Random Irreducible Quadrangulations

Simon Bergant
St Edmund Hall
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2019

Dedicated to my family.

Acknowledgements

Foremost, I would like to thank my supervisor Marc Lackenby for introducing me to hyperbolic geometry of alternating links. His support, guidance and encouragement over the last four years have been invaluable. I also very much appreciated his open-mindedness which gave me the freedom to explore the topics that I found to be most interesting.

I would like to express my deepest gratitude to my Mum, my Dad and my twin sister Ana as well as all of my family and friends for their constant support and encouragement on my way to finish my thesis. Among my friends I would especially like to mention Maïa Foulis, Prerna Bakshi, Mari Johnson, Emily Russell, Maren Fichter, Petra Poklukar and Saša Marolt. Thank you very much.

Abstract

In the thesis we study random irreducible quadrangulations. We aim to extend known results on local limits of random planar maps and the perceived universality to the class of irreducible quadrangulations.

The first part of the thesis establishes a multitude of enumeration results using standard generating function techniques. We introduce various classes of irreducible quadrangulations and we make a detailed study of their generating functions and asymptotics. The material of this chapter forms the backbone of the subsequent calculations.

The last chapter is conceptually divided into two parts. The first part extends known sharp concentration results for maximum vertex degree and vertex degree distribution to the class of irreducible quadrangulations. In the second part we describe the skeleton decomposition of irreducible hulls and introduce a formal operator to evaluate statistical sums over the semi-layers of the decomposition.

The concluding section of the thesis exploits the properties of the hull operator to provide a new proof of the existence of the Uniform Infinite Irreducible Planar Quadrangulation (UIIPQ). We also explore the asymptotics of the iterates of the hull operator to derive limiting probabilities for the perimeter and volume of r-hulls in the UIIPQ. Our results are (unsurprisingly) in agreement with the conjectured universal behaviour observed in other random models such as the UIPT and UIPQ.

Statement of Originality

Chapter 2 and sections 3.1, 3.2 and subsections 3.3.1 and 3.3.2 of chapter 3 contain an overview of the background material, written by the author but relying on the sources cited within. Chapters 1, 4 and the remainder of chapter 3 constitute the author's own work.

To the best of my knowledge all citations are complete and accurate.

Contents

1 Introduction **1**

2 Preliminaries **11**

2.1 Planar maps 11

2.2 Quadrangulations 17

2.3 Ideal Right-Angled Hyperbolic Polyhedra 26

3 Enumeration of Irreducible Quadrangulations **29**

3.1 Singularity Analysis 29

3.1.1 Lagrange inversion 29

3.1.2 Flajolet-Odlyzko transfer theorem 34

3.2 Enumeration of Irreducible quadrangulations 45

3.2.1 Substitution technique 45

3.2.2 Known enumeration results 54

3.3 Enumeration of irreducible quadrangulations with simple boundary . 55

3.3.1 Generating function $Q(t, x)$ 55

3.3.2 Generating function $\Phi(t, x)$ 64

3.3.3 Irreducible cores 82

3.3.4 Generating function $\Phi(t, s, x)$ 89

3.4 Enumeration of 2-neighbourhoods 117

3.4.1 Cut-and-paste technique 117

3.4.2 Generating function $N_2(t, z, x)$ 126

4 Random Quadrangulations **133**

4.1 Random Quadrangulations 133

4.1.1 Uniform probabilities 133

4.1.2 Maximum degree concentration results 138

4.2 Hulls of Irreducible Quadrangulations 146

4.2.1 Layer decomposition of hulls 147

4.2.2 Hulls of irreducible quadrangulations 158
4.2.3 The hull operator 160
4.3 The Hull Operator $\boldsymbol{A}_q$ 163
4.3.1 Resolvent of the hull operator 172
4.3.2 The hull boundary operator 178
4.3.3 Asymptotic analysis of the perturbation determinant 180
4.4 Limiting Distributions 192
4.4.1 One-endedness of the UIIPQ 192
4.4.2 Hull boundary length limiting distribution 197
4.4.3 Hull volume limiting distribution 203

A q-Uniform Estimates **217**
A.1 q-Uniform convergence 217
A.2 q-Uniform Wiener Algebra 222

Chapter 1

Introduction

A planar map is a proper embedding in the 2-sphere of a finite connected planar graph considered up to orientation preserving homeomorphisms of the sphere. Planar maps are a basic object of study in the field of combinatorics and graph theory. One of the fundamental questions concerning planar maps is the problem of enumeration of various classes (e.g. planar triangulations of different types, quadrangulations, bipartite maps etc.). The problem was initially considered unassailable, apart from a few simple classes of trees or polygon dissections, due to the presence of symmetries of the underlying maps. The real breakthrough began in 1960's with the publication of W.Tutte's seminal "census" papers [60]. The pivotal idea introduced in these papers is the introduction of a *root* in the map, viz. a distinguished oriented edge, which informally speaking "destroys" any symmetries of the map. Tutte's papers deal with enumeration of various rooted classes of planar maps and are based mostly on recursive decomposition.

Since the introduction of rooted classes, enumerative combinatorics has become an active area of research. Most notably Tutte's original idea of recursive decomposition has been supplemented with the development of rich bijective approaches. These bijections have provided further insight and understanding of the often simple enumerative formulas of complicated classes of planar maps, by putting them in bijection with a simpler class of decorated trees. It has been observed that most classes of planar maps admit the same "universal behaviour": the generating functions admit the same critical exponent.

Planar maps are not only studied in combinatorics, but are a central object also in theoretical physics. It was realized that planar maps can be considered as a good approximation of $2D$ random surfaces and constitute a model of 2-dimensional *quantum gravity*. The observed universal behaviour of random planar maps has in physics literature become known as the class of "pure $2D$ quantum gravity". In recent years

a lot of effort has been concentrated on making this connection mathematically rigorous. The first mathematical model of random planar geometry was described in 2003 by O. Angel and G. Schramm. They studied uniform triangulations of the sphere and proved this sequence of random maps converges in the local topology towards the Uniform Infinite Planar triangulation (UIPT), a certain probability distribution on infinite planar triangulations. Soon after similar results (although with rather different proofs) were obtained for various other models of random maps, such as uniform planar quadrangulations (by M. Krikun [42]). This has spawned an active area of research of local limits of random maps.

We now give an overview of the thesis and state some of our main results. In chapter 2 we recall the basis notions from the theory of planar maps and irreducible planar quadrangulations, which will be the central objects of the thesis. We also include a very brief motivation for studying irreducible quadrangulations due to their intimate relation with the class of ideal right-angled hyperbolic polyhedra. None of the results of this chapter are new and are well-known to experts. The class of irreducible quadrangulations is defined as (we refer to section 2.2 for precise definitions).

Definition 1. An irreducible planar quadrangulations is a properly embedded planar map in the 2-sphere (up to orientation preserving homeomorphisms) such that every face of the embedding is of degree 4, the underlying graph is simple and every 4-cycle forms the boundary of some face of the embedding.

The main focus of this thesis is therefore twofold. On the one hand, we aim to extend our understanding of local properties of irreducible quadrangulations. We focus in particular on the distribution of vertex degrees. It has been observed that in many classes of planar maps belonging to the "universal class" hinted at above, the random variable counting the number of vertices of a given degree satisfies a central limit theorem (see [19]). It has been conjectured that a similar result should also hold for 3-connected planar maps and by extension (via Tutte's bijection) for irreducible quadrangulations. Although we were unable to provide a proof of this conjecture, we provide new compelling evidence that support the existence of a central limit theorem. We also investigate the distribution of the maximum vertex degree, deriving sharp concentration results of the kind that are already known for the class of all planar maps ([29]) and triangulations ([28]).

On the other hand, having observed the same universal behaviour for irreducible quadrangulations we investigate their Benjamin-Schramm limit with respect to the

local topology. We provide a new proof of the existence of the Uniform Infinite Irreducible Planar Quadrangulation and study some of its local properties.

Chapter 3 is devoted to studying the generating functions of various classes of irreducible quadrangulations. It establishes the fundamental enumerative results needed to study the Benjamin-Schramm limit of uniform irreducible quadrangulations as well as its local properties in the sequel. The first two sections merely provide an overview of the basic enumerative techniques and asymptotic analysis. The primary source for these sections are the books [31] and [25]. In section 3.3 we introduce two classes of rooted irreducible quadrangulations with a simple boundary:

(I) Proper rooted irreducible quadrangulations with a simple boundary.

(II) Rooted irreducible quadrangulations with a simple boundary satisfying property (3.3.1) as defined on page 66.

We provide a new derivation of the generating function of proper rooted irreducible quadrangulations with a simple boundary. The analogous simple class had already featured prominently in M. Krikun's article [42], which is one of the main motivating factors for studying such irreducible quadrangulations in the first place. Similarly, the second class satisfying the peculiar property (see 3.3.1), has rather unexpectedly likewise already appeared in literature before, although in a completely unrelated context. The class was introduced by E. Guitter in [32] as an auxiliary tool to studying the two-point function of quadrangulations. We carefully describe his recursive decomposition of the class and use substitution to pass to the more restrictive set of irreducible quadrangulations. These results are therefore not particularly new, but merely re-parametrizations (through face substitution) of what has already been established before. We end the subsection with a careful analysis of the singularity structure of the generating functions and use it to prove the following central limit theorem

Theorem A. *Let a_{nk} be the number of rooted irreducible quadrangulations with root vertex of degree k and $n+k$ faces. Suppose $\varepsilon > 0$ then*

$$a_{nk} \sim \frac{1}{2\pi n^2} \frac{r(r^2-r-1)}{(r^2-2)(r+1)^2} \sqrt{\frac{(r+1)(r^2-4)(r^2-3)^3(r^2-r-1)}{(r^2-1)^2(r^5-5r^4+15r^3-25r^2+12r+6)}} \times \left[\left(\frac{r^2-2}{(r^2-r-1)^2} \right)^{\mu} \frac{(r^2-2)^2}{r^2-3} \right]^n \tag{1.1}$$

uniformly as $n, k \to \infty$ *provided* $\mu \in (\varepsilon, 1-\varepsilon)$*, where* $\mu = k/n$ *and parameter* $r \in (2, \infty)$ *is implicitly defined by the equation*

$$k/n = \frac{r(r-2)(r^2-r-1)}{(r^2-3)(r-1)^2}.$$

Despite appearing as seemingly uncorrelated, the classes (I) and (II) are intimately connected. This connection is explored in the next subsection 3.3.3 with the introduction of the so-called irreducible cores. Put simply it is possible to pass from one class to the other through the operation of face substitution in a certain underlying submap, which we call the irreducible core. We provide a description of the possible cores, by putting them in bijection with a class of vertex-bicoloured planar trees. Using Lagrange's theorem we provide explicit connection formulas between the generating functions of the two classes.

The main bulk of chapter 2 begins with subsection 3.3.4. Due to the pesky restriction on separating 4-cycles in the definition of irreducible quadrangulations the (essentially) only known approach to counting them (apart from the bijective proof with a certain subset of binary trees described by A direct bijective approach has been described by E. Fusy, D. Poulalhon and G. Schaeffer in [27]) so far has been through face-substitution (see chapter 3 for a more in-depth discussion). This is because in recursive decompositions it becomes very laborious and technical to describe all the possible scenarios, where a separating 4-cycle may occur. On the other hand, face substitution makes it increasingly more difficult to keep track of the various combinatorial parameters other than the basic ones such as the number of faces.

The novelty of our approach is the introduction of a direct counting procedure for irreducible quadrangulations that does not rely on substitution. It is a certain modification of the recursive decomposition introduced by E. Guitter in [32], adapted to the setting of irreducible quadrangulations. Using the new recursive decomposition we derive exact generating functions for pointed rooted irreducible quadrangulations satisfying property (3.3.1), defined on page 66, counted according to the number of black and white vertices, the degree of the root vertex and the degree of the additional marked vertex.

Finally, we end the chapter by investigating the structure of neighbourhoods of irreducible quadrangulations that are used to define the local metric. Recall that the local topology on a set of planar maps is an ultrametric generated by the set of combinatorial balls, where a combinatorial ball of radius r is the submap composed of all the faces having at least one vertex at distance strictly less than r from the root vertex. The definition needs to be supplemented by specifying the embedding of the

ball in the 2-sphere. Here we adopt Krikun's convention (as opposed to the convention used in say [16, 17]): we first augment the r-ball by drawing in all the face diagonals between vertices at distance r from the root-vertex and cut the resulting map along these "diagonal cycles", then we glue a simple triangular face back along each diagonal (keeping the faces disjoint) and erase the diagonals. Our final result in chapter 2 deals with the enumeration of 2-neighbourhoods of irreducible quadrangulations. Using a cut-and-paste technique we provide the exact generating function for irreducible quadrangulations with a simple boundary counted according to the number of faces, boundary length and the exact structure of the boundary, which we encode with a certain composition $\underline{\pi}$. As such we introduce two separate notions of boundary length:

(i) The length $|\underline{\pi}|_0$ of the composition parameter $\underline{\pi}$, which gives the number of vertices of distance $r-1$ from the root vertex incident with the boundary.

(ii) The sum $|\underline{\pi}|_1$ of the composition parameter $\underline{\pi}$, which gives half the number of faces incident to the boundary.

In chapter 4 we use the enumerative results of chapter 3 to investigate the local properties of random irreducible quadrangulations. We first use singularity analysis to deduce strong concentration results for the maximum vertex degree:

Theorem B. *Let Δ_n denote the maximum vertex degree of a random rooted irreducible quadrangulation with n faces, then*

$$\mathbb{E}(\Delta_n) = \frac{\log n - \frac{1}{2}\log\log n}{\log 2} + \mathcal{O}(1), \qquad \mathbb{V}(\Delta_n) = \mathcal{O}(\log n). \tag{1.2}$$

Furthermore, if $\Omega_n \to \infty$ denotes any function then

$$\mathbb{P}\left(\left|\Delta_n - \frac{\log n - \frac{1}{2}\log\log n}{\log 2}\right| \leq \Omega_n\right) = 1 - \mathcal{O}(1/\log n + (1/2)^{\Omega_n}). \tag{1.3}$$

Let ζ_k denote the number of vertices of degree k in an n-faced random irreducible quadrangulation. The result strongly suggest the maximum vertex degree Δ_n of random irreducible quadrangulations is distributed according to the extremal Gumbel distribution of type 1. However, a proof of this stronger result would require knowledge of higher moments of the variables ζ_k, which are so far out of our reach. We are, however, able to describe at least the first and second moments of ζ_k. More precisely, we also prove the following theorem:

Theorem C. *Let $T = (1-t/2)^{1/2}$ and define (for $k > 2$):*

$$b_k = [t^{k-1}]\frac{\left(5T - \sqrt{24+T^2}\right)^2\left(T+\sqrt{24+T^2}\right)^3\left(7T+\sqrt{24+T^2}\right)}{2^8 3^6 T^3\sqrt{24+T^2}} \text{ and } \mu_k = \frac{3^6}{2k}b_k.$$

Let ζ_k denote the number of vertices of degree k of a random rooted irreducible quadrangulation with n faces, then uniformly for all $k = \mathcal{O}(\log n)$ we have

$$\mathbb{E}(\zeta_k) = n\mu_k\left(1+\mathcal{O}\left(\frac{\log^{10} n}{n}\right)\right), \text{ and } \mathbb{V}(\zeta_k) = n\mu_k + (n\mu_k)^2\mathcal{O}\left(\frac{\log^{10} n}{n}\right).$$

To study the existence and properties of the local limit of random irreducible quadrangulations we use another tool introduced initially by M. Krikun [42, 41]. We present in section 4.2 the *skeleton decomposition* of irreducible quadrangulations. Before we describe the decomposition, we need to introduce the *hull*. A combinatorial r-ball B_r, composed of all faces having a vertex at distance strictly less than r from the root vertex, may have several edge-disjoint boundary components. However, it has been observed (we compute the exact limiting probabilities in section 4.1) that with high probability the complement of exactly one boundary component will contain the majority of the quadrangulation. The r-hull $B_r^\bullet$ is obtained by "filling-in" the holes of all the boundary components except the maximal one. The skeleton decomposition is then traditionally defined in terms of layers, which are sets of the form $B_r^\bullet \backslash B_{r-1}^\bullet$, and a collection of independent free quadrangulations with a boundary.

Unfortunately, this approach is not well-adapted to the study of irreducible quadrangulations. Roughly, speaking the quadrangulations filling-in the holes (or interstices) of the decompositions, which are usually filled with independent Boltzmann quadrangulations depending only on the hole's perimeter length, no longer depend simply on the boundary length but on the overall structure of the boundary (the same structure that has been encoded in chapter 3 by the composition parameter $\underline{\pi}$). As such we are forced to somewhat diverge from the usual skeleton decomposition by introducing what we call a semi-layer. Using semi-layers we are again able to put irreducible hulls in a bijective correspondence with a class of rooted planar forests and a collection of independent free irreducible quadrangulations with a simple boundary and appropriate composition parameter.

Theorem D. *The skeleton decomposition is a bijection between irreducible hulls of radius r, upper composition parameter $\underline{\lambda}$ and lower boundary cycle of length p and pairs consisting of an $(r, \underline{\lambda}, p)$-admissible forest F and a collection $\{(M_v, \underline{\pi}_v)\}_{v\in F^*}$, where for each node $v \in F^*$, $\underline{\pi}_v$ is the composition parameter describing the genealogy*

structure of the node v in F and M_v is an admissible irreducible quadrangulation with parameter $(1, \underline{\pi}_v)$.

Just as in the case of triangulations of type II, considered in [41], the decomposition is complicated by the fact that the admissible forests arising in the decomposition do not coincide with the set of all possible planar forests due to exceptions (E_0), (E_1) and (E_2) (which are put in place to prevent the occurrence of separating 4-cycles). Following Krikun's idea we introduce a hull operator that allows us to compute the statistical sum over all possible semi-layers:

Theorem E. *Let $p \geq 3$ denote the length of the lower boundary of a semi-layer L and let $n = n(L)$ denote the number of inner quadrilateral faces in L. Suppose we take a function $G(z)$ that allows expansion $G(z) = \sum_{k\geq 3} c_k z^{k-1}$, then we can write the sum over all semi-layers $L \in \Lambda'$ as*

$$\sum_L c_p x^n z^{p-1} = (\boldsymbol{A}(x)G)(z)$$

where $\boldsymbol{A}(x)$ is a linear operator defined as follows

$$\begin{aligned}(\boldsymbol{A}(x)G)(z) = R'(z,x)G(R(z,x)) - (R(z,x) - x + (1+z)R'(z,x))G(x)\\ - zR'(0,x)(R'(0,x)G'(x) - 2G(x))\end{aligned} \tag{1.4}$$

where prime denotes differentiation with respect to z and we have made the following notation convention $R(z,x) = (1+z)z^{-1}Z(z,x)$ (series Z is defined in lemma (4.2.3)).

In section 4.3 we investigate the properties of the hull operator $\boldsymbol{A}(x) = \boldsymbol{A}_q$. We do this only in a formal context, although we faithfully indicate at each step how our results could equivalently also be translated in the language of Hilbert spaces (more specifically Hardy spaces of analytic function over an appropriate domain). Analysing the properties of the main composition function $R(z,x)$ in the definition of the operator, we prove that through a simple change of variables it is possible to transform the otherwise complicated algebraic operator $\boldsymbol{A}_q$ to a simple operator $\boldsymbol{B}_q$ that involves only rational functions. We continue working with the transformed operator $\boldsymbol{B}_q$ and introduce its resolvent, which can be thought of formally as the generating function of the sequence of iterates $\{\boldsymbol{B}_q^n\}_{n=0}^{\infty}$. Using the second resolvent identity we derive a "simple" formula for the resolvent, expressing it in terms of resolvents $\Phi(z,w)$ and $\Psi(z,w)$ of certain coefficients (and their w-derivatives), as well as (what we call) the perturbation determinant $\Delta_q(z)$.

The remainder of the section is then devoted to studying the analytic properties of $\Phi(z)$ and $\Psi(z)$ and the perturbation determinant. The idea is to deduce an asymptotic relation for the iterates of the hull operator $\boldsymbol{A}_q$ using singularity analysis and the transfer theorems of Flajolet and Odlyzko. The results established in this section form the backbone of our calculations, basically providing us with all the necessary technical assumptions enabling us the use of asymptotic analysis.

The final section of chapter 4, uses the techniques developed so far to finally deduce the existence of the Uniform Infinite Irreducible Quadrangulation (UIIPQ).

The existence of the UIIPQ has already been proven by L. Addario-Berry in [1]. His approach is radically different from ours. Recall, that according to Skorokhod's representation theorem a weakly convergent sequence of probability measures μ_n with limit μ_∞, defined on some metric space M and such that the support of μ_∞ is separable, can be realized as a sequence of random variables X_n defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the law of X_n is μ_n for all n and X_n converges $\mathbb{P}$-almost surely to X_∞. Using the Fusy-Poulalhon-Schaeffer "closure" bijection between binary trees and irreducible quadrangulations of the hexagon [27] and a growth procedure for binary trees [45], he constructs a growing sequence of random rooted quadrangulations $\mathbf{q}_n$ such that $\mathbf{q}_n \to \mathbf{q}_\infty$ almost surely and shows that $\mathbf{q}_\infty$ is the distributional local limit of uniformly random rooted irreducible quadrangulations. Our proof is instead based on an idea of M. Krikun to investigate the limits of r-hulls and use them to "build" the (UIIPQ).

Theorem F (L. Addario-Berry, [1])**.** *For every $n \geq 1$, let μ_n denote the uniform probability measure on the space of finite rooted irreducible quadrangulations with n faces, then the sequence $\{\mu_n\}_{n\geq 1}$ converges to a probability measure μ, in the sense of weak convergence, in the space of probability measures on $(\mathbf{Q}, d_{\mathbf{Q}})$. Moreover, μ is supported on the set of infinite rooted irreducible quadrangulations of the plane.*

Having established the existence of the (UIIPQ) we turn next to the investigation of some of its local properties. This was first initiated by O. Angel in [5] for the (UIPT) and his investigation has since been extended to various other model of random maps. There are several interesting characteristics one can examine, e.g perimeter of hulls (and balls), volumes of hulls (and balls), isoperimetric inequalities (see [44]), distances, random walks, percolation etc. We have focused mainly on perimeter length and volume of irreducible hulls $B_r^\bullet$ with respect to both notions of boundary length (as described above).

Examining initially the limiting distribution of perimeters of irreducible hulls, we prove using singularity analysis:

Theorem G. *Let $|\gamma_r|_1$ denote the boundary length of the r-hull under distribution μ, then the random variable $14/5|\gamma_r|_1 r^{-2}$ converges in distribution to the $\Gamma(3/2)$ law.*

Furthermore, adapting the ideas we obtain the same scaling limit (although with a different scaling constant) also for the second notion of boundary length, viz.

Theorem H. *Let $|\gamma_r|_0$ denote the number of parts of the r-hull under distribution μ, then the random variable $6|\gamma_r|_0 r^{-2}$ converges in distribution to the $\Gamma(3/2)$ law.*

We turn our attention next to volumes of irreducible hulls. The singularity analysis approach is no longer applicable in this case, so we instead determine the limiting distributions using what is essentially Tannery's theorem. We provide the exact version of the theorem needed in our work and we set out to verify its conditions are met. In doing so, we need to verify the coefficients of the (multiplicative) inverse $1/\Delta_q(w)$ of the perturbation determinant are sufficiently fast q-uniformly convergent. Although, Faa di Bruno's formula would in principle allow us to compute these coefficients explicitly, the sheer complexity of the computation makes it impossible to deduce any useful (uniform) bounds. Consequently, we extend the classical Levy-Wiener theorem on absolutely convergent power series to the class of normally convergent power series depending on a parameter q. We do not follow Wiener's original proof of the theorem, but instead rely on Gelfand's reformulation of the theorem in the context of commutative Banach algebras with unit. We initially adapt Newman's elegant proof of the classical Wiener theorem to the uniform setting and then extend Brandenburger's results using the eponymous lemma. The proofs of these various auxiliary results are delegated to appendix A. Once established, it is a simple (albeit somewhat technical) matter of checking all the conditions of Tannery's theorem are satisfied, which finally gives us the following result (here and below $x_0 = 1/4$)

Theorem I. *Let $\lambda > 0$, then*

$$\lim_{r\to\infty} \mathbb{E}\left(e^{-\lambda \operatorname{Vol}(B_r^\bullet)/r^4}\right) = 3^{3/2}\frac{\cosh(x_0\lambda)^{1/4}}{(\cosh^2(x_0\lambda)^{1/4}+2)^{3/2}}. \tag{1.5}$$

An extension of the above mentioned techniques also allows us to determine the limiting distribution of volumes of hulls conditioned on boundary length. Our results are again in agreement with the expected universal behaviour that has been observed in other classes of random maps:

Theorem J. *Let $\lambda, \ell > 0$, then*

$$\lim_{r\to\infty} \mathbb{E}\left(e^{-\lambda \operatorname{Vol}(B_r^\bullet)/r^4} \middle| |\partial B_r^\bullet|_1 = [\ell r^2]\right) = (x_0\lambda)^{3/4} \frac{\cosh(x_0\lambda)^{1/4}}{\sinh^3(x_0\lambda)^{1/4}} \exp\left(-\frac{14}{5}\ell\left((x_0\lambda)^{1/2}\left(\coth^2(x_0\lambda)^{1/4} - \frac{2}{3}\right) - 1\right)\right).$$

Similarly, we deduce the limiting distribution of volume conditioned on the number of parts of the boundary to be

Theorem K. *Let $\lambda, p > 0$, then*

$$\lim_{r\to\infty} \mathbb{E}\left(e^{-\lambda \operatorname{Vol}(B_r^\bullet)/r^4} \middle| |\partial B_r^\bullet|_0 = [pr^2]\right) = (x_0\lambda)^{3/4} \frac{\cosh(x_0\lambda)^{1/4}}{\sinh^3(x_0\lambda)^{1/4}} \exp\left(-6p\left((x_0\lambda)^{1/2}\left(\coth^2(x_0\lambda)^{1/4} - \frac{2}{3}\right) - 1\right)\right).$$

Chapter 2

Preliminaries

2.1 Planar maps

A multigraph is usually defined as an abstract combinatorial structure, however, it is often convenient to regard multigraphs as topological spaces, in which vertices are points and edges are interior-disjoint paths between their endpoints rather than strict combinatorial objects. More formally, we will define multigraphs embedded in a surface. Let Σ be an orientable closed surface - according to the fundamental classification theorem for closed surfaces Σ is homeomorphic to the 2-sphere $\mathbb{S}^2$ or a finite connected sum $\mathbb{T}_g$ of $g > 0$ copies of the 2-torus $\mathbb{T} = \mathbb{T}_1$. An *oriented edge* in Σ is a continuous mapping $e : [0, 1] \to \Sigma$ such that e is either injective or the restriction of e to the half-closed interval $[0, 1)$ is injective with $e(0) = e(1)$ (in the latter case, e is also called an *oriented loop*). We will always consider oriented edges only up to reparametrization by an increasing homeomorphism $[0, 1] \to [0, 1]$ and any quantity associated with an oriented edge should be invariant under such reparametrization. The *origin* of an oriented edge e is the point $e^- = e(0)$ and the *target* of e is the point $e^+ = e(1)$, the two together are called the *extremities* of e. The *reversal* of e is the oriented edge $\bar{e}$ defined by $\bar{e}(t) = e(1 - t)$. An *edge* $\mathbf{e}$ is defined as a 2-set $\{e, \bar{e}\}$ consisting of an oriented edge and its reversal. The *interior* of an edge $\mathbf{e}$ is defined as $e((0, 1))$.

An *embedded multigraph* in Σ is a multigraph $G = (V, E)$ such that the following conditions hold:

- V is a finite subset of Σ.
- E is a finite subset of edges in Σ.
- The vertices incident to an edge $\mathbf{e} = \{e, \bar{e}\}$ are $e^-, e^+ \in V$.

- The interior of an edge $\mathbf{e} \in E$ does not intersect V or the edges of E distinct from $\mathbf{e}$.

It should be noted that the definition of an embedded multigraph does not depend only on the combinatorial structure of the underlying multigraph but also on the choice of the embedding in the surface Σ (as well as the choice of surface Σ). For example the complete graph K_4 on 4 vertices can be embedded in $\mathbb{S}^2$ or $\mathbb{T}$ to form two different embedded multigraphs. Secondly, restricting our attention to embedded multigraphs means no loss of generality since any multigraph can be embedded in some closed orientable surface Σ.

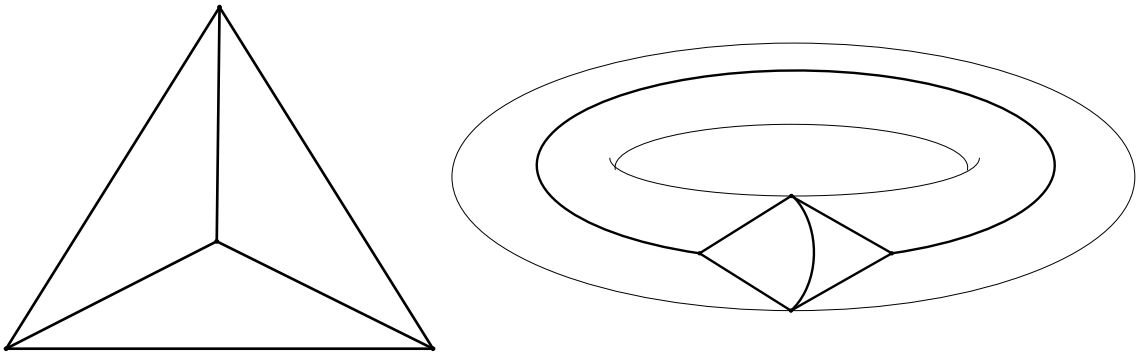


Figure 2.1: Embedding of the complete graph K_4 in the plane $\mathbb{R}^2$ (which is equivalent to an embedding in the 2-sphere $\mathbb{S}^2$ via stereographic projection) and in the torus $\mathbb{T}$.

Let $G = (V, E)$ be an embedded graph in the surface Σ. The *support* of G is the set

$$\operatorname{supp} G = V \cup \cup_{\mathbf{e} \in E} \operatorname{Int}(e),$$

and a *face* of G is a (path) connected component of $\Sigma \setminus \operatorname{supp}(G)$. We denote with $F(G)$ the set of all faces of G.

Definition 2. Let Σ be an orientable closed surface. A *map* on Σ is an embedded graph whose faces are all homeomorphic to the open unit disk in $\mathbb{R}^2$ (such an embedding is called 2-*cell*). A map on the 2-sphere $\mathbb{S}^2$ is also called a *plane map*.

It is easily established that a map on any orientable closed surface Σ must necessarily have a connected underlying multigraph. Furthermore, on the 2-sphere $\mathbb{S}^2$ it actually follows from the Jordan-Schönflies theorem that planar maps are exactly connected embedded graphs. This is no longer true for surfaces of higher genus.

Let $\mathbf{m} = (V, E)$ be a map embedded in an orientable closed surface Σ, and let $\vec{E} = \{e \in \mathbf{e} : \mathbf{e} \in E\}$ be the set of all oriented edges of $\mathbf{m}$. Since Σ is orientable, it

is possible to define, for every oriented edge $e \in \vec{E}$, a unique face f_e of $\mathbf{m}$ located to the left of the oriented edge e. We call f_e the face *incident* to e. The edges incident to a given face f form a closed curve in Σ, although not necessarily a Jordan curve since it can happen that $f_e = f_{\bar{e}}$ for some oriented edge e. The *degree of a face* f is defined as

$$\deg f := \#\{e \in \vec{E} : f_e = f\}.$$

Clearly, the degree of a face as just defined agrees with the usual graph-theoretic notion of degree of a face of (not necessarily embedded) planar graphs. The oriented edges incident to a given face f are arranged cyclically in counterclockwise order around the face in what we call the *facial ordering*.

Associated with every oriented edge e of a map $\mathbf{m}$ is also a *corner* incident to e, which is defined as a small simply connected neighbourhood of e^- intersected with f_e such that the corners of any two distinct oriented edges do not intersect.

The *degree of a vertex* $u \in V$ is the usual graph-theoretic notion $\deg = \#\{e \in \vec{E} : e^- = u\}$ and similarly as for faces, the outgoing edges from a vertex u are arranged cyclically in counterclockwise order around u.

Theorem 2.1.1 (Euler's polyhedral formula)**.** *Let* $\mathbf{m}$ *be a map embedded in a closed orientable surface* Σ_g *of genus* g *and denote with* $V(\mathbf{m})$*,* $E(\mathbf{m})$*,* $F(\mathbf{m})$ *the sets of all vertices, edges and faces of* $\mathbf{m}$*, respectively, then*

$$\#V(\mathbf{m}) - \#E(\mathbf{m}) + \#F(\mathbf{m}) = \chi(\Sigma_g) = 2 - 2g. \tag{2.1}$$

Furthermore, the following relations hold

(i) $\frac{1}{2}\sum_f \deg f = \#E(\mathbf{m})$*, the summation is over the set* $F(\mathbf{m})$ *of faces of* $\mathbf{m}$*.*

(ii) $\frac{1}{2}\sum_u \deg u = \#E(\mathbf{m})$*, the summation is over the set* $V(\mathbf{m})$ *of vertices of* $\mathbf{m}$*.*

The last result is frequently referred to as the Handshaking lemma in literature.

We say two maps $\mathbf{m}$ and $\mathbf{m}'$ on Σ are *isomorphic* if there exists an orientation-preserving homeomorphism $h \in \mathrm{Homeo}^+(\Sigma)$ of the surface Σ onto itself such that h induces a graph isomorphism of $\mathbf{m}$ with $\mathbf{m}'$. An *automorphism* of a map $\mathbf{m}$ is an isomorphism of $\mathbf{m}$ with itself (it should be interpreted as a symmetry of a map). Two maps $\mathbf{m}$ and $\mathbf{m}'$ are *combinatorially isomorphic* if there exists an incidence preserving isomorphism of their underlying graphs, i.e. there exists a bijection $\sigma : (V, E, F) \to (V', E', F')$ which preserves the incidence of vertices, edges and faces. It is possible to show that two maps are combinatorially isomorphic if and only if there exists a

homeomorphism $h \in \mathrm{Homeo}(\Sigma)$ (not necessarily orientation-preserving) of the surface Σ onto itself such that h induces a graph isomorphism of $\mathbf{m}$ with $\mathbf{m}'$. This clearly underlines the difference between an isomorphism of maps and an isomorphism of its underlying multigraphs: the second is only required to preserve incidence of vertices and edges, whereas the first must also preserve incidence of edges and faces.

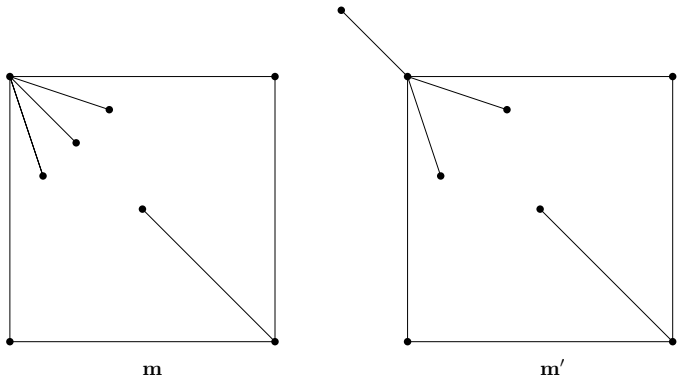


Figure 2.2: The underlying multigraph of the embedded graphs $\mathbf{m}$ and $\mathbf{m}'$ are isomorphic, but the embeddings are not even combinatorially isomorphic.

We will always identify isomorphic maps $\mathbf{m}$ and $\mathbf{m}'$, which of course implies that the underlying combinatorial graphs associated with $\mathbf{m}$ and $\mathbf{m}'$ are isomorphic. As such we will think of maps as equivalence classes modulo orientation-preserving homeomorphisms of the surface of embedding.

An important notion in the theory of (planar) maps is duality. Let Σ be an orientable closed surface and $\mathbf{m} = (V, E)$ a map on Σ. The *dual map* $\mathbf{m}^*$ is obtained by exchanging the roles of vertices and faces and keeping the incidence relations. More precisely, to construct the dual map $\mathbf{m}^*$ of $\mathbf{m}$ place a vertex f^* in each face f of $\mathbf{m}$ and connect two such vertices by a dual edge e^* across each edge of $\mathbf{m}$. It is possible to show that the construction described above can be carried out in such a way that distinct dual edges do not intersect, so that we obtain an embedded graph $\mathbf{m}^*$ with vertex-set $V^* = \{f^* : f \in F(\mathbf{m})\}$ and edge-set E^* equal to the set of dual edges, which is in fact a map. Duality has the following properties:

Lemma 2.1.1 (Duality) *Let Σ be an orientable closed surface and $\mathbf{m}$ a map on Σ.*

(i) The dual map $\mathbf{m}^*$ is a map on Σ.

(ii) $\#V(\mathbf{m}^*) = \#F(\mathbf{m})$, $\#E(\mathbf{m}^*) = \#E(\mathbf{m})$ and $\#F(\mathbf{m}^*) = \#V(\mathbf{m})$.

(iii) Duality is an involution on the set of maps.

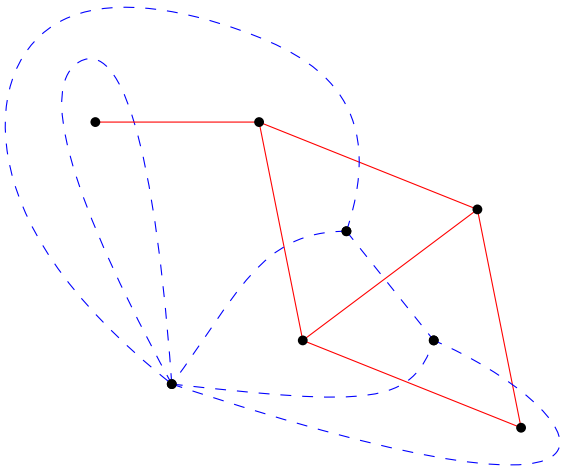


Figure 2.3: The construction of a dual map of a planar map (the dual of a simple planar map is not necessarily simple).

We have already observed that multigraphs can (in general) be embedded in multiple (closed, orientable) surfaces. On the other hand, it is well-known that the complete graph K_5 on 5 vertices cannot be embedded in the 2-sphere $\mathbb{S}^2$ (although it can be embedded in the torus). Given a (connected) multigraph G, the minimal genus of a surface Σ that admits a 2-cell embedding of G is called the *genus* of G. Multigraphs of genus 0 are also called *planar multigraphs*. It should be emphasized here that planar multigraphs have to be distinguished from embedded planar multigraphs, since a particular planar multigraph may admit several non-isomorphic embeddings in the 2-sphere. However, certain multigraphs clearly admit a unique embedding (for example the embedding of K_4). This naturally raises the question of which planar multigraphs have a unique embedding (up to orientation-preserving homeomorphisms of the 2-sphere). H. Whitney was the first to provide a complete characterization of planar multigraphs with a unique embedding in the 2-sphere:

Theorem 2.1.2 (H. Whitney, [61])**.** *A planar multigraph G has a unique embedding in the 2-sphere $\mathbb{S}^2$ if and only if it is a subdivision of a 3-connected planar graph.*

Consequently, when speaking of 3-connected planar graphs it makes no difference whether we have in mind the underlying graph or a particular embedding. In lieu of Whitney's result it intrinsically makes sense to speak of the faces of a 3-connected planar map.

One of the fundamental questions in combinatorics is the problem of map enumeration. A particularly important problem that remained unsolved for over a century was the enumeration of combinatorially inequivalent 3-connected planar maps due to

their connection with 3-dimensional convex polyhedra (via Steinitz's theorem). The problem was considered especially difficult and not much progress was made apart from enumeration of various classes of trees. The main obstacle to counting maps was the presence of non-trivial symmetries. However, in 1960's William Tutte published a series of papers detailing enumeration results of various classes of rooted planar maps. Traditionally a map has been thought of as a connected unlabelled graph embedded on a 2-sphere $\mathbb{S}^2$ (as defined above). In order to circumvent the issue of non-trivial symmetries, Tutte introduced a new class of maps, called *rooted planar maps*, as a way to desymmetrize the map. Let $\mathbf{m} = (V, E)$ be a planar map,

- A *pointed (planar) map* is a pair $(\mathbf{m}, v_0)$, where $\mathbf{m} = (V, E)$ is a planar map and $v_0 \in V$ is a distinguished vertex called the *pointed vertex*.
- A *(planar) map with a distinguished face* is a pair $(\mathbf{m}, f_0)$, where $\mathbf{m} = (V, E)$ is a planar map and f_0 is a distinguished face of $\mathbf{m}$ called the *exterior face*. Any planar map with a distinguished face admits a canonical embedding in the plane $\mathbb{R}^2$ (obtained by stereographic projection), in which the unique infinite face of the embedding corresponds to the distinguished face f_0. The relatively compact faces are called *inner faces* of the map.
- A *rooted (planar) map* is a pair $(\mathbf{m}, \vec{e})$, where $\mathbf{m} = (V, E)$ is a map and $\vec{e} \in \vec{E}$ is a distinguished oriented edge called the *root* or *root edge*. We often omit the mention of the root edge $\vec{e}$ in notation and refer simply to $\mathbf{m}$ as a rooted map. The face f_e incident to the root edge $\vec{e}$ is often called the *root face* (or the *external face*), while the remaining faces are called *internal*. Since a rooted map has a distinguished face, namely the root face, we will adopt the standard convention of embedding rooted maps in the plane $\mathbb{R}^2$ with the root face the unique infinite face of the embedding and appearing now on the right of the root edge. The initial vertex e^- of the root edge is also called the *root vertex* (or sometimes ambiguously the *root*).[1]

The following folklore result by W. Tutte explains why it is easier to count rooted maps - informally, rooting destroys the symmetries of the map.

Theorem 2.1.3 (W. Tutte [60, p. 252])**.** *An automorphism of a planar map* $\mathbf{m}$ *that fixes an oriented edge fixes all the oriented edges.*

[1] A second equivalent definition of rooted maps that is also often found in literature defines a rooted map as a pair $(\mathbf{m}, c)$ consisting of a planar map $\mathbf{m}$ and a distinguished corner c; the two definitions are equivalent since we can associate to a corner the unique oriented edge incident with it and vice-versa.

When speaking of automorphisms of planar maps with a distinguished element we will therefore always require that the automorphism fixes the distinguished element. As such Tutte's theorem (2.1.3) can equivalently be restated as saying that rooted planar maps have trivial automorphism groups.

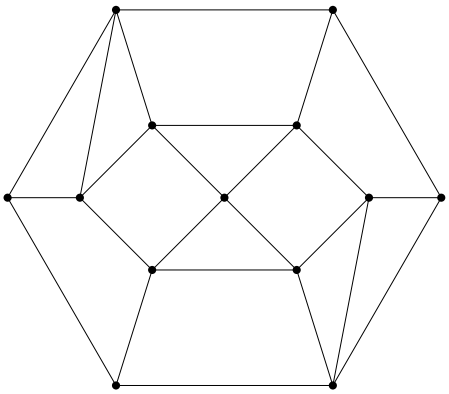


Figure 2.4: A planar map with a non-trivial symmetry (rotation by 180° about the central vertex).

Naturally the number $r_n(\mathcal{P})$ of rooted planar maps with say n edges belonging to a certain class $\mathcal{P}$ is not the same as the number $s_n(\mathcal{P})$ of unrooted maps of $\mathcal{P}$. If we knew *a priori* that a certain member of the class has a trivial symmetry group and n edges then by rooting we would obtain $4n$ inequivalent rooted copies. Although not every map has a trivial automorphism group it has been shown that for various classes of planar maps (e.g k-connected planar maps for $k = 1, 2, 3$) almost every member of the class has no non-trivial symmetries. Hence asymptotically

$$s_n(\mathcal{P}) \sim r_n(\mathcal{P})/4n \text{ as } n \to \infty.$$

Moreover, using Polya's theory it sometimes even possible to recover exact counting results for the number of unrooted maps with n edges belonging to class $\mathcal{P}$. However, we will mostly be interested merely in asymptotic estimates.

2.2 Quadrangulations

We will introduce in this section for us the most important subclass of planar maps - namely, the class of planar quadrangulations. It turns out that planar maps can equivalently be encoded as planar quadrangulations in a one-to-one correspondence called Tutte's bijection (which will be recalled below).

Definition 3. A (rooted) planar map $\mathbf{q}$ is a *(rooted) planar quadrangulation* if all its faces are of degree 4 (in particular, a quadrangulation contains no loops but may contain multiple edges). Since we will be dealing exclusively with planar quadrangulations, we will from now on refer to them simply as quadrangulations and will always implicitly assume them to be planar.

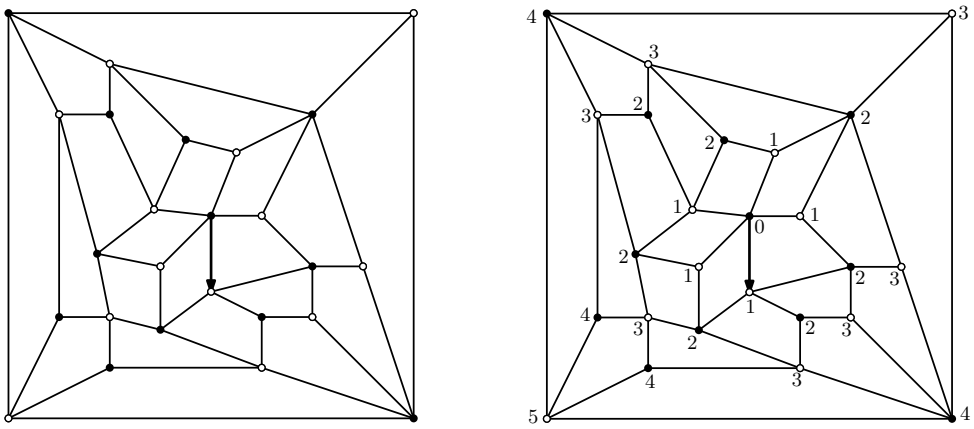


Figure 2.5: A simple planar quadrangulation with its canonical bicolouration of vertices (on the left) and equipped with the induced distance labelling (on the right).

Let $\mathbf{q}$ be a quadrangulation then the number of vertices, edge and faces of $\mathbf{q}$ are related by $\#V(\mathbf{q}) = n$, $\#E(\mathbf{q}) = 2n$ and $\#V(\mathbf{q}) = n + 2$, an easy consequence of Euler's relation. Suppose $(\mathbf{q}, v_0)$ is a pointed quadrangulation with a marked vertex v_0. We define the labelling φ of the map $\mathbf{q}$ by $\varphi(x) = d_{\mathbf{q}}(x, v_0)$ for each vertex x, where $d_{\mathbf{q}}(x, y)$ denote the graph-distance of $\mathbf{q}$.

