Singular Instantons
and
Parabolic Bundles
over
Complex Surfaces

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Thesis submitted in partial fulfillment
of the requirements of the degree of
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Abstract

Let (X, Σ) be a pair consisting of a Riemannian 4–manifold and an embedded surface and G a compact connected Lie group. A singular G–instanton on $X \backslash \Sigma$ is one not extending to X and the singularity is encoded in a conjugacy class in G, the holonomy along Σ.

When $X = \mathbb{C}^2$ and Σ is a complex line we show that if the holonomy along Σ has a certain property there is a natural correspondence between euclidean G–instantons on $\mathbb{C}^2 \backslash \Sigma$ and suitable holomorphic objects on $(\mathbb{P}^2, \Sigma, D_\infty)$ called parabolic framings, where $D_\infty = \mathbb{P}^2 \backslash \mathbb{C}^2$. As a corollary we obtain a classification of G–instantons on $S^4 \backslash S^2$ for the round metric in terms of parabolic framings which in the trivial case of no singularities gives a new proof of Donaldson Theorem on framed bundles [D1], and a classification of hyperbolic G–monopoles with arbitrary mass and charge in terms of based rational maps of $\mathbb{P}^1$ into flag manifolds proving conjectures of Atiyah [A1] and Garland–Murray [GM].

Using the techniques developed and Simpson existence Theorem, [S], when (X, Σ) is a compact complex Kähler pair and $G = U(n)$ or $SU(n)$ we also obtain a natural correspondence between irreducible instantons on $X \backslash \Sigma$ and stable parabolic vector bundles on (X, Σ) proving the "Parabolic bundles conjecture" of Kronheimer [K].

We then construct moduli spaces of stable parabolic and framed parabolic bundles and of singular instantons and carry out a comparison of the deformation theories where the correspondence is proved showing that there always is an infinitesimal isomorphism which is a diffeomorphism at smooth points. As a consequence we obtain the smoothness of all moduli spaces of irreducible G–instantons on $S^4 \backslash S^2$, sufficient topological conditions for the existence of G–instantons on $S^4 \backslash S^2$ with prescribed holonomy along S^2 and necessary and sufficient topological conditions when $G = SU(n)$ proving a conjecture of Kronheimer–Mrowka [KM1]. We also deduce that the extended moduli spaces of instantons on $S^4 \backslash S^2$, where the holonomy along S^2 is allowed to vary with fixed centralizer, are products so that the diffeomorphism type of the moduli space depends on the holonomy along S^2 only up to its centralizer in G. Corresponding results for hyperbolic monopoles also follow.

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Introduction

0.1 An overview

Let (X,ω) be a complete complex Kähler surface. Over the last fifteen years there has ben a considerable amount of effort aimed at understanding the relation between instantons on (X,ω)—connections on a bundle on X which are finite action solutions to the anti–self–duality equations—and the holomorphic geometry of X or a completion of it, $\overline{X}$ (in the algebro–geometric sense). The crucial link was provided by the fact that in this case the anti–self–duality equations are equivalent to another system of equations, the Hemitian–Yang–Mills (H.Y.M.) equations, which are very natural from the point of view of Kähler geometry.

When (X,ω) is compact this relation is by now well understood, at least for structure group $U(n)$, and for irreducible $U(n)$–solutions there is a satisfactory body of knowledge concerning the solvability of the equations, number of moduli and properties of the parameter space in terms of stable holomorphic vector bundles on X. This is the so called Hitchin–Kobayashi correspondence of [D4], [UY] and [S]. When (X,ω) is not compact the picture is not clear in general but there is a large class of examples where $Vol(X)$ has a certain specific growth, in particular it is infinite, for which a good understanding of what is to be expected and a correspondence with suitable holomorphic bundles on $\overline{X}$ exist. These are the so called cone–like metrics on complex surfaces X which are the complement of a smooth curve D_∞ (thought of as a divisor at infinity) with positive self–intersection number in a compact complex surface $\overline{X}$ ([D1], [Ba]). In each of these cases the correspondence can be phrased in terms of framed bundles—bundles equipped with an isomorphism to a given holomrphic bundle on D_∞. In particular in the case of Donaldson Theorem ([D1]) these are holomorphic bundles with a fixed holomorphic trivialisation on a complex line in $\mathbb{P}^2$.

We are then naturally led to ask whether a correspondence with suitable holomorphic objects can be established when the Kähler surface considered is non–complete, at least in the case where the non–completeness is of a very specifc type, namely it arises from having removed a complex submanifold (more generally subvariety) Σ from a complete (X,ω). When Σ is zero–dimensional—a set of isolated points—we are reduced to the complete case due to Uhlenbeck removable singularities Theorem. When Σ is a complex curve, however, new singular solutions not extending to X may arise due to the appearance of a non–trivial obstruction to extending the connection across Σ, the holonomy along Σ (Definition 0.1). One of the central aims of this thesis is to understand some aspects of the relation between these solutions and suitable holomorphic objects on X or $\overline{X}$ precisely through the holonomy along Σ.

While the anti–self–duality equations are special to real dimension 4 the H.Y.M. equations make sense in all complex dimensions and a good idea of what to expect of instan-

tons with complex codimesion 1 singularities, at least on a compact X, can be garnered by studying the corresponding problem in one dimension less, namely H.Y.M. connections with point singularities on a curve. For these (more precisely for the irreducible ones) there is an interpretation in terms of parabolic stable bundles ([B1], [B2], [S]). The concepts of parabolic vector bundle and stable parabolic vector bundle readily generalize to higher dimension (see §0.1, §2.2 and §2.5) and in the case of a compact X we can therefore expect the latter to be the holomorphic objects on X in terms of which instantons on $X\backslash\Sigma$ can be classified. In the $SL(2)$–case this is the "Parabolic bundles conjecture" of [K] which has been proved in [KM2] in the special case of rational holonomies along Σ. Using Simpson existence Theorem, [S], and Råde recent local regularity result, [R1], we shall extend this to the $GL(n)$–case for arbitrary holonomies when $\Sigma \cdot \Sigma \geqslant 0$. An independent and more general proof has been given also by Biquard [B4].

In the case of non–compact X of the above form $X = \overline{X}\backslash D_\infty$ we shall also assume that Σ and D_∞ meet transversely. Whereas in complex dimension 1, due to the conformal equivalence between a punctured disc and a semi–infinite cylinder, there is no difference between the behaviour of solutions near a puncture or "at infinity", in complex dimension 2 the restriction of ω to a tubular neighbourhood of Σ is not in general conformal to a cone–like metric so we can expect the behaviour of the solutions "at infinty", that is along D_∞, to be fairly different from that along Σ, in particular, away from the intersection we expect the behaviuor of the solutions along Σ to resemble that of instantons on $X\backslash\Sigma$ for a compact X and that along D_∞ to resemble the behaviour of instantons on a complete cone–like surface with some compatibility condition at the intersection. We are thus led to introduce the notion of a framed parabolic bundle or parabolic framing on $(\overline{X}, \Sigma, D_\infty)$ (Definition 0.9) as a natural candidate for the objects in terms of which solutions on $\overline{X}\backslash\Sigma\cup D_\infty$ can be classified. This notion makes sense at the principal bundle level and for later applications we develop the theory for general complex reductive Lie groups. Here we shall prove that a correspondence with these objects exists only in the case of euclidean instantons on $\mathbb{C}^2\backslash\Sigma$, where Σ is a complex line and the triple involved is $(\mathbb{P}^2, \Sigma, D_\infty)$, with $D_\infty = \mathbb{P}^2\backslash\mathbb{C}^2$. Thanks to the very explicit control that one has on the growth of the euclidean metric on one hand and to the topological simplicity of ND_∞^*, the punctured normal bundle of D_∞, on the other this case turns out to be particularly simple and somewhat degenerate, but most of the conceptual features of the "general" case are already present in it. In particular the two key ingredients of the proof, the solvability of the H.Y.M. equation on $\mathbb{C}^2\backslash\Sigma$ using Donaldson local result [D3] and the extension of H.Y.M. holomorphic bundles across $\Sigma\cup D_\infty$ using [R1], should not be too difficult to generalize once [R1] is brought to bear to understand the behaviour of singular instantons on a punctured half–cylinder(see §1.3). We shall explain in the Conclusions how it should then be possible to extend the correspondence to a natural class of surfaces which are in a sense more general than the cone–like metrics.

A perhaps surprising feature of both correspondences which we couldn't satisfactory explain except for the $SU(n)$–case is that they do not seem to hold for instantons having certain particular holonomies along Σ if one wishes to formulate them at the level of G–bundles rather than just for vector bundles (in the case of stable bundles G is presently restricted to $U(n)$ or $SU(n)$). For this reason we found it necessary to introduce a notion

of admissibility (Definition 0.3 (ii)) for an element in a compact Lie group G and it is only for instantons with admissible holonomy along Σ that we could prove the Theorems at the G–bundle level, that is the correspondences are in terms of holomorphic $G^{\mathbb{C}}$–bundles.

In the case of compact (X,ω) moduli spaces of irreducible instantons on one hand and of stable vector bundles on the other exist and the correspondence is known to refine to an isomorphism of real ringed spaces. Similarly, for the non–compact case, moduli spaces of instantons on the one hand ([T], [DFK]) and of framed bundles on $(\overline{X}, D_\infty)$ on the other ([Le], [Lu]) are known to exist under suitable conditions and in the case of Donaldson Theorem the correspondence is again an isomorphism of ringed spaces. We therefore found it natural to ask whether moduli spaces of the objects on both sides of the correspondences we are proving exist and whether these correspondences also admit refinement to isomorphisms of ringed spaces. Generalizing the results of [KM1] on the side of connections and developing a suitable gauge theoretic framework on the side of the parabolic bundles and framings we answer affirmatively the first of these questions, but unfortunately we can only give a partial answer to the second in that we only show that in the case of the correspondences we prove there is an infinitesimal isomorphism of the deformation theories which is a diffeomorphism at smooth points. This is however quite adequate for a number of applications, especially in the case of framings, which gave some of the main motivation for studying the problem in the first place.

Two such applications were of particular interest to us. The first was the study of singular G–istantons on $S^4\backslash S^2$ with the round metric, where S^2 is a standard smoothly embedded 2–sphere and G a compact connected Lie group. Much as for instantons on S^4 in the case of ordinary gauge theory these are in some sense the building blocks of gauge theory with real codimension 2 singularities and we wanted to obtain results analogous to those of [AHS] on the non–emptyness, smoothness and dimension on the moduli spaces of irreducible G–instantons on $S^4\backslash S^2$ with prescribed singularity along S^2 and given topological invariants (these are defined in §1.4). At present these information would seem quite difficult to obtain directly by purely real methods, but become rather more accessible when sought in terms of analogous properties of the moduli spaces of parabolic framings on $(\mathbb{P}^2, \Sigma, D_\infty)$ via the refinement of the correspondence at the level of ringed spaces and the identifications of instantons on $S^4\backslash S^2$ with euclidean instantons on $\mathbb{C}^2\backslash\Sigma$.

The second main application was to obtain a new and more natural proof of the well–known result of Atiyah, [A1], that framed hyperbolic G–monopoles with G a classical group and integral mass $[i\alpha]$ (see §0.3 for the definition of this) correspond to based rational maps $\mathbb{P}^1 \to G/C(i\alpha)$, where $C(i\alpha)$ is the centralizer of $i\alpha$ under the adjoint action, which would extend to all masses and compact connected groups, thereby proving conjectures of Atiyah, [A1], and Garland–Murray, [GM]. Since instantons on $\mathbb{R}^4\backslash\mathbb{R}^2$ which are S^1–invariant under the S^1–action by rotation fixing $\mathbb{R}^2$ can be identified with hyperbolic monopoles (see §0.3), whereas $\mathbb{C}^*$–equivariant parabolic framings can be easily identified with based rational maps into suitable flag manifolds (see Chapter 6, Corollary 6.13), the desired proof can be obtained from the correspondence Theorem provided one shows that the S^1–equivariant objects on one hand and the $\mathbb{C}^*$–equivariant ones on the other correspond. From the more refined version of the correspondence it can then also be deduced that this is an isomorphism of ringed spaces yelding information on the smoothness, non–emptyness and

dimension on the moduli spaces of hyperbolic G–monopoles as well.

0.2 Some definitions

Before we can state precisely the results proved it is necessary to introduce a few definitions. Let (X^4, Σ^2) be a pair consisting of a connected 2–manifold inside a Riemannian 4–manifold and let A be a finite action connection on a principal G–bundle $E \to X\backslash\Sigma$, where G is a compact connected Lie group. Fix a point p$\in \Sigma$ and let D_p be a normal disc to Σ at p. On D_{p} consider polar coordinates (r, θ) and let γ_r be the circle of radius r centred at p. Let $[g^r_{\mathrm{p}}]$ be the conjugacy class in G determined by the parallel transport of A along γ_r. Under the finite action assumption

$$[J(A)](\mathrm{p}) = \lim_{r\to 0}[g^r_{\mathrm{p}}]. \tag{0.1}$$

exists for almost all p, is independent of p, [SS], and it is called the holonomy of A along Σ. Clearly A has an extension to a G–connection on a bundle on X only if $[J(A)](\mathrm{p}) = 1$ for all p and this is the obstruction in terms of which the classification of instantons on $X\backslash\Sigma$ will be achieved. We shall denote by

$$M^{G,J}(X,\Sigma), \qquad M^{G,J,*}(X,\Sigma), \tag{0.2}$$

the set of (smooth) gauge equivalence classes of G–instantons and of G–instantons with stabilizer $Z(G)$ on $X\backslash\Sigma$ respectively with holonomy J along Σ. We shall often omit G and the pair (X, Σ) from the notation. As we remarked above in order for the correspondence Theorems to hold at the level of G–bundles it is necessary to impose some restrictions on the holonomy along Σ.

Definition 0.3. *Let G be a compact connected Lie group with lie algebra $\mathfrak{g}$.*

(i) *An element $i\alpha \in \mathfrak{g}$ is said to be admissible if it can be written in the form*

$$i\alpha = i\alpha' + im,$$

where $ad(\alpha')$ has all its eigenvalues in $(-1, 1)$ and $m \in Ker(\exp(2\pi i -))$.

(ii) *An element $J \in G$ is said to be admissible if there exist an admissible element $i\alpha \in \mathfrak{g}$ with $\exp(2\pi i\alpha) = J$.*

It will follow from Lemma 2.8 that the representation in (i) is actually unique. Note that if an element in a conjugacy class in G or $\mathfrak{g}$ (under the adjoint action) is admissible then all elements in it are and we say that the conjugacy class is admissible. It thus makes sense to say that a connection has admissible holonomy along Σ.

To every conjugacy class $[J]$ in G we uniquely associate a logarithm in $\mathfrak{g}$, as follows. Choose once and for all a pair $(\mathfrak{h}, \mathfrak{h}^+)$ consisting of a Cartan subalgebra (C.S.A.) $\mathfrak{h} \subseteq \mathfrak{g}$ and a closed positive Weyl chamber $\mathfrak{h}^+$. For any conjugacy class $[J]$ in G there is a unique element of the form $J = \exp(-2\pi i\alpha)$ with $i\alpha \in \mathfrak{h}^+$ and $ad(\alpha)$ having all its eigenvalues in $[-1, 1]$ (this will be proved in Lemma 2.8) and we call this preferred $i\alpha$ the logarithm of J,

$$i\alpha = \log(J). \tag{0.4}$$

A conjugacy class $[J]$ is then admissible if and only if $i\alpha = \log(J)$ is such that α has all its eigenvalues in $(-1, 1)$. Whereas for $G = U(n)$ such an α can always be found (see §1.4), that is every element is admissible, for general G this fails to be true.

Now let $G^{\mathbb{C}}$ be a complex reductive Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Given an element $i\alpha \in \mathfrak{g}^{\mathbb{C}}$ lying in some compact form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ we can associate to it a parabolic subgroup $P(\alpha)$ as follows. Let $\mathfrak{p}(\alpha) \subseteq \mathfrak{g}^{\mathbb{C}}$ be defined by $\mathfrak{p}(\alpha) = \mathfrak{c}(\alpha) \oplus \mathfrak{u}(\alpha)$ where

(i) $\mathfrak{c}(\alpha) = Lie(C(i\alpha)) = Ker(ad(\alpha))$,

(ii) $\mathfrak{a}d((\alpha))_{|\mathfrak{u}(\alpha)} > 0$.

We explain in §2.1 why this is actually a parabolic subalgebra and we let

$$P(\alpha) = \exp(\mathfrak{p}(\alpha)), \tag{0.5}$$

be the associated parabolic subgroup.

We can now introduce the holomorphic objects that will play the role of the framed bundles of Donaldson theorem [D1] in our situation. Let (X, Σ) be a pair consisting of a compact complex manifold X^n (we shall be interested in $n = 1$ or 2) and a codmension one submanifold Σ—a smooth divisor—having irreducible components $\{\Sigma^j\}$. Let $G^{\mathbb{C}}$ be a complex reductive Lie group and let $P = \{P^j\}$ be a collection of parabolic subgroups of $G^{\mathbb{C}}$ one for each Σ^j.

Definition 0.6. A P–quasi–parabolic bundle on (X, Σ) consists of a pair $(\mathcal{E}, s)$ where

(i) $\mathcal{E}$ is a holomorphic $G^{\mathbb{C}}$–bundle on X

(ii) $s = (s^1, \cdots, s^j)$ are holomorphic reductions of the structure group of $\mathcal{E}_{|\Sigma^j}$ to P^j, namely holomorphic sections of the associated analytic $G^{\mathbb{C}}/P^j$–fibre bundle on Σ^j.

When we want to stress what the structure group of $\mathcal{E}$ is we shall refer to a P–quasi–parabolic bundle as to a $(G^{\mathbb{C}}, P)$–quasi–parabolic bundle.

Definition 0.7. An α–parabolic bundle on (X, Σ) consists of a triple $(\mathcal{E}, s, \alpha)$, where $i\alpha = \{i\alpha^j\} \subseteq \mathfrak{g}^{\mathbb{C}}$ is a set of elements belonging to some compact form of $\mathfrak{g}^{\mathbb{C}}$ and $(\mathcal{E}, s)$ is a $P(\alpha) = \{P(\alpha^j)\}$–quasi–parabolic bundle on (X, Σ).

When $G^{\mathbb{C}} = GL(n)$ these definitions are easily seen to be equivalent to the ordinary definitions of quasi–parabolic and α–parabolic vector bundle (see §2.2). In particular for $G^{\mathbb{C}} = GL(n)$ or $SL(n)$ the notion of α–stability can be defined generalizing the definition of [K] for $SL(2)$, see §2.5.

For fixed P two P–quasi–parabolic bundles $\mathcal{E}$, $\mathcal{E}'$ are isomorphic if there is an isomorphism of the underlying holomorphic $G^{\mathbb{C}}$–bundles $g : \mathcal{E} \cong \mathcal{E}'$ intertwining the reductions, that is

$$g_{|\Sigma}(s) = s'.$$

We denote by

$$\mathcal{M}^P(X, \Sigma), \qquad \mathcal{M}^\alpha(X, \Sigma), \qquad \mathcal{M}^{\alpha\text{-}S}(X, \Sigma), \tag{0.8}$$

the set of equivalence classes of P–quasi–parabolic, α–parabolic and stable α–parabolic bundles respectively.

Now assume that $n = 2$, let $D_\infty \subseteq X$ be a further smooth divisor intersecting Σ transversely and let $\mathcal{E}^0$ be a P–quasi–parabolic bundle or α–quasi parabolic bundle on $(D_\infty, \Sigma \cap D_\infty)$. Note that given a P–quasi–parabolic or α–parabolic bundle on (X, Σ) we obtain by restriction a P–quasi–parabolic or α–parabolic bundle on $(D_\infty, \Sigma \cap D_\infty)$.

Definition 0.9.

(i) An $\mathcal{E}^0$–framed quasi–parabolic bundle on (X, Σ, D_∞) consists of a triple $(\mathcal{E}, s, \varphi)$, where $(\mathcal{E}, s)$ is a P–quasi–parabolic bundle on (X, Σ) and φ is an isomorphism of P–quasi–parabolic bundles on $(D_\infty, \Sigma \cap D_\infty)$,

$$\varphi : \mathcal{E}_0 \cong \mathcal{E}_{|D_\infty}.$$

(ii) An $\mathcal{E}^0$–framed parabolic bundle (or parabolic framing) on (X, Σ, D_∞) consists of data $(\mathcal{E}, s, \varphi, \alpha)$, where $(\mathcal{E}, s, \alpha)$ is an α–parabolic bundle on (X, Σ) and φ is an isomorphism of α–parabolic bundles on $(D_\infty, \Sigma \cap D_\infty)$,

$$\varphi : \mathcal{E}_0 \cong \mathcal{E}_{|D_\infty}.$$

For fixed $\mathcal{E}^0$ two framed quasi–parabolic principal bundles $(\mathcal{E},\ s,\ \varphi)$ and $(\mathcal{E}',\ s',\ \varphi')$ are equivalent if there is an isomorphism of the underlying quasi–parabolic bundles $g : \mathcal{E} \cong \mathcal{E}'$ intertwining the framings, that is

$$g_{|D_\infty} \circ \varphi = \varphi'$$

The same notion of equivalence holds for parabolic framings and we denote the set of equivalence classes of such objects by

$$\widetilde{\mathcal{M}}^P(X, \Sigma), \qquad \widetilde{\mathcal{M}}^\alpha(X, \Sigma),$$

respectively. In the sequel we shall often omit the pair (X, Σ) from the notation.

It is worth remarking at this point that $\widetilde{\mathcal{M}}^\alpha(X, \Sigma)$ depends on α only through $P(\alpha)$, so in effect it can be identified with $\widetilde{\mathcal{M}}^{P(\alpha)}(X, \Sigma)$. As it will become apparent in the sequel, however, the extra choice of the compact element $i\alpha$ in $\mathfrak{g}^{\mathbb{C}}$ is essential to establish a correspondence theorem and in particular Theorem 1.

Let P be a parabolic subgroup of $G^{\mathbb{C}}$. One family of parabolic framings we shall be particularly interested in is that of framings on the triple $(\mathbb{P}^2, \Sigma, D_\infty)$, where Σ and D_∞ are distinct complex projective lines, and $\mathcal{E}^0$ is the trivial framing consisting of the product bundle $G^{\mathbb{C}} \times D_\infty$ with the product subbundle $P \times D_\infty$.

When (X, Σ) is a compact connected orientable pair with Σ irreducible and G is a compact connected simple Lie group or $U(n)$, we shall explain in §1.4 and §2.3 respectively that the moduli spaces (0.2) and (0.8) are partitioned by topological types in terms of invariants

$$(\kappa, c, \ell) \in H^4(X, \mathbb{Z}) \times H^2(X, \pi_1(G)) \times \overset{t}{\oplus} H^2(\Sigma, \mathbb{Z}),$$

as

$$M^J = \bigcup_{\kappa,c,\ell} M^J_{\kappa,c,\ell}, \qquad \mathcal{M}^P = \bigcup_{\kappa,c,\ell} \mathcal{M}^P_{\kappa,c,\ell}.$$

and similarly in the case of framings. In particular for the pair (S^4, S^2) and the above triple $(\mathbb{P}^2, \Sigma, D_\infty)$ with the trivial framing c always vanishes so there is only a pair of topological invariants κ, ℓ which can be identified with a pair

$$(\kappa, \ell) \in \pi_3(G) \times \pi_2(G/C(J)).$$

Exploiting the conformal invariance of the anti–self–duality equations we define topological invariants for instantons on $\mathbb{C}^2 \setminus \Sigma$ by identifying them with instantons for the round metric on $S^4 \setminus S^2$.

0.3 Statement of the Results

We are now in the position to state the main results. Let $\widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)$ denote the moduli spaces of $\mathcal{E}^0$–framed parabolic bundles where $\mathcal{E}^0$ is the trivial product framing described at the end of the previous section. Our first main Theorem is then

Theorem 1.

(i) *For every compact connected Lie group G and every admissible $J \in G$ there is a natural bijection*

$$M^J(\mathbb{C}^2, \Sigma) \cong \widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)/C(i\alpha),$$

where $i\alpha = \log(J)$.

(ii) *When G is simple for every topological type $\kappa, \ell \in \pi_3(G) \times \pi_2(G/C(J))$ the natural bijection in (i) restricts to a natural bijection*

$$M^J_{\kappa,\ell}(\mathbb{C}^2, \Sigma) \cong \widetilde{\mathcal{M}}^\alpha_{\kappa,\ell}(\mathbb{P}^2, \Sigma)/C(i\alpha)$$

As a corollary of Theorem 1 and of the conformal equivalence $(S^4, round) \sim (\mathbb{C}^2, euc)$ we then immediately deduce

Corollary 1.

(i) *For every compact connected Lie group G and every admissible $J \in G$ there is a natural bijection*

$$M^J(S^4, S^2) \cong \widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)/C(i\alpha),$$

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(ii) *When G is simple for every topological type $\kappa, \ell \in \pi_3(G) \times \pi_2(G/C(J))$ the natural bijection in (i) restricts to a natural bijection*

$$M^J_{\kappa,\ell}(S^4, S^2) \cong \widetilde{\mathcal{M}}^\alpha_{\kappa,\ell}(\mathbb{P}^2, \Sigma)/C(i\alpha)$$

We explain in §1.4 how, at least for $SU(n)$, the case of connections with non–admissible holonomy can be reduced to the admissible one thus classifying all singular $SU(n)$–connections. Note that in the trivial case, that is $J = 1$ and $P = G^{\mathbb{C}}$, we recover Donaldson Theorem on framed bundles, [D1], thus obtaining a new direct analytical proof of it.

As a consequence of some of the the techniques used in the proof of Theorem 1 and of Simpson existence Thorem we then deduce our second main Theorem.

Theorem 2. *Let (X, Σ, ω) be a compact complex Kähler pair such that $\Sigma \cdot \Sigma \geqslant 0$. Then*

(i) *There is a natural bijection*

$$M^{J,*}(X, \Sigma) \cong \mathcal{M}^{\alpha,S}(X, \Sigma),$$

where $i\alpha = \log(J)$.

(ii) *For any admissible $J \in SU(n)$ there is a natural bijection*

$$M^{SU(n),J,*}(X, \Sigma) \cong \mathcal{M}^{SL(n),\alpha,S}(X, \Sigma),$$

where $i\alpha = \log(J)$.

(iii) *For every topological type κ, c, ℓ the above bijection restricts to a bijection*

$$M^{J,*}_{\kappa,c,\ell}(X, \Sigma) \cong \mathcal{M}^{\alpha,S}_{\kappa,c,\ell}(X, \Sigma),$$

with the $SU(n)$–connections on the left corresponding to parabolic bundles with $c = c_1 = -Tr(\alpha)[\Sigma]$, where $i\alpha = \log(J)$. For admissible J an analogous statement can be formulated for $SU(n)$–topological invariants ($c = 0$).

As a corollary of Theorem 2 and of the Chern–Weil formula of Chapter 4 we also obtain

Corollary 2. *Let (X, Σ) be a compact connected complex Kähler pair such that $\Sigma \cdot \Sigma > 0$. Then every unitary representation of $\pi_1(X \backslash \Sigma)$ is the sum of irreducible projectively unitary representations of $\pi_1(X)$.*

The stratum $k = 0$, $\widetilde{\mathcal{M}}^{\alpha}_0(\mathbb{P}^2, \Sigma) = \bigcup_{0,\ell} \widetilde{\mathcal{M}}^{\alpha}_{0,\ell}(\mathbb{P}^2, \Sigma)$, is of special interest. On the one hand, for every $i\alpha \in \mathfrak{g}$, we can identify $\widetilde{\mathcal{M}}^{\alpha}_0(\mathbb{P}^2, \Sigma)$ with $\mathcal{R}^*(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))$, the space of based rational maps from $\mathbb{P}^1$ into $G^{\mathbb{C}}/P(\alpha)$, (Chapter 6, Proposition 6.12). On the other, when $\alpha = \alpha' + m$ with $m \in ker(\exp(2\pi i -))$ and $ad(\alpha')$ having all its eigenvalues in $[-1, 1]$, we show in Proposition 6.7 that $\widetilde{\mathcal{M}}^{\alpha}_0(\mathbb{P}^2, \Sigma)$ can be identified with the elements in a suitable $\widetilde{\mathcal{M}}^{\alpha'}_{k,\ell}(\mathbb{P}^2, \Sigma)$ whose underlying bundle is equivariant with respect to a weight m $\mathbb{C}^*$–action covering the $\mathbb{C}^*$–action on $\mathbb{P}^2$ obtained by extending the $\mathbb{C}^*$–action on $\mathbb{C}^2$ fixing Σ which complexifies the S^1–action by rotation in the planes normal to Σ. Because of the $\mathbb{C}^*$–equivariance of the holomorphic bundles, it is in turn possible to show that the corresponding connections are S^1–equivariant with respect to a weight m action. Let $\mathbb{H}^3$ denote hyperbolic 3–space of constant curvature –1. Let $E \to \mathbb{H}^3$ be a principal G–bundle. Recall that a hyperbolic G– monopole is a pair $(\tilde{A}, \phi)$ consisting of a G–connection on E and a section of the adjoint bundle $\mathfrak{g}_E$ of E satisfying the Bogomolny equations

$$d_{\tilde{A}}\phi = *F_{\tilde{A}},$$

and having finite Yang–Mills–Higgs action

$$\int_{\mathbb{H}^3} |F_{\tilde{A}}|^2 + |\nabla_A \phi|^2 < \infty.$$

Finite action implies (see §6.3) that in a suitable gauge the limit

$$\phi(\infty)\lim_{r\to\infty}\phi(r\underline{x})$$

exists for every direction $\underline{x}$ in $\mathbb{H}^3$ (thought of as being the hyperbolic 3-ball) and its conjugacy class $[\phi(\infty)]$ is independent of $\underline{x}$. We call it the mass of the monopole. A G–monopole is said to have integral mass if $\phi(\infty)$ belongs to the integral lattice of $\mathfrak{g}$. As for the limit holonomy of an instanton it makes sense to say that the mass of a monopole is admissible. We denote by $M^{\alpha}(\mathbb{H}^3)$ the set of equivalence classes of Hyperbolic monopoles with mass $[i\alpha]$ modulo gauge transformations. As it is well known, [A1], using the conformal equivalence

$$(\mathbb{R}^4\backslash\mathbb{R}^2, euc)\sim r^2(\mathbb{H}^3\times S^1, hyp\times round)$$

an S^1–equivariant G–instanton A on $(\mathbb{R}^4\backslash\mathbb{R}^2)$ descends to a hyperbolic G–monopole $(\widetilde{A},\phi)$ and vice–versa. A monopole has then admissible mass if and only if the corresponding S^1–equivariant instanton has admissible holonomy. Hyperbolic monopoles and rational maps both have a single topological invariant. For the former this is the magnetic charge, given by the class that the Higgs field determines in $\pi_2(G/C(i\alpha))$, and for the latter this is the class that the map determines in $\pi_2(G^{\mathbb{C}}/P(\alpha))$. Using the correspondence outlined above between rational maps and S^1–equivariant instantons we then deduce our third main Theorem.

Theorem 3.

(i) *For every compact connected goup G and every conjugacy class $[i\alpha]\in\mathfrak{g}$ there is a natural bijection*

$$M^{\alpha}(\mathbb{H}^3)\cong\mathcal{R}^*(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))/C(i\alpha).$$

(ii) *The above natural bijection restricts to a bijection*

$$M^{\alpha}_{\ell}(\mathbb{H}^3)\cong\mathcal{R}^*_{\ell}(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))/C(i\alpha).$$

The proof just outlined is best suited to the case of monopoles with admissible mass. In the non–admissible case some extra work is actually required to apply Theorem 1 and in Chapter 6 we shall also give a more direct approach to Theorem 3 which is insensitive to admissibility questions and best shows why $M^{\alpha}(\mathbb{H}^3)$ should depend on $\phi(\infty)$ only up to its centralizer. Since every connected subgroup C of a Lie group G as above is of the form $C(i\alpha)$ for some $i\alpha\in\mathfrak{g}$, this settles the conjecture made in [GM] that for every such C there should exist hyperbolic monopoles with C–symmetry breaking at infinity (that is such that the normalizer of $[\phi(\infty)]$ is C) and that their parameter space should be $\mathcal{R}^*(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))/C(i\alpha)$.

We then pass to consider moduli problems for the various objects considered. Besides the moduli space of singular instantons we consider a slightly different moduli space, $\widetilde{M}^J$, fibring over M^J with fibre $C(J)$ (see §1.6 for the definition) which will enter naturally in the comparison between the deformation thory of instantons on $\mathbb{C}^2\backslash\Sigma$ and that of parabolic framings on $(\mathbb{P}^2,\Sigma,D_{\infty})$. Generalizing the results of [KM1] we prove.

Theorem 4. *Let (X,Σ) be a compact connected orientable pair and G a compact connected simple Lie group or $U(n)$. Then*

(i) *For any Riemannian metric on X the moduli space of G–instantons with holonomy centralized by $Z(G)$ and admissible holonomy J along Σ, $M^{J,*}_{\kappa,c,\ell}$, has a natural real analytic space structure. If $G = SU(n)$ or $U(n)$ and $i\alpha = \log(J)$ has eiegenvalues with multiplicities $\{n_i\}$ and $\ell = (\ell_1,\cdots,\ell_t)$ with respect to the basis (2.5) for $[\Sigma, G/C(J)]$, its dimension at smooth points is given by*

$$\begin{aligned} d = dim(M^{J,*}_{\kappa,c,\ell}(X,\Sigma)) =& 4n\kappa + 2c^2 + (n^2-1)(1-b^1(X)+b^+(X)) \\ &+ 2\sum_{i=1}^{t}(n_{i+1}+n_i)\ell_i + (\sum_i^t n_i)c\cdot\Sigma + (n^2 - \sum_i^t n_i^2)\frac{\chi(\Sigma)}{2} \end{aligned}$$

and $d-(1-b_1(X))$ respectively.

(ii) *For any Riemannian metric on X the moduli space of framed G–instantons with admissible holonomy J along Σ, $\widetilde{M}^J_{\kappa,c,\ell}$, has a natural real analytic space structure. If $G = SU(n)$ or $U(n)$ and $i\alpha = \log(J)$ its dimension at smooth points is given by*

$$\begin{aligned} \widetilde{d} = dim(\widetilde{M}^J_{\kappa,c,\ell}(X,\Sigma)) =& 4n\kappa + 2c^2 + (n^2-1)(1-b^1(X)+b^+(X)) \\ &+ 2\sum_{i=1}^{t}(n_{i+1}+n_i)\ell_i + (\sum_i^t n_i)c\cdot\Sigma + (n^2 - \sum_i^t n_i^2)\frac{\chi(\Sigma)}{2} \\ &+ (\sum_i^t n_i^2) - 1. \end{aligned}$$

and $\widetilde{d}+b_1(X)$ respectively.

The two analytic space structures are related in the sense that the structure in (i) is the quotient of the structure in (ii) by the analytic action of $Ad(C(J))$ defined from (1.45).

For any closed connected subgorup $C \leqslant G$ we also define extended moduli spaces of instantons with C–holonomy along Σ (see §4.4) and an anlogous statement to Theorem 4 can be proved for these spaces.

On the holomorphic side we shall prove the following Theorem. See Chapters 2 and 5 for the notation.

Theorem 5. *Let (X,Σ,D_∞) be a complex triple where X is a compact complex surface, Σ and D_∞ are smooth irreducible curves meeting transversely, $G^{\mathbb{C}}$ a complex simple Lie group or $GL(n)$, P a parabolic subgorup of it and $\mathcal{E}^0$ a P–quasi–parabolic framing on $(D_\infty, \Sigma\cap D_\infty)$.*

(i) *If (X,ω) is Kähler and Σ is such that $\Sigma\cdot\Sigma \geqslant 0$, the set of stable $SL(n)$ or $GL(n)$ α–parabolic vector bundles on (X,Σ), $\mathcal{M}^{\alpha,S}_{k,c,\ell}$, has a natural complex space structure which is a coarse moduli space for α–stable parabolic bundles of the given topological type. If α has eigenvalues with multiplicities $\{n_i\}$ and $\ell = (\ell_1,\cdots,\ell_t)$ with respect to the basis of (2.5) for $[\Sigma, G^{\mathbb{C}}/P(\alpha)]$, at any point $[\mathcal{V}_0]$ such that $H^2((\mathcal{T}_\mathcal{E})_0)$ vanishes, $\mathcal{M}^{\alpha,S}_{k,c,\ell}$ equipped with the above complex space structure is*

locally a complex manifold of dimension

$$d = dim(\mathcal{M}^{\alpha,S}_{k,c,\ell}(X,\Sigma)) = 2nk - \frac{n^2-1}{2}(1 - b^1(X) + b^+(X)) + (1 + h^{0,2}(X)) + \sum_i^t (n_i + n_{i+1})\ell_i + (\sum_i^t n_i)c\cdot\Sigma + (n^2 - \sum_{i\leqslant j}^t n_i n_j)\frac{\chi(\Sigma)}{2}$$

(ii) *If for all* $k \geqslant 1$ $H^0(\mathcal{T}_0[-kD_\infty]_{|D_\infty}) = 0$ *the moduli space of* $\mathcal{E}^0$*–quasi–parabolic framings on* (X,Σ,D_∞), $\widetilde{\mathcal{M}}^P_{k,c,\ell}$, *has a natural complex space structure which is a coarse moduli space for parabolic framings of the given topological type. At any point* $[\overline{\partial}_0]$ *such that* $H^2((\widetilde{\mathcal{T}}_\mathcal{E})_0)$ *vanishes,* $\widetilde{\mathcal{M}}^P_{k,c,\ell}$ *equipped with the above complex space structure is locally a complex manifold of dimension*

$$\widetilde{d} = dim(\widetilde{\mathcal{M}}^P_{k,c,\ell}(X,\Sigma)) = 2C_{G^\mathbb{C}}k - \frac{dim(G^\mathbb{C})}{2}(1 - b^1(X) + b^+(X)) + C_P(\ell) + dim(G^\mathbb{C}/P)\frac{\chi(\Sigma)}{2} + dim(G^\mathbb{C})\frac{\chi(D_\infty)}{2} - \sum_{p\in\Sigma\cap D_\infty} dim(G^\mathbb{C}/P),$$

when $G^\mathbb{C}$ *is simple and*

$$\widetilde{d} + h^{0,2}(X) + C_P(c\cdot\Sigma),$$

when $G^\mathbb{C} = GL(n)$. *Here* C_P *is an integer valued functional on* $[\Sigma, G^\mathbb{C}/P(\alpha)]$ *depending only on* P *and* $C_{G^\mathbb{C}}$ *a constant depending only on* $G^\mathbb{C}$ *(see Lemma 2.18 and table (5.20)).*

A statement analogous to (ii) holds also for general complex reductive groups.

We then have the following partial comparison Theorem for the deformation theories giving the moduli spaces of Theorems 4 and 5. See Capters 5 and 7 for the definition.

Theorem 6. *Let* (X,Σ) *be a compact complex Kähler pair or* $(\mathbb{C}^2,\Sigma)$ *with the euclidean metric and,* $A \in M^{J,*}_{\kappa,c,\ell}(X,\Sigma)$ *or* $\widetilde{M}^J_{\kappa,\ell}(\mathbb{C}^2,\Sigma)$ *and* $\mathcal{E} \in \mathcal{M}^{\alpha,S}_{\kappa,c,\ell}(X,\Sigma)$ *or* $\widetilde{\mathcal{M}}^\alpha_{\kappa,\ell}(\mathbb{C}^2,\Sigma)$ *the associated parabolic bundle or framing by Theorems 1 or 2. Then*

(i) *The maps of Theorems 1 and 2 induce isomorphisms between the cohomology groups* H^i_A *or* $\widetilde{H}^i_A$ *of the anti–self–duality deformation complex and those,* $H^i(\mathcal{T}_\mathcal{E})$ *or* $H^i(\widetilde{\mathcal{T}}_\mathcal{E})$, *of the sheaves of infintesimal parabolic automorphisms or framed infinitesimal parabolic automorphisms of* $\mathcal{E}$.

(ii) $[A]$ *is a smooth point of the appropriate moduli space of instantons if and only if* $\mathcal{E}$ *is a smooth point of the corresponding moduli space of parabolic bundles or framing and in this case the maps given by Thoerem 1 or 2 are local diffeomorphisms.*

As an application of Theorem 6 and of the study of some concrete examples of moduli spaces of parabolic framings and parabolic stable bundles in Chapter 5 we shall now derive a number of corollaries concerning the non–emptyness, smoothness and dimension of the moduli spaces $M^J_{\kappa,\ell}(X,\Sigma)$ and $M^{J,*}_{\kappa,\ell}(X,\Sigma)$ for some special pairs. We identify a class $\ell \in [\Sigma, G^\mathbb{C}/P]$ with an integer vector $(\ell_1,\cdots,\ell_t)$ via the basis of (2.5).

Corollary 3.

(i) *For each admissible $J \in SU(n)$, where $i\alpha = \log(J)$ has eigenvalues of multiplicities $\{n_i\}$, and any topological invariants κ, ℓ such that*

$$k \geqslant 0, \qquad \forall\, i,\ \ell_i + k \geqslant 0, \tag{0.10}$$

$\widetilde{M}^{SU(n),J}_{\kappa,\ell}(S^4, S^2)$ is a non–empty smooth manifold of dimension

$$dim(\widetilde{M}^{SU(n),J}_{\kappa,\ell}(S^4, S^2)) = 4nk + 2\Sigma(n_i + n_{i+1})\ell_i$$

which is naturally diffeomorphic to the complex manifold $\widetilde{\mathcal{M}}^{P(\alpha)}_{\kappa,\ell}(\mathbb{P}^2, \Sigma)$, while it is empty for any topological invariants not satisfying (0.10).

(ii) *In the same situation as in (i), for any topological invariants which further satisfy*

$$\forall i, \qquad (\kappa_i, \ell_i) \neq (0,0), \qquad \kappa_i + \ell_i \geqslant n_{i+1},$$

$M^{SU(n),J,*}_{\kappa,\ell}(S^4, S^2)$ is a non–empty smooth manifold of dimension

$$dim(M^{SU(n),J,*}_{\kappa,\ell}(S^4, S^2)) = 4nk + 2\Sigma(n_i + n_{i+1})\ell_i - (\sum_{i=1}^{t+1} n_i^2 - 1).$$

(iii) *Let G be any compact simple simply connected Lie group and $J \in G$ any admissible element, where $i\alpha = \log(J)$. Then for any topological invariants $\kappa = \sum_{i=1}^{t} \kappa_i$ and $\ell = (\ell_1, \cdots, \ell_t)$ satisfying*

$$\forall i, \qquad \kappa_i \geqslant 0, \qquad \kappa_i + \ell_i \geqslant 0, \tag{0.11}$$

the moduli space $\widetilde{M}^{G,J}_{\kappa,\ell}$ is a smooth non–empty manifold of dimension $2C_G\kappa + C_C(\ell)$ which is naturally diffeomorphic to the complex manifold $\widetilde{\mathcal{M}}^{P(\alpha)}_{\kappa,\ell}(\mathbb{P}^2, \Sigma)$.

(iv) *Let G be any simple compact simply connected Lie group and $J \in G$ any admissible regular element. Then for any topological invariants $\kappa = \sum_{i=1}^{t} \kappa_i$ and $\ell = (\ell_1, \cdots, \ell_t)$ satisfying (0.11) and such that further $\forall i,\ (\kappa_i, \ell_i) \neq (0,0)$, the moduli space $M^{G,J,*}_{\kappa,\ell}$ is a smooth manifold of dimension*

$$dim(M^{G,J,*}_{\kappa,\ell}(S^4, S^2)) = 2C_G k + 4 \sum_{i=1}^{rk(G)} \ell_i - rk(G),$$

containing a non–empty open set consisting of irreducible connections.

As a consequence of Theorem 3 we have a corresponding result for hyperbolic monopoles

Corollary 4.

(i) *For every compact simple simply connected Lie group G, any $i\alpha \in \mathfrak{g}$ and for each topological type ℓ satisfying condition (0.10) the moduli space of framed hyperbolic monopoles of mass α and charge ℓ, $\widetilde{M}_\ell^{G,\alpha}(\mathbb{H}^3)$, equipped with its natural real analytic space structure is a non–empty smooth manifold of dimension $C_C(\ell)$ which is naturally diffeomorphic to the complex manifold $\mathcal{R}_\ell^*(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))$.*

(ii) *Let $0 \neq i\alpha \in \mathfrak{su}(n)$ and let $\{n_i\}$ be the multiplicities of the eigenvalues of α. Then for any $\ell \in \pi_2(SU(n)/C(i\alpha))$ such that*

$$\forall i, \qquad \ell_i \geqslant n_{i+1},$$

the moduli spaces $M_\ell^{SU(n),\alpha,*}(\mathbb{H}^3)$ are smooth manifolds of dimension

$$dim(M_\ell^{SU(n),\alpha,*}(\mathbb{H}^3)) = 2\sum_{i=1}^{t}(n_i + n_{i+1})\ell_i - (\sum_{i=1}^{t+1} n_i^2 - 1),$$

containing a non–empty open set consisting of irreducible monopoles.

(iii) *Let G be any compact simple simply connected Lie group and $0 \neq i\alpha \in \mathfrak{g}$ a regular element. Then for any $\ell \in \pi_2(G/C(i\alpha))$ such that*

$$\forall i, \qquad \ell_i \geqslant 1,$$

the moduli spaces $M_\ell^{G,\alpha,*}(\mathbb{H}^3)$ are smooth manifolds of dimension

$$dim(M_\ell^{G,\alpha,*}(\mathbb{H}^3)) = 4\sum_{i=1}^{t}\ell_i - rk(G),$$

containing a non–empty open set of irreducible monopoles.

One more special feature of the moduli spaces of the last two corollaries is the triviality of the extended moduli spaces defined in §4.4 and §6.5.

Corollary 5. *For every close connected subgroup $C \leqslant G$ the extended moduli spaces of framed singular instantons with C–holonomy along S^2 and given topological invariants, $\widetilde{\widetilde{M}}_{k,\ell}^{C}(S^4, S^2)$, and that of framed hyperbolic monopoles with C–symmetery breakdown at infinity and given charge, $\widetilde{\widetilde{M}}_\ell^{C}(\mathbb{H}^3)$, with their natural real analytic space structures are smooth manifolds diffeomorphic to the products*

$$Reg^+(C) \times \widetilde{\mathcal{M}}_{k,\ell}^{P(J)}(\mathbb{P}^2, \Sigma), \qquad Reg^+(\mathfrak{c}) \times \mathcal{R}_\ell^*(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha)),$$

respectively.

Finally as a consequence of Theorems 2 and 6 we shall obtain

Corollary 6. *Let $(X,[H])$ be a minimal rational surface with Kähler class given by a hyperplane class H with respect to which stability is defined and Σ a smooth curve in $[mH]$. Then for any integers κ,ℓ satisfying*

$$k>0, \qquad 2\ell > 2(g(\Sigma)-1),$$

and any $J\in SU(2)$ such that $i\alpha=\log(J)$ is of the form

$$\alpha=\begin{pmatrix}\beta & 0\\ 0 & -\beta\end{pmatrix}, \qquad 0<\beta<\frac{1}{\deg(\Sigma)},$$

the moduli space $M_{\kappa,\ell}^{SU(2),J,*}$, with its natural real analytic space structure is diffeomorphic to the complex manifold $\mathcal{M}_{\kappa,\ell}^{\alpha,S}(X,\Sigma)$, where $i\alpha=\log(J)$. In particular it is a non–empty real manifold independent of α of dimension

$$dim(M_{\kappa,\ell}^{SU(2),J,*}(X,\Sigma))=8\kappa-3+4\ell+2(1-g(\Sigma)).$$

0.4 Outline

The thesis is subdivided into seven chapters. In Chapter 1 we gather some fundamental facts on the local theory of singular connections from [R1], [R3] and [SS] in a form suitable for applications to hyperbolic monopoles and genereal groups, study the behaviour at infinity of instantons on $\mathbb{C}^2\backslash\Sigma$ and give a topological classification of G–instantons on $X\backslash\Sigma$ for arbitrary compact connected oriented pairs (X,Σ) and compact connected simple Lie groups G. In the cese of (S^4,S^2) we also examine the case of a general compact conected G. Finally we define (as topological spaces) the moduli spaces of singular instantons and framed singular instantons of given holonomy and topological type. Chapter 2 is devoted to setting up the theory of principal (quasi–) parabolic bundles and its relation to the more familiar notion of parabolic vector bundle. We give a topological classification of parabolic bundles and develop the differential geometric aspect of the theory giving a parametrisation of the moduli spaces in terms of certain $\overline{\partial}$–operators particularly well–suited to the construction of the moduli spaces. Finally we prove a key property of stable parabolic bundles. Chapter 3 which is the core of the thesis is dedicated entirely to the proof of the two main theorems, Theorems 1 and 2. In Chapter 4 we generalize the construction of the moduli space of singular instantons of [KM1] to general compact connected G applying the results of [R1] and use this to construct moduli spaces of framed singular instantons. We then generalize the Chern–Weil formula from [KM1] to the $U(n)$–case and use it to give a first prove of the necessity of Condition (0.10), to show that in the case of a compact Kähler pair (X,Σ) the unitary irreducible representations of $\pi_1(X\backslash\Sigma)$ all come from irreducible representations of $\pi_1(X)$ and also, when $G=U(n)$ or $SU(n)$, to compute the dimension of the moduli spaces at smooth points extending the methods of $[KM1]$ via the excision principle. In Chapter 5 we carry out the construction of the moduli spaces of quasi–parabolic bundles or framings for a general complex reductive $G^{\mathbb{C}}$ exploiting the

parametrisation given in Chapter 2 to apply the methods of Kuranishi theory. At the local level we give a criterion for smoothness and calculate the dimension at smooth points for simple $G^{\mathbb{C}}$ or $GL(n)$. We then show that when $G^{\mathbb{C}} = GL(n)$ or $SL(n)$ global coarse moduli spaces of stable parabolic bundles exist on Kähler pairs (X, Σ) such that $\Sigma \cdot \Sigma \geqslant 0$ and that for general complex reductive $G^{\mathbb{C}}$ global coarse moduli spaces of parabolic framings exist on triples (X, Σ, D_∞) provided the framing object satisfies Condition 5.11 and Σ and D_∞ are smooth irreducible divisors intersecting transversely. We next study in some detail two families of examples, namely the framings on $(\mathbb{P}^2, \Sigma, D_\infty)$ appearing in the statement of Theorm 1 and $SL(2)$–parabolic stable bundles on $(X, \Sigma, [mH])$, where X is a rational surface, H a hyperplane class and Σ a multiple of mH. Finally we explain how the given construction of the moduli spaces can be adapted to the case where Σ and D_∞ are divisors with at most ordinary singularities meeting transversely at smooth points and generalize the dimension formulae to this case. In Chapter 6 we prove Theorem 3 in two slightly different ways. The first method which works best for monopoles with admissible mass is to apply directly Theorem 1 identifying the moduli spaces of (framed) hyperbolic monopoles with certain subvarieties of the moduli space of parabolic framings consisting of elements invariant under a $\mathbb{C}^*$–action. The second method, which is insensitive to whether the mass is admissible or not, gives the clearest explanation as to why the moduli spaces should depend on the mass $[i\alpha]$ only up to $C(i\alpha)$ and requires proving a slightly more general form of Theorem 1 in the invariant case. We then define moduli spaces and extended moduli of monopoles as ringed spaces and using the sharp estimates of [R1] also give a construction of the moduli spaces of high mass monopoles via a slice theorem. The last chapter is dedicated to the comparison between the deformation theories constructed in Chapters 4 and 5 and we prove that the natural bijections of Theorems 1 and 2 induce infinitesimal isomorphisms between them which are in fact diffeomorphisms at smooth points of the moduli spaces. We then use this in conjunction with the examples studied in Chapter 5 to obtain more detailed information concerning the smoothness and non–emptyness of the moduli spaces of irreducible singular G–instantons on (S^4, S^2), that of irreducible hyperbolic monopoles, and that of singular $SU(2)$–instantons on (X, Σ) when this is as stated in Corollary 6.

0.5 Notation

The Notation used throughout the thesis is the standard notation of gauge theory, as found for instance in [DK], and in particular of gauge theory with real codimension two singularities as developed in [B1], [B2], [B3] and [KM1]. Chapters are divided into sections and an internal reference of the form §?.! indicates the chapter and the section in it e.g. §2.4 refers to Section 4 in Chapter 2. Within chapters there is only one numbering of the items, that is equations, Lemmas, Propositions, Theorems ecc. all appear in the same sequence and a reference in brackets of the form (?.!!) gives the chapter and the number of the equation within that chapter, e.g. (5.23) indicates equation 23 in Chapter 5.

Chapter 1

Singular instantons

In this Chapter we gather some facts about connections on the complement of a two–submanifold Σ in a Riemannian four–manifold (X, g) in a form suitable for applications to bundles with general structure group and hyperbolic monopoles. These include Råde's gauge fixing result. See [SS], [KM1], [R1] and [R2] for more details. We then give a topological classification of singular G–instantons for G compact simply connected simple on a compact connected orientable pair (X, Σ). Finally we define the moduli spaces $M^J_{k,c,\ell}$ of Theorems 1 and 2. Since the idea of a K–structure or a K–connection on a G–bundle where K is a subgroup of G will be central throughout the thesis we begin by briefly recalling these notions.

1.1 K–structures

Let G be a Lie group and let K be a closed subgroup. Let E be a principal G–bundle. We recall that a K–structure or K–reduction on E consists of a principal K–bundle E' together with a fibrewise injective map

$$\lambda : E' \longrightarrow E,$$

commuting with the respective right actions

$$\forall x \in X,\ \forall y \in E'_x,\ \forall k \in K \qquad \lambda(yk) = \lambda(y)k.$$

If we think of the inclusion as giving a faithful left action of K on G, E is the associated G–bundle

$$E = E' \times_K G.$$

This is then naturally equipped with a K–structure and we can recover E' by passing to the bundle of K–frames. If E' and E'' are G–bundles with K–structures, we say that an isomorphism $\psi : E' \cong E''$ is a K–isomorphism if it intertwines the K–structures. A particularly useful way to charachterise K–structures is by cocycles. A G–bundle E is the same as a cocycle of G–valued functions $\{g_{ij}\}$ for a trivializing open cover of X, the transition functions for E. The bundle will admit a K–structure if and only if it admits an equivalent cocycle of K–valued transition functions. Now let A be a connection on a K–bundle E'. For any representation ρ this induces one, $\rho(A)$, with holonomy lying in K on the associated G–bundle E and $\rho(A)$ is said to be a K–connection. Vice–versa let A be a K–connection on E. The holonomy then gives a K–structure on E and A can be considered as a connection on a K–bundle. Locally a connection A on a G–bundle is a K–connection if and only if there is a gauge g such that A is of the form

$$g(A) = d + a, \qquad a \in \Omega^1(\mathfrak{k}),$$

where $\mathfrak{k} = Lie(K)$. Globally any connection for which there is a trivializing open cover and gauges in which A has the form above is a K–connection on E. We shall be particularly interested in two cases,
1) $\rho : G \to U(n)$ is a faithful unitary representation, then the associated $U(n)$–bundle can be identified with the unitary vector bundle

$$V = E \times_G \mathbb{C}^n,$$

and a connection A on E induces a connection $\rho(A)$ and a covariant derivative

$$d_{\rho(A)} : \Omega^0(V) \to \Omega^1(V).$$

2) $\rho = Ad$ is the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Then the associated bundle can be identified with the adjoint bundle $\mathfrak{g}_E$.

In the last case we have an induced $Ad(G)$–structure on $\mathfrak{g}_V$ given by the subbundle $\rho(\mathfrak{g}_E)$. If ρ or at least the associated infinitesimal representation is faithful we identify $\rho(\mathfrak{g}_E)$ with $\mathfrak{g}_E$ and denote by $\Omega^i(\mathfrak{g}_E)$ the distinguished subspaces of the $\Omega^i(\mathfrak{g}_V)$'s induced by $\mathfrak{g}_E$.

1.2 Analytical properties of singular connections

We recall here the salient analytical features of the theory of singular connections. These are of a local nature and since an embedded 2–manifold inside a Riemannian 4–manifold locally approximates the model pair (B^4, B^2) consisting of a 2–ball in a Euclidean 4–ball it will suffice to study this. The first basic fact is the existence of the limit holonomy (0.1). As in the Introduction let (r, θ) be polar coordinates in the normal discs D_p to B^2. If $E = G \times B^4$ is a trivial G–bundle, for each $i\alpha \in \mathfrak{g}$, the constant connection form

$$A_\alpha = i\alpha d\theta,$$

has holonomy $\exp(-2\pi i\alpha)$ around every positively oriented loop at the origin on the normal discs to B^2. We shall now assume G to be compact.

Lemma 1.1. *Let A be a finite action connection on a principal G–bundle $E \longrightarrow B^4 \backslash B^2$, then the limit holonomy (0.1) exists for almost every $p \in B^2$ and its conjugacy class in G is independent of p. If this class is $[J(A)] = [\exp(-2\pi i\alpha)]$, we can always find a gauge in which for almost every $p \in B^2$ and $\theta \in D_p$,*

$$(1.2) \qquad J(A) = \lim_{r \to 0} J_p(r) = \exp(-2\pi i\alpha), \qquad \phi(A) = \lim_{r \to 0} A_\theta(r, \theta) = i\alpha.$$

$[J(A)]$ is called the holonomy of A along B^2.

Proof. Let $\rho : G \to U(n)$ be a faithful unitary representation. We then obtain a finite action connection $\rho(A)$ on the associated complex vector bundle V to which the proof of [SS] for the $SU(2)$ case can be easily extended giving a unitary limit $J(\rho(A))$ with the

required properties. Since $\rho(A)$ is a G–connection and $\rho(G)$ is closed in $U(n)$ it follows that in any particular gauge where the limits (1.2) exist they must be of the form

$$J(\rho(A)) = \rho(J(A)), \qquad \phi(\rho(A))) = \rho(\phi(A)),$$

for some $J(A) \in G$, $\phi(A) \in \mathfrak{g}$ and that the conjugacy class $[J(A)]$ is well defined in G. □

As remarked in the Introduction, while the parallel transport of A completely specifies $[J(A)]$, $[\phi(A)]$ is determined only modulo the integral lattice Λ of $\mathfrak{g}$. Indeed it can be changed by applying gauge transformations of the form $g = \exp(2\pi\lambda\theta)$, $\lambda \in \Lambda$. For instance, in a given gauge, the flat constant models with holonomy J are parametrised by Λ. While for local purposes the precise choice is unimportant it has some significance at the global level and we shall return to this point in §1.4.

We now define the function spaces in terms of which singular connections can be locally understood. Let V^n be a trivial complex vector bundle with constant fibre metric on the model pair (B^4, B^2), let $i\alpha$ be any element in $\mathfrak{u}(n)$, $\{\alpha_i\}$ the eigenvalues of α, $\{\alpha_{ij} = \alpha_i - \alpha_j\}$ those of $[\alpha, -]$ and consider the corresponding eigenbundle decomposition

$$V = \underset{\alpha_i}{\oplus} V_{\alpha_i}, \qquad End(V) = \underset{\alpha_{ij}}{\oplus} Hom(V_{\alpha_j}, V_{\alpha_i}). \tag{1.3}$$

We also let

$$End(V)_C = \underset{\alpha_{ij}\in\mathbb{Z}}{\oplus} Hom(V_{\alpha_j}, V_{\alpha_i}), \qquad End(V)_U = \underset{\alpha_{ij}\notin\mathbb{Z}}{\oplus} Hom(V_{\alpha_j}, V_{\alpha_i}).$$

These induce analogous decompositions on forms and we can accordingly write an $End(V)$–valued i–form f as

$$f = f_C \oplus f_U, \qquad f_C \in End_C(V),\ f_U \in End_U(V). \tag{1.4}$$

Let r be the euclidean distance function from B^2. If α is a diagonalisable endomorphism of V with real eigenvalues $\{\alpha_i\}$ we shall often use the notation r^α to denote the endomorphism with corresponding eigenvalues $\{r^{\alpha_i}\}$. We denote by d_α the constant unitary flat connection on V with holonomy $[\exp(-2\pi i\alpha)]$ and also the induced connection on $End(V)$,

$$d_\alpha = d + [i\alpha, -]d\theta : \Omega^0(End(V)) \to \Omega^1(End(V)). \tag{1.5}$$

Let ∇_α be the covariant derivative induced by d_α on the tensor bundles $\Omega^i(End(V))$. Following [KM1], for $p > 1$ and k a non–negative integer we define Sobolev spaces on B^4

$$L^p_{k,\alpha} = \{a \in L^p_{k,loc}(B^4\backslash B^2, End(V)) | a, \cdots \nabla^k_\alpha a \in L^p\} \tag{1.6}$$

$$W^p_k = \{f \in L^p_{k,loc}(B^4\backslash B^2) \,|\, r^k f, \cdots, \nabla^k f \in L^p\}, \tag{1.7}$$

the latter extended by components to vector valued functions. When $k = 0$ we omit it from the notation. On B^4 we consider the standard S^1 action fixing B^2. The lifts of this to $End(V)$ are parametrised by the representation they induce on the fibres of $V_{|B^2}$. This is (up to conjugation) of the form

$$\theta \mapsto \exp(2\pi i m\theta),$$

for some $m \in \ker(\exp(2\pi i(-)))$ called the weight of the action. Let $\{m_i\}$ be the eigenvalues of m and let $w(p,k) = k - 4/p$ denote the conformal weight of the ordinary L^p_k norms on B^4. The next Lemma summarizes the salient facts about these spaces as proved in §3 of [KM1] for the $SU(2)$ case.

Lemma 1.8.

(i) *Let $f \in \Omega^i(End(V))$ and $f = (f_{ij})$ its decomposition as in (1.4). Then $f \in L^p_{k,\alpha}$ if and only if*

$$\begin{aligned} &\forall \alpha_{ij} \in \mathbb{Z}, \quad \exp(-2\pi i \alpha_{ij}\theta) f_{ij} \in L^p_k, \\ &\forall \alpha_{ij} \notin \mathbb{Z}, \qquad\qquad f_{ij} \in W^p_k. \end{aligned}$$

If f is weight m S^1–equivariant the latter holds for all $\alpha_{ij} \notin \{m_{ij}\} \cup \{0\}$.

(ii) *The forms*

$$\mathcal{C}^\infty(\Omega^i(End(V)_C)) \oplus \mathcal{C}^\infty_0(B^4 \backslash B^2, \Omega^i(End(V)_U)),$$

are dense in $L^p_{k,\alpha}$.

(iii) *If $l \leqslant k$ and $w(p,k) \geqslant w(q,l)$, there is a continuous embedding*

$$L^p_{k,\alpha} \hookrightarrow L^q_{l,\alpha}.$$

If $w(p,k) > j \geqslant 0$, the $\exp(-2\pi i \alpha_{ij}) f_{ij} \in \mathcal{C}^{j+\delta}(B^4)$ when $\alpha_{ij} \in \mathbb{Z}$ and $f_{\alpha_{ij}}/r^{j+\delta} \in \mathcal{C}_0$ when $\alpha_{ij} \notin \mathbb{Z}$ for every $0 < \delta < w(p,k) - j$.

(iv) *If $m \leqslant \min\{k,l\}$ the combination of Lie bracket and wedge product gives continuous multiplications*

$$L^p_{k,\alpha} \times L^q_{l,\alpha} \to L^r_{m,\alpha},$$

provided one of the following conditions hold.

$$\begin{aligned} &(i) \quad 0 \leqslant w(r,m) \leqslant \min\{w(p,k), w(q,l)\}, \\ &(ii) \quad w(p,k) > 0, \ w(r,m) \leqslant w(q,l) \leqslant 0, \\ &(iii) \quad w(p,k) < 0, \ w(q,l) < 0, \ \text{and } w(p,k) + w(q,l) > w(r,m). \end{aligned}$$

(v) *Let $a \in L^p_{k,\alpha}$ and consider the connection $A_\alpha = d_\alpha + [a,-]$ on $End(V)$ and its covariant derivative ∇_{A_α}. Then $f \in L^p_{k+1,\alpha}$ if and only if*

$$\forall\ 0 \leqslant j \leqslant k+1, \qquad \nabla^j_{A_\alpha} f \in L^p.$$

Proof. (i) Let $\gamma \in \mathbb{R}$ and let ∇_γ be the covariant derivative on a trivial vector bundle over B^4 induced by the flat connection $d + i\gamma d\theta$ with γ acting as scalar multiplication. By the invertibility of the operator on functions on the circle

$$\partial_\gamma = \frac{\partial}{\partial\theta} + i\gamma, \qquad \gamma \notin \mathbb{Z} \tag{1.9}$$

it follows as in Lemma 4.2 of [KM1] that for $\gamma \notin \mathbb{Z}$,

$$\forall f \in \mathcal{C}^\infty(B^4), \quad \int_{B^4\backslash B^2} r^{p(i-k)} |f|^p \leqslant C \int_{B^4\backslash B^2} r^{pi} |\nabla^k_\gamma f|^p, \tag{1.13}$$

thus $L^p_{k,\gamma} = W^p_k$. Since ∇_α acts on $\Omega^i(V_{\gamma_i})$ as ∇_{γ_i} the claim for $\alpha_i \notin \mathbb{Z}$ follows. When $\gamma \notin \{m_i\} \cup \{0\}$ the operator (1.9) has no kernel in the S^1–equivariant part so (1.10) holds for the equivariant sections too. If $\alpha_{ij} \in \mathbb{Z}$,

$$\nabla_{\alpha_{ij}}(f) = \exp(2\pi i \alpha_{ij}\theta)\nabla(\exp(-2\pi i \alpha_{ij}\theta)f)),$$

hence

$$\nabla_{\alpha_{ij}}(f) \in L^p \iff \exp(-2\pi i \alpha_{ij}\theta)f) \in L^p_1,$$

and by iteration

$$\nabla^k_{\alpha_{ii}}(f) \in L^p \iff \exp(-2\pi i \alpha_{ij}\theta)f \in L^p_k.$$

(ii) By Lemma 3.1 of [KM1], W^p_k is the completion of $\mathcal{C}^\infty_0(B^4\backslash B^2, \Omega^i)$ in the norm

$$\|f\|_{W^p_k} = \|r^{-k}f\|_{L^p} + \|r^{1-k}\nabla f\|_{L^p} + \cdots + \|\nabla^k f\|_{L^p},$$

so the claim follows from (i).
(iii) Using (i) this immediately follows by applying the standard Sobolev embedding theorems.
(iv) This follows by (i) and Lemma 3.4 of [KM1].
(iv) Given (iv) this is immediate from the definition of the $L^p_{k,\alpha}$ spaces. □

We shall also need some regularity for the operator d_α.

Lemma 1.11. *Let $d_\alpha + [a, -]$ be a connection on $End(V)$ with $a \in L^p_{k,\alpha}$, and $k \geqslant 1$. If $f \in Ker(d_\alpha + [a, -]) \cap L^\infty$ then $f \in L^p_{k+1,\alpha}$.*

Proof. By assumption

$$d_\alpha f = [f, a]. \tag{1.12}$$

Since $f \in L^\infty$ and by (i) of the previous Lemma

$$\|a\|_{L^{4+\delta,1}} \leqslant \|a\|_{L^p_{k,\alpha}},$$

for some $\delta > 0$, it follows that

$$\|f\|_{L^{4+\delta}_{1,\alpha}} \leqslant C\|a\|_{L^p_{k,\alpha}}. \tag{1.13}$$

Applying ∇_α to both sides of (1.12) we obtain

$$\nabla^2_\alpha(f) = \{\nabla_\alpha f \otimes a\} + \{f \otimes \nabla_\alpha a\},$$

which by the multiplication $L^p_{k,\alpha} \times L^{4+\delta}_{1,\alpha} \hookrightarrow L^p_{k,\alpha}$ and (1.13) gives

$$\|f\|_{L^p_{2,\alpha}} \leqslant C\|a\|_{L^{p,1}_{k,\alpha}}.$$

Now iterate to get $L^p_{k,\alpha}$ bounds if $k \geqslant 2$. $\square$

We now come to the most fundamental fact concerning singular connections which is one of the main analytical inputs of the thesis, the gauge fixing Theorem of [SS],[R1], and [R3]. We remark that the published proof of the gauge fixing theorem, [SS], contains, as far as we can see, a major conceptual fault. A new proof will appear in [R3]. Given $i\alpha \in \mathfrak{u}(n)$ let $\{\alpha_i\}$ be its eigenvalues and $\{\alpha_{ij} = \alpha_1 - \alpha_j\}$ the eigenvalues of $[\alpha, -]$. Put $\tilde{\alpha}_{ij} = \min_{m\in\mathbb{Z}}\{\,|\alpha_{ij} + m|\,\}$, and let $\alpha_0 = \min_{\tilde{\alpha}_{ij}\neq 0}\{\tilde{\alpha}_{ij}\}$. Let h denote the product metric of the standard hyperbolic metric on $\mathbb{H}^3$ and the round metric on S^1 so that h is conformally equivalent to the euclidean metric and on i–forms

$$|a|_h = r^i|a|.$$

Denote by $L^{p,k}_h$ the ordinary L^p_k spaces for h on $\mathbb{H}^3 \times S^1$ and as in §3 of [R1] define

$$L^{p,k}_{h,w}(\mathbb{H}^3 \times S^1) = \{f \in L^p_h | wf, w\nabla_h f, \cdots, w\nabla^k_h f \in L^p_h\}, \qquad w = r^{4/p-2}.$$

Theorem 1.14. *Let $p \in [2,\infty] \cap (2 + \alpha_0/6)^{-1}, (1/2 - \alpha_0/6)^{-1})$.*

(i) *([R1], Corollary 3.4) Let $k \geqslant 0$, $\gamma \in \mathbb{R}\backslash\mathbb{Z}$, and $(1/2+\gamma_0/6)^{-1} < p < (1/2-\gamma_0/6)^{-1}$. Then the operator with trivial coefficients*

$$\Delta_\gamma = d_\gamma d^{*_h}_\gamma \oplus d^{*_h}_\gamma d_\gamma : L^{p,k}_{h,w}(\mathbb{H}^3 \times S^1) \to L^{p,k-2}_{h,w}(\mathbb{H}^3 \times S^1)$$

is invertible.

(ii) *([R1], Theorem 1)There are constants ε and C such that for a unitary connection $d+A$ on the trivial vector bundle $V \longrightarrow B^4\backslash B^2$ with $\|F_A\|_{L^2} < \varepsilon$, $A \in L^p_{1,loc}(B^4\backslash B^2)$ and for any $i\alpha \in \mathfrak{u}(n)$ such that A has holonomy $[\exp(-2\pi i\alpha)]$ along B^2, there is a gauge transformation g such that*

$$g(A) = d_\alpha + a, \quad a \in L^p_{1,\alpha}, \qquad \|a\|_{L^p_{1,\alpha}} < C\|F_A\|_{L^p},$$

and

$$d^*(a_C) = 0, \qquad d^{*_h}_\alpha(a_U) = 0.$$

(iii) *([R1], Theorem 2) If further A is A.S.D. , g can be chosen so that letting $a = \oplus_{ij} a_{ij}$ be the decomposition of a according to the eigenvalues of $[\alpha, -]$ one has*

$$\forall j \leqslant k, \forall \alpha_{ij} \in \mathbb{Z}, \qquad |\nabla^j \exp(-2\pi i\alpha_{ij})a_i| \leqslant \frac{C_k}{r^{j-1}}\|F_A\|_{L^2},$$

$$\forall j \leqslant k, \forall \alpha_{ij} \notin \mathbb{Z}, \qquad |\nabla^j a_i| \leqslant \frac{C_k}{r^{j-\tilde{\alpha}_{ij}}}\|F_A\|_{L^2}.$$

If A is a G–connection the gauges can be chosen to be G–gauges so that $a \in \Omega^1(\mathfrak{g})$. On sections which are S^1–equivariant with respect to the weight m S^1–action the operator in (i) remains invertible provided $\gamma \neq m$. If A is an S^1–equivariant connection the gauge of (ii) can be chosen so that a is an S^1–equivariant form and the second estimate in (iii) holds for all the components where $\alpha_{ij} \notin \{m_i - m_j\} \cup \{0\}$ with $[|(\alpha_{ij} - m_{ij})|] + 1$ in place of $\tilde{\alpha}_{ij}$, where $[x]$ denote the largest integer $< x$.

Proof. See [R1] and [R3]. We only prove the clauses concerning G and S^1–equivariant gauges. First observe that given a connection of the form

$$A = d_{\rho(\alpha)} + a, \quad i\alpha \in \mathfrak{g}, \quad a \in L^p_{1,\alpha}(\Omega^1(\mathfrak{g})),$$

using the closedness of $\Omega^i(\mathfrak{g})$ in $\Omega^i(\mathrm{End}(V))$ and of $\mathcal{G}$ in the gauge group of V, we can apply the continuity method to obtain a singular Hodge G–gauge with respect to $d_{\rho(\alpha)}$ as in [R1], Theorem 1. Now let

$$A = d + a', \qquad a' \in \Omega^1(\mathfrak{g}),$$

be a G–connection with holonomy J and $i\alpha \in \mathfrak{g}$ any element such that $\exp(-2\pi i\alpha) = J$. With the above information the proof of (ii) given in [R3] shows that the gauge transformation achieving (ii) for $\rho(A) = d + \rho(a')$ and $d_{\rho(\alpha)}$ can be taken G–valued so that $a \in \Omega^1(\mathfrak{g})$. It follows that any finite action G–connection with holonomy J admits a singular Hodge G–gauge with respect to $d_{\rho(\alpha)}$. When A is an instanton the regularity part of [R1] then applies for this gauge giving (iii). For the clause concerning S^1–equivariant connections, observe that since the operator in (i) remains invertible on sections which are S^1–equivariant with respect to the weight m S^1–action provided $\gamma \notin m$, starting from an S^1–equivariant gauge the gauge fixing problem can be solved restricting to the S^1–equivariant parts of the function spaces involved. We then obtain an S^1–equivariant gauge transformation achieving (ii) in which the connection matrix will still be S^1–equivariant. □

1.3 The behaviour at infinity of Instantons on $\mathbb{C}^2\backslash\Sigma$

An essential ingredient in the proof of Theorem 1 is understanding the behaviour at infinity of instantons on $\mathbb{C}^2\backslash\Sigma$. To this purpose we regard $\mathbb{P}^2$ as the blow up of S^4 at a point p_∞ on the S^2 corresponding to Σ and consider a finite action G–connection A on $\mathbb{C}^2\backslash\Sigma$ with holonomy J along Σ as one on $S^4\backslash S^2$. Let (u, v) be local coordinates at a point in D_∞ such that $D_\infty : \{u = 0\}$ and if the point is $\mathrm{p} = D_\infty \cap \Sigma$ then also $\Sigma : \{v = 0\}$ so that $|u|$ and $|v|$ give distance functions on Σ and D_∞ respectively. We denote by D_v and D_u the normal discs to D_∞ and Σ respectively. At p define spaces $L^p_{k,\alpha}$ with respect to the euclidean metric for these coordinates taking B^2 to be a disc on Σ centred at p and on the rest of D_∞ let L^p_k be the ordinary Sobolev spaces for the euclidean metric of the local coordinates.

Lemma 1.15. *Let $A = d + a$ be a unitary connection on the trivial vector bundle $V \longrightarrow B^4\backslash B^2$ with holonomy J, $i\alpha \in \mathfrak{u}(n)$ such that $\exp(-2\pi i\alpha) = J$ and L^p curvature with $p \in [2, (1/2 - \alpha_0/6)^{-1})$. Then*

(i) *At each point of $D_\infty\backslash p$ there is a gauge in which*

$$\pi^*(A) = d - \frac{1}{2}\alpha\{\frac{du}{u} - \frac{d\overline{u}}{\overline{u}}\} + a^u + a^v,$$

with

$$a^u \in L^p_{1,loc}(D^2\backslash D_\infty), \quad \nabla^v(a^u_U) \in L^p, \quad a^v \in L^p_1, \quad \frac{a^v}{|u|} \in L^p,$$

and also

$$\forall v, \quad \|a^u\|_{L^p(D_v)} \leqslant C\|a\|_{L^p_{1,\alpha}},$$

(ii) *At p there is a gauge in which*

$$A = d + \frac{1}{2}\alpha\{\frac{dv}{v} - \frac{d\overline{v}}{\overline{v}}\} - \frac{1}{2}\alpha\{\frac{du}{u} - \frac{d\overline{u}}{\overline{u}}\} + a^u + a^v,$$

where the bounds as in (i) hold and further $a^v \in L^p_{1,\alpha}$.

(iii) *If A is an instanton, the above bounds can be replaced by the* L^∞ *bounds*

$$|a^u| = O(1), \qquad |a^v| = O(|u|).$$

If A is a G–connection the gauges can be chosen so that a^u, $a^v \in \Omega^1(\mathfrak{g})$.

Proof. Let (z, w) be complex coordinates at p_∞ agreeing with the global orientation of S^4 and such that $S^2 : \{z = 0\}$. From 1.14 (ii) we know that for p as stated there is a G–gauge at p_∞ such that

$$A = d_\alpha + a = d_\alpha + a^z + a^w, \qquad a \in L^p_{1,\alpha}, \tag{1.16}$$

where $a^z = a^z dz + a^{\overline{z}} d\overline{z}$ and similarly for a^w, and if A is an instanton with a also satisfying the bounds of 1.14 (iii). We pull–back this gauge spelling out in local coordinates the definition of the blow–up map $\pi : \mathbb{P}^2 \to S^4$. Consider complex coordinates $(\overline{w}, z)$ with the opposite orientation and perform the classical blow up. This gives an orientation reversing map $\pi : \Delta \longrightarrow S^4$, where Δ is the unit disc bundle of the tautological bundle over the exceptional divisor D_∞, and $\mathbb{P}^2$ is orientation preserving diffeomorphic to $S^4\backslash\{\mathrm{p}_\infty\} \cup_\pi \Delta$ provided we change the orientation of Δ. Let Σ be the proper transform of S^2 and take local coordinates (u, v) at a point in D_∞ compatible with the $\mathbb{P}^2$ orientation, with $D_\infty : \{u = 0\}$ and if the point is $\mathrm{p} = \Sigma \cap D_\infty$ also $\Sigma : \{v = 0\}$. We can assume that $(u, v) = (w, \frac{z}{w})$ or $(u, v) = (\frac{w}{z}, z)$ and then π has a local expression at a point $\mathrm{q} \in D_\infty$

$$\begin{aligned} &\pi : (u,\, v) \longmapsto (\overline{u}v,\, u) = (w,\, z), \quad \text{if } \mathrm{q} \notin \Sigma, \\ &\pi : (u,\, v) \longmapsto (u,\, \overline{u}v) = (w,\, z), \quad \text{if } \mathrm{q} = \mathrm{p}. \end{aligned} \tag{1.17}$$

Let (r, θ) be polar coordinates in D_{p_∞} as in the introduction so that $w = r\exp(2\pi i\theta)$. Then $\arg(w) = \arg(\overline{u}v) = \arg(v) - \arg(u)$, hence $\pi^*(d\theta) = d\theta_\Sigma - d\theta_{D_\infty}$ and

$$\begin{aligned} \pi^*(d_\alpha) &= d + i\alpha d\theta_\Sigma - i\alpha d\theta_{D_\infty} = d + \frac{1}{2}\alpha\{\frac{dv}{v} - \frac{d\overline{v}}{\overline{v}}\} - \frac{1}{2}\alpha\{\frac{du}{u} - \frac{d\overline{u}}{\overline{u}}\}, \quad \text{if } \mathrm{q} = \mathrm{p}, \\ \pi^*(d_\alpha) &= d - i\alpha d\theta_{D_\infty} \qquad\qquad = d - \frac{1}{2}\alpha\{\frac{du}{u} - \frac{d\overline{u}}{\overline{u}}\}, \qquad\qquad \text{if } \mathrm{q} \neq \mathrm{p}, \end{aligned}$$

where θ_Σ is the positive angular coordinate on a normal plane to Σ while θ_{D_∞} is the positive angular coordinate on a normal plane to D_∞ with respect to the $\mathbb{P}^2$ orientation.

Observe that there is a continuous Sobolev restriction

$$L_1^p \to L^p(D),$$

for every disc through p_∞. Also, if $p > 2$, $L^p_{1,\alpha} \hookrightarrow L^{4+\epsilon}$ for every ϵ such that $1 - 2/p - 4/(4+\epsilon) \geqslant 0$ so denoting by s the radial distance to the origin, there are q, r, $1/q + 1/r = 1$ such that

$$\int_{B^4} \frac{|a|^p}{|s|^2} \leqslant \int_{B^4} |a|^{pq} \int_{B^4} \frac{1}{|s|^{2r}} < \infty.$$

Since s pulls back to $|u|$,

$$|du| = |dw|, \qquad |dv| = |u||dz|, \qquad |Vol(z,w)| = |u|^2 |Vol(u,v)|,$$

and the discs through p_∞ pull back to the normal discs to D_∞, writing $\pi^*(a) = a_u + a_v$, the claim immediately follows pulling back (1.16). □

In particular we see from that A has non–trivial holonomy both along Σ and D_∞ and these are mutually inverse. There is a more conceptual way to understand this pointing in the right direction towards generalizing Theorem 1 to other open complex surfaces (see the conclusions) which we now sketch. Consider a tubular neighbourhood $N(D_\infty)$ of D_∞. The punctured tubular neighbourhood $N^*(D_\infty) = N(D_\infty)\backslash D_\infty$ with the metric induced by the euclidean metric is conformal to a half–infinite cylinder

$$N^*(D_\infty) \sim [0,\infty) \times Y,$$

where $Y = S^3$ is the total space of the circle bundle of $N^*(D_\infty)$ and

$$N^*(D_\infty)\backslash\Sigma = [0,\infty) \times Y^*, \tag{1.18}$$

where $Y^* = S^3\backslash S^1$ is the total space of the circle bundle minus the fibre corresponding to $\mathrm{p} = \Sigma \bigcap D_\infty$. It can be checked that $\pi_1(Y^*)$ has the presentation

$$\pi_1(Y^*) =< c, \gamma \,|\, c = \gamma^{-1} > \cong \mathbb{Z}, \tag{1.19}$$

where, with respect to the orientation induced by the complex structure of $N^*(D_\infty)$, c corresponds to the positive generator of π_1 of the fibre of Y^* and γ is a positively oriented loop 1–linking the missing fibre. Thus for any Lie group G, a representation $\sigma : \pi_1(Y^*) \to G$ is completely determined by $\sigma(c)$ or $\sigma(\gamma)$. If G is compact, fixing a maximal torus $\mathbb{T} \leqslant G$ we can thus identify G–representations of $\pi_1(Y^*)$ up to conjugacy with $\mathbb{T}$. These are the same as flat G–connections up to gauge equivalence so that for each $J \in \mathbb{T}$ there is, up to gauge, a unique flat connection Γ_J on Y^* whose parallel transport is J along c and J^{-1} along γ. Notice that flat connections on $[0,\infty) \times Y^*$ are the same as flat connections on Y^* or more precisely the pull–back of these which we also denote by Γ_J. Subdivide the cylinder into bands of fixed length $I_N = [N, N+1+\epsilon]$ and let

$I = [0, 1+\epsilon] \times Y^*$ be a model band. We say that a G–connection A on $[0,\infty) \times Y^*$ has B–limit Γ_J on Y^* if there are G–gauges g_N in which the sequence

$$a_N = (g_N^*(A_{|I_N}) - \Gamma_J)_{|I} \to 0 \qquad \text{in } B,$$

where $\{A_{|I_N}\}$ is pulled back to a sequence on I and B is some function space of 1–forms. Following (1.27), for α exponentiating to J we can define spaces $L^p_{k,\alpha}$ on I. If B is one of them and A has B–limit Γ_J it follows that A has holonomy J along D_∞ and then A must have holonomy J^{-1} along $[0,\infty) \times S^1$. Vice–versa, if a connection with holonomy J^{-1} along $[0,\infty) \times S^1$ has a B-limit on Y^* for such B, the limit can only be Γ_J. Thus the singularity of a connection on $\mathbb{C}^2 \setminus \Sigma$ with B–limit at infinity determines and is determined by the limit. In the case of the unpunctured cylinder $[0,\infty) \times Y$, Y a compact circle bundle, it is well known ([T2]) that (at least under suitable assumptions on the sign of Y) for instantons there is essentially only one notion of convergence to a flat connection on Y, namely $\mathcal{C}^\infty$ exponential convergence. It is therefore natural to ask whether the same holds on the punctured cylinder (with possibly several $[0,\infty) \times S^1$ deleted). We can't address the general question here but we explain in what sense this is true for the specific case considered. Write I_N as the union of a standard pair (B^4, B^2) and a standard 4–ball. Applying 1.14 (ii) to an instanton with holonomy J^{-1} along $[0,\infty) \times S^1$ on the standard pair, the ordinary Uhlenbeck theorem on the 4–ball and patching and then patching between adjacent I_N we obtain a global gauge on $[0,\infty) \times Y^*$ such that on every sub cylinder $[T,\infty) \times Y^*$

$$A = d_\alpha + a, \qquad \forall j \leqslant k, \ |\nabla^j a| \leqslant C_k \frac{1}{r^{j-\alpha_0}} \|F_A\|_{L^2}, \tag{1.20}$$

where r is the distance from $[0,\infty) \times S^1$ in the cylinder metric. Using the pointwise bounds given by (1.20) and spectral properties of the operator d_α restricted to a suitable subspace of $L^p_{1,\alpha}(S^3 \setminus S^1)$ (this being defined in the obvious way), one can adapt the argument in §4.2 of [DFK] to prove α_0–exponential decay of the curvature functional $E_T = \int_{[T,\infty)} |F_A|^2$ and deduce from (1.20) that

$$\forall k, \ \exists C_k, \ \forall j \leqslant k, \qquad \|\nabla^j a\|_{\mathcal{C}^0}(\{T\} \times Y^*) \leqslant C_k \frac{1}{r^{j-\alpha_0}} \exp(-\alpha_0 T), \tag{1.21}$$

which in view of the fact that the bounds of 1.14 (ii) along $[0,\infty) \times S^1$ are sharp is the strongest type of convergence to be hoped for and the analogue of the $\mathcal{C}^\infty$ exponential convergence in the case with no singularities. In this case, the derivation of (1.21) just sketched, though more natural is unnecessary. Exploiting the conformal equivalence

$$[0,\infty) \times Y^* \sim B^4 \setminus B^2,$$

for any finite action instanton A on $[0,\infty) \times Y^*$ regarded as one on $B^4 \setminus B^2$ we can pull–back the gauge of 1.14 (ii) to the cylinder obtaining even stronger bounds than (1.21), namely with $1 + \alpha_0$ in place of α_0. However bounds of the form (1.21) are the best to be hoped for in general and all that one needs for the link with the holomorphic theory.

