$  and  $\Delta = \cos(\eta)$  are free parameters of the model. By construction we recover the spin-1/2 Heisenberg XXZ Hamiltonian if we set all inhomogeneities to zero  $\xi_2 = \xi_4 = \dots = \xi_L = 0$ . In the following we mainly consider the case where  $\xi_{2k}$  are independent random variables drawn from a flat distribution

$$P_W(\xi) = \frac{1}{2W} \theta(W - |\xi|). \quad (3)$$

The derivation of the Hamiltonian (1) is summarized in Appendix A. The model (1) is a variant of a class of disordered impurity models previously studied by Klümper and Zvyagin<sup>49</sup>, who in particular determined thermodynamic properties<sup>49–53</sup>. Yang-Baxter integrability imposes severe restrictions on the form of the Hamiltonian. This results in all three kinds of interactions involving the same random parameters and can be viewed as short-range correlated disorder in a model of interacting spins.

As a sufficiently strong next-nearest neighbour exchange can induce dimerization our model can in some sense be considered as an interacting analogue of the “dimer models” mentioned above.

### A. Higher conservation laws

As shown in Appendix A the Hamiltonian (1) is related to the transfer matrix  $\tau(\mu)$  of an inhomogeneous six-vertex model. This connection is useful for constructing higher conservation laws, which are a characteristic feature of Yang-Baxter integrable models. In the case at

hand they can be obtained by taking logarithmic derivatives of the transfer matrix at  $\mu = 0$

$$Q^{(n)} = i^n \frac{d^{n-1}}{d\mu^{n-1}} \bigg|_{\mu=0} \ln(\tau(\mu)), \quad n = 2, 3, \dots \quad (4)$$

The Hamiltonian is by construction proportional to  $Q^{(2)}$

$$H = -2i \sin \eta Q^{(2)}. \quad (5)$$

Importantly the higher conservation laws are also (ultra)local in the following sense: they can be expressed in the form

$$Q^{(n)} = \sum_{j=1}^L Q_j^{(n)}, \quad (6)$$

where  $Q_j^{(n)}$  act non-trivially only on a finite number of neighbouring sites. We note that the structure of these conservation laws is very different from that on the “1-bits” in many-body localized systems.

In the following we will make use of the first higher conservation law  $Q^{(3)}$ . To that end we require an explicit expression in terms of the L-operator (A2) and its derivatives. For the operator  $Q^{(2)}$  this is readily done:

$$Q^{(2)} = - \sum_{j=1}^{L/2} Q_{2j-1,2j}^{(2,1)} + Q_{2j-1,2j,2j+1}^{(2,2)}, \quad (7)$$

where

$$\left[ Q_{1,2,3}^{(2,2)} \right]_{\alpha_1 \alpha_2 \alpha_3}^{\beta_1 \beta_2 \beta_3} = \left[ L(-x_2) \right]_{\alpha_2 d}^{\alpha_1 c} \left[ L'(0) \right]_{\alpha_3 e}^{c \beta_3} \left[ L(x_2) \right]_{d \beta_2}^{e \beta_1},$$

$$\left[ Q_{1,2}^{(2,1)} \right]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \left[ L'(-x_2) \right]_{\alpha_2 d}^{\alpha_1 c} \left[ L(x_2) \right]_{d \beta_2}^{c \beta_1}. \quad (8)$$

The conservation law  $Q^{(3)}$  can be expressed as a sum of terms that involve spin interactions on two, three, four and five neighbouring sites respectively

$$Q^{(3)} = -i \sum_{j=1}^{L/2} \left[ Q_{2j-1,2j}^{(3,1)} + Q_{2j-1,2j,2j+1}^{(3,2)} \right. \\ \left. + Q_{2j-1,2j,2j+1,2j+2}^{(3,3)} + Q_{2j-1,2j,2j+1,2j+2,2j+3}^{(3,4)} \right], \quad (9)$$

where

$$\left( Q_{1,2}^{(3,1)} \right)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \left[ L''(-x_2) \right]_{\alpha_2 d}^{\alpha_1 c} \left[ L(x_2) \right]_{d \beta_2}^{c \beta_1} \\ - \left[ Q_{1,2}^{(2,1)} Q_{1,2}^{(2,1)} \right]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2},$$

$$\left( Q_{1,2,3}^{(3,2)} \right)_{\alpha_1 \alpha_2 \alpha_3}^{\beta_1 \beta_2 \beta_3} = 2 \left[ L'(-x_2) \right]_{\alpha_2 d}^{\alpha_1 c} \left[ L'(0) \right]_{\alpha_3 e}^{c \beta_3} \left[ L(x_2) \right]_{d \beta_2}^{e \beta_1} \\ - \left[ Q_{1,2}^{(2,1)} Q_{1,2,3}^{(2,2)} + Q_{1,2,3}^{(2,2)} Q_{1,2}^{(2,1)} \right]_{\alpha_1 \alpha_2 \alpha_3}^{\beta_1 \beta_2 \beta_3},$$

$$Q_{1,2,3,4}^{(3,3)} = Q_{3,4}^{(2,1)} Q_{1,2,3}^{(2,2)} - Q_{1,2,3}^{(2,2)} Q_{3,4}^{(2,1)},$$

$$Q_{1,2,3,4,5}^{(3,4)} = Q_{3,4,5}^{(2,2)} Q_{1,2,3}^{(2,2)} - Q_{1,2,3}^{(2,2)} Q_{3,4,5}^{(2,2)}. \quad (10)$$

The operator  $Q^{(3)}$  can be expressed in terms of Pauli matrices using (A2), but this is not particularly useful for our purposes.

### III. NON-INTERACTING LIMIT

The particular case  $\eta = \pi/2$  maps to non-interacting spinless fermions by means of a Jordan-Wigner transformation<sup>54</sup>. The resulting Hamiltonian (1) is block-diagonal  $H = P_+ H_+ + P_- H_-$ , where  $P_{\pm} = \frac{1}{2}(1 \pm (-1)^F)$  are projection operators onto the subspaces with even and odd numbers of fermions respectively ( $F$  is the fermion number operator). We find

$$H_+ = \sum_{j < k=1}^L c_j^\dagger A_{jk} c_k + \text{h.c.}, \quad (11)$$

where  $A_{1,L-1} = 2i \tanh(\xi_L)$ ,  $A_{1,L} = \frac{2}{\cosh(\xi_L)}$  and

$$A_{2j \pm 1, 2j} = -\frac{2}{\cosh(\xi_{2j})}, \quad A_{2j-1, 2j+1} = 2i \tanh(\xi_{2j}), \quad (12)$$

Fermions tunnel between neighbouring sites with amplitudes that are random apart from the constraints  $A_{2j-1, 2j} = A_{2j, 2j+1}$ . In addition there is a next-nearest neighbour hopping on the sublattice of all odd sites. Importantly the corresponding tunneling amplitudes  $A_{2j-1, 2j+1}$  are not independent random variables, but are related to the amplitudes  $A_{2j-1, 2j}$ . The fermion hopping (12) can therefore be thought of as realizing a particular kind of *correlated disorder*. As we will see, this has important consequences for the physical properties of energy eigenstates. Single-particle energy eigenstates are constructed as  $|\Psi_n\rangle = \sum_{j=1}^L \phi_{n,j} c_j^\dagger |0\rangle$ , where  $\phi_n$  are the (orthonormal) eigenvectors of the matrix  $A$  and  $|0\rangle$  is the state without fermions. In order to investigate whether the model (11) is localized we have determined the inverse participation ratio of single-particle energy eigenstates  $I_n = \sum_{j=1}^L |\phi_{n,j}|^4$ . We have considered several probability distributions of the random parameters  $\xi_{2j}$ , all of which lead to the same conclusion. We therefore focus on (3). In Fig. 2 we show normalized histograms of  $I_n$  averaged over 1000 disorder realizations for  $W = 3$  and two different system sizes. We see that the inverse participation ratios are strongly peaked at a value that we find to scale inversely with system size as  $1/L$ . This indicates that the eigenstates are not localized. At this point the question arises whether the model (11), (12) is delocalized as a result of fine tuning, or whether it is representative of a broader class of theories. To investigate this issue we have considered free fermion models of the type (11) with tunneling amplitudes

$$A_{2j \pm 1, 2j} = -2|x_{2j}|, \quad A_{2j-1, 2j+1} = 2is \operatorname{sgn}(x_{2j}) \sqrt{1 - x_{2j}^2} + s'y_{2j}, \quad (13)$$

