

# Powerfree Values of Polynomials

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## 1 Introduction

Let  $f(X) \in \mathbb{Z}[X]$  be an irreducible polynomial of degree  $d$ . It is conjectured that, for any integer  $k \geq 2$ , the polynomial  $f(n)$  takes infinitely many  $k$ -th power free values, providing that  $f$  satisfies the obviously necessary congruence conditions. Thus for every prime  $p$  we need to assume that there is at least one integer  $n_p$  for which  $p^k \nmid f(n_p)$ . This problem appears to become harder as the degree  $d$  increases, but easier as  $k$  increases. Thus in 1933 Ricci [11] handled the case  $k \geq d$ , and even proved an asymptotic formula

$$N_{f,k}(x) \sim A(f, k)x \quad (x \rightarrow \infty) \tag{1}$$

where

$$N_{f,k}(x) := \#\{n \in \mathbb{N} : n \leq x, f(n) \text{ } k\text{-free}\}.$$

Here the constant  $C(f, k)$  is given as

$$C(f, k) := \prod_p (1 - \rho_f(p^k)p^{-k})$$

where

$$\rho_f(d) := \#\{n \bmod d : d \mid f(n)\}.$$

Further progress was made twenty years later by Erdős [3] who showed that one could obtain  $k$ -free values for  $k = d - 1$ , as soon as  $d \geq 3$ . For such  $k$  the asymptotic formula (1) was later obtained by Hooley [9]. The next development was due to Nair [10] who established (1) for  $k \geq (\sqrt{2} - \frac{1}{2})d$ . In particular Nair's result shows that  $k = d - 2$  is admissible for  $d \geq 24$ . The author [5, Theorem 16] then showed how the “determinant method” could be applied to the problem, and demonstrated that the asymptotic formula remained valid for  $k \geq (3d + 2)/4$ , so that one may take  $k = d - 2$  providing only that  $d \geq 10$ . Indeed by using methods of Salberger (to

appear), Browning [2, Theorem 1] has shown that one can replace these inequalities by  $k \geq (3d + 1)/4$  and  $d \geq 9$  respectively.

In this paper we show that further progress is possible for irreducible polynomials of the form  $f(X) = X^d + c$ . For these we establish the following result.

**Theorem 1** *Let  $f(X) = X^d + c \in \mathbb{Z}[X]$  be an irreducible polynomial, and suppose that  $k \geq (5d + 3)/9$ . Then there is a constant  $\delta(d)$  such that*

$$N_{f,k}(x) = C(f, k)x + O(x^{1-\delta(d)}).$$

*The implied constant may depend on  $f$  and  $k$ .*

For comparison with the earlier results we point out that this will allow  $k = d - 2$  as soon as  $d \geq 6$ . The result of Erdős handles the case of cubic polynomials taking square-free values, and the most interesting open question then concerns quartic polynomials taking square-free values. We would therefore like to handle  $k = d - 2$  for  $d = 4$ , and one can track our progress towards this goal through the historical discussion above.

There is a related question concerning powerfree values of  $f$  at prime arguments. Here there is a natural condition that for every prime  $p$  there should be an integer  $n_p$ , coprime to  $p$ , and such that  $p^k \nmid f(n_p)$ . With this in mind one defines

$$N'_{f,k}(x) := \#\{p \text{ prime} : p \leq x, f(p) \text{ } k\text{-free}\}$$

and

$$C'(f, k) := \prod_p (1 - \rho'_f(p^k) \phi(p^k)^{-1})$$

where

$$\rho'_f(d) := \#\{n \bmod d : \text{g.c.d.}(n, d) = 1, d \mid f(n)\}.$$

The corresponding asymptotic formula

$$N'_{f,k}(x) \sim C'(f, k)\pi(x) \quad (x \rightarrow \infty)$$

has been proved for  $k = d$  by Uchiyama [12], by a method that also handles the case  $k > d$ . However it remains an open problem to establish this in the case  $k = d - 1$  considered for the previous problem by Erdős and Hooley. None the less, important progress has been made by Helfgott [7] and [8], showing in particular that the asymptotic formula holds in the case  $k = 2$  and  $d = 3$ .

