# Valuation of Inflation-Indexed Derivatives with three factor model




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One must learn by doing the thing,
for though you think you know it,
you have no certainty until you try.
*Aristotle.*

This thesis is dedicated to
my parents and my sister
for all their support, love and care.

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# Chapter 1

# Introduction

In a world facing an aging population, the priorities of investors are changing. The investors saving for their retirement are interested in sacrificing current spending for future consumption, only if the real worth of their money is guaranteed. The uncertainty of future inflation and its effect in the reduction of acquiring power is one of the driving forces that is boosting the inflation indexed markets.

Inflation-indexed derivatives are a relatively new class of instruments in the global financial markets. Their characteristics resemble those of the interest rate and foreign exchange derivatives. In the past ten years, the inflation-indexed market has risen steadily in Euro-zone and in other countries where there is an inflation-indexed securities market. The big investment banks and financial firms still consider it as having a large potential market growth worldwide.

The main objective of this thesis is to implement a non-arbitrage model for pricing inflation-indexed derivatives, specifically inflation-indexed swaptions in their European and Bermudan version. In the first chapter, an introduction of inflation-related concepts is presented together with the main motivation to research this topic. The second chapter presents an overview of the theoretical background necessary to develop the model used for option pricing. It is worth mentioning that the model presented is suitable for a market where some inflation-indexed securities are traded[1], but where there is not a liquid market for inflation-indexed derivatives. At the end of this chapter, a review of two more complex models is presented as well as the reasons as why to they were not chosen for implementation. In the third chapter the pricing formulas for some plain vanilla instruments (swaps) are developed, focusing on the pricing of an European inflation-indexed swaption. Chapter four focus on the implementation issues of the model, the estimation of parameters and analysis of results for the European swaption. In the last chapter, we use the Longstaff and Schwartz model for the pricing of a bermudan inflation-indexed swaption. Finally both conclusions and further extensions of the work are presented.

[1] It is possible to build term structure of real and nominal rates

To help this paper be a self contained work, there is a glossary with some key concepts or financial "jargon" that were not defined in the body of the paper which I consider important for the understanding of some of the theorems or explanations presented[2].

## 1.1 Inflation

The first question that might arise is what is inflation and what causes it? Inflation can be defined as a sustained increase in the general price level of goods across the economy, if the prices decrease then it is called deflation. Inflation diminishes the acquiring power of the population in general. To explain what causes inflation, economists have produced several theories using macroeconomic and microeconomic analysis. One of the postulated underlying causes of inflation is the level of monetary demand in the economy - how much money is being spent. Inflation tends to rise when, at the current price level, demand for goods and services in the economy is greater than the economy's ability to produce them. Another factor that increases inflation is the expectation of it. Expected inflation matters for wages and prices because future price rises reduce the amount of goods and services that today's wage settlement can buy. One of core purposes of the Central Banks is to maintain stability of prices (i.e. low inflation) which they control through their monetary policy. Expected inflation is affected by the monetary policy and how much people believe in the ability and commitment of the authorities - the government and the central bank - to achieve their inflation objectives.

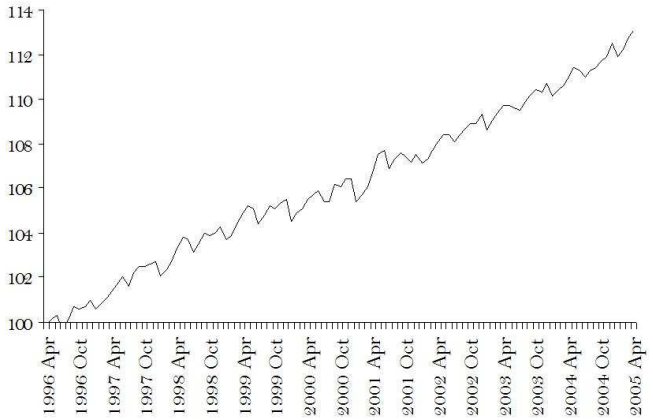



Figure 1.1: UK CPI time series 1996-2005, CPI April 1996=100. Source: National Statistics UK

[2]We do not pretend to present an extensive glossary, therefore there might be concepts that are not defined and was of interest of the reader to know their definition.

Consumer behaviour has seasonal features: for example consumption rises around Christmas, followed by a period of price discounting in January; the demand of energy and warm clothing tends to be higher in cold winter months than in the summer, and so on. To some extent, this causes prices to fluctuate which is reflected in the consumer price indices.

In the past fifty years, most developed countries have experienced decreasing levels of inflation. However we might be at a significant inflexion point in terms of anti-inflation policy. Deflation is just as much of a social evil as inflation and it has changed global policy makers stance. The efforts to avoid deflation also suggest that the political pressure for central banks to contain inflation has temporarily reduced. The central bankers are being encouraged to take a more balanced view with regards to the trade-off between inflation risks and growth. It is not certain that this change in emphasis is going to be beneficial or not, but it is generating a higher uncertainty over price levels in the next 10 years than the uncertainty in the past 10 years.

Inflation is measured in many ways. The most familiar measure in the UK is the Retail Prices Index (RPI), but monetary policy is now based on the Consumer Prices Index ; (CPI) (See figure 1.1)). Both measure the prices of products and services that consumers buy. A price index is made up of the prices of hundreds of goods and services - from basic items like bread to new products, such as PCs. Prices are sampled up and down the country every month; in supermarkets, petrol stations, travel agents, insurance companies and many other places. All these prices are combined together to produce an overall index of prices. Inflation rate is then a measure of the average change in prices across the economy over a specified period, most commonly 12 months. If, say, the annual rate of inflation in January this year was 3%, then prices overall would be 3% higher than in January last year. So a typical basket of goods and services costing, say, £100 last January would cost £103 this January.

The standard approach for modelling inflation is based on econometric models [11]. Their objective is to forecast inflation rate given a time series of data. Usually they relate inflation rate to other macro economic variables such as short term interest rates and monetary policy and/or past data of inflation (autoregressive models). The econometric approach nevertheless is not useful for pricing derivatives. Econometric models are derived under the historical probability measure, while the modern theory for option pricing requires the use of risk neutral probability which guarantees the market is arbitrage free and the existence of a replicating hedging strategy[3].

## 1.2 Inflation-indexed securities

Even in a stable low-inflationary environment, in the long term there remains a considerable uncertainty about the real value of nominal bond returns, £100 now would have the purchasing power of £74 in 30 years time if inflation averaged 1% but only £48 if inflation averaged 2.5%, more than

[3]See Chapter 2 Martingale approach for option pricing.

50% less[4]. While an individual's consumption basket might not change in the same proportion as the change in the relevant inflation index, no other financial asset can give close to the real value certainty of inflation-linked products.

Inflation-indexed securities are instruments that protect their buyers from changes in the general level of prices in the economy. They guarantee that the security holder (buyer of the security) is going to have the same purchasing power plus a fixed return. On the other hand, the borrower (seller of security) is going to have a constant cost for their debt in real terms.

The use of indexed securities can be traced back as far as the 18th century when the state of Massachusetts issued bills of public credit linked to the cost of silver [8]. The experience showed that indexation based in one single good was not a good idea and more complex indexation methods have been developed throughout history based on a basket of products that would reflect the prices of the whole economy. It was until very recently (the second half of the 20th century) that indexed debt became more popular in financial markets, particularly amongst governments after periods of high and volatile inflation. For example countries in Latin America such as Argentina (1972-1989), Brazil (1964) and Mexico (1989) have issued such securities as means of maintaining the acceptability of long-term contracts. Countries with commitment to low inflation have made use of indexed debt to lower their costs and to increase the credibility of their economic policies within the markets. Recent issues include UK Gilt (1981), Australia CIBs (1985), Canada RRB (1991), Sweden (1994), US TIPS (1997), France OATi (1998), Greece SBIL (2003) and Italy BTP(2003). Most of the issuers of inflation-index securities are governments but corporations could also benefit from it. Corporate balance sheets are full of real assets, so offsetting these with real liabilities could be appealing. Large company cash flows also tend to have a considerable inflation element to them. For instance, supermarkets sales will be similar to the inflation basket and so their prices will rise in a similar vein. Some utilities and public infrastructure projects may have more direct inflation linked revenues, which is strongly in their interest to hedge. Having inflation exposure hedged within the portfolio of a corporate acts as a revenue stabiliser.

There are a large number of cash flow structures for inflation indexed securities, the most common are capital indexed bond, interest indexed bond, current pay indexed, indexed annuity and indexed zero-coupon bonds[5].

Unique features of inflation bonds include their ability to provide low or negative return correlation with other assets, long duration[6] with respect to real interest rates, and low yield volatility. Benefits (1)protect assets and future income against inflation (2)better match liabilities and assets when both are affected by inflation (3)provide diversification in combination with other assets.

[4]The current inflation target of the Bank of England is 2% annually, the Mexican Central Bank inflation target is 3.5% annually.

[5]For a description of structures Deacon et. al (2004) Chapter 2.

[6]Look in the glossary for Duration.

According to the view of Barclays Capital [2], demographic pressures are such that it is quite conceivable for the inflation-linked market to continue growing at its current rate. For instance, if US pension funds were to allocate the same percentage of their assets to linkers as have UK pension funds, this allocation would be larger than the current global market ($600bn). Even UK pension funds have offset less than only a quarter of their total inflation-linked liabilities with inflation-linked bonds. Japanese pension funds have a higher percentage of their liabilities with inflation linkages and are further along the demographic transition but are only just starting to address this exposure. In spite of its growth in the past years, this market is still young and as a consequence, has some drawbacks such as lack of liquidity.

### 1.2.1 Nominal Rates, Real Rates and Expected Inflation

The yield on an inflation-indexed security represents the real return that the investor could realize by holding the security to maturity. In nominal terms, the investor earns this real return plus additional compensation for any inflation realized over the life of the security. By comparison, the total nominal return realized by holding a nominal security to maturity is simply equal to its yield. That yield embeds the return that the investor demands to compensate him for expected future inflation and the risk associated with that inflation. An indexed security offers the investor protection against unanticipated changes in inflation, while a nominal security does not.

In an environment of complete and efficient nominal and real debt markets, on any particular day, the ex ante nominal and real rates are directly observable. If there are some investors in both markets, the difference between nominal and real rates must reflect their expectation of future inflation. Because investors might be compensated for bearing inflation risk, the yield on the nominal security may include an inflation risk premium. If this risk premium is positive, as is often assumed, inflation compensation will exceed the expected rate of inflation. In a world of deterministic interest nominal and real rates, their difference is exactly equal to the certain inflation rate over the same period.

The prices of real and nominal bonds not only depend on expectations of real and nominal interest but also on tax regimes, market liquidity, the choice of index, the indexation method for real bonds and so on. These factors are going to be ignored in this thesis in order to keep our model simple and parsimonious.

There are a number of articles regarding the relationship between nominal and real rates, the expectation of future inflation and inflation risk premium in several economies using the econometric approach. For example: Sack (2000) [20] analyses US nominal and real yields and finds that the rate of inflation expected over the next ten years fell from 3% in mid-1997 to 2.5% at the beginning of 2000. Deacon and Derry [7] described a number of techniques to produce an inflation a term structure

of from the UK government bonds. They concluded that their preferred was one developed by Arak and Keicher which was used by Bank of England to produce its inflation report.

## 1.3 Inflation-indexed derivatives

The market for inflation-indexed or inflation-linked (IL) derivatives has grown considerably since its beginning in the early 1990s, in countries like UK and Euro-zone. Derivatives are usually designed and used to fill in the gaps and to produce synthetically complicated payoff structures that the underlying market can not produce solely in order to fulfill the demand of issuers or investors. It is not the exception with IL derivatives, they have added flexibility to the underlying inflation-indexed bond markets and have opened up opportunities to achieve financial objectives unavailable through the use of indexed-bonds solely. Their hedging capability against variety of risks is one of their most appealing characteristics in reality: to match the timing and frequency of cash flows, index matching, maturity matching among others.

The most common products traded nowadays are IL swaps (ILSs). Options (caps and floors), swaptions and other derivatives have started trading in small amounts. They are likely to grow on a more standardized basis, but are likely to develop in parallel with hedging requirements of the end users (e.g. the currently trading in the UK market LPI[7] swaps which are based on annual RPI inflation but with a floor of 0% and a cap of 5%) rather than a replication of what trades in the interest rate market[8].

Deacon, Derry and Mirfenderesky [8] present four categories for the inflation-indexed market according to its development. Their categorisation is:

**Level I** : In this market there are no tradable market instruments linked to the index of interest. Swap prices are made on the basis of matched trades, supply and demand and/or taking of basis risk by using instruments from other markets (Level II or III) as hedges. An example of this market is the Spanish inflation swap market.

**Level II** : These markets are those with one or more tradable market instruments, typically sovereign inflation-index bonds, which serve to "anchor" a few points on the inflation or real yield curve. However, due to scarcity of maturities of interest, supply and demand still drives prices at the "missing" points along the yield curve. Examples of these markets include the Euro-zone and the domestic French inflation swap markets.

**Level III** : These markets have "many" tradable market instruments and so a near complete and near arbitrage free inflation or real swap curve can be constructed. In such cases swap reset risks, repo costs and convexity issues are the dominant factors that determine the swap rates

[7] Limited Price Indexation.
[8] Barclays Capital (2004) [2]

relative to the more liquid securities along the yield curve. Specific supply and demand in the swap market becomes a secondary issue that more finely defines the inflation swap prices within these bounds. The UK Retail Price Index (RPI) swap market falls into this category. On the other hand, the US CPI swap market has full and liquid Treasury Inflation Protection Securities (TIPS) curve but it is an early stage of development.

**Level IV** : This level is currently hypothetical because all the markets in the world fall into one of the categories mentioned before, nevertheless it is expected to appear in the near future. This market has reached a level of maturity, liquidity and stability analogous to the major nominal interest rate swaps markets, such that inflation-indexed swaps trade int heir own right but side-by-side with the inflation-indexed bonds. The ILSs would themselves be basic tradable market instruments and their prices set independently of any underlying sovereign or other index-linked bond market, and as a nominal counterparts.

### 1.3.1 Examples of Inflation-liked payout structures

- **Inflation cap**

  This product pays out if inflation (as measured by percentage appreciation in the consumer price index) exceeds a certain threshold $K$ over a given period from $(0, T)$. The payout $C_T$ is given by:

  $$C_T = X \max\left[\left(\frac{I_T}{I_0} - 1\right) - K, 0\right],$$

  where X is the notional in currency (pounds, dollars, pesos) and $I_t$ is the value of the inflation index at time $t$ (e.g. CPI, RPI).

- **Zero-coupon swap** (ZCIIS)

  This is a product used most commonly for hedging real annuity exposures. In its payer version the investor pays the inflation return $\frac{I_T}{I_0} - 1$ versus receiving a pre-agreed zero coupon rate $(1 + Z_c)^T - 1$, where $T$ is the agreed maturity date; See figure (1.2).

- **Year-on-year inflation swap (YYIIS)**

  In its payer version it pays the annualized inflation index return. At time $T_{i+1}$, the inflation leg pays $\frac{I_{T_{i+1}}}{I_{T_i}} - 1$ versus receiving a fixed leg paying $\frac{NZ_c}{m}$ where $m$ is the periodicity of coupon in one year; See figure (1.3).

- **Real annuity swap**

  This product is very similar to the YYIIS; the difference is that the fixed swap rate $Z_c$ is compounded so the investor would receive $(1 + Z_c)^{\frac{m}{12}}$ at every reset date.

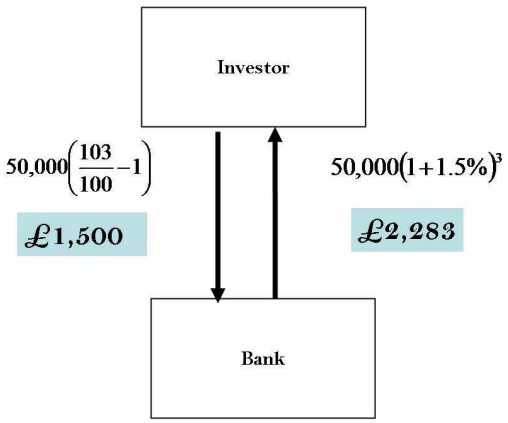



Figure 1.2: Illustrative cash flow chart for Zero-coupon swap with maturity 3 months over notional of £50,000, I(3)=103 I(0)=100, $Z_c$ 1%

- **Knockout options**

  A typical structure, might pay if the total inflation exceeds a certain threshold $K$ at time $T$. Knockout would occur if the total inflation drops below a certain specified critical level $K'$ between time $t$ and $T$:

$$\begin{aligned} K_T &= N \max\left[\left(\frac{I_T}{I_0} - 1\right) - K', 0\right] \text{ unless} \\ \left(\frac{I_T}{I_0} - 1\right) - K' &\leq 0 \text{ at some time } t \leq m \leq T, \text{ in which case } K_T = 0 \end{aligned}$$

There are many variations on this structure. The idea is to create a cheaper product by giving up positive payout in certain circumstances.