Lemma 2.2.1 (Properties of the distance function on $\mathbf{q}$) *Let $\mathbf{q}$ be a pointed quadrangulation with marked vertex v_0.*

(i) If x and y are adjacent vertices of $\mathbf{q}$ then $|\varphi(x) - \varphi(y)| = 1$.

(ii) A quadrangulation is a bipartite map, so we may equip a pointed quadrangulation $(\mathbf{q}, v_0)$ with a canonical bicolouration of its vertices where a vertex x is coloured white if $\varphi(x)$ is odd and is coloured black if $\varphi(x)$ is even.

(iii) If x_1, y_1, x_2, y_2 are the vertices of a face of $\mathbf{q}$ in a facial walk, then at least one of the two equalities is satisfied $\varphi(x_1) = \varphi(x_2)$ or $\varphi(y_1) = \varphi(y_2)$. A face of a quadrangulation is called simple if one equality is satisfied and confluent otherwise.

We will also work with certain subsets of quadrangulations and planar maps that are near-quadrangulations. Before we state the relevant definitions we need to introduce a few related concepts. Let's recall that the *girth* of a planar map is the minimal

length (the number of edges) of its cycles, if the map contains at least one cycle, and ∞ otherwise. By this definition, maps of infinite girth are exactly (planar) trees. In a map not reduced to a tree, the degree of every face is larger than or equal to the girth, so a planar map is *loopless* if and only if it has girth at least 2 and is *simple* if and only if it has girth at least 3.

Let d be a non-negative integer. Following J. Bouttier and E. Guitter [10], we define a rooted map to be *strongly d-irreducible* [2] if its girth is at least d and any cycle of length d is the boundary of an inner face of degree d. Several remarks are in place regarding this definition. Firstly, the definition of irreducibility just introduced does not agree with what has historically been called a d-irreducible map (the difference in definitions will be discussed shortly). Secondly, the definition applies only to rooted maps (or maps with a distinguished face) since it requires the faces of maps to be distinguished into inner faces and an outer face. Finally, note that every rooted map is strongly 0-irreducible and a strongly d-irreducible map with outer degree smaller than or equal to d is either a tree (with one face of even degree $\leq d$) or a d-cycle delimiting two faces of degree d.

In the way strong d-irreducibility has been defined so far, there is a unique rooted map with an outer face of degree d, namely the d-cycle (indeed, the boundary of the outer face forms a cycle of length d so it must bound an inner face). A weaker and somewhat more natural definition of d-irreducibility among planar maps consists in simply picking those planar maps of girth d such that every cycle of length d in the map forms the boundary of an arbitrary (i.e. inner or outer) face of degree d. We will call such a map *d-irreducible*.

Definition 4. A planar map $\mathbf{q}$ is called a *quadrangulation with holes* if all its faces have degree 4, except for a number of distinguished faces which can be of arbitrary *even* degree[3]. We call these distinguished faces the *holes* (or sometimes *boundaries*) of the quadrangulation $\mathbf{q}$. A quadrangulation (with or without holes) is called *simple* if does not contain any multiple edges.

A quadrangulation with holes is said to have *simple boundaries* if all its boundary components are disjoint and form a cycle (i.e. vertices on the boundaries are distinct).

A quadrangulation is *irreducible* if it is simple and has no separating 4-cycles (i.e. every 4-cycle delimits a face)- equivalently, it must be 4-irreducible.

[2] We diverge somewhat from their terminology, so as to align our definitions with what has historically been called an irreducible map. Note that what we call a strongly d-irreducible map is a d-irreducible map in [10], and our d-irreducible map are called weakly d-irreducible in [10]

[3] The restriction to even degree distinguished faces is put in place to ensure every quadrangulation with holes is a bipartite map.

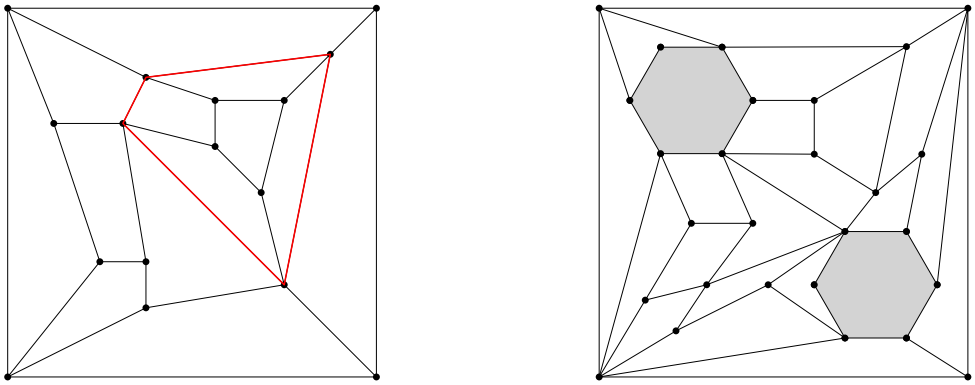


Figure 2.6: A simple quadrangulation with a separating 4-cycle (shown in red) and a quadrangulation with holes having simple boundaries.

We will also have occasion to deal with infinite (planar) quadrangulations. Infinite quadrangulations can naturally be introduced in several distinct ways, however, we will follow the approach developed by M. Krikun in [42]. Let $\mathbf{Q}_n$ denote the set of rooted quadrangulations with n faces of a class of quadrangulations (either all, simple quadrangulations or irreducible quadrangulations) and denote with $\mathbf{Q}_f = \cup_{n\geq 1}\mathbf{Q}_n$ the set of all rooted quadrangulations with finitely many faces. For every pair $\mathbf{q}, \mathbf{q}' \in \mathbf{Q}_n$ we define a metric

$$d_{\mathbf{Q}}(\mathbf{q}, \mathbf{q}') = \frac{1}{1 + \sup\{r : B_{\mathbf{Q},r}(\mathbf{q}) = B_{\mathbf{Q},r}(\mathbf{q}')\}}$$

where, for $r \geq 1$, $B_{\mathbf{Q},r}(\mathbf{q})$ denotes the planar map whose edges (resp. vertices) are all edges (resp. vertices) incident to a face of $\mathbf{q}$ having at least one vertex at distance strictly smaller than r from the root vertex, and equality of two such maps is to be interpreted as an isomorphism of two rooted planar maps, and we take $\sup \emptyset = 0$ by convention. We call $B_{\mathbf{Q},r}(\mathbf{q})$ the *combinatorial r-neighbourhood* of the root vertex in $\mathbf{q}$. We will study the structure of combinatorial neighbourhoods in (irreducible) quadrangulations much more in depth later on - for now let us only remark that by definition $B_{\mathbf{Q},r}(\mathbf{q})$ is a quadrangulation with holes having only simple faces at the boundary.

It is easily verified that $d_{\mathbf{Q}}$ defines a metric on $\mathbf{Q}_f$. However, the metric space $(\mathbf{Q}_q, d_{\mathbf{Q}})$ is not a complete metric space (see figure below). We define $(\mathbf{Q}, d_{\mathbf{Q}})$ to be the completion of this space. We call *infinite quadrangulations* the elements of $\mathbf{Q}$ that are not finite quadrangulations and we denote the set of all such quadrangulations by $\mathbf{Q}_\infty = \mathbf{Q}\backslash\mathbf{Q}_f$. Since metric completion is functorial, it is easily shown that the function $q \in \mathbf{Q}_f \mapsto B_{\mathbf{Q},r}(\mathbf{q})$ uniquely extends to a continuous function, which we will again denote simply by $B_{\mathbf{Q},r}$, on $\mathbf{Q}$ (furthermore, the extended function $B_{\mathbf{Q},r}$ is

naturally interpreted as the union of faces of $\mathbf{q}$ that have a vertex of distance strictly smaller than r from the root). The following lemma is easy to prove and describes some basic properties of the completed metric space $(\mathbf{Q}, d_{\mathbf{Q}})$:

Lemma 2.2.2 *The metric space $(\mathbf{Q}, d_{\mathbf{Q}})$ is Polish. Furthermore, a subset $F \subseteq \mathbf{Q}$ is relatively compact if and only if for any $r \geq 0$,*

$$\#\{B_{\mathbf{Q},r}(\mathbf{q}) : \mathbf{q} \in F\} < \infty \iff \sup\{\deg v : v \in V(B_{\mathbf{Q},r}(\mathbf{q})),\, \mathbf{q} \in F\} < \infty.$$

We provide next a convenient description of infinite quadrangulations. An infinite quadrangulation $\mathbf{q}$ defines a unique infinite graph G with a root edge, together with a consistent family of planar embeddings $\{B_{\mathbf{Q},r}; r \geq 1\}$ of the combinatorial r-neighbourhoods of the root vertex in G. Conversely, any sequence $\mathbf{q}_1, \mathbf{q}_2, \ldots$ of rooted quadrangulations with holes, such that $\mathbf{q}_r = B_{\mathbf{Q},r}(\mathbf{q}_{r+1})$ for every $r \geq 1$, specifies a unique infinite quadrangulations $\mathbf{q}$ whose combinatorial r-neighbourhood of the root vertex is $\mathbf{q}_r$, for every $r \geq 1$.

Definition 5. An infinite quadrangulation $\mathbf{q} \in \mathbf{Q}_\infty$ is called a *quadrangulation of the plane* if it has one end, that is, for every $r \geq 1$ the graph $\mathbf{q} \backslash B_{\mathbf{Q},r}(\mathbf{q})$ has exactly one infinite connected component.

Remark 1 Alternatively, it is not hard to convince oneself that quadrangulations of the plane also coincide with equivalence classes of certain proper embeddings of infinite graphs in the plane $\mathbb{R}^2$, viewed up to orientation preserving homeomorphisms of the plane. Namely, quadrangulations of the plane are exactly proper embeddings χ of locally finite (meaning vertices have finite degree) planar graphs such that

(i) every compact subset of $\mathbb{R}^2$ intersects at most finitely many edges of χ.

(ii) the connected components of the union of edges of χ in $\mathbb{R}^2$ are all bounded topological discs and have (face) degree 4.

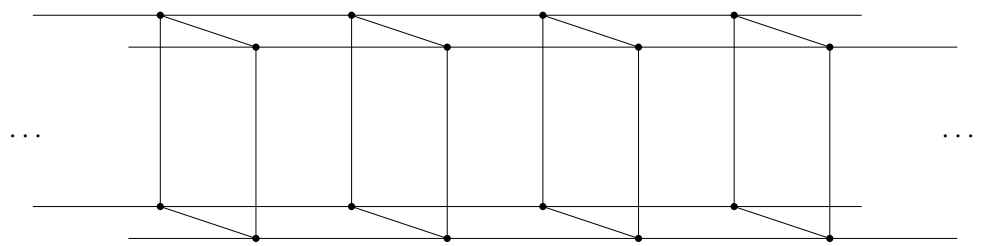


Figure 2.7: The infinite quadrangulation obtained by continuing the sequence indefinitely in both directions is clearly not planar.

A generic element of $\mathbf{Q}_\infty$ is not necessarily a quadrangulation of the plane (see figure above). However, any element of $\mathbf{Q}_\infty$ can be seen as an infinite quadrangulation

of a certain non-compact surface as we will explain now. We will follow closely the excellent exposition on infinite quadrangulations by N. Curien et al. [14]. Recall that an element $\mathbf{q}$ of $\mathbf{Q}_\infty$ is a sequence of compatible maps with holes $(\mathbf{q}_1, \mathbf{q}_2, \ldots)$, in the sense that $\mathbf{q}_r = B_{\mathbf{Q},r}(\mathbf{q}_{r+1})$ for every $r \geq 1$. This sequence defines a unique cell complex $S_\mathbf{q}$ up to homeomorphism, with an infinite number of 2-cells which are quadrangles. The cell complex $S_\mathbf{q}$ is an orientable, connected, separable topological surface such that every compact connected sub-surface is planar. It is known, see I. Richards [54], that the topology of $S_\mathbf{q}$ is uniquely characterized by its *end space*, which is a certain totally disconnected compact space. Intuitively, ends of a space should be interpreted as ideal points at infinity, e.g. the end space of the hyperbolic d-space is exactly its sphere at infinity. Formally, ends are defined as follows. A *boundary component* of $S_\mathbf{q}$ is defined to be a descending sequence $U_1 \supset U_2 \supset \cdots$ of non-empty connected open subsets of $S_\mathbf{q}$ such that

- The boundary ∂U_i of each set U_i is compact.
- $\cap_{i \geq 1} \overline{U}_i = \emptyset$ (meaning no $\overline{U}_i$ can be compact).

Two boundary components $(U_1, U_2, \ldots)$ and $(V_1, V_2, \ldots)$ are called *equivalent* if for for every $i \geq 1$ there exists a $j \geq 1$ such that $V_j \subseteq U_i$, and vice-versa. An *end* is an equivalence class of boundary components. We denote the set of ends of $S_\mathbf{q}$ by $\mathscr{E}_\mathbf{q}$. Note that the cell complex $S_\mathbf{q}$ has at least one end. Indeed, a connected locally-finite cell complex with infinitely many cells is locally compact and σ-compact, so it admits an exhaustion $\{K_n\}$ by compact sets whose interiors form a cover. Because the complex is infinite, we can define a boundary component by considering descending sequences of open connected components of the complementary sets $S_\mathbf{q} \backslash K_i$.

We can equip the set of ends with a topology as follows. For every open set $U \subseteq S_\mathbf{q}$ with compact boundary, we let V_U be the set of all ends whose corresponding boundary components are sequences of sets which are eventually contained in U. It is not difficult to check that the sets V_U form a basis for topology. The set $\mathscr{E}_\mathbf{q}$ equipped with the topology generated by the sets V_U is called the *end space* of $S_\mathbf{q}$.

Conversely, it is clear that a rooted quadrangulation of an orientable, connected, separable, non-compact planar surface[4] defines an element of $\mathbf{Q}_\infty$ - simply take the sequence of combinatorial balls centred at the root vertex with the same definition as above (separability of the surface ensures that the collection of balls exhausts the whole surface). We thus have

[4]A 2-surface S is *planar* if every compact connected subsurface of S is of reduced genus 0 (i.e. planar) - equivalently, a 2-surface is planar if and only if any closed Jordan curve separates it.

Theorem 2.2.1 (N. Curien et al., [14])**.** *The elements of* $\mathbf{Q}_\infty$ *are exactly quadrangulations of orientable, connected, separable, non-compact planar surfaces considered up to orientation-preserving homeomorphisms.*

We have already remarked above that a compact exhaustion of $S_{\mathbf{q}}$ whose interiors form a cover, naturally gives rise to ends of the complex. Moreover, it is not difficult to show that the entire ends space consists exactly of the boundary components that arise from any such compact exhaustion. This observation allows us to better understand what ends space looks like in the context of infinite quadrangulations, since the complex $S_{\mathbf{q}}$ comes equipped with a natural compact exhaustion by combinatorial balls.

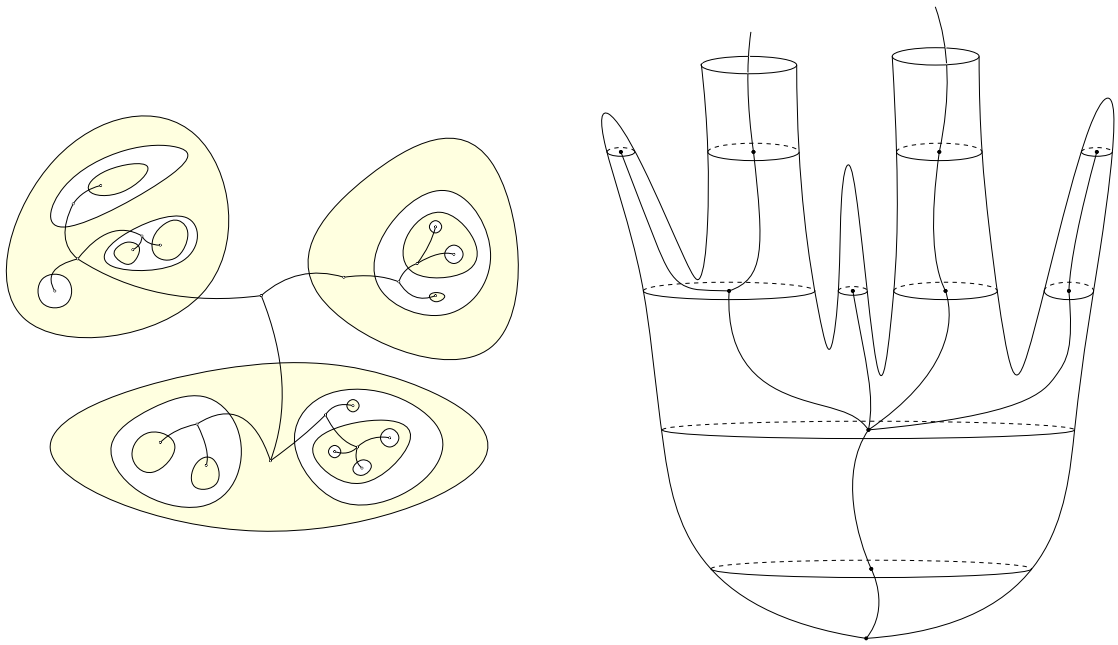


Figure 2.8: A schematic representation of the tree $\mathcal{T}_{\mathbf{q}}$ associated to an infinite quadrangulation $\mathbf{q}$.

We can associate to $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots) \in \mathbf{Q}_\infty$ an infinite tree $\mathcal{T}_{\mathbf{q}}$ as follows. The vertices x of this tree are the holes of $\mathbf{q}_1, \mathbf{q}_2, \ldots$ and an edge links the vertices x and y if there exists an $r \geq 1$ such that x is a hole of $\mathbf{q}_r$, y is a hole of $\mathbf{q}_{r+1}$, and y is included in the face determined by x. Furthermore, we connect all the holes of $\mathbf{q}_1$ by an edge to an extra root vertex. Let's define $\partial\mathcal{T}_{\mathbf{q}}$ (the end space of the infinite tree $\mathcal{T}_{\mathbf{q}}$) to be the set of all infinite injective paths (also called *spines*) in $\mathcal{T}_{\mathbf{q}}$ starting from the root and equip it with the topology generated by the sets W_x made of all the spines that pass through the vertex x of $\mathcal{T}_{\mathbf{q}}$ (note that $\mathcal{T}_{\mathbf{q}}$ is a locally-finite tree, so its ends space will be a compact totally disconnected space). It is now easy to see

that $\mathscr{C}_{\mathbf{q}}$ is homeomorphic to $\partial\mathscr{T}_{\mathbf{q}}$. In particular, when $\mathscr{T}_{\mathbf{q}}$ has a unique spine, then $\mathscr{C}_{\mathbf{q}}$ is reduced to a point, which means that the topology of $S_{\mathbf{q}}$ is that of the plane $\mathbb{R}^2$ - in other words, we recover the usual definition of a planar infinite quadrangulation.

Finally, let us review some constructions relating the classes of (rooted) planar maps and (rooted) quadrangulations. Let $\mathbf{m}$ be a planar map with vertex set $V = V(\mathbf{m})$ and edge set $E = E(\mathbf{m})$.

- The *derived map* of the map $\mathbf{m}$ is obtained by superimposition of the map $\mathbf{m}$ with its dual $\mathbf{m}^*$; more precisely, to construct the derived map $\Delta(\mathbf{m})$ of $\mathbf{m}$ place a vertex f^* in each face f of $\mathbf{m}$ and add an edge between two such vertices across each edge of $\mathbf{m}$, with 4-valent vertices created at the intersection of dual edges.
- The *medial map* of the map $\mathbf{m}$ is the map $\mathscr{M}(\mathbf{m})$ whose vertex set is E and with an edge between two such vertices if the corresponding edges share a vertex and belong to the same face of $\mathbf{m}$.
- The *incidence map* of the map $\mathbf{m}$ is the map $\mathbb{Q}(\mathbf{m})$ whose vertex set is $V + V^*$ and with one edge for every oriented edge $e \in \vec{E}(\mathbf{m})$ between e^- and f_e^*.

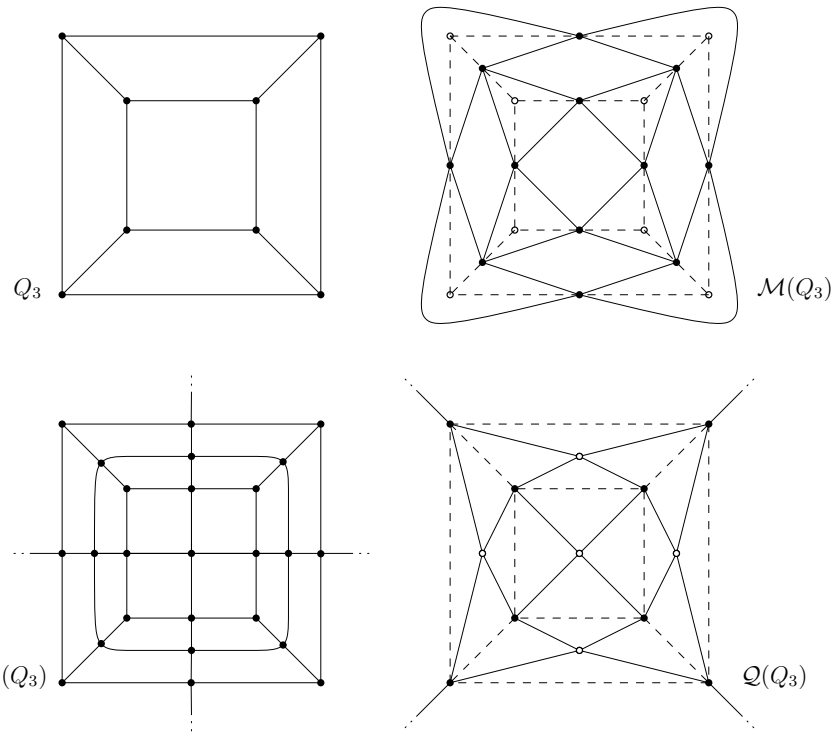


Figure 2.9: Example of derived map, medial map and incidence map of the cube Q_3.

The constructions of the derived map, medial map and associated quadrangulation of a planar map $\mathbf{m}$ are presented below. Note that the derived map $\Delta(\mathbf{m})$ of a planar

map $\mathbf{m}$ is a planar quadrangulation equipped with a natural partition $(W, W^*, \overline{W})$ of its vertex set such that $(W + W^*, \overline{W})$ is a bipartition of $\Delta(\mathbf{m})$ and each vertex of $\overline{W}$ is 4-valent. The medial map $\mathscr{M}(\mathbf{m})$ of a planar map is a 4-regular planar map equipped with a natural bicolouration of its faces. Finally, the incidence map $\mathbb{Q}(\mathbf{m})$ of a map $\mathbf{m}$ is a planar quadrangulation equipped with a natural vertex bicolouration. Furthermore, it is easily verified from the definition that $\mathbb{Q}(\mathbf{m}^*) = \overline{\mathbb{Q}(\mathbf{m})}$, where the bar indicates exchanging the colours of the vertices, and $\mathbb{Q}(\mathbf{m})^* = \mathscr{M}(\mathbf{m})$.

Theorem 2.2.2 (Incidence map, [60], [12], [49])**.**

(i) The incidence map Q establishes a one-to-one correspondence between the class of planar maps and the class of planar quadrangulations.

(ii) The incidence map Q restricts to a one-to-one correspondence between the class of 2-connected planar maps and the class of simple planar quadrangulations.

(iii) The incidence map Q restricts to a one-to-one correspondence between the class of 3-connected planar map and the class of irreducible planar quadrangulations.

Furthermore, all the maps extend naturally to bijections between the corresponding rooted classes.

The extension of the incidence map to rooted classes is obtained by rooting the quadrangulation at the unique corner corresponding to the root of the original map. Note that Tutte's bijection associates to a rooted n-edged planar map $\mathbf{m}$ a rooted n-faced quadrangulation $\mathbf{q}$. Furthermore, if $V_\circ(\mathbf{q}), V_\bullet(\mathbf{q})$ denotes the set of white, respectively black, vertices of a rooted quadrangulations $\mathbf{q}$ whose root vertex is black $\bullet$, then we also have the relations

$$\#V(\mathbf{m}) = \#V_\bullet(\Delta(\mathbf{m})) \text{ and } \#F(\mathbf{m}) = \#V_\circ(\Delta(\mathbf{m})).$$

We have discussed at the end of the previous section symmetries of planar maps and mentioned that for the class of 3-connected planar maps it is known that almost every map has no non-trivial symmetries. More precisely the following result is known to hold:

Theorem 2.2.3 (R. Wormald [55])**.** *Let r_n denote the number of rooted 3-connected planar maps with n edges and m_n the number of unrooted 3-connected planar maps with n edges, then*

$$m_n = \frac{1}{4n} r_n \left[1 + o(0.998)^n\right].$$

Using Tutte's bijection we will deduce below an analogous result for n-faced irreducible quadrangulations. Let's call an irreducible quadrangulation symmetric if it admits a non-trivial automorphism, then we have

Theorem 2.2.4. *Let q_n denote the number of rooted irreducible quadrangulations with n faces, then the number of symmetric irreducible quadrangulations with n faces is $o(c^n q_n)$ where c denotes some positive constant strictly less than* 1*.*

Proof. Let $\mathbf{q}$ be a planar quadrangulation equipped with a bicolouration of its vertices and $\mathbf{m}$ the unique map corresponding to $\mathbf{q}$ with respect to Tutte's bijection, then $\operatorname{Aut}\mathbf{q} = \mathbb{Z}_d \times \operatorname{Aut}\mathbf{m}$ where $d = 2$ if $\mathbf{m}$ is self-dual and $d = 1$ otherwise. Indeed, let $h \in \operatorname{Aut}\mathbf{q}$ be an automorphism of $\mathbf{q}$, then h either preserves or interchanges the bicolouration of vertices of $\mathbf{q}$. If h preserves the bicolouration of vertices, then it clearly descends to a well-defined automorphism of $\mathbf{m}$ and $\mathbf{m}^*$. On other hand, if h interchanges the bicolouration of vertices it defines an isomorphism of $\mathbf{m}$ and $\mathbf{m}^*$.

Note that $q_n = r_n$ since n-faced rooted irreducible quadrangulations correspond in a one-to-one fashion to n-edged rooted 3-connected planar maps under Tutte's bijection. Accordingly to prove the number of symmetric irreducible quadrangulations is $o(c^n q_n)$ it suffices to show that the number of rooted self-dual 3-connected planar maps is $o(c^n r_n)$. However, rooted self-dual k-connected planar maps for $k = 1, 2, 3$ have already been enumerated by S. Kitaev et al. The following asymptotic estimate, which is a particular case of their result, in combination with theorem (2.2.3) establishes the claim:

Lemma 2.2.3 (S. Kitaev et al. [38]) *The number of rooted self-dual* 3*-connected planar maps with n edges is asymptotically $o(r_n^{1/2})$.* □

2.3 Ideal Right-Angled Hyperbolic Polyhedra

Our interest in irreducible quadrangulations has really arose due to their intimate connection with a certain class of hyperbolic polyhedra. The geometry of hyperbolic polyhedra is far richer and less understood than that of spherical or Euclidean polyhedra. Moreover, there are several anomalies unique solely to the hyperbolic space such as the existence of unbounded polyhedra that are nevertheless of finite hyperbolic volume. We will provide in the this section a very brief outline of the problem and its connection with irreducible quadrangulations. We assume the reader is familiar with the basic concepts of hyperbolic geometry. An excellent reference is the book [53].

Let $\mathbb{H}^d$ denote the d-dimensional hyperbolic space. Let us recall that a non-empty subset C of $\mathbb{H}^d$ is said to be *convex* if for every pair of distinct points $x, y \in C$ the (unique) geodesic segment $[x, y]$ is completely contained in C. The least integer m such that C is contained in a geodesic m-plane of $\mathbb{H}^d$ is called the *dimension* of C. A *side* of a convex subset C of $\mathbb{H}^d$ is a non-empty maximal convex subset of its (topological) boundary ∂C. Each side is contained in a unique geodesic plane, called the *supporting plane* of the convex set C.

Definition 6. A *convex polyhedron* P^m in $\mathbb{H}^d$ is a non-empty, closed, convex subset of $\mathbb{H}^d$ such that the collection S of its sides is locally-finite in $\mathbb{H}^d$. Since $\mathbb{H}^d$ is a separable locally compact metric space every locally-finite collection of subsets of $\mathbb{H}^d$ is at most countable.

Let P^m be a convex polyhedron in $\mathbb{H}^d_b$ (the *Poincaré ball model* of $\mathbb{H}^d$). An *ideal point* of P is a point of $\overline{P} \cap S^{d-1}$, where $\overline{P}$ is the closure of P in $\mathbb{R}^d$. A side of P is said to be incident with an ideal point u of P if u belongs to the closure of the side in $\mathbb{R}^d$. A *horopoint* of P in $\mathbb{H}^d_b$ is an ideal point u of P for which there exists a closed horoball B based at u such that B intersects non-trivially only the sides of P which are incident with u. Finally, an *ideal vertex* of P in $\mathbb{H}^d_b$ is a horopoint whose link is compact. A *generalized vertex* of a convex polyhedron in $\mathbb{H}^d$ is either an actual vertex (i.e. 0-dimensional side) or an ideal vertex of the polyhedron. A convex polyhedron in $\mathbb{H}^d$ is called *ideal* if all of its (at most finitely many) vertices are ideal vertices on the sphere at infinity.

Definition 7. Let P^m be a convex polyhedron in $\mathbb{H}^d$. Distinct sides S and T are said to be adjacent if $S \cap T$ is a side of both S and T. The *dihedral angle* between adjacent sides S and T of a convex polyhedron is a number $\vartheta(S, T)$ defined as follows: let π_S and π_T denote the supporting $(m-1)$-plane of S and T and let π_P denote the unique geodesic m-plane containing the polyhedron P^m. The geodesic plane π_S and π_T subdivide the m-plane π_P into four regions, one of which contains the polyhedron P. Notice also that $\pi_S \cap \pi_T = \pi_{S \cap T}$ is exactly the supporting plane of the common side. Let x be a point of $S \cap T$ and define geodesic lines $\lambda, \mu : \mathbb{R} \to \pi_P$ such that

- $\lambda(0) = x = \mu(0)$.
- λ and μ are normal to π_S and π_T, respectively.
- $\lambda'(0)$ and $\mu'(0)$ are directed away from the half-spaces of π_P containing P.

Let α be the angle between λ and μ at the point x. It can be checked that the value α depends neither on the point x nor on the choice of the geodesic lines λ and μ, but is instead an intrinsic property of the polyhedron P. The *dihedral angle* of P between S and T is defined to be the angle $\vartheta(S,T) = \pi - \alpha \in (0,\pi)$.

Definition 8. An *abstract polyhedron* X^d is a regular cell complex on the d-sphere such that the intersection of any two closed cells is a closed cell (possibly empty)-in other words, the complex has the *intersection property*. A *labelling* of an abstract polyhedron X^d is a function ϑ that assigns a real number from the interval $(0,\pi)$ (to be interpreted as dihedral angle) to each $(d-1)$-face of the complex. The pair (X,ϑ) is called a *labelled abstract polyhedron*.

Any (geometric) polyhedron P^d in $\mathbb{H}^d$ determines a unique labelled abstract polyhedron (with the labelling function given by the dihedral angles). We say a labelled abstract polyhedron (X^d,ϑ) is *realizable* if there exists a convex polyhedron P^d in $\mathbb{H}^d$ whose associated labelled cell complex is combinatorially label preserving isomorphic to (X^d,ϑ). The problem of the possible combinatorial types of convex hyperbolic polyhedra and the admissible dihedral angle assignments of a fixed combinatorial type is a difficult one. In general, it is not even known if a convex hyperbolic polyhedron is uniquely determined by its combinatorial structure and dihedral angles. However, in dimension 3 and with the additional restriction that the dihedral angles be acute, it is a consequence of the deep theorems of E. Steintz [59], P. Mani [46] and E. M. Andreev [4, 3] that the answers to all three questions are known.

Let $\mathscr{R}^\infty$ denote the class of all ideal right-angled convex hyperbolic polyhedra in $\mathbb{H}^3$. According to Andreev's theorem each polyhedron of the set is uniquely determined (up to hyperbolic isometries of $\mathbb{H}^3$) by its 1-skeleton and the set of 1-skeleta admits the following combinatorial characterization

$$\Gamma = \mathrm{Skel}_1 P \text{ for } P \in \mathscr{R}^\infty \iff \Gamma^* \text{ is an irreducible quadragulation.}$$

What this simple observation says is that the complicated geometry of an ideal right-angled convex hyperbolic polyhedron is (somehow) completely recorded in its combinatorial 1-skeleton. Although the explicit connection between the geometry of the polyhedron and the combinatorics of its 1-skeleton is still not well-understood, we hope to be able to deduce some interesting geometric properties of polyhedra of class $\mathscr{R}^\infty$ from the local combinatorial properties of irreducible quadrangulations (which are the planar duals of their 1-skeleta). We are particularly, interested in the connection between the hyperbolic volume of a polyhedron (which is a well-known isometry invariant) and the structure of large irreducible quadrangulations.

Chapter 3

Enumeration of Irreducible Quadrangulations

3.1 Singularity Analysis

An important tool in our investigation of random irreducible quadrangulations will be asymptotic analysis of the series coefficients of various generating functions connected with irreducible quadrangulations. We will review in the following section the basic techniques and theorems of asymptotic analysis of analytic functions.

3.1.1 Lagrange inversion

Recursively defined combinatorial structures often lead to functional equations whose solutions may be analysed analytically around the dominant singularities. An important and in practice often encountered special case is when the generating function is defined by inversion. A common pattern observed in this context is the appearance of a square-root type singularity. Before describing the asymptotic analysis of such an implicit scheme, let's remind ourselves first of a classical result by Lagrange. We will be working at first in a purely formal context (meaning we will not be concerned with issues of convergence and analycity). Let's set up our notation and terminology. The reader is referred to the excellent book by I. P. Goulden and D. M. Jackson, [31], for further details on combinatorial enumeration.

Let k be a commutative ring (with unity). We denote by $k[t]$ the set of polynomials in t with coefficients in k. If k is a field, we will denote by $k(t)$ the field of fractions in t with coefficients in k. We will denote by $\bar{k}$ the algebraic closure of k.

We will consider formal series of the form $c(t) = \sum_{i \geq i_0} c_i t^{i/d}$ where $i_0 \in \mathbb{Z}$ is an integer, the coefficients a_i belong to the field k and we assume the initial coefficient $a_{i_0} \neq 0$ is non-zero and $d \in \mathbb{N} \backslash \{0\}$ is a positive integer. A series $c(t)$ satisfying the

given assumptions is called a *Puiseux series* (or a *fractional Laurent series*). The set of all Puiseux series with coefficients in k will be denoted by $k\{\{t\}\}$ and comes equipped with a natural valuation $\mathrm{val} : k\{\{t\}\}\backslash\{0\} \to k$ that maps a non-zero element $c(t)$ to the lowest exponent i_0/d that appears in the series expansion of $c(t)$. We use the standard notation for the coefficients of a series, namely $[t^{i/d}]c(t) = c_i$. The following subsets of $k\{\{t\}\}$ will often be used:

- $k[\![t]\!]$ is the set of all formal *power series* in t with coefficients in k.
- $k(\!(t)\!)$ is the set of all formal *Laurent series* in t with coefficients in k.
- $k\{t\}$ is the set of all formal *fractional power series* in t with coefficients in k.

Each of these sets is a commutative ring (with unity), and $k(\!(t)\!)$ and $k\{\{t\}\}$ are fields provided k is a field. More precisely, if k is a field then $k(\!(t)\!)$ is the fraction field of $k[\![t]\!]$ and $k\{\{t\}\}$ if the fraction field of $k\{t\}$. Regrading algebraic closure, the following result is well known:

Theorem 3.1.1 (Newton-Puiseux theorem)**.** *Let k be an algebraically closed field of characteristic 0, then $k\{\{t\}\} = \overline{k(\!(t)\!)}$ is an algebraically closed field.*

All the notions introduced naturally extend to series in several indeterminates. Note especially the following inclusions:

$$k[t,u] \subseteq k[\![t]\!][u] \subseteq k[u][\![t]\!] \subseteq k[\![t,u]\!] = k[\![t]\!][\![u]\!],$$

where in the notation we assume associativity to the left, so $k[\![t]\!][u]$ is the set of all formal polynomials in u whose coefficients are formal power series in u and $k[u][\![t]\!]$ is the set of all formal power series in t whose coefficients are polynomial in u. When speaking of formal power series with multiple indeterminates, we will frequently abbreviate the notation by writing simply $k[\![\underline{t}]\!]$, where $\underline{t} = \{t_1, t_2, \ldots\}$ denotes a set of commutative indeterminates and $\underline{t}^{\underline{i}}$ will denote the product $t_1^{i_1} \cdots t_k^{i_k}$ (the exponent $\underline{i}$ will always have at most finitely many non-zero entries).[1] Given a formal power series $\phi \in k[\![\underline{t}]\!]$ the *constant term* of ϕ is the coefficient $[\underline{t}^{\underline{0}}]\phi$ often denoted simply by $\phi(\underline{0})$. The following subsets will be particularly important:

$$k[\![\underline{t}]\!]_0 = \{\phi \in k[\![\underline{t}]\!] : \phi(\underline{0}) = 0\}$$

[1]Formally, the set of formal power series "with infinitely many variables“ is defined as the *projective limit* of $\{k[\![t_1, \ldots, t_n]\!]\}_{n\geq 1}$ defined by

$$k[\![\underline{t}]\!] = \{(c_n)_{n\geq 1} \in \prod_{n\geq 1} k[\![t_1, \ldots, t_n]\!] : c_n = c_{n+k}(t_1, \ldots, t_n, 0, \ldots, 0) \text{ for all } n, k \geq 0\}.$$

$$k[\![\underline{t}]\!]_1 = \{\phi \in k[\![\underline{t}]\!] : \phi(\underline{0}) \text{ is a unit in } k\}.$$

We should also set up our conventions used when dealing with infinite sums (or products) and compositions of formal power series. A set $\Phi = \{\phi_j \in k[\![\underline{t}]\!]\}_{j\geq 0}$ is called a *summable* family of power series if each monomial in $k[\![\underline{t}]\!]$ occurs (with non-zero coefficient) in at most finitely many members of Φ. For a summable family Φ, product is distributive over addition, addition is associative and commutative and the infinite sum $\sum_j \phi_j$ and infinite product $\prod_j (1+\phi_j)$ for $\phi_j \in k[\![\underline{t}]\!]_0$ can be defined analogously to the finite case as follows. Since every monomial appears at most finitely many times in ϕ_j, the infinite sum is obtained by summing over all monomials in $k[\![\underline{t}]\!]$, a sum that is well-defined. On the other hand, each monomial in $k[\![\underline{t}]\!]_0$ has only finitely many ordered factorizations into monomials in $k[\![\underline{t}]\!]_0$. Since Φ is summable, only finitely many ϕ_j's have a non-zero coefficients for any particular monomial factor, so the infinite product is obtained by summing over all ordered factorizations of monomials in $k[\![\underline{t}]\!]$, a sum that is again well-defined.

Let $\phi = \sum_{\underline{i}} c_{\underline{i}} \underline{t}^{\underline{i}} \in k[\![\underline{t}]\!]$ be a formal power series and $\underline{\psi} = (\psi_1, \psi_2, \ldots)$, where $\psi_j \in k[\![\underline{t}]\!]$, a sequence of formal power series. The *composition* of ϕ and $\underline{\psi}$ (obtained by *substitution* of $\underline{\psi}$ for $\underline{t}$) is $\sum_{\underline{i}} c_{\underline{i}} \underline{\psi}^{\underline{i}}$ and is denoted by $\phi(\underline{\psi}(\underline{t}))$. We say that the composition (or equivalently, substitution) is *admissible* if $\{c_{\underline{i}} \underline{\psi}^{\underline{i}}\}$ is a summable family. The following two situations occur frequently in practice:

1. If $\underline{t}$ is finite and $\psi_j \in k[\![\underline{t}]\!]_0$, then composition is admissible for each $\phi \in k[\![\underline{t}]\!]$.
2. If $\phi \in k[\underline{t}][\![\underline{s}]\!]$ then the substitution of $\underline{\psi}$ for $\underline{t}$ in ϕ is admissible for every $\underline{\psi}$.

Let $\phi \in tk[\![t]\!]_1$ then there exists a unique formal power series $\psi \in k[\![t]\!]_0$, called the *compositional inverse* of ϕ, such that $\phi(\psi(t)) = t = \psi(\phi(t))$. On the other hand, if $\phi \in k[\![\underline{t}]\!]_1$, then there exists a unique power series $\psi \in k[\![\underline{t}]\!]_1$, called the *multiplicative inverse* of ϕ, such that $\psi(\underline{t})\phi(\underline{t}) = 1 = \phi(\underline{t})\psi(\underline{t})$. Moreover, the multiplicative inverse of ϕ exists if and only if $\phi \in k[\![\underline{t}]\!]_1$.

We can now state Lagrange's inversion theorem:

Theorem 3.1.2 (Lagrange's inversion theorem, [31])**.** *Let k be a commutative ring (with unity) and $\phi(\lambda) \in k[\![\lambda]\!]_1$ a formal power series in λ with unit constant term, then there exists a unique formal power series $\alpha(x) \in k[\![x]\!]_0$ with vanishing constant term such that $\alpha = x\phi(\alpha)$. Moreover, if $F(\lambda) \in k(\!(\lambda)\!)$ then*

$$[x^n]F(\alpha) = \begin{cases} \frac{1}{n}[\lambda^{n-1}]F'(\lambda)\phi^n(\lambda) & \text{for } n \geq \operatorname{val} F,\ n \neq 0 \\ [\lambda^0]F(\lambda) + [\lambda^{-1}]F'(\lambda)\log(\phi(\lambda)\phi^{-1}(0)) & \text{for } n = 0. \end{cases}$$

The multivariate analogue of Lagrange's inversion theorem is:

Theorem 3.1.3 (Multivariate Lagrange's inversion theorem, [31]). *Let k be a commutative ring (with unity) and $\phi_1(\underline{\lambda}), \ldots, \phi_r(\underline{\lambda}) \in k[\![\underline{\lambda}]\!]_1$ formal power series in $\underline{\lambda}$ with unit constant term, where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$, then there exist unique formal power series $\alpha_1(\mathbf{x}), \ldots, \alpha_r(\mathbf{x}) \in k[\![\underline{x}]\!]_0$ such that $\alpha_i = x_i\phi_i(\underline{\alpha})$ for all i. Moreover, if $F(\underline{\lambda}) \in k(\!(\underline{\lambda})\!)$ then*

$$[\underline{x}^{\underline{k}}]F(\underline{\alpha}) = [\underline{\lambda}^{\underline{k}}]\left\{F(\underline{\lambda})\underline{\phi}^{\underline{k}}(\underline{\lambda}) \det\left[\delta_{ij} - \lambda_j\phi_i^{-1}(\underline{\lambda})\tfrac{\partial}{\partial\lambda_j}\phi_i(\underline{\lambda})\right]_{ij}\right\}. \tag{3.1}$$

We will discuss the asymptotics of solutions obtained by Lagrange's inversion in the next section. However, before moving on let's briefly discuss solvability of certain algebraic equations involving formal power series that frequently arise in enumerative combinatorics. It is often the case, when trying to determine the generating function $M(x)$ of a class of rooted planar maps with respect to a certain combinatorial parameter, that the problem simplifies if one introduces a second auxiliary parameter (a role often played by the degree of the root vertex) and leads to a polynomial equation involving the bivariate generating function $M(x, y)$ and its specialization $M(x, 1) = M(x)$. Following D. Zilbirger's terminology (see, [62, p. 457]), the newly introduced variable y is called a *catalytic variable* (it serves as a catalyst that is to be discarded, i.e. replaced by 1, at the end of the reaction). Furthermore, despite only having one functional equation for two unknown functions, under some mild restrictions the polynomial equation will nevertheless uniquely determine both generating functions $M(x, y)$ and $M(x, 1)$. This was first observed by W. Tutte and W. G. Brown [13], for the case of quadratic equations, who developed a constructive way to solve such an equation, called the *quadratic method*. Their approach was later significantly generalized by M. Bousquet-Mélou and A. Jehanne [9], who designed a general way of solving (up to reasonable technical assumptions) polynomial equations with one catalytic variable.

Theorem 3.1.4 (M. Bousquet-Mélou and A. Jehanne, [9]). *Let k be a field of characteristic 0, $F(y) := F(x, y) \in k[y][\![x]\!]$ a formal power series in X whose coefficients are polynomials in y, $F_0(y) \in k[\![y]\!]$ a formal power series in y, and let $Q(y_0, y_1, \ldots, y_r, x, u)$ be a polynomial in $r + 3$ indeterminates with coefficients in the field k. Denote with Δ the operator of divided difference (also called the discrete derivative) with respect to y, and consider the functional equation with one catalytic variable y,*

$$F(y) \equiv F(x, y) = F_0(y) + Q(F(y), \Delta F(y), \ldots, \Delta^{(r)}F(y), x, y).$$

The formal power series $F(x, y)$ in $k[y][\![x]\!]$ defined by it is unique and algebraic over $k(x, y)$ (the field of fractions in x, y with coefficients in k). Furthermore, there exists an explicit algorithm to determine the algebraic equation satisfied by $F(x, y)$.

We will not describe the general algorithm explicitly and instead refer the reader to the article [9]. In our application, the most commonly occurring equations with catalytic variables will be quadratic, so we will instead describe in more detail the quadratic method. The method is based on the following result of W. G. Brown, whose proof seems to contain a mistake as noted by M. Bousquet-Mélou and A. Jehanne, and was corrected by them in [9].

Theorem 3.1.5 (W. G. Brown [13], M. Bousquet-Mélou and A. Jehanne [9])**.** *Let k be a field and $\Delta(t, u) \in k[\![t]\!][u]$.*

(i) Δ has a square-root in $k[\![t, u]\!]$ if and only if it can be factored as $\Delta(t, u) = RQ^2$, where $R \in 1 + u\, k[\![t]\!][u]$ and $Q \in k[\![t]\!][u]$.

(ii) Δ has a square-root in $k[u][\![t]\!]$ if and only if it can be factored as $\Delta = RQ^2$, where $R \in 1 + ut\, k[\![t]\!][u]$ and $Q \in k[\![t]\!][u]$.