1.4 Topological classification of singular connections

In this section we examine some global properties of finite action singular G–connections on a general pair (X, Σ) with X^4 and Σ^2 compact connected and orientable and G a compact connected simple Lie group or $U(n)$. These are the topological invariants of a $\mathcal{C}^\infty$ bundle on X with a reduction to the centralizer of the holonomy on Σ on which the singular connection has a form which is the global analogue of 1.14 (ii). The choice of $i\alpha \in \mathfrak{g}$ such that $\exp(-2\pi i\alpha) = J$ was not particularly important for the local theory, however to make sense of (1.14) globally we need to have a preferred choice for the local flat model. In the unitary case there is a natural candidate. Observe that in any conjugacy class $[J] \subseteq U(n)$ there is a unique element of the form

$$J = \exp(-2\pi i\alpha), \tag{1.22}$$

where

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \alpha_1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \alpha_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & \alpha_t \end{pmatrix}, \tag{1.23}$$

$$0 \leqslant \alpha_t < \alpha_{t-1} < \cdots < \alpha_1 < 1.$$

We can then always choose the gauges of Theorem 1.14 so that the limit holonomy and the limit $\phi(A)$ of the rotational component A_θ are (1.22) and $i\alpha$ with α given by (1.23). In general we can choose the α in 1.14 to be the logarithm of $[J]$, $i\alpha = \log(J)$, as defined in (0.4). While in the $U(n)$ case $ad(\alpha)$ has all its eigenvalues in $(-1, 1)$, as we already remarked in the introduction in general this is not always possible. Recall that we defined an element $J \in G$ to be admissible if there is an element $i\alpha \in \mathfrak{g}$ with $\exp(-2\pi i\alpha) = J$ such that $ad(\alpha)$ has all its eigenvalues lying in $(-1, 1)$ (Definition 0.3). We shall see that the failure of this condition is effectively an obstruction to extending across Σ a G–bundle with connection in a manner which is both compatible with the connection and the G–structure.

We recall that when X and G are as above a G–bundle E on X is topologically determined by a pair of classes κ, c where $\kappa \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ is an integer and $c \in H^2(X, \pi_1(G))$. There is some freedom so as to what normalization of κ to choose, we shall take

$$\kappa = \frac{1}{C_G} \mathrm{p}_1(\mathfrak{g}_E),$$

where p_1 is the first Pontrjagyn class of the adjoint bundle and C_G a constant depending on G only (see [AHS] or table (5.20)). When G is simply connected κ is the only invariant and it can be identified with the second Chern class of a topological $SU(2)$–reduction on E and when $G = U(n)$, $c = c_1(E)$ and $\kappa = c_2(E) - \frac{1}{2}c_1(E)^2$. When $X = S^4$ we have in any case

$$[S^4, BG] = \pi_3(G) \cong \mathbb{Z}.$$

so the unique invariant is a class in $\pi_3(G)$ which can be naturally identified with the κ above. Now let $J \in G$ and let $C = C(J)$ be the centralizer of J. Consider the fibration

$$G/C \longrightarrow EG/C \longrightarrow EG/G. \tag{1.24}$$

Since all the spaces are simply connected, when Σ is a compact connected orientable 2–dimensional manifold, we obtain an exact sequence of groups (not just pointed spaces)

$$0 \longrightarrow [\Sigma, G/C] \longrightarrow [\Sigma, EG/C] \longrightarrow [\Sigma, EG/G] \longrightarrow 0. \tag{1.25}$$

Since $EG/C = BC$, $EG/G = BG$, and since $[\Sigma, BC]$ and $[\Sigma, BG]$, classify C and G–bundles on Σ respectively, we see that topological C–reductions on a given G–bundle on Σ are parametrised by $[\Sigma, G/C]$ and we can think of a C–reduction as of a map $\Sigma \to G/C$. There are identifications

$$[\Sigma, G/C] = [\Sigma, \bigvee_i^t S_i^2] = \bigoplus_i^t H^2(\Sigma, \mathbb{Z}),$$

so that the topological type of a C–reduction is classified by an element ℓ in $\bigoplus_i^t H^2(\Sigma, \mathbb{Z})$. The identification depends on the choice of the spheres S_i^2 and we shall fix a preferred choice in §2.1. When $\Sigma = S^2$ we also have a natural identification

$$[S^2, G/C] = \pi_2(G/C),$$

so that ℓ can be naturally identified with an element in $\pi_2(G/C)$.

We show that when A is an $U(n)$–connection on $V \to X \backslash \Sigma$ with holonomy J and L^p curvature, $p > 2$, there is a topologically unique $U(n)$–bundle $\overline{V}$ extending V to X with a $C = C(J)$–reduction s on Σ and a connection $\overline{A}$ gauge equivalent to A off Σ determined by the property that $\phi(\overline{A}) = \log(J)$ in almost every fibre of $\mathfrak{g}_{\overline{V}}$. When A is a G–connection and J is admissible, $(\overline{V}, s)$ has a topologically unique G–structure. When J is not admissible this fails to be true both for topological reasons, as the $SU(n)$–case shows when $[\Sigma] \neq 0$, and group theoretic ones as the case of general groups shows for $(X, \Sigma) = (S^4, S^2)$.

To make this precise let $i\alpha \in U(n)$ and $(\overline{V}, s)$ a $U(n)$–bundle on X with a $C(i\alpha)$–reduction on Σ. Here

$$C(i\alpha) = \times_i^t U(n_i),$$

where n_i are the multiplicities of the eigenvalues of α, so a $C(i\alpha)$–reduction is the same as an orthogonal decomposition

$$\overline{V}_{|\Sigma} = \bigoplus_i^{t+1} \overline{V}_i, \qquad \forall i, \quad rk(\overline{V}_i) = n_i,$$

Following [KM1], §2 (i), to a vector bundle with such an orthogonal decomposition we attach a model singular connection A_α, to be thought of as the global analogue of d_α as

defined in (1.5). The orthogonal decomposition can be pulled–back to one along the disc fibres of the unit normal bundle of Σ identified with a tubular neighbourhood $N(\Sigma)$ so that

$$(1.26) \qquad \overline{V}_{|N(\Sigma)} \cong \overset{t}{\underset{i}{\oplus}} \overline{V}_i, \qquad End(\overline{V})_{|N(\Sigma)} \cong (\bigoplus_{i}^{t+1} End(\overline{V}_i) \oplus (\bigoplus_{i\neq j}^{t+1} Hom(\overline{V}_i, \overline{V}_j).$$

For each summand $\overline{V}_i$ we consider a smooth connection d_{a_i} on $N(\Sigma)$ pulled back from Σ. Let η be a closed 1–form on $N(\Sigma)\backslash\Sigma$ restricting to $d\theta$ on each disc fibre, for instance η is the pull–back of a connection on the circle bundle of $N(\Sigma)\backslash\Sigma$. Let β be a cut–off function which is one at the origin and zero further away and r a distance function from Σ which locally approximates the euclidean distance from B^2 in B^4. Put

$$A_{\alpha|N(\Sigma)} = \oplus_i d_{a_i} + i\beta\alpha\eta$$

and extend it to a connection A_α on $X\backslash\Sigma$. Notice that since the decomposition is compatible with the eigenspaces of α, α is well defined as an endomorphism of $\overline{V}_{|N(\Sigma)}$ so we may define $End(\overline{V})_C$ and $End(\overline{V})_U$ as in (1.4). We then define Sobolev spaces as in (1.6), also denoted $L^p_{k,\alpha}$, using the covariant derivative of this model singular connection

$$(1.27) \qquad L^p_{k,\alpha} = \{f \in L^p_{k,loc}(X\backslash\Sigma, \Omega^i(End(\overline{V})))|a, \cdots, \nabla^k_{A_\alpha} a \in L^p\},$$

$$(1.28) \qquad W^p_k(\Omega^i) = \{f \in L^p_{k,loc}(X\backslash\Sigma, \Omega^i)|r^{-k}f, \cdots, \nabla^k f \in L^p\}.$$

When $(\overline{V}, s)$ has a G–structure $(\overline{E}, s)$ associated to a faithful representation $\rho : G \to U(n)$, $\tilde{J} = \rho(J)$, $\tilde{\alpha} = \rho(\alpha)$ and $C(J) = \underset{j}{\times} C_j$, in the definition of $A_{\tilde{\alpha}}$ we can take the connections on the $\overline{V}_i$ to be of the form $d_{\rho(a_j)}$ where $\oplus a_j$ is a $\underset{j}{\times} C_j$–connection form. We then say that $A_{\tilde{\alpha}}$ is a model singular G–connection. Decomposition (1.26) on $End(\overline{V})$ then induces ones on the $\Omega^i(\mathfrak{g}_{\overline{E}})$'s

$$\Omega^i(\mathfrak{g}_{\overline{E}}) = \Omega^i((\mathfrak{g}_{\overline{E}})_C) \oplus \Omega^i((\mathfrak{g}_{\overline{E}})_U),$$

where $(\mathfrak{g}_{\overline{E}})_C$ is a bundle of Lie algebras isomorphic to $Lie(C(i\alpha))$ and the (1.27) have distinguished subspaces consisting of the completions of the $\Omega^i(\mathfrak{g}_{\overline{E}})$'s in the induced $L^p_{k,\tilde{\alpha}}$ norms.

Proposition 1.29. *Let $p > 2$ be as in 1.14. Given a $U(n)$–connection A on a vector bundle $V \longrightarrow X\backslash\Sigma$ with $A \in L^p_{1,loc}(X\backslash\Sigma)$, $F_A \in L^p$, holonomy J and $i\alpha = \log(J)$ there is extension data $(\overline{V}, \overline{A}, A_\alpha)$ such that*

(i) *$(\overline{V}, \overline{A})$ extends (V, A) and $\overline{V}_{|\Sigma}$ is naturally equipped with an orthogonal decomposition of the form (1.26).*

(ii) *A_α is a model singular connection on $\overline{V}$ and $A_\alpha - \overline{A} \in L^p_{1,\alpha}$.*

The data is unique in the sense that if $g(V', A') \cong (V, A)$ *on* $X\backslash\Sigma$, *has extension data* $(\overline{V}', \overline{A}', A'_\alpha)$ *also satisfying (i) and (ii), then* g *extends to an* $L^p_{2,\alpha}$ *isomorphism,* $\overline{g}(\overline{V}, \overline{A}) = (\overline{V}', \overline{A}')$. *If* (V, A) *comes from a* G*–bundle* E, $\tilde{J} = \rho(J)$ *for some admissible* $J \in G$ *and* $\tilde{\alpha} = \rho(\alpha)$ *for some* α *exponentiating to* J *with* $ad(\alpha)$ *having all its eigenvalues in* $(-1, 1)$ *then* $\overline{V}$ *has a* G*–structure extending* E, *the decomposition in (i) comes from a* $C(J)$*–reduction on* E, $A_{\tilde{\alpha}}$ *and* $\overline{A}$ *can be taken to be* G*–connections and the data is* G*–unique.*

Proof. Let $\{U_i\}$ be a cover of X by geodesic balls small enough to be approximately euclidean such that U_i is disjoint from Σ or $(U_i, U_i \cap \Sigma)$ is an approximately standard pair and in any case $\|F_A\|_{L^p(U_i)}$ is small enough for the ordinary Uhlenbeck gauge fixing Theorem or 1.14 (ii) to apply with $i\alpha = \log(J)$. Thus we can find gauge transformations g_i such that

$$g_i(A) = d + a_i, \quad a_i \in L^p_1, \quad \text{or} \quad g_i(A) = d_\alpha + a_i, \quad a_i \in L^p_{1,\alpha}. \tag{1.30}$$

If U_i and U_j intersect trivially on Σ it follows by (1.30) and regularity for d that $g_{ij} \in L^p_2$ so it is continuous. When U_i and U_j intersect non trivially on Σ, $g_{ij} = g_i g_j^{-1}$ satisfies the equation

$$dg_{ij} + iad(\alpha)(g_{ij}) + \overline{a}_i g_{ij} - g_{ij}\overline{a}_j = 0, \tag{1.31}$$

and since the only integer eigenvalue of $ad(\alpha)$ is 0, applying Lemma 1.11 to $End(V)$, (1.31) gives that $g_{ij} \in L^p_{2,\alpha}$. By Lemma 1.8 the g_{ij} are continuous across Σ and have all their components relative to the non–zero eigenvalues of $[\alpha, -]$ vanishing on Σ thus satisfy $Ad(g_{ij})(\alpha) = \alpha$ on Σ. Since $C(i\alpha) = C(J)$ it follows that they lie pointwise in $C(J)$ on Σ. Since $\{g_{ij}\}$ satisfies the cocycle condition, it gives data for a $U(n)$–bundle $\overline{V}$ across Σ equipped with a $C(J)$–reduction on Σ—an orthogonal decomposition of the form (1.26)—and tautologically isomorphic to V off Σ. Notice that once we know the g_{ij}'s are continuous across Σ, we may deduce that they lie in $C(J)$ on Σ directly from the fact that by choice, on both U_i and U_j the limit holonomy is J. Then g_{ij} must intertwine the two limits, that is $Ad(g_{ij})(J) = J$ on Σ, so it belongs to $C(J)$. By construction of $\overline{V}$, the forms $\{\overline{A}_i = A_\alpha + \overline{a}_i\}$ give a connection $\overline{A}$ on $\overline{V}$ and the $\{g_i\}$'s a bundle map $g : V \cong \overline{V}_{|X\backslash\Sigma}$ such that $g(\overline{A}) = A$. This $\overline{A}$ will have holonomy limit on almost all fibres $\overline{V}_p$ defining the $C(J)$–reduction and differ by a model singular connection A_α on $(\overline{V}, s)$ by a global $L^p_{1,\alpha}$ form because this was true for the A_i's. If g, (V', A') and $(\overline{V}', s', \overline{A}', A'_\alpha)$ are as specified, it follows by Lemma 1.11 applied to $Hom(V, V')$ that in the gauges of 1.14 (ii) g is represented by $L^p_{2,\alpha}$ matrices thus, by the construction, g will extend to an $L^p_{2,\alpha}$ isomorphism $\overline{g}(\overline{V}, \overline{A}) = (\overline{V}', \overline{A}')$.

Now let $\rho : G \to U(n)$ be a faithful representation and suppose that $(V, \tilde{A}) = (E \times_G \mathbb{C}^n, \rho(A))$ for some G–connection on E with admissible holonomy J. Since J is admissible, using the relevant clauses in the Uhlenbeck gauge fixing theorem or in Theorem 1.14 (ii) we can find a G–valued gauge transformation g_i such that (1.30) now becomes

$$g_i(A) = d + a_i, \quad a_i \in L^p_1(\Omega^1(\mathfrak{g})), \quad \text{or} \quad g_i(A) = d_{\rho(\alpha)} + a_i, \quad a_i \in L^p_{1,\alpha}(\mathfrak{g}), \tag{1.32}$$

where $i\alpha \in \mathfrak{g}$ exponentiates to J and $ad(\alpha)$ has all its eigenvalues in $(-1,1)$. From this we obtain an $L^p_{2,\rho(\alpha)}$ G–valued cocycle g_{ij}. Since $ad(\rho(\alpha))$ acts on $\rho(\mathfrak{g})$ as $ad(\alpha)$ on $\mathfrak{g}$, it has all its eigenvalues in $(1,-1)$ on it, so it follows from 1.11 that $\{g_{ij}\}$ is continuous across Σ and lies in $C(\rho(J))$ on Σ showing that $(\overline{V},s)$ has the required G–structure. We denote it $(\overline{E},s)$. It is then tautological that $A_{\rho(\alpha)}$ and $\overline{A}$ will induce G–connections for this G–structure. For the G–uniqueness let (E',A') be a second G–bundle with connection such that $(E',A') = g(E,A)$ on $X\backslash\Sigma$ and let ρ' be a second unitary faithful representation with associated bundle V'. We obtain an induced isomorphism

$$f(\mathfrak{g}_E, Ad(A)) = (\mathfrak{g}_{E'}, Ad(A'))$$

where $\mathfrak{g}_E$ and $\mathfrak{g}_{E'}$ are thought of as subbundles of $\mathfrak{g}_V$ and $\mathfrak{g}_{V'}$ respectively. Let α' be a second element in $\mathfrak{g}$ exponentiating to J such that $ad(\alpha')$ has all its eigenvalues in $(-1,1)$. In the gauges (1.32) f and its inverse are then represented by L^∞ matrices which must lie in $L^p_{2,\alpha*\alpha'}(\Omega^0(End(\mathfrak{g}))$, where

$$\alpha * \alpha'(f) = [\alpha,-]\circ f - f\circ[-,\alpha'].$$

From Lemma 1.8 we know that the blocks relative to the non-integer eigenvalues of $\alpha*\alpha'$ vanish on Σ. Pulling back by an element in the Weyl group of G we can assume that in these gauges f is an $ad(\mathfrak{g})$ valued function in $End(\mathfrak{g})$ and $[\alpha,-]$ and $[\alpha',-]$ are diagonal matrices with their eigenvalues on $\mathfrak{g}$ ordered in decreasing fashion. Since these are all in $(1,-1)$, if $[\alpha,-]_{|\mathfrak{g}} \neq [\alpha',-]_{|\mathfrak{g}}$ it follows that there is a row or a column of f or of f^{-1} on which $\alpha*\alpha'$ acts with eigenvalues lying all in $(0,1)$ or $(-1,0)$ on the non–zero components of f so that either $\det(f)_{|\Sigma} = 0$ or $\det(f^{-1})_{|\Sigma} = 0$ contradicting the boundedness of the two. Hence $[\alpha,-]_{|\mathfrak{g}} = [\alpha',-]_{|\mathfrak{g}}$ and f extends to an $L^p_{2,\alpha}$ G–isomorphism $\overline{f}(\overline{E},\overline{A}) = (\overline{E'},\overline{A'})$. □

It follows that at the topological level an L^p–curvature G–connection A on $X\backslash\Sigma$ with admissible holonomy J is completely determined by J and the topological type of the pair $(\overline{E},s)$ consisting of a G–bundle with $C(J)$–reduction on Σ associated to it by the proof. By the discussion at the beginning of the section this is determined by the classes k, c and ℓ which we therefore take as invariants of the connection. We shall see in §4.2 how, at least in the $U(n)$–case, k, c and ℓ can be recovered from Chern–Weil type formulae.

Remarks.

1) It follows from the proof that for almost all $p \in \Sigma$, $J = J(A) = \exp(-2\pi i\alpha)$ in $End(\overline{V})_p$, thus the orthogonal decomposition in (i) may be thought as being the eigenbundle decomposition of the limit holonomy. This is precise when A is an instanton since then by (iii) of 1.14 the limit holonomy of $\overline{A}$ will exists in every fibre in the gauges used for the construction.

2) If α is zero and only in this case A has an extension as an ordinary L^p_2 connection to X and hence, if it is an instanton, is gauge equivalent to a smooth one. So the holonomy is indeed the only obstruction to extensibility.

3) Since finite action euclidean instantons on $\mathbb{C}^2\backslash\Sigma$ are the same as finite action instantons on $S^4\backslash\Sigma$ for the standard round metric we can use the proposition to attach a pair of topological invariants to a finite action euclidean G–instantons on $\mathbb{C}^2\backslash\Sigma$. This could have

also been done more directly using the analysis of instantons on a punctured half–cylinder outlined in §1.3.

4) In the special case when $(X, \Sigma) = (S^4, S^2)$ and $G = SU(n)$ or a quotient of it we can still define extension data and topological invariants for the non–admissible holonomies but the procedure is not natural. Since the non admissible elements are those whose logarithms lie on the outer wall of the fundamental domain for the exponential in $(\mathfrak{h}, \mathfrak{h}^+)$ of Lemma 2.8 (the hyperplanes where the highest root takes value one), and the vertices of this correspond to the generators of the central lattice of $SU(n)$, it suffices to pass to the adjoint group to ensure that every element in $Ad(SU(n))$ is admissible. We can then apply the previous proposition to obtain topological invariants for the connection as an $Ad(SU(n))$–connection. Since the cohomology of S^4 in dimension different from zero and four vanishes for arbitrary coefficients, while the conjugacy classes $[J\zeta]$ with $\zeta \in Z(SU(n))$ are all distinct, we see that the extension data lifts to $SU(n)$ and the lifting are parametrised by the non–trivial elements in $Z(SU(n))$ (see §1.5). By the same reason, given a G–connection with central holonomy (this is the most degenerate case since then $ad(\alpha)$ has only $\{-1, 0, 1\}$as eigenvalues), the extension data attached to the associated $Ad(G)$–connection can be lifted to G.

More concretely this amounts to twisting by a suitable flat line bundle. Indeed, suppose that J is not admissible, and let $i\alpha = \log(J)$. As observed above the vertices of the outer wall of the fundamental domain $\mathfrak{f}$ in $\mathfrak{h}^+$ are the generators of the central lattice of $SU(n)$. This implies that there are unique elements $i\alpha'$ and w in the central lattice and Weyl group of $SU(n)$ such that

$$i\alpha'' = Ad(w)(i\alpha + i\alpha') \in \mathfrak{f},$$

with $ad(\alpha'')$ lying on one of the walls of $\mathfrak{h}^+$ and having all its eigenvalues in $(-1, 1)$. Since $[S^2] = 0$ there exist a global angular coordinate θ in the normal planes to S^2 so there is a trivial line bundle with flat connection

$$A' = d + i\gamma\, d\theta,$$

on $S^4 \backslash S^2$, where $\exp(-2\pi i\gamma)1 = \exp(-2\pi i\alpha')$. The tensor product connection

$$A'' = A \otimes A'$$

has then holonomy $J\exp(-2\pi i\alpha'')$ up to conjugation. Since A' is flat, A was an $SU(n)$–connection and $J\exp(-2\pi i\alpha') \in SU(n)$, it follows that A'' is still an $SU(n)$–connection and an instanton if A was. Since $J\exp(-2\pi i\alpha')$ is admissible by construction, we can take the extension data and invariants of A to be those attached to A'' by the previous Proposition. Using an associated bundle the same construction can be applied to all those G–connections whose holonomy becomes admissible when passing to the adjoint group.

Notice that since J and $J\exp(-2\pi i\alpha')$ belong to different conjugacy classes in $SU(n)$, A and A'' can not be gauge equivalent. It is in this sense that the above method of defining $SU(n)$–extension data and hence invariants for a connection with non admissible holonomy is not natural.

There is one more remark we wish to make vis–a–vis applications of Proposition 1.29 to Theorems 1 and 2. When $[\Sigma]$ is divisible by n there exist a flat singular line bundle

connection Γ on $X\backslash\Sigma$ with holonomy $\exp(2\pi i 1/n)$ around Σ, then for any connection A satisfying the hypothesis of 1.29 $\tilde{A} = A \otimes \Gamma$ also does and we can ask what is the relation between the topological invariants of A and those of $\tilde{A}$. The full answer will be given in Lemma 4.20, here we relate the first Chern classes of the two.

Lemma 1.33. *Let the hypotheses and notation be as in Proposition 1.29. Then*

(i) *If $[\Sigma]$ is divisible by n and $\tilde{V}$ is the extension associated to $\tilde{A}$ by 1.29 then*

$$c_1(\tilde{V}) = c_1(\overline{V}) - Tr(\tilde{\alpha} - \alpha)[\Sigma],$$

where $i\tilde{\alpha} = \log(J(\tilde{A}))$.

(ii) *If A is an $SU(n)$-connection*

$$c_1(\overline{V}) = -Tr(\alpha)[\Sigma].$$

Proof. (i) Let $\gamma = \tilde{\alpha} - \alpha$. Then the two cocycles for A and $\tilde{A}$ constructed in Proposition 1.29 can be taken to be related by

$$\tilde{g}_{ij} = \begin{cases} Ad(w)(\exp(2\pi i\gamma\theta_i) g_{ij} \exp(-2\pi i\gamma\theta_j)), & \text{if } U_i \cap U_j \cap \Sigma \neq 0 \\ w \exp(2\pi i\gamma\theta_i) g_{ij}, & \text{if } U_i \cap \Sigma \neq 0, U_j \cap \Sigma = 0 \\ g_{ij} \exp(-2\pi i\gamma\theta_j)) w^{-1}, & \text{if } U_i \cap \Sigma = 0, U_j \cap \Sigma \neq 0 \\ g_{ij}, & \text{if } U_i \cap \Sigma = U_j \cap \Sigma = 0 \end{cases} \tag{1.34}$$

for some element w of the Weyl group. Notice that though g_{ij} and g'_{ij} are single valued, $\exp(2\pi i\gamma\theta_i)$ is not. Similarly, we can take a set of transition functions for $[\Sigma]$ (from U_i to U_j) to be given by

$$\zeta_{ij} = \begin{cases} \exp(2\pi i(\theta_i - \theta_j), & \text{if } U_i \cap U_j \cap \Sigma \neq 0 \\ \exp(2\pi i\theta_i), & \text{if } U_i \cap \Sigma \neq 0, U_j \cap \Sigma = 0 \\ \exp(-2\pi i\theta_j), & \text{if } U_i \cap \Sigma = 0, U_j \cap \Sigma \neq 0 \\ 1, & \text{if } U_i \cap \Sigma, U_j \cap \Sigma = 0. \end{cases}$$

Since g_{ij} is the transition function from U_j to U_i (we are making gauge transformations act by pull-back), we deduce from (1.34) that a cocycle of transition functions for $\det(\tilde{V})$ and one for $\det(\overline{V})$ are related by

$$\det(\tilde{g}_{ij}) = \zeta_{ij}^{-t/n} \det(g_{ij}), \qquad t = Tr(\gamma),$$

hence $c_1(\tilde{V}) = c_1(\overline{V}) - Tr(\gamma)[\Sigma]$.

(ii) Since we assume A to be an $SU(n)$-connection $J \in SU(n)$ and there are traceless elements in $\mathfrak{u}(n)$ exponentiating to J. Then $i\alpha = \log_{U(n)}(J)$ differs from such elements by an element in $Ker(\exp(2\pi i(-)))$ so $Tr(\alpha)$ is a positive integer. Now regard A as an

$SU(n)$–connection, let $t = Tr(\alpha)$, let $\alpha' = \alpha - \Delta(1, \cdot\cdot, 1_t, 0, \cdot\cdot, 0)$ and consider the $SU(n)$–cocycle $\{g'_{ij}\}$ obtained from (1.31) for α'. The cocycle obtained from (1.32) on $N(\Sigma)$ is then of the form

$$g_{ij} = \exp(2\pi i \Delta(1, \cdot\cdot, 1_t, 0, \cdot\cdot, 0)\theta_i) g'_{ij} \exp(-2\pi i \Delta(1, \cdot\cdot, 1_t, 0, \cdot\cdot, 0)\theta_j),$$

so that

$$\det(g_{ij}) = \zeta_{ij}^{-t/n} \det(g''_{ij}), \quad g''_{ij} = \zeta_{ij}^{-t/n},$$

hence $c_1(\overline{V}) = -t[\Sigma] = -Tr(\alpha)[\Sigma]$ as required. □

Thus for $SU(n)$–connections $c_1(\overline{V})$ is completely determined by α, hence by J.

1.5 General structure groups on (S^4, S^2)

We now proof the claim made in the introduction that, much as in the ordinary case, the classification of finite action G–instantons on $S^4 \backslash S^2$ for an arbitrary compact connected Lie group G can be reduced to the case of a simple simply connected G. We need a regularity Lemma for L^2–harmonic forms on the complement of a submanifold of codimension $\geqslant 2$.

Lemma 1.35. *Let X be a compact Riemannian n–manifold, $n \geqslant 2$, and let Σ be a submanifold of codimension $\geqslant 2$, then there is an isomorphism*

$$\mathcal{K}^*(X) \cong L^2\mathcal{K}^*(X \backslash \Sigma)$$

between harmonic forms on X and L^2–harmonic (i.e. closed and coclosed) forms on $X \backslash \Sigma$.

Proof. Clearly $\mathcal{K}^*(X)$ embeds in $L^2\mathcal{K}^*(X \backslash \Sigma)$. To see that this is a surjection, observe that if φ is an L^2 harmonic k–form on $X \backslash \Sigma$, then $d \oplus d^*(\varphi)$ extends to a distribution in $L^2_{-1}(\Omega^{n-k-1}(X))$ supported on Σ. We want to show that it is 0. It suffices to show that given a smooth test $(n-k-1)$–form ψ supported near a point in Σ, there is a sequence ψ_i in $\mathcal{C}_0(X \backslash \Sigma)$ converging to ψ in L^2_1 so that

$$d \oplus d^*(\varphi)(\psi) = \int \varphi \wedge d \oplus d^*(\psi) = \int \varphi \wedge d \oplus d^*(\psi_i) = \int d \oplus d^*(\varphi) \wedge \psi_i = 0.$$

Then φ is weakly harmonic on X hence strongly harmonic by elliptic regularity. Let r be a smooth distance function from Σ and let ψ be a smooth 2–form supported near a point in Σ. Let χ be the Green's function for Σ near the point so that χ is approximately $\log(r)$ and consider a truncation χ_N of it so that $\chi_N = N$ near Σ. Let $A(1, 1/N)$ be the annulus at the origin in $\mathbb{R}^2$ with radii 1 and $1/N$. Then, as $N \to \infty$,

$$\begin{aligned} \int |\nabla((1 - \tfrac{\chi_N}{N})\psi) - \nabla\psi|^2 &\leqslant \tfrac{C}{N}\{\int |\nabla(\chi_N)|^2 + \int |\chi_N|^2\} \\ &\leqslant \tfrac{C'}{N}\{\int_{A(1,1/N)} \tfrac{1}{r^2} r dr + \int |\log r|^2 r dr\} \to 0 \end{aligned}$$

□

Proposition 1.36. *Let G be a compact connected Lie group and let $\overset{t}{\underset{i}{\times}} G_i$ be the simple factors of its universal covering group. There is a one to one correspondence between finite action G and $\overset{t}{\underset{i}{\times}} G_i$-instantons on $S^4\backslash S^2$ up to the choice of an element in $Z(\underset{i}{\times} G_i)$.*

Proof. First of all we prove that a finite action G–instanton always reduces to a semisimple subgroup. If A is a G–instanton on $S^4\backslash S^2$ decompose the curvature in central and semisimple components. It is well known that the central component is a direct sum of harmonic 2–forms which are L^2 by assumption. By the previous lemma these are harmonic on S^4 therefore zero. Thus A has semisimple curvature and so reduces to a semisimple subgroup of G which has universal covering group contained in $\underset{i}{\times} G_i$. Since the second cohomology of $S^4\backslash S^2$ vanishes for arbitrary coefficients A has a lifting to a connection on the universal covering group bundle [AHS]. Since A already reduced to the semisiple part the liftings must reduce to $\underset{i}{\times} G_i$ and are parametrised by $H^1(S^4\backslash S^2, Z(\underset{i}{\times} G_i)) = Z(\underset{i}{\times} G_i)$. □

Since each G_i is compact simple and simply connected we see that for the pair (S^4, S^2) it suffices to classify instantons with structure group of this kind. In view of Proposition 1.36 it follows that an instanton on $S^4\backslash S^2$ with compact structure group G and admissible holonomy has t pairs of topological invariants k_i, ℓ_i attached, one for each G_i.

1.6 Moduli spaces and framed singular instantons

We now define the moduli spaces $M^{*,G,J}_{\kappa,c,\ell}(X,\Sigma)$ and $\widetilde{M}^{G,J}_{\kappa,\ell}(\mathbb{C}^2,\Sigma)$ of Theorems 2 and 1 respectively. Let G be a Lie group as specified in §1.4 and J an admissible element in G. Let ρ be a faithful unitary representation and $(\overline{V}, s)$ a $U(n)$–bundle on X with $C(\rho(J))$–reduction on Σ having a G–structure $(\overline{E}, s)$ of topological type (k, c, ℓ). Let $i\alpha \in \mathfrak{g}$ be an element exponentiating to J with all the eigenvalues of $ad(\alpha)$ lying in $(-1, 1)$ and $A_{\rho(\alpha)}$ a model singular G–connection with holonomy J on $(\overline{V}, s)$. In this situation we defined the subspaces $L^p_{k,\alpha}(\mathfrak{g}_{\overline{E}})$, consisting of the $\mathfrak{g}_{\overline{E}}$–valued sections. These are preserved by the covariant derivative $\nabla_{A_{\rho(\alpha)}}$.

In order to define the moduli spaces we need first of all global versions of Lemmas 1.8 and 1.11 for the spaces (1.27).

Lemma 1.37. *Let $i\alpha \in \mathfrak{u}(n)$.*

(i) *Let $f \in \Omega^i(End(\overline{V}))$. Then $f \in L^p_{k,\alpha}$ if and only if f is in $L^p_{k,loc}$ off Σ and*

$$f_C \in L^p_k, \qquad f_U \in W^p_k, \quad \text{on } N(\Sigma),$$

and the smooth i–forms f such that

$$f_{|N(\Sigma)} \in \mathcal{C}^\infty(\Omega^i(End(V)_C)) \oplus \mathcal{C}^\infty_0(X\backslash\Sigma, \Omega^i(End(V)_U)),$$

are dense in $L^p_{k,\alpha}$.

(ii) *The embeddings and multiplications of 1.8 (ii) and (iii) hold for the $L^p_{k,\alpha}$ spaces (1.27).*

(iii) *The spaces* $L^p_{k,\alpha}$ *can be defined also by taking the covariant derivative of any connection of the form* $A_\alpha + [a, -]$ *with* $a \in L^p_{k-1,\alpha}$ *in (1.27).*

(iv) *Let* $d_{A_\alpha} + a$ *be a connection on* V *with* $a \in L^p_{k,\alpha}$. *If* $f \in L^\infty \cap Ker(d_{A_\alpha} + a)$ *then* $f \in L^p_{k+1,\alpha}$.

(v) *For* α *of the form (1.23) the spaces* $L^p_{k,\alpha}$ *depend on* α *only up to* $C(i\alpha)$ *and the operator norm of any continuous family of the form*

$$d_{A_{\alpha_t}} + [a_t, -] : L^p_{k,\alpha} \to L^p_{k-1,\alpha}, \qquad k - 1 \geqslant 0,$$

varies continuously provided $C(i\alpha_t)$ *remains constant.*

Proof. In a gauge compatible with (1.26), A_α differs from d_α by a smooth $End(V)_C$–valued form on $N(\Sigma)$. By (iii) of Lemma 1.8 it follows that the (1.6) can be defined using any connection of the form $d_\alpha + a$, $a \in L^p_{k,\alpha}$ so the (1.27) coincide with the (1.6) in this gauge which by the decomposition (1.26) implies (i) to (iv) from the corresponding statements in Lemmas 1.8 and 1.11. For α of the form (1.23) $End(V)_C = C(i\alpha)$ so the last claim follows from (i) noting that the operator norm of $d_{A_\alpha} - d_{A_{\alpha'}}$ is controlled by $|\alpha - \alpha'|$. □

As remarked several times if $(\overline{V}, s)$ has a G–structure and $\tilde{\alpha} = \rho(\alpha)$, then $[\tilde{\alpha}, -]$ acting on $\mathfrak{g}_{\overline{E}}$ has the same eigenvalues as $[\alpha, -]$, thus, while $\rho(\alpha)$ may not be of the form (1.26), if $ad(\alpha)$ has all its eigenvalues in $(-1, 1)$, $[\rho(\alpha), -]_{\mathfrak{g}_{\overline{E}}}$ behaves as if $\rho(\alpha)$ was of this form. In particular, for such an $\tilde{\alpha}$ Lemma 1.37 holds for the $\mathfrak{g}_{\overline{E}}$ sections.

Given $i\alpha \in \mathfrak{g}$ and letting $\{\alpha_{ij}\}$ be the eigenvalues of $ad(\alpha)$ on $\mathfrak{g}$ we defined

$$\alpha_0 = \min_{\alpha_{ij} \notin \mathbb{Z},\, n \in \mathbb{Z}} \{\, |\alpha_{ij} + n| \,\}. \tag{1.38}$$

For any $J \in G$ we then let

$$J_0 = \alpha_0,$$

where $i\alpha = \log(J)$ and for any p, $2 < p < (1/2 - J_0/6)^{-1}$ put

$$\mathcal{A}^p_J = \{\, A_\alpha + a \,|\, a \in L^p_{1,\alpha}(\mathfrak{g}_{\overline{E}}) \,\}. \tag{1.39}$$

and let $\mathcal{A}^{p,*}_J$ be the open subset consisting of connections whose holonomy (on $X \backslash \Sigma$) is centralized by $Z(G)$, the centre of G. We let

$$\mathcal{A}^{p,+}_J, \qquad \mathcal{A}^{p,+,*}_J,$$

consist of the A.S.D. elements in $\mathcal{A}_J$ and $\mathcal{A}^{p,*}_J$ respectively. It follows by the last Lemma and the remarks following it that when $k \geqslant 1$ the elements of $L^p_{k+1,\alpha}(\mathfrak{g}_{\overline{E}})$ are continuous so can be pointwise exponentiated and we define the gauge group to be

$$\mathcal{G}^p_J = \{\, \exp(g) \,|\, g \in L^p_{2,\alpha}(\mathfrak{g}_{\overline{E}}) \,\}. \tag{1.40}$$

This consists of continuous gauge transformations and by the multiplication theorems in 1.37 (ii) can be given a topological group structure and can be made to act smoothly on $\mathcal{A}_J^p$ by pull-back. Notice that by (i) of Lemma 1.37 $\mathcal{G}_J^p$ depends on J only through $C(J)$. We let

$$\mathcal{B}_J^p = \mathcal{A}_J^p/\mathcal{G}_J^p,$$

denote the set of equivalence classes of singular connections with holonomy J of class $L^p_{1,\alpha}$ equipped with the quotient topology and $\mathcal{B}_J^{p,*}$ the subset induced by connections with holonomy centralized by $Z(G)$. Define the moduli spaces $M^{G,\,J}_{\kappa,c,\ell}(X,\Sigma)$ and its part consisting of connections centralized by $Z(G)$, $M^{*,G,\,J}_{\kappa,c,\ell}(X,\Sigma)$, to be

$$M^{G,\,J}_{\kappa,c,\ell}(X,\Sigma) = \mathcal{A}_J^{p,+}/\mathcal{G}_J, \qquad M^{G,\,J,*}_{\kappa,c,\ell}(X,\Sigma) = \mathcal{A}_J^{p,+,*}/\mathcal{G}_J \tag{1.41}$$

with the topology induced from $L^p_{1,\alpha}$. By Proposition 1.29, κ, c and ℓ are indeed the topological invariants attached to any connection in $M^{G,\,J}_{\kappa,c,\ell}(X,\Sigma)$. In the sequel we shall usually suppress the pair (X,Σ) and the group G from the notation.

Our definition of the moduli spaces is a priori depending on the choice of the representation ρ, on the model connection A_α and on p. Because of (iii) of 1.14 one would expect the definition to be independent of p provided $2 < p < (1/2 - J_0/6)^{-1}$. We shall prove this in Proposition 4.6 so that our notation is justified.

For fixed p and ρ we see by (iii) of Lemma 1.37 that the definition is independent of the choice of A_α because if A'_α is another model singular G–connection relative to (1.26) the two differ by an $L^p_{2,\alpha}$–form and we can apply (iv) of 1.37. It was shown in Proposition 1.29 that, up to ordering, the eigenvalues of $ad(\alpha)$ restricted to $\rho(\mathfrak{g})$ are in fact independent of the choice of $i\alpha \in \mathfrak{g}$ exponentiating to J or of the representation chosen. It follows that the spaces $L^p_{k,\alpha}(\mathfrak{g}_{\overline{E}})$, and hence for fixed k and p the definition of the J–moduli spaces, is intrinsic. Indeed by the argument proving the G–uniqueness of the extension data, we see that if $(\overline{V},\, s)$ and $(\overline{V}',\, s')$ are two pairs with G–structures and the same topological data, a smooth G–isomorphism intertwining the reductions will induce a homeomorphism between any two moduli spaces with the same J.

Clearly every element in $M^{G,\,J}_{\kappa,c\,\ell}$ is contained in some gauge equivalence class of finite action G–instantons with holonomy J on $X\backslash\Sigma$. Vice–versa, by Proposition 1.29, every finite action G–instanton on $X\backslash\Sigma$ with admissible holonomy J is gauge equivalent to one in some $M^{G,\,J}_{\kappa,c,\,\ell}$. For fixed κ, c, ℓ and J, there is a family of these spaces which we can parametrise by varying $C(J)$–reductions of type ℓ on a fixed G–bundle with invariants κ, c, but as we just explained they are all homeomorphic so we think of $M^{J}_{\kappa,c\,\ell}$ as a single moduli space, and then gauge equivalent instantons on $X\backslash\Sigma$ correspond to the same gauge equivalence class in it. Let $M^J = \mathcal{A}_J^+/\mathcal{G}$ denotes the set of (smooth) gauge equivalence classes of finite action G–instantons on $X\backslash\Sigma$ with holonomy J along Σ and $M^{J,*} = \mathcal{A}_J^{+,*}/\mathcal{G}$ the subset determined by connections with holonomy centralized by $Z(G)$. We can summarise this discussion and §1.4 in the following statement.

Proposition 1.42. *Let X be a compact connected orientable Riemannian 4–manifold, Σ a compact connected orientable 2–submanifold, G a compact connected simple Lie group*

or $U(n)$ and $J \in G$ an admissible element. Any finite action A.S.D. G–connection A on $X\backslash\Sigma$ with holonomy J along Σ is gauge equivalent to one in $M^J_{\kappa,c,\ell}$ for some (k, c, ℓ) uniquely determined by A and we have identifications

$$M^J = \bigcup_{\kappa,\, c,\, \ell} M^J_{\kappa,c,\ell}\,, \qquad M^{J,*} = \bigcup_{\kappa,\, c,\, \ell} M^{J,*}_{\kappa,c,\ell}\,. \tag{1.43}$$

Now let p be a point in Σ and let C be a fixed copy of $C(J)$ in G. A framed singular connection is a pair (A, φ) where $A \in \mathcal{A}^p_J$ and φ is an isomorphism such that

$$\varphi : G \cong \overline{E}_{\mathrm{p}}, \quad \varphi(C(J)) = s(\mathrm{p}).$$

Consider the product

$$\mathcal{A}^p_J \times Hom(s(\mathrm{p}),\, C)).$$

This has an obvious product topology induced from the topology on $\mathcal{A}^p_J$. Since the gauge transformations in $\mathcal{G}^p_J$ are continuous we can make $\mathcal{G}^p_J$ act on it by

$$g(A, \varphi) = (g(A), Ad(g_{\mathrm{p}})(\varphi)).$$

This action is continuous with respect to the above topology and we define the set of equivalence classes of framed singular connections to be

$$(\mathcal{A}^p_J \times Hom(s(\mathrm{p}),\, C))/\mathcal{G}^p_J, \tag{1.44}$$

with the quotient topology. The moduli space of framed singular instantons is then the subset induced by $\mathcal{A}^{p,+}_J$

$$\widetilde{M}^J_{\kappa,c,\ell} = (\mathcal{A}^{p,+}_J \times Hom(s(\mathrm{p}),\, C))/\mathcal{G}^p_J, \tag{1.45}$$

with the induced topology. Again we show in Chapter 4 that (1.44) is independent of p. The independence on ρ and A_α follows from that of $\mathcal{A}^p_J$ proved above. There a different description of this space is given which is more convenient for analytical purposes.

There is an obvious action of C on $(\mathcal{A}^p_J \times Hom(s(\mathrm{p}),\, C))/\mathcal{G}^p_J$ by

$$\forall g \in C, \qquad g[A, \varphi] = [A, Ad(g)(\varphi)],$$

and the quotient by this action is $\mathcal{B}^p_J$. The action is free away from irreducible connections where the natural map is a $C(Ad(J))$–fibration so to understand the relation between $(\mathcal{A}^p_J \times Hom(s(\mathrm{p}),\, C))/\mathcal{G}^p_J$ and $\mathcal{B}^p_J$ hence between $\widetilde{M}^J_{\kappa,c,\ell}$ and $M^J_{\kappa,c,\ell}$, we need to understand the fibre at reducible connections.

Proposition 1.46. *Let $A \in \mathcal{A}^p_J$ then*

(i) $Ker(d_A) \cap L^p_{2,\alpha}(\mathfrak{g}_{\overline{E}}) = Lie(Stab(A)) \subseteq Lie(C(J))$.

(ii) *If $Ker(d_A) = Lie(C(J))$ A extends across Σ as a $C(Ad(J))$–connections of class L^p_1 and as a smooth connection if it is an instanton.*

Proof. (i) Fix a point p $\in \Sigma$ and let $Ker(d_A)(\mathrm{p})$ be the set of values of elements in $Ker(d_A)$ at p. By Lemma 1.37 we know that sections of $L^p_{k,\alpha}$ are continuous, so that $Ker(d_A)(\mathrm{p})$ is well defined, and lies in $(\mathfrak{g}_{\overline{E}})_C$ on Σ, a bundle of Lie algebras isomorphic to $Lie(C(J))$, so that $Ker(d_A)(\mathrm{p}) \subseteq Lie(C(J))$. Since $A \in \mathcal{A}^p_J$, by Fubini's Theorem it induces a continuous connection on almost every path through p and we can apply the usual parallel transport argument to deduce that a covariant constant section is completely determined by its value at p along these paths. Thus the elements in $Ker(d_A)(\mathrm{p})$ are determined almost everywhere by their value at p hence everywhere by continuity so that $Ker(d_A)$ is isomorphic to a subalgebra of $Lie(C(J))$.
(ii) Since $\exp(Ker(d_A))$ is a subgroup of the stabilizer of the holonomy of A and since $C(J)$ is self–stabilizing, if $Ker(d_A) = Lie(C(J))$ it follows that A must have holonomy reducing to $C(J)$. Thus A is a direct sum of connections with central holonomy along Σ and must therefore extend as a $C(J)$–connection of class L^p_1 on X and by the usual regularity arguments as a smooth $C(J)$–connection if it is an instanton.
□

Let $\{d_\alpha\}$ denote the set of all flat connections with holonomy $J = \exp(-2\pi i\alpha)$ along Σ.

Corollary 1.47. *When (X, Σ) is such that $\mathcal{H}^-(X) = 0$ and $C(J)$ is a maximal torus in G any $A \in \mathcal{A}^+_J$ such that $Stab(A) = C(J)$ is flat. If also $[\Sigma] = 0$ and $(\kappa, c, \ell) \neq 0$ or $[\Sigma] \neq 0$ and $\alpha_{ij}[\Sigma]$ is not an integral class for some eigenvalue α_{ij} of $ad(\alpha)$, where $i\alpha = \log(J)$, no such A exists. In particular when $G = SU(2)$ or $SO(3)$ and $J \neq \pm 1$*

$$\mathcal{A}^{+,*}_J = \mathcal{A}^+_J \backslash \{d_\alpha\},$$

and if the additional hypothesis hold the natural map

$$(\mathcal{A}^p_J \times Hom(s(p),\, C))/\mathcal{G}^p_J \xrightarrow{\pi} \mathcal{B}^p_J,$$

is a principal fibration.

Proof. By (i) of the previous Proposition when $Stab(A) = C(J)$ the connection reduces to a sum of abelian connections. By (ii) of the previous Proposition the curvature of a finite action G-instanton on $X \backslash \Sigma$ reducing to a torus extends smoothly across Σ as a direct sum of S^1 2–forms hence as a sum of A.S.D. harmonic forms on X and must be 0 if $\mathcal{H}^-(X) = 0$. Thus the connection must have been a direct sum of flat singular S^1–connections. If $[\Sigma] = 0$ the S^1–bundles on X arising from the components by Proposition 1.29 must all be trivial by Lemma 1.33 so that κ, c and ℓ are all 0. When $[\Sigma] \neq 0$ Lemma 1.33 shows that, denoting by α_{ij} the eigenvalues of $ad(\alpha)$ where $i\alpha = \log(J)$, $\alpha_{ij}[\Sigma]$ must be an integral class since it is the first Chern class of a vector bundle on X. Thus under the stated assumptions no such reduction can occur. If $G = SU(2)$ or $SO(3)$ and $J \neq \pm 1$, $C(J) = S^1$ so the only possible reductions fall into the above category and hence they must be flat. Since the action of $C(J)$ on $(\mathcal{A}^p_J \times Hom(s(\mathrm{p}),\, C))$ is free, when the additional assumptions hold it follows that π is a principal fibration. □

Chapter 2

Principal parabolic bundles and parabolic vector bundles

As it will be seen in Chapter 3 the analytical significance of Theorem 1 appears most clearly at the level of vector bundles and it is only at this level that the necessary analysis can be naturally carried out. The same can be said about the construction of the moduli spaces of parabolic and framed parabolic bundles. In order to bring this to bear and obtain the proof of Theorems 1 and 6 for general groups we therefore need to have a good understanding of the relation between holomorphic principal bundles and associated vector bundles in the framework of parabolic bundles. Though elementary this relation has to be spelled out with care and we devote the chapter to elucidate it together with its differential geometric interpretation and also to give the topological classification of quasi–parabolic bundles. We begin by recalling a few facts about parabolic subgroups.

2.1 Parabolic subgroups and flag manifolds

Let $G^{\mathbb{C}}$ be a connected complex reductive Lie group so that $G^{\mathbb{C}}$ is the complexification of a compact G, and let P be a parabolic subgroup of $G^{\mathbb{C}}$. Let $C^{\mathbb{C}}$ be a Levi subgroup of P, that is a maximal reductive subgroup, it then holds that $P \sim C^{\mathbb{C}}$, in particular $C^{\mathbb{C}}$ is connected. Let G be a compact form of $G^{\mathbb{C}}$ and $C = G \cap P$ the corresponding compact Levi subgroup of P. Since there is an obvious homotopy equivalence $C^{\mathbb{C}} \sim C$ the inclusion induces an equivalence $G/C \sim G^{\mathbb{C}}/P$, hence given a compact connected oriented two–manifold Σ,

$$[\Sigma, G/C] = [\Sigma, G^{\mathbb{C}}/P]. \tag{2.1}$$

As explained in §1.4 these are groups and to identify them we need some basic facts concerning the topology of G/C, namely its skeleton in dimension 1 and 2. Since the quotient G/C can be realized with G simply connected ([BH1]) it follows from the homotopy sequence of the fibration

$$C \to G \to G/C, \tag{2.2}$$

that G/C is simply connected. To determine the 2–skeleton we shall describe an explicit basis for $\pi_2(G/C)$ which (2.2) shows to be isomorphic to $\pi_1(C)$. Let $\mathfrak{c} = Lie(C)$ and fix a C.S.A. $\mathfrak{h} \subseteq \mathfrak{c}$. From the simple connectivity of G it follows that ([BR], Corollary 3.16)

$$\pi_1(C) \cong \Lambda_{\mathfrak{g}}(\mathfrak{h})/\Lambda_{[\mathfrak{c},\mathfrak{c}]}(\mathfrak{h} \cap [\mathfrak{c},\mathfrak{c}]) \cong \Lambda_{Lie(Z(C))}, \tag{2.3}$$

where for a Lie group K with a fixed C.S.A. $\mathfrak{t}$, $\Lambda_K(\mathfrak{t}) = \{u \in \mathfrak{t} | \exp_K(u) = 1\}$ is the integral lattice of $\mathfrak{t}$, showing that $\pi_1(C)$ is a free group. Now let $R = \{\beta\}$ denote the roots of the

chosen C.S.A., R_+ the roots which are positive for a Borel subgroup contained in P, so that P is the standard parabolic associated to some $I_{\mathfrak{p}} \subseteq R_+$, that is

$$\mathfrak{p} = Lie(P) = (\bigoplus_{\beta \in I_{\mathfrak{p}}} \mathfrak{g}^{\beta}) \oplus (\bigoplus_{\beta \in R_+ \setminus I_{\mathfrak{p}}} \mathfrak{g}^{\beta} \oplus \mathfrak{g}^{-\beta}),$$

and R_+^S the associated simple system so that $R_+^S \setminus I_{\mathfrak{p}}^S$ is a system of positive simple roots for $[\mathfrak{c}, \mathfrak{c}]$. For each $\beta \in R_+^S$ let J_β be the unique element in $\mathfrak{h}^{\mathbb{C}}$ such that

$$\forall H \in \mathfrak{h}, \qquad \beta(H) = K(H, J_\beta),$$

and define

$$H_\beta = 2i < \beta, \beta >^{-1} J_\beta, \tag{2.4}$$

where $<,>$ is the dual of the Killing form on $\mathfrak{g}^*$. Then $\{H_\beta\}$ generates the integral lattice of $\mathfrak{h}$ and $\{H_\beta\}_{\beta \in R_+^S \setminus I_{\mathfrak{p}}^S}$ that of $\mathfrak{h} \cap [\mathfrak{c}, \mathfrak{c}]$ so (2.3) immediately gives that $\{H_{\beta_i}\}_{\beta \in I_{\mathfrak{p}}^S}$ is a basis for $\pi_1(C)$, hence for $\pi_2(G/C)$. It is now easy to obtain a description of the 2–skeleton of G/C in terms of homogeneous 2–spheres which is also more natural from the point of view of $G^{\mathbb{C}}/P$. For each root β we obtain a copy of $\mathfrak{sl}(2)$ in $\mathfrak{g}^{\mathbb{C}}$ by taking

$$\mathfrak{sl}(2)_\beta = H_\beta \oplus \mathfrak{g}^{\beta} \oplus \mathfrak{g}^{-\beta}.$$

For each i

$$\mathfrak{sl}(2)_{\beta_i} \cap \mathfrak{p} = \mathfrak{p}_{\beta_i},$$

is a standard parabolic so at the group level it induces in the quotient a holomorphically embedded 2–sphere $S^2_{\beta_i}$

$$S^2_{\beta_i} = SL(2)_{\beta_i}/P_{\beta_i} \subseteq G^{\mathbb{C}}/P.$$

representing the homotopy class $[H_{\beta_i}]$. With this description we see that the 2–skeleton of $G^{\mathbb{C}}/P$ is the wedge of spheres $\bigvee_i S^2_{\beta_i}$ and we shall choose

$$-[S^2_{\beta_i}] = -[H_{\beta_i}] \tag{2.5}$$

as preferred basis for $\pi_2(G/C)$. We then have the identifications

$$[\Sigma, G^{\mathbb{C}}/P] = [\Sigma, G/C] = [\Sigma, \bigvee_i S^2_{\beta_i}] = \bigoplus_i [\Sigma, S^2_{\beta_i}] = \bigoplus_i H^2(\Sigma, \mathbb{Z}) \cong \overset{t}{\oplus}\mathbb{Z}, \tag{2.6}$$

with the isomorphism being given by associating to a map $\Sigma \to G^{\mathbb{C}}/P$ the degrees of the maps $\Sigma \to -[S^2_{\beta_i}]$ induced by it at the homotopy level. In the sequel $[\Sigma, G/C]$ or $[\Sigma, G^{\mathbb{C}}/P]$ shall always be identified with $\overset{t}{\oplus}\mathbb{Z}$ in this way.

While in general it is not true that $\pi_1(C)$ is free, it follows from (2.2) that $\pi_2(G/C)$ always embeds in $\pi_1(C)$. When $\pi_1(G)$ is free and C connected, so that $C = C(J)$ for some $J \in G$, it then follows that that $\pi_1(C)$ is also free so that there are decompositions

$$\begin{aligned} \mathfrak{c} &= \bigoplus_{i=1}^{t} \mathfrak{c}_i, \qquad \forall i, \qquad Z(\mathfrak{c}_i) = \mathbb{R}, \\ C &= \bigtimes_{i=1}^{t} C_i, \qquad \forall i. \quad \pi_1(C_i) = \mathbb{Z}. \end{aligned} \tag{2.7}$$

Let G be a compact connected Lie group, $W(G)$ its Weyl group, $(\mathfrak{h}, \mathfrak{h}^+)$ a pair consisting of a C.S.A. and a closed positive Weyl chamber. We claimed in the Introduction and in §1.4 that given any conjugacy class $[J]$ in G there is a unique element $i\alpha \in \mathfrak{h}^+$ exponentiating to an element $J \in [J]$ and with the property that $ad(\alpha)$ has all its eigenvalues in $[-1, 1]$. We now prove this. Notice that the statement for a compact connected G immediately follows once we have it for a simple G so we shall restrict our attention to this case.

Lemma 2.8. *Let G be a compact connected simple Lie group, $(\mathfrak{h}, \mathfrak{h}^+)$ as above and T the maximal torus of $\mathfrak{h}$. Then there is a unique fundamental region $\mathfrak{f} \subseteq \mathfrak{h}$ (i.e.$\exp(2\pi\mathfrak{f}) = T$ and exp is injective on the interior of $\mathfrak{f}$) such that the roots take values in $[-1, 1]$ on it and in $(-1, 1)$ away from the outer walls, further exp is bijective onto $T/W(G)$ when restricted to $\mathfrak{f} \bigcap \mathfrak{h}^+$.*

Proof. If G is of the form $G = \tilde{G}/Z$ with $\tilde{G}$ the universal covering group, $Z \leqslant Z(\tilde{G})$ and $\mathfrak{f}_{\tilde{G}}$ the desired fundamental region for $\tilde{G}$, we will show how to obtain the region for G by suitably restricting $\mathfrak{f}_{\tilde{G}}$. We therefore begin by proving the assertion for a simply connected G. Let K be the Killing form. Let $D(G) = \{H \in \mathfrak{h} \,|\exists \beta_i,\ \beta_i(H) \in \mathbb{Z}\}$. This is the union of all translates of the root hyperplanes in $\mathfrak{h}$ by multiples of the dual vectors J_{β_i} defined by (2.4). Let $\mathfrak{f}_0$ be a component of $\mathfrak{h} \backslash D(G)$ in $\mathfrak{h}^+$ whose closure contains the origin and put

$$\mathfrak{f} = \bigcup_{w \in W(G)} w \cdot \overline{\mathfrak{f}_0},$$

so that $\mathfrak{f}$ is the convex polyhedron such that all the roots take values in $[-1, 1]$ on it and in $(-1, 1)$ on its interior. The vertices of $\mathfrak{f}$ generate a maximal rank lattice Λ_D containing the integral lattice Λ and we have a natural homomorphism of tori $\Theta : \mathfrak{h}/\Lambda \longrightarrow \mathfrak{h}/\Lambda_D$ where $\mathfrak{h}/\Lambda = T$. Clearly exp maps $\mathfrak{f}$ onto the second torus. Since Θ is not constant it must be a surjection being a group homomorphism between compact tori of the same dimension and indeed a finite map of degree $|\Lambda_D/\Lambda|$. To prove that exp surjects $\mathfrak{f}$ onto T it thus suffice to show that $\mathfrak{f}$ maps onto $\exp(\Lambda_D/\Lambda)$ which follows from the very definition of Λ_D. To prove injectivity of exp on $\overset{\circ}{\mathfrak{f}}$ observe as recalled above that when G is simply connected, the integer lattice is generated by $\{H_{\beta_i}\}$. Then we can choose generators $\{\omega_i\}$ for Λ_D, such that any generator of Λ is of the form $\omega = \sum n_i \omega_i$ with the n_i's integers of the same sign and one of them greater than 2 in modulus if all the others are zero. It follows that

$$(\overset{\circ}{\mathfrak{f}} + \Lambda) \bigcap \overset{\circ}{\mathfrak{f}} = \varnothing$$

giving the injectivity of exp . Now consider $\overline{\mathfrak{f}_0}$. The surjectivity of exp restricted to $\overline{\mathfrak{f}_0}$ follows from the fact that the Weyl group $W(G)$ is simply transitive on the set of closed positive Weyl chambers and the fact that for any $w \in W(G)$, $H \in \mathfrak{h}$, we have that $\exp(Ad(w)(H)) = Ad(w)(\exp(H))$. The injectivity is guaranteed by the observation above concerning the form of the elements in Λ. Now let $G = \tilde{G}/Z$ with torus $T = \tilde{T}/Z$, and $\{1\} \neq Z \leqslant Z(\tilde{G})$. Put $\Lambda_Z = \{ H \in \mathfrak{h} \,|\, \exp(2\pi H) \in Z \}$ so that Λ_Z is a lattice contained in Λ_D and containing Λ. To keep exp injective on $\overset{\circ}{\mathfrak{f}}$ we may need to shrink the fundamental region obtained for $\tilde{G}$. In fact, because Λ is a proper subgroup of Λ_Z, we can replace D by a finer configuration of hyperplanes D' obtained by requiring that the roots take some suitable rational values on these and use it to define $\mathfrak{f}_G$ as above which gives a fundamental region with the required properties for $\tilde{T}/Z$. $\square$

Given $[J] \subseteq G$ we called the element $i\alpha \in \mathfrak{g}$ of the Lemma such that $\exp(-2\pi i\alpha)$ is the unique element in $\exp(\mathfrak{f} \cap \mathfrak{h}^+) \cap [J]$ the logarithm of $[J]$. We claimed in the introduction that given an element $i\alpha$ in a compact subalgebra $\mathfrak{g} \subseteq \mathfrak{g}^{\mathbb{C}}$, $P(\alpha)$ as defined by (0.3) is actually a parabolic subgroup. To see why this is true choose $(\mathfrak{h}, \mathfrak{h}^+)$ so that $i\alpha \in \mathfrak{h}^+$ and let $I_{\mathfrak{p}}$ be the subsets of positive roots such that $ad(\alpha)$ acts by strictly positive eigenvalues on the corresponding roots spaces. It follows from Jacobi's identity that $\mathfrak{p}(\alpha)$ is the standard parabolic subalgebra associated to $I_{\mathfrak{p}}$ with nilradical $\mathfrak{u}(\alpha)$ and Levi part $\mathfrak{c}(\alpha)$ so that $P(\alpha)$ is indeed a parabolic subgroup.

Given an admissible conjugacy class $[J] \subseteq G$ let $i\alpha = \log(J)$. Notice that since $ad(\alpha)$ has no integer eigenvalues other than 0, $C(i\alpha)^{\mathbb{C}}$, the centralizer of α under the adjoint action equals $C(J)^{\mathbb{C}}$. This is the crucial point that makes $P(\alpha)$ the "right" parabolic subgroup and allows us to relate the differential geometric with the holomorphic theory. When J is not admissible however $P(\alpha)$ no longer has this property because then $Lie(C(J)^{\mathbb{C}})$ is the sum of all integral eigenspaces of $ad(\alpha)$ whereas $\mathfrak{c}(\alpha) = Lie(C(i\alpha)^{\mathbb{C}}) = ker(ad(\alpha))$ consists of the zero eigenspace alone.

One case where the parabolic subgroup $P(\alpha)$ has a particularly simple description is when $G = GL(n)$. Recall that the type of a flag of subspaces of a vector space V^n

$$0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_l \subsetneq V^n,$$

is the set of integers $\{n_i\}$ where

$$rk(F_i) = \sum_{k \leqslant i} n_k. \tag{2.9}$$

If we take the compact form of $\mathfrak{h}^+$ to consist of purely imaginary diagonal matrices $i\alpha$ with eigenvalues $\{i\alpha_i\}$ for given multiplicities $\{n_i\}$ and the α_i decreasing $P(\alpha)$ is now the parabolic subgroup stabilizing the standard flag of type $\{n_i\}$.

To prove Theorem 1 we shall need to relate principal parabolic bundles and parabolic vector bundles (see §2.2). This will be achieved via the following elementary Lemma.

Lemma 2.10. *Let $G^{\mathbb{C}}$ be a complex simple Lie group or $GL(n)$, $\rho(G^{\mathbb{C}}) \to GL(m)$ a faithful representation, $i\alpha$ an element in some compact form of $\mathfrak{g}^{\mathbb{C}}$. Then*

$$P(\alpha) = \rho^{-1}(P(\rho(\alpha))).$$

Proof. By definition

$$P(\rho(\alpha)) = Ker(ad(\rho(\alpha)) \oplus \mathfrak{u}(\rho(\alpha)),$$

where $\mathfrak{u}(\rho(\alpha))$ consists of the strictly positive eigenspace of $ad(\rho(\alpha)$ on $\mathfrak{gl}(m)$. Since $ad(\rho(\alpha))$ acts on $d\rho(\mathfrak{g}^{\mathbb{C}})$ with the same eigenvalues as $ad(\alpha)$ on $\mathfrak{g}^{\mathbb{C}}$ it follows that

$$d\rho(\mathfrak{c}(\alpha)) \subseteq \mathfrak{c}(\rho(\alpha)), \qquad d\rho(\mathfrak{u}(\alpha)) \subseteq \mathfrak{u}(\rho(\alpha)),$$

hence $d\rho(\mathfrak{p}(\alpha) \subseteq \mathfrak{p}(\rho(\alpha))$ and so $\rho(P(\alpha)) \subseteq P(\rho(\alpha)) \cap \rho(G^{\mathbb{C}})$. Similarly the reverse inclusions must hold so $\rho(P(\alpha)) = \rho(G^{\mathbb{C}}) \cap P(\rho(\alpha))$. □

2.2 K–structures and parabolic bundles

Let $G^{\mathbb{C}}$ be a complex reductive Lie group and let $\mathcal{E} \to X$ be a principal holomorphic $G^{\mathbb{C}}$–bundle on a complex manifold X. Let $K \leqslant G^{\mathbb{C}}$ be a complex subgroup. Proceeding as in §1.1 but now requiring that all the maps and the bundles considered be holomorphic, we define holomorphic K–structures or reductions, associated holomorphic $G^{\mathbb{C}}$–bundles and K–isomorphism. In this case we can locally characterize the $G^{\mathbb{C}}$–bundles with K–structures as being exactly those whose defining holomorphic cocycle is equivalent to one of holomorphic K–valued transition functions. We shall be interested in the case where $\rho : G^{\mathbb{C}} \to GL(n)$ is a linear representation, then the associated $G^{\mathbb{C}}$–bundle can be identified with the holomorphic vector bundle

$$\mathcal{V} = \mathcal{E} \times_{G^{\mathbb{C}}} \mathbb{C}^n.$$

We shall now examine this in the contest of parabolic bundles. We begin by recalling the definitions of parabolic and framed parabolic vector bundles. Let (X, Σ) be a pair consisting of an n–dimensional complex manifold (we shall consider the case n=1 or 2) and a codimension 1 complex submanifold—a smooth divisor—Σ with irreducible components

$$\Sigma = \bigcup_j^s \Sigma^j.$$

Let $\{n_i^j\}$ be positive integers such that for every j, $\{n_i^j\}$ is a partition of n.

Definition 2.11.

(i) *An $\{n_i^j\}$–quasi–parabolic vector bundle on (X, Σ) is a pair $(\mathcal{V}, \{\mathcal{F}_i^j\})$, where $\mathcal{V}$ is a rank n holomorphic vector bundle on X and for each j, $\{\mathcal{F}_i^j\}$ is a filtration of $\mathcal{V}_{|\Sigma^j}$ by holomorphic subbundles*

$$0 \subsetneq \mathcal{F}_1^j \subsetneq \mathcal{F}_2^j \cdots \subsetneq \mathcal{F}_{t_j}^j \subsetneq \mathcal{V}_{|\Sigma^j}, \qquad rk(\mathcal{F}_i^j) = \sum_{k \leqslant i} n_k^j.$$

(ii) *An α–parabolic vector bundle on (X, Σ) consists of an $\{n_i^j\}$–quasi–parabolic vector bundle on (X, Σ) together with the choice of elements $\alpha = \{\alpha^j\}_{i=1}^s$ in $i\mathfrak{u}(n)$ where the eigenvalues $\{\alpha_i^j\}$ of α^j have multiplicities $\{n_i^j\}$ and satisfy*

$$\forall j, \qquad \alpha_{t_j}^j < \alpha_{t_j-1}^j \cdots < \alpha_1^j.$$

The $\{\alpha_i^j\}$ are called the weights of the filtration. Notice that unlike in most of the literature we do not assume the weights to lie all in an interval of the form $[C, C+1)$. This restriction is natural for the $U(n)$ case but not for parabolic vector bundles induced from a principal parabolic bundle via a representation. Two $\{n_i^j\}$–quasi–parabolic vector bundles are isomorphic if the underlying holomorphic vector bundles are isomorphic by an isomorphism preserving the filtrations and two parabolic bundles are isomorphic if their underlying quasi–parabolic bundles are isomorphic and have the same weights. We denote by

$$\mathcal{M}^P(X, \Sigma), \qquad \mathcal{M}^\alpha(X, \Sigma), \tag{2.12}$$

the set of isomorphism classes of $\{n_i^j\}$–quasi–parabolic and α–parabolic vector bundles on (X, Σ) respectively where $P = \{P^j\}$ and for each j, P^j is the parabolic subgroup of $GL(n)$ (unique up to conjugation) stabilizing a flag of type $\{n^j\}_{i=1}^{t_j}$. This notation will be justified in a moment. In the sequel we shall usually denote parabolic principal or vector bundles simply by their underlying holomorphic bundles on X and omit the pair (X, Σ) from the notation for the various moduli spaces.

Let $\mathcal{V}$ be an α–parabolic bundle with a $G^{\mathbb{C}}$–structure and suppose $\alpha^j = \rho(\beta^j)$ for elements $i\beta^j \in \mathfrak{g}^{\mathbb{C}}$ lying in some compact form of $\mathfrak{g}^{\mathbb{C}}$. Notice that the filtration on $\mathcal{V}_{|\Sigma^j}$ naturally induces a holomorphic reduction to $P(\alpha^j)$ on the holomorphic frame bundle of $\mathcal{V}_{|\Sigma^j}$ by considering only the α^j–filtered holomorphic frames, that is those frames $\{e_k^j\}$ such that

$$\forall i, \quad < \{\iota_{\Sigma^j}^*(e_k^j) \,|\, k \leqslant \sum_{l \leqslant i} n_l^j\} >= \mathcal{F}_i^j, \tag{2.13}$$

where ι_{Σ^j} is the inclusion of Σ^j. In particular an $\{n_i^j\}$–quasi–parabolic vector bundle is a P–quasi parabolic principal bundle. We shall say that the $G^{\mathbb{C}}$–structure is compatible with the parabolic structure if at each point in Σ^j there is a holomorphic $G^{\mathbb{C}}$–frame which is α^j–filtered. In general, given a G–bundle with a K and a K'–structure defined on submanifolds Σ and Σ', we say that the two are compatible if at each point of $\Sigma \cap \Sigma'$ there is a frame which is both a K and a K'–frame.

Proposition 2.14. *Let $\rho : G^{\mathbb{C}} \to GL(n)$ be a faithful linear representation and let $\{\alpha^j = \rho(\beta^j)\}$ where $i\beta^j$ are elements in a compact form of $\mathfrak{g}^{\mathbb{C}}$. There is a one–to–one correspondence between $\{\beta^j\}$–parabolic bundles and $\{\alpha^j\}$–parabolic vector bundles on (X, Σ) with a compatible $G^{\mathbb{C}}$–structure.*

Proof. Let $\mathcal{E}$ be a $\{\beta^j\}$–parabolic bundle. Passing to the associated ρ–vector bundle we obtain a holomorphic vector bundle $\mathcal{V}$ with a $G^{\mathbb{C}}$–structure and a holomorphic reduction on Σ^j to any subgroup of $GL(n)$ containing $\rho(P(\beta^j))$. In particular $\mathcal{V}_{|\Sigma^j}$ has a reduction to $P(\alpha^j)$. It follows that $\mathcal{V}_{|\Sigma^j}$ must a have a cocycle of holomorphic $P(\alpha^j)$–valued transition functions. Since by construction $P(\alpha^j)$ is the stabilizer of the standard type $\{n_i^j\}$ flag this means that for every j, $\mathcal{V}_{|\Sigma^j}$ ie equipped with a filtration by holomorphic subbundles $\{\mathcal{F}_i^j\}$ with $rk(\mathcal{F}_i^j) = n_i^j$ and so $\{\alpha_i^j\}$ can be taken as weights for the filtration to give it the

structure of an α–parabolic vector bundle with a $G^{\mathbb{C}}$–structure which is by construction compatible. Vice–versa, let $\mathcal{V}$ be an α–parabolic vector bundle with a compatible $G^{\mathbb{C}}$–structure. As we explained before the beginning of the proof, the parabolic filtrations give holomorphic $P(\alpha^j)$–reductions on $\mathcal{V}_{|\Sigma^j}$. By the compatibility assumption at a point in Σ we can choose an α^j–filtered frame which is also a $G^{\mathbb{C}}$–frame. The transition functions between two such frames can then be chosen $G^{\mathbb{C}}$–valued and must lie in $P(\alpha^j)$ because the two frames where filtered. By Lemma 2.10 we know that $G^{\mathbb{C}} \cap P(\alpha^j) = P(\beta^j)$ so that we obtain for $\mathcal{V}$ a cocycle of holomorphic $G^{\mathbb{C}}$–valued transition functions lying in $P(\beta^j)$ on Σ. The bundle of holomorphic $G^{\mathbb{C}}$–frames is thus naturally equipped with $P(\beta^j) = G^{\mathbb{C}} \cap P(\alpha^j)$–reductions on the Σ^j's and is therefore a β–parabolic principal bundle. The procedures of going from a $\{\beta^j\}$–parabolic principal bundle to an $\{\alpha^j\}$–parabolic vector bundle and then back as just described are mutually inverse proving the claim. □

In particular we have the (obvious) consequence that when $G^{\mathbb{C}} = GL(n)$ and ρ is the fundamental representation the notion of parabolic principal and vector bundle coincide.

Now let D_∞ be a further smooth divisor meeting Σ transversely so that $\Sigma \cap D_\infty$ is a smooth divisor in D_∞, and let $\mathcal{V}^0$ be a quasi–parabolic or α–parabolic vector bundle on $(D_\infty, \Sigma \cap D_\infty)$. Note that a quasi–parabolic or parabolic vector bundle $\mathcal{V}$ on (X, Σ) induces by restriction a quasi–parabolic or parabolic vector bundle on $(D_\infty, \Sigma \cap D_\infty)$.

Definition 2.15.

(i) *A $\mathcal{V}^0$–framed quasi–parabolic vector bundle on (X, Σ, D_∞) is a quasi–parabolic vector bundle $\mathcal{V}$ on (X, Σ) together with a framing—an isomorphism of quasi – parabolic vector bundles*

$$\varphi : \mathcal{V}_{|D_\infty} \xrightarrow{\simeq} \mathcal{V}^0.$$

(ii) *A $\mathcal{V}^0$–framed parabolic vector bundle on (X, Σ, D_∞) is a parabolic vector bundle $\mathcal{V}$ on (X, Σ) together with a framing—an isomorphism of α–parabolic vector bundles*

$$\varphi : \mathcal{V}_{|D_\infty} \xrightarrow{\simeq} \mathcal{V}^0.$$

Much as in the case of parabolic framed bundles two framed quasi–parabolic or parabolic vector bundles are isomorphic if the underlying quasi–parabolic or parabolic vector bundles are isomorphic by an isomorphism intertwining the framings. As in the Introduction let $\mathcal{E}^0$ be a β–parabolic principal bundle on $(D_\infty, \Sigma \cap D_\infty)$. Letting $\rho : G^{\mathbb{C}} \to GL(n)$ be a faithful representation and applying Proposition 2.14 to $\mathcal{E}^0$ we obtain an α–parabolic vector bundle $\mathcal{V}^0$ on $(D_\infty, \Sigma \cap D_\infty)$ with a compatible $G^{\mathbb{C}}$–structure, where $\alpha = \rho(\beta)$. A $\mathcal{V}^0$–framed parabolic vector bundle $\mathcal{V}$ is said to have a $G^{\mathbb{C}}$–structure if it has a compatible $G^{\mathbb{C}}$–bundle structure and the isomorphism

$$\varphi : \mathcal{V}_{|D_\infty} \xrightarrow{\simeq} \mathcal{V}^0.$$

is a $G^{\mathbb{C}}$–isomorphism. Proceeding as in Proposition 2.14 we then obtain

Proposition 2.16. *Let $\rho : G^{\mathbb{C}} \to GL(n)$ be a faithful representation. There is a one–to–one correspondence between $\mathcal{E}^0$–framed parabolic principal bundles and $\mathcal{V}^0$–framed parabolic vector bundles on (X, Σ, D_∞) with a compatible $G^{\mathbb{C}}$–structure.*

As recalled above K–structures have also an interpretation in terms of cocycles of transition functions and we can correspondingly translate the definitions of quasi–parabolic (parabolic) and framed quasi–parabolic (parabolic) principal bundles in this language. Thus a P–quasi–parabolic bundle on (X, Σ) is the same as a cocycle $\{g_{ij}\}$ of holomorphic $G^{\mathbb{C}}$–valued functions taking values in P^j on Σ^j and a $\mathcal{E}^0$–framed quasi parabolic bundle is a pair consisting of a cocycle $\{g_{ij}\}$ as above and a $G^{\mathbb{C}}$–cocycle $\{f_i\}$ on D_∞ such that

$$(g_{ij})_{|D_\infty} = f_i^{-1} g_{ij}^0 f_j,$$

where $\{g_{ij}^0\}$ is the cocycle defining the framing bundle $\mathcal{E}^0$. As noticed in the proof of 2.14 an $\{n_i^j\}$–quasi–parabolic vector bundle is the same as a $GL(n)$–holomorphic cocycle restricting to $\mathcal{F}^j$ on Σ^j and we can thus reinterpret both 2.14 and 2.16 in terms of cocycles of transition functions.

2.3 Topological classification

In this section we give a topological classification of principal quasi–parabolic bundles in terms of the classes κ, c and ℓ mentioned in the Introduction.

We begin by introducing the notions of quasi–parabolic and framed quasi parabolic bundles in the $\mathcal{C}^\infty$ category. These are as in Definition 0.6 and 0.7 omitting the word holomorphic. Similarly for the definition of isomorphism. In an analogous manner we define $\mathcal{C}^\infty$ quasi–parabolic vector bundles and framings. We shall usually denote by $E^{\mathbb{C}}$, V and F_i respectively topological $G^{\mathbb{C}}$–bundles, vector bundles and the elements in the filtration of a $\mathcal{C}^\infty$ quasi–parabolic vector bundle. Let $G^{\mathbb{C}}$ a complex reductive group. As recalled in §2.1 there is a homotopy equivalence $G^{\mathbb{C}} \sim G$ so that

$$[X, BG^{\mathbb{C}}] = [X, BG],$$

and the topological type of a $G^{\mathbb{C}}$–bundle is the same as that of any G–reduction on it. As recalled in §1.4, when X is a compact connected orientable 4–manifold and G a compact connected simple Lie group or $GL(n)$, G–bundles on X are classified by a pair of classes $(\kappa, c) \in H^4(X, \mathbb{Z}) \times H^2(X, \pi_1(G))$. Now let P be a parabolic subgroup of $G^{\mathbb{C}}$, G a compact form of $G^{\mathbb{C}}$, $C = G \cap P$ the corresponding compact Levi subgroup for P and Σ a compact connected Riemann surface. As recalled in §2.1 there is an homotopy equivalences $G^{\mathbb{C}}/P \sim G/C$ so that inclusion induces homotopy equivalences between the fibration

$$G^{\mathbb{C}}/P \longrightarrow EG^{\mathbb{C}}/P \longrightarrow EG^{\mathbb{C}}/G^{\mathbb{C}},$$

and fibration (1.24). It follows that topological P–reductions on a given $G^{\mathbb{C}}$–bundle on Σ are parametrised by $[\Sigma, G/C]$ and since the natural map $\pi_2(G/C) \to \pi_2(G^{\mathbb{C}}/P)$ can be thought of as the identity, the class corresponding to a given P–reduction is the same as

that corresponding to the C–reduction induced by any G–reduction on the $G^{\mathbb{C}}$–bundle. Thinking of a P–reduction on a $G^{\mathbb{C}}$–bundle as of a map $\Sigma \to G^{\mathbb{C}}/P$, which by (2.7) is topologically determined by a class $\ell \in \overset{t}{\oplus} H^2(\Sigma, \mathbb{Z})$, we see that for such $G^{\mathbb{C}}$ the topological types of quasi–parabolic bundles on (X, Σ) are in one–to–one correspondence with triples (κ, c, ℓ) as described. In the case of framings, since the bundle is isomorphic to a given one on D_∞ the framing isomorphism is topologically trivial so that κ, c and ℓ also classify topological types of framed quasi–parabolic bundles. From this we obtain partitions of the set of equivalence classes of P–quasi–parabolic bundles on (X, Σ) and of $\mathcal{E}^0$–framed quasi–parabolic bundles on (X, Σ, D_∞)

$$\mathcal{M}^P = \bigcup_{\kappa,c,\ell} \mathcal{M}^P_{\kappa,c,\ell}, \qquad \widetilde{\mathcal{M}}^P = \bigcup_{\kappa,c,\ell} \widetilde{\mathcal{M}}^P_{\kappa,c,\ell}. \tag{2.17}$$

When G is simply connected $c = 0$ so that $\mathcal{M}^P$ and $\widetilde{\mathcal{M}}^P$ are partitioned by the invariants κ and ℓ only. When $X = \mathbb{P}^2$, D_∞ a complex line and we take $\mathcal{E}^0$ to be holomorphically trivial $G^{\mathbb{C}}$–bundle, an $\mathcal{E}^0$–framing always has a a unique holomorphic lifting to the universal covering of $G^{\mathbb{C}}$ so c must vanish and we may assume G simply connected. Furthermore, much as for singular instantons on (S^4, S^2), a principal $(G^{\mathbb{C}}, P)$–parabolic bundle trivial over D_∞ with $G^{\mathbb{C}}$ a reductive group has a unique lifting to a product of $(G_i^{\mathbb{C}}, P_i)$–parabolic bundles where $\underset{i}{\times} G_i^{\mathbb{C}}$ are the simple factors of the universal covering group of $G^{\mathbb{C}}$ and $P_i = G_i^{\mathbb{C}} \cap P$. Thus the holomorphic classification of these objects can be reduced to the case of complex simple simply connected structure groups.

Notice that in (2.17) some of the components may well be empty and the moduli spaces considered in §5.2 give examples of this. When $\Sigma = \overset{t}{\bigcup} \Sigma^j$ and we consider P–quasi–parabolic bundles for $P = \{P^j\}$, instead of a single class $\ell \in \overset{t}{\oplus} H^2(\Sigma^j, \mathbb{Z})$, as in (2.17) there will be a vector

$$\underline{\ell} \in \pi_2(G^{\mathbb{C}}/P_1) \times \cdots \times \pi_2(G^{\mathbb{C}}/P_t),$$

parametrising topological types of P–quasi–parabolic bundles. We shall return to the case of more general divisors Σ and D_∞ in §5.3. From now on and until then Σ and D_∞ are always smooth irreducible curves.

Let $E^{\mathbb{C}}$ be a P–quasi–parabolic bundle with invariants k, c, ℓ so that the adjoint bundle $\mathfrak{g}_{E^{\mathbb{C}}}$ has a subbundle of parabolic subalgebras $\mathfrak{p}$. In Chapter 5 we shall give a formula for the dimension of the moduli space of quasi–parabolic bundles at smooth points in terms of κ, c, ℓ and the topological invariant of X and Σ (and also D_∞ in the framed case) and to this aim we need an explicit formula for $\deg(\mathfrak{g}_{E^{\mathbb{C}}}/\mathfrak{p})$ in terms of these invariants which we now give.