### 1.3.2 Use of swaptions for hedging Mortgage backed securities portfolios

This work was motivated by the needs of a mortgage bank in Mexico (Sociedad Hipotecaria Federal). This bank was created in 2001 to promote the primary and secondary mortgage market in Mexico. In particular, this mortgage bank has its mortgages linked to an inflation index (UDIS) and its liabilities are mostly in nominal floating rate notes. This mismatch between the duration and index of their assets and liabilities leaves the bank exposed to several risks. A mortgage portfolio is defined with the mortgage assets and the bonds or other debt equity used to finance the assets. Many financial firms use market prices for their assets and liabilities for the market value of their equity. Interest rates changes is the major source of risk for the market value of their equity. Mismatch between the duration of their assets and liabilities has caused many companies to suffer bankruptcy or major financial losses during the 80's to Savings and Loans in the US. Unexpected prepayment is also a source of potential losses in the market value of portfolio. Many portfolio managers decide to issue

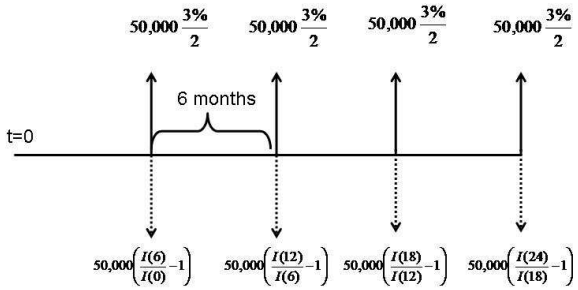



Figure 1.3: Illustrative scheme on payers version of year-on-year swap with coupon every 6 months over notional of £50,000

callable bonds to finance their portfolio and offset this risk. Nevertheless, portfolio managers tend not to hedge perfectly their balance because it would significantly reduce their profits.

Inflation linked swaps and swaptions provide a cost-effective strategy to transform short-term nominal debt into synthetical long-term inflation indexed debt and match the duration of their portfolio.

In practice, the model developed in this work is going to be used for risk management and asset liability analysis. The derivatives (swaptions) are going to be valued individually as well as a part of the structured and trading portfolio of the bank.

# Chapter 2

# Pricing of Derivatives

The first part of this chapter serves as introduction to martingale pricing established by Harrison and Kreps and Harrison and Pliska, and a brief overview of the Heath Jarrow and Morton framework. The second part of the chapter reviews the most popular models for pricing inflation-index derivatives including some discussion about their assumptions and implementation suitability in a market with characteristics of level II defined in Chapter 1.

## 2.1 The Martingale approach

Modern Financial theory has developed two fundamental theorems that state the conditions to have an arbitrage free and complete market suitable for option pricing. Apart from their powerful results they have allowed the development of very efficient computational implementations. The mathematical proofs are going to be omitted due to their length and the requirement of deep results from functional analysis[1].

First we consider a market model consisting of the asset price processes $S_0, S_1, ..., S_N$ on the time interval $[0, T]$. The "numeraire process" $S_0$ is assumed to be strictly positive.

**Definition 2.1.1** *A probability measure $Q$ on $\mathcal{F}_T$ is called an equivalent martingale measure for the market model, the numeraire $S_0$ and the time interval if it has the following properties:*

- *$Q$ is equivalent[2] to $P$ on $\mathcal{F}_T$*
- *All price processes $S_0, S_1, ..., S_N$ are martingales under $Q$ on the time interval $[0, T]$.*

An equivalent martingale measure is often referred as "a martingale measure" or "an EMM". If $Q \sim P$ has the property that $S_0, S_1, ..., S_N$ are local martingales, then $Q$ is called a local martingale measure. The numeraire process $S_0$ is always a martingale, and it is often referred as "the EMM".

[1] see Bjork [5, chapter 10 and 11]; he presents some of the proofs and further references.

[2] A probability measure $Q$ is equivalent to $P$ if they have the same null sets, that is for all $A \in \mathcal{F}, Q(A) = 0$ if and only if $P(A) = 0$.

**The First Fundamental Theorem 2.1.1** *The model is arbitrage free if and only if there exists a martingale measure* ($Q \sim P$) *such that the processes*

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, ..., \frac{S_N(t)}{S_0(t)}$$

*are (local) martingales under* $Q$.

If the numeraire $S_0$ is the money market account, i.e. $dS_0(t) = r(t)S_0(t)dt$, so that

$$S_0(t) = B(t) = e^{-\int_0^t r(s)ds} \tag{2.1}$$

where $r$ is the (possibly stochastic) short rate, and if we assume that all processes are Wiener driven, then a measure $Q \sim P$ is a martingale measure if all assets have the short rate as their local rates of return, i.e. the $Q$-dynamics are of the form

$$dS_i(t) = S_i(t)r(t)dt + S_i(t)\sigma_i(t)dW_i^Q(t) \ , \ i = 1, ..., N,$$

where $W^Q$ is a (multidimensional) $Q$-Wiener process.

**Second Fundamental Theorem 2.1.2** *Assuming absence of arbitrage, the market model is complete if and only if the martingale measure* $Q$ *is unique.*

### 2.1.1 Martingale Pricing

If we want to price a contingent claim maturing at $T$ given by $X$, first we consider the "primary" market $S_0, ..., S_N$ as given a priori. We have to determine a price process $\Pi(t; X)$ for $X$ assuming that the primary market is arbitrage free. There are two main approaches:

- The derivative should be priced consistently with the prices of the underlying assets, i.e. the extended market $\Pi(\cdot; X), S_0, ..., S_N$ is free of arbitrage.
- If the claim is attainable, with hedging portfolio $h$, then the only reasonable price is given by $\Pi(t, X) = V(t; h)$

The first approach demands that there should exists a martingale measure for the extended market. If we let $Q$ as such measure, then the arbitrage free price process for the claim $X$ is

$$\Pi(t; X) = S_0(t)E_Q\left[\frac{X}{S_0(T)}\Big|\mathcal{F}_t\right]. \tag{2.2}$$

where $Q$ is the martingale measure for the a priori given market with $S_0$ as numeraire. If we choose the bank account $B(t)$ as the numeraire equation (2.1) the pricing formula reduces to

$$\Pi(t; X) = E_Q\left[e^{-\int_t^T r(s)ds}X\Big|\mathcal{F}_t\right] \tag{2.3}$$

For the second approach we assume that $X$ can be replicated by $h$. Since the holding of the derivative contract and the holding of the replicating portfolio are equivalent, then the price of the derivative must be given by

$$\Pi(t;X) = V(t;h)$$

in order to avoid arbitrage. Different choices of hedging portfolios (if such exist) will produce the same price process.

There are two fundamental theorems that allow us to use the martingale approach to arbitrage theory in a very simple way.

- The martingale representation theorem which shows that in a Wiener world every martingale can be written as a stochastic integral with respect to the underlying Wiener process.
- Girsanov Theorem, which gives complete control of all absolutely continuous measure transformations in a Wiener world.

**The Martingale Representation Theorem 2.1.3** *Let $W$ be a d-dimensional Wiener process, and assume that $\mathcal{F}_t$ is the filtration generated by the Wiener process $W_t$. Let $M$ be any $\mathcal{F}_t$-adapted martingale. Then there exist uniquely determined $\mathcal{F}_t$-adapted processes $h_1, ..., h_d$ such that $M$ has the representation*

$$M(t) = M(0) + \sum_{d}^{i=1} \int_0^t h_i(s) dW_i(s) \ , \ t \in [0,T]. \tag{2.4}$$

*If the martingale $M$ is square integrable, then $h_1, ..., h_d$ are in $\mathcal{H}^2$.*

**The Girsanov Theorem 2.1.4** *Let $W^P$ be a d-dimensional standard $P$-Wiener process on $(\Sigma, \mathcal{P}, P, \underline{\mathcal{F}})$ and let $\varphi$ be any d-dimensional adapted column vector process. Choose a fixed $T$ and define the process $L$ on $[0,T]$ by*

$$\begin{aligned} dL_t &= \varphi_t^* L_t dW_t^P, \\ L_0 &= 1, \end{aligned}$$

*i.e.*

$$L_t = e^{\int_0^t \varphi_s^* dW_s^P - \frac{1}{2}\int_0^t \|\varphi_s\|^2 ds.}$$

*Assume that*

$$E_{\mathbb{P}}[L_T] = 1,$$

*and define the new probability measure $Q$ on $\mathcal{F}_T$ by*

$$L_T = \frac{dQ}{dP} \ , \ on \ \mathcal{F}_T.$$

*Then*

$$dW_t^P = \varphi_t dt + dW_t^Q,$$

*where $W^Q$ is a $Q$-Wiener process.*

The process $\varphi$ above will often be referred to as the Girsanov kernel of the measure transformation, and is such that

$$E_Q[e^{\frac{1}{2}\int_0^T \|\varphi_t\|^2 dt}] < \infty;$$

$L$ is called the likelihood function or the Radon-Nikodym derivative. An equivalent way (and more popular) of formulating the conclusion of Girsanov's Theorem is to say that the process $W^Q$, defined by

$$W_t^Q = W_t^P - \int_0^t \varphi_s ds \tag{2.5}$$

is a standard $Q$-Wiener process.

## 2.2 Heath-Jarrow-Morton Framework (HJM)

There are a number of interest rate models where the short rate $r$ is the only explanatory variable[3]. In the framework proposed by Heath-Jarrow-Morton (HJM) the entire curve of forward rates evolves simultaneously, according to a set of volatility curves. In principle any interest rate model with a continuous forward rate curve can be embedded into an HJM model.

We assume that, for every fixed $T > 0$, the forward rate $f(\cdot, T)$ has a stochastic differential which under the objective measure $P$ is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\bar{W}(t), \tag{2.6}$$

$$f(0,T) = f^*(0,T), \tag{2.7}$$

where $d\bar{W}(t)$ is a ($d$-dimensional) $P$-Wiener process whereas $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted processes, the $*$ denotes it is a market value. Equation (2.6) is one stochastic differential equation in the $t$-variable for each fixed maturity $T$. The initial condition is the observed forward rate curve denoted as $\{f^*(0,T); T \geq 0\}$, which will automatically fit the observed and the theoretical bond prices at $t = 0$. The pricing formula for any contingent claim will be given by equation (2.3). If we have specified $\alpha, \sigma$ and initial forward curve, by the relation

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}, \tag{2.8}$$

then we have specified the entire term structure $\{P(t,T); T > 0, 0 \leq t \leq T\}$.

Since there are $d$ sources of randomness (one for each component of the $d$-dimensional Wiener process), and an infinite number of traded assets (one bond for each maturity $T$), an arbitrage opportunity has been introduced into the bond market. The induced system of bond prices admits no arbitrage if $\alpha$ and $\sigma$ fulfill certain conditions.

[3]For introduction to interest rate modelling and short rate models refer to Bjork[5, ch.21 and 22]

**HJM drift condition 2.2.1** *Assume that the family of forward rates given by (2.6) and that the induced bond market is arbitrage free. Then there exists a d-dimensional column vector process*

$$\lambda(t) = [\lambda_1(t), ..., \lambda_d(t)]'$$

*with the property that for all* $T \geq 0$ *and for all* $t \leq T$ *we have*

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)' ds - \sigma(t,T)\lambda(t). \tag{2.9}$$

If we now take the approach of martingale modelling, we assume that the forward rates are specified under a martingale measure $Q$ as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{2.10}$$

$$f(0,T) = f^*(0,T), \tag{2.11}$$

where $W$ is a ($d$-dimensional) $Q$-Wiener process. A martingale measure automatically provides arbitrage free prices, but there are different formulas for bond prices which require a consistency relation between $\alpha$ and $\sigma$ in the forward rate dynamics.

**HJM drift condition under $Q$ 2.2.2** *Under the martingale measure $Q$, the processes $\alpha$ and $\sigma$ for every $t$ and every $t \leq T$ must satisfy*

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)' ds. \tag{2.12}$$

Under a specific choice of $\alpha$ and $\sigma$ and initial forward term structure, the forward rates are expressed as

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s).$$

We can compute the bond prices using formula (2.8), and use this bond prices to compute the prices of derivatives.

### 2.2.1 Extended Vasicek Model

As shown above, an HJM model is fully described by the instantaneous forward rate volatility. Several standard functional forms for $\sigma(t,T)$ have been investigated and used in practice. If we define the volatility function as

$$\sigma(t,T) = \sigma e^{\lambda(T-t)} \text{ , } \sigma, \lambda \text{ constants}, \tag{2.13}$$

the Vasicek type volatility.

The extended Vasicek model has the short rate process

$$dr(t) = [\theta(t) - a(t)r(t)]dt + \sigma(t)dW(t). \tag{2.14}$$

Furthermore if we assume $a(t)$ and $\sigma(t)$ are positive constants we recover the extended Vasicek model of Hull and White (1994) [12]. The short rate $r(t)$ reverts to a time-dependent mean $\theta(t)/a$, with the process

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW(t). \tag{2.15}$$

The time dependent function $\theta(t)$ is chosen to exactly fit the term structure of interest rates currently observed in the market. If we denote $f^*(0,T)$ and $P^*(0,T)$ as the market instantaneous forward rate and market discount factor, respectively, at time 0 for the maturity $T$, then

$$f^*(0,T) = -\frac{\partial \ln P^*(0,T)}{\partial T} \tag{2.16}$$

it can be shown that

$$\theta(t) = \frac{\partial f^*(0,T)}{\partial T} + af^*(0,T) + \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{2.17}$$

The price at time $t$ of a pure discount bond paying off 1 at time $T$ is given by the following equation

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{2.18}$$

where

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \tag{2.19}$$

$$A(t,T) = \frac{P^*(0,T)}{P^*(0,t)} \exp\left\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2\right\} \tag{2.20}$$

Equation (2.18) is going to be frequently used in Chapter 3 in the development of the pricing formulas for swaps.

The extended Vasicek model is one of the most popular Affine type of models of short rate. This type of models as seen above have explicit solutions for bond prices and some for bond option prices and it is relatively straightforward to price other instruments[4].

## 2.3 Notation

Before the presentation of the models we are going to introduce some notation and concepts that will be used throughout this thesis.

- Let the subscript $r$ stand for real, $n$ for nominal and $I$ for inflation.
- Let $r_k(t) = f_k(t,t)$ be the continuous spot rate $k \in \{r, n\}$
- Let $B_k(t)$ be the time $t$ money market account value $k \in \{r, n\}$

$$B_k(t) = \exp\left\{\int_0^t r_k(t)dv\right\}. \tag{2.21}$$

[4] More discussion about Affine type models see [13, Ch. 7].

- Let $I(t)$ be the value of the inflation index at time $t$ $(CPI, CPI-U, INPC)$[5]
- Let $P_n(t,T)$ be the time $t$ price of a nominal zero coupon bond maturing at time $T$ in the nominal currency (pounds, dollars, pesos).
- Let $P_r(t,T)$ be the time $t$ price of a real zero coupon bond maturing at time $T$ expressed in inflation-index units. To obtain the price in nominal terms at time $t$ we would have to multiply this price times the value of the index at time $t$.
- Let $f_k(t,T)$ for $k \in \{n,r\}$ be the value at time $t$ of forward rates for date $T$; then

$$P_k(t,T) = \exp\left\{-\int_t^T f_k(t,u)du\right\}. \tag{2.22}$$

## 2.4 Models for pricing Inflation-Indexed derivatives

The following models are the majority (if not all) of the models in literature for pricing inflation-indexed derivatives. They are presented in chronological order of appearance which coincides with their level of complexity. The emphasis is going to be done in model of Jarrow and Yildrim (2003) in which all the mathematical formulation is going to be presented. For the rest of the models we present only a brief description and critique[6].

### 2.4.1 Jarrow and Yildrim Model (2003)

The main reference in literature for pricing inflation-indexed derivatives is the paper of Jarrow and Yildrim (2003)[15]. In their paper they use a HJM model to price TIPS and derivative securities. The idea of the three-factor arbitrage-free term structure model is an analog of HJM of foreign currency model [14]. They considered the nominal dollars as the domestic currency, the real dollars as foreign currency and the inflation index as the spot exchange rate. The fluctuation of the real and nominal interest rates and the inflation rate are allowed to be correlated.

Let the uncertainty of the economy be characterised by a probability space $(\Omega, F, P)$. The Brownian Motions $\bar{W}_k(t)$ $k = n, r, I$ $t = 0, ..., T$ are initialised at zero and have correlations given by:

$$d\bar{W}_n(t)d\bar{W}_r(t) = \rho_{n,r}dt \tag{2.23}$$

$$d\bar{W}_n(t)d\bar{W}_I(t) = \rho_{n,I}dt \tag{2.24}$$

$$d\bar{W}_r(t)d\bar{W}_I(t) = \rho_{r,I}dt \tag{2.25}$$

[5] See Chapter 1 for the definition of price index or the glossary in the appendix

[6] Further mathematical development would increase considerably the extension of this thesis and could confuse which model would be used.

The evolution of the nominal and real $T$-maturity forward rate is defined as:

$$df_k(t,T) = \alpha_k(t,T)dt + \sigma_k(t,T)d\bar{W}_k(t),\ k \in \{r,n\} \tag{2.26}$$

where $\alpha_i$ is random and $\sigma_k$ is a deterministic function of time subject to some technical conditions of smoothness and boundedness[7]. The inflation index's evolution is described by

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)d\bar{W}_I(t) \tag{2.27}$$

where $\mu_I(t)$ is random and $\sigma_I(t)$ is deterministic function subject to smoothness and boundedness conditions. In this model the forward rates have a normal distribution, and the inflation index is a lognormal distribution. Therefore, in this model it is possible to have negative nominal and real rates.