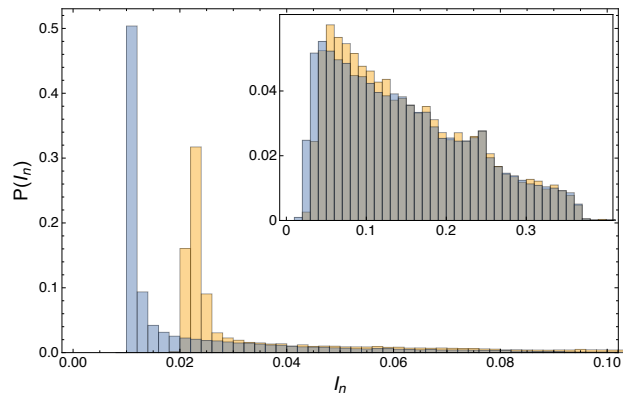


FIG. 2. Histograms of the inverse participation ratios for single-particle energy eigenstates for system sizes  $L = 64$  (yellow) and  $L = 128$  (blue) averaged over 1000 disorder realizations with probability distribution for inhomogeneities given by the box distribution with  $W = 3$ . Inset: same for eigenstates of (11), (13) with  $s = 0$ ,  $s' = 0.2$ .

where we take  $x_{2j}$  and  $y_{2j}$  to be independent random variables with probability distribution  $P_1(x)$  (3). The tuning parameters  $0 \leq s, s' \leq 1$  allow us to interpolate between the “Yang-Baxter” case in which the next-nearest neighbor tunneling amplitudes  $A_{2j-1, 2j+1}$  are fixed in terms of the  $A_{2j, 2j+1}$  and the limit in which they become independent random variables. We have analyzed IPRs for a range of values  $s$  and  $s'$ . The results suggest that for  $s \approx 1$  and small values of  $s'$ , i.e. Hamiltonians close to the Yang-Baxter point, eigenstates are delocalized. On the other hand for small values of  $s$  and  $s'$ , i.e. weak uncorrelated next-nearest neighbour tunneling, the data is more consistent with localization as shown in the inset of Fig. 2. This suggests that the “Yang-Baxter” model (11), (12) does not correspond to an isolated point in parameter space but is representative of a delocalized region that arises as a result of the correlation between the nearest neighbor and next-nearest neighbor tunneling.

#### A. Local Quantum Quench

A second way of investigating localization properties in energy eigenstates is by considering the spreading of correlations after a local quantum quench. We prepare the system in the initial finite energy density state and then overturn two neighboring spins. This choice of initial state allows us to work in the even fermion parity sector of the Hilbert space,  $(-1)^{\hat{F}} = 1$ . In order to investigate the spreading of correlations we determine the expectation value of the  $z$ -component of spin at site  $\ell$ . Using Wick’s theorem we obtain compact expressions for  $S_\ell^z(t)$  that can be evaluated numerically for systems of

hundreds of spins. In Fig. (3) we show results for a representative example, where a system of size  $L = 128$  is initially prepared in an energy eigenstate corresponding to inverse temperature  $\beta = 1$ . We see that the per-

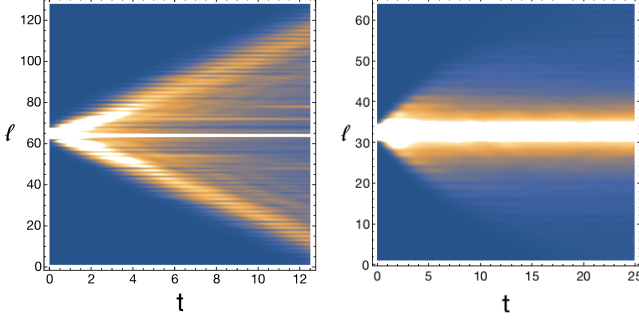


FIG. 3. Left plot:  $\langle S_\ell^z(t) \rangle$  averaged over 30 disorder realizations from the box probability distribution for a system of size  $L = 128$  and initial thermal state with  $\beta = 1$ . There is a clear light cone effect. Right plot: the same for the modified free fermion model (13) with  $s = s' = 0$ ,  $L = 64$ . Picture is consistent with localization.

turbation, which is initially localized at sites  $L/2$  and  $L/2 + 1$ , propagates ballistically through the system, as can be seen from the presence of a “light-cone” outside of which our observable remains negligibly small. The velocity characterizing this ballistic propagation depends on the disorder distribution and can be determined exactly in the thermodynamic limit. The spreading of a local perturbation in energy eigenstates of the modified free fermion model (13) can be analyzed in an analogous way. As shown in Fig. 3, for small values of  $s$  and  $s'$  the perturbation remains localized at sites  $L/2$  and  $L/2 + 1$  in an extended time window even though a weak light-cone effect occurs at early times. This again indicates that the modified free fermion model is localized at small values of  $s$ ,  $s'$ .

#### IV. STRONGLY INTERACTING REGIME

Examination of the IPR of (1) away from the free fermion point for small system sizes  $L = 10, 12$  is compatible with delocalized behaviour of energy eigenstates. We also have studied the spreading of local perturbations in energy eigenstates. (i) We have considered a single spin flip at an odd site on top of the saturated ferromagnetic state. Representative results for the subsequent dynamics on an  $L = 100$  site system are shown in Fig. (4). There is a clear light-cone effect that signals ballistic spreading of the perturbation. (ii) We have flipped two neighbouring spins in the ground state, *cf.* Ref. 55 for a discussion of the analogous protocol in the clean system. In this case however our numerics is limited however to small systems of up to  $L = 16$ . We find that there again is a clear light cone effect, see Fig. 5.

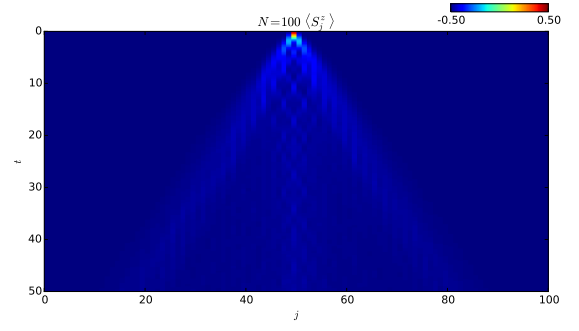


FIG. 4. Spreading of a single spin flip on top of the saturated ferromagnetic state for  $L = 100$  and  $\eta = 0$  in (1), averaged over 50 disorder realizations with distribution  $P_1(\xi)$ .