Let's illustrate how Brown's theorem may be used to solve a quadratic equation with one catalytic variable. We start by setting up our problem; we have an $(r + 1)$-tuple of (unknown) formal power series, say $F(u), F_1, \ldots, F_r$, such that $F(u) \in k[u][\![t]\!]$ and $F_i \in k[\![t]\!]$ for all i, that satisfies a quadratic equation of the form

$$(2aF(u) + b)^2 = \Delta(t, u) =: \Delta(u), \tag{3.2}$$

where a, b and Δ are polynomials in $F_1, \ldots, F_r, t$ and u with coefficients in k. Note that there is no loss of generality in assuming our polynomial equation is of the above form, since any quadratic equation can be brought to this form by completing the square. Obviously Δ has a square-root in $k[u][\![t]\!]$ (namely, the series $2aF(u)+b$), so the second part of Brown's theorem (3.1.5) applies: the canonical factorization of Δ contains a squared factor. Adopting the notation of Brown's theorem we can write $\Delta = RQ^2$ (where both factors R and Q are polynomials in u). In order to determine the degree of u in each factor, one has to determine how many roots of Δ are infinite (resp. finite) when $t = 0$. This can easily be accomplished using Newton's diagrams. One then introduces a new set of indeterminates, namely the coefficients of the polynomials R and Q, and obtains a system of polynomials equations for the coefficients by comparing the powers u^j in Δ and its factorization for all j.

In particular, if there is just one unknown function F_1, then equation (3.2) suggests any fractional power series $U = U(t)$ that cancels $2aF(t)+b = 0$ also cancel $\Delta(u)$ and by differentiating with respect to u, we see that U must actually be a multiple root of Δ. Hence the discriminant of $\Delta(u)$ with respect to u must be zero, which gives an algebraic equation satisfied by the unknown power series F_1. We have recovered exactly Brown's quadratic method.

3.1.2 Flajolet-Odlyzko transfer theorem

We will review in this section the basic theory of asymptotic analysis. The material presented here is drawn heavily from P. Flajolet and R. Sedgewick's book [25]. Let's quickly set up our conventions on Landau's asymptotic notation.

Definition 9. Let Ω be a set, $s_0 \in \Omega$ a distinguished element of Ω and $f, g : \Omega\backslash\{s_0\} \to \mathbb{C}$ (or $\mathbb{R}$) two functions. We say$f$ *is of order not exceeding* g and write $f = \mathcal{O}(g(s))$ (called big-oh notation) as $s \to s_0$ if the ratio $|f(s)/g(s)|$ is bounded in a neighbourhood of s_0, or equivalently if there exists a neighbourhood U of s_0 and a constant $C > 0$ such that

$$|f(s)| \leq C|g(s)|, \forall s \in U\backslash\{s_0\}.$$

We say f *is of order less than* g and write $f(s) = o(g(s))$ (called little-oh notation) as $s \to s_0$ provided $\lim_{s\to s_0} |f(s)/g(s)| = 0$ or equivalently, if for any $\varepsilon > 0$ there exists a neighbourhood U_ε of s_0 (depending on ε) such that

$$|f(s)| < \varepsilon|g(s)|, \forall s \in U_\varepsilon\backslash\{s_0\}.$$

Finally, we say f and g are *asymptotically equivalent* and write $f(s) \sim g(s)$ as $s \to s_0$ provided $\lim_{s\to s_0} |f(s)/g(s)| = 1$.

These notions also have uniform counterparts for families of functions depending on an auxiliary parameter. We merely require the choice of constant (or the limits) to hold uniformly with respect to the auxiliary parameter.

In the previous section we considered power series from a purely formal point of view. However, a lot of valuable information on the behaviour of the coefficients of a power series can be obtained (using the machinery of complex analysis) if the function defined by a power series is analytic in an open disk about the origin. Let $f(z) = \sum_n c_n z^n$ be a formal power series in z such that $f(z)$ is analytic in an open disk $\mathbb{D}(r)$ of radius $r > 0$ centred about the origin. Using analytic continuation the domain of definition of f can be extended to larger and larger disks until we encounter

a singularity of $f(z)$. The maximal radius of such a disk (centred at the origin) is called *the radius of convergence* and can be determined (using Cauchy-Hadamard's theorem) as $R^{-1} = \limsup_n |c_n|^{1/n}$. Since generating functions considered in combinatorial enumeration always have non-negative coefficients the following theorem of Pringsheim is especially relevant as it locates at least one of the singularities with minimal modulus of the generating function:

Theorem 3.1.6 (Pringsheim's theorem)**.** *Let $f(z) = \sum_n c_n z^n$ be a power series with radius of convergence $R > 0$ and non-negative coefficients, then $z = R$ is a singularity of $f(z)$.*

Note that the theorem does not preclude other points of the circle of convergence to also be singularities of $f(z)$ nor does it say anything about the nature of the singularity at $z = R$. We should also remark that (provided it is possible) by analytically continuing $f(z)$ beyond the circle of convergence may introduce other singularities whose modulus is strictly larger than R. However, it turns out that the singularities of minimal modulus, which are called *dominant singularities*, determine (under reasonable technical assumptions) the asymptotic behaviour of the coefficients of the power series defining $f(z)$ whereas singularities outside of the circle of convergence do not influence the asymptotics. We will describe the needed technical assumptions.

A more general approach to asymptotic analysis was developed by P. Flajolet and A. Odlyzko in [24]. It is based on the observation that an asymptotically dominant term of an analytic function about its dominant singularity will frequently also determine the asymptotically dominant term of its coefficients. The comparison principle is based on the known asymptotic analysis of the binomial function $(1-z)^\alpha$, as follows:

Theorem 3.1.7 (Asymptotic analysis of $(1-z)^{-\alpha}$, [24])**.** *Let α be an arbitrary complex number in $\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$, then the coefficient of z^n in $f(z) = (1-z)^{-\alpha}$ admits a complete asymptotic expansion in descending powers of n,*

$$\begin{aligned}[z^n]f(z) &\sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \frac{1}{n}e_1 + \frac{1}{n^2}e_2 + \cdots\right)\\ &\sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \frac{1}{2n}\alpha(\alpha-1) + \frac{1}{24n^2}\alpha(\alpha-1)(\alpha-2)(3\alpha-1) + \cdots\right),\end{aligned}$$

where e_k is a polynomial in α of degree $2k$.

Moreover, the basic principles underlying the proof of the asymptotic expansion of $[z^n](1-z)^{-\alpha}$ can be extended to a wider class of singular functions, most notably

ones that also involve logarithmic terms. We have compiled below a list of known asymptotic expansions of singular functions involving logarithmic terms of the form

$$f_{\alpha,\beta}(z) \equiv f(z) = (1-z)^{-\alpha} \overbrace{\left[\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right]^{\beta}}^{\lambda(z)}.$$

We denote by $\mathcal{S} = \{(1-z)^{-\alpha}\lambda(z)^{\beta} : \alpha, \beta \in \mathbb{C}\}$ the set of all such singular functions, which are sometimes called *algebraic-logarithmic functions*. The asymptotic expansions are:

$$\alpha \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0},\ \beta \in \mathbb{C}\backslash\mathbb{Z}_{\geq 0} \quad [z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}\left(1 + \frac{a_1}{\log n} + \frac{a_2}{\log^2 n} + \cdots\right)$$

$$\alpha \in \mathbb{Z}_{\leq 0},\ \beta \in \mathbb{C}\backslash\mathbb{Z}_{\geq 0} \quad [z^n]f(z) \sim n^{\alpha-1}(\log n)^{\beta}\left(\frac{b_1}{\log n} + \frac{b_2}{\log^2 n} + \cdots\right)$$

$$\alpha \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0},\ \beta \in \mathbb{Z}_{\geq 0} \quad [z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(E_0(\log n) + \frac{E_1(\log n)}{n} + \frac{E_2(\log n)}{n^2} + \cdots\right)$$

$$\alpha \in \mathbb{Z}_{\leq 0},\ \beta \in \mathbb{Z}_{\geq 0} \quad [z^n]f(z) \sim n^{\alpha-1}\left(F_0(\log n) + \frac{F_1(\log n)}{n} + \frac{F_2(\log n)}{n^2} + \cdots\right).$$

The coefficients are determined as follows $b_k = \binom{\beta}{k} \frac{d^k}{ds^k}\frac{1}{\Gamma(s)}\Big|_{s=\alpha}$ and $a_k = \Gamma(\alpha)b_k$,

$$\begin{aligned} E_k &= \text{ polynomial of degree } k \\ F_k &= \text{ polynomial of degree } k-1 \end{aligned}$$

Definition 10. Given two number φ, R such that $R > 1$ and $\varphi \in (0, \frac{1}{2}\pi)$, the open domain $\Delta(\varphi, R)$ is defined as

$$\Delta(\varphi, R) = \{z \in \mathbb{C} : |z| < R,\ z \neq 1 \text{ and } |\arg(z-1)| > \varphi\}.$$

A domain is called a Δ-*domain* at 1 if it is a $\Delta(\varphi, R)$ domain for some R and φ. For a complex number $\zeta \neq 0$, a Δ-domain at ζ is the image by the analytic mapping $z \mapsto \zeta z$ of a Δ-domain at 1. A function is called Δ-*analytic* if it is analytic in some Δ-domain.

The class of Δ-analytic functions is amenable to singularity analysis; the following transfer theorem is the underlying tool to deriving an asymptotic expansion for the coefficients of a Δ-analytic function.

Theorem 3.1.8 (Transfer theorem for $\mathcal{O}$, o and $\sim$, P. Flajolet and A. Odlyzko, [24])**.** *Let α, β be arbitrary real numbers and let $f(z)$ be a function that is Δ-analytic at* 1*.*

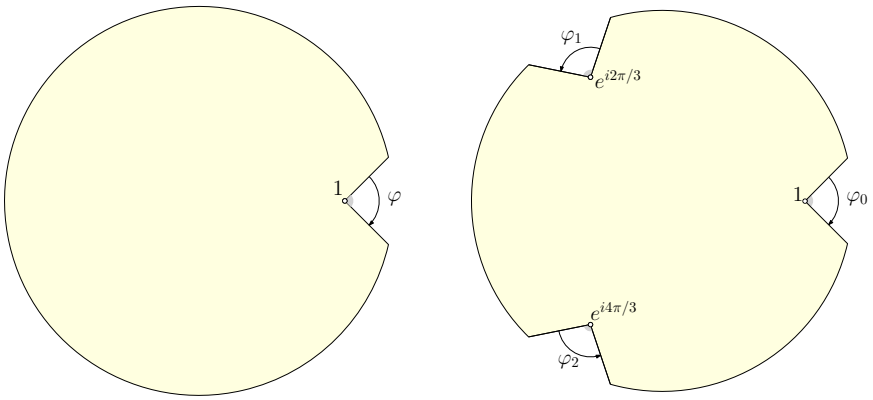


Figure 3.1: A Δ-domain (on the left) and a Δ-domain with several indentations corresponding to generating functions with multiple singularities of minimal modulus.

(i) Suppose $f(z)$ satisfies in the intersection of a neighbourhood of 1 with its Δ-domain the condition $f(z) = \mathcal{O}(f_{\alpha,\beta})$, then the coefficient of z^n in $f(z)$ satisfies $[z^n]f(z) = \mathcal{O}(n^{\alpha-1}\log^\beta n)$.

(ii) Suppose $f(z)$ satisfies in the intersection of a neighbourhood of 1 with its Δ-domain the condition $f(z) = o(f_{\alpha,\beta})$, then the coefficient of z^n in $f(z)$ satisfies $[z^n]f(z) = o(n^{\alpha-1}\log^\beta n)$.

(iii) Suppose $f(z)$ satisfies in the intersection of a neighbourhood of 1 with its Δ-domain the condition $f(z) \sim Kf_\alpha$ for $\alpha \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}$, then the coefficient of z^n in $f(z)$ satisfies $[z^n]f(z) \sim Kn^{\alpha-1}/\Gamma(\alpha)$.

The tools developed allow us, starting from an expansion of a function around its dominant singularity, also called a *singular expansion*, to transfer term-by-term an approximation of the function to an asymptotic estimate of its coefficients. More precisely, we have

Theorem 3.1.9 (Singularity analysis, single singularity)**.** *Let $f(z)$ be a function analytic at 0 with a singularity ζ such that $f(z)$ can analytically continued to a domain of the form $\zeta \cdot \Delta_0$, for a Δ-domain Δ_0 at 1. Suppose there exist functions σ, τ, where σ is a (finite) linear combination of functions from $\mathcal{S}$ and $\tau \in \mathcal{S}$ such that $f(z) = \sigma(z/\zeta) + \mathcal{O}(\tau(z/\zeta))$ as $z \to \zeta$ in $\zeta \cdot \Delta_0$. Then the coefficient of z^n in $f(z)$ satisfies the asymptotic estimate*

$$[z^n]f(z) = \zeta^{-n}\sigma_n + \mathcal{O}(\zeta^{-n}\tau_n^*),$$

where $\sigma_n = [z^n]\sigma(z)$ has its coefficients determined by the scheme above and $\tau_n^* = n^{\alpha-1}\log^\beta n$ if $\tau(z) = (1-z)^{-\alpha}\lambda(z)^\beta$.

In case of multiple singularities, the separate contributions from each of the singularities, as given by the basic singularity analysis process described above, are to be added up. More precisely, we have the following theorem:

Theorem 3.1.10 (Singularity analysis, multiple singularities)**.** *Let $f(z)$ be analytic in $|z| < \rho$ and have a finite number of singularities on the circle $|z| = \rho$ at points $\zeta_j = \rho \exp(i\theta_j)$ for $j = 1, \ldots, r$. Assume that there exists a Δ-domain Δ_0 such that $f(z)$ is analytic in the indented disc $D = \cap_{j=1}^{r} \zeta_j \cdot \Delta_0$. Suppose there exist r functions $\sigma_1, \ldots, \sigma_r$, each a linear combination of elements from $\mathcal{S}$, and a function $\tau \in \mathcal{S}$ such that $f(z) = \sigma_j(z/\zeta_j) + \mathcal{O}(\tau(z/\zeta_j))$ as $z \to \zeta_j$ in D. Then the coefficient of z^n in $f(z)$ satisfies the asymptotic estimate*

$$[z^n]f(z) = \zeta_1^{-n}\sigma_{1,n} + \cdots + \zeta_r^{-n}\sigma_{r,n} + \mathcal{O}(\rho^{-n}\tau_n^*),$$

where each $\sigma_{j,n} = [z^n]\sigma_j(z)$ has its coefficients determined by the scheme above, and $\tau_n^* = n^\alpha \log^\beta n$ if $\tau(z) = (1-z)^{-\alpha}\lambda(z)^\beta$.

Following P. Flajolet and A. Sedgewick [25] we say that a function is *amenable to singularity analysis* if it satisfies the conditions of singularity analysis as expressed by theorem (3.1.9) (for a single dominant singularity) or theorem (3.1.10) (for multiple dominant singularities). The property of being amenable to singularity analysis is preserved by several basic operations such sums and products. We will briefly review the closure properties of functions amenable to singularity analysis with respect to various constructions frequently encountered in enumerative combinatorics. Let $f(z)$ and $g(z)$ be two functions amenable to singularity analysis with singular expansions

$$f(z) \sim c(1-z)^{-\alpha} \text{ and } g(z) \sim d(1-z)^{-\beta}$$

as $z \to 1$ such that $\alpha, \beta \notin \mathbb{Z}_{\leq 0}$, where without loss of generality $\alpha \geq \beta$.

- The sum $f(z) + g(z)$ is amenable to singularity analysis with expansion

$$f(z) + g(z) \sim \begin{cases} c(1-z)^{-\alpha} & \text{if } \alpha > \beta \\ (c+d)(1-z)^{-\alpha} & \text{if } \alpha = \beta \text{ and } c + d \neq 0 \end{cases}$$

- The product $f(z)g(z)$ is amenable to singularity analysis with expansion

$$f(z)g(z) \sim cd(1-z)^{-(\alpha+\beta)}.$$

We have already remarked that recursively defined structures (such as various classes of trees) often lead to functional equations defined by inversion and are especially amenable to singularity analysis (up to reasonable technical assumptions). Let y be a function implicitly defined by an equation of the form $y(z) = z\varphi(y(z))$. We impose the following conditions:

Condition (H_1) The function $\varphi(u)$ is analytic at $u = 0$ and satisfies $\varphi(0) \neq 0$, $[u^n]\varphi(u) \geq 0$ and $\varphi(u) \neq \varphi_0 + \varphi_1 u$ (consequently, the inversion problem is well-defined locally about the origin - the non-linearity condition on φ only excludes the trivial case whose solution is $y = \varphi_0 z/(1 - \varphi_1 z)$ that can easily be analysed directly).

Condition (H_2) Within the *open* disc of convergence of φ at 0, say $|u| < R$, there exists a (necessarily unique) positive solution τ of the *characteristic equation* $\varphi(\tau) - \tau\varphi'(\tau) = 0$ (e.g. a sufficient condition for existence is $\lim x\varphi'(x)/\varphi(x) > 1$ as $x \to R^-$).

The conditions imply that the radius of convergence of the solution $y(x)$ is the corresponding positive value ρ of z such that $y(\rho) = \tau$, that is $\rho = \tau/\varphi(\tau) = 1/\varphi'(\tau)$.

Let's also introduce the following definition. A function φ is called *p-periodic* if $\varphi(u) = u^r g(u^p)$ for some power series g and $p \geq 2, r \geq 0$; it is *aperiodic* otherwise.

Theorem 3.1.11 (Singular inversion, [25]). *Let φ be a nonlinear function satisfying conditions (H_1) and (H_2), and the let $y(z)$ be the (unique) solution of the functional equation $y = z\varphi(y)$. Then, the quantity $\rho = \tau/\varphi(\tau)$ is the radius of convergence of $y(z)$ at 0 (with τ the root of the characteristic equation), and the singular expansion of $y(z)$ near ρ is of the form*

$$y(z) = \tau - \sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}}\sqrt{1 - z/\rho} + \sum_{j \geq 2}(-1)^j d_j (1 - z/\rho)^{j/2},$$

where d_j are some computable constants. If we additionally assume φ is aperiodic, then the coefficient of z^n in $y(z)$ satisfies the asymptotic estimate

$$[z^n]y(z) \sim \sqrt{\frac{\varphi(\tau)}{2\pi\varphi''(\tau)}}\rho^{-n} n^{-3/2}\left(1 + \frac{1}{n}e_1 + \frac{1}{n^2}e_2 + \cdots\right),$$

where e_k is a family of computable constants.

We have introduced above the class of aperiodic generating functions. We can generalize the notion to multivariate functions as follows. A multivariate function $F(\underline{z})$ analytic in a neighbourhood of the origin that admits an expansion of the form $\sum_{\underline{n}} a_{\underline{n}} \underline{z}^{\underline{n}}$ is *periodic* if there exists integers $k_i \geq 0$ such that $\gcd \underline{k} = 1$ and $F(t^{k_1}, \ldots, t^{k_d})$ is periodic (as a univariate function); it is *aperiodic* otherwise. It can be difficult in practice to check whether a given multivariate power series is aperiodic or not, so we will provide first a simple necessary and sufficient condition.

Lemma 3.1.1 *Let $F(\underline{z})$ be analytic in a neighbourhood of the origin and admit an expansion of the form $\sum_{\underline{n}} a_{\underline{n}} \underline{z}^{\underline{n}}$, then F is aperiodic if and only if there exists a non-empty finite set J of multiindices such that $a_{\underline{j}} \neq 0$ for $\underline{j} \in J$ and*

$$\gcd\{\det[\underline{j}_1 - \underline{j}_0, \ldots, \underline{j}_d - \underline{j}_0] | \, \underline{j}_i \in J\} = 1.$$

Proof. Let us denote by I the set of all multiindices $\underline{n} \in \mathbb{N}^d$ such that $a_{\underline{n}} \neq 0$. We will prove first the contrapositive of the reverse implication, which now takes the form

$$F(\underline{z}) \text{ periodic } \Rightarrow \gcd\{\det[\underline{n}_1 - \underline{n}_0, \ldots, \underline{n}_d - \underline{n}_0] | \, \underline{n}_i \in I\} > 1.$$

Suppose therefore that $F(\underline{z})$ is periodic, then there exists a coprime vector $\underline{k} \in \mathbb{Z}^d$ and some positive integers $p \geq 2$ and $r \geq 0$ such that $F(t^{k_1}, \ldots, t^{k_d}) = t^r G(t^p)$ for some power series G. It follows that for any multiindices $\underline{n}, \underline{n}' \in \mathbb{Z}^d$ such that $a_{\underline{n}} a_{\underline{n}'} \neq 0$ we have $\underline{k} \cdot (\underline{n} - \underline{n}') \equiv 0 \mod p$. Let $\underline{n}_0, \underline{n}_1, \ldots, \underline{n}_d \in I$ be any $(d+1)$ distinct multiindices, then upon successively eliminating the variables k_i, the system of congruence equations $\underline{k} \cdot (\underline{n}_i - \underline{n}_0) \equiv 0 \mod p$ is equivalent to

$$\det[\underline{n}_1 - \underline{n}_0, \ldots, \underline{n}_d - \underline{n}_0] k_i \equiv 0 \mod p \text{ for } i = 1, \ldots, d.$$

Since we assumed $\gcd \underline{k}_i = 1$, by Bezout's identity there exists some $\underline{l} \in \mathbb{Z}^d$ such that $\underline{k} \cdot \underline{l} = 1$. Multiplying the ith equation by l_i and summing gives $\det[\underline{n}_1 - \underline{n}_0, \ldots, \underline{n}_d - \underline{n}_0] \equiv 0 \mod p$. Since the vectors $\underline{n}_i$ were arbitrary, it follows that the greatest common divisor of the set of determinants is $\geq p > 1$.

Conversely, suppose $F(\underline{z})$ is aperiodic, then it follows from the definition that for every coprime vector $\underline{k} \in \mathbb{Z}^d$ and any $\underline{n}' \in I$ we have $\gcd\{\underline{k} \cdot (\underline{n} - \underline{n}') | \underline{n} \in I\} = 1$. We will prove by induction on $d \geq 1$ that $\gcd\{\det[\underline{n}_1 - \underline{n}_0, \ldots, \underline{n}_d - \underline{n}_0] | \, \underline{n}_i \in I\} = 1$ (it follows easily that there exists a finite subset $J \subseteq I$ with the same property). The statement is trivially true for $d = 1$. Suppose it holds in dimension $d \geq 1$ and

consider any $(d+2)$ distinct multiindices $\underline{n}_0, \underline{n}_1, \ldots, \underline{n}_d, \underline{n} \in I$. Using Cauchy-Binet formula we can expand the determinant with respect to the last column as

$$\det[\underline{n}_1 - \underline{n}_0, \ldots, \underline{n}_d - \underline{n}_0, \underline{n} - \underline{n}_0] = \sum_i (-1)^{i+d} \det[\underline{n}_1^i - \underline{n}_0^i, \ldots, \underline{n}_d^i - \underline{n}_0^i](\underline{n} - \underline{n}_0)_i$$

where $\underline{m}^i = (m_1, \ldots, \hat{m}_i, \ldots, m_d)$ denotes the vector without the ith coordinate. If we choose the vector $\underline{k}$ with coordinates

$$k_i = \frac{(-1)^{i+d} \det[\underline{n}_1^i - \underline{n}_0^i, \ldots, \underline{n}_d^i - \underline{n}_0^i]}{\gcd_i\{\det[\underline{n}_1^i - \underline{n}_0^i, \ldots, \underline{n}_d^i - \underline{n}_0^i]\}}$$

then $\gcd \underline{k}_i = 1$ so by aperiodicity of $F(\underline{z})$ we deduce (keeping $\underline{n}_0, \underline{n}_1, \ldots, \underline{n}_d$ fixed)

$$\gcd\{\det[\underline{n}_1 - \underline{n}_0, \ldots, \underline{n}_d - \underline{n}_0, \underline{n} - \underline{n}_0] | \underline{n} \in I\} = \gcd_i\{\det[\underline{n}_1^i - \underline{n}_0^i, \ldots, \underline{n}_d^i - \underline{n}_0^i]\}.$$

Let D_{d+1} denote the set of $(d+1) \times (d+1)$ determinants considered, then noting that gcd is commutative we deduce

$$\begin{aligned}
\gcd D_{d+1} &= \gcd\{\gcd_i\{\det[\underline{n}_1^i - \underline{n}_0^i, \ldots, \underline{n}_d^i - \underline{n}_0^i]\} | \, \underline{n}_j \in I\} \\
&= \gcd_i\{\gcd\{\det[\underline{n}_1^i - \underline{n}_0^i, \ldots, \underline{n}_d^i - \underline{n}_0^i] | \, \underline{n}_j \in I\}\} \\
&= \gcd_i\{\gcd\{\det[\underline{m}_1^i - \underline{m}_0^i, \ldots, \underline{m}_d^i - \underline{m}_0^i] | \, \underline{m}_j \in I_i\}\},
\end{aligned}$$

where $I_i = \{\underline{n}^i | \underline{n} \in I\}$. Taking vectors $\underline{k}$ with ith coordinate zero, it follows that I_i satisfies the aperiodicity condition so by inductive hypothesis the greatest common divisor of all the $d \times d$ determinants is 1. □

We can now prove the following multivariate extension of Z. Gao and N. Wormald's lemma that will be used frequently in the sequel:

Lemma 3.1.2 (Z. Gao and N. Wormald [30]) *Let $F(\underline{z})$ have a power series expansion about the origin convergent in a closed polydisk $\overline{\mathbb{D}}(R_1) \times \cdots \overline{\mathbb{D}}(R_d)$ and with non-negative coefficients $a_{\underline{n}} \geq 0$. Suppose F is aperiodic then $|F(\underline{z})| \leq F(\underline{r})$ for every $|z_i| \leq r_i \leq R_i$ with equality if and only if $\underline{z} = \underline{r}$.*

Proof. Note that it suffices to prove the inequality on the circle $|z| = r$, since $F(\underline{r})$ is monotonically increasing in $\underline{r}$ (due to the non-negativity of the coefficients), so we necessarily have strict inequality for any interior point of the polydisk. We may assume without loss of generality (re-scaling the variables if necessary) that $r_i = 1$ and $F(\underline{r}) = 1$. Since the coefficients are non-negative, by triangle inequality for power series, the left-hand side does not exceed the right-hand side, so $|F(\underline{z})| \leq 1$ for $|\underline{z}| = 1$.

Hence we may assume equality occurs on the unit polydisk, so we must consider points of the form $z_i = e^{i\theta_i}$ for some $\theta_i \in [0, 2\pi)$. If $|F(\underline{z})| = 1$, then by absolute convergence and Cauchy's multiplication theorem we have

$$0 = 1 - |F(\underline{z})|^2 = \sum_{\underline{n} \neq \underline{m}} a_{\underline{n}} a_{\underline{m}} (1 - e^{i\underline{\theta} \cdot (\underline{n} - \underline{m})}) = \sum_{\underline{n} \neq \underline{m}} a_{\underline{n}} a_{\underline{m}} (1 - \cos(\underline{\theta} \cdot (\underline{n} - \underline{m})))$$

where $\cdot$ denotes the dot product. Since all the terms of the latter series are non-negative, equality can only occur if every term is zero. Hence for any pair of indices $(\underline{n}, \underline{m})$ such that $a_{\underline{n}} a_{\underline{m}} \neq 0$ we must have $\cos(\underline{\theta} \cdot (\underline{n} - \underline{m})) = 1$ so

$$\underline{\theta} \cdot (\underline{n} - \underline{m}) \equiv 0 \mod 2\pi.$$

Sine we assumed $F(\underline{z})$ is aperiodic there exists a non-empty finite set J of multiindices such that $a_{\underline{j}} > 0$ for $\underline{j} \in J$ and $\gcd\{\det[\underline{j}_1 - \underline{j}_0, \ldots, \underline{j}_d - \underline{j}_0] | \, \underline{j}_i \in J\} = 1$. Let $\underline{j}_0$, $\underline{j}_1 \ldots \underline{j}_d \in J$ be any $d+1$ multiindices then the congruence system $\underline{\theta} \cdot (\underline{j}_i - \underline{j}_0) \equiv 0 \mod 2\pi$ is (after successive elimination of variables) equivalent to

$$\det[\underline{j}_1 - \underline{j}_0, \ldots, \underline{j}_d - \underline{j}_0]\theta_i \equiv 0 \mod 2\pi \text{ for } i = 1, \ldots, d.$$

Applying Bezout's identity to each congruence relation we deduce that $\theta_i \equiv 0 \mod 2\pi$ for each $i = 1, \ldots, 2\pi$ which implies $z_i = 1$. □

Another important class of functions amenable to singularity analysis (that we shall encounter a lot in chapter 4) is the class of polylogarithms. Informally, polylogarithmic functions are the prototypical generating functions of sequences involving powers and logarithms, e.g. $\sqrt{n}$ or $\log n$. The precise formal definition is as follows:

Definition 11. Let $\alpha \in \mathbb{C}$ be an arbitrary complex number and $r \in \mathbb{Z}_{\geq 0}$ a non-negative integer. The *polylogarithm* $\mathrm{Li}_{\alpha,r}(z)$ is defined by the following series

$$\mathrm{Li}_{\alpha,r}(z) := \sum_{n \geq 1} (\log n)^r \frac{z^n}{n^\alpha}, \tag{3.3}$$

which is absolutely and uniformly convergent on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. It follows that polylogarithms are *a priori* analytic on $\mathbb{D}$. The name comes from the simple observation that the quantity $\mathrm{Li}_{1,0}(z)$ reduces to the usual logarithm $\log(1-z)^{-1}$. We shall follow the established notational convention and use the abbreviation $\mathrm{Li}_{\alpha,0}(z) \equiv \mathrm{Li}_\alpha(z)$. The polylogarithmic functions admit a complete analytic continuation to the entire complex plane slit along the positive part $[1, +\infty)$ of the real axis (the point 1 is in general a logarithmic branch-point of the polylogarithm), as was first established by W. B. Ford [26].

It is known that the polylogarithms are amenable to singularity analysis [22] and their singular expansion involves the *Riemann zeta function* defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re s > 1$, and by analytic continuation elsewhere. The following result is taken from P. Flajolet and R. Sedgewick [25]:

Theorem 3.1.12 (Singularity expansion of polylogarithms)**.** *Let $\alpha \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 0}$, then the polylogarithm $\mathrm{Li}_{\alpha,r}(z)$ is analytic in the slit complex plane $\mathbb{C}\backslash[1, +\infty)$ and for $\alpha \notin \{1, 2, \ldots\}$ it admits a singular expansion (with logarithmic terms for $r > 0$) of the form:*

$$\mathrm{Li}_{\alpha}(z) \sim \Gamma(1-\alpha) w^{\alpha-1} + \sum_{j \geq 0} \frac{(-1)^j}{j!} \zeta(\alpha - j) w^j, \tag{3.4}$$

where $w = -\log z = (1-z) + \frac{1}{2}(1-z)^2 + \cdots$ and $\mathrm{Li}_{\alpha,r}(z) = (-1)^r \partial_{\alpha}^r \mathrm{Li}_{\alpha}(z)$ for $r \geq 0$. Finally, if $\alpha = m \in \mathbb{Z}_{\geq 1}$ the following singularity expansion holds:

$$\mathrm{Li}_{m}(z) \sim \frac{(-1)^m}{(m-1)!} w^{m-1} (\log w - H_{m-1}) + \sum_{j \geq 0, j \neq m-1} \frac{(-1)^j}{j} \zeta(m-j) w^j, \tag{3.5}$$

where H_m is the harmonic number.

We consider next the closure properties of singularity analysis with respect to functional composition. Let $f(z)$ and $g(z)$ again be functions amenable to singularity analysis that are analytic at the origin and have non-negative coefficients. We form the composition $h(z) = f(g(z))$, assuming $g(0) = 0$. Let R_f, R_g and R_h denote the radii of convergence and introduce the notation $\tau_f = f(R_f)$ and $\tau_g = f(R_g)$. We need to distinguish between three cases depending on the value of τ_g in comparison to the value of R_f.

(i) *Supercritical case* arises when $\tau_g > R_f$. In this case, when z increases from 0 there exists a value r strictly less than ρ_g such that $g(r)$ attains the value ρ_f, which induces a singularity of the composition $f \circ g$. In other words, $r = \rho_h = g^{-1}(\rho_f)$ is the dominant singularity of the composition and around this point g is analytic, so a singular expansion of $f \circ g$ is obtained by composition of the singular expansion of f with the regular expansion of g at r. *The singularity type is that of the outer function f.*

(ii) *Subcritical case* arises when $\tau_g < R_f$. This is the dual situation of the supercritical case, the dominant singularity of the composition $f \circ g$ is R_g, so we have $R_h = R_g$. The singular expansion of $f \circ g$ is obtained by composition of the regular expansion of f at τ_g with the singular expansion of g at r_g. *The singularity type is that of the inner function g.*

(iii) *Critical case* arises when $\tau_g = R_f$. This is the boundary case and corresponds to a confluence of the singularities. We have $r_h = r_f$ and the singular expansion of $f \circ g$ is obtained by composition of the singular expansions of f and g. *The singularity type is a mix of the types of the inner and outer functions f, g.*

Finally, let us remark that the class of functions amenable to singularity analysis is also closed under differentiation, integration and Hadamard products. The general rule of thumb is as follows: derivatives and primitives of functions that are amenable to singularity analysis admit singular expansions obtained term by term via formal differentiation and integration. The analysis of Hadamard products is more involved. We refer the reader to the article by J. A. Fill, P. Flajolet and N. Kapur [21] for more details.

Before moving on let us briefly touch upon the topic of singularity analysis of multivariate functions. It is notoriously more difficult and technical to analyse the coefficients of multivariate generating functions. One of the most well understood cases addresses the multivariate generating functions whose coefficients obey a central limit theorem. The common theme to all of these is the following method: treat the sequence $\{a_{\underline{r}}\}$ as a sequence of $(d-1)$-dimensional arrays indexed by r_d and show that the nth $(d-1)$-dimensional generating function is roughly the nth power of a given generating function; use this approximation to invert the characteristic function and obtain a central limit theorem. The downside of the approach is the rather limited application and vast number of technical assumptions.

Theorem 3.1.13 (E. A. Bender, L. B. Richmond [6])**.** *Let $f(\underline{x}, z) = \sum a_n(\underline{k})\underline{x}^{\underline{k}} z^n$ be a generating function of non-negative sequence $a_n(\underline{k})$. Let R be a compact subset of $(0, \infty)^d$ and suppose there exists $\varepsilon > 0$, a rational $q \neq -1, -2, \ldots$ and functions $A(\underline{s})$ and $r(\underline{s})$ such that the following conditions hold:*

(i) $A(\underline{s})$ is continuous and non-zero for $\underline{s} \in N(R, \varepsilon)$.

(ii) $r(\underline{s})$ is non-zero and its third-partial derivatives are continuous for $\underline{s} \in N(R, \varepsilon)$.

(iii) For $\underline{s} \in N(R, \varepsilon)$ *and* $|z| < |r(\underline{s})|(1+\varepsilon)$*, the function*

$$[1 - z/r(\underline{s})]^q f(e^s, z) - [A(\underline{s})/(1 - z/r(\underline{s}))]$$

is analytic and bounded.

(iv) The Hessian matrix $B(\underline{s}) = [\partial^2 \log \lambda(e^{\underline{s}})/\partial s_i \partial s_j]_{1 \le i,j \le d}$ *with* $\lambda(\underline{s}) = 1/r(\underline{s})$ *is non-singular for all* $\underline{s} \in N(R, \varepsilon)$*.*

(v)[2] $f(e^{\underline{s}}, z)$ *is analytic and bounded for* $|z| \le |r(\Re \underline{s})|(1+\varepsilon)$ *and* $\varepsilon \le |\Im s_i| \le \pi$*.*

Then, if $\underline{k}/n = -\nabla \log r(\underline{\alpha})$ *has a solution for* $\underline{\alpha} \in R$*, we have*

$$a_n(\underline{k}) \sim \frac{n^q}{\Gamma(q+1)} e^{-\underline{\alpha}.\underline{k}} r(\underline{\alpha})^n A(\underline{\alpha}) \sqrt{(2\pi n)^d |B(\underline{\alpha})|}$$

uniformly for all such $\underline{k}$*. Furthermore, we also have a local central limit theorem*

$$a_n(\underline{j}) \sim e^{\underline{\alpha}.(\underline{k}-\underline{j})} a_n(\underline{k}) \left(\exp\left\{ -\tfrac{1}{2} \underline{u} B(\underline{\alpha})^{-1} \underline{u}^t \right\} + o(1) \right)$$

uniformly where $\underline{u} = (\underline{j} - \underline{k})/\sqrt{n}$*.*

The other set of methods used to analyze multivariate generating functions hinges on the multivariate analogue of Cauchy's integral representation and saddle point techniques. This analytic approach is most successful when applied to multivariate rational functions. However, it can also be used to analyze multivariate algebraic functions, that frequently arise in enumerative combinatorics, by the method of extracting the diagonal. It is a little known result by K. V. Safonov (2000, [58]) that every algebraic function in d variables arises as a generalized diagonal of a rational function in $d+1$ variables. The analytic methods can then be used to analyze the resulting rational function in the appropriate direction.

3.2 Enumeration of Irreducible quadrangulations

3.2.1 Substitution technique

The topic of combinatorial enumeration really began with the work of W. Tutte (e.g. see, [60]) in 1960's and the introduction of the class of rooted planar maps. The counting problem was often approached by W. Tutte using recursive decomposition of maps into simple pieces. We will thus briefly describe the language of combinatorial

[2] It was demonstrated in [6] that this condition absolutely cannot be omitted from the theorem. However, it is also this condition that is often the most difficult to prove in practice.

classes and techniques that allow us to translate a recursive decomposition into a functional relation between the generating functions. An excellent reference to the subject is the book Analytic Combinatorics [25], where most of the material of the present section is drawn from.

Let $\mathscr{C}$ be a set. A *statistic* is a function $|\cdot|_{\mathscr{C}} : \mathscr{C} \to \mathbb{N}$. A class $\mathscr{C}$ equipped with the statistic $|\cdot|_{\mathscr{C}}$ (usually called the *size*) is called a *combinatorial class* provided the subset $\mathscr{C}_n$ of all objects of size n is finite for all $n \in \mathbb{N}$. We will frequently omit the index $\mathscr{C}$ from the notation of the statistic for the sake of brevity. Two fundamental combinatorial classes are the class $\mathscr{E}$ consisting of a single object of size 0 and $\mathscr{Z}$ consisting of a single object of size 1. A *variable scheme* is a set of pairs (st_i, x_i) consisting of a statistic on $\mathscr{C}$ and an indeterminate x_i. Given a combinatorial class $\mathscr{C}$ and a variable scheme $\{(\mathrm{st}_1, x_1), \ldots, (\mathrm{st}_k, x_k)\}$ there are two types of formal power series we can define:

- **Ordinary generating function** $F_{\mathscr{C}}(t, x_1, \ldots, x_k) = \sum_{c \in \mathscr{C}} t^{|c|} x_1^{\mathrm{st}_1(c)} \cdots x_k^{\mathrm{st}_k(c)}$.
- **Exponential generating function** $F_{\mathscr{C}}(t, x_1, \ldots, x_k) = \sum_{c \in \mathscr{C}} \frac{t^{|c|}}{|c|!} x_1^{\mathrm{st}_1(c)} \cdots x_k^{\mathrm{st}_k(c)}$.

We say that the variable x_i *marks* the statistic st_i. The reason for defining two types of generating functions is that the cardinality of the class $\mathscr{C}_n$ of objects of size n can exhibit different rates of increase - in particular, when the growth is faster than exponential the ordinary generating function will only be convergent at the origin (meaning it will not define an analytic function in any neighbourhood of 0), so it is more convenient to consider instead the exponential generating function. The latter case often occurs when enumerating labelled combinatorial classes.

Given two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ there are several possible constructions that can be performed to obtain a new combinatorial class whose generating function is expressible in terms of the generating functions of the classes $\mathscr{A}$ and $\mathscr{B}$. Below we only list connections between ordinary generating functions since exponential generating functions will not feature in this work:

(i) **Disjoint union**: the disjoint union of combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is the combinatorial class $\mathscr{C}$ whose underlying set is the disjoint union of the underlying sets of $\mathscr{A}$ and $\mathscr{B}$ with size function $|c|_{\mathscr{C}} = \mathbb{1}_{\mathscr{A}}(c)|c|_{\mathscr{A}} + \mathbb{1}_{\mathscr{B}}(c)|c|_{\mathscr{B}}$. It is usually denoted by $\mathscr{A} + \mathscr{B}$ and its generating function is

$$F_{\mathscr{C}}(t) = F_{\mathscr{A}}(t) + F_{\mathscr{B}}(t).$$

(ii) **Cartesian product**: the Cartesian product of combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is the combinatorial class $\mathscr{C}$ whose underlying set is the Cartesian product of the

underlying sets of $\mathscr{A}$ and $\mathscr{B}$ with size function $|(a,b)|_{\mathscr{C}} = |a|_{\mathscr{A}} + |b|_{\mathscr{B}}$. It is usually denoted by $\mathscr{A} \cdot \mathscr{B}$ or simply $\mathscr{A}^2$ for the (Cartesian) product of class with itself (with similar notation for higher powers) and its generating function is

$$F_{\mathscr{A}\cdot\mathscr{B}}(t) = F_{\mathscr{A}}(t) \cdot F_{\mathscr{B}}(t).$$

(iii) **Sequence construction**.: the sequence construction of a combinatorial class $\mathscr{A}$ is the combinatorial class $\mathscr{C}$ whose underlying set is the set $\{(a_1, \ldots, a_k) : k \in \mathbb{N}_0,\, a_i \in \mathscr{A}\}$ of all finite sequences of elements of $\mathscr{A}$ (including the empty sequence) with size function $|(a_1, \ldots, a_k)|_{\mathscr{C}} = \sum_{i=1}^{k} |a_i|_{\mathscr{A}}$. It is usually denoted by $\mathrm{Seq}(\mathscr{A})$. Note that the sequence construction can equivalently be thought of as a shorthand for $\mathscr{C} + \sum_{i\geq 1} \mathscr{A}^i$. Its generating function is therefore given by

$$F_{\mathrm{Seq}(\mathscr{A})} = (1 - F_{\mathscr{A}})^{-1}.$$

(iv) **Pointing construction**: the pointing construction of a combinatorial class $\mathscr{A}$ is the combinatorial class $\mathscr{C}$ whose underlying set is the set $\cup_{i\geq i}\mathscr{A}_i \times \{1, 2, \ldots, i\}$ with size function $|(a,i)|_{\mathscr{A}} = |a|_{\mathscr{A}}$. Elements of this set may be thought of as objects of $\mathscr{A}$ with a distinguished building block (counted by the size statistic). It is usually denoted by $\mathscr{A}^\bullet$ and its generating function is

$$F_{\mathscr{A}^\bullet}(t) = t\frac{\partial}{\partial t} F_{\mathscr{A}}(t).$$

(v) **Substitution construction**: the substitution construction (or composition) of combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is the combinatorial class $\mathscr{C} = \sum_{k\geq 0} \mathscr{A}_k \times \mathscr{B}^k$ and is usually denoted by $\mathscr{A} \circ \mathscr{B}$. Objects of $\mathscr{A} \circ \mathscr{B}$ can be thought of as obtained from an object $a \in \mathscr{A}$ by replacing each of its distinguished building blocks (counted by the size statistic) by an arbitrary object of $\mathscr{B}$, while preserving the underlying structure of a. This interpretation implicitly assumes that the atoms (i.e. distinguished building blocks) of an object can eventually be distinguished from each other. The generating functions are related by

$$F_{\mathscr{A}\circ\mathscr{B}}(t) = F_{\mathscr{A}}(F_{\mathscr{B}}(t)).$$

Several other constructions are well-known (such as multi-set, power set and cyclic sequence constructions) and appear often in combinatorial enumeration. It should be noted that all the generating function identities above have been written only with respect to the trivial variable scheme. It is possible to extend these relations to also hold for general variable schemes $\{(\mathrm{st}_1, x_1), \ldots, (\mathrm{st}_k, x_k)\}$ provided all the statistics are additive. A statistic st defined on all the powers $\mathscr{C}^n$ of a combinatorial class $\mathscr{C}$

is called *additive* if it satisfies $\mathrm{st}(c_1, \ldots, c_k) = \sum_{i=1}^{k} \mathrm{st}(c_i)$ (note that the size statistic is additive by definition). Many naturally occurring statistics are additive, e.g. the number of faces of degree k in a planar map or the number of vertices of a planar map. However, not all statistics are additive. A prominent example of a statistic that is usually not additive is the statistic marked by a catalytic variable (e.g. the degree of the root vertex in a rooted planar map).

There are essentially three main techniques for enumerating classes of rooted planar maps. The *recursive approach* was introduced by Tutte and used to great effect to enumerate classes of connected and 2-connected rooted planar maps. Given a class $\mathscr{M}$ of rooted planar maps the idea is to describe the elements of $\mathscr{M}$ in a recursive manner that can be translated to a functional equation involving the generating function. It is often the case that the recursive decomposition of $\mathscr{M}$ does not directly lead to a functional equation, unless a second (auxiliary) combinatorial parameter is introduced. The resulting relation will then involve a catalytic variable and its solutions can be obtained using the methods described in the previous section. The recursive approach is especially useful when enumerating classes of rooted trees, since the class of trees comes naturally equipped with a recursive structure (namely, the decomposition of a tree into subtrees).

The recursive nature of tree classes is what has lead to the development of the *bijective approach*. It was noted that several complicated classes of (rooted) planar maps posses algebraic generating functions whose coefficients admit unusually simple enumeration formulas, which can be explained by putting the class in a bijective correspondence with an appropriate class of (rooted, decorated or otherwise) trees. There is in general no recipe for constructing such bijections and most known bijections today have been obtained in a rather *ad hoc* manner. The bijective approach also frequently yields an algorithmic description of the class and has proved instrumental in studying the metric properties of the maps as well as solving statistical mechanics models on maps.

The final approach to map enumeration is through *substitution*. Since this will be the technique used to enumerate most classes of maps considered in this thesis, we will describe the substitution technique in more detail. The underlying theme is the recurrent observation that a complicated planar map can frequently be decomposed (in a unique manner) into simpler maps, whose generating function is already known. On the level of generating functions such decomposition leads to a composition scheme relating the generating functions of the two classes of maps. We distinguish between two types of substitutions:

- *Edge-substitution*: given a loopless map-class $\mathscr{A}$ (which might be unrooted, pointed or rooted) and a rooted class $\mathscr{B}$, the class $\mathscr{C} = \mathscr{A} \circ_e \mathscr{B}$ is the class of graphs obtained by taking a graph $\gamma \in \mathscr{A}$, called the *core graph*, and substituting each labelled edge $e = uv$ (which is implicitly given an orientation) of γ by a graph $\gamma_e \in \mathscr{B}$, thereby identifying the origin of the root of γ_e with u and the target of the root of γ_e with v. After the identification the root edge of γ_e is deleted. The connection between generating functions in which x marks edges and y marks vertices is $C(x, y) = A(B(x, y), y)$.