By the First Fundamental Theorem the three processes are arbitrage free if there exist a measure $Q_n$ equivalent to $P$ such that

$$\frac{P_n(t,T)}{B_n(t)}, \frac{I(t)P_r(t,T)}{B_n(t)}, \frac{I(t)B_r(t)}{B_n(t)} \tag{2.28}$$

are $Q_n$-martingales. By Girsanov's theorem given that $\bar{W}_i$ $i = n, r, I$ are $P$-Brownian Motions there exist market prices of $(\lambda_n(t), \lambda_r(t), \lambda_I(t))$ such that

$$dW_k(t) = \bar{W}_k(t) - \int_0^t \lambda_k(s)ds \text{ for } i = n, r, I \tag{2.29}$$

are $Q_n$-Brownian Motions. There following conditions ensure that the relative prices in Eq. (2.28) are $Q_n$-martingales.

$$\alpha_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)ds - \lambda_n(t)\right) \tag{2.30}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)ds - \sigma_I(t)\rho_{r,I} - \lambda_r(t)\right) \tag{2.31}$$

$$\mu_I(t) = r_n(t) - r_r(t) - \sigma_I(t)\lambda_I(t) \tag{2.32}$$

Eq. (2.30) is the arbitrage-free forward rate drift restriction in the original HJM model. Eq. (2.31) is analogous to drift restriction for the real rates adjusted by the correlation and volatility of the real rate and inflation. The last expression is called the Fisher equation. With the use of Ito's Lemma and the above mentioned martingales they obtained expressions for the price processes

[7] $\alpha_n(v,T)$ is $\mathcal{F}_t$-adapted and jointly measurable with $\int_0^T |\alpha_n(v,T)|dv < \infty$ P-a.s. and $\sigma_n(v,T)$ satisfies $\int_0^T \sigma_2(v,T)dv < \infty$ P-a.s.

under probability measure $Q_n$:

$$
\begin{aligned}
df_n(t,T) &= \sigma_n(t,T)\int_t^T \sigma_n(t,s)ds + \sigma_n(t,T)dW_n(t) && (2.33)\\
df_r(t,T) &= \sigma_r(t,T)\int_t^T \left[\sigma_r(t,s)ds - \rho_{r,I}\sigma_I(t)\right]dt + \sigma_r(t,T)dW_r(t) && (2.34)\\
\frac{dI(t)}{I(t)} &= [r_n(t) - r_r(t)]dt + \sigma_I(t)dW_I(t) && (2.35)\\
\frac{dP_n(t,T)}{P_n(t,T)} &= r_n(t)dt - \int_t^T \sigma_n(t,s)dW_n(t) && (2.36)\\
\frac{dP_r(t,T)}{P_r(t,T)} &= \left[r_r(t) - \rho_{r,I}\sigma_I(t)\int_t^T \sigma_r(t,s)ds\right]dt \\
&\quad - \left[\int_t^T \sigma_r(t,s)ds\right]dW_r(t) && (2.37)
\end{aligned}
$$

Assuming an exponentially decreasing volatility of the form $\sigma_k(t,T) = \sigma_k e^{-a_k(T-t)}$ and that $\sigma_k$ is constant for $k \in \{n,r\}$, we obtain the extended Vasicek model for the short rates

$$
\begin{aligned}
dr_n(t) &= [\theta_n(t) - a_n r_n(t)]dt + \sigma_n dW_n(t), && (2.38)\\
dr_r(t) &= [\theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r_r(t)]dt + \sigma_r dW_r(t), && (2.39)\\
\frac{dI(t)}{I(t)} &= [r_n(t) - r_r(t)]dt + \sigma_I dW_I(t), && (2.40)
\end{aligned}
$$

where $\theta_k(t)$ are deterministic functions to be used to exactly fit the term structures of nominal and real rates. As already remarked, these are given by

$$\theta_k(t) = \frac{\partial f_k^*(0,T)}{\partial T} + a_k f_k^*(0,t) + \frac{\sigma_k^2}{2a_k}(1 - e^{-2a_k t}), k \in \{n,r\}. \quad (2.41)$$

In their paper, Jarrow and Yildrim calibrated their model to US data. They tested its validity by means of hedging analysis which resulted to be satisfactory. Furthermore they presented closed formulas for valuation of European call option on the inflation index.

There are several shortcomings with this model, the first and most important one is that the parameters are not directly observable in the market. The second drawback is that it does not allow a link between instruments that are traded such as zero-coupon and year-on-year products. This last point should not be considered if we are interested in pricing instruments in markets where there is not a liquid market for this products.

### 2.4.2 Fabio Mercurio Market Model (2004)

In his recent paper (2004) Mercurio developed two market models alternative to JY (2003) and equivalent to Belgrade et al.(2004) for pricing YYIIS. His first market model recovers the lognormal LIBOR model for the nominal and the real rates and a geometric brownian motion for the forward inflation index. The YYIIS price depends on: the (instantaneous) volatilities of nominal and real

forward rates and their correlations, for each maturity of the cash flows; the correlations between real forward rates and forward inflation indices, again for each maturity of the cash flows. Compared with the JY formula for YYIIS, with this model it is more complicated both in terms of input parameters and in terms of the calculations involved. It can be solved using numerical integration. As is typical in a market model, the input parameters can be determined more easily than in the JY approach. This approach has the drawback that the volatility of real rates may be hard to estimate. Given this deficiency he developed a second market model to overcome this estimation issue.

The second model uses the fact that the forward inflation index at time $T_i$ is a martingale under the $T_i$-forward measure, then he develops the process of the inflation index $dI(T_{i-1})$ under the $T_i$-forward measure. The result is a pricing formula for YYIIS similar to that in the JY case and may be preferred to the one in the LIBOR market model, since it combines the advantage of a fully-analytical formula with that of a market-model approach. The price of YYIIS depends on the (instantaneous) volatilities of the forward inflation indexes and their correlations, the (instantaneous) volatilities of nominal forward rates and the instantaneous correlations between forward inflation indices and nominal forward rates. Moreover, this formula does not depend on the volatility of real rates, which is typically difficult to estimate.

The weakness of this model is that it is based in an approximation that affects long maturities, specially when the correlations between the forward nominal rates and inflation are different from zero.

Mercurio presented a comparison between his models and JY(2003) and concluded that the three of them produced similar results when calibrating with market data although he found that they differed when away-from-the money are considered.

### 2.4.3 Belgrade-Benhamou-Koehler Market Model (2004)

This model developed by Belgrade, Benhamou and Koehler (2004)[3] tries to incorporate information coming from the zero-coupon and year-on-year swaps market to produce a no arbitrage model analog to Brace-Gatarek-Musiela (BGM)[8]. This model has two main objectives, to be simple, i.e. to have only few parameters; and to be robust i.e. to replicate market prices. Their framework is to assume that the market model for inflation considers forward inflation index return as a diffusion with deterministic volatility structure. Under the risk neutral probability measure $Q$ this index follows geometric Brownian Motion with deterministic drift and volatility. In their paper they consider three different functional forms of volatility (constant, exponentially decaying and adjusted exponentially). They present a method to parameterise the volatility structure to include the market data of caps/floors (vol cube information). They also perform a convexity adjustment of the inflation swaps derived from the difference of martingale measures between the numerator and the

[8]See Joshi(2004)for more references on BGM[16].

denominator. Given that it is not possible to estimate implicit correlations from the market data, they suggest some boundary conditions which for certain model hypothesis (for example constant volatility structure) result unrealistic. This model could be used only in markets where there is enough information from zero-coupon and year-on-year swaps. It is important to be aware that to derive the model model some approximations where done in the process so the answer is not exact. Another drawback of this model is that it is computationally intensive.

# Chapter 3

# Pricing derivatives using Jarrow and Yildrim

Given the interest in pricing inflation-indexed derivatives for the Mexican Market[1], we take the view that the most suitable model is that of Jarrow and Yildrim (2003). The following section the pricing formulas for plain vanilla swaps and a fixed real vs. floating nominal swap are going to be developed. Once we obtain this results the next chapter is going to cover the implementation of the model and analysis of the results.

## 3.1 Pricing formulas for basic inflation swaps

The most actively traded inflation-indexed instruments are the Inflation-indexed Swap (IIS), especially the zero coupon swap (ZCIIS) and the year-on-year swap (YYIIS).

### 3.1.1 Zero-coupon swap

In this contract Party A agrees to pay Party B the inflation rate over a fixed period of time on exchange of a fixed rate $K$ over a notional of $N$ pounds. Let us call as the floating the leg, the leg paying inflation and fixed the leg, the leg paying the fixed rate. Assume the maturity of the contract is equal to $M$ years. The inflation will be measured as the percentage increase of the inflation index, denoted by $I(t)$, from time $t = T_0 = 0$ to $t = T_M$. At maturity the fixed leg of the contract pays

$$N[(1+K)^M - 1],$$

while the floating leg pays

$$N\left[\frac{I(T_M)}{I(T_0)} - 1\right].$$

By standard no-arbitrage pricing theory the price of the floating leg[2] at time $t$ equals

$$ZCIIS_f(t, T_M, I(T_0), N) = NE_n\left\{e^{-\int_t^{T_M} r_n(u)du}\left[\frac{I(T_M)}{I(T_0)} - 1\right]\Big|\mathcal{F}_t\right\}. \tag{3.1}$$

[1] A description of Mexican market presented in the next chapter when we cover the implementation issues.
[2] Denoted by $ZCIIS_f$ the subscript $f$ stands for floating

Recall $E_n\{\cdot|\mathcal{F}_t\}$ denotes the expectation under the risk neutral martingale measure $Q_n$, and $\mathcal{F}_t$ is the $\sigma$-algebra generated by the underlying processes up to time $t$. By the foreign-currency analogy, the nominal price of a real zero-coupon bond equals the nominal price of the contract paying off one unit of the inflation index at the bond maturity, in mathematical terms for $t < T_M$:

$$I(t)P_r(t,T_M) = I(t)E_r\left\{e^{-\int_t^{T_M} r_r(u)du}|\mathcal{F}_t\right\} = E_n\left\{e^{-\int_t^{T_M} r_n(u)du}I(T_M)|\mathcal{F}_t\right\}. \tag{3.2}$$

If we expand the term in equation (3.1) and substitute the expectation $E_n\{\cdot\}$ in equation (3.2) we obtain a model-independent formula for the price of the floating leg given by

$$ZCIIS_f(t,T_M,I(T_0),N) = N\left[\frac{I(t)}{I(T_0)}P_r(t,T_M) - P_n(t,T_M)\right]. \tag{3.3}$$

In particular the value at time $t = 0$

$$ZCIIS_f(0,T_M,I(T_0),N) = N\left[P_r(0,T_M) - P_n(0,T_M)\right],$$

As noted by Mercurio [19] this price is not based on specific assumptions on the evolution of the interest rate market, but follow from the absence of arbitrage, which enable one to strip the real zero-coupon bond prices from the quoted prices of zero-coupon inflation indexed swaps.

Furthermore the fixed rate $K$ at which the swap has zero value at time zero is given by

$$K = \left[\frac{P_r(0,M)}{P_n(0,M)}\right]^{1/M} - 1.$$

Given market quotes of $K_M = K(T_M)$ for given maturities $T_M$, and the nominal bond prices it would be possible to derive the discount factors in the real economy (if not known in the market) using the last formula.

### 3.1.2 Year-on-year Swap

Continuing from the definition presented in Chapter 1, assume the year-on-year swap we want to price have $m$ number of periodic payments at $t = T_1, ...T_i, ..., T_M$ for $i = 1, ..., m$ we denote $\varphi_i$ the fraction of the year between $[T_{i-1}, T_i]$, $T_0 = 0$ and a notional of $N$. It is important to notice that the year-on-year swap fixes the initial value of the inflation index each payment date, in contrast with using always the initial index value $I(T_0)$. The value at time $t < T_i$ of the floating leg[3] that will be exchanged at time $T_i$ is equal to

$$YYIIS_f(t,T_{i-1},T_i,\varphi_i,N) = N\varphi_i E_n\left\{e^{-\int_t^{T_i} r_n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]\Big|\mathcal{F}_t\right\}, \tag{3.4}$$

If $t > T_{i-1}$ equation (3.4) reduces to pricing a the floating leg of a ZCIIS with equation (3.3). In the case where $t < T_{i-1}$, and using a result from iterative expectations

$$YYIIS_f(t,T_{i-1},T_i,\varphi_i,N) = N\varphi_i E_n\left\{e^{-\int_t^{T_{i-1}} r_n(u)du}E_n\left[e^{-\int_{T_{i-1}}^{T_i} r_n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]\Big|\mathcal{F}_{T_{i-1}}\right]\Big|\mathcal{F}_t\right\}, \tag{3.5}$$

[3]Denoted by $YYIIS_f$ subscript $f$ stands for floating.

The inner expectation is equation (3.1) with $ZCIIS_f(T_{i-1}, T_i, I(T_{i-1}), 1)$, so the last equation is equal to

$$N\varphi_i E_n\left\{e^{-\int_t^{T_{i-1}} r_n du}[P_r(T_{i-1},T_i) - P_n(T_{i-1},T_i)]|\mathcal{F}_t\right\}.$$

$$= N\varphi_i E_n\left\{e^{-\int_t^{T_{i-1}} r_n du}P_r(T_{i-1},T_i)|\mathcal{F}_t\right\} - N\varphi_i P_n(T_{i-1},T_i). \tag{3.6}$$

The expectation is the nominal discounted price of a payoff equal to the real zero-coupon bond price $P_r(T_{i-1}, T_i)$. If the real rates were deterministic, this would be equal to the present value in nominal terms of the forward price of the real bond,

$$E_n\left\{e^{-\int_t^{T_{i-1}} r_n du}P_r(T_{i-1},T_i)|\mathcal{F}_t\right\} = P_r(T_{i-1},T_i)P_n(t,T_{i-1}) = \frac{P_r(t,T_i)}{P_r(t,T_{i-1})}P_n(t,T_{i-1}).$$

Since the real rates in this model are stochastic (3.6) is model dependent. Under the dynamics defined by JY (2003) in equation (2.38), the forward price of the real bond is corrected by a factor depending on both the nominal and real interest rates volatilities and their correlation.

If we choose as numeraire the $T_i$-maturing nominal bond, the resulting nominal $T_i$-forward martingale measure is $Q_n^{T_i}$ and its expectation $E_n^{T_i}$ for a general $T$. Equation (3.6) under this martingale measure is equal to

$$YYIIS_f(t,T_{i-1},T_i,\varphi_i,N) = N\varphi_i E_n^{T_{i-1}}\{P_r(T_{i-1},T_i)|\mathcal{F}_t\} - N\varphi_i P_n(T_{i-1},T_i). \tag{3.7}$$

In order to obtain the value of the expectation in the last equation, it is necessary to recall the zero-coupon bond price formula in the Hull and White model presented in chapter 2 (2.18).

By the change-of-numeraire technique developed by Geman et al.(1995) and Brigo and Mercurio (2001) the real instantaneous rate evolves under $Q_n^{T_{i-1}}$ according to

$$dr_r(t) = [\theta_r(t) - a_r(t) - \rho_{r,I}\sigma_r\sigma_I - \rho_{n,r}\sigma_n\sigma_r B_n(t,T_{i-1})]dt + \sigma_r dW_r^{T_{i-1}}(t), \tag{3.8}$$

with $W_r^{T_{i-1}}$ a $Q_n^{T_{i-1}}$-Brownian motion. The real rate $r_r(T_{i-1})$ is a normal random variable under this forward measure satisfying[4]

$$\begin{aligned}
E[r_r(T_{i-1})|\mathcal{F}_t] &= r_r(t)e^{-a_r(T_{i-1}-t)} + \alpha(T_{i-1}) - \alpha(t)e^{-a_r(T_{i-1}-t)} - \rho_{r,I}\sigma_I\sigma_r B_r(t,T_{i-1}) \\
&- \frac{\rho_{n,r}\sigma_n\sigma_r}{a_r+an}[B_r(t,T_{i-1}) + a_r B_n(t,T_{i-1})B_r(t,T_{i-1}) - B_n(t,T_{i-1})] \\
\mathrm{Var}[r_r(T_{i-1})|\mathcal{F}_t] &= \frac{\sigma_r^2}{2a_r}[1 - e^{-2a_r(T_{i-1}-t)}]
\end{aligned}$$

where

$$\alpha(t) = f_r^*(0,t) + \frac{\sigma_r^2}{2a_r^2}(1 - e^{-a_r t})^2.$$

[4]Derivation is similar to that done by Brigo and Mercurio (2001) for Hull White p.65

So the real bond price $P_r(T_{i-1}, T_i)$ is lognormally distributed, Mercurio (2004) obtained the closed form formula for equation (3.7) as

$$YYIIS_f(t, T_{i-1}, T_i, \varphi_i, N) = N\varphi_i P_n(t, T_{i-1}) \frac{P_r(T, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} - N\varphi_i P_n(t, T_i), \tag{3.9}$$

where

$$\begin{aligned} C(t, T_{i-1}, T_i) &= \sigma_r B_r(T_{i-1}, T_i)[B_r(t, T_{i-1})(\rho_{r,I}\sigma_I - \frac{1}{2}\sigma_r B_r(t, T_{i-1}) \\ &+ \frac{\rho_{n,r}\sigma_n}{a_n + a_r}(1 + a_r B_n(t, T_{i-1}))) - \frac{\rho_{n,r}\sigma_n}{a_n + a_r} B_n(r, T_{i-1})]. \end{aligned}$$

This formula takes into account the current forward price of the real bond multiplied by a the correction factor that depends on the (instantaneous) volatilities of the nominal rate, the real rate and the inflation index as well as the (instantaneous) correlations between nominal and real rates and between the real rate and inflation index. If $\sigma_r = 0$ this term disappears and we get the deterministic formulation for the real rate.