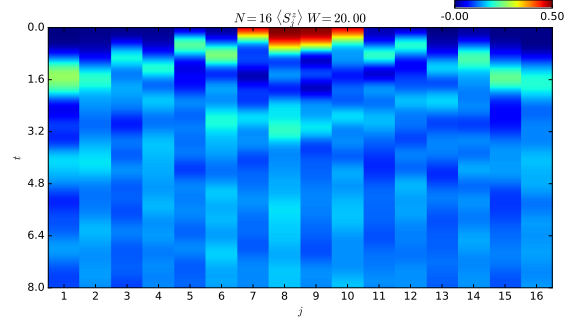


FIG. 5. (b) Spreading of a spin flips at two neighbouring sites on top of the ground state for  $N = 16$  and  $\cos(\eta) = 2$  in (1), averaged over 20 disorder realizations with distribution  $P_{20}(\xi)$ .

#### V. BOUNDS ON SPIN AND ENERGY TRANSPORT

We will now demonstrate that the eigenstates of (1) exhibit ballistic energy and spin transport for any anisotropy  $\eta$  and disorder strength  $W$ . We employ a combination of two methods: the first is based on Mazur’s inequality<sup>56</sup> and was previously employed to establish the existence of a finite temperature Drude weight in the clean case<sup>57</sup>, while the second is based on the recently developed hydrodynamic approach to transport in integrable models<sup>58–61</sup>. The starting point of the first approach is the existence of a set of conserved quantities  $[H, Q_n] = 0$  that are orthogonal in the sense that  $\langle Q_n Q_m \rangle_\beta = \delta_{n,m} \langle Q_n^2 \rangle_\beta$ . Here  $\langle \cdot \rangle_\beta$  denotes a thermal expectation value. As the z-component of total spin  $\sigma^z = \sum_{j=1}^L \sigma_j^z$  is a conserved quantity in our model we employ a magnetic field term to fix the magnetization in our thermal ensemble. Given an operator  $A = A^\dagger$  with  $\langle A \rangle_\beta = 0$  the following inequality due to Mazur<sup>56</sup> then holds

$$\lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \int_0^{T_0} dt \langle A(t) A \rangle_\beta \geq \sum_n \frac{\langle A Q_n \rangle_\beta^2}{\langle (Q_n)^2 \rangle_\beta}. \quad (14)$$

A positive bound for the right-hand side of (14) implies that the autocorrelation function of the operator  $A$  does not decay to zero at late times. This implies that the Fourier transform has a non-vanishing (generalized) Drude weight

$$\frac{1}{L} \int_0^\infty dt \cos(\omega t) \langle A(t) A \rangle_\beta = 2\pi D_A \delta(\omega) + \dots \quad (15)$$

When  $A$  is the spin current or the energy current operator the non-decay of the autocorrelation functions shows that the system is an ideal conductor of spin/energy. The Hamiltonian (1) has an extensive number of integrals of motion  $Q^{(n)}$  (4). The conservation laws relevant to us here have local densities and we focus on the most local of these,  $Q^{(3)}$ , which involves interactions between spins on at most five neighbouring sites, cf. eqn (9). We furthermore constrain our discussion to infinite temperatures  $\beta = 0$ . For local operators the corresponding thermal average equals the expectation value in typical energy eigenstates at the associated energy density, which allows us to draw conclusions about the local properties of the eigenstates of (1). In order to use the Mazur inequality (14) we carry out a subtraction  $\mathcal{Q}_3 = Q^{(3)} - \langle Q^{(3)} \rangle_{\beta=0}$ ,

which ensures that the expectation value of  $\mathcal{Q}_3^2$  is extensive, i.e.  $\lim_{L \rightarrow \infty} L^{-1} \langle \mathcal{Q}_3^2 \rangle_{\beta=0} = a_1 > 0$ . The expression for  $a_1$  is very cumbersome so that we do not report it here.

The spin and energy current operators  $J^{S,E}$  associated with the Hamiltonian (1)  $H = \sum_j H_{2j-1,2j,2j+1}$  are obtained from the continuity equations

$$\begin{aligned} i \sum_{j=-\infty}^{\ell} [\sigma_j^z, H] &= J_\ell^S, \\ i \sum_{j=-\infty}^{\ell} [H_{2j-1,2j,2j+1}, H] &= J_\ell^E. \end{aligned} \quad (16)$$

Evaluating the commutators and then summing over all sites gives

$$\begin{aligned} J^E &= 4i \sin^2(\eta) \sum_j Q_{2j-1,2j,2j+1,2j+2}^{(3,3)} \\ &\quad + Q_{2j-1,2j,2j+1,2j+2,2j+3}^{(3,4)}, \end{aligned} \quad (17)$$

where  $Q^{(3,3)}$  and  $Q^{(3,4)}$  are given in (10). The spin current operator can be written in the form

$$\begin{aligned} J^S &= 2 \sum_j J_{2j}^{(1)} (T_{2j-1,2j}^{xy} - T_{2j-1,2j}^{yx} + T_{2j,2j+1}^{xy} - T_{2j,2j+1}^{yx}) + 2 J_{2j}^{(2)} (T_{2j-1,2j+1}^{xy} - T_{2j-1,2j+1}^{yx}) \\ &\quad - \frac{2K_{2j}}{\Delta_{2j}} (T_{2j-1,2j,2j+1}^{xx} + T_{2j-1,2j,2j+1}^{yy}) + K_{2j} (T_{2j-1,2j,2j+1}^{zx} + T_{2j-1,2j,2j+1}^{zy} + T_{2j-1,2j,2j+1}^{yz} + T_{2j-1,2j,2j+1}^{zx}), \end{aligned} \quad (18)$$

where we have defined

$$T_{j_1, \dots, j_n}^{\alpha_1, \dots, \alpha_n} = \prod_{k=1}^n \sigma_{j_k}^{\alpha_k}. \quad (19)$$

We find that in contrast to the homogeneous case, the energy current is not conserved, i.e.  $[H, J^E] \neq 0$ .

At infinite temperature and finite magnetization  $m$  a tedious but straightforward calculation gives the following result for the overlap of the spin current with the third conserved charge

$$\langle J^S \mathcal{Q}_3 \rangle_{\beta=0} = \frac{\Delta}{1 - \Delta^2} \sum_n \frac{4m(1 - 4m^2) f(\xi_{2n})}{[\cosh(2\xi_{2n}) - (\cos(2\eta))]^3}, \quad (20)$$

where

$$\begin{aligned} f(z) &= \cos(2\eta) \cosh(6z) \\ &\quad - 2(\cos(4\eta) - \cos(2\eta) + 3) \cosh(4z) \\ &\quad + (6\cos(4\eta) - \cos(6\eta) + 10) \cosh(2z) \\ &\quad - 18\cos(2\eta) + 2\cos(4\eta) + 6. \end{aligned} \quad (21)$$

For a very large system we may replace the sum by an

integral so that

$$\begin{aligned} \langle J^S \mathcal{Q}_3 \rangle_{\beta=0} &= a_S L + o(L), \\ a_S &= \frac{\Delta}{1 - \Delta^2} \int d\xi \frac{4m(1 - 4m^2) f(\xi) P(\xi)}{(\cosh(2\xi) - \cos(2\eta))^3}. \end{aligned} \quad (22)$$

Here  $P(\xi)$  is the probability distribution on the random variables  $\xi_{2n}$ . Importantly we have  $a_S \neq 0$  unless we fine-tune the probability distribution. This in turn provides a positive bound for the Mazur inequality

$$\lim_{L \rightarrow \infty} \lim_{T_0 \rightarrow \infty} \frac{1}{T_0 L} \int_0^{T_0} dt \langle J^S(t) J^S \rangle_{\beta=0} \geq \frac{a_S^2}{a_1}. \quad (23)$$

In the case of the energy current for simplicity we consider the zero magnetization sector  $m = 0$ . Applying Mazur's inequality we find

$$\begin{aligned} &\lim_{L \rightarrow \infty} \lim_{T_0 \rightarrow \infty} \frac{1}{T_0 L} \int_0^{T_0} dt \langle J^E(t) J^E \rangle_{\beta=0} \\ &\geq \lim_{L \rightarrow \infty} \frac{1}{L} \frac{\langle J^E \mathcal{Q}_3 \rangle_{\beta=0}^2}{\langle \mathcal{Q}_3^2 \rangle_{\beta=0}} = \frac{64(2 + 2\cos(2\eta))^2}{16\sin^4(\eta)a_1}. \end{aligned} \quad (24)$$

Interestingly the bound (24) is independent of the inhomogeneities. The generalization to  $m \neq 0$  is very tedious



but straightforward and provides a non-zero bound as well.