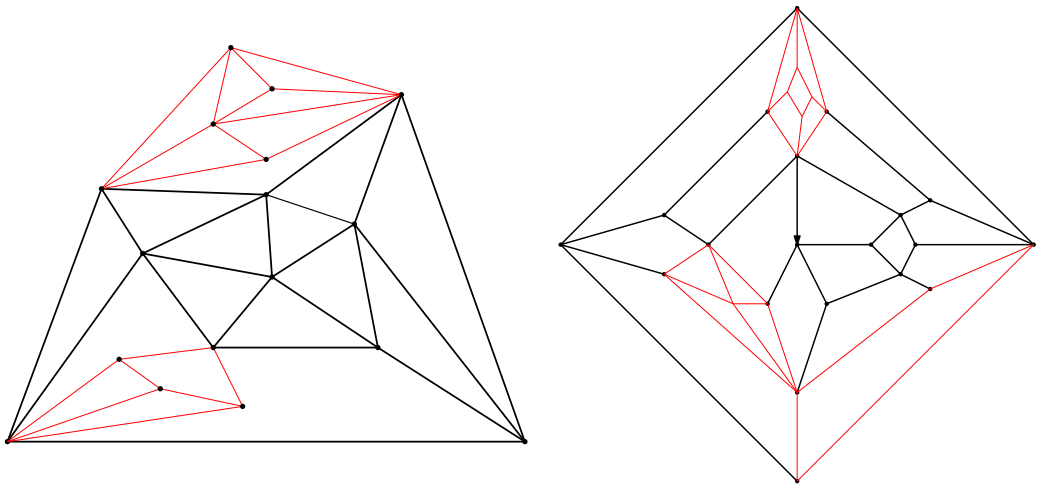


Figure 3.2: A schematic representation of edge and face substitution - the dark edges are part of the core while the red edges belong to graphs of class $\mathscr{B}$ and have been substituted in the core by replacing either an edge (on the left) or a face (on the right).

- *Face-substitution*: given a rooted loopless map-class $\mathscr{A}$ and a rooted class $\mathscr{B}$, the class $\mathscr{C} = \mathscr{A} \circ_f \mathscr{B}$ is the class of graphs obtained by taking a graph $\gamma \in \mathscr{A}$, called the *core graph*, and substituting each face f (which is implicitly given a rooting [3]) of γ by a graph $\gamma_e \in \mathscr{B}$ whose root face degree is equal to $\deg f$, thereby identifying the root face and the root of γ_e with f and the root of f.

Caveat: The composition identities of bivariate generating functions for vertex- and face-substitution hold only for labelled classes of graphs (i.e. graphs in which all the non-distinguished vertices and non-root edges carry a unique label). However, by Tutte's theorem (2.1.3) rooted planar maps are so called *asymmetric* objects. That is, any object with n vertices may be labelled in precisely $n!$ ways, using a fixed n-element set of labels. Hence it makes no difference, whether we treat planar maps as

[3]The core graph γ is rooted, so we can deterministically assign a root to each face γ. The exact assignment is not important as long as substitution in γ is always performed according to the same rule.

labelled or unlabelled. The identities will continue to hold for classes of (unlabelled) rooted planar maps as long as the decomposition of a map of class $\mathscr{C}$ into its core and maps of class $\mathscr{B}$ is unique.

We will now describe the (main) substitution approach for irreducible quadrangulations. The following lemma due to R. C. Mullin and P. J. Schellenberg [49] is crucial to the construction.

Lemma 3.2.1 (R. C. Mullin, P. J. Schellenberg [49], J. Bouttier, E. Guitter [10]) *Let* $\mathbf{q}$ *be a rooted simple quadrangulation with simple boundary embedded in the plane* $\mathbb{R}^2$ *such that the root face corresponds to the unique infinite face of the embedding. A bounded open region in* $\mathbb{R}^2$ *whose boundary is the closure of the union of four edges of* $\mathbf{q}$ *is called a quad of* $\mathbf{q}$*, and a quad is said to be maximal if it is not properly contained in any other quad of* $\mathbf{q}$*. Then two maximal quads of* $\mathbf{q}$ *are either equal or disjoint.*

Proof. Let $\mathbf{q}$ be a rooted simple quadrangulation with a simple boundary embedded in the plane such that the root face corresponds to the unique infinite face of the embedding. Let $\mathscr{C}(\mathbf{q})$ denote the set of maximal quads, i.e. those cycles of length 4 whose interior is not strictly included in the interior of another cycle of length 4. For $C \in \mathscr{C}(\mathbf{q})$, the edges and vertices of $\mathbf{q}$ contained in the interior of C form a map $\mathbf{m}_C$ which is again a quadrangulation with simple boundary of length 4 (we will assign its rooting below). Suppose two distinct cycles in $\mathscr{C}(\mathbf{q})$, say C_1 and C_2, overlap (i.e. the intersection of their interiors is non-empty). Since the interior of one cycle cannot be included in the interior of the other by maximality, there exists two vertices v and v' at the intersection of C_1 and C_2 such that one of the parts of C_1 between v and v' lies in the interior of C_2 and one of the parts of C_2 between v and v' lies in the interior of C_1. Let us denote the lengths of these parts by n_1 and n_2, respectively.

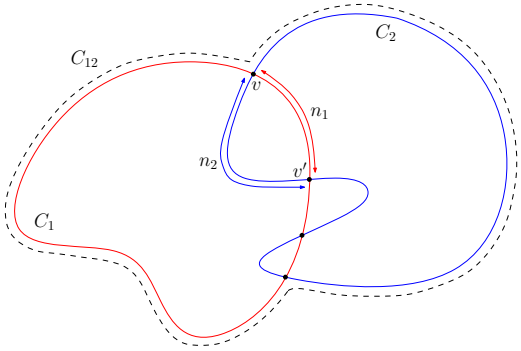


Figure 3.3: The figure is taken from [10].

Note that a simple quadrangulation $\mathbf{m}_{C_i}$ is of girth 4; hence the length of both parts must satisfy $n_i \geq 2$. Indeed, the part of the boundary of C_2 between v and v'

is an inner path in $\mathbf{m}_{C_1}$, so if we add to it the shorter boundary between v and v' in $\mathbf{m}_{C_1}$ of length say m, we obtain a closed path of length $n_2 + \min(m, 4-m)$. The closed path is not necessarily simple, but removing any double edges leaves us with a simple cycle of length at most $n_2 + \min(m, 4-m)$. However, as the map $\mathbf{m}_{C_1}$ has girth 4, we deduce that $n_2 \geq 4 - \min(m, 4-m) = \max(m, 4-m) \geq 2$.

However, we can then build a cycle C_{12} by following the outermost boundary of the union of interiors of C_1 and C_2 (note that the union is necessarily connected). This cycle has length at most $(4-n_1)+(4-n_2) \leq 4$, hence it has length 4 from the girth condition on $\mathbf{q}$. But that implies that the interiors of C_1 and C_2 lies strictly in the interior of the cycle C_{12} of length 4, contradicting the maximality of C_i. Hence maximal quads cannot overlap. □

We can now prove there exists a one-to-one correspondence between, on the one hand, the set of rooted simple quadrangulations with a simple boundary of length $2n \geq 6$ and, on the other hand, the set of pairs of the form $(\mathbf{q}, \{\mathbf{m}_f\}_{f \in F(\mathbf{q})})$ where $\mathbf{q}$ is an irreducible quadrangulation with simple boundary of length $2n$, $F(\mathbf{q})$ is the set of inner faces of $\mathbf{q}$ and for $f \in F(\mathbf{q})$, $\mathbf{m}_f$ is a rooted simple quadrangulation.

Starting from such a pair $(\mathbf{q}, \{\mathbf{m}_f\}_{f \in F(\mathbf{q})})$, we define a rooted map $\mathbf{q}'$ by "gluing" inside each face $f \in F(\mathbf{q})$ the map $\mathbf{m}_f$. More precisely, we identify each edge of $\mathbf{m}_f$ incident with the root face with an edge of $\mathbf{q}$ incident with f, starting from the root edge of $\mathbf{m}_f$ which is identified with an edge of f selected in some deterministic manner (e.g. by breadth-first search from the root of $\mathbf{q}$). Note that (by our assumption) the boundary of f and that of the root face of $\mathbf{m}_f$ are both simple, thus by identification they yield a 4-cycle of $\mathbf{q}'$. Clearly, $\mathbf{q}'$ is a simple rooted quadrangulation (with the same root as $\mathbf{q}$) of girth 4. Furthermore, by an application of the encircling lemma [10, p. 924] it is easily seen that any cycle of length 4 in $\mathbf{q}'$ necessarily remains within a single face of $\mathbf{q}$, so the boundaries of quadrilateral faces of $\mathbf{q}$ can be identified as precisely the maximal quads of $\mathbf{q}'$. As such starting from $\mathbf{q}'$ we can recover $\mathbf{q}$ by erasing all edges and vertices contained in a maximal quad. Hence the correspondence which maps $(\mathbf{q}, \{\mathbf{m}_f\}_{f \in F(\mathbf{q})})$ to $\mathbf{q}'$ is injective.

Conversely, suppose we start with an arbitrary rooted simple quadrangulation $\mathbf{q}'$ with a simple boundary of length $2n \geq 6$. Let $\mathscr{C}(\mathbf{q}')$ denote the set of all maximal quads of $\mathbf{q}'$, then for any maximal quad $C \in \mathscr{C}(\mathbf{q}')$ the edges and vertices lying within C evidently form a simple quadrangulation $\mathbf{q}_C$. Since distinct maximal quads do not overlap, we may unambiguously erase the content of each quad resulting in a rooted irreducible quadrangulation $\mathbf{q}$. Hence we may identify the set of faces $F(\mathbf{q})$ of $\mathbf{q}$ with the set of maximal quads $\mathscr{C}(\mathbf{q}')$. Finally, we need to select a root for

each quadrangulation $\mathbf{q}_C$ determined by a maximal quad. We can achieve this by selecting an edge of every face f of the irreducible quadrangulation $\mathbf{q}$ using the same procedure as above, which provides us with a canonical rooting of the associated simple quadrangulation $\mathbf{q}_C \equiv \mathbf{q}_f$. Clearly, the procedure just described is inverse to the mapping above, so the correspondence is one-to-one. We summarize our result below:

Lemma 3.2.2 *Let $n \geq 3$ be a fixed positive integer. Face substitution establishes a one-to-one correspondence between the class of rooted simple quadrangulation with a simple boundary of length $2n$ and the set of pairs of the form $(\mathbf{q}, \{\mathbf{m}_f\}_{f \in F(\mathbf{q})})$ where $\mathbf{q}$ is an irreducible quadrangulation with simple boundary of length $2n$, $F(\mathbf{q})$ is the set of inner faces of $\mathbf{q}$ and for $f \in F(\mathbf{q})$, $\mathbf{m}_f$ is a rooted simple quadrangulation.*

We have assumed so far that boundary length of the root face is > 4. Indeed, the substitution approach as described above fails to establish an analogous one-to-one correspondence between the classes of simple and irreducible quadrangulations due to the possible presence of root and "anti-root“ diagonals (below is an example of a simple quadrangulation with a diagonal at the root vertex):

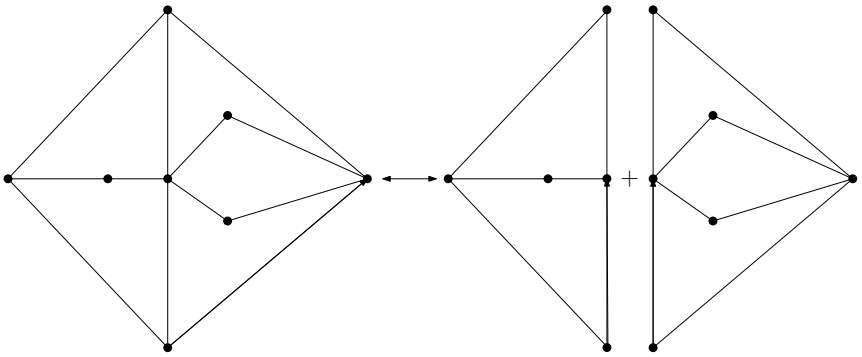
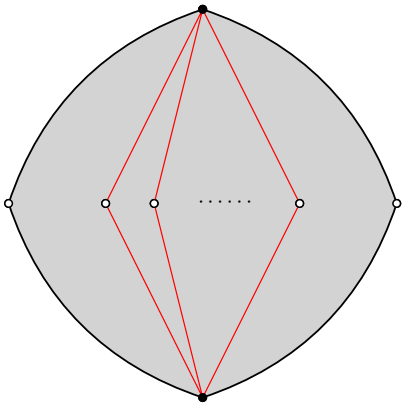


Figure 3.4: Decomposition at a diagonal.

Let us examine more closely again the face substitution approach described above. Since the root face will now likewise form a cycle of length 4 when determining the maximal quads we have to impose the restriction that the outer boundary itself is *not* considered as a cycle of length 4. As such two special cases arise that could not occur above, namely: (i) the trivial 4-cycle (considered as an irreducible quadrangulation) has a single antecedent which is again the 4-cycle (this time considered as a simple quadrangulation) and (ii) some maximal quads may now overlap, so that replacing a maximal quad with a face in a simple quadrangulation is not well-defined. Let us analyse first when two maximal quads can overlap. Examining the proof of lemma

(3.2.1), the only time we (implicitly) used the assumption that the outer boundary has length > 4 was to deduce the cycle C_{12} (in the notation of the lemma) was contained strictly inside the map. In the present case, it may happen that C_{12} exactly coincides with the outer boundary of the root face. If this is the case, since C_{12} has length 4 we must necessarily have $n_1 = n_2 = 2$ (recall that n_i was at least 2) so that vertices v and v' are antipodal in the outer boundary of the root face and the internal paths connecting them form two mutually avoiding *diagonal paths* of length 2.



Clearly, a simple quadrangulation cannot have more than one pair of antipodal vertices connected by a diagonal path. Otherwise, the two diagonal paths would cross at an inner vertex, which would create a cycle of length < 4 contradicting the assumption that the map is a simple quadrangulation (hence of girth 4). It follows that (as we have already remarked above) overlapping of maximal quads can only occur if there exist two antipodal vertices on the outer boundary of the root face that are connected by an arbitrary number $p \geq 2$ of mutually vertex-disjoint diagonal paths. Hence, following the convention of Mullin and Schellenberg, we can treat separately the two cases depending on whether the simple quadrangulation contains no diagonals between antipodal vertices of the root face or else (we include the case of a single diagonal in the second set since it allows for a more uniform treatment, although according to the presentation above a single diagonal does not give rise to overlapping maximal quads). The substitution approach works with no change for the first set of quadrangulations and the decomposition at the first diagonal (see figure (3.4)) leads to a recursion for the generating function of the second set. We leave the details to the reader (or else see [10, pp. 964–966]).

3.2.2 Known enumeration results

We will collect in the present section known enumeration results for rooted simple and irreducible quadrangulations. We will not dwell on their respective proofs, since most of the results will be side products of our theorems (or rather their proofs).

Theorem 3.2.1 (W. G. Brown, [12]). *Let $\tilde{q}(x)$ denote the generating function of rooted simple quadrangulations with a weight x per each inner face, then*

$$\tilde{q}(x) = \frac{u(1-2u)}{(1-u)^2} \text{ and } x = u(1-u)^2. \tag{3.6}$$

Explicitly, the coefficient $[x^n]\tilde{q}(x)$ giving the number of rooted simple quadrangulations with n faces admits the following closed-form expression

$$[x^n]\tilde{q}(x) = \frac{12(3n+2)!}{n!(2n+4)!} \sim 2\sqrt{\frac{3}{\pi}}\left(\frac{27}{4}\right)^{n+1} n^{-5/2}.$$

Theorem 3.2.2 (W. G. Brown, [12]). *Let $\tilde{q}(t,x)$ denote the generating function of rooted simple quadrangulations with simple boundary of arbitrary length $2p$ ($p \geq 2$) with a weight x per each inner face and a weight $\sqrt{t}$ per boundary edge, then*

$$\tilde{q}(t,x) = \frac{1-2u-(1-u)v}{(1-u)^4(1-v)^4} \text{ and } x = u(1-u)^2,\ t = v(1-v)^2(1-u)^3. \tag{3.7}$$

Explicitly, the coefficient $[t^p x^n]\tilde{q}(t,x)$ giving the number of rooted simple quadrangulations with a simple boundary of length $2p$ and n inner faces admits the following closed-form expression

$$[t^p x^n]\tilde{q}(t,x) = \frac{3(3p+4)!(3n+3p+2)!}{(2p+3)!p!n!(2n+3p+4)!} \sim \frac{(3p+4)!}{2p!(2p+3)!}\sqrt{\frac{3}{\pi}}\left(\frac{27}{8}\right)^{p}\left(\frac{27}{4}\right)^{n+1} n^{-5/2},$$

where the asymptotic relation holds for fixed p as $n \to \infty$.

Theorem 3.2.3 (W. Tutte, [60]). *Let $c(x)$ denote the generating function of rooted irreducible quadrangulations with a weight x per each inner face, then*

$$c(x) = 2x + \frac{2x^3}{1+x} - \frac{u}{(1+u)^3} \text{ and } x = u(1-u), \tag{3.8}$$

where we have adopted J. Bouttier and E. Guitter's convention in [10] whereby the quadrangulations K_4 and $K_{2,3}$ are considered irreducible (although W. Tutte did not follow this convention since the derived maps of K_4 and $K_{2,3}$ are not considered 3-connected). The coefficient $[x^n]c(x)$ does not admit a simple closed-form, we have for $n \geq 6$

$$[x^n]xc(x) = 2(-1)^n + \sum_{i=0}^{n-3} \frac{(-1)^i 2^{i-1}(i+1)(i+2)^2}{n-2}\binom{2n-1}{n-i-3} \sim \frac{2}{243\sqrt{\pi}} n^{-5/2} 4^n.$$

Remark 2 Note that in ([60]) W. Tutte does not enumerate rooted irreducible quadrangulations but rather rooted 3-connected planar maps. However, due to Tutte's bijection (2.2.2) the generating functions are the same.

Theorem 3.2.4 (R. C. Mullin, P. J. Schellenberg [49])**.** *Let $b(t,x)$ denote the bivariate generating function of rooted irreducible quadrangulations with a simple boundary with a weight x per each inner face and a weight $\sqrt{t}$ per each boundary edge, then*

$$b(t,x) = xt^2\frac{1-2u-(1-u)v}{(1-2u)(1-v)^4} \text{ and } x = \frac{u(1-2u)}{(1-u)^2},\ t = \frac{(1-u)v(1-v)^2}{u}$$

Explicitly, the coefficient $[t^p x^n]c(t,x)$ giving the number of rooted irreducible quadrangulations with a simple boundary of length $2p$ and n inner faces admits the following closed-form expression

$$[t^p x^n]b(t,x) = \frac{(3p-3)!(2n-p-1)!}{(p-3)!(2p-1)!n!(n-p+1)!} \sim \frac{(3p-3)!}{(p-3)!(2p-1)!}\frac{1}{\sqrt{\pi}}2^{-p-1}4^n n^{-5/2},$$

where the asymptotic relation holds for fixed p as $n \to \infty$.

3.3 Enumeration of irreducible quadrangulations with simple boundary

3.3.1 Generating function $Q(t,x)$

Let's recall first the definition of a combinatorial ball in a rooted quadrangulation.

Definition 12. Let $\mathbf{q}$ be a rooted quadrangulation and $r \geq 1$ a positive integer. The *combinatorial ball* $B_r(\mathbf{q})$ of radius r is the submap of $\mathbf{q}$ consisting of all the faces of $\mathbf{q}$ that contain at least one vertex at distance strictly less than r from the root vertex. With this definition there are only simple faces of type $(r-1, r, r+1, r)$ at the boundary of a combinatorial ball $B_r(\mathbf{q})$ and an r-ball will be a quadrangulation with holes. We will use the terms (combinatorial) r-ball and r-neighbourhood synonymously.

Lemma 3.3.1 *Let $\mathbf{q}$ be a quadrangulation and $r \geq 1$ a positive integer, then distinct boundary cycles of the combinatorial r-ball $B_r(\mathbf{q})$ are edge-disjoint simple cycles.*

Proof. Recall that the boundary faces of an r-neighbourhood are all r-simple meaning that with respect to the canonical distance labelling from the root the face is of type $(r, r+1, r, r-1)$. Let's draw a diagonal between the corners labelled r in each boundary face of $B_r(\mathbf{q})$. Let v be a vertex incident to such an r-diagonal (i.e. v has label r in $\mathbf{q}$ and is incident with an r-simple face), then the number of r-diagonals incident with v must be even. Hence the collection of all r-diagonals is a union of a collection of simple edge-disjoint cycles. □

As defined an r-neighbourhood is embedded in the ambient quadrangulation. However, we would like to consider r-balls as separate planar maps, so as to able to estimate the probability of a (irreducible) quadrangulation having an r-neighbourhood with the given combinatorial type. There are two distinct ways of "cutting out" the r-ball from a (irreducible) quadrangulation - one introduced by M. Krikun in [42] for general quadrangulations and the more commonly used definition used especially with other classes of planar maps (e.g. triangulations).

- Let $\mathbf{q}$ be a rooted irreducible quadrangulation with combinatorial ball $B_r(\mathbf{q})$ of radius r about the root. Draw all the diagonals between boundary vertices labelled r in $B_r(\mathbf{q})$ and cut the map along the diagonals. The resulting map $\mathbf{k}_0$ has only triangular faces at the boundary and the boundary cycles consist exactly of vertices labelled r and their diagonals. Attach a triangle at each boundary edge of $\mathbf{k}_0$ and erase the diagonals. The map $\mathbf{k}$ thus obtained is called the *r-neighbourhood* of $\mathbf{q}$ and is an irreducible quadrangulation with holes.

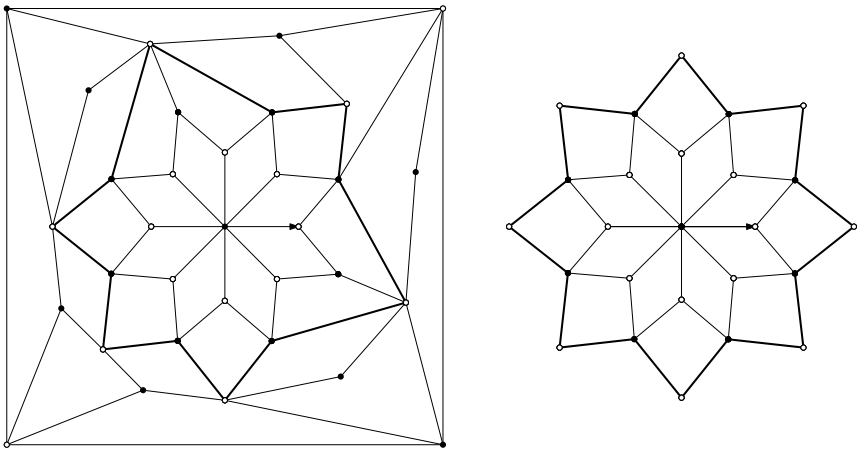


Figure 3.5: On the left the embedded 2-neighbourhood of the given quadrangulation (its boundary is shown in bold) and on the right the associated embedding of the 2-neighbourhood in the sense of Krikun.

- Let $\mathbf{q}$ be a rooted irreducible quadrangulation with combinatorial ball $B_r(\mathbf{q})$ of radius r about the root. Denote with $c_1, \ldots, c_r$ the boundary cycles of $B_r(\mathbf{q})$ consisting alternately of vertices labelled r and $r+1$. Cut the map along the cycles c_i. The resulting map $\mathbf{k}'$ is called the *embedded r-neighbourhood* and is an irreducible quadrangulation with holes.

Note that in both definitions the r-neighbourhood is a subdivision of a 3-connected planar graph, so by Whitney's theorem (2.1.2) it has an essentially unique embedding

in the 2-sphere up to orientation preserving homeomorphisms. It is important to distinguish between the two definitions - although from a probabilistic point view the difference seems to be negligible. Indeed, having an embedded r-neighbourhood of a given combinatorial type is a much more restrictive condition on a (irreducible) quadrangulation $\mathbf{q}$ than having an r-neighbourhood with a fixed combinatorial type. We will assume from now, whenever we speak of r-neighbourhoods, that they are embedded in the plane in Krikun's sense, as described in the first definition above. Should the need arise to use the second definition, we will refer to the corresponding map as the embedded r-neighbourhood.

Definition 13. An irreducible quadrangulation of the $2p$-gon is called *proper* if all the vertices of the boundary are distinct (i.e. the boundary is a simple cycle) and the boundary vertices can be bi-partitioned into two edge-disjoint subsets $b_\circ$ and $b_\bullet$ such that each vertex of $b_\bullet$ has degree 2. Although a 4-cycle C_4 formally satisfies these requirements we will not consider it a proper quadrangulation, while $K_{2,3}$ will be included in the set of proper irreducible dissections as a special convention (that simplifies the subsequent calculations). We will adopt the standard convention that a rooted proper irreducible quadrangulation with boundary is rooted on the boundary and oriented so that the outer face is on the right, with root vertex of degree 2.

Note that an r-neighbourhood in an irreducible quadrangulation with a single boundary component is exactly a proper irreducible quadrangulation with boundary of length $2p$, for some $p \geq 3$. Indeed, it is clear by definition that the result of cutting out an r-neighbourhood of an irreducible quadrangulation $\mathbf{q}$ with a single boundary component will produce an irreducible quadrangulation with proper simple boundary of length $2p$ for some positive integer p. Clearly, $p \geq 2$ otherwise the r-neighbourhood (and by extension $\mathbf{q}$) will contain multiple edges. We also cannot have $p = 2$ since that would introduce a separating 4-cycle in $\mathbf{q}$.

We will devote this section to the enumeration of rooted proper irreducible quadrangulations with arbitrary boundary length $2p$ for $p \geq 3$. Our approach will use the substitution relation, so we will first have to enumerate the class of rooted proper simple quadrangulations with arbitrary boundary length $2p$ for $p \geq 2$. Note that the class has already been enumerated by M. Krikun in [42], however it will be instructive to provide an independent proof of his result. The following theorem summarizes our result:

Theorem 3.3.1 (M. Krikun, [42])**.** *Let $\tilde{Q}(t, x)$ denote the bivariate generating function enumerating rooted proper simple quadrangulations with a simple boundary of*

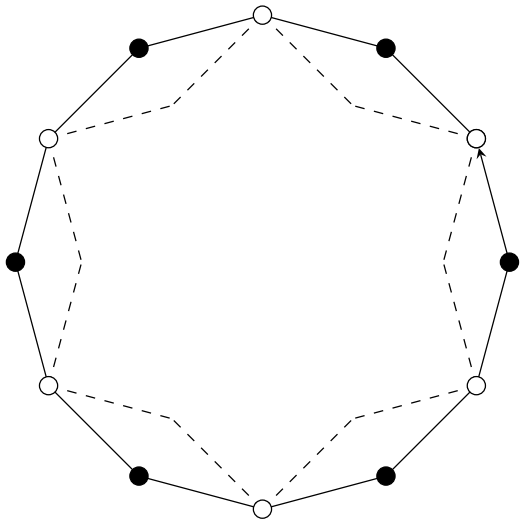


Figure 3.6: A proper irreducible quadrangulation with its bicolouration of boundary vertices.

arbitrary length $2p$ *(*$p \geq 2$*) with a weight* x *per inner face and a weight* $\sqrt{t}$ *per boundary edge, then* $\tilde{Q}(t)$ *is fully determined by the following quadratic equation:*

$$\tilde{Q}(t) = x\frac{x(1-t) + x\tilde{q} - t\tilde{Q}(t)}{(1-tx)(1-t) - tx\tilde{q} + t^2\tilde{Q}(t)}, \tag{3.9}$$

where $\tilde{q} \equiv \tilde{q}(x)$ *denotes the generating function of simple quadrangulations with a weight* x *per inner face.*

Proof. A proper simple quadrangulation with simple boundary has at least two inner faces, so we may look at the face immediately on the left of the root-edge and call v the black vertex other than the root vertex (recall that we decided for convenience to bi-colour the map in black and white, the root-vertex being black). Let us draw all the edges incident to v and cut the map along them.

After cutting, the face to the left of the root edge gets disconnected and contributes a weight tx to $t^2\tilde{Q}(t)$. The rest of the maps form a sequence of blocks. Each block is either a single face or a proper simple quadrangulation with boundary. Furthermore, the boundary of each block consists of two segments - the part that belongs to the original boundary of the map and the part that consists of the internal edges of the map. We may consider a sequence $a_0, a_1, \ldots, a_n$ $(n \geq 1)$ of numbers of (counterclockwise) successive connections of v to (white) boundary vertices. We define $a_i - 1$ to be the number of edges between two consecutive connections of v to white boundary vertices (in other words, if we label the connections of v to the white boundary vertices from 0 to n starting with the connection to the white-endpoint

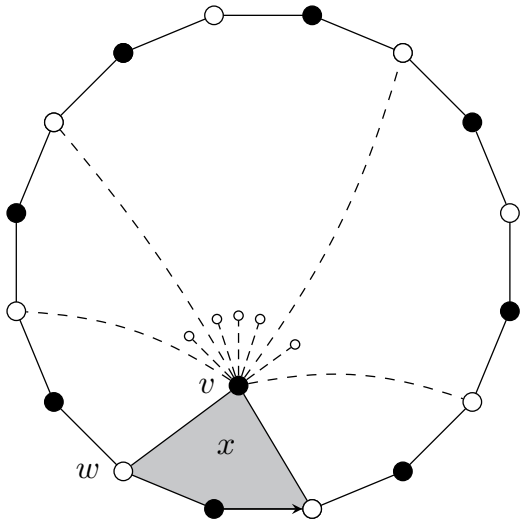


Figure 3.7: A proper irreducible quadrangulation with all the edges from v drawn. The map is decomposed by removing the face (in grey - weight x) on the left of the root edge and cutting along all the connections of the black vertex v (incident with the grey face) to the boundary vertices. In the figure the sequence of connections is $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 4, a_5 = 2, a_6 = 1, a_7 = 1$.

of the root vertex, then $a_i - 1$ is exactly the number of edges in the interior of the map to the left of the i-connection). We have $a_0 = 1$ since the first connection is from v to the white end-point of the root-edge on the boundary and there are no other connections to the left, while $a_n \geq 1$. Now the m-th block has boundary of total length $2j$ for some $j \geq 2$ with $2j - 2a_m$ edges on the original boundary of the map. Let us again denote by $\tilde{q}_{2p}(x)$ the generating function of rooted proper simple quadrangulations with a simple boundary of length $2p$ and a weight x per each inner face, then a block must be given a weight $w(a_m)$ with

$$w(a) = \delta_{a,1} xt + \sum_{j>a} \tilde{q}_{2j}(x) t^{j-a} = \delta_{a,1} tx + t^{2-a} \tilde{Q}_a(t),$$

where we have introduce the notation $\tilde{Q}_a(t)$ to denote the series on the left-hand side. The contribution to $t^2\tilde{Q}(t)$ of a sequence of n blocks (together with the face on the left of the root-edge) is therefore

$$\sum_{a_0=1} tx \prod_{m=1}^{n} (\delta_{a_m,1} tx + t^{2-a_m} \tilde{Q}_{a_m}) = tx \left\{ tx + \sum_{i\geq 1} t^{2-i} \tilde{Q}_i \right\}^n = tx \left\{ tx + \frac{t\tilde{Q}(1) - t^2\tilde{Q}}{1-t} \right\}^n.$$

Summing over $n \geq 1$ yields the contribution

$$tx \frac{tx(1-t) + t\tilde{Q}(1) - t^2\tilde{Q}(t)}{(1-tx)(1-t) - (t\tilde{Q}(1) - t^2\tilde{Q}(t))} \tag{3.10}$$

to $t^2\tilde{Q}(t)$, hence (after dividing by t^2) to the functional equation (3.9) provided we identify the term $\tilde{Q}(1)$.

One can easily convince themselves that $\tilde{Q}(1)$ is simply the generating function of rooted simple quadrangulations with a weight x per each face - in other words, $\tilde{Q}(1) = x\tilde{q}$. Indeed, we may relate $\tilde{q}$ and $\tilde{Q}(1)$ via the following argument: consider a rooted simple quadrangulation enumerated by $x\tilde{q}$. Assume the root vertex has degree $p \geq 2$ and draw all the connections from the root vertex. Cutting along these connections creates a map with a boundary of length $2p$ which is easily seen to be a proper simple quadrangulation with a weight x per each inner face. Such maps are enumerated by $\tilde{Q}(1)$ and since the correspondence is one-to-one this yields the relation $\tilde{Q}(1) = x\tilde{q}$. Upon substitution in relation (3.10) and cancellation of terms, this gives exactly the quadratic equation (3.9) for $\tilde{Q}(t)$.

We have been lucky enough to be able to recognize the term $\tilde{Q}(1)$ in terms of an already known generating function. However, we could also have determined $\tilde{Q}(1)$ directly from the relation (3.10) and obtain along the way a tractable expression for $\tilde{Q}$ using the following standard trick (this will also act as a check of our computations). We may write

$$\tilde{Q}(1) = \frac{x^2(t-1) + \tilde{Q}(t)(1-t+t^2x) + t^2\tilde{Q}^2(t)}{x + t\tilde{Q}(t)}. \tag{3.11}$$

Upon differentiating with respect to t (recall that $\tilde{Q}(1)$ does not depend on t) this yields

$$\begin{aligned} 0 = \tilde{Q}'(t)\{x(1-t+tx) + 2xt^2\tilde{Q}(t) + t^3\tilde{Q}^2\} \\ + \{x^3 + (2tx + x - 1)x\tilde{Q}(t) + (t^2x + 2tx - 1)\tilde{Q}^2(t) + t^2\tilde{Q}^3(t)\}. \end{aligned}$$

This equation is satisfied in particular if we let t vary on a line $t = t(x)$ where the two terms between brackets in the above expression vanish. Cancelling these two terms and solving for x and $\tilde{Q}$ gives the following two possible solutions

$$\begin{aligned} x &= \frac{t(x)-1}{t(x)^3}, \quad \tilde{Q}(t(x)) = \frac{(t(x)-1)^2}{t(x)^4} \\ x &= \frac{1}{t(x)(t(x)-1)}, \quad \tilde{Q}(t(x)) = \frac{1}{t(x)(t(x)-1)} \end{aligned}$$

which selects two lines $t = t(x)$ where we know the value of $\tilde{Q}$. Plugging these values in (3.11) gives

$$\tilde{Q}(1) = x(t(x)-1)(2-t(x))$$

for the first choice and $\tilde{Q}(1) = -1 - t(x)$ for the second choice. The latter result is clearly not satisfactory (recall that $\tilde{Q}(1)$ enumerates all rooted proper simple quadrangulations with boundary of arbitrary length $2p$ ($p \geq 2$) with a weight x per inner face) so we are left with the first choice. In other words, we deduce from our particular solution the parametric expression for $\tilde{Q}(1)$ (we make the substitution $t \to 1 + u$ which simplifies the parametrization):

$$\tilde{Q}(1) = xu(1-u) \text{ and } x = \frac{u}{(1+u)^3} \tag{3.12}$$

(here $u(x) = t(x) - 1$ should be viewed as a simple parametrization of x) which implicitly determine $\tilde{Q}(1)$ as a function of x. Moreover, equations (3.12) agree with the parametrization of $x\tilde{q}$ obtained by Brown (3.2.1) if we make the further substitution of variables $v = u/(1+u)$, confirming that $\tilde{Q}(1)$ does indeed count rooted simple quadrangulations. □

Lemma 3.3.2 (Fundamental relation) *Let $Q(t, x)$ denote the bivariate generating function enumerating rooted proper irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) with a weight x per inner face and a weight $\sqrt{t}$ per boundary edge, then the substitution relation translates to the following composition scheme relating $Q(t)$ and $\tilde{Q}(t)$,*

$$t^2\{\tilde{Q}(t,x) - \tilde{q}_4(x)\} = Q\left(tx\frac{1+\tilde{q}(x)}{\tilde{q}(x)}, \tilde{q}(x)\right) - t^2x^2(1+\tilde{q}(x))^2, \tag{3.13}$$

where $\tilde{q}_4(x) = [t^2]\tilde{Q}(t,x) = x^2(1+\tilde{q}(x))$.

Proof. A proper simple quadrangulation of the $2p$-gon is obtained from an irreducible one by replacing each inner face not adjacent to the boundary by a general simple quadrangulation and replacing each of the the p faces adjacent to the boundary with a simple quadrangulation with root vertex of degree 2, provided the boundary is of length $2p$ with $p \geq 3$. The correspondence is well-defined and one-to-one by lemma (3.2.1). Relation (3.13) is now an easy consequence of the substitution relation just described:

$$\sum_{n,p\geq 3} \tilde{a}_{n,p}x^n t^p = \sum_{n,p\geq 3} a_{n,p}\tilde{q}(x)^{n-p}[x(1+\tilde{q}(x))]^p t^p.$$

Note that $x(1+\tilde{q}(x))$ is exactly the generating function of rooted simple quadrangulations with root vertex of degree 2 and the expression for $\tilde{q}_4(x)$ follows from the quadratic equation (3.9) by plugging in $t = 0$. □

A combination of lemmas (3.3.1) and (3.3.2) and a simple reparametrization yields:

Theorem 3.3.2. *The bivariate generating function $Q(t,x)$ enumerating rooted proper irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) is given by*

$$Q(t) = \frac{t^2x^3}{1+x} - \frac{1}{2u(1+x)^2}\Big((1+u)^3x^2t^2 - (1+x)(1+3u+4u^2)xt + u(1+x)^2 \\ - (u(1+x) - (1+u)^2xt)\sqrt{(1+u)^2x^2t^2 - 2(1+3u)(1+x)xt + (1+x)^2}\Big)$$

It turns out that the coefficients $q_{2p}(x) = [t^p]Q(t,x)$ can be expressed in closed-from in terms of Legendre polynomials. Let us first remind ourselves how Legendre polynomials are defined and what are their main properties. We will introduce the polynomials as eigenfunctions of a certain Sturm-Liouville operator, although there are many other possible definitions.

Definition 14. The *Legendre differential equation* is the second-order ordinary differential equation

$$\mathscr{L}y = \frac{d}{dz}\left[(1-z^2)\frac{dy}{dz}\right] = -\lambda y \Leftrightarrow (1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \lambda y = 0 \tag{3.14}$$

where $\lambda \in \mathbb{C}$ is an arbitrary complex parameter. The equation is of Fuchsian type with regular singular points $\{-1, 1\}$ (note that $z = 0$ is an ordinary point).

The (unique) values $\lambda_n = n(n+1)$ with $n = 0, 1, 2, \ldots$, for which the Legendre equation admits a bounded solution, are called the *eigenvalues* of the Sturm-Liouville operator $\mathscr{L}$ and the corresponding (bounded) *eigenfunctions* are the *Legendre polynomials* $P_n(z)$. The following properties of the polynomials are well-known and can be proved using the differential equation (3.14). For each $n \geq 0$, $P_n(z)$ is a polynomial of degree n and satisfies the three-term recurrence relation

$$(n+1)P_{n+1}(z) = (2n+1)zP_n(z) - nP_{n-1}(z),$$

with initial conditions $P_0(z) = 1$ and $P_1(z) = z$. Furthermore, the generating function $G(t,z) = \sum_{n\geq 0} P_n(z)t^n$ has the following closed-form

$$G(t,z) = \frac{1}{\sqrt{1-2zt+t^2}}.$$

We can now derive the following lemma giving an explicit closed-form solution for the generating functions $q_{2p}(x)$:

Lemma 3.3.3 *Let* $u(x) = \frac{1}{2}\{1 - \sqrt{1-4x}\}$ *denote the generating function of the shifted Catalan numbers, then for* $p \geq 2$

$$q_{2(p+1)}(x) = \frac{1}{2}\left(\frac{x(1+u)}{1+x}\right)^{p+1}\left[\frac{1+u}{u}g_{p-1}\left(\frac{1+3u}{1+u}\right) - g_p\left(\frac{1+3u}{1+u}\right)\right], \tag{3.15}$$

where $g_n(z)$ *is a polynomial of degree* $n+1$ *for each* $n \geq -1$ *and satisfies the three-term recurrence relation*

$$(n+2)g_{n+1}(z) = (2n+1)zg_n(z) - (n-1)g_{n-1}(z),$$

with initial conditions $g_{-1}(z) = -1$ *and* $g_0(z) = z$*. The polynomials* $\{g_n\}$ *are related to the Legendre polynomials by the equation* $n(n+1)g_n(z) = (z^2-1)P_n'(z)$ *(where prime denotes differentiation with respect to* z*) for* $n \geq 1$*.*

Proof. The generating function of the polynomials $\{g_n\}$ introduced in the lemma by the equation $n(n+1)g_n(z) = (z^2-1)P_n'(z)$ is easily checked to have the following closed-form expression

$$F(z,t) = \sum_n g_n(z)t^n = \frac{1}{2}(1-z^2)t^2 + \frac{1}{t}\left(1 - zt - \sqrt{1-2zt+t^2}\right).$$

On the other hand, using the explicit formula for the generating function $Q(t,x)$ of rooted proper irreducible quadrangulations we can express it in terms of $F(z,t)$ as

$$Q(t) = x^2t^2 - \frac{1}{2w}(w-\tau)\tau F(2w+1,\tau), \text{ where } w = \frac{u}{1+u},\ \tau = \frac{x(1+u)}{1+x}t.$$

Note that the second term is of order $\mathcal{O}(t^3)$, so assuming $p \geq 2$ only the second term contributes to the coefficient $q_{2(p+1)}(x) = [t^{p+1}]Q(t)$. Formula (3.15) now follows by a simple calculation. □

Using the explicit formula for $q_{2p}(x)$ we can easily calculate the first few terms of the expansion. We have

$$\begin{aligned}
q_6(x) &= x^3 + x^6 + 3x^8 + 4x^9 + 15x^{10} + 39x^{11} + 120x^{12} + \mathcal{O}(x^{13}) \\
q_8(x) &= x^4 + 2x^5 + 4x^7 + 5x^8 + 14x^9 + 38x^{10} + 100x^{11} + 301x^{12} + \mathcal{O}(x^{13}) \\
q_{10}(x) &= x^5 + 5x^6 + 5x^7 + 10x^8 + 30x^9 + 61x^{10} + 185x^{11} + 500x^{12} + \mathcal{O}(x^{13}) \\
q_{12}(x) &= x^6 + 9x^7 + 21x^8 + 34x^9 + 105x^{10} + 258x^{11} + 719x^{12} + \mathcal{O}(x^{13}).
\end{aligned}$$

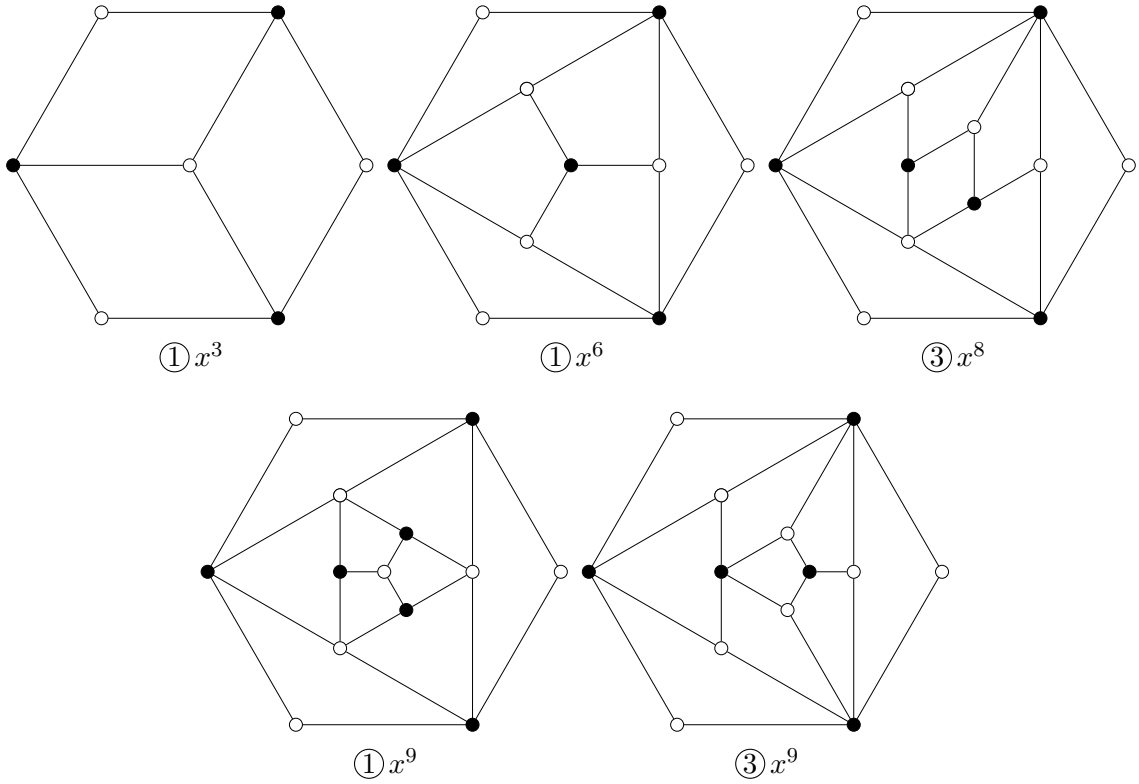


Figure 3.8: A direct inspection of the maps enumerated by q_6 and with up to 9 inner faces. To obtain all the acceptable maps, each of the maps presented here must also be rotated around its centre (and re-rooted at its bottom edge). According to the symmetry of the map at hand, the number of *distinct* maps obtained by these rotations is 1, 2 or 3 as indicated. Summing these numbers yields respectively $1, 1, 3$ and 4 distinct maps with $3, 6, 8$ and 9 inner faces, explaining the first four terms of the expansion.

3.3.2 Generating function $\Phi(t, x)$

We have defined in the previous section combinatorial r-balls in a rooted irreducible quadrangulation by a certain cutting procedure. The process can be reversed (to an extent) by the operation of connected sum as described below.