The value at time $t$ of the inflation-indexed leg of the swap is the sum of all the floating payments

$$\begin{aligned} YYIIS_f(t, \mathcal{T}, \Psi, N) &= N\varphi_{i(t)} \left[ \frac{I(t)}{I(T_{i(t)-1})} P_r(t, T_{i(t)}) - P_n(t, T_{i(t)}) \right] \\ &+ N \sum_{i=i(t)+1}^{M} \varphi_i \left[ P_n(t, T_{i-1}) \frac{P_r(T, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} - P_n(t, T_i) \right] \end{aligned}$$

where $\mathcal{T} = \{T_1, ..., T_M\}, \Phi = \{\varphi_1, ..., \varphi_M\}$ and $i(t) = \min\{i : T_i > t\}$.

## 3.2 Pricing a fixed real vs. floating nominal swap

We assume for generality that the swap is going to start at some time in the future $T_\alpha$ (forward-start) and the payments at periodic times $T_i$ until its maturity $T_M$. In this swap Party A pays a fixed rate $S$ over a variable notional $N_i$ that is going to be adjusted by the inflation occurred between the beginning of the swap $T_\alpha$ and the reset date $T_i$.

$$S \frac{I(T_i)}{I(T_0)} N_i \varphi_i$$

and at maturity

$$\frac{I(T_i)}{I(T_0)} N_i$$

Party B at each reset date $T_i$ pays a floating nominal rate fixed at $T_{i-1}$ over the variable notional not adjusted by inflation namely

$$N_i \varphi_i f_n(T_{i-1}, T_i)$$

and at maturity

$$N_i.$$

In this type of swaps and also in currency swaps, the market practice is that notionals have to be exchanged at the maturity of the swap to reduce the "exchange risk" for the leg receiving inflation. In some forward starting swaps the notionals are exchanged also at the beginning of the swap.

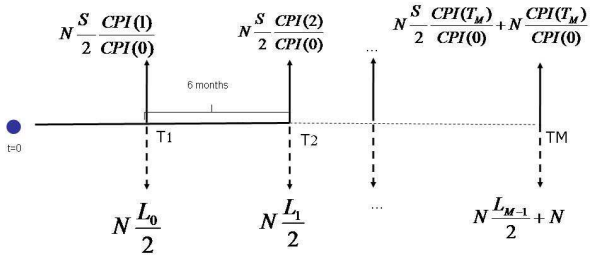



Figure 3.1: Illustrative cash flow chart for receiver's swap starting at $\alpha = 0$ and maturing at $T_M$.

As in the formulas developed before $\varphi_i$ is the fraction of the year elapsed between $[T_{i-1}, T_i]$. Figure (3.1) is a scheme of the payoffs at each reset date. Standard no arbitrage theory implies that the value at time $0 \leq t \leq T_M$ of the swap is given by

$$E_n\left\{\sum_{i=i(t)}^{M} S\varphi_i N_i \frac{I(T_i)}{I(T_0)} e^{-\int_t^{T_i} r_n(u)du} + N_M \frac{I(T_M)}{I(T_0)} e^{-\int_t^{T_M} r_n(u)du}\Big|\mathcal{F}_t\right\} -$$
$$E_n\left\{\sum_{i=i(t)}^{M} \varphi_i N_i f_n(T_{i-1}, T_i) e^{-\int_t^{T_i} r_n(u)du} + N_M e^{-\int_t^{T_M} r_n(u)du}\Big|\mathcal{F}_t\right\},$$

where $i(t) = \min\{i : T_i > t\}$.

We can use the same argument as the one used for the price of the floating leg of the zero-coupon inflation swap Eq. (3.2) to find the value of $E_n[I(T_i)e^{-\int_t^{T_i} r_n(u)du}|\mathcal{F}_t]$. Finally

$$\sum_{i=1}^{M} \varphi_i N_i \frac{SI(t)}{I(T_0)} P_r(t, T_i) + N_M \frac{I(t)}{I(T_0)} P_r(t, T_M) - \sum_{i=1}^{M} \varphi_i N_i f_n(T_{i-1}, T_i) P_n(t, T_i) + N_M P_n(t, T_M). \quad (3.10)$$

The forward swap rate $S_{\alpha,M}(t)$ at time $t$ is the rate in the indexed leg that makes the swap a fair contract at the present time, in other words it is the fixed rate $K$ for which left hand side in expression (3.10) equals the right hand side. If we solve for the swap rate we find that

$$S_{\alpha,M}(t) = \frac{\sum_{i=\alpha+1}^{M} N_i \left(P_n(t, T_{i-1}) - P_n(t, T_i)\right) + N_M \left(P_n(t, T_M) - \frac{I(t)}{I(T_0)} P_r(t, T_M)\right)}{\sum_{i=\alpha+1}^{M} \frac{I(t)}{I(T_0)} N_i \varphi_i P_r(t, T_i)}. \quad (3.11)$$

More intuitively, a payer version of this swap can be replicated by buying a floating nominal rate bond and selling a fixed rate real bond. The swap rate we are looking for is the fixed rate that one would have to pay to make the value of this portfolio zero at time zero.

This formula is also model independent and was derived solely with the principles of no arbitrage. It is worth to mention that for its calculation it involves no correlations or volatilities between the nominal and real rates or inflation. In the next section we present the pricing formula for a European swaption, where the volatilities and correlations play an important role in the pricing formula.

## 3.3 Pricing European swaption

A swaption is an option to enter a swap. An option on the right to pay the fixed rate in a swap is called a *payer's swaption*. An option on the right to pay the floating rate i.e. receive the fixed rate, is called a *receiver's swaption*. When there is only one exercise date (generally the beginning of a forward starting swap or a reset date to break a swap), it is called a European swaption. To obtain the payoff of a payer's swaption, we assume that the underlying swap has a fixed strike rate of $K$, and the maturity date is the same as the beginning of the forward starting swap $T_\alpha$. If we substitute in equation (3.10), we know that the option will be exercised if

$$\sum_{i=1}^{M} N_i \varphi_i \quad f(T_\alpha, T_{i-1}, T_i) P_n(T_\alpha, T_i) - K \frac{I(T_\alpha)}{I(T_0)} P_r(T_\alpha, T_i) \quad + N_M \quad P_n(T_\alpha, T_M) - \frac{I(T_\alpha)}{I(T_0)} P_r(T_\alpha, T_M) \quad > 0 \quad (3.12)$$

If we use the positive-part operator and the annuity factor

$$C(t) = \sum_{i=\alpha+1}^{M} \varphi_i N_i P_r(t, T_i) \frac{I(t)}{I(T_0)} \tag{3.13}$$

then the payoff of a payer's swaption is

$$V(T_\alpha) = (S_{\alpha,M}(T_\alpha) - K)^+ C(T_\alpha), \tag{3.14}$$

where $S_{\alpha,M}(t)$ is the market swap rate at time $t$ of a swap that starts at $T_\alpha$ and matures at $T_M$. Similarly, the pay-off of a receiver's swaption is

$$(K - S_{\alpha,M}(T_\alpha)^+ C(T_\alpha).$$

The value at time $t$ of an European swaption maturing at $T_\alpha$ under the $Q_n$ is given by

$$V(t) = E_n \left[ V(T_\alpha) \exp\left( - \int_t^{T_\alpha} r_n(s) ds \right) \Big| \mathcal{F}_t \right], \tag{3.15}$$

this involves the simulation of an integral which is rather cumbersome. There is an alternative approach that simplifies the calculations. If we change from the risk neutral measure to the $T$-forward measure, i.e. we use the nominal discount bond $P_n(t,T)$ with $T \geq T_\alpha$ as numeraire instead of the nominal market account; the price of the swaption is given by

$$V(t) = P_n(t,T) E_n^T \left[ \frac{(S_{\alpha,M}(T_\alpha) - K)^+}{P_n(T_\alpha, T)} C(T_\alpha) \right]. \tag{3.16}$$

To calculate the last expectation it is necessary to find the dynamics of the factors under the $T$-forward measure[5]. Using the change of numeraire technique developed by Brigo and Mercurio (2001) we find the processes for the nominal rate, real rate and inflation index under the $T$-forward measure

$$
\begin{aligned}
dr_n(t) &= [\theta_n - a_n r_n(t) - \sigma_n^2 B_n(t,T)]dt + \sigma_n dW_n^T(t) && (3.17)\\
dr_r(t) &= [\theta_r - a_r r_r(t) - \rho_{r,I}\sigma_I\sigma_r - \sigma_r\sigma_n\rho_{n,r}B_n(t,T)]dt + \sigma_r dW_r^T(t) && (3.18)\\
\frac{dI(t)}{I(t)} &= [r_n(t) - r_r(t) - \sigma_I\sigma_n\rho_{n,I}B_n(t,T)]dt + \sigma_I dW_I^T, && (3.19)
\end{aligned}
$$

where $B_n(t,T) = \frac{1}{a_n}(1 - e^{-a_n(T-t)})$.

Equations (3.17)-(3.18) can be expressed as Ornstein-Ulhenbeck process and can be easily integrated up to time $t$ conditional on $\mathcal{F}_s, 0 \le s \le t$

$$
\begin{aligned}
r_n(t) &= r_n(s)e^{-a_n(t-s)} + \int_s^t e^{-a_n(t-u)}\left(\theta_n(u) - \sigma_n^2 B_n(u,T)\right)du && (3.20)\\
&\quad +\sigma_n\int_s^t e^{-a_n(t-u)}dW_n^T(u)\\
r_r(t) &= r_r(s)e^{-a_r(t-s)} + \int_s^t e^{-a_r(t-u)}\left(\theta_r(u) - \rho_{r,I}\sigma_I\sigma_r - \sigma_n\sigma_r\rho_{n,r}B_n(u,T)\right)du && (3.21)\\
&\quad +\sigma_r\int_s^t e^{-a_r(t-u)}dW_r^T(u)
\end{aligned}
$$

Therefore, $r_n(t)$ and $r_r(t)$ conditional on $\mathcal{F}_s$ are normally distributed with mean and variance given respectively by

$$
\begin{aligned}
E[r_n(t)|\mathcal{F}_s] &= r_n(s)e^{-a_n(t-s)} + \vartheta_n(t) - e^{-a_n(t-s)}\vartheta_n(s) && (3.22)\\
&\quad +\frac{\sigma_n^2}{2a_n}\left[e^{-a_n(t-s)}B_n(s,T) - B_n(t,T) - B_n(s,t)\right],\\
\mathrm{Var}[r_n(t)|\mathcal{F}_s] &= \frac{\sigma_n^2}{2a_n}[1 - e^{-2a_n(t-s)}], && (3.23)
\end{aligned}
$$

where

$$\vartheta_n(t) = f_n^*(0,t) + \frac{\sigma_n^2}{2a_n^2}(1 - e^{-a_n t})^2.$$

$$
\begin{aligned}
E[r_r(t)|\mathcal{F}_s] &= r_r(s)e^{-a_r(t-s)} + \vartheta_r(t) - e^{-a_r(t-s)}\vartheta_r(s) - B_r(s,t)\rho_{r,I}\sigma_I\sigma_r + && (3.24)\\
&\quad +\frac{\rho_{n,r}\sigma_n\sigma_r}{a_n + a_r}\left[e^{-a_r(t-s)_\alpha}B_n(s,T) - B_n(t,T) - B_r(s,t)\right],\\
\mathrm{Var}[r_r(t)|\mathcal{F}_s] &= \frac{\sigma_r^2}{2a_r}[1 - e^{-2a_r(t-s)}]. && (3.25)
\end{aligned}
$$

where

$$\vartheta_r(t) = f_r^*(0,t) + \frac{\sigma_r^2}{2a_r^2}(1 - e^{-a_r t})^2.$$

[5]Recall the dynamics of the real and nominal rates and inflation index under the nominal money market account are given by formulas (2.38)-(2.40).

The logarithm of the inflation index is normally distributed with mean and variance given by

$$E[\log I(t)|\mathcal{F}_s] = \log I(T_s)\left\{r_n(t) - r_r(t) - \rho_{n,I}\sigma_I\sigma_n B_n(t,T) - \frac{1}{2}\sigma_I^2\right\}(t-s) \quad (3.26)$$

$$\mathrm{Var}[\log I(t)|\mathcal{F}_s] = \sigma_I^2. \quad (3.27)$$

Once we have the correct distribution of our factors under the $T$-forward measure it is straightforward to price the European swaption at any time. If we are interested in the value at time zero $t = s = 0$ of the swaption, we use the bond maturing at $T_\alpha$ as numeraire. The pricing formula (3.16) reduces to

$$V_0 = P_n(0,T_\alpha)E_n^{T_\alpha}[V(T_\alpha)], \quad (3.28)$$

since $P_n(0,T_\alpha)$ is known at time zero and $P_n(T_\alpha,T_\alpha) = 1$.

# Chapter 4

# Estimation of parameters and Valuation using quasi-Monte Carlo Simulation

## 4.1 Mexican Inflation Linked market

We are going to use data from Mexico to implement the model presented in the previous chapter. The government of Mexico started issuing inflation indexed debt already in the 1970's. More recently, on May 30 1996, the government introduced inflation-linked securities called *Udibonos*; these are coupon indexed bonds that pay semi-annual coupons linked to the (UDI), an inflation-linked unit of value defined in terms of the Mexican CPI. They were issued to help extend the maturity of public debt, to lower the cost of funding and to increase the range of public financing instruments. Later in the 1990's, the Mexican government issued the so called Pics which are similar to *Udibonos* but having maturities up to 25 years. Given the existence of these inflation-linked bonds, it is possible to build a real term structure. Since our focus in the following is on pricing options on inflation-linked swaps, we assume that both the nominal and real interest rate curves have already been constructed.

The derivatives market on the other hand is not developed, there are a few over-the-counter operations but there are no quotes in the market for caps or swap volatilities neither in nominal market nor inflation-linked market. Therefore, it is not possible to calibrate the model to existing prices or volatilities in the market.

For the estimation of volatilities and correlations involved in the Jarrow and Yildrim model we use historical data of bond prices and inflation index.

### 4.1.1 Estimating parameters for nominal and real markets

"History tends to repeat itself." Some risk analyst believe in this and that is the reason why it is a common practice to test the performance of portfolios in historical conditions and/or stress scenarios to assess how much money the company can loose if it does. For pricing, the approach is to use the

market data to estimate the fair price implied in the markets assuming there is no arbitrage. Our approach is based in historical data for the estimation of the volatility of the rates and inflation and correlation between the three factors given that the main interest is for risk management purposes and not for speculation.

We used daily data of the nominal and real term structure curves from 01-January-2001 to 08-June-2005[1]. We recall we considered a one-factor volatility function for the nominal and real rates of the form

$$\sigma_k(t,T) = \sigma_k e^{-a_k(T-t)} \ , \ k = \{r, n\} \tag{4.1}$$

where $\sigma_i$ and $a_i$ for $i = r, n$ are constants, which allowed us to recover the Extended Vasicek model for short rates. If we use the dynamics of bond prices under the nominal market measure (2.36) and (2.37) we can derive that the bond returns evolve according to the following normal distribution:

$$\frac{\Delta P_k(t,T)}{P_k(t,T)} - [\mu_k]\Delta t \sim N\left(0, \left(\int_t^T \sigma_i(t,s)ds\right)^2 \Delta t\right). \tag{4.2}$$

If we use daily observations (trading days) $\Delta t = \frac{1}{252}$, the expected return on the bond is small relative to the standard deviation so it is possible to neglect it from the estimation procedure [15]. So the variance of the zero-coupon bond prices over the time interval $[t, t+\Delta]$ satisfies the equation

$$var\left(\frac{\Delta P_k(t+\Delta,T)}{P_k(t,T)}\right) = \frac{\sigma_k^2(e^{-a_k(T-t)-1)^2})\Delta t}{a_r^2} \tag{4.3}$$

The correlation between nominal and real rates is also crucial for option pricing, we can estimate it with formula

$$\widehat{\rho}_{r,n} = \text{Corr}(\Delta r_r(t), \Delta r_n(t)) \tag{4.4}$$

If we use the observations of the real and nominal zero coupon prices generated from the term structures we can compute the sample variance as an estimate of left side of last expression. We can use the cross-sectional data from different maturities of the zero-coupon bonds and with non-linear regression estimate the parameters.

The resulting estimates of these parameters are:

| | |
|---|---|
| $\widehat{a}_r = 0.04933$ | $\widehat{\sigma}_r = 0.01543$ |
| $\widehat{a}_n = 0.13690$ | $\widehat{\sigma}_n = 0.02668$ |

| |
|---|
| $\widehat{\rho}_{n,r} = 0.17750$ |

[1]Source Price Provider Company Valmer Inc.