Our methods are sufficiently robust that they apply immediately to powerfree values of  $f(p)$ . We have the following result.

**Theorem 2** *Let  $f(X) = X^d + c \in \mathbb{Z}[X]$  be an irreducible polynomial, and suppose that  $k \geq (5d + 3)/9$ . Suppose that for every prime  $p$  there is an integer  $n_p$ , coprime to  $p$ , and such that  $p^k \nmid f(n_p)$ . Then for any fixed  $A > 0$  we have*

$$N'_{f,k}(x) = C'(f, k)\pi(x) + O_A(x(\log x)^{-A}).$$

*In particular this holds for  $k = d - 1$  and every  $d \geq 3$ .*

The preliminary manoeuvres for these problems are straightforward. We shall fix the polynomial  $f$  (and hence also  $d$ ) throughout, so that all order constants may depend tacitly on  $f$  and  $d$ . The key fact we shall use is that

$$\sum_{b^k | f(n)} \mu(b) = \begin{cases} 1, & f(n) \text{ is } k\text{-free,} \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$N_{f,k}(x) = \sum_b \mu(b)N(b, x)$$

with

$$N(b, x) = \#\{n \leq x : b^k | f(n)\},$$

and similarly that

$$N'_{f,k}(x) = \sum_b \mu(b)N'(b, x)$$

with

$$N'(b, x) = \#\{p \leq x : b^k | f(p)\}.$$

Clearly  $N'(b, x) = N(b, x) = 0$  for  $b \gg x^{d/k}$ . If we denote the solutions to  $f(n) \equiv 0 \pmod{b^k}$  by  $n_1, \dots, n_r$ , where  $r = \rho_f(b^k)$ , then

$$\begin{aligned} N(b, x) &= \sum_{i \leq r} \#\{n \leq x : n \equiv n_i \pmod{b^k}\} \\ &= \sum_{i \leq r} (xb^{-k} + O(1)) \\ &= xb^{-k}\rho_f(b^k) + O(\rho_f(b^k)), \end{aligned}$$

and similarly, providing that  $b \leq (\log x)^{2A}$  we have

$$\begin{aligned} N'(b, x) &= \sum_{i \leq r} \#\{p \leq x : p \equiv n_i \pmod{b^k}\} \\ &= \sum_{i \leq r, (n_i, b)=1} \pi(x; b^k, n_i) \\ &= \frac{\pi(x)}{\phi(b^k)} \rho'_f(b^k) + O_A(\rho_f(b^k)x(\log x)^{-4A}), \end{aligned}$$

by the Siegel-Walfisz Theorem. Now, for any  $\xi > 0$  we find that

$$\sum_{b \leq \xi} \mu(b) N(b, x) = x \sum_{b \leq \xi} \frac{\mu(b) \rho_f(b^k)}{b^k} + O\left(\sum_{b \leq \xi} \rho_f(b^k)\right).$$

The function  $\rho_f$  is multiplicative, with  $\rho(p^k) \ll 1$ , whence

$$\rho(b^k) \ll_{\varepsilon} b^{\varepsilon} \quad (2)$$

for any  $\varepsilon > 0$  and any square-free  $b$ . If  $k \geq 2$  it follows on taking  $\varepsilon = 1/2$  that

$$\sum_{b \leq \xi} \frac{\mu(b) \rho_f(b^k)}{b^k} = \sum_{b=1}^{\infty} \frac{\mu(b) \rho_f(b^k)}{b^k} + O\left(\sum_{b > \xi} b^{1/2-k}\right) = C(f, k) + O(\xi^{-1/2})$$

and

$$\sum_{b \leq \xi} \rho_f(b^k) \ll \xi^{3/2}.$$

In particular if we set  $\xi = x^{1/2}$  we see that

$$\sum_{b \leq \xi} \mu(b) N(b, x) = C(f, k)x + O(x^{3/4}).$$

In precisely the same way, if we take  $\xi = (\log x)^{2A}$ , then

$$\sum_{b \leq \xi} \mu(b) N'(b, x) = C'(f, k)\pi(x) + O_A(x(\log x)^{-A}).$$

We now consider the range  $\xi < b \leq x^{1-\eta}$ , where  $\eta$  is a small positive constant. Here we have

$$\sum_{\xi < b \leq x^{1-\eta}} \mu(b) N(b, x) \ll \sum_{\xi < b \leq x^{1-\eta}} N(b, x) \ll \sum_{\xi < b \leq x^{1-\eta}} \left(\frac{x}{b^k} + O(1)\right) \rho_f(b^k).$$