Lemma 2.18. *Let $G^{\mathbb{C}}$ be a simple connected complex Lie group, $E^{\mathbb{C}}$ a P–quasi–parabolic bundle with topological invariants κ, c, ℓ. R_+ any system of positive roots such that P is the standard parabolic associated to the subsystem $I_{\mathfrak{p}} \subseteq R_+$ and $Z(\mathfrak{c}^{\mathbb{C}})$ the centre of the distinguished Levi subalgebra of $\mathfrak{p}$ for I. Then*

$$C_P(-) = \sum_{\beta \in I_{\mathfrak{p}}} i\beta(-) : Z(\mathfrak{c}^{\mathbb{C}}) \to \mathbb{C}.$$

descends to an integer valued functional on $[\Sigma, G^{\mathbb{C}}/P]$ *independent of the choice of* R_+ *and*

$$\deg(\mathfrak{g}/\mathfrak{p}) = \sum_{\beta \in I_{\mathfrak{p}}} i\beta(\ell).$$

Proof. Fix a pair $(\mathfrak{h}, \mathfrak{h}^+)$ as in Lemma 2.8 so that P is the standard parabolic for some subsystem $I_{\mathfrak{p}}$ of the positive roots R_+. Let G be the compact form of $G^{\mathbb{C}}$ such that the corresponding compact Levi subgroup $C = G \cap P$ of P has $Lie(C) = \mathfrak{c}$. As in §2.1 we may assume that $G^{\mathbb{C}}$ is actually simply connected so that by (2.7) $\mathfrak{c}$ decomposes as a direct sum $\mathfrak{c} = \oplus_{i=1}^t \mathfrak{c}_i$, where each $\mathfrak{c}_i$ has centre $\mathbb{R}$ and C as a product $C = \times_{i=1}^t C_i$ with $\pi_1(C_i) = \mathbb{Z}$ for each C_i. Thus $Z(\mathfrak{c}) = \oplus_{i=1}^t \mathbb{R}$. We can consider the functional

$$C_C = \sum_{\beta \in I_{\mathfrak{p}}} i\beta(-) : Z(\mathfrak{c}) \longrightarrow \mathbb{R}. \tag{2.19}$$

This is integral valued on $\Lambda_{Z(\mathfrak{c})}$ and we claim that it is in fact independent of the choice of $\mathfrak{h}$ and in fact of $\mathfrak{c}$ as well so that it induces an intrinsically defined element in $H^1(P)$. First let $\mathfrak{h}'$ be a second C.S.A. contained in $\mathfrak{c}$ and fix a set of positive roots R'_+ for it associated to a Borel subgroup contained in P. We can find an element $g \in C$ such that

$$Ad(g)(\mathfrak{h}) = \mathfrak{h}'.$$

and such that $I'_{\mathfrak{p}} = Ad(g)(I_{\mathfrak{p}})$. Since $Z(\mathfrak{c})$ is central in $\mathfrak{c}$, $Ad(g)$ acts trivially on it therefore

$$\forall H \in Z(\mathfrak{c}), \quad \sum_{\beta' \in I'_{\mathfrak{p}}} i\beta'(H) = \sum_{\beta' \in I'_{\mathfrak{p}}} i\beta'(Ad(g)(H)) = \sum_{\beta \in I_{\mathfrak{p}}} i\beta(H),$$

proving that C_C is independent of the choice of $\mathfrak{h} \subseteq \mathfrak{c}$. By (2.3) we can naturally identify $\pi_1(C)$ with the integral lattice in $Z(\mathfrak{c})$. thus C_C induces an integral valued functional on $\pi_1(C)$. As recalled in §2.1 for any choice of compact form G of $G^{\mathbb{C}}$ and corresponding Levi subgroup $C = G \cap P$ we can identify $\pi_1(C)$ with $\pi_1(P)$ thus for each choice of C (2.19) gives an induced functional, C_C, on $\pi_1(P)$ and it follows that this functional is independent of C if we can show that for any two Levi subgroups C and C' of P such that $Ad(g)(C) = C'$

$$\forall H \in P, \quad C_{C'}(H) = C_C(Ad(g)(H)).$$

Since we have just shown that C_C and $C_{C'}$ are independent of the choice of the C.S.A. used to define them, we can compute them taking C.S.A.'s $\mathfrak{h}$ and $\mathfrak{h}' = Ad(g)(\mathfrak{h})$ and positive root systems such that P is the standard parabolic for $I_{\mathfrak{p}}$ and $I'_{\mathfrak{p}} = Ad(g)(I_{\mathfrak{p}})$. We then have

$$\begin{aligned} \forall H \in P, \quad C_{C'}(H) &= \sum_{\beta' \in I'_{\mathfrak{p}}} i\beta'(H) \\ &= \sum_{\beta \in I_{\mathfrak{p}}} i\beta(Ad(g)(H)) \\ &= C_C(Ad(g)(H)). \end{aligned}$$

as required. As recalled in §2.1 $\pi_2(G^{\mathbb{C}}/P)$ embeds in $\pi_1(P)$ so that C_P induces an element in $(\pi_2(G^{\mathbb{C}}/P))^* = H^2(G^{\mathbb{C}}/P)$. As already remarked, at the homotopy level we can identify $[\Sigma, G^{\mathbb{C}}/P]$ with $[S^2, G^{\mathbb{C}}/P] = \pi_2(G^{\mathbb{C}})$ so that we obtain a functional on $[\Sigma, G^{\mathbb{C}}/P] = [\Sigma, BP]$. It is easily shown that the induced functional is independent of the identification since this amounts to an unimodular automorphism in the lattice $\pi_2(G^{\mathbb{C}})$. Let $\ell \in [\Sigma, G^{\mathbb{C}}/P]$ be the element classifying the given P–reduction on $E^{\mathbb{C}}$ identified with $\sum_{\beta_i \in I_{\mathfrak{p}}^S} \ell_i(-H_{\beta_i})$. We show that

$$\deg(\mathfrak{g}_{E^{\mathbb{C}}}/\mathfrak{p}) = C_P(\ell).$$

From the P–reduction on $E^{\mathbb{C}}_{|\Sigma}$ we obtain a C–reduction thus $E^{\mathbb{C}}_{|\Sigma} = \times E_i^{\mathbb{C}}$ is a product of $\times_{i=1}^t C_i$–bundles. Let R_+^S be a system of positive simple roots with corresponding generators $\{H_\beta\}$ for the integral lattice in $Z(\mathfrak{c})$ as in (2.4) so that they also give a basis for $\pi_1(C)$. Since Σ is 2–dimensional

$$[\Sigma, BC] = \overset{t}{\underset{i}{\times}}[\Sigma, BC_i] = \overset{t}{\underset{i}{\times}}[\Sigma, BU(1)],$$

and each of the C_i–bundles reduces further to an S^1–bundle where S^1 with its positive orientation can be identified with $\exp(2\pi t H_{\beta_i})$, the 1–parameter subgroup of H_{β_i}. Then the inclusion induces an isomorphism

$$\pi_1(S^1) \overset{\cong}{\hookrightarrow} \pi_1(C_i),$$

and with this choice the integer giving the topological type of the C_i–bundle, the first Chern class of this S^1–reduction, is $-\ell_i$. By Chern–Weil theory, on $E^{\mathbb{C}}$ we can find a connection $a = \oplus a_i$ with each a_i an S^1–connection and curvature $b = \oplus b_i$ such that

$$\forall_i, \qquad \frac{i}{2\pi}\int_\Sigma b_i = c_1(E_i^{\mathbb{C}})(H_{\beta_i}) = -\ell_i.$$

Again by Chern–Weil theory we have that

$$deg(\mathfrak{g}_{E^{\mathbb{C}}}/\mathfrak{p}) = -\deg(\mathfrak{p}) = -\frac{i}{2\pi}\int Tr(ad(b)_{|\mathfrak{p}}),$$

where since $b \in \Omega^2(\mathfrak{c})$ and $\mathfrak{p}$ is self–normalizing $ad(b)$ induces an endomorphism of $\mathfrak{p}$, and it follows that

$$\begin{aligned} deg(\mathfrak{g}_{E^{\mathbb{C}}}/\mathfrak{p}) =& -iTr(ad(\sum_{i=1}^t \ell_i(H_{\beta_i}))_{|\mathfrak{p}}) \\ =& i\sum_{i=1}^t \ell_i Tr(ad(-H_{\beta_i})_{|\mathfrak{p}}) \\ =& i\sum_{i=1}^t \ell_i(\sum_{\pm\beta\in R_+\setminus I_{\mathfrak{p}}} \beta(-H_{\beta_i}) + \sum_{\beta\in I_{\mathfrak{p}}} \beta(-H_{\beta_i})) \\ =& i\sum_{\beta\in I_{\mathfrak{p}}} \beta(\ell) \\ =& C_P(\ell), \end{aligned} \tag{2.20}$$

as we wanted. □

When $G^{\mathbb{C}} = GL(n)$, it is clear from the definition of parabolic vector bundle that the topological type is also determined by κ, c and the multi–integer $(c_1, c_2, \cdots, c_t)$, where c_i is the first Chern class of the element F_i in the filtration. We now show that when the basis (2.6) is chosen for $[\Sigma, G^{\mathbb{C}}/P]$, $-(c_1, c_2, \cdots, c_t)$ coincides with the multi–integer determined by the class ℓ corresponding to the reduction determined by the filtration. Note that $c \cdot \Sigma$ is the class in $[\Sigma, BGL(n)]$ that classifies the restricition of a complex vector bundle on X to Σ. For $G^{\mathbb{C}} = GL(n)$ it follows from (1.24) that $[\Sigma, BP] = [\Sigma, GL(n)/P] \oplus [\Sigma, BGL(n)]$. Since C_P above was in fact defined as a functon on $\pi_1(P)$, in a similar way as we regarded it as a function on $[\Sigma, G^{\mathbb{C}}/P]$ we can also regard it as a function on $[\Sigma, BP]$ thus in particular $C_P(\ell + c \cdot \Sigma)$ is defined.

Lemma 2.20. *Let $P \leqslant GL(n)$ be the parabolic subgroup stabilizing a flag of type $\{n_i\}_{i=1}^t$, $E^{\mathbb{C}}$ a P–quasi–parabolic bundle with topological invariants (κ, c, ℓ), $\ell = \sum_i \ell_i[-H_{\beta_i}] \in [\Sigma, GL(n)/P]$ the element classifying the P–reduction written with respect to the basis (2.5) and V the associated $\{n_i\}$–quasi parabolic vector bundle on (X, Σ) with filtration $\{F_i\}$ by the fundamental representation. Then*

$$\forall i, \qquad c_i = c_1(F_i) = -\ell_i.$$

and

$$\deg(\mathfrak{g}_{E^{\mathbb{C}}}/\mathfrak{p}) = C_P(\ell + c \cdot \Sigma).$$

Proof. Choose a Hermitian metric on V. A representative for the class of the C–reduction corresponding to the P–reduction given by the filtration $\{F_i\}$ can be obtained from the orthogonal splitting

$$V_{|\Sigma} = \bigoplus_{i=1}^{t} F_{i+1}/F_i. \tag{2.21}$$

Let $k_i = rk(F_i)$. Without loss of generality we may assume that

$$\forall i, \quad 1 \leqslant i \leqslant t+1, \qquad F_i = \bigoplus_{i=1}^{k_i - 1} \mathcal{C}^{\infty} \oplus L(c_i),$$

where by $L(c_i)$ we mean a $\mathcal{C}^{\infty}$ line bundle of degree $c_i = c_1(F_i)$ and we are taking $F_{t+1} = V_{|\Sigma}$. Since

$$C = \bigtimes_{i=1}^{t+1} U(n_i),$$

we have as in the previous Proposition that

$$[\Sigma, BC] = \bigtimes_{i=1}^{t+1} [\Sigma, BU(1)].$$

Let $\{E_i\}$ be the standard positive roots for $GL(n)$ so that $\{E_i - E_{i+1}\}$ are the standard positive roots for $SL(n)$. Identifying the Lie algebra with its dual via the Killing form we can take the vectors H_β of (2.5) for P to be

$$H_{k_i} = i(E_{k_i} - E_{k_i+1}),$$

whereas we can take the 2–spheres $S^2_{H_{k_i}}$ determined by the H_{k_i} in BC to represent the universal first Chern classes for the summands in (2.21) and iE_n to represent the universal first Chern class of a $GL(n)$–bundle. With reference to the sequence (1.24) we can then decompose the class in $[\Sigma, BC]$ corresponding to (2.21) as

$$\sum_{i=1}^{t}(-c_i)(-i(E_{k_i+1} - E_{k_i})) + c_{t+1}iE_n,$$

where we identify with $i(E_{k_i+1} - E_{k_i})$ and iE_n the degree one elements in $[\Sigma, S^2_{H_{k_i}}]$ and $[\Sigma, BGL(n)]$. If $c = c_1(V)$ we then see that $c_{t+1} = c \cdot \Sigma$ and $\ell_i = -c_i$ for each i. The argument of the previous Proposition applied to (2.21) now gives that

$$\deg(\mathfrak{g}_{E^{\mathbb{C}}}/\mathfrak{p}) = C_P(\ell + c \cdot \Sigma).$$

□

2.4 Parabolic bundles and $\overline{\partial}$–operators

We shall now examine the differential geometric interpretation of holomorphic bundles in the contest of parabolic bundles. Let $E^{\mathbb{C}} \to X$ be a smooth $G^{\mathbb{C}}$–bundle over a complex manifold X. A holomorphic structure on $E^{\mathbb{C}}$ is an involutive $G^{\mathbb{C}}$–equivariant distribution $I_{\mathcal{E}} \subseteq (TE^{\mathbb{C}}) \otimes \mathbb{C}$ (often called an integrable semi–connection) satisfying

$$I_{\mathcal{E}} \oplus \overline{I_{\mathcal{E}}} = (TE^{\mathbb{C}}) \otimes \mathbb{C}. \qquad I_{\mathcal{E}} \cap \overline{I_{\mathcal{E}}} = 0.$$

Let $\rho : G^{\mathbb{C}} \to GL(n)$ be a linear representation. A semi–connection I_0 on $E^{\mathbb{C}}$ induces, much in the same way as a connection induces a covariant derivative, a $\overline{\partial}$–operator $\overline{\partial}_0$ on the associated vector bundle V, a first order differential operator

$$\overline{\partial}_0 : \Omega^0(V) \longrightarrow \Omega^{0,1}(V),$$

satisfying the Leibniz rule. Vice–versa such a $\overline{\partial}$–operator on V determines a semi–connection I_0 on $E^{\mathbb{C}}$. If I_1 is a second semi–connection then

$$I_1 = I_0 + a, \qquad a \in \Omega^{0,1}(X, \mathfrak{g}_{E^{\mathbb{C}}}),$$

and

$$\overline{\partial}_1 = \overline{\partial}_0 + \rho(a). \tag{2.22}$$

Conversely, given $a \in \Omega^{0,1}(\rho(\mathfrak{g}_{E^{\mathbb{C}}}))$, $\overline{\partial}_0 + a$ determines a semi–connection on $E^{\mathbb{C}}$. If ρ or at least the associated infinitesimal representation are faithful, it follows that there is a one–to–one correspondence between semi–connections on $E^{\mathbb{C}}$ and $\overline{\partial}$–operators of the form (2.22). We let $\mathcal{G}_{E^{\mathbb{C}}}$ denote the gauge group of $E^{\mathbb{C}}$. This acts by pull–back on the set of semi–connections and two semi–connections are isomorphic if they lie in the same $\mathcal{G}_{E^{\mathbb{C}}}$–orbit. Similarly the gauge group of V, $\mathcal{G}_V$, acts on the set of $\overline{\partial}$–operators by pull–back and two $\overline{\partial}$–operators coming from semi–connections define isomorphic $\overline{\partial}$–operators if they lie in the same orbit by the subgroup $\rho(\mathcal{G}_{E^{\mathbb{C}}})$ of $\mathcal{G}_V$. Thus there is a one–to–one correspondence between the set of isomorphism classes of semi–connections and the set of $\rho(\mathcal{G}_{E^{\mathbb{C}}})$–equivalence classes of $\overline{\partial}$–operators of the form (2.22). When I_0 is a holomorphic structure on $E^{\mathbb{C}}$, $\overline{\partial}_0$ is an integrable $\overline{\partial}$–operator, that is satisfies the integrability equation

$$\overline{\partial}_0^2 = 0,$$

and we call such a $\overline{\partial}$–operator a $\rho(G^{\mathbb{C}})$–holomorphic structure on V. In the other direction, the semi connection I_1 corresponding to an integrable $\overline{\partial}$–operator of the form (2.22) is integrable so that in the above correspondence the integrable subsets are preserved. It follows for instance by the integrability theorem for holomorphic structures on vector bundles in the form proved in Proposition 3.48 when $\alpha = \beta = 0$. that an integrable I is the same as equipping $E^{\mathbb{C}}$ with a holomorphic bundle structure. that is a cocycle of holomorphic transition functions. Two holomorphic bundles with isomorphic underlying topological bundles can then be regarded as different holomorphic structures on $E^{\mathbb{C}}$ and they are isomorphic if and only if the corresponding holomorphic structures are gauge equivalent. Thus for any ρ whose infinitesimal representation is faithful there is a one–to–one correspondence, descending at the level of isomorphism classes. between principal holomorphic $G^{\mathbb{C}}$–bundles with underlying topological bundle isomorphic to $E^{\mathbb{C}}$ and holomorphic $\rho(G^{\mathbb{C}})$–structures on V.

Assume now that $E^{\mathbb{C}}$ is a smooth P–quasi–parabolic bundle and let s denote its P–reduction on Σ. A holomorphic structure I on $E^{\mathbb{C}}$ makes $E^{\mathbb{C}}$ into a quasi–parabolic bundle if and only if s is holomorphic with respect to it. This means that

$$Im(ds) \subseteq I,$$

pointwise and in this case we call I a quasi–parabolic structure on $E^{\mathbb{C}}$. Let ρ be a linear representation whose infinitesimal representation is faithful. The associated $\overline{\partial}$–operator will preserve the $\rho(P)$–reduction induced by s on $V_{|\Sigma}$ and vice–versa any holomorphic structure on V of the form (2.22) preserving the $\rho(P)$–reduction induced by s defines a quasi–parabolic structure on $E^{\mathbb{C}}$. More explicitly, letting $\mathfrak{p} = Lie(P)$, these $\overline{\partial}$–operators are the ones that in any $\rho(P)$–filtered frame for V at a point on Σ have the form

$$\overline{\partial}_I = \overline{\partial} + a, \qquad i_{\Sigma}^*(a) \in \Omega^{0,1}(\mathfrak{p}), \tag{2.23}$$

where i_{Σ} is the inclusion of Σ, and a frame $\{e_i\}$ is P–filtered if it is filtered in the sense (2.13) for the flag of some parabolic subgroup $P' \leqslant GL(n)$ such that $P' \cap \rho(G^{\mathbb{C}}) = \rho(P)$. We call such a $\overline{\partial}$–operator a $\rho(G^{\mathbb{C}})$–quasi–parabolic structure on V. At the global level

the P–reduction on $E^{\mathbb{C}}$ induces a subbundle of parabolic Lie algebras of $\mathfrak{g}_{E^{\mathbb{C}}}$, also denoted $\mathfrak{p}$, and given a base $G^{\mathbb{C}}$–quasi–parabolic structure $\overline{\partial}_0$ we see as in (2.22) that all the other ones are of the form

$$\overline{\partial}' = \overline{\partial}_0 + a, \qquad \overline{\partial}'^{\,2} = 0, \qquad a \in \Omega^{0,1}(\mathfrak{g}_{E^{\mathbb{C}}}), \quad i_\Sigma^*(a) \in \Omega^{0,1}(\mathfrak{p}). \tag{2.24}$$

Two quasi–parabolic structures on $E^{\mathbb{C}}$ will be isomorphic if there is an element of $\mathcal{G}_{E^{\mathbb{C}}}$ pulling back one to the other and preserving s and correspondingly two $\rho(G^{\mathbb{C}})$–quasi–parabolic structures on V will be isomorphic if they lie in the same orbit by the subgroup of $\rho(\mathcal{G}_{E^{\mathbb{C}}})$ consisting of those element that preserve the induced reduction. Applying the aforementioned integrability theorem it follows as for ordinary holomorphic bundles that P–quasi–parabolic bundles with underlying $\mathcal{C}^\infty$ quasi–parabolic bundle isomorphic to $E^{\mathbb{C}}$ are the same as quasi–parabolic structures on $E^{\mathbb{C}}$ so that for each of the moduli spaces (2.17) we have the parametrisation

$$\mathcal{M}^P_{\kappa,c,\ell} = \frac{\{I \,|\, Im(ds) \subseteq I_{|\Sigma}\}}{\{g \in \mathcal{G}^{\mathbb{C}} \,|\, g(s) \subseteq s\}},$$

where κ, c and ℓ are the invariants of $E^{\mathbb{C}}$. From the previous discussion we thus obtain that once a base structure $\overline{\partial}_0$ is fixed, for any ρ whose infinitesimal representation is faithful there is a one–to–one correspondence between P–quasi–parabolic bundles with underlying $\mathcal{C}^\infty$ quasi–parabolic bundle isomorphic to $E^{\mathbb{C}}$ and $G^{\mathbb{C}}$–quasi–parabolic structures on V and a parametrisation of $\mathcal{M}^P_{\kappa,c,\ell}$ (we are identifying the $\Omega^{0,i}(\mathfrak{g}_{E^{\mathbb{C}}})$'s with the subspaces of the $\Omega^{0,i}(\mathfrak{g}_V)$'s they determine under ρ)

$$\mathcal{M}^P_{\kappa,c,\ell} = \frac{\{a \in \Omega^{0,1}(\mathfrak{g}_{E^{\mathbb{C}}}) \,|\, (\overline{\partial}_0 + a)^2 = 0, \quad i_\Sigma^*(a) \in \Omega^{0,1}(\Sigma, \mathfrak{p})\}}{\{g \in \mathcal{G}_{E^{\mathbb{C}}} \,|\, g(\mathfrak{p}) \subseteq \mathfrak{p}\}}. \tag{2.25}$$

Assume further that $E^{\mathbb{C}}$ is a smooth E^0–framed quasi–parabolic bundle and let $\mathcal{E}^0 = (E^0, I_{E^0})$ be a fixed holomorphic structure. The previous discussion shows that $\mathcal{E}^0$–framed quasi–parabolic bundles whose smooth underlying quasi–parabolic bundle is isomorphic to $E^{\mathbb{C}}$ are the same as quasi–parabolic structures on $E^{\mathbb{C}}$ making the isomorphism $\varphi : E^{\mathbb{C}}_{|D_\infty} \cong \mathcal{E}^0$ holomorphic for $\mathcal{E}^0$ and the set of isomorphism classes of such quasi–parabolic framings can be identified with

$$\widetilde{\mathcal{M}}^P_{\kappa,c,\ell} = \frac{\{I \,|\, Im(ds) \subseteq I_{|\Sigma} \quad d\varphi(I_{|D_\infty}) \subseteq I_{E^0}\}}{\{g \in \mathcal{G}_{E^{\mathbb{C}}} \,|\, g(s) \subseteq s,\ g_{|D_\infty} = 1\}}.$$

For any representation ρ as above we then obtain a parametrization of $\widetilde{\mathcal{M}}^P_{\kappa,c,\ell}$ as

$$\widetilde{\mathcal{M}}^P_{\kappa,c,\ell} = \frac{\{a \in \Omega^{0,1}(\mathfrak{g}_{E^{\mathbb{C}}}) \,|\, (\overline{\partial}_0 + a)^2 = 0,\ i_\Sigma^*(a) \in \Omega^{0,1}(\Sigma, \mathfrak{p}),\ i_{D_\infty}^*(a) = 0\}}{\{g \in \mathcal{G}_{E^{\mathbb{C}}} \,|\, g(\mathfrak{p}) \subseteq \mathfrak{p},\ g_{|D_\infty} = 1\}}, \tag{2.26}$$

where i_{D_∞} is the inclusion of D_∞. In both (2.25) and (2.26) we can equip the moduli spaces with the quotient topology of the $\mathcal{C}^\infty$ topology on $\Omega^{0,1}(\mathfrak{g}_{E^{\mathbb{C}}})$ which we take as the natural

topology for the moduli spaces. Observe that in (2.25) and (2.26) we have implicitly restricted ourselves to $G^{\mathbb{C}}$ simple or $GL(n)$ but the discussion clearly gives analogous parametrisations for the moduli spaces of $(G^{\mathbb{C}}, P)$–quasi–parabolic bundles or framings in the case of a general complex reductive group.

Two obvious representations to consider for the purpose of studying the moduli spaces are the adjoint representation when $G^{\mathbb{C}}$ is a complex simple Lie group and the fundamental representation if $G^{\mathbb{C}} = GL(n)$ or $SL(n)$ and we shall now focus on these two cases. As for ordinary holomorphic bundles the construction of the moduli spaces will be by Kuranishi Theory, however to apply this successfully in the present situation we first need to develop the suitable analytical framework. This will be based on a more refined parametrisation of the moduli spaces which we now describe. Again what we shall say applies to complex reductive groups but the exposition will be for complex simple groups or $GL(n)$. Let $\mathcal{E} = (E, I_0)$ be a quasi–parabolic or framed quasi–parabolic bundle and let $\overline{\partial}_0$ be the induced integrable $\overline{\partial}$–operator on the adjoint bundle $\mathfrak{g}_{E^{\mathbb{C}}}$ making it into a holomorphic bundle $\mathfrak{g}_{\mathcal{E}}$. The subbundle of parabolic Lie algebras $\mathfrak{p}$ induced by the holomorphic P–reduction on Σ is now a holomorphic subbundle so that the sheaf of sections of $\mathfrak{g}_{\mathcal{E}}$ has now a natural subsheaf, $\mathcal{T}_{\mathcal{E}}$, consisting of the sections that lie in $\mathfrak{p}$ on Σ. This is the sheaf of infinitesimal bundle automorphism preserving the parabolic structure. In the framed case one can consider the subsheaf of $\mathcal{T}_{\mathcal{E}}$ consisting of the section vanishing along D_∞. These is the sheaf of infinitesimal bundle automorphism which also preserve the framing and we shall denote it by $\widetilde{\mathcal{T}}_{\mathcal{E}}$. Then by definition

$$\widetilde{\mathcal{T}}_{\mathcal{E}} = \mathcal{T}_{\mathcal{E}}[-D_\infty],$$

and both $\mathcal{T}_{\mathcal{E}}$ and $\widetilde{\mathcal{T}}_{\mathcal{E}}$ are locally free sheaves of $\mathcal{O}_X$–modules. To see this choose local coordinates near Σ such that $\Sigma : \{z = 0\}$ and a holomorphic trivialisation of $\mathfrak{g}_{\mathcal{E}}$ which extend a trivialization of $\mathfrak{p}$ on Σ. Locally we can extend $\mathfrak{p}$ to a holomorphic subbundle on X and letting $\mathfrak{u}^-$ be a local holomorphic complement for it we have

$$\mathcal{T}_{\mathcal{E}} = (< z > \mathfrak{u}^-) \oplus \mathfrak{p}.$$

The abstract version of the same argument is to observe that $\mathcal{T}_{\mathcal{E}}$ can be defined as the kernel of the exact sequence of sheaves

$$0 \longrightarrow \mathcal{T}_{\mathcal{E}} \longrightarrow \mathfrak{g}_{\mathcal{E}} \xrightarrow{r_{|\Sigma}} \mathfrak{g}_{\mathcal{E}}/\mathfrak{p} \longrightarrow 0. \tag{2.27}$$

Applying $\mathcal{E}xt(-, \mathcal{O}_X)$ we see that $\mathcal{T}_{\mathcal{E}}$ has projective dimension 0 hence it is locally free. It follows that $\mathcal{T}_{\mathcal{E}}$ and $\widetilde{\mathcal{T}}_{\mathcal{E}}$ correspond to holomorphic vector bundles. Let T and $\widetilde{T}$ denote their $\mathcal{C}^\infty$ underlying complex vector bundles respectively. At the vector bundle level, the inclusion $\mathcal{T} \hookrightarrow \mathfrak{g}_{\mathcal{E}}$ induces injective maps on sections

$$\Omega^{0,i}(\widetilde{T}) \longrightarrow \Omega^{0,i}(T) \longrightarrow \Omega^{0,i}(\mathfrak{g}_{E^{\mathbb{C}}}),$$

for all i. In local holomorphic trivialisations compatible with the above direct sum structure this is given by the identity followed by multiplying by z the $\mathfrak{u}^-$ component, and in the

framed case by further multiplying by w in local coordinates where $D_\infty = \{w = 0\}$. Thus we can charachterise the images of the $\Omega^{0,i}(T)$'s in the $\Omega^{0,i}(\mathfrak{g}_{E^{\mathbb{C}}})$'s as the subspaces consisting of sections whose $\mathfrak{u}^-$ components belong to the ideal sheaf $< z >$, and the images of the $\Omega^{0,i}(\widetilde{T})$ as the subspaces of sections that also lie in the ideal sheaf $< w >$. This description is invariant of the choice of holomorphic trivialisations or coordinates, provided they are compatible with $\mathfrak{p}$ on Σ and the divisors Σ and D_∞. It follows from this description that $\overline{\partial}_0$ preserves the subspaces $\Omega^{0,i}(T)$ and if it is a framing also the $\Omega^{0,i}(\widetilde{T})$'s giving subcomplexes of the $\overline{\partial}_0$–complex

$$\{\Omega^{0,i}(T), \overline{\partial}_0\}, \qquad \{\Omega^{0,i}(\widetilde{T}), \overline{\partial}_0\}. \tag{2.28}$$

Note that the spaces $\Omega^{0,i}(T)$ and $\Omega^{0,i}(\widetilde{T})$ are closed under Lie brackets and wedge products in the $\Omega^i(\mathfrak{g}_{E^{\mathbb{C}}})$'s so that letting

$$\mathcal{A}_{\mathfrak{p}} = \{\overline{\partial}_0 + a | a \in \Omega^{0,1}(T)\}, \qquad \widetilde{\mathcal{A}}_{\mathfrak{p}} = \{\overline{\partial}_0 + a | a \in \Omega^{0,1}(\widetilde{T})\},$$

any $\overline{\partial}$–operator in them gives an operator on the complexes (2.28). We thus call $\mathcal{A}_{\mathfrak{p}}$ and $\widetilde{\mathcal{A}}_{\mathfrak{p}}$ the sets of parabolic and framed parabolic $\overline{\partial}$–operators respectively. By the same reason, the pull–back action of the gauge groups

$$\mathcal{G}_{\mathfrak{p}} = \exp(\Omega^0(T)), \qquad \widetilde{\mathcal{G}}_{\mathfrak{p}} = \exp(\Omega^0(\widetilde{T})),$$

preserves $\mathcal{A}_{\mathfrak{p}}$, and $\widetilde{\mathcal{A}}_{\mathfrak{p}}$, respectively so we can consider the quotients

$$\mathcal{B}_{\mathfrak{p}} = \mathcal{A}_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}}, \qquad \widetilde{\mathcal{B}}_{\mathfrak{p}} = \widetilde{\mathcal{A}}_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}}, \tag{2.29}$$

equipped with the quotient topology of the $\mathcal{C}^\infty$ topology. We let

$$\mathcal{H}_{\mathfrak{p}} = \{\overline{\partial} \in \mathcal{A}_{\mathfrak{p}} \,|\, \overline{\partial}^2 = 0\}, \qquad \widetilde{\mathcal{H}}_{\mathfrak{p}} = \{\overline{\partial} \in \widetilde{\mathcal{A}}_{\mathfrak{p}} \,|\, \overline{\partial}^2 = 0\},$$

be the respective integrable parts. These induce distinguished subsets

$$\mathcal{H}_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}}, \qquad \widetilde{\mathcal{H}}_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}}. \tag{2.30}$$

which inherit the induced topology and applying Kuranishi theory to the complexes (2.28) we will equip them with complex analytic spaces structures. A priori it is however not obvious that the (2.30) can be identified with the moduli spaces (2.25) and (2.26). The inclusions induce natural maps

$$\mathcal{H}_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}} \longrightarrow \mathcal{M}^P_{\kappa,c,\ell}, \qquad \widetilde{\mathcal{H}}_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}} \longrightarrow \widetilde{\mathcal{M}}^P_{\kappa,c,\ell},$$

and what needs to be shown is therefore that these maps are actually homeomorphisms with respect to the quotient topologies. This is achieved by the next two Lemmas.

Lemma 2.31. *Let $\overline{\partial}_1 = \overline{\partial}_0 + a$ be a quasi-parabolic structure on $E^{\mathbb{C}}$. There is an element in $\{g \in \mathcal{G}_{E^{\mathbb{C}}} \mid Ad(g)(\mathfrak{p}) \subseteq \mathfrak{p}\}$ such that*

$$Ad(g)^*(\overline{\partial}_1) \in \mathcal{H}_{\mathfrak{p}}.$$

If further $\overline{\partial}_1$ makes $(E^{\mathbb{C}}, \varphi)$ into a quasi-parabolic framing then g can be chosen to be 1 on D_∞ and

$$Ad(g)^*(\overline{\partial}_1) \in \widetilde{\mathcal{H}}_{\mathfrak{p}}.$$

Proof. Choose a tubular neighbourhood $N(\Sigma)$ of Σ with a smooth fibration on Σ such that the fibres are holomorphic discs with respect to the given complex structure on X and also the fibration is holomorphic except on finitely many very small open sets $\{W_i\}$ of Σ. The W_i can be chosen sufficiently small so that there is a cover of Σ by open sets $\{U_i\}$ with each W_i strictly contained in some U_i and these small enough for there to be open sets V_i in $N(\Sigma)$ projecting onto the U_i's, covering a possibly smaller tubular neighbourhood of Σ and complex gauge transformations g_i with

$$g_i = 1 + u_i, \qquad u_{i|\Sigma} \in \Omega^0(\mathfrak{p}) \tag{2.32}$$

and $\mathcal{C}^0$ small in V_i, such that

$$g_i^*(\overline{\partial}_0 + a) = \overline{\partial}_0. \tag{2.33}$$

Let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$ and such that each ψ_i is constant on W_i. Pull it back to one on $N(\Sigma)$ constant along the disc fibres. Note that where the fibration is not holomorphic ψ_i will be constant in the whole neighbourhood of Σ. Define

$$g = 1 + \sum \psi_i u_i.$$

Because the way the u_i's are chosen g is a complex gauge transformation which respects $\mathfrak{p}$ on Σ and we extend it in any way to X. Since the disc fibres D_w of $N(\Sigma)$ have been chosen holomorphic, locally in a neighbourhood of a point p in Σ we can always choose complex coordinates (z, w) so that $D_{\mathrm{p}} = \{w = 0\}$, $\Sigma = \{z = 0\}$. Let $\tilde{a} = g^*(\overline{\partial}_0 + a_1) - \overline{\partial}_0$. By the definition of the ψ_i's it follows that whatever the holomorphic coordinates chosen are

$$\overline{\partial}^z(\psi_i) = 0,$$

in a neighbourhood of Σ so by (2.32) and (2.33)

$$g\tilde{a}^z = \overline{\partial}_0^z g + a^z g = \sum \psi_i \{\overline{\partial}_0^z g_i + a_z g_i\},$$

vanishes on D_{p}, and from this equation a straightforward calculation shows that $\overline{\partial}_0^w \tilde{a}^z$ also vanishes on D_{p}. By the integrability condition on $\overline{\partial}_1$ it follows that the $\tilde{a}_w$ is $\overline{\partial}_0^z$–holomorphic on D_{p}. Since $g_{|\Sigma}$ preserves $\mathfrak{p}$ and since by the compatibility condition with the P–reduction $\iota_\Sigma^*(a^w) \in \Omega^{0,1}(\mathfrak{p})$, we deduce that $\tilde{a}^w \in \Omega^{0,1}(T)$.

If $\overline{\partial}_1$ is also a quasi–parabolic framing, $\iota^*_{D_\infty}(a) = 0$. Let $N(D_\infty)$ be a tubular neighbourhood of D_∞ with a fibration having analogous properties as that in $N(\Sigma)$ and choose both fibrations so that the neighbourhoods where they are not holomorphic are disjoint from $D_\infty \cap \Sigma$. Then choosing the cover $\{U_i\}$ of Σ such that any point in $\Sigma \cap D_\infty$ is contained in the interior of one of its elements, the complex gauge transformation of (2.33) can be assumed to satisfy $g_{|D_\infty} = 1$ and applying the previous argument to the rest of a tubular neighbourhood of D_∞ of $\widetilde{a}$, we can obtain a complex gauge transformation $\widetilde{g}$ which is also 1 on D_∞ and is such that $\widetilde{\widetilde{a}} = \widetilde{g}^*(\overline{\partial}_0 + \widetilde{a}) - \overline{\partial}_0$ belongs to $\Omega^{0,1}(\widetilde{T})$. □

Lemma 2.34. *Let $\overline{\partial}_i = \overline{\partial}_0 + a_i$, $i = 1, 2$ be quasi parabolic–structures on $E^{\mathbb{C}}$ and let $\overline{\partial}_0 + \widetilde{a}_i$ be their representatives given by Lemma 2.31. If the $\overline{\partial}_i$'s are gauge equivalent then the $\overline{\partial}_0 + \widetilde{a}_i$'s are gauge equivalent by a complex gauge transformation $\widetilde{g} \in \mathcal{G}_{\mathfrak{p}}$. If further the $\overline{\partial}_i$'s are quasi–parabolic framed structures on $(E^{\mathbb{C}}, \varphi)$ we can also ensure that $\widetilde{g} \in \widetilde{\mathcal{G}}_{\mathfrak{p}}$.*

Proof. Let $g^*(\overline{\partial}_0 + a_1) = \overline{\partial}_0 + a_2$, let g_i be the gauge transformations bringing each of the $\overline{\partial}_i$'s into the gauge of previous Lemma and put $\widetilde{g} = g_2 \circ g \circ g_1^{-1}$, so that $\widetilde{g}^*(\overline{\partial}_0 + \widetilde{a_1}) = \overline{\partial}_0 + \widetilde{a_2}$ then

$$\overline{\partial}_0 \widetilde{g} = \widetilde{g}\widetilde{a_2} - \widetilde{a_1}\widetilde{g}. \tag{2.35}$$

By construction we had $\widetilde{a}_i^z$ vanishing along the disc fibres, hence from (2.35) $\overline{\partial}_0^z \widetilde{g} = 0$ on them. Since both the g_i's and g preserved the quasi–parabolic structure it follows that $\widetilde{g} \in \mathcal{G}_{\mathfrak{p}}$. Finally if the $\overline{\partial}_i$'s are quasi–parabolic framings so that $\widetilde{a}^w$ vanishes along the disc fibres of $N(D_\infty)$, (2.35) also gives that $\overline{\partial}_0^w \widetilde{g} = 0$ along such fibres and since both the g_i's and g are 1 along $D_\infty = 1$ it follows that $\widetilde{g} \in \widetilde{\mathcal{G}}_{\mathfrak{p}}$. □

The previous two lemmas show that there are bijections between the moduli spaces (2.25), (2.26) and the quotients (2.30). It follows by the proofs of the lemmas inserting the appropriate estimates at each place that these are in fact homeomorphisms. In particular we see that if we had taken a different quasi–parabolic or framed quasi–parabolic structure than $\overline{\partial}_0$ to define the quotients (2.30) we would have obtained homeomorphic spaces so that the parametrisation obtained is independent of the choice of $\overline{\partial}_0$ and the construction of the moduli spaces can now be carried out on the (2.30) as we shall do in Chapter 5.

2.5 Stable parabolic vector bundles

We recall here the notion of stable α–parabolic vector bundle (or α–stable parabolic vector bundle) appearing in the statement of Theorem 2. Let $\mathcal{S}$ be a coherent sheaf over a complex manifold X. It is well known (see for instance [K]) that the set of singular points of $\mathcal{S}$, that is the points at which $\mathcal{S}$ is not locally free, has codimension at least 1 so, if X is connected, its rank is well defined as the rank of the locally free sheaf it restricts to out of the singular set. A coherent subsheaf $\mathcal{S}$ of a holomorphic vector bundle $\mathcal{V}$ is said to be saturated if $\mathcal{V}/\mathcal{S}$ is torsion–free. In particular $\mathcal{S}$ must be a reflexive sheaf (isomorphic to its double dual). In case X is a compact complex surface it follows that $\mathcal{S}$ is a vector bundle with a sheaf injection into $\mathcal{V}$ whose image (still denoteed $\mathcal{S}$) and cokernel are a subbundle and a quotient bundle of $\mathcal{V}$ out of a finite number of points. Assume now that

(X, ω) is a compact connected Kähler surface. Since the first Chern class is supported in codimension 2, it is easy to see that any torsion–free coherent sheaf $\mathcal{S}$ has a well defined first Chern class. For instance we can consider the first Chern form of any connection on $\mathcal{S}$ away from its singular points. Since this is a 2–form it follows that it extends to a well defined cohomology class on X which we define to be the first Chern class of $\mathcal{S}$. The ($[\omega]$–)degree of $\mathcal{S}$ is then

$$\deg(\mathcal{S}) = \int_X c_1(\mathcal{S}) \wedge \omega.$$

The degree of a divisor Σ is the degree of the associated line bundle $[\Sigma]$. Let $\mathcal{V}$ be an α–parabolic vector bundle. The α–degree of $\mathcal{V}$ is defined to be

$$\deg_\alpha(\mathcal{V}) = \deg(\mathcal{V}) + Tr(\alpha)\deg(\Sigma), \tag{2.36}$$

and the slope

$$\mu_\alpha(\mathcal{V}) = \frac{\deg_\alpha(\mathcal{V})}{rk(\mathcal{V})}. \tag{2.37}$$

Notice that the definitions of α–degree and slope involve only topological data so they make sense also for $\mathcal{C}^\infty$ α–parabolic vector bundles and then $\deg_\alpha(\mathcal{V}) = \deg_\alpha(V)$ where V is the $\mathcal{C}^\infty$ underlying parabolic bundle of $\mathcal{V}$.

Now let $\mathcal{S}$ be a proper non–zero saturated subsheaf of $\mathcal{V}$. Let

$$0 \subsetneq \mathcal{F}_{j_1} \subsetneq \cdots \subsetneq \mathcal{F}_{j_i} \subsetneq \cdots \subsetneq \mathcal{F}_{j_s} \subsetneq \mathcal{V}_{|\Sigma},$$

be the elements in the filtration of $\mathcal{V}_{|\Sigma}$ such that in the generic fibre on Σ

$$0 = \mathcal{F}_1 \cap \mathcal{S}_{|\Sigma} = \cdots \subsetneq \mathcal{F}_{j_1} \cap \mathcal{S}_{|\Sigma} = \cdots \subsetneq \mathcal{F}_{j_i} \cap \mathcal{S}_{|\Sigma} = \cdots \subsetneq \mathcal{F}_{j_s} \cap \mathcal{S}_{|\Sigma} = \cdots = \mathcal{V}_{|\Sigma} \cap \mathcal{S}_{|\Sigma}. \tag{2.38}$$

Notice that the above intersections and their ranks are well defined since $\mathcal{S}$ is a subbundle of $\mathcal{V}$ except for finitely many points. As observed we can think of $\mathcal{S}$ as of a vector bundle with an injective sheaf mapping $\iota : \mathcal{S} \to \mathcal{V}$ which fails to be a bundle injection at finitely many points. The inverse images of the $\mathcal{F}_i$ under ι are then subbundles of $\mathcal{S}$ so they define on it an α'–parabolic bundle structure in the ordinary sense where if $n' = rk(\mathcal{S})$, $\alpha' \in iu(n')$ has eigenvalues

$$\alpha'_i = \alpha_{j_i},$$

with multiplicities

$$n'_i = rk(\mathcal{F}_{j_i} \cap \mathcal{S}_{|\Sigma}) - rk(\mathcal{F}_{j_{i-1}} \cap \mathcal{S}_{|\Sigma}),$$

(we take $\mathcal{F}_{j_0} = 0$). We define the induced α–degree and slopes of $\mathcal{S}$ to be

$$\deg_\alpha(\mathcal{S}, \mathcal{V}) = \deg(\mathcal{S}) + Tr(\alpha')\deg(\Sigma), \qquad \mu_\alpha(\mathcal{S}, \mathcal{V}) = \frac{\deg_\alpha(\mathcal{S}, \mathcal{V})}{rk(\mathcal{S})}, \tag{2.39}$$

respectively. This are of course the same as the α'–degree and slope of $\mathcal{S}$ thought of as a parabolic vector bundle in the way described.

Definition 2.40. *An α–parabolic vector bundle $\mathcal{V}$ on (X, Σ) is stable if for every non–zero proper saturated subsheaf $\mathcal{S}$*

$$\mu_\alpha(\mathcal{S}, \mathcal{V}) < \mu(\mathcal{V}).$$

Let $\pi : \mathcal{E} \to \mathcal{U}$ be the projection map to a proper non–zero torsion–free quotient sheaf and let

$$\mathcal{F}_{k_1} \subsetneq \cdots \mathcal{F}_{k_i} \subsetneq \cdots \subsetneq \mathcal{F}_{k_r}$$

be the elements in the filtration of $\mathcal{V}_{|\Sigma}$ such that on the generic fibre

$$0 = \pi(\mathcal{F}_1) = \cdots \subsetneq \pi(\mathcal{F}_{k_1}) = \cdots \subsetneq \pi(\mathcal{F}_{k_i}) = \cdots \subsetneq \pi(\mathcal{F}_{k_r}) = \cdots = \pi(\mathcal{V}_{|\Sigma}).$$

Again $\{k_i\}$ and $rk(\pi(\mathcal{F}_{k_i}))$ are well defined since $\mathcal{U}$ is a quotient bundle out of finitely many points. Let $n'' = rk(\mathcal{U})$ and let $\alpha'' \in iu(n'')$ have eigenvalues

$$\alpha''_i = \alpha_{k_i},$$

with multiplicities

$$n''_i = rk(\pi(\mathcal{F}_{k_i})) - rk(\pi(\mathcal{F}_{k_{i-1}})).$$

(we take $\mathcal{F}_{k_0} = 0$). We define the induced α–parabolic degree and slope of $\mathcal{U}$ to be

$$\deg_\alpha(\mathcal{E}, \mathcal{U}) = \deg(\mathcal{U}) + Tr(\alpha'')\deg(\Sigma), \qquad \mu_\alpha(\mathcal{E}, \mathcal{U}) = \frac{\deg_\alpha(\mathcal{E}, \mathcal{U})}{rk(\mathcal{U})}. \tag{2.41}$$

A basic property of α–stability which will be very important in the construction of the moduli spaces, is that as with ordinary stability it can be characterized also in terms of torsion–free quotients.

Proposition 2.42. *An α–parabolic bundle $\mathcal{V}$ is stable if and only if for every proper non–zero torsion–free quotient sheaf $\mathcal{U}$*

$$\mu_\alpha(\mathcal{E}, \mathcal{U}) > \mu_\alpha(\mathcal{V}).$$

Proof. By definition the saturated subsheaves of $\mathcal{V}$ are exactly the kernels of the projections to torsion–free quotient sheaves $\pi : \mathcal{V} \twoheadrightarrow \mathcal{U}$ so every such $\mathcal{U}$ is of the form $\mathcal{U} = \mathcal{V}/\mathcal{S}$ for some $\mathcal{S}$ and hence fits into the exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{V} \longrightarrow \mathcal{U} \longrightarrow 0.$$

By the additivity of the first Chern class in exact sequences we have

$$\deg(\mathcal{V}) = \deg(\mathcal{S}) + \deg(\mathcal{U}).$$

and by a combinatorial argument applied to a generic fibre on Σ one sees that letting α' and α'' be the matrices appearing in the definitions of induced α–slopes of $\mathcal{S}$ and $\mathcal{U}$ as given by (2.39) and (2.41),

$$Tr(\alpha) = Tr(\alpha') + Tr(\alpha'').$$

Putting together these two it follows that

$$rk(\mathcal{S})(\mu_\alpha(\mathcal{V}) - \mu_\alpha(\mathcal{S}, \mathcal{V}) = -rk(U)(\mu_\alpha(\mathcal{V}) - \mu_\alpha(\mathcal{V}, U),$$

showing that

$$\mu_\alpha(\mathcal{V}) > \mu_\alpha(\mathcal{S}, \mathcal{V})$$

for every proper non–zero saturated subsheaf if and only if

$$\mu_\alpha(\mathcal{V}) < \mu_\alpha(\mathcal{V}, U)$$

for every proper non–zero torsion–free quotient sheaf. □

Notice that isomorphic α–parabolic bundles are both stable or both unstable so that the subset of the set of α–parabolic bundles of given topological type consisting of the stable ones descends to a subset of $\mathcal{M}^{\alpha}_{\kappa,c,\ell}$ denoted

$$\mathcal{M}^{\alpha,S}_{\kappa,c,\ell}.$$

Finally we recall that given α–parabolic bundles $\mathcal{V}'$. $\mathcal{V}''$ with filtrations $\{\mathcal{F}'_i\}$, $\{\mathcal{F}''_i\}$, a morphism $\psi : \mathcal{V}' \to \mathcal{V}''$ is a parabolic morphism if

$$\forall i, \qquad \psi(\mathcal{F}'_i) \subseteq \mathcal{F}''_i. \tag{2.43}$$

Chapter 3

Correspondence Theorems

In this chapter we prove Theorem 1. As a consequence of some of the techniques developed and of Simpson existence Theorem [S] we also derive a proof of the Parabolic bundles conjecture of [KM1], Theorem 2. As pointed out in the introduction Theorem 1 fits as a somewhat degenerate case into a wider contest. These developments will appear in [M2] and we shall discuss in the Conclusions some aspects of a more general theory. In §5.3 we discuss some aspects of the purely holomorphic side. We begin by recalling some basic facts of Hermitian differential geometry that will be used in the sequel.

3.1 Hermitian principal bundles

Let $G^{\mathbb{C}}$ be a complex reductive group and let $\mathcal{E}$ be a holomorphic principal $G^{\mathbb{C}}$–bundle over a complex manifold X. As in §2.4 we think of $\mathcal{E}$ as of a smooth $G^{\mathbb{C}}$–bundle $E^{\mathbb{C}}$ with a holomorphic structure $I_{\mathcal{E}}$. An Hermitian structure on $E^{\mathbb{C}}$ consists of a pair $(I_{\mathcal{E}}, \lambda)$ where λ is a $\mathcal{C}^\infty$ G–structure on $E^{\mathbb{C}}$. Given any such structure on $E^{\mathbb{C}}$, there is a unique connection $A_{\mathcal{E},\lambda}$, the Chern connection of $(I_{\mathcal{E}}, \lambda)$, compatible with both in the sense that A is a G–connection and its horizontal distribution consists of complex subspaces. The curvature of A is of type (1,1) and vice–versa given a G–connection on $E^{\mathbb{C}}$ with curvature of type (1,1) there is a unique I_A of which A is the Chern connection. Again for analytical purposes it is convenient to describe the situation in terms of the covariant derivative on an associated vector bundle. Let $\rho : G^{\mathbb{C}} \to GL(n)$ be a linear representation and let $\overline{\partial}_{\mathcal{E}}$ be the $\overline{\partial}$–operator induced by $I_{\mathcal{E}}$ on the associated vector bundle $\mathcal{V}$. A G–reduction λ on $E^{\mathbb{C}}$ induces a G–reduction on V, which, since $\rho(G)$ is a subgroup of $U(n)$, must be induced by some Hermitian metric h on V. In this case we say that the metric is compatible with the $G^{\mathbb{C}}$–structure or that (V, h) has a $G^{\mathbb{C}}$–structure. Since for any complex reductive subgroup $G^{\mathbb{C}}$ of $GL(n)$ there are copies of $U(n)$ such that $G^{\mathbb{C}} \cap U(n) = G$ is a maximal compact subgroup of $G^{\mathbb{C}}$, on V we can always find metrics compatible with the $G^{\mathbb{C}}$–structure. If $f \in \mathcal{G}_{E^{\mathbb{C}}}$ then $g = \rho(f)^h \circ \rho(f)$ is a positive self–adjoint $G^{\mathbb{C}}$–automorphism of V, hence if h is a Hermitian metric compatible with the $G^{\mathbb{C}}$–structure gh also is and all Hermitian metrics compatible with the $G^{\mathbb{C}}$–structure are obtained in this way. When ρ is a faithful representation this gives in turn all the G–reductions on $E^{\mathbb{C}}$. If now (X, ω) is Kähler, the existence of a H.Y.M. reduction on $(E^{\mathbb{C}}, I_{\mathcal{E}})$ will follow from that of a H.Y.M. metric on $\mathcal{V} = (V, \overline{\partial}_{\mathcal{E}})$ compatible with the $G^{\mathbb{C}}$–structure. An Hermitian structure $(\overline{\partial}_{\mathcal{V}}, h)$ on a vector bundle determines uniquely a connection $A_{\mathcal{V},h}$, also called the Chern connection, compatible with both. The Chern connection of $\rho(A_{\mathcal{E},\lambda})$ is then that of $(\overline{\partial}_{\mathcal{E}}, h)$ and we can concretely represent $\rho(A_{\mathcal{E},\lambda})$ as $\rho(A_{\mathcal{E},\lambda}) = A^{1,0} + A^{0,1}$ where

$$A^{0,1} = \overline{\partial}_{\mathcal{E}}, \qquad A^{1,0} = \partial^h = *_h \circ \overline{\partial}_{\mathcal{E}} \circ *_h.$$

We also recall the variation formula for the curvature of the Chern connection when the

metric h is changed to $h' = hg$, where g is a positive h–self–adjoint complex gauge transformation of V. If a holomorphic structure is fixed we denote by A_h and F_h the Chern connection and its curvature, then,

$$F_{h'} - F_h = \overline{\partial}_{\mathcal{E}}(g^{-1}\partial_h g). \tag{3.1}$$

When a metric h is fixed and the holomorphic structure $\mathcal{V}$ is varied we shall instead denote the Chern connection and its curvature by $A_{\overline{\partial}_{\mathcal{V}}}$ and $F_{\overline{\partial}_{\mathcal{V}}}$ respectively.

If we regard holomorphic bundles of the same topological type as different holomorphic structures on the same underlying C^∞ $G^{\mathbb{C}}$–bundle $E^{\mathbb{C}}$ as in §2.4, we can identify H.Y.M. bundles with this topological type as the Hermitian structures on $E^{\mathbb{C}}$ whose Chern connection is H.Y.M. . Fix a G–reduction λ_0 on $E^{\mathbb{C}}$. Given an H.Y.M. bundle $(I_{\mathcal{E}}, \lambda)$ of this topological type let g be the complex bundle automorphism pulling back λ to λ_0, then the Chern connection of $g(I_{\mathcal{E}}, \lambda)$ is the λ_0–connection

$$A_{g(I_{\mathcal{E}}),\lambda_0} = g^{-1} \circ A_{I_{\mathcal{E}},\lambda} \circ g.$$

and of course an H.Y.M. connection. By the definition of isomorphism of H.Y.M. bundles it follows that this procedure associates isomorphic H.Y.M. bundles to gauge equivalent λ_0–instantons and then the assignment

$$(I_{\mathcal{E}}, \lambda) \mapsto A_{g^*(I_{\mathcal{E}}),\lambda_0}$$

gives an identification between the set of equivalence classes of H.Y.M. bundles of a given topological type and the set of gauge equivalence classes of G–instantons on $E^{\mathbb{C}}$.

In the case $E^{\mathbb{C}}$ is a bundle on $X\backslash\Sigma$ and we are considering finite action H.Y.M. bundles it is clear that $A_{I_{\mathcal{E}},\lambda}$ and $A_{g^*(I_{\mathcal{E}}),\lambda_0}$ have the same holonomy so we can identify the set of equivalence classes of H.Y.M. bundles on $X\backslash\Sigma$ with constant zero whose Chern connection has holonomy J along Σ with the set of equivalence classes of finite action G–instantons on $X\backslash\Sigma$ with the same holonomy. In particular when X is compact we can use §1.4 and 1.6 to attach topological invariants to a H.Y.M. bundle and partition the moduli space accordingly.

In the case where $X = \mathbb{C}^2\backslash\Sigma$, either using the conformal equivalence $(S^4, round) \sim (\mathbb{C}^2, euc)$ or pulling back by the blow–up map $\pi : \mathbb{P}^2 \to S^4$ as in §1.3 we identify instantons on $S^4\backslash S^2$ for the round metric with euclidean instantons on $\mathbb{C}^2\backslash\Sigma$ and we thus inherit a partition of $M^J(\mathbb{C}^2, \Sigma)$ and $\widetilde{M}^J(\mathbb{C}^2, \Sigma)$ into topological types parametrised by the classes κ, ℓ

$$M^J_{\kappa,\ell}(\mathbb{C}^2, \Sigma) = \pi^*(M^J_{\kappa,\ell}(S^4, S^2)), \qquad \widetilde{M}^J_{\kappa,\ell}(\mathbb{C}^2, \Sigma) = \pi^*(\widetilde{M}^J_{\kappa,\ell}(S^4, S^2)). \tag{3.2}$$

Using the cylinder analysis outlined in §1.3 there is a more direct way to define this moduli spaces which is also more natural vis–a–vis the comparison with the complex geometry, but we shall not pursue this here.

3.2 Existence of H.Y.M. metrics on $\mathbb{C}^2\backslash\Sigma$

Let Σ be a smooth complex curve in $\mathbb{P}^2$ (this shall later be taken to be a line), D_∞ a complex line and consider the associated affine curve in $\mathbb{C}^2 = \mathbb{P}^2\backslash D_\infty$. To prove the existence of a H.Y.M. metric on $\mathcal{V} \longrightarrow \mathbb{C}^2\backslash\Sigma$ with respect to the restriction of the euclidean metric on $\mathbb{C}^2$ we proceed by a simple exhaustion method. We first solve a boundary value problem on a compact domain $D_R \subseteq \mathbb{C}^2\backslash\Sigma$. We then show that if D_R expands to fill up $\mathbb{C}^2\backslash\Sigma$ in a suitable manner, and the boundary value on each ∂D_R is the restriction of a global Hermitian metric h_0 satisfying suitable asymptotic conditions, the solutions converge in $\mathcal{C}^\infty$ on $\mathbb{C}^2\backslash\Sigma$ and have the desired behaviour along Σ and D_∞. Let $G^{\mathbb{C}}$ a connected complex semisimple Lie group. If the boundary conditions and holomorphic structure came from a principal $G^{\mathbb{C}}$–bundle via some faithful representation we can make sure that the eventual solution is also $G^{\mathbb{C}}$–compatible giving a H.Y.M. G–reduction on the original principal bundle.

We begin by showing that $\mathbb{C}^2\backslash\Sigma$ admits exhaustion functions of the appropriate kind.

Lemma 3.3. *$\mathbb{C}^2\backslash\Sigma$ admits a plurisubharmonic exhaustion function ψ bounded by the Kähler form of the euclidean metric.*

Proof. Let η be a strictly positive potential function for the Kähler form of the Euclidean metric, let $\sigma \in H^0([\Sigma])$ be such that $Div(\sigma) = \Sigma$ and let $\beta(r)$ be a cut–off function which is one in a neighbourhood of the origin and zero further away. Choose a metric k on $[\Sigma]$ and for some large constant C put

$$\psi = C\eta - \beta(|\sigma|_k)\log|\sigma|_k.$$

A direct computation shows that for C and C' sufficiently large

$$0 \leqslant -i\overline{\partial}\partial\psi \leqslant C'\omega.$$

□

Let $\rho : G^{\mathbb{C}} \longrightarrow GL(n)$ be a faithful linear representation. Using the variation formula (3.2) for the curvature the existence of a H.Y.M. metric on a holomorphic $G^{\mathbb{C}}$ bundle $\mathcal{E} \to X$ is then reduced to finding a positive self–adjoint $G^{\mathbb{C}}$–valued gauge transformation g on the associated vector bundle $\mathcal{V}$ such that

$$-\hat{F}_{h_0} = i\Lambda\overline{\partial}_{\mathcal{E}}(g^{-1}(\partial_h(g)).$$

Let $\{D_R\}$ be the family of domains $D_R = \psi^{-1}([0, R])$. The existence of H.Y.M. reductions on a holomorphic $G^{\mathbb{C}}$–bundle on D_R is guaranteed by the following

Proposition 3.4. *Let $\mathcal{V} \longrightarrow D_R$ be a holomorphic vector bundle. For any given metric h_0 on ∂D_R there is a complex gauge transformation g_R such that $h_R = h_0 g_R$ is the unique solution h_R to the Dirichlet problem*

$$(*_R) \qquad \begin{cases} h_{|\partial D_R} = h_0 \\ \hat{F}_h = 0. \end{cases}$$

Further if h_0 extends to a metric on D_R we can ensure that $\det(h_R) = \det(h_0)$ and if $\mathcal{V} = \mathcal{E} \times_{G^{\mathbb{C}}} \mathbb{C}^n$ and h_0 comes from a G–reduction on $\mathcal{E}$ the same is true for h_R.

Proof. The first part follows as in [D3]. For the last clause, suppose that h_0 comes from a G–reduction on $\mathcal{E}$ and let $\mathcal{G}_{E^{\mathbb{C}}}$ be the gauge group of $\mathcal{E}$. Using the closedness of $\Omega^i(\mathfrak{g}_{E^{\mathbb{C}}})$ in $\Omega^i(\mathrm{End}(V))$ and that of $\mathcal{G}_{E^{\mathbb{C}}}$ in the complex gauge group of V, we can apply the continuity or the heat equation method as in [D3] to obtain a gauge transformation $g \in \mathcal{G}_{E^{\mathbb{C}}}$ such that gh_0 solves $(*R)$. Since $\mathfrak{g}_{E^{\mathbb{C}}}$ lies in the semisimple part of End $(\mathcal{V})$, g can also be taken to have determinant one. □

Denote by $|\cdot|_h$ the fibre norm defined by an Hermitian metric h. To prove $\mathcal{C}^0$–convergence of the g_R's given by the proposition as $R \to \infty$, we use a version of the differential inequality in [D2], Lemma 3.4. Let g be a positive self–adjoint endomorphism of a Hermitian vector space, let $\sigma(g)$ denote its spectrum and define

$$\Psi(g) = \log \max(\sigma(g)).$$

If g is a positive self–adjoint complex gauge transformation of a Hermitian vector bundle $(\mathcal{V}, h)$ on a Kähler surface (X, ω), $\Psi(g)$ becomes a continuous function satisfying

Lemma 3.5. *Let $h' = hg$ then there is a positive constant C such that*

(i)
$$\Delta\Psi(g)(x) \leqslant C\{|\hat{F}_h(x)|_h + |g(x)|_h|\hat{F}_{h'}(x)|_{h'}\}$$
weakly on X. Similarly for $\Psi(g^{-1})$.

(ii) *There is an $\varepsilon > 0$ such that if $\Psi(g)(x)$, $\Psi(g^{-1})(x) < \varepsilon$, then*
$$\{|g(x) - I|_h, |g(x) - I|_{h'}\} < C|\Psi(g)(x)|.$$

We now proceed to prove $\mathcal{C}^0$ convergence. Let r denote the euclidean distance from the origin in $\mathbb{C}^2$.

Proposition 3.6. *Let h_0 be an initial Hermitian metric on a $\mathcal{V} \longrightarrow \mathbb{C}^2 \backslash \Sigma$ such that*

(i)
$$\sup_{x \in \mathbb{C}^2 \backslash \Sigma} |\hat{F}_{h_0}(x)|_{h_0} < \infty$$

(ii)
$$(1 + r^3)|\hat{F}_{h_0}(x)|_{h_0} = O(1),$$
as $r = |x| \to \infty$ in $\mathbb{C}^2 \backslash \Sigma$.

Then the solutions to $(*_R)$ given by Proposition 3.4 converge in $\mathcal{C}^0$ over compact sets in $\mathbb{C}^2 \backslash \Sigma$ to a limit h_∞. Furthermore the complex gauge transformation g_∞ such that $h_\infty = h_0 g_\infty$ satisfies,

$$(1 + r)|g_\infty - 1|_{h_0} = O(1), \quad \text{as } r = |x| \to \infty \text{ in } \mathbb{C}^2 \backslash \Sigma.$$

If h_0 comes from a G–reduction h_∞ also does.

Proof. Let $\{g_R\}$ be the positive self–adjoint complex gauge transformations giving the solutions to the local Dirichlet problems $(*_R)$. Extend each g_R trivially to the identity on the rest of $\mathbb{C}^2\setminus\Sigma$. By assumption there is a constant $M>0$ such that

$$\forall x\in\mathbb{C}^2\setminus\Sigma,\qquad |\hat{F}_{h_0}(x)|_{h_0}<\frac{M}{1+r^3}.$$

Taking the convolution product of $\dfrac{M}{1+r^3}$ with a suitable multiple of the Green's function of the euclidean Laplacian we obtain a positive smooth function Γ on $\mathbb{C}^2\setminus\Sigma$ such that

$$\Delta\Gamma=\frac{M}{1+r^3},\qquad \Gamma=O(\frac{M}{1+r}),\quad \text{on } \mathbb{C}^2. \tag{3.7}$$

Now consider the following linear Dirichlet problem on the domains D_R defined above.

$$\begin{cases}\varphi_{|\partial D_R}=0\\ \Delta\varphi=\Delta\Gamma.\end{cases} \tag{3.8}$$

Let ψ_R be the solution to the homogeneous Dirichlet problem

$$\begin{cases}\psi_{|\partial D_R}=\Gamma_{|\partial D_R}\\ \Delta\psi=0.\end{cases} \tag{3.9}$$

and let $\varphi_R=\Gamma-\psi_R$. This solves (3.7) on D_R. We have by construction that for every R

$$\Delta(-\varphi_R)=-\frac{M}{1+r^3},\qquad -\varphi_R=0\quad\text{on } \partial D_R,$$

so $-\varphi_R$ is subharmonic and by the maximum principle it achieves its maximum on ∂D_R, thus $-\varphi_R\leqslant 0$ on D_R. Fix $\overline{R}$. For $R_2>R_1\geqslant\overline{R}$ we have

$$\Delta(\psi_{R_2}-\psi_{R_1})=0.\quad\text{on } D_{\overline{R}}. \tag{3.10}$$

and

$$(\psi_{R_2}-\psi_{R_1})_{|\partial D_{R_1}}=(\psi_{R_2}-\Gamma)_{|\partial D_{R_1}}=-\varphi_{R_2|\partial D_{R_1}}. \tag{3.11}$$

and so by (3.10) and (3.11) $\psi_{R_2}-\psi_{R_1}\leqslant 0$ on $D_{\overline{R}}$. Thus $\{\psi_R\}_{R\geqslant\overline{R}}$ forms a decreasing sequence on $D_{\overline{R}}$. Further, ψ_R achieves its maximum on ∂D_R where it equals Γ and since Γ is bounded it follows that for any $x\in D_R$, $\{|\psi_R(x)|\}_{R\geqslant\overline{R}}$ is a bounded sequence. By Harnack's theorem $\{\psi_R\}$ converges in $\mathcal{C}^\infty$ to a bounded harmonic function on D_R so $\{\psi_R\}$ actually converges in $\mathcal{C}^\infty$ on compact subsets of $\mathbb{C}^2\setminus\Sigma$. Since the $\varphi_R=\Gamma-\psi_R$ are bounded it follows that φ_R converges in $\mathcal{C}^\infty$ on compact subsets of $\mathbb{C}^2\setminus\Sigma$ to a bounded limit φ. We use this to prove the uniform convergence of the g_R on compact subsets of $\mathbb{C}^2\setminus\Sigma$. Again

it suffices to consider the D_R's. Fix $\overline{R}$ and $\delta > 0$. Let $N = \sup_R\{\sup_{D_R} |g_R|\}$. N is finite because by (i) of the lemma

$$\forall R, \quad \Delta\Psi(g_R) \leqslant C|\hat{F}_{h_0}|_{h_0} \leqslant \frac{M}{1+r^3},$$

and similarly for $\Psi(g_R^{-1})$, so by construction of the φ_R,

$$\begin{cases} \Psi(g_R) - \varphi_R \leqslant 0, & \text{on } \partial D_R \\ \Delta(\Psi(g_R) - \varphi_R) \leqslant 0. & \text{on } D_R, \end{cases}$$

and similarly for $\Psi(g_R^{-1})$, hence

$$\{\Psi(g_R),\ \Psi(g_R^{-1})\} \leqslant \varphi_R \leqslant \varphi \leqslant \Gamma = O(\frac{1}{1+r}). \tag{3.12}$$

Since there is some positive constant T such that for all positive self adjoint endomorphisms B of a vector space, $|B| \leqslant T(\Psi(B))$, uniform boundedness of the $\{g_R\}$ follows.
Let now ε and C be as in (ii) of Lemma 3.5. By (i) of 3.5 applied this time to $g_{R_1}^{-1}g_{R_2}$ such that

$$h_{R_2} = h_0 g_{R_2} = h_{R_1} g_{R_1}^{-1} g_{R_2}. \quad \Delta(\Psi(g_{R_1}^{-1}g_{R_2}) \leqslant 0, \text{ on } D_{R_1},$$

hence

$$\Delta(\Psi(g_{R_1}^{-1}g_{R_2}) - (\varphi_{R_2} - \varphi_{R_1})) \leqslant 0, \text{ on } D_{R_1}.$$

But on ∂D_{R_1}, $\Psi(g_{R_2}g_{R_1}^{-1}) = \Psi(g_{R_2})$, $\varphi_{R_1} = 0$ and $\Psi(g_{R_2}) \leqslant \varphi_{R_2}$ so

$$\Psi(g_{R_1}^{-1}g_{R_2}) - (\varphi_{R_2} - \varphi_{R_1})_{|\partial D_{R_1}} = \Psi(g_{R_2}) - \varphi_{R_2|\partial D_{R_1}} \leqslant 0$$

so

$$\Psi(g_{R_1}^{-1}g_{R_2}) \leqslant \varphi_{R_2} - \varphi_{R_1} \quad \text{on } D_{R_1}.$$

Similarly, applying the argument to $\Psi(g_{R_2}^{-1}g_{R_1})$ one shows that

$$\Psi(g_{R_2}^{-1}g_{R_1}) \leqslant \varphi_{R_2} - \varphi_{R_1} \quad \text{on } D_{R_1},$$

hence on $D_{\overline{R}}$, $\{\Psi(g_{R_1}^{-1}g_{R_2}),\ \Psi(g_{R_2}^{-1}g_{R_1})\} < \varepsilon$ and by (ii) of Lemma 3.5

$$\sup_{D_{\overline{R}}} |g_{R_1}^{-1}g_{R_2} - I| \leqslant C\frac{\min\{\varepsilon, \delta\}}{CN} \leqslant \frac{\delta}{N},$$

but

$$\sup_{\overline{D_R}} |g_{R_2} - g_{R_1}| \leqslant \sup_{D_{\overline{R}}}\{|g_{R_1}||g_{R_1}^{-1}g_{R_2} - I|\} \leqslant \delta,$$

which gives the desired uniform convergence. In fact it follows from (3.12) that if g_∞ is this limit

$$|g_\infty - I| = O(\frac{1}{1+r}), \quad \text{on } \mathbb{C}^2\backslash\Sigma.$$

The clause about h_∞ coming from a G–reduction if h_0 does follows from the closedness of $\mathcal{G}_{E^\mathbb{C}}$ in the gauge group of $\mathcal{V}$. □

Let ∇ denote the gradient of A_{h_0} on $\mathcal{V} \to \mathbb{C}^2\backslash\Sigma$ and L_k^p the Sobolev spaces defined by it.

Lemma 3.13. *At each $q \in \mathbb{C}^2 \setminus \Sigma$ there is a compact neighbourhood U such that the solutions g_R to $(*_R)$ are bounded in $L_2^p(U)$, uniformly in R and for any $p \geqslant 1$.*

Proof. This follows from Lemma 6.4 of [S] and the remark following it. □

Corollary 3.14. *The metric h_∞ given by Proposition 3.6 is actually smooth on $\mathbb{C}^2 \setminus \Sigma$, satisfies the H.Y.M. equation and $\det(h_\infty) = \det(h_0)$.*

Proof. By the Lemma $\{\nabla h_R\}$ is locally uniformly bounded in L_1^p for any $p \geqslant 1$. Choosing $p > 4$, the compact embedding $L_1^p \hookrightarrow \mathcal{C}^0$ implies that the ∇h_R's have a strong limit in $\mathcal{C}^0$. The same proof as in the previous lemma shows that the $|\nabla h_R|$'s are locally uniformly bounded in L_2^p for any $p \geqslant 1$ and hence that $\{\nabla^2 h_R\}$ has a strong limit in $\mathcal{C}^0$. Continuing this bootstrapping procedure one finally obtains $\mathcal{C}^\infty$–convergence. The limit h_∞ has therefore a well defined curvature and the convergence will preserve the H.Y.M. equation so that h_∞ is H.Y.M. The statement about determinants follows from the corresponding statement for the h_R's. □

3.3 Properties of solutions

Proposition 3.15. *The H.Y.M. metric h_∞ constructed above is the unique H.Y.M. metric $h = h_0 g$ on $\mathcal{V} \to \mathbb{C}^2 \setminus \Sigma$ such that*

(i) *h is mutually bounded with respect to h_0.*
(ii) *g extends continuously to the identity on D_∞*
(iii) $\det(h) = \det(h_0)$

Proof. By construction $|g_\infty|_{h_0}$ and $|g_\infty^{-1}|_{h_0}$ are bounded so h_∞ and h_0 are mutually bounded and $\det(h_\infty) = \det(h_0) g_\infty$, further h_∞ satisfies (ii) because we have shown in Proposition 3.6 that

$$|g_\infty - I| = O(\frac{1}{1+r}).$$

Let δ be a positive number and put $C_\delta(\Sigma) = \{ x \in \Sigma^2 \mid d(x, \Sigma) < \delta \}$, where d is the distance function of the euclidean metric. Now observe that if a $\mathcal{C}^\infty$ function f on $\mathbb{C}^2 \setminus \Sigma$ satisfies

(1) f is bounded
(2) $\Delta f \leqslant 0$ on $\mathbb{C}^2 \setminus \Sigma$
(3) $f \geqslant 0$
(4) $\forall \varepsilon, \delta > 0$, $\exists R$ such that $f(x) < \varepsilon$, on $\mathbb{C}^2 \setminus B(0, R) \bigcup C_\delta(\Sigma)$,

then f is identically 0. To see this notice that a function f on $\mathbb{C}^2 \setminus \Sigma$ satisfying (1) and (2) must satisfy (2) in the weak sense on the whole of $\mathbb{C}^2$ by the argument used in Proposition 2.2 of [S]. After possibly redefining f on Σ, f is a subharmonic function on $\mathbb{C}^2$ and so using the mean value inequality,

$$\forall x \in \mathbb{C}^2 \setminus \Sigma, \quad f(x) \leqslant \frac{1}{r^3} \int_{\partial B(x,r)} f(x) d\partial B.$$

Fix $x \in \mathbb{C}^2$. Given any ε and δ positive, using (4) we can find R such that

$$f(x) < \varepsilon, \text{ on } \partial B(x, R) \backslash C_\delta(\Sigma),$$

and since f is bounded and nonnegative,

$$0 \leqslant f(x) \leqslant \frac{1}{R^3} \int_{\partial B} \varepsilon d\partial B + \frac{\sup(f)}{R^3} \int_{B(\delta)} dB(\delta) \leqslant \varepsilon + \delta^3 \sup(f).$$

Letting ε and δ tend to 0 it follows that $f(x) = 0$ so f is identically 0. Suppose $h = h_0 g$ is another solution satisfying (i), (ii) and (iii). We know by (i) of Lemma 3.5 that

$$\Delta(\Psi(g_\infty g^{-1})) \leqslant 0, \qquad \text{on } \mathbb{C}^2 \backslash \Sigma.$$

Also, since both metrics are mutually bounded with respect to h_0 and have the same determinant, $\Psi(g_\infty g^{-1})$ is non negative and bounded on $\mathbb{C}^2 \backslash \Sigma$ and because of (iii), $\Psi(g_\infty g^{-1})$ satisfies also condition (4) above so it is identically zero on $\mathbb{C}^2$ and we deduce that $g_\infty = g$. $\square$

If K is a compact Lie group acting holomorphically isometrically on $(\mathbb{C}^2 \backslash \Sigma, \omega)$ with a holomorphic lift of the action to $\mathcal{V}$ so that K preserves the H.Y.M. equation it follows by this uniqueness that h_∞ will be K-invariant whenever h_0 is. We now show that if h_0 has finite curvature action, h_∞ also does so it is a H.Y.M. metric.

Proposition 3.16. *If the initial metric h_0 satisfies the hypothesis of Proposition 3.6 and further there is a positive δ such that*

$$(1 + r^{2+\delta})|F_{h_0}(x)|_{h_0} \longrightarrow 0, \quad \text{as} \quad r = |x| \to \infty \text{ in } \mathbb{C}^2 \backslash \Sigma,$$

then h_∞ has square integrable curvature and furthermore we have that

$$\int_{\mathbb{C}^2 \backslash \Sigma} Tr(F_{h_\infty} \wedge F_{h_\infty}) \leqslant \int_{\mathbb{C}^2 \backslash \Sigma} Tr(F_{h_0} \wedge F_{h_0}).$$

Proof:. Let ψ be the exhaustion function given by Lemma 3.3. Let

$$f_R = \max\{0, 1 - \frac{1}{R}\psi\}.$$

Let $D_R = \psi^{-1}[0, R]$ and let $\{h_R\}$ be the family of local solutions to the H.Y.M. equations on the domains $\{D_R\}$ given by Proposition 3.4. We can proceed as in Lemma 3.3 of [S] to show that since ψ is plurisubharmonic and bounded by the Kähler form there is a positive constant C independent of R such that

$$\int_{\mathbb{C}^2 \backslash \Sigma} f_R Tr(F_{h_R} \wedge F_{h_R}) \leqslant \int_{\mathbb{C}^2 \backslash \Sigma} f_R Tr(F_{h_0} \wedge F_{h_0}) + \frac{C}{R} \int_{D_R} |s_R||F_{h_0}|_{h_0} \omega^2. \tag{3.17}$$

where the complex gauge transformation g_R such that $h_R = h_0 g_R$ can be written as $g_R = \exp(s_R)$. Fix a compact subset Z of $\mathbb{C}^2 \setminus \Sigma$. Since for each R h_R is H.Y.M. ,

$$Tr(F_{h_R} \wedge F_{h_R}) = |F_{h_R}|^2_{h_R},$$

and hence for all large enough R we have

$$\int_{\mathbb{C}^2 \setminus \Sigma} f_R Tr(F_{h_R} \wedge F_{h_R}) = \int_{\mathbb{C}^2 \setminus \Sigma} f_R |F_{h_R}|^2_{h_R} \geqslant C'(R) \int_Z |F_{h_R}|^2_{h_R}, \tag{3.18}$$

with $C'(R)$ a constant independent of Z and which can be taken to tend to 1 as $R \to \infty$. Since $h_R \to h_\infty$ in $\mathcal{C}^\infty$ of compact subsets of $\mathbb{C}^2 \setminus \Sigma$ it follows that

$$\lim_{R \to \infty} \int_Z |F_{h_R}|^2_{h_R} = \int_Z |F_{h_\infty}|^2_{h_\infty}. \tag{3.21}$$

From (3.17), (3.18) and (3.19) we then obtain the bound

$$\int_Z |F_{h_\infty}|^2_{h_\infty} \leqslant \frac{1}{C'(R)} \{ \lim_{R \to \infty} [\int_{\mathbb{C}^2 \setminus \Sigma} f_R Tr(F_{h_0} \wedge F_{h_0}) + \frac{C}{R} \int_{D_R} |s_R| |F_{h_0}|_{h_0})] \}. \tag{3.20}$$

As $R \to \infty$, $f_R \to 1$ so the first integral on the right hand side of (3.20) is finite because

$$|Tr(F_{h_0} \wedge F_{h_0})| \leqslant |F_{h_0}|^2_{h_0}.$$

which by the assumptions is clearly integrable on $\mathbb{C}^2 \setminus \Sigma$. For the second integral note that the proof of Proposition 3.6 gives that $|g_R - I| = O(\frac{1}{1+R})$ as $R \to \infty$, hence $|s_R| = O(\frac{1}{1+R})$. By the assumptions on $|F_{h_0}|$ it thus follows that

$$\frac{1}{R} \int_{D_R} |s_R| |F_{h_0}|_{h_0} \omega^2 \leqslant \frac{1}{R} Vol(D_R) \frac{K}{R^{3+\delta}} \leqslant K \frac{1}{R^\delta},$$

with K some constant independent of R. Letting R tend to ∞ this gives

$$\int_Z |F_{h_\infty}|^2_{h_\infty} \leqslant \int_{\mathbb{C}^2 \setminus \Sigma} Tr(F_{h_0} \wedge F_{h_0}).$$

We now let $Z \to \mathbb{C}^2 \setminus \Sigma$ to obtain the required result. □

We now relate the asymptotic behaviour of A_{h_∞} along Σ to that of A_{h_0}. This is a local fact so we can restrict ourselves to a model complex pair (D^2, D) consisting of a disc inside a 2-disc.

Proposition 3.21. *Let h, h' be mutually bounded smooth metrics on $\mathcal{V} \to D^2\backslash D$ with F_h, $F_{h'} \in L^2$ then A_h and $A_{h'}$ have the same holonomy along D.*

Proof. With 1.14 (ii) at hand the claim easily follows however we can also give a proof not requiring the sophisticated analysis going into that result. By Lemma 1.1 it suffices to check (i) at a point on D. We can find a transversal disc to D through p where both A_h and $A_{h'}$ restrict to have L^2 curvature. On this disc choose unitary radial gauges for the two connections. From the proof of 1.1 it follows that in these gauges the limit holonomy exists on the disc for both connections and by a further unitary change of gauge we can assume that

$$A_h = d + \alpha d\theta + a, \qquad A_{h'} = d + \alpha' d\theta + a',$$

with

$$\|a\| \in L^2, \qquad \|a'\| \in L^2, \tag{3.22}$$

and α, α' of the form (1.23). Let g be the positive h–self–adjoint complex gauge transformation such that $h' = gh$. Since h and h' are mutually bounded g is bounded with its inverse, so in the gauges (3.22) both are represented by bounded matrices, and by definition of the Chern connection g must satisfy

$$g^*(A_h^{1,0}) = A_{h'}^{1,0} \tag{3.23.}$$

Consider a sequence of concentric discs D_N of radius $1/N$ and pull–back the two connections restricted to these discs to the unit disc to obtain sequences $\{A_h^N\}$ and $\{A_{h'}^N\}$ and a corresponding sequence of bounded complex gauge transformations $\{g^N\}$. The bounds in (3.22) immediately imply that the sequences tend to d_α and $d_{\alpha'}$ in L^2 respectively so that letting $z = r\exp(2\pi i\theta)$, $A_h^{1,0}$ $A_h^{1,0}$ converge to

$$\overline{\partial}_\alpha = \overline{\partial} - \alpha\frac{d\overline{z}}{2\overline{z}}, \qquad \overline{\partial}_{\alpha'} = \overline{\partial} - \alpha'\frac{d\overline{z}}{2\overline{z}},$$

respectively. Since g^N is bounded with its inverse it has a subsequence converging to a limit $\tilde{g}$ with bounded inverse which by (3.23) satisfies

$$\overline{\partial}\tilde{g} + \alpha * \alpha(\tilde{g}) = \overline{\partial}\tilde{g} + \{\alpha\tilde{g} - \tilde{g}\alpha'\}\frac{d\overline{z}}{2\overline{z}} = 0. \tag{3.24}$$

If $\alpha \neq \alpha'$, it follows that there is one column of $\tilde{g}$ or of $\tilde{g}^{-1}$ on which $\alpha * \alpha'$ acts with all non–zero eigenvalues strictly between 1 and –1. Without loss of generality assume this happens for $\tilde{g}$. Since

$$Ker(\overline{\partial} + \gamma\frac{d\overline{z}}{2\overline{z}}) = \{r^{-\gamma}f \mid f \in Hol(D^*)\},$$

for $\gamma \notin \mathbb{Z}$ and $\tilde{g}$ is bounded it follows from (3.24) that $det(\tilde{g}) = 0$ contradicting the fact that $\tilde{g}^{-1}$ is bounded. Thus $\alpha = \alpha'$. □

We deduce that A_{h_∞} has the same holonomy along Σ as A_{h_0} provided h_0 has L^2 curvature. We summarise the results of the last two sections in the following theorem.

Theorem 3.25. *Let $\mathcal{V}$ be a holomorphic vector bundle on $\mathbb{C}^2\setminus\Sigma$. If h_0 is a metric satisfying the conditions of Proposition 3.6 and 3.16, there exist a unique H.Y.M. metric h_∞ on $\mathcal{V}$, mutually bounded with respect to h_0 such that $\det(h_\infty)=\det(h_0)$ and $h_0^{-1}h_\infty$ extends continuously as the identity across D_∞. This metric has the same holonomy as h_0. If $\mathcal{V}=\mathcal{E}\times_{G^{\mathbb{C}}}\mathbb{C}^n$ and h_0 comes from a G–reduction on $\mathcal{E}$ or is invariant under the action of a compact group K on $\mathcal{V}$ covering a holomorphic isometric action on $\mathbb{C}^2\setminus\Sigma$, h_∞ also does.*

As a final remark we note that in the case where Σ is a complex line A_{h_∞} and A_{h_0} thought of as connections on $[0,\infty)\times Y^*$, the half cylinder (1.18), have the same limit on Y^*. Once the properties of connections on a punctured half–cylinder, outlined in §1.3 for (1.18), are developed in general this should hold for all curves Σ in $\mathbb{P}^2$.

Proposition 3.26. *Let Σ be a complex line in $\mathbb{P}^2$ and let h, h' be mutually bounded smooth metrics on $\mathcal{V}\to\mathbb{C}^2\setminus\Sigma$ such that F_h, $F_{h'}\in L^2$ and the complex gauge transformation g relating them, $h'=hg$, tends to 1 along D_∞ then A_h and $A_{h'}$ have the same limit on Y^* up to $U(n)$.*

Proof. By the discussion in §1.3 we know that in this case the limit at infinity and the holonomy along Σ determine one another completely so from Proposition 3.21 it already follows that A_h and $A_{h'}$ both have $L^2_{1,\alpha}$-limit Γ_J, where $J=\exp(2\pi i\alpha)$ is the inverse of the holonomy, up to complex gauge equivalence. The extra information we have here is that since g tends to 1 along D_∞ the limits are the same up to a constant in $U(n)$. □

3.4 Initial Metrics

With Theorem 3.25 and Simpson's existence theorem ([S], Theorem 1. See §3.7, Theorem 3.72 for a statement of this result) at hand one implication of both Theorems 1 and 2 easily follows provided we can construct suitable initial metrics. For Theorem 1 we need to show that given a principal α–parabolic framing $\mathcal{E}$ on $(\mathbb{P}^2,\Sigma,D_\infty)$ there is an associated vector bundle with a Hermitian metric induced by a G–reduction whose Chern connection satisfies the conditions of Propositions 3.6 and 3.16 and has limit $\Gamma_{\rho(J^{-1})}$ on Y^*. For Theorem 2 we need to show that given an α–parabolic vector bundle $\mathcal{V}$ on a general compact complex pair (X,Σ) with α–degree zero there is an initial metric on it whose Chern connection has curvature orthogonal to the Kähler form, satisfying condition (i) of Proposition 3.6 and square integrability of the curvature. It turns out that making condition (i) of Proposition 3.6 satisfied does not require any specific property of the pair (X,Σ) nor of the parabolic bundle other than the fact that they are holomorphic. We shall therefore begin by considering the case of principal α–parabolic bundles on general pairs and use it as a stepping stone for the case of framings on $(\mathbb{P}^2,\Sigma,D_\infty)$. It is worth remarking that essentially the same method works to produce initial metrics with the analogous decay property for framings on more general (X,Σ,D_∞) provided the framing bundle $\mathcal{E}^0$ is suitably flat and the base metric has a suitable growth along D_∞.