The above calculation proves that at energy densities corresponding to infinite temperature the model (1) exhibits (i) a non-zero Drude weight at any finite magnetization; (ii) ballistic energy transport.

## VI. SPIN AND ENERGY TRANSPORT FROM GENERALIZED HYDRODYNAMICS

Generalized Drude weights (15) can be analyzed in full by means of the approach introduced in Ref. 58. The starting point is the existence of a basis of local charges  $\hat{Q}_i$  and associated currents  $J_i$ . Using these charges a *generalized Gibbs ensemble* is defined by the density matrix  $\rho_{\text{GGE}} \sim \exp(-\sum_n \mu_n \hat{Q}_n)$ , where  $\mu_i$  are “chemical potentials”. The generalized Drude weights  $D_A$  are then obtained from appropriate expectation values in this ensemble and are determined by using the thermodynamic Bethe ansatz (TBA) method<sup>62</sup>. According to Ref. 58, in integrable models  $D_A$  can be expressed as

$$D_A = \sum_n \int d\lambda \frac{\eta_n(\lambda)}{\rho_n^{\text{tot}}(\lambda)} \left( \frac{\epsilon'_n(\lambda) q_A^{\text{eff}}(\lambda)}{2\pi(1 + \eta_n(\lambda))} \right)^2, \quad (25)$$

where  $\eta_n(\lambda) = \bar{\rho}_n(\lambda)/\rho_n(\lambda)$  is the ratio of hole and particle densities,  $\rho_n^{\text{tot}}(\lambda, \{\xi_{2j}\}) = \rho_n + \bar{\rho}_n$ ,  $\epsilon_n(\lambda)$  are the energies of n-string excitations over the state of thermal equilibrium<sup>63</sup> and  $q_A^{\text{eff}} = \partial_{\mu_A} \log \eta_n$  are effective transport charges. The implementation of this approach in our “inhomogeneous” case reveals (see Appendix B for more details) that the disorder merely renormalizes the Drude weight through the disorder-dependence of the velocity of the elementary excitations over the equilibrium state under consideration, which enters (15) via the factor  $1/\rho_n^{\text{tot}}(\lambda)$ . It follows then that the disorder average can be exchanged with the integration and summation in (25). The disorder averaged Drude weight is then given by

$$\bar{D}_A = \sum_n \int d\lambda [\overline{\rho_n^{\text{tot}}(\lambda)}]^{-1} \eta_n(\lambda) \left[ \frac{\epsilon'_n(\lambda) q_A^{\text{eff}}(\lambda)}{2\pi(1 + \eta_n(\lambda))} \right]^2, \quad (26)$$

where  $[\overline{\rho_n^{\text{tot}}(\lambda)}]^{-1} = \int P(\{\xi\}) \frac{1}{\rho_n^{\text{tot}}(\lambda, \{\xi\})}$  denotes the disorder average with probability distribution function  $P(\xi)$ . As the total density  $\rho_n^{\text{tot}}(\lambda)$  is a positive quantity this average is non-zero for generic  $P(\xi)$ . Therefore, the Drude weight is only renormalized due to the disorder dependence of string particle and hole densities. We note that in contrast to the Mazur bound calculation the TBA approach takes into account the full set of conserved quantities. These observations can be universally extended to any integrable model with disorder of the type described here.

## VII. CONCLUSIONS

In this paper we studied a Yang-Baxter integrable interacting spin system with controllable short-range correlated disorder. Using a combination of diagnostics we have demonstrated the absence of many-body localization. We find that the model is in fact an ideal conductor for both energy and magnetization. For particular parameter values the model can be mapped to non-interacting fermions and we have established the absence of Anderson localization in this case. In contrast, a sufficiently strong deformation of the free-fermion Hamiltonian away from the Yang-Baxter point shows signatures of localization. We expect that in the interacting case small perturbations away from the Yang-Baxter point will lead to diffusive behaviour, while sufficiently strong deformations will be required to induce an MBL transition.

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## Appendix A: Inhomogeneous XXZ chain

The Quantum Inverse Scattering Method (QISM)<sup>36</sup> provides a simple way of introducing “impurities” into Yang-Baxter integrable models. This has been used in the literature to construct a variety of models with impurities embedded in both non-interacting and correlated hosts<sup>64–73</sup>, as well as models with “disorder”<sup>49–53</sup>. Here we focus on the simplest case, which is related to the spin-1/2 Heisenberg XXZ chain. The basic ingredients in the QISM are the R-matrix  $R(\mu) \in \text{End}(V_A \otimes V_A)$  and the L-operator  $L(\mu) \in \text{End}(V_A \otimes V_Q)$ , where  $V_A$  and  $V_Q$  are finite-dimensional “auxiliary” and “quantum” vector spaces. In the cases we are interested in the Yang-Baxter relations read

$$R(\lambda - \mu) [L(\lambda) \otimes L(\mu)] = [L(\mu) \otimes L(\lambda)] R(\lambda - \mu). \quad (\text{A1})$$

In the case of the spin-1/2 XXZ chain we have<sup>36</sup>

$$\begin{aligned} (L(\lambda))_{\alpha\beta}^{ab} &= \frac{1 + \tau_{ab}^z \sigma_{\alpha\beta}^z}{2} + b(\lambda) \frac{1 - \tau_{ab}^z \sigma_{\alpha\beta}^z}{2} \\ &\quad + c(\lambda) (\tau_{ab}^- \sigma_{\alpha\beta}^+ + \tau_{ab}^+ \sigma_{\alpha\beta}^-), \\ b(\lambda) &= \frac{\sinh(\lambda)}{\sinh(\lambda + i\eta)}, \quad c(\lambda) = \frac{i \sin(\eta)}{\sinh(\lambda + i\eta)} \end{aligned} \quad (\text{A2})$$

where  $\eta$  is a free parameter and  $\tau^\alpha, \sigma^\alpha$  are Pauli matrices acting on the auxiliary and quantum spaces respectively. The QISM provides a commuting family of transfer matrices  $[\tau(\mu), \tau(\lambda)] = 0$  of the form

$$\tau(\mu)_{\alpha_1, \dots, \alpha_L}^{\beta_1, \dots, \beta_L} = \prod_{j=1}^L \left[ L(\mu - \xi_j) \right]_{\alpha_j \beta_j}^{c_j c_{j+1}}, \quad (\text{A3})$$

where the free parameters  $\xi_j$  are known as “inhomogeneities” and where we have defined  $c_{L+1} = c_1$ . In order to obtain a local Hamiltonian we now set

$$\xi_{2j+1} = 0, \quad (\text{A4})$$

and then take the logarithmic derivative of the transfer matrix at  $\mu = 0$

$$H = 2i \sin \eta \frac{d}{d\mu} \Big|_{\mu=0} \ln(\tau(\mu)). \quad (\text{A5})$$

The explicit expression for the resulting Hamiltonian is given by (1).