If we use the bound (2) with  $\varepsilon = \frac{1}{2}\eta \leq \frac{1}{2}$  this yields

$$\sum_{\xi < b \leq x^{1-\eta}} \mu(b) N(b, x) \ll x \xi^{-1/2} + x^{1-\eta/2} \ll x^{1-\eta/2}.$$

This bound is satisfactory for Theorem 1. Since  $N'(b, x) \leq N(b, x)$  we will get exactly the same bound in the estimation of  $N'_{f,k}(x)$ , and again this is satisfactory for Theorem 2.

To complete the proof of the two theorems it will now be enough to show that

$$\sum_{x^{1-\eta} < b \ll x^{d/k}} N(b, x) \ll x^{1-\delta}$$

for some  $\delta > 0$ , providing that  $\eta$  is small enough. By a suitable dyadic subdivision we then see that it will suffice to establish the following estimate.

**Lemma 1** *Let  $f(X) = X^d + c \in \mathbb{Z}[X]$  be an irreducible polynomial. For any  $N, A, B \in \mathbb{N}$  define*

$$F(N; A, B) := \#\{(n, a, b) \in \mathbb{N}^3 : f(n) = ab^k, N < n \leq 2N, A < a \leq 2A, B < b \leq 2B\}.$$

*Then if  $(5d+3)/9 \leq k \leq d-1$  there is a constant  $\delta$  depending on  $d$  such that*

$$F(N; A, B) \ll_f N^{1-\delta}$$

*for  $B \geq N^{1-\delta}$ .*

We have now reduced our problem to one of counting solutions to a Diophantine equation  $f(n) = ab^k$ , inside a suitable box. A general procedure for such questions is provided by the “determinant method” developed in the author’s paper [4]. The efficiency of the method depends on the dimension of the associated algebraic variety. For  $f(n) = ab^k$  we are counting integer points on an affine surface. Thus far we have made no use of the special shape of the polynomial  $f$ , but if we observe that  $f(n) = n^d + O(1)$  we see that  $(n, a, b)$  lies close to the weighted projective curve  $X_0^d = X_1 X_2^k$ , where  $X_0$  and  $X_2$  are given weight 1, and  $X_1$  has weight  $d-k$ . Thus the particular form of the polynomial  $f$  allows us to consider points close to a curve, rather than points on a surface. Reducing the dimension in this way is the key to our saving. The procedure is discussed in more detail in the author’s work [6], to which the interested reader should be directed.

## 2 The Determinant Method

Since  $f(n) = n^d + O(1)$  we will have

$$N^d B^{-k} \ll A \ll N^d B^{-k} \tag{3}$$

for large  $N$ . Moreover, since  $a \geq 1$  we may assume that  $B^k \ll N^d$ , and indeed we shall assume that

$$N^{1-\eta} \ll B \ll N^{d/k} \tag{4}$$

for some positive constant  $\eta$ . We will choose a parameter  $K \geq 1$  having

$$1 \ll \frac{\log K}{\log N} \ll 1, \quad (5)$$

and divide the available range for  $n/b$  into  $O(K)$  subintervals

$$I = (m_0 N/BK, (m_0 + 1)N/BK]$$

with endpoints defined by integers  $m_0$  in the range

$$K \ll m_0 \ll K. \quad (6)$$

We use  $F_I(N; A, B)$  to denote the corresponding contribution to  $F(N; A, B)$ . Since  $f(n) = n^d + O(1)$  we have  $n^d = ab^k + O(1)$  and

$$(n/b)^d = a/b^{d-k} + O(B^{-d}).$$

It will be convenient to put  $k = d - j$  so that

$$(n/b)^d = a/b^j + O(B^{-d}). \quad (7)$$

We now begin the determinant method by listing the points  $(n_r, a_r, b_r)$  contributing to  $F_I(N; A, B)$ . Thus the index  $r$  runs from 1 to