Construction 3.27.

(i) Parabolic bundles on a general (X, Σ). We set up the construction specifically for the case of principal parabolic bundles with structure group a complex simple group $G^{\mathbb{C}}$ or $GL(n)$. For given k, c, ℓ and an element $i\alpha \in \mathfrak{g}$ let $E^{\mathbb{C}}$ be the $\mathcal{C}^\infty$ principal α–parabolic bundle on (X, Σ) of this topological type. Let $\rho : G^{\mathbb{C}} \to GL(n)$ be a faithful linear representation and let V be the associated $\rho(\alpha)$–parabolic vector bundle as in Proposition 2.14. As observed in §3.1 we can choose on V a Hermitian metric h compatible with the $G^{\mathbb{C}}$–structure and we then consider the orthogonal decomposition or $C(\rho(i\alpha))$–reduction

$$V_{|\Sigma} \cong \bigoplus_i F_i/F_{i+1}. \tag{3.28}$$

induced by h and the parabolic filtration. We equip a tubular neighbourhood $N(\Sigma)$ of Σ in X with a fibred structure $N(\Sigma) \xrightarrow{\nu} \Sigma$ such that ν is holomorphic along the disc fibres. Since

$$V_{|N(\Sigma)} \cong \nu^*(V_{|\Sigma}).$$

we can pull–back (3.28) via ν to an orthogonal decomposition along the disc fibres. Let $\sigma \in H^0(X, [\Sigma])$ be such that

$$\mathrm{Div}(\sigma) = \Sigma.$$

For any given 2–form η representing the Poincare' dual of Σ we can choose a Hermitian metric j on $[\Sigma]$ so that

$$\frac{i}{2\pi}\overline{\partial}\partial \log(|\sigma|_j^2) = \eta. \tag{3.29}$$

Let $\beta(r)$ be a cut–off function which is 1 at the origin and 0 further away and let f_α be the complex gauge transformation of $V_{|X\setminus\Sigma}$

$$f_\alpha = \exp(\beta(|\sigma|_j)\log(|\sigma|_j)\rho(\alpha)). \tag{3.30}$$

Notice that this makes sense since the decomposition (3.28) is a reduction to $C(\rho(i\alpha))$ so $\rho(\alpha)$ is defined as an endomorphism of V on $N(\Sigma)$. For suitable choices of the 2–form η of (3.29) the Hermitian metric

$$h_\alpha = f_\alpha^*(h). \tag{3.31}$$

will be the desired initial metric on $X\setminus\Sigma$ for any $G^{\mathbb{C}}$–parabolic structure on V for Theorem 2.

(ii) Parabolic framings on $(\mathbb{P}^2, \Sigma. D_\infty)$. Again, fix a linear faithful representation $\rho : G^{\mathbb{C}} \to GL(n)$ giving framing vector bundle $\mathcal{V}^0 = D_\infty \times \mathbb{C}^n$. Fix once and for all a constant Hermitian metric on this compatible with the $G^{\mathbb{C}}$–structure. The parabolic filtration at $\mathrm{p} = D_\infty \cap \Sigma$ then gives a holomorphic orthogonal direct sum decomposition on $\mathcal{V}^0$. As in (i), for a given topological type k, ℓ we fix a $\mathcal{C}^\infty$ α–parabolic framing $(E^{\mathbb{C}}, \varphi)$ with these

topological invariants and let V be the associated framed $\rho(\alpha)$–parabolic vector bundle as in Proposition 2.16. The isomorphism

$$\varphi : V_{|D_\infty} \cong \mathcal{V}^0$$

transports the above holomorphic orthogonal decomposition to $V_{|D_\infty}$ and we choose a Hermitian metric h on V compatible with the $G^{\mathbb{C}}$–structure and coinciding with the given one on D_∞. As in (i) we equip $N(\Sigma)$ and $N(D_\infty)$ with fibred structures holomorphic along the disc fibres and also such that $\nu^{-1}(\mathrm{p}) = D_\infty$ and both fibred structures are holomorphic in a neighbourhood of p. We now obtain an orthogonal decomposition on $N(\Sigma) \cup N(D_\infty)$ as in (i) with the extra requirement that it coincides with the one already given by the framing on D_∞, it is the pull–back of this one along the disc fibres of $N(D_\infty)$ and its definition on $N(\Sigma)$ coincides with the one on $N(D_\infty)$ on $N(\Sigma) \cap N(D_\infty)$. Let $\sigma,\ \tau \in H^0(\mathbb{P}^2,\ \mathcal{O}(1))$ be such that

$$\mathrm{Div}(\sigma) = \Sigma, \qquad \mathrm{Div}(\tau) = D_\infty.$$

We can choose Hermitian metrics $j,\ k$ on $\mathcal{O}(1)$ so that

$$\overline{\partial}\partial \log(|\tau|_j) = \overline{\partial}\partial \log(|\sigma|_k).$$

Let $\beta(r)$ be a cut–off function as in (i) and let f_α be the complex gauge transformation of $V_{|\mathbb{C}^2 \setminus \Sigma}$

$$f_\alpha = \exp(\beta(|\sigma|_j|\tau|_k)\log(\frac{|\sigma|_j}{|\tau|_k})\rho(\alpha)). \tag{3.32}$$

As in (i) this definition makes sense. The Hermitian metric

$$h_\alpha = f_\alpha^*(h). \tag{3.33}$$

is the desired initial metric on $V_{|\mathbb{C}^2 \setminus \Sigma}$ for Theorem 3.25. □

As in §2.4 we identify parabolic bundles or framings with suitable $\overline{\partial}$–operators on an associated vector bundle of the $\mathcal{C}^\infty$ parabolic bundle or framing with the same topological invariants, for instance the associated vector bundle of Construction 3.27. For later purposes we now prove a proposition that contains more than just checking that h_α has the right properties. In the case of parabolic bundles over a compact pair (X, Σ) we shall also assume that X is equipped with a Kähler metric ω.

Proposition 3.34. *For a given topological type of α–parabolic vector bundle or framing let (V, h, s) be the data consisting of a unitary vector bundle with orthogonal decomposition (3.28) as from Construction 3.27. Then*

(i) *Let $i\alpha \in \mathfrak{u}(n)$ be of the form (1.23). There is a pair $(\overline{V},\ \overline{s})$, $\mathcal{C}^\infty$–isomorphic to (V, s) on which h_α extends as a smooth Hermitian metric and in the case of framings $\overline{V}$ has a smooth trivialisation over D_∞. If $i\alpha \in \mathfrak{g}$ is such that $ad(\alpha)$ has all its eigenvalues*

in $(-1,1)$ *and* (V,s,h) *has a* G*–structure* E *via some faithful representation* ρ, $(\overline{V},\overline{s},h_\alpha)$ *also has a* $G^{\mathbb{C}}$*–structure,* $\overline{E}$*, isomorphic to* E *off* Σ $(\Sigma\cup D_\infty)$.

(ii) *If* $i\alpha\in\mathfrak{u}(n)$ *is of the form (1.23) then for any* α*–parabolic structure* $\mathcal{V}$ *on* V

$$\sup_{x\in N\setminus\Sigma}|\hat{F}_{\overline{\partial}_{\mathcal{V}},h_\alpha}(x)|<\infty \tag{3.35}$$

and

$$A_{\overline{\partial}_{\mathcal{V}},h_\alpha}-A_\alpha\in L^p_{1,\alpha},\qquad \forall p\in[2,(1/2-\alpha_0/6)^{-1}) \tag{3.36}$$

where A_α *is a model singular connection for* $(\overline{V},\overline{s})$. *If* $\deg_\alpha(V)=0$ *and we take* $\eta=\frac{i}{2\pi}Tr(F_{\overline{\partial}_{\mathcal{V}},h})$ *in (3.29), the curvature of* $A_{\overline{\partial}_{\mathcal{V}},h_\alpha}$ *has zero component along the Kähler form and if* $c_1(V)=-Tr(\alpha)[\Sigma]$, *it is an* $SU(n)$*–connection and* $A_{\overline{\partial}_{\mathcal{V}},h_\alpha}-A_\alpha$ *is an* $\mathfrak{su}(n)$*–valued form.*

(iii) *If* $i\alpha\in\mathfrak{g}$ *is such that* $ad(\alpha)$ *has all its eigenvalues in* $(-1,1)$ *and* (V,s,h) *has a* G*–structure through some faithful representation* ρ, *then for any* $\rho(\alpha)$*–parabolic framed structure* $\mathcal{V}$ *on* V *coming from an* α*–parabolic framed structure,* $F_{\overline{\partial}_{\mathcal{V}},h_\alpha}$ *satisfies (3.36) and the conditions of Propositions 3.6 and 3.16,* A_α *can be taken to be a model singular* G*–connection and* $A_{\overline{\partial}_{\mathcal{V}},h_\alpha}$ *is a* G*–connection with* $A_{\overline{\partial}_{\mathcal{V}},h_\alpha}-A_\alpha\in L^p_{1,\alpha}(S^4\backslash S^2,\mathfrak{g}_{\overline{E}})$ *for all* $p\in[2,(1/2-\alpha_0/6)^{-1})$.

Proof. (i) Let

$$End(V)=(\bigoplus_{i\leqslant j}Hom(V_j,V_i)\oplus\bigoplus_i End(V_i,V_i)\oplus(\bigoplus_{i\leqslant j}Hom(V_i,V_j)), \tag{3.37}$$

be the decomposition induced on $N(\Sigma)$ by that on V of Construction 3.27 (on $N(\Sigma)\cup N(D_\infty)$ in the framed case). This is the decomposition according to the the eigenvalues $\alpha_i-\alpha_j$ of $ad(\alpha)$. Let $\{g_{ij}\}$ be a unitary cocycle compatible with the orthogonal decomposition on $N(\Sigma)$ $(N(\Sigma)\cup N(D_\infty))$ defining V relative to some open cover. The compatibility means that the components of g_{ij} relative to the non–zero eigenvalues vanish on $N(\Sigma)$ $(N(\Sigma)\cup N(D_\infty))$ and in the framed case also that g_{ij} restricts to the identity on D_∞. Since the $\alpha_i-\alpha_j$ are all contained in $(-1,1)$, it follows from the definition of h_α that

$$g^{ij}_\alpha=f_\alpha^{-1}\circ g_{ij}\circ f_\alpha, \tag{3.38}$$

is a smooth h_α–unitary cocycle across Σ, lies in $C(i\alpha)$ on $N(\Sigma)$ $(N(\Sigma)\cup N(D_\infty))$ and in the framed case is the identity on D_∞. It therefore defines a $\mathcal{C}^\infty$ unitary vector bundle with $C(i\alpha)$–reduction on Σ, $(\overline{V},\overline{s},h_\alpha)$, and in the framed case also a smooth trivialisation on D_∞. Since $Ad(f_\alpha)$ acts trivially on the $C(i\alpha)$–components of decomposition (3.37) it follows that the pair $(\overline{V},\overline{s})$ has the same topological type as (V,s). If (V,s,h) (and the framing) has a $G^{\mathbb{C}}$–structure we can take $\{g_{ij}\}$ to be a G–cocycle. Since α' has all its eigenvalues in $(-1,1)$ and $ad(\alpha)$ acts on $\rho(\mathfrak{g})$ with the same eigenvalues as $ad(\alpha')$ on $\mathfrak{g}$ it

follows from the definition of h_α that g_α^{ij} is a smooth G–cocycle for $(\overline{V}, \overline{s}, h_\alpha)$, giving it a G–structure $\overline{E}$ which by construction is G–conjugate to that on V off Σ ($\Sigma \cup D_\infty$).

(ii) Out of $N(\Sigma)$ h is smooth so we restrict our attention to it. From decomposition (3.28) we have that f_α acts on V as multiplication by $|\sigma|^{\alpha_i}$ on V_i so $Ad(f_\alpha)$ acts on $End(V)$ as multiplication by $|\sigma|_j^{\alpha_i - \alpha_j}$ on $Hom(V_j, V_i)$. From (3.37) we obtain induced decompositions on $\Omega^{0,i}(End(V))$–forms with $Ad(f_\alpha)$ acting in the same way. By Lemma 2.31 we can assume every parabolic structure to be holomorphically trivial along the disc fibres of ν making the orthogonal decompositions of (3.28) and (3.37) flat along them. Let (z, w) denote complex coordinates at a point in Σ so that $\{w = 0\}$ is a fibre of ν and z is a coordinate normal to Σ and in a local gauge write

$$\overline{\partial}_\nu + a = (\overline{\partial}_z + a_z) + (\overline{\partial}_w + a_w). \tag{3.39}$$

Since the orthogonal splitting (3.28) is flat along the disc fibres of $N(\Sigma)$ at each point in Σ we can find an h–gauge such that the matrix of the $\overline{\partial}$–operator has the form

$$a_z = zw\tilde{a}_z, \qquad a_w = (\bigoplus_{i \geqslant j} a_w^{ij}) \oplus \bigoplus_i a_w^{ii} \oplus (z \bigoplus_{i \leqslant j} a_w^{ji}). \tag{3.40}$$

Notice, however, that we can not assume the fibration to be holomorphic, in particular we can't assume the ν fibres to be given by $\{w = const.\}$ except for that at a chosen point. At each point of Σ we thus obtain h_α–unitary gauges where the $\overline{\partial}$–operator has matrix

$$\begin{aligned} a_z^\alpha &= f_\alpha^{-1}\overline{\partial} f_\alpha + zw Ad(f_\alpha)(\tilde{a}_z), \\ a_w^\alpha &= Ad(f_\alpha)(a_w) = (\bigoplus_{i>j} a_w^{ij}(|\sigma|_j^{\alpha_i - \alpha_j})) \oplus \bigoplus_i a_w^{ii} \oplus (z \bigoplus_{i<j} a_w^{ji}(|\sigma|_j^{\alpha_j - \alpha_i}). \end{aligned} \tag{3.41}$$

By the way σ was chosen we have that

$$\overline{\partial}(f_\alpha^{-2}\partial f_\alpha^2) = \frac{\pi}{i}\eta_\alpha. \tag{3.42}$$

Since

$$F_{\overline{\partial}_\nu, h_\alpha} = Ad(f_\alpha)(F_{\overline{\partial}_\nu, h}) + Ad(f_\alpha)(\overline{\partial}_\nu(f_\alpha^{-2}\partial_h(f_\alpha^2))), \tag{3.43}$$

(3.41), (3.42) and (3.43) imply the following pointwise bounds on $N(\Sigma)$

$$|(F_{\overline{\partial}_\nu, h_\alpha})_{w\overline{w}}|_{h_\alpha} = |(F_{\overline{\partial}_\nu, h_\alpha})_{z\overline{z}}| = O(1), \ |(F_{\overline{\partial}_\nu, h_\alpha})_{z\overline{w}}|_{h_\alpha} = |(F_{\overline{\partial}_\nu, h_\alpha})_{w\overline{z}}|_{h_\alpha} = O(|\sigma|_j^{-\alpha_0}).$$

This together with (3.41) immediately gives (3.35) and (3.36). When $\deg_\alpha(V) = 0$,

$$\deg(V) = \int_X \frac{i}{2\pi} Tr(F_{\overline{\partial}_\nu, h}) \wedge \omega = -Tr(\alpha)\deg(\Sigma) = \int_X -Tr(\alpha)\eta \wedge \omega,$$

for some form η Poincare' dual to Σ. Taking traces in (3.43) gives

$$Tr(F_{\overline{\partial}_{\mathcal{V}},h_\alpha}) = Tr(F_{\overline{\partial}_{\mathcal{V}},h}) + Tr(\overline{\partial}(f_\alpha^2 \partial f_\alpha^{-2})) = \frac{2\pi}{i}\eta Tr(\alpha),$$

so taking the same η in (3.29) it follows that

$$\int_{X\setminus\Sigma} \frac{i}{2\pi} Tr(F_{\overline{\partial}_{\mathcal{V}},h_\alpha}) \wedge \omega = 0.$$

Similarly, when $c_1(V) = -Tr(\alpha)[\Sigma]$ we can ensure that

$$\frac{i}{2\pi} Tr(F_{\overline{\partial}_{\mathcal{V}}}) = Tr(\alpha)\eta - Tr(\alpha)\eta = 0,$$

so that $A_{\overline{\partial}_{\mathcal{V}}}$ is an $SU(n)$–connection. That A_α can be chosen so that $A_{\overline{\partial}_{\mathcal{V}}} - A_\alpha$ is an $\mathfrak{su}(n)$–form can be extracted from Proposition 1.29.

(iii) Using the calculations of (ii) on $N(\Sigma)\setminus N(D_\infty)$, where the euclidean metric has locally the same asymptotics of any smooth metric, it suffices to restrict our attention to $N(D_\infty)$. We first estimate the curvature with respect to a smooth metric on $\mathbb{P}^2$ and then translate this into euclidean bounds. From decomposition (3.28) of $V_{|N(D_\infty)}$ under the action of α we have that f_α acts on V as multiplication by $(\frac{|\sigma|_j}{|\tau|_k})^{\alpha_i}$ on V_i and so $Ad(f_\alpha)$ acts on $End(V)$ as multiplication by $(\frac{|\sigma|_j}{|\tau|_k})^{\alpha_i-\alpha_j})$ on $Hom(V_j, V_i)$ and similarly on the $\Omega^{0,i}(End(V))$'s. By the relevant part of Lemma 2.31 we can assume every parabolic framed structure to be holomorphically trivial along the disc fibres of $N(D_\infty)$, along D_∞ and in a neighbourhood of p so that the orthogonal decomposition of (3.28) and (3.37) are flat there. Let (z,w) denote complex coordinates at a point in D_∞ so that w is normal to D_∞ and along Σ and locally write the $\overline{\partial}$–operator of $\mathcal{V}$ as in (3.39). By the flatness of h at each point in D_∞ and since in this case we can actually assume the fibration on $N(D_\infty)$ to be holomorphic, we can find an h–gauge compatible with the $G^{\mathbb{C}}$–structure where the $\overline{\partial}$–operator has $\mathfrak{g}^{\mathbb{C}}$–valued matrix of the form

$$a_w = 0, \qquad a_z = w\widetilde{a}_z, \tag{3.44}$$

and we can also assume that a_z vanishes in a neighbourhood of p. At each point of D_∞ we thus have a gauge where the $\overline{\partial}$–operator has matrix

$$a^\alpha = f_\alpha^{-1}\overline{\partial} f_\alpha + a_z^\alpha. \quad a_z^\alpha = w(\bigoplus_{i\geqslant j} a_z^{ij}(\frac{|\sigma|_j}{|\tau|_k})^{\alpha_i-\alpha_j})) \oplus a_z^{ii} \oplus (z\bigoplus_{i\leqslant j} a_z^{ji}(\frac{|\sigma|_j}{|\tau|_k})^{\alpha_j-\alpha_i})). \tag{3.45}$$

By the way σ and τ were chosen we have that

$$\overline{\partial}(f_\alpha^{-2}\partial(f_\alpha^2)) = 0. \tag{3.46}$$

(this holds also along Σ).

Using (3.43), (3.44), (3.45), (3.46) and the fact that since $ad(\alpha)$ had all its eigenvalues lying in $(-1,1)$ and the matrix of the $\overline{\partial}$-operator was $\mathfrak{g}^{\mathbb{C}}$-valued $ad(\rho(\alpha))$ acts non-trivially only where its eigenvalues lie in $(-1,1)$, direct calculations imply the bounds on $N(D_\infty)$

$$|(F_{\overline{\partial}_{\mathcal{V}},h_\alpha})_{z\overline{z}}|_{h_\alpha},\ |(F_{\overline{\partial}_{\mathcal{V}},h_\alpha})_{w\overline{w}}| = O(1),\ |(F_{\overline{\partial}_{\mathcal{V}},h_\alpha})_{z\overline{w}}|_{h_\alpha} = |(F_{\overline{\partial}_{\mathcal{V}},h_\alpha})_{w\overline{z}}|_{h_\alpha} = O(|\sigma|_j^{-\alpha_0}+|\tau|_k^{-\alpha_0}).$$

To translate this into euclidean bounds take euclidean coordinates along D_∞ to be given by $(1/z,\ w)$ or $(z,\ 1/w)$ to obtain

$$(1+r^3)|\hat{F}_{\overline{\partial}_{\mathcal{V}},h_\alpha}|_{h_\alpha} = O(1), \quad r_\Sigma{}^{\alpha_0}(1+r^{3-2\alpha_0})|F_{\overline{\partial}_{\mathcal{V}},h_\alpha}|_{h_\alpha} = O(1), \text{ on } \mathbb{C}^2,$$

which since $0 < \alpha_0 < 1$ shows that $A_{\overline{\partial}_{\mathcal{V}},h_\alpha}$ satisfies the conditions of Propositions (3.6) and (3.16). That $A_{\overline{\partial}_{\mathcal{V}},h_\alpha}$ is a G-connection when $(\overline{V},s,h_\alpha)$ has a G-structure follows from the construction while the claim on A_α and $A_{\overline{\partial}_{\mathcal{V}},h_\alpha} - A_\alpha$ follows from Proposition 1.29. The last claim follows from (3.41) and (3.45). □

As already remarked in §1.3, $A_{\overline{\partial}_{\mathcal{V}},h_\alpha} - A_\alpha \in L^p_{1,\alpha}(S^1)$ implies that $A_{\overline{\partial}_{\mathcal{V}},h_\alpha}$ has $\mathcal{C}^\infty$-exponential limit Γ_{J-1} on Y^* in the sense (1.21).

3.5 Extension of H.Y.M. vector bundles

The main ingredient in the proof of the other direction of both Theorems 1 and 2 is a local extension theorem for H.Y.M. vector bundles with L^2-curvature on a punctured polydisc. Using Theorem 1.14 (iii) this can be reduced to a relatively simple problem.

Equip B^4 with a complex structure making it into a polydisc D^2 with complex coordinates (z,w) so that $D_z = \{z\}\times D$, $D_w = D\times\{w\}$, let $r_z = |z|$, $r_w = |w|$ and consider the $\overline{\partial}$-operator on a trivial vector bundle V and the corresponding one on $End(V)$

$$\begin{aligned}\overline{\partial}_{\alpha,\beta} &= \overline{\partial} - \alpha\frac{d\overline{z}}{2\overline{z}} - \beta\frac{d\overline{w}}{2\overline{w}}\\ \overline{\partial}_{\alpha,\beta} &= \overline{\partial} - [\alpha\frac{d\overline{z}}{2\overline{z}},-] - [\beta\frac{d\overline{w}}{2\overline{w}},-],\end{aligned} \tag{3.47}$$

where α and β are real diagonal matrices with eigenvalues $\{\alpha_i\}$, $\{\beta_j\}$.

Proposition 3.48. *Let $(\mathcal{V},h)$ be a trivialised Hermitian vector bundle on $D^*\times D^*$ such that $\overline{\partial}_{\mathcal{V}} = \overline{\partial}_{\alpha,\beta} + a$ with $a\in L^\infty$. There exist a continuous complex gauge transformation g such that (on a possibly smaller polydisc)*

$$g^*(\overline{\partial}_{\mathcal{V}}) = \overline{\partial}_{\alpha,\beta}.$$

If $a_w = O(r_w^k)$ and $a_z = O(r_w^{k+1})$ we can choose g of the form $g = 1 + O(r_w^{k+1})$. If $(\mathcal{V},h)$ has a G-structure then g can be chosen to be $G^{\mathbb{C}}$-valued.

Proof. We have to find g such that $g^*(\overline{\partial}_{\alpha,\beta}+a) = \overline{\partial}_{\alpha,\beta}$. This is equivalent to finding an invertible solution to the $\overline{\partial}$-equation

$$\overline{\partial}g - [\alpha,g]\frac{d\overline{z}}{2\overline{z}} - [\beta,g]\frac{d\overline{w}}{2\overline{w}}] + ag = 0. \tag{3.49}$$

Writing $g = 1 + f$ it suffices to find f close to zero satisfying

$$\overline{\partial} f - [\alpha, f]\frac{d\overline{z}}{2\overline{z}} - [\beta, f]\frac{d\overline{w}}{2\overline{w}}] + af = a$$

and since we only need to find the solution in a small neighbourhood of the origin we can multiply a by a cut–off function which is one near the origin to obtain a form compactly supported in D^2 and preserve integrability on a smaller polydisc. Using the pointwise bounds on a we can separate variables and use the contraction mapping principle as in [DK] p. 51. We first solve the equation

$$\overline{\partial}_z f - [\alpha, f]\frac{d\overline{z}}{2\overline{z}} + a_z f = a_z. \tag{3.50}$$

Let $K(u, v)$ be the Cauchy kernel on $\mathbb{C}$. Let $[\gamma]$ be the integer part of $|\gamma|$ if $\gamma \notin \mathbb{Z}$, $\gamma - 1$ if γ is a non–negative integer and $|\gamma|$ if γ is a strictly negative integer. For any bounded functions a, b and f and any $\gamma \geqslant 0$, the operator

$$T_z^\gamma(f)(z, w) = \frac{r_z^\gamma}{z^{[\gamma]}} \int_{D_w} [K(z, u)\frac{u^{[\gamma]}}{r_u^\gamma}](a(u, w) - b(u, w)f(u, w)) \tag{3.51}$$

is defined and since the Cauchy kernel is the fundamental solution to the $\overline{\partial}$–equation in one variable it satisfies $\overline{\partial}_\gamma(T_\gamma(f)) = a - af$. Similarly, if $\gamma < 0$, for the operator

$$T_z^\gamma(f)(z, w) = \frac{z^{1+[\gamma]}}{r_z^{-\gamma}} \int_{D_w} [K(z, u)\frac{r_u^{-\gamma}}{u^{1+[\gamma]}}](a(u, w) - b(u, w)f(u, w)). \tag{3.52}$$

Let $\{\alpha_i\}$ be the eigenvalues of α so that $ad(\alpha)$ acts as multiplication by $\alpha_i - \alpha_j$ on the ij–th component of a matrix $f = (f^{ij})$. We can define an operator T_z^α on matrix valued function $f = (f^{ij})$ by components letting T_z^α act on the $(\alpha_i - \alpha_j)$–eigenspace as in (3.52) if $\alpha_i - \alpha_j \geqslant 0$ and as in (3.54) if $\alpha_i - \alpha_j < 0$ with the term in brackets in the integrands being replaced by

$$a_z^{ij} + \sum_k a_z^{ik} f^{kj}.$$

It follows that if f is a fixed point of T_z^α it satisfies equation (3.50). One can check that for some constant C independent of a and f

$$\|T_z^\alpha(f)\|_{L^\infty} \leqslant C\{\|a\|_{L^\infty}\|f\|_{L^\infty} + \|a\|_{L^\infty}\}, \tag{3.53}$$

thus if $\|a\|_{L^\infty}$ is small T_z^α is a contraction of L^∞ into itself. Because of the pointwise bounds on a we can certainly achieve this after possibly applying a scaling of the coordinates. The contraction mapping principle now implies that T_z^α has a fixed point f with small L^∞ norm. Then $g = 1 + f$ satisfies $g^*(\overline{\partial}_{\alpha,\beta} + a) = \overline{\partial}_{\alpha,\beta} + a'_w$ and by the integrability assumption a'_w must now be holomorphic in z. Also a'_w has small L^∞ norm. We now solve the equation

$$\overline{\partial}_w f + [\beta, f]\frac{d\overline{w}}{2\overline{w}} + a'_w f = a'_w. \tag{3.54}$$

We define operators T_w^γ as in (3.51) and (3.52) by replacing r_z with r_w, a_z with a_w and integrating along the $\{D_z\}$ discs and a corresponding operator $T_w^{-\alpha}$. As above the contraction mapping principle implies that $T_w^{-\alpha}$ has a fixed point f' with small L^∞ norm solving (3.54). Since a'_w was holomorphic in z, f' also is and thus $g = 1 + f'$ satisfies

$$g^*(\overline{\partial}_{\alpha,\beta} + a'_w) = \overline{\partial}_{\alpha,\beta}.$$

The composite $g'' = g' \circ g$ is now the required solution to (3.49). Since g'' is bounded its continuity follows from the L^∞ bounds on a and (3.49) provided $ad(\alpha)$ ad $ad(\beta)$ don't have any non–zero integer eigenvalues or, in general, from the integral representation of f and f'. Suppose now that $a_w = O(r_w^k)$, $a_z = O(r_w^{k+1})$. Consider the $\overline{\partial}$–operator

$$\overline{\partial}_{\alpha,\beta} + \frac{a}{w^{k+1}}.$$

This is integrable and by the assumptions on a_z we see that a_z/w^{k+1} is still bounded so we can proceed as above to find a continuous solution f to the corresponding equation of (3.50) for a_z/w^{k+1}. Then $w^{k+1}f$ solves (3.50) and $g = 1 + w^{k+1}f$ is of the form $1 + O(r_w^{k+1})$. It follows that the resulting matrix a'_w is $O(r_w^k)$ and we we can solve (3.54) with a'_w/w^{k+1} in place of a'_w. This is no longer bounded but since we are now solving in the direction normal to the singular locus T_w^β can still be made into a contraction so that we obtain a continuous solution f' to the equation. Then $w^{k+1}f'$ solves (3.54) and $g' = 1 + w^{k+1}f'$ is of the form $1 + O(r_w^{k+1})$. As above $g'' = g' \circ g$ is now the required solution and $g'' = 1 + O(r_w^{k+1})$ by construction. Finally, if $(\mathcal{V}, h)$ has a G–structure and we start from a G–gauge, invoking as in Proposition 3.4 the closedness of $\Omega^i(\mathfrak{g}_\mathcal{E})$ in $\Omega^i(\text{End }(\mathcal{V}))$, we can take the above complex gauge transformation to be $G^{\mathbb{C}}$–valued. □

With this local result at hand we now proceed to the proof of the global extension theorem.

Theorem 3.55.

(i) *Let $(\mathcal{V}, h)$ be a H.Y.M. vector bundle on $X\backslash\Sigma$ such that A_h has holonomy J. Then $\mathcal{V}$ has an extension to an α–parabolic vector bundle $\overline{\mathcal{V}}$ on (X, Σ), where $i\alpha = \log(J)$, with $\deg_\alpha(\overline{\mathcal{V}}) = 0$ and such that*

$$\forall x \in \Sigma, \quad \mathcal{F}_i(x) =< \{s(x) | s \in \mathcal{O}(\overline{\mathcal{V}}),\ h(s) = O(r_\Sigma^{\alpha_i})\} > . \tag{3.56}$$

If further A_h is an $SU(n)$–connection $\det(\overline{\mathcal{V}}) = -Tr(\alpha)[\Sigma]$.

(ii) *Let $(\mathcal{V}, h)$ be a H.Y.M. vector bundle on $\mathbb{C}^2\backslash\Sigma$ such that A_h has holonomy J along Σ. Then $\mathcal{V}$ has an extension to an α–parabolic vector bundle $\overline{\mathcal{V}}$ on $(\mathbb{P}^2, \Sigma)$, where $i\alpha = \log(J)$, with a flat $C(i\alpha)$–structure on D_∞. $\overline{\mathcal{V}}_{|D_\infty} = \oplus_i \overline{\mathcal{V}}_i$ such that (3.56) holds away from D_∞ and*

$$\forall x \in D_\infty, \quad \overline{\mathcal{V}}_i(x) =< \{s(x) \,|\, s \in \mathcal{O}(\overline{\mathcal{V}}),\ h(s) \sim r_\Sigma^{\alpha_i} r_{D_\infty}^{-\alpha_i}\} > . \tag{3.57}$$

In (i) the extension is unique in the sense that if $(\mathcal{V}', h')$ is another H.Y.M. vector bundle with an extension $\overline{\mathcal{V}}'$ to an α'–parabolic bundle on (X, Σ) satisfying (3.56) and $\mathcal{V}' \overset{f}{\cong} \mathcal{V}$, is bounded with its inverse, then $\alpha = \alpha'$ and f extends to a parabolic isomorphism $\overline{f}$ on X. In (ii) the extension is unique in the sense that if $(\mathcal{V}', h')$ is another H.Y.M. vector bundle with an extension $\overline{\mathcal{V}}'$ to an α'–parabolic bundle on $(\mathbb{P}^2, \Sigma)$ and flat $C(\alpha')$–structure on D_∞ satisfying (3.56) and (3.57) and f is as above and further $f^*(h)/h' \to 1$ along D_∞, then $\alpha = \alpha'$ and f extends to a parabolic isomorphism $\overline{f}$ on $\mathbb{P}^2$ intertwining the flat structures on D_∞. If $(\mathcal{V}, h)$ has a $G^{\mathbb{C}}$–structure and $i\alpha = \rho(\log(J))$ for some admissible $J \in G$ in both (i) and (ii) above $\overline{\mathcal{V}}$ can be chosen to have $G^{\mathbb{C}}$–structure extending that on $\mathcal{V}$ and compatible with the parabolic structure so that (3.56) and (3.57) hold for this α, and the extension is $G^{\mathbb{C}}$–unique.

Proof. (i) We begin by extending the holomorphic structure across Σ. We have to exhibit at each point a local holomorphic frame. We proceed in two steps. First we apply Proposition 1.29 to obtain a $\mathcal{C}^\infty$ vector bundle $\overline{V}$ with a $C(i\alpha)$–reduction on Σ where the Chern connection A_h is gauge equivalent to one $\overline{A}_h$ such that $\overline{A}_h = A_\alpha + a$ with a satisfying the global equivalent of the bounds of 1.14 (iii). Considering this as a $\mathcal{C}^\infty$ α–parabolic vector bundle with the filtration induced by the $C(i\alpha)$ reduction we know by Lemma 1.37 that $\deg_\alpha(\overline{V}) = 0$ and further $\det(\overline{V}) = -Tr(\alpha)[\Sigma]$ if A_h is an $SU(n)$–connection. It follows that in a gauge f_i compatible with the $C(i\alpha)$–reduction

$$\overline{\partial}_{\overline{A}_h} = \overline{\partial}_\alpha + a, \qquad a \in L^\infty(\Omega^{0,1}(End(\overline{V}))),$$

so applying Proposition 3.48 we obtain, on a possibly smaller neighbourhood, a continuous complex gauge transformation g_i such that

$$g_i^*(\overline{\partial}_{\overline{A}_h}) = \overline{\partial}_\alpha. \tag{3.58}$$

The gauge transformation r_z^α satisfies $(r_z^\alpha)^*(\overline{\partial}_\alpha) = \overline{\partial}$ hence

$$g_i' = r_z^\alpha \circ g_i, \tag{3.59}$$

gives a holomorphic frame. We show that the extensions are compatible. By construction of $\overline{V}$ the gauge transformation $f_{ij} = f_i \circ f_j^{-1}$ relating two gauges compatible with s is in $L^p_{2,\alpha}$, for all $p \in [2, (1/2 - J_0/6)^{-1})$. In particular it is continuous across Σ and hence so is the composite $g_{ij}' = g_i \circ f_{ij} \circ g_j^{-1}$, where g_i, g_j are the two solutions to (3.58). By construction g_{ij}' satisfies

$$\overline{\partial} g_{ij}' + [\alpha, g_{ij}'] \frac{d\overline{z}}{\overline{z}} = 0. \tag{3.60}$$

Since

$$Ker(\overline{\partial} + \gamma \frac{d\overline{z}}{2\overline{z}}) = \{r_z^{-\gamma} f | f \in Hol(D^* \times D)\}, \tag{3.61}$$

it follows that g'_{ij} has a block representation of the form $(r_\Sigma^{\alpha_k-\alpha_l}g'^{kl}_{ij})$. Since α is of the form (1.23), $|\alpha_k-\alpha_l|<1$ for all kl so the g'^{kl}_{ij}'s are holomorphic across Σ and vanish on Σ if $\alpha_k-\alpha_l<0$. It follows that

$$g''_{ij}=r_z^{ad(-\alpha)}\circ g'_{ij}, \tag{3.62}$$

is holomorphic across Σ, lies in $P(\alpha)$ on Σ and hence gives a holomorphic cocycle with a $P(\alpha)$–reduction on Σ extending $\mathcal{V}$ which by Proposition 2.14 is an α–parabolic bundle $\overline{\mathcal{V}}$ on (X,Σ). By construction $\deg_\alpha(\overline{\mathcal{V}})=\deg_\alpha(\overline{V})=0$. When A_h is an $SU(n)$–connection we can ensure that $A_{\overline{A}_h}-A_\alpha$ is an $\mathfrak{su}(n)$–valued 1–form and then the solution to (3.58) can be taken in $SL(n)$ so

$$\det(g''_{ij})=\det(r_z^\alpha g_i f_{ij} g_j^{-1} r_z^{-\alpha})=det(f_{ij}),$$

which since $\det(\overline{V})=-Tr(\alpha)[\Sigma]$ implies that $\{\det(g''_{ij})\}$ is a holomorphic cocycle for $-Tr(\alpha)[\Sigma]$. Since $\{\det(g''_{ij})\}$ is also a cocycle for $\det(\overline{\mathcal{V}})$ this gives that $\det(\overline{\mathcal{V}})=-Tr(\alpha)[\Sigma]$. The growth charachterisation (3.57) immediately follows from (3.59) since the g_i's are continuous.

(ii) Using (i) we only need to extend the holomorphic structure across D_∞. We begin by applying Lemma 1.15 to Proposition 1.29 to obtain a $\mathcal{C}^\infty$–vector bundle $\overline{V}$ on $\mathbb{P}^2$ and a connection $\overline{A}_h$ gauge equivalent to A_h on $\mathbb{C}^2\backslash\Sigma$ which have the same properties as in Proposition 1.29 and such that further $\overline{V}$ is trivial on D_∞ and at each point q $\in D_\infty\backslash$p there is a gauge where

$$\overline{\partial}_{\overline{A}_h}=\overline{\partial}_{-\alpha}+a,\qquad a_u=O(r_u),\quad a_v=O(1),$$

while at p there is a gauge where

$$\overline{\partial}_{\overline{A}_h}=\overline{\partial}_{\alpha,-\alpha}+a,\qquad a_u=O(r_u),\quad a_v=O(1).$$

Proposition 3.48 now gives continuous solutions of the form $g=1+O(|u|)$ to

$$g^*(\overline{\partial}_{\overline{A}_h})=\overline{\partial}_{-\alpha},\qquad g^*(\overline{\partial}_{\overline{A}_h})=\overline{\partial}_{\alpha,-\alpha}. \tag{3.63}$$

Since

$$(r_u^{-\alpha})^*(\overline{\partial}_{-\alpha})=\overline{\partial},\qquad (r_v^\alpha r_u^{-\alpha})^*(\overline{\partial}_{\alpha,-\alpha})=\overline{\partial},$$

the complex gauge transformations

$$g''=r_u^{-\alpha}\circ g,\qquad g''=r_v^\alpha r_u^{-\alpha}g, \tag{3.64}$$

give holomorphic frames.

We show that the extensions are compatible along D_∞. Let U_i and U_j be polydiscs intersecting non–trivially on D_∞ and not containing p. Since $\overline{V}$ is trivial on D_∞ the $L^p_{2,\alpha}$–gauge transformation $f_{ij}=f_i\circ {f_j}^{-1}$ relating two unitary gauges on D_∞ compatible with

the $C(i\alpha)$–reduction can be taken constant there. The composite $g'_{ij} = g_i \circ f_{ij} \circ g_j$, where g_i and g_j are the two solutions to (3.63), is then also continuous, coincides with f_{ij} on D_∞ and has block off–diagonal components satisfying

$$\sup |\frac{g_{ij}^{kl'}}{r_u}| < \infty.$$

Since the transition functions

$$g''_{ij} = r_u^{ad(-\alpha)}(g'_{ij}), \qquad g''_{ij} = r_u^{ad(-\alpha)} r_v^{ad(\alpha)}(g'_{ij}),$$

are by construction holomorphic off D_∞, they must extend holomorphically across it as constants in $C(i\alpha)$. We thus obtain a cocycle extending $\mathcal{V}$ holomorphically across $\Sigma \cup D_\infty$ which lies in $P(\alpha)$ on Σ and is locally constant in $C(i\alpha)$ on D_∞ giving the required α–parabolic structure on Σ and flat $C(i\alpha)$–structure on D_∞. The growth characterisation (3.57) is clear from (3.64) because the g_i's are continuous and restrict to the identity on D_∞ while (3.58) follows from the construction of the cocycle.

We prove the uniqueness. In each case let $(\mathcal{V}', h')$, $\overline{\mathcal{V}'}$ and f be as stated. For (i) let s_i and s_j be filtered holomorphic frames for the extensions at a point in Σ. Let (f_{ij}), $(\tilde{f}_{ij})$ represent f and f^{-1} in these frames. By (3.56)

$$\{r_z^{-\alpha}(s_i)\}, \qquad \{r_z^{-\alpha'}(s'_i)\},$$

are bounded frames and since f makes the metrics mutually bounded

$$r_z^{\alpha * \alpha'}(f_{ij}), \qquad r_z^{\alpha' * \alpha}(\tilde{f}_{ij})$$

which represent f and f^{-1} must be bounded. The argument used to conclude Proposition 3.21 shows that this is possible if and only if $\alpha = \alpha'$. It follows as in the proof of the compatibility of the extensions that (f_{ij}) is holomorphic across Σ and lies in $P(\alpha)$ there so that f extends to a parabolic isomorphism on X.

For (ii) observe first that by (3.56) and (3.57) it follows that

$$\bigoplus_{j \leqslant i} \overline{\mathcal{V}}_i(\mathrm{p}) = \mathcal{F}_i(\mathrm{p}), \qquad \bigoplus_{j \leqslant i} \overline{\mathcal{V}}'_i(\mathrm{p}) = \mathcal{F}'_i(\mathrm{p}),$$

so we can choose holomorphic frames compatible with both the parabolic filtrations and the $C(i\alpha)$–structures. Then we can find a local representation of f by a bounded matrix unitary on D_∞ of the form

$$f_{ij} = (r_v^{\alpha_j - \alpha_i} r_u^{\alpha_i - \alpha_j} f'_{ij}),$$

with (f'_{ij}) holomorphic off $D_\infty \cup \Sigma$. Again (f_{ij}) extends holomorphically to an element in $P(-\alpha)$ across D_∞ and in $P(\alpha)$ across Σ thus it must lie in $C(i\alpha)$ at p since f_{ij} was unitary on D_∞. This holds for all choices of cocycles compatible with the $C(i\alpha)$–structures and filtrations. Since for one such choice $\overline{f}_{|D_\infty}$ is represented by a constant, this constant is in $C(i\alpha)$ so $\overline{f}_{|D_\infty}$ intertwines the $C(i\alpha)$–structures.

For the last statement observe that when $J \in G$ is admissible it follows by Proposition 1.29 that $\overline{V}$ has a G–structure extending that on (V, h) and there is a connection $A_{\overline{A}_h}$ on $\overline{V}$ gauge equivalent to A_h off Σ $(\Sigma \cup D_\infty)$ such that $A_{\overline{A}_h} = A_\alpha + a$ where $i\alpha = \rho(\log(J))$, A_α is a model singular G–connection and $a \in L^\infty(\Omega^1(\mathfrak{g}_{\overline{E}}))$. By the last clause in 3.48 the complex gauge transformation g_i solving (3.63) can now be chosen $G^{\mathbb{C}}$–valued so that the holomorphic cocycle constructed is a $G^{\mathbb{C}}$–cocycle. Since J was admissible $ad(\alpha)$ has all its eigenvalues in $(-1, 1)$ and since, as observed many times, $ad(\rho(\log(J)))$ acts on $\rho(\mathfrak{g})$ with the same eigenvalues as $ad(\alpha)$ on $\mathfrak{g}$ it follows as above that the $G^{\mathbb{C}}$–cocycle constructed is holomorphic across Σ $(\Sigma \cup D_\infty)$. Further, since the $G^{\mathbb{C}}$–frames used in the extension procedure are by definition filtered for the α–parabolic structure being constructed it follows that the $G^{\mathbb{C}}$–structure and the α–parabolic structures are compatible. Similarly for the framing. By construction it also follows that, in the notation of Proposition 3.34, the resulting $G^{\mathbb{C}}$–structure $\overline{\lambda}^{\mathbb{C}}$ is related to the original one $\lambda^{\mathbb{C}}$ by

$$\overline{\lambda}^{\mathbb{C}} = Ad(f_\alpha g)(\lambda^{\mathbb{C}}),$$

for some complex gauge transformation g so they are isomorphic off $\Sigma \cup D_\infty$. The same equation relates f and $\overline{f}$ so that when f is a $G^{\mathbb{C}}$–isomorphism $\overline{f}$ is a $G^{\mathbb{C}}$–isomorphism between the extended $G^{\mathbb{C}}$–structures. □

3.6 Proof of Theorem 1

As remarked in the introduction the framings considered here are somewhat special. To better illustrate Theorem 1 we now explain why.

Let $\mathcal{E}^0$ be a quasi–parabolic bundle on $(D_\infty, D_\infty \cap \Sigma)$. According to definition (0.9) an $\mathcal{E}^0$–quasi–parabolic framing on (X, Σ, D_∞) is a quasi–parabolic bundle on (X, Σ) such that the induced quasi–parabolic bundle on $(D_\infty, \Sigma \cap D_\infty)$ is isomorphic to $\mathcal{E}^0$. When $\mathcal{E}^0$ is holomorphically trivial we can think of the quasi–parabolic structure as the choice of preferred copies of P in $G^{\mathbb{C}}$ one for each point in $\Sigma \cap D_\infty$. If these copies all coincide this is the same as asking that $\mathcal{E}^0$ has a reduction to P over D_∞. Thus, when $P = P(\alpha)$ for some $i\alpha \in \mathfrak{g}$ we see as in Construction 3.27 that fixing a flat G–structure on D_∞ on an $\mathcal{E}^0$–framing is the same as fixing a flat $C(i\alpha)$–structure compatible with the $P(\alpha)$–reduction on Σ and vice–versa, a flat $C(i\alpha)$–structure on D_∞ compatible with the $P(\alpha)$–reduction on Σ determines an $\mathcal{E}^0$–framing up to $C(i\alpha)$. On the moduli space of framings there is a natural action of the group of parabolic automorphisms of $\mathcal{E}^0$ by

$$g[(\mathcal{E}, \varphi)] = [(\mathcal{E}, g \circ \varphi)].$$

Here this group is $P(\alpha)$ which has $C(i\alpha)$ as a subgroup and it follows by the previous discussion that $\widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)/C(i\alpha)$ is the same as the set of equivalence classes of $P(\alpha)$–parabolic bundles with a compatible flat $C(i\alpha)$–structure on D_∞. In general, when $i\alpha = \log(J)$ for some admissible J, this case will arise from instantons whose limit on the circle bundle at infinity in the cylindrical picture of §1.3 is a flat connection which does not pull–back from the basis. Since the fibre loops are central in the fundamental group of the

circle bundle the limit has to reduce to $C(J)$, where J is the holonomy along the fibres. As observed there for the triple we are restricting to this is always the case whenever the instanton considered has holonomy J^{-1} along Σ, which explains Theorem 1

Part (i) of Theorem 1 is now easily deduced.

Proposition 3.65. *Let J be an admissible element in G, $[(\mathcal{E}, \lambda)] \in M^J(\mathbb{C}^2, \Sigma)$, $(\mathcal{V}, h)$ the associated Hermitian vector bundle through some faithful unitary representation ρ, $i\alpha = \log(J)$ and $\overline{\mathcal{V}}$ the $\rho(\alpha)$–parabolic vector bundle with flat $C(i\alpha)$–structure on D_∞ $(\mathcal{V}, h)$ extends to by Theorem 3.56. The assignment*

$$(\mathcal{E}, \lambda) \mapsto \overline{\mathcal{V}},$$

induces a well–defined map

$$\Psi : M^J(\mathbb{C}^2, \Sigma) \longrightarrow \widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)/C(J).$$

Proof. By the last clause in Theorem 3.55 we know that $\overline{\mathcal{V}}$ is equipped with a compatible holomorphic $G^{\mathbb{C}}$–structure so that by Lemma 2.16 it corresponds to a principal α–parabolic bundle $\overline{\mathcal{E}}$ with a compatible flat $C(i\alpha)$–structure on D_∞, thus an element in $\widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)/C(J)$. Suppose that there is an isomorphism of H.Y.M. bundles $g : (\mathcal{E}, \lambda) \cong (\mathcal{E}', \lambda')$. By Theorem 3.55 we know that the induced isomorphism of H.Y.M. vector bundles extends to a parabolic $G^{\mathbb{C}}$–isomorphism of the extensions intertwining the flat structures on D_∞ hence it lifts to a parabolic isomorphism $\overline{g}$ of principal parabolic bundles intertwining the $C(J)$–reductions. Thus the assignment

$$(\mathcal{E}, h) \mapsto \overline{\mathcal{E}}$$

descends to a map between $M^J(\mathbb{C}^2, \Sigma)$ and $\widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)/C(J)$. □

To prove (i) of Theorem 1 we only need to exhibit an inverse to Ψ. Let $\mathcal{E}$ be an α–parabolic framing where $i\alpha = \log(J)$ for some admissible $J \in G$ and $\mathcal{V}$ the corresponding $\rho(\alpha)$–parabolic framed vector bundle for some faithful representation ρ. Let $(h_\alpha)_\infty$ be the $G^{\mathbb{C}}$–metric on $\mathcal{V}$ associated by Theorem (3.25) to the metric h_α of Construction 3.27 and let $\{\alpha_i\}$ be the eigenvalues of $\rho(\alpha)$ in descending order.

Proposition 3.66. *The metric $h = (h_\alpha)_\infty$ is the unique H.Y.M. $G^{\mathbb{C}}$–metric on $\mathcal{V}_{|\mathbb{C}^2 \setminus \Sigma}$ such that*

(i)

$$\forall x \in \Sigma, \quad \mathcal{F}_i(x) =< \{s(x) | s \in \mathcal{O}(\mathcal{V}).\ h(s) = O(r_\Sigma^{\alpha_i} r_{D_\infty}^{-\alpha_i})\} > .$$

(ii)

$$\forall x \in D_\infty, \quad \mathcal{V}_i(x) =< \{s(x) | s \in \mathcal{O}(\mathcal{V}).\ h(s) \sim r_\Sigma^{-\alpha_i} r_{D_\infty}^{\alpha_i}\} > .$$

The uniqueness is in the sense that if $\mathcal{E} \overset{f}{\cong} \mathcal{E}'$ *is an isomorphism of principal parabolic framed bundles and* h' *is a H.Y.M.* $G^{\mathbb{C}}$*–metric on an associated bundle of the latter through some faithful representation, satisfying (i) and (ii), then* f *restricts to an isomorphism of H.Y.M. bundles,* $f^*(\mathcal{E}_{|\mathbb{C}^2\setminus\Sigma}, \lambda) = (\mathcal{E}'_{|\mathbb{C}^2\setminus\Sigma}, \lambda')$*, where* λ *and* λ' *are the* G*–reductions* h *and* h' *came from.*

Proof. By construction h_α satisfies conditions (i) and (ii). Since $(h_\alpha)_\infty$ and h_α are mutually bounded and $(h_\alpha)_\infty^{-1}(h_\alpha)$ extend as the identity across D_∞ it follows that $(h_\alpha)_\infty$ also does. Now suppose that f is an isomorphism of parabolic framed bundles. As in Proposition 1.29 we have an induced isomorphism of the adjoint bundles thought of as subbundles of the endomorphism bundles of the associated vector bundles. Denote by h and h' also the metrics on the endomorphism bundles. Since f intertwines the framings and parabolic structures, the assumptions on h' imply that $f^*(h)$ and h' are mutually bounded H.Y.M. metrics on $\mathfrak{g}_{\mathcal{E}}$ with trivial determinant and $g = (f^*(h))^{-1}h'$ extends to the identity on D_∞ thus by the uniqueness clause in 3.25 they must be the same H.Y.M. metric and f lifts to an isomorphism of H.Y.M. bundles $f^*(\mathcal{E}_{|\mathbb{C}^2\setminus\Sigma}, \lambda) = (\mathcal{E}'_{|\mathbb{C}^2\setminus\Sigma}, \lambda')$. □

Letting λ_∞ denote the G–reduction h_∞ comes from it follows that the assignment $\mathcal{E} \longmapsto (\mathcal{E}_{|\mathbb{C}\setminus\Sigma}, (\lambda)_\infty)$ descends to a well defined map of moduli sets $\Phi : \widetilde{\mathcal{M}}^\alpha(\mathbb{P}^2, \Sigma)/C(J) \longrightarrow M^J(\mathbb{C}^2, \Sigma)$.

Proposition 3.67. *The maps* Ψ *and* Φ *given by 3.65 and 3.66 are mutually inverse.*

Proof. Let $\mathcal{E}$ be a parabolic framing, $(\mathcal{E}_{|\mathbb{C}^2\setminus\Sigma}, \lambda_\infty) = \Phi(\mathcal{E})$ and $\mathcal{E}' = \Psi(\mathcal{E}_{|\mathbb{C}^2\setminus\Sigma}, \lambda_\infty)$. The growth characterisation in terms of h_∞ of the filtrations and framings on an associated parabolic framed vector bundles through a faithful representation given by 3.55 and 3.66 respectively implies, by the uniqueness clause in 3.55, that the identity extends to a parabolic isomorphism intertwining the flat $C(J)$–structures on D_∞, hence a framed parabolic isomorphism up to an element in $C(J)$. Vice–versa, let $(\mathcal{E}, \lambda)$ be a H.Y.M. bundle, $\overline{\mathcal{E}} = \Psi(\mathcal{E}, \lambda)$, and $\Phi(\overline{\mathcal{E}}) = (\mathcal{E}', \lambda')$. The characterisation of the metrics induced by λ and λ' given by 3.55 and 3.66 respectively both in terms of the framings and filtrations on an associated parabolic framed vector bundle, imply by the uniqueness clause in 3.25 that the identity on $\overline{\mathcal{E}}$ restricts to an isomorphism of H.Y.M. bundles on $\mathbb{C}^2\setminus\Sigma$. □

Part (ii) of Theorem 1 follows from the next proposition. The notation is as in Chapters 1 and 2.

Proposition 3.68. *For any admissible* $J \in G$ *and any topological invariants* κ, ℓ*,* Ψ *and* Φ *induce bijections*

$$\Psi : M^J_{\kappa,\ell}(\mathbb{C}^2, \Sigma) \longrightarrow \widetilde{\mathcal{M}}^\alpha_{\kappa,\ell}(\mathbb{P}^2, \Sigma)/C(J)$$

and

$$\Phi : \widetilde{\mathcal{M}}^\alpha_{\kappa,\ell}(\mathbb{P}^2, \Sigma)/C(J) \longrightarrow M^J_{\kappa,\ell}(\mathbb{C}^2, \Sigma).$$

Proof. Let $(\overline{E}, \overline{s})$ be the $G^{\mathbb{C}}$–bundle with $C(J)^{\mathbb{C}}$–reduction on Σ trivial on D_∞ having topological invariants κ, ℓ so that $M^J_{\kappa,\ell}(\mathbb{C}^2, \Sigma)$ is defined on it. Let $(\overline{V}, \overline{s})$ be an associated vector bundle with $C(\rho(i\alpha))^{\mathbb{C}}$–reduction on Σ through a faithful representation ρ where

$i\alpha = \log(J)$. The parabolic extension $\overline{\mathcal{V}}$ of Theorem 3.55 was constructed from it by exhibiting suitable holomorphic $G^{\mathbb{C}}$–frames. Patching together the relevant local complex gauge transformations we obtain a global complex gauge transformation of the form

$$f_\alpha = r_\Sigma^{-\rho(\alpha)} r_{D_\infty}^{\rho(\alpha)} g,$$

with f a continuous, $G^{\mathbb{C}}$–valued complex gauge transformation on $\overline{V}$ respecting $\overline{s}$, equal to the identity on D_∞ and such that $f_\alpha(\overline{\partial}_{\mathcal{V}})$ extends as the $\overline{\partial}$–operator defining the holomorphic structure on $\overline{\mathcal{V}}$ and making the parabolic filtration holomorphic. We then see that the relation between $(\overline{V}, \overline{s})$ and $\overline{\mathcal{V}}$ with the $C(i\alpha)^{\mathbb{C}}$–reduction induced by the parabolic filtration is the same as that between the pairs denoted $(\overline{V}, \overline{s})$ and (V, s) in Proposition 3.34 (i). In particular they are topologically isomorphic as $G^{\mathbb{C}}$–bundles with $C(i\alpha)^{\mathbb{C}}$–reductions so κ and ℓ are preserved.

For the converse, let $\mathcal{E}$ be an α–parabolic framing in $\widetilde{\mathcal{M}}^{\alpha}_{k,\ell}(\mathbb{P}^2, \Sigma)$, $\mathcal{V}$ the associated $\rho(\alpha)$–parabolic framed vector bundle, h_α the initial metric of Construction 3.27, and $(\overline{V}, \overline{s})$ the pair given by Proposition 3.34 (i). By the proof of 3.34 (iii), at a point in $\Sigma \cup D_\infty$, in a gauge compatible with the $C(i\alpha)$–reduction, $A_{\mathcal{V}, h_\alpha}$ has the form $d_\alpha + a$, $d_{-\alpha} + a$ or $d_{\alpha, -\alpha} + a$ with $a \in L^\infty$. Let g be the complex gauge transformation relating $A_{\mathcal{V}, h_\alpha}$ and $A_{\mathcal{V}, h_\infty}$ as in (3.25). Representing $A_{\mathcal{V}, h_\alpha}$ in the gauge of 1.14 (iii) we can find by Proposition 3.48 continuous complex gauge transformations such that the resulting representative of g satisfies

$$\overline{\partial} g - [\alpha, g]\frac{d\overline{z}}{\overline{z}} = 0. \quad \text{or} \quad \overline{\partial} g + [\alpha, g]\frac{d\overline{u}}{\overline{u}} - [\alpha, g]\frac{d\overline{v}}{\overline{v}}] = 0.$$

It follows as in 3.55 that g is continuous and lies in $C(i\alpha)^{\mathbb{C}}$ along Σ. Since g extends to the identity on D_∞, it gives a topological $G^{\mathbb{C}}$–isomorphism between $(\overline{V}, \overline{s})$ and the pair associated to $A_{g(\mathcal{I}_\mathcal{E}), h_\alpha}$ by Proposition 1.29. We thus obtain a H.Y.M. bundle in some $M^J_{k,\ell}(\mathbb{C}^2, \Sigma)$. As explained in §1.6 we can now pull this back to a fixed such moduli space (for instance the one defined on $(\overline{V}, \overline{s})$) giving the required map Ψ. □

Corollary 1 now immediately follows from the discussion of §3.1 and (3.2).

3.7 Proof of Theorem 2

We now assume that (X, ω) is a compact Kähler surface and that Σ is a smooth irreducible divisor. Let $\mathcal{V}$ be a holomorphic vector bundle on $X \backslash \Sigma$ with a Hermitian metric h such that $F_h \in L^1$. The h–degree of $\mathcal{V}$ is defined to be

$$\deg_h(\mathcal{V}) = \frac{i}{2\pi} \int_{X \backslash \Sigma} Tr(F_h) \wedge \omega. \tag{3.69}$$

and the h–slope

$$\mu_h(\mathcal{V}) = \frac{\deg_h(\mathcal{V})}{rk(\mathcal{V})}.$$

For every saturated subsheaf $\mathcal{S}$ of $\mathcal{V}$ we have an induced metric h' out of the points where $\mathcal{S}$ is not a subbundle and we define the induced h–degree to be

$$\deg_h(\mathcal{V},\mathcal{S}) = \frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_{h'})\wedge\omega, \tag{3.70}$$

and the induced h–slope

$$\mu_h(\mathcal{V},\mathcal{S}) = \frac{\deg_h(\mathcal{V},\mathcal{S})}{rk(\mathcal{S})}.$$

We define $(\mathcal{V},h)$ to be stable (analytically stable) if for every proper non–zero saturated subsheaf $\mathcal{S}$

$$\mu_h(\mathcal{V}) > \mu_h(\mathcal{V},\mathcal{S}). \tag{3.71}$$

Simpson's existence Theorem can then be stated as follows for our purposes.

Theorem 3.72. *([S], Theorem 1) Let (X,ω) be a compact Kähler surface and let Σ be a smooth irreducible divisor in X such that $\Sigma\cdot\Sigma \geqslant 0$. Let $\mathcal{V}$ be a holomorphic vector bundle on $X\setminus\Sigma$ with a metric h_0 such that $\sup_{X\setminus\Sigma}|\hat{F}_{h_0}| < \infty$, $\|F_{h_0}\|_{L^2} < \infty$ and $(\mathcal{V},h_0)$ is stable. Then there is a unique H.Y.M. metric h_∞ on $\mathcal{V}$ mutually bounded with respect h_0 and with $\det(h_\infty) = \det(h_0)$.*

In order to apply this to Theorem 2 we first need to relate parabolic stability and analytic stability.

Let $i\alpha$ be of the form (1.23), V a $\mathcal{C}^\infty$ α–parabolic vector bundle on (X,Σ) with a Hermitian metric h and S a vector bundle with a map $\iota : S \to V$ which except for finitely many points is a bundle injection so that the $\iota^*(F_i)$'s are subbundles of S. Such an S has an induced α'–parabolic structure as in (2.38) and defines an endomorphism π of V which is the h–projection onto $\iota(S)$ defined where $\iota(S)$ is a subbundle. Let h_α be the metric of Construction 3.27 $(\overline{V},\overline{s})$ the pair with model singular connection A_α given by Proposition 3.34 so that we have a corresponding map $\iota : S \to \overline{V}$ and h_α–projection $\overline{\pi}$. For every connection A on $\overline{V}$ we then have an induced connection $\overline{\pi}A$ on $\iota(S)$.

Proposition 3.73. *Let $i\alpha \in \mathfrak{u}(n)$ be of the form (1.23) and V a $\mathcal{C}^\infty$ α–parabolic vector bundle. Then*

(i) *For every connection $A = A_\alpha + a$ with $a \in L^2_{1,\alpha}$, we have the equation of currents on X*

$$[\frac{i}{2\pi}Tr(F_A)] = c_1(V) + Tr(\alpha)[\Sigma].$$

in particular

$$\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_A)\wedge\omega = \deg_\alpha(V).$$

(ii) *With respect to decomposition (1.26) of $\overline{V}$ on $N(\Sigma)$ let A_i denote the connection induced on $\overline{V}_i$. We have the equation of currents with compact support in $N(\Sigma)$*

$$\forall i,\qquad [\frac{i}{2\pi}Tr(F_{A_i})] = c_1(\overline{V}_i) + Tr(\alpha_i)[\Sigma].$$

(iii) *For every vector bundle S with a map $\iota : S \to V$ as above and for every connection A as in (i) we have the equation of currents on X*

$$[\frac{i}{2\pi} Tr(F_{\overline{\pi}A})] = c_1(S) + Tr(\alpha')[\Sigma],$$

in particular

$$\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_{\overline{\pi}A}) \wedge \omega = \deg_\alpha(V, S).$$

Proof. (i) We begin by proving the claim for a connection A of the form

$$A = A_\alpha + a.$$

with a a smooth form compatible with the orthogonal splitting (1.26) on $N(\Sigma)$. Since A_α is compatible with the splitting on $N(\Sigma)$, there is some connection A_0 on $\overline{V}$ smooth on the whole of X and compatible with the splitting on $N(\Sigma)$ such that

$$A - A_0 = \beta(r)\eta\alpha.$$

thus

$$F_A - F_{A_0} = d_{A_0}(\beta(r)\eta\alpha). \tag{3.74}$$

By Chern–Weil theory we know that for every 2 dimensional De Rham cohomology class $[u]$

$$\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_{A_0}) \wedge u = \int_X c_1(\overline{V}) \wedge u = \int_X c_1(V) \wedge u = c_1(V) \cup [u].$$

and we have to evaluate

$$\frac{i}{2\pi}\int_{X\setminus\Sigma} d(\beta(r)\eta) \wedge u.$$

This may be done directly, however, observing that $\frac{i}{2\pi}(d(\beta(r)\eta)$ is the Thom class of $N\Sigma_{|X}$, so that as a cohomology class can be identified with the Poincare' dual of Σ, ([BT], p. 74), we deduce that the integral is $[\Sigma] \cup [u]$ which together with (3.74) gives

$$\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_A) \wedge u = (c_1(V) + Tr(\alpha)[\Sigma]) \cup [u].$$

Now let A be of the form

$$A = A_\alpha + a. \qquad a \in L^2_{1,\alpha}.$$

By Lemma 1.41 (i) we can find a sequence of connections $\{A_i = A_\alpha + a_i\}$ converging to A in $L^2_{1,\alpha}$ with the a_i's smooth and respecting the splitting (1.26) on $N(\Sigma)$. Since the functional $\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_{(-)}) \wedge u$ is continuous on $L^2_{1,\alpha}$ it follows that

$$\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_A) \wedge u = \lim_{i-\infty}\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_{A_i}) \wedge u = (c_1(V) + Tr(\alpha)[\Sigma]) \cup [u],$$

as required. The proof of (ii) is similar.

(iii) We begin by proving the formula in the case where S is actually a subbundle of V with the property that the induced filtration (2.38) is a filtration of S by subbundles. Let h be a fixed metric on V from which to begin Construction 3.27. By the assumptions on S we can give $V_{|N(\Sigma)}$ a $\mathcal{C}^\infty$–filtration finer than the parabolic filtration. Let $\{i_j\}_{j=1}^k$ be the indices appearing in (2.38). We then consider

$$F_1 \subsetneq \cdots \subsetneq F_{i_1} \cap S + F_{i_1-1} \subsetneq \cdots F_{i_2} \cap S + F_{i_2-1} \subsetneq \cdots \subsetneq F_{i_k} \cap S + F_{i_k-1} \subsetneq \cdots F_t = V_{|N(\Sigma)}.$$

From this we construct an orthogonal decomposition subsuming (3.28) and such that as unitary vector bundles with the induced metric of h, $V_{N(\Sigma)} = S \oplus V/S$ with

$$S_{|N(\Sigma)} = \bigoplus_j S \cap F_{i_j}/S \cap F_{i_{j-1}},$$

$$(V/S)_{|N(\Sigma)} = (\bigoplus_{i \notin \{i_j\}} F_i/F_{i-1}) \oplus (\bigoplus_j F_{i_j}/(S \cap F_{i_j} + F_{i_j-1}).$$

Translating this to the pair $(\overline{V}, \overline{s})$ of Proposition 3.34 (i) it follows that we can take the model singular connection A_α to be a direct sum

$$A_\alpha = A_{\alpha'} \oplus A_{\alpha''},$$

with α', α'' the weights of the induced parabolic structures on S and V/S as in §2.5 and $A_{\alpha'}$, $A_{\alpha''}$ model singular connections on S and V/S respectively. In particular this implies that for every $A \in L^2_{1,\alpha}$ the induced connection $\overline{\pi}A$ is in $L^2_{1,\alpha'}$. Since

$$\deg_\alpha(V, S) = \deg_{\alpha'}(S),$$

we can now apply (i) to $\overline{\pi}A$ regarded as an $L^2_{1,\alpha'}$–connection on S to deduce the required formula.

Now let ι be any map as specified and consider the corresponding projection π defined out of the points where ι is degenerate. A local analysis shows that near a point where ι is degenerate the components of π are of the form

$$\frac{f}{s^n g}. \qquad f, g \in \mathcal{C}^\infty,$$

where f has a zero of order n at the origin, g is non–vanishing and s is the radial distance from the origin. In particular $\pi \in L^2_1$ and we see from the previous case that it still holds that the induced projection $\overline{\pi}$ for $\overline{V}$ belongs to $L^2_{1,\alpha}$. Since ι is assumed to have only isolated degenerate points we can find a sequence of smooth subbundles S_i on which the filtration on V induces filtrations by subbundles as in the first case, such that the corresponding projections π_i converge to π in L^2_1 and

$$\forall i, \qquad \deg_\alpha(V, S_i) = \deg_\alpha(V, S).$$

We then obtain a corresponding sequence $\overline{\pi}_i$ converging to $\overline{\pi}$ in $L^2_{1,\alpha}$. Since for any fixed $A \in L^2_{1,\alpha}$ and 2–dimensional De Rham class $[u]$ the functional

$$\int_{X\setminus\Sigma} Tr(F_{(-)A}) \wedge u,$$

is continuous on the $L^2_{1,\alpha}$–subbundles of $\overline{V}$ (self–adjoint sections $\pi \in L^2_{1,\alpha}(End(\overline{V}))$ such that $\pi^2 = \pi$), we obtain

$$\frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_{\overline{\pi}A}) \wedge u = \lim_{i-\infty} \int_{X\setminus\Sigma} Tr(F_{\overline{\pi}_i A}) \wedge u = (c_1(S) + Tr(\alpha')[\Sigma]) \cup [u],$$

as required. □

We can now prove the key lemma needed to apply Theorem 3.72.

Proposition 3.75. *Let $\mathcal{V}$ be a stable α–parabolic bundle with $i\alpha \in \mathfrak{u}(n)$ of the form (1.23) and let h_α be the metric of Construction 3.27 then $(\mathcal{V}_{|X\setminus\Sigma}, h_\alpha)$ is stable.*

Proof. We have to prove that for every non–zero proper saturated subsheaf $\mathcal{S}$ of $\mathcal{V}_{|X\setminus\Sigma}$,

$$\mu_{h_\alpha}(\mathcal{V}, \mathcal{S}) < \mu_{h_\alpha}(\mathcal{V}).$$

Since $\mathcal{V}$ is assumed to be stable it will suffice to show that for every $\mathcal{S}$ for which $\deg_{h_\alpha}(\mathcal{V}, \mathcal{S})$ is not $-\infty$, $\mathcal{S}$ extends to a saturated subsheaf of $\mathcal{V}$ on X such that

$$\deg_\alpha(\mathcal{V}, \mathcal{S}) = \deg_{h_\alpha}(\mathcal{V}, \mathcal{S}). \tag{3.76}$$

Let π denote orthogonal projection onto $\mathcal{S}$ using h_α. From Lemma 3.2 of [S] we know that for any h

$$\deg_h(\mathcal{V}, \mathcal{S}) = \frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(\pi F_{h_\alpha}) \wedge \omega - \frac{1}{2\pi}\int_{X\setminus\Sigma} |\overline{\partial}_{\mathcal{V}}\pi|^2_h. \tag{3.77}$$

When the left hand side is finite, taking $h = h_\alpha$ it follows by regularity for $\overline{\partial}_{\mathcal{V}}$ that $\pi \in L^2_1$ so it extends as a L^2_1–holomorphic subbundle of $\mathcal{V}$ on X (a self–adjoint section $\pi \in L^2_1(End(\mathcal{V}))$ such that $\pi^2 = \pi$ and $(1-\pi)\overline{\partial}_{\mathcal{V}}\pi = 0$). We can then apply the Uhlenbeck–Yau theorem on holomorphic L^2_1–subbundles [UY] to deduce that $\mathcal{S}$ has an extension to a non–zero poper saturated subsheaf of $\mathcal{V}$ still denoted $\mathcal{S}$. Let S be the underlying $\mathcal{C}^\infty$ parabolic bundle of $\mathcal{S}$ regarded as an α'–parabolic bundle with a morphism $\iota : \mathcal{S} \to \mathcal{V}$ which fails to be a bundle injection at most at a finite number of points. By (ii) of Proposition 3.34 we know that $A_{\overline{\partial}_{\mathcal{V}}, h_\alpha} \in L^2_{1,\alpha}$ so applying (iii) of Proposition 3.73 it follows that

$$\deg_h(\mathcal{V}, \mathcal{S}) = \frac{i}{2\pi}\int_{X\setminus\Sigma} Tr(F_{\pi(A_{\overline{\partial}_{\mathcal{V}}, h_{\alpha'}})}) = \deg_{\alpha'}(S) = \deg_{\alpha'}(\mathcal{S})$$

which since

$$\deg_{h_\alpha}(\mathcal{V}, \mathcal{S}) = \deg_{\alpha'}(\mathcal{S}).$$

gives (3.76). □

Finally we need to know that if the H.Y.M. bundle we start from has irreducible Chern connection then the extension obtained in (i) of Theorem 3.55 is stable.

Proposition 3.78. *Let $(\mathcal{V},h)$ be a H.Y.M. bundle on $X\backslash\Sigma$ whose Chern connection A_h has holonomy J and let $i\alpha = \log(J)$. If A_h is irreducible then the α–parabolic extension $\overline{\mathcal{V}}$ given by Theorem 3.55 is stable.*

Proof. Let $\mathcal{S}$ be a proper non–zero saturated subsheaf. By (ii) of Proposition 1.29 we know that A_h is gauge equivalent to some $\overline{A}_h$ with $\overline{A}_h - A_\alpha \in L^2_{1,\alpha}$. Applying (iii) of Proposition 3.73 it now follows that

$$\deg_\alpha(\mathcal{V},\mathcal{S}) = \deg_{\alpha'}(\mathcal{S}) = \frac{i}{2\pi}\int_{X\backslash\Sigma} Tr(F_{i^*(A_h)}) = \deg_h(\mathcal{V},\mathcal{S}),$$

which together with (3.77) implies that

$$\deg_\alpha(\mathcal{V},\mathcal{S}) = \frac{i}{2\pi}\int_{X\backslash\Sigma} Tr(\pi F_h)\wedge\omega - \frac{1}{2\pi}\int_{X\backslash\Sigma} |\overline{\partial}_{\mathcal{V}}\pi|^2_h,$$

where π is the projection onto $\mathcal{S}$. Since now A_h is H.Y.M. applying (i) of Proposition 3.73

$$\frac{i}{2\pi}\int_{X\backslash\Sigma} Tr(\pi F_h)\wedge\omega = \frac{rk(\mathcal{V})}{rk(\mathcal{S})}deg_\alpha(\mathcal{V}),$$

which shows that $\mu(\mathcal{V},\mathcal{S}) < \mu(\mathcal{V})$, unless $\overline{\partial}_{\mathcal{V}}(\pi) = 0$. This would mean that the orthogonal splitting given by $\mathcal{S}$ is holomorphic contradicting irreducibility of A_h hence strict inequality holds. □

We can now derive Theorem 2

Proof of Theorem 2.

Proof. (i) By Theorem 3.57 (i) to any H.Y.M. bundle $(\mathcal{V},h)$ with constant zero having finite action Chern connection with holonomy J along Σ we can associate an α–parabolic extension $\overline{\mathcal{V}}$, where $i\alpha = \log(J)$, such that $\deg_\alpha(\overline{\mathcal{V}}) = 0$. By Proposition 3.78, when the connection is irreducible $\overline{\mathcal{V}}$ is α–stable and by the uniqueness clause in Theorem 3.55 the isomorphism class of $\overline{\mathcal{V}}$ as an α–parabolic bundle is independent of the choice of the representative in $[(\overline{\mathcal{V}},h)]$ so the assignment

$$\Psi : (\mathcal{V},h) \mapsto \overline{\mathcal{V}}.$$

descends to a well–defined map of the moduli spaces. Going in the other direction, we know by Proposition 3.75 that for every stable α–parabolic bundle $\mathcal{V}$ on (X,Σ), $(\mathcal{V}_{|X\backslash\Sigma}, h_\alpha)$ is stable, where h_α is the metric of Construction 3.27. If $\deg_\alpha(\mathcal{V}) = 0$ we know that $\hat{F}_{h_\alpha} = 0$ and applying Simpson's Theorem in the form recalled above it follows that we obtain a H.Y.M. metric h_∞ on it mutually bounded with respect to h_α and with the same determinant, so in particular satisfying $\hat{F}_{h_\infty} = 0$. Let g be the complex gauge transformation such that $h_\infty = gh_\alpha$. The H.Y.M. bundle $(g(\overline{\partial}_{\mathcal{V}}), h_\alpha)$ has Chern connection which is therefore an instanton with the same holonomy and by the uniqueness clause in 3.55 the α–parabolic extension it gives raise to must be isomorphic to the one that

$(\mathcal{V}_{X\setminus\Sigma}, h_\alpha)$ would give raise to, namely $\mathcal{V}$ itself. If $A_{g(\overline{\partial}_{\mathcal{V}}),h_\alpha}$ was reducible it would follow that this parabolic extension is parabolic split, that is it is a parabolic bundle whose parabolic filtration and weights are the union of the filtrations and weights of two parabolic bundles of which it is the direct sum, which clearly contradicts stability. Thus $A_{g(\overline{\partial}_{\mathcal{V}}),h_\alpha}$ and so $A_{\overline{\partial}_{\mathcal{V}},h_\infty}$ have to be irreducible and by the uniqueness clause in Simpson Theorem it follows much as in Proposition 3.66 that the assignment

$$\Phi : \mathcal{V} \mapsto (\mathcal{V}_{X\setminus\Sigma}, (h_\alpha)_\infty),$$

descends to a well defined map of moduli spaces which is a two sided inverse for Ψ.

It follows by the last clause of (i) of Theorem 3.55 that when h has trivial determinant so that $A_{\overline{\partial}_{\mathcal{V}},h}$ is an $SU(n)$–connection, $\det(\overline{V}) = -Tr(\alpha)[\Sigma]$. Vice–versa (ii) of Proposition 3.34 implies that when $\det(V) = -Tr(\alpha)[\Sigma]$, the metric h_α of Construction 3.27 has trivial determinant so that the solution given by Simpson Theorem will also have.
(ii) Analogous versions of Propositions 3.73 to 3.78 can be proved for stable $SL(n)$ α–parabolic bundles, where $i\alpha = \log(J)$ for some admissible $J \in SU(n)$. The rest of the argument used in (i) together with the the relevant clauses in Theorem 3.55 and Proposition 3.34 concerning general groups now gives the desired $SL(n)$ -version of the correspondence.
(iii) This follows as in the proof of Proposition 3.68 using (i) of Proposition 3.34 and Theorem 3.55. □

As for Theorem 1 the reinterpretation of the correspondence in terms of instantons rather than H.Y.M. bundles follows from the discussion in §3.1.

Chapter 4

Moduli spaces of singular instantons and of framed singular instantons

In this Chapter we generalize the construction of the moduli space of singular instantons from [KM1] to $M^J_{\kappa,c,\ell}$ and use it to obtain a moduli space structure on $\widetilde{M}^J_{\kappa,c,\ell}$. We then show how to extend the Chern–Weyl formula of [KM1] to the $U(n)$ case and use this in conjunction with the results from Chapter 3 to derive the necessity of Condition 0.10 on the topological invariants of an $SU(n)$–singular instanton on $S^4\backslash S^2$ and a classification of flat unitary connections on $X\backslash\Sigma$ for complex pairs in terms of stable vector bundles on X. Finally we generalize the index computation of [KM1] again to the $U(n)$ case. This will be used in Chapter 7 to carry out the infinitesimal comparison between the deformation theories of singular and framed singular instantons and that of parabolic and framed parabolic bundles respectively.

4.1 Construction of the moduli spaces $M^J_{\kappa,c,\ell}$

Throughout the Chapter we shall assume that J is admissible. Under this assumption in §1.6 we defined $M^J_{\kappa,c,\ell}$ as a topological space depending on the choice of the Sobolev space used to model the space of finite action instantons with holonomy J and given topological invariants. When this choice is made it follows as for ordinary instantons that the infinitesimal deformations around a base point $A = A_\alpha + a$ are described by a 2–step complex

$$B_0 \xrightarrow{d_A} B_1 \xrightarrow{d_A^+} B_2^+, \tag{4.1}$$

where B_i are the chosen Sobolev space completions of the $\mathfrak{g}$–valued forms, A_α a model singular G–connection with holonomy J and invariants κ, c, ℓ and $a \in B_1$. We took

$$B_i = L^p_{2-i,\alpha}(\mathfrak{g}_{\overline{E}}), \qquad 2 < p < (1/2 - J_0/6)^{-1}. \tag{4.2}$$

This is weak enough to be a reasonable infinitesimal model in view of 1.14, but as of now it seems not to be strong enough to make possible a general comparison between the deformation theory of instantons and that of parabolic bundles. For this purpose it seems necessary to introduce Sobolev spaces with different regularity in the directions normal and parallel to Σ. This however would require the existence of a gauge with sharper control on the parallel derivatives of the connection matrix than that of 1.14 (iii).

We now show that the definition of the moduli space is independent of p provided $2 < p < (1/2 - J_0/6)^{-1}$ so that in particular we obtain the same moduli spaces as in [KM1]. From 1.14 (iii) it would seem natural that $2 + J_0$ rather than $(1/2 - J_0/6)^{-1}$ should be the natural upper bound for p. The choice of $(1/2 - J_0/6)^{-1}$ is dictated by

the use of Corollary 3.4 of [R1] for the construction of the moduli spaces. We can not prove this in general, however we show in §6.5 that in the case of hyperbolic monopoles of "high mass", namely those such that the value $[i\alpha]$ of the Higgs field at infinity has the property that $ad(\alpha)$ has all its non–zero eigenvalues strictly larger than 1 in modulus, at each point $[A]$ the moduli space of monopoles admit a local model all of whose elements globally satisfy the bounds of Theorem 1.14 (iii).

We denote by $\mathcal{A}_J^{p,+}$, $\mathcal{G}_J^p$ and $M_{\kappa,c,\ell}^{p,J}$ the affine spaces, gauge group and moduli spaces defined by using the $L^p_{2-i,\alpha}$ spaces. Let $2 < p < p'$. The inclusion

$$\mathcal{A}_J^{p',+} \hookrightarrow \mathcal{A}_J^{p,+}, \tag{4.3}$$

induces a natural map

$$M_{\kappa,c,\ell}^{p',J} \xrightarrow{\iota_{p'}} M_{\kappa,c,\ell}^{p,J}. \tag{4.4}$$

Proposition 4.5. *For any p' such that $2 < p < p' < (1/2 - J_0/6)^{-1}$ the composite $\iota_{p'}$ is a homeomorphisms with respect to the quotient topologies.*

Proof. Since $\iota_{p'}$ is induced by the continuous inclusion (4.4) they descend to continuous maps of the quotients. Let

$$A_1, A_2 \in \mathcal{A}_J^{p',+}, \qquad g \in \mathcal{G}_J^p,$$

be such that $g(A_1) = A_2$. It follows by Lemma 1.37 (iv) that $g \in \mathcal{G}_J^{p'}$ so ι is injective. To prove that ι is a homeomorphism let $[A] \in M_{\kappa,c,\ell}^J$. We proceed as in Proposition 1.29 using 1.14 (iii) to deduce that there exist $g' \in \mathcal{G}_J^p$ with

$$g'(d_B + a) = d_B + a', \qquad \|a'\|_{\in L^{p'}_{2,\alpha}} \leqslant \|a\|_{\in L^p_{1,\alpha}},$$

and in fact satisfying the global equivalent of the bounds of 1.14 (iii). We cover (X, Σ) by approximately euclidean geodesic balls U_i small enough so that U_i is either disjoint from Σ or $(U_i, U_i \backslash \Sigma)$ is an approximately standard pair (B^4, B^2) and on each of them the ordinary Uhlenbeck theorem or (ii) of Theorem 1.14 can be applied. By the anti–self–duality of A we obtain in the first case an L^p_2–gauge transformation g_i close to the identity such that

$$g_i(d + a) = d + \tilde{a}, \qquad \tilde{a} \in \mathcal{C}^\infty,$$

with $\mathcal{C}^k$ norm controlled by $\|a\|_{L^p_{1,\alpha}}$ for all k, and in the second an $L^p_{2,\alpha}$–gauge transformation g_i close to the identity such that

$$g_i(d_\alpha + a) = d_\alpha + \tilde{a},$$

where $\tilde{a}$ satisfies 1.14 (iii) with the bounds in term of $\|a\|_{L^p_{1,\alpha}}$. We can now patch together the g_i to obtain a global gauge transformation $g' \in \mathcal{G}_J^p$ with

$$g'(d_B + a) = d_B + a', \qquad \|a'\|_{L^{p'}_{1,\alpha}} \leqslant C' \|a'\|_{L^p_{1,\alpha}},$$

and a' actually satisfying the pointwise bounds of 1.14 (iii). Thus $\iota_2 \circ \iota_{p'}$ is surjective and a homeomorphism. □

The construction of the moduli spaces can now be carried out from (4.1) as in [KM1] taking $B_i = L^p_{2-i,\alpha}$ for any $2 < p < (1/2 - J_0/6)^{-1}$. We shall denote by $H_A^{i,p}$ the cohomology groups of (4.1) where $B_i = L^p_{k+1-i,\alpha}$. Following [KM1] we first examine a model complex with trivial complex coefficients on S^4.

Proposition 4.6. *Let $\gamma \in \mathbb{R}\backslash\mathbb{Z}$, $(1/2+\gamma_0/6)^{-1} < p < (1/2-\gamma_0/6)^{-1}$ and consider the complex*

$$W_2^p(\Omega^0(S^4)) \xrightarrow{d_\gamma} W_1^p(\Omega^1(S^4)) \xrightarrow{d_\gamma^+} W^p(\Omega^+(S^4)). \tag{4.7}$$

Then

(i) $d_\gamma^+ : Ker(d_\gamma^*(r^{-2}(-))) \to W^p(\Omega^+(S^4))$ *and* $r^{-2}d_\gamma^{*h} : Im(d_\gamma) \to: W^p(\Omega^0(S^4))$*are invertible.*

(ii) $Ker(d_\gamma^*(r^{-2}(-)))$ *gives a closed complement to* $Im(d_\gamma)$ *in* $W_1^p(\Omega^1(S^4))$.