### 1. Spectral properties

The Hamiltonian (1) is readily diagonalized by Algebraic Bethe Ansatz<sup>36</sup>. The energy eigenvalues are given by

$$E = - \sum_{j=1}^N \frac{4 \sin^2(\eta)}{\cosh(2\lambda_j) - \cos(\eta)}, \quad (\text{A6})$$

where the rapidities  $\lambda_1, \dots, \lambda_N$  are solutions of the Bethe Ansatz equations

$$\begin{aligned} & \left( \frac{\sinh(\lambda_j + i\eta/2)}{\sinh(\lambda_j - i\eta/2)} \right)^{\frac{L}{2}} \prod_{k=1}^{L/2} \frac{\sinh(\lambda_j - \xi_{2k} + i\eta/2)}{\sinh(\lambda_j - \xi_{2k} - i\eta/2)} \\ &= \prod_{k \neq j} \frac{\sinh(\lambda_j - \lambda_k + i\eta)}{\sinh(\lambda_j - \lambda_k - i\eta)}, \quad j = 1, \dots, N. \end{aligned} \quad (\text{A7})$$

Equations (A6) and (A7) establish a peculiar property of the model (1): the spectrum is invariant under arbitrary permutations of the inhomogeneities  $\{\xi_2, \xi_4, \dots, \xi_L\}$ , i.e.

$$\text{spec } H[\{\xi_2, \dots, \xi_L\}] = \text{spec } H[\{\xi_{P(2)}, \dots, \xi_{P(L)}\}] \quad (\text{A8})$$

for any permutation  $P$  of the integers  $2, 4, \dots, L$ . This property is not apparent from the explicit expression (1) and Hamiltonians corresponding to different permutations of the inhomogeneities generally do not commute.

#### a. Free Fermion Point

The Hamiltonian (1) has a free fermion point at  $\eta = \frac{\pi}{2}$ . The corresponding Hamiltonian is

$$\begin{aligned} H &= \sum_{j=1}^{L/2} \frac{1}{\cosh(\xi_{2j})} \sum_{\alpha=x,y} [\sigma_{2j-1}^\alpha \sigma_{2j}^\alpha + \sigma_{2j}^\alpha \sigma_{2j+1}^\alpha] \\ &\quad - \sum_{j=1}^{L/2} \tanh(\xi_{2j}) [\sigma_{2j-1}^y \sigma_{2j}^z \sigma_{2j+1}^x - \sigma_{2j-1}^x \sigma_{2j}^z \sigma_{2j+1}^y]. \end{aligned} \quad (\text{A9})$$

By applying the Jordan-Wigner transformation one can bring the Eq. (A9) into the form of the Eq.

#### b. Isotropic (XXX) Limit

The SU(2) invariant versions of the Hamiltonian and the Bethe Ansatz equations are obtained by redefining

$$\lambda_j = \frac{\eta}{2} \Lambda_j, \quad \xi_{2j} = \frac{\eta}{2} \gamma_j, \quad (\text{A10})$$

and then taking the limit  $\eta \rightarrow 0$ . This gives a Hamiltonian of the form

$$\begin{aligned} H &= \sum_j \frac{4}{\gamma_j^2 + 4} [\vec{\sigma}_{2j-1} \cdot \vec{\sigma}_{2j} + \vec{\sigma}_{2j} \cdot \vec{\sigma}_{2j+1} - 2] \\ &\quad - \sum_j \frac{2\gamma_j}{\gamma_j^2 + 4} \vec{\sigma}_{2j} \cdot (\vec{\sigma}_{2j-1} \times \vec{\sigma}_{2j+1}) \\ &\quad + \sum_j \frac{\gamma_j^2}{\gamma_j^2 + 4} (\vec{\sigma}_{2j-1} \cdot \vec{\sigma}_{2j+1} - 1). \end{aligned} \quad (\text{A11})$$

The Bethe Ansatz equations become

$$\left( \frac{\Lambda_j - i}{\Lambda_j + i} \right)^{\frac{M}{2}} \prod_{k=1}^{M/2} \frac{\Lambda_j - \gamma_k - i}{\Lambda_j - \gamma_k + i} = \prod_{k \neq j} \frac{\Lambda_j - \Lambda_k - 2i}{\Lambda_j - \Lambda_k + 2i}. \quad (\text{A12})$$

The energy corresponding to a solution of (A12) is

$$E = - \sum_j \frac{8}{\Lambda_j^2 + 1}. \quad (\text{A13})$$

### Appendix B: Drude weights from the TBA calculations

Let us consider first the case of  $|\Delta| > 1$ . While any eigenstate in a finite system of size  $L$  is assigned a unique set of rapidities  $\{\lambda_j\}_{k=1}^N$  taken from solutions of Bethe equations (A7), in the thermodynamic limit (defined as  $L \rightarrow \infty, N \rightarrow \infty$  with  $N/L$  finite), the solutions to Bethe equations organize into regular patterns which indicate the presence of well-defined particle excitations.

These correspond to magnons and their bound states, so-called Bethe strings<sup>62</sup>. A general string solution reads  $\{\lambda_\alpha^{k,m}\} = \{\lambda_\alpha^k + (k+1-2m)\frac{i\eta}{2}\}$ , where  $m = 1, 2, \dots, k$  and  $\alpha$  numerates different  $k$ -strings and  $m$  runs over internal rapidities. Scattering of different magnonic particles are characterized by the amplitudes

$$S_j = \frac{\sin(\lambda - j\frac{i\eta}{2})}{\sin(\lambda + j\frac{i\eta}{2})},$$

$$S_{jk} = \prod_{m=-\frac{k-1}{2}}^{\frac{k-1}{2}} \prod_{n=-\frac{j-1}{2}}^{\frac{j-1}{2}} S_{2m+2n+2}$$

$$= S_{|j-k|} S_{j+k} \prod_{m=1}^{\min(j,k)-1} S_{|j-k|+2m}^2 \quad (\text{B1})$$

with convention that  $S_0 \equiv 1$ . In the thermodynamic limit particle rapidities become densely distributed along the real axis in the rapidity plane. This permits to introduce distributions  $\rho_k(\lambda)$  of  $k$ -string particles, along with the dual hole distributions  $\bar{\rho}_k(\lambda)$  (holes are solutions to Bethe ansatz equations which differ from Bethe roots  $\lambda_k$ ). The discrete Bethe equations (A7) get replaced by the integral Bethe-Yang equations. Assuming validity of the string solution in the presence of  $M$  inhomogeneities ( $M/N \leq 1/2$ ), we can write these integral equations for the densities of string particles and holes in the thermodynamic limit of the inhomogeneous case. The Bethe-Yang equations for particles  $\rho_n(\lambda)$  and holes  $\bar{\rho}_n(\lambda)$  are given by

$$\frac{1}{N} \left( \sum_{j=1}^M a_n(\lambda + \xi_j) + (N-M)a_n(\lambda) \right) = \bar{\rho}_n(\lambda) + A_{nm} \star \rho_m(\lambda). \quad (\text{B2})$$

Here, the explicit form of the functions  $a_n(\lambda)$  and  $A_{nm}(\lambda) = \delta_{nm}\delta(\lambda) + a_{nm}$ , which depend on the anisotropy parameter  $\Delta$ , can be obtained from the following relations

$$a_n(\lambda) = \frac{1}{2\pi i} \partial_\lambda \log S_n(\lambda),$$

$$a_{nm}(\lambda) = \frac{1}{2\pi i} \partial_\lambda \log S_{nm}(\lambda) \quad (\text{B3})$$

where indexes  $n, m$  label corresponding stringy content. The  $\star$  operation refers to the convolution with the kernel  $A_{nm}$

$$A_{mn} \star \rho_m(x) \equiv \sum_m \int_{-Q}^Q dy A_{mn}(x-y) \rho_m(y) \quad (\text{B4})$$

where the integration and summation limits depend on the value of anisotropy parameter. Explicitly, for  $\Delta > 1$  we have

$$a_n(\lambda) = \frac{1}{2\pi} \frac{\eta \sinh(n\eta)}{\cosh(n\eta) - \cos(\eta\lambda)}. \quad (\text{B5})$$

For the isotropic (XXX) situation, when  $\eta \rightarrow 0$ , the driving function and the kernel are given by

$$a_n(\lambda) = \frac{1}{\pi} \frac{n}{(n^2) + \lambda^2}, \quad (\text{B6})$$

$$A_{nm}(\lambda) = \delta(\lambda)\delta_{nm} + (1 - \delta_{nm})a_{|n-m|}(\lambda) + 2a_{|n-m|+2}(\lambda) + \dots + 2a_{n+m-2}(\lambda) + a_{n+m}(\lambda) \quad (\text{B7})$$

while in this case  $Q = \infty$  and sum runs to infinity as well.