Definition 15. Let $\mathbf{k}$ be a proper irreducible quadrangulation with n faces and, for simplicity, a simple connected boundary of length $2p$ rooted at an arbitrary edge. Let $\mathbf{s}$ be any other rooted proper irreducible quadrangulation with simple connected boundary of length $2p$. The *connected sum* of $\mathbf{k}$ and $\mathbf{s}$, denoted by $\mathbf{k}\#\mathbf{s}$, is the quadrangulation obtained as follows:

1. Recall that the boundary vertices of $\mathbf{k}$ (resp. $\mathbf{s}$) can be bipartitioned into two edge-disjoint sets $b_\circ$ and $b_\bullet$ such that every black vertex is of degree 2. Construct the map $\mathbf{k}'$ (resp. $\mathbf{s}'$) by dividing each boundary quadrilateral in $\mathbf{k}$ by a diagonal

connecting the white boundary vertices and cutting the map along the diagonals to produce triangular faces at the boundary.

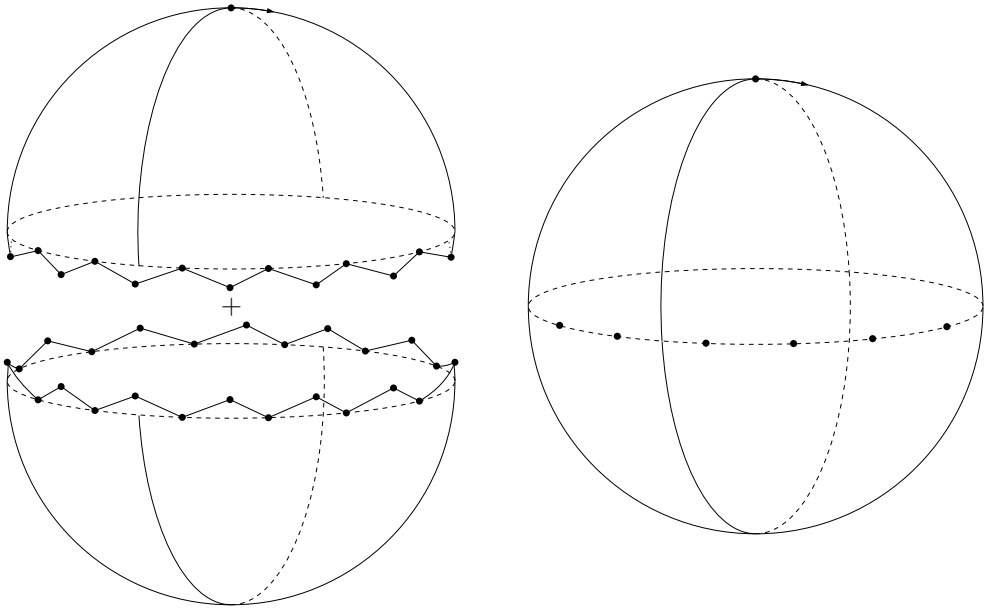


Figure 3.9: A schematic representation of the connected sum of two proper quadrangulations with simple boundary.

2. Identify the boundaries of $\mathbf{k}'$ and $\mathbf{s}'$ (each of length p)[4] and erase the identified boundary edges. The resulting map is the connected sum of $\mathbf{k}$ and $\mathbf{s}$.

The connected sum of $\mathbf{k}$ and $\mathbf{s}$ has p faces less than $\mathbf{k}$ and $\mathbf{s}$ had together, i.e. if $\mathbf{s}$ has $N - n + p$ faces then $\mathbf{k}\#\mathbf{s}$ would have N faces.

If we denote by $\mathbf{Q}_{n,2p}$ the class of all rooted proper quadrangulations with n faces and a simple connected boundary of length $2p$ and by $\mathbf{Q}_n$ the associated class of all rooted quadrangulations with n faces, then connected sum is a well-defined binary operation $\# : \mathbf{Q}_{n,2p} \times \mathbf{Q}_{N-n+p,2p} \to \mathbf{Q}_N$. Moreover, the process is clearly reversible. Indeed, take a quadrangulation $\mathbf{q}$ with a root neighbourhood $\mathbf{k}$, cut it in two along the boundary of $\mathbf{k}'$, and add p triangles to each part. This will give $\mathbf{k}$ and some quadrangulation with a simple boundary, which can then be used to reconstruct $\mathbf{q}$. Hence the restriction $\#_{\mathbf{k}} : \mathbf{Q}_{N-n+p,2p} \to \mathbf{Q}_N$ is a one-to-one mapping whose image consists precisely of all rooted quadrangulations with root neighbourhood $\mathbf{k}$.

[4]There are p distinct ways of identifying the boundaries of $\mathbf{k}'$ and $\mathbf{s}'$, but since $\mathbf{k}$ is rooted we can choose (in some deterministic way) one of the diagonals on the boundary of $\mathbf{k}'$ and require that when gluing, it is identified with the diagonal of the rooted face of $\mathbf{s}'$; this way the ambiguity is eliminated.

Unfortunately, the restriction of $\#_{\mathbf{k}}$ to the subset of all rooted proper irreducible quadrangulations of the $2p$-gon with $(N-n+p)$ faces, while still injective, no longer maps into the set of irreducible quadrangulations with root neighbourhood $\mathbf{k}$ (while the connected sum operation clearly precludes multiple edges it may introduce separating 4-cycles in the resulting quadrangulations). Fortunately separating 4-cycles of a connected sum can only occur in a very restricted manner as described below:

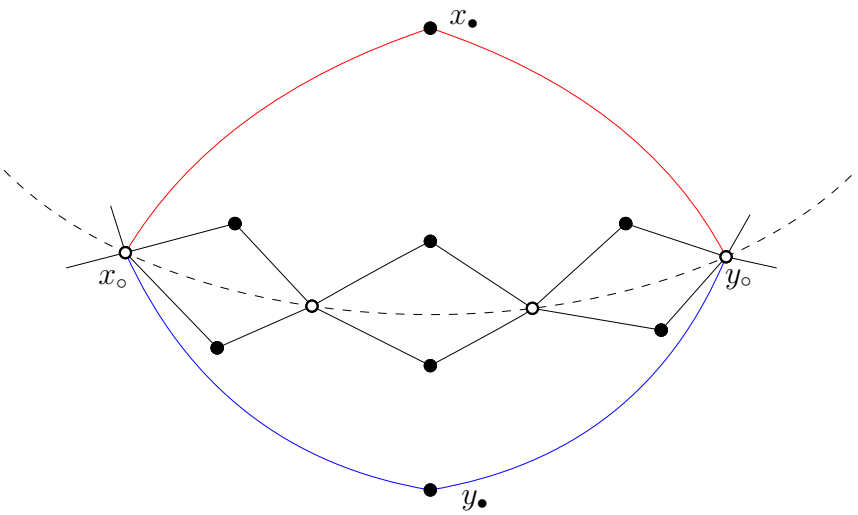


Figure 3.10: The (unique) situation described in lemma (3.3.4), which produces a separating 4-cycle in the connected sum of two proper irreducible quadrangulations.

Lemma 3.3.4 *Let* $\mathbf{k}$, $\mathbf{s}$ *be rooted proper irreducible quadrangulations of the* $2p$*-gon and denote with* $\mathbf{q} = \mathbf{k}\#\mathbf{s}$ *their connected sum. If there exists a separating* 4*-cycle in* $\mathbf{q}$ *with vertices* $x_\circ x_\bullet y_\circ y_\bullet$ *and indicated bicolouration, then* $x_\circ$ *and* $y_\circ$ *are boundary vertices of* $\mathbf{k}, \mathbf{s}$*,* $x_\bullet$ *is an inner vertex of* $\mathbf{k}$ *and* $y_\bullet$ *is an inner vertex of* $\mathbf{s}$ *(or vice-versa).*

In other words, a separating 4-cycle can occur in the connected sum of $\mathbf{k}$ and $\mathbf{s}$ if and only if there exist two white boundary vertices in $\mathbf{k}$ and $\mathbf{s}$ not adjacent to a common face (that are identified in the gluing operation) which have a common (black) adjacent vertex inside both maps- such a two-step connection between two white boundary vertices will be called a *diagonal*. Note that any two white boundary vertices adjacent to a common boundary face are trivially connected by a two-step path (by definition). We have excluded such two-step connections from the definition of diagonals and we will sometimes refer them as the *trivial diagonals*. This motivates the following definition (with the same assumptions on $\mathbf{k}$ and $\mathbf{s}$ as in lemma (3.3.4)):

Definition 16. A rooted proper quadrangulation $\mathbf{s}$ is said to be $\mathbf{k}$-*admissible* if their connected sum $\mathbf{k}\#\mathbf{s}$ is an irreducible quadrangulation. We will denote by $c_{\mathbf{k}}(N)$ the number of $\mathbf{k}$-admissible quadrangulations with N faces.

The definition implies that for each of the $c_{\mathbf{k}}(N-n+p)$ maps $\mathbf{s}$ with $N-n+p$ faces we get a different N-faced irreducible quadrangulation, and every N-faced irreducible quadrangulation with root neighbourhood $\mathbf{k}$ is obtained in this way. In other words,

$$\mathbb{P}(B(\mathbf{q}) = \mathbf{k}) = \frac{c_{\mathbf{k}}(N-n+p)}{c(N)},$$

where $c(N)$ denotes the number of all rooted irreducible quadrangulations with N faces. The number $c_{\mathbf{k}}(N)$ will strongly depend on the number and location of the diagonals of $\mathbf{k}$. We will investigate in the subsequent sections the dependence of $c_{\mathbf{k}}(N)$ on the combinatorial structure of $\mathbf{k}$.

An extremal case occurs when $\mathbf{k}$ is the so-called *quadrangular star* s_p *of order* p for some $p \geq 3$, which is defined as the unique proper irreducible quadrangulation with total boundary length $2p$ and a unique interior vertex (see figure below). Clearly, the connected sum $s_p\#\mathbf{s}$ will contain no separating 4-cycles if and only if $\mathbf{s}$ has no diagonals. The class of proper irreducible quadrangulations with no diagonals turns out to be particularly significant since a proper quadrangulation with (possibly some) diagonals can be uniquely decomposed along the diagonals into a collection of blocks with no diagonals. Hence we will enumerate first the rooted proper irreducible quadrangulations $\mathbf{q}$ with no diagonals and a simple boundary of length $2p$. More precisely, we will remove the boundary quadrilaterals from $\mathbf{q}$, which will leave us with an irreducible quadrangulation $\mathbf{q}'$ with simple boundary of length $2p$ ($p \geq 2$). The new quadrangulation $\mathbf{q}'$ is rooted at the other extremity of the root edge of $\mathbf{q}$ so that the unbounded face of the embedding appears on the right of the new root edge and we switch the bicolouration of vertices in $\mathbf{q}'$ to align with our convention of colouring the root vertex black $\bullet$. The newly created map $\mathbf{q}'$ satisfies the following two properties:

Property 3.3.1

- *Two boundary vertices cannot be linked by an edge lying strictly inside the map.*
- *Two black boundary vertices cannot have a common (white) adjacent vertex strictly inside the map (i.e. the map* $\mathbf{q}'$ *has no black diagonals).*

It is interesting to note that simple (not necessarily irreducible) quadrangulations with simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) have

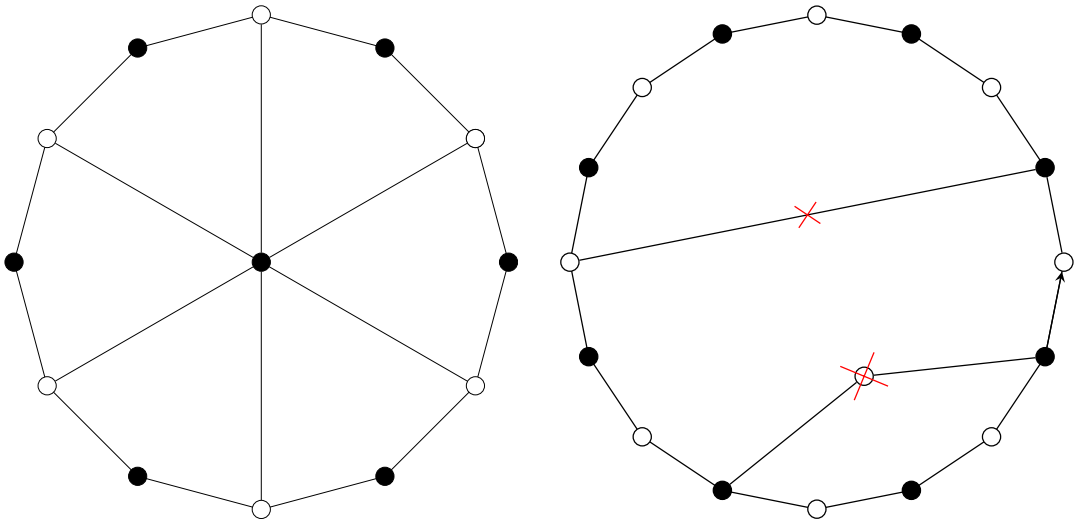


Figure 3.11: A quadrangular star and the forbidden connections in an irreducible quadrangulation $\mathbf{q}'$ obtained from a proper irreducible quadrangulation $\mathbf{q}$, with no diagonals, after removing all the boundary quadrilaterals.

already arisen in literature before in an unrelated context. We are referring here to the article by E. Guitter [32], in which he derives the distance-dependent two-point function of quadrangulations by directly solving a recursion relation for the associated slice generating function. We will closely follow his proof of counting simple quadrangulations satisfying property (3.3.1) and again use the substitution technique to derive the corresponding result for the irreducible ones.

Theorem 3.3.3 (E. Guitter, [32])**.** *Let $t^2\tilde{\Phi}(t,x)$ denote the bivariate generating function enumerating rooted simple quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) with a weight x per inner face and a weight $\sqrt{t}$ per boundary edge, then $\tilde{\Phi}(t)$ is fully determined by the following quadratic equation:*

$$\tilde{\Phi}(t) = x + \frac{x}{t}\left\{\frac{(1+t)\tilde{\Phi}(t)}{1-(1+t)\tilde{\Phi}(t)} - \frac{\tilde{h}_4}{1-\tilde{h}_4}\right\}, \tag{3.16}$$

where $\tilde{h}_4 = \tilde{\Phi}(0) = \tilde{q}/(1+\tilde{q})$.

Proof. The proof of the quadratic relation again hinges on a certain decomposition procedure obtained by looking at the face incident with the root vertex. The quadrangulation may be reduced to a single face (with a boundary of length 4), which leads to the first contribution t^2x to $t^2\tilde{\Phi}(t)$, hence (after dividing by t^2) to the first term x in (3.16). In all other cases, we may look at the face immediately to the left of the root-edge and call v and w its black and white incident vertices other than the extremities of the root-edge (once again let us remind the reader that we have

decided for convenience to bicolour the map in black and white, the root vertex being black). The (black) vertex v *cannot lie on the boundary* as otherwise, the white vertex w (which cannot lie on the boundary in this case because of the first requirement of property (3.3.1)) would be a common neighbour to v and to the root-vertex, thus violating the second requirement of property (3.3.1). Hence vertex v lies strictly inside the map. On the other hand, the (white) vertex w may either lie on the boundary, case (a), or not, case (b).

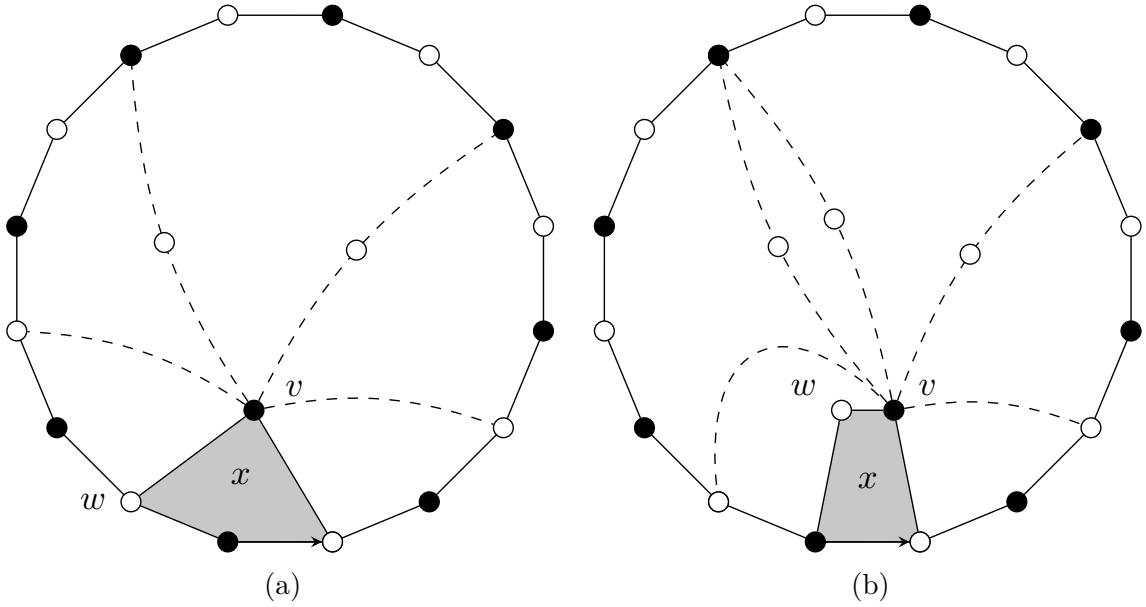


Figure 3.12: The decomposition of a map enumerated by $\tilde{\Phi}(t)$ (and not reduced to a single face). The map is decomposed by removing face (in grey - weight x) on the left of the root-edge and cutting along all the connections of the black vertex v (incident to the grey face and strictly inside the map) to the boundary vertices by either simple edges or by two-step paths.

In the latter case, w *cannot* be connected to a vertex of the boundary other than the root vertex, otherwise the second requirement of property (3.3.1) would again be violated. To the contrary, the vertex v is connected by simple edges to a number of white vertices of the boundary (including the endpoint of the root edge as well as vertex w in case (a)), and may be connected by two-step paths to black vertices of the boundary (including the root vertex in case (b)). Let us draw all these connections and cut the map along them. After cutting, the face to the left of the root edge get disconnected and contributes a weight tx to $t^2\tilde{\Phi}(t)$ in case (a) and a weight $\sqrt{t}x$ in case (b). The rest of the maps form a sequence of blocks.

We may consider the sequence $a_0, a_1, \ldots, a_n$ ($n \geq 1$) of the lengths 1 and 2 of the (counterclockwise) successive connections of v to the boundary vertices, where 1

indicates a simple edges connecting to a white vertex and 2 indicates a two-step path connecting to a black vertex. We have $a_0 = 1$ since the first connection is from v to the white end-point of the root edge, whereas $a_n = 1$ in case (a) and $a_n = 2$ in case (b). Now the m-th block has boundary of total length $2j$ for some $j \geq 2$, with $2j - a_{m-1} - a_m$ edges on the original boundary of the map. It must thus be given a weight $w(a_{m-1}, a_m)$ with

$$w(a, a') = \sum_{j \geq 2} \tilde{h}_{2j}(x)(\sqrt{t})^{2j-a-a'} = (\sqrt{t})^{4-a-a'}\tilde{\Phi}(t).$$

Considering both cases (a) and (b), the final contribution to $t^2\tilde{\Phi}(t)$ of a sequence of n blocks (together with that of the face to the left of the root edge) is therefore

$$\begin{aligned}
&\tilde{\Phi}^n(t)\left(\sum_{\substack{a_0=1,a_n=1\\ a_1,\dots,a_{n-1}\in\{1,2\}}} tx\prod_{m=1}^{n}(\sqrt{t})^{4-a_{m-1}-a_m} + \sum_{\substack{a_0=1,a_n=2\\ a_1,\dots,a_{n-1}\in\{1,2\}}} \sqrt{t}x\prod_{m=1}^{n}(\sqrt{t})^{4-a_{m-1}-a_m}\right)\\
&= tx\tilde{\Phi}^n(t)\left(\sum_{\substack{a_n=1\\ a_1,\dots,a_{n-1}\in\{1,2\}}}\prod_{m=1}^{n} t^{2-a_m} + \sum_{\substack{a_n=2\\ a_1,\dots,a_{n-1}\in\{1,2\}}}\prod_{m=1}^{n} t^{2-a_m}\right)\\
&= tx\tilde{\Phi}^n(t)\sum_{a_1,a_2,\dots,a_n\in\{1,2\}}\prod_{m=1}^{n} t^{2-a_m}\\
&= tx\tilde{\Phi}^n(t)(1+t)^n.
\end{aligned}$$

To go from the first to the second line, we simply use the fact that $a_0 = a_n$ for sequences of the first sum and $a_0 = a_n - 1$ for sequences of the second sum. Finally, summing over $n \geq 1$ yields the contribution

$$tx\frac{(1+t)\tilde{\Phi}(t)}{1-(1+t)\tilde{\Phi}(t)}$$

to $t^2\tilde{\Phi}(t)$, hence (after dividing by t^2) to the second term in (3.16).

So far in our decomposition, we have not yet enforced the condition that in $\tilde{\Phi}(t)$ the total boundary length $2i$ of maps must satisfy $i \geq 2$. In our block decomposition there is a situation in case (b) where this condition is violated. It arises when the first block has boundary length 4 with boundary vertices v, w and the two extremities of the root edge (and with exactly 1 edge on the original boundary of the whole map) followed by an arbitrary sequence of blocks of size 4 whose boundaries are made of two-step paths from v to the root vertex (these block do not contribute to the

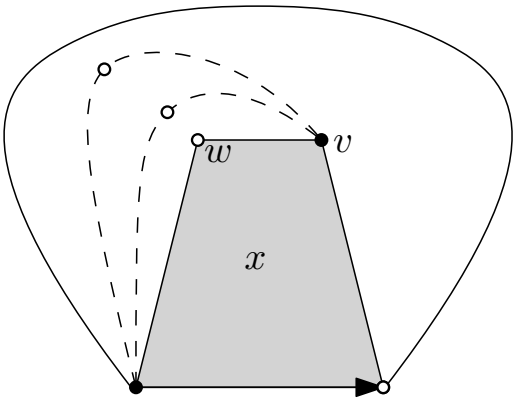


Figure 3.13: A representation of maps of total boundary length 2, which can arise in the block decomposition but do not contribute weight to the generating function.

original boundary length). These maps made of $n \geq 1$ blocks that are enumerated by $\tilde{h}_4$ contribute

$$tx \sum_{n \geq 1} \tilde{h}_4^n = tx \frac{\tilde{h}_4}{1 - \tilde{h}_4}$$

to the previous term and must be subtracted to properly recover $t^2 \tilde{\Phi}(t)$. This explains (again after dividing by t^2) the final term in (3.16).

Finally, we need to determine $\tilde{h}_4(x)$ as a function of the face weight x to be able to determine $\tilde{\Phi}(t)$. We will derive the following relation:

$$\tilde{h}_4(x) = \frac{\tilde{q}(x)}{1 + \tilde{q}(x)}. \tag{3.17}$$

Recall that $\tilde{h}_4$ is the generating function of simple quadrangulations without two-step paths from the root vertex. A map enumerated by $\tilde{h}_4$ may have no two-step connections between boundary vertices - let's denote the generating functions of these maps by G. Else there are $p \geq 1$ two-step paths connecting the white vertex incident with the root edge to its antipodal vertex on the outer face. Cutting along all the diagonal paths gives a sequence of $p + 1 \geq 2$ rooted simple quadrangulations. These latter maps cannot have any two-step paths connecting two antipodal white boundary vertices, but are otherwise arbitrary simple quadrangulations. It follows that their generating function is simply given by $\tilde{q} - (\tilde{h}_4 - G)$, so we have the following relation

$$\tilde{h}_4 - G = \frac{(\tilde{q} - (\tilde{h}_4 - G))^2}{1 - (\tilde{q} - (\tilde{h}_4 - G)} \implies \tilde{h}_4 = G + \frac{\tilde{q}^2}{1 + \tilde{q}}. \tag{3.18}$$

Finally, note that the set of simple quadrangulations is partitioned into two disjoint subsets: the set of simple quadrangulations without two-step connections between

boundary vertices (enumerated by G) and the set of simple quadrangulations with a two-step connection between boundary vertices (either the connection exists between the root vertex and its antipode or else between the white vertex adjacent to the root vertex and its antipode; the two possibilities are mutually exclusive). The latter set is enumerated by $2(\tilde{h}_4 - G)$ (the factor 2 accounts for the symmetry). Hence we can write $\tilde{q} = 2(\tilde{h}_4 - G) + G$ from which it follows that

$$G = \frac{\tilde{q}(1-\tilde{q})}{1+\tilde{q}} \text{ and } \tilde{h}_4 = \frac{\tilde{q}}{1+\tilde{q}}.$$

Finally, let us recall that we have already derived in the previous section a convenient parametrization of the generating function of rooted simple quadrangulations. Namely, $\tilde{q}(x) = u(1-u)$ and $x = u/(1+u)^3$. □

Lemma 3.3.5 (Fundamental relation) *Let $\Phi(t,x)$ denote the bivariate generating function enumerating rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) with a weight x per inner face and a weigh $\sqrt{t}$ per boundary edge, then the substitution relation translates to the following composition scheme relating $\Phi(t)$ and $\tilde{\Phi}(t)$,*

$$\tilde{\Phi}(t,x) - \tilde{h}_4(x) = \Phi(t, \tilde{q}(x)) - \tilde{q}(x). \tag{3.19}$$

Proof. A simple quadrangulation of the $2p$-gon satisfying property (3.3.1) is obtained from an irreducible one by replacing each inner face by a general simple quadrangulation, provided the boundary is of length $2p$ with $p \geq 3$. The correspondence is again well-defined and one-to-one by lemma (3.2.1) and gives exactly the fundamental relation (3.19). □

A combination of lemmas (3.3.3) and (3.3.5) and a simple reparametrization yields:

Theorem 3.3.4. *The bivariate generating function $\Phi(t,x)$ enumerating rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) is given by*

$$\Phi(t,x) = \frac{x^2}{1+x} + \frac{1}{1+t} + \frac{1}{2t(1+t)(1+u)^3}\Big\{ut^2 - (1+3u+4u^2)t - u(1+u-u^2) \\ -(t-u)\sqrt{u^2t^2 - 2(1+3u+3u^2)ut + (1+u-u^2)^2}\Big\}, \tag{3.20}$$

where the function $u = u(x)$ is determined implicitly by equation $x = u(1-u)$ and initial condition $u(0) = 0$.

Corollary 3.3.1 *The generating function $c(x)$ of rooted irreducible quadrangulations with a weight x per inner face is given by*

$$c(x) = x(1+x) + x\Phi(x,x).$$

The preceding corollary makes use of the fact that rooted irreducible quadrangulations correspond in a one-to-one fashion with the set of rooted irreducible dissections satisfying property (3.3.1), as described in the following lemma:

Lemma 3.3.6 *Let $p \geq 3$ be a positive integer and denote with $\mathbf{s}_p$ the quadrangular star of order p, then the map*

$$\mathbf{q} \mapsto \mathbf{s}_p \# \mathbf{q},$$

where $\mathbf{s}_p\#\mathbf{q}$ inherits its root from the quadrangular star, establishes a one-to-one correspondence between the set of all rooted irreducible quadrangulations $\mathbf{q}$ of the $2p$-gon satisfying property (3.3.1) and the set of all rooted irreducible quadrangulations with root vertex of degree p.

Hence $\Phi(t,x)$ is also the generating function of rooted irreducible quadrangulations with a weight x per face not adjacent to the root vertex and a weight t per quadrilateral face adjacent to the root vertex (equivalently, the exponent k of the term t^k in the expansion of $\Phi(t,x)$ is equal to the degree of the root vertex). We will derive a uniform asymptotic expansion of its coefficients with respect to the parameters (n,k). We deduce first the singularities of $\Phi(t,x)$. Apart from the possible fixed poles at $x=-1$ and $x=-2$, the only source of singularities is the square-root. The polynomial under the square-root can be written in the from

$$\sigma(t,x) = u^2t^2 - 2(1+3u+3u^2)ut + (1+u-u^2)^2,$$

where we have introduced (for brevity of notation) the auxiliary parameter $u = u(x)$ implicitly defined by the equation $u(1-u) = x$ and the initial condition $u(0) = 0$ (it is expressible in terms of radicals involving $\sqrt{1-4x}$). Let's denote by $x_0 = 1/4$ the branch-point of $u(x)$, which is to be noted is always a branch-point of $\Phi(t,x)$, and let $t_0 = 1/2$. We will investigate the singularities of minimal modulus and to that end we can prove the following lemma:

Lemma 3.3.7 (Dominant singularity structure of $\Phi(t,x)$) *Let $b > 0$ be given and let $x(\beta)$ denote a singularity of $\Phi(\beta,x)$ with least absolute value. Then $x(b)$ is unique and a positive real number. Furthermore, $|\beta| = b$ and $|x(\beta)| \leq x(b)$ implies $\beta = b$. In other words $|x(\beta)|$ has a unique minimum on the circle $|\beta| = b$ and the singularity*

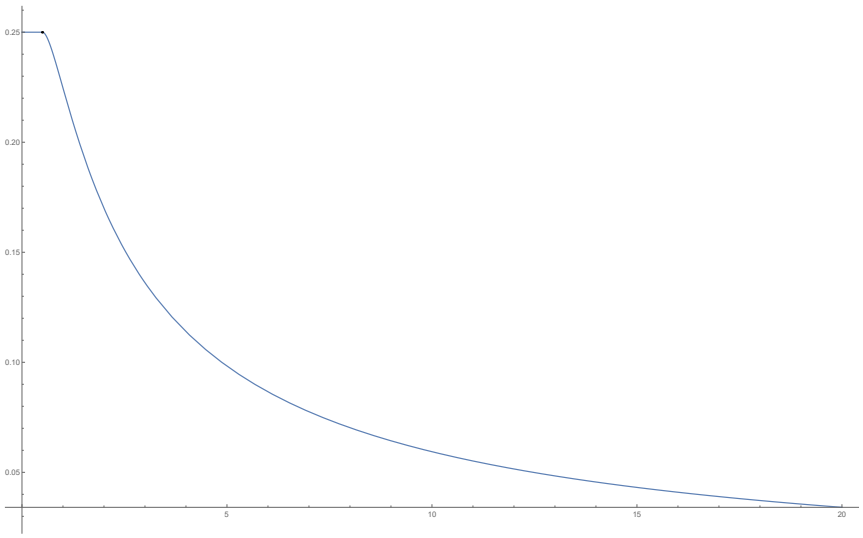


Figure 3.14: The graph of the minimum modulus function $|x|(b)$.

at the point $x(b)$ is a square-root type branch-point such that $x(b) = x_0$ on $(0, t_0]$ and the restriction $x(b) \in \mathscr{C}^3(t_0, +\infty)$.

Note that an immediate consequence of lemma (3.3.7) is the observation that for $t \in \mathbb{D}_{t_0}$ the dominant singularity of $\Phi(t, x)$ is independent of t and is achieved uniquely at the point x_0. A weaker version of this result had already been observed by M. Drmota et al. in [18], where they prove that at least for t sufficiently close to the real axis and satisfying $|t| \leq t_0$ the above conclusion is true.

The most important consequence of the lemma is the observation that for $t \geq t_0$ the function $\Phi(t, x)$ has a unique singularity of minimal modulus attained at a point of the positive real axis and the singularity is a positive real number defined by a non-constant (three-times) continuously differentiable function $x(b)$.

Proof. We start with a few preliminary remarks. Note the appearance of square-root $\sqrt{1-4x}$ in the definition of the function $\Phi(t, x)$. When talking about square-roots we will always assume that the principal branch of the square-root is being used with a branch cut along the negative real axis. It follows that the correct determination of $\sqrt{1-4x}$ will have a branch-cut along the half-closed interval $[x_0, \infty)$ of the real axis.

Using the combinatorial interpretation of the generating function $\Phi(t, x)$ it follows that $q_{nk} \geq 0$ and the coefficient of x^n is a finite-degree polynomial in t (of degree $\deg = n - 2$ to be exact), so $\Phi(b, x)$ has a singularity with $x > 0$ by Pringsheim's theorem. Let's call the smallest $x(b)$. Furthermore, conditions $q_{nk} \geq 0$ and $|\beta| = b$ necessarily imply $|x(\beta)| \geq x(b)$. It remains to show that equality can occur only when $\beta = b$ and that $x(b)$ is unique. Note that $\Phi(x, t)$ has four sources of singularities.

Namely, a simple pole at $x = -1$, a pole of degree 3 at $x = -2$, a fixed branch-point at x_0 and roots of equation $\sigma(t, x) = 0$. We can disregard the poles since they can never be minimal singularities of $\Phi(t, x)$, so we need only to consider solutions of $\sigma(\beta, x) = 0$ that satisfy $|x| \leq x_0$ (provided any exist). Equation $\sigma(t, x) = 0$ is quadratic in the variable t and can be solved to yield the following two roots

$$t_{\pm}(u) = \frac{1 + 3u + 3u^2}{u} \pm \frac{2(1+u)}{u}\sqrt{u(1+2u)}.$$

Considering only values of x for which $|x| \leq x_0$, we will identify first the root of minimal modulus. We will prove that $|t_-(x)| \leq |t_+(x)|$ for $|x| \in (0, x_0]$ with equality if and only if $x \in (-x_0, 0)$; in particular, it follows that the root $t(x)$ of least absolute value is unique for $x \in x_0\overline{\mathbb{D}}\backslash[-x_0, 0]$ and (always) satisfies $t(x) = t_-(x)$.

In order to prove the stated inequality, we introduce an auxiliary function $a = a(u)$ as follows:

$$a(u) = \frac{1 + 3u + 3u^2}{2(1+u)\sqrt{u(1+2u)}}.$$

The inequality $|t_-(x)| \leq |t_+(x)|$ can be re-arranged to $|1 - a(u)| \leq |1 + a(u)|$ since the denominator of $a(u)$ is non-zero for $x \in x_0\overline{\mathbb{D}}^*$, which is clearly equivalent to the condition $\Re a(u) \geq 0$. The map $a(u)$ has two branch-points at 0 and $-t_0$ (which correspond to the x-values x_0 and $-3x_0$, respectively) and with the principal square-root has a branch-cut along the interval $[-t_0, 0]$. On the other hand, the relation $u(1-u) = x$ implies that the image of the disc $x_0\overline{\mathbb{D}}$ is a Bernoulli leminscate with centre t_0 and focii 0 and 1, contained entirely in the right half-plane $\Re u \geq -x_0$. If we make the following substitution

$$u(z) = -\frac{1}{4} + \frac{1+2z}{4(1-2z)} = \frac{z}{1-2z}, \quad z(u) = \frac{u}{1+2u},$$

then the right-half plane $\Re u \geq -x_0$ is mapped onto the disc $t_0\overline{\mathbb{D}}$ and the mapping $a(u)$ assumes the following form (in the new variable z):

$$a(z) = -\frac{1}{2}\left\{z^{1/2} - z^{-1/2} + \frac{1}{z^{1/2} - z^{-1/2}}\right\}.$$

However, formulas $\Re(z - z^{-1}) = (r - r^{-1})\cos\theta$ and $\Re z^{-1} = |z|^{-2}\Re z$ easily imply that $\Re a(u) \geq 0$ for any $z \in t_0\overline{\mathbb{D}}$. Hence $\Re a(u)$ is non-negative in the right-half plane $\Re u \geq -x_0$ with equality if and only if $z(u) < 0$, which is equivalent to $u \in (-t_0, 0)$ (or $x \in (-x_0, 0)$)[5].

[5]Not much more effort is needed to show that the inequality $\Re a(u) \leq 0$ is in fact equivalent to $u \in \mathbb{D}(-2/3, 1/3) \cup (-t_0, 0)$.

Having identified the root minimal modulus, we will prove next the following two auxiliary inequalities for values of $x \in x_0\overline{\mathbb{D}}$,

$$\left|\frac{u(x)(1+3u(x)+3u(x)^2)}{(1+x)^2}\right| \leq \frac{u(|x|)(1+3u(|x|)+3u^2(|x|))}{(1+|x|)^2} \tag{3.21}$$

$$\left|\frac{4u^3(x)(1+u(x))^2(1+2u(x))}{(1+x)^4}\right| \leq \frac{4u(|x|)^3(1+u(|x|))^2(1+2u(|x|))}{(1+|x|)^4}. \tag{3.22}$$

Although these inequalities look formidable, we will find they follow as an immediate consequence of our assumptions from a simple result regarding the behaviour of the power series expansions of the functions involved around the origin. Let us denote the functions appearing on the left of the inequalities by $A(z)$ and $B(z)$, respectively. Then we can state the inequalities in much simpler terms merely as $|A(z)| \leq A(|z|)$ and $|B(z)| \leq B(|z|)$. Let us also recall that the function $u = u(x)$ has been defined as the unique solution of equation $u(1-u) = x$ such that $u(0) = 0$ and we have chosen the branch-cut as $[x_0, \infty)$ corresponding to the principal value of the square-root $\sqrt{1-4x}$. Finally, the coefficients of $u(z)$ are just the Catalan numbers $[z^n]u(z) = \mathrm{Cat}_n$. We will now prove the following:

Claim 1 *The power series expansions of $A(z)$ and $B(z)$ around the origin have radius of convergence x_0 and non-negative coefficients.*

The first part of the claim is clear; obviously the singularity of minimal modulus of both functions is the branch-point at $z = x_0$ (note that the zeros of the denominator are $-\varphi$ and $-\bar{\varphi}$, where φ is the golden ratio). To prove the second part of the claim it is tempting to evaluate the coefficients of the functions $A(z)$ and $B(z)$ explicitly using either Lagrange's theorem (3.1.2) or directly using Cauchy's multiplication theorem for power series (judiciously using the identity $u(1-u) = x$). However, both approaches yield an alternating sum for the coefficients, which we have been unable to express either in closed form or in terms of known functions, making it difficult to prove it is positive. We therefore employ the following trick: we introduce an auxiliary function $c(z) = u(z)/(1+z)$. The next identities are then easily checked:

$$A(z) = u(z) + c^2(z)\{1 + (4+z)u(z) + u^2(z)\}$$
$$B(z) = 4c^2(z)\{u(z) + c^2(z)(2 + (6+z)u(z) + 3u^2(z)\}.$$

The class of power series (convergent about the origin) with non-negative coefficients is clearly closed under multiplication and linear combinations with positive coefficients. Therefore, it suffices to prove that $c(z)$ with radius of convergence x_0 has non-negative

coefficients. Note that $[z^n]c(z) = \sum_{0\leq k\leq n}(-1)^{n-k}\,\mathrm{Cat}_k$ is the alternating sum of Catalan numbers. We can write

$$[z^n]c(z) = \begin{cases} 1 + \sum_{k=1}^{m}(\mathrm{Cat}_{2k} - \mathrm{Cat}_{2k-1}) & n = 2m \\ \sum_{k=0}^{m}(\mathrm{Cat}_{2k+1} - \mathrm{Cat}_{2k}) & n = 2m+1. \end{cases}$$

The identity $(k+1)\{\mathrm{Cat}_k - \mathrm{Cat}_{k-1}\} = 3(k-1)\,\mathrm{Cat}_{k-1}$, then implies the coefficients are non-negative, which completes the proof of the claim. Inequalities (3.21) and (3.22) are now an immediate consequence of the triangle inequality for power series. Finally, we would like to settle the question of equality. Note that both $u(z)$ and $c(z)$ have a dominant singularity at x_0 with singular expansion of the form $c_0 - c_1(1 - 4x)^{1/2} + o(1-4x)^{1/2}$ for some positive constants $c_0, c_1 > 0$. Hence according to the transfer theorem of Flajolet and Odlyzko their coefficients are asymptotically of order $\sim Kn^{-3/2}4^n$ for some positive constant $K > 0$, which in particular implies both power series expansions about the origin are absolutely and uniformly convergent on the closed disk $x_0\overline{\mathbb{D}}$. Hence we may apply lemma (3.1.2) to deduce $|u(z)| = u(|z|)$ if and only if $z = |z|$ and similarly for $c(z)$. Consequently, equality in (3.21) and (3.22) can only occur at $x = |x|$ for any $x \in x_0\overline{\mathbb{D}}$.

Let's introduce next the following matrix $M = M(x)$ defined by

$$M = \begin{bmatrix} A(x) & 1 \\ B(x) & A(x) \end{bmatrix} = \frac{1}{(1+x)^4}\begin{bmatrix} u(1+3u+3u^2)(1+x)^2 & (1+x)^4 \\ 4u^3(1+u)^2(1+2u) & u(1+3u+3u^2)(1+x)^2 \end{bmatrix}.$$

It is easily checked that the characteristic polynomial $p(\lambda) = \det(M - \lambda I)$ is equal to $\lambda^{-2}\sigma(\lambda^{-1}, x)$ so the eigenvalues of the matrix are just the roots $t_{\pm}^{-1}(x)$. Inequalities (3.21) and (3.22) imply $|M(x)| \leq M(|x|)$ for all $|x| \leq x_0$. Since $M(|x|)$ is a positive matrix, it follows by Perron-Frobenius theorem that the eigenvalues of the matrix $M(x)$ are dominated in absolute value by the Perron-Frobenius root of the matrix $M(|x|)$ with equality if and only if the matrices are equal (see [47, pp. 126, 5.7.5]). In other words, we must have $|t_{\pm}(x)| \geq t_{-}(|x|)$ for all $x_0\overline{\mathbb{D}}$ with equality if and only if $x = |x|$.

Finally, note that the map $u : (0, x_0) \xrightarrow{\sim} (0, t_0)$ is a one-to-one correspondence (since $u'(x) = (1-4x)^{-1/2} > 0$) and the functions $r_{\pm}(u)$ are easily shown to be positive, strictly decreasing and satisfying $t_0 < r_{-}(u) < r_{+}(u)$ on $(0, u_0)$. It follows that for $|t| \leq t_0$ the unique singularity of minimal modulus is the fixed branch-point x_0, while for $|t| \geq t_0$ the unique singularity of minimal modulus is parametrized by $(t_{-}(a), a)$ for $a \in (0, x_0)$. Furthermore, by an easy application of inverse function theorem to the function $t = t_{-}(x)$ we deduce the inverse $x = x(t)$ is (at least) three-times continuously differentiable on $(0, x_0)$. $\square$

Theorem 3.3.5 (Central limit theorem for Φ)**.** *Let a_{nk} be the number of rooted irreducible quadrangulations with root vertex of degree k and $n+k$ faces. Suppose $\varepsilon > 0$ then*

$$a_{nk} \sim \frac{1}{2\pi n^2} \frac{r(r^2-r-1)}{(r^2-2)(r+1)^2} \sqrt{\frac{(r+1)(r^2-4)(r^2-3)^3(r^2-r-1)}{(r^2-1)^2(r^5-5r^4+15r^3-25r^2+12r+6)}} \times \left[\left(\frac{r^2-2}{(r^2-r-1)^2} \right)^{\mu} \frac{(r^2-2)^2}{r^2-3} \right]^n \tag{3.23}$$

uniformly as $n, k \to \infty$ provided $\mu \in (\varepsilon, 1-\varepsilon)$, where $\mu = k/n$ and parameter $r \in (2, \infty)$ is implicitly defined by the equation

$$k/n = \frac{r(r-2)(r^2-r-1)}{(r^2-3)(r-1)^2}.$$

The maximum value of q_{nk} occurs for $k = \alpha_0 n + o(n^{1/2})$, where α_0 is the unique positive solution of equation $16x^4+16x^3-8x^2+4x-1=0$. If $k = \alpha_0 n + x n^{1/2}\beta_0$, where x is bounded and β_0 is an algebraic number with value approximately 16.0275, then

$$a_{nk} \sim \frac{(r_0+1)^{-2}}{r_0^2-2} \sqrt{\frac{r_0^2(r_0+1)(r_0^2-4)(r_0^2-3)^3(r_0^2-r_0-1)^3}{(r_0^2-1)^2(r_0^5-5r_0^4+15r_0^3-25r_0^2+12r_0+6)}} \left(\frac{(r_0^2-2)^2}{r_0^2-3} \right)^n \frac{e^{-x^2/2}}{2\pi n^2},$$

where r_0 is the unique solution of $x^4-2x^3-2x^2+2x+3=0$ in the interval $(2,\infty)$.