Proof. By Theorem 1.14 (i) $\Delta_\gamma : L_{k,w}^{p,h} \to L_{k-2,w}^{p,h}$ is invertible. Since

$$(\Delta_\gamma)_{|\Omega^+} = (1+*_h)(d_\gamma d_\gamma^{*h}) = 2d_\gamma^+ d_\gamma^{*h},\ (\Delta_\gamma)_{|Ker(d^{*h})} = d_\gamma^{*h}d_\gamma$$

we see that

$$d_\gamma^+ d_\gamma^{*h} : L_{k,w}^{p,h} \to: L_{k-2,w}^{p,h},$$

is also invertible. Similarly we see that

$$\begin{aligned} d_\gamma d_\gamma^{*h} + d_\gamma^{*h} d_\gamma^+ &: L_{k,w}^{p,h} \to L_{k-2,w}^{p,h}, \\ d_\gamma^{*h} d_\gamma &: L_{k,w}^{p,h} \to L_{k-2,w}^{p,h}, \end{aligned}$$

are invertible and since

$$(d_\gamma d_\gamma^{*h} + d_\gamma^{*h} d_\gamma^+)_{|Ker(d^{*h})} = d_\gamma^{*h} d_\gamma^+,$$

that also $(d_\gamma^{*h} d_\gamma^+)_{|Ker(d_\gamma^{*h})}$ is. It follows as in ordinary Hodge theory that we have the closed decomposition

$$L_{1,w}^{p,h}(\Omega^1(\mathbb{H}^3 \times S^1)) = Im(d_\gamma(d_\gamma^{*h}d_\gamma)^{-1}d_\gamma^{*h}) \oplus Im(d_\gamma^{*h}(d_\gamma^+ d_\gamma^{*h})^{-1}d_\gamma^+),$$

which since $L_{1,w}^{p,h}(\Omega^1(\mathbb{H}^3 \times S^1)) = W_1^p(\Omega^1(S^4))$ and $r^{-2}L_w^{p,h}(\Omega^0(\mathbb{H}^3 \times S^1)) = W^p(\Omega^0(S^4))$, gives (i) and (ii). □

Let r_α^{-2} denote the endomorphism on a $\mathfrak{g}_{\overline{E}}$ which acts as multiplication by r^{-2} on $(\mathfrak{g}_{\overline{E}})_U$ and as the identity on $(\mathfrak{g}_{\overline{E}})_C$.

Proposition 4.8. *Let $2 \leqslant p < (1/2 - J_0/6)^{-1}$, let d_α be the flat singular G–connection with $i\alpha = \log(J)$ and consider the complex*

$$L_{2,\alpha}^p(\Omega^0(\mathfrak{g}_{\overline{E}})) \xrightarrow{d_\alpha} L_{1,\alpha}^p(\Omega^1(\mathfrak{g}_{\overline{E}})) \xrightarrow{d_\alpha^+} L^{p,+}(\Omega^+(\mathfrak{g}_{\overline{E}})).$$

Then

(i) $d_\alpha^+ : Ker(d_\alpha^*(r_\alpha^{-2}(-))) \to L^p(\Omega^+(\mathfrak{g}_{\overline{E}}))$ *is invertible.*

(ii) $Ker(d_\alpha^*(r_\alpha^{-2}(-)))$ *gives a closed complement to* $Im(d_\alpha)$ *in* $L_{1,\alpha}^p(\Omega^1(\mathfrak{g}_{\overline{E}}))$.

Proof. The complex splits into the direct sum of two complexes. The first, the $\Omega^i((\mathfrak{g}_{\overline{E}})_C)$ complex on which d_α acts like the trivial connection, and the second, the $\Omega^i((\mathfrak{g}_{\overline{E}})_U)$ complex, which is the sum of complexes of the form (4.7) one for each non–zero eigenvalue of $[\alpha, -]$. On the $\Omega^i((\mathfrak{g}_{\overline{E}})_C)$ the claim follows by ordinary Hodge theory on S^4, while on the second the claim follows by the previous proposition and the assumption that $ad(\alpha)$ has all its eigenvalues in $(-1, 1)$. □

Proposition 4.9. *(cfr.[KM1] §5) Let $2 \leqslant p < (1/2 - J_0/6)^{-1}$ and $A \in \mathcal{A}_J^{+,p}$. Then*

(i) *the operator* $d_A^+ : L^p_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})) \to: L^p(\Omega^+(\mathfrak{g}_{\overline{E}}))$ *has a parametrix.*

(ii) $H_A^{2,p}$ *is isomorphic to the space of self–dual d_A–harmonic forms $\mathcal{H}^+ \subseteq L^p(\Omega^+(\mathfrak{g}_{\overline{E}}))$ and $Im(d_A^+)$ equals its annahilator $\mathcal{H}^{\perp,p}$ in the L^2 inner product.*

(iii) d_A^+ *has a right inverse $Q_A : \mathcal{H}^{\perp,p} \to L^p_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}}))$.*

(iv) *If $A \in \mathcal{A}_J^{+,p'}$ and d_A^+ is thought of as an operator $L^p_{1,\alpha}(\mathfrak{g}_{\overline{E}}) \to L^p(\mathfrak{g}_{\overline{E}})$ for $2 \leqslant p \leqslant p'$, the operator Q_A and representatives for the cohomology groups of the complex can be taken to be the same for all p and the closed complements of (iii) can be chosen to be compatible with the inclusion $L^{p'}_{1,\alpha} \hookrightarrow L^p_{1,\alpha}$.*

Proof. (i) Using Proposition 4.8 and the same patching argument as in Lemma 5.2 of [KM1] with the multiplication theorems 1.11 (iii) we obtain the desired global parametrix.
(ii) This is Lemma 5.4 of [KM1]
(iii) From (i) $\mathcal{H}^\perp$ has finite codimension and the existence of the right inverse follows from the open mapping theorem.
(iv) Since the inverse operator for d_α^+ in (4.8) is the same for all p (it is the restriction of the inverse on L^2), when $A \in L^{p'}_{1,\alpha}$, using the multiplication theorems 1.11 (iii) we see that the parametrix can be constructed to map $L^p(\Omega^+(\mathfrak{g}_{\overline{E}})))$ into $L^p_{1,\alpha}$ for $2 \leqslant p \leqslant p'$. By (ii) we also see that $\mathcal{H}^+$ is the same for all such p and then $\mathcal{H}^{\perp,p'} \hookrightarrow \mathcal{H}^{\perp,p}$ and we can take Q_A to be the right inverse on $\mathcal{H}^{\perp,2}$ so that that the images of Q_A are all contained in one another. Since the closed complements to $Im(d_A)$ are given by $Im(Q_A) \oplus H_A^{1,p}$ for any representatives of the cohomolgy groups in $L^p_{1,\alpha}$, the closed complements in (iii) can be chosen compatibly with the inclusions if representatives for the first cohomology groups can be chosen simultaneously. Let $H_A^{1,p'}$ be a representative in $L^{p'}_{1,\alpha}$. By Lemma 1.37 we see that this injects in $(Ker(d_A^+)/Im(d_A))^p$ for all p. To show that it surjects it suffices to show that they have the same dimension, which follows if for any $b \in Ker(d_A^+)$ with $\|b\|_{L^p_{1,\alpha}}$ small we can find $a \in L^{p'}_{1,\alpha}$ and $c \in L^p_{1,\alpha}$ with

$$b - a = d_A(c).$$

Using the invertibility of

$$d_\alpha^*(r_\alpha^{-2}(-)) \oplus d_\alpha^+ : L^p_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})) \to L^p(\Omega^0(\mathfrak{g}_{\overline{E}})) \oplus L^p(\Omega^+(\mathfrak{g}_{\overline{E}})),$$

on S^4 we can use the method of [DS] §6, together with the procedure of Proposition 4.5 to see that if $b \in L^p_{1,\alpha} \cap Ker(d_A^+)$ there is a smooth family of gauge transformations

$$g_t \in \mathcal{G}_J^p, \qquad t \in [0,1], \qquad g_0 = 1,$$

such that

$$g_t(d_A + tb) - d_A = b_t \in L^{p'}_{1,\alpha}.$$

Differentiating at $t = 0$ gives

$$d_A u + b = c, \qquad u \in L^p_{1,\alpha}, \qquad c \in L^{p'}_{1,\alpha},$$

as required. □

At this point the construction of the moduli spaces is standard. The existence of the transversal $T_{[A]}$ of (iv) to $Im(d_A)$ shows that $\mathcal{B}^p_J = \mathcal{A}^p_J/\mathcal{G}^p_J$ is a Hausdorff space and $\mathcal{B}^{p,*}_J = \mathcal{A}^{p,*}_J/\mathcal{G}^p_J$ an analytic Banach manifold. The space $L^p(\Omega^+(\mathfrak{g}_{\overline{E}}))$ then descends to a Banach vector bundle on $\mathcal{B}^{p,*}_J$ and by (iv) of the previous Proposition it follows that the section

$$\Theta : [A] \mapsto F^+_A, \tag{4.10}$$

is a Fredholm section and the moduli space $M^{J,p,*}_{\kappa,c,\ell}$ appears as its zero set $Z(\Theta)$. This equips $M^{J,p,*}_{\kappa,c,\ell}$ with a real analytic space structure which can be locally described by any finite dimensional model for Θ. This consist of representatives for the cohomology groups $H^{i,p}_A$ and an analytic map

$$\theta : H^{1,p}_A \to H^{2,p}_A,$$

exhibiting the ringed space structure of $Z(\Theta)$ at $[A]$ as that of $\mathcal{O}_{H^{1,p}_A}/\mathcal{I}_\theta$, where $\mathcal{I}_\theta$ is the ideal sheaf of θ. In particular $M^{J,p,*}_{\kappa,c,\ell}$ is regular at points where H^2_A vanishes.

When $G = U(n)$, the only non simple group we are considering, the adjoint representation decomposes $\mathfrak{u}(n)$ into simple and central parts, $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus i\mathbb{R}$ so correspondingly (4.1) breaks up into the direct sum of the A.S.D. De Rham complex and a complex of $\mathfrak{su}(n)$–valued forms,

$$B^0_0 \oplus L^p_2(\Omega^0) \xrightarrow{d_A \oplus d} B^0_1 \oplus L^p_1(\Omega^1) \xrightarrow{d^+_A \oplus d^+} B^{0,+}_2 \oplus L^p(\Omega^+), \tag{4.11}$$

where

$$B^0_i = L^p_{2-i,\alpha}(\mathfrak{su}(n)).$$

In particular the Banach bundle of self–dual two forms just considered breaks up as a direct sum and we can consider Θ_0, the composition of Θ with the projection onto the $B^{0,+}_2$ summand. Then

$$\Theta_0^{-1}(0) = \Theta^{-1}(0),$$

so in this case the complex space structure on the moduli space is that of $Z(\Theta_0)$ and, denoting by $H^{i,p,0}_A$ the cohomology groups of the B^i_0 complex, local models can be taken to be of the form

$$\theta : H^{1,p}_A \to H^{2,p,0}_A,$$

so that the regularity of the moduli spaces at $[A]$ is equivalent to the vanishing of $H^{2,p,0}_A$.

From Proposition 4.5 we know that the moduli spaces $M^{J,p,*}_{\kappa,c,\ell}$ are independent of p as topological spaces. From (iv) of the previous proposition it is easy to see that the same holds at the ringed space level.

Proposition 4.12. *Let p as in Proposition 4.9. Then the ringed space structure defined on $M^{J,p,*}_{\kappa,c,\ell}$ as the zero set of Θ is independent of p.*

Proof. It suffices to show that at a given point $[A]$ the same model can be taken to describe $Z(\Theta)$ for all p's. To this purpose we use Proposition 4.6 and (iv) of the previous Proposition to see that at a given $[A] \in M^{J,p,*}_{\kappa,c,\ell}(k)$ and for any $2 < p \leqslant p' < (1/2 - J_0/6)^{-1}$, we can find a representative $A \in L^{p'}_{1,\alpha}$ for $[A]$, common representatives for the cohomology groups $H^{i,p}_A$ and transversals T^p_A compatible with the inclusions. The moduli space are now all cut out on this transversal by the local representative of Θ at A, namely the Fredholm map

$$A + b \mapsto d^+_A(b) + [b,b]^+,$$

and the finite dimensional models can all be taken to coincide with the one in $L^{p'}_{1,\alpha}$,

$$\theta : Ker(d_A) \cap T^{p'}_A \to \mathcal{H}^+,$$

so the local ringed space structure for $2 < p \leqslant p'$ are all isomorphic to $\mathcal{O}_{H^{1,p'}_A}/\mathcal{I}_\theta$. □

As in the case of ordinary instantons we can also describe the structure at the reducible connections as a finite dimensional analytic model of the form

$$\theta : H^1_A/H^0_A \to H^2_A,$$

where now, by Proposition 1.46, $H^0_A \subseteq Lie(C(J))$.

We next apply this to the construction of the moduli space of framed instantons which will be used to prove Theorem 6. We begin by rephrasing the definition. Given a point $\mathrm{p} \in \Sigma$ and denoting by s a distance function from p which approximates the euclidean distance from the origin in B^4 and is bounded further away we define for any $p > 1$,

$$\widetilde{L}^p_{k,\alpha}(\Omega^i(\mathfrak{g}_{\overline{E}})) = \{f \in L^p_{k,\alpha} | \frac{f}{s^{2/p(2-i)}}, \cdots, \frac{\nabla f}{s^{2/p(2-i-k)}} \in L^p\}.$$

For $p > 2$ we correspondingly obtain subspaces

$$\widetilde{\mathcal{A}}^p_J, \qquad \widetilde{\mathcal{A}}^{p,+}_J,$$

of $\mathcal{A}^p_J$ and its A.S.D. part and a subgroup

$$\widetilde{\mathcal{G}}^p_J,$$

of the gauge group (1.40). Consider the quotient

$$\widetilde{\mathcal{B}}^p_J = \widetilde{\mathcal{A}}^p_J/\widetilde{\mathcal{G}}^p_J, \tag{4.13}$$

equipped with the quotient topology. Fix an isomorphism

$$\varphi_0 : Ad(G) \cong \overline{E}_{\mathrm{p}}, \quad \varphi(C(Ad(J))) = s(\mathrm{p}).$$

This gives an inclusion $\widetilde{\mathcal{A}}^p_J \hookrightarrow \mathcal{A}^p_J \times Hom(s(p), C(Ad(J)))$ by

$$\iota : A \mapsto (A, \varphi_0).$$

Lemma 4.14. *For $p > 2$ the above inclusion induces a homeomorphisms*

$$\widetilde{\mathcal{B}}^p_J \cong (\mathcal{A}^p_J \times Hom(s(p), C(Ad(J))))/\mathcal{G}^p_J.$$

Proof. Clearly ι descends to a continuous map of the quotients. If $A_1, A_2 \in \widetilde{\mathcal{A}}^p_J$ and $g \in \mathcal{G}^p_J$ satisfies $g(\iota(A_1)) = \iota(A_2)$ it follows by definition that $Ad(g_{\rm p})(\varphi_0) = \varphi_0$ so that being continuous g_p must be the identity at p. Since $p > 2$ it follows that $g - 1 = O(s^\delta)$ for some small $\delta > 0$ hence $(g-1)/s^{4/p} \in L^p$. Notice that since

$$\int_{B^4} \frac{|f|^p}{s^p} \leqslant \int_{B^4} |f|^{up} \int_{B^4} s^{vp},$$

where $1/u + 1/v = 1$, if $f \in L^p_k$ with $k - 4/p > 0$ we can choose u and v so that the integrals on the right converge. It follows that $\widetilde{L}^p_{k-i,\alpha} = L^p_{k-i,\alpha}$ for $i = 0, 1$, hence $g \in \widetilde{\mathcal{G}}^p_J$ and $\widetilde{\mathcal{A}}^p_J = \mathcal{A}^p_J$ so that i is injective and surjectivity is then trivial because in every $\mathcal{G}^p_J$ orbit $[A, \varphi]$ there is an element of the form (A, φ_0). □

It follows by the argument of Proposition 4.6 that the subset

$$\widetilde{\mathcal{A}}^{p,+}_J/\widetilde{\mathcal{G}}^p_J, \tag{4.15}$$

is independent of p as long as this is as specified there. Since ι maps it onto $\widetilde{M}^J_{\kappa,c,\ell}$ as defined by (1.45), where (κ, c, ℓ) are the topological invariants of $\overline{E}$, the bundle with $Ad(J)$–reduction on Σ supporting $\mathcal{A}^p_J$, it follows that we can unequivocally refer to (4.13) and (4.15) as the set of equivalence classes of framed singular connections and moduli space of framed singular instantons with holonomy J. Let $\{\widetilde{B}_i, d_A\}$ be the complex with the same operators as (4.1) where

$$\widetilde{B}_i = \widetilde{L}^p_{2-i,\alpha}(\mathfrak{g}_{\overline{E}}). \tag{4.16}$$

We then have an obvious map of complexes given by the inclusion

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{B}_0 & \xrightarrow{d_A} & \widetilde{B}_1 & \xrightarrow{d_A^+} & \widetilde{B}_2 & \longrightarrow & 0 \\
 & & \downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \\
0 & \longrightarrow & B_0 & \xrightarrow{d_A} & B_1 & \xrightarrow{d_A^+} & B_2 & \longrightarrow & 0,
\end{array}$$

and we can apply the results for the complex of unframed singular instantons to the top complex to equip $\widetilde{M}^J_{\kappa,c,\ell}$ with a real analytic space structure. We denote by $\widetilde{H}^i_A$ the cohomology groups of the top complex.

Lemma 4.17. *The cohomology groups of the two complexes are related by*

$$\widetilde{H}^0_A = 0, \qquad \widetilde{H}^1_A = H^1_A \oplus (Lie(C(J))/H^0_A), \qquad \widetilde{H}^2_A = H^2_A.$$

Proof. (i) If $u \in Ker(d_A) \cap \widetilde{B}_0$ we know that $u \in \mathcal{C}^0$ and that u vanishes at p. Since $A \in \mathcal{A}^p_J$ it follows by Fubini's Theorem that A induces a continuous connection on almost all paths through p and then the usual parallel transport argument shows that $u = 0$ almost everywhere hence everywhere since u is continuous (we are assuming X connected). From the above map of complexes

$$\widetilde{H}^1_A/H^1_A \cong Im(d_A)/Im(d_{A|\widetilde{B}_0}) \cong \frac{Lie(C(J)) + H^0_A}{H^0_A} \cong Lie(C(J))/H^0_A,$$

since we know by Proposition 1.46 that H^0_A is a subgroup of $Lie(C(J))$, hence $\widetilde{H}^1_A = H^1_A \oplus (Lie(C(J))/H^0_A)$. Finally, by Lemma 4.14 $\widetilde{B}_i = B_i$ for $i = 1, 2$ so $H^2_A = \widetilde{H}^2_A$. □

Using this we can apply Proposition 4.9 to the complex of framed instantons obtaining at every $[A] \in \widetilde{\mathcal{A}}^p_J$ a transversal

$$\widetilde{T}_{[A]} = T_{[A]} \oplus Lie(C(J))/H^0_A,$$

to the action of $\widetilde{\mathcal{G}}^p_J$. This equips $\widetilde{\mathcal{B}}^p_J$ with an analytic Banach manifold structure and as in the unframed case we obtain a Banach vector bundle with an analytic Fredholm section defined by (4.10) whose ideal sheaf can be used to obtain the analytic space structure on $\widetilde{M}^J_{\kappa,c,\ell}$. Using Propositions 4.12 we obtain a model with the same regularity as for the unframed instantons showing that this is independent p. From description (1.45) of $\widetilde{M}^J_{\kappa,c,\ell}$ we now see that the action of $C(Ad(J))$ described there is analytic and the analytic space structure on $M^J_{\kappa,c,\ell}$ is the quotient structure of this action giving an alternative model for $M^J_{\kappa,c,\ell}$ at reducible connections.

In particular this equips $\widetilde{M}^J_{\kappa,\ell}(\mathbb{C}^2, \Sigma)$ as defined in (3.2) with an analytic space structure. Once more we remark that this could have been done more directly via cylinder analysis.

4.2 The Chern-Weil formula

To calculate the index of (4.1) we shall generalize the method of §6 of [KM1] to the $U(n)$-case. One of the ingredients in their calculations is the Chern–Weil formula for singular instantons and its behaviour upon tensoring by certain flat singular line bundle connections. We also give some direct applications of this formula.

Proposition 4.18. *Let $A \in \mathcal{A}^p_J$, $2 < p < (1/2 - J_0/6)^{-1}$, have topological invariants κ, c, ℓ, let $i\alpha = \log(J)$ where α has eigenvalues $1 > \alpha_1 > \alpha_2 \cdots \alpha_{t+1} \geqslant 0$ with multiplicities $n_1, \cdots, n_t$ and let $\ell = (\ell_1, \ell_2, \cdots, \ell_t)$ with respect to the basis of (2.5) for $[\Sigma, GL(n)/P(J)]$, then*

$$\frac{1}{8\pi^2}\int Tr(F_A \wedge F_A) = \kappa - \sum_{i=1}^{t}(\alpha_{i+1} - \alpha_i)\ell_i - \alpha_{t+1}c \cdot \Sigma - \frac{1}{2}\sum_{i=1}^{t+1} n_i\alpha_i^2\Sigma \cdot \Sigma. \tag{4.18}$$

Proof. Recall that by Proposition 2.20 $-\ell_i = c_1(F_i)$ where $F_i = \bigoplus_{j\leqslant i} \overline{V}_j$ with respect to decomposition (1.26). Following [KM1] we begin to prove the formula in the case when (1.26) holds globally on X so that the model singular connection A_α can be taken globally reducible.

$$A_\alpha = \bigoplus_j d_{b_j} + \beta(r) i\alpha\eta,$$

where d_{b_i} restricts to a connection on $\overline{V}_j$ such that $[\frac{i}{2\pi} Tr(F_{b_j})] = c_1(\overline{V}_j)$, in particular

$$\frac{i}{2\pi}\int_\Sigma \sum_{j\leqslant i} Tr(F_{b_j}) = -\ell_i.$$

As recalled in the proof of Proposition 3.73 $[\frac{i}{2\pi} d(\beta\eta)] = [\Sigma]$ so it follows that

$$\frac{i}{2\pi} F_{A_\alpha} = \bigoplus_j (\frac{i}{2\pi} F_{b_i} + \alpha_i [\Sigma] 1_{n_j}),$$

and

$$\begin{aligned}
\frac{1}{8\pi^2}\int Tr(F_A \wedge F_A) =& \sum_j \int Tr(F_{b_i} \wedge F_{b_i}) - \sum_j \alpha_i c_1(\overline{V}_j) - \frac{1}{2}\sum_j n_j \alpha_j^2 \Sigma\cdot\Sigma \\
=& \sum_j (c_2(\overline{V}_j) - \frac{1}{2} c_1(\overline{V}_j)^2) - \overset{t}{\Sigma}(\alpha_{i+1} - \alpha_i)\ell_i - \alpha_{t+1} c\cdot\Sigma - \frac{1}{2}\sum_j n_j\alpha_j^2 \Sigma\cdot\Sigma \\
=& \kappa - \overset{t}{\Sigma}(\alpha_{i+1} - \alpha_i)\ell_i - \alpha_{t+1} c\cdot\Sigma - \frac{1}{2}\sum_j n_j\alpha_j^2 \Sigma\cdot\Sigma,
\end{aligned}$$

since in this case

$$\kappa = c_2(\overline{V}) - \frac{1}{2} c_1(\overline{V}) = \sum_j (c_2(\overline{V}_j) - \frac{1}{2} c_1(\overline{V}_j)^2).$$

We can now deduce the general case exactly as in [KM1]. □

A case of special interest is when A is flat. Then the invariants of A are completely determined by α and $\Sigma\cdot\Sigma$. For future reference we isolate this as a separate Lemma.

Lemma 4.19. *Let A be a flat connection on $X\backslash\Sigma$ with holonomy J such that $i\alpha = \log(J)$. Then*

$$\begin{aligned}
c =& -Tr(\alpha)[\Sigma], \\
\ell_i =& \sum_{j\leqslant i} \alpha_j n_j \Sigma\cdot\Sigma, \\
\kappa =& \sum_{j\leqslant i} (\alpha_{i+1} - \alpha_i)\alpha_j n_j \Sigma\cdot\Sigma - \alpha_{t+1} Tr(\alpha)\Sigma\cdot\Sigma + \frac{1}{2}\overset{t+1}{\Sigma} n_i\alpha_i^2 \Sigma\cdot\Sigma.
\end{aligned}$$

Proof. Since A is flat the formulae for c and ℓ_i follow from Proposition 3.73 and we then deduce the formula for κ from the previous Proposition. □

The next Lemma is computational.

Lemma 4.20. *Let $A \in \mathcal{A}_J$ have topological invariants κ, c, ℓ, $i\alpha = \log(J)$ where α has eigenvalues $1 > \alpha_1 > \alpha_2 \cdots \alpha_{t+1} \geqslant 0$ with multiplicities $n_1, \cdots, n_t$, $\ell = (\ell_1, \ell_2, \cdots \ell_t)$ with respect to the basis (2.5) for $[\Sigma, GL(n)/P(J)]$, and let $[\Sigma]$ be divisible by n so that there is a flat line bundle singular connection Γ with holonomy $\zeta = \exp(2\pi i 1/n)$ on $X \backslash \Sigma$. Then the invariants $\tilde{\kappa}, \tilde{\ell}, \tilde{c}$ and $\tilde{\alpha} = -i \log(\zeta J)$ of $\tilde{A} = A \otimes \Gamma$ are related to those of A as follows*

$$\tilde{\alpha} = (\alpha'' + \frac{n-1}{n}) \oplus (\alpha' - \frac{1}{n}),$$

where $\alpha' = Diag(\alpha_1, \cdots \alpha_{s+1})$ consists of the eigenvalues of α which are larger than $1/n$ and $\alpha'' = Diag(\alpha_{s+2}, \cdots \alpha_{s+1+r+1})$ of the ones which are strictly smaller than $1/n$.

$$\begin{aligned}
\tilde{\kappa} &= \kappa + \ell_{r+1} + \frac{1}{n} c \cdot \Sigma + \{\frac{1}{n} - \sum^{r+1} n_{s+1+i}\} \frac{\Sigma \cdot \Sigma}{2} \\
\tilde{c} &= c - \{1 - \sum^{r+1} n_i\}[\Sigma], \\
\tilde{\ell}_j &= \ell_{s+1+j} - \ell_{s+1} - \{\sum_{i \geqslant j+1}^{r+1} \frac{n-1}{n} n_{s+1+i} - \sum^{r+1} \frac{n-1}{n} n_{s+1+i}\} \Sigma \cdot \Sigma, \quad 1 \leqslant j \leqslant s, \\
\tilde{\ell}_{r+1} &= -\ell_{s+1} - c \cdot \Sigma + \sum^{r+1} \frac{n-1}{n} n_{s+1+i} \Sigma \cdot \Sigma, \\
\tilde{\ell}_{r+1+j} &= \ell_j - \ell_{s+1} - c \cdot \Sigma - \{\sum^{j} \frac{n_i}{n} - \sum^{r+1} \frac{n-1}{n} n_{r+1+i}\} \Sigma \cdot \Sigma, \quad 1 \leqslant j \leqslant r.
\end{aligned}$$

Proof. Since $\tilde{A}$ has holonomy ζJ the formula for $\tilde{\alpha}$ is immediate and the statement about the first Chern classes follows by Lemma 1.33 (i). Since Γ is flat A and $\tilde{A}$ have the same curvature so the Chern–Weil integrals for A and $\tilde{A}$ coincide and by the previous Propostion we obtain the identity

$$\begin{aligned}
\kappa - \sum^{t} (\alpha_{i+1} - \alpha_i) \ell_i - \alpha_{t+1} c \cdot \Sigma - \frac{1}{2} \sum_j n_j \alpha_j^2 \Sigma \cdot \Sigma = \tilde{\kappa} &- \sum^{r+1} (\alpha_{r+1+i+1} - \alpha_{r+1+i}) \tilde{\ell}_i \\
&- (\alpha_1 - \alpha_{r+1} - 1) \tilde{\ell}_{r+1} \\
&- \sum^{s} (\alpha_{i+1} - \alpha_i) \tilde{\ell}_{r+1+i} \\
&+ (\alpha_{s+1} - 1/n) \tilde{c} \cdot \Sigma.
\end{aligned} \tag{4.21}$$

The derivation of the Chern-Weil formula shows that for α's of the form (1.26) it is continuous in α provided $C(i\alpha)$ is constant. If for some special degenerate values of α (4.21) gives relations between the invariants of A and those of $\tilde{A}$ it then follows by the integrality of the invariants that the relations must hold for α close to the degenerate value and hence for all α's with the same $C(i\alpha)$. Letting $\alpha_i \to 1/n$ for all i keeping $C(i\alpha)$ constant we obtain

$$\kappa - \frac{1}{n} c \cdot \Sigma - \frac{1}{2n} \Sigma \cdot \Sigma = \tilde{\kappa} + \tilde{\ell}_{r+1} - \sum^{r+1} n_{s+1+i} \frac{\Sigma \cdot \Sigma}{2}. \tag{4.22}$$

Letting

$$\alpha_i \to 1/n, \ \forall 1 \leqslant i \leqslant s+1, \qquad \alpha_i \to 0, \ \forall s+1 < i \leqslant s+1+r+1,$$

we obtain

$$\ell_{s+1} = -\tilde{\ell}_{r+1} - c \cdot \Sigma + \sum^{r+1} \frac{n-1}{n} n_{s+1+i} \Sigma \cdot \Sigma. \tag{4.23}$$

For a given $r+1 < j \leqslant s+1+r$ let

$$\alpha_i \to 1/n, \ \forall i \leqslant j, \quad \alpha_i \to 0, \ \forall i > j,$$

to obtain

$$\ell_{s+1+j} = \tilde{\ell}_j - \tilde{\ell}_{r+1} + \sum_{i \geqslant j+1}^{r+1} \frac{n-1}{n} n_{s+1+i} \Sigma \cdot \Sigma. \tag{4.24}$$

Finally, for a given $1 \leqslant j \leqslant s$ let

$$\alpha_i \to 2/n, \ \forall i \leqslant j, \quad \alpha_i \to 1/n, \ \forall j < i \leqslant s+1, \quad \alpha_i \to 0, \ \forall i > s+1,$$

to obtain

$$\ell_j = \tilde{\ell}_{r+1+j} - \tilde{\ell}_{r+1} + \sum^{j} \frac{n_i}{n} \Sigma \cdot \Sigma. \tag{4.25}$$

Putting (4.22) to (4.25) together gives the desired formulae. □

As a first consequence of the Chern Weil–formula and Theorem 1 we deduce the necessity of Condition (0.10).

Corollary 4.26. *Let A be a finite action singular instanton on $S^4 \backslash S^2$ with holonomy J. Then its invariants κ and $\ell = (\ell_1, \cdots, \ell_t)$ satisfy*

$$\kappa \geqslant 0, \qquad \forall i, \ \kappa + \ell_i \geqslant 0.$$

Proof. We apply the degeneration argument of the previous Proposition. Assume to the contrary that there exist a singular $SU(n)$–instanton on $S^4 \backslash S^2$ with holonomy J such that $\kappa < 0$ or for some i, $\kappa + \ell_i < 0$. Let $i\alpha = \log(J)$. By Theorem 1 expressed in terms of vector bundles this corresponds to an framed α–parabolic vector bundle on $(\mathbb{P}^2, \Sigma, D_\infty)$ with the same topological invariants. As we observed in Chapter 2 this is also a framed α'–parabolic vector bundle for any α' such that $P(\alpha) = P(\alpha')$ hence for any such α' we can use the other direction of Theorem 1 to construct an $SU(n)$–instanton with holonomy $J' = \exp(-2\pi i \alpha')$ and the original topological invariants. We can let α' degenerate to zero while keeping $P(\alpha') = P(\alpha)$. Then, since the connections considered are all instantons, for α' sufficiently close to 0 the Chern–Weil formula implies the contradiction

$$0 < \|F_A\|_{L^2} = \kappa < 0.$$

Let $\{\alpha_i\}$ be the eigenvalues of α and $\{n_i\}$ their multiplicities. For a fixed j such that $\kappa + \ell_j < 0$ let α' degenerate to the diagonal matrix with 1 in the first $\sum_j^j n_i$ places and 0 on the rest of the diagonal while keeping $P(\alpha') = P(\alpha)$. Reasoning as above the Chern–Weil formula now implies the contradiction

$$0 < \|F_A\|_{L^2} = \kappa + \ell_j < 0.$$

□

A second Corollary we can deduce along the same lines (see also Corollary 5.8 of [KM1]) is the following

Corollary 4.27. *Let (X, Σ) be a compact connected complex Kähler pair such that $\Sigma \cdot \Sigma > 0$. Then every unitary representation of $\pi_1(X \backslash \Sigma)$ is the sum of irreducible $PU(n)$–representations of $\pi_1(X)$.*

Proof. It suffices to prove that every irreducible $U(n)$–representation comes from an irreducible $PU(n)$–representations of $\pi_1(X)$. Suppose that ρ is such a representation of $\pi_1(X \backslash \Sigma)$ and let A be the corresponding flat connection. Let $J = \exp(-2\pi i \alpha)$ be the holonomy of A along Σ. If this is central then A extends as a $PU(n)$–connection across Σ, a fortiori flat, hence corresponds to an irreducible $PU(n)$–representation of $\pi_1(X)$. We can thus assume that $i\alpha = \log(J)$ is not central in $\mathfrak{u}(n)$, so it must have at least two distinct eigenvalues. By Theorem 2 we know that A corresponds to a stable α–parabolic vector bundle $\mathcal{V}$ on (X, Σ) whose topological invariants are given by Lemma 4.19. It follows from the definition of α–stability that $\mathcal{V}$ is also $\alpha + \varepsilon$–stable for ε sufficiently close to 0 such that $C(\alpha + \varepsilon) = C(i\alpha)$. Thus by the converse we can construct an A.S.D. connections on $\mathcal{V}$ with holonomy $\exp(-2\pi i(\alpha + \varepsilon))$ provided $\deg_{\alpha+\varepsilon}(\mathcal{V}) = 0$. By assumption α has at least two distinct eigenvalues $\alpha_i > \alpha_j$. Varying them along the line $\{tv | v = (n_j, -n_i)\}$ in $\mathbb{R}^2$ keeps $\deg_{\alpha+tv}(\mathcal{V}) = 0$ so that we obtain a family A_t of A.S.D. connection with $A_0 = A$, the flat connection we started from, and $\mathcal{V}$ as corresponding parabolic extension.

Write the curvature of a connection A as the sum of its semi–simple and central component $F_A = F_A^S + F_A^C$. For every finite action instanton on $X \backslash \Sigma$ we then have the Bogomolov–Gieseker inequality

$$E(A) = \int_{X\backslash\Sigma} Tr(F_A \wedge F_A) - \frac{1}{2}\int_{X\backslash\Sigma} Tr(F_A) \wedge Tr(F_A) = \|F_A^S\|_{L_2}^2 \geqslant 0, \tag{4.28}$$

with equality if and only if $F_A^S = 0$, and we consider the function of t, $E(t) = E(A_t)$. Using Propositions 4.18 and 3.73, both integrals in (4.28) can be expressed in terms of the topological invariants of $\mathcal{V}$ given by Lemma 4.19 yielding the explicit formula for $E(t)$,

$$E(t) = \frac{\Sigma \cdot \Sigma}{2}[(n_i^2 - n_i)t^2 + (n_j^2 - n_j)\frac{n_i^2}{n_j^2}t^2 - 2n_i n_j \frac{n_i}{n_j}t^2] = -\frac{\Sigma \cdot \Sigma}{2}t^2[n_i + \frac{n_i}{n_j}].$$

By assumption $\Sigma \cdot \Sigma > 0$ so the last equation together with (4.28) gives the contradiction

$$\forall t \neq 0, \qquad 0 > E(t) \geqslant 0.$$

Thus A must have been reducible in the first place unless its holonomy along Σ was central. □

From this and Theorem 2 we can deduce one further corollary.

Corollary 4.29. *Let the hypothesis be as in Corollary 4.27 and let α be of the form (1.23) with eigenvalues $\{\alpha_i\}$ having multiplicities $\{n_i\}$. There is a one to one correspondence between gauge equivalence classes of unitary flat connections with holonomy $\exp(-2\pi i\alpha)$ along Σ and isomorphism classes of α–parabolic vector bundles on (X,Σ) which are globally the direct sum of semi–stable holomorphic vector bundles $\mathcal{V}=\oplus_i \mathcal{V}_i$ where*

$$\forall i,\quad c_1(\overline{\mathcal{V}}_i)=-\alpha_i n_i[\Sigma],\qquad c_2(\overline{\mathcal{V}}_i)=\alpha_i^2(n_i^2-n_i)\frac{\Sigma\cdot\Sigma}{2},$$

and the parabolic filtration is given by $\mathcal{F}_i=\oplus_{j\leqslant i}(\overline{\mathcal{V}}_j)_{|\Sigma}$.

Proof. By the previous Corollary we know that given a finite action A.S.D. connection A as in the hypothesis this must reduce to a direct sum of finite action $U(n_i)$–instantons A_i with central holonomy $\exp(-2\pi i\alpha_i)$ along Σ. If A_i is not irreducible we may decompose it further $A_i=\oplus A_{ij}$ so that each A_{ij} is an irreducible $U(n_{ij})$–instanton. Applying Theorem 3.55 and Lemma 4.18 to each of them we obtain an α_i–stable holomorphic bundle $\overline{\mathcal{V}}_{ij}$ with

$$\forall i,\quad c_1(\overline{\mathcal{V}}_{ij})=-\alpha_i n_{ij}[\Sigma],\qquad c_2(\overline{\mathcal{V}}_{ij})=\alpha_i^2(n_{ij}^2-n_{ij})\frac{\Sigma\cdot\Sigma}{2},$$

so that each $\overline{\mathcal{V}}_i$ will have the required Chern classes. Also, by the definition of α_i–stability we see that for every proper non–zero saturated subsheaf $\mathcal{S}$ of $\overline{\mathcal{V}}_{ij}$,

$$\frac{deg(\mathcal{S})}{rk(\mathcal{S})}+\alpha_i\deg(\Sigma)=\mu_{\alpha_i}(\overline{\mathcal{V}}_{ij},\mathcal{S})<\mu_{\alpha_i}(\overline{\mathcal{V}}_{ij})=\frac{deg(\overline{\mathcal{V}}_{ij})}{rk(\overline{\mathcal{V}}_{ij})}+\alpha_i\deg(\Sigma),\tag{4.30}$$

hence

$$\frac{deg(\mathcal{S})}{rk(\mathcal{S})}<\frac{deg(\overline{\mathcal{V}}_{ij})}{rk(\overline{\mathcal{V}}_{ij})},$$

showing that $\overline{\mathcal{V}}_{ij}$ is stable in the ordinary sense. Vice–versa, starting from a parabolic bundle of the type described, we see that either each $\overline{\mathcal{V}}_i$ is stable or else it decomposes as a direct sum of stable bundles $\overline{\mathcal{V}}_{ij}$. By (4.30) each of them will be an α_i–stable parabolic bundle with filtration consisting of $(\overline{\mathcal{V}}_{ij})_{|\Sigma}$ only so by Theorem 3.55 we obtain an irreducible instanton A_{ij} with holonomy $\exp(-2\pi i\alpha_i)$ along Σ and such that by the Chern–Weil formula and Proposition 3.73

$$\|F^C_{A_{ij}}\|^2_{L^2}=-\int_{X\setminus\Sigma}Tr(F^C_{A_{ij}})\wedge Tr(F^C_{A_{ij}})=-[Tr(F^C_{A_{ij}})]\cup[Tr(F^C_{A_{ij}})]=0,$$

$$\|F^S_{A_{ij}}\|^2_{L^2}=E(A_{ij})=\alpha_i^2(n_{ij}^2-n_{ij})\frac{\Sigma\cdot\Sigma}{2}-\alpha_i^2n_{ij}^2\frac{\Sigma\cdot\Sigma}{2}+\alpha_i^2n_{ij}\Sigma\cdot\Sigma-\alpha_i^2n_{ij}\frac{\Sigma\cdot\Sigma}{2}=0,$$

showing that A_{ij} must be flat. Since Theorem 3.55 respects gauge equivalence classes we obtain the required statement. □

The point of the last two corollaries was to highlight the fact that there are essentially no new flat connections on $X\setminus\Sigma$. All of them extend to X, possibly after passing to $PU(n)$, and their classification can be reduced to ordinary stable bundles on X. This is due to the fact that Σ is a smooth irreducible curve. Relaxing either or both of these conditions should certainly provide new interesting flat connections since in this way we can make $\pi_1(X\setminus\Sigma)$ arbitrarily large. We shall have no time to pursue this here.

4.3 The index formula

We now come to the index formula for the dimension of $M^J_{\kappa,c,\ell}(X,\Sigma)$. We follow closely §6 of [KM1] and adapt their proof using the excision principle to the $U(n)$ case. By Propositions 4.9 (iv) this is independent of p in $[2,(1/2-J_0/6)^{-1})$ since we have common representatives for the cohomology groups. As recalled in §4.1 when $G=U(n)$ (4.1) breaks up into the direct sum of the A.S.D. De Rham complex and a complex of $\mathfrak{su}(n)$–valued forms, $\{B^0_i, d_A\}$. By Hodge theory we know that the Euler characteristic of the A.S.D. De Rham complex is $1-b^1+b^+$ so it suffices to calculate the Euler characteristic of the $\mathfrak{su}(n)$–complex, which we can interpret as deformations of instantons whose determinant connection (this has curvature extending smoothly across Σ by Lemma 1.35) is fixed. In particular it describes deformations of $SU(n)$–singular instantons. We let

$$\bar{\ell}_i = \ell_i - \frac{n_i}{n_{i+1}+n_i} c\cdot\Sigma. \tag{4.31}$$

Denoting by $H^{0,i}_A$ the cohomolgy groups of this complex what then we wish to prove that for every A in $M^J_{\kappa,c,\ell}(X,\Sigma)$,

$$\begin{aligned} -\sum_{i=0}^{2}(-1)^i dim(H^{0,i}_A) =& 4n\kappa + 2c^2 + dim(SU(n))(1-b^1+b^+) \\ &+ 2\sum_{i=1}^{t}(n_{i+1}+n_i)\bar{\ell}_i + dim(U(n)/C(i\alpha))\frac{\chi(\Sigma)}{2}. \end{aligned} \tag{4.32}$$

By Proposition 4.9 ((i) to (iii) of which hold also if A is not an instanton), for any connection $A\in\mathcal{A}^p_J$ with topological invariants κ, c, ℓ the complex

$$B^0_0 \xrightarrow{\delta_A} B^0_1 \xrightarrow{d^+_A} B^{0,+}_2,$$

where $\delta_A = d_A - Q_A F^+_A$ and Q_A is the right inverse of Proposition 4.9, is a Fredholm complex so its Euler characteristic, $\iota_{\kappa,c,\ell}(X,\Sigma)$, is defined. It is shown in [KM1] that ι is independent of Q, $A\in\mathcal{A}^p_J$ and it remains unchanged when the operator d^+_A is defined with respect to a metric which is an orbifold metric along Σ (and the Sobolev spaces of a smooth metric). As observed in (v) of Lemma 1.37 the operators d_{A_t} and $d^+_{A_t}$ vary continuously in operator norm when α of the form (1.23) is varied keeping $C(J)=C(i\alpha)$ constant, thus ι is independent of α as well for fixed $C(i\alpha)$. Let

$$-\iota_{\kappa,c}(X) = 4n\kappa + 2c^2 + dim(SU(n))(1-b^1+b^+),$$

be the index of the usual anti self duality deformation complex on X for $U(n)$–connections with fixed determinant. Let $g(\Sigma)$ denote the genus of Σ The main step in proving (4.32) is then

Proposition 4.33. *There are integer constants C_i, D E and F such that*

$$-(\iota_{\kappa,c,\ell}(X,\Sigma) - \iota_{\kappa,c}(X)) = \sum^{t} C_i\bar{\ell}_i + Dg(\Sigma) + E\,\Sigma\cdot\Sigma + F. \tag{4.34}$$

Proof. The proof by excision of Lemma 6.6 of [KM1] applies with no modification in our contest to show that given sets of data $(X_i, \Sigma_i, \kappa_i, \ell_i, c_i)$, $i = 1, 2$ where (X_i, Σ_i) are compact connected orientable pairs and (κ_i, ℓ_i, c_i) topological invariants for a $U(n)$–bundle with a $C(i\alpha)$–reduction on Σ_i,

$$\iota_{\kappa_1,c_1,\ell_1}(X_1, \Sigma_1) - \iota_{\kappa_1,c_1}(X_1) = \iota_{\kappa_2,c_2,\ell_2}(X_2, \Sigma_2) - \iota_{\kappa_2,c_2}(X_2), \tag{4.35}$$

provided $c_1 \cdot \Sigma = c_2 \cdot \Sigma$, $\ell_1 = \ell_2$ and Σ_1 and Σ_2 have diffeomorphic neighbourhoods.

For each i, let $\overline{V}^{1_i}$ be a vector bundle with $C(i\alpha)$–reduction on S^4 with invariants $\kappa = 0$, $c = 0$, $\ell = (0, \cdots, 0, 1_i, 0, \cdots, 0)$ so that with respect to decomposition (1.26)

$$c_1(\overline{V}_j) = \delta_{ij}, \ \ 1 \leqslant j \leqslant t, \qquad c_1(\overline{V}_{t+1}) = -1,$$

and let A^{1_i} be a model singular connection for this bundle so that we can take $i\alpha\beta\eta = i\alpha d\theta$

$$d_{a_j} = 0, \ \ 1 \leqslant j \neq i \leqslant t, \qquad a_{t+1} = -a_i.$$

We can also assume A^{1_i} to be flat in the neigbourhood of a point $\mathrm{p} \in S^2$. Using the same procedure as in [KM1] we can glue $(\overline{V}^{1_i}, A^{1_i})$ to any $(\overline{V}, A)$ where A is a connection with holonomy $\exp(-2\pi i\alpha)$ and invariants $\kappa, c, \ell = (\ell_1, \cdots \ell_t)$ on a pair (X, Σ) to obtain a new bundle and connection $(\overline{V}', A')$ with the same holonomy κ and c but with

$$\overline{\ell}' = (\overline{\ell}_1, \cdots, \overline{\ell}_{i-1}, \overline{\ell}_i + 1, \overline{\ell}_{i+1}, \cdots, \overline{\ell}_t).$$

Applying the excision principle it follows that when ℓ_i is increased by 1 the index changes by a quantity which is independent of (X, Σ) and of κ, c, ℓ (and hence $\overline{\ell}$). It follows that each $\overline{\ell}_i$ enters linearly in the index formula with a constant coefficient C_i.

The argument of [KM1] applies directly to our case to show that we can increase $g(\Sigma)$ and $\Sigma \cdot \Sigma$ by 1 without changing κ, c, ℓ or $\Sigma \cdot \ell$. Then the index changes by quantities which are independent of (X, Σ) and of κ, c, ℓ so that $g(\Sigma)$ and $\Sigma \cdot \Sigma$ must also enter linearly with constant coefficients in the index formula which by (4.35) gives (4.34). □

We now pass to calculate the constants appearing in (4.34) by analysing specific examples.

Lemma 4.36. *The constants in (4.34) have the following values.*

$$\begin{aligned} \forall i, \quad C_i &= 2(n_i + n_{i+1}), \\ D &= -\dim(U(n)/C(i\alpha)), \\ E &= 0, \\ F &= \dim(U(n)/C(i\alpha)). \end{aligned}$$

Proof. As in [KM1] we can calculate the C_i's via internal consistency considerations. Consider a bundle $\overline{V}$ with orthogonal decomposition of the form (1.26) on the standard pair (S^4, S^2) with invariants κ, ℓ and ranks $\{n_i\}_{i=1}^{t+1}$. In particular $t+1$ and $\{n_i\}$ are the number

and multiplicities of the eigenvalues of α where $\exp(-2\pi i\alpha)$ is the holonomy of a singular connection compatible with the orthogonal decomposition of $\overline{V}_{|S^2}$. Choose values $\{\alpha^i\}_{i=1}^t$ for α such that the eigenvalues $\{\alpha^i_j\}$ of α^i satisfy

$$\forall j \leqslant i, \quad \alpha^i_j \geqslant 1/n,$$
$$\forall j > i, \quad \alpha^i_j < 1/n.$$

For each α^i consider a model singular connection A^i on $\overline{V}$ with the given invariants and holonomy $\exp(-2\pi i\alpha^i)$. Exploiting the independence of the index on α up to $C(i\alpha)$ we now that the indices of the complexes (4.1) for each of the α_i must be the same. Since $[S^2] = 0$ we can find line bundle singular connections with holonomy $\exp(2\pi i/n)$ on $S^4\backslash S^2$(explicitly $d - i/nd\theta$). We now apply Lemma 4.19 to each A^i obtaining corresponding relations between the invariants of A^i and those of $\tilde{A}^i$. In each case, though the invariants change, the associated Lie algebras bundles must be isomorphic since Γ is a flat connection, so in particular the corresponding complexes (4.1) have the same index so we obtain identities

$$\forall k, \ 1 \leqslant k \leqslant t, \qquad \iota_{\kappa,\ell} = \iota_{\tilde{\kappa}^i,\tilde{\ell}^i}.$$

Substituting the explicit values given by Lemma 4.18 we obtain the system

$$\frac{n_{t+1}}{n_1}\sum_{i=1}^{k} C_i + \left(\frac{n_{t+1}n_{k+1} - n_1 n_{k+2}}{n_1(n_{k+1} + n_{k+2})}\right) C_{k+1} - \sum_{k+2}^{t} C_i = -4\left(n - (\frac{n_{t+1} + n_1}{n_1})\sum_{i=1}^{k+1} n_1\right).$$

Direct calculation shows that the solution is

$$\forall i, \qquad C_i = 2(n_{i+1} + n_i).$$

We next calculate D and F. For D we consider again S^4 but now let Σ be a standard embedded surface of genus g. Let $\overline{V}$ be a rank n trivial vector bundle with an orthogonal decomposition of the form (1.26) on the whole of S^4 so that the invariants κ, c and ℓ are all zero. Since $[\Sigma] = 0$ we can find on $S^4\backslash\Sigma$ flat singular line bundle connections of the form $d + i\alpha d\theta_\Sigma$ for any real number α. We then put on $\overline{V}$ a flat singular connection d_α sum of line bundles ones with α having eigenvalues

$$1 > a_1/n > a_2/n \cdots a_t/n > a_{t+1}/n \geqslant 0, \qquad a_i \in \mathbb{Z}, \quad n > a_1 > \cdots > a_{t+1} \geqslant 0,$$

so that the adjoint bundle $\mathfrak{g}_{\overline{V}}$ has an induced decomposition as in (1.26) with the ij–summand having an induced flat connection of the form

$$d + i\frac{(a_i - a_j)}{n} d\theta_\Sigma.$$

We know by the previous Lemma that the index of the d_α–deformation complex, say with respect to the round metric, is

$$-\iota_{0,0,0}(S^4, \Sigma) = -\dim(SU(n)) + Dg + F,$$

and we shall find D and F by computing the index in a different manner. By the remarks preceding Proposition 4.34 we know that the index of the d_α–complex with respect to an orbifold metric is the same so we take an orbifold metric with cone angle n along Σ on S^4 and let $\tilde{X} \xrightarrow{\pi} S^4$ be the n–fold branched cover of S^4, branched along Σ which has a smooth metric. The A.S.D. De Rham complex on $\tilde{X}$ decomposes under the action of the covering transformations, the n–th roots of unity, as a sum of n–complexes, each corresponding to one of the complexes with trivial real coefficients on S^4

$$(4.37,) \qquad W^p_{2,2}(S^4) \xrightarrow{d_{i/n}} W^{p,1}_{1,2}(S^4) \xrightarrow{d^+_{i/n}} W^{p,2}_{0,2}(\Omega^+(S^4)), \qquad 0 \leqslant i \leqslant n-1$$

defined with respect to the orbifold metric. Consequently its index $\iota(\tilde{X})$ decomposes as

$$\iota(\tilde{X}) = \iota_0 + \iota_1 + \cdots + \iota_{n-1},$$

where ι_i is the index of (4.37). In particular ι_0 is just the index of the ordinary A.S.D. De Rham complex on S^4, hence its value is 1. As observed above the operator norm of

$$d_\alpha = d + i\alpha d\theta : W^p_{2-i}(\Omega^i(S^4)) \to W^p_{2-(i+1)}(\Omega^i(S^4)),$$

acting on the trivial line bundle varies continuously with α as long $0 < \alpha < 1$ so the complexes (4.37) have all the same index for $i \neq 0$. Because of the existence of a global angular coordinate on $S^4 \backslash \Sigma$ there are gauge transformations $\exp(2\pi i m \theta_\Sigma)$ for every $m \in \mathbb{Z}$, giving isomorphisms of the the d_α complex, $0 < \alpha < 1$, to the $d_{\alpha+m}$ complex for any $m \in \mathbb{Z}$ so that the index is in fact the same for all $\alpha \notin \mathbb{Z}$. Since $\tilde{X}$ has an orientation reversing diffeomorphism, $\sigma(\tilde{X})$, the signature of $\tilde{X}$ is zero, further

$$\begin{aligned} e(\tilde{X}) =& ne(S^4) - (n-1)e(\Sigma) \\ =& 2n + (n-1)(2g-2) \\ =& 2 + 2(n-1)g, \end{aligned}$$

and so

$$\begin{aligned} \iota(\tilde{X}) =& 1 - b^1 + b^+ \\ =& \frac{1}{2}(\Sigma + e) \\ =& 1 + (n-1)g. \end{aligned}$$

We thus obtain

$$\forall 1 \leqslant j \leqslant n-1, \qquad \iota_j = \frac{\iota(\tilde{X}) - 1}{n-1} = g.$$

Lifting $\mathfrak{g}_V$ to $\tilde{X}$ we obtain an equivariant vector bundle with flat connection extending across $\pi^{-1}(\Sigma)$. The covering transformations act along the fibre by $\exp(2\pi i(i-j)/n)$ on the $\mathfrak{g}^{ij}_V$ summand and it follows that

$$\begin{aligned} -\iota_{0,0,0}(S^4, \Sigma) =& -\iota_0(\dim(\mathfrak{g}^C_V) - 1) - \sum_{i \neq j} \iota_{(i-j)} \dim(\mathfrak{g}^{ij}_V) \\ =& -\dim(C(i\alpha) - 1) - g \dim(U(n)/C(i\alpha)). \end{aligned}$$

We thus obtain that for all g

$$-\dim(SU(n)) + Dg + F = -\dim(C(i\alpha)) + 1 - g\dim(U(n))/C(i\alpha),$$

so $D = -\dim(U(n)/C(i\alpha))$ and $F = \dim(U(n)/C(i\alpha))$.

It only remains to show that $E = 0$. To prove this we consider a case where $\Sigma \cdot \Sigma \neq 0$ such as a smooth curve of degree n in $\mathbb{P}^2$. We set up the calculation as in the previous example considering a vector bundle $\overline{V}$ which is globally a direct sum as in (1.26) and equip it with a flat singular connection d_α which on each $\overline{V}_i$ is the sum of flat line bundle singular connections. We now take α to have eigenvalues

$$\forall i, \qquad \alpha_i = \frac{t+1-i}{n},$$

so that on $\overline{V}_i$ the connection is

$$d_{\alpha_i} = d + i\frac{t+1-i}{n}d\theta,$$

and it follows from Lemma 1.33 that each $\overline{V}_i$ has $c_1 = -n_i(t+1-i)[\Sigma]$. Since $\Sigma \cdot \Sigma = n^2$ we obtain from Lemma 4.18 that the invariants of d_α are

$$\begin{aligned} c &= -\sum_i \frac{(t+1-i)}{n} n_i[\Sigma], \\ \ell_i &= \sum_{j \leqslant i} \frac{(t+1-i)}{n} n_i \Sigma \cdot \Sigma, \\ k &= -\frac{1}{n}\sum_i^t \ell_i + \sum_i^{t+1} \frac{(t+1-i)^2}{2} n_i. \end{aligned}$$

By the previous calculations we know that the index has the form

$$\begin{aligned} -\iota_{\kappa,c,\ell}(\mathbb{P}^2, \Sigma) =& 2n\kappa - \dim(SU(n))(1 - b^1 + b^+) \\ &+ 2\sum_i^t (n_{i+1} + n_i)\overline{\ell_i} + \dim(U(n)/C(i\alpha))\frac{\chi(\Sigma)}{2} + E\,\Sigma \cdot \Sigma, \end{aligned}$$

thus substituting the invariants of this specific case we obtain

$$\begin{aligned}
-\iota_{\kappa,c,\ell}(\mathbb{P}^2,\Sigma) =& 2n\sum_i^{t+1}[(n_{i+1}+n_i-2)\sum_{j\leqslant i}(t+1-i)n_j] \\
&+2n\sum_i^{t+1}(t+1-i)^2n_i+2(\sum_i(t+1-i)n_i)^2 \\
&-2n(\sum_i^t n_i)(\sum_i^{t+1}(t+1-i)n_i) \\
&-2(n^2-1)+(1-\frac{(n-1)(n-2)}{2})(n^2-\sum_i^{t+1}n_i^2)+n^2\,E \\
&=2n\{\sum_{j\leqslant i}^{t+1}(n_{i+1}+n_i-2)n_j-(\sum_i^t n_i)(\sum_i^{t+1}(t+1-i)n_i)\} \\
&+2n\{\sum_i^{t+1}(t+1-i)^2n_i-2\sum_{i\leqslant j}^{t+1}(t+1-j)n_j\}+2(\sum_i^{t+1}(t+1-i)n_i)^2 \\
&-\sum_{j<i}^{t+1}[n^2-3n+4]n_in_j-2(\sum_i^{t+1}n_i^2-1)+n^2\,E \\
&=2n\sum_{j<i}^{t+1}(i-j)n_in_j-2\sum_{j<i}^{t+1}(i-j)^2n_in_j \\
&-\sum_{j<i}^{t+1}[n^2-3n+4]n_in_j-2(\sum_i^{t+1}n_i^2-1)+n^2E \\
&=-\sum_{j<i}^{t+1}[n^2-3n+4+2(i-j)^2-2n(i-j)]n_in_j-2(\sum_i^{t+1}n_i^2-1)+n^2\,E.
\end{aligned}$$

We shall now calculate this index in a different manner to prove that $E=0$. We apply the same method as in the previous case passing to a branched cover. Let $\overline{X}$ be the n–fold branched cover of $\mathbb{P}^2$ branched along Σ ($\overline{X}$ exists and is unique since Σ is divisible by n in $Pic(\mathbb{P}^2)$ and this has no torsion). Equip $\overline{X}$ with a Kähler metric so that $\mathbb{P}^2$ has an induced orbifold metric with cone angle n along Σ and consider the deformation complex on $\mathfrak{g}_{\overline{V}}$ for the same d_α but with respect to the orbifold metric. In the notation of the previous example we have

$$-\iota_{\kappa,c,\ell}(\mathbb{P}^2,\Sigma)=-\iota_0(\dim(\mathfrak{g}_{\overline{V}}^C)-1)-\sum_{i\neq j}\iota_{(i-j)}\dim(\mathfrak{g}_{\overline{V}}^{ij}), \tag{4.38}$$

and we need to compute the indices ι_{i-j}. Unlike in the previous case these are not all the same due to the fact that the flat line bundle singular connections $d_{(i-j)/n}$ are now acting on topologically different real line bundles L_{i-j}, namely having complexification with first Chern class $-(i-j)/n[\Sigma]$. Notice, however, that by Lemma 1.33 the $d_{-(i-j)}$ complex on $L_{-(i-j)}$ is isomorphic to the $d_{n-(i-j)}$ complex on $L_{n-(i-j)}$ so in particular it follows that

$$\iota_{-(i-j)}=\iota_{n-(i-j)}.$$

Since the metric considered on $\tilde{X}$ is Kähler we can relate in the familiar way the A.S.D.

De Rham complex and the Dolbeault complex.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d^+} & \Omega^+ & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \\ 0 & \longrightarrow & \Omega^0 \otimes \mathbb{C} & \xrightarrow{\overline{\partial}} & \Omega^{0,1} & \xrightarrow{\overline{\partial}} & \Omega^{0,2} & \longrightarrow & 0, \end{array}$$

where ι is the inclusion and j_1 and j_2 the projection onto the (0,1) and (0,2) components respectively, and we know by Hodge theory that $\iota(\tilde{X}) = 2\chi(\mathcal{O}_{\tilde{X}})$, where $\chi(\mathcal{O}_{\tilde{X}})$ is the Euler characteristic of the $\overline{\partial}$–complex. The vertical maps are compatible with the action of the covering transformations, which is holomorphic, so that the $\overline{\partial}$–complex also breaks up into a sum of complexes equivariant with respect to the $\exp(2\pi ij/n)$–lifting of the action of the covering transformations and the equivariant parts correspond. Accordingly $\chi(\mathcal{O}_{\tilde{X}})$ breaks up into a sum

$$\chi(\mathcal{O}_{\tilde{X}}) = \chi_0 + \chi_1 + \cdots + \chi_{n-1},$$

and $\iota_j = 2\chi_j$. Since a holomorphic function in a neighbourhood of a point in $\pi^{-1}(\Sigma)$ is $\exp(2\pi i\, j/n)$–equivariant if and only if it vanishes to order j along $\pi^{-1}(\Sigma)$, the $\exp(2\pi i\, j/n)$–equivariant part of the $\overline{\partial}$–complex on $\tilde{X}$ can be identified with the $\overline{\partial}$–complex of $\mathcal{O}(-j/n\,\Sigma)$ on $\mathbb{P}^2$. We therefore conclude by Riemann-Roch that

$$\iota_j = 2\chi_j = j^2 - 3j + 2,$$

hence from (4.34) we obtain

$$\begin{aligned} -\iota_{\kappa,c,\ell}(\mathbb{P}^2, \Sigma) &= -2(\Sigma n_i^2 - 1) \\ &\quad - \sum_{i>j} [(i-j)^2 - 3(i-j)^2] n_i n_j \\ &\quad + \sum_{i>j} [(n-(i-j))^2 - 3(n+(i-j)) + 4] n_i n_j \\ &= -\sum_{i>j} [n^2 - 3n + 4 + 2(i-j)^2 - 2n(i-j)] n_i n_j - 2(\sum_i^{t+1} n_i^2 - 1). \end{aligned}$$

Comparing this with the previous computations we see that E must vanish as required. □

This concludes the proof of of 4.32. From Lemma 4.17 we immediately deduce the index formula for the $\mathfrak{su}(n)$–part of the deformation complex of framed instantons to be

$$\begin{aligned} -\widetilde{\iota}_{\kappa,c,\ell} =& 4n\kappa + 2c^2 + dim(SU(n))(1 - b^1 + b^+) \\ &+ 2\sum_{i=1}^{t} (n_{i+1} + n_i)\overline{\ell}_i + dim(U(n)/C(i\alpha))\frac{\chi(\Sigma)}{2} \\ &+ dim(C(i\alpha)) - 1. \end{aligned} \tag{4.39}$$

To obtain the index of (4.1) in the unframed case we now only need to substitute $SU(n)$ with $U(n)$ in (4.39) and in the framed case also add a +1 to the right hand side of (4.39). We summarise the results of the last three sections in the following Theorem which appeared as Theorem 4 in §0.3.

Theorem 4.40. *Let (X,Σ) be a compact connected orientable pair and G a compact connected simple Lie group or $U(n)$. Then*

(i) *For any Riemannian metric on X the moduli space of G–instantons with holonomy centralized by $Z(G)$ and admissible holonomy J along Σ, $M^{J,*}_{\kappa,c,\ell}$, has a natural real analytic space structure. If $G = SU(n)$ or $U(n)$ its dimension at smooth points is given by (4.32) and (4.32) $-(1-b_1)$ respectively.*

(ii) *For any Riemannian metric on X the moduli space of framed G–instantons with admissible holonomy J along Σ, $\widetilde{M}^{J}_{\kappa,c,\ell}$, has a natural real analytic space structure. If $G = SU(n)$ or $U(n)$ its dimension at smooth points is given by (4.39) and (4.39) $+ b_1$ respectively.*

The two analytic space structures are related in the sense that the structure in (i) is the quotient of the structure in (ii) by the analytic action of $Ad(C(J))$ defined from (1.45) where this action is free.

In the case where $(X,\Sigma) = (S^4, S^2)$ we shall compute the dimension of the moduli spaces for general G in Chapter 7 from holomorphic geometry.

4.4 The extended moduli spaces

To conclude the Chapter we generalize the definition of the extended moduli spaces of [KM1]. In Chapter 7 we shall prove that when $(X,\Sigma) = (S^4, S^2)$ the extended moduli spaces are actually a product.

Let C be a closed connected subgroup of a compact connected simple Lie group G or $U(n)$. As in §2.2 fix a pair $(\mathfrak{h},\mathfrak{h}^+) \subseteq Lie(C)$ consisting of a C.S.A. and a closed positive Weyl chamber and let $\mathfrak{f}$ be the corresponding fundamental domain for the exponential given by Lemma 2.8. We know by the Lemma that the exponential gives a bijection

$$\exp(2\pi -) : \mathfrak{h}^+ \cap \mathfrak{f} \to \{[g] | g \in C\},$$

so that every admissible element in C is conjugate to a unique element in $\exp(2\pi(\mathfrak{h}^+ \cap \mathfrak{f}))$. Let

$$\begin{aligned} Reg(C) &= \{g \in C | C_G(g) = C\} \\ Reg^+(C) &= \{g \in \exp(2\pi(\mathfrak{h}^+ \cap \mathfrak{f})) | C_G(g) = C\}, \end{aligned} \tag{4.41}$$

where C_G denotes the centralizer of g in G. We also define

$$Reg(\mathfrak{c}) = \{u \in \mathfrak{c} | C_G(u) = C\},$$

$$Reg^+(\mathfrak{c}) = Reg(\mathfrak{c}) \cap \mathfrak{h}^+.$$

Note that $Reg(C)$ consists of the regular elements in $Z(C)$. We define the extended space of connections with L^p curvature action, C–holonomy on Σ and fixed topological invariants to be

$$\widehat{\mathcal{A}}^p_C = \bigcup_{J \in Reg^+(C)} \mathcal{A}^p_J,$$

similarly we define the extended space of instantons with C–holonomy along Σ and the subset of those having holonomy centralized by $Z(G)$

$$\widehat{\mathcal{A}}^{p,+}_C, \qquad \widehat{\mathcal{A}}^{p,+,*}_C,$$

by considering the relevant parts of $\widehat{\mathcal{A}}^p_C$. The gauge group $\mathcal{G}^p_J$, which by Lemma 1.37 (iv) is independent of $J \in C$ provided $C_G(J) = C$, acts on each slice of $\widehat{\mathcal{A}}^p_C$ by the pull–back action and the quotient is the extended moduli space of connections with C–holonomy along Σ and fixed topological invariants

$$\widehat{\mathcal{B}}^p_C = \widehat{\mathcal{A}}^p_C/\mathcal{G}^p_J.$$

As in the case of a single moduli space, the extended moduli space of instantons

$$\widehat{\mathcal{M}}^C_{\kappa,c,\ell} = \widehat{\mathcal{A}}^{p,+}_C/\mathcal{G}^p_J,$$

can be topologized with the induced topology since it naturally appears as the zero set of a section of a Banach vector bundle on $\widehat{\mathcal{B}}^p_C$. Recall that by Lemma 1.37 (iv), the spaces $L^p_{k,\alpha}$ are independent of $i\alpha \in \mathfrak{g}$ as long as this varies in $Reg^+(\mathfrak{c}) \cap \mathfrak{f}$. We consider a complex $\{\widehat{B}_i\}$ with $\widehat{B}_i = B_i$ as defined above for $i = 0, 2$,

$$\widehat{B}_1 = L^p_{1,\alpha} \oplus Z(\mathfrak{c}) < \beta\eta >,$$

where $\beta\eta$ is as in §1.4 and the differential in the degree 1 at a base point A is

$$d^+_A(a, \alpha'\beta\eta) = d^+_A(a + \alpha'\beta\eta),$$

which maps in L^p since $\alpha' \in Z(\mathfrak{c})$ and $d(\beta\eta)$ is the Thom class of $N\Sigma$. Applying Proposition 4.9 to this complex which is just a finite dimensional modification of (4.2) we can equip $\widehat{\mathcal{B}}^{p,*}_C$ with a smooth manifold structure locally modelled on a neighbourhood at the origin in $T \oplus Z(\mathfrak{c})$, where T is the transversal to $Im(d_A)$ given by Proposition 4.9, and a regular map

$$\pi : \widehat{\mathcal{B}}^{p,*}_C \to Reg^+(C),$$

inducing a real analytic map

$$\pi : \widehat{\mathcal{M}}^{C,*}_{\kappa,c,\ell} \to Reg^+(C), \tag{4.42}$$

whose fibres are the moduli spaces $\mathcal{M}^{J,*}_{\kappa,c,\ell}$ for $J \in Reg^+(C)$. For these spaces we thus obtain local models given by $\theta^{-1}(0) \oplus Z(\mathfrak{c})$, where as before θ is a map at the origin $\theta : H^1_A \to H^{2,0}_A$. Similarly we can define and construct the extended moduli spaces of framed instantons with C–holonomy along Σ and given topological invariants,

$$\widehat{\widetilde{\mathcal{M}}}^C_{\kappa,c,\ell},$$

and equip them with a regular map $\widetilde{\pi}$ to $Reg^+(C)$). The two are related in the same way as any two slices, that is $\widehat{\mathcal{M}}^{C,*}_{\kappa,c,\ell}$ is the quotient of $\widehat{\widetilde{\mathcal{M}}}^C_{\kappa,c,\ell}$ by the analytic action of $Ad(C)$ on each slice (where this action is free).

Chapter 5

Moduli spaces of parabolic bundles and parabolic framings

In this chapter we shall show that for a general Kähler pair (X, Σ) such that $\Sigma \cdot \Sigma \geqslant 0$ the moduli space of stable α–parabolic bundles on (X, Σ), $\mathcal{M}^{\alpha,S}_{k,c,\ell}$, has the structure of a complex analytic space and that, under suitable assumptions, for a general compact connected complex triple (X, Σ, D_∞) where D_∞ is any smooth divisor transversal to Σ, that of framed quasi–parabolic bundles on (X, Σ, D_∞) also is. The construction is very natural from the holomorphic point of view but difficult to compare with $M^{\alpha,*}_{k,c,\ell}$ and $\widetilde{M}^{J}_{k,c,\ell}$. We shall carry out part of the comparison between the two deformation theories in Chapter 7 by giving what essentially amounts to a different construction of these moduli spaces. The construction given here applies equally well to the case where both Σ and D_∞ are divisors with ordinary singularities. We shall outline in §5.3 how this can be done. For the moment we focus on the case where both Σ and D_∞ are smooth and irreducible.

5.1 Holomorphic construction

We will adapt the methods of Kuranishi theory to construct local complex space structures at simple points (see Definition 5.8) of the moduli space of $(G^{\mathbb{C}}, P)$–quasi–parabolic and framed quasi parabolic bundles of a given topological type where $G^{\mathbb{C}}$ is a complex reductive Lie group. Recall that when $G^{\mathbb{C}}$ is simple or $GL(n)$ these are denoted by $\mathcal{M}^{P}_{k,c,\ell}$ and $\widetilde{\mathcal{M}}^{P}_{k,c,\ell}$ respectively. As in Chapter 2 the exposition will be for the case where $G^{\mathbb{C}}$ is simple or $GL(n)$, but all of what we say except for the statements concerning stable bundles and the dimension formulae of Proposition 5.18 holds word for word in the case of arbitrary complex reductive Lie groups. From §2.4 and using the same notation we know that equipping the moduli spaces with the quotient topology induced by the $\mathcal{C}^\infty$ topology on $\Omega^{0,1}(\mathfrak{g}_{E^{\mathbb{C}}})$, there are homeomorphisms.

$$\mathcal{M}^{P}_{k,c,\ell} \cong \mathcal{H}_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}}, \qquad \widetilde{\mathcal{M}}^{P}_{k,c,\ell} \cong \widetilde{\mathcal{H}}_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}}, \tag{5.1}$$

so the construction of the moduli spaces can be carried out on the spaces on the right in (5.1). Let $E^{\mathbb{C}}$ be the $\mathcal{C}^\infty$ P–quasi–parabolic bundle or framing with topological invariants k, c, ℓ so that the (5.1) are defined on it. Let $\overline{\partial}_0$ be any $\overline{\partial}$–operator in $\mathcal{H}_{\mathfrak{p}}$ or $\widetilde{\mathcal{H}}_{\mathfrak{p}}$. As explained in §2.4, $\overline{\partial}_0$ respects the $\Omega^{0,i}(T)$ or $\Omega^{0,i}(\widetilde{T})$ so we obtain complexes

$$\{\,\Omega^{0,i}(T), \overline{\partial}_0\,\}, \qquad \{\,\Omega^{0,i}(\widetilde{T}), \overline{\partial}_0\,\}. \tag{5.2}$$

It follows that equipping T or $\widetilde{T}$ with Hermitian fibre metrics there is a good Hodge theory for $\overline{\partial}_0$ thought of as a $\overline{\partial}$–operator on them. We can obtain this Hodge theory also more

directly, as follows. As in §2.4 let $\mathfrak{p}$ be the subbundle of parabolic Lie algebras of $\mathfrak{g}_{E^{\mathbb{C}}}$ induced by the P–reduction on Σ. Pull back $\mathfrak{p}$ to a subbundle of $\mathfrak{g}_{E^{\mathbb{C}}}$ along the disc fibres of a tubular neighbourhood $N(\Sigma)$ of Σ as in Lemma 2.31. Consider a G–reduction on $E^{\mathbb{C}}$, let $\mathfrak{u}^-$ be the corresponding orthogonal complement of $\mathfrak{p}$ on Σ and pull–back along the disc fibres the orthogonal splitting

$$\mathfrak{g}_{E^{\mathbb{C}}|\Sigma} = \mathfrak{p} \oplus \mathfrak{u}^-.$$

Letting $\sigma \in H^0([\Sigma])$ be such that $Div(\sigma) = \Sigma$ we can then more invariantly describe the images of the $\Omega^{0,i}(T)$'s in $\Omega^{0,i}(\mathfrak{g}_{E^{\mathbb{C}}})$ as the subspaces consisting of the elements of

$$\Omega^{0,i}(\mathfrak{p}) \oplus < \frac{1}{\sigma} > \otimes \Omega^{0,i}(\mathfrak{u}^-),$$

smooth on $N(\Sigma)$ and letting $\tau \in H^0([D_\infty])$ be such that $Div(\tau) = D_\infty$, the images of the $\Omega^{0,i}(\widetilde{T})$'s as the subspaces of the $\Omega^{0,i}(\mathfrak{g}_{E^{\mathbb{C}}})$'s consisting of the elements of

$$< \frac{1}{\tau} > \otimes (\Omega^{0,i}(\mathfrak{p}) \oplus < \frac{1}{\sigma} > \otimes \Omega^{0,i}(\mathfrak{u}^-)),$$

smooth on $N(\Sigma) \cup N(D_\infty)$. Let h' and h'' be the metrics induced on the orthogonal summands of $\mathfrak{g}_{E^{\mathbb{C}}|\Sigma}$, let k be a metric on $[\Sigma]$ and define a new metric h on $\mathfrak{g}_{E^{\mathbb{C}}}$ by putting on $N^*(\Sigma)$

$$h = h' \oplus \frac{1}{|\sigma|_k^2} h'',$$

and extending it smoothly to the rest of $X \backslash \Sigma$. For the framed case let k' be a metric on $[D_\infty]$ and define a metric $\widetilde{h}$ by putting

$$\widetilde{h} = \frac{1}{|\tau|_{k'}^2} h.$$

Thinking of the $\Omega^{0,i}(T)$'s and $\Omega^{0,i}(\widetilde{T})$'s as being subspaces of the $\Omega^{0,i}(\mathfrak{g}_{E^{\mathbb{C}}})$'s we see that h and $\widetilde{h}$ induce bounded metrics on them and one can show that with this metric there is a good Hodge theory for the restricted $\overline{\partial}$–complexes (5.2). Note that these metrics have the same growth behaviour as metrics pushed forward from T or $\widetilde{T}$ would and if the right choices are made we can consider them as such so we are just recovering the ordinary Hodge theory for $\overline{\partial}_0$ on T or $\widetilde{T}$. Taking either point of view we therefore have the following proposition.

Proposition 5.3. *Let the spaces $\{\Omega^{0,i}(T)\}$ and $\{\Omega^{0,i}(\widetilde{T})\}$ be equipped with the fibre metrics h and $\widetilde{h}$ respectively. Then*

(i) *The resulting L_k^2–completions $L_k^2(\Omega^{0,i}(T))$ and $L_k^2(\Omega^{0,i}(\widetilde{T}))$, $k \geqslant 0$ enjoy the usual embedding, compactness and multiplication properties of the ordinary L_k^2 spaces.*

(ii) *For any $\overline{\partial}_0$ in $\mathcal{A}_{\mathfrak{p}}$ or $\widetilde{\mathcal{A}}_{\mathfrak{p}}$ the operators*

$$\overline{\partial}_0 \oplus \overline{\partial}_0^{*_h}, \quad \overline{\partial}_0 \oplus \overline{\partial}_0^{*_{\tilde{h}}}, \quad \Delta_{\overline{\partial}_0}^{h}, \quad \Delta_{\overline{\partial}_0}^{\tilde{h}},$$

enjoy the same elliptic theory as the analogous operators with trivial scalar coefficients.

(iii) *For any $\overline{\partial}_0$ in $\mathcal{H}_{\mathfrak{p}}$ or $\widetilde{\mathcal{H}}_{\mathfrak{p}}$ the complexes $\{\Omega^{0,i}(T), \overline{\partial}_0\}$ and $\{\Omega^{0,i}(\widetilde{T}), \overline{\partial}_0\}$ are elliptic complexes and have an L^2–Hodge theory. In particular the spaces of $\Delta^h_{\overline{\partial}_0}$ and $\Delta^{\tilde{h}}_{\overline{\partial}_0}$ – harmonic sections consist of smooth elements.*

We denote by $H^{0,i}(T), H^{0,i}(\widetilde{T})$ the cohomology groups for these complexes and by $\mathcal{H}^{0,i}(T), \mathcal{H}^{0,i}(\widetilde{T})$ the spaces of harmonic sections. Since the complexes (5.2) are fine resolutions of the sheaves $\mathcal{T}_{\mathcal{E}}$ and $\widetilde{\mathcal{T}}_{\mathcal{E}}$, we obtain that

$$\forall i, \qquad H^i(T) \cong H^{0,i}(T) \cong \mathcal{H}^{0,i}(T), \tag{5.4}$$

and similarly for $\widetilde{T}$. As for the case of ordinary holomorphic bundles, in order to obtain good, that is Hausdorff, deformations it is necessary to introduce some rigidity condition even in local families. To this purpose we introduce the following notions.

Definition 5.5.

(i) *A parabolic bundle $\mathcal{E}$ is said to be simple if $H^0(\mathcal{T}_{\mathcal{E}}) = Lie(Z(G^{\mathbb{C}}))$.*

(ii) *A framed parabolic bundle $\mathcal{E}$ is said to be simple if $H^0(\widetilde{\mathcal{T}}_{\mathcal{E}}) = 0$.*

(iii) *A parabolic $\overline{\partial}$–operator $\overline{\partial}_0 \in \mathcal{A}_{\mathfrak{p}}$ is simple if $Ker(\overline{\partial}_0) = Lie(Z(G^{\mathbb{C}}))$ and a parabolic framed $\overline{\partial}$–operator $\overline{\partial}_0 \in \widetilde{\mathcal{A}}_{\mathfrak{p}}$ is simple if $Ker(\overline{\partial}_0) = 0$.*

Notice the difference between (i) and (ii): in the case of framings the centre acts non–trivially so one has to exclude it. The local rigidification is then achieved by restricting to the simple sets. We denote by $\mathcal{A}^*_{\mathfrak{p}}$ and $\widetilde{\mathcal{A}}^*_{\mathfrak{p}}$ the subsets of the spaces defined in §2.4 consisting of the simple elements and similarly for $\mathcal{H}^*_{\mathfrak{p}}$ and $\widetilde{\mathcal{H}}^*_{\mathfrak{p}}$. A basic fact concerning these subspaces is that they are open.

Proposition 5.6. *The simple sets $\mathcal{A}^*_{\mathfrak{p}}$ and $\widetilde{\mathcal{A}}^*_{\mathfrak{p}}$ are open in $\mathcal{A}_{\mathfrak{p}}$ and $\widetilde{\mathcal{A}}_{\mathfrak{p}}$ respectively.*

Proof. It suffices to show that the subsets

$$\{\overline{\partial} \in \mathcal{A}_{\mathfrak{p}} | Ker(\overline{\partial}) \neq 0\}, \qquad \{\overline{\partial} \in \widetilde{\mathcal{A}}_{\mathfrak{p}} | Ker(\overline{\partial}) \neq 0\},$$

are closed in $\mathcal{A}_{\mathfrak{p}}$ and $\widetilde{\mathcal{A}}_{\mathfrak{p}}$ respectively. With Proposition 5.3 at hand this is a standard semi–continuity argument in both cases. We give it for the unframed case. Let $\{\overline{\partial}_i\} \subseteq \mathcal{A}_{\mathfrak{p}}$ be a sequence converging to $\overline{\partial}_0$ in $\mathcal{A}_{\mathfrak{p}}$ such that

$$\forall i, \quad \exists\, 0 \neq f_i \in \Omega^0(T), \qquad \overline{\partial}_i(f_i) = 0.$$

Consider the sequence $\{\lambda_i\}$ of the $\mathcal{C}^0$ norms of the f_i's, then

$$\forall i, \quad \forall k, \quad \exists\, C^k_i, \qquad \|f_i\|_{L^2_k} \leqslant C^k_i \lambda_i.$$

Since $\overline{\partial}_i \to \overline{\partial}_0$ in $\mathcal{C}^\infty$ it follows that $C^k_i \to C^k$ for all k. Fix some large k. The sequence f_i/λ_i is bounded in $L^2_{k+1}(\Omega^0(T))$ and of $\mathcal{C}^0$ norm 1 so some subsequence of it will have a

non–zero strong limit $f \in L^2_k(\Omega^0(T))$ such that $\overline{\partial}_0(f) = 0$. Elliptic regularity then gives that $f \in \Omega^0(T)$. $\square$

This implies on one hand that the corresponding subsets

$$\mathcal{B}^*_{\mathfrak{p}} = \mathcal{A}^*_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}}, \qquad \widetilde{\mathcal{B}}^*_{\mathfrak{p}} = \widetilde{\mathcal{A}}^*_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}},$$

are open in $\mathcal{B}_{\mathfrak{p}}$ and $\widetilde{\mathcal{B}}_{\mathfrak{p}}$, and on the other that $\mathcal{H}^*_{\mathfrak{p}}$, $\mathcal{H}^*_{\mathfrak{p}}$ are open in the integrable sets so that

$$\mathcal{M}^{P,*}_{k,c,\ell} = \mathcal{H}^*_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}}, \qquad \widetilde{\mathcal{M}}^{P,*}_{k,c,\ell} = \widetilde{\mathcal{H}}^*_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}}$$

are open in the moduli spaces. We call them the moduli spaces of simple quasi–parabolic and simple framed quasi–parabolic structures respectively.

We now define L^2_k versions $\mathcal{A}_{\mathfrak{p}}(k)$ and $\mathcal{G}_{\mathfrak{p}}(k)$ of $\mathcal{A}_{\mathfrak{p}}$ and $\mathcal{G}_{\mathfrak{p}}$ modelled on $L^2_k(\Omega^{0,1}(T))$ and $L^2_{k+1}(\Omega^0(T))$ respectively with $k \geqslant 3$. Then $\mathcal{G}_{\mathfrak{p}}(k)$ is a Hilbert complex Lie group acting holomorphically on $\mathcal{A}_{\mathfrak{p}}(k)$ and we denote by $\mathcal{B}_{\mathfrak{p}}(k)$ the set of equivalence classes of parabolic $\overline{\partial}$–operators of class L^2_k and $\mathcal{B}^*_{\mathfrak{p}}(k)$ the subset of simple elements. Similarly we define $\widetilde{\mathcal{B}}_{\mathfrak{p}}(k)$ and $\widetilde{\mathcal{B}}^*_{\mathfrak{p}}(k)$. By the above Proposition $\mathcal{B}^*_{\mathfrak{p}}(k)$ and $\widetilde{\mathcal{B}}^*_{\mathfrak{p}}(k)$ are open. Let

$$\mathcal{S}_k(T) = \{a \in L^p_k\Omega^{0,1}(T) \,|\, \overline{\partial}^{*_h}_0(a) = 0\}.$$

Using Proposition 5.3 it follows by the usual arguments ([Ko], pp 243-262 or [DK] pp 237-239), that if $\overline{\partial}_0$ is any simple parabolic $\overline{\partial}$–operator the natural projection

$$\pi : \overline{\partial}_0 + S_k(T) \to \mathcal{B}^*_{\mathfrak{p}}(k),$$

gives a homeomorphism of a neighbourhood of the origin in $S_k(T)$ with a neighbourhood of $[\overline{\partial}_0]$ in $\mathcal{B}^*_{\mathfrak{p}}(k)$ which has therefore locally the structure of a Banach complex manifold. Similarly for $\widetilde{\mathcal{B}}^*_{\mathfrak{p}}(k)$. While the latter will be shown to be a Hausdorff space under reasonably general assumptions, so that the local charts give a Banach complex manifold structure, $\mathcal{B}^*_{\mathfrak{p}}(k)$ usually fails to be Hausdorff so that the charts may not necessarily glue together.