Classification of the particle content in the gapless regime  $|\Delta| < 1$  is more involved, details can be found in<sup>74, 62</sup>. Here, in addition to the magnon type label  $k$ , an extra parity label  $v \in \pm$  is required. Importantly, integers  $k$  now no longer coincide with the length of a string, i.e. a number of magnons forming a bound state. Instead, the  $k$ -th particle consists of  $n_k$  Bethe roots and carries parity  $v_k$  (see<sup>74</sup> for further details). Setting  $\Delta = \cos(\gamma)$ , where  $\gamma/\pi = m/l$  (with  $m, l$  co-prime integers) is a root of unity, the number of distinct particles in the spectrum is finite. Changing the parametrization  $\lambda \rightarrow i\lambda$ ,  $\eta \rightarrow i\gamma$  and incorporating the additional parity label, the elementary scattering amplitudes and kernels read

$$S_k(\lambda) \rightarrow S_{(n_j, v_j)} = \frac{\sinh[\lambda - n_j \frac{i\gamma}{2} + (1 - v_j) \frac{i\pi}{4}]}{\sinh[u + n_j \frac{i\gamma}{2} + (1 - v_j) \frac{i\pi}{4}]} \quad (\text{B8})$$

and the whole set of scattering kernels is obtained, as in the case of  $\Delta > 1$ , from Eqs. (B1, B3). The Bethe-Yang equations gets modified,

$$a_j(x) = \text{sign}(q_j)(\rho_j + \bar{\rho}_j) + a_{jk} \star \rho_k \quad (\text{B9})$$

where the summation in the convolution expression runs from 1 to  $m_l$  defined as  $m_0 = 0$ ,  $m_i = \sum_{k=1}^i \nu_k$  and numbers  $\nu_1, \dots, \nu_{l-1} \geq 1$ ,  $\nu_l \geq 2$  participate in the continuum fraction expression for  $\gamma/\pi$ , e.g.  $\gamma/\pi = 1/(\nu_1 + 1/(\nu_2 + 1/(\nu_3 + \dots)))$ . Numbers  $q_j$  are defined recursively as<sup>62</sup>,  $q_0 = \pi/\gamma$  and

$$q_j = \frac{1}{2}[(1 - \delta_{m_i, j})q_{j-1} + q_{j+1}], \quad m_i \leq j \leq m_{i+1} - 2$$

$$q_j = (1 - 2\delta_{m_{i-1}, j})q_{j-1} + q_{j+1}, \quad j = m_i - 1, i < l$$

Explicitly, the kernels  $a_j(\lambda)$  are given by

$$a_j(\lambda) = \frac{1}{2\pi} \frac{\gamma \sin(\gamma q_j)}{\cosh(\gamma\lambda) + \cos(\gamma q_j)} \quad (\text{B10})$$

The most important thing to notice here is that the left hand side (driving terms) of the Bethe-Yang equations depends on the inhomogeneities while the right hand side (convolution kernel) *does not* depend on inhomogeneities. This can also be checked by explicit re-derivation of steps leading to these equations (B2).

The second set of equations is derived using the variation of the free energy (per particle)  $f = e - Ts$  with respect to  $\rho_n$  and  $\bar{\rho}_n$ . Here

$$e = 2\pi \sum_n a_n(\lambda) \rho_n(\lambda) \quad (\text{B11})$$



is the energy density and the entropy density is

$$s = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\lambda \left[ (\rho_n + \bar{\rho}_n) \ln(\rho_n + \bar{\rho}_n) - \rho_n \ln \rho_n - \bar{\rho}_n \ln \bar{\rho}_n \right]. \quad (\text{B12})$$

Variation of (B2) leads to the relationship between  $\delta\rho_n$  and  $\delta\bar{\rho}_n$ ,

$$\delta\bar{\rho}_n = -A_{nm} \star \delta\rho_n. \quad (\text{B13})$$

which finally leads to the second TBA equation

$$\ln(1 + \eta_n) = \frac{2\pi J}{T} a_n + A_{nm} \star \ln(1 + \eta_m^{-1}), \quad (\text{B14})$$

where  $\eta_n = \bar{\rho}_n/\rho_n$ . Importantly, since the right hand side of (B2) does not depend on  $\delta\rho_n$  or on  $\delta\bar{\rho}_n$ , eqn (B14) and hence  $\eta_n$  is *independent of the inhomogeneities*. It is customary to re-cast (B14) in terms of the dressed energies defined by  $\varepsilon_n = T \ln(\eta_n)$

$$\frac{\varepsilon_j}{T} = \frac{\varepsilon_j^{(0)}}{T} + a_{nm} \star \ln(1 + e^{-\varepsilon_m/T}), \quad (\text{B15})$$

where the bare energies are  $\varepsilon_j^{(0)} = 2\pi J a_n$ . In<sup>58</sup> a hydrodynamic approach to the Drude weight(s) has been formulated based on the TBA approach. The starting point is the existence of a basis of local charges  $\hat{Q}_i$  and associated currents  $J_i$ . Using these charges a *generalized Gibbs ensemble* is defined by the density matrix

$$\rho_{\text{GGE}} \sim \exp\left(-\sum_n \mu_n \hat{Q}_n\right), \quad (\text{B16})$$

where  $\mu_i$  are “chemical potentials”. The generalized Drude weights  $D_A$  are then obtained from appropriate expectation values in this ensemble and are determined by using the thermodynamic Bethe ansatz (TBA)

method<sup>62</sup>. According to Ref.<sup>58</sup> in integrable models  $D_A$  can be expressed as

$$D_A = \sum_n \int d\lambda \frac{\eta_n(\lambda)}{\rho_n^{\text{tot}}(\lambda)} \left( \frac{\varepsilon'_n(\lambda) q_A^{\text{eff}}(\lambda)}{2\pi(1 + \eta_n(\lambda))} \right)^2, \quad (\text{B17})$$

where

$$q_A^{\text{eff}} = \partial_{\mu_A} \log \eta_n \quad (\text{B18})$$

are effective transport charges. The functions  $\varepsilon'_n$  are derivatives of the energies of elementary excitations over the state of thermal equilibrium and were calculated in Ref<sup>63</sup>. They are obtained from the dressed energies by solving a set of linear integral equations

$$\varepsilon'_j \star (1 - K)_{jk}(\lambda) = \frac{d\varepsilon_k^{(0)}(\lambda)}{d\lambda}, \quad (\text{B19})$$

$$K_{jk}(x, y) = -\text{sgn}(q_j) a_{jk}(x - y) (1 + e^{\varepsilon_j/T})^{-1}.$$