$$R := F_I(N; A, B).$$

We choose an integer parameter  $D \geq 1$  and consider the monomials

$$m(n, a, b) = n^u a^v b^w$$

for which  $u + jv + w = D$ . Thus we may consider  $D$  as the weighted degree of the monomial, where the variables  $(n, a, b)$  are given weights  $(1, j, 1)$ . The number of such monomials will be

$$H := \sum_{v \leq D/j} (D - jv + 1) = \frac{D^2}{2j} + O(D) \quad (8)$$

and we label them as  $m_1(n, a, b), \dots, m_H(n, a, b)$ . We now proceed to consider the  $R \times H$  matrix  $M$  say, whose  $(r, h)$  entry is  $m_h(n_r, a_r, b_r)$ . The strategy of the determinant method is to show that  $M$  has rank strictly less than  $H$ , if the parameters  $K$  and  $D$  are suitably chosen. If this can be achieved, there will be a non-zero integer vector  $\mathbf{c}$  such that  $M\mathbf{c} = \mathbf{0}$ . This vector will

depend on the interval  $I$ , that is to say it will depend on  $m_0$ . It provides the coefficients of a weighted homogeneous polynomial

$$C_I(n, a, b) = \sum_h c_h m_h(n, a, b)$$

such that

$$C_I(n_r, a_r, b_r) = 0, \quad (r \leq R). \quad (9)$$

If  $R < H$  the matrix  $M$  automatically has rank less than  $H$ . Otherwise it suffices to show that any  $H \times H$  sub-determinant vanishes, and it will be enough to consider the determinant formed from the first  $H$  rows of  $M$ , which we shall denote by  $\Delta$ . Clearly  $\Delta$  is an integer, and our strategy is to show that  $|\Delta| < 1$  so that  $\Delta$  must vanish.

We proceed to divide the  $r$ -th row of  $M$  by  $b_r^D B^{-D}$  for each  $r \leq R$ , and similarly to divide the column corresponding to the monomial  $n^u a^v b^w$  by  $N^u A^v B^w$ . Since

$$n^u a^v b^w = \left(\frac{b}{B}\right)^D \left(\frac{nB}{bN}\right)^u \left(\frac{aB^j}{b^j A}\right)^v N^u A^v B^w$$

for  $u + jv + w = D$ , this produces a new determinant  $\Delta_1$  whose entries are of the form  $m_h(nB/bN, aB^j/b^j A, 1)$ . Moreover we have

$$|\Delta| = |\Delta_1| \prod_{r \leq H} (b_r/B)^D \prod_{u,v,w} N^u A^v B^w \leq 2^{HD} P |\Delta_1|, \quad (10)$$

where

$$P = \prod_{u+jv+w=D} N^u A^v B^w.$$

If we write  $B = N^\beta$  then we have  $\log A = (d - k\beta) \log N + O(1)$ , by (3). It follows that

$$\begin{aligned} \log P &= (\log N) \sum_{u+jv+w=D} (u + v(d - k\beta) + w\beta) + O_D(1) \\ &= (\log N) \left\{ \frac{D^3}{6j} (1 + (d - k\beta)j^{-1} + \beta) + O(D^2) \right\} + O_D(1). \end{aligned} \quad (11)$$

We now write

$$\frac{n_r B}{b_r N} = \frac{m_0}{K} + s_r, \quad \text{and} \quad \frac{a_r B^j}{b_r^j A} = \frac{N^d}{AB^k} \left( \frac{m_0}{K} + s_r \right)^d + t_r.$$

Since  $n_r/b_r \in (m_0 N/BK, (m_0 + 1)N/BK]$  it follows that

$$s_r \ll K^{-1}.$$

Moreover (3) and (7) yield

$$\frac{a_r B^j}{b_r^j A} = \frac{N^d}{AB^k} \left( \frac{n_r B}{b_r N} \right)^d + O(N^{-d}),$$

and hence

$$t_r \ll N^{-d}.$$

The  $(r, h)$  entry of  $\Delta_1$  will now be a polynomial in  $s_r$  and  $t_r$ , taking the shape

$$f_h(s_r, t_r) = (m_0 K^{-1} + s_r)^u (N^d A^{-1} B^{-k} (m_0 K^{-1} + s_r)^d + t_r)^v.$$