The spaces $L^p_{k-1}(\Omega^{0,2}(T))$ and $L^p_{k-1}(\Omega^{0,2}(\widetilde{T}))$ are acted on via adjoint action by $\mathcal{G}_{\mathfrak{p}}(k)$ and $\widetilde{\mathcal{G}}_{\mathfrak{p}}(k)$ respectively so descend to Banach holomorphic vector bundles on $\mathcal{B}^*_{\mathfrak{p}}(k)$ and $\widetilde{\mathcal{B}}^*_{\mathfrak{p}}(k)$. The $\Omega^{0,i}(T)$'s and $\Omega^{0,i}(\widetilde{T})$'s are closed under Lie brackets so they are preserved by the integrability map $\overline{\partial} \mapsto \overline{\partial}^2$, which therefore descends to a holomorphic section

$$\Psi : [\overline{\partial}] \mapsto \overline{\partial}^2, \tag{5.7}$$

of the above bundles and the moduli spaces (5.1) now appear as its zero set $Z(\Psi)$. We can thus equip them with the (possibly non–Hausdorff) Banach ringed space structure of the ideal sheaf of Ψ. Since $\mathcal{A}_{\mathfrak{p}}$ and $\widetilde{\mathcal{A}}_{\mathfrak{p}}$ are dense in $\mathcal{A}_{\mathfrak{p}}(k)$, $\widetilde{\mathcal{A}}_{\mathfrak{p}}(k)$ the argument of Proposition 4.2.16 of [DK] using the elliptic theory given by Proposition 5.3 shows that $Z(\Psi)$ is represented by smooth elements, so that its definition as a set is independent of k, and Ψ is a Fredholm map at $[\overline{\partial}_0]$, so that in fact $Z(\Psi)$ is finite dimensional and admits

finite dimensional models. These consists of representatives for the cohomology groups of the complexes (5.2) and holomorphic maps

$$\psi : H^{0,1}(T) \to H^{0,2}(T), \quad \text{or} \quad \psi : H^{0,1}(\widetilde{T}) \to H^{0,2}(\widetilde{T}),$$

between them such that the ringed space structure on $Z(\Psi)$ at $[\overline{\partial}_0]$ is equivalent to the germ of complex space at the origin given by $\mathcal{I}_\psi$, the ideal sheaf of ψ,

$$(Z(\Psi), [\overline{\partial}_0]) \cong (\psi^{-1}(0), \mathcal{O}_{H^{0,1}}/\mathcal{I}_\psi, 0).$$

In particular we can take the model obtained by considering the lifting

$$\mathcal{Q} : S(T) \to \Omega^{0,2}(T), \qquad \mathcal{Q}(a) = \overline{\partial}_0 a + a \wedge a. \tag{5.8}$$

This is equivalent by a biholomorphic map at the origin in $\Omega^{0,1}(T)$, θ, to the zero set Y of a holomorphic map

$$\psi : \mathcal{H}^{0,1}(T) \longrightarrow \mathcal{H}^{0,2}(T).$$

As noticed in Proposition 5.3 the harmonic spaces $\mathcal{H}^{0,i}(T)$ consist of smooth elements showing that the ringed space structure too is independent of k. Similarly in the case of framings. When $G^{\mathbb{C}}$ is reductive but not semi–simple the adjoint representation decomposes as $\mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}}$ where $\mathfrak{g}_0^{\mathbb{C}}$ is the semi–simple part and $\mathfrak{t}^{\mathbb{C}} = Z(Lie(\mathfrak{g}^{\mathbb{C}}))$ so that for a holomorphic $G^{\mathbb{C}}$–bundle $\mathcal{E}$, $\mathfrak{g}_\mathcal{E}$ splits holomorphically as $(\mathfrak{g}_\mathcal{E})_0 \oplus (\bigoplus \mathcal{O}_X)$. Correspondingly $\mathcal{T}_\mathcal{E}$ has a distinguished direct summand $(\mathcal{T}_\mathcal{E})_0$ and in particular, when $G^{\mathbb{C}} = GL(n)$ the defining sequence of $\mathcal{T}$ gives

$$\mathcal{T}_\mathcal{E} = (\mathcal{T}_\mathcal{E})_0 \oplus \mathcal{O}, \qquad T = T_0 \oplus \mathbb{C},$$

so that the complex (5.2) splits up into the direct sum of $\{\Omega^{0,i}(T_0), \overline{\partial}_0\}$ and the scalar Dolbeault complex. The bundle of T–valued $(0,2)$–forms on $\mathcal{B}^*$ decomposes accordingly and letting Ψ_0 be Ψ followed by projection onto $\Omega^{0,2}(T_0)$,

$$\Psi^{-1}(0) = \Psi_0^{-1}(0).$$

It follows that the ringed space structure of the moduli spaces is in fact that of the zero set of Ψ_0 and a local model for the moduli spaces is in this case given by the ideal sheaf structure at the origin of a holomorphic map

$$\psi : \mathcal{H}^{0,1}(T) \longrightarrow \mathcal{H}^{0,2}(T_0).$$

We thus have the following Proposition.

Proposition 5.9. *Let (X, Σ, D_∞) a complex triple as specified at the beginning of the Chapter, $G^{\mathbb{C}}$ a complex simple Lie group or $GL(n)$ and P a parabolic subgroup of $G^{\mathbb{C}}$.*

(i) *The moduli space of simple P–quasi–parabolic bundles of a given topological type on (X, Σ), $\mathcal{M}^{P,*}_{k,c,\ell}$, is locally equipped with a natural complex space structure as the zero set of the section (5.7).*

(ii) *At a point* $[\overline{\partial}_0]$ *this is isomorphic to the ideal sheaf structure at the origin of a holomorphic map*

$$\psi : H^1(\mathcal{T}_\mathcal{E}) \to H^2((\mathcal{T}_\mathcal{E})_0),$$

where $\mathcal{E} = (E^{\mathbb{C}}, \overline{\partial}_0)$, *so that* $H^1(\mathcal{T}_\mathcal{E})$ *can be identified with the Zariski tangent space to* $\mathcal{M}^{P,*}_{k,c,\ell}$ *at* $[\overline{\partial}_0]$ *and if* $H^2((\mathcal{T}_\mathcal{E})_0)$ *vanishes* $\mathcal{M}^{P,*}_{k,c,\ell}$ *has a complex manifold structure near* $[\overline{\partial}_0]$.

(iii) *The same holds for the moduli space of simple framed quasi–parabolic bundles on* (X, Σ, D_∞), $\widetilde{\mathcal{M}}^{P,*}_{k,c,\ell}$, *with* $\widetilde{\mathcal{T}}_\mathcal{E}$ *in place of* $\mathcal{T}_\mathcal{E}$.

Analogous statements hold for a general complex reductive Lie group.

We next turn to the question of Hausdorfness. As we already remarked, at the global level simpleness on its own may not guarantee the existence of Hausdorff moduli spaces. In the framed case a natural way to obtain a global rigidification is to assume some rigidity on the framing object. This can be brought to bear via the following Proposition analogous to those of [Le] and [Lü] for the case of ordinary holomorphic bundles. Let $\mathcal{T}^0$ denote the sheaf of infinitesimal parabolic automorphism of the framing bundle $\mathcal{E}^0$. Let $\overline{\partial}'$ and $\overline{\partial}''$ be elements in $\mathcal{A}_\mathfrak{p}$ and let $\overline{\partial}_0$ be a base point so that $\overline{\partial}' = \overline{\partial}_0 + a', \overline{\partial}'' = \overline{\partial}_0 + a''$. We define the operator $\overline{\partial}' \otimes \overline{\partial}''$ on $\Omega^0(End(T))$ by

$$\overline{\partial}' \otimes \overline{\partial}''(f) = \overline{\partial}_0(f) + [a', -] \circ f + f \circ [-, a''].$$

Similarly for any two elements in $\widetilde{\mathcal{A}}_\mathfrak{p}$ we can define $\overline{\partial}' \otimes \overline{\partial}''$ on $End(\widetilde{T})$. In either case the kernel sheaf, $\mathcal{K}er(\overline{\partial}' \otimes \overline{\partial}'')$ is a sheaf of $\mathcal{O}_X$–modules whose group of global section, $Ker(\overline{\partial}' \otimes \overline{\partial}'')$, has finite dimension since $\overline{\partial}' \otimes \overline{\partial}''$ is elliptic overdetermined. We denote by $Ker(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty]$ the global sections vanishing along D_∞.

Proposition 5.10.

(i) *If for all* $k \geqslant 1$,

$$H^0(\mathcal{T}^0[-kD_\infty]_{|D_\infty}) = 0, \tag{5.11}$$

then for every framed parabolic $\overline{\partial}$*–operator* $Ker(\overline{\partial})[-D_\infty] = 0$ *and for any two such operator* $\overline{\partial}'$, $\overline{\partial}''$, $Ker(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty] = 0$

(ii) *If* $D_\infty \cdot D_\infty \geqslant 0$ *then condition (5.11) is equivalent to* $H^0(\mathcal{T}^0[-D_\infty]_{|D_\infty}) = 0$.

Proof. (i) Since $\overline{\partial}'_{|D_\infty} = \overline{\partial}''_{|D_\infty} = \overline{\partial}^0$,

$$\mathcal{K}er(\overline{\partial}' \otimes \overline{\partial}'')_{|D_\infty} \cong End(\mathcal{T}^0).$$

Notice that $\mathcal{T}^0$ is a subsheaf of $End(\mathcal{T}^0)$ so we can define the sheaf $\mathcal{K}er^0(\overline{\partial}' \otimes \overline{\partial}'')$ as the kernel of the sequence

$$0 \longrightarrow \mathcal{K}er^0(\overline{\partial}' \otimes \overline{\partial}'') \longrightarrow \mathcal{K}er(\overline{\partial}' \otimes \overline{\partial}'') \xrightarrow{r_{|D_\infty}} End(\mathcal{T}^0)/\mathcal{T}^0 \longrightarrow 0.$$

We then have the exact sequence

$$0 \longrightarrow \mathcal{K}er^0(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty] \longrightarrow \mathcal{K}er^0(\overline{\partial}' \otimes \overline{\partial}'') \xrightarrow{r_{|D_\infty}} \mathcal{T}^0 \longrightarrow 0.$$

Tensoring by $[-kD_\infty]$ and taking cohomology (5.14) then gives isomorphisms

$$\forall k \geqslant 1, \qquad Ker^0(\overline{\partial}' \otimes \overline{\partial}'')[-kD_\infty]) \cong Ker^0(\overline{\partial}' \otimes \overline{\partial}'')[-(k+1)D_\infty]).$$

If $Ker^0(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty] \neq 0$ it follows that $\mathcal{K}er^0(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty]$ has global sections vanishing to any order along D_∞, in particular $Ker^0(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty]$ must have infinite dimension which is impossible. Thus $Ker^0(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty] = 0$ and since

$$Ker^0(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty] = Ker(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty],$$

also $Ker(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty] = 0$. Taking $\overline{\partial}' = \overline{\partial}''$ we see in particular that $\overline{\partial}'$ must be a simple parabolic framed $\overline{\partial}$–operator.
(ii) If $D_\infty \cdot D_\infty \geqslant 0$ then for all k there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{T}^0[-(k+1)D_\infty]_{|D_\infty} \longrightarrow \mathcal{T}^0[-kD_\infty]_{|D_\infty} \longrightarrow \mathcal{T}^0[-kD_\infty]_{|k(D_\infty \cdot D_\infty)} \longrightarrow 0,$$

thus taking cohomology we see that

$$\forall k, \qquad H^0(\mathcal{T}^0[-(k+1)D_{\infty|D_\infty}]) \geqslant H^0(\mathcal{T}^0[-kD_{\infty|D_\infty}]),$$

so condition (5.11) holds if and only if $H^0(\mathcal{T}^0[-D_\infty]_{|D_\infty}) = 0$. □

In the unframed case for $G^{\mathbb{C}} = GL(n)$, we now fix the choice of an element α of the form (1.23) such that $P = P(\alpha)$, then a P–quasi–parabolic bundle can be regarded as an α–parabolic vector bundle and $\mathcal{M}^P_{k,c,\ell}$ has the distinguished subset consisting of the stable bundles, $\mathcal{M}^{\alpha,S}_{k,c,\ell}$. The next proposition proves a key property of stable α–parabolic bundles that guarantees Hausdorfness of the stable set $\mathcal{M}^{\alpha,S}_{k,c,\ell}$.

Proposition 5.12.

(i) *Let $\mathcal{V}'$, $\mathcal{V}''$ be stable α–parabolic bundles with the same degree then every non–zero parabolic morphism $\psi : \mathcal{V}' \longrightarrow \mathcal{V}''$ is a parabolic isomorphism.*
(ii) *Every stable α–parabolic bundle is simple.*

Proof. (i) Assume that $\psi : \mathcal{V}' \longrightarrow \mathcal{V}''$ is a non–zero parabolic morphism as defined in (2.43) and consider the induced morphism $\det(\psi) : \det(\mathcal{V}') \longrightarrow \det(\mathcal{V}'')$. Since $\mathcal{V}'$ and $\mathcal{V}''$ have the same degree $\det(\psi)$ is a global holomorphic section of a line bundle of degree zero hence must be either nowhere vanishing, in which case ψ is an isomorphism, or identically zero. It thus suffices to show that ψ is an isomorphism at at least one point. Tensoring by suitable line bundles we can assume that ψ vanishes only at isolated points. The image $\mathcal{I}m(\psi)$ is then a torsion–free subsheaf of $\mathcal{V}''$ of rank $\geqslant 1$ such that $\mathcal{V}''/\mathcal{I}m(\psi)$ is torsion–free. If $rk(\psi)$ is maximum at some point then we are done so we can assume that

$rk(\mathcal{I}m(\psi)) \leqslant rk(\mathcal{V}'') - 1$. Since $\mathcal{V}''/\mathcal{I}m(\psi)$ is torsion–free $\mu_\alpha(Im(\psi), \mathcal{V}'')$ is defined and by stability of $\mathcal{V}''$

$$\mu_\alpha(\mathcal{V}'', \mathcal{I}m(\psi)) - \mu_\alpha(\mathcal{V}'') < 0.$$

On the other hand $\mathcal{I}m(\psi)$ is a torsion–free quotient sheaf of $\mathcal{V}'$ so $\mu_\alpha(\mathcal{V}', \mathcal{I}m(\psi))$ is defined and since $\mathcal{V}'$ is stable it follows by Proposition 2.40 that

$$\mu_\alpha(\mathcal{V}', \mathcal{I}m(\psi)) - \mu_\alpha(\mathcal{V}') > 0.$$

However, since ψ is a parabolic morphism one can check that

$$\mu_\alpha(V', \mathcal{I}m(\psi)) \leqslant \mu_\alpha(\mathcal{I}m(\psi), V''),$$

which implies that

$$\mu_\alpha(\mathcal{V}') < \mu_\alpha(\mathcal{V}'').$$

Since by the assumptions $\mu_\alpha(\mathcal{V}') = \mu_\alpha(\mathcal{V}'')$ we obtain a contradiction thus ψ must have been an isomorphism.
(ii) Let $\mathcal{V}$ be a stable α–parabolic bundle and let $0 \neq \psi \in End(\mathcal{V})$. Pick any point $\mathrm{p} \in X$ such that $\psi(\mathrm{p}) \neq 0$ and let λ be one of the non–zero eigenvalues of $\psi(\mathrm{p})$. By (i) it follows that

$$\psi - \lambda 1,$$

must vanish identically since it vanishes at p thus $\psi = \lambda 1$. □

Proposition 5.13.

(i) *If condition (5.11) is satisfied then $\widetilde{\mathcal{B}}^*$ is a Hausdorff space.*
(ii) *The α–stable set $\mathcal{M}^{\alpha,S}_{k,c,\ell}$ is a Hausdorff space.*

Proof. In either case the argument is the standard one proving that the diagonal is closed in the product. We show that if the sequence $([\overline{\partial}'_i], [\overline{\partial}''_i])$ converges to $([\overline{\partial}'], [\overline{\partial}''])$ then $[\overline{\partial}'] = [\overline{\partial}'']$.
(i) By assumption we can find sequences

$$\{\overline{\partial}'_i\} \subseteq \widetilde{\mathcal{A}}_{\mathfrak{p}}, \qquad \{f_i = Ad(g_i)\} \subseteq \widetilde{\mathcal{G}}_{\mathfrak{p}},$$

such that

$$\overline{\partial}'_i \to \overline{\partial}', \qquad \overline{\partial}''_i = f_i^*(\overline{\partial}'_i) \to \overline{\partial}'', \tag{5.14}$$

in the $\mathcal{C}^\infty$ topology. Consider the sequence $\{\lambda_i\}$ of the $\mathcal{C}^0$ norms of the f_i's. Since the $\overline{\partial}$–operators $\overline{\partial}'_i \otimes \overline{\partial}''_i$ on $End(\widetilde{T})$ converge in $\mathcal{C}^\infty$ to $\overline{\partial}' \otimes \overline{\partial}''$, the same argument as in Proposition 5.9 shows that if $\{\lambda_i\}$ or some subsequence of it is bounded, the f_i will have a $\mathcal{C}^\infty$–limit f and since

$$\forall i \qquad \det(f_i) = 1, \qquad f_{i|D_\infty} = 1,$$

f also satisfies these conditions. Furthermore since the inner automorphism of a Lie algebra $\mathfrak{g}$ are closed in $GL(\mathfrak{g})$ this isomorphism is induced by a gauge transformation on the principal bundle which implies that $[\overline{\partial}'] = [\overline{\partial}'']$. We can therefore assume that $\lambda_i \to \infty$. The sequence $f_i' = f_i/\lambda_i$ is then a bounded $\mathcal{C}^0$–sequence of solutions to the same elliptic equation as the f_i's so again by the elliptic estimate it converges in $\mathcal{C}^\infty$ to f'. Since the f_i' have $\mathcal{C}^0$ norms 1 f' is non–zero, further, since the f_i's were 1 on D_∞ and since $\lambda_i \to \infty$, it follows that f' vanishes on D_∞ so that

$$0 \neq f' \in Ker(\overline{\partial}' \otimes \overline{\partial}'')[-D_\infty],$$

contradicting (i) of Proposition 5.10.

(ii) Let $\overline{\partial}'$ and $\overline{\partial}''$ be the $\overline{\partial}$–operators corresponding to two stable α–parabolic bundles. We consider the corresponding sequences of the (5.15) and arguing as in (i) we obtain a non–zero parabolic morphism $f' : \mathcal{V}' \to \mathcal{V}''$. By Proposition 5.12 f' must be a parabolic isomorphism giving again a contradiction. □

By (ii) of Proposition 5.12 we know that every stable α–parabolic bundle is simple so that $\mathcal{M}^{\alpha,S}_{k,c,\ell} \subseteq \mathcal{M}^{P,*}_{k,c,\ell}$. If we knew that $\mathcal{M}^{\alpha,S}_{k,c,\ell}$ was actually open it would follow that the local complex space structure given at a point in $\mathcal{M}^{P,*}_{k,c,\ell}$ by Proposition 5.9 is actually a complex structure on $\mathcal{M}^{\alpha,S}_{k,c,\ell}$. We thus need to prove the openness of the stable set and we shall do this using the comparison between the deformation theory of singular instantons and that of stable parabolic bundles. This will be carried out in part in Chapter 7 so we postpone until then the proof of the following proposition (see Corollary 7.20 and the remarks following it).

Proposition 5.15. *If $\Sigma \cdot \Sigma \geqslant 0$ the stable set $\mathcal{M}^{\alpha,S}_{k,c,\ell}$ is open in $\mathcal{M}^{P,*}_{k,c,\ell}$.*

We finally show that the quotients (5.1) parametrise locally universal families. For this we need to introduce the notion of based family of quasi–parabolic structures (framed quasi–parabolic structures) and quasi–parabolic bundles (framings). Let (Z, z_0) be a based germ of complex space so that there is an open polydisc D^n at the origin in $\mathbb{C}^n$ and a holomorphic function $f : D^n \to \mathbb{C}^m$ such that as a ringed space

$$(Z, z_0) \cong (f^{-1}(0), \mathcal{O}/\mathcal{I}_f, 0).$$

Let $\overline{\partial}_0$ be a P–quasi–parabolic structure (framed quasi–parabolic structure) on $E^{\mathbb{C}}$, $\mathbf{E}^{\mathbb{C}}$ the pull–back of $E^{\mathbb{C}}$ to $D^n \times X$ so that we obtain a $\mathcal{C}^\infty$ P–quasi parabolic bundle on $(D^n \times X, D^n \times \Sigma)$ and the adjoint bundle $\mathfrak{g}_{\mathbf{E}^{\mathbb{C}}}$ has a subbundle of parabolic lie algebras $\mathfrak{P}$ on $D^n \times \Sigma$. Pull–back $\overline{\partial}_0$ to a $\overline{\partial}$–operator on $\mathfrak{g}_{\mathbf{E}^{\mathbb{C}}}$. By a deformation of P–quasi parabolic structures (framings) at $\overline{\partial}_0$ parametrised by (Z, z_0) we mean a smooth section

$$a \in \Omega^{0,1}(D^n \times X, \mathfrak{g}_{\mathbf{E}^{\mathbb{C}}}),$$

such that

$$\overline{\partial} a + a \wedge a = 0, \quad mod\ \mathcal{I}_Z, \qquad a_{|\{0\}\times X} = 0, \qquad \iota^*_{D^n \times \Sigma}(a) \in \Omega^{0,1}(\mathfrak{P}),$$

(and also $\iota^*_{D^n\times D_\infty}(a)=0$ in the framed case) and the map

$$Z\to\Omega^{0,1}(X,\mathfrak{g}_{E^{\mathbb{C}}}),\qquad z\mapsto a_z=a_{|\{z\}\times X},$$

is holomorphic. By a deformation of P–quasi parabolic bundles (framings) at $\mathcal{E}_0$ parametrised by (Z,z_0) we mean a $\mathcal{C}^\infty$ cocycle of transition functions $\mathbb{E}=\{g_{ij}\}$ for the $\mathcal{C}^\infty$ P–quasi–parabolic bundle $\mathbf{E}^{\mathbb{C}}$, that is

$$\{g_{ij}\}_{|D^n\times\Sigma}\in P,$$

(and also $\{g_{ij}\}_{|D^n\times D_\infty}=1$ in the framed case) such that

$$\mathbb{E}_{|\{0\}\times X}\cong\mathcal{E}_0,\qquad\{i^*_{Z\times X}(g_{ij})\}\in G^{\mathbb{C}}(\mathcal{O}_Z).$$

Proposition 5.16. *Let $[\overline{\partial}_0]$ be a simple quasi–parabolic (framed quasi–parabolic) structure, then a neighbourhood Y ($\widetilde{Y}$) of $[\overline{\partial}_0]$ in $\mathcal{M}^{P,*}_{k,c,\ell}$ ($\widetilde{\mathcal{M}}^{P,*}_{k,c,\ell}$) is the basis for the universal deformation of P–quasi–parabolic (framed quasi–parabolic) bundles at $\mathcal{E}_0=(E^{\mathbb{C}},\overline{\partial}_0)$.*

Proof. We first check that the family parametrised by Y actually exists ad that it has the property that its restriction to each $\{y\}\times X$ slice is isomorphic to the quasi–parabolic bundle (framing) corresponding to y. Pulling–back $E^{\mathbb{C}}$ to a $\mathcal{C}^\infty$ P–quasi–parabolic bundle $\mathbf{E}^{\mathbb{C}}$ on $(Y\times X,\,Y\times\Sigma)$ this amounts to finding a cocycle of holomorphic transition functions for $\mathbf{E}^{\mathbb{C}}$ on $Y\times X$ lying in P on $Y\times\Sigma$ (and equal to the identity on $Y\times D_\infty$ in the case of framings) whose restriction to each $\{y\}\times X$ slice coincides with the quasi–parabolic (framed quasi–parabolic bundle) defined by $\overline{\partial}_y$.
The essence of the argument, as explained in [DK] p.230, consists in extending the integrability theorem for families of $\overline{\partial}$–operators on vector bundles to the case of non reduced deformations. This is done in [My] and we use its methods applied to the adjoint bundle in the contest of parabolic bundles. We refer to that paper for more details. Take the ideal sheaf at the origin of (5.8) as a model for Y so that Y is a germ of complex space at the origin of the harmonic space $\mathcal{H}^{0,1}(T)(\mathcal{H}^{0,1}(\widetilde{T}))$ and denote by D^n a polydisc at the origin in this. The above description of $\Omega^{0,1}(T)$ shows that we have a deformation $a(y)=\theta^{-1}(y)$, $y\in\mathcal{H}^1(T)$, of P–quasi–parabolic structures at $\overline{\partial}_0$ on $\mathbf{E}^{\mathbb{C}}$ parametrised by $(\psi^{-1}(0),0)\subseteq(D^n,0)$ with the extra property that

$$a_{\mathrm{u}\text{-}}\in\mathcal{I}_{D^n\times\Sigma},$$

(and $a\in\mathcal{I}_{D^n\times D_\infty}$ for framings). Since a is holomorphic in the Y direction it follows by Propositon 2.3 of [My] that for a small disc Δ in X we can find a complex gauge transformation g on $D^n\times\Delta$ such that

$$g(\overline{\partial}_0+a)=\overline{\partial}_0+\widetilde{a},\qquad\widetilde{a}=0\ mod\ \mathcal{I}_{Y\times\Delta},$$

and since a is $\mathfrak{g}^{\mathbb{C}}$–valued and satisfies the above extra condition we can choose g to be $Ad(G^{\mathbb{C}})$–valued and such that

$$g=1+f,\qquad f_{\mathrm{u}\text{-}}\in\mathcal{I}_{Y\times\Sigma}.$$

The analytic restriction to $Y \times \Sigma$ of the transition functions between two such holomorphic trivialisation on product neighbourhoods $Y \times \Delta_i$ with Δ_1 and Δ_2 intersecting non–trivially on Σ is then manifestly a $\mathfrak{p}$–valued holomorphic function for the induced complex space structure on $Y \times \Sigma$ giving the required parabolic structure. In the case of framings we use the above description of $\Omega^{0,1}(\widetilde{T})$ to obtain a deformation of framed parabolic structures at $\overline{\partial}_0$ with the extra condition that

$$a \in \mathcal{I}_{D^n \times D_\infty},$$

and then g can be chosen to be of the form

$$g = 1 + f, \qquad f \in \mathcal{I}_{Y \times D_\infty}, \quad f_{\mathfrak{u}^-} \in \mathcal{I}_{Y \times \Sigma}.$$

This gives the required cocycles.

To prove local universality let $\mathbb{E}$ be a deformation of P– quasi–parabolic (framed quasi–parabolic) bundles at $\mathcal{E}_0$. On each $\{z\} \times X$ slice $i^*_{\{z\} \times X}(\mathbb{E})$ is a holomorphic vector bundle hence determines a holomorphic structure $\overline{\partial}_z = \overline{\partial}_0 + a_z$ defining a P–quasi–parabolic bundle (framing). By the holomorphicity of $\mathbb{E}$ it follows again by Propositon 2.3 of [My] that $\{\overline{\partial}_z\}$ induces a deformation of P–quasi–parabolic structures at $\overline{\partial}_0$ parametrised by some subset of $(\mathcal{Q}^{-1}(0) \cap U, \mathcal{O}_U/\mathcal{I}_\mathcal{Q}, 0)$ where U is a neighbourhood at the origin in $S(T)$ $(S(\widetilde{T}))$. Using the extension of the integrability theorem outlined above this induces a deformation of parabolic bundles (framings) isomorphic to $\mathbb{E}$. It follows that the assignment

$$z \mapsto \overline{\partial}_z$$

induces a holomorphic map

$$(Z, z_0) \to (Y, 0)$$

pulling back to $\mathbb{E}$ the family constructed above for $(Y, 0)$, which is unique if $[\overline{\partial}_0]$ defines a simple quasi–parabolic bundle(framing). □

Putting together the last few propositions we obtain the global counterpart of 5.9.

Theorem 5.17. *Let (X, Σ, D_∞) be a complex triple as at the beginning of the Chapter, $G^{\mathbb{C}}$ a complex simple Lie group or $GL(n)$ and P a parabolic subgroup of it.*

(i) *If (X, ω) is Kähler and Σ is such that $\Sigma \cdot \Sigma \geqslant 0$, the set of stable α–parabolic vector bundles on (X, Σ), $\mathcal{M}^{\alpha, S}_{k,c,\ell}$ has a natural complex space structure which is a coarse moduli space for stable α–parabolic bundles of the given topological type.*

(ii) *If for all $k \geqslant 1$, $\mathbf{H}^0(\mathcal{T}^0[-kD_\infty]_{|D_\infty}) = 0$, the moduli space of $\mathcal{E}^0$–quasi parabolic parabolic framings on (X, Σ, D_∞), $\widetilde{\mathcal{M}}^P_{k,c,\ell}$, has a natural complex space structure which is a coarse moduli space for parabolic framings of the given topological type.*

A statement analogous to (ii) holds also for general complex reductive groups.

Proof. (i) Given the local result 5.9, the existence of the required global complex space structure follows from 5.13 and 5.15. As for the last claim, observe that given 5.16 and the fact that $\mathcal{M}^{\alpha, S}_{k,c,\ell}$ has the bijective property on reduced points for a coarse moduli functor, it is mechanical to check that $\mathcal{M}^{\alpha, S}_{k,c,\ell}$ must be a coarse moduli space.

(iii) By Proposition 5.13 (i) we know that under this assumptions $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}^*$ is Hausdorff so $\widetilde{\mathcal{M}}^P_{k,c,\ell} = \widetilde{\mathcal{M}}^{P,*}_{k,c,\ell}$ also is, which together with the local result of 5.9 shows that it has the required complex space structure. That this is a coarse moduli space follows from 5.16 for the same reasons as in (i). □

Before we progress to study some examples of moduli spaces we give two more elements of general theory. A formula for the dimension of the moduli space at smooth points and a criterion that guarantees smoothness in a number of interesting cases.

Proposition 5.18. *Let (X, Σ, D_∞) be a complex triple as at the beginning of the Chapter, $G^{\mathbb{C}}$ a complex simple Lie group or $GL(n)$, P a parabolic subgroup of it and also assume that X is a Kähler surface. Then*

(i) *At any point $[\overline{\partial}_0]$ such that $H^2((\mathcal{T}_\mathcal{E})_0)$ vanishes, $\mathcal{M}^{P,*}_{k,c,\ell}$ equipped with the natural complex space structure of Proposition 5.9 is locally a complex manifold of dimension*

$$d = 2C_{G^{\mathbb{C}}}k - \frac{dim(G^{\mathbb{C}})}{2}(1 - b^1(X) + b^+(X)) + C_P(\ell) + dim(G^{\mathbb{C}}/P)\frac{\chi(\Sigma)}{2},$$

when $G^{\mathbb{C}}$ is simple and

$$d + (1 + h^{0,2}(X)) + C_P(c \cdot \Sigma),$$

when $G^{\mathbb{C}} = GL(n)$, with $C_{G^{\mathbb{C}}} = C_{SL(n)}$.

(ii) *At any point $[\overline{\partial}_0]$ such that $H^2((\widetilde{\mathcal{T}}_\mathcal{E})_0)$ vanishes, $\widetilde{\mathcal{M}}^{P,*}_{k,c,\ell}$ equipped with the natural complex space structure of Proposition 5.9 is locally a complex manifold of dimension*

$$\widetilde{d} = d + dim(G^{\mathbb{C}})\frac{\chi(D_\infty)}{2} - \sum_{p \in \Sigma \cap D_\infty} dim(G^{\mathbb{C}}/P),$$

when $G^{\mathbb{C}}$ is simple and

$$\widetilde{d} + h^{0,2}(X) + C_P(c \cdot \Sigma),$$

when $G^{\mathbb{C}} = GL(n)$.

Here $C_{G^{\mathbb{C}}}$ is a positive integer depending on $G^{\mathbb{C}}$ only (see table (5.20)) and C_P is the element of $[\Sigma, G^{\mathbb{C}}/P]^*$ of Lemma 2.18.

Proof. By Proposition 5.9 (ii) we need to compute $H^1(\mathcal{T})$ and $H^1(\widetilde{\mathcal{T}}) = H^1(\mathcal{T}[-D_\infty])$. Under the given assumptions one has

$$dim(H^1(\mathcal{T})) = -\chi(\mathcal{T}), \qquad dim(H^1(\mathcal{T}[-D_\infty]) = -\chi(\mathcal{T}[-D_\infty]),$$

for the simple complex Lie groups and

$$dim(H^1(\mathcal{T})) = -\chi(\mathcal{T}) + 1 + h^{0,2}(X), \qquad dim(H^1(\mathcal{T}[-D_\infty]) = -\chi(\mathcal{T}[-D_\infty]) + h^{0,2}(X),$$

when $G^{\mathbb{C}} = GL(n)$. By the additivity of the Euler characteristic in short exact sequences of sheaves, it follows from (2.27) that

$$-\chi(\mathcal{T}) = -\chi(\mathfrak{g}_{\mathcal{E}}) + \chi(\mathfrak{g}_{\mathcal{E}}/\mathfrak{p}).$$

Hirzebruch–Riemann–Roch now gives

$$-\chi(\mathfrak{g}_{\mathcal{E}}) = c_2(\mathfrak{g}_{\mathcal{E}}) - (dim(G^{\mathbb{C}}))\chi(\mathcal{O}_X),$$

$$\chi(\mathfrak{g}_{\mathcal{E}}/\mathfrak{p}) = -\deg(\mathfrak{p}) + dim(G^{\mathbb{C}}/P)\frac{\chi(\Sigma)}{2}.$$

If X is Kähler $2\chi(\mathcal{O}_X) = 1 - b^1(X) + b^+(X)$. It can be shown that $c_2(\mathfrak{g}_{\mathcal{E}}) = 2C_{G^{\mathbb{C}}}k$, where k is the 4–dimensional class of $\mathcal{E}$ and $C_{G^{\mathbb{C}}}$ is a constant depending only on $G^{\mathbb{C}}$ and not on k or on the base manifold ([AHS], below we recall the values of $C_{G^{\mathbb{C}}}$) so the formula for d follows from Lemmas 2.18 and 2.20 according to which

$$\deg(\mathfrak{p}) = -C_P(\ell),$$

when $G^{\mathbb{C}}$ is simple and

$$\deg(\mathfrak{p}) = -C_P(\ell + c \cdot \Sigma),$$

if $G^{\mathbb{C}} = GL(n)$. In the framed case we have the exact sequence of sheaves

$$0 \longrightarrow \widetilde{\mathcal{T}} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}^0 \longrightarrow 0,$$

with $\mathcal{T}^0$ satisfying the exact sequence on D_∞

$$0 \longrightarrow \mathcal{T}^0 \longrightarrow \mathfrak{g}_{\mathcal{E}^0} \longrightarrow \bigoplus_{p \in \Sigma \cap D_\infty} \mathfrak{g}_{\mathcal{E}^0}/\mathfrak{p} \longrightarrow 0.$$

It follows that

$$\chi(\mathcal{T}^0) = \chi(\mathfrak{g}_{\mathcal{E}^0}) - \chi(\bigoplus_{p \in \Sigma \cap D_\infty} \mathfrak{g}_{\mathcal{E}^0}/\mathfrak{p}) = dim(G^{\mathbb{C}})\frac{\chi(D_\infty)}{2} - \sum_{p \in \Sigma \cap D_\infty} dim(G^{\mathbb{C}}/P).$$

Substituting this into

$$-\chi(\widetilde{\mathcal{T}}) = -\chi(\mathcal{T}) + \chi(\mathcal{T}^0),$$

we obtain the required formula. □

We shall now compute explicitly the terms depending on the topology of the bundle and the P–reduction in the dimension formula for the simple groups in the case where $P = B$ is a Borel subgroup and then when $G^{\mathbb{C}} = SL(n)$ (or $GL(n)$)for all parabolic subgroups.

Example 5.19. We assume that $G^{\mathbb{C}}$ is a simple complex Lie group and $P = B$ is a Borel subgroup. We express ℓ with respect to the basis (2.5) for $[\Sigma, G^{\mathbb{C}}/P]$ as

$$\ell = \sum_{\beta_i \in R_+^s} \ell_i(-H_{\beta_i}),$$

where $\{\beta_i\}$ are the simple roots for a positive system R_+ and we are identifying H_{β_i} with the degree one elements in $[S^2, S^2_{\beta_i}]$. For a simple root β_i one can check directly (or see [Bou] p.168 Proposition 29 (iii)) that denoting by $<,>$ the dual of the Killing form

$$\sum_{\beta \in R_+} <\beta, \beta_i> = <\beta_i, \beta_i> .$$

Substituting this in the formula for $\deg(\mathfrak{p})$ given by Lemma 2.18, where in this case since P is a Borel subgroup $I_{\mathfrak{p}} = R_+$, we obtain that

$$\begin{aligned} C_P(\ell) &= \sum_{\beta \in R_+} i\beta(\ell) = \sum_i \ell_i \sum_{\beta \in R_+} i\beta(-H_{\beta_i}) \\ &= -\sum_i \ell_i \sum_{\beta \in R_+} 2 <\beta_i, \beta_i>^{-1} i\beta(iJ_{\beta_i}) \\ &= \sum_i \ell_i \sum_{\beta \in R_+} 2 <\beta_i, \beta_i>^{-1} <\beta, \beta_i> \\ &= 2\sum_i \ell_i. \end{aligned}$$

From [AHS] we know that $C_{G^{\mathbb{C}}}$ is the square length of the highest root. Using the table given there we then obtain the following table.

(5.20)

$G^{\mathbb{C}}$	$C_{G^{\mathbb{C}}}$	$2C_{G^{\mathbb{C}}}k + C_P(\ell)$
$SL(n)$	n	$2nk - 2\sum_i^{n-1} \ell_i$
$Spin(2n+1, \mathbb{C})$	$n-2$	$2nk - 2\sum_i^{n} \ell_i$
$Sp(n, \mathbb{C})$	$n-2$	$2nk - 2\sum_i^{n} \ell_i$
$Spin(2n, \mathbb{C})$	$n-1$	$2nk - 2\sum_i^{n} \ell_i$
G_2	4	$8k - 2\sum_i^{2} \ell_i$
F_4	9	$18k - 2\sum_i^{4} \ell_i$
E_6	12	$24k - 2\sum_i^{6} \ell_i$
E_7	18	$36k - 2\sum_i^{7} \ell_i$
E_8	30	$60k - 2i\sum_i^{8} \ell_i$

Notice that the elements $\{-H_{\beta_i}\}$ chosen to represent the basis of $[\Sigma, G^{\mathbb{C}}/P]$ have length $<\beta_i, \beta_i>^{-1}$ so are not necessarily all of the same length.

Example 5.21. We shall now assume that $G^{\mathbb{C}}$ is either $SL(n)$ or $GL(n)$ and allow P to be any parabolic subgroup. Let $\{n_i\}_{i=1}^t$ be the type of the maximal flag stabilized by P. As in Lemma 2.20 we then identify the basis $\{H_{\beta_i}\}$ as

$$H_{\beta_i} = i(E_{k_i} - E_{k_i+1}), ,$$

where $k_i = \sum\limits_{j \leqslant i} n_j$. We now have

$$I_{\mathfrak{p}} = \{E_j - E_k | n_i \leqslant j < n_{i+1},\ n_{i+1} \leqslant k\},$$

and a direct calculation shows that

$$\sum_{\beta \in I_{\mathfrak{p}}} i\beta(H_i) = n_i + n_{i+1},$$

so we obtain

$$-\deg(\mathfrak{p}) = \sum_i^t \ell_i(n_i + n_{i+1}) + (\sum_i^t n_i)c \cdot \Sigma,$$

and using the above table

$$2C_G k + C_P(\ell) = 2nk + \sum_i^t (n_i + n_{i+1})\ell_i + (\sum_i^t n_i)c \cdot \Sigma.$$

We can use this example to check the validity of the formula. As in Lemma 2.20 we identify a P–quasi–parabolic bundle $\mathcal{E}$ with a type $\{n_i\}$ quasi–parabolic vector bundle $\mathcal{V}$ and the components $\{\ell_i\}$ of the element $\ell \in [\Sigma, G^{\mathbb{C}}/P]$ classifying the P reduction with minus the first Chern classes of the elements in the corresponding parabolic filtration

$$0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_t \subsetneq \mathcal{V}_{|\Sigma}, \qquad rk(\mathcal{F}_i) - rk(\mathcal{F}_{i-1}) = n_i,$$

and obtain a topological isomorphism

$$End(\mathcal{V})/\mathfrak{p} = \bigoplus_i^t (Hom(\mathcal{F}_i/\mathcal{F}_{i-1},\ \mathcal{V}/\mathcal{F}_i)).$$

From this we obtain

$$\deg(End(\mathcal{V})/\mathfrak{p}) = \sum_i (n_i(\ell_i + c_{t+1}) - (n - \sum_j^i n_j)(c_i - c_{i-1})) = \sum_i^t (n_i + n_{i+1})\ell_i + (\sum_i^t n_i)c_{t+1},$$

where $c_{t+1} = c_1(\mathcal{V}_{|\Sigma}) = c \cdot \Sigma$.

We now give a simple smoothness criterion for the moduli space which we shall use to study the examples of §5.2.

Proposition 5.22. *Assume that either*

(i) $-\mathcal{K}_X \geqslant 0$ *and*

$$\begin{cases} H^0((\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_\Sigma]) = 0, \text{ if } \Sigma\cdot\Sigma \geqslant 0 \\ H^0(\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_\Sigma - \Sigma\cdot\Sigma]) = 0, \text{ if } \Sigma\cdot\Sigma < 0, \end{cases}$$

resp.

$$\begin{cases} H^0((\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_\Sigma + \Sigma\cdot D_\infty]) = 0, \text{ if } \Sigma\cdot\Sigma \geqslant 0 \\ H^0(\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_\Sigma + \Sigma\cdot(D_\infty - \Sigma)]) = 0, \text{ if } \Sigma\cdot\Sigma < 0 \end{cases}$$

and

$$\begin{cases} H^0(\mathcal{T}_0^0[\mathcal{K}_{D_\infty}]) = 0, \text{ if } D_\infty\cdot D_\infty \geqslant 0, \\ H^0(\mathcal{T}_0^0[\mathcal{K}_{D_\infty} - D_\infty\cdot D_\infty]) = 0, \text{ if } D_\infty\cdot D_\infty < 0, \end{cases}$$

or

(ii) $-(\mathcal{K}_X + \Sigma) \geqslant 0$ *(resp.* $-(\mathcal{K}_X + \Sigma + 2D_\infty) \geqslant 0$*)*

then $\mathcal{M}^{P,*}_{k,c,\ell}$ *is locally a complex manifold (resp.* $\widetilde{\mathcal{M}}^{P,*}_{k,c,\ell}$ *is a complex manifold) with respect to the complex space structure defined by Proposition 5.9.*

Proof. By (ii) of Proposition 5.9 we have to show that $H^2((\mathcal{T}_\mathcal{E})_0) = 0$ at each point $[\mathcal{E}] = [(E^\mathbb{C}, \overline{\partial})]$. Assume that (i) holds. By Kodaira–Serre duality,

$$H^2((\mathcal{T}_\mathcal{E})_0) \cong (H^0((\mathcal{T}_\mathcal{E})_0^*[\mathcal{K}_X]))^*.$$

From the description of $\mathcal{T}$ given in §2.4 we can deduce the exact sequence

$$0 \longrightarrow (\mathcal{T}_\mathcal{E})_0 \longrightarrow (\mathcal{T}_\mathcal{E})_0^* \xrightarrow{r_{|\Sigma}} (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^* \oplus (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\Sigma] \longrightarrow 0, \tag{5.23}$$

Tensoring this by $\mathcal{K}_X$ gives

$$0 \longrightarrow (\mathcal{T}_\mathcal{E})_0[\mathcal{K}_X] \longrightarrow (\mathcal{T}_\mathcal{E})_0^*[\mathcal{K}_X] \xrightarrow{r_{|\Sigma}} (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_X]_{|\Sigma} \oplus (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\Sigma + \mathcal{K}_X]_{|\Sigma} \longrightarrow 0. \tag{5.24}$$

Using the adjunction formula we have that

$$(\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_X]_{|\Sigma} \cong (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[K_\Sigma - \Sigma\cdot\Sigma],\ (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\Sigma + \mathcal{K}_X]_{|\Sigma} \cong (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[K_\Sigma].$$

If (i) holds $-\mathcal{K}_X$ is effective. Since $\mathcal{E}$ is simple, it follows that $H^0((\mathcal{T}_\mathcal{E})_0[\mathcal{K}_X]) = 0$. The remaining assumptions now imply that

$$H^0((\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_X]_{|\Sigma} \oplus (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\Sigma + \mathcal{K}_X]_{|\Sigma}) = 0,$$

hence, by the long exact cohomology sequence associated to (5.24), also that $\mathrm{H}^0((\mathcal{T}_\mathcal{E})_0^*[\mathcal{K}]) = 0$.

Now suppose that (ii) is satisfied. Since there is an injection

$$(\mathcal{T}_\mathcal{E})_0^* \hookrightarrow (\mathcal{T}_\mathcal{E})_0[\Sigma], \tag{5.25}$$

tensoring with $\mathcal{K}_X$ and taking global sections we obtain one

$$H^0((\mathcal{T}_\mathcal{E})_0^*[\mathcal{K}_X]) \hookrightarrow H^0((\mathcal{T}_\mathcal{E})_0[\Sigma + \mathcal{K}_X]).$$

Since $-(\Sigma+\mathcal{K}_X)$ is effective and $\mathcal{E}$ simple it follows that $\mathrm{H}^0((\mathcal{T}_\mathcal{E})_0[\Sigma+\mathcal{K}_X])$ vanishes. For the framed case the obstruction to smoothness lies in

$$H^2((\widetilde{\mathcal{T}}_\mathcal{E})_0) = H^2((\mathcal{T}_\mathcal{E})_0[-D_\infty]) \cong (H^0((\mathcal{T}_\mathcal{E})_0[\mathcal{K}_X + D_\infty]))^*.$$

When (i) holds the argument above then works with (5.24) being replaced by the sequence

$$0 \longrightarrow (\mathcal{T}_\mathcal{E})_0[-D_\infty] \longrightarrow (\mathcal{T}_\mathcal{E})_0^*[D_\infty] \xrightarrow{r_{|\Sigma\cup D_\infty}} (\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[D_\infty]\oplus(\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\Sigma+D_\infty]\oplus\mathcal{T}_0^0\oplus\mathcal{T}_0^0[D_\infty] \longrightarrow 0.$$

When (ii) holds we tensor (5.25) by $[\mathcal{K}_X + D_\infty]$. Taking global sections we obtain an inclusion

$$H^0((\mathcal{T}_\mathcal{E})_0^*[D_\infty + \mathcal{K}_X]) \hookrightarrow H^0((\mathcal{T}_\mathcal{E})_0[\Sigma + \mathcal{K}_X + D_\infty]).$$

Since we assume that $-(\Sigma + \mathcal{K}_X + 2D_\infty)$ is effective and that $H^0((\mathcal{T}_\mathcal{E})_0[-D_\infty])$ vanishes, $H^0((\mathcal{T}_\mathcal{E})_0[\Sigma + \mathcal{K}_X + D_\infty])$ also vanishes. □

5.2 Examples of moduli spaces

We shall now study in more detail some particular moduli spaces, the $SL(n)$–moduli spaces $\widetilde{\mathcal{M}}^P_{k,\ell}(\mathbb{P}^2, \Sigma)$, whose properties will be needed for the proof of Corollaries 3, 4 and 5, and the $SL(2)$–moduli spaces $\mathcal{M}^{\alpha,S}_{k,\ell}(X, \Sigma)$, where X is a minimal rational surface and $[\Sigma] = [mH]$ the multiple of a hyperplane class H with respect to which stability is defined, whose properties will give Corollary 6.

(a) The moduli spaces $\widetilde{\mathcal{M}}^P_{k,\ell}(\mathbb{P}^2, \Sigma)$. As a first illustration of the general theory of the previous section we consider the moduli spaces $\widetilde{\mathcal{M}}^P_{k,\ell}(\mathbb{P}^2, \Sigma)$ where Σ and D_∞ are distinct complex lines and $\mathcal{E}^0$ is the trivial framing as in Theorem 1.

Proposition 5.26. *Let P be a parabolic subgroup of a complex simple Lie group $G^{\mathbb{C}}$. Then with respect to local the complex space structure of Proposition 5.9 the moduli spaces $\widetilde{\mathcal{M}}^P_{k,\ell}(\mathbb{P}^2, \Sigma)$ (if not empty) are smooth complex manifolds of dimension*

$$2C_{G^{\mathbb{C}}}k + C_P(\ell).$$

In the case where $G^{\mathbb{C}} = SL(n)$ and P is the parabolic subgroup stabilizing a flag of type $\{n_i\}$ this equals

$$2nk + \sum_i^t \ell_i(n_i + n_{i+1}).$$

Proof. Provided the conditions of Proposition 5.18 are satisfied the first statement about the dimension follows directly by inserting in formula (ii) of Proposition 5.18 the topological

invariants of $\mathbb{P}^2$, Σ and D_∞ and the fact that Σ and D_∞ have only one point in common and the second by Example 5.25. Since in this case $\mathcal{E}^0 = D_\infty \times G^{\mathbb{C}}$, Condition (5.11) is certainly satisfied so we obtain that every parabolic framing in this moduli space is simple and the space is Hausdorff. Since $\mathcal{K}_X + \Sigma + 2D_\infty = -3H + H + 2H = \mathcal{O}_{\mathbb{P}^2}$, criterion (ii) in Proposition 5.22 applies so the moduli spaces are smooth. □

We now pass to prove non–emptyness of these moduli spaces provided condition (0.10) is satisfied in the case where $G^{\mathbb{C}} = SL(n)$. This will be used in Chapter 7 to obtain the sufficiency of condition (0.11) for general groups. As usual, when $G^{\mathbb{C}} = SL(n)$ and the maximal flag stabilized by P is of type $\{n_i\}$, a P–quasi–parabolic bundle $\mathcal{E}$ is the same as a type $\{n_i\}$ quasi–parabolic vector bundle $\mathcal{V}$. In this case since $\Sigma = S^2$ we can naturally regard the element ℓ classifying the P–reduction as a class in $\pi_2(G^{\mathbb{C}}/P)$ and as in Example 5.21 we identify it with the multi–integer $(-c_1, \ldots, -c_t)$ where $c_i = c_1(\mathcal{F}_i)$. Recall that by Grothendieck's Theorem every holomorphic $G^{\mathbb{C}}$–bundle on $\mathbb{P}^1$ admits a holomorphic reduction to the maximal torus, thus for $G^{\mathbb{C}} = SL(n)$ it splits into a direct sum of line bundles

$$(\oplus \mathcal{O}(a_i))) \oplus (\oplus \mathcal{O}(-b_j)) \oplus (\oplus \mathcal{O}), \qquad \sum a_i = \sum b_j, \qquad a_i,\, b_j > 0.$$

By the semicontinuity of cohomology there is a Zariski open set in the dual projective space $(\mathbb{P}^2)^\vee$ on which this splitting type is constant and it is called the type of the bundle. Again by semicontinuity a bundle which is trivial on a line must be trivial on the generic line so it is of type $(0, \ldots, 0)$. Given an $SL(n)$–bundle $\mathcal{V}$ on $\mathbb{P}^2$ we obtain a function

$$(\mathbb{P}^2)^\vee \mapsto \mathbb{N},$$

by

$$\delta : \Sigma \mapsto \Sigma a_i(\Sigma),$$

and letting δ_0 be the value that δ takes on the generic line, for every line Σ one has that $\delta_0 \leqslant \delta(\Sigma)$. We now give an upper bound on δ in the case we are interested in.

Lemma 5.27. *Let $\mathcal{V}$ be an $SL(n)$–vector bundle on $\mathbb{P}^2$ of type $(0, \ldots, 0)$ and having $c_2(\mathcal{V}) = k$, then on any line $\Sigma \subseteq \mathbb{P}^2$,*

$$\delta(\Sigma) \leqslant k.$$

Proof. Let Σ be a line in $\mathbb{P}^2$ and observe that for any bundle

$$\delta(\Sigma) = dim(H^1(\mathcal{V}(-1)_{|\Sigma})),$$

thus we need to show that

$$dim(H^1(\mathcal{V}(-1)_{|\Sigma})) \leqslant k.$$

We therefore consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{V}(-2) \longrightarrow \mathcal{V}(-1) \longrightarrow \mathcal{V}(-1)_{|\Sigma} \longrightarrow 0,$$

and take the associated cohomology sequence which gives in part

$$\cdot \longrightarrow H^0(\mathcal{V}(-1)) \longrightarrow H^0(\mathcal{V}(-1)_{|\Sigma}) \longrightarrow H^1(\mathcal{V}(-2)) \xrightarrow{\psi(\Sigma)} H^1(\mathcal{V}(-1)) \longrightarrow \cdot$$

Since $\mathcal{V}$ is of type $(0,\ldots,0)$ it can not have a non–zero section vanishing on a divisor therefore $H^0(\mathcal{V}(-k)) = 0$ for all $k > 0$. The sequence thus reduces to

$$0 \longrightarrow H^0(\mathcal{V}(-1)_{|\Sigma}) \longrightarrow H^1(\mathcal{V}(-2)) \xrightarrow{\psi(\Sigma)} H^1(\mathcal{V}(-1)) \longrightarrow \cdot$$

Note that $\mathcal{V}$ is of type $(0,\ldots,0)$ if and only if its dual $\mathcal{V}^\vee$ is, hence

$$\forall k > 0, \qquad H^0(\mathcal{V}^\vee(-k)) = 0,$$

and it follows by Serre duality that

$$H^2(\mathcal{V}(-2)) \cong (H^0(\mathcal{V}^\vee(-1)))^* = 0.$$

Riemann–Roch then gives that

$$dim(H^1(\mathcal{V}(-2))) = -\chi(\mathcal{V}(-2)) = c_2(\mathcal{V}) = k,$$

so by the last sequence above $dim(H^1(\mathcal{V}(-1)_{|\Sigma})) \leqslant k$. □

We thus obtain

Corollary 5.28. *Condition (0.10) is necessary for the moduli spaces $\widetilde{\mathcal{M}}^P_{k,\ell}(\mathbb{P}^2, \Sigma)$ to be non–empty.*

Proof. Notice first of all that by (5.27) an $SL(n)$–bundle of type $(0,\ldots,0)$ must have non–negative second Chern class. By the previous lemma we know that for any type $(0,\ldots,0)$ $SL(n)$–bundle $\mathcal{V}$ with $c_2(\mathcal{V}) = k$, $\delta_\Sigma(\mathcal{V}) \leqslant k$. Let $\mathcal{F} \subseteq \mathcal{V}_{|\Sigma}$ be a subbundle and notice that

$$c_1(\mathcal{F}) \leqslant \delta_\Sigma(\mathcal{F}) = dim(H^0(\mathcal{F}(-1))).$$

Since we have an injection in cohomology

$$H^0(\mathcal{F}(-1)) \hookrightarrow H^0(\mathcal{V}(-1)_{|\Sigma}),$$

it follows that $c_1(\mathcal{F}) \leqslant k$ so that if $(\ell_1,\ldots,\ell_t)$ are minus the first Chern classes of the elements in a parabolic filtration we must have

$$\forall i, \qquad \ell_i + k \geqslant 0.$$

□

We now show the sufficiency. We begin by proving the existence of $SL(n)$–bundles trivial on D_∞ with given c_2 and certain very non–generic splitting type on a given line. The difficulty in obtaining filtrations of all the types required stems from the fact that $\mathcal{O}(\ell_i)$ admits no holomorphic maps to $\mathcal{O}(\ell_{i+1})$ if $\ell_i > \ell_{i+1}$.

Proposition 5.29. *Let $-\ell_1 \geqslant -\ell_2 \geqslant \cdots \geqslant -\ell_{n-1} \geqslant 0$ and assume $k+\ell_1 \geqslant 0$, then there exist an $SL(n)$–bundle trivial on D_∞ and such that*

$$\mathcal{V}_{|\Sigma} = \mathcal{O}(-\ell_1) \oplus (\bigoplus_{i=1}^{n-2})\mathcal{O}(\ell_i - \ell_{i+1}) \oplus \mathcal{O}(\ell_{n-1}).$$

Proof. To prove the claim we slightly generalize Serre's construction for $SL(2)$–bundles with prescribed Chern classes on a complex surface (see [GH] pp. 726-729 for instance). Let $Z_1, \ldots, Z_t$ be finite disjoint sets of simple points on a complex surface X such that

$$\forall i < t, \qquad |Z_i| = \ell_{i+1} - \ell_i, \qquad |Z_t| = \ell_t + (k - \ell_1),$$

and suppose we can find a locally free extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{V} \longrightarrow \oplus_i \mathcal{I}_{Z_i} \longrightarrow 0, \tag{5.30}$$

where $\mathcal{I}_{Z_i}$ is the ideal sheaf of Z_i. Then $\mathcal{V}$ is an $SL(n)$–bundle with $c_2(\mathcal{V}) = k$ and if further $X = \mathbb{P}^2$ and

$$\bigcup_{i=1}^{t-1} Z_i \subseteq \Sigma,$$

with Z_t having its first ℓ_t points on Σ and the remaining $k-\ell_1$ points off $\Sigma \cup D_\infty$, it follows that (5.30) restricts to the trivial extension on D_∞ and to the exact sequence

$$0 \longrightarrow \mathcal{O}(\ell_1) \longrightarrow \mathcal{V}_{|\Sigma} \longrightarrow (\bigoplus_{i=1}^{i=n-2} \mathcal{O}(\ell_{i+1} - \ell_i) \oplus \mathcal{O}(-\ell_{n-1}) \longrightarrow 0,$$

on Σ. Since

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{n-1} \geqslant 0,$$

the sequence must split on Σ and so $\mathcal{V}$ has the required splitting type on Σ. The extension (5.30) corresponds to a class $e \in Ext^1(X, \oplus_i \mathcal{I}_{Z_i}, \mathcal{O})$ and by the spectral sequence relating local and global Ext's, this induces e_{p} in each stalk $\mathcal{E}xt^1_{\mathrm{p}}(\oplus_i \mathcal{I}_{Z_i}, \mathcal{O})$. Since the Z_i's are assumed disjoint and simple we see that at each point p

$$\mathcal{E}xt^1_{\mathrm{p}}(\oplus_i \mathcal{I}_{Z_i}, \mathcal{O}) = \mathcal{E}xt^1_{\mathrm{p}}(\mathcal{I}_{\mathrm{p}}, \mathcal{O}) = \mathbb{C}, \quad \text{if p} \in \bigcup Z_i,$$
$$\mathcal{E}xt^1_{\mathrm{p}}(\oplus_i \mathcal{I}_{Z_i}, \mathcal{O}) = 0, \quad \text{if p} \notin \bigcup Z_i,$$

and the class e represents a locally free extension if and only if e_{p} is a unit for each $\mathrm{p} \in \bigcup Z_i$. We thus need to prove that there is a class $e \in Ext^1(X, \oplus_i \mathcal{I}_{Z_i}, \mathcal{O})$ with this property. Looking at the spectral sequence relating local and global Ext's we find that one obstruction whose vanishing guarantees the solvability of the problem is

$$E_2^{2,0} = H^2(X, \mathcal{E}xt^1(\oplus_i \mathcal{I}_{Z_i}, \mathcal{O})) \cong \oplus_i H^2(X, \mathcal{O}).$$

Since on $\mathbb{P}^2$ this vanishes we see that our problem can be solved there. □

From this it is easy to manufacture parabolic framings for all the topological types required by condition (0.10).

Proposition 5.31. *For any k and $\ell = (\ell_1, \dots, \ell_t)$ satisfying condition (0.10) and for any type $\{n_i\}$ there is an $SL(n)$-bundle $\mathcal{V}$ on $\mathbb{P}^2$, trivial on D_∞ with $C_2(\mathcal{V}) = k$ and admitting a type $\{n_i\}$ filtration*

$$\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_t \subsetneq \mathcal{V}_{|\Sigma},$$

with $c_1(\mathcal{F}_i) = -\ell_i$.

Proof. Let $\ell_{t+1} = c_1(\mathcal{V}_{|\Sigma}) = 0$ and order the ℓ_i's

$$\ell_{i_{t+1}} \succ \cdots \succ \ell_{i_1},$$

where $\ell_{i_{j+1}} \succ \ell_{i_j}$ if $i_{j+1} < i_j$ and $\ell_{i_{j+1}} \geqslant \ell_{i_j}$ and the ordering is the obvious one otherwise. We look for the first "inversion", namely the least j such that

$$\ell_{i_{j+1}} \geqslant \ell_{i_j}, \qquad i_{j+1} < i_j.$$

Call this j_1. Let j_2 be the least j such that an inversion after j_1 occurs at $\ell_{i_{j_1+j}}$, that is

$$\ell_{i_{j_1+j_2}} \geqslant \ell_{i_{j_1+j_2+1}}, \qquad i_{j_1+j_2+1} < i_{j_1+j_2}.$$

Then let j_3 be the least j such that an inversion occurs after $j_1 + j_2$ at $\ell_{i_{j_1+j_2+j}}$ and so on until we find a j_s such that no more inversions occur after it (notice that $s \leqslant t$). Now consider those j_i's for which $-\ell_{j_i}$ is strictly greater than zero, say $j_1, \dots, j_r$ (of course there may be none). By Proposition 5.29 we can find an $SL(r+1)$-bundle $\mathcal{V}'$ trivial on D_∞ with $c_2(\mathcal{V}') = k$ and

$$\mathcal{V}'_{|\Sigma} = \mathcal{O}(-\ell_{j_1}) \oplus \Big(\bigoplus_{i=1}^{r-1} \mathcal{O}(\ell_{j_i} - \ell_{j_{i+1}})\Big) \oplus \mathcal{O}(\ell_{j_r}).$$

Let

$$\mathcal{V} = \mathcal{V}' \oplus \Big(\bigoplus^{n-(r+1)} \mathcal{O}\Big).$$

Then $\mathcal{V}$ is trivial on D_∞, has the same Chern classes as $\mathcal{V}'$ and we show that $\mathcal{V}_{|\Sigma}$ will have a filtration by holomorphic subbundles of the required ranks and Chern classes. We let

$$\mathcal{F}_1 = \mathcal{O}(\ell_1) \oplus \Big(\bigoplus^{k_1-1} \mathcal{O}\Big),$$

$$\mathcal{F}_2 = \begin{cases} \mathcal{O}(\ell_2) \oplus \Big(\bigoplus^{k_2-1} \mathcal{O}\Big), & \text{if } j_1 \geqslant 2, \\ \mathcal{F}_1 \oplus \mathcal{O}(\ell_2 - \ell_1) \oplus \Big(\bigoplus^{k_2-(k_1+1)} \mathcal{O}\Big), & \text{if } j_1 = 1 \end{cases}$$

$$\mathcal{F}_3 = \begin{cases} \mathcal{O}(\ell_3) \oplus \Big(\bigoplus^{k_3-1} \mathcal{O}\Big), & \text{if } j_1 \geqslant 3, \\ \mathcal{F}_2 \oplus \mathcal{O}(\ell_3 - \ell_2) \oplus \Big(\bigoplus^{k_3-(k_2+1)} \mathcal{O}\Big), & \text{if } j_1 + j_2 = 2, \text{or } j_2 = 2, \\ \mathcal{F}_1 \oplus \mathcal{O}(\ell_3 - \ell_1) \oplus \Big(\bigoplus^{k_3-(k_1+1)} \mathcal{O}\Big), & \text{if } j_1 = 1,\ j_1 + j_2 \geqslant 3 \end{cases}$$

$$\vdots = \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\mathcal{F}_i = \begin{cases} \mathcal{O}(\ell_i) \oplus \Big(\bigoplus^{k_i-1} \mathcal{O}\Big), & \text{if } j_1 \geqslant i, \\ \mathcal{F}_{j(i)} \oplus \mathcal{O}(\ell_i - \ell_{j(i)}) \oplus \Big(\bigoplus^{k_i-(k_{j_i}+1)} \mathcal{O}\Big), & \text{if } j(i) = \max\{j_h \mid j_h < i\}. \end{cases}$$

It can be checked that the $\mathcal{F}_i$'s do indeed give a filtration of $\mathcal{V}_{|\Sigma}$ and by construction they have the right ranks and Chern classes. □

As a simple corollary we obtain the existence of principal framed quasi–parabolic bundles of certain special topological types which we shall use in Chapter 7 to prove the existence of G–instantons with topological invariants satisfying Condition (0.11) for simple G.

Proposition 5.32. *Let $G^{\mathbb{C}}$ be a complex simple connected Lie group and P a parabolic subgroup then for every pair κ, ℓ_i such that $\kappa \geqslant 0$ and $k + \ell_i \geqslant 0$, there exist principal quasi–parabolic framings of topological type (κ, ℓ) on $(\mathbb{P}^2, \Sigma, D_\infty)$ where*

$$\ell = (0, \cdots, 0, \ell_i, 0 \cdots, 0),$$

with respect to the basis (2.5).

Proof. Notice first that as remarked in §2.3 here we can assume $G^{\mathbb{C}}$ to be simply connected. By the previous Proposition, for each pair κ, ℓ_i satisfying the stated conditions we can construct an $SL(2)$–quasi–parabolic framing $\mathcal{E}$ on $(\mathbb{P}^2, \Sigma, D_\infty)$ with these topological invariants. Using the basis (2.5) for $\pi_2(G^{\mathbb{C}}/P)$ we can embed copies of $SL(2)$ in $G^{\mathbb{C}}$ so that the embeddings are of index 1 at the topological level, that is the generator of $\pi_3(SL(2))$ maps onto the generator of $\pi_3(G^{\mathbb{C}})$. Also notice that if β is one of the simple roots in $I_{\mathfrak{p}}$ defining the basis (2.5), regarded as a root in the $\mathfrak{sl}(2)$ it spans, the dual vectors associated to it by the two Killing forms as in §2.1 are related by

$$J_\beta^{\mathfrak{sl}(2)} = \frac{K_{\mathfrak{sl}(2)}}{K_{\mathfrak{g}^{\mathbb{C}}}} J_\beta^{\mathfrak{g}^{\mathbb{C}}}.$$

It follows that H_β as defined by (2.4) is the same whether the Killing form of $\mathfrak{sl}(2)$ or that for $\mathfrak{g}^{\mathbb{C}}$ are used. This implies that the induced inclusion

$$\mathbb{P}^1 = SL(2)/B \hookrightarrow G^{\mathbb{C}}/P,$$

maps the generator of $\pi_2(\mathbb{P}^1)$ to one of the elements of the basis (2.5) for $\pi_2(G^{\mathbb{C}}/P)$. Considering the induced bundle of $\mathcal{E}$ through the embedding

$$SL(2)_{\beta_i} \hookrightarrow G^{\mathbb{C}},$$

we now obtain a P–quasi–parabolic framing with the desired invariants. □

(b) The $SL(2)$–moduli spaces $\mathcal{M}^{\alpha,S}_{k,0,\ell}(X, \Sigma)$. We now look at an example of moduli spaces of parabolic stable bundles, namely the case of $SL(2)$ stable α–parabolic vector bundles on a pair (X, Σ) where $i\alpha$ is an element in $\mathfrak{su}(2)$ of the from

$$\alpha = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \qquad 0 < \beta < 1/2,$$

X is minimal rational and Σ is a multiple of a hyperplane class H with respect to which stability is defined. In this case the quasi–parabolic structure of an α–parabolic vector bundle $\mathcal{V}$ is given simply by a holomorphic line subbundle $\mathcal{F} \subseteq \mathcal{V}_{|\Sigma}$, so that ℓ can be identified with minus the first Chern class of this line bundle.

Proposition 5.33. *Let $2\ell > 2(g(\Sigma)-1)$, then $\mathcal{M}^{\alpha,S}_{k,0,\ell}(X,\Sigma)$ (if not empty) is a smooth complex manifold of dimension*

$$d = 4k - 3 + 2\ell + (1 - g(\Sigma)).$$

Proof. The last claim follows by inserting the topological invariants for a rational surface in the formula for $SL(n)$ of Proposition 5.18 (i) and taking $SL(2)$ in the calculation of Example 5.21. For the first claim notice that since X is assumed to be minimal rational $K_X < 0$. By (i) of Proposition 5.22 the moduli spaces will then be smooth as soon as $H^0((\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_\Sigma]) = 0$. In this case $(\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_\Sigma]$ is a line bundle on Σ so to ensure smoothness it suffices to show that under the stated conditions it has negative degree. Since $\mathfrak{g}_\mathcal{E}/\mathfrak{p} \cong \det(\mathcal{V}) \otimes \mathcal{F}^{-2}$,

$$\deg(\mathfrak{g}_\mathcal{E}/\mathfrak{p}) = 2\ell,$$

and it follows that

$$\deg((\mathfrak{g}_\mathcal{E}/\mathfrak{p})^*[\mathcal{K}_\Sigma]) = -2\ell + 2(g(\Sigma)-1),$$

which is negative precisely when

$$2\ell > 2(g(\Sigma)-1).$$

□

We now prove that for $k > 0$ and α suitably restricted the stated condition is also sufficient to guarantee non-emptyness.

Proposition 5.34. *If $2\ell > 2(g(\Sigma)-1)$, $k > 0$ and $0 < \beta < \dfrac{1}{\deg(\Sigma)}$ the moduli spaces $\mathcal{M}^{\alpha,S}_{k,0,\ell}(X,\Sigma)$ are non–empty.*

Proof. We begin by showing that there exist quasi parabolic bundles of the required topological type. Since $H^2(\mathcal{O}_X) = H^0(\mathcal{K}_X) = 0$ the obstruction in the Serre construction of Proposition 5.29 vanishes so there is a rank 2 holomorphic vector bundle $\mathcal{V} \to X$ arising from the exact sequence of sheaves

$$0 \to \mathcal{O}_X \xrightarrow{\sigma} \mathcal{V} \to \mathcal{I}_Z \to 0,$$

where Z is a set of k simple points disjoint from Σ. Thus $\mathcal{V}$ is an $SL(2)$–bundle with $c_2(\mathcal{V}) = k$. It follows by Riemann–Roch that

$$dim(H^0(\mathcal{V}_{|\Sigma} \otimes \mathcal{F}^\vee)) \geqslant 2\ell + 2(1 - g(\Sigma)),$$

hence under the stated assumptions $dim(H^0(\mathcal{V}_{|\Sigma} \otimes \mathcal{F}^\vee)) \geqslant 2$. In particular, when $\ell = 0$ we can assume $\mathcal{F}$ to be chosen distinct from the restriction of the subsheaf of the defining sequence of $\mathcal{V}$. Since a non–zero section ψ with zero divisor D can be considered as a global section of $\mathcal{V}_{|\Sigma} \otimes \mathcal{F}^\vee[-D]$, letting $s \in H^0([D])$ be such that $Div(s) = [D]$, $\psi \otimes s$ gives a nowhere vanishing global section of $\mathcal{V}_{|\Sigma} \otimes \mathcal{F}^\vee$. This is the same as a line subbundle

$\mathcal{F} \hookrightarrow \mathcal{V}_{|\Sigma}$ thus giving a quasi–parabolic bundle of the required topological type. Since by assumption $0 < \beta < 1/\deg(\Sigma)$, for any non–zero proper saturated subsheaf $\mathcal{L}$ we have

$$\mu_\alpha(\mathcal{V}, \mathcal{L}) \leqslant \frac{1}{2}(c_1(\mathcal{L}) \cdot H + \beta \deg(\Sigma)) < 0,$$

provided we know that $\mathcal{L} \cdot H < 0$. Here such a subsheaf is the same as a holomorphic line bundle $\mathcal{L}$ with a holomorphic map $\sigma' : \mathcal{L} \to \mathcal{V}$ vanishing at a finite number of points. Suppose that there is one such line bundle with $\mathcal{L} \cdot H \geqslant 0$. Since $Pic(X) = \mathbb{Z} < B > \oplus \mathbb{Z} < F >$, where B and F are rational curves equivalent to the base and the fibre of the ruling of X respectively and $H = aF + bB$ for some strictly positive integers a and b, it follows that $\mathcal{L} \cdot F \geqslant 0$ or $\mathcal{L} \cdot B \geqslant 0$, hence there is some rational curve C disjoint from Z such that $\mathcal{L} \cdot C \geqslant 0$. By the way $\mathcal{V}$ was constructed, from $\sigma' : \mathcal{L} \to \mathcal{V}$ we obtain an exact sequence of the form

$$0 \to \mathcal{L} \xrightarrow{\sigma'} \mathcal{V} \to \mathcal{L}^\vee \otimes \mathcal{I}_Z \to 0.$$

Restricting this sequence to C we find that

$$\mathcal{V}_{|C} \cong \mathcal{O}(a) \oplus \mathcal{O}(-a),$$

for some $a \geqslant 0$. On the other hand, restricting to C the defining sequence of $\mathcal{V}$ we see that

$$\mathcal{V}_{|C} \cong \mathcal{O} \oplus \mathcal{O},$$

giving a contradiction unless $a = 0$. In the latter case we can assume that $\mathcal{L} \cdot F = 0$ and $\mathcal{L} \cdot B = 0$ otherwise we can use the same argument to get a contradiction choosing C suitably. This means that $\mathcal{L} = \mathcal{O}_X$ and it follows that $\sigma' : \mathcal{L} \to \mathcal{V}$ must actually coincide with the subsheaf in the defining sequence of $\mathcal{V}$ up to a constant. As we observed above, when $\ell = 0$ we can choose $\mathcal{F}$ to be generically distinct from $\mathcal{L}$ so that

$$\mu_\alpha(\mathcal{V}, \mathcal{L}) = -\beta \deg(\Sigma) < 0.$$

Thus the quasi–parabolic bundle constructed is α–stable. □

Also observe that the moduli spaces will be independent of α if this is as in the corollary. Indeed if a bundle $\mathcal{V}$ is α–stable and $\alpha < \alpha' < 1/\deg(\Sigma)$ it follows that it must also be α'–stable. For if $\mathcal{L}$ is a destabilizing line bundle, so that $\mu_{\alpha'}(\mathcal{V}, \mathcal{L}) \geqslant 0$, either $\mathcal{L}$ has induced parabolic weight $-\alpha'$ and then $c_1(\mathcal{L}) \cdot H \geqslant 1$ or else $\mathcal{L}$ has induced parabolic weight α' and then $c_1(\mathcal{L}) \cdot H \geqslant 0$. In either case it follows that $\mu_\alpha(\mathcal{V}, \mathcal{L}) \geqslant 0$ so that $\mathcal{L}$ destabilizes $\mathcal{V}$ also as an α–parabolic bundle. Similarly a bundle that is α–stable will be stable for all $0 < \alpha' < \alpha$.

Corollary 5.35. *Let X be a minimal rational surface and $[\Sigma] = [mH]$ where H is a hyperplane class with respect to which stability is defined. For every pair of integers k, ℓ such that*

$$k > 0, \qquad 2\ell > 2(g(\Sigma) - 1),$$

and every $i\alpha \in \mathfrak{su}(n)$ *of the above form, the moduli space* $\mathcal{M}^{\alpha,S}_{k,0,\ell}(X,\Sigma)$ *of stable* α*–parabolic bundles is a smooth complex manifold of dimension*

$$d = 4k - 3 + 2\ell + (1 - g(\Sigma))$$

which is independent of α.

5.3 Divisors with ordinary singularities

We now extend the results of §5.1 to the case where Σ and D_∞ are divisors with ordinary singularities meeting transversely at smooth points. By a divisor with ordinary singularities we mean a reduced divisor such that at any one of its singular points the number of tangents equals its multiplicity at that point. We first show how to carry out the construction of the moduli spaces in the case where Σ is a union of smooth disjoint curves and D_∞ has simple normal crossings (s.n.c), namely it is the finite union of smooth irreducible curves meeting transversely, and then reduce the case of ordinary singularities to this. We begin by generalising the definitions of the introduction to the case where Σ and D_∞ have at most singularities of the type described above. Let Σ and D_∞ be reduced divisors in a complex surface X with components

$$\Sigma = \bigcup_i^s \Sigma^i, \qquad D_\infty = \bigcup_j^t D^j_\infty,$$

respectively and let $\{\mathrm{p}\} = Sing(\Sigma)$. Let $P = \{P^i\}$ be parabolic subgroups of $G^{\mathbb{C}}$ one for each Σ^i and $K = \{K_\mathrm{p}\}$ a set where, denoting by $\{i_\mathrm{p}\}$ the indices corresponding to the components of Σ meeting at p, $K_\mathrm{p} = \{P^{i(\mathrm{p})}_\mathrm{p}\}$ is a collection of parabolic subgroups of $G^{\mathbb{C}}$ one for each $\mathrm{p} \in Sing(\Sigma)$ with

$$P^{i(\mathrm{p})}_\mathrm{p} \cong P^{i(\mathrm{p})}.$$

Definition 5.36. *A* (P, K)*–quasi–parabolic bundle on* (X, Σ) *consists of a pair* $(\mathcal{E}, s)$ *where* $\mathcal{E}$ *is a holomorphic* $G^{\mathbb{C}}$*–bundle on* X *and* $s = (s^1, ..., s^s)$ *are holomorphic* P^i*–reductions one for each* Σ^i *such that identifying the fibre of* $\mathcal{E}$ *at a point with* $G^{\mathbb{C}}$

$$\forall p \in Sing(\Sigma),\ \exists \varphi_\mathrm{p} \in Aut(G^{\mathbb{C}}),\ \forall i(\mathrm{p}) \qquad \varphi_p(s^{i(\mathrm{p})}(\mathrm{p})) = P^{i(\mathrm{p})}_\mathrm{p}.$$

Now let $i\alpha = \{i\alpha^i\}$ be elements in $\mathfrak{g}$ one for each Σ^i.

Definition 5.37. *An* α*–parabolic bundle on* (X, Σ) *is a* (P, K)*–quasi–parabolic bundle on* (X, Σ) *where* $P = \{P(\alpha^i)\}$.

Fix a P–quasi–parabolic (α–parabolic) $G^{\mathbb{C}}$–bundle $\mathcal{E}^0$ on D^∞ with P^i–quasi–parabolic structure at the points in $\Sigma \cap D^\infty$. As in Definition 0.9 we then let a (P, K)–quasi parabolic framing (α–parabolic framing) consist of a triple $(\mathcal{E}, s, \varphi)$, where $(\mathcal{E}, s)$ is a (P, K)–quasi parabolic bundle and φ is a parabolic isomorphism $\varphi : \mathcal{E}_{|D_\infty} \cong \mathcal{E}^0$. The notion of isomorphism is defined as before by requiring that the parabolic reductions and framing be preserved.

As in §2.3 we can consider the notion of parabolic bundles and framings also at the topological level by omitting the word holomorphic from the previous definitions. The fact that the components Σ^j of Σ are no longer assumed smooth is immaterial for the purpose of classifying topological P^j–reductions on a $G^{\mathbb{C}}$–bundle on Σ^j so, once K is specified, it follows by the discussion of §2.3 that when $\Sigma = \bigcup_j \Sigma^j$, topological types of $(P,\ K)$–quasi–parabolic bundles are parametrised by triples

$$(k, c, \underline{\ell}) \in H^4(X, \mathbb{Z}) \times H^2(X, \pi_1(G)) \times \prod_j [\Sigma, G^{\mathbb{C}}/P^j].$$

Notice that for fixed $(k, c, \underline{\ell})$ different choices of K, that is specifying different configurations for the elements in K_{p_i}, determines distinct topological types. If we were to fix only $(k, c, \underline{\ell})$, the moduli space of P–quasi–parabolic or framed quasi parabolic bundles with topological type $(k, c, \underline{\ell})$ would turn out to be the disjoint union of components one for each choice of K. In line with the previous notation we denote the set of equivalence classes of P–quasi–parabolic bundles or framings on $(X, \Sigma,\ D_\infty)$ of given topological type by

$$\mathcal{M}^{P,K}_{k,c,\underline{\ell}}(X,\ \Sigma), \qquad \widetilde{\mathcal{M}}^{P,K}_{k,c,\underline{\ell}}(X,\ \Sigma),$$

respectively. For $G^{\mathbb{C}} = GL(n)$ we have the associated notion of an α–parabolic vector bundle $\mathcal{V}$ on $(X,\ \Sigma)$, and we define the α–degree and slope of $\mathcal{V}$ to be

$$\deg_\alpha(\mathcal{V}) = \deg(\mathcal{V}) + \sum_i Tr(\alpha^i)\deg(\Sigma^i), \qquad \mu_\alpha(\mathcal{V}) = \frac{\deg_\alpha(\mathcal{V})}{rk(\mathcal{V})}.$$

We define the induced α–degree of a saturated non–zero proper subsheaf to be

$$\deg(\mathcal{S},\ \mathcal{V}) = \deg(\mathcal{S}) + \sum_i Tr(\alpha^{i'})\deg(\Sigma^i),$$

where the $\alpha^{i'}$ are defined by considering the induced filtration (2.38) on each component of Σ, and $\mathcal{V}$ is stable if for every such subsheaf

$$\mu_\alpha(\mathcal{S},\ \mathcal{V}) < \mu_\alpha(\mathcal{V}).$$

The proofs of 2.40 and 5.12 go through in the case where Σ and D_∞ have ordinary singularities (in fact for any reduced divisors Σ and D_∞) to give the corresponding statements.

Let now assume that Σ is a union of disjoint smooth curves and D_∞ is a s.n.c. divisor. Of course in this case the condition K on the parabolic reductions on intersecting components is empty. Fix a topological type $k, c, \underline{\ell}$ and let $E^{\mathbb{C}}$ be a $\mathcal{C}^\infty$ P–quasi–parabolic bundle of this type. Keeping the same notation as in §5.1 we see as in §2.4 that once a base point $\overline{\partial}_0$ is fixed we can parametrize the set of equivalence classes of parabolic bundles or framings of this topological type by

$$\mathcal{M}^P_{k,c,\underline{\ell}} = \frac{\{a \in \Omega^{0,1}(\mathfrak{g}_{E^{\mathbb{C}}}) \,|\, (\overline{\partial}_0 + a)^2 = 0,\ \forall j,\ i^*_{\Sigma^j}(a) \in \Omega^{0,1}(\Sigma^j,\ \mathfrak{p}^j)\}}{\{g \in \mathcal{G}^{\mathbb{C}} \,|\, \forall j,\ Ad(g)(\mathfrak{p}^j) \subseteq \mathfrak{p}^j\}},$$

$$\widetilde{\mathcal{M}}^P_{k,c,\underline{\ell}} = \frac{\{a \in \Omega^{0,1}(\mathfrak{g}_{E^{\mathbb{C}}}) \,|\, (\overline{\partial}_0 + a)^2 = 0,\ \forall j,\ i^*_{\Sigma^j}(a) \in \Omega^{0,1}(\Sigma^j,\ \mathfrak{p}^j),\ i^*_{D_\infty}(a) = 0\}}{\{g \in \mathcal{G}^{\mathbb{C}} \,|\, \forall j,\ Ad(g)(\mathfrak{p}^j) \subseteq \mathfrak{p}^j,\ g_{|D_\infty} = 1\}},$$

where $\mathfrak{p}^j$ are the holomorphic subbundles of parabolic Lie algebras induced on $\mathfrak{g}_{\mathcal{E}}$ by the parabolic reductions on each Σ^j. The sheaf of infinitesimal quasi–parabolic automorphisms $\mathcal{T}$ can now be defined by the exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathfrak{g}_{\mathcal{E}} \longrightarrow \oplus_i \mathfrak{g}_{\mathcal{E}}/\mathfrak{p}^j \longrightarrow 0,$$

and the sheaf of framed infinitesimal quasi–parabolic automorphism is related to $\mathcal{T}$ by $\widetilde{\mathcal{T}} \cong \mathcal{T}[-D_\infty]$. Mutatis mutandis in (2.28) and keeping the same notation we define the complexes $\{\Omega^{0,i}(T), \overline{\partial}_0\}, \{\Omega^{0,i}(\widetilde{T}), \overline{\partial}_0\}$ and starting from these the spaces

$$\mathcal{A}_{\mathfrak{p}}, \quad \widetilde{\mathcal{A}_{\mathfrak{p}}}, \quad \mathcal{H}_{\mathfrak{p}}, \quad \widetilde{\mathcal{H}}_{\mathfrak{p}}, \quad \mathcal{G}_{\mathfrak{p}}, \quad \widetilde{\mathcal{G}}_{\mathfrak{p}}, \quad ,$$

and the quotients

$$\mathcal{B}_{\mathfrak{p}}, \quad \widetilde{\mathcal{B}}_{\mathfrak{p}}, \quad \mathcal{H}_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}}, \quad \widetilde{\mathcal{H}}_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}}.$$

We then have injections

$$\mathcal{H}_{\mathfrak{p}}/\mathcal{G}_{\mathfrak{p}} \longrightarrow \mathcal{M}^P_{k,c,\underline{\ell}}, \qquad \widetilde{\mathcal{H}}_{\mathfrak{p}}/\widetilde{\mathcal{G}}_{\mathfrak{p}} \longrightarrow \widetilde{\mathcal{M}}^P_{k,c,\underline{\ell}}.$$

We can generalize Lemmas 2.31 and 2.34 to show that the above injections are actually homeomorphisms.

Lemma 5.38.

(i) *Let $\overline{\partial}_1$ be a quasi–parabolic structure on $E^{\mathbb{C}}$. There is an element g in $\{g \in \mathcal{G}^{\mathbb{C}} \,|\, \forall j,\ Ad(g)(\mathfrak{p}^j) \subseteq \mathfrak{p}^j\}$ such that*

$$Ad(g)^*(\overline{\partial}_1) \in \mathcal{A}_{\mathfrak{p}}.$$

If further $\overline{\partial}_1$ makes $E^{\mathbb{C}}$ into a quasi–parabolic framing then g can be chosen to be 1 on D_∞ and

$$Ad(g)^*(\overline{\partial}_1) \in \widetilde{\mathcal{A}_{\mathfrak{p}}}.$$

(ii) *Let $\overline{\partial}_i = \overline{\partial}_0 + a_i$, $i = 1, 2$ be P–quasi–parabolic structures on E and let $\overline{\partial}_0 + \widetilde{a}_i$ be their representatives given by (i). If the $\overline{\partial}_i$'s are gauge equivalent then the $\overline{\partial}_0 + \widetilde{a}_i$'s are gauge equivalent by a complex gauge transformation $\widetilde{g} \in \mathcal{G}_{\mathfrak{p}}$. If further the $\overline{\partial}_i$'s give quasi–parabolic framings we can also ensure that $\widetilde{g} \in \widetilde{\mathcal{G}}_{\mathfrak{p}}$.*

Proof. (i) This follows as in Lemma 2.31 since the Σ^j's are disjoint and smooth. In the case of framings at a normal crossing on D_∞ we can take the complex gauge transformations satisfying (2.32) and (2.33) to be 1 on both divisors and do the gluing away from the crossing points to obtain the result.