### 4.1.2 Estimation of parameters of Inflation

Jarrow and Yildrim assume that the inflation index follows a lognormal distribution. We are going to analyse the time series of inflation to test if the inflation has seasonality and to examine the possible side effects of our simplification. For estimation we use the Mexican $INPC$ or $CPI$ inflation index. This index is published by the Mexican Central Bank with a lag of 1 month. So the monthly data points are linearly interpolated to obtain a daily index called $UDI$ which has two weeks lag with respect to the $CPI$. In 2001 Mexico changed its monetary policy to one of target inflation and this caused a considerable change in the trend and volatility of the inflation see figure (4.1). Given that for valuation purposes we are interested in the future inflation, the data before 2001 is not used for the estimation of the parameters. Nevertheless it is highly recommended to use the history before 2001 to do some stress testing of the model. As was mentioned before, one characteristic of inflation

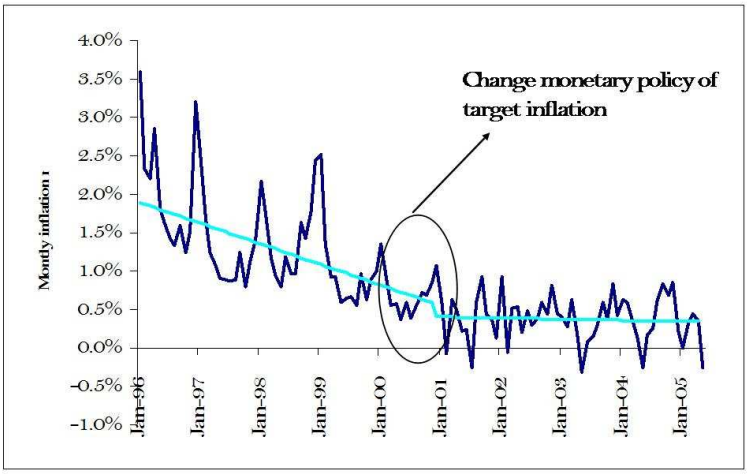



Figure 4.1: Monthly inflation measured as percentage increase of $CPI$ of Mexico.

is its seasonality. Since the work of Box and Jenkins (1976), it is a common practice to analyse the sample autocorrelation function in order to identify the presence of seasonal patterns. Analysing the autocorrelation of the monthly inflation (figure (B.3) in appendix B), we see that inflation is significantly correlated with the observation a year before but less correlated with the observation 6 months earlier, suggesting there is seasonality in this series.

The evolution of the real and nominal zero-coupon bond prices are directly linked to the volatility of inflation rate, $\sigma_I$ which we assume to be constant; and the correlations between the inflation index and the real and spot rate of interest, $\rho_{r,I}$, $\rho_{n,I}$ and between the real and nominal sport interest

rates $\rho_{n,r}$. Using the sample moments, we can compute the estimates of these parameters

$$\widehat{\sigma}_I = \left\{ \frac{1}{\Delta} var\left( \frac{\Delta I(t)}{I(t)} \right) \right\}^{1/2} \tag{4.5}$$

$$\widehat{\rho}_{r,I} = corr\left( \Delta r_r(t), \frac{\Delta I(t)}{I(t)} \right) \tag{4.6}$$

$$\widehat{\rho}_{n,I} = corr\left( \Delta r_n(t), \frac{\Delta I(t)}{I(t)} \right) \tag{4.7}$$

$$\tag{4.8}$$

Here we use $\Delta = \frac{1}{12}$ so as not to introduce bias[2] in our estimation we use the same periodicity for the changes in the inflation index and the changes in the rates values. The estimated parameters are shown in the following table.

| | |
|---|---|
| $\widehat{\sigma}_I$ | 0.000881492 |
| $\widehat{\rho}_{r,I}$ | -0.12853 |
| $\widehat{\rho}_{n,I}$ | 0.18230 |

In appendix B Figure (B.1) we show the nominal and real short rates and the evolution of the inflation index in Mexico for the sample period chosen.

### 4.1.3 The effect of seasonality of inflation in option pricing

The model of Jarrow and Yildrim (2003) assumes the process for the inflation index is a geometric brownian motion. This model is the most commonly used for inflation in spite it fails to model other effects present in inflation such as seasonality and interventions. Belgrade and Benhamou(2004) [4] have recently analysed the effect of seasonality in the pricing of derivatives and proposed a simple static pattern to reshape the forward curve of inflation index (CPIs). Hence, it is only the forward curve that is modified while the inflation dynamics stays unchanged.

The seasonality is taken in a monthly basis. They use a vector of yearly seasonal up and down bumps $\{\mathcal{B}(i)\}_{i=1,..12}$ indexed by their corresponding months $i$ with the convention that January equals 1. The seasonality estimation comes to the finding of a vector of 12 up and down (additive or multiplicative) bumps from historical $CPI$ data. They used a parametric (Least-Squares) and a non-parametric method ($X11$) for the estimation of this effects.

For the pricing of inflation-swaps this effect is not of second order considering the tight bid-ask margins in which these instruments are traded. Belgrade and Benhamou (2004) estimated that for a 10 year inflation zero coupon swap in the French Market the impact of seasonality is of $-1.5$ to 2.0 basis points.

[2] the inflation index is not observed in reality in monthly basis and the daily values are just linear interpolations from monthly observations.

Nevertheless, if the focus of the model is to pricing of inflation-linked options, the seasonality becomes of second order given that the main draw of the price is the volatility of the rates and inflation and their correlations.

## 4.2 Quasi-Monte Carlo simulation

The pricing formula for the European swaption under the $T$-forward measure involves the calculation of an expected value given in equation (3.28). The value of this expectation can be found by direct integration, solving the partial differential equation that the relative price satisfies or by Monte Carlo integration. In this case, we decided to use Monte Carlo simulation for several reasons: given that the main goals of the implementation is to use the model for risk management and scenario analysis it is the appropriate technique used for this purposes; consistency with the Longstaff and Schwartz approach for pricing the Bermudan swaption; and if we decide to extend this model in the future to a multifactor model such as Libor Market Model, it would have to use Monte Carlo for the pricing.

The basic Monte Carlo method is very simple, it essentially uses the law of large number to evaluate the expectation. A central feature of Monte Carlo that the order of its error is $\mathcal{O}(N^{-1/2})$. In order to add one decimal place of precision to the estimation of the price it requires 100 times as much points, as a consequence to obtain high levels of accuracy one needs a large number of samples. Monte Carlo simulation works because it tries to cover the unit interval in an even manner. However, in the short term, the values may cluster around certain values which is why the simulation takes a long time to converge. As Monte Carlo simulation is slow to converge, a lot of research has gone into methods for increasing its speed of convergence, such as variance reduction techniques [16]. If instead we use a deterministic sequence of numbers that covers the interval, a low-discrepancy sequence, the rate of convergence reduces to $\mathcal{O}(N^{-1-\epsilon}), \epsilon > 0$, where $\epsilon$ depends on the dimension of the problem. Using low-discrepancy sequences to carry out Monte Carlo, is sometimes called quasi-Monte Carlo[3].

### 4.2.1 Quasi-Random Numbers: Faure Sequence

There are several types of low-discrepancy sequences: Halton sequences, Sobol' sequences and Faure sequences. In practice both Faure sequence and Sobol' sequence are superior to the Halton sequence [13]. Faure sequence has certain advantages for the valuation of high dimensional integrals [17]. We describe the algorithm presented in Boyle and Tan (96) for $k$-dimensions. Let $p$ be the smallest prime greater than $\max(k, 2)$, $n$ the number of simulation. For each $n, k$ Faure numbers are generated

[3]The term quasi-Monte Carlo can be misleading because the numbers are deterministically generated and do not pretend to be random.

recursively. Suppose that $n = \sum_{j=0}^{\infty} a_j^1 p^j, 0 \leq a_j^1 < p$. Set

$$\phi_p^1(n) = \sum_{j=0}^{\infty} a_j^1 p^{-j-1}. \tag{4.9}$$

For $1 \leq s \leq k, \phi_p^s$ is defined as

$$\phi_p^s(n) = \sum_{j=0}^{\infty} a_j^s p^{-j-1}. \tag{4.10}$$

where $a_j^s(n)$ are defined recursively. Suppose all $a_j^{s-1}(n)$ are known, then

$$a_j^s(n) = \sum_{i \geq j}^{\infty} {}^iC_j a_j^{s-1}(n) \text{mod } p, \tag{4.11}$$

where ${}^iC_j = \frac{i!}{j!(i-j)!}$. Thus the next level of coefficients is obtained by multiplying by an upper triangular matrix. The sequence of $k$-dimensional vectors $\{\phi_p(n)\}_{n=1,2,\ldots}$ is the $k$-dimensional Faure sequence.

In order to generate the sequence, it is required to provide an initial point or "seed" to start the recursions. This starting index must be greater than or equal to 1. Fox [9] recommends to use as starting point $p^4 - 1$ to improve uniformity.

The code used in our implementation is described in Fox (86) [9]. In Appendix B Table D we show a table of the first 12 points generated for three dimensions in base 3 with initial point equal to 1. In figure D.1 we show the points in plots onto two dimensions and in the cube, where we can appreciate how they are evenly distributed to cover it. We can notice that the projections in $[1, 0)^2$ are identical when the distance between the factors is equal and the projection of Factor 3 and Factor 1 is equal but exchanging the axes. This phenomenon is caused by the cyclic properties of the generator matrices, so the projections of the points in the $i$th and $j$th coordinates depends only on the distance $j - i$, module the base $p$; further discussion of this result can be found in [10].

### 4.2.2 Generating Multivariate Normal Variables

To apply the Faure sequence to Monte Carlo simulation based on Normal random variables, we map the point in the $k$-dimensional cube by applying the inverse normal function to the sequence $\Phi^{-1}(\phi_n)$. For the inversion of the Normal distribution we use the algorithm given by Moro [16][4].

A multivariate normal distribution $N(\mu, \Sigma)$ is specified by its mean vector $\mu$ and covariance matrix $\Sigma$. We know that if $Z \sim N(0, I)$ and $X = \mu + AZ$, then $X \sim N(\mu, AA^\intercal)$. With the method described above we can generate independent standard normal random variables obtained with the inversion of the normal distribution on the Faure sequence. Thus, the problem of sampling $X$ reduces to finding the matrix $A$ for which $AA^\intercal = \Sigma$. A Cholesky factorization of $\Sigma$ would give us this matrix $A$. If $\Sigma$ is positive definite, it has a Cholesky factorization and the matrix $A$ is unique up to changes in sign [10].

[4] See the algorithm in Appendix D.1

We can simulate the nominal rate, the real rate and the logarithm of the inflation index as multivariate normal distribution. Using the historical correlations and the mean and variance of the three factors given by equations (3.22)-(3.25) and (3.26)-(3.27).

### 4.2.3 Analysis of Results

We are going to assume a flat term structure of real and nominal rates, using as reference the market rates from 8 July 2005 in Mexico, the nominal spot rate equal to 12% and real spot rate equal to 4.5%. The inflation index used as reference is today's index value assume equals $I_{T_0} = 1$. All this analysis (unless stated contrary) are based on a payer's swaption that matures in one year. Its underlying swap is a fixed real vs. floating nominal forward[5] starting swap that starts also in one year, has cash flow payments every six months and has a maturity of 5 years. We fixed the strike of the swaption equal to the forward swap rate. In appendix C we present some graphs with the results.

First we used 10,000 simulations (in theory to obtain close to four digits of precision). We obtained the histograms and sample mean, standard deviation and correlation of each factor. We verified that the sample values were very close to the theoretical or real values, see figure (C.11). We also analysed the histogram of the payoff of the option at maturity as well as the underlying swap rate at the maturity of the option. We find the results are consistent, all the swap rates were below our strike would represent a payoff of zero in the option, see figure (C.12).

The next thing we analyse in an informal way is the the convergence of the method. It is not possible to do a more formal analysis because we do not know the exact value of the price through an analytical formula[6]. Graphically we see in (figure C.1) that the price of the option converges very rapidly as we increase the number of simulations. In figure (C.2) we compare the price obtained Monte Carlo with a variance reduction technique (antithetic sampling) for the same number of simulations, we find the quasi-Monte Carlo is superior even with very few number of simulations.

Another interesting analysis is how the price changes as we increase or decrease the strike price. This is because a hedger has to decide whether to hedge completely a position or use an option strike out-of-the money and save on the cost of the option (figure C.3). The strike runs from $K = .01$ to $K = .04$. As the strike is more in-the-money the option is more expensive because we would have the right to enter a swap to pay a small fixed real rate in exchange of floating nominal rate, therefore it is more likely to be exercised.

We also analysed what happens when we increase the maturity date of the option as we keep the maturity date of the swap, i.e. we price European options for underlying swaps with less cash flows each time (C.4). The underlying swap rate becomes very small and in some cases becomes negative

[5] It is described in section (3.2).

[6] If we had the analytical solution for the price of the option we could plot the convergence and test if it is of order $\mathcal{O}(1/n^1 - \epsilon)$ for $\epsilon > 0$ depends on the number of dimensions [10].

as we evolve in the future. This is a consequence of the dynamics assumed for the inflation index. If we recall the inflation index is modelled as a geometric brownian motion and its drift is derived from the difference between the nominal and the real rates and a correction factor given by the change of numeraire. When the difference between the nominal rate and the real rate is large (as is the case for Mexico which is around 8%), the index grows exponentially and this sole effect causes that the swap rate has to be negative in order make the present value of its cash flows equal to the value of the floating nominal bond. This effect is not observed when the implied inflation (difference between nominal and real rates) is not very big or when we are pricing a short dated swaption. So this is a major drawback of this model, it is not suitable for pricing long dated instruments or with large implied inflation.

One of the most important tests are how the price of the option changes as we change the volatilities and correlations. We varied the volatility of each factor from .001 to .40 with increments of .05 with constant correlations (historical). This analysis is presented graphically as surfaces of the price of the option in figures (C.5)- (C.7) for each fixed value of the volatilities. We can conclude that the price of the option is very sensitive to the volatility of the real rate, secondly to the volatility of the inflation index and at last the volatility of the nominal rate. The volatility of the real rate has the major effect because of the annuity term is involved in the pricing formula. As we mentioned before, the level of the inflation index a determinant factor in the price of the option, and its volatility has an effect in the simulated values of inflation. It is worth to mention that using historical data, the volatility of the nominal rate is twice as big as the volatility of the real rates and almost 3 times bigger than the volatility of the inflation index. In periods of economic crisis the volatility of inflation usually rises and this would have a major effect in the price of the option, more than the effect of an increase in the volatility of the nominal rate.

We did a similar analysis to correlations, we changed the correlation between each pair of factors from -.75 to .75 with increments of .25. In this case we had to be careful that the resulting covariance matrix was still positive definite to be able to have a Cholesky factorization of it. When the resulting covariance matrix was not positive definite, we assumed the price of the option was zero. The shapes of the resulting surfaces are more complex than the analysis we presented for volatilities; see figures (C.8)-(C.10). Graphically we can see that the shape of the surfaces is the same when the fix the correlation between the nominal rate and inflation and the real rates and inflation. This means that a change in their correlation would produce similar effects on the price of the option. On the other hand, when we fix the correlation between the nominal and real rates, the surfaces have a different pattern, which tends to have more stable behaviour. The price of the option reaches its maximum when the correlation between nominal rates and the inflation is large and negative, and when there is no correlation among the factors. This is a very interesting result as in periods of crisis, the correlations may change sign or become apparently independent from one and another. The price

of the option on the other hand is small when we have positive correlation between real rates and inflation, and when the correlation of the nominal rate and inflation index is either low or zero. In this case, when the real rates and inflation index have positive correlation[7], it means that if the inflation increases, the real rate would also increase. Hence, the cash flow of the inflation indexed leg will increase because of these two effects, the swap rate will decrease and it is less likely that the the option will be exercised.

[7]Recall the historical correlation is negative.

# Chapter 5

# Bermudan Swaption

Bermudan swaptions are options to enter a swap not only in a single date, but at a sequence of dates.

For pricing Bermudan swaptions with the Jarrow and Yildrim (2003) we are going to use Longstaff Schwartz "Monte Carlo Regression" or "Least Squares Monte Carlo" (LSMC). This method is quite general and has been used in practice to price interest rate Bermudan swaptions with LIBOR Forward Models (see Brigo and Mercurio [6]).

## 5.1 Longstaff and Schwartz Approach

Longstaff and Schwartz (2001) developed a method to value American and Bermudan-style options by simulation. This approach is readily applicable in path-dependent and multifactor situations where traditional finite difference techniques or lattice methods cannot be used. At any exercise time, the holder of the Bermudan-option optimally compares the payoff from immediate exercise with the expected payoff from continuation, and then exercises if the immediate payoff is higher. Therefore, the optimal exercise strategy is determined by the conditional expectation of the payoff from continuing to keep the option alive. This method estimates the conditional expectation from cross-sectional information in the simulation by using least squares. Specifically, the ex-post realised payoffs from continuation are regressed on functions of the values of the state variables. The fitted value from this regression provides a direct estimate of the conditional expectation function. By estimating the conditional expectation function for each exercise date, it is possible to obtain a complete specification of the optimal exercise along each path.

### 5.1.1 Valuation framework

The underlying LSM algorithm is based on the general derivative pricing paradigm of Black and Scholes (1973) and Heath, Jarrow and Morton (1992), and others. We assume given a complete probability space $(\Omega, \mathcal{F}, P)$ and finite time horizon $[0, T]$, the sample space $\Omega$ is the set of all possible realisations of the stochastic economy between time 0 and $T$ and has typical $\omega$ representing a sample

path, $F$ is the sigma field of distinguishable events at time $T$, and $P$ is a probability measure defined on the elements of $\mathcal{F}$. We define $F = \{\mathcal{F}_t; t \in [0,T]\}$ to be the augmented filtration generated by the price processes for the securities in the economy, and assume that $\mathcal{F}_T = \mathcal{F}$. Consistent with the no-arbitrage paradigm, we assume the existence of an equivalent measure $Q$ for this economy.