Proof. We will obtain first a more explicit parametrization of the dominant singularity. Due to the presence of the square-root $\sqrt{1-4x}$ it is difficult to handle equation $\sigma(t,x)=0$ directly. Let's eliminate first the square-root to obtain a polynomial equation involving only t and x. The result of elimination of parameter u between equations $\sigma(t,x)=0$ and $x=u(1-u)$ (in other words, the resultant Res_u of the polynomials) gives a polynomial equation $F(t,x)=0$, where $F(t,x) \in \mathbb{Q}[\![t,x]\!]$ is the quartic polynomial

$$x^2t^4 - 2x(1+9x)t^3 + (1+28x+45x^2+34x^3)t^2 - 2(7-x-23x^2-15x^3)t + (1+x)^4.$$

The polynomial $F(t,x)$ is irreducible over $\mathbb{C}[\![t,x]\!]$. However, note that in the field extension $\mathbb{Q}[\![t,u]\!]$ it is reducible and factors as $F(t,u) = \sigma(t,u)\sigma(t,\bar{u})$ where $\bar{u} = 1-u$ denotes the conjugate root. Naturally, we must keep in mind that the affine variety $\mathscr{V}(F)$ properly contains the zero set $\mathscr{V}(\sigma)$. The (affine) algebraic curve $C = \mathscr{V}(F)$ is of genus 0 and can therefore be parametrized by rational functions. The following pair is easily checked to constitute a proper rational parametrization of C:

$$t(p) = \frac{(p-\varphi)^2(p-\bar{\varphi})^2}{(p-\sqrt{2})(p+\sqrt{2})},\ x(p) = \frac{(p-\sqrt{3})(p+\sqrt{3})}{(p-\sqrt{2})^2(p+\sqrt{2})^2}, \tag{3.24}$$

where $\varphi = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio and $\bar{\varphi} = \frac{1}{2}(1-\sqrt{5})$ is the conjugate golden ratio. The curve C is singular and its singularity set is easily determined to be

$$\mathrm{Sing}(C) = \left\{(0,-1), (-1,-2), \left(\tfrac{25}{2}, \tfrac{1}{4}\right), \left(\tfrac{4}{3}(5 \pm 2\sqrt{7}), \tfrac{1}{9}(11 \mp 4\sqrt{7})\right)\right\}$$

along with two points at infinity (given by $Q_0 = [0:1:0]$ and $Q_1 = [1:0:0]$ in projective coordinates). If we denote the singular points by P_0, P_1, P_1 and $P_\pm$, then P_1 and and $P_\pm$ are ordinary double points, P_2 is a cusp and P_0, Q_i are tacnodes. The nec-

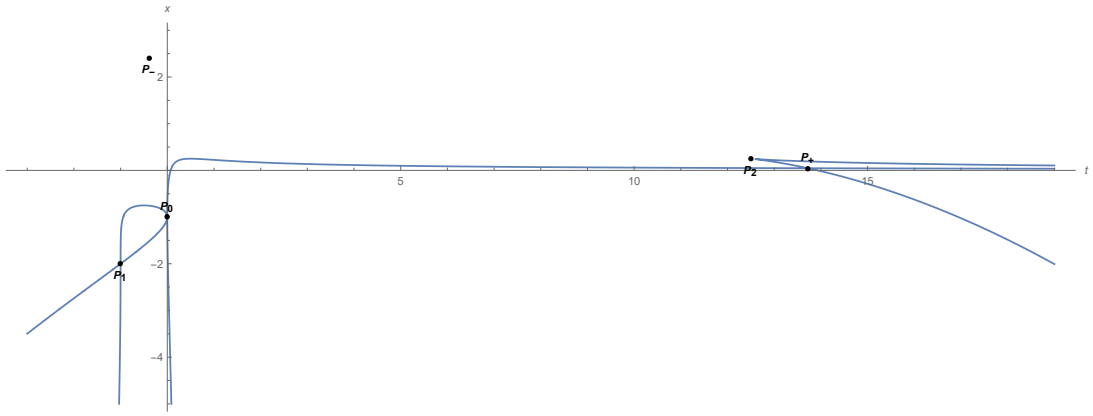


Figure 3.15: The affine curve $C = \mathscr{V}(F)$ with marked singular points. Note that the branch of the curve through P_- is complex.

essary and sufficient condition for a point $(t(p), x(p))$ to belong to the intersection of the zero sets $\mathscr{V}(\sigma) \cap \mathscr{V}(F)$ is (let's also write $p = \xi + i\eta$):

$$\Re\left(\frac{p^2-4}{p^2-2}\right) \geq 0 \;\Leftrightarrow\; \eta^4 + 2(\xi^2+3)\eta^2 + (\xi^2-2)(\xi^2-4) \geq 0. \tag{3.25}$$

Let's denote with $\mathscr{R}$ the region determined by inequality (3.25), see figure. Naturally, the parametrization (3.24) describes all four roots of equation $F(t,x) = 0$ (away from the singular points). The possible regions of univalence on which parameter p properly describes a single branch are delimited by the following (singular) elliptic curve E:

$$(x-1)y^4 + (2x^3 - 2x^2 + 4x - 3)y^2 + (x-1)^2(x+2)(x^2-x-1) = 0. \tag{3.26}$$

The plane curve E has two real branches, which separate the complex plane into four regions $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$ and $\mathscr{D}_4$ such that the restriction $t_i(p) : \mathscr{D}_i \to \mathbb{C}$ is one-to-one (and a biholomorphism onto an appropriate slit plane) for each i. Consequently, each restriction $x_i(p) : \mathscr{D}_i \to \mathbb{C}$ gives a branch of the algebraic function defined by the equation $F(t,x) = 0$ (corresponding to the value $t = t_i(p)$). The new parametrization of the unique (positive) minimal solution is then $(t(r), x(r))$ for $r \in (2, \infty)$ (it can

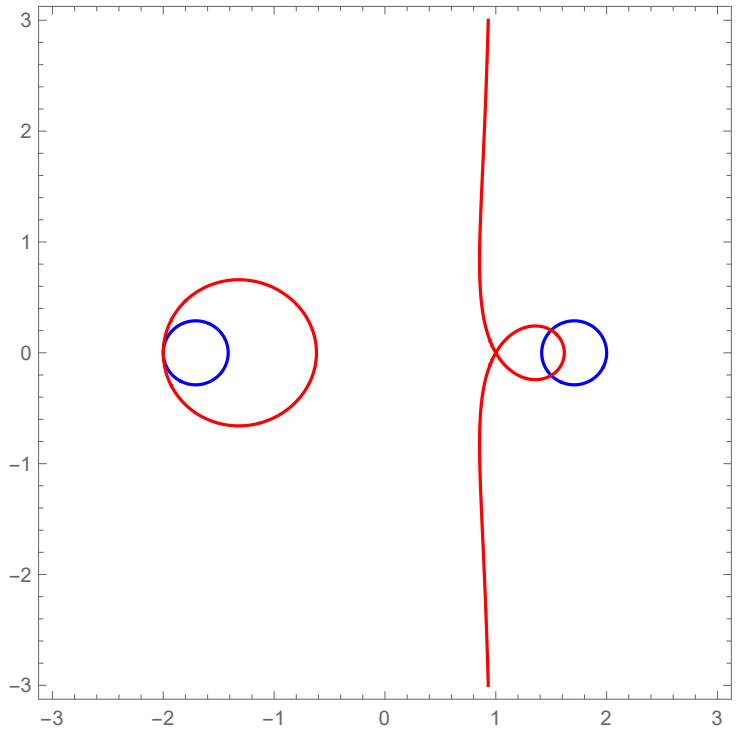


Figure 3.16: The region $\mathscr{R}$ determined by inequality (3.25) is the exterior of the blue curves, while the elliptic curve E delimiting the regions of univalence is shown in red.

easily be checked that r must exactly be restricted to the given interval in order to recover the solution $t_-(x)$ found above for $x \in (0, x_0)$). It follows that

$$\mu = k/n = -\frac{d\log x}{d\log t} = -\frac{t}{x}\frac{dx}{dt} \text{ and } \sigma^2 = -\frac{d^2\log x}{(d\log t)^2} \tag{3.27}$$

and the derivatives are evaluated on $(t(r), x(r))$ for $r \in (2, \infty)$. We can easily evaluate all the derivative involved in terms of the auxiliary parameter r, to yield the following parametrization of μ and σ:

$$\mu = \frac{r(r-2)(r^2-r-1)}{(r^2-3)(r-1)^2} \text{ and } \sigma^2 = \frac{t(r)(r^5-5r^4+15r^3-25r^2+12r+6)}{2x(r)(r+2)(r^2-3)(r-1)^5}.$$

It is easily checked that $\sigma(r) \neq 0$ for $r \in (2, \infty)$ and $\mu(r) = \mu$ has a unique (positive) solution $r \in (2, \infty)$ for every $\mu \in (0, 1)$ (the ratio k/n is bounded above by 1 by A. Kolpakov's theorem ([40, Proposition 1])). It follows from theorem (3.1.13) that $a_{nk}t^k / \sum a_{ni}t^i \sim 1/\sqrt{2\pi n\sigma^2}$ evaluated at the point $t = t(\mu)$, so we obtain

$$a_{nk} \sim \sqrt{\frac{(r-1)^5(r+2)(r^2-3)^2}{\pi(r^5-5r^4+15r^3-25r^2+12r+6)}} \frac{n^{-1/2}(r^2-2)^{k-1}}{(r^2-r-1)^{2k+1}} \sum a_{ni}t(r)^i \tag{3.28}$$

where $\mu = k/n$, r is implicitly defined by the equation above and k, n go to infinity in such a way that $t(\mu)$ stays away from 0 and ∞, i.e. μ stays away from 0 and 1. We use

Darboux's theorem to estimate the final sum. Setting $t = t(r)$ we study $\Phi(t, x)$ as $x \to x(r)$. Apart from some linear terms the main contribution to the dominant singular term comes from the square-root, which gives

$$\Phi(t(r), x) \sim -\frac{r(r^2-2)^2}{(r-1)} \sqrt{\frac{(r^2-3)(r-2)}{(r^2-r-1)^3(r^2-1)^5}} \sqrt{1 - x/x(r)}$$

as $x \to x(r)$. Applying Darboux's theorem gives

$$\sum a_{ni} t(r)^i \sim -t(r)^2 \frac{r(r^2-2)^2}{(r-1)} \sqrt{\frac{(r^2-3)(r-2)}{(r^2-r-1)^3(r^2-1)^5}} \frac{n^{-3/2}(r^2-2)^{2n}}{\Gamma(-1/2)(r^2-3)^n}.$$

Recall that $\Gamma(-1/2) = -2\sqrt{\pi}$. Substituting into formula (3.28) and simplifying gives the result. The final part of the theorem follows from the observation that the maximum of a_{nk} occurs at that μ for which $t(\mu) = 1$ and that the variation near the mean is given by a normal curve. □

According to the lemma on minimal singularities of $t^2\Phi(t, x)$ the critical value of the parameter t is $t_0 = 1/2$ and for $|t| < t_0$ the minimal singularity of the generating function $t^2\Phi(t)$ is fixed and equal to $x_0 = 1/4$. Using say `Mathematica`, we can easily deduce the singular expansion about the point x_0 to be of the form

$$t^2\Phi(t, x) = \alpha_0(t) + \alpha_1(t)(1 - 4x) + \alpha_3(t)(1 - 4x)^{3/2} + \mathcal{O}((1 - 4x)^2),$$

where the first few coefficients are given in terms of the auxiliary function $w(t) = (2t - 1)(2t - 25)$ as follows

$$\begin{aligned}
\alpha_0(t) &= t\frac{67t^2 + 287t - 50 - 10(2t-1)\sqrt{w(t)}}{540(1+t)} \\
\alpha_1(t) &= t\frac{200(2t^2 - 17t - 10) - (929t - 400)\sqrt{w(t)}}{8100\sqrt{w(t)}} \\
\alpha_3(t) &= 4t\frac{16t^3 - 132t^2 - 96t + 25 - (4t-5)(2t-1)\sqrt{w(t)}}{729(2t-1)\sqrt{w(t)}}.
\end{aligned} \tag{3.29}$$

The dominant singular term is of order $3/2$ and the minimal singularity of the coefficient $\alpha_3(t)$ clearly occurs at $t_0 = 1/2$. Again using `Mathematica` we deduce its singular expansion about t_0 as

$$\alpha_3(t) = \frac{2}{243} + \frac{1}{81}\sqrt{6}(1 - 2t)^{-3/2} - \frac{145}{3888}\sqrt{6}(1 - 2t)^{-1/2} + \mathcal{O}(\sqrt{1 - 2t}).$$

Let us denote $a_p = [t^p]\alpha_3(t)$, then applying the transfer theorem to $[t^p]t^2\Phi(t,x)$ for a fixed value of $p \geq 3$, we deduce that the coefficients $[x^n t^{p-2}]\Phi(t,x)$ admit the following asymptotic expansion:

$$[x^n t^{p-2}]\Phi(t,x) = \frac{3}{4\sqrt{\pi}} a_p n^{-5/2} 4^n \left(1 + \mathcal{O}(n^{-1/2})\right) \text{ as } n \to \infty \tag{3.30}$$

$$a_p = \frac{2\sqrt{6}}{81\sqrt{\pi}} p^{1/2} 2^p \left(1 + \mathcal{O}(p^{-1})\right) \text{ as } p \to \infty$$

3.3.3 Irreducible cores

We will describe a relation between the generating function $\tilde{Q}(t,x)$, enumerating proper simple quadrangulations with a weight x per inner face and a weight $\sqrt{t}$ per boundary edge, and the generating function $\tilde{\Phi}(t,x)$, enumerating rooted irreducible quadrangulations satisfying property (3.3.1) with a weight x per inner face and a weight $\sqrt{t}$ per boundary edge. We will associate to each proper simple quadrangulation $\mathbf{q}$ a rooted plane tree equipped with a bicolouration of its vertices rooted at a black leaf $\bullet$ and having only black leaves - such a tree will be called *balanced*.

Starting from a rooted proper simple quadrangulation $\mathbf{q}$ with a root vertex v_0, and where the root vertex is considered white $\circ$ (so we take the conjugate bicolouration of the conventional one), one obtains a balanced tree $\mathbf{t}$ as follows:

- A two-step path between two black vertices of the boundary is called a *diagonal*. Remove any interior edge of $\mathbf{q}$ that is not part of a diagonal. Let $\mathbf{c}$ denote the remaining map - called the *core* of the quadrangulation $\mathbf{q}$. Note that the core of any proper simple quadrangulation is a proper bipartite map with exterior faces of degree $2p$ equipped with a canonical bicolouration of its vertices (which it inherits from $\mathbf{q}$) such that every interior vertex is white $\circ$ (property 0).
- Let f be an inner face of $\mathbf{c}$; a black vertex v_f is placed in the interior of the face f and an edge is added inside the face between the newly created vertex v_f and each interior white vertex on the boundary of f, in such a way that the various edges do not intersect (property 1).
- The edges and vertices of $\mathbf{q}$ are deleted. The resulting map is a (bipartite) tree $\mathbf{t}$ with bicoloured vertices and monochromatic leaves (property 2).

The resulting tree $\mathbf{t}$ is rooted at the black vertex corresponding to the quadrangular face adjacent to the root edge of $\mathbf{q}$. The following three properties validate the construction of $\mathbf{t}$ (the proof are easy, so we omit them):

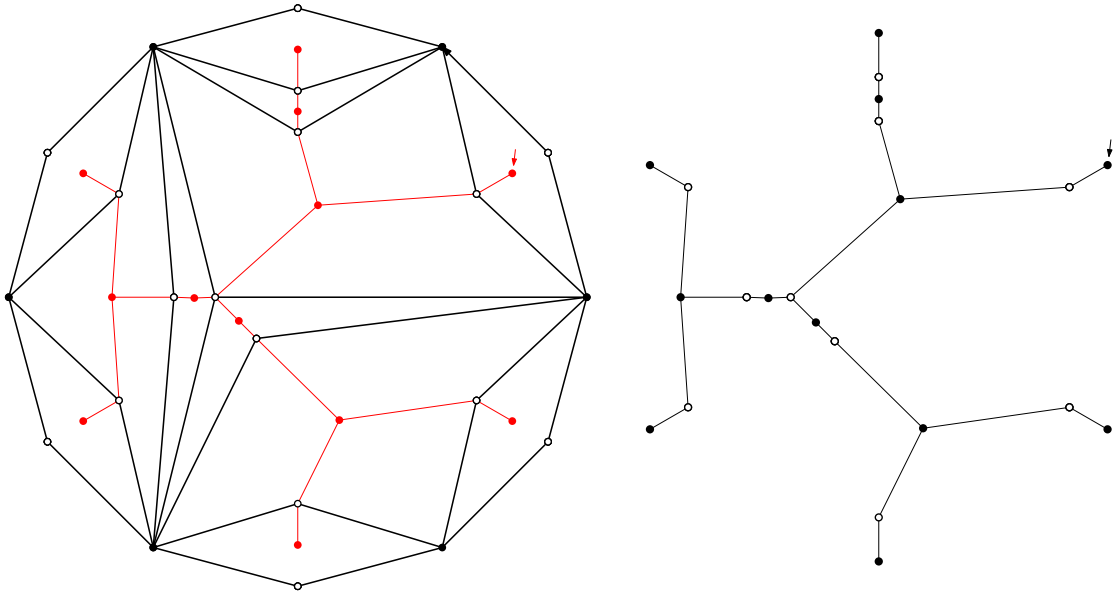


Figure 3.17: The construction of a balanced tree from an irreducible core.

Lemma 3.3.8 (Property 0) *There is a one-to-one correspondence between cores of rooted proper simple quadrangulations with (simple) boundary and rooted proper bipartite maps with simple boundary and monochromatic interior vertices.*

Lemma 3.3.9 (Property 1) *The edges $v_f v_\circ$, where $v_\circ \in \partial f$ do not intersect.*

Lemma 3.3.10 (Property 2) $\mathbf{t}$ *is a bicoloured tree with monochromatic leaves.*

Conversely, suppose $\mathbf{t}$ is a bicoloured tree with black leaves that is rooted as the leaf v_0. We will describe how to construct the core $\mathbf{c}$ from the tree $\mathbf{t}$. Denote with c_{2p} a simple cycle of length $2p$ with vertices alternatively coloured black $\bullet$ and white $\circ$.

- Append an additional white vertex to each black leaf of $\mathbf{t}$ and call the resulting tree, the *augmented tree* $\mathbf{t}_0$. Re-root $\mathbf{t}_0$ at the white vertex appended to the root vertex of $\mathbf{t}$.
- Place the augmented tree $\mathbf{t}_0$ in the interior of the cycle c_{2p} and identify its leaves with the white boundary vertices of the cycle (preserving the root vertex). Denote the resulting map by $\mathbf{q}_0$. Note that a simple counting argument establishes that $\mathbf{q}_0$ is a dissection of the $2p$-gon with p interior faces such that every black boundary vertex $v_\bullet$ belongs to the boundary of a unique interior face $f_\bullet$.
- An edge is added inside the face $f_\bullet$ between the boundary vertex $v_\bullet$ and each white interior vertex of $\mathbf{q}_0$ (in other words, each white non-leaf vertex of $\mathbf{t}$) on the boundary of $f_\bullet$ in such a way that the various edges do no intersect.

- The edges and black vertices of $\mathbf{t}$ are deleted. The resulting map is a bipartite dissection of the $2p$-gon with monochromatic interior vertices.

The two constructions are clearly inverse. Therefore, we have established a one-to-one correspondence between rooted balanced trees and cores of rooted proper irreducible quadrangulations with boundary. We summarize the results here:

Theorem 3.3.6. *The construction gives a bijection between rooted cores and rooted balanced trees with the following properties:*

1. *The length of the boundary of* $\mathbf{c}$ *equals twice the number of leaves of* $\mathbf{t}$.
2. *The number of interior edges of* $\mathbf{c}$ *equals the number of edges of* $\mathbf{t}$.
3. *The number of faces of* $\mathbf{c}$ *equals the number of black vertices of* $\mathbf{t}$.
4. *The degree of a boundary non-adjacent face* f *of* $\mathbf{c}$ *equals twice the degree of the black vertex* v_f *of* $\mathbf{t}$.

A rooted proper simple quadrangulation with boundary is uniquely determined by its core and, for each interior face f, the corresponding rooted simple quadrangulations $\mathbf{q}_f$ with boundary satisfying property (3.3.1) that has been "cut out“ in forming the core. Furthermore, the total boundary length of $\mathbf{q}_f$ is equal to degree of the face f. Using the bijection established in theorem (3.3.6), this is equivalent to specifying a rooted balanced tree $\mathbf{t}$ and the set $\{\mathbf{q}_\bullet\}$ (one for each black vertex of $\mathbf{t}$) of rooted simple quadrangulations satisfying property (3.3.1). We will determine now the generating function of rooted balanced trees.

Let $R \equiv R(t;\underline{x}) = R(t;x_1,x_2,\ldots)$ denote the ordinary multivariate generating function of (bipartite) rooted plane trees equipped with a bicolouration of their vertices and having monochromatic leaves (we will adopt the convention that the leaves are black $\bullet$) rooted at an (arbitrary) white vertex $\circ$ with a weight t per each leaf and a weight x_i per each black vertex of degree $i+1 \geq 2$. Similarly, we will denote by $S \equiv S(t;\underline{x})$ the ordinary generating function of (bipartite) rooted balanced trees rooted at a unique white leaf having at least one black vertex with a weight t per each non-root leaf and a weight x_i per each black vertex of degree $i+1 \geq 1$. The generating functions R and S will determined using decomposition at the root, as is standard (see figure (3.18) and (3.19) for illustration of the decomposition)

Theorem 3.3.7. *The generating functions of rooted trees and rooted balanced trees where* t *marks leaves and* x_i *marks black vertices of degree* $i+1 \geq 2$ *are given by*

equations:

$$R = t + \frac{S^2}{1-S} \text{ and } S = t + \sum_{i\geq 1} x_i (R - t + S)^i. \tag{3.31}$$

Furthermore, if R_a denotes the generating function of rooted balanced trees with a weight t per each leaf and a weight x_i per each black vertex of degree $i+1 \geq 2$, rooted at a black vertex of degree a then $R_1 = t + t(S + S^2 + \cdots)$ and $R_{a+1} = x_a R^{a+1}$.

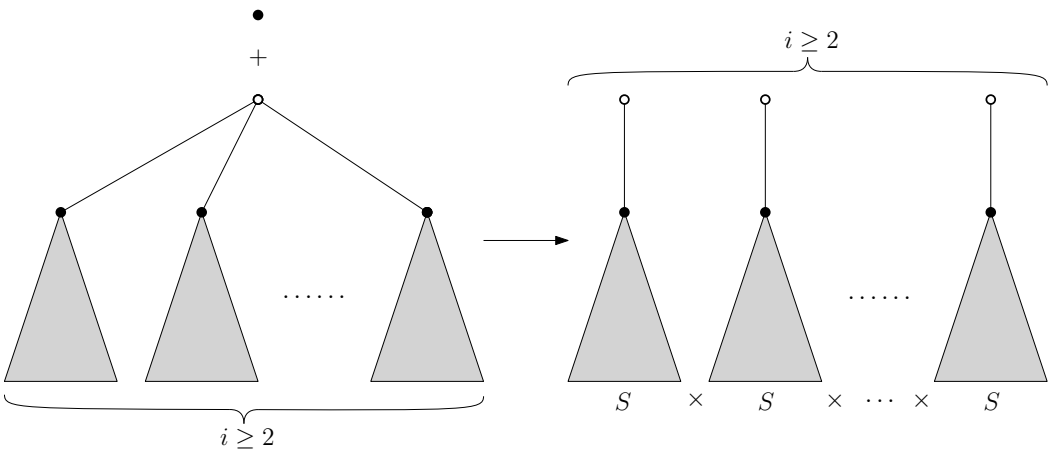


Figure 3.18: Decomposition at the root of the trees counted by R.

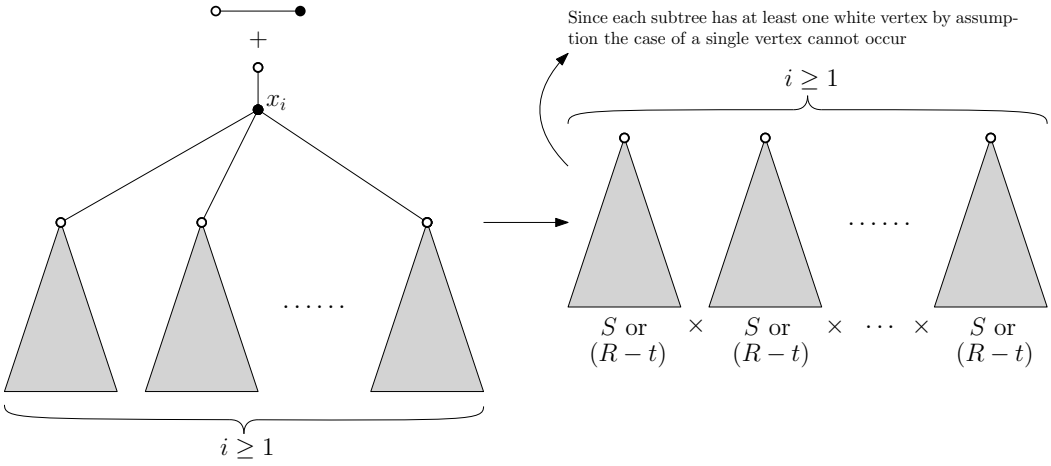


Figure 3.19: Decomposition at the root of the trees counted by S.

The system can easily be used to determine the first few terms of R and S

$$R = t + (1+\tilde{x}_1)^2 t^2 + (1+\tilde{x}_1)^3\{1 + 2\tilde{x}_1 + 2x_2(1+\tilde{x}_1)\}t^3 + \mathcal{O}(t^4)$$
$$S = (1+\tilde{x}_1)t + (1+\tilde{x}_1)^2\{\tilde{x}_1 + x_2(1+\tilde{x}_1)\}t^2 + \mathcal{O}(t^3),$$

where we have introduced for brevity the following notational convention $\tilde{a} = a/(1-a)$. We will continue using this notational abbreviation for the remainder of the

section. We can also express the generating function S implicitly using the system (3.31) (by elimination of R) as follows:

$$S = t + \sum_{i\geq 1} x_i \frac{S^i}{(1-S)^i} \Rightarrow S = t\left\{1 - \sum_{i\geq 1} x_i \frac{S^{i-1}}{(1-S)^i}\right\}^{-1}. \tag{3.32}$$

Equation (3.32) is not yet suitable for Lagrange's inversion. Let's introduce the auxiliary generating function $\varphi(\zeta) = \sum_{i\geq 1}(x_i + x_{i+1})\zeta^i$. Note that $\varphi(0) = 0$ and it can easily be verified that equation (3.32) is equivalent to the equation

$$S = \frac{t}{1 - x_1 - \varphi(\tilde{S})}.$$

It follows that the conditions of Lagrange's inversion theorem are met, so it may be used to determine the coefficients of S. Before using the theorem let us first describe the set up and make some additional comments on the variables. The initial terms of S already suggest that the variable x_1 (counting 4-faces of the core or equivalently 2-valent vertices of the tree) will behave differently from the remaining variables x_i for $i > 1$. Indeed, for a fixed number p of leaves the degree of x_i for $i > 1$ is bounded since an easy application of the identities in theorem (2.1.1) give the relation $p-2 = \lambda_3 + 2\lambda_4 + \cdots$, where λ_i is the number of i-valent vertices of the tree, which implies $\lambda_{i+1} \leq [p/i]$. On the other hand, no such upper bound can exist for the number λ_2 of 2-valent vertices since any edge of a balanced tree can be replaced by a path of arbitrary length, while preserving the balancedness of the tree and the number of leaves, introducing arbitrarily many 2-valent vertices. We will have to treat the variable x_1 separately, so we will denote by $\underline{x}$ the *truncated variable*, namely $\underline{x} = (x_2, x_3, \ldots)$. Let us make a change of variables $t \to x_1 t$ and determine the coefficient of $t^{-1}(R_1 - x_1 t) = x_1\tilde{S}$:

$$\begin{aligned}
[\underline{x}^{\underline{\lambda}} t^{p+1}] x_1\tilde{S} &= \frac{x_1\tilde{x}_1^{p+1}}{p+1}[\underline{x}^{\underline{\lambda}}\zeta^p]\left\{1 - (1+\tilde{x}_1)\varphi(\tilde{\zeta})\right\}^{-(p+1)}(1+\tilde{\zeta})^2 \\
&= \frac{x_1\tilde{x}_1^{p+1}}{p+1}\sum_{i\geq s}\binom{p+i}{p}(1+\tilde{x}_1)^i[\underline{x}^{\underline{\lambda}}\zeta^p]\varphi(\tilde{\zeta})^i(1+\tilde{\zeta})^2 \\
&= \frac{x_1\tilde{x}_1^{p+1}}{p+1}\sum_{i\geq s}\binom{p+i}{p}\binom{i}{s}\binom{s}{\underline{\lambda}}(1+\tilde{x}_1)^s\tilde{x}_1^{i-s}[\zeta^p]\tilde{\zeta}^{i-s+N}(1+\tilde{\zeta})^{s+2} \\
&= \frac{\tilde{x}_1^{p+2}(1+\tilde{x}_1)^{s-1}}{p+1}\binom{s}{\underline{\lambda}}\sum_{i\geq s}\binom{p+i}{p}\binom{i}{s}\binom{p+s+1}{i+N+1}\tilde{x}_1^{i-s},
\end{aligned} \tag{3.33}$$

where we have introduce the notation $s = \lambda_2 + \cdots + \lambda_{k+1}$ and $N = \lambda_2 + \cdots + k\lambda_{k+1}$. Note that the expression is non-zero if and only if $p \geq N$ and the summation is finite

(it extends only to $p-N+s$). The final sum can also be expressed in terms of Jacobi polynomials, but for the sake of brevity we will refrain from doing so.

We have so far been concerned with cores of simple quadrangulations. However, the cores of simple quadrangulations and their irreducible counterparts are related via the substitution relation, whereby every non-trivial path in the balanced tree (corresponding to a nested sequence of 4-faces) is replaced by a simple edge. Consequently, relation (3.33) can be used to deduce connection formulas between q_{2p} and h_{2i} for $p \geq 3$. On the level of generating functions the substitution relation dictates introducing the change of variables $\tilde{x}_1 \to x$ and $x_i \to h_{2(i+1)}$ for $i > 1$. Evaluating the sums (3.33) for small values of p gives the first few relations:

$$\begin{aligned}
q_6(x) &= x^3 + x^3 h_6 \\
q_8(x) &= x^4(1+2x) + 4x^4(1+x)h_6 + x^4 h_8 + 2x^4(1+x)h_6^2 \\
q_{10}(x) &= x^5(1+5x+5x^2) + 5x^5(1+x)(2+3x)h_6 + 5x^5(1+x)h_8 \\
&\quad + 15x^5(1+x)^2 h_6^2 + x^5 h_{10} + 5x^5(1+x)h_6 h_8 + 5x^5(1+x)^2 h_6^3 \\
q_{12}(x) &= x^6(1+2x)(1+7x+7x^2) + x^6 h_{12} + 6x^6(1+x)h_{10} + 3x^6(1+x)(5+7x)h_8 \\
&\quad + 3x^6(1+x)h_8^2 + 2x^6(1+x)(10+35x+28x^2)h_6 + 6x^6(1+x)h_6 h_{10} \\
&\quad + 42x^6(1+x)^2 h_6 h_8 + 21x^6(1+x)^2(3+4x)h_6^2 + 21x^6(1+x)^2 h_6^2 h_8 \\
&\quad + 56x^6(1+x)^3 h_6^3 + 14x^6(1+x)^3 h_6^4.
\end{aligned}$$

Clearly, q_{2p} is expressed as a polynomial in h_{2p}. The constant term can be explicitly described in terms of the so-called Schröder polynomials, which we introduce next. We will provide the combinatorial definition in terms of lattice paths.

Definition 17. A *Schröder path* is a lattice path from $(0,0)$ to (n,n) composed of three kinds of steps $(0,1)$ (east), $(1,0)$ (north) and $(1,1)$ (northeast), that does not go above the line $y = x$. Let u, v, w be commuting variables. We define the *weighted Schröder number* $s_n^{u,v,w}$ as the total weight of all Schröder paths from $(0,0)$ to (n,n), where each east step has weight u, north step has weight v and northeast step has weight w. We define the *Schroöder polynomials* by $s_n(x) = s_n^{1,x,-1}$ for $n \geq 0$, then $s_n^{u,v,w} = (-w)^n s_n(-uv/w)$ and we have

$$s_n(x) = \sum_{j=0}^{n} (-1)^{n-j} \operatorname{Cat}_j \binom{n+j}{n-j} x^j \tag{3.34}$$

$$S(t,x) = \sum_{n \geq 0} s_n(x) t^n = \frac{1 + t - \sqrt{(1+t)^2 - 4xt}}{2xt}.$$

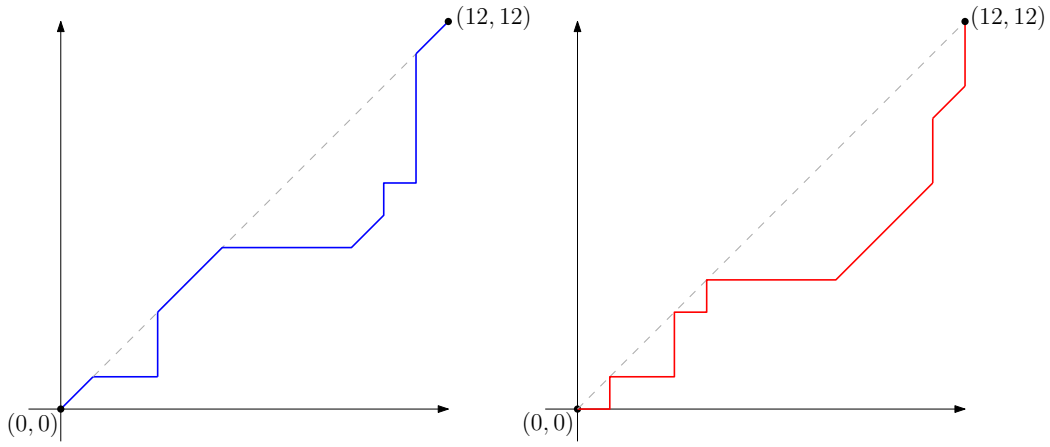


Figure 3.20: A Schröder path and a modified Schröder path.

The *modified Schröder polynomials* $\tilde{s}_n(x)$ correspond to Schröder paths from $(0,0)$ to (n,n) that do not have any (northeast) step contained in the line $y = x$. The following relation is easily established $s_n(x) = (x-1)\tilde{s}_n(x)$ for $n \geq 1$, so we have

$$\tilde{s}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n}(-1)^j\binom{n-1}{j}\binom{2n-j}{n}x^{n-1-j} \tag{3.35}$$

$$\tilde{S}(t,x) = \sum_{n\geq 1} s_n(x)t^n = \frac{1-(2x-1)t-\sqrt{(1+t)^2-4xt}}{2x(x-1)t}.$$

It is easily checked that the constant term is expressible as $x^{p+2}(-1)^{p-1}\tilde{s}_p(-x)$.

The relations can also be inverted yielding connection formulas between h_{2p} and q_{2p}. Indeed, if we denote by a_i the coefficient of t^i in $\tilde{S}(x_1 t; \underline{x})$ then we can inductively express x_j for $j \geq 1$ as a function of a_i; in particular, the initial coefficient is just $\tilde{x}_1$. Let $x_i = b_i(a_1, \ldots, a_i)$ and denote the generating function $T(t; \underline{y}) = \sum_i b_i(\underline{y})t^i$. Using relation (3.32) we easily deduce the following functional equation for T:

$$t = \sum_{i\geq 1}\frac{y_i}{x_1^i}\left\{\frac{t}{t+1} - T(t;\underline{y})\right\}^i \Rightarrow \hat{T} = t\left\{\sum_{i\geq 0} y_{i+1}x_1^{-i}\hat{T}^i\right\}^{-1}, \text{ where } \hat{T} = \frac{t}{1+t} - T(t;\underline{y}).$$

To simplify notation we again introduced an auxiliary function $\psi(\zeta) = \sum_{i\geq 1} y_{i+1}\zeta^i$. Using Lagrange's inversion we find the coefficients of $\hat{T}$; as before the variable y_1 has to be treated separately, so we will denote by $\underline{y}$ the *truncated variable*. Since the initial coefficient is simply $\tilde{x}_1$ the first connection formula is just $y_1 = \tilde{x}_1$. Let us not forget that in determining the coefficients of $t^{-1}(R_1(x_1 t) - x_1 t)$ we also multiplied the function $\tilde{S}$ by x_1, which corresponds in the "inverse" generating function T to the

substitution $y_i \to x_1^{-1} y_i$. Performing these necessary modification yields

$$\begin{aligned}[\underline{y}^{\underline{\mu}} t^{p+1}]\hat{T} &= \frac{(1+\tilde{x}_1)^{-p-1}}{p+1}[\underline{y}^{\underline{\mu}}\zeta^p]\left(1+\frac{\psi(\zeta)}{1+\tilde{x}_1}\right)^{-(p+1)} \\ &= \frac{(-1)^s}{p+1}\binom{p+s}{p}\binom{s}{\underline{\mu}}\tilde{x}_1^{-(2s+N)}(1+\tilde{x}_1)^{s-1}\delta_{N,p}\end{aligned}$$

where δ_{ij} is the *Kronecker delta* and we have introduced the notation $s = \mu_2 + \cdots + \mu_{k+1}$ and $\mu_2 + \cdots + k\mu_{k+1} = N$. Using the same substitution relation between irreducible cores and simple cores as described above, we can easily determine the first few connection formulas:

$$\begin{aligned}h_6(x) &= x^{-3}q_6 - 1 \\ h_8(x) &= x^{-4}q_8 - 2x^{-6}(1+x)q_6^2 + 1 \\ h_{10}(x) &= x^{-5}q_{10} - 5x^{-7}(1+x)q_6q_8 + 5x^{-9}(1+x)^2q_6^3 - 1 \\ h_{12}(x) &= x^{-6}q_{12} - 3x^{-8}(1+x)q_8^2 - 6x^{-8}(1+x)q_6q_{10} \\ &\quad + 21x^{-10}(1+x)^2q_6^2q_8 - 14x^{-12}(1+x)^3q_6^4 + 1\end{aligned}$$

3.3.4 Generating function $\Phi(t, s, x)$

We will go one step further in this section and determine the generating function of pointed rooted irreducible quadrangulations. Using the first and second moment method this will allow us to determine exact concentration results for the distribution of the maximum vertex degree in rooted irreducible quadrangulations in chapter 4.

Definition 18. Let d be a positive integer. A *d-pointed rooted irreducible quadrangulation* is a $(d+1)$-tuple $(\mathbf{q}; v_1, \ldots, v_d)$ consisting of a rooted irreducible quadrangulation $\mathbf{q}$ and d pairwise distinct distinguished vertices $v_1, \ldots, v_d$ different from the root vertex. Note that a linear order has been imposed on the set of distinguished vertices, whereby two d-pointed rooted irreducible quadrangulations with the same set of marked vertices but different order are considered distinct. Any rooted irreducible quadrangulation equipped with additional marked vertices will simply be called a *multipointed rooted irreducible quadrangulation* and a 1-pointed rooted irreducible quadrangulation will simply be called a pointed rooted irreducible quadrangulation.

Let's introduce the following quantities

$$\Phi_\bullet(t, s) = \Phi_\bullet(t, s, x) = \sum_{p,k\geq 2} h^\bullet_{2p,k}(x) t^{p-2} s^k$$

$$\Phi_\circ(t,s) = \Phi_\circ(t,s,x) = \sum_{p,k\geq 2} h^\circ_{2p,k}(x) t^{p-2} s^k$$
$$\Phi_i(t,s) = \Phi_i(t,s,x) = \sum_{p,k\geq 2} h^i_{2p,k}(x) t^{p-2} s^k$$

then $t^2\Phi_\bullet(t,s,x)$ enumerates pointed rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) whose distinguished vertex is a black boundary vertex $\bullet$, $t^2\Phi_\circ(t,s,x)$ enumerates pointed rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) whose distinguished vertex is a white boundary vertex $\circ$ and $t^2\Phi_i(t,s,x)$ enumerates pointed rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) whose distinguished vertex is an inner vertex, where (in all three cases) there is a weight

(i) x per each inner face,

(ii) $\sqrt{t}$ per each boundary edge,

(iii) s per each *inner face* adjacent to the distinguished vertex.

Note that the exponent of s is equal to $\deg v_* - 1$ if v_* is a boundary vertex and $\deg v_*$ otherwise. Before continuing let us remark on our choice of partitioning the generating function with respect to the position and colour of the distinguished vertex. We would ultimately like to determine the generating function of pointed rooted irreducible quadrangulations. Recall that an arbitrary rooted irreducible quadrangulation $\mathbf{q}$ with root vertex of degree p is obtained from a rooted irreducible dissection $\mathbf{q}_p$ of the $2p$-gon satisfying property (3.3.1) by connected sum with the quadrangular star of degree p, $\mathbf{q} = \mathbf{q}_p \# \mathbf{s}_p$, which establishes a one-to-one correspondence between the classes. Furthermore, note that connected sum leaves the valency of a distinguished vertex unaltered if the vertex is inner, increases it by 2 if it is a black boundary vertex and by 1 if it is a white boundary vertex. Therefore, if $G(t,s,x)$ denotes the multivariate generating function of pointed rooted irreducible quadrangulations with a weight x per face, weight t per each face incident with the root vertex and a weight s per each face incident with the distinguished vertex, then the relation between the generating functions is:

$$G(t,s,x) = \Phi_i(xt,s,x) + s^2\Phi_\bullet(xt,s,x) + s\Phi_\circ(xt,s,x). \tag{3.36}$$

Hence the division of generating function according to position and colour of the distinguished vertex is imperative if we are to correctly recover the generating function G in the end. Since we are already counting separately boundary distinguished vertices

according to colour and it does not present any more difficulties to do the same for inner distinguished vertices, we further partition the generating function $\Phi_i(t,s,x)$ into two parts:

$$\Phi_i(t,s,x) = \Phi_{i,\bullet}(t,s,x) + \Phi_{i,\circ}(t,s,x)$$

according to the colour of the distinguished vertex (with obvious change in notation). We record now the following simple observation, which will somewhat simplify the succeeding discussion:

Lemma 3.3.11 *Let $t^2\varphi(t,s)$ and $t^2\psi(t,s)$ denote the generating functions of those pointed rooted irreducible quadrangulations with a simple boundary of length $2p$ ($p \geq 2$) satisfying property (3.3.1), whose distinguished vertex is the same as the black $\bullet$, respectively white $\circ$ extremity of the root edge with a weight x per inner face, $\sqrt{t}$ per boundary edge and s per each inner face incident with the distinguished vertex, then*

$$\Phi_\bullet(t,s) = 2\varphi(t,s) + t\varphi_t(t,s) \text{ and } \Phi_\circ(t,s) = 2\psi(t,s) + t\psi_t(t,s). \tag{3.37}$$

Proof. Suppose the distinguished vertex is a black boundary vertex $\bullet$, then for each rooted irreducible quadrangulations with a simple boundary of total length $2p$, there are p ways to distinguish a black boundary vertex. Hence the relation between the generating functions is

$$\Phi_\bullet(t,s) = t^{-1}\frac{d}{dt}\left[t^2\varphi(t,s)\right] = 2\varphi(t,s) + t\varphi_t(t,s).$$

The same argument applies verbatim to $\Phi_\circ(t,s)$. □

We have so far been counting classes of rooted irreducible quadrangulations exclusively using the substitution approach. The same approach would in principle also yield an expression for the generating functions Φ_*. However, using substitution has some major drawbacks when trying to generalize the counting procedure to pointed quadrangulations or otherwise. Indeed, there are two main ingredients needed to be able to use the substitution technique successfully. Firstly, we need to be able to describe in a deterministic manner the class under consideration in terms of a simpler class and keep track of changes of the combinatorial parameters of interest with respect to substitution. Secondly, we need to already know the corresponding generating functions of the simpler class.

In the present case, using substitution would amount to determining first the corresponding generating functions $\tilde{\Phi}_*$ for the pointed rooted simple quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1)

as well as the generating function $\tilde{q}(s,x)$ of rooted simple quadrangulations with a pointed vertex. Finally, we would have to use the fundamental relations between the generating functions to obtain a tractable expression for Φ_*. Although it is possible to do so, it would lead to some very involved computations. Luckily, the decomposition procedure described in theorem (3.3.3) can be suitably adapted so as to also apply to rooted irreducible quadrangulations. We will describe now a direct approach to counting rooted irreducible quadrangulations satisfying property (3.3.1) that does not rely on substitution.

The recursion hinges once again on the cutting procedure described in the proof of theorem (3.3.3) for the simple analogues. However, there are three marked differences:

1. Gluing two irreducible quadrangulations satisfying property (3.3.1) along a boundary edge or a two-step boundary path can produce a separating 4-cycle in the resulting map.
2. The special case of the cutting procedure (in either case (a) or (b)) when vertex v has no other connections to boundary vertices apart from the *trivial* ones (i.e. connections to the extremities of the root edge in case (a) and connection to the white vertex of the root edge in case (b)) has to be treated separately. Indeed, for simple quadrangulations removing the marked face incident with the root edge left us with a generic simple quadrangulation with the required properties counted simply by $\tilde{\Phi}$. However, for irreducible quadrangulations that is no longer the case - the resulting irreducible quadrangulations with boundary obtained by removing the marked face are <u>not</u> counted by Φ.
3. The exceptional case (that will need to be subtracted at the end) will consist of those gluings of blocks whose resulting map has boundary length ≤ 4 since the total boundary length of a dissection has been assumed a priori ≥ 4 and the only irreducible map satisfying property (3.3.1) of total boundary length 4 is by our convention the 4-cycle that has already been accounted for initially.

We can easily resolve problem (1). Indeed, gluing any two rooted irreducible quadrangulations with simple boundary satisfying property (3.3.1) along a boundary edge or two-step path $\bullet - \circ - \bullet$ can produce a separating 4-cycle only if both maps are 4-cycles and have been glued as depicted below - referred to respectively, as exceptions (I) and (II).

Due to problem (2) the cutting procedure as described in theorem (3.3.3) does not immediately yield a tractable recursive decomposition of irreducible quadrangulations. We will describe in what follows the necessary modifications. As is standard

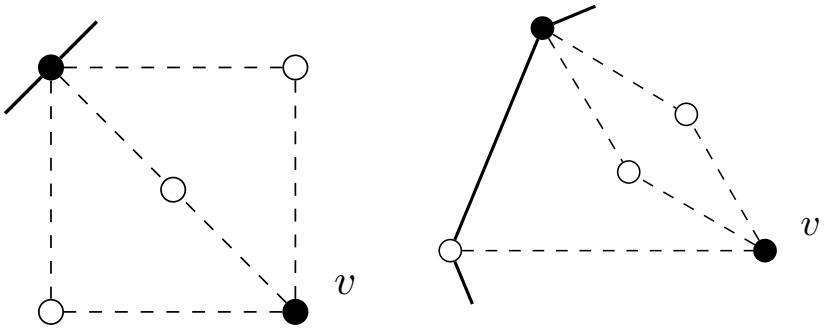


Figure 3.21: Exceptions (I) and (II) which produce a separating 4-cycle when gluing two blocks of the decomposition.

we start the decomposition by looking at the unique face to the left of the root edge. The quadrangulation may be reduced to a single faces (with a boundary length 4), which is the exceptional case (E) that needs to be treated separately. Otherwise, we are in the position as described in the proof of theorem (3.3.3), whereby we have to distinguish between two cases depending on whether the white vertex incident to the revealed face, other than the white extremity of the root edge, belongs to the boundary of the quadrangulation or not. We will denote the black and white vertices of the revealed face, other than the extremities of the root edge, by v_1 and w_1 respectively.

As already remarked the black vertex v_1 incident with the revealed face can happen to have no non-trivial connections to boundary vertices. If this is the case we look at the next (unique) inner face incident with v_1 and the white extremity of the root edge and call v_2 and w_2 its black and white incident vertices other than v_1 and the white extremity of the root-edge. The black vertex v_2 *cannot lie on the boundary* because of the first requirement of property (3.3.1), while the white vertex w_2 cannot lie on the boundary or be connected to a vertex of the boundary otherwise v_1 would have a non-trivial connection to a boundary vertex, violating our assumption. We consider all non-trivial connections of v_2 to boundary vertices (noting that the decomposition of theorem (3.3.3) goes through unchanged), unless none exist. In the latter case, we apply the reasoning described for vertex v_1 to v_2.