Clearly  $f_h$  may depend on  $h, m_0, K, D$  and  $d$ , but it is independent of  $r$ . Moreover the degree of  $f_h$  will be at most  $dD$ . It follows from (3) and (6) that  $N^d A^{-1} B^{-k} \ll 1$  and  $m_0 K^{-1} \ll 1$ , whence we have the bound  $\|f_h\| \ll_D 1$  for the height of  $f_h$ .

In order to estimate the size of  $\Delta_1$  we will use Lemma 3 of the author's work [6]. For each of the monomials  $s^u t^v$  we write

$$\|s^u t^v\| = K^{-u} N^{-dv},$$

and we list them in order as  $m_1, \dots, m_H$  with  $\|m_1\| \geq \|m_2\| \geq \dots$ . Then according to [6, Lemma 3] we have

$$\Delta_1 \ll_D (\max \|f_h\|)^H \prod_{h=1}^H \|m_h\| \ll_D \prod_{h=1}^H \|m_h\|. \quad (12)$$

To proceed further we shall write  $K = N^\kappa$ , and note that  $1 \ll \kappa \ll 1$ , by (5). If we now write  $m(\lambda)$ , say, for the number of monomials  $m_r = s^u t^v$  with  $\|m_r\| \geq N^{-\lambda}$ , then

$$m(\lambda) = \#\{(u, v) \in \mathbb{Z}^2 : u, v \geq 0, \kappa u + dv \leq \lambda\} = \frac{\lambda^2}{2\kappa d} + O(\lambda) + O(1).$$

If  $\|m_H\| = N^{-\lambda_0}$  then  $m(\lambda_0) \geq H$ , while for any  $\varepsilon > 0$  we will have

$$m(\lambda_0 - \varepsilon) \leq H - 1.$$

We may therefore deduce that

$$\lambda_0 = \sqrt{2\kappa d H} + O(1).$$

We then find that

$$\prod_{h=1}^H \|m_h\| = N^{-\mu}$$



with

$$\begin{aligned}
\mu &= \sum_{\kappa u + dv \leq \lambda_0} (\kappa u + dv) + O(\lambda_0^2) + O(1) \\
&= \frac{\lambda_0^3}{3\kappa d} + O(\lambda_0^2) + O(1) \\
&= \frac{2^{3/2}}{3} (\kappa d)^{1/2} H^{3/2} + O(H).
\end{aligned}$$

In view of (8), (10), (11) and (12) we may now conclude that

$$\frac{\log |\Delta|}{\log N} \leq \frac{D^3}{6j} (1 + (d - k\beta)j^{-1} + \beta) - \frac{2^{3/2}}{3} (\kappa d)^{1/2} H^{3/2} + O_D((\log N)^{-1}) + O(D^2).$$

Thus (8) yields

$$\begin{aligned}
\frac{\log |\Delta|}{D^3 \log N} &\leq \frac{1}{6j} (1 + (d - k\beta)j^{-1} + \beta) - \frac{2^{3/2}}{3} (\kappa d)^{1/2} (2j)^{-3/2} \\
&\quad + O_D((\log N)^{-1}) + O(D^{-1}).
\end{aligned}$$

We therefore choose

$$\kappa = \frac{j}{4d} \left( 1 + \frac{d - k\beta}{j} + \beta \right)^2 + \eta, \tag{13}$$

with the same small constant  $\eta$  as in (4). Then (5) will be satisfied, and we will have

$$\frac{\log |\Delta|}{D^3 \log N} < 0$$

providing that we first choose  $D = D(f, d, \eta)$  sufficiently large, and then ensure that  $N$  is sufficiently large in terms of  $f, d$  and  $\eta$ .

We therefore deduce that  $\Delta = 0$  when  $K = N^\kappa$ . With this choice the matrix  $M$  introduced at the beginning of the section will have rank strictly less than  $H$ , so that all solutions  $(n_r, a_r, b_r)$  counted by  $F_I(N; A, B)$  satisfy the auxiliary equation (9).