The only quantities in (B17) that depend on the inhomogeneities  $\xi_{2j}$  are the total densities  $\rho_n^{\text{tot}}(\lambda)$ . This can be seen from (B2) once the disorder-independent equations (B14) for  $\eta_n$  have been solved. It follows that the disorder averaging of the generalized Drude weights can be interchanged with the integration and summation in (B17). Introducing

$$\overline{\left( \frac{1}{\rho_n^{\text{tot}}(\lambda)} \right)} = \int P(\{\xi\}) \frac{1}{\rho_n^{\text{tot}}(\lambda, \{\xi\})} \quad (\text{B20})$$

where  $P(\xi)$  is a disorder probability distribution we then can express disorder averaged Drude weights in the form

$$\bar{D}_A = \sum_n \int d\lambda \overline{\left( \frac{1}{\rho_n^{\text{tot}}(\lambda)} \right)} \eta_n(\lambda) \left( \frac{\varepsilon'_n(\lambda) q_A^{\text{eff}}(\lambda)}{2\pi(1 + \eta_n(\lambda))} \right)^2.$$

As the total density  $\rho_n^{\text{tot}}(\lambda)$  is a positive quantity  $D_A$  is only renormalized due to the dependence of the string particle and string hole densities on disorder, and will not vanish unless the disorder probability distribution is fine-tuned.

<sup>1</sup> P.W. Anderson, Absence of diffusion in certain random lattices, *Phys. Rev.* **109**, 1492 (1958).

<sup>2</sup> N.F. Mott, W.D. Twose, The theory of impurity conduction, *Adv. Phys.* **10**, 107 (1961).

<sup>3</sup> K. Ishii, Localization of eigenstates and transport phenomena in the one-dimensional disordered system, *Progr. Theor. Phys. Suppl.* **53**, 77 (1973).

<sup>4</sup> K.B. Efetov, Supersymmetry and theory of disordered metals, *Adv. Phys.* **32**, 53 (1983).

<sup>5</sup> M.Ya. Azbel, Delocalization transition in one dimension, *Solid State Commun.* **37**, 789 (1981).

<sup>6</sup> M.Ya. Azbel, Eigenstates and properties of random systems in one dimension at zero temperature, *Phys. Rev. B* **28**, 4106 (1983).

<sup>7</sup> I. Goldshtein, S. Molchanov, and L. Pastur, A pure point spectrum of the stochastic one-dimensional Schringer operator, *Funct. Anal. Appl.* **11**, 1 (1977).

<sup>8</sup> R. Johnston, B. Kramer, Localization in one dimensional correlated random potentials, *Z. Phys. B: Condens. Matter* **63** (1986).

<sup>9</sup> J. C. Flores, Transport in models with correlated diagonal and off-diagonal disorder, *J. Phys. Condens. Matter* **1**, 8479 (1989).

<sup>10</sup> B. Kramer and A. MacKinnon, Localization: theory and experiment, *Rep. Prog. Phys.* **56**, 1469 (1993).

<sup>11</sup> F. M. Izrailev and A. A. Krokhnin, Localization and the mobility edge in one-dimensional potentials with correlated disorder, *Phys. Rev. Lett.* **82**, 4062 (1999).

- <sup>12</sup> F. A. B. F. de Moura and M. L. Lyra, Delocalization in the 1D Anderson model with long-range correlated disorder, *Phys. Rev. Lett.* **81**, 3735 (1998).
- <sup>13</sup> J. W. Kantelhardt, S. Russ, A. Bunde, S. Havlin, and I. Webman, *Phys. Rev. Lett.* **84**, 198 (2000).
- <sup>14</sup> H. Shima, T. Nomura, and T. Nakayama, Localization-delocalization transition in one-dimensional electron systems with long-range correlated disorder, *Phys. Rev. B* **70**, 075116 (2004).
- <sup>15</sup> V. Bellani, E. Diez, R. Hey, L. Toni, L. Tarricone, G. B. Parravicini, F. Domínguez-Adame, and R. Gómez-Alcalá, Experimental Evidence of Delocalized States in Random Dimer Superlattices, *Phys. Rev. Lett.* **82**, 2159 (1999).
- <sup>16</sup> P. Carpena, P. Bernaola-Galvan, P. Ch. Ivanov, H. E. Stanley, Metal-insulator transition in chains with correlated disorder, *Nature* **418**, 955 (2002).
- <sup>17</sup> P. Carpena, P. Bernaola-Galvan, P. Ch. Ivanov, H. E. Stanley, Metal-insulator transition in chains with correlated disorder, *Nature* **421**, 764 (2003).
- <sup>18</sup> D. H. Dunlap, H.-L. Wu, and P. Phillips, Absence of localization in a random-dimer model, *Phys. Rev. Lett.* **65**, 88 (1990).
- <sup>19</sup> H.-L. Wu, P. Phillips, Polyaniline is a random-dimer model: a new transport mechanism for conducting polymers, *Phys. Rev. Lett.* **66**, 1366 (1991).
- <sup>20</sup> P. Phillips and H.-L. Wu, Localization and its absence: a new metallic state for conducting polymers, *Science* **252**, 1805 (1991).
- <sup>21</sup> J. C. Flores, M. Hilke, Absence of localization in disordered systems with local correlations, *J. Phys. A: Math. Gen.* **26**, L1255 (1993).
- <sup>22</sup> D. Basko, I. Aleiner, and B. Altshuler, Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states, *Ann. Phys.* **321**, 1126 (2006).
- <sup>23</sup> I. V. Gornyi, A. D. Mirlin and D. G. Polyakov, Interacting Electrons in Disordered Wires: Anderson Localization and Low- $T$  Transport, *Phys. Rev. Lett.* **95**, 206603 (2005).
- <sup>24</sup> V. Oganessian and D. A. Huse, Localization of interacting fermions at high temperature, *Phys. Rev. B* **75**, 155111 (2007).
- <sup>25</sup> A. Pal and D. A. Huse, Many-body localization phase transition, *Phys. Rev. B* **82**, 174411 (2010).
- <sup>26</sup> B. Bauer and C. Nayak, Area laws in a many-body localized state and its implications for topological order, *J. Stat. Mech.* **09005** (2013).
- <sup>27</sup> J. Z. Imbrie, On Many-Body Localization for Quantum Spin Chains, *J. Stat. Phys.* **163**, 998 (2016).
- <sup>28</sup> M. Serbyn, Z. Papić, and D. A. Abanin, Local Conservation Laws and the Structure of the Many-Body Localized States, *Phys. Rev. Lett.* **111**, 127201 (2013).
- <sup>29</sup> D. A. Huse, R. Nandkishore, and V. Oganessian, Phenomenology of fully many-body-localized systems, *Phys. Rev. B* **90**, 174202 (2014).
- <sup>30</sup> V. Ros, M. Mueller and A. Scardicchio, Integrals of motion in the Many-Body localized phase, *Nucl. Phys. B* **891**, 420 (2015).
- <sup>31</sup> R. Nandkishore and D. A. Huse, Many body localization and thermalization in quantum statistical mechanics, *Ann. Rev. Cond. Matt.* **6**, 15 (2015).
- <sup>32</sup> E. Altman and R. Vosk, Universal dynamics and renormalization in many body localized systems, *Ann. Rev. Cond. Matt.* **6**, 83 (2015).
- <sup>33</sup> F. Alet and N. Laflorencie, Many-body localization: an introduction and selected topics, [arXiv:1711.03145](https://arxiv.org/abs/1711.03145).
- <sup>34</sup> D. A. Abanin and Z. Papić, Recent progress in many-body localization, *Ann. Phys.* **529**, 1700169 (2017).
- <sup>35</sup> S. A. Parameswaran, R. Vasseur, Many-body localization, symmetry, and topology, [arXiv:1801.07731](https://arxiv.org/abs/1801.07731).
- <sup>36</sup> V. E. Korepin, A. G. Izergin, and N. M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge University Press, 1993).
- <sup>37</sup> M. Gaudin, J. S. Caux, *The Bethe Wavefunction* (Cambridge University Press, 2014).
- <sup>38</sup> F. Buccheri, A. de Luca and A. Scardicchio, On the structure of typical states of a disordered Richardson model and many-body localization, *Phys. Rev. B* **84**, 094203 (2011).
- <sup>39</sup> C. Dasgupta, S.-K. Ma, Low-temperature properties of the random Heisenberg antiferromagnetic chain, *Phys. Rev. B* **22**, 1305 (1980).
- <sup>40</sup> D. S. Fisher, *Phys. Rev. Lett.* **69**, 534 (1992).
- <sup>41</sup> R. Vosk and E. Altman, Many-body localization in one dimension as a dynamical renormalization group fixed point, *Phys. Rev. Lett.* **110**, 067204, (2013).
- <sup>42</sup> R. Vasseur, A. J. Friedman, S. A. Parameswaran and A. C. Potter, Particle-hole symmetry, many-body localization, and topological edge modes, *Phys. Rev. B* **93**, 134207 (2016).
- <sup>43</sup> I. V. Protopopov, W. W. Ho and D. A. Abanin, Effect of  $su(2)$  symmetry on many-body localization and thermalization, *Phys. Rev. B* **96**, 041122 (2017).
- <sup>44</sup> P. Prelovsek, O. S. Barisic and M. Znidaric, Absence of full many-body localization in the disordered Hubbard chain, *Phys. Rev. B* **94**, 241104 (2016).
- <sup>45</sup> A. C. Potter and R. Vasseur, Symmetry constraints on many-body localization, *Phys. Rev. B* **94**, 224206 (2016).
- <sup>46</sup> J. A. Hoyos, N. Laflorencie, A. P. Vieira and T. Vojta, Protecting clean critical points by local disorder correlations, *Europhys. Lett.* **93**, 30004 (2011).
- <sup>47</sup> I. Mondragon-Shem, A. Pal, T. L. Hughes and C. R. Laumann, Many-body mobility edge due to symmetry-constrained dynamics and strong interactions, *Phys. Rev. B* **92**, 064203 (2015).
- <sup>48</sup> J. C. Getelina, F. C. Alcaraz and J. A. Hoyos, Entanglement properties of correlated random spin chains and similarities with conformal invariant systems, *Phys. Rev. B* **93**, 045136 (2016).
- <sup>49</sup> A. Klümper and A. A. Zvyagin, Exact Thermodynamics of Disordered Impurities in Quantum Spin Chains, *Phys. Rev. Lett.* **81**, 4975 (1998).
- <sup>50</sup> A. Klümper and A. A. Zvyagin, Disordered magnetic impurities in uniaxial critical quantum spin chains, *J. Phys. Cond. Mat.* **12**, 8705 (2000).
- <sup>51</sup> A. A. Zvyagin, Universal low-energy behavior of disordered quantum spin chains: Exact analytic results, *Phys. Rev. B* **62**, R6069(R) (2000).
- <sup>52</sup> A. A. Zvyagin, Bethe ansatz solvable multi-chain quantum systems, *J. Phys. A: Math. Gen.* **34**, R21 (2001).
- <sup>53</sup> A. A. Zvyagin, Low-energy dynamical characteristics of a quantum spin chain with magnetic impurities, *Phys. Rev. B* **82**, 104403 (2010).
- <sup>54</sup> E. Lieb, T. Schultz and D. Mattis, Two soluble models of an antiferromagnetic chain, *Ann. Phys.* **16**, 407 (1961).
- <sup>55</sup> M. Ganahl, E. Rabel, F. H. L. Essler and H.-G. Evertz, Observation of Complex Bound States in the Spin-1/2 Heisenberg  $XXZ$  Chain Using Local Quantum Quenches, *Phys. Rev. Lett.* **108**, 077206 (2012).