We are interested in valuing Bermudan-style derivative securities with random cash flows occurring during the period $[0,T]$. We restrict our attention to the derivatives with payoffs that are elements of the space of square-integrable or finite variable functions $L^2(\Omega, \mathcal{F}, Q)$. This condition is justified if we assume the conditional expectation is an element of the $L^2$ space of square-integrable functions relative to some measure, since $L^2$ is a Hilbert space, it has a countable orthonormal basis and the conditional expectation can be represented as a linear function of the elements of the basis. The value of an American-Bermudan option equals the maximized value of the discounted cash flows from the option, where the maximum is taken over all stopping times with respect to the filtration $F$. Let $C(\omega, s; t, T)$ denote the path of the cash flows generated by the option, conditional on the option not being exercised at or prior to time $t$ and on the optionholder following the optimal stopping strategy for all $s, t < s \leq T$. The objective of the LSM algorithm is to provide a pathwise approximation to the optimal stopping rule that maximises the value of the American-Bermudan option. In the Bermudan case, the option can only be exercised at the $L$ discrete times $0 < t_1 \leq t_2 \leq ... \leq t_L = T$, and consider the optimal stopping policy at each exercise date. At the final expiration date of the option, the investor exercises the option if it is in the money, or allows it to expire if it is out of the money. At exercise time prior to the final expiration date, the optionholder must choose whether to exercise immediately or to continue the life of the option and consider the exercise decision at the next exercise time. The value of the option is maximised pathwise, and hence unconditionally, if the investor exercises as soon as the immediate exercise value is greater than or equal to the value of continuation. At time $t_l$, the cash flow and value of immediate exercise is the same and it is known to the investor; the cash flows from continuation are of course not known. No arbitrage valuation theory implies that the option cannot be exercised until time $t_l$ and is estimated by the expectation of the remaining discounted cash flows $C(\omega, s; t_l, T)$ with respect to the risk-neutral pricing measure $Q$. The value from continuation at time $t_l$ can be expressed as

$$F(\omega, t_l) = E_Q \left[ \sum_{j=l+1}^{L} \exp\left( - \int_{t_l}^{t_j} r(\omega, s) ds \right) C(\omega, t_j; t_l, T) | \mathcal{F}_{t_l} \right] \tag{5.1}$$

where $r(\omega, t)$ is the stochastic riskless (nominal or real) discount rate, and the expectation is taken conditional on the information set $\mathcal{F}_{t_l}$ at time $t_l$. So the problem of optimal exercise reduces to comparing the immediate exercise value with this conditional expectation, and then exercising as soon as the immediate exercise value is positive and greater than or equal to the conditional expectation.

The LSM approach uses least squares to approximate the conditional expectation function at $t_{L-1}, t_{L-2}, ..., t_1$. The algorithm works backwards since the path of cash flows $C(\omega, s; t, T)$ generated by the option is defined recursively; $C(\omega, s; t_l, T)$ can differ from $C(\omega, s; t_{l+1}, T)$ since it may be optimal to stop at time $t_{l+1}$, thereby changing all subsequent cash flows along a realized path $\omega$. Specifically at time $t_{l+1}$, we assume that the unknown functional form of $F(\omega; t_{L-1})$ in Equation (5.1) can be represented as a linear combination of a countable set of $\mathcal{F}_{t_{L-1}}$-measurable basis functions.

There are many possible choices of basis functions for the regression. When the value of the asset underlying the option is a Markovian process only current values of the state variables are necessary, when they are non-Markovian, both current and past realisations of the state variables can be included in the basis functions and the regressions. Longstaff and Schwartz suggest as one possible choice of basis functions the set of (weighted) Laguerre polynomials $L_n = \exp(-X/2)\frac{e^X}{n!}\frac{d^n}{dX^n}(X^n e^{-X})$. With this specification, $F(\omega, t_{L-1})$ can be represented as

$$F(\omega, t_{L-1}) = \sum_{j=0}^{\infty} a_j L_j(X), \tag{5.2}$$

where the $a_j$ coefficients are constants.

To implement the LSM approach, first it is necessary to approximate $F(\omega, t_{L-1})$ using the first $L < \infty$ basis functions, denoted as $F_L(\omega, t_{L-1})$. Once this subject of basis functions has been specified,$F_L(\omega, t_{L-1})$ is estimated by projecting or regressing the discounted values of $C(\omega, s; t_{L-1}, T)$ onto the basis functions for the paths where the options is in the money at time $t_{L-1}$. Only in the money paths are used in the estimation since the exercise decision is only relevant when the option is in the money. By focusing on in-the-money paths, the region where the conditional expectation has to be estimated is limited consequently less basis functions are needed to obtain an accurate approximation to the conditional expectation function. Since the values of the basis functions are independently and identically distributed across paths, the fitted value of the regression $\hat{F}_L(\omega, t_{L-1})$ converges in mean square and in probability to $F_L(\omega, t_{L-1})$ as the number of in-the-money paths in the simulation goes to infinity.

Once the conditional expectation function at time $t_{L-1}$ is estimated, it is possible to determine whether early exercise at time $t_{L-1}$ is optimal for an in-the-money path $\omega$ by comparing the immediate exercise value with $\hat{F}_L(\omega, t_{L-1})$ and repeating for each in-the-money path. Once the exercise decision is identified, the option cash flow paths $C(\omega, s; t_{L-2}, T)$ can be approximated. The recursion proceeds by rolling back to time $t_{L-2}$ and repeating the procedure until the exercise decision at each exercise time along each path have been determined.

The Bermudan option is the valued by starting at time zero, moving forward along each path until the first stopping time occurs, discounting the resulting cash flow from exercise back to time zero, and then taking the average over all paths $\omega$. When there are two state variables $X$ and $Y$, the set of basis functions should include terms in $X$ and $Y$, as well as cross-products of these terms. Similarly

for higher-dimensional problems. Longstaff and Schwartz through some numerical examples suggest that the number of basis functions necessary to approximate the conditional expectation grow slowly than exponentially.

Theoretically, when the American option is continuously exercisable, for any finite choice of $M, L$ and vector $\theta$ in $R^{M\times(L-1)}$ representing the coefficients for the $M$ basis functions at each of the $L-1$ early exercise dates, let $LSM(\omega; M, L)$ denote the discounted cash flow resulting from following the LSM rule of exercising when the immediate exercise value is positive and greater than or equal to $\hat{F}_L(\omega_i, t_{L-1})$ as defined by $\theta$. Then the following inequality holds almost surely,

$$V(X) \geq \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} LSM(\omega_i, M, L). \tag{5.3}$$

The LSM algorithm results in a stopping rule for an American-style option. The value is based on the stopping rule that maximises the value of the option; all the other stopping rules, including the stopping rule implied by the LSM algorithm, result in values less than or equal to that implied by the optimal stopping rule. This criterion then provides guide in determining the number of basis functions needed to obtain an accurate approximation; simply increase the number of basis functions $M$ until the value implied by the LSM algorithm no longer increases. But by its nature providing a general convergence result for the LSM algorithm is difficult since it is needed to consider limits as the number of discretisation points $L$, the number of basis functions $M$, and the number of paths $N$ go to infinity.

### 5.1.2 Least-Square Monte Carlo for Bermudan Swaption

Originally Longstaff and Schwartz tested their method considering a version of a *string model* for swaptions. In the Bermudan swaption considered here, the underlying swap starts at the first exercise date of the option. The swaption holder has the right to exercise the option at some fixed dates and enter the swap whose life span decreases as time moves forward. This option could be used to break the underlying swap in fixed reset dates.

As Longstaff and Schwartz suggested in their paper, we use a the quasi-Monte-Carlo technique (Faure sequence) described in the last chapter in conjunction with LSMC to improve the computational speed and efficiency. Faure sequences are still suitable for this problem since we are dealing with a small number of dimensions (in our case a maximum of 12 dimensions=3 Factors x 4 time steps), this problem is similar to the swaption priced by Joy et al.[17]. For a larger number of exercise dates we support the recommendation of Glasserman [10] of using Sobol' sequences or the principal component algorithm developed by Akesson and Lehoczky [1].

The underlying swap we used is the same used in our European case. It matures in five years, resets every six months, and exercise can occur once a year, one year included and five years excluded.

There are therefore five exercise dates and at each exercise date the underlying swap has two less coupons than in the previous exercise date.

For Bermudan case, it is necessary to simulate the factors at discrete time steps (each exercise date). A convenient choice of numeraire in this case is the nominal bond that matures at the last exercise date of the Bermudan swaption (say $T_{M-2}$). The distribution of the parameters at each time step is derived from equations (3.22)-(3.27) using proper values for $s = 0, T_i, T_{i+1}, T_{M-1}$ as exercise dates and previous values of the factors $\{r_n(0), r_r(0), I(0), ..., r_n(T_M - 2), r_r(T_M - 2), I(T_M - 2)\}$. Once we run the algorithm and we estimated the optimal stopping time, we can estimate the price of the option

$$V_0 = P_n(0, T_{M-2}) \sup_i E^{T_{M-2}} \left[ \frac{V_{T_i}}{P_n(T_i, T_{M-2})} \right] \ , \ i = 1, ..., T_{M-2}. \tag{5.4}$$

This is a straightforward calculation given that the Hull-White model has an analytical expression for $P_n(t, T)$. For the estimation of the conditional expectation we used as basis functions a constant,the first two powers of the swap rate, the factors and the cross product of the factors, resulting in 4 for one factor model, 6 for two factor model and 9 for the three factor model.

Using the values of the Mexican Market, we found that the optimal stopping time was always in the first exercise date or never to exercise the option. This is a result of the model for the inflation index, it is growing un proportionally and the future swap rates result very small or even negative, therefore the option is not exercised (figure C.13). When we changed the current nominal rate and real rate to decrease the gap between them, we found there were some paths with different stopping times than the first exercise date.

We analysed the effect of a reduction in the number of factors in the price of the Bermudan swaption. For this exercise we changed the term structure of rates to be 6% the nominal rate and 4.5% the real rate in order to obtain some stopping times different from the first exercise date. It is important to consider a reduction in the number of parameters because if we were interested in increasing the number of exercise dates, the problem would soon become of very high dimensionality. The inflation index has a small historical volatility compared to the volatility of the other two factors, therefore when we assume its behaviour is deterministic. We can observe in figure (C.15) that the price does not get significantly affected with the reduction to two factors, especially when we increase the number of simulations. On the other hand when we assume both the real rate and the inflation index have deterministic behaviour the price of the option drops considerably (figure C.14). Therefore, we do not suggest to use a one factor model for the pricing of this option.

# Chapter 6

# Conclusion

To price inflation indexed derivatives one needs to know the theory behind interest rate modelling and foreign-currency modelling, combined with some economic intuition. To produce a sensible model one has to consider the dynamics of the nominal and real markets and the inflation index. The three factor HJM model (foreign-currency analogy) allows one to treat the real rate the rates in a foreign economy, and the inflation index (CPI) as the spot exchange rate in a framework consistent with no arbitrage. We assumed a volatility function as in an Extended Vasicek model for the nominal and real rates, as in Jarrow and Yildrim (2003). We developed the pricing formulas for an European Swaption and implemented the pricing model using quasi-Monte Carlo simulation based on historical data for the estimation of the parameters of the model. Finally we implemented the Longstaff and Schwartz algorithm for the pricing of a Bermudan swaption. All the algorithms were programmed for Matlab.

We confirmed that the Gaussian affine model used (Extended Vasicek) is very tractable and has a number of characteristics that enables us to have good numerical valuation procedures, although the estimation of its parameters is not straightforward. For the estimation of the parameters we used historical data of the past five years of the Mexican economy. We used daily data for the estimation of the volatilities of the nominal and real rates, and a non-linear least squares algorithm for the estimation of the mean reversion of the nominal and real spot rates. The volatility of the inflation index and the correlation with the nominal rate and real rate was using monthly observations. This models permit rates to become negative with non-zero probability, but as is discusses in [13] we found that when this model is fitted to "ordinary" term structures this probability is very low[1]. Furthermore, for markets like Mexican where there is no active trading of options there is no need to calibrate the model with market quotes and one factor models for the nominal and real interest rates are adequate. We suggest as a natural extension for this model to incorporate the multifactor nature of the term structure of the nominal and real rates as in the Libor Market Model.

[1] Although this might not be the case in markets with low levels interest rates.

We found that the geometric brownian motion model for the inflation index is not realistic because historically there are other effects present in the series of the inflation index such as seasonality and interventions; however, assuming it is a simple geometric brownian motion provides a good level of tractability to the model. More specifically for option pricing there is one major shortcoming of this model. When the drift (estimated as the difference between the nominal rates and real rates) is large, the model is inappropriate for pricing long dated options (probably not more than 2 years), even with a small implied inflation (around 2%) the model would not be applicable for options dated more than 5 years. It would cause a significant underpricing of the option (in the payer's version) and an overpricing for the receiver's version. The improvement in the dynamics of the inflation index should be one of the main areas for future work. The new models should incorporate jump diffusion processes (for interventions), seasonality (deterministic or stochastic), mean reversion for the inflation and/or include not only short term but long term interest rates on its dynamics to improve the modelling of its drift.

The parameters of the model that need to be monitored closely are the volatilities and correlations between the factors. The price or value of the option is very sensitive to these parameters, specially to the volatility of real rates and inflation, and the correlation between the nominal rates and inflation. In periods of crisis in the markets this parameters tend to change rapidly and it has been often the case that the correlations change sign or the factors behave independently. This situation should be considered very closely in markets where there the parameters have to be estimated historically. It might take a few observations before the estimated parameters take the change in volatilities or correlations, so it is important to have alternative methods to identify rapidly these changes and use them for pricing the option.

Monte Carlo models and specifically quasi-Monte carlo techniques are very convenient for risk management, scenario analysis and valuation of large portfolios as mentioned by Joy et at.[17]. We confirmed that quasi-Monte Carlo methods produce substantially more precise values than ordinary Monte Carlo for the pricing of the European Swaption. For calculations that need to be repeated often with only minor change in parameters (such as sensitivity of price to some parameters or price an option everyday) quasi-Monte Carlo is very time effective. However there is an important feature of quasi-Monte Carlo methods that has to be considered, its dependence on the dimension of the problem. In our examples we treated low dimensional problems (3-12 dimensions) with Faure sequences, but it we advice to analyse the benefits of using other low discrepancy series such as Sobol' if one desires to increase significantly the dimensionality of the problem (increasing the number of factors or time steps). Furthermore, we suggest to derive the semi-analytical formulas for the European swaption or implement finite difference methods and compare the results obtained with the three methods.

For the pricing of the Bermudan swaption the Longstaff and Schwartz approach is in principle simple to implement. One has to choose a convenient numeraire, generate the paths for the underlying of the option, and choose the basis functions for the estimate of the conditional expectation (using the least-squares). In all the models we used as basis functions one constant, the first three powers of the underlying swap, the factors and the crossed products of the factors. As it is mentioned in [18] we included more basis functions but we did not find an increase in the fitness of the model. Finally we find the optimal stopping time for each path and average the discounted cash flows. Using the market rates for Mexico, the optimal stopping time was in the first exercise date or not exercise the option at any time. This is a consequence of the effect of the exponential growth of the inflation index in time. We consider the price of the Bermudan option with out algorithm should be taken as a lower bound to the price of the Bermudan swaption when dealing with long dated swaptions or large implied inflation.

For the pricing of the Bermudan swaption with multiple exercise dates, one should consider the benefits of decreasing the number of factors in terms of the desired accuracy of the results with the tractability of the model. We found that the difference from using a one factor model to a three factor model can be considerable (15% with our data) and this might increase if the volatility of the factors that we are eliminating is bigger than in our example. A more detailed analysis of the sensitivity of the price of the Bermudan option as we change the volatilities and correlations should be considered before going from the three factor model to some simplification of it.

# Appendix A

# Glossary

**Affine** : $f$ is an affine function of $X$ if $f(X) = a + bX$.

**Arbitrage** :Arbitrage is guaranteed way of making money without risk.

**BTP** :Italian Buoni Poliennali del Tesoro. The inflation-indexed BTP is often referred to as a BTPEi, indexed to Euro-HICP excluding tobacco.

**CIBs** : Australian Capital Indexed Bond.

**CPU-I** :non-seasonally adjusted US City Average All Items Consumer Price Index for All Urban Consumers.

**Duration** : is defined as a weighted average of the coupon dates of a bond where the discounted values of the coupon payments are used as weights,and it will provide the "mean time to coupon payment". It is a measure of the sensitivity of the bond price with respect to changes in the yield. For a bond with coupons $c_i$ with price at $t = 0$ equal to $P$, yield to maturity $y$ and coupon payments at $T_i$, the duration $D$ is defined as:

$$D = \frac{\sum_{i=1}^{n} T_i c_i e^{-yT_i}}{P}$$

**Euro-HICP** : Euro-zone Harmonised Index of Consumer Prices.

**Filtration** : A set $\{\mathcal{F}_t\}_{0 \leq t \leq T_{max}}$ of $\sigma$-algebras such that $\mathcal{F}_{t1} \subseteq \mathcal{F}_{t2}$ for $t_1 \leq t_2$, representing the increase in information available by observing a stochastic system through time.

**Gilt** :UK government bonds.

**In-the-money** : Term commonly used in finance to say that the price of the underlying of an option is in the region where it would be exercised giving a positive value for the optionholder.

**Hilbert space** :A Hilbert space is a vector space $H$ with an inner product $< f, g >$ such that the norm defined by $|f| = \sqrt{(} < f, f >)$ turns $H$ into a complete metric space. If the metric defined by the norm is not complete, then $H$ is instead known as an inner product space.