(ii) The proof for quasi–parabolic bundles is the same as in Lemma 2.34. In that notation, $\widetilde{a}_i$ now vanishes in a neighbourhood of a normal crossing on D_∞ so $\overline{\partial}_0 \widetilde{g} = 0$ near it. Since both the g_i's and g are 1 on all components of D_∞ it follows that $\widetilde{g} \in \widetilde{\mathcal{G}}_{\mathfrak{p}}$. □

We then obtain the corresponding versions of Propositions 5.6 and 5.9 and the same holds for Propositions 5.10, 5.13, and 5.16 thus we can generalize Theorem 5.17 (ii) to this

case. For α–stable parabolic bundles note that since the components of Σ are assumed to be smooth and disjoint the extension part of the correspondence Theorem immediately generalizes to this case. Since Simpson's existence Theorem only depends on the notion of analytic stability which is insensitive to what Σ is as long as it is reduced and $\Sigma \cdot \Sigma \geqslant 0$, and since the argument of Proposition 3.75 goes through in this contest to relate parabolic and analytic stability, the existence part also generalizes so Proposition 5.15 and thus (i) of 5.17 follow as in the case of an irreducible Σ. The formulae of proposition 5.18 for the dimension of $\mathcal{M}^{P,*}_{k,c,\underline{\ell}}(X, \Sigma)$ are now replaced (under the same assumptions, so in particular X is Kähler) by

$$d = 2C_G k - \frac{dim(G^{\mathbb{C}})}{2}(1 - b^1(X) + b^+(X)) + \sum_i (C_{P^i}(\ell_i) + dim(G^{\mathbb{C}}/P^i)\frac{\chi(\Sigma^i)}{2})$$

in the case of a simple $G^{\mathbb{C}}$ and

$$d + (1 + h^{0,2}(X)) + \sum_i C_{P^i}(c \cdot \Sigma^i),$$

when $G^{\mathbb{C}} = GL(n)$. The formulae for the dimension of $\widetilde{\mathcal{M}}^{P,*}_{k,c,\underline{\ell}}(X, \Sigma)$ are then

$$\widetilde{d} = d + dim(G^{\mathbb{C}})\frac{\chi(D_\infty)}{2} - \sum_i \sum_{p \in \Sigma^i \cap D_\infty} dim(G^{\mathbb{C}}/P^i)$$
$$\widetilde{d} + h^{0,2}(X) + \sum_i C_{P^i}(c \cdot \Sigma^i)$$

in the case of simple groups and $GL(n)$ respectively. Note that $\chi(D_\infty)$ is obtained as

$$\chi(D_\infty) = \sum_j \chi(D^j_\infty) - \sum_{p \in Sing(D_\infty)} dim(G^{\mathbb{C}}).$$

We now explain how to reduce the case of divisors with ordinary singularities to this. The same procedure should generalize to handle general singularities, but we shall not attempt to do it.

Let $\{p\} = Sing(\Sigma)$ and $\{q\}$ be the set of singular points of D_∞ which are not normal crossings of two distinct components. Since all these points are assumed to be distinct tangent singularities for Σ and D_∞ and since Σ and D_∞ are assumed to intersect transversely, a single blow–up at each of them reduces this case to the one considered above. Namely there is a compact complex surface $\widehat{X}$ and a morphism

$$\pi : \widehat{X} \longrightarrow X,$$

which is the blow–up at each point in $\{p, q\}$ and a biholomorphism of $\widehat{X} \backslash \pi^{-1}(\{p, q\})$ onto $X \backslash \{p, q\}$, such that the pair $(\widehat{X}, \pi)$ is unique up to isomorphism compatible with the projections and inducing the identity on X, the reduced proper transforms of Σ and of each of the components of D_∞ are smooth curves and the reduced total transform of

$\Sigma \cup D_\infty$ is a s.n.c. divisor. A holomorphic bundle $\mathcal{E}$ on X pulls–back to a holomorphic bundle $\widehat{\mathcal{E}}$ on $\widehat{X}$ trivial on the reduced exceptional divisor

$$E = \pi^{-1}(\{\mathrm{p},\, \mathrm{q}\}).$$

A P^i–reduction s^i on $\mathcal{E}_{|\Sigma^i}$ pulls–back to a reduction $\widehat{s}_i$ on $\widetilde{\Sigma}^i$ and the topological type of $(\mathcal{E},\, s)$ can be identified with that of $(\widehat{\mathcal{E}},\, \widehat{s})$. A framing bundle $\mathcal{E}^0$ pulls back to a quasi–parabolic bundle on the reduced total transform of $(D_\infty, D_\infty \cap \Sigma)$, $(\pi^{-1}(D_\infty), \pi^{-1}(D_\infty \cap \Sigma))$, and a framing $\varphi : \mathcal{E}_{|D_\infty} \cong \mathcal{E}^0$ to one $\widehat{\mathcal{E}}_{|\pi^{-1}(D_\infty)} \cong \pi^*(\mathcal{E}^0)$ for the pull–back quasi–parabolic structures. Let E_Σ consist of the components of the exceptional divisor corresponding to points in Σ and notice that $(\widehat{\mathcal{E}}_{|E_\Sigma},\, \widehat{s}_{|E_\Sigma})$ is naturally a quasi–parabolic bundle with structure on $\widetilde{\Sigma} \cap E_\Sigma$. We let $\widehat{\mathcal{E}}^0$ be a quasi–parabolic bundle on $(\pi^{-1}(D_\infty \cup E), \pi^{-1}(D_\infty \cap \Sigma) \cup (\widetilde{\Sigma} \cap E_\Sigma))$ coinciding with $\pi^*(\mathcal{E}^0)$ on the reduced total transform of D_∞ and isomorphic to $(\widehat{\mathcal{E}}_{|E},\, \widehat{s}_E)$ on E_Σ. We consider the moduli space of $\mathcal{E}^0$–quasi parabolic framings of a given topological type on $(X,\, \Sigma,\, D_\infty)$

$$\widetilde{\mathcal{M}}^{P,K}_{k,c,\underline{\ell}}(X,\, \Sigma)$$

and also the moduli space

$$\widetilde{\mathcal{M}}^{P}_{k,c,\underline{\ell}}(\widehat{X},\, \widetilde{\Sigma})$$

of $\widehat{\mathcal{E}}^0$–quasi–parabolic framings of the same topological type on $(\widehat{X},\, \widetilde{\Sigma},\, \pi^{-1}(D_\infty))$. Any element in $\widetilde{\mathcal{M}}^{P,K}_{k,c,\underline{\ell}}(X,\, \Sigma)$ pulls back to an element in $\widetilde{\mathcal{M}}^{P}_{k,c,\underline{\ell}}(\widehat{X},\, \widetilde{\Sigma})$ up to the choice of an isomorphism $\varphi_\Sigma : \widehat{\mathcal{E}}_{|E_\Sigma} \cong \widehat{\mathcal{E}}^0_{|E_\Sigma}$ compatible with the quasi–parabolic structures. Vice–versa, the group $Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma})$ of quasi–parabolic automorphisms of $\widehat{\mathcal{E}}^0_{|E_\Sigma}$ acts on $\widetilde{\mathcal{M}}^{P}_{k,c,\underline{\ell}}(\widehat{X},\, \widetilde{\Sigma})$ and it follows that π induces a bijection

$$\widetilde{\mathcal{M}}^{P}_{k,c,\underline{\ell}}(\widehat{X},\, \widetilde{\Sigma})/Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma}) \cong \widetilde{\mathcal{M}}^{P,K}_{k,c,\underline{\ell}}(X,\, \Sigma). \tag{5.39}$$

Taking D_∞ to be empty the right hand side is just $\mathcal{M}^{P,K}_{k,c,\underline{\ell}}(X,\, \Sigma)$, the set of equivalence classes of (P, K)–quasi–parabolic bundles of a given topological type. As in §2.4 we can naturally identify the right hand side with a suitable moduli space of $\overline{\partial}$–operators and equip it with the quotient topology of this. This makes the bijection a homeomorphism. It thus suffices to equip the left hand side space with a moduli space structure, study its properties and analyse the effect of quotienting by $Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma})$. Since $\widetilde{\Sigma}$ is a union of smooth disjoint curves while $\pi^{-1}(D_\infty \cup E_\Sigma)$ is a s.n.c. divisor and the two meet transversely at smooth points, the previous case applies. Let $\mathcal{E}$ be a (P, K)–quasi–parabolic bundle. Note that the definition of $\mathcal{T}$ makes sense whenever Σ and D_∞ are reduced so the sheaves $\mathcal{T}_\mathcal{E}$ and $\widetilde{\mathcal{T}}_\mathcal{E}$ are defined and we can define simpleness as in 5.5.

Proposition 5.40. *Let (X, Σ, D_∞) be a complex triple as specified above, $G^{\mathbb{C}}$ a complex simple Lie group or $GL(n)$, $P = \{P^j\}$ parabolic subgroups of it one for each component*

of Σ and $K = \{K_{\mathrm{p}}\}$ collections of parabolic subgroups of $G^{\mathbb{C}}$ one for each $p \in Sing(\Sigma)$ as specified above. Then

(i) *$\mathcal{M}^{P,K,*}_{k,c,\underline{\ell}}(X, \Sigma)$ has locally a natural complex space structure. The Zariski tangent space at a point $[\mathcal{E}] = (E^{\mathbb{C}}, \overline{\partial}_0)$ can be identified with $H^1(\mathcal{T}_{\mathcal{E}})$and the structure is locally regular if and only if $H^2((\mathcal{T}_{\mathcal{E}})_0)$ vanishes.*

(ii) *$\widetilde{\mathcal{M}}^{P,K,*}_{k,c,\underline{\ell}}(X, \Sigma)$ has a natural complex space structure. The Zariski tangent space at a point $[\mathcal{E}] = (E^{\mathbb{C}}, \overline{\partial}_0)$ can be identified with $H^1(\widetilde{\mathcal{T}_{\mathcal{E}}})$and the structure is locally regular if and only if $H^2((\widetilde{\mathcal{T}_{\mathcal{E}}})_0)$ vanishes.*

Proof. Note that the simple sets in the moduli spaces on the two sides of (5.39) correspond via π thus if $[\overline{\partial}_0]$ is a simple quasi–parabolic bundle or framing on X, at $[\pi^*(\overline{\partial}_0)]$ there is locally a complex space structure. In fact it can be easily shown that there is such a structure on the inverse image of the whole of a neighbourhood of $[\overline{\partial}_0]$ and by the simpleness assumption the action of $Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma})$ on this set, which is proper, is also free so that the quotient has naturally also a complex space structure. If we let $\widehat{\mathcal{T}}$ and $\widehat{\widehat{\mathcal{T}}}$ denote the sheaves of infinitesimal quasi–parabolic and framed automorphisms on a point in the inverse image of $[\overline{\partial}_0]$ then there is an exact sequence of sheaves on X

$$0 \longrightarrow \widehat{\mathcal{T}} \longrightarrow \mathcal{T}_{\mathcal{E}} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q}$, the Lie algebra of $Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma})$, is a skyscraper sheaf concentrated on the set $\{\mathrm{p}\}$ and similarly in the framed case. It follows that $H^1(\widehat{\mathcal{T}})/H^0(\mathcal{Q}) = H^1(\mathcal{T}_{\mathcal{E}})$ and $H^2((\widehat{\mathcal{T}})_0) = H^2((\mathcal{T}_{\mathcal{E}})_0)$. Since we observed above that the action of $Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma})$ is free and since $H^1(\widehat{\mathcal{T}})$, $H^2((\widehat{\mathcal{T}})_0)$ are the Zariski tangent space and the obstruction to smoothness at $[\pi^*(\overline{\partial}_0)]$ the claim follows $\square$

When X is Kähler We can now derive dimension formulae for these moduli spaces from those of the previous case. Each component of E_Σ contributes separately to $Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma})$, that is

$$Aut(\widehat{E}^0_{|E_\Sigma}) = \underset{\mathrm{p}\in Sing(\Sigma)}{\times} Aut(\widehat{E}^0_{|E_{\mathrm{p}}}),$$

and from the definition of (P, K)–quasi–parabolic bundle it follows that

$$Aut(\widehat{\mathcal{E}}^0_{|E_\Sigma}) = \underset{\mathrm{p}\in Sing(\Sigma)}{\times} (\cap P_{\mathrm{p}}^{i(p)}).$$

Using the relation

$$\chi(\Sigma^i) = \chi(\tilde{\Sigma}^i) - \sum_{\mathrm{p}\in Sing(\Sigma^i)} (\nu_{\mathrm{p}}(\Sigma^i) - 1),$$

we thus obtain for (P, K)–quasi–parabolic bundles in the case of simple groups and $GL(n)$

respectively

$$d = 2C_{G^{\mathbb{C}}}k - \frac{dim(G^{\mathbb{C}})}{2}(1 - b^1(X) + b^+(X)) +$$
$$\sum_i (C_{P^i}(\ell_i) + dim(G^{\mathbb{C}}/P^i)\frac{\chi(\Sigma^i)}{2})$$
$$+ \sum_{p \in Sing(\Sigma)} (dim(G^{\mathbb{C}}) - \sum_{p = \cap \Sigma^{i(p)}} dim(G^{\mathbb{C}}/P^{i(p)}) - dim(\cap P_p^{i(p)})),$$
$$d + (1 + h^{0,2}(X)) + \sum_i C_{P^i}(c \cdot \Sigma^i),$$

and in the framed case

$$\tilde{d} = d + dim(G^{\mathbb{C}})\frac{\chi(D_\infty)}{2} - \sum_i \sum_{p \in \Sigma^i \cap D_\infty} dim(G^{\mathbb{C}}/P^i),$$
$$\tilde{d} + h^{0,2}(X) + \sum_i C_{P^i}(c \cdot \Sigma^i).$$

When a Kähler structure is chosen, to obtain a complex space structure on the stable set $\mathcal{M}^{\alpha,S}_{k,c,\underline{\ell}}$ it only remains to prove its openness in $\mathcal{M}^{P,K,*}_{k,c,\underline{\ell}}$. This follows as in (i) of 5.17 provided we can generalize Theorem 2 to this framework (at least for rational weights). The existence as remarked above poses no difficulty provided the condition $\Sigma \cdot \Sigma$ is satisfied. What is needed for the extension part is a proper development of the cylinder analysis outlined in §1.3. Finally the smoothness criteria of Proposition 5.22 become now.

Proposition 5.41. *Assume that either*

(i) $-\mathcal{K}_X \geqslant 0$ *and*

$$\forall i, \quad \begin{cases} H^0((\mathfrak{g}_\mathcal{E}/\mathfrak{p}^i)^*[\mathcal{K}_{\Sigma^i}]) = 0, \text{ if } \Sigma^i \cdot \Sigma^i \geqslant 0 \\ H^0(\mathfrak{g}_\mathcal{E}/\mathfrak{p}^i)^*[\mathcal{K}_{\Sigma^i} - \Sigma^i \cdot \Sigma^i]) = 0, \text{ if } \Sigma^i \cdot \Sigma^i < 0, \end{cases}$$

or, in the framed case

$$\forall i, \quad \begin{cases} H^0((\mathfrak{g}_\mathcal{E}/\mathfrak{p}^i)^*[\mathcal{K}_{\Sigma^i} + \Sigma^i \cdot D_\infty]) = 0, \text{ if } \Sigma^i \cdot \Sigma^i \geqslant 0 \\ H^0(\mathfrak{g}_\mathcal{E}/\mathfrak{p}^i)^*[\mathcal{K}_{\Sigma^i} + \Sigma \cdot (D_\infty - \Sigma^i)]) = 0, \text{ if } \Sigma^i \cdot \Sigma^i < 0 \end{cases}$$

and

$$\begin{cases} H^0(\mathcal{T}_0^0[\mathcal{K}_{D_\infty}]) = 0, \text{ if } D_\infty \cdot D_\infty \geqslant 0, \\ H^0(\mathcal{T}_0^0[\mathcal{K}_{D_\infty} - D_\infty \cdot D_\infty]) = 0, \text{ if } D_\infty \cdot D_\infty < 0, \end{cases}$$

or

(ii) $-(\mathcal{K}_X + \Sigma) \geqslant 0$ *is effective (resp.* $-(\mathcal{K}_X + \Sigma + 2D_\infty) \geqslant 0$*)*

then $\mathcal{M}^{P,K,*}_{k,c,\ell}$ *is locally a complex manifold (resp.* $\widetilde{\mathcal{M}}^{P,K,*}_{k,c,\ell}$ *is a complex manifold) with respect to the complex space structure of Proposition 5.40*

Proof. In (i) the short exact sequences (5.23) and (5.24) now become four steps sequences whose extra step is given by the incidence relations K. Since the third terms are

$$\bigoplus_i (\mathfrak{g}_\mathcal{E}/\mathfrak{p}^i \oplus \mathfrak{g}_\mathcal{E}/\mathfrak{p}^i[\Sigma^i]), \qquad \bigoplus_i (\mathfrak{g}_\mathcal{E}/\mathfrak{p}^i[D_\infty] \oplus \mathfrak{g}_\mathcal{E}/\mathfrak{p}^i[\Sigma^i + D_\infty]) \oplus \mathcal{T}^0 \oplus \mathcal{T}^0[D_\infty]),$$

the stated conditions suffice (and could in fact be replaced by weaker but more cumbersome ones) for the argument of (i) of Proposition 5.22 to apply. In (ii) exactly the same argument as in (ii) of Proposition 5.22 applies. □

Chapter 6

Hyperbolic monopoles and rational maps

We shall give two ways to prove Theorem 3 from the material of the previous Chapters. The first, which follows as an application of Theorem 1 has the interesting feature of exhibiting the monopoles moduli spaces as subvarieties of moduli spaces of parabolic framings all of whose underlying bundles are $\mathbb{C}^*$–equivariant but works "naturally" only for monopoles with admissible mass. The second, which is more direct and best shows why the moduli space of monopoles with $\phi(\infty) = [i\alpha]$ should depend on α only through $C(i\alpha)$, is insensitive to wether or not the mass is admissible but requires proving a slightly more general result than Theorem 1 in the case of $\mathbb{C}^*$–invariant bundles.

6.1 $\mathbb{C}^*$–equivariant bundles

For both approaches we need to establish a few facts about $\mathbb{C}^*$–equivariant bundles. For our purposes these will be holomorphic $G^{\mathbb{C}}$–bundles with a $\mathbb{C}^*$–action covering the $\mathbb{C}^*$–action on $\mathbb{P}^2$ obtained by complexifying the S^1–action on $\mathbb{C}^2$ by rotation on the planes normal to Σ (with respect to some fixed choice of complex coordinates) and extending it to $\mathbb{P}^2$. The extension, which is unique, has a further fixed point $\mathrm{q} \in D_\infty$ distinct from p_∞. Since Σ is fixed for the action, at each point of Σ we obtain a representation η of $\mathbb{C}^*$ in $G^{\mathbb{C}}$. Its conjugacy class is independent of the point and if a trivialisation of the bundle at a base point on Σ is chosen we can identify $\mathbb{C}^*$ with a subgroup of $G^{\mathbb{C}}$ and the representation is then of the form

$$\eta : \lambda \mapsto \exp(2\pi m \log(\lambda)), \tag{6.1}$$

for some $m = m(\eta) \in \ker(\exp(2\pi i))$ depending on η only and not on the choices called the weight of the action and the bundle is a weight m $\mathbb{C}^*$–equivariant bundle. Let $\mathcal{V}$ be the associated holomorphic vector bundle via some faithful representation ρ. This has an induced $\mathbb{C}^*$–action which at the base point we can identify with

$$\lambda \mapsto \exp(2\pi\rho(m)\log(\lambda)) = \lambda^{\rho(m)}, \tag{6.2}$$

making it into a weight $\rho(m)$ $\mathbb{C}^*$–equivariant holomorphic vector bundle and we identify the eigenvalues $\{m_i\}$ of $\rho(m)$ with integers. Let $\mathrm{q} \in D_\infty$ be such that the fibres of the line bundle structure obtained identifying $\mathbb{P}^2\backslash\mathrm{q}$ with the total space of $\mathcal{O}(1)$ coincide with the $\mathbb{C}^*$–orbits and let $\{U\}$ be a cover of $\mathcal{O}(1)$ obtained by pulling back along the fibres a cover of Σ.

Lemma 6.3. *Let $\mathcal{V}$ be a weight m $\mathbb{C}^*$–equivariant holomorphic vector bundle and let $\{m_i\}$ be the eigenvalues of m. Then*

(i) *$\mathcal{V}_{|\Sigma}$ has a holomorphic splitting according to the eigenvalues of m*

$$\mathcal{V}_{|\Sigma} = \underset{m_i}{\oplus} \mathcal{V}^{m_i}.$$

(ii) *$End(\mathcal{V})_{|\mathcal{O}(1)}$ has an algebraic cocycle such that the ij–component relative to the decomposition induced by (i) pulled back along the fibres is homogeneous of degree $m_i - m_j$. Vice–versa, a vector bundle with such a cocycle is weight m $\mathbb{C}^*$–equivariant.*

(iii) *A principal holomorphic $G^{\mathbb{C}}$–bundle $\mathcal{E}$ is weight m $\mathbb{C}^*$–equivariant if and only if the associated vector bundle $\mathcal{V}$ through a faithful representation ρ has a $G^{\mathbb{C}}$–cocycle satisfying (ii) for $\rho(m)$. In this case $\mathcal{E}_{|\Sigma}$ has a holomorphic reduction to $C(im)^{\mathbb{C}}$.*

Proof. (i) This is immediate.
(ii) On each open set of the cover $\{ U \}$, $\mathcal{V}_{|\Sigma}$ decomposes as a direct sum of weight m_i $\mathbb{C}^*$–representations. Let e_i and e'_i denote subsets of holomorphic frames for $\mathcal{V}$ on two intersecting sets U, V of the cover such that $< e_i >_{|\Sigma}$ and $< e'_i >_{|\Sigma}$ generate these weight m_i $\mathbb{C}^*$–representations and $\{e_i\}$, $\{e'_i\}$ are holomorphic frames for $\mathcal{V}$. By G.A.G.A. both the bundle and the action are algebraic so the frames and the transition functions g^{VU} can be chosen algebraic. Consider their block decomposition (g^{VU}_{ij}) relative to the frames. We use the same symbol to denote a block and any of its elements and manipulate blocks as numbers. Let z be a coordinate along the $\mathcal{O}(1)$–fibres in the intersection so that $\Sigma = \{ z = 0 \}$. We show that

$$\forall \lambda \in \mathbb{C}^*, \quad \lim_{z \to 0} \frac{g^{VU}_{ij}(\lambda z)}{g^{VU}_{ij}(z)} = \lambda^{m_i - m_j}, \tag{6.4}$$

which clearly implies the homogeneity. Define f and h by

$$\lambda(e'_i(z)) = \sum_j f_{ji}(z, \lambda) e'_j(\lambda z), \quad \lambda(e_i(z)) = \sum_j h_{ji}(z, \lambda) e_j(\lambda z). \tag{6.5}$$

Since e_i and e'_i restrict to weight m_i–representations on Σ,

$$\lim_{z \to 0} f(z, \lambda) = \lim_{z \to 0} h(z, \lambda) = \lambda^m.$$

Now, $e'_j(z) = \sum g^{VU}_{ij}(z) e_i(z)$ and it follows from (6.5) that

$$g^{VU}(\lambda z) f(z, \lambda) = h(z, \lambda) g^{VU}(z),$$

so

$$\lim_{z \to 0} g(\lambda z) = \lambda^{ad(-m)} (\lim_{z \to 0} g(z)),$$

which looking at the components proves (6.4). Suppose now that a holomorphic vector bundle $\mathcal{V}$ on $\mathcal{O}(1)$ has an algebraic cocycle $\{ g^{VU} \}$ satisfying the homogeneity property. Let $\{e_i\}$, $\{e'_i\}$ be two local algebraic frames along the $\mathcal{O}(1)$–fibres related by the cocycle so that $e_j(z) = \sum g^{VU}_{ij} e'_i(z)$. We define a weight m $\mathbb{C}^*$–action by

$$\lambda(e_i(z)) = \lambda^{m_i} e_i(\lambda z), \tag{6.6}$$

and extend it by linearity. Because of the homogeneity property of $\{ g^{VU} \}$ we have

$$\begin{aligned}\lambda(e'_j(z)) = \lambda^{m_j} e'_j(\lambda z) &= \lambda^{m_j} \sum g_{ij}^{VU}(\lambda z) e_i(\lambda z) \\ &= \lambda^{m_j} \sum \lambda^{m_i - m_j} g_{ij}^{VU}(z) e_i(\lambda z) \\ &= \sum g_{ij}^{VU}(z) \lambda(e_i(z)),\end{aligned}$$

so (6.6) is independent of the choice of the frame and defines a weight m $\mathbb{C}^*$–action on $\mathcal{O}(1)$. In particular $\mathcal{V}$ is an algebraic $\mathbb{C}^*$–equivariant bundle on $\mathbb{C}^2 \setminus \{0\} = \mathcal{O}(1) \setminus \Sigma$. Since all such bundles extend uniquely to $\mathbb{C}^*$–equivariant bundles over the whole $\mathbb{C}^2$ we get an extension of $\mathcal{V}$ to a weight m $\mathbb{C}^*$–equivariant bundle on $\mathbb{P}^2$.
(iii) The first statement follows from (ii), because the frames giving the cocycle of (ii) and used to define (6.6) can be chosen to be $G^{\mathbb{C}}$–frames. Then the cocycle of (ii) is a $G^{\mathbb{C}}$–cocycle, the action of (6.6) is a $G^{\mathbb{C}}$–compatible action and the same will hold for the extension. In particular the splitting of (i) is a $G^{\mathbb{C}}$–splitting which means that $\mathcal{E}_{|\Sigma}$ has a holomorphic $C(im)^{\mathbb{C}}$–reduction. □

6.2 $\mathbb{C}^*$–equivariant parabolic framings and rational maps

As recalled in §2.3, the topological type of a $C(im)^{\mathbb{C}}$–reduction on a $G^{\mathbb{C}}$–bundle on a complex line Σ is determined by a class $\ell' \in \pi_2(G/C(im))$ and taking $C = C(im)$ in (1.24) we can regard this as a class in $\pi_1(C(im)) = H_1(C(im))$. The representation (6.1) is a map $\mathbb{C}^* \to C(im)$ so determines a class in $H_1(C(im))$. If $\mathcal{E}$ is weight m $\mathbb{C}^*$–equivariant it must be trivial on any of the $\mathcal{O}(1)$–fibres of Lemma 6.3, in particular it follows as in §2.2 that we can restrict ourselves to the case where $G^{\mathbb{C}}$ is simple. Equivariant cohomology also shows that its 4–dimensional class can be identified with $k =< m, \ell' >$, where the pairing is induced by the Killing form of G on $H_1(C(im))$, so ℓ' is the only topological invariant of $\mathcal{E}$. Let $i\alpha'$ be an element in a compact form of $\mathfrak{g}^{\mathbb{C}}$ such that $ad(\alpha')$ has all its eigenvalues in $[-1, 1]$ and the eigenvalues of $ad(m)$ on the ± 1 eiegenspace of $ad(\alpha')$ are different from ± 1. Denote by $\widetilde{\mathcal{M}}^{\alpha'}_{m,\ell''}(\mathbb{P}^2, \Sigma)$ the subset of $\widetilde{\mathcal{M}}^{\alpha'}_{k,\ell}(\mathbb{P}^2, \Sigma)$ consisting of elements whose underlying bundle is weight m $\mathbb{C}^*$–equivariant with respect to an action such that the resulting $C(im)^{\mathbb{C}}$–reduction is compatible with the $P(\alpha')$–reduction. Since $C(im + i\alpha') = C(i\alpha') \cap C(im)$ in this case combining the two reductions we obtain a topological $C(m + \alpha')^{\mathbb{C}}$–reduction whose topological type ℓ'' is determined by ℓ' and ℓ". Vice–versa in this situation ℓ and ℓ' can be recovered from ℓ''. Recall that according to Definition 0.3 an element $i\alpha$ in a compact Lie algebra $\mathfrak{g}$ is admissible if it can be written in the form

$$i\alpha = im + i\alpha',$$

where $m \in \ker(\exp(2\pi i -))$ and α' is such that $ad(\alpha')$ has all its eigenvalues in $(-1, 1)$. It follows from Lemma 2.8 that this decomposition is unique. If α is not admissible a decomposition can be found with $ad(\alpha)$ having its eigenvalues in $[-1, 1]$. This is not unique, but it is uniquely determined if we fix a closed positive Weyl chamber $\mathfrak{h}^+$ and choose $m \in \mathfrak{h}^+$ to be the element farthest away from the origin with α' satisfying this

condition. In this case $ad(m)$ has eigenvalues greater or equal to two in modulus on the ± 1 eigenspaces of $ad(\alpha')$ and it still holds that $C(i\alpha' + im) = C(i\alpha') \cap C(im)$.

Proposition 6.7. *Let $i\alpha = im + i\alpha'$ be an element lying in some compact form of $\mathfrak{g}^{\mathbb{C}}$ with m and α' such that $m \in Ker(\exp(2\pi i -))$, $ad(\alpha')$ has its eigenvalues in $[-1, 1]$ and the ± 1 eiegenspaces of $ad(m)$ and $ad(\alpha')$ have trivial intersection. There is a natural identification*

$$\widetilde{\mathcal{M}}^{\alpha'}_{m,\,\ell''}(\mathbb{P}^2, \Sigma) \cong \widetilde{\mathcal{M}}^{\alpha}_{0,\,\ell''}(\mathbb{P}^2, \Sigma).$$

Proof. Since the unique holomorphic $G^{\mathbb{C}}$–bundle whose 2 and 4–dimensional classes vanish is the trivial one, the objects in the moduli space on the right hand side consist of $P(\alpha)$–reductions on the trivialised bundle $G^{\mathbb{C}} \times \mathbb{P}^2$ with prescribed value at p_∞. Consequently an object in the moduli space of associated ρ–vector bundles consists of a trivialised vector bundle $\mathcal{V}$ with a $\rho(\alpha)$–filtration on Σ pointing in a given direction at p_∞. Let σ and τ be as in Construction 3.27 and notice that $|\frac{\sigma}{\tau}|$ is homogeneous of degree 1 along the $\mathbb{C}^*$–orbits different from D_∞. Let $m_1 > \cdots > m_t$ be the eigenvalues of $\rho(m)$, $\oplus_i \alpha_i$ the corresponding decomposition of $\rho(\alpha')$ and $\alpha_{i1} > \cdots > \alpha_{it_i}$ the eigenvalues of α_i so that the $\rho(\alpha)$–filtration has elements $\{\mathcal{F}_i^j\}$ and $\mathcal{F}_i^j$ has weight $m_i + \alpha_{ij}$. Let $\{U\}$ be a cover of Σ with filtered frames $\{e_i^U\}$ for $\mathcal{V}_{|\Sigma}$ so that $< e_i^U >= \oplus_j \mathcal{F}_i^j / \mathcal{F}_i^{j+1}$. Note that at p_∞ there is a preferred splitting given by the framing and if U and V overlap at D_∞ we can assume the splittings chosen so that the transition function g^{VU} is 1. By the triviality of $\mathcal{V}$ the frames and the resulting cocycle pull–back to frames and cocycle for $\mathcal{V}$ along the $\mathcal{O}(1)$–fibres. We define a new sheaf on $\mathcal{O}(1)$ by

$$\Gamma(\widetilde{\mathcal{V}})_{|U} = \mathcal{O} < |\frac{\sigma}{\tau}|^{-m_i} e_i^U > . \tag{6.8}$$

This definition gives rise to a weight m $\mathbb{C}^*$–equivariant bundle on $\mathbb{P}^2$ with compatible $G^{\mathbb{C}}$–structure and α'–framed parabolic bundle structure. Indeed $\{g^{VU}\}$ lies in $P(\rho(m + \alpha'))$, is constant along the $\mathcal{O}(1)$–fibres and is the identity near D_∞ thus, letting $(g^{VU}_{ii'})$ be its block decomposition, and $(g^{VU}_{jj'})$ the block decomposition of $g^{VU}_{ii'}$,

$$g^{VU}_{ii'} = 0 \quad \text{if } m_i < m_{i'}, \quad g^{VU}_{jj'} = 0 \quad \text{if } \alpha_{ij} < \alpha_{ij'}. \tag{6.9}$$

The transition functions we obtain for $\widetilde{\mathcal{V}}$,

$$\tilde{g}^{VU} = (\tilde{g}^{VU}_{ij}) = (|\frac{\sigma}{\tau}|^{m_i - m_j} g^{VU}_{ij}), \tag{6.10}$$

are then holomorphic across Σ and by the properties of g^{VU}_{ij} give a cocycle satisfying (ii) of 6.3 which is the identity on D_∞. Thus the sheaf defines a framed weight m $\mathbb{C}^*$–equivariant bundle. In particular we have the splitting of 6.3 (i) and because of (6.9) each $\tilde{\mathcal{V}}_{m_i}$ has an α_i–filtration $\{\mathcal{F}_i^j\}$. Ordering the α_{ij} we can define a compatible $P(\alpha')$–reduction by letting

$$\tilde{\mathcal{F}}_{t(ij)} = \bigoplus_h \bigoplus_{\alpha_{hk} \leqslant \alpha_{ij}} \mathcal{F}_h^k.$$

The $G^{\mathbb{C}}$–structure is immediate because the frames and transition functions used in (6.8) and (6.9) can be taken to be $G^{\mathbb{C}}$ ones if $\mathcal{V}$ comes from a principal framed parabolic bundle. For the topological type observe that by the way the cocycle $\tilde{g}^{VU}$ is defined the type of the $C(\rho(im + i\alpha'))$–reduction on Σ will coincide with that of the original $P(\rho(m + \alpha'))$–reduction by the argument of Proposition 3.34 (ii). It is clear how to define an inverse transform starting from an α'–parabolic framed weight $\rho(m)$ $\mathbb{C}^*$–equivariant bundle $\widetilde{\mathcal{V}}$. The splitting of 6.3 (i) implies that we can define α_i by looking at the filtration induced by the $P(\alpha')$–filtration on $\mathcal{V}_{m_i}$. We choose an open cover $\{ U \}$ with frames $\{ \tilde{e}_i^U \}$ along the fibres intertwined by the cocycle $\{ \tilde{g}^{VU} \}$ of 6.3 and compatible with the combined m and α'–filtrations and define

$$\Gamma(\widetilde{\widetilde{\mathcal{V}}})_{|U} = \mathcal{O} < |\frac{\sigma}{\tau}|^{m_i} \tilde{e}_i^U > . \tag{6.11}$$

The cocycle $\{ \tilde{g}^{VU} \}$ then gives a cocycle

$$\tilde{\tilde{g}}^{VU} = (\tilde{\tilde{g}}_{ij}^{VU}) = (|\frac{\sigma}{\tau}|^{m_j - m_i} \tilde{g}_{ij}^{VU}). \tag{6.12}$$

which, by the homogeneity properties of $\{ \tilde{g}^{VU} \}$, is holomorphic across Σ, constant along the $\mathcal{O}(1)$–fibres, and one on D_∞. By 6.3 (ii) this defines a framed weight zero $\mathbb{C}^*$–equivariant bundle, that is a bundle with a holomorphic trivialisation on D_∞. From (6.12) and the compatibility with the α_i–filtrations we also see that the cocycle lies in $P(\rho(m + \alpha'))$ on Σ, giving the required filtration, and again by 3.34 (ii) that the topological type of the $C(im + i\alpha')$–reduction is preserved. As above the construction can be made so as to obtain a compatible $G^{\mathbb{C}}$–structure if $\tilde{\mathcal{V}}$ has a $G^{\mathbb{C}}$–structure by taking the cocycle of 6.3 (ii) to be a $G^{\mathbb{C}}$–cocycle. □

As recalled in §2.1 for a compact connected simple Lie group G and a subgroup of the form $C(i\alpha)$, $[\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha)] = \pi_2(G/C(i\alpha))$ so the topological type of a rational map $\mathbb{P}^1 \to G^{\mathbb{C}}/P(\alpha)$ is determined by a class $\ell \in \pi_2(G/C(i\alpha))$. Let $\mathcal{R}_\ell^*(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))$ denote the space of based rational maps of topological type ℓ into the complex homogeneous space $G^{\mathbb{C}}/P(\alpha)$, that is rational maps taking a fixed point in $\mathbb{P}^1$ onto a fixed point in $G^{\mathbb{C}}/P(\alpha)$. There is a simple interpretation of $\widetilde{\mathcal{M}}_{0,\ell}^{P(\alpha)}(\mathbb{P}^2, \Sigma)$ in terms of these maps.

Proposition 6.13. *For every $i\alpha$ lying in some compact form of $\mathfrak{g}^{\mathbb{C}}$ and every $\ell \in \pi_2(G^{\mathbb{C}}/P(\alpha))$ there is a natural identification*

$$\widetilde{\mathcal{M}}_{0,\ell}^{G^{\mathbb{C}}, P(\alpha)}(\mathbb{P}^2, \Sigma) \cong \mathcal{R}_\ell^*(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha)).$$

Proof. As remarked in the proof of Proposition 6.7, an object in $\widetilde{\mathcal{M}}_{0,\ell}^{G^{\mathbb{C}}, P(\alpha)}(\mathbb{P}^2, \Sigma)$ is a $P(\alpha)$–reduction s on Σ of the trivialised bundle $G^{\mathbb{C}} \times \mathbb{P}^2$ taking a prescribed value at p_∞. The reduction is a holomorphic section of the associated $G^{\mathbb{C}}/P(\alpha)$–analytic fibre bundle which in this case is trivial so gives a holomorphic and hence, by G.A.G.A., rational map $s : \mathbb{P}^1 \to G^{\mathbb{C}}/P(\alpha)$ based at p_∞. The topological type of this map is of course the same as that of s as a reduction. Vice–versa a based rational map of a certain topological type into $G^{\mathbb{C}}/P(\alpha)$ gives tautologically a reduction of $G^{\mathbb{C}} \times \mathbb{P}^2$ to $P(\alpha)$ of the same topological type taking a prescribed value at p_∞. □

We thus obtain from Proposition 6.7

Corollary 6.14. *For every $i\alpha = im + i\alpha'$ lying in some compact form of $\mathfrak{g}^{\mathbb{C}}$ as in Proposition 6.7 there is a natural identification*

$$\widetilde{\mathcal{M}}^{G^{\mathbb{C}},\alpha'}_{m,\ell''}(\mathbb{P}^2,\Sigma) \cong \mathcal{R}^*_{\ell''}(\mathbb{P}^1,\, G^{\mathbb{C}}/P(\alpha))$$

6.3 The equivariant approach to Theorem 3

We now give the first proof of Theorem 3. First, however, a few remarks concerning the relation between hyperbolic monopoles and S^1–equivariant instantons. If $(\tilde{A},\, \phi)$ is a pair consisting of a connection on a (necessarily trivial) G–bundle $E \longrightarrow \mathbb{H}^3$ and a section of the adjoint bundle, we can pull–back E to a bundle on $\mathbb{R}^4\backslash\mathbb{R}^2$ so that $A = \tilde{A} + \phi d\theta$ is an S^1–equivariant connection with respect to the trivial S^1–action on E, that is S^1–invariant. Vice–versa, given an S^1–equivariant connection A on an S^1–equivariant G–bundle $E \longrightarrow \mathbb{R}^4\backslash\mathbb{R}^2$ which has connection matrix $A = \tilde{A} + \phi d\theta$, in an S^1–equivariant gauge, the pair $(\tilde{A},\, \phi)$ is S^1–invariant so is a hyperbolic monopole on $E/S^1 \longrightarrow \mathbb{H}^3$. By the appropriate clause of Theorem 1.14 the gauge of 1.14 (ii) can be chosen S^1–invariant and the limit $\phi(A)$ is then actually the limit $\phi(\infty)$ of the Higgs field at infinity which is therefore well defined as a conjugacy class in $\mathfrak{g}$ and exists along almost every direction to ∞ in $\mathbb{H}^3$, in fact every direction for hyperbolic monopoles.

As in Lemma 2.8 fix a pair $(\mathfrak{h},\, \mathfrak{h}^+)$ consisting of a C.S.A. and a closed positive Weyl chamber. As remarked before Proposition 6.7, in any class $[i\alpha] \in \mathfrak{g}$ there is a unique representative $i\alpha = i\alpha' + im \in \mathfrak{h}^+$ with $ad(\alpha')$ having all its eiegenvalues in $[-1, 1]$ and $m \in \mathfrak{h}^+$ as far as possible from the origin. Applying 1.14 (iii) to the S^1–equivariant instanton coming from a hyperbolic monopole with $\phi(\infty) = [i\alpha]$, at each point we can find a gauge where $\phi(A) = i\alpha'$ is as above on every normal disc to $\mathbb{R}^2$. It follows that in this gauge the S^1–action must induce a weight m representation of S^1 on the fibres on $\mathbb{R}^2$, that is it can be identified up to conjugacy with the restriction of (6.1) to S^1 and that on an associated vector bundle with the restriction of (6.2) to S^1.

Vice–versa if A is an S^1–equivariant instanton with residue α' and weight m action in the gauge of 1.14 (iii), pulling back by $\exp(2\pi i m\theta)$ we obtain an equivariant gauge so that the corresponding hyperbolic monopole has $\phi(\infty) = [i\alpha' + im]$. Associated to A is a weight m S^1–equivariant H.Y.M. bundle (that is both the G–reduction and holomorphic structure are S^1–equivariant) whose Chern connection has holonomy $J = \exp(-2\pi i\alpha')$. We let $M^J_m(\mathbb{C}^2,\, \Sigma)$ denote the moduli space of such objects. For future use we restate Proposition 1.29 in the S^1–equivariant case.

Proposition 6.15. *Let A be a connection on a $U(n)$–bundle $V \to S^4\backslash S^2$ with holonomy $J = \exp(-2\pi i\alpha)$ as in Proposition 1.29 and now also assume that A is equivariant with respect to a weight m S^1–action with $ad(m)$ having eiegenvalues different from ± 1 on the ± 1 eiegenspaces of $ad(\alpha)$. Then the extension data $(\overline{V}, \overline{A}, A_\alpha)$ can be chosen to be S^1–equivariant for a weight m action extending that on V. The extension is the unique one satisfying (i) and (ii) of Proposition 1.29 which is also S^1–equivariant in the sense that if $g(V, A) \cong (V', A')$ is a weight m' S^1–equivariant bundle on $S^4\backslash S^2$ with extension*

data $(\overline{V}, \overline{A}, A_{\alpha'})$ *satisfying the same properties then* g *extends to an* S^1*–equivariant* $L^p_{2,\alpha}$*–isomorphism* $\overline{g}(\overline{V}', \overline{A}') \cong (\overline{V}, \overline{A})$. *If* (V, A) *had an* S^1*–equivariant* G*–structure,* $(\overline{V}, \overline{A}, A_\alpha)$ *can be chosen also to have one extending that on* (V, A) *and the data is then* G*–unique.*

Proof. The main point is to check that the extension is now defined also in the case when α is not admissible. Observe that allthough now $ad(\alpha)$ might have ± 1 among its eiegenvalues, by the assumptions on m this happens on eigenspaces where $ad(m)$ has eiegenvalues different from ± 1. It follows that 1.8 and 1.11 can be applied so that the G–cocycle constructed in Proposition 1.29, which there was continuous only when $ad(\alpha)$ had all its eiegenvalues in $(-1, 1)$, is now continuous across Σ in the non–admissible case too and lies in $C(i\alpha)$ so that the extension data of Proposition 1.29 is defined. Further, since as already observed the gauge transformations carrying the connection in the gauge of 1.14 (iii) can be taken to be S^1–equivariant, we can ensure that the cocycle will be S^1–equivariant so that $\overline{V}$ and $\overline{A}$ will be weight m S^1–equivariant bundles and connection respectively. As for A_α, notice that in these gauges the model flat connection d_α is weight m S^1–equivariant so we can construct a weight m S^1–equivariant model singular connection A_α on $\overline{V}$ such that $\overline{A} - A_\alpha \in L^p_{1,\alpha}$.
For the uniqueness the same proof as in 1.29 will work by using the clause on S^1–equivariant sections in 1.8 and 1.11 to prove that f will extend also when $ad(\alpha)$ has ± 1 among its eigenvalues.
The remark about the G–structure follows as in 1.29 □

To handle the case of non–admissible holonomies we improve Construction 3.27 using the following fact.

Lemma 6.16. *Let* C *be a rational curve in a complex surface* X *such that* $C \cdot C \leqslant 1$. *If a holomorphic* $G^{\mathbb{C}}$*–bundle* $\mathcal{E}$ *on* X *is holomorphically trivial on* C *then it is also holomorphically trivial on* C_1, *the first infinitesimal neighbourhood of* C *in* X.

Proof. The essential point is to check that under the stated conditions, the restriction map gives an isomorphism $\mathrm{Pic}(C_1) \cong \mathrm{Pic}(C)$. To see this observe that one has the following exact sequence of sheaves on C_1

$$0 \longrightarrow \mathcal{O}_C(-C) \xrightarrow{\exp} \mathcal{O}^*_{C_1} \xrightarrow{|_C} \mathcal{O}^*_C \longrightarrow 0$$

Indeed one has tautologically that the kernel of the restriction map above is $\mathcal{J}_C/{\mathcal{J}_C}^2$, the ideal sheaf of C in C_1 but as a sheaf on C_1 this is isomorphic to $\mathcal{O}_C(-C)$ and the sequence follows. Since $\ell = C \cdot C \leqslant 1$, $H^1(C_1, \mathcal{O}_C(-C)) = H^1(\mathbb{P}^1, \mathcal{O}(-\ell)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2+\ell))^* = 0$ so from the cohomology sequence associated to the above exact sequence we deduce the desired isomorphism between $\mathrm{Pic}(C_1)$ and $\mathrm{Pic}(C)$. The same argument actually shows that restriction induces an isomorphism on the cohomology spaces of each line bundle so, passing to the associated bundle $\mathcal{V}$ and proceeding exactly as for $\mathbb{P}^1$ ([OSS], p 22), one shows that $\mathcal{V}$ has a $G^{\mathbb{C}}$–compatible reduction to a direct sum of invertible sheaves. This reduction will commute with the restriction so, by the above isomorphism, if all the summands of $\mathcal{V}_{|C}$ are trivial the same must be true on C_1 so $\mathcal{E}_{|C_1}$ is trivial. □

The following Proposition together with Corollary 6.13 gives Theorem 3. Recall that $M^\alpha(\mathbb{H}^3)$ denotes the moduli space of hyperbolic monopoles with $\phi(\infty) = [i\alpha]$.

Proposition 6.17. *For every conjugacy class* $[i\alpha] = [im + i\alpha'] \in \mathfrak{g}$ *with* α' *and* m *as above, there is a bijection*

$$M^{\alpha}(\mathbb{H}^3) \cong \widetilde{\mathcal{M}}_m^{\alpha'}(\mathbb{P}^2, \Sigma)/C(i\alpha).$$

Proof. The previous remarks identify $M^{\alpha}(\mathbb{H}^3)$ and $M_m^J(\mathbb{C}^2, \Sigma)$, where $J = \exp(-2\pi i\alpha')$ thus we need to identify the latter with $\widetilde{\mathcal{M}}_m^{\alpha'}(\mathbb{P}^2, \Sigma)/C(i\alpha)$. This amounts to showing that in the equivariant case Theorem 1 extends to non–admissible holonomies and is compatible with the actions. We begin by extending Theorem 3.55. Since Propositions 3.48 holds for any connection of the form $d_\alpha + a$ where $i\alpha \in \mathfrak{g}$ provided $a \in L^\infty$ it follows that the existence part of 3.55 goes through for any α' such that $ad(\alpha')$ has all its eigenvalues in $[-1, 1]$. Next we show that given a weight m S^1–equivariant H.Y.M. bundle, the extension of the associated H.Y.M. vector bundle $(\mathcal{V}, h)$ is $\rho(m)$ $\mathbb{C}^*$–equivariant. By the previous Proposition we know that that the topological extension of Proposition 1.29 can be taken weight m S^1–equivariant. Starting from this and considering gauges where the bounds of 1.14 (iii) hold and the action is of weight m, the complex gauge transformation $\tilde{g}_i$ of (3.59) giving the holomorphic trivialisation on an open set U_i has the form $\tilde{g}_i = r^{\rho(\alpha')} \circ f_i$ with f_i continuous. By the compactness of S^1 and the equivariance of the connection, f_i can be chosen S^1–equivariant. The gauge transformation relating two such unitary gauges is also S^1–equivariant so $f_i g_{ij} f_j^{-1}$ is S^1–equivariant. Thus the holomorphic cocycle $\tilde{g}_{ij} = r^{ad(-\rho(\alpha'))}(f_i g_{ij} f_j^{-1})$ defining the extension is S^1–equivariant and intertwines holomorphic gauges where the action is of weight $\rho(m)$ hence complexifies to a weight $\rho(m)$ $\mathbb{C}^*$–action. Then $\tilde{g}_{ij}$ extends to a cocycle along the $\mathbb{C}^*$–fibres satisfying 6.3 (ii).
Finally we extend the uniqueness of 3.55 (ii). We prove that if $\widehat{\alpha} = \widehat{m} + \widehat{\alpha}'$ is another element in $\mathfrak{g}$ as above and $f : (\mathcal{E}, \lambda) \cong (\widehat{\mathcal{E}}, \widehat{\lambda})$ ian isomorphism of $\mathbb{C}^*$–equivariant H.Y.M. bundles where $(\widehat{\mathcal{E}}, \widehat{\lambda})$ has a weight $\widehat{m}$ $\mathbb{C}^*$–equivariant extension satisfying 3.55 for $\widehat{\alpha}'$, then $m = \widehat{m}$, and f extends to an equivariant isomorphism as in 3.55. As usual let $(\mathcal{V}, h)$ and $(\widehat{\mathcal{V}}, \widehat{h})$ be the associated H.Y.M. vector bundles through some faithful representations. As in Proposition 6.11 choose local frames compatible with the combined $P(\rho(\alpha'))$–$C(\rho(m))^{\mathbb{C}}$ and $P(\widehat{\rho}(\widehat{\alpha}'))$–$C(\widehat{\rho}(\widehat{m}))^{\mathbb{C}}$–filtrations on the associated vector bundles and on which the actions are standard of weight $\rho(m)$ and $\widehat{\rho}(\widehat{m})$, that is given by (6.2). Applying the gauge transformations $r^{\rho(\alpha')}$ and $r^{\widehat{\rho}(\widehat{\alpha}')}$ we obtain gauges in which the induced map g between the respective adjoint bundles thought of as subbundles of $End(\mathcal{V})$ and $End(\widehat{\mathcal{V}})$ respectively, is bounded with its inverse, intertwines standard weight $ad(m)$ and $ad(\widehat{m})$ actions and satisfies the equation

$$\overline{\partial} f + [ad(\widehat{\alpha}') \circ g - g \circ ad(\alpha')]\frac{d\overline{z}}{2\overline{z}} = 0.$$

The boundedness of g^{-1} implies that $m = \widehat{m}$. Since a bounded S^1–equivariant solution of $\overline{\partial} g + \gamma \frac{d\overline{z}}{\overline{z}} f = 0$ is continuous provided γ is not the weight of the action and since $ad(\alpha')$, $ad(\widehat{\alpha}')$ have all their eigenvalues in $[-1, 1]$ while $ad(m)$ has eigenvalues strictly greater than one in moduluis on the ± 1 eigenspaces of $ad(\alpha')$ and $ad(\widehat{\alpha}')$, it follows as in 3.55 that $\alpha' = \widehat{\alpha}'$ and f extends to a $\mathbb{C}^*$–equivariant parabolic isomorphism of the extensions. Similarly at infinity.

In view of Theorem 3.26 to prove the converse it suffices to show that on a framed $\mathbb{C}^*$–equivariant α'–parabolic bundle $\mathcal{E}$, the initial metric of Construction 3.27 on an associated bundle through a faithful representation $\rho : G^{\mathbb{C}} \to GL(n)$ can be chosen S^1–equivariant and satisfying Proposition 3.34 also when $ad(\alpha')$ has ± 1 among its eigenvalues. The constant Hermitian metric fixed on D_∞ in (ii) of 3.27 is automatically S^1–equivariant and the corresponding orthogonal decomposition can be extended S^1–equivariantly and compatibly with the $G^{\mathbb{C}}$–structure on $\mathcal{E}_{|\Sigma}$. We pull–back this along the $\mathcal{O}(1)$ fibres to obtain an S^1–equivariant $G^{\mathbb{C}}$–metric h' on $\mathbb{P}^2\backslash\{\mathrm{q}\}$. Let q' be a point on Σ different from p_∞. Now consider the $\mathcal{O}(1)$–structure on $\mathbb{P}^2\backslash\{\mathrm{q}'\}$ obtained by projecting onto D_∞ from q'. This projection is S^1–equivariant and so we can pull–back the reduction on D_∞ to an S^1–equivariant reduction h_t on $\mathbb{P}^2\backslash\{\mathrm{q}'\}$. We now choose the cut–off function β of Construction 3.27 taking care that it be rotationally symmetric with respect to the angular coordinate in the $\mathbb{C}^*$–fibres and glue the two reductions as done there. Finally we introduce $\rho(\alpha')$ singularities along Σ and $-\rho(\alpha')$ singularities along D_∞ again as in Construction 3.28. This can be done equivariantly because the function $\chi = |\frac{\sigma}{\tau}|$ is constant along the S^1–orbits. Since $\mathcal{E}$ is trivial on D_∞, by the previous Lemma it is also trivial on the first infinitesimal neighbourhood. The above metric h' extending that on D_∞ can thus be assumed constant and compatible with the orthogonal holomorphic splitting induced by the parabolic filtration and h_t near p_∞ up to first order. We can then improve (3.45) to

$$a_z = |w|^2 \tilde{a}_z.$$

Further, (iii) of Lemma 6.3 means that near Σ we can choose a unitary gauge so that in the decomposition induced by $ad(\rho(m + \alpha'))$ analogous to (3.37), the components $a^{ij}_{\bar{w}}$ relative to non–zero eigenvalues of $ad(\rho(m))$ vanish identically. By choice of the decomposition $\alpha = m + \alpha'$, $ad(m)$ has eigenvalues strictly greater than one in modulus on the ± 1 eigenspace of $ad(\alpha')$ so that if $ad(\alpha')$ has ± 1 among its eiegenvalues, in the ensuing estimates we can replace -1 with $-\alpha'_0$ along Σ and with $1 - \alpha'_0$ along D_∞, thus conditions 3.6 and 3.16 are still satisfied. □

We shall prove in the next section the statement concerning the topological invariants.

6.4 The invariant approach to Theorem 3

We now describe a slightly different and more direct approach to Theorem 3. For this we need yet another addendum to Proposition 1.29 in the S^1–invariant case.

Proposition 6.18. *Let $(\tilde{A}, \phi)$ be a hyperbolic monopole with $\phi(\infty) = [i\alpha]$ on a $U(n)$–bundle $V \to \mathbb{H}^3$ and let A be the corresponding S^1–invariant instanton on $S^4\backslash S^2$. Then there is S^1–invariant extension data $(\overline{V}, \overline{A}, A_\alpha)$ such that (i) and (ii) of 1.29, the uniqueness among S^1–invariant extensions as in 6.16 and the clause on the G–structure hold.*

Proof. As in 6.16 the main point is to show that the extension can still be defined though now the eigenvalues of $ad(\alpha)$ are arbitrary. As remarked above, on a standard pair $(U_i, U_i \cap \Sigma)$ we can find an S^1–invariant gauge where the connection has the form

$$g_i(A) = d_\alpha + a_i,$$

with a_i S^1–invariant and satisfying the bounds of 1.14 for monopoles. Because of the S^1–invariance we can apply the relevant clauses in Lemmas 1.8 and 1.11 to construct the extension data of Proposition 1.29 starting from the gauges above. Using the stronger pointwise bounds for invariant monopoles given by 1.14 (iii) we now obtain control on the off–diagonal blocks of the cocycle used to construct $\overline{E}$, in term of the eigenvalues of α, though these may contain arbitrary integers, to obtain that the cocycle of 1.29 is continuous across S^2, lies in $C(i\alpha)$ there and as in Proposition 6.15 can be chosen S^1–equivariant. The remaining claims are as in Proposition 6.15. □

The alternative proof of (i) of Theorem 3 will follow from the next Proposition and Corollary 6.13.

Proposition 6.19. *For every conjugacy class* $[i\alpha] \in \mathfrak{g}$ *there is a bijection*

$$M^{\alpha}(\mathbb{H}^3) \cong \widetilde{\mathcal{M}}_0^{\alpha}(\mathbb{P}^2, \Sigma)/C(i\alpha).$$

Proof. Let $(\tilde{A}, \phi)$ be a hyperbolic monopole with $\phi(\infty) = [i\alpha]$, let A the associated S^1–invariant instanton and apply the previous Proposition to obtain S^1–invariant topological extension data. In particular, since the bundle $\overline{E}$ is S^1–invariant, it will be topologically trivial. Consider a gauge compatible with the $C(i\alpha)$–reduction and where $\overline{A}$ satisfies the bounds of 1.14 (iii) for monopoles. Since α and β were assumed to be arbitrary real diagonal matrices in Proposition 3.48, we can now apply that proposition to the $\overline{\partial}$–operator on an associated vector bundle via some faithful representation $\rho : G \to G^{\mathbb{C}}$ in this gauge to solve the $\overline{\partial}$–equation. By the S^1–invariance of a and the compactness of S^1 we obtain an S^1–invariant continuous $G^{\mathbb{C}}$–valued complex gauge transformation f_i pulling–back the $\overline{\partial}$–operator to

$$\overline{\partial} - \rho(\alpha)\tfrac{d\overline{z}}{2\overline{z}}$$

so applying the gauge transformation $r^{\rho(\alpha)}$ we obtain a holomorphic frame. Similarly at points on D_{∞}. Because of the pointwise bounds in terms of the eigenvalues of $\rho(\alpha)$ on the connection matrix we can use the relevant clause in 3.48 to ensure that the holomorphic cocycle

$$\tilde{g}_{ij} = r_u^{ad(\rho(-\alpha))} r_v^{ad(\rho(\alpha))}(f_i g_{ij} f_j^{-1}),$$

we obtain as in Theorem 3.55, where g_{ij} is the S^1–invariant cocycle of the topological extension given by the previous Proposition, still lies in $P(\rho(\alpha))$ on Σ and in $C(\rho(i\alpha))$ on D_{∞}. This is then a $G^{\mathbb{C}}$–valued cocycle holomorphic across $\Sigma \cup D_{\infty}$, $\mathbb{C}^*$–invariant, lying in $P(\rho(\alpha))$ on Σ and in $C(\rho(i\alpha))$ on D_{∞} and the corresponding bundle will satisfy (3.59) for $\rho(\alpha)$. Since the extension is now $\mathbb{C}^*$–invariant, that is equivariant with respect to a weight 0 $\mathbb{C}^*$–action, it must be holomorphically trivial so we obtain an object in $\widetilde{\mathcal{M}}_0^{\alpha}(\mathbb{P}^2, \Sigma)/C(i\alpha)$. Going the other direction, we start from an element in $\widetilde{\mathcal{M}}_0^{\alpha}(\mathbb{P}^2, \Sigma)$ so, as observed in 6.11, a trivialised holomorphic $G^{\mathbb{C}}$–bundle with a holomorphic $P(\alpha)$–reduction on Σ compatible with the framing. As in the previous proposition we can make sure that the metric h of Construction 3.28 on an associated bundle via a faithful representation ρ is S^1–invariant. Exploiting the triviality of the underlying bundle we can substitute (3.41) and (3.45) with

$$\bigoplus_{i<j} a_w^{ij} = 0, \qquad a_z = 0,$$

and thanks to this the metric h_α defined as in 3.28 will still satisfy Proposition 3.34 irrespectively of how large the eigenvalues of α may be so that Theorem 3.26 can be applied. As in Proposition 6.17 we can use the S^1–invariance to prove that the two constructions are mutually inverse for any α. □

With either approach, to conclude the proof of Theorem 3 we have to prove that the topological invariant of the rational map can be identified with that of the monopole it corresponds to. Let A be an S^1–invariant instanton with $\phi(A) = [i\alpha] = [im + i\alpha']$ as above so that it corresponds to a weight m S^1–equivariant instanton with holonomy $J = \exp(-2\pi i\alpha')$. In the invariant setting Proposition 6.19 assigns to A a pair $(\overline{E}, s)$ consisting of the trivial G–bundle and a $C(i\alpha)$–reduction. In the weight m equivariant setting, Proposition 6.16 assigns to A a pair $(\overline{E}, s)$ with $\overline{E}$ a weight m S^1–equivariant G–bundle and s an $C(\alpha')$–reduction. As in §6 (ii) the action induces a $C(m)$–reduction which combines with s to give a $C(i\alpha)$–reduction and the topological type of the pair can be recovered from this and the weight of the action. The $C(i\alpha)$–reductions one obtains in the two setups are the same. Indeed the cocycles used to construct them lie in $C(i\alpha)$ on S^2 and by the triviality of the normal bundle of S^2 in S^4 we can assume they are related by gauge transformations of the form $Ad(\exp(2\pi im\theta))$ which act trivially on them on S^2. The argument of Proposition 3.34, together with that used in Proposition 6.17 to extend 3.55 to the case of non–admissible holonomies or the one in §6.2 for arbitrary α, will guarantee that the topological type of the $C(i\alpha)$–reduction is preserved in the correspondence in either of the two setups. To prove (ii) of Theorem 3 we thus only need to identify the magnetic charge of a hyperbolic monopole with $\phi(\infty) = [i\alpha]$ with the topological type of the $C(i\alpha)$–reduction associated to the corresponding S^1–invariant instanton.

Lemma 6.20. *The magnetic charge of a monopole with $\phi(\infty) = [i\alpha]$ is the same as the topological type of the $C(i\alpha)$–reduction associated to the corresponding S^1–invariant instanton on $S^4\backslash S^2$.*

Proof. We recall the definition of the magnetic charge. Applying 1.14 (iii) to the S^1–invariant instanton corresponding to a monopole with $\phi(\infty) = [i\alpha]$, we can construct a global S^1–invariant gauge where the Higgs field has limit along every direction at infinity. This is obtained by gluing local S^1–invariant gauges where the limit of the Higgs field is $i\alpha$ along each direction with rotationally symmetric cut–off functions and we can assume that only two gauges are needed. We can ensure that given a base point on S^2 the limit is actually $i\alpha$ on that direction. The limit along any other direction x in $\mathbb{H}^3$ is then of the form

$$\lim_{r\to\infty} \phi(rx) = Ad(g(x))(i\alpha), \quad g(x) \in G.$$

The assignment $x \mapsto g(x)$ descends to a map

$$\phi : S^2 \longrightarrow G/C(i\alpha) \tag{6.21}$$

and its homotopy class, which is independent of the choices made, is the magnetic charge. In associating the pair $(\overline{E}, s)$ to the S^1–invariant instanton as in Propositiobn 6.18 we can also assume that only two invariant gauges are used to cover S^2. Let g_1 and g_2 be the corresponding gauge transformations so that $g_{12} = g_1 \circ g_2^{-1}$ lies in $C(i\alpha)$ on S^2. The

topological type of the $C(i\alpha)$–reduction is determined by a class in $\pi_1(C(i\alpha))$ which can be concretely realized as the homotopy class of the restriction of g_{12} to an equatorial $S^1 \subseteq S^2$. To define the magnetic charge we can start from the gauge obtained gluing the g_i's and the map of (6.21) is then induced by a map in G of the form

$$\exp(\beta u_1 + (1-\beta)u_2),$$

where β is a cut–off function and $\exp(u_i) = g_i$. Taking $C = C(i\alpha)$ in (1.24) we have the exact sequence

$$0 \longrightarrow \pi_2(G/C(i\alpha)) \longrightarrow \pi_1(C(i\alpha)) \longrightarrow \pi_1(G) \longrightarrow 0,$$

where the injection can be identified with restriction to an equatorial $S^1 \subseteq S^2$. It follows from the form given for the map inducing ϕ that its restriction and that of g_{12} to an equatorial S^1 are homotopic. □

6.5 Moduli spaces of hyperbolic monopoles

Using Propositions 6.15 and 6.18 and the results of Chapter 4 we can construct moduli spaces of hyperbolic monopoles of a given mass and magnetic charge. As for the proof of Theorem 3 we can follow two slightly different routes. The first is to identify monopoles of mass $[i\alpha] = [im + i\alpha']$, where m and α' are as at the beginning of §6.3, with weight m S^1–equivariant instantons having holonomy $J = \exp(-2\pi i\alpha')$. By Proposition 6.15 we can attach to the instanton coming from a monopole a pair of topological invariants κ, ℓ' which by Lemma 6.22 and the remarks at the beginning of §6.3 are equivalent to knowledge of the mass m and magnetic charge ℓ. Using the fact that the extension pair $(\overline{E}, \overline{A})$ and model singular connection $A_{\alpha'}$ can be chosen S^1–equivariant, for S^1–equivariant connections we have addenda to Lemma 1.37 analogous to those of Lemma 1.8 and we denote by $L^{p,m}_{k,\alpha'}(\Omega^i(\mathfrak{g}_{\overline{E}}))$ the space of S^1–equivariant $L^p_{1,\alpha'}$ i–forms. Note in particular that this is now defined also when $ad(\alpha')$ has ± 1 among its eigenvalues. We define the space of weight m S^1–equivariant $L^p_{1,\alpha'}$ connections to be

$$\mathcal{A}^{p,m}_J = \{\, A_\alpha + a \,|\, a \in L^{p,m}_{1,\alpha'}(\Omega^1(\mathfrak{g}_{\overline{E}}))\,\},$$

and correspondingly denote the space of weight m S^1–equivariant instantons and the part whose elements are stabilized by $Z(G)$ by

$$\mathcal{A}^{p,m,+}_J, \qquad \mathcal{A}^{p,m,+,*}_J.$$

The subgroup of weight m S^1–equivariant gauge transformations is defined to be

$$\mathcal{G}^{p,m}_J = \{\, \exp(g) \,|\, g \in L^{p,m}_{2,\alpha'}(\Omega^0(\mathfrak{g}_{\overline{E}}))\,\},$$

and we can define the set of equivalence classes of weight m S^1–equivariant connections and instantons with holonomy J of class $L^p_{1,\alpha'}$ to be

$$\mathcal{B}^{p,m}_J = \mathcal{A}^{p,m}_J/\mathcal{G}^{p,m}_J, \qquad M^J_{m,\ell'}(S^4, S^2) = \mathcal{A}^{p,m,+}_J/\mathcal{G}^{p,m}_J.$$

Similarly for the subsets of elements stabilized by $Z(G)$. Denoting by $M^{\alpha}_{\ell}(\mathbb{H}^3)$ the set of gauge equivalence classes of hyperbolic monopoles of mass α and magnetic charge ℓ we therefore have an identification

$$M^{\alpha}_{\ell}(\mathbb{H}^3) = M^{J}_{m,\ell'}(S^4, S^2),$$

in particular it follows by Proposition 1.42 that every hyperbolic monopole of mass α belongs to $M^{J}_{m,\ell'}(S^4, S^2)$ for some ℓ'. We then consider the S^1–equivariant part of the complex (4.1) and observe that the right inverse to d^{+}_{A} of Proposition 4.9 can be chosen to preserve the S^1–equivariant part of the complex to equip $M^{J}_{m,\ell'}$ with a real analytic space structure. By the remarks of §1.6 and Proposition 4.12 this is independent of the various choices implicit in the definition in the same way as the moduli space of instantons were. In particular it is independent of p as long as $p \in (2, (1/2 - J_0/6)^{-1})$. Defining in an analogous manner

$$\widetilde{\mathcal{A}}^{p,m,+}_{J},$$

we can identify the moduli space of framed monopoles of mass m and charge ℓ as

$$\widetilde{M}^{\alpha}_{\ell}(\mathbb{H}^3) = \widetilde{M}^{J}_{m,\ell'}(S^4, S^2).$$

Considering the S^1–equivariant part of the complex (4.16) we can also equip these with a real analytic space structure and $M^{J}_{m,\ell'}$ and $\widetilde{M}^{J}_{m,\ell'}$ are related in the same way as the corresponding moduli spaces of instantons. In §7.5 we shall use Theorem 6 and the calculations of Chapters 4 and 5 to obtain a formula for the dimension of these moduli spaces.

The second route is to observe that in the S^1–invariant case, using Proposition 6.19 and the same remarks as above one can define spaces $L^{p,0}_{k,\alpha}(\Omega^i(\mathfrak{g}_{\overline{E}}))$ and hence spaces of S^1–invariant connections and instantons of class $L^{p}_{1,\alpha}$ and given topological invariants for any $i\alpha \in \mathfrak{g}$,

$$\mathcal{A}^{p,0}_{\alpha}, \qquad \mathcal{A}^{p,0,+}_{\alpha}.$$

The group of S^1–invariant gauge transformations is defined to be

$$\mathcal{G}^{p,0}_{\alpha} = \{\exp(u) \,|\, u \in L^{p,0}_{2,\alpha}(\Omega^0(\mathfrak{g}_{\overline{E}}))\}.$$

This act on the S^1–invariant connections and instantons and we obtain moduli spaces of S^1–invariant connections and instantons with given topological invariant

$$\mathcal{B}^{p,0}_{\alpha} = \mathcal{A}^{p,0}_{\alpha}/\mathcal{G}^{p,0}_{\alpha}, \qquad M^{\alpha}_{0,\ell} = \mathcal{A}^{p,0,+}_{\alpha}/\mathcal{G}^{p,0}_{\alpha}.$$

We then identify the moduli spaces of monopoles of mass $[i\alpha]$ and charge ℓ with $M^{\alpha}_{0,\ell}$ and that of framed monopoles with

$$\widetilde{M}^{\alpha}_{0,\ell} = \widetilde{\mathcal{A}}^{p,0,+}_{\alpha}/\mathcal{G}^{p,0}_{\alpha}.$$

As recalled in proving the last clause of 1.14, in the S^1–invariant case the Laplacian Δ_γ of 1.14 (i) remains invertible for any $\gamma \neq 0$. For any given S^1–invariant $L^{p}_{1,\alpha}$ connection

this allows one to construct a right inverse to $d_A^+ : L^{p,0}_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})) \to L^{p,0}(\Omega^+(\mathfrak{g}_{\overline{E}}))$ and use the S^1–invariant parts of the complexes (4.1) and (4.16) to equip $M^\alpha_{0,\ell}$ and $\widetilde{M}^\alpha_{0,\ell}$ with real analytic space structures. Conjugation by the gauge transformation $\exp(-2\pi i m\theta)$ gives natural isomorphisms

$$L^{p,0}_{i,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})) \cong L^{p,m}_{i,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})),$$

therefore isomorphisms of the deformation complexes we get in the two setups which induce isomorphisms of the moduli spaces so that the ringed space structures we obtain in the two ways coincide.

We say that a pair $(\tilde{A}, \phi)$ with finite Y.M.H. action has high mass if the value $[i\alpha]$ of the Higgs field at infinity has the property that all the non–zero eigenvalues of $ad(\alpha)$ are strictly greater than one in modulus (this property is conjugation invariant). For high mass monopoles, exploiting the stronger bounds given by 1.14 (iii) in the S^1–invariant case, we can give a more natural construction of the moduli spaces than that obtained from Chapter 4, namely via a slice theorem. Notice that the decomposition

$$\mathfrak{g} = \mathfrak{g}_C \oplus \mathfrak{g}_U,$$

is orthogonal with respect to the Killing form so from this we can define a new fibre metric on $\Omega^i(\mathfrak{g}_{\overline{E}})$ putting together the metric induced by the Lie algebra inner product

$$K^i = K_{|C} \oplus \frac{1}{r^{4-2i}} K_{|U},$$

with the base metric for the forms. This is to be thought of as considering the hyperbolic rather than the euclidean metric on $\Omega^i(\mathfrak{g}_{\overline{E}})_U$ and we denote by $L^{p,h}_{w,k}$ the corresponding L^p–spaces. Locally these agree with the $L^{p,h}_{w,k}$ spaces of [R1] on $\Omega^i(\mathfrak{g}_{\overline{E}})_U$ and with the ordinary L^p_k spaces on $\Omega^i(\mathfrak{g}_{\overline{E}})_C$. Let $d_A \in \mathcal{A}^p_J$, denote by d^h_A its formal adjoint with respect to the L^2 inner product induced by this fibre metric,

$$d^h_A = r^4_\alpha d^*_A r^{-2}.$$

In general d^h_A does not map $L^{p,h}_{w,1}(\Omega^1(\mathfrak{g}_{\overline{E}}))$ into $L^{p,h}_w(\Omega^0(\mathfrak{g}_{\overline{E}}))$. This happens when for instance $a_U = O(r^{1+\epsilon})$ for any $\epsilon > 0$. Let Δ^h_A be the associated Laplacian and $\mathcal{H}^{i,h}$ the corresponding harmonic spaces.

Proposition 6.22. *Let $d_A = d_B + a$ where B is a model singular S^1–invariant G–connection whose Higgs field has high mass $[i\alpha]$ and $a \in L^{p,0}_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}}))$ satisfies the global equivalent of the pointwise bounds of 1.14 (iii) for monopoles, where $2 < p < (1/2 - \alpha_0/6)^{-1}$. Then the operator*

$$r^{-2}_\alpha \circ \Delta^h_A : L^{p,0}_{2,\alpha}(\Omega^0(\mathfrak{g}_{\overline{E}})) \to L^{p,0}(\Omega^0(\mathfrak{g}_{\overline{E}})),$$

is Fredholm and there is a decomposition orthogonal with respect to the ordinary L^2 inner product

$$L^{p,0}(\Omega^0(\mathfrak{g}_{\overline{E}})) = r^{-2}_\alpha \mathcal{H}^{0,h} \oplus Im(r^{-2}_\alpha \circ \Delta^h_A).$$

Proof. By the assumptions on d_A and the above remarks d_A^h maps $L^{p,h}_{w,1}(\Omega^1(\mathfrak{g}_{\overline{E}}))$ into $L^{p,h}_w(\Omega^0(\mathfrak{g}_{\overline{E}}))$ so Δ_A^h is defined as an operator

$$\Delta_A^h : L^{p,h}_{w,2}(\Omega^0(\mathfrak{g}_{\overline{E}})) \to L^{p,h}_w(\Omega^0(\mathfrak{g}_{\overline{E}})).$$

Note that

$$r_\alpha^{-2} L^{p,h}_w(\Omega^0(\mathfrak{g}_{\overline{E}})) = L^p(\Omega^0(\mathfrak{g}_{\overline{E}})), \qquad L^p_{2,\alpha}(\Omega^0(\mathfrak{g}_{\overline{E}})) = L^{p,h}_{w,2}(\Omega^0(\mathfrak{g}_{\overline{E}})),$$

thus the claim follows if we show that Δ_A^h is Fredholm and there is a decomposition orthogonal with respect to the $L^{2,h}$ inner product

$$L^{p,h}_w(\Omega^0(\mathfrak{g}_{\overline{E}})) = \mathcal{H}^{0,h} \oplus Im(\Delta_A^h).$$

Since $H^0(S^4) \cong \mathbb{R}$, the Laplacian

$$\Delta : L^p_2(\Omega^0(S^4)) \to L^p(\Omega^0(S^4)),$$

is invertible modulo constants and by 1.14 (i) the Laplacian

$$\Delta_\gamma^h : L^{p,h}_{w,2}(\Omega^0(\mathbb{H}^3 \times S^1)) \to L^{p,h}_w(\Omega^0(\mathbb{H}^3 \times S^1)),$$

is invertible whenever $\gamma \neq 0$ on S^1–invariant sections. It follows that

$$\Delta_\alpha^h : L^{p,h,0}_{w,2}(\Omega^0(\mathfrak{g}_{\overline{E}})_U) \to L^{p,h,0}_w(\Omega^0(\mathfrak{g}_{\overline{E}})_U),$$

is invertible and hence that

$$\Delta_\alpha^h : L^{p,h,0}_{w,2}(\Omega^0(\mathfrak{g}_{\overline{E}})) \to L^{p,h,0}_w(\Omega^0(\mathfrak{g}_{\overline{E}})),$$

is invertible modulo $\mathfrak{g}_C$. Next consider a connection $d_A = d_\alpha + a$, $a \in L^{p,0}_{1,\alpha}$. One sees that the operator norm difference between the associated Laplacian Δ_A^h and Δ_α^h is controlled by $\|a\|_{L^{p,0}_{1,\alpha}}$, thus Δ_A^h is invertible provided $\|a\|_{L^{p,0}_{1,\alpha}}$ is sufficiently small. Now let $d_A = d_B + a$ be a connection as stated in the hypothesis. Cover $(\mathbb{R}^4, \mathbb{R}^2)$ by small balls $\{U_i\}$ so that either U_i is an approximately standard B^4 or $(U_i, U_i \cap \Sigma)$ is an approximately standard pair. Because of the pointwise bounds on the connection matrix of d_A in a local S^1–invariant gauge, given a constant ϵ and any p, $2 \leqslant p < (1/2 - \alpha_0/6)^{-1}$, we can ensure that the U_i are small enough to admit a subordinate partition of unity such that on each patch we obtain cut-off connections of the form

$$d_{\tilde{A}} = d + \tilde{a}, \qquad \|\tilde{a}\|_{L^p_1} < \epsilon,$$

on the U_i disjoint from Σ, and of the form

$$d_{\tilde{A}} = d_\alpha + \tilde{a}, \qquad \|\tilde{a}\|_{L^p_{1,\alpha}} < \epsilon,$$

otherwise, and such that $d_{\tilde{A}}$ coincides with d_A on smaller balls U_i' still covering X. In each case it follows by the above that $\Delta^h_{\tilde{A}}$ is invertible modulo constants. We now patch up these local inverses as in Lemma 5.2 of [KM1] to obtain a two sided parametrix for Δ^h_A showing that it is Fredholm. The usual integration by parts argument together with the formal self–adjointness of Δ^h_A now gives the required orthogonal decomposition for $p = 2$ with $Ker(\Delta^h_A)$ defined weakly in $L^{2,h}$. It can be shown that $\Delta^h_A(v) = 0$ weakly implies the equation strongly and then local regularity arguments using the invertibility of Δ_α and the bounds on the connection matrix of d_A imply that $Ker(\Delta^h_A) \subseteq L^{p,0}_{2,\alpha}$ for $2 < p < (1/2 - \alpha_0/6)^{-1}$ (and in fact satisfy the corresponding pointwise bounds of 1.14 (iii) for 0–forms), so there is an induced decomposition for all these p as well. □

At each point $[A]$ of the moduli spaces of high mass monopoles we can now obtain a local model for the moduli spaces all of whose elements satisfy the global analogue of the bounds of 1.14 (iii).

Corollary 6.23. *For high mass α, at each point $[A] \in M^{\alpha,*}_\ell(\mathbb{H}^3)$ there is a local model for the moduli spaces and a representative for the Zariski tangent space all of whose elements satisfy the global equivalent of the pointwise bounds of 1.14 (iii). In particular the ringed space structure defined on $M^{\alpha,*}_\ell(\mathbb{H}^3)$ as the zero set of Θ is independent of p provided $2 < p < 2 + \alpha_0$.*

Proof. By Proposition 6.22 at each point $[A]$ in $M^{\alpha,*}_\ell(\mathbb{H}^3)$ we can find an element d_A of the form $d_A = d_B + a$ where d_B is a model singular G–connection and $a \in L^{p,0}_{1,\alpha}$ satisfies the global equivalent of 1.14 (iii) for monopoles of the given mass. By the previous proposition we can now solve the gauge Coulomb equation for $r_\alpha^{-2} \circ d^h_A$, that is, for all $b \in L^{p,0}_{1,\alpha}$ sufficiently close to zero we can find $g = g(b) \in \mathcal{G}^p_J$, $g(0) = 1$, such that

$$r_\alpha^{-2} \circ d^h_A(g(d_A + b) - d_A) = 0,$$

and $g(b)$ depends smoothly on b. Starting from the model constructed using the transversal given by Proposition 4.9 we thus obtain an equivalent local model for the moduli spaces as the zero set of the Fredholm map

$$\mathcal{A}^{p,0}_\alpha \ni b \mapsto d^h_A(b) \oplus d_A(b)^+ + [b,b]^+,$$

(or as a quotient of it by $Stab(A)$ when A is not irreducible) and for the tangent space at the origin, $(H^1_A)^0$ (the latter denoting the S^1–invariant part of complex (4.1)), as the kernel of $d^h_A \oplus d^+_A$. Using the bounds on a, the same techniques as in §6 of [R1] applied locally give the required bounds both for the solutions to the linear equation $d^h_A(b) \oplus d^+_A(b) = 0$ and the non linear equation $d^h_A(b) \oplus d_A(b)^+ + [b,b]^+ = 0$. □

6.6 Extended moduli spaces of hyperbolic monopoles

Before concluding the Chapter, for a given closed connected subgroup C of a compact connected simple Lie group G we shall define the moduli space of monopoles with C–symmetry breakdown at infinity. By this we mean monopoles such that the centralizer

of the value $\phi(\infty)$ of the Higgs field at infinity under the adjoint action is isomorphic to C. Let $\mathfrak{c} = Lie(C)$. Recall that given a closed positive Weyl chamber $\mathfrak{h}^+ \subseteq \mathfrak{g}$ in §4.4 we defined

$$Reg^+(\mathfrak{c}) = \{u \in \mathfrak{c} | C_G(u) = C\} \cap \mathfrak{h}^+.$$

We define the extended space of S^1–invariant connections with $\mathfrak{c}$–limit on S^2 to be

$$\widehat{A}^{p,0}_{\mathfrak{c}} = \bigcup_{i\alpha \in Reg^+(\mathfrak{c})} \mathcal{A}^{p,0}_{\alpha},$$

and similarly for the extended moduli space of S^1–invariant instantons. Since $L^{p,0}_{1,\alpha}$ is the same space for all $i\alpha \in Reg^+(\mathfrak{c})$ these have an induced topology from the product

$$Reg^+(\mathfrak{c}) \times L^{p,0}_{1,\alpha}.$$

The gauge group $\mathcal{G}^{p,0}_{\alpha}$, which depends on α only up to $Reg^+(\mathfrak{c})$, acts on each slice by the pull–back action and we can define the extended moduli space of S^1–invariant connections with $\mathfrak{c}$–limit to be

$$\widehat{\mathcal{B}}^{p,0}_{\mathfrak{c}} = \widehat{\mathcal{A}}^{p,0}_{\mathfrak{c}} / \mathcal{G}^{p,0}_{\alpha}.$$

The moduli space of monopoles with C–symmetry breakdown at infinity and given magnetic charge is then

$$\widehat{M}^C_{\ell}(\mathbb{H}^3) = \widehat{\mathcal{A}}^{p,0,+}_{\mathfrak{c}} / \mathcal{G}^{p,0}_{\alpha},$$

with the induced topology of $\widehat{\mathcal{B}}^{p,0}_{\mathfrak{c}}$ as in §4.4. Similarly one defines the moduli space of framed monopoles with given magnetic charge and C–symmetry breakdown at infinity to be

$$\widetilde{\widehat{M}}^C_{\ell}(\mathbb{H}^3) = \widetilde{\widehat{\mathcal{A}}}^{p,0,+}_{\mathfrak{c}} / \widetilde{\mathcal{G}}^{p,0}_{\alpha}.$$

Chapter 7

Comparison of the deformation theories

As we said in the Introduction we didn't succeed in obtaining a general comparison Theorem between the deformation theories of singular instantons and framed parabolic or parabolic stable bundles. Here we shall show that in the cases covered by Theorems 1 and 2 there are natural isomorphism between the cohomology groups of the deformation complexes (4.1) and (5.2) and that at smooth points these can be identified with the differentials of the maps of sets given by Theorems 1 and 2 so that at these points the moduli spaces are indeed diffeomorphic. We then apply these results to obtain more refined informations on the moduli spaces of irreducible singular instantons on (S^4, S^2) and (X, Σ), where (X, Σ) is as in Example (b) of §5.2. We begin by identifying the deformation theories at the infinitesimal level.