- <sup>56</sup> P. Mazur, Non-ergodicity of phase functions in certain systems, *Physica* **43**, 533 (1969).
- <sup>57</sup> X. Zotos, F. Naef and P. Prelovsek, Transport and conservation laws, *Phys. Rev. B* **55**, 11029 (1997).
- <sup>58</sup> E. Ilievski and J. de Nardis, Ballistic transport in the one-dimensional Hubbard model: the hydrodynamic approach, *Phys. Rev. B* **96**, 081118 (2017).
- <sup>59</sup> E. Ilievski, J. De Nardis, Microscopic origin of ideal conductivity in integrable quantum models, *Phys. Rev. Lett.* **119**, 020602 (2017).
- <sup>60</sup> B. Bertini, M. Collura, J. De Nardis and M. Fagotti, Transport in Out-of-Equilibrium XXZ Chains: Exact Profiles of Charges and Currents, *Phys. Rev. Lett.* **117**, 207201 (2016).
- <sup>61</sup> O. A. Castro-Alvaredo, B. Doyon and T. Yoshimura, Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium, *Phys. Rev. X* **6**, 041065 (2016).
- <sup>62</sup> M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, 1999).
- <sup>63</sup> L. Bonnes, F.H.L. Essler and A.M. Läuchli, Light-cone dynamics after quantum quenches in spin chains, *Phys. Rev. Lett.* **113**, 187203 (2014).
- <sup>64</sup> N. Andrei and H. Johannesson, Heisenberg chain with impurities (an integrable model), *Phys. Lett. A* **100**, 108 (1984).
- <sup>65</sup> H. J. de Vega and F. Woynarovich, New integrable quantum chains combining different kinds of spins, *J. Phys. A* **25**, 4499 (1992).
- <sup>66</sup> H. J. de Vega, L. Mezincescu and R. Nepomechie, Thermodynamics of integrable chains with alternating spins, *Phys. Rev. B* **49**, 13223 (1994).
- <sup>67</sup> P. A. Bares, Exact results for a one-dimensional  $t-J$  model with impurity, *cond-mat/9412011* (unpublished).
- <sup>68</sup> P. Schmitteckert, P. Schwab and U. Eckern, Quantum Coherence in an Exactly Solvable One-Dimensional Model with Defects, *Europhys. Lett.* **30**, 543 (1995).
- <sup>69</sup> G. Bedürftig, F.H.L. Essler and H. Frahm, Integrable Impurity in the Supersymmetric  $t-J$  Model, *Phys. Rev. Lett.* **77**, 5098 (1996).
- <sup>70</sup> G. Bedürftig, F.H.L. Essler and H. Frahm, Exact solution of a  $t-J$  chain with impurity, *Nucl. Phys. B* **489**, 697 (1997).
- <sup>71</sup> H.-P. Eckle, A. Punnoose and R. A. Römer, Absence of backscattering at integrable impurities in one-dimensional quantum many-body systems, *Europhys. Lett.* **39**, 239 (1997).
- <sup>72</sup> M. Bortz and A. Klümper, Lattice path integral approach to the one-dimensional Kondo model, *J. Phys. A* **37**, 6413 (2004).
- <sup>73</sup> R. Yahagi, J. Sato and T. Deguchi, Finite-temperature behavior of an impurity in the spin-1/2 XXZ chain, *J. Stat. Mech.* (2014) P11020.
- <sup>74</sup> M. Takahashi and M. Suzuki, One Dimensional Anisotropic Heisenberg Model at Finite Temperatures, *Prog. Theor. Phys.* **48**, 2187 (1972).