**INPC** :Indice Nacional de Precios al Consumidor, or Mexican CPI Index. This index is published by the Central bank every two weeks with a lag of 1 month when the inflation actually occurred.

**Fisher Equation** :Formulated by Alan Fisher in the inter-war period before the start of the inflation indexed bond markets. It states that a nominal bond yield is made up of three components: inflationary expectations, a required real yield that investors require over and above those expectations and a "premium" that reflects the assumption that investors want additional compensation for accepting additional inflationary risks when holding nominal bonds.

**Martingale** :A stochastic process is a martingale if $\mathbb{E}_t[S^T] = S_t$, for all times $t \leq T$. That is the expected value in the future conditioned on what is known now is always identical to today's value.

**OATi** :French inflation indexed bonds or Obligations assimilables du Trésor. Its indexation is based on the final non seasonally-adjusted French CPI ex-tobacco index.

**RRB** :Canadian Real Return Bond. They are coupon indexed bonds that pay semi-annual coupons, the indexation is based on cumulative movements in the CPI.

**SBIL** : Greek inflation indexed bonds or State Bonds Index Linked.

**TIPS** : US Treasury Inflation-Protected Securities, are securities whose principal is tied to the inflation rate. They bear a stated interest rate, and the owner receives semi-annual interest payments. They are issued in terms of 5, 10, and 20 years.

**Wiener process** : A stochastic process $W$ that meets the following conditions

1. $W(0) = 0$
2. The process $W$ has independent increments, i.e. if $r < s \leq t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.
3. For $s < t$ the stochastic variable W(t)-W(s) has the Gaussian distribution $N(0, \sqrt{t-s})$
4. W has continuous trajectories
5. Stop rule

# Appendix B

# Analysis of Historical Data

We include some graphs to illustrate the data we are using in this project.

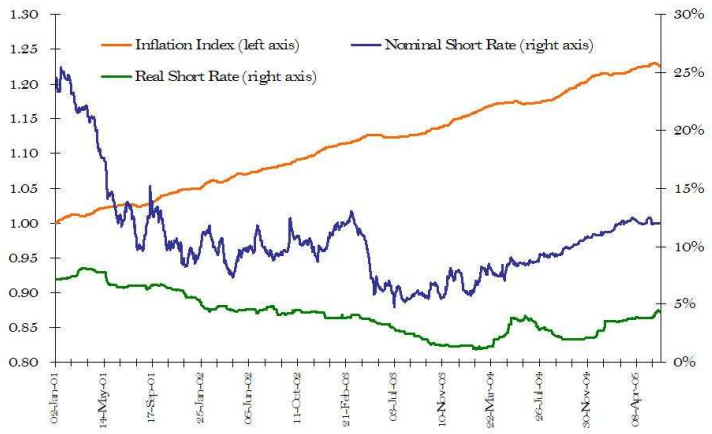



Figure B.1: Plot of nominal and real short rates, and inflation index in Mexico from 02-01-2001 to 10-06-2005. The inflation index (INPC) was standardised to be 1 at the beginning of the series. Source: Banco de Mexico and Valmer.

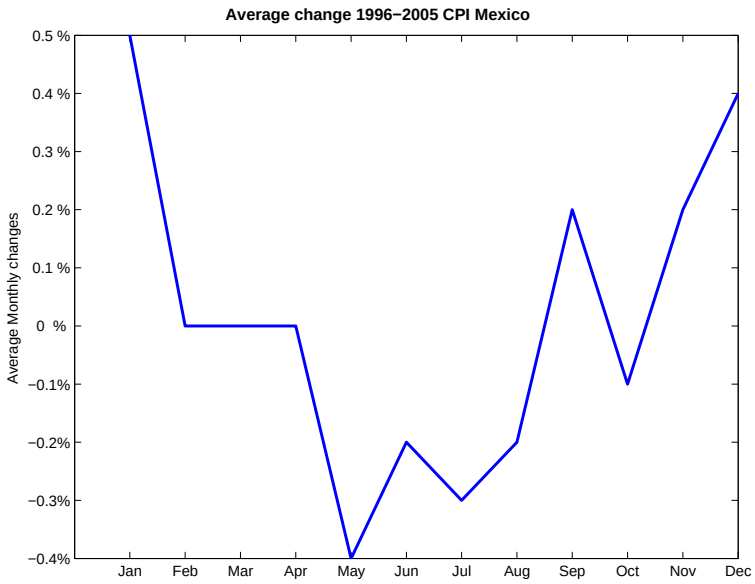



Figure B.2: Average monthly variation inflation index in Mexico 1996-2005. Source:Banco de Mexico.

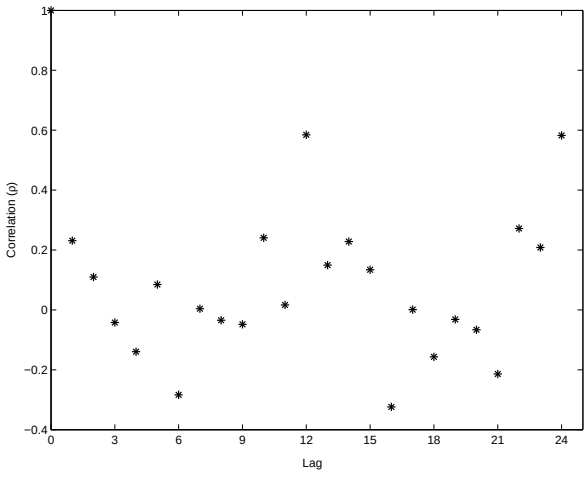



Figure B.3: Auto-correlation function of Inflation, shows correlation of the variable with lagged values.

# Appendix C

# Analysis of Results

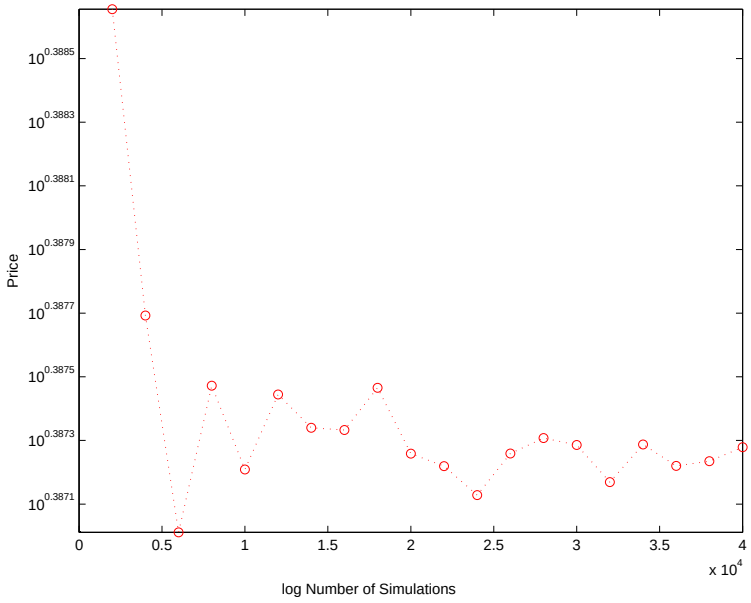



Figure C.1: Convergence price of European swaption increasing number of simulations up to 40,000 using quasi-Monte Carlo (Faure sequences) in 3 dimensions.

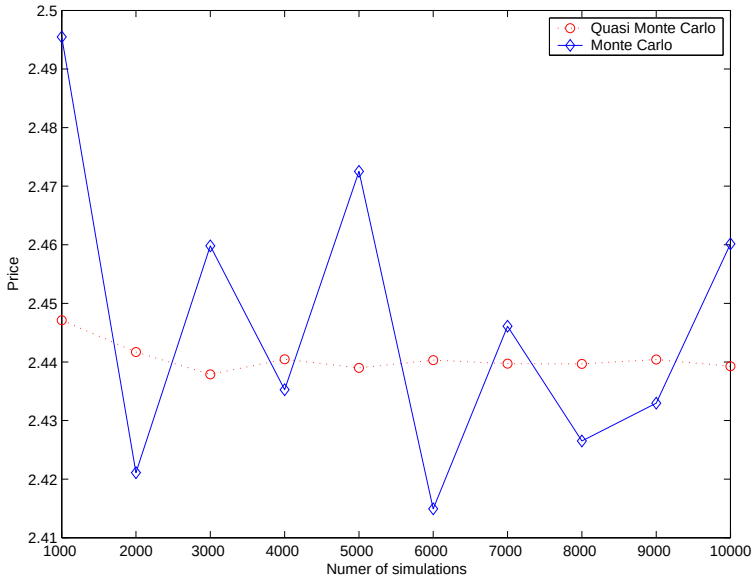



Figure C.2: Convergence price of European swaption increasing number of simulations. Monte Carlo with variance reduction technique (antithetic sampling) and quasi-Monte Carlo (Faure sequences)

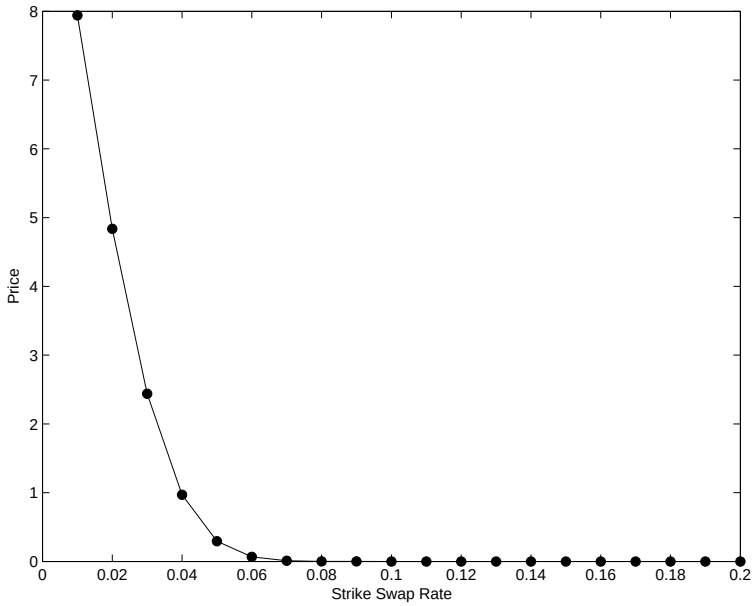



Figure C.3: Price of swaption increasing the strike K.

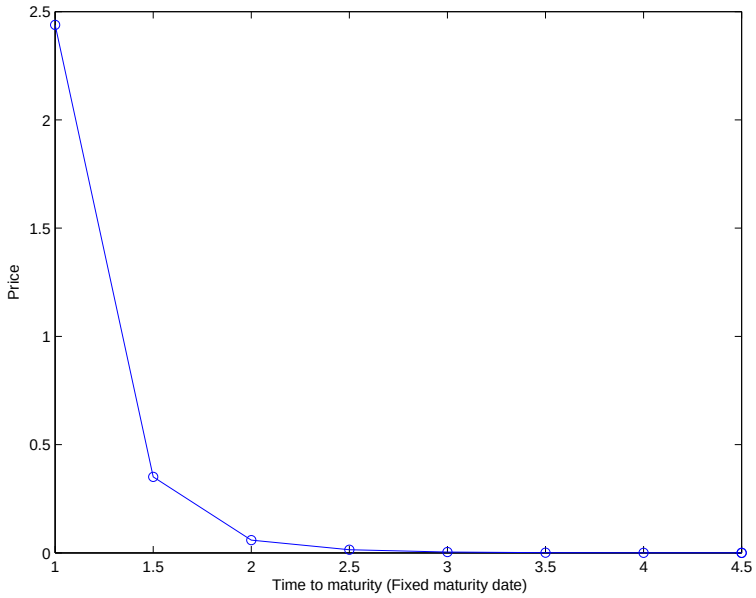



Figure C.4: Price of swaption increasing the maturity of swaption, fixed maturity date of underlying swap.

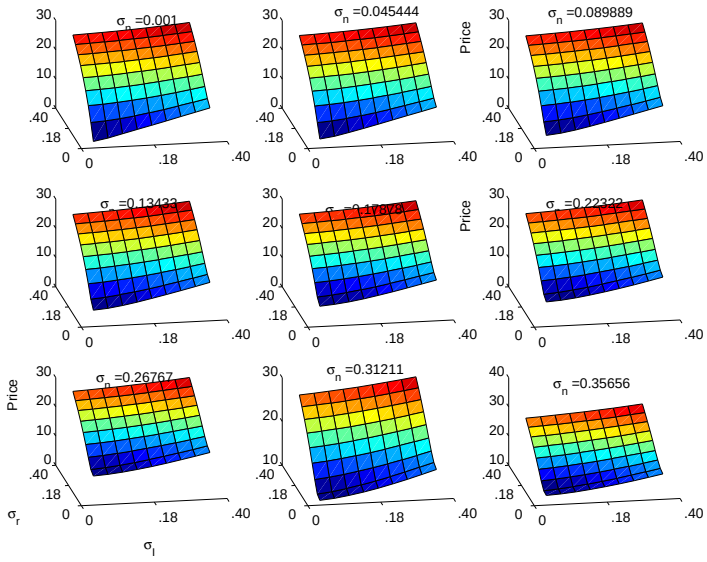



Figure C.5: Surface of price varying the volatility of the 3-Factors. Each surface corresponds to different value of volatility of the nominal rate, x-axis is volatility of the inflation index,y-axis volatility of the real rate.

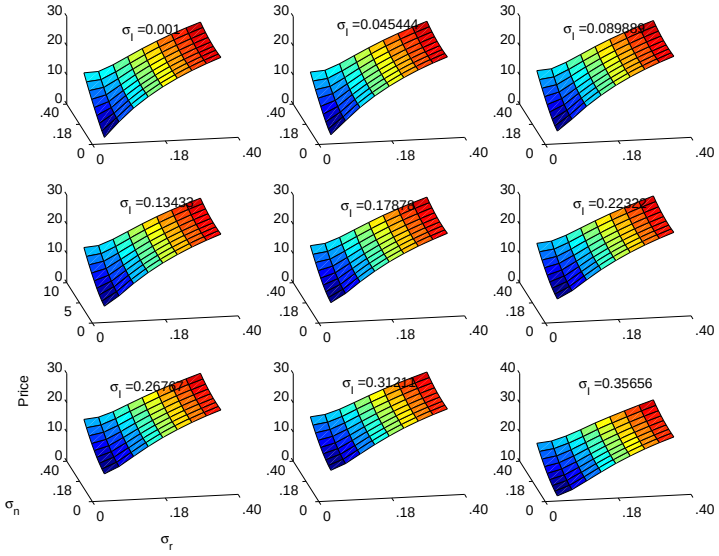



Figure C.6: Surface of price varying the volatility of the 3-Factors. Each surface corresponds to different value of volatility of inflation, x-axis is volatility of the real rate,y-axis volatility of the nominal rate.

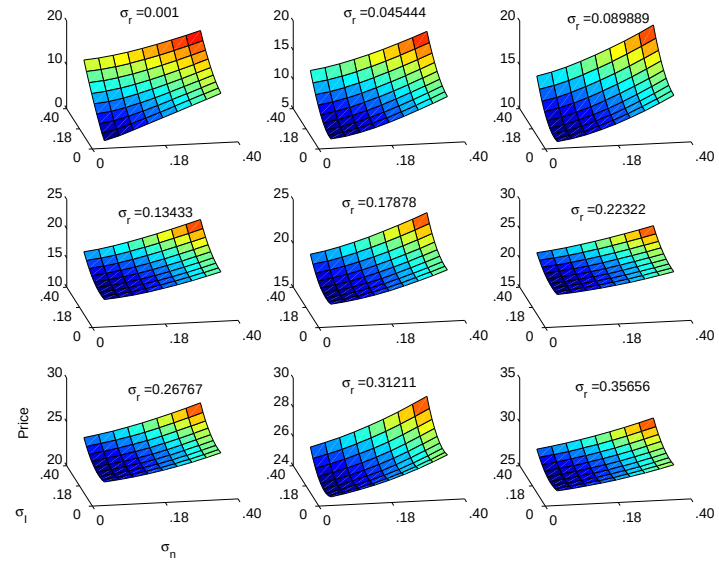



Figure C.7: Surface of price varying the volatility of the 3-Factors. Each surface corresponds to different value of volatility of the real rate, x-axis is volatility of the nominal rate,y-axis volatility of the inflation index.

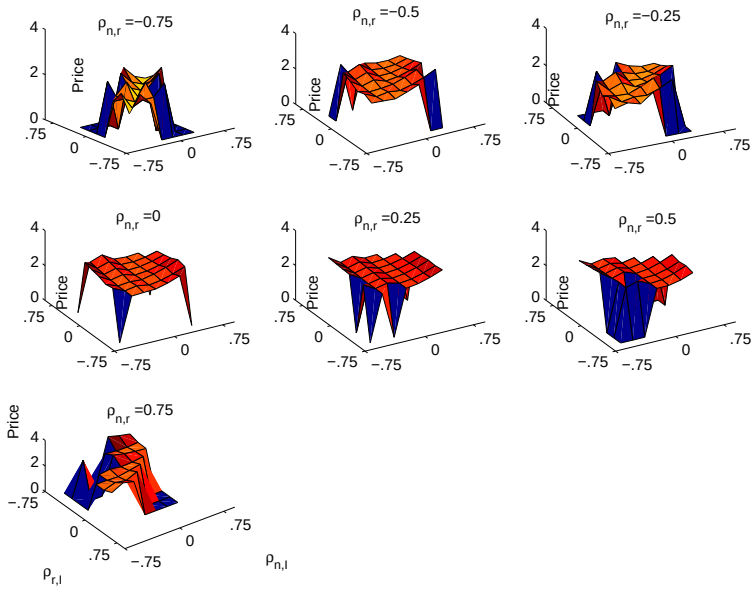



Figure C.8: Surface of price varying the correlation of the 3-Factors. Each surface corresponds to different value of correlation between the nominal and real rates.