We continue revealing faces in this manner until we reach a black vertex with a non-trivial connection to a boundary vertex. Since all the revealed faces are incident with the white extremity of the root-edge and its degree is finite, we will ultimately reach the final inner black vertex incident with the extremity which will necessarily have a non-trivial connection to either a black or white boundary vertex, so the process

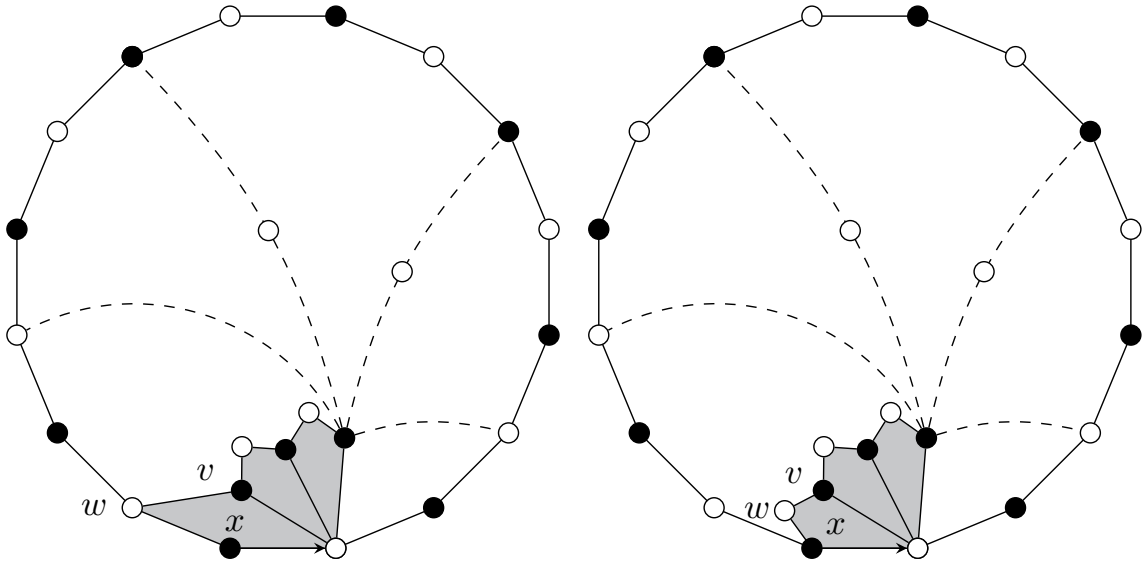


Figure 3.22: The decomposition of maps enumerated by $\Phi(t)$ (and not reduced to a single face). We continue revealing faces incident with the white extremity of the root edge until we reach a black vertex with a non-trivial connection to a boundary vertex.

will terminate after finitely many, say $d \geq 1$, steps. We will denote these cases by (A_d) and (B_d), respectively. The inner black and white vertices of ith revealed face are denoted by v_i and w_i, respectively, and satisfy the following properties:

- The black vertices $v_1, \ldots, v_{d-1}$ have no connections (single or two-step) to non-incident boundary vertices, while the last black vertex v_d has at least one non-trivial connection to a boundary vertex.
- The white vertices $w_1, \ldots, w_d$ have no connections to black boundary vertices.

Note that the initial and final block of the decomposition (which are necessarily distinct by property (i)) are the only blocks incident with any of the revealed faces. Moreover, due to properties (i) and (ii) the final block will be a generic rooted irreducible dissection satisfying property (3.3.1) and counted by Φ.

We again consider the sequence $a_0, a_1, \ldots, a_k$ $(k \geq 2)$ of the lengths 1 and 2 of the (counterclockwise) successive connections of v_d to the boundary vertices, where 1 indicates a simple edge connecting to a white vertex and 2 indicates a two-step path connecting to a black vertex. We have $a_0 = 1$ since the first connection is from v_d to the white end-point of the root edge, whereas $a_n = 2d - 1$ in case (A_d) and $a_n = 2d$ in case (B_d). Unlike for simple quadrangulations whose individual blocks were independent, the weights of blocks arising in the decomposition of irreducible

quadrangulations are not independent due to exceptions (I) and (II). We will therefore partition the sequence $a_0, a_1, \ldots, a_k$ of connections as follows:

$$\underbrace{11\cdots1}_{r_1}\underbrace{22\cdots2}_{s_1}\cdots\underbrace{11\cdots1}_{r_k}\underbrace{22\cdots2}_{s_k}\underbrace{11\cdots1_d}_{r_{k+1}} \text{ in case } (A_d)$$

$$\underbrace{11\cdots1}_{r_1}\underbrace{22\cdots2}_{s_1}\cdots\underbrace{11\cdots1}_{r_{k+1}}\underbrace{22\cdots2_d}_{s_{k+1}} \qquad \text{in case } (B_d)$$

where 1_d and 2_d indicates d faces have been revealed, and the parameters satisfy $r_i, s_j \geq 1$ (to avoid double counting), $\sum_i r_i + \sum_j s_j > 2$ (since v_d has at least one non-trivial connection) and $k \geq 0$. Let us examine how the weights are assigned to a sequence taking into account exceptions (I) and (II).

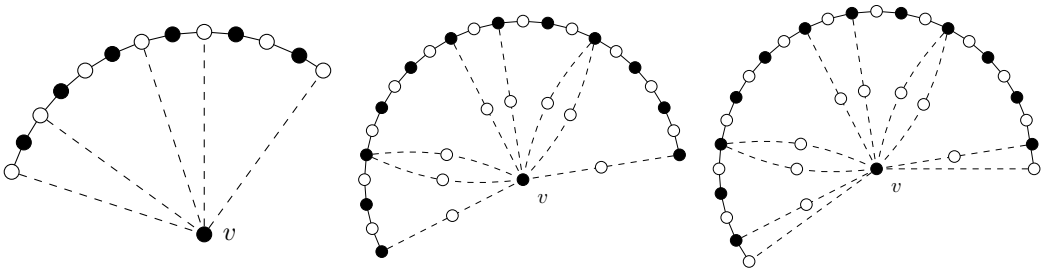


Figure 3.23: The grouping of blocks in the decomposition. From left to right are examples of a white monochromatic sequence, a black monochromatic sequence and an augmented monochromatic sequence.

It will be instructive to separate the generating function Φ into two parts as $\Phi(t) = x + \Phi_0(t)$, where $\Phi_0(t)$ is the corresponding generating function of those rooted irreducible quadrangulations satisfying property (3.3.1) whose total boundary length is ≥ 6. We will assign the weight of a monochromatic sequence (where by abuse of language we will refer to a sequence consisting solely of 1's or 2's as *monochromatic*). Furthermore, we will append the blocks arising from a consecutive one-step and two-step connection to the adjacent sequence of 2's (referring to it as an *augmented monochromatic sequence*) since exception (II) affects only the weight of sequence of 2's but not a sequence of 1's. Let's denote by $w_\circ(r)$ and $w_\bullet(s), \tilde{w}_\bullet(s)$ the corresponding weights of monochromatic sequences and fully augmented monochromatic sequence of 2's without exceptional blocks, where a block containing the marked vertex (either as an interior or boundary vertex) will be called *exceptional*. The following lemma describes the weight assignments and their properties:

Lemma 3.3.12 *The weight of a monochromatic sequence $w_\circ(r)$ and $w_\bullet(s)$ without exceptional blocks is given explicitly by*

$$w_\circ(r) = t^{r-1}(x + \Phi_0)^{r-1} \tag{3.38}$$

$$w_\bullet(s) = \sum_{i\geq 0} \binom{s-i}{i} x^i \Phi_0^{s-1-i}. \tag{3.39}$$

Let $w_\bullet(s;x,y)$ denote the polynomial determined by the weight of a sequence of 2's of length s, then it satisfies the recurrence relation[6] $w_\bullet(s+1) = yw_\bullet(s) + xyw_\bullet(s-1)$ *with initial conditions $w_\bullet(1) = 1$ and $w_\bullet(2) = x+y$ and is given explicitly in the following closed-form (where F_s are the Fibonacci polynomials):*

$$yw_\bullet(s) = (\sqrt{xy})^s F_{s+1}(\sqrt{y/x}) = \frac{(y+\sqrt{4xy+y^2})^{s+1} - (y-\sqrt{4xy+y^2})^{s+1}}{2^{s+1}\sqrt{4xy+y^2}}. \tag{3.40}$$

The weight of fully augmented monochromatic sequence is simply $\tilde{w}_\bullet(s) = tw_\bullet(s+2)$. Furthermore, denote with $W_\circ(t,x)$ and $W_\bullet(t,x)$ the generating functions of the fully augmented monochromatic weights, then

$$W_\circ(t,x) = \sum_{r\geq 1} w_\circ(r) = \frac{1}{1-t(x+\Phi_0)}$$
$$W_\bullet(t,x) = \sum_{s\geq 1} \tilde{w}_\bullet(s;x,\Phi_0) = \frac{t(1+x)^2\Phi_0}{1-(1+x)\Phi_0} - t\Phi_0.$$

Proof. Let us start with a monochromatic sequence of 1's of length $r+1$. Since the exceptions (I) and (II) do not affect the weight of a block coming from two successive one-step connections of v_d to a white boundary vertex, the blocks are independent. Hence the total weight of the sequence is just the product of weights of all of its r individual blocks:

$$w_\circ(r) = \prod_{i=1}^{r-1} \left\{ \sum_{j\geq 2} h_{2j}(x)(\sqrt{t})^{2j-2} \right\} = t^{r-1}\Phi^{r-1}$$

On the other hand, to correctly assign the weight to a monochromatic sequence of 2's, we need to take into account exceptions (I) and (II). Given a sequence of 2's of length s, every two consecutive connections determine a block with weight x (if the block is reduced to a face) or Φ_0 (if the block has boundary length ≥ 6) and there are $s+1$ blocks. Exceptions (I) and (II) are equivalent to the condition that no two consecutive blocks receive weight x. It will therefore be helpful to refine our weight assignment according to the number of blocks with weight x.

Suppose $s-1-i$ blocks receive weight Φ_0, where $i \geq 0$. The remaining i blocks will receive weight x and must therefore be non-consecutive. Hence each of the s

[6]The weight polynomials have only been defined for positive s; however, the recurrence relation may be used to extend the definition to all integers. The extended "polynomials" are $w_\bullet(0;x,y) = y^{-1}$ and $w_\bullet(-s-1;x,y) = (-1)^{s-1}(xy)^{-s}w_\bullet(s-1;x,y)$.

spaces between the blocks weighted by Φ_0 can be filled by 0 or 1 block with weight x. This can be done in $\mathrm{Binom}(s-i,i)$ way, so the number of sequences of length s with i blocks of weight x is

$$\binom{s-i}{i} x^i \prod_{i=1}^{s-1-i} \left\{ \sum_{j\geq 3} h_{2j}(x)(\sqrt{t})^{2j-4} \right\} = \binom{s-i}{i} x^i \Phi_0^{s-1-i}.$$

Summing over all $i \geq 0$ gives (3.39). Note that the sum is actually finite, since by standard convention the binomial coefficient $\mathrm{Binom}(n,k)$ is non-zero only for $0 \leq k \leq n$. The last non-zero term is thus the $\lfloor s/2 \rfloor$-th term.

The closed-form expression for the weight polynomials $w_\bullet(s;x,y)$ follows from the observation that the polynomials can be expressed in terms of the Fibonacci polynomials $F_s(x)$ defined by the recurrence relation $F_{s+1}(x) = xF_s(x) + F_{s-1}(x)$ with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. Alternatively, it can be checked that the polynomials $w_\bullet(s;x,y)$ and the right-hand side of (3.40) both satisfy the same recurrence relation and initial conditions.

Finally, note that verbatim the same argument also determines the weight of a fully augmented sequence of 2's of length s while taking into account exceptions (I) and (II). The only difference is that each term in the sum is also multiplied by the extra factor t that comes from the initial and final block of the augmented monochromatic sequence. The expressions for the weight generating functions are an easy application of the formula for the geometric series. □

Finally, the weight of the final block and a monochromatic sequence containing an exceptional block or the final block (possibly both) will have to be treated completely separately. We denote the weight of the final block by $w^*_\circ(d,g,a)$ and $w^*_\bullet(d,g,a)$, where the index indicates the colour of the boundary vertex incident with the root face and adjacent to the final block, a is the length of the initial connection of the final block and g is the generating function of the exceptional block (we will include the contribution of the marked faces in the weight of the final block). The weight of a monochromatic sequence will be determined on a case-by-case basis, since it crucially depends on the type of distinguished vertex. We have

Lemma 3.3.13 *The weights $w^*_\circ(d,a,g)$ and $w^*_\bullet(d,a,g)$ of the final block are*

$$w^*_\circ(d,a,g,r) = t \sum_{j\geq d+1} g_{2j}(x)(\sqrt{t})^{2j-2d+1-a}(xr)^d = (\sqrt{t})^{7-2d-a}(xr)^d G_{d+1}(t) \tag{3.41}$$

$$w^*_\bullet(d,a,g,r) = \sqrt{t} \sum_{j\geq d+2} g_{2j}(x)(\sqrt{t})^{2j-2d-a} = (\sqrt{t})^{5-2d-a}(xr)^d G_{d+2}(t) \tag{3.42}$$

where $[t^p]g(t) = g_{2p}(t)$. *The generating functions of the weights (summed over all* $d \geq 1$*) can be expressed in terms of the generating function* $g(t)$ *as follows:*

$$\sum_{d\geq 1} w^*_\circ(d,a,g,r) = xr(\sqrt{t})^{5-a}\frac{xrg(xr)-tg(t)}{xr-t}$$
$$\sum_{d\geq 1} w^*_\bullet(d,a,g,r) = xr(\sqrt{t})^{5-a}\frac{g(xr)-g(t)}{xr-t}$$

We have successfully resolved problems (1) and (2). It remains to consider the final problem. Note that the conventions set up so far prevent the creation of a map of total boundary length 2, so we need only to worry about maps of boundary length 4. We can approach the issue either as in the proof of theorem (3.3.3), whereby at the end we examine the decomposition for possible cases giving rise to quadrangulations of total boundary length 4 and subtract their contribution, or we already impose the restriction on the boundary during the counting procedure. We will do the latter since it is easier to keep track of all the degenerate cases for each case separately rather than trying to correctly describe them at the end (without omission or double counting). There are exactly two degenerate cases for each $d \geq 1$ as depicted below.

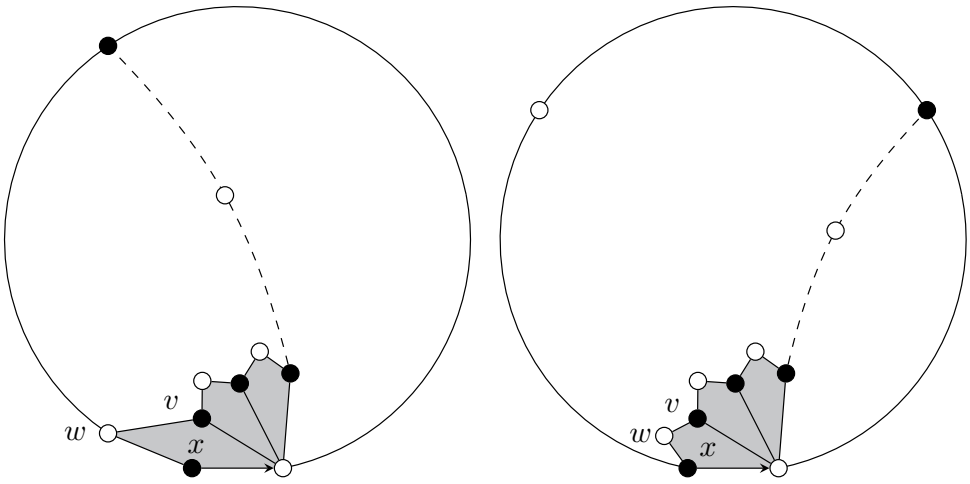


Figure 3.24: The degenerate cases of the decomposition that yield a map of total boundary length 4.

The following lemma will be used repeatedly to determine the total contribution of a sequence of monochromatic blocks. It is essentially a generalization of the binomial theorem and follows quite easily from the definition of multinomial coefficients (with possibly restricted parts):

Theorem 3.3.8 (S. Eger [20])**.** *Let $\mathscr{R} = (R, +, \cdot, \mathbf{0}, \mathbf{1})$ be a commutative ring with addition $+$, multiplication $\cdot$ and additive and multiplicative identities $\mathbf{0}$ and $\mathbf{1}$, respectively, and $S \subseteq \mathbb{N}_0$ a subset of non-negative integers. An integer composition π is said to be S-restricted if every part of π belongs to S; we denote with $\mathscr{C}_S(n,k)$ the set of all S-restricted compositions of n with k. Let $w : S \to R$ be a given weight, then*

$$\sum_{\pi \in \mathscr{C}_s(n,k)} w(x_1) \cdots w(x_k) = [t^n] \left\{ \sum_{s \in S} w(s) t^s \right\}^k, \tag{3.43}$$

where $\pi = (x_1, \ldots, x_k)$ in the initial summation.

We are now in a position to determine the full contribution of the decomposition. We will consider first the case of marked boundary vertices.

Theorem 3.3.9. *The generating functions $\Phi_\bullet(t,s)$ and $\Phi_\circ(t,s)$ are completely determined by (3.37) and the functional equations obtained upon setting $r = 1$ in*[7]

$$\begin{aligned}
\varphi_0(t) &= \frac{xsr(1+t)(x+(1+x)\Phi_0(t))}{1-tx-(1+t)(1+x)\Phi_0(t)} \Delta_{xr}[t\Phi_0(t) + \varphi_0(t)] \\
&\quad + \frac{x^2 sr(tx+(1+t)(1+x)\Phi_0(t))}{1-tx-(1+t)(1+x)\Phi_0(t)} - xs(xr\Phi_0(xr) + \varphi_0(xr)) \qquad (3.44) \\
\psi_0(t) &= \frac{xr(1+t)(x+(1+x)\Phi_0(t))}{1-tx-(1+t)(1+x)\Phi_0(t)} \Delta_{xr}[(1+ts)\psi_0(t)] \\
&\quad + \frac{x^2 s^2 r(tx+(1+t)(1+x)\Phi_0(t))}{1-tx-(1+t)(1+x)\Phi_0(t)} - x(1+xsr)\psi_0(xr) \qquad (3.45)
\end{aligned}$$

where $\varphi_0(t) = \varphi(t) - xs$, $\psi_0(t) = \psi(t) - xs$ and $\Delta_a F(t)$ denotes the discrete derivative with respect to t,

$$\Delta_a F(t) = \frac{F(t) - F(a)}{t - a}.$$

Proof. Let us start with making a few observations. The separation into cases (A_d) and (B_d), although convenient in the decomposition of simple quadrangulations, is less natural from the point of view of binary strings of fully augmented monochromatic sequences. A binary string essentially belongs to one of the following four classes:

$$1\underbrace{11\cdots1}_{r_1}\underbrace{22\cdots2}_{s_1}\cdots\underbrace{11\cdots1}_{r_k}\underbrace{22\cdots2}_{s_k}\underbrace{11\cdots1}_{r_{k+1}}e_d$$
$$1\underbrace{22\cdots2}_{s_1}\underbrace{11\cdots1}_{r_1}\cdots\underbrace{11\cdots1}_{r_k}\underbrace{22\cdots2}_{s_{k+1}}\underbrace{11\cdots1}_{r_{k+1}}e_d$$

[7]The meaning of the mysterious additional indeterminate r is easily deduced from the proof of the theorem; its exponent in the generalized generating functions $\Phi_*(t,s,r)$ is equal to the number of inner black vertices. We have decided to include it, since the refinement presents no more difficulty.

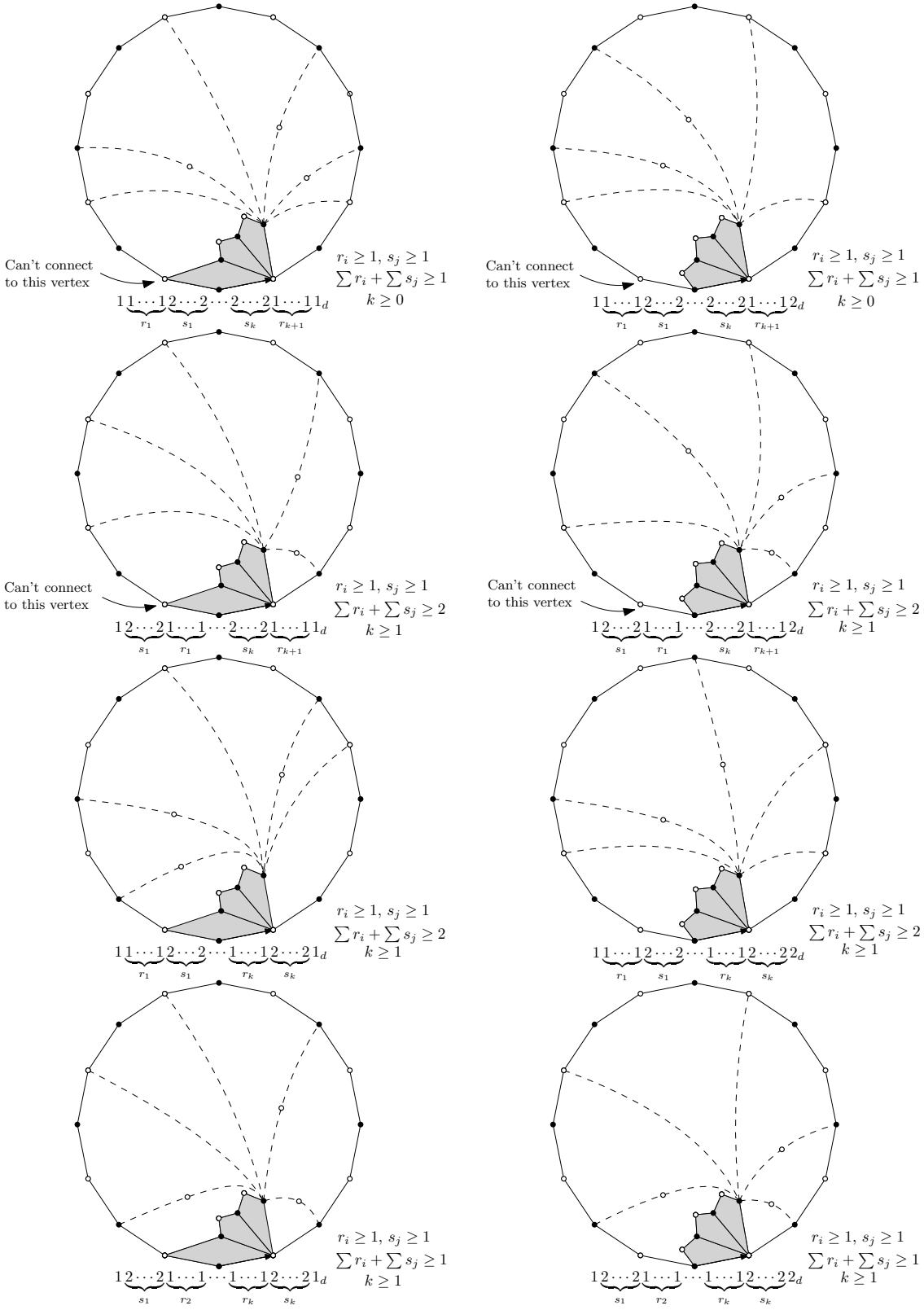


Figure 3.25: The case distinction.

$$1\underbrace{11\cdots1}_{r_1}\underbrace{22\cdots2}_{s_1}\cdots\underbrace{22\cdots2}_{s_k}\underbrace{11\cdots1}_{r_{k+1}}\underbrace{22\cdots2}_{s_{k+1}}e_d$$
$$1\underbrace{22\cdots2}_{s_1}\underbrace{11\cdots1}_{r_1}\cdots\underbrace{22\cdots2}_{s_k}\underbrace{11\cdots1}_{r_k}\underbrace{22\cdots2}_{s_{k+1}}e_d,$$

where the final term is 1_d in case (A_d) and 2_d in case (B_d). Each class will have to be treated separately and for $k = 0$ we will have to modify the contributions of the first and last class, since the representation of the strings as described in the diagram is not unique. We introduce the notation $(E_*(a), F_*(a,d), G_*(a,d))$ for the weight of a (fully augmented) monochromatic sequence of type $*$ and length a that contains an exceptional block, the final block with d marked faces or both, respectively. We will also write $\tilde{G}_*(a,d)$ for the weight of a (fully augmented) monochromatic sequence of type $*$ and length a that contains both the exceptional and final block and is the unique sequence arising in the decompositions (i.e. when $k = 0$ in the first and last case above). The exact expressions will depend on the generating functions and will be determined in the course of the proof.

Suppose the marked vertex is a boundary vertex. Using lemma (3.3.11) it suffices to assume that the distinguished vertex is an extremity of the root edge. If the marked vertex is black $\bullet$, this considerably simplifies the analysis of the total contribution, since the final block is the only possible exceptional block of the decomposition. On the other hand, if the marked vertex is white $\circ$ we could assume the distinguished vertex is the white extremity of the root edge, which would somewhat simplify the discussion. However, as the proof of (3.3.11) shows we may also assume the distinguished vertex is the white extremity of the opposite edge of the root (in a clockwise description of the boundary); the resulting generating function is again $\psi(t,s)$. The added value of the latter assumption is that the only exceptional block of the decomposition is again the final block. According to the case distinction the contribution of a sequence of blocks is then in both cases:

$$\begin{aligned}\Sigma_* = \sum_{d\geq1}\sum_{r\geq1} G_\circ(r,d) &+ \sum_{d\geq1}\sum_{k\geq1}{}^{*}\sum_{r_i,s_j\geq1} w_\circ(r_1+1)F_\circ(r_{k+1},d)\prod_{i=2}^{k} w_\circ(r_i)\prod_{i=1}^{k} w_\bullet(s_i)\\ &+ \sum_{d\geq1}\sum_{k\geq0}\sum_{r_i,s_j\geq1} \tilde{w}_\bullet(s_1)F_\circ(r_{k+1},d)\prod_{i=1}^{k} w_\circ(r_i)\prod_{i=2}^{k+1} w_\bullet(s_i)\\ &+ \sum_{d\geq1}\sum_{k\geq0}\sum_{r_i,s_j\geq1} w_\circ(r_1+1)F_\bullet(s_{k+1},d)\prod_{i=2}^{k+1} w_\circ(r_i)\prod_{i=1}^{k} w_\bullet(s_i)\\ + \sum_{d\geq1}\sum_{s\geq1} G_\bullet(s,d) &+ \sum_{d\geq1}\sum_{k\geq1}{}^{*}\sum_{r_i,s_j\geq1} \tilde{w}_\bullet(s_1)F_\bullet(s_{k+1},d)\prod_{i=1}^{k} w_\circ(r_i)\prod_{i=2}^{k} w_\bullet(s_i).\end{aligned}$$

Note that the initial terms of the first and final sum have been written out separately to avoid double counting. We can simplify the sums using lemma (3.3.8) as follows:

$$\Sigma_* = \left\{\sum_{r,d\geq 1} F_\circ(r,d)\right\}\frac{W_\circ W_\bullet}{1-W_\circ W_\bullet} + \left\{\sum_{s,d\geq 1} F_\bullet(s,d)\right\}\frac{W_\circ W_\bullet + W_\circ - 1}{1-W_\circ W_\bullet} \\ + \sum_{r,d\geq 1} G_\circ(r,d) + \sum_{s,d\geq 1} G_\bullet(s,d). \tag{3.46}$$

It remains to determine the weights $F_*(a,d)$ and $G_*(a,d)$. The weight assignments are derived in a manner similar to the proof of lemma (3.3.12), so we will simply write the appropriate expressions and leave the details of their derivation to the reader. For the sake of brevity we will aim to present a uniform treatment of $F_*(a,d)$ and $G_*(a,d)$ irregardless of the colour of the boundary vertex; we should point out that the general "form" of the weight is always the same, the only difference being whether the revealed faces in case (B_d) contribute a factor s to the weight of the marked vertex (if it is black $\bullet$) or not (if it is white $\circ$). We have

$$F_\circ(a,d) = w_\circ(a)\left\{sw^*_\circ(d,1,G,r) + Sw^*_\bullet(d,1,g,r)\right\}$$
$$F_\bullet(a,d) = \sqrt{t}w_\bullet(a+1)[sw^*_\circ(d,2,G,r) + Sw^*_\bullet(d,2,g,r)] - \delta_{d,1}st^2rx^2G(0)\Phi_0 w_\bullet(a-1)$$

where $(S,G,g) = (s,\Phi,\varphi)$ if the marked vertex is black $\bullet$ and $(S,G,g) = (1,\psi,\psi)$ if the marked vertex is white $\circ$. The combined weights are simply $G_\circ(a,d) = F_\circ(a+1,d)$ (since there are at least two blocks in the decomposition) and

$$G_\bullet(a,d) = F_\bullet(a,d) - \delta_{a,1}t^2r^dx^{d+1}(sG_{2(d+1)} + Sg_{2(d+2)}) + \delta_{a,1}\delta_{d,1}st^2rx^2G(0)$$

where the first term has to be subtracted because it yields in the decomposition a quadrangulation with total boundary length 4 (see figure 3.24) and the final term is added by the inclusion-exclusion principle to correctly recover the weight. We have also introduced the notation $g_{2p}(x) = [t^{p-2}]g(t,x)$ and $G_{2p}(x) = [t^{p-2}]G(t,x)$. Substituting the expressions into formula (3.46) and using lemmas (3.3.12) and (3.3.13) (the contribution $W_\bullet$ should be treated with caution, since it is the generating function of the fully augmented weights $\tilde{w}_\bullet(s)$ not the monochromatic weights $w_\bullet(s)$) to simplify the result gives (we continue working with (S,G,g) to avoid having to perform, what are essentially the same, calculations twice):

$$\sum_{r,d\geq 1} F_\circ(r,d) = xrt^2W_\circ\Delta_{xr}[stG(t) + Sg(t)]$$
$$\sum_{s,d\geq 1} F_\bullet(s,d) = \frac{xrt^2(x+(1+x)\Phi_0)}{1-(1+x)\Phi_0}\Delta_{xr}[stG(t) + Sg(t)] - \frac{st^2rx^2G(0)}{1-(1+x)\Phi_0}$$

$$\sum_{r,d\geq 1} G_\circ(r,d) = xrt^2(W_\circ - 1)\Delta_{xr}[stG(t) + Sg(t)]$$
$$\sum_{s,d\geq 1} G_\bullet(s,d) = \frac{xrt^2(x+(1+x)\Phi_0)}{1-(1+x)\Phi_0}\Delta_{xr}[stG(t)+Sg(t)] - \frac{st^2rx^2G(0)}{1-(1+x)\Phi_0}$$
$$- t^2x(xsrG(xr) + Sg(xr) - Sg(0)) + st^2rx^2G(0)$$

After substituting into (3.46) and simplification, we obtain the following formula of the total contribution

$$\Sigma_* = \frac{xr(1+t)(x+(1+x)\Phi_0)}{1-tx-(1+t)(1+x)\Phi_0}\Delta_{xr}[stG(t)+Sg(t)] - \frac{x^2srG(0)}{1-tx-(1+t)(1+x)\Phi_0}$$
$$- xS(g(xr)-g(0)) - x^2sr(G(xr)-G(0))$$

Replacing (S, G, g) with appropriate values depending on the colour of the marked vertex and simplifying we obtain the functional equations (3.44) and (3.45). □

As a check of our computations we may substitute $s = 1$ in the functional equation determining the generating function $\varphi(t, s, r)$ (or $\psi(t, s, r)$). Since the position of the distinguished vertex of maps counted by φ is fixed, as opposed to the maps counted by $\Phi_{i,\bullet}$ and $\Phi_{i,\circ}$, we recover the generating function $\Phi(t, r)$. After substituting $s = 1$ and rearranging the functional equation becomes

$$\Phi_0(t) + x\Phi_0(xr) = \frac{xr(x+(1+x)\Phi_0(t))(1+(1+t)(1+xr)\Delta_{rx}\Phi_0(t))}{(1+xr)(1-tx-(1+t)(1+x)\Phi_0(t))} - \frac{x^2r}{1+xr}$$

Multiplying both sides by the common denominator and simplifying yields a quadratic equation satisfied by $\Phi(t, r)$ and $\Phi(xr, r)$ (it should be kept in mind that we have written the generating function of all dissections in the form $\Phi = x+\Phi_0$). However, we prefer to continue working with Φ_0. After completing the square we obtain an equation of the form $(2a\Phi_0+b)^2 = \Delta(t, x, r)$. The right-hand side $2a\Phi_0+b = 0$ admits a unique solution $t = \alpha(x, r) \in \mathbb{C}[\![x, r]\!]$, so α is a formal power series such that $\Delta|_{t=\alpha} = 0$ and $\partial_t\Delta_{t=\alpha} = 0$. Eliminating the unknown function $\Phi_0(xr)$ between the equations gives the resultant $16r\alpha^2(1+\alpha)^2x^4(1+x)^8(1+xr)^8(r(1+x)-\alpha)(\alpha^4+2xr\alpha^2-r\alpha+x(1+x)r^2)$. Clearly, $\alpha \neq 0, 1$ and $\alpha(0, r) = 0$ so we conclude that α is a root of the quartic factor. There are many possible parametrization, but introducing the functions $R = R(x, y)$ and $S = S(x, y)$, where $y = xr$, defined implicitly by the system (the explicit series

representations are easily derived using multivariate Lagrange's theorem)[8]

$$\begin{cases} R = x(1+S)^2, & R(x,y) = \displaystyle\sum_{i\geq 1}\sum_{j\geq 0} \frac{1}{2j+1}\binom{2j+1}{i}\binom{2i}{j}x^i y^j \\ S = y(1+R)^2, & S(x,y) = \displaystyle\sum_{i\geq 0}\sum_{j\geq 1} \frac{1}{2i+1}\binom{2i+1}{j}\binom{2j}{i}x^i y^j \end{cases} \tag{3.47}$$

gives the most compact solution. Substituting into the quartic equation gives the solution $\alpha = S/(1+R)$, which upon substitution into $\Delta|_{t=\alpha} = 0$, say, and simplification gives two possibilities for $\Phi_0(xr)$. The correct solution is determined by examining the first term of the expansion. We find the value of $\Phi_0(xr)$ and associated singular value $\Phi(\alpha)$ to be

$$\begin{aligned} \Phi_0(xr, r) &= \frac{1}{1+xr} + \frac{1}{1+x} - 1 - \frac{RS}{x^2 r(1+R+S)^3} \\ \Phi_0\left(\frac{S}{1+R}\right) &= \frac{R^2S^2}{(1+R+S)^2(R+(1+S)^2)} = -\frac{x}{1+x} + \frac{R(1+R)}{(1+R+S)^2} \end{aligned} \tag{3.48}$$

The first equation is especially suggestive - after substitution $(x, r) \to (x_\circ, x_\circ^{-1}x_\bullet)$ and multiplication by $x_\circ x_\bullet$ we recover exactly R. C. Mullin and P. J. Schellenberg's formula for the bivariate generating function of rooted irreducible quadrangulations with bicoloured vertices. Similarly, solving the resulting quadratic equation for $\Phi(t, r)$ and making a similar substitution of variables gives exactly M. Drmota's formula (see [18]) for the trivariate generating function of rooted irreducible quadrangulations with bicoloured vertices and a catalytic variable marking the degree of the root vertex. We leave the details of verification to the reader.

However, the parametrization in terms of functions (R, S) is unfortunately inconvenient for asymptotic analysis. We have found the following alternative parametrization (u, v) to be more amenable to singularity analysis. Furthermore, as $u(1, x) = u(x)$ where $u(x)$ is the familiar generating function of Catalan numbers we will use the same notation u for its bivariate generalization. Hopefully this will not cause much confusion; the meaning will always be clear from the context. We have

$$\begin{cases} u = x\dfrac{1+v}{1-uv}, & u(x,y) = \displaystyle\sum_{i\geq 1}\sum_{j\geq 0} \frac{1}{2i-1}\binom{2i-1}{j}\binom{2j}{i-1}x^i y^j \\ v = y\dfrac{1+u}{1-uv}, & v(x,y) = \displaystyle\sum_{i\geq 0}\sum_{j\geq 1} \frac{1}{2j-1}\binom{2i}{j-1}\binom{2j-1}{i}x^i y^j \end{cases}$$

[8]Incidentally, the functions (R, S) have a well-known combinatorial interpretation. $R = R(x, y)$ is the generating function of black-rooted bicoloured binary trees counted according to the number of black (weighted by x) and white (weighted by y) vertices and $R = R(x, y)$ is the corresponding generating function of white-rooted bicoloured binary trees.

The closed formulas for the coefficients are easily obtained by using the connection formulas between the parametrizations

$$(R,S)=\left(\frac{u(1+v)}{1-uv},\frac{v(1+u)}{1-uv}\right)\ \text{and}\ (u,v)=\left(\frac{R}{1+S},\frac{S}{1+R}\right).$$

Similarly, we can express the generating function of rooted irreducible quadrangulations $\Phi_0(xr)$ in terms of (u,v). If we were to perform Lagrange inversion to determine the coefficient of $x^n y^m$ we would obtain exactly E. A. Bender and N. C. Wormald's formula for the number of rooted convex polyhedra (see [7]):

$$\Phi_0(xr)=\frac{1}{1+x}+\frac{1}{1+xr}-1-\frac{uv(1-uv)}{x^2r(1+u)^2(1+v)^2}.$$

We will provide a somewhat simpler rational parametrization of the generating function. Having anticipated the subsequent discussion we have already been referring to the point $(\alpha,\Phi_0(\alpha))$ as a singular point. Indeed, the quadratic equation determining $\Phi(t)$, can be viewed as an affine algebraic curve $\mathscr{C}$ in (t,Φ) with parameters (u,v) (or equivalently, (r,x) but to avoid radicals it is more convenient to work with the parametrization). It is easily shown that $\mathscr{C}$ is a quartic curve with a unique affine singular point $(\alpha,\Phi_0(\alpha))$ and two singular points at infinity $[1:0:0]$ and $[0:1:0]$, all of multiplicity 2 and with delta invariant 1 (meaning all singular points of $\mathscr{C}$ are ordinary double points), which by the genus formula implies $\mathscr{C}$ is of genus 0. Consequently, it is birationally equivalent to $\mathbb{P}^1$. We have found the following rational parametrization gives an especially compact expression for the generating functions

$$\begin{aligned}t&=\frac{\tau(1+v-uv-(1+v)\tau)}{u(1+v-\tau)}\\ \Phi_0(t)&=-\frac{u(1-uv)}{(1+u)(1+v-uv)}+\frac{\tau(1-uv-\tau)}{t(1+u)(1+v)(1-\tau)}\end{aligned}$$

The correct branch of the function $\tau=\tau(t)$ is determined by condition $\tau(0)=0$ and satisfies $\tau(xr)=uv(1+u)^{-1}$. We can further simplify $\Phi_0(t)$ as follows

$$\boxed{\Phi_0(t)=\frac{1}{1+x}+\frac{1}{1+t}-1-\frac{x^2r\tau(1+v+\tau)}{tuv(1-uv)^2(u+\tau)}.}\tag{3.49}$$

We leave the details to the reader. We should also like to point out that although the simplification $(\Phi_\bullet,\Phi_\circ)\to(\varphi,\psi)$ was beneficial from a computational point of view, it was also necessary. Indeed, as we shall see the functional equations determining the generating functions $\Phi_{i,\bullet}$ and $\Phi_{i,\circ}$ will depend on φ and ψ not on $\Phi_\bullet,\Phi_\circ$.

Each equation involves an unknown generating function $\Phi_*(t,s)$ and its specialization $\Phi_*(x,s)$, so the indeterminate t marking the boundary edges acts as a catalytic variable. Note that the equations are linear in the unknown functions, so Tutte's kernel method is applicable. The next results give exact expressions for the generating functions:

Theorem 3.3.10 (Generating function $\Phi_\bullet(t,s)$)**.** *The multivariate generating function $\Phi_\bullet(t,s,r,x)$ enumerating rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) and a marked black boundary vertex with a weight x per inner face, $\sqrt{t}$ per boundary edge, s per inner face adjacent to the marked vertex and r per inner black vertex is given by (3.37) with*

$$\varphi_0(t) = \frac{ts}{(1+t)(1+s)} + \frac{\tau\zeta(\tau\zeta - \tau - \zeta + 1 - uv)}{uv(1+t)(1+s)(1-\tau-\zeta)} \tag{3.50}$$

where $\tau = \tau(t,x,r)$, $\zeta = \zeta(s,x,r)$ are implicitly determined by the following equations and conditions $\tau(0) = 0$ and $\zeta(0) = 0$,

$$t = \frac{\tau(1+v-uv-(1+v)\tau)}{u(1+v-\tau)} \text{ and } xs = \frac{\zeta(1+u-uv-(1+u)\zeta)}{v(1+u-\zeta)}. \tag{3.51}$$

Proof. According to lemma (3.3.11) it suffices to find an expression for the corresponding generating function $\varphi(t,s,r)$ whose marked black boundary vertex is the root vertex. We have found in theorem (3.3.9) a functional equation satisfied by the unknown generating functions $\varphi(t,s,r)$ and $\varphi(xr,s,r)$ in which the indeterminate t acts as a catalytic variable. Note that the functional equation is linear, so it can be solved using Tutte's kernel method. However, we can prove bijectively the following identity

$$\begin{aligned}\varphi(xr,s,r,x) &= xs + \Phi_0(xs, r^{-1}, xr) \\ &= xs + r^2sx^4 + r^3s(3+s)x^6 + r^3s(1+2s+3r)x^7 + \mathcal{O}(x^8).\end{aligned} \tag{3.52}$$

Indeed, the first term on the right-hand side accounts for the degenerate case of the 4-cycle. Note that $x^2s^2r^2\varphi_0(xr,s,r,x)$ is exactly the generating function of all rooted irreducible quadrangulations with a black root $\bullet$ and a weight x per face, r per black vertex and s per face adjacent with the root vertex. On the other hand, $x^2s^2r^2\Phi_0(xsr,r,x)$ is the generating function of all rooted irreducible quadrangulations with a white root $\circ$ and a weight x per face, r per black vertex and s per face adjacent with the root vertex. Let $n_\bullet, n_\circ$ and n denote the number of black, white vertices and faces of an irreducible quadrangulation, respectively, then $n_\bullet + n_\circ = n+2$. It follows (by an appropriate change of variables) that $x^2s^2r^2\Phi_0(xs, r^{-1}, xr)$ is exactly the generating function $x^2s^2r^2\varphi_0(xr,s,r,x)$, which proves the relation.

Alternatively, the same relation could also have been obtained using the kernel method. Indeed, re-writing the functional equation (3.44) in the form $K(t,r)\varphi(t,r) = L(\varphi(xr),t,r)$, it is easily checked that the *kernel* $K(t,r) = 0$ admits a unique power series solution $t = \beta(r,s) \in \mathbb{C}[\![r,s]\!]$ (it suffices to compare the coefficients of $r^n s^m$). Unfortunately, the kernel is not a polynomial in the indeterminate t so it requires some work to find the solution β explicitly. We have determined that the generating function $\Phi(t,r)$ is a solution of a quadratic equation and can be expressed parametrically in terms of (u,v). Since both the kernel and the quadratic equation are polynomials in the variables (t,Φ) we can calculate their resultant with respect to Φ, which yields

$$v(1+u)^2(1+u-uv)(1+v-uv)^4(1+t)((1+u)t - v(1-uv))p(\beta,s,R,S),$$

where p is a quadratic polynomial in β (and s). It is easily checked that β cannot be a zero of the first two factors, so it has to be a zero of the quadratic polynomial p. After some straightforward algebraic calculations we find the correct solution (there are *a priori* two possibilities, but the correct determination can be found by examining the initial terms of a series expansion about the origin) to be

$$\beta(r,s) = \frac{(1-\xi)((1+v)\xi - uv)}{u(v+\xi)}, \text{ where } xs = \frac{\xi(1+u-uv-(1+u)\xi)}{v(1+u-\xi)}$$

The unknown function $\varphi_0(xr)$ can be expressed in terms of the formal power series $\beta(r,s)$ as

$$\varphi_0(xr) = \frac{\beta(\beta - xr)}{(1+xs)(1+\beta)(\beta - xr + xsr)} - \frac{xr}{1+x}(x + (1+x)\Phi_0(xr)).$$

Having identified the unknown series $\varphi(xr)$, we can substitute the expression in the functional equation (3.44) and solve for $\varphi(t)$. Note that in terms of the parametric functions (u,v) introduced above the substitution $(r,x) \to (r^{-1}, xr)$ corresponds to $(u,v) \to (v,u)$. After some tedious algebraic calculations we obtain (3.50). □

Theorem 3.3.11 (Generating function $\Phi_\circ(t,s)$)**.** *The multivariate generating function $\Phi_\bullet(t,s,r,x)$ enumerating rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) and a marked white boundary vertex with a weight x per inner face, $\sqrt{t}$ per boundary edge, s per inner face adjacent to the marked vertex and r per inner black vertex is given by*

$$\psi_0(t) = -\frac{xs}{1+x} + \frac{\tau\zeta(\tau\zeta - (1+v)(\tau+\zeta) + 1 + v - uv)}{v(1+v)(u(1+v) - u(\tau+\zeta) - \tau\zeta)} \tag{3.53}$$

where $\tau = \tau(t,x,r), \zeta = \zeta(s,x,r)$ *are implicitly determined by the following equations and conditions* $\tau(0) = 0$ *and* $\zeta(0) = 0$*,*

$$t = \frac{\tau(1+v-uv-(1+v)\tau)}{u(1+v-\tau)} \text{ and } xsr = \frac{\tau(1+v-uv-(1+v)\zeta)}{u(1+v-\zeta)}. \tag{3.54}$$

Proof. Again by lemma (3.3.11) it suffices to determine the corresponding generating function $\psi(t,s,r)$ whose marked white boundary vertex is the white extremity of the root edge. The functional equation (3.45) satisfied by the unknown generating functions $\psi(t,s,r)$ and $\psi(xr,s,r)$ is linear, but we can again avoid using the kernel method by proving bijectively the following identity

$$\psi(xr,s,r,x) = xs + s\Phi_0(xsr,r,x) \tag{3.55}$$

As before the first term on the right-hand side accounts for the degenerate case of the 4-cycle. We find that $x^2sr^2\psi(xr,s,r,x)$ is exactly the generating function of all rooted irreducible quadrangulations with a white root $\circ$ and a weight x per face, r per black vertex and s per face adjacent with the root vertex. The relation follows. Finally, substituting all the parametrizations in equation (3.45) and simplifying gives the result. □

Let us consider next the case when the marked vertex is an inner vertex. We can derive the following functional equations for the corresponding generating functions

Theorem 3.3.12. *The generating functions* $\Phi_{i,\bullet}(t,s)$ *and* $\Phi_{i,\circ}(t,s)$ *are completely determined by equations (3.37) and the following system of functional equations obtained upon setting* $r = 1$*:*

$$\begin{aligned}
\Phi_{i,\bullet}(t,s,r) &= \frac{(1+t)x(1+x)r((1+x)\Delta_{xr}[(1+t)\Phi(t)]-x^2)}{(1-tx-(1+t)(1+x)\Phi_0)^2}\Phi_{i,\bullet} \\
&+ \frac{(1+t)xr(x+(1+x)\Phi_0)}{1-tx-(1+t)(1+x)\Phi_0}\Delta_{xr}[(1+t)\Phi_{i,\bullet}(t)] - x(1+xr)\Phi_{i,\bullet}(xr) \\
&+ \frac{(1+t)x^2s^2r^2(x+(1+x)\Phi_0)}{1-tx-(1+t)(1+x)\Phi_0}\Delta_{xr}^2[(1+t)\varphi(t)] + xs^2\varphi_0(xr) \\
&+ \frac{xsr(1+t)(xs+(1+xs)\varphi_0)}{1-txs-(1+t)(1+xs)\varphi_0}\Delta_{xr}[(1+t)\varphi_0(t)] - x^2s^2r(1+xr)\varphi'(xr) \\
&+ \frac{x^2s^2r(txs+(1+t)(1+xs)\varphi_0)}{1-txs-(1+t)(1+xs)\varphi_0} - xs^2(1+xr)\varphi_0(xr)
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
\Phi_{i,\circ}(t,s,r) &= \frac{(1+t)x(1+x)r((1+x)\Delta_{xr}[(1+t)\Phi(t)]-x^2)}{(1-tx-(1+t)(1+x)\Phi_0)^2}\Phi_{i,\circ} \\
&+ \frac{(1+t)xr(x+(1+x)\Phi_0)}{1-tx-(1+t)(1+x)\Phi_0}\Delta_{xr}[(1+t)\Phi_{i,\circ}(t)] - x(1+xr)\Phi_{i,\circ}(xr)
\end{aligned} \tag{3.57}$$

$$
\begin{aligned}
&+\frac{xr((1+x)\psi-x^2s)}{1-tx-(1+t)(1+x)\Phi_0}\Delta_{xr}[(1+t)\psi(t)]\\
&+\frac{(1+t)x^2sr^2(x+(1+x)\Phi_0)}{1-tx-(1+t)(1+x)\Phi_0}\Delta_{xr}^2[(1+t)\psi(t)]-xs(1+xr)\psi(xr)\\
&+\frac{(1+t)xsr(x+(1+x)\Phi_0)}{1-tx-(1+t)(1+x)\Phi_0}\Delta_{xr}\psi(t)-x^2sr(1+xr)\psi'(xr)\\
&+\frac{(1+t)xr(x+(1+x)\Delta_{xr}[(1+t)\Phi_0(t)])\psi_0(2xs+(1+x)\psi_0)}{(1-tx-(1+t)(1+x)\Phi_0)^2}\\
&-\frac{x^3sr(s+\psi_0)}{1-tx-(1+t)(1+x)\Phi_0}+x^2s^2(1+xr)
\end{aligned}
$$

Proof. We will apply the case distinction and use the notation introduced in the proof of theorem (3.3.9).