### 3 Completion of the Proof

We now complete our estimation of  $F_I(N; A, B)$  by considering how many triples  $(n, a, b)$  can satisfy both the original equation  $f(n) = ab^k$  and the additional equation (9). The procedure here will follow precisely that used in the author's paper [5, §5.3]. Since  $C_I$  is homogeneous with exponent weights

$(1, j, 1)$  any factor would have to be similarly weighted-homogeneous. Since  $f(x) - yz^k$  is irreducible and is not weighted-homogeneous it follows that  $C_I(x, y, z)$  cannot have a non-trivial factor in common with  $f(x) - yz^k$ . As in [5, pages 84 and 85] we find that either

$$F_I(N, A, B) \ll_{\varepsilon} (1 + N/B)N^{\varepsilon} \quad (14)$$

or that there is an irreducible polynomial  $G_I(X, Y) \in \mathbb{Z}[X, Y]$ , with degree bounded in terms of  $d$  and  $\varepsilon$ , but at least  $d$ , such that

$$G_I(n, b) = 0 \quad (15)$$

for every triple  $(n, a, b)$  counted by  $F_I(N, A, B)$ .

For a given interval  $I$  we will have

$$n/b \in I = (m_0 N/BK, (m_0 + 1)N/BK]$$

It therefore follows that

$$\left| n - \frac{m_0 N}{BK} b \right| \leq \frac{2N}{K}, \quad B < b \leq 2B. \quad (16)$$

It will be convenient to define a linear mapping  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$T\mathbf{x} := \begin{pmatrix} K(2N)^{-1}x_1 - (2B)^{-1}m_0x_2 \\ (2B)^{-1}x_2 \end{pmatrix}$$

and to consider the lattice

$$\Lambda = \{T\mathbf{x} : \mathbf{x} \in \mathbb{Z}^2\}$$

of determinant  $K(4NB)^{-1}$ . Then if  $\mathbf{x} = (n, b)$  satisfies (16) we produce a point  $T\mathbf{x} = (\alpha_1, \alpha_2) \in \Lambda$  falling in the square

$$S = \{(\alpha_1, \alpha_2) : \max(|\alpha_1|, |\alpha_2|) \leq 1\}.$$

Let  $\mathbf{g}^{(1)}$  be the shortest non-zero vector in the lattice and  $\mathbf{g}^{(2)}$  the shortest vector not parallel to  $\mathbf{g}^{(1)}$ . These vectors will form a basis for  $\Lambda$ . Moreover we have  $\lambda_1 \mathbf{g}^{(1)} + \lambda_2 \mathbf{g}^{(2)} \in S$  only when  $|\lambda_1| \ll |\mathbf{g}^{(1)}|^{-1}$  and  $|\lambda_2| \ll |\mathbf{g}^{(2)}|^{-1}$ . These constraints may be written in the form  $|\lambda_i| \leq L_i$ , for appropriate bounds  $L_1, L_2$ . Since  $|\mathbf{g}^{(2)}| \geq |\mathbf{g}^{(1)}|$  and  $|\mathbf{g}^{(1)}| \cdot |\mathbf{g}^{(2)}| \ll \det(\Lambda) \ll K(NB)^{-1}$  we will have  $L_1 \gg L_2$  and  $L_1 L_2 \gg NBK^{-1}$ . We now write  $\mathbf{h}^{(i)} = T^{-1} \mathbf{g}^{(i)}$  for  $i = 1, 2$ . These vectors will then be a basis for  $\mathbb{Z}^2$ , and if  $\mathbf{x} = \lambda_1 \mathbf{h}^{(1)} + \lambda_2 \mathbf{h}^{(2)}$  is in the region (16) then we will have  $|\lambda_i| \leq L_i$  for  $i = 1, 2$ . This allows us to

make a change of basis, replacing  $(x_1, x_2)$  by  $(\lambda_1, \lambda_2)$  so that our constraints on  $n, b$  are replaced by the conditions  $|\lambda_i| \leq L_i$ .