7.1 Identification of the cohomologies

As in §1.6, given a compact connected orientable pair (X, Σ), a fixed topological type k, c, ℓ and a given admissible element $J \in G$, let $(\overline{E}, s)$ be the $\mathcal{C}^\infty$ G–bundle with $C(J)$–reduction on Σ having these topological invariants. Throughout this section we fix a $2 < p < (1/2 - J_0/6)^{-1}$ so that we shall denote the affine space $\mathcal{A}^p, \widetilde{\mathcal{A}}^p_J$ simply by $\mathcal{A}_J$ and $\widetilde{\mathcal{A}}_J$ respectively. Assume now that (X, ω) is a Kähler surface and (X, Σ) a complex pair which is either compact or else is the pair $(\mathbb{C}^2, \Sigma)$ of Theorem 1 with the euclidean metric. In the latter case we identify connections on $\mathbb{C}^2 \setminus \Sigma$ with connections on $S^4 \setminus S^2$ via the blow–up map at a point p_∞ in S^4 and as in (3.2) the space $\widetilde{\mathcal{A}}_J$ for $(\mathbb{C}^2, \Sigma)$ and the moduli spaces are defined by pulling back the space $\widetilde{\mathcal{A}}_J$ and the framed moduli spaces on (S^4, S^2) with framing at p_∞ (once again we could proceed more directly and define this spaces along the lines of §1.3). As usual we think of $\overline{E}$ as being the G–reduction of a $G^{\mathbb{C}}$–bundle $\overline{E}^{\mathbb{C}}$. Equip $\mathfrak{g}_{\overline{E}^{\mathbb{C}}}$ with a metric h on induced from the G–structure on $\overline{E}$. Using the Kähler structure we then have the familiar algebraic isomorphisms

$$
\begin{aligned}
\Omega^0(\mathfrak{g}_{\overline{E}}) \otimes \mathbb{C} &\overset{j_0}{\simeq} \Omega^0(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}) \\
\Omega^1(\mathfrak{g}_{\overline{E}}) &\overset{j_1}{\simeq} \Omega^{0,1}(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}) \\
\Omega^2(\mathfrak{g}_{\overline{E}}) &\overset{j_2}{\simeq} \Omega^{0,2}(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}) \oplus \mathbb{C} < \omega >,
\end{aligned}
\tag{7.1}
$$

where the maps are the obvious ones for complex pairs and are the composites of these with the pull–back via the blow–up map in the case of forms pulled back from S^4. Corre-

spondingly we obtain Banach space isomorphisms

$$
\begin{aligned}
L^p_{1,\alpha}(\Omega^0(\mathfrak{g}_{\overline{E}}))\otimes\mathbb{C} &\overset{j_0}{\simeq} L^p_{1,\alpha}(\Omega^0(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}))\\
L^p_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})) &\overset{j_1}{\simeq} L^p_{1,\alpha}(\Omega^{0,1}(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}))\\
L^p(\Omega^+(\mathfrak{g}_{\overline{E}})) &\overset{j_2}{\simeq} L^p(\Omega^{0,2}(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}))\oplus L^p(\mathbb{C}<\omega>).
\end{aligned}
\tag{7.2}
$$

and similarly for the spaces $\widetilde{L}^p_{1,\alpha}$. Let $A\in\mathcal{A}_J$, and let $\overline{\partial}_A$ its $\overline{\partial}$–operator. We define the Banach spaces

$$
\begin{aligned}
Dom(\overline{\partial}_A) =&\{f\in L^p_{1,\alpha}(\Omega^0(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}))|\overline{\partial}_A(f)\in L^p_{1,\alpha}(\Omega^{0,1}(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}))\}\\
\widetilde{Dom}(\overline{\partial}_A) =&\{f\in \widetilde{L}^p_{1,\alpha}(\Omega^0(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}))|\overline{\partial}_A(f)\in \widetilde{L}^p_{1,\alpha}(\Omega^{0,1}(\mathfrak{g}_{\overline{E}^{\mathbb{C}}}))\},
\end{aligned}
$$

equipped with the graph norm. We thus obtain differential complexes $\{C_i,\overline{\partial}_A\}$, where

$$
C_0 = Dom(\overline{\partial}_A) \text{ or } \widetilde{Dom}(\overline{\partial}_A),\quad C_i = L^p_{2-i,\alpha}(\Omega^i(\mathfrak{g}_{\overline{E}^{\mathbb{C}}})) \text{ or } \widetilde{L}^p_{2-i,\alpha}(\Omega^i(\mathfrak{g}_{\overline{E}^{\mathbb{C}}})),\ i=1,2.
$$

We let $\mathcal{A}^{0,1}_J$ and $\widetilde{\mathcal{A}}^{0,1}_J$ be the spaces of $\overline{\partial}$–operators associated to connections in $\mathcal{A}_J$ and $\widetilde{\mathcal{A}}_J$ respectively. We begin by proving a basic regularity property for some special $\overline{\partial}$–operators in these spaces. As usual decompose $\mathfrak{g}_{\overline{E}}$ as $(\mathfrak{g}_{\overline{E}})_C\oplus(\mathfrak{g}_{\overline{E}})_U$ and write $f=f_U+f_C$ for a section of it.

Lemma 7.3. *Let $\overline{\partial}_A\in\mathcal{A}^{0,1}_J$ be of the form $\overline{\partial}_b+a$ where $\overline{\partial}_b$ is the $\overline{\partial}$–operator of a model singular connection and $a\in L^\infty\cap L^p_{1,\alpha}$. For any $f\in L^p$ such that $f_U/r_\Sigma\in L^p$ and $\overline{\partial}_A(f)\in L^p_{1,\alpha}$, $f\in Dom(\overline{\partial}_A)$. If further $\overline{\partial}_A\in\widetilde{\mathcal{A}}^{0,1}_J$, $\overline{\partial}_A(f)\in\widetilde{L}^p_{1,\alpha}$, $f/r_{D_\infty}\in L^p(\mathbb{P}^2)$, and $f_U/r_\Sigma r_{D_\infty}\in L^p(\mathbb{P}^2)$ then $f\in\widetilde{Dom}(\overline{\partial}_A)$. The spaces $Dom(\overline{\partial}_A)$ and $\widetilde{Dom}(\overline{\partial}_A)$ both consist of continuous sections.*

Proof. This is a local matter so we consider a local gauge on a polydisc D^2 which is either disjoint from Σ or such that $(D^2,\Sigma\cap D^2)$ is a standard complex pair and in this case the gauge is compatible with the $C(J)$–reduction. In the first case we can write $\overline{\partial}_A=\overline{\partial}+a$ with $a\in L^p_{1,\alpha}$. By assumption

$$
\overline{\partial}f+[a,f]=b,\qquad b\in L^p_1,
$$

so by elliptic theory for $\overline{\partial}$ it follows that $f\in L^p_2$ hence lies in $Dom(\overline{\partial}_A)$. Also, since $w(k,p)=2-4/p>0$ we have the embedding $L^p_2\hookrightarrow\mathcal{C}^0$ giving the continuity of f.

On polydiscs of the second type we can write $\overline{\partial}_A=\overline{\partial}_\alpha+a$ with $a\in L^p_{1,\alpha}$ and then by definition

$$
\overline{\partial}_\alpha f+[a,f]=b,\qquad b\in L^p_{1,\alpha}.
\tag{7.4}
$$

Off Σ, $\overline{\partial}_\alpha$ is elliptic overdetermined so as above we obtain that $f\in\mathcal{C}^0(D^2\backslash\Sigma)$. We now show that f extends continuously across Σ. By assumption $[\alpha\dfrac{d\overline{z}}{2\overline{z}},f]\in L^p$, thus it follows

from (7.4) and the assumptions on a and b that $\overline{\partial} f \in L^p$ so $f \in L^p_1$ and since we assume $f_U \in L^p$ we obtain by Lemma 1.11 that $f \in L^p_{1,\alpha}$ thus $f \in Dom(\overline{\partial}_A)$. Since $L^p_{1,\alpha} \hookrightarrow L^{4+\epsilon}$ for some $\epsilon > 0$, for f_C we now have an equation of the form

$$\overline{\partial} f_C = b', \qquad b' \in L^{4+\epsilon},$$

hence $f_C \in L^{4+\epsilon}_1 \subseteq \mathcal{C}^0$. Multiplying f by a cut–off function we can now assume that f is compactly supported in U without changing the problem. As in §3.5 denote by D_w and D_z the discs normal and parallel to Σ respectively. Since f, a, and b belong to $L^p_{1,\alpha}$ which is contained in L^p_1, by the continuous Sobolev restriction

$$L^p_1(D^2) \to L^p_1(D),$$

for all discs $D \subseteq D^2$, we obtain that a^w and a^z have uniformly L^p restrictions to all the D_z and D_w discs and there is a constant C independent of z or w such that for each D_z and D_w

$$\begin{aligned} &\|f\|_{L^p(D_z)} \leqslant C\|f\|_{L^p_{1,\alpha}}, \qquad &&\|\overline{\partial}^w f + [a^w, f]\|_{L^p(D_z)} \leqslant C\|b\|_{L^p_{1,\alpha}} \\ &\|f\|_{L^p(D_w)} \leqslant C\|f\|_{L^p_{1,\alpha}}, \qquad &&\|\overline{\partial}_\alpha f + [a^z, f]\|_{L^p(D_w)} \leqslant C\|b\|_{L^p_{1,\alpha}}. \end{aligned} \tag{7.5}$$

Since in complex dimension 1 we have the embedding $L^p_1 \hookrightarrow \mathcal{C}^0$, it follows by elliptic theory for $\overline{\partial}^w$ applied to the D_z discs and the first of the (7.5) that $f \in L^\infty(D^2)$. From (7.4) it follows by Cauchy's integral formula that on almost all the D_z discs

$$\frac{f_U(z,w)}{z} = \frac{i}{2\pi}\int_{D_z} \frac{([a^w, f])_U + b^w_U(W,z)}{z}\frac{dW d\overline{W}}{W-w},$$

hence

$$\forall w, \quad \int_{D_w} \frac{|f_U(z,w)|^p}{|z|^p} dz d\overline{z} \leqslant \int_{D_w} |\int_{D_z} \frac{[a^w, f]_U + b^w(z,W)}{z}\frac{dW d\overline{W}}{W-w}|^p,$$

and since $p > 2$, applying Hölder's inequality to the inner integral,

$$\forall w, \quad \int_{D_w} \frac{|f_U(z,w)|^p}{|z|^p} dz d\overline{z} \leqslant \int_{D^2} |\frac{[a^w, f]_U + b^w_U}{r}|^p < \infty,$$

since $b_U \in L^p_{1,\alpha}$ and

$$[a^w, f]_U = \{a^w_U, f_C\} + \{a^w_C, f_U\} + \{a^w_U, f_U\},$$

with both a and f in L^∞. This shows that $f_U/|z| \in L^p(D_w)$ for all w which together with $\overline{\partial}^z_\alpha \in L^p(D_w)$ implies that

$$\forall w, f_U \in L^p_1(D_w), \qquad \frac{f_U}{|z|} \in L^p(D_w),$$

so f_U is continuous on all D_w and vanishes on D. Together with the continuity on $D^2\backslash D$ this implies that f_U extends continuously as zero across D.

For the framed case we have to consider what happens along D_∞. For this we work with complex coordinates (u,v) centered at points on D_∞ such that $D_\infty = \{u=0\}$ and also $\Sigma = \{v=0\}$ if the point is $\Sigma \cap D_\infty$. Using Lemma 1.15 we now have an equation of the same form as (7.4) with $\overline{\partial}_{-\alpha}$ or $\overline{\partial}_{\alpha,-\alpha}$ replacing $\overline{\partial}_\alpha$ and b now only in $L^p(\mathbb{P}^2)$. Since we assume $f/|u| \in L^p(\mathbb{P}^2)$, $f_U/|v| \in L^p$ and $a \in L^\infty$ we deduce that $f \in L^p_1(\mathbb{P}^2)$ which by the calculations of Lemma 1.15 implies $f \in \widetilde{L}^p_{1,\alpha}$, hence $f \in \widetilde{Dom}(\overline{\partial}_A)$. Since by Lemma 1.15 b has finite L^p restrictions to all the D_u and D_v discs uniformly bounded by its $\widetilde{L}^p_{1,\alpha}$ norm, and $a^v/|u|, b^v/|u| \in L^p(\mathbb{P}^2)$ the same argument as the one used above for f_U will now give that f extends continuously as zero across $D_\infty\backslash\Sigma$. Finally on the disc D_v corresponding to Σ we have

$$\int_{D_v} \frac{|f(u,v)|^p}{|u|^p} du d\overline{u} \leqslant \int_{D_v} | \int_{D_u} \frac{[a^v, f] + b^v(z,W)}{u} + \frac{-[\alpha d\overline{v}, f_U]}{\overline{V}u} \frac{dV d\overline{V}}{V-v} |^p.$$

Since $p > 2$, applying Hölder's inequality to the inner integral, the given assumptions show that $f/|u| \in L^p(D_v)$, which together with $\overline{\partial}_{-\alpha}(f) \in L^p(D_v)$ implies that f is continuous and vanishes there so f extends continuously as zero across D_∞ as well. □

We know by Theorem 3.55 that for an instanton A, $(\overline{E}^{\mathbb{C}}, \overline{\partial}_A)$ extends to a parabolic bundle or framing. As remarked in §3.6, when the framing bundle $\mathcal{E}^0$ is trivial and the copies of P, one for each point in $\Sigma \cap D_\infty$, determining the quasi–parabolic structure are all the same, the quasi–parabolic structure is the same as a reduction to P over the whole of D_∞. While at the global level there is no difference between these two ways of viewing the structure on $\mathcal{E}^0$, at the sheaf theoretic level we can now consider two distinct objects. One is $\mathcal{T}^0$ as usual, the other is $\mathcal{T}^{\mathfrak{p}}$, the sheaf of infinitesimal automorphisms preserving the parabolic reduction on the whole of D_∞. Correspondingly for a given framing $\mathcal{E}$, alongside with $\widetilde{\mathcal{T}}_{\mathcal{E}} = \mathcal{T}_{\mathcal{E}}[-D_\infty]$, we also have the sheaf $\mathcal{T}'_{\mathcal{E}}$ of infinitesimal automorphisms which preserve the induced parabolic reduction on D_∞. The two are related by the following map of sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}'_{\mathcal{E}} & \longrightarrow & \mathcal{T}_{\mathcal{E}} & \xrightarrow{r_{|D_\infty}} & \mathcal{T}^{\mathfrak{p}} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathcal{T}}_{\mathcal{E}} & \longrightarrow & \mathcal{T}_{\mathcal{E}} & \xrightarrow{r_{|D_\infty}} & \mathcal{T}^0 & \longrightarrow & 0, \end{array}$$

where the right map is the injection and hence the left map a surjection. Note that $\mathcal{T}^{\mathfrak{p}}$ and $\mathcal{T}^0$ have the same cohomology groups so that from the long exact sequences associated to this diagram we obtain

$$\forall i, \qquad H^i(\mathcal{T}'_{\mathcal{E}}) = H^i(\widetilde{\mathcal{T}}_{\mathcal{E}}), \tag{7.6}$$

as one would expect. From the differential geometric point of view this type of structure on D_∞ corresponds in general to singular instantons whose limit on the circle bundle at

infinity reduces to the maximal torus in the cylindrical end picture of the discussion in §1.3. It is therefore not surprising that for the specific triple we are restricting to, where all instantons are of this kind, it is the sheaf $\mathcal{T}'_{\mathcal{E}}$ rather than $\widetilde{\mathcal{T}}_{\mathcal{E}}$ that naturally arises from $(\overline{E}^{\mathbb{C}}, \overline{\partial}_A)$. We shall need a concrete description of $\mathcal{T}'_{\mathcal{E}}$. Let (u, v) be the usual complex coordinates centred at a point in D_∞ and consider local frames for $\mathfrak{g}_{\mathcal{E}}$ compatible with $\mathfrak{p}$ both on Σ and D_∞ so that $\mathcal{T}^{\mathfrak{p}}$ can be identified with $\mathfrak{p}$ on D_∞. From the above sequence for $\mathcal{T}'_{\mathcal{E}}$ we then see that

$$\mathcal{T}'_{\mathcal{E}} =< u > \mathfrak{p} \oplus < v > (\mathfrak{g}_{\mathcal{E}}/\mathfrak{p}).$$

Lemma 7.7. *Let $\overline{\partial}_A$ be the $\overline{\partial}$–operator associated to an instanton $A \in \mathcal{A}^+_J$ or $\widetilde{\mathcal{A}}^+_J$, then the cohomology groups of the complexes $\{C_i, \overline{\partial}_A\}$ are isomorphic to the cohomology groups of the parabolic bundle or framing determined by $(\overline{E}^{\mathbb{C}}, \overline{\partial}_A)$.*

Proof. Notice that the cohomology groups of $\{C_i, \overline{\partial}_A\}$ are invariant in the orbit of $\overline{\partial}_A$ under the gauge group $\mathcal{G}_J$. By Proposition 4.5 we can find $g \in \mathcal{G}_J$ such that $g(d_A) = d_b + a$, with d_b a model singular connection in $\mathcal{A}$ and $a \in L^\infty$ thus

$$g(\overline{\partial}_A) = \overline{\partial}_b + a^{0,1}, \qquad a^{0,1} \in L^\infty,$$

and it suffices to prove the statement for $\overline{\partial}_A$ of this form.

Recall that the sheafifications of the complexes $\{C_i, \overline{\partial}_A\}$ consist of the differential complexes of sheaves $\{\mathcal{C}_i, \overline{\partial}_A\}$ whose sections on an open set U are defined in the same way as the global sections C_i substituting the spaces $L^p_{1,\alpha}$ with the spaces $(L^p_{1,\alpha})_{loc}$. Let $\mathcal{E}$ be the holomorphic object determined by $(\overline{E}^{\mathbb{C}}, \overline{\partial}_A)$. We shall prove that the $\{\mathcal{C}_i, \overline{\partial}_A\}$ are fine resolutions of $\mathcal{T}_{\mathcal{E}}$ and $\mathcal{T}'_{\mathcal{E}}$ in the unframed and framed case respectively. We begin with the unframed case. The fineness is standard and it suffices to prove acyclicity of the complex with respect to an open cover by polydiscs $\{U_i\}$ such that each U_i is either disjoint from Σ or else $(U_i, U_i \cap \Sigma)$ is a standard complex pair. Consider first the second case. Let (z, w) be complex coordinates as in the previous Lemma and $b \in Ker(\overline{\partial}_A)(U)$. By Proposition 3.48 we can find a continuous complex gauge transformation g on a possibly smaller polydisc such that

$$g(\overline{\partial}_A) = \overline{\partial}_\alpha, \qquad \sup |\frac{g_U}{|z|^{\alpha_0}}| < \infty,$$

then, by the previous Lemma

$$f \in Ker(\overline{\partial}_A) \iff Ad(g)(f) \in Ker(\overline{\partial}_\alpha) \cap \mathcal{C}_0,$$

which as observed many times in §3.5 implies that $|z|^{ad(\alpha)}(Ad(g)(f))$ extends holomorphically across Σ and lies in $\mathfrak{p}$ on it so that $\mathcal{K}er(\overline{\partial}_A) \cong \mathcal{T}_{\mathcal{E}}$.

Now let $b \in \mathcal{C}^1(U)$ be $\overline{\partial}_A$–closed and let $b' = Ad(g)(b)$ so that $b' \in \mathcal{C}^0$ and $b'/r_\Sigma \in L^p$. Put

$$f'(z, w) = \int K^z_\alpha b'^z dZ d\overline{Z} + \int K^w b^w dW d\overline{W},$$

where K^w is the ordinary Cauchy kernel in the W direction and K^z_α is the Cauchy kernel in the z–direction on f'_C and the kernel defined as for the operators T^γ_z of Proposition

3.48 on the non–zero eigenspaces of $ad(\alpha)$. Because of the $\overline{\partial}_\alpha$–closedness of b' it follows as in Proposition 3.48 that $\overline{\partial}_\alpha(f') = b'$ and by the properties of b' also that $f' \in \mathcal{C}^0$ and $f'_U/|z| \in L^p$. Then $f = Ad(g^{-1})(f')$ satisfies

$$\overline{\partial}_A(f) = b, \qquad f \in \mathcal{C}^0, \qquad \frac{f_U}{|z|} \in L^p,$$

so by the previous Lemma $f \in Dom(\overline{\partial}_A)$.

To prove the exactness at the second step we use the right inverse to the model singular connection d_α on the pair (S^4, S^2) given by Proposition 4.8. Let $c \in \mathcal{C}^2(U)$. By the algebraic isomorphisms (7.1) we obtain a form

$$\gamma = c - c^h \in \Omega^+(\mathfrak{g}_E)(U).$$

Locally write $d_A = d_\alpha + A$, where $A = a - a^h$. Multiplying A and γ by a cut–off function we regard them as a singular connection and self–dual form on S^4 for a conformal structure close to the standard one (we can assume that the metric on U is close to the round metric on S^4). Consider the operator

$$T : L^p_{1,\alpha}(\Omega^1(S^4)) \to L^p(\Omega^+(S^4)),$$

$$T(b) = d^+_\alpha(b) + [\psi A, b]^+.$$

The difference in operator norm between T and d_α is controlled by the L^{2p} norm of ψA which because of the L^∞ bounds on A we can make small by choosing ψ to be concentrated at a point on S^2. The right invertibility of d^+_α for the standard metric implies right invertibility of the same operator defined by sufficiently close metrics so that T can also be made right invertible and we can find $\delta \in L^p_{1\alpha}$ such that $T(\delta) = \psi\gamma$. Writing $\delta = b - b^h$, type considerations now imply that on a possibly smaller open set than U

$$\overline{\partial}_A(b) = c.$$

The exactness away from Σ is classical and can anyway be proved by the above method using $\overline{\partial}$ and d^+ instead of $\overline{\partial}_\alpha$ and d^+_α.

We now consider the framed case. It suffices to consider a cover which in addition to the above properties is such that each polydisc U_i is either disjoint from D_∞ or it is disjoint from Σ and $(U_i, D_\infty \cap U_i)$ is a standard complex pair, or $(U_i, (\Sigma \cup D_\infty) \cap U_i)$ is the standard complex pair with normal crossings as in §3.5. By the previous part we only need to prove exactness along D_∞. Let U_i be disjoint from Σ and consider complex coordinates (u, v) as in Lemma 7.3. By Proposition 3.48 on a possibly smaller open set we can find a continuous complex gauge transformation g such that

$$g(\overline{\partial}_A) = \overline{\partial}_{-\alpha}, \qquad \sup |\frac{g}{|u|}| < \infty,$$

then

$$Ad(g)(f) \in Ker(\overline{\partial}_{-\alpha}) \cap \mathcal{C}^0, \qquad \frac{Ad(g)(f)}{|u|} \in L^p,$$

which implies that $|u|^{ad(-\alpha)}(Ad(g)(f))$ extends holomorphically across D_∞ and lies in $\mathfrak{g}_\mathcal{E}/\mathfrak{p}$ on it. By the above description of $\mathcal{T}'_\mathcal{E}$ this means that $|u|^{ad(-\alpha)}(Ad(g)(f)) \in \mathcal{T}'_\mathcal{E}$ showing that $\mathcal{K}er(\overline{\partial}_A) \cong \mathcal{T}'_\mathcal{E}$.

Now let $b \in \mathcal{C}^1(U)$ be $\overline{\partial}_A$–closed. Using the same method as above together with the bounds of Lemma 1.15 we can find a continuous solution f' to $\overline{\partial}_{-\alpha}(f') = b' = Ad(g)(b)$ and since $b'/|u| \in L^p$ it now also follows that $f'/|u| \in L^p$. Then $f = Ad(f')$ satisfies $\overline{\partial}_A(f) = b$ on a possibly smaller open set and belongs to $\widetilde{Dom}(\overline{\partial}_A)$ by Lemma 7.3. The exactness at the second step follows as above regarding a (0,2) form as a self–dual form on $\mathbb{R}^4$ hence on S^4 by conformal invariance.

We finally analyse what happens at the normal crossing. It follows as above using Proposition 3.48 and the local description of $\mathcal{T}'_\mathcal{E}$ that $\mathcal{K}er(\overline{\partial}_A) \cong \mathcal{T}'_\mathcal{E}$. For the exactness at the first step we proceed as above using Lemma 1.15, Lemma 7.3 and the kernel

$$b \mapsto \int K^v_\alpha b^v dV d\overline{V} + \int K^u_{-\alpha} b^u dU d\overline{U}.$$

The exactness at the second step is as above. □

Remark. This implies in particular that the $\{C^i, \overline{\partial}_A\}$'s are Fredholm complexes and the reader may at this point ask himself whether we can use them to give an alternative construction of the holomorphic moduli spaces. Although $Dom(\overline{\partial}_A) \subseteq \mathcal{C}^0$, there is neither a Banach multiplication property for this space nor a bracket action on $L^p_{1,\alpha}$. Thus, though we can define a complex gauge group by

$$\mathcal{G}_J^{\overline{\partial}_A} = \exp(Dom(\overline{\partial}_A)),$$

this will neither be a Banach Lie group nor act on $L^p_{1,\alpha}$. On the other hand there is a well defined action of the complexified gauge group $\mathcal{G}_J^\mathbb{C}$ on the $L^p_{1,\alpha}$ (0,1)–forms. However if f_α as in (3.28), we see that letting $\mathcal{G}_\mathfrak{p}$ be the parabolic gauge group as in §2.4, $Ad(f_\alpha)(\mathcal{G}_\mathfrak{p})$ is not a subgroup of $\mathcal{G}_J^\mathbb{C}$ since when $0 < \alpha < 1$, the function $|z|^\alpha / r^2_\Sigma$ is not in L^p for any $p \geqslant 2$. We are thus forced to enlarge the complex gauge group and in particular to take an infinitesimal model where the highest power of r_Σ such that $f_U / r^k_\Sigma \in L^p$ is one. Since $Ad(f_\alpha)(\mathcal{G}_\mathfrak{p})$ is contained in $\mathcal{G}_J^{\overline{\partial}_A}$, $Dom(\overline{\partial}_A)$ is a suitable candidate. Lemma 7.3 shows that in this $L^p_{1,\alpha}$ framework for the connections it is the weakest possible candidate, while the last lemma implies that it is also the strongest topology one can take in degree zero if the first cohomology group of the complex $\{C_i, \overline{\partial}_A\}$ is to be isomorphic to $H^1(\mathcal{T}_\mathcal{E})$.

It is precisely this double failure that prevents us from obtaining a full comparison theorem between the differential geometric and holomorphic moduli spaces in this setup. Note also that whereas $\mathcal{G}_J$ was independent of $i\alpha = \log(J)$ as long as this varied in $Reg^+(C(J))$, the function $|z|^\alpha, 0 < \alpha < 1/2$ shows that $\mathcal{G}_J^{\overline{\partial}_A}$ does instead depend on it.

We shall denote here by $\{B_i, d_A\}$ both the complexes (4.1) and (4.16). Because of the

isomorphisms (7.2) the complexes $\{B_i, d_A\}$ and $\{C_i, \overline{\partial}_A\}$ are related by maps of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_0 & \xrightarrow{d_A} & B_1 & \xrightarrow{d_A^+} & B_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \\ 0 & \longrightarrow & C_0 & \xrightarrow{\overline{\partial}_A} & C_1 & \xrightarrow{\overline{\partial}_A} & C_2 & \longrightarrow & 0, \end{array}$$

in the unframed ad framed case respectively. At the Banach space level j_0 is a closed injection, j_1 is an isomorphisms and j_2 is a surjection. These maps induce maps of the respective cohomologies and we have the following analogue of the Hodge theory isomorphism for the case of instantons over a compact Kähler surface.

Lemma 7.8. *The maps induce isomorphisms*

$$\begin{array}{lll} H_A^0 \otimes \mathbb{C} \cong H_{\overline{\partial}_A}^0 & H_A^0 \cong H_{\overline{\partial}_A}^{0,1} & H_A^2 \cong H_{\overline{\partial}_A}^{0,2} \oplus H_{\overline{\partial}_A}^0 < \omega > \\ \widetilde{H}_A^0 = 0 = \widetilde{H}_{\overline{\partial}_A}^0 & \widetilde{H}_A^1 \cong \widetilde{H}_{\overline{\partial}_A}^{0,1} & \widetilde{H}_A^2 \cong H_{\overline{\partial}_A}^{0,2} \end{array}$$

Proof. Recall that by Lemmas 4.17 and 7.7

$$\widetilde{H}_A^0 = 0 = \widetilde{H}_{\overline{\partial}_A}^0.$$

Clearly $H_A^0 \otimes \mathbb{C} \hookrightarrow \cong H_{\overline{\partial}_A}^0$. Let $g \in C_0 \bigcap Ker(\overline{\partial}_A)$. By the Kähler identities it follows that

$$\Delta_A g = 0 = \Delta_A(g - g^h) = 0$$

with $g - g^h \in L_{1,\alpha}^p$ thus by (ii) of Lemma 1.37 we can find a sequence ψ_i with $(\psi_i)_U \in C_0^\infty(X \backslash \Sigma)$ converging to $g - g^h$ in $L_{1,\alpha}^p$ and hence

$$\begin{aligned} \|d_A(g - g^h)\|_{L^2}^2 &= \int < d_A(g - g^h),\, d_A(g - g^h) > \\ &= \lim_{i \to \infty} \int < d_A(g - g^h),\, d_A \psi_i > \\ &= \lim_{i \to \infty} \int < \Delta_A(g - g^h), \psi_i >= 0 \end{aligned} \tag{7.9}$$

Thus $d_A(g - g^h) = 0$ and since by Lemma 7.3 $g - g^h \in C^0$, it follows by Lemma 1.11 that $g - g^h \in L_{2,\alpha}^p$, thus the assignment $g \mapsto g - g^h$ shows that $H_A^0 \otimes \mathbb{C} \cong H_{\overline{\partial}_A}^0$.
We now show that j_1 induces an injection in cohomology while j_2 induces a surjection. For the first statement let $a \in B^1$ belong to $Ker(d_A^+)$. Suppose that $a^{0,1} = \overline{\partial}_A b$, where $b \in C_0$. Then

$$d_A(b - b^h) = a^{0,1} - (a^{0,1})^h + \partial_A b - \overline{\partial}_A(b^h). \tag{7.10}$$

Since $F_A^+ = 0$ and $d_A^+ a = 0$, (7.10) implies that

$$(7.11) \qquad d_A^+ d_A(b - b^h) = d_A^+(\partial_A b - \overline{\partial}_A(b^h)) = \Lambda(\overline{\partial}_A \partial_A b - \partial_A \overline{\partial}_A(b^h)) \cdot \omega.$$

Using the Kähler identities (7.11) then gives

$$\Delta_A(b + b^h) = 0.$$

Since $b + b^h \in C_0$, so in particular it lies in $L^p_{1,\alpha}$ or $\widetilde{L}^p_{1,\alpha}$, we can integrate by parts as in (7.9) to deduce that $d_A(b + b^h) = 0$, hence

$$b = -b^h, \qquad mod\ Ker(d_A) \otimes \mathbb{C},$$

and it follows that

$$\frac{1}{2} d_A(b - b^h) = \partial_A(b^h) + \partial_A(b) = a,$$

with $b - b^h$ real. Since by Lemma 7.3 $b \in \mathcal{C}^0$, it follows by Lemma 1.11 that $b - b^h \in B^0$. Hence $a \in Im(d_A)$.
Since j_2 induces a surjection at the Banach space level it is obvious that it induces a surjection in cohomology.

We now use the index calculations of §4.3 to deduce that these are in fact isomorphisms. First consider the $U(n)$ case and take $\mathfrak{g}_{\overline{V}}$ to be the adjoint bundle of a unitary vector bundle $\overline{V}$. By Lemma 7.7 we know that the index of the $\overline{\partial}$–complex is the same as the Euler characteristic of either $\mathcal{T}_\mathcal{E}$ or $\widetilde{\mathcal{T}}_\mathcal{E}$. Comparing the formulae (4.32) and (4.39) for the index of the anti–self–duality complex in this case with the formulae of Proposition 5.21 for $GL(n)$ we obtain that

$$\begin{aligned} h^1_A &= 2h^1_{\overline{\partial}_A}, & \widetilde{h}^1_A &= 2\widetilde{h}^1_{\overline{\partial}_A}, \\ h^2_A &= h^2_{\overline{\partial}_A} + h^0_{\overline{\partial}_A}, & \widetilde{h}^2_A &= 2\widetilde{h}^2_{\overline{\partial}_A}, \end{aligned}$$

thus

$$H^1_A \cong H^1_{\overline{\partial}_A}, \qquad \widetilde{H}^i_A \cong \widetilde{H}^i_{\overline{\partial}_A}, \qquad i = 1, 2,$$

and $dim(Ker(j_2)) = dim(H^0_A)$. Since $H^0_A \cdot < \omega > \subseteq Ker(j_2)$ it follows that $H^2_A \cong H^2_{\overline{\partial}_A} \oplus H^0_{\overline{\partial}A} \cdot < \omega >$.

Let now G be a compact connected simple group and assume that $\mathfrak{g}_{\overline{V}}$ has a G–structure, $\mathfrak{g}_{\overline{E}}$, and that A is a G–connection so that $\{B_i, d_A\}$ has a distinguished subcomplex consisting of the $\mathfrak{g}_{\overline{E}}$–valued forms. The right inverse of Proposition 4.9 can then be assumed to respect this subcomplex and since for H^0_A and H^2_A we have harmonic representatives it follows that the cohomology groups of this subcomplex inject into the cohomology groups of the larger one. Similarly the complex $\{C_i, \overline{\partial}_A\}$ will have a distinguished subcomplex consisting of the $\mathfrak{g}_{\overline{E}^\mathbb{C}}$–valued forms and the same considerations as above show that its cohomology groups also inject into the ones of the larger complex. Further the splittings

for the two complexes can be chosen compatibly because if Q_A is the right inverse of d_A^+ given by Proposition 4.9, the isomorphisms (7.7) give us that

$$\Pi^{0,1} \circ Q_A \circ \Pi^{0,2},$$

is a right inverse for $\overline{\partial}_A$ modulo $H^{0,2}_{\overline{\partial}_A}$. Since the j_i's restrict to map of the subcomplexes, with these representatives for the cohomology groups it clearly follows that the above relations also hold for the cohomology groups of the subcomplexes. □

Remark. The isomorphism $H^0_A \otimes \mathbb{C} \cong H^0_{\overline{\partial}_A}$ may at first seem unexpected since on one hand we know by Proposition 1.46 that $H^0_A \subseteq Lie(C(J))$, while on the other $H^0_{\overline{\partial}_A} \cong H^0(\mathcal{T})$ by Lemma 7.7, and $\mathcal{T}$ is the sheaf of infinitesimal parabolic automorphisms. A concrete example sheds light on the general mechanism. Let (X, Σ) be a complex Kähler pair and A an $SU(2)$–instanton with holonomy $\exp(-2\pi i\alpha)$ along Σ where

$$\alpha = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \qquad 0 < \beta < 1/2,$$

reducing to $S^1 = C(i\alpha)$ (here the only non–trivial possible reduction) so that $H^0_A \cong \mathbb{R}$. By Lemma 1.33 we know that A gives raise to a split $SU(2)$–bundle

$$\overline{V} = L \oplus L^{-1},$$

with

$$c_1(L) = -\beta[\Sigma], \qquad c_1(L^{-1}) = \beta[\Sigma].$$

This splitting extends in fact holomorphically and L becomes the line sub–bundle of an α–parabolic bundle $\mathcal{V}$ on (X, Σ). We see that

$$End(\mathcal{V}) = \begin{pmatrix} \mathcal{O} & -2\beta[\Sigma] \\ 2\beta[\Sigma] & \mathcal{O} \end{pmatrix},$$

with

$$\mathcal{T}_{\mathcal{V}} = \begin{pmatrix} \mathcal{O} & -2\beta[\Sigma] \\ (2\beta - 1)[\Sigma] & \mathcal{O} \end{pmatrix}, \qquad Tr(-) = 0.$$

Since Σ is effective

$$H^0((2\beta - 1)[\Sigma]) = 0 = H^0(-2\beta[\Sigma]),$$

so $H^0(\mathcal{T}_{\mathcal{V}}) \cong \mathbb{C}$.

7.2 Continuity of the correspondence

We now prove that the bijections given by Theorems 1 and 2

$$\begin{aligned} \Phi &: M^{J,*}_{\kappa,c,\ell}(X, \Sigma) \longrightarrow \mathcal{M}^{\alpha,S}_{\kappa,c,\ell}(X, \Sigma), \\ \widetilde{\Phi} &: \widetilde{M}^{J}_{\kappa,\ell}(\mathbb{C}^2, \Sigma) \longrightarrow \widetilde{\mathcal{M}}^{\alpha}_{\kappa,\ell}(\mathbb{P}^2, \Sigma), \end{aligned} \tag{7.12}$$

where $i\alpha = \log(J)$, are continuous, depend continuously on $J \in Reg^+(C(J))$ and using the result of the previous section that at smooth points they are in fact diffeomorphisms depending smoothly on $J \in Reg^+(C(J))$. To emphasize their dependence on α, throughout this section we shall denote these maps by Φ_α and $\widetilde{\Phi}_\alpha$ respectively. The proof shall consist in giving a global description of these maps using the gauge theoretic parametrisation (5.1) of the holomorphic moduli spaces.

Let $E^{\mathbb{C}}$ be the $\mathcal{C}^\infty$ P–quasi–parabolic bundle with the same invariants as $(\overline{E}, s)$ of the previous section so that the moduli spaces (5.1) are defined on it. Equip $E^{\mathbb{C}}$ with the G–reduction h_α of Construction 3.27 (i). By Proposition 3.34 we then obtain a $\mathcal{C}^\infty$ G–bundle with $C(J)$–reduction on Σ which can be identified with $(\overline{E}, s)$ above and such that

$$f_\alpha = \exp(\beta(|\sigma|_j)\alpha \log(|\sigma|_j)),$$

pulls back $\mathcal{A}_\mathfrak{p}$ and $\mathcal{H}_\mathfrak{p}$ to subsets of $\mathcal{A}_J^{0,1}$ and its integrable part respectively. In the case of framings we equip $E^{\mathbb{C}}$ with the reduction of Construction 3.27 (ii) and again by Proposition 3.34 we obtain a G–bundle with $C(J)$–reduction on Σ such that

$$f_\alpha = \exp(\beta(|\sigma|_j|\tau|_k)\alpha \log(\frac{|\sigma|_j}{|\tau|_j})),$$

pulls back $\widetilde{\mathcal{A}}_\mathfrak{p}$ and $\widetilde{\mathcal{H}}_\mathfrak{p}$ to subsets of $\widetilde{\mathcal{A}}_J^{0,1}$ and its integrable part respectively.

Let $[A] \in M^{J,*}_{\kappa,c,\ell}$ or $\widetilde{M}^J_{\kappa,\ell}$. and let $\overline{\partial}_A$ be the corresponding $\overline{\partial}$–operator. By Lemma 4.5 we can choose for $[A]$ a representative of the form $d_B + a$, where d_B is a model singular connection and a satisfies the global equivalent of the pointwise bounds of 1.14 (iii).

We now show that this can be done "smoothly" in a neighbourhood of $([A], J)$ when this is considered as a point in the extended moduli spaces.

Lemma 7.13. *Let $A = d_B + a$ be an instanton of the above form, $U \times V \subseteq L^p_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})) \oplus Z(\mathfrak{c}(i\alpha)) < \beta\eta >$ a sufficiently small ball at $(0,0)$, where $i\alpha = \log(J)$, and $T \subseteq U \times V$ a local model for the extended moduli spaces at $[A]$ as constructed from Proposition 4.12. For every element $(b, i\gamma\beta\eta) \in U \times V$ there is a gauge transformation $g(b,\gamma) \in \mathcal{G}_J$, where $g(0,0) = 1$, such that if $(b, i\gamma\beta\eta) \in T$*

$$b' = g(b,\gamma)(d_A + b + i\gamma\beta\eta) - (d_A + i\gamma\beta\eta)$$

satisfies globally the bounds of Theorem 1.14 (iii). The family $g(b,\gamma)$ depends smoothly on b and γ in the sense that if $(b_t, i\gamma_t\beta\eta)$ is a path at d_A in $\mathcal{A}_J \times Reg^+(\mathfrak{c}(i\alpha))$, the path $g(b_t,\gamma_t)(d_A + b_t + i\gamma\eta\beta) - (d_A + i\gamma\beta\eta)$ is at least as regular as $(b_t, i\gamma\beta\eta)$.

Proof. We apply the same argument as in Proposition 4.5 only now for a family of connections with varying holonomies. Let $\{U_i\}$ be a cover of (X, Σ) by approximately euclidean geodesic balls small enough so that U_i is either disjoint from Σ or $(U_i, U_i \backslash \Sigma)$ is an approximately standard pair (B^4, B^2) and on each of them the ordinary Uhlenbeck theorem or (ii) of 1.14 can be applied for all the connections in $d_A + U \times V$ restricted to the U_i's. On each of the U_i we find an L^p_2 or $L^p_{2,\alpha}$ gauge transformation g_i carrying $d_A = d_\alpha + a$

into $d+a'$ with a' in ordinary gauge Coulomb hence smooth, or into $d_\alpha + a'$ with a' in a singular gauge Coulomb hence satisfying the bounds of 1.14 (iii). In particular g_i is either smooth or has $\nabla_\alpha^k g_i$ satisfying the bounds of 1.14 (iii). Because of the independence of the spaces $L^p_{k,\alpha}$ on α as long as $i\alpha$ varies in $Reg^+(\mathfrak{c}(i\alpha)) \cap \mathfrak{f}$ which is open in $Z(\mathfrak{c})$ and the smooth dependence of all the operators involved in the proof of Theorem 1 in [R1] on α when this varies in the same range, the openness step in the proof of that Theorem can be extended to include small changes of the holonomy about $\exp(-2\pi i\alpha)$ as a parameter in $Z(\mathfrak{c}(i\alpha))$. Using this for the model pairs and the openness in the usual gauge fixing theorem on the approximately standard balls, we can then find gauge transformations $g_i(b,\gamma)$ close to the identity with $g_i(0) = 1$ and depending smoothly on $(b, i\gamma) \in U \times V$ putting an element in $d + a' + Ad(g_i)(b + i\gamma\beta\eta)$ or $d_\alpha + a' + Ad(g_i)(b + i\gamma\beta\eta)$ in Coulomb gauge with respect to d or $d_{\alpha+\gamma}$. Using ordinary elliptic theory or Theorem 2 of [R1] we see that in these gauges when $(b, i\gamma) \in T$ each $g_i(b,\gamma)(d + a' + b + i\gamma\beta\eta) - d$ will be smooth and each $g_i(b,\gamma)(d_\alpha + a' + b + i\gamma\eta\beta) - d_{\alpha+\gamma}$ will satisfy pointwise bounds analogous to those of 1.14 (iii). We can now pull–back by g_i^{-1} to obtain a family $g_i^{-1}g_i(b,\gamma)$ with the required properties on U_i. Now consider a partition of unity subordinate to the cover $\{U_i\}$ and patch together the local gauge transformations to obtain the desired family of global gauge transformations. The smoothness claim follows from the fact that on each U_i the local solutions $g_i(b,\gamma)$ are obtained from the implicit function theorem so depend smoothly on (b,γ). □

This gives a map

$$(b, i\gamma\beta\eta) \mapsto (g(b,\gamma)(d_A + b + i\gamma\beta\eta) - (d_A + i\gamma\beta\eta), i\gamma\beta\eta), \tag{7.14}$$

carrying the original model T at $([A], J)$ into a new model (as a topological space) $d_A + T'$ where for each $i\gamma\beta\eta \in \pi_2(T')$, each b in the γ–slice in T' satisfies the global equivalent of the bounds of 1.14 (iii). From this we can now give a global description of the map Φ.

Lemma 7.15. *Let $d_A = d_B + a$ be the above instanton and $\overline{\partial}_A$ its $\overline{\partial}$–operator. For all sufficiently small $(b, i\gamma\beta\eta) \in T'$ there is a complex gauge transformation $g'(b,\gamma)$ with $Ad(f_\gamma)(g'(b',\gamma)) \in \mathcal{G}_J^{\overline{\partial}_A}$ and satisfying componentwise the same bounds as a such that*

$$g'(b, i\gamma)(\overline{\partial}_A + (b + i\gamma\beta\eta)^{0,1}) \in f_{\alpha+\gamma}(\mathcal{H}_\mathfrak{p}).$$

The family $g'(b,\gamma)$ depends smoothly on (b,γ) in the sense that if $(b_t, i\gamma_t\beta\eta)^{0,1}$ is a path at $\overline{\partial}_A$ in $\overline{\partial}_A + (T')^{0,1}$, the path $g'(b_t,\gamma_t)$ is at least as regular as $(b_t, i\gamma_t\beta\eta)$. If $\overline{\partial}_A \in \widetilde{\mathcal{A}}_J^+$ we can also assume that $sup|g'(b,i\gamma)/r_{D_\infty}| < \infty$ and $g'(b,i\gamma)(\overline{\partial}_A + b^{0,1} + (i\gamma\beta\eta)^{0,1}) \in f_{\alpha+\gamma}(\widetilde{\mathcal{H}}_\mathfrak{p})$.

Proof. Observe that for all γ sufficiently close to 0 in $Z(\mathfrak{c}(i\alpha))$

$$f_{\alpha+\gamma}(\mathcal{A}_\mathfrak{p}) = \overline{\partial}_B + (i\gamma\beta\eta)^{0,1} + Ad(f_{\alpha+\gamma})(\Omega^{0,1}(T)).$$

We thus want a family of complex gauge transformations bringing each $\overline{\partial}_A + (b + i\gamma\beta\eta)^{0,1}$ sufficiently close to $\overline{\partial}_A$ into this form. Because of the uniform pointwise bounds on the b's in T' we can proceed as in Lemma 2.31. Choose a fibred structure on a tubular

neighbourhood $N(\Sigma)$ as in that Lemma and cover $N(\Sigma)$ by small polydiscs $\{U_i\}$ such that each $(U_i, \Sigma \cap U_i)$ is a standard complex pair. For each $(b,\gamma) \in T'$ we can now apply Proposition 3.48 and find complex gauge transformations $g_i(b,\gamma) = 1 + u_i(b,\gamma)$ with the $u_i(b,\gamma)$'s $\mathcal{C}^0$–close to the identity and satisfying the required bounds such that

$$g_i(b,\gamma)(\overline{\partial}_A + b + (i\gamma\beta\eta)^{0,1}) = \overline{\partial}_{\alpha+\gamma}. \tag{7.17}$$

Since we assume that the b's are $L^p_{1,\alpha}$–uniformly close to zero and the pointwise bounds on T' are controlled by the $L^p_{1,\alpha}$ norm, we can choose the cover to be sufficiently fine so that (7.17) can be solved simultaneously for all small enough $(b, i\gamma\beta\eta)$ on each U_i with an explicit control on the $\mathcal{C}^0$–closedness of the $g_i'(b,\gamma)$ to the identity. As in the proof of Theorem 3.55 the transition functions between two such gauges are continuous, $\overline{\partial}_{\alpha+\gamma}$–holomorphic and $\mathcal{C}^0$–close to the identity. We can now patch up the local solutions using a fixed partition of unity subordinate to $\{U_i\}$ constant along the disc fibres and use the standard gluing argument of Proposition 4.4.10 of [DK] to obtain the desired family of gauge transformation. Similarly in the framed case.

In this situation, the family of local solutions $g_i'(b,\gamma)$ to (7.17) obtained by Proposition 3.48 on each U_i, depends smoothly on (b,γ) since given by applying the contraction mapping principle to a family of contractions which is manifestly smooth in (b,γ). It follows that the family of global gauge transformations $g'(b,\gamma)$ will also depend smoothly on (b,γ). $\square$

Composing (7.14) with projection onto the $(0,1)$–component and the map

$$(b^{0,1}, (i\gamma\beta\eta)^{0,1}) \mapsto (f_{\alpha+\gamma} g'(b,\gamma))(\overline{\partial}_A + (b + i\gamma\beta\eta)^{0,1}), \tag{7.18}$$

we thus obtain a continuous map of the original model into $\mathcal{H}_{\mathfrak{p}}$ for each $\alpha + \gamma$ sufficently close to α depending smooothly on γ. It follows that this map descends to a continuous map of the extended moduli spaces into the moduli spaces of parabolic stable or parabolic framed bundles (we could define extended moduli spaces for these too, but there isn't really much point in doing it since in the framed case these would triviallly be a product whereas in the stable case they would at least be a fibration, thus locally a product, by the openness in α of the stability conditon). Comparing with the proof of Theorem 3.55 it is clear that for each small γ this map coincides with the map $\Phi_{\alpha+\gamma}$ constructed there showing that Φ_γ is indeed continuous and depends continuously on $\gamma \in Z(\mathfrak{c}(i\alpha))$ as long as this is close to zero. The same applies to $\widetilde{\Phi}$. Finally we show that at smooth points these maps are in fact diffeomorphisms depending smoothly on α which together with Lemma 7.8 concludes the proof of Theorem 6.

Corollary 7.19. *At smooth points of the moduli spaces Φ_α and $\widetilde{\Phi}_\alpha$ are diffeomorphisms depending smoothly on α.*

Proof. By the previous Lemma when $[A]$ is a smooth point so that $([A], J)$ is a smooth point of the extended moduli spaces, we obtain that the map constructed is smooth as a map of the above local model for the extended moduli spaces at $([A], J)$ into $\mathcal{A}_{\mathfrak{p}}$. Since by Lemmas 7.7 and 7.8 we know that $\mathcal{E} = (E^{\mathbb{C}}, (f_\alpha \circ g'(0,0))(\overline{\partial}_A))$ is then a smooth point of

the holomorphic moduli spaces the map descends to a smooth map of the extended moduli space into the holomorphic moduli space. Explicitly, since the transversal T from which we began can now be assumed to be a ball at the origin in $\mathbb{R}^n \times Z(\mathfrak{c}(i\alpha))$ we obtain the differential of Φ_α by differentiating along a smooth path $(b_t, i\gamma_t\beta\eta)$ at the origin. Let

$$\frac{d}{dt}(b_t, i\gamma_t\beta\eta)_{|t=0} = (b, i\gamma\beta\eta), \qquad \frac{d}{dt}(g(b_t, \gamma_t))_{|t=0} = u.$$

The differential of (7.14) is then given by

$$(b, i\gamma\beta\eta) \mapsto (b + d_A(u), i\gamma\beta\eta),$$

which is clearly an isomorphism of $\widehat{H}^1_A$. Similarly, letting $(b_t, i\gamma\beta\eta)$ be a path at $(0, i\gamma\beta\eta)$ in T' and

$$\frac{d}{dt}(b_t)_{|t=0} = b, \qquad \frac{d}{dt}(g'(b_t, \gamma)))_{|t=0} = u',$$

we see that the differential at $(0, i\gamma\beta\eta)$ of the composite of projection onto the $(0,1)$–component with (7.18) in the $L^p_{1,\alpha}$–direction is given by

$$\begin{aligned} b \mapsto Ad(f_{\alpha+\gamma})(Ad(g'(0,\gamma))(b^{0,1} - [u'(g')^{-1}(0,\gamma), (i\gamma\beta\eta)^{0,1}]) + \\ (g')^{-1}(0,\gamma)[\overline{\partial}_A(u') - u'(g'(0,\gamma)(\overline{\partial}_A) - \overline{\partial}_A)]), \end{aligned}$$

which depends smoothly on γ. For $\gamma = 0$ this equals

$$b \mapsto Ad(f_\alpha)(Ad(g'(0,0))(b^{0,1}) + (g')^{-1}(0,0)[\overline{\partial}_A(u') - u'(g'(0,0)(\overline{\partial}_A) - \overline{\partial}_A)]),$$

which gives a map $H^1_A \to H^1(\mathcal{T}_\mathcal{E})$ whose injectivity follows as in Lemma 7.8. By that Lemma the two cohomology groups, which are now the tangent spaces, have the same dimension so that the differential must in fact be surjective. □

One immediate application of this is to singular instantons and stable parabolic bundles over Kähler surfaces.

Corollary 7.20. *Let (X, Σ) be a compact connected complex pair such that $\Sigma \cdot \Sigma \geqslant 0$ and (X, ω) is Kähler. Then*

(i) *At a smooth point $[A]$, $M^{J,*}_{k,c,\ell}(X, \Sigma)$ is naturally a complex manifold.*

(ii) *The extended moduli spaces $\widehat{M}^{J,*}_{k,c,\ell}(X, \Sigma)$ is a continuous fibration with respect to the projection π restricted to the subset of $Reg^+(C(J))$ for which the fibres are not empty, smooth when $\widehat{M}^{J,*}_{k,c,\ell}(X, \Sigma)$ is a smooth manifold.*

(iii) *At a smooth stable point $[\mathcal{V}]$ of $\mathcal{M}^{P(\alpha)}_{k,c,\ell}(X, \Sigma)$, $\mathcal{M}^{\alpha,S}_{k,c,\ell}(X, \Sigma)$ is open in $\mathcal{M}^{P(\alpha)}_{k,c,\ell}(X, \Sigma)$ so has a complex manifold structure.*

Proof. By the previous Corollary at smooth points Φ_α is a diffeomorphism so on one hand it naturally pulls–back the complex manifold structure of $\mathcal{M}^{\alpha,S}_{k,c,\ell}$ to $M^{J,*}_{k,c,\ell}$ giving (i) and on the other it carries a neighbourhood of a given point $[A]$ in $M^{J,*}_{k,c,\ell}(X, \Sigma)$ onto

a neighbourhood of the corresponding point in $\mathcal{M}^{\alpha,S}_{k,c,\ell}(X,\Sigma)$, where $i\alpha = \log(J)$, which proves (iii). For (ii) observe that by the continuous dependence of $\Phi_{\alpha'}$ on α' as long as this is $\log(J')$ for some $J' \in Reg^+(C(J))$ and the independence of $\mathcal{M}^{\alpha,S}_{k,c,\ell}(X,\Sigma)$ under small deformations of α in the same range (when not empty!), we obtain a family of continuos identifications of the fibres of π close to $\pi^{-1}(i\alpha)$ with $\mathcal{M}^{\alpha,S}_{k,c,\ell}(X,\Sigma)$ continuously depending on α' hence a locally trivial structure. By the previous Lemma we know that when all points are smooth the $\Phi_{\alpha'}$'s are in fact a family of diffeomorphisms smoothly depending on α' as long as this in the stated range so that the local product structure is smooth. □

At points where our comparison of the deformation theories fails, i.e. non smooth points, we can still deduce the openness of $\mathcal{M}^{\alpha,S}_{k,c,\ell}(X,\Sigma)$ in $\mathcal{M}^{P(\alpha)}_{k,c,\ell}(X,\Sigma)$ by using the openness of the stability condition in α. Let α' be a rational approximation of α, that is a matrix of the form (1.23) with all rational eigenvalues orderly close to those of α and such that $P(\alpha') = P(\alpha)$. For α' sufficiently close to α, the notions of α and α'–stability coincide so that in particular $\mathcal{M}^{\alpha,S}_{k,c,\ell}(X,\Sigma)$ and $\mathcal{M}^{\alpha',S}_{k,c,\ell}(X,\Sigma)$ coincide and it suffices to prove the openness in $\mathcal{M}^{P(\alpha)}_{k,\ell}(X,\Sigma)$ of the latter. This can be done by reducing to the standard comparison theorem between irreducible $U(n)$–instantons and stable holomorphic vector bundles via orbifold techniques as done in [KM2] for the $SU(2)$ case.

7.3 Applications to singular instantons on (S^4, S^2)

With the results of the previous section and of Chapter 5 in our hands we can now pass to study in greater detail the moduli spaces $\widetilde{M}^J_{\kappa,\ell}(S^4,S^2)$ and $M^{J,*}_{\kappa,\ell}(S^4,S^2)$. We begin by proving an existence result for general simple simply connected compact groups.

Proposition 7.21. *Let G be any compact simple simply connected Lie group. For any admissible $J \in G$ and any topological invariants κ, ℓ such that $\kappa = \sum_{i=1}^{t} \kappa_i$ and with respect to the basis (2.5) of $\pi_2(G/C(J))$, $\ell = (\ell_1, \cdots, \ell_t)$ satisfies*

$$\forall i, \qquad \kappa_i \geqslant 0, \qquad \kappa_i + \ell_i \geqslant 0,$$

$\widetilde{M}^J_{\kappa,\ell}(S^4,S^2)$ is not empty.

Proof. By Proposition 5.38 we know that for every pair κ_i, ℓ_i as in the hypothesis there is an α–parabolic framing on $(\mathbb{P}^2, \Sigma, D_\infty)$ with topological invariants κ_i, $(0, \cdots, \ell_i, \cdots, 0)$, where $i\alpha = \log(J)$. Applying one implication of Theorem 1 we then know that for each i, $\widetilde{M}^J_{\kappa_i,(0,\cdots,\ell_i,\cdots,0)}(S^4,S^2)$ is not empty and we shall use this information to obtain that $\widetilde{M}^J_{\kappa,\ell}(S^4,S^2)$ is not empty.

Observe that by Propositions 5.31 and 7.8, for all admissible holonomies J and all possible topological invariants κ, ℓ the moduli spaces $\widetilde{M}^J_{\kappa,\ell}(S^4,S^2)$ are cut out smoothly by the anti–self–duality equation. In other words at any point $[A]$ of these moduli spaces H^2_A vanishes. It follows that the operator

$$d^+_A : L^p_{1,\alpha}(\Omega^1(\mathfrak{g}_{\overline{E}})) \to L^p(\Omega^+(\mathfrak{g}_{\overline{E}})),$$

is right invertible.

Pick t points $\{[A_1], \cdots, [A_t]\}$ one for each moduli space $\widetilde{M}^J_{\kappa_i,(0,\cdot\cdot,\ell_i,\cdot\cdot,0)}(S^4, S^2)$. We can now apply the method of §7.2 of [DK] to construct solutions in the moduli space $\widetilde{M}^J_{\kappa_1+\cdots+\kappa_i,(\ell_1,\cdot\cdot,\ell_i,\cdot\cdot,0)}(S^4, S^2)$ for each i by gluing and perturbation. We start from the data $(X_i, \Sigma_i, \overline{E}_i, A_i)$ where for each i $(X_i, \Sigma_i) = (S^4, S^2)$. Choose a point on S^2 and a small ball of radius s around it so that we can assume that for each of the A_i's there are G–gauges where $A_i = d_\alpha + a_i$ with a_i satisfying the bounds of Theorem 1.14 (iii). We start by gluing A_1 and A_2 as explained in §7 of [KM1] to obtain a connection A'_{12} with topological invariants $\kappa_1 + \kappa_2$ and $(\ell_1, \ell_2, 0, \cdots, 0)$. Notice first of all that we can glue copies of S^4 with the round metric so that the resulting conformal structure is still equivalent to the standard one. By the argument of §7.2.1 of [DK] we can ensure that the gluing is made so that $\|F^+(A'_{12})\|_{L^2} \leqslant Cs^2$ with C a constant depending say only on the L^∞ norms of the a_i. Since $d^+_{A_1}$ and $d^+_{A_2}$ are right invertible, when s is sufficiently small the argument of §7.2.3 of [DK] shows that $d^+_{A'_{12}} : L^p_{1,\alpha} \to L^p$ is right invertible for a fixed p, $2 < p < (1/2 - J_0/6)^{-1}$, and then the method of §7.2.2 of [DK] constructs a solution to the equation in the form

$$A_{12} = A'_{12} + a, \qquad a \in L^p_{1,\alpha},$$

in particular A_{12} has the same topological invariants as A'_{12}. Since $d^+_{A_{12}}$ and $d^+_{A_3}$ are right invertible we can iterate to obtain the required solution. □

Let $C_G = 2C_{G^{\mathbb{C}}}$, $C_C = 2C_P$. We then obtain

Corollary 7.22.

(i) *For each $J = \exp(-2\pi i\alpha) \in U(n)$ where $i\alpha$ is of the form (1.23) and for each topological type κ, ℓ satisfying condition (0.10) the moduli space of framed singular instantons $\widetilde{M}^{U(n),J}_{k,\ell}(S^4, S^2)$ equipped with its natural real analytic space structure is a non–empty smooth manifold of dimension $4nk + 2\Sigma(n_i + n_{i+1})\ell_i$, which is naturally diffeomorphic to the complex manifold $\widetilde{\mathcal{M}}^{P(\alpha)}_{k,\ell}(\mathbb{P}^2, \Sigma)$.*

(ii) *For every compact simply connected simple Lie group G, any admissible $J \in G$ and any topological type κ, ℓ satisfying condition (0.11) the moduli space of framed singular instantons $\widetilde{M}^{G,J}_{k,\ell}(S^4, S^2)$ equipped with its natural real analytic space structure is a non–empty smooth manifold of dimension $2C_G k + C_C(\ell)$, which is naturally diffeomorphic to the complex manifold $\widetilde{\mathcal{M}}^{P(\alpha)}_{k,\ell}(\mathbb{P}^2, \Sigma)$, where $i\alpha = \log(J)$.*

(iii) *The extended moduli space of framed singular instantons with C–holonomy and given topological invariants $\widetilde{\widetilde{M}}^C_{k,\ell}(S^4, S^2)$ with its natural real analytic space structure is a smooth manifold diffeomorphic to the product*

$$Reg^+(C) \times \widetilde{\mathcal{M}}^{P(J)}_{k,\ell}(\mathbb{P}^2, \Sigma).$$

In particular in (i) and (ii) the smooth structure depends on J only up to $Reg^+(C(J))$.

Proof. (i) This follows from Propositions 5.31 and 5.37 via Theorems 1 and 7.
(ii) This follows by applying Theorems 4 and the previous Proposition.

(iii) By the definition of $P(\alpha)$ the moduli spaces $\widetilde{\mathcal{M}}^{P(\alpha)}_{k,\ell}(\mathbb{P}^2,\Sigma)$ depend on α only up to $C(i\alpha)$. In particular, since $P(\alpha)$ is the same for all α such that $i\alpha = \log(J)$ with $J \in Reg^+(C)$, it follows that for any given closed connected subgroup C of G the above moduli spaces are the same for all $J \in Reg^+(C)$ and are further smooth manifolds. By Corollary 7.19 we know that for each given $J \in Reg^+(C)$, the maps Φ_α are a family of diffeomorpisms smoothly depending on α provided this remains close to $\log(J)$. This means that the pojection $\widetilde{\pi}$ of (4.42) is locally trivial, that is a fibration, and since $Reg^+(C)$ is contractible it must actually be a product fibration. □

We shall now use the previous Corollary and the proof of Proposition 7.21 to obtain information on the moduli space of irreducible singular instantons and in particular on the existence of irreducible singular instantons for suitable holonomies and topological invariants in the spirit of [AHS]. We begin by considering the $SU(n)$ case.

Proposition 7.23. *Let $J \in SU(n)$ be any admissible element and let $\{n_i\}_{i=1}^{t+1}$ be the multiplicities of the eigenvalues of J. Then for any topological invariants κ, ℓ satisfying Condition 0.10 and such that further*

$$\forall 1 \leqslant i \leqslant t, \qquad (\kappa_i,\ell_i) \neq (0,0), \qquad \kappa_i + \ell_i \geqslant n_{i+1},$$

there exist irreducible $SU(n)$–instantons with this holonomy and these topological invariants. In particular the moduli spaces $M^{SU(n),J,*}_{\kappa,\ell}$ are smooth manifolds of dimension

$$dim(M^{SU(n),J,*}_{\kappa,\ell}) = 4n\kappa + 2\sum_{i=1}^{t}(n_i + n_{i+1})\ell_i - (\sum_{i=1}^{t+1} n_i^2 - 1),$$

containing a non–empty open subset consisting of irreducible connections

Proof. Assuming the non–emptyness the rest of the claim immediately follows by (ii) of the previous Corollary either by using Lemma 4.17 or by recalling that in this case the quotient map

$$\widetilde{M}^{SU(n),J}_{\kappa,\ell} \to M^{SU(n),J}_{\kappa,\ell}$$

by the $C(J)$–action is a principal fibration at points $[A] \in \widetilde{M}^{SU(n),J}_{\kappa,\ell}$ such that $Ker(d_A) = 0$. To prove the non–emptyness we proceed inductively. In the $SU(2)$–case this follows immediately from the previous Corollary since by Corollary 1.47, for every $J \in SU(2)$ different from ± 1, all points in $\widetilde{M}^{SU(2),J}_{\kappa,\ell}$ are orbits of irreducible connections.
Let now $J \in SU(n)$ be any admissible element and fix topological invariants κ, ℓ as in the hypothesis. Choose a copy of $S(U(n-1)\times U(1))$ in $SU(n)$ containing J. We may assume this to be the standard one so that writing J according to the product as

$$J = J' \times J'', \qquad J' = K\delta, \qquad K \in SU(n-1),\ \delta \in U(1),$$

and letting $t' = rk(\pi_2(SU(n-1)/C(K)))$, in the notation of §2.1

$$\pi_2(SU(n)/C(J)) = \pi_2(SU(n-1)/C(K)), \qquad \text{if } t' = t,$$
$$\pi_2(SU(n)/C(J)) = \pi_2(SU(n-1)/C(K)) \oplus < -[S^2_{e_{n-1}-e_n}] >, \qquad \text{if } t' = t-1,$$

and the basis (2.5) for $\pi_2(SU(n)/C(J))$ contains that for $\pi_2(SU(n-1)/C(K))$. By the inductive assumption we can find an irreducible $SU(n-1)$ instanton with holonomy K and topological invariants κ', ℓ' where $\kappa' = \kappa$, $\ell' = \ell$ if $t' = t$ and $\kappa' = \kappa_1 + \cdots + \kappa_{t'}$, $\ell' = (\ell_1, \cdots, \ell_{t'})$ if $t' = t-1$. Tensoring by a flat line bundle connection with holonomy δ we obtain a $U(n-1)$–instanton with holonomy J' and the same topological invariants and taking the direct sum connection with a flat line bundle connection with holonomy J'' we obtain an $S(U(n))$–instanton A with holonomy J, stabilizer $U(1)$ and topological invariants κ', ℓ' if $t' = t$ and $(\ell', 0)$ if $t' = t-1$. Considering A first as an $SU(n-1)$–connection and then as an $SU(n)$–connection and observing that if J has eigenvalues with multiplicities $\{n_i\}$

$$dim(C(J)) = \sum_{i=1}^{t+1} n_i^2 - 1,$$

we then have that the dimensions of the first cohomology groups of the complexes (4.1) for $SU(n-1)$ and $SU(n)$–connections at d_A are related by

$$\begin{aligned}
dim(H^1_A(\mathfrak{su}(n))) = \\
&1 + 4n\kappa + 2\sum_i^{t-1}(n_i + n_{i+1})\ell_i + 2(n_t + n_{t+1})\ell_t - dim(C(J)) > \\
&1 + 4(n-1)\kappa + 2\sum_i^{t-1}(n_i + n_{i+1})\ell_i + 2(n_t + n_{t+1} - 1)\ell_t - (dim(C(K)) = \\
&1 + 4(n-1)\kappa + 2\sum_i^{t-1}(n_i + n_{i+1})\ell_i + 2(n_t + n_{t+1} - 1)\ell_t + \\
&- (dim(C(J)) + 1 - 2n_{t+1}) = \\
&1 + dim(H^1_A(\mathfrak{su}(n-1))),
\end{aligned}$$

provided the hypothesis are satisfied. We can now apply the second argument in §8 of [AHS] to deduce that there must be irreducible $SU(n)$–connections in the neighborhood of $[A]$ which solves the problem for $t' = t$. Next assume that $t' = t-1$. As in Proposition 7.21 we can construct an instanton A'' with holonomy J and topological invariants κ'', $(0, \ell_t)$ starting from the existence of irreducible solutions for $SU(2)$ with topological invariants κ', ℓ_t and then using Theorem 1 and the embedding $SL(2)_{\beta_t} \leqslant SL(n)$. We now form the connected sum $A' = A \# A''$. This has the required topological invariants and holonomy and further, since A was an irreducible instanton and A'' has structure group not smaller than $SU(2)_{\beta_t}$, A' must be irreducible. Using the argument of Proposition 7.21 we can now obtain a nearby instanton with the same holonomy and topological numbers. By the semicontinuity property for the holonomy of a connection explained in [AHS] it follows that the resulting instanton must also be irreducible. □

Next we examine the case of general groups. For simplicity we restrict here to the case where the holonomy is a regular element so that $C(J)$ is a maximal torus in G.

Proposition 7.24. *Let G be any simple compact simply connected Lie group and $J \in G$ any admissible element. Then the moduli spaces $M^{G,J,*}_{\kappa,\ell}$ are, if not empty, smooth manifolds of dimension*

$$dim(M^{G,J,*}_{\kappa,\ell}) = 2C_G k + C_C(\ell) - dim(C(J)).$$

If J is also a regular element then for any topological invariants κ, ℓ satisfying Condition 0.11 and such that further $\forall i$, $(\kappa_i, \ell_i) \neq (0,0)$ the moduli space $M^{G,J,*}_{\kappa,\ell}$ is a smooth manifold of dimension

$$dim(M^{G,J,*}_{\kappa,\ell}) = 2C_G k + 4 \sum_{i=1}^{rk(G)} \ell_i - rk(G),$$

containing a non–empty open set consisting of irreducible connections.

Proof. The first claim follows from (ii) of Corollary 7.22 and Lemma 4.17 as in the previous Proposition. The proof of the second is based on the gluing procedure of Proposition 7.21. Observe that when G_i, $i = 1,2$ are subgroups of G and A_i, are irreducible G_i–connections regarded as G–connections, the connected sum connection A_{12} has holonomy $G_{12} =< G_1, G_2 >$, the smallest subgroup containing both G_i's. In particular, if G is simply connected, $\{SU(2)_{\beta_i}\}_{\beta_i \in R+}$ are the copies of $SU(2)$ in G associated to the simple roots of a fixed positive system as in §2.1, and A_i irreducible $SU(2)$–connections (on $S^4 \backslash S^2$) considered as G–connections through these embeddings, the connected sum connection will be an irreducible G–connection. Given a regular admissible element $J \in G$ and topological invariants $\kappa = \kappa_1 + \cdots + \kappa_n$, $\ell = (\ell_1, \cdots, \ell_n)$ satisfying the hypothesis, we can construct as in Proposition 7.21 instantons A_i with holonomy J, invariants κ_i, ℓ_i and structure group $< SU(2)_{\beta_i}, J >$. By the above observations the connected sum connection is an irreducible G–connection close to an instanton. By the semicontinuity for the holonomy we deduce that the nearby instanton obtained as in the proof of Proposition 7.21 is also an irreducible G–connection. □

Finally we can go backwards and use these results to prove the non–emptyness of the moduli space of parabolic framings for general structure group when Condition (0.11) is satisfied and the existence of irreducible bundles with fairly non–generic (i.e. holomorphically non–trivial) $\mathbb{C}^*$–reduction types on Σ.

Corollary 7.25.

(i) *For any complex simple simply connected Lie group $G^{\mathbb{C}}$ and parabolic subgroup P and for all topological types κ, ℓ satisfying Condition 0.11, $\widetilde{\mathcal{M}}^P_{\kappa,\ell}(\mathbb{P}^2, \Sigma)$ is not empty.*

(ii) *If κ, ℓ satisfy the conditions of Proposition 7.24 and $P = B$ is a Borel subgroup of $G^{\mathbb{C}}$ these moduli spaces contain elements whose underlying holomorphic bundle $\mathcal{E}$ has structure group which can not be reduced compatibly with the parabolic structure. When $G^{\mathbb{C}} = SL(n)$ the same holds for P any parabolic subgroup provided the topological invariants satisfy the Conditions of Proposition 7.23.*

In the case where the holonomy J is not regular it would be rather more difficult to obtain a precise statement on the existence of irreducible instantons with this holonomy since to apply the argument of Proposition 7.23 for the $SU(n)$ case one would need to calculate explicitly the function C_G and the dimension of $C(J)$ appearing in the index formula. Observe however that from Lemma 2.18 we can at least say that, with respect to the basis (2.5) for $\pi_2(G/C)$, C_G can be identified with a vector of positive integers. Using the argument of Proposition 7.23 and the maximal dimension inclusions given in table 8.3

of [AHS] we can then certainly deduce that for κ and ℓ such that $C_G\kappa + C_C(\ell)$ is large enough there are irreducible instantons with this holonomy and topological invariants.

7.4 Applications to hyperbolic monopoles

We shall now derive results analogous to those in the previous section for Hyperbolic monopoles. As usual we can proceed along two routes, namely the equivariant and the invariant one.

We begin by remarking that the identification obtained in Chapter 6 between the moduli space of $\mathbb{C}^*$–invariant parabolic framings and that of based rational maps is natural, that is it is an isomorphism of ringed spaces and in fact we can take the ringed space structure on $\mathcal{R}^*_\ell(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))$ to be that of $\widetilde{\mathcal{M}}^{G^{\mathbb{C}},P(\alpha)}_{0,\ell}(\mathbb{P}^2,\Sigma)$. Similarly, if we construct the moduli space $\widetilde{\mathcal{M}}^{\alpha'}_{m,\ell''}(\mathbb{P}^2,\Sigma)$ by considering the $\mathbb{C}^*$–equivariant part of the complex (5.1) one can show that the identification with $\widetilde{\mathcal{M}}^{G^{\mathbb{C}},P(\alpha)}_{0,\ell}(\mathbb{P}^2,\Sigma)$ given by Proposition 6.11 is an isomorphism of ringed spaces. The comparison of the deformation theories carried out in §7.1 and §7.2 can now be restricted to the equivariant parts, when using $M^J_{m,\ell'}(S^4,S^2)$ and to the invariant parts when using $M^\alpha_{0,\ell}(S^4,S^2)$ as defined in §6.4 to parametrize hyperbolic monopoles of given mass and charge. In fact, using Corollary 6.23, for high mass monopoles we can give a tidier proof of the fact that the map Φ is a smooth map than the one in §7.2. We thus obtain the corresponding version of Corollary 7.22.

Corollary 7.26.

(i) *For every compact simply connected simple Lie group G, any $i\alpha \in \mathfrak{g}$ and for each topological type ℓ satisfying condition (0.10) the moduli space of framed hyperbolic monopoles of mass α and charge ℓ, $\widetilde{M}^{G,\alpha}_\ell(\mathbb{H}^3)$, equipped with its natural real analytic space structure is a non–empty smooth manifold of dimension $C_C(\ell)$ which is naturally diffeomorphic to the complex manifold $\mathcal{R}^*_\ell(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))$.*

(ii) *For every closed connected subgroup $C \leqslant G$. The extended moduli space of framed hyperbolic monopoles with C–symmetry breakdown at infinity and given charge, $\widehat{\widetilde{M}}^C_\ell(\mathbb{H}^3)$, with its natural real analytic space structure is a smooth manifold diffeomorphic to the product*

$$Reg^+(\mathfrak{c}) \times \mathcal{R}^*_\ell(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha)).$$

In particular the smooth structure in (i) depends on α only up to $Reg^+(\mathfrak{c})$.

Proof. (i) Since Condition 0.10 means here that all ℓ_i are non–negative we can use the method of Proposition 7.21 to prove non–emptyness directly in the S^1–invariant set up. The rest of the claim then follows from Proposition 5.31, via Theorems 3 and Corollary 7.19 and the remarks made at the beginning of this section.

(iii) By the definition of $P(\alpha)$ the moduli spaces $\widetilde{\mathcal{M}}^{P(\alpha)}_{0,\ell}(\mathbb{P}^2,\Sigma) = \mathcal{R}^*_\ell(\mathbb{P}^1, G^{\mathbb{C}}/P(\alpha))$ depend on α only up to $P(\alpha)$ which is the same for all $i\alpha \in Reg^+(\mathfrak{c})$. The claim now follows as in (iii) of Corollary 7.22. □

As in the previous section we shall now address the question of the existence of irreducible monopoles.

Corollary 7.27.

(i) *Let $0 \neq i\alpha \in \mathfrak{su}(n)$ and let $\{n_i\}$ be the multiplicities of the eigenvalues of α. Then for any $\ell \in \pi_2(SU(n)/C(i\alpha))$ such that*

$$\forall i, \qquad \ell_i \geqslant n_{i+1},$$

the moduli spaces $M_\ell^{SU(n),\alpha,*}(\mathbb{H}^3)$ are smooth manifolds of dimension

$$dim(M_\ell^{SU(n),\alpha,*}(\mathbb{H}^3)) = 2\sum_{i=1}^{t}(n_i + n_{i+1})\ell_i - (\sum_{i=1}^{t+1} n_i^2 - 1),$$

containing a non–empty open set consisting of irreducible monopoles.

(ii) *Let G be any simple compact simply connected Lie group and $0 \neq i\alpha \in \mathfrak{g}$. Then the moduli spaces $M_\ell^{G,\alpha,*}(\mathbb{H}^3)$ are, if not empty, smooth manifolds of dimension*

$$dim(M_\ell^{G,\alpha,*}(\mathbb{H}^3)) = C_C(\ell) - dim(C(i\alpha)).$$

If α is a regular element, then for any $\ell \in \pi_2(G/C(i\alpha))$ such that

$$\forall i, \qquad \ell_i \geqslant 1,$$

the moduli spaces $M_\ell^{G,\alpha,*}(\mathbb{H}^3)$ are smooth manifolds of dimension

$$dim(M_\ell^{G,\alpha,*}(\mathbb{H}^3)) = 4\sum_{i=1}^{t}\ell_i - rk(G),$$

containing a non–empty open set of irreducible monopoles.

(iii) *Let G be as in (i) but now only assume that $i\alpha \in \mathfrak{g}$ is non–zero. Then for any $\ell \in \pi_2(G/C(i\alpha))$ such that $C_C(\ell)$ is large enough there exist irreducible G–monopoles with these charge and mass.*

Proof. This follows by putting together Propositions 7.23 and 7.24 and the remarks following it. □

Since hyperbolic monopoles of charge ℓ and mass α for the hyperbolic metric of constant curvature $-\epsilon$ can be reinterpreted as monopoles of the same charge having mass $\epsilon\alpha$ for the hyperbolic metric of constant curvature -1, using (ii) of the previous Corollary and a close analysis of how the norms of elements obtained via Theorem 1 behave when the same parabolic framing is used to construct monopoles with any mass $i\alpha \in Reg^+(\mathfrak{c})$, it should be possible to give a rigorous proof of Atiyah's conjecture that the moduli space of hyperbolic monopoles of given charge converges to the moduli space of euclidean monopoles of the same charge when we let the curvature of hyperbolic space tend to zero. However we shall not pursue this here.

7.5 Stable bundles and singular instantons on rational surfaces

With Theorems 2 and 6 at hand we can now use Example (b) of §5.2 to obtain information on the moduli spaces of singular $SU(2)$–instantons on a pair (X, Σ) where X is a minimal rational surface and $\Sigma = [mH]$ a multiple of a hyperplane class $[H]$ defining the metric and the stability condition. From Corollary 5.35 we immediately deduce.

Corollary 7.28. *Let (X, Σ), $[H]$ and (κ, ℓ) be as in Corollary 5.35 and let $J \in SU(2)$ be of the form $J = \exp(-2\pi i\alpha)$ where*

$$\alpha = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \qquad 0 < \beta < \frac{1}{\deg(\Sigma)},$$

then the moduli space $M_{\kappa,\ell}^{SU(2),J,*}$ with its natural real analytic space structure is diffeomorphic to the complex manifold $\mathcal{M}_{\kappa,\ell}^{\alpha,S}(X, \Sigma)$, where $i\alpha = \log(J)$. In particular it is a non–empty real manifold independent of α of dimension

$$dim(M_{\kappa,\ell}^{SU(2),J,*}(X, \Sigma)) = 8\kappa - 3 + 4\ell + 2(1 - g(\Sigma)).$$

Proof. This follows immediately from Corollary 5.35 through Theorems 2 and 6. □

A few observations on this example. The first is that for α in the stated range it follows by Corollary 7.19 that the extended moduli spaces of irreducible instantons with holonomy are a product. The second is that in the special case when $X = \mathbb{P}^2$ and $\Sigma = mH$, where H is the class of a line, it follows by Corollary 1.47 that there are no reducible connections with this holonomy along Σ thus the same statements holds for $M_{\kappa,\ell}^{SU(2),J}(X, \Sigma)$. Finally we note that for $X = \mathbb{P}^2$, $\mathcal{M}_{\kappa,\ell}^{\alpha,S}(X, \Sigma)$ can be computed explicitly at least in simple cases such as $\kappa = 1$ and $\deg(\Sigma) \leqslant 3$. It would then be interesting to give a more concrete interpretation of the correspondence in the spirit of the examples in §4.1 of [DK] for at least one such case, however we have no time to do it here.

Conclusions

The reader might have guessed from our preoccupation with semi–infinite cylinders at a number of places that we would have liked to present Theorem 1 as an example of a general result rather than as an isolated case were a correspondence between instantons on a non–complete infinite volume Kähler surface and parabolic framings holds. Apart from the technical obstacles, one of the main problems in achieving this was that whereas the case appearing in Theorem 1 may be regarded as "god–given", at first it was not clear (at least not to us) what a sufficiently general but yet equally natural class of triples and metrics for which one could try to prove a correspondence Theorem along the lines of Theorem 1 should be. Neither was it clear what the behaviour at infinity and hence the framing conditions should be in general: the conditions at infinity in Theorem 1 turn out to be very degenerate and hence do not give the best guidance towards generalization. Similarly, on the local side, it was not clear what to expect near a singular point of Σ if this is allowed to have singularities worse than normal crossings. After some thought we came up with what we think is a good class of candidates for generalizing the correspondence and as a consolation for the fact that we could not present this as a Theorem here we shall spend the last few paragraphs of the thesis illustrating this broader framework for the correspondence and how Theorem 1 fits into it.

Consider a compact complex surface $\overline{X}$ with divisors Σ and D_∞ such that

(i) D_∞ is a smooth irreducible divisor with $D_\infty \cdot D_\infty > 0$.
(ii) Σ is a reduced divisor with at most ordinary singularities such that $\Sigma \cdot \Sigma \geqslant 0$.
(iii) Σ and D_∞ intersect transversely.
(iv) The projective embedding associated to the linear system $|D_\infty|$,

$$\phi_{|D_\infty|} : \overline{X} \hookrightarrow \mathbb{P}^N,$$

embeds $\overline{X}$ with at most ordinary singularities (the tangent cone at a singular point of $\phi(\overline{X})$ is a smooth irreducible plane curve).
(v) The exceptional locus E of $\phi_{|D_\infty|}$ is disjoint from D_∞, intersects Σ transversely and $D_\infty = \phi^{-1}_{|D_\infty|}(H)$ is a hyperplane section.

Let *euc* be the euclidean metric on $\mathbb{C}^N = \mathbb{P}^N \backslash H$ and equip $X = \overline{X} \backslash (\Sigma \cup D_\infty \cup E)$ with the Kähler structure

$$\omega = \phi^*(euc).$$

It is for Kähler surfaces of the form (X, ω) that we wish to prove a correspondence with parabolic framings over the triple $(\overline{X}, \Sigma, D_\infty \cup E)$. In order to state this precisely we need a few observations concerning the representation of the fundamental group of circle bundles over punctured Riemann surfaces. For simplicity we shall restrict here to the $GL(n)$–case. Let Σ be a Riemann surface with marked points $\{\mathrm{p}\}$, let Y be a circle bundle over Σ and

let Y^* be the punctured circle bundle obtained by deleting the S^1–fibres at the marked points. As usual we can identify representations $\pi_1(Y^*) \to U(n)$ up to conjugation with flat $U(n)$–connections up to gauge equivalence and thanks to the parabolic Hitchin–Kobayashi correspondence on punctured Riemann surfaces recalled in the Introduction these can be in turn identified with suitable parabolic bundles on $(\Sigma, \{\mathrm{p}\})$ uniquely up their stabilizer.

We begin by considering the blow–up $\hat{\overline{X}} \to \overline{X}$ at the points in $Sing(\Sigma)$ as in §5.3 and denote by E_Σ the exceptional divisor. By the assumptions on D_∞, Σ, E and the metric, the restrictions of ω (or rather of its pull–back to $\hat{\overline{X}}$) to the punctured tubular neighbourhoods $N^*(D_\infty) = N(D_\infty)\backslash(D_\infty \cup \Sigma)$, $N^*(E^i) = N(E^i)\backslash E \cup \Sigma$ and $N^*(E^i_\Sigma)\backslash E^i_\Sigma \cup \widetilde{\Sigma}$ ($E = \cup E^i$ and $E_\Sigma = \cup E^i_\Sigma$ are the irreducible components of E and E_Σ respectively) are conformally equivalent to ones exponentially close to cylinder metrics on punctured half–cylinders $[0, \infty) \times Y^*$, where Y^* is the circle bundle over the punctured Riemann surfaces obtained from D_∞, E_i or E^i_Σ. From the results of [T2] concerning circle bundles of non–zero degree over Riemann surfaces and using [R1] as sketched in §1.3 we would expect a finite action instanton on $(X\backslash\Sigma, \omega)$ to determine flat limits Γ_{D_∞}, Γ_{E_i}, Γ_{p} one for each of these circle bundles. Much as in the case of a compact X to an instanton on $X\backslash\Sigma$ we attached the holonomy along Σ and considered the moduli space of instantons with given holonomy along Σ, given a collection of flat connections $\Gamma = \{\Gamma_{D_\infty}, \Gamma_{E_i}, \Gamma_{\mathrm{p}}\}$, we can now define the moduli space of instantons with Γ–limit. Notice that by the assumptions on D_∞ and Σ this determines the limit holonomy $J^i = \exp(-2\pi i \alpha^i)$ along each component Σ^i of Σ. By the previous observations $\{\Gamma_{D_\infty}, \Gamma_{E_i}, \Gamma_{\mathrm{p}}\}$ can be uniquely identified with a parabolic bundle $\mathcal{V}^0$ over $(D_\infty \cup E, (D_\infty \cup E \cup E_\Sigma) \cap \Sigma)$ up to stabilizers. We denote by $\widetilde{\mathcal{M}}^\alpha(\overline{X}, \Sigma)$ the set of equivalence classes of $\mathcal{V}^0$–framed $\alpha = \{\alpha^i\}$–parabolic bundles on $(\overline{X}, \Sigma, D_\infty)$. The Theorem we are aiming for ([M2]) can then be stated as

Conjecture. *Let (X, ω) and $(\overline{X}, \Sigma, D_\infty \cup E)$ as described above, then for any choice of Γ and corresponding framing bundle $\mathcal{V}^0$, there is a natural bijection*

$$M^\Gamma(X, \Sigma) \cong \widetilde{\mathcal{M}}^\alpha(\overline{X}, \Sigma)/Stab(\Gamma_{D_\infty}) \times Aut(\mathcal{V}^0_{|E \cup E_\Sigma}).$$

The expected difference in the behaviours along D_∞ and $E \cup E_\Sigma$ in the reducible cases is due to the opposite signs of the circle bundles on D_∞ and each of the components of $E \cup E_\Sigma$. Taking $\overline{X} = \mathbb{P}^2$, Σ, D_∞ to be distinct complex lines, ω the euclidean metric on $\mathbb{C}^2 = \mathbb{P}^2\backslash D_\infty$ and Γ any non–zero flat connection on Y^*, the circle bundle of $N(D_\infty)$ with the fibre corresponding to $\mathrm{p} = \Sigma \cap D_\infty$ deleted which as we know has $\pi_1 = \mathbb{Z}$ so that $Stab(\Gamma) = S^1$, we are of course recovering the $GL(n)$–version of Theorem 1.

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