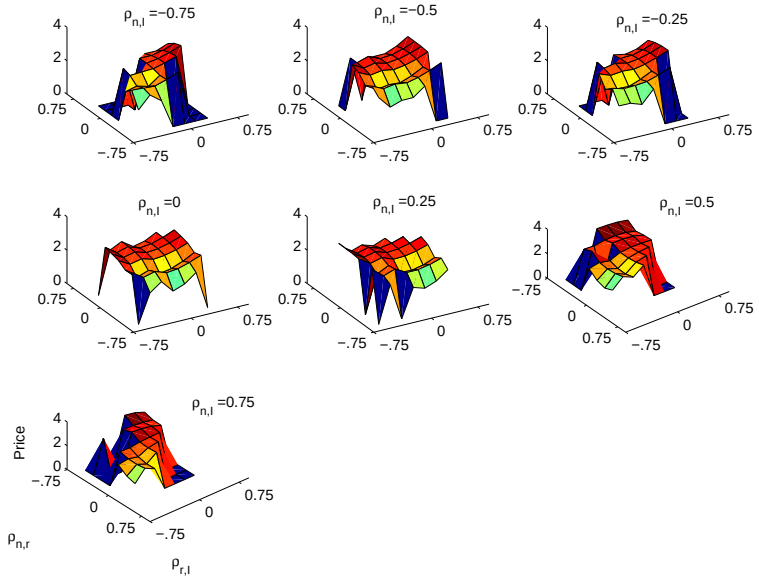



Figure C.9: Surface of price varying the correlation of the 3-Factors. Each surface corresponds to different value of correlation between the nominal and inflation index.

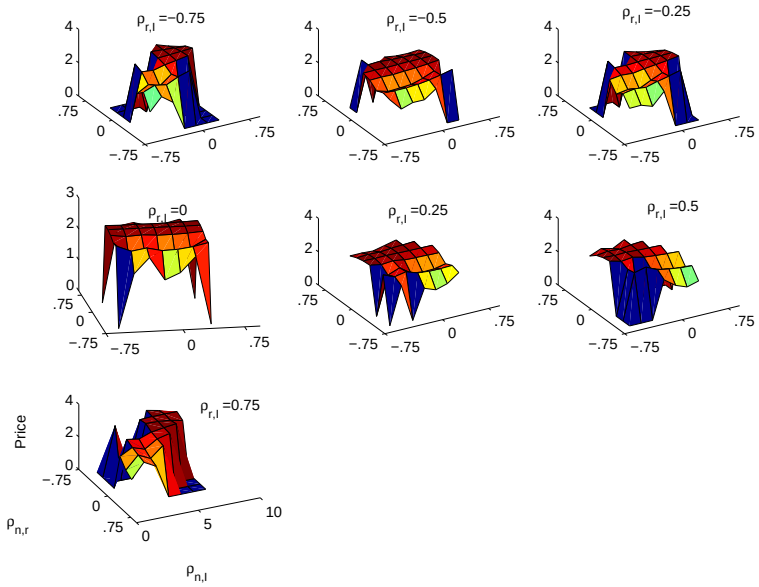



Figure C.10: Surface of price varying the correlation of the 3-Factors. Each surface corresponds to different value of correlation between the real rate and the inflation index.

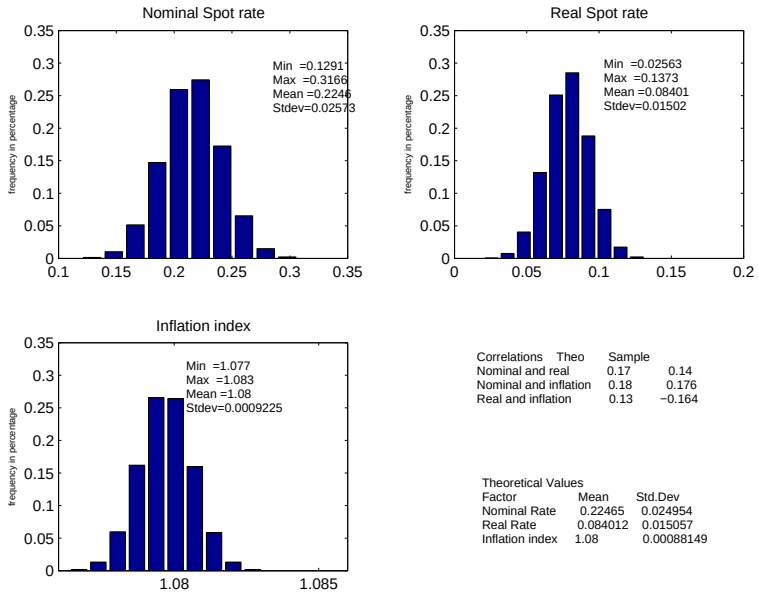



Figure C.11: Histogram of nominal short rate, real short rate and inflation index at maturity of option (1y), strike 3%, maturity of swap 5y.

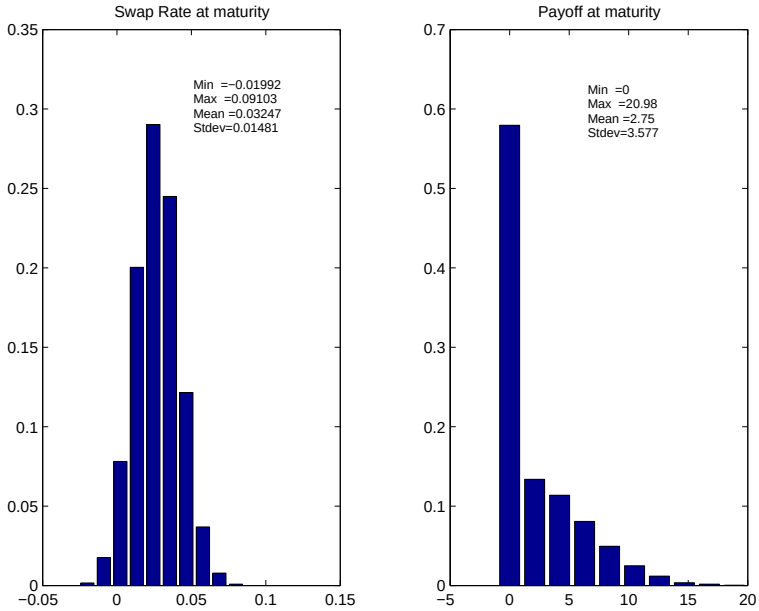



Figure C.12: Histogram of simulated payoff of the option and swap rate at the maturity of the option. Maturity of option (1y), strike 3%, maturity of swap 5y.

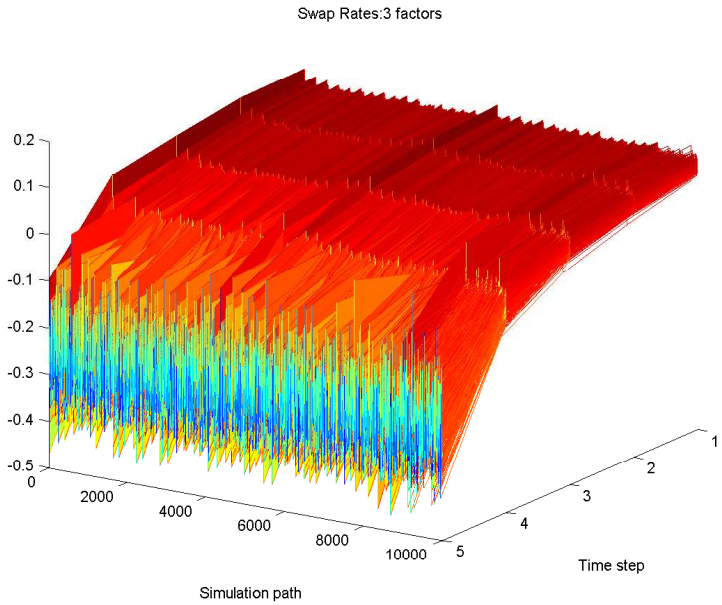



Figure C.13: Estimated paths of the swap rate used for the pricing of the Bermudan Swaption using 3 factors and market data.

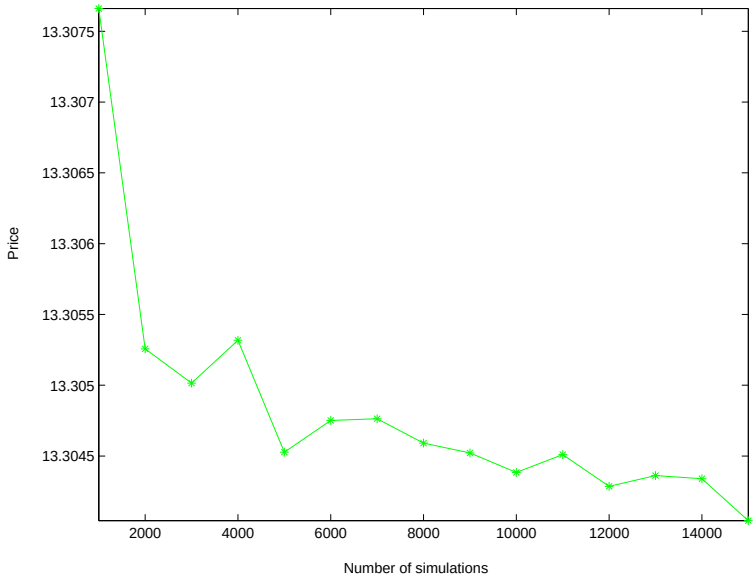



Figure C.14: Convergence of price of Bermudan swaption increasing the number of simulations in one factor model (nominal rate). The swaption has 5 annual exercise dates and strike 3%.

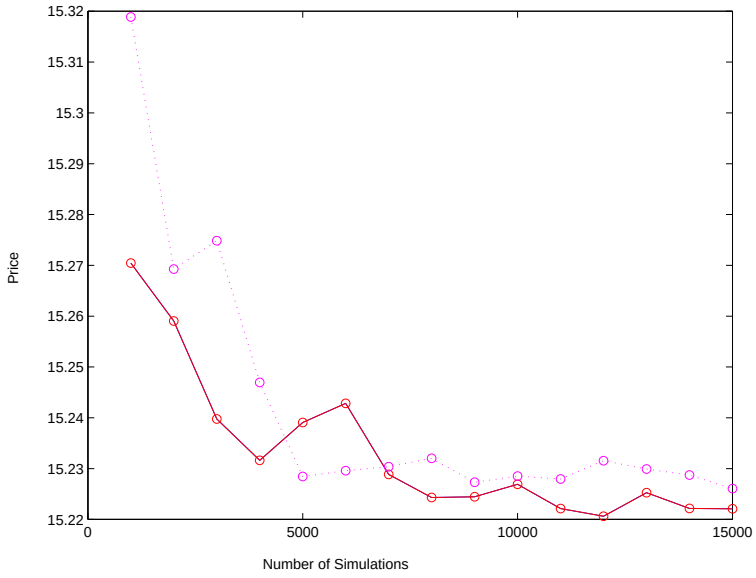



Figure C.15: Convergence of price of Bermudan swaption increasing the number of simulations in two factor model (nominal and real rates) and three factor model (nominal rate,real rate and inflation index).

# Appendix D

# Example Faure sequences

The following table shows the first 12 Faure sequences for three dimensional problem using base 3.

| $n$ | $F1$ | $F2$ | $F3$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1/3 | 1/3 | 1/3 |
| 2 | 2/3 | 2/3 | 2/3 |
| 3 | 1/9 | 4/9 | $\frac{7}{9}$ |
| 4 | 4/9 | $\frac{7}{9}$ | 1/9 |
| 5 | $\frac{7}{9}$ | 1/9 | 4/9 |
| 6 | 2/9 | $\frac{8}{9}$ | 5/9 |
| 7 | 5/9 | 2/9 | $\frac{8}{9}$ |
| 8 | $\frac{8}{9}$ | 5/9 | 2/9 |
| 9 | 1/27 | $\frac{16}{27}$ | $\frac{13}{27}$ |
| 10 | $\frac{10}{27}$ | $\frac{25}{27}$ | $\frac{22}{27}$ |
| 11 | $\frac{19}{27}$ | $\frac{7}{27}$ | $\frac{4}{27}$ |

The algorithm implemented to generate the Faure sequence can be found in [10] or in [9].

In order to compare the results obtained with a traditional Monte Carlo simulation (random uniform variates) we include Figure D.2. As we can visually notice the points tend to cluster and it would be necessary to generate more points to cover the cube.

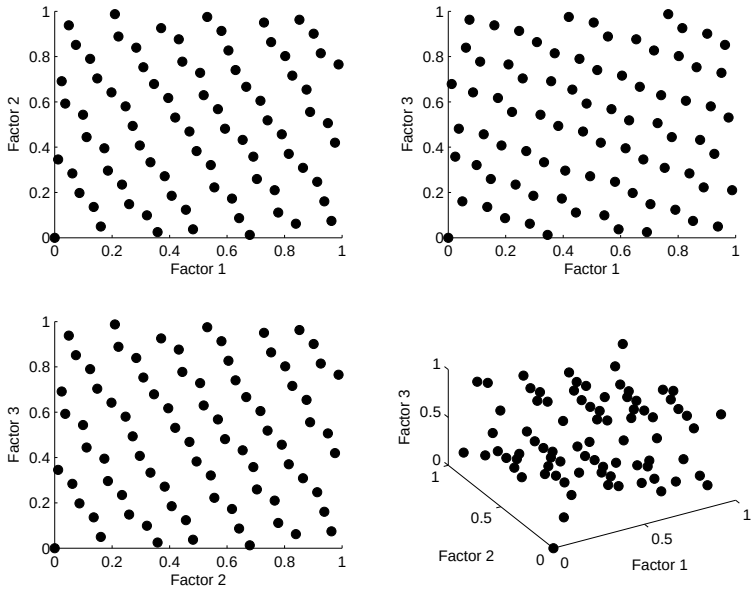



Figure D.1: Projections of 81 Faure points in 3 dimensions using base 3. The the figures show coordinates 1 and 2, 1 and 3, 2 and 3. Last 3-d scatter plot of points in the unit cube.

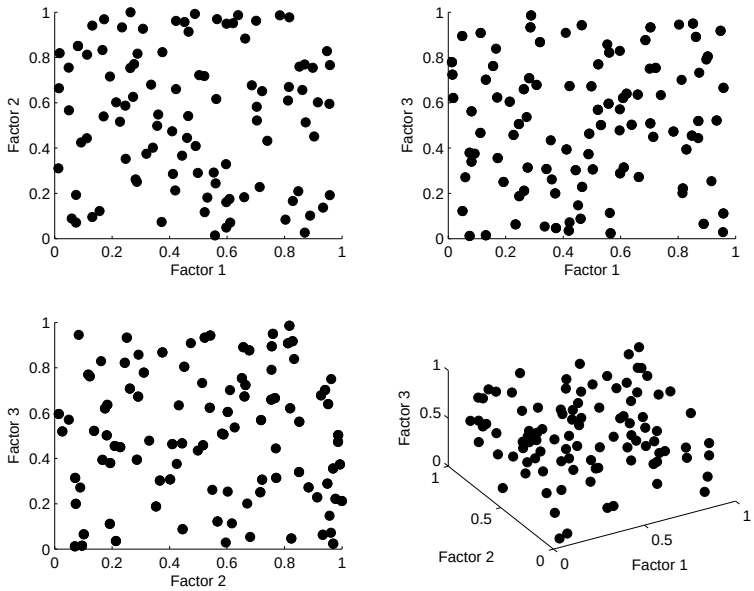



Figure D.2: Projections of 81 random uniform points in 3 dimensions. The figures show coordinates 1 and 2, 1 and 3, 2 and 3. Last 3-d scatter plot of points in the unit cube.

## D.1 The inverse cumulative normal function

The following is the code for Matlab of the algorithm proposed by Moro shown in [16] to evaluate the inverse cumulative normal function for $x \in (0, 1)$.

```
function w=inv_norm(x)
%This is B.Moro's algorithm for inversion of the normal distribution
% x vector defined between 0 and 1

a(1)=2.50662823884; a(2)=-18.61500062529; a(3)=41.39119773534;
a(4)=-25.44106049637; b(1)=-8.47351093090; b(2)=23.08336743743;
b(3)=-21.06224101826; b(4)=3.13082909833;

c(1)=0.3374754822726147; c(2)=0.9761690190917186;
c(3)=0.1607979714918209; c(4)=0.0276438810333863;
c(5)=0.0038405729373609; c(6)=0.0003951896511919;
c(7)=.0000321767881768; c(8)=0.0000002888167364;
c(9)=0.0000003960315187;

for i=1:length(x)
    y=x(i)-0.5;
    if(abs(y)<0.42)
        num=y.^2*[0:3]*a';
        den=y.^2*[1:4]*b'+1;
        w(i)=y*num/den;
    else
        if y<0
            r=x(i);
        else
            if (x(i)==1)
                r=1^(-10);
            else
            r=1-x(i);
        end
        end
    s=log(-log(r));
    t=s.^[0:8]*c';
    if x(i)>.5
       w(i)=t;
    else
       w(i)=-t;
    end
    end
end
```

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