We therefore proceed to substitute  $\lambda_1, \lambda_2$  for  $n, b$  in (15). We may then use the bound of Bombieri and Pila [1, Theorem 5] to show that the number of possible pairs  $\lambda_1, \lambda_2$  is  $\ll_\varepsilon \max(L_1, L_2)^{1/d+\varepsilon} \ll_\varepsilon L_1^{1/d+\varepsilon}$ , since the degree of  $G_I$  is at least  $d$ . Thus

$$F_I(N, A, B) \ll_\varepsilon L_1^{1/d+\varepsilon}.$$

The number  $L_1$  depends on the interval  $I$ , which is determined by  $m_0$ . We therefore write  $L_1 = L_1(m_0)$  accordingly. In view of the alternative (14) we then see that

$$F(N, A, B) \ll_\varepsilon K(1 + N/B)N^\varepsilon + \sum_{K \ll m_0 \ll K} L_1(m_0)^{1/d+\varepsilon}, \quad (17)$$

the range for  $m_0$  being given by (6).

We proceed to investigate the number of choices for  $m_0$  which produce a value  $L_1(m_0)$  lying in a given dyadic interval  $(L, 2L]$  say. In the notation above, if  $(n, b) = (x_1, x_2)$  corresponds to  $\mathbf{g}^{(1)}$  then

$$L_1 \left( x_1 - \frac{m_0 N}{BK} x_2 \right) \ll \frac{N}{K}$$

and  $L_1 x_2 \ll B$ . Moreover we will have  $\text{g.c.d.}(x_1, x_2) = 1$ . Thus the number of intervals  $I$  for which  $L < L_1 \leq 2L$  is at most the number of triples  $(x_1, x_2, m_0) \in \mathbb{Z}^3$  with  $\text{g.c.d.}(x_1, x_2) = 1$ , for which

$$L \left( x_1 - \frac{m_0 N}{BK} x_2 \right) \ll \frac{N}{K}, \quad Lx_2 \ll B, \quad \text{and} \quad K \ll m_0 \ll K. \quad (18)$$

We proceed to consider whether the value  $x_2 = 0$  can occur. If  $x_2 = 0$  the first of the conditions above would yield  $Lx_1 \ll N/K$ . However we cannot have  $x_1 = x_2 = 0$ , so that we must have  $L \ll N/K$  whenever  $x_2 = 0$ . We now recall that  $L_1 \gg L_2$  and that  $L_1 L_2 \gg NBK^{-1}$ , whence

$$L^2 \gg NBK^{-1}. \quad (19)$$

It follows that if  $x_2 = 0$  then  $(N/K)^2 \gg L^2 \gg NBK^{-1}$  and hence that  $BK \ll N$ . However, since  $K = N^\kappa$  with  $\kappa$  given by (13), we see from (4) that  $BK/N$  tends to infinity with  $N$ , which ensures that the case  $x_2 = 0$  cannot arise.

We now see in particular that the second condition of (18) yields  $L \ll B$ . If we rewrite the first of the conditions (18) to say that

$$m_0 x_2 = N^{-1} BK x_1 + O(BL^{-1})$$

we then see that each choice for  $x_1$  restricts the product  $m_0 x_2$  to an interval of length  $\ll B/L$ , with  $B/L \gg 1$ . Moreover  $m_0 x_2$  is never zero. Thus a divisor function estimate shows that there are  $O_\varepsilon(N^\varepsilon B L^{-1})$  possible pairs  $(x_2, m_0)$  for each value of  $x_1$ . The conditions (18) show that  $x_1 \ll N/L$ , so that  $x_1$  takes  $O(1 + N/L)$  values. This allows us to conclude that the number of integers for  $m_0$  which produce a value  $L_1(m_0)$  in the range  $L < L_1 \leq 2L$  is  $O_\varepsilon((1 + N/L)N^\varepsilon B L^{-1})$ .

We can now feed this information into (17), using a dyadic subdivision for the values of  $L_1(m_0)$  to obtain

$$F(N, A, B) \ll_\varepsilon K(1 + N/B)N^\varepsilon + \sum_L L^{1/d+\varepsilon}(1 + N/L)N^\varepsilon B L^{-1},$$

in which  $L$  runs over powers of 2, subject to the condition  $L \gg (NBK^{-1})^{1/2}$  given by (19). It then follows that

$$F(N, A, B) \ll_\varepsilon K(1 + N/B)N^\varepsilon + L_0^{1/d+\varepsilon}(1 + N/L_0)N^\varepsilon B L_0^{-1},$$

where  $L_0 := \max\{1, (NBK^{-1})^{1/2}\}$ . On taking  $\varepsilon = \eta$  we deduce from (4) that

$$F(N, A, B) \ll_\eta N^{2\eta}\{K + L_0^{1/d+\eta}(1 + N/L_0)B L_0^{-1}\}.$$

We proceed to analyse our estimate for  $F(N, A, B)$  in terms of the variable  $t = (\log B)/(\log N)$  by defining

$$\rho(t) = \frac{j}{4d} \left(1 + \frac{d - kt}{j} + t\right)^2$$

and  $q(t) = \rho(t) + 1 - t$ . Then

$$q'(t) = \frac{j}{d} \left(1 + \frac{d - kt}{j} + t\right) \left(1 - \frac{k}{j}\right) - 1.$$

This is clearly negative if  $k \geq j$  and  $0 \leq t \leq d/k$ . Hence if  $k \geq d/2$  we have

$$q(t) \geq q(d/k) = \frac{j^3}{4dk^2} \geq 0$$

for  $0 \leq t \leq d/k$ . It therefore follows that  $KN \leq B$ , and hence that  $L_0 \leq N$  for the relevant range of  $B$ . Our estimate now simplifies to give

$$F(N, A, B) \ll_\eta N^{2\eta}\{K + L_0^{-2+1/d+\eta}NB\}.$$

This will be of order  $N^{1-\eta}$  if  $\eta > 0$  is sufficiently small, and if

$$\sup_{1 \leq t \leq d/k} \rho(t) < 1 \quad \text{and} \quad \sup_{1 \leq t \leq d/k} Q(t) < 0$$

for

$$Q(t) = \left(-2 + \frac{1}{d}\right) \frac{1+t-\rho(t)}{2} + t.$$

To handle the condition on  $\rho(t)$  we note that the function attains its supremum at either  $t = 1$  or  $t = d/k$ . Moreover if  $v = k/d$  satisfies  $5/9 < v < 1$  we find that  $\rho(1) = 9(1-v)/4 < 1$  and

$$\rho(d/k) = \frac{(1+v)(1-v^2)}{4v^2}.$$

This latter function is decreasing with respect to  $v$ , and takes the value  $196/225 < 1$  at  $v = 5/9$ . It follows that the supremum is strictly less than 1 if  $5/9 < k/d < 1$ .

To verify the condition on  $Q(t)$  we note that if  $1 \leq t \leq d/k$  then

$$\begin{aligned} Q'(t) &= \left(-2 + \frac{1}{d}\right) \frac{1}{2} \left\{ 1 - \frac{j}{2d} \left( 1 + \frac{d-kt}{j} + t \right) \left( -\frac{k}{j} + 1 \right) \right\} + 1 \\ &= \frac{1}{2d} - \left( 1 - \frac{1}{2d} \right) \frac{2k-d}{2d} \left( 1 + \frac{d-kt}{j} + t \right) \\ &\leq \frac{1}{2d} - \left( 1 - \frac{1}{2d} \right) \frac{2k-d}{2d} \left( 1 + \frac{d}{k} \right) \\ &< 0 \end{aligned}$$

for  $k > d/2$ . Thus

$$Q(t) \leq Q(1) = \frac{9j}{4d} \left( 1 - \frac{1}{2d} \right) - \left( 1 - \frac{1}{d} \right),$$

which is strictly negative for

$$j < \frac{4d(2d-2)}{9(2d-1)}.$$

This condition is equivalent to

$$k > \frac{10d^2 - d}{18d - 9} = \frac{5d+2}{9} + \frac{2}{18d-9}.$$

Thus it is necessary and sufficient that

$$k \geq \frac{5d+3}{9}.$$

This completes the proof of Lemma 1, and hence also of our two theorems.

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