Suppose the distinguished vertex is an inner vertex. There is unfortunately no analogue of lemma (3.3.11) that would simplify the enumeration, so the computations will be rather more involved. Still we would like to treat both cases as uniformly as possible, irregardless of colour, to avoid spurious repeated calculations. However, let us point out from the very start that the exceptional weights (E_*, F_*, G_*) will not allow a uniform treatment (even their form will depend crucially on the colour of the marked vertex) and there is an exceptional case of the decomposition that can arise only if the marked vertex is black $\bullet$ and has to be treated completely separately. We are referring here to the instance where v_d is the marked vertex, so every block of the decomposition is exceptional. We will postpone its analysis to the end. Excluding this possibility, there is exactly one monochromatic sequence containing the distinguished vertex. We will write the total contribution of each of the four cases separately.

$$
\begin{aligned}
\Sigma_{(1)} = &\sum_{r,d\geq 1}\tilde{G}_\circ(r,d)+\left\{\sum_{r,d\geq 1}F_\circ(r,d)\right\}\left\{\sum_{s\geq 1}E_\bullet(s)\right\}\frac{W_\circ-1}{(1-W_\circ W_\bullet)^2}\\
&+\left\{\sum_{r,d\geq 1}F_\circ(r,d)\right\}\left\{\sum_{r\geq 1}E_\circ(r)\right\}\frac{W_\bullet(1-W_\bullet)}{(1-W_\circ W_\bullet)^2}+\left\{\sum_{r,d\geq 1}G_\circ(r,d)\right\}\frac{(W_\circ-1)W_\bullet}{1-W_\circ W_\bullet}.
\end{aligned}
$$

Instead of writing out the long summations involved in arriving at this expression, let us rather explain how the counting is done. The initial and final term of the binary string is a white monochromatic sequence and exactly one term of the string contains the marked vertex by the foregoing discussion. We therefore have 4 contributions:

(i) The string consists a single white monochromatic sequence that necessarily contains the marked vertex (the first term); otherwise, the string contains at least three terms and at least one sequence of 2's.

(ii) The marked vertex belongs to a monochromatic sequence consisting of 2's (the second term);

(iii) The marked vertex belongs to a non-final monochromatic sequence consisting of 1's (the third term) - caution must be taken in this case, to distinguish whether the marked vertex belongs to the first term of the string or not, since the first monochromatic sequence is always of length ≥ 2.

(iv) Finally, the marked vertex may belong to the final term of the string (which accounts for the final term).

The contribution of the second case is:

$$\Sigma_{(2)} = \left\{ \sum_{r,d\geq 1} F_\circ(r,d) \right\} \left\{ \sum_{s\geq 1} E_\bullet(s) \right\} \frac{1}{(1-W_\circ W_\bullet)^2} + \left\{ \sum_{r,d\geq 1} F_\circ(r,d) \right\} \left\{ \sum_{r\geq 1} E_\circ(r) \right\} \frac{W_\bullet^2}{(1-W_\circ W_\bullet)^2} + \left\{ \sum_{r,d\geq 1} G_\circ(r,d) \right\} \frac{W_\bullet}{1-W_\circ W_\bullet}.$$

The initial term of the binary string is a black monochromatic sequence, while the final term is still a white sequence. The three terms of the sum correspond respectively to cases (ii), (iii) and (iv) (no additional consideration is needed in case (iii) since the initial term of the string is a fully augmented sequence of 2's). Moving on, the contribution of the third case is:

$$\Sigma_{(3)} = \left\{ \sum_{s,d\geq 1} G_\bullet(s,d) \right\} \frac{W_\circ - 1}{1-W_\circ W_\bullet} + \left\{ \sum_{s,d\geq 1} F_\bullet(s,d) \right\} \left\{ \sum_{s\geq 1} E_\bullet(s) \right\} \frac{(W_\circ - 1)W_\circ}{(1-W_\circ W_\bullet)^2} + \left\{ \sum_{s,d\geq} F_\bullet(s,d) \right\} \left\{ \sum_{r\geq 1} E_\circ(r) \right\} \frac{1-W_\bullet}{(1-W_\circ W_\bullet)^2}$$

Finally, the contribution of the last case is

$$\Sigma_{(4)} = \sum_{s,d\geq 1} \tilde{G}_\bullet(s,d) + \left\{ \sum_{s,d\geq 1} F_\bullet(s,d) \right\} \left\{ \sum_{s\geq 1} E_\bullet(s) \right\} \frac{W_\circ}{(1-W_\circ W_\bullet)^2} + \left\{ \sum_{s,d\geq 1} F_\bullet(s,d) \right\} \left\{ \sum_{r\geq 1} E_\circ(r) \right\} \frac{W_\bullet}{(1-W_\circ W_\bullet)^2} + \left\{ \sum_{s,d\geq 1} G_\bullet(s,d) \right\} \frac{W_\circ W_\bullet}{1-W_\circ W_\bullet}$$

We deal next with the exceptional case when v_d is a marked vertex. Since all the blocks are of the same type, there are no "exceptional" blocks, so the total contribution is the same as we would have if we were to use the decomposition to derive the functional equation for the ordinary generating function $\Phi(t)$. To avoid introducing even more unnecessary notation, we will retain the same notation as we have used so far, keeping

in mind that every occurrence of Φ should be replaced with φ. The total contribution in this case is thus

$$\Sigma_{(5)} = \sum_{r,d\geq 1} G_\circ(r,d) + \sum_{s,d\geq 1} G_\bullet(s,d) + \left\{\sum_{r,d\geq 1} F_\circ(r,d)\right\} \frac{W_\circ W_\bullet}{1-W_\circ W_\bullet} + \left\{\sum_{s,d\geq 1} F_\bullet(s,d)\right\} \frac{W_\bullet W_\circ + W_\circ - 1}{1-W_\circ W_\bullet}$$

The formulas of the exceptional weights are the same as in the case of a marked boundary vertex with $(S,G,g) = (s,\varphi,\varphi)$.

To complete the proof we need to assign the exceptional weights $(E_*, F_*, G_*, \tilde{G}_*)$ and substitute the expressions into the formulas of the contributions. The weight of a non-final white sequence containing the marked vertex is obtained from the ordinary weight $w_\circ(a)$ by pointing at Φ_0, which corresponds to taking the derivative with respect to y, and multiplying the result by $\Phi_{i,*}(t,s)$. The same reasoning also works for the weight of a black sequence if the marked vertex is black $\bullet$. On the other hand, if the marked vertex is white $\circ$ we need to also take into account the possibility that it coincides with a white vertex forming a two-step connection of the sequence (meaning two blocks will contain the marked vertex). We obtain

$$\begin{aligned} E_\circ(a) &= (a-1)t^{a-1}(x+\Phi_0)^{a-2}\Phi_{i,*}(t,s) \\ E_\bullet(a) &= \Phi_{i,\bullet}(t,s)\,\frac{\partial}{\partial y}\tilde{w}_\bullet(a;x,y)\Big|_{y=\Phi_0} \end{aligned}$$

On the other hand, if the marked vertex is a white inner vertex the second weight is to be modified as

$$E_\bullet(a) = t\left\{\psi_0^2(t,s)\sum_{k=1}^{a} w_\bullet(k)w_\bullet(a-k+1) + 2xs\psi_0(t,s)\Phi_0\sum_{k=1}^{a} w_\bullet(k)w_\bullet(a-k)\right\} + \Phi_{i,\circ}(t,s)\,\frac{\partial}{\partial y}\tilde{w}_\bullet(a;x,y)\Big|_{y=\Phi_0}$$

where $\psi_0(t,s) = \psi(t,s) - sx$. The convolutions in the last formula can further be simplified by using the representation of the monochromatic weights $w_\bullet$ in terms of the Fibonacci polynomials. However, since we are ultimately interested only in the generating function of the exceptional weight $E_\circ$ we will not carry out the simplifications. Note that we have committed a slight faux pax by not restricting the summation of the exceptional weight $E_\circ(r)$ to indices $r \geq 2$, since assuming the marked vertex belongs to the sequence implies the sequence contains at least one block. Fortunately, the formula for $E_\circ(a)$ correctly returns the value 0 for $a = 1$.

The weight assignment of the final monochromatic sequence that does not contain the marked vertex is the same in both cases and is simply

$$F_\circ(a,d) = w_\circ(a)\left\{w^*_\circ(d,1,\Phi,r) + w^*_\bullet(d,1,\Phi,r)\right\}$$
$$F_\bullet(a,d) = \sqrt{t}w_\bullet(a+1)\left\{w^*_\circ(d,2,\Phi,r) + w^*_\bullet(d,2,\Phi,r)\right\} - \delta_{d,1}t^2rx^3\Phi_0 w_\bullet(a-1).$$

It remains to determine the weights G_* and $\tilde{G}_*$. We will have to treat separately the cases when the marked vertex is a black inner vertex $\bullet$ or a white inner vertex $\circ$. When the final monochromatic sequence is white, there are three possibilities for the location of the marked vertex (corresponding in this order to the three summands). Either the marked vertex belongs to a non-final block, to the interior of the final block or is incident with a revealed face (we have suppressed r from the notation of the final weights to conserve on space):

$$\begin{aligned} G_\circ(a,d) = E_\circ(a)\left\{w^*_\circ(d,1,\Phi) + w^*_\bullet(d,1,\Phi)\right\} + w_\circ(a)\left\{w^*_\circ(d,1,\Phi_{i,*}) + w^*_\bullet(d,1,\Phi_{i,*})\right\} \\ + w_\circ(a)s^{b+1}\left\{(d-1)w^*_\circ(d,1,g) + (d-b)w^*_\bullet(d,1,g)\right\} \end{aligned}$$

where $(b,g) = (1,\varphi)$ if the marked vertex is black $\bullet$ and $(b,g) = (0,\psi)$ is the marked vertex is white $\circ$. Again the combined weight of a white monochromatic sequence is simply $\tilde{G}_\circ(a) = G_\circ(a+1)$ (since there are at least two blocks in the decompositions). Finally, when the final monochromatic sequence is black, there is apart from the three possibilities for the location of the marked vertex described above a fourth option. If the distinguished vertex is white, it can again coincide with a white vertex forming a two-step connection of the sequence and the two-step path is either disjoint from the final block or incident with it. As such the weight assignment is

$$\begin{aligned} G_\bullet(a,d) = \sqrt{t}\Phi_{i,\bullet}\left\{w^*_\circ(d,2,\Phi,r) + w^*_\bullet(d,2,\Phi,r)\right\}\frac{\partial}{\partial y}w_\bullet(a+1;x,y)\Big|_{y=\Phi_0} \\ - \delta_{d,1}\Phi_{i,\bullet}t^2rx^3\frac{\partial}{\partial y}yw_\bullet(a-1;x,y)\Big|_{y=\Phi_0} \\ + \sqrt{t}w_\bullet(a+1)\left\{w^*_\circ(d,2,\Phi_{i,\bullet},r) + w^*_\bullet(d,2,\Phi_{i,\bullet},r)\right\} \\ + \sqrt{t}w_\bullet(a+1)s^2(d-1)\left\{w^*_\circ(d,2,\varphi,r) + w^*_\bullet(d,2,\varphi,r)\right\}. \end{aligned}$$

if the marked vertex is black $\bullet$. While

$$\begin{aligned} G_\bullet(a,d) = \sqrt{t}\Phi_{i,\circ}\left\{w^*_\circ(d,2,\Phi,r) + w^*_\bullet(d,2,\Phi,r)\right\}\frac{\partial}{\partial y}w_\bullet(a+1;x,y)\Big|_{y=\Phi_0} \\ - \delta_{d,1}\Phi_{i,\circ}t^2rx^3\frac{\partial}{\partial y}yw_\bullet(a-1;x,y)\Big|_{y=\Phi_0} \end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{t} w_\bullet(a+1) \left\{ w_\circ^*(d,2,\Phi_{i,\circ},r) + w_\bullet^*(d,2,\Phi_{i,\circ},r) \right\} \\
&+ \sqrt{t} w_\bullet(a+1) s \left\{ (d-1) w_\circ^*(d,2,\psi,r) + d w_\bullet^*(d,2,\psi,r) \right\} \\
&+ \psi_0^2(t,s) \sum_{k=1}^{a-1} w_\bullet(k) F_\bullet(a-k-1) + xs\psi_0(t,s)\Phi_0 \sum_{k=1}^{a-1} w_\bullet(k) F_\bullet(a-k-2) \\
&+ xs\psi_0(t,s)\Phi_0 \sum_{k=1}^{a-1} w_\bullet(k-1) F_\bullet(a-k-1) \\
&+ \sqrt{t} \{ \psi_0(t,s) w_\bullet(a) + xs\Phi_0 w_\bullet(a-1) \} \left\{ w_\circ^*(d,2,\psi,r) + w_\bullet^*(d,2,\psi,r) \right\} \\
&- \delta_{d,1} t^2 x^3 s^2 r \Phi_0 w_\bullet(a-1)
\end{aligned}
$$

if the marked vertex is white $\circ$. The combined weight $\tilde{G}_*$ of a black monochromatic sequence is then obtained from G_* by subtracting the degenerate cases with total boundary length 4 in order to properly recover the generating function. Hence

$$
\begin{aligned}
\tilde{G}_\bullet(a,d) = G_\bullet(a,d) &- \delta_{a,1} t^2 r^d x^{d+1} \{ h^{i,*}_{2(d+1)} + h^{i,*}_{2(d+2)} \} \\
&- \delta_{a,1} t^2 r^d x^{d+1} s^{1+b} \{ (d-1) g_{2(d+1)} + (d-b) g_{2(d+2)} \} \\
&- (1-b) \delta_{a,1} t^2 r^d x^{d+1} s \{ \psi_{2(d+1)} + \psi_{2(d+2)} \} + (1-b) \delta_{a,1} \delta_{d,1} s^2 t^2 r x^3
\end{aligned}
$$

where $(b,g) = (1,\varphi)$ if the marked vertex is black $\bullet$ and $(b,g) = (0,\psi)$ if the marked vertex is white $\circ$, and the last term vanishes if the marked vertex is black since it corresponds to the case when the marked vertex coincides with the white vertex forming a two-step connection, so it only arises if the distinguished vertex is white $\circ$.

Finally, all the weight assignment are to be substituted (with their appropriate values depending on the colour of the marked vertex) into the formulas for the total contributions $\Sigma_{(1)}, \Sigma_{(2)}, \Sigma_{(3)}$ and $\Sigma_{(4)}$ of the four cases and added up. If the marked vertex is black $\bullet$, the exceptional case $\Sigma_{(5)}$ must also be added to the final sum. After a lot of tedious algebraic simplifications we obtain the functional equations (3.56) and (3.57), respectively. □

Theorem 3.3.13 (Generating function $\Phi_{i,\bullet}(t,s)$)**.** *The multivariate generating function $\Phi_{i,\bullet}(t,s,r)$ enumerating rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) and a marked black inner vertex with a weight x per inner face, $\sqrt{t}$ per boundary edge, s per inner face adjacent to the marked vertex and r per inner black vertex is given by*

$$
\Phi_{i,\bullet}(t) = \frac{r\tau_t \tau^2 \zeta^3}{tu^2 v^2 (uv - \tau\zeta)} \tag{3.58}
$$

where $\tau = \tau(t,x,r)$ and $\zeta = \zeta(s,x,r)$ are defined as in the statement of theorem (3.3.10) and τ_t denotes the partial derivative of $\tau = \tau(t,x,r)$ with respect to t.

Proof. We have determined in theorem (3.3.12) the functional equation satisfied by the unknown generating functions $\Phi_{i,\bullet}(t,s,r)$ and $\Phi_{i,\bullet}(xr,s,r)$ in which the indeterminate t acts as a catalytic variable. Although the equation can be solved using the kernel method, the calculations are rather involved. We can prove the following formula bijectively

$$\Phi_{i,\bullet}(xr,s,r) = \frac{(1+u+v)\zeta^3}{uv(1+u)(1-uv)(1+u-2uv-u^2v)(1+u-\zeta)} \tag{3.59}$$

It suffices to show the following pointing formula (parametrization (3.59) follows by integration and simplification)

$$s\frac{\partial}{\partial s}\left[(xr)^2\Phi_{i,\bullet}(xr)\right] = 2x\frac{\partial}{\partial x}\left[(xsr)^2\Phi_0(xs,r^{-1},xr)\right] - \frac{\partial}{\partial t}\left[(xsrt)^2\Phi_\bullet(xs,t,r^{-1},xr)\right]_{t=1}.$$

We will prove the identity by introducing an auxiliary class of doubly rooted irreducible quadrangulations. Indeed, we can interpret the left-hand side as the generating function of doubly rooted irreducible quadrangulations with a white root $\circ$ and a black root $\bullet$ non-adjacent to the white root with a weight x per face, r per black vertex and s per face adjacent to the black root vertex. On the other hand, we know that $x^2s^2r^2\Phi(xs,r^{-1},xr)$ is the generating function of rooted irreducible quadrangulations with a black root $\bullet$ and a weight x per face, r per black vertex and s per face adjacent with the root vertex, so the first term on the right-hand side is just the generating function of all doubly rooted irreducible quadrangulations rooted at a black and white vertex with s keeping track of the black root degree. In order to correctly recover the left-hand side we need to subtract all doubly rooted irreducible quadrangulations with incident black and white root vertices - it is easily checked that the last term on the right-hand side enumerates exactly such maps.

Alternatively, we could have determined the root $\gamma(s,r)$ of the kernel $L(t,r)$. The calculations are done using similar ideas as in the proof of theorem (3.3.10), although in the present case they are rather more involved. □

Despite the complicated functional equation for $\Phi_{i,\bullet}(t,s)$ we have obtained a pretty simple expression for the generating function when expressed in terms of the auxiliary maps τ and ζ. Similar considerations will also yield a surprisingly simple formula for $\Phi_{i,\circ}(t,s)$. Note that via Tutte's bijection, the generating function $\Phi_{i,\circ}(t,s)$ is in a sense a "proper" generalization of $\Phi(t)$ since the corresponding 3-connected planar maps counted by it now have two marked vertices whose degrees are kept track of.

Theorem 3.3.14 (Generating function $\Phi_{i\circ}(t,s)$)**.** *The multivariate generating function $\Phi_\bullet(t,s,r)$ enumerating rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) and a marked white inner vertex with a weight x per inner face, $\sqrt{t}$ per boundary edge, s per inner face adjacent to the marked vertex and r per inner black vertex is given by*

$$\Phi_{i,\circ}(t) = \frac{x^2s^2}{(1+x)^2} + \psi_t(t) - \frac{\tau_t\tau\zeta^2}{u^2v^2t} + \frac{\tau_t t^{-2}u^{-2}v^{-1}(1+u+v)\tau^2\zeta^2(1-\tau)}{(1+v-\tau)(u(1+v)-u(\tau+\zeta)-\tau\zeta)} \tag{3.60}$$

where $\tau = \tau(t,x,r)$ and $\zeta = \zeta(s,x,r)$ are defined as in the statement of theorem (3.3.11) and τ_t denotes the partial derivative of $\tau = \tau(t,x,r)$ with respect to t.

Proof. We will use the functional equation (3.57) satisfied by the unknown generating function $\Phi_{i,\circ}(t,s,r)$ and $\Phi_{i,\circ}(xr,s,r)$ in which the indeterminate t again acts as a catalytic variable. We being by proving bijectively the following identity

$$\Phi_{i,\circ}(xr) = \frac{x^2s^2}{(1+x)^2} - \frac{\zeta^2 w(\zeta,u,v)}{v^2(1+v)^2(1-uv)(1+u-2uv-u^2v)(1-\zeta)^2} \tag{3.61}$$

where $w(\zeta,u,v) = (1+v)(1+u+v)\zeta(\zeta+u-1)+(1+u)(uv-u-1)\zeta-(uv-1)(1+u-2uv-u^2v)$, which is a direct consequence of the following pointing formula

$$\begin{aligned} s\frac{\partial}{\partial s}\left[x^2r^2\Phi_{i,\circ}(xr)\right] = 2x\frac{\partial}{\partial x}\left[x^2s^2r^2\Phi_0(xsr,r,x)\right] - \frac{\partial}{\partial t}\left[x^2s^2r^2t\Phi_\circ(xsr,t,r,x)\right] \\ - s\frac{\partial}{\partial s}\left[x^2s^2r^2\Phi_0(xsr,r,x)\right] \end{aligned}$$

We again introduce the class of doubly rooted irreducible quadrangulations. Clearly, the left-hand side can be interpreted as the generating function of doubly rooted irreducible quadrangulations with two white roots $\circ$ that are not adjacent to a common face with a weight x per face, r per black vertex and s per face adjacent to the second root vertex. On the other hand, we have already shown that $x^2s^2r^2\Phi_0(xsr,r,x)$ is the generating function of all rooted irreducible quadrangulations with a white root and a weight x per face, r per black vertex and s per face adjacent with the root vertex so the first term on the right-hand side is simply the generating function of all doubly rooted irreducible quadrangulations rooted at two (distinct) white vertices with s keeping track of the degree of the second white root. We need to subtract the double rooted maps whose white root vertices are adjacent or equal, which are accounted by the second and the third term respectively. □

We can refine relation (3.36) to account for the colour of the marked vertex, giving the system of relations

$$G_\bullet(t,s,r) = t^2x^2r^2\Phi_{i,\bullet}(txr,s,r) + t^2x^2s^2r^2\Phi_\bullet(txr,s,r)$$

$$G_\circ(t,s,r) = t^2x^2r^2\Phi_{i,\circ}(txr,s,r) + t^2x^2sr^2\Phi_\circ(txr,s,r).$$

Using the explicit formulas for the generating functions derived in the previous theorem this allows us to express the multivariate generating functions $G_\bullet(t,s)$ and $G_\circ(t,s)$ enumerating rooted irreducible quadrangulations with a marked black $\bullet$ (respectively white $\circ$) vertex and a weight x per face, t per face adjacent to the root vertex, s per face adjacent to the marked vertex and r per black vertex. Since the formulas are rather involved we will refrain from writing them out.

We are finally in a position to derive an expression for the generating function of all rooted pointed irreducible quadrangulations. We will no longer distinguish between black and white vertices since we are allowing the distinguished vertex to be of arbitrary colour. Furthermore, it would lead to messy formulas because the parametrization of the indeterminate marking the degree of the pointed vertex depends on the colour of the distinguished vertex. Specializing the last formulas to $r = 1$ amounts to the substitution $v = u$, which yields

Corollary 3.3.2 (Generating function $G(t,s)$) *The multivariate generating function $G(t,s)$ of rooted irreducible quadrangulations with a marked vertex (distinct from the root vertex) with a weight x per face, t per face adjacent to the root vertex and s per face adjacent to the marked vertex is given by*

$$G(t,s) = t^2x^2s^2\Phi_\bullet(tx,s) + t^2x^2s\Phi_\circ(tx,s) + t^2x^2\Phi_{i,\bullet}(tx,s) + t^2x^2\Phi_{i,\circ}(tx,s)$$

where $\tau = \alpha(t,x)$ and $\zeta = \alpha(s,x)$ with functions $\alpha = \alpha(t,x)$ and $u = u(x)$ defined implicitly by the following system of equations and conditions $\alpha(0) = 0, u(0) = 0$:

$$xt = \alpha\frac{1+x-(1+u)\alpha}{1+u-\alpha} \text{ and } x = u(1-u) \tag{3.62}$$

Unfortunately the direct decomposition does not seem to be well suited to counting multipointed rooted irreducible quadrangulations with arbitrarily many distinguished vertices. The main problem stems from the separation of vertices into inner and boundary, which necessitates a distinction of cases. As soon as there is more than one pointed vertex (apart from the root) due to two-step connections arising in the decomposition, depending on the location, a pointed vertex may change from inner to boundary. What's more if some inner distinguished vertices become boundary vertices of a block, their relative position is predetermined, e.g. when counting irreducible quadrangulations with two marked vertices one of the cases in the decomposition requiring special consideration is when both marked vertices are inner, white

and coincide with the white vertices of two consecutive two-step connections of the decomposition - the corresponding block will now have two marked white boundary vertices whose relative distance on the boundary is equal to 2. It is unclear how fixing the relative position of marked boundary vertices affects the generating function.

As far as we can tell the decomposition procedure described above for rooted irreducible quadrangulations is also one of the first direct approaches to counting irreducible quadrangulations. A direct bijective approach has been described by E. Fusy, D. Poulalhon and G. Schaeffer in [27] through a bijection with (a subset of) bicoloured binary tress. Unfortunately, we have been unable to keep track of vertex degrees under the bijection, so our results do not seem to be derivable directly from the bijection. Another related result is the direct decomposition of 3-connected planar maps (also called c-nets) described by M. Bodirsky et al. in [8] which would in principle, via Tutte's bijection, yield a direct decomposition of irreducible quadrangulations. However, their analysis is rather complicated and relies on several auxiliary generating functions so it would be messy and cumbersome to translate it into the language of quadrangulations.

3.4 Enumeration of 2-neighbourhoods

3.4.1 Cut-and-paste technique

Recall that the probability of observing a given neighbourhood $\mathbf{k}$ depends crucially on the location of its diagonals, i.e. inner two-step connections between boundary vertices labelled r that are not incident to the same boundary quadrilateral. We will say that a diagonal between boundary vertices u and v with inner vertex x is *trivial* if x is also connected to every boundary vertex labelled r between u and v in a counterclockwise direction around the boundary of $\mathbf{k}$. It is a simple consequence of irreducibility, that a root neighbourhood of an irreducible quadrangulation has only trivial diagonals at the boundary. Note that the result excludes 1-neighbourhoods. However, a 1-neighbourhood of the root vertex in an irreducible quadrangulation is just a quadrangular star $\mathbf{s}_p$ of degree $p = \deg e_-$.

Lemma 3.4.1 *Let* $\mathbf{k}$ *be the* r*-neighbourhood of a rooted irreducible quadrangulation* $\mathbf{q}$ *with a simple boundary, then* $\mathbf{k}$ *has no non-trivial diagonals provided* $r > 1$*.*

Proof. The proof is purely topological. A diagonal separates the neighbourhood into two connected components by Jordan's curve theorem and the root is contained in the interior of exactly one of the components since we assumed $r > 1$. It follows

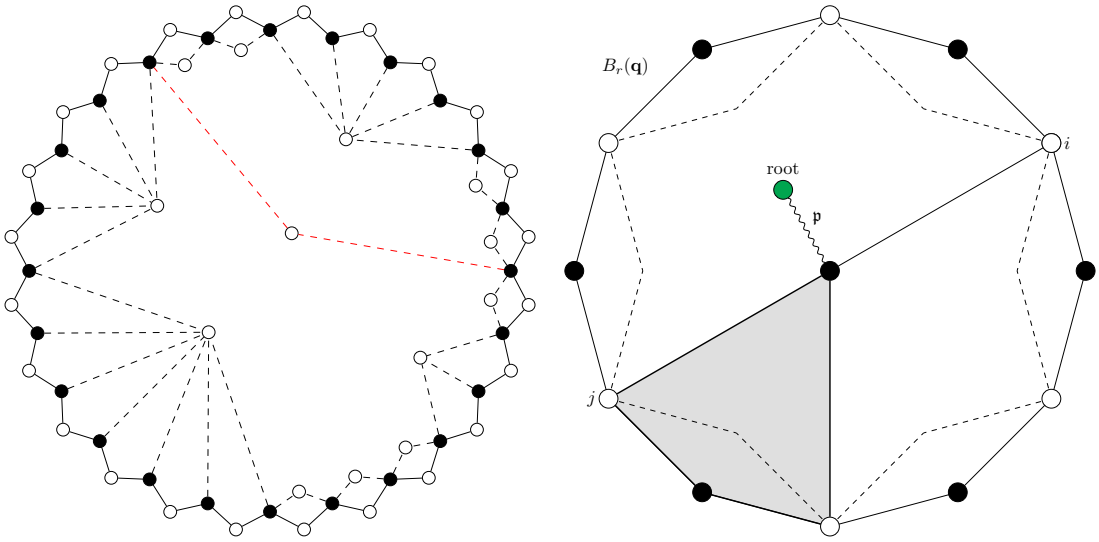


Figure 3.26: On the left is an example of a non-trivial diagonal (in red) and trivial diagonals (in black). Notice that the trivial diagonals are "nested" and the outermost one delimits a sequence of quadrangular faces with a common apex, which will be called a μ-*fan* (where μ is the number of faces making up the fan). On the right is the figure accompanying the proof that a root neighbourhood has no non-trivial diagonals.

that all the boundary vertices of the second component have to be connected by a simple edge to the interior vertex of the diagonal, otherwise their distance to the root vertex would be $> r$ contradicting the assumption that $\mathbf{k}$ is an r-neighbourhood. Now irreducibility reduces all the resulting quadrangles to faces, meaning the diagonal is trivial (see figure below). □

We have described in the previous section a direct decomposition approach to counting rooted irreducible quadrangulations satisfying property (3.3.1), which seems tempting to use to enumerate rooted neighbourhoods according to their diagonal structure. However, the decomposition is not well suited to counting rooted irreducible quadrangulation with simple boundary satisfying property (3.3.1) according to the degree sequence of all of its black boundary vertices (the blocks in the decomposition would no longer be independent, which would lead to messy formulas and a generating function depending on infinitely many variables). We will therefore approach the problem differently.

We will associate to each r-neighbourhood a certain integer composition that will exactly encode the (diagonal) structure of its boundary. Integer compositions and partitions are well-studied and ubiquitous in enumerative combinatorics. Before we

describe the encoding we will provide for convenience sake a short recap of their main properties and set up our notation.

Definition 19. A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ such that $\sum_i \lambda_i = n$. The λ_i are called the *parts* of the partition. We will denote the partition $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ by $\underline{\lambda}$, and we shall write $\underline{\lambda} \vdash n$ to denote "$\underline{\lambda}$ is a partition of n". Let $e_i(\underline{\lambda})$ be the number of parts of $\underline{\lambda}$ that are equal to i, then we will write $\underline{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ and $\bar{\lambda} = (1^{e_1}, \ldots, k^{e_k})$ to distinguish whether the partition is given in terms of its parts or exponents, respectively.

An *integer composition* of a positive integer n is a finite sequence of positive integers $\pi_1, \pi_2, \ldots, \pi_\ell$ such that $\sum_i \pi_i = n$. The π_i are called the *parts* of the composition. We will denote the composition $(\pi_1, \pi_2, \ldots, \pi_\ell)$ by $\underline{\pi}$, and we shall write $\underline{\pi} \models n$ to denote "$\underline{\pi}$ is a composition of n". We will denote by $\mathscr{C}(n, \ell)$ the set of all integer compositions of n with ℓ parts and by $\mathscr{C}(n)$ the set of all integer compositions of n; it is well-known that their cardinalities are

$$\#\mathscr{C}(n, \ell) = \binom{n-1}{\ell-1} \text{ and } \#\mathscr{C} = 2^{n-1}.$$

We can associate with every integer composition $\underline{\pi}$ a unique integer partition $\underline{\lambda}$ obtained by arranging the parts of the composition in a non-increasing order of magnitude. The partition $\underline{\lambda}$ (or $\bar{\lambda}$) will be called the *type* of the composition $\underline{\pi}$. We denote by $\mathscr{C}(e_1, \ldots, e_k)$ the set of all integer compositions of type $\bar{\lambda}$ (we have the relations $n = e_1 + 2e_2 + \cdots + ke_k$ and $\ell = e_1 + e_2 + \cdots + e_k$); its cardinality is

$$\#\mathscr{C}(e_1, \ldots, e_k) = \frac{n}{\ell}\binom{\ell}{\bar{\lambda}} = \frac{n}{\ell}\binom{\ell}{e_1, \ldots, e_k}.$$

Given two integer compositions $\underline{\mu}$ and $\underline{\tau}$ of the same length ℓ, we write $\underline{\tau} \leq \underline{\mu}$ if $\tau_i \leq \mu_i$ for every $i = 1, 2, \ldots, \ell$. The multivariate generating function $C_\ell(\underline{t}; z)$ of compositions of length ℓ with a weight t_i marking the size of the ith part and weight z marking the sum of the parts of the composition is easily determined to be

$$C_\ell(t_1, \ldots, t_\ell; z) = \frac{t_1 \cdots t_\ell z^\ell}{(1 - zt_1) \cdots (1 - zt_\ell)}.$$

Recall that $\mathbf{q}$ is a rooted irreducible quadrangulation and $\mathbf{k}$ is an r-neighbourhood of $\mathbf{k}$ with radius $r > 1$. Since $\mathbf{k}$ has no non-trivial diagonals its boundary consists of a sequence of quadrangular fans. Since $\mathbf{q}$ is rooted we can mark (in some deterministic way) one of the quadrangular fans and root the neighbourhood at the unique left-most vertex of the boundary corresponding to the marked fan (the precise location of the

root edge is not really important as long as it is chosen in a deterministic way). We associate to $\mathbf{k}$ the *composition parameter* composed of the sizes of fans, enumerated in a counterclockwise direction, starting with the marked fan (see figure below). We will always assume from now on that any root neighbourhood has been equipped with this additional data.

According to lemma (3.3.4) the admissible complements of a given rooted proper irreducible quadrangulation $\mathbf{k}$ depend only on the diagonal structure of $\mathbf{k}$, which is now encoded in terms of its composition parameter. In other words, the set of admissible complements of any two proper irreducible dissections $\mathbf{k}$ and $\mathbf{s}$ with the same composition parameter are equal. We will say a rooted proper irreducible quadrangulation $\mathbf{s}$ is $\underline{\pi}$-*admissible* if the connected sum $\mathbf{k}\#\mathbf{s}$ is an irreducible quadrangulation for any proper irreducible dissection $\mathbf{k}$ with composition parameter $\underline{\mu}$.

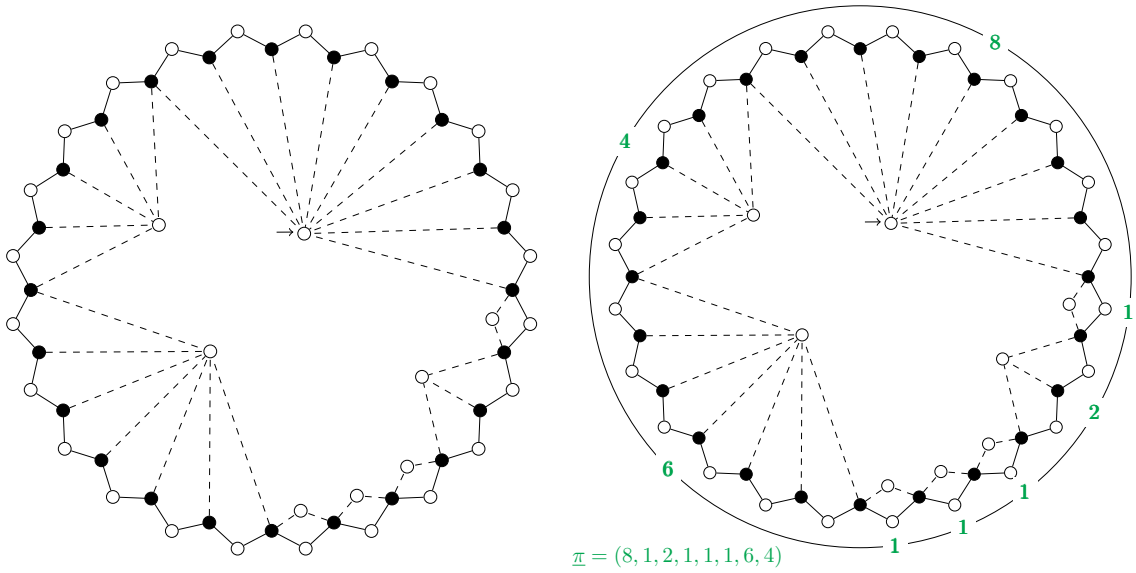


Figure 3.27: The assignment of a composition parameter, where the arrow indicates the distinguished initial fan.

Definition 20. Let $p \geq 3$ be a positive integer and $\underline{\pi} \models p$ a composition of p of length ℓ. We denote by $q(\underline{\pi}; x)$ the generating function of all rooted $\underline{\pi}$-admissible irreducible dissections of the $2p$-gon and we introduce the following quantity

$$Q_\ell(t_1, \ldots, t_\ell; x) = \sum_p \sum_{\underline{\pi} \in \mathscr{C}(p,\ell)} q(\underline{\pi}; x) t_1^{\pi_1} \cdots t_\ell^{\pi_\ell} \tag{3.63}$$

the multivariate generating function of all $\underline{\pi}$-admissible rooted irreducible quadrangulations with combination parameter of length ℓ with a weight x per inner face and a weight t_i per each term i of the composition $\underline{\pi}$.

The main result of this section is the following relation between the multivariate generating function $Q_\ell(\underline{t};x)$ of $\underline{\pi}$-admissible rooted irreducible quadrangulations and the multivariate generating function $C_\ell(\underline{t};z)$ of compositions. Before stating the theorem let us pause to introduce a final crucial operation on formal power series, namely the Hadamard product.

Definition 21. Let $A(z)$ and $B(z)$ be (formal) power series about the origin with series expansions

$$A(z)=\sum_{\nu\geq 0} a_\nu z^\nu \text{ and } B(z)=\sum_{\nu\geq 0} b_\nu z^\nu.$$

The *Hadamard product* of $A(z)$ and $B(z)$, usually denoted by $A(z)\odot B(z)$, is the formal power series

$$A(z)\odot B(z)=\sum_{\nu\geq 0} a_\nu b_\nu z^\nu.$$

Although Hadamard product is not easily evaluated for the case of general generating functions $A(z)$ and $B(z)$, in the case, $A(z)$ say, is a rational function partial fraction decomposition and the following easily proven lemma allows for a simple computation of Hadamard products.

Lemma 3.4.2 *Let $F(z)$ be a formal power series about the origin , $t\in\mathbb{C}$ an arbitrary complex number and $r\geq 0$ a non-negative integer, then*

$$F(z)\circ_z \frac{z^r}{(1-zt)^{r+1}}=\frac{z^r}{r!}F^{(r)}(zt).$$

Theorem 3.4.1. *The multivariate generating function $Q_\ell(\underline{t};x)$ of all $\underline{\pi}$-admissible rooted irreducible quadrangulations with combination parameter of length ℓ with a weight x per inner face and a weight t_i per each term i of $\underline{\pi}$ is given by*

$$Q_\ell(x^{-1}\Psi(zt_1),\ldots,x^{-1}\Psi(zt_\ell))=Q(z)\odot_z C_\ell(t_1,\ldots,t_\ell;z) \tag{3.64}$$

where $\Psi(t,x)=(1+x^{-1})t^{-1}Q(t,x)-x^2t$ and $\odot$ denotes the Hadamard product of generating functions.[9]

Proof. Let $\mathbf{k}$ be a rooted proper irreducible dissection of the $2p$-gon with combination parameter $\underline{\pi}$. We will assume that the black boundary vertices of $\mathbf{k}$ are labelled from 1 to p in a counterclockwise direction around the boundary starting with the black extremity of the root edge. We will also adopt the convention that the boundary

[9]The meaning of the indeterminate z appearing in formula (3.64) can be deduced from the proof; its exponent is equal to half the total length of the boundary of an admissible complement.

vertices of any $\mathbf{k}$-admissible rooted irreducible quadrangulation $\mathbf{s}$ inherit the induced labelling of $\mathbf{k}$ coming from the connected sum operation (note that the root-edge of $\mathbf{s}$ is thus uniquely specified).

The proof of equation (3.64) is based on the following simple observation: an arbitrary proper irreducible quadrangulations $\mathbf{s}$ of the $2p$-gon is either $\mathbf{k}$-admissible or else it has a certain set of diagonals in common with $\mathbf{k}$. We assign to $\mathbf{k}$ the set $\mathrm{Diag}(\mathbf{k})$, which we call the *principal diagonal set*, defined as the Cartesian product

$$\mathrm{Diag}(\mathbf{k}) = \mathscr{C}(\pi_1) \times \cdots \times \mathscr{C}(\pi_\ell),$$

where $\underline{\pi} = (\pi_1, \ldots, \pi_\ell)$ and $\mathscr{C}(\pi)$ denotes the set of all compositions of π. We will justify the name of the set $\mathrm{Diag}(\mathbf{k})$ by associating to each element $(\underline{d}_1, \ldots, \underline{d}_\ell)$ a unique set of diagonals of $\mathbf{k}$. According to lemma (3.4.1) the dissection $\mathbf{k}$ has no non-trivial diagonals, so it suffices to justify that the set of all possible diagonals of a μ-fan is in one-to-one correspondence with the set $\mathscr{C}(\mu)$ of all combinations of μ. However, the set of diagonals of a fan is just the set of all subtrees containing the apex which can equivalently be encoded by compositions.

Let $\underline{d} \in \mathrm{Diag}(\mathbf{k})$ be a diagonal set. We introduce the auxiliary generating function $e(\underline{d}; x)$ of all rooted proper irreducible quadrangulations of the $2p$-gon, which have apart from the diagonals between consecutive white boundary vertices, whose existence follows the definition, exactly the diagonals $\underline{d}$ in common with $\mathbf{k}$ with a weight x per inner face. Since $\mathrm{Diag}(\mathbf{k})$ is in one-to-one correspondence with all possible subsets of diagonals of $\mathbf{k}$, the observation gives the following relation between the generating functions

$$q_{2p}(x) = \sum_{\underline{d} \in \mathrm{Diag}(\mathbf{k})} e(\underline{d}; x). \tag{3.65}$$

It remains to determine the auxiliary sequence $\{e(\underline{d})\}$ of generating functions. We use the cut-and-paste technique depicted below. Let k_i denote the length of composition $\underline{d}_i$ of the diagonal set $\underline{d}$, then the auxiliary generating function is given by

$$e(\underline{d}; x) = x^{-(k_1 + \cdots + k_\ell)} q((k_1, \ldots, k_\ell); x) \prod_i w(\underline{d}_i) \tag{3.66}$$

where the weight is given by $w(a) = [t^a]\Psi(t, x)$ with $\Psi(t, x) = t^{-1}(1 + x^{-1})Q(t, x) - tx^2$ and extended to sequence termwise as $w(a_1, \ldots, a_n) = w(a_1) \cdots w(a_n)$.

Plugging (3.66) into the recurrence relation (3.65) and grouping together all the terms with the same q_* we obtain

$$q_{2p}(x) = \sum_{\underline{1}_\ell \leq \underline{\tau} \leq \underline{\pi}} q(\underline{\tau}; x) \prod_i \sum_{\mathscr{C}(\pi_i, \tau_i)} x^{-\tau_i} w(a_{i1}) \cdots w(a_{i\tau_i})$$

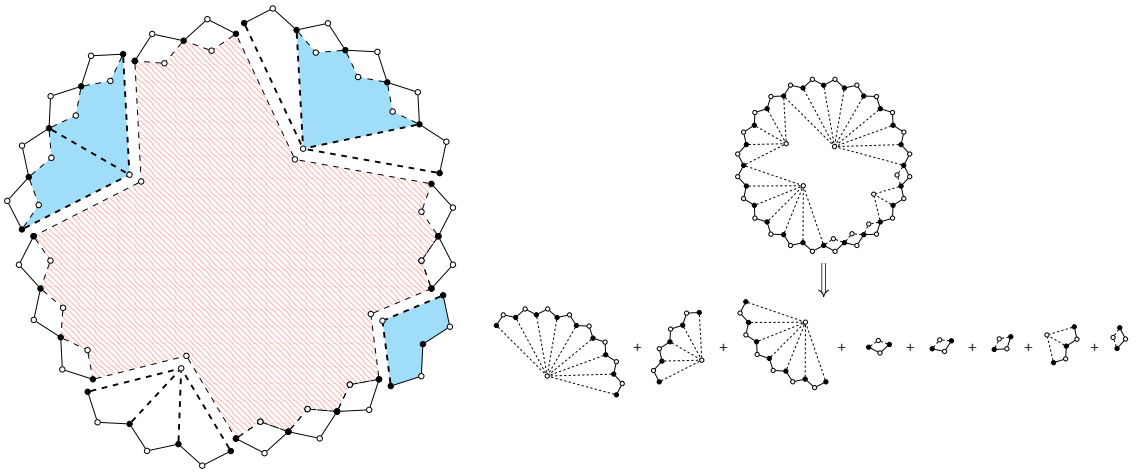


Figure 3.28: The cut and paste decomposition of the auxiliary sequence $e(\underline{d}; x)$ of generating functions. The inner region (shown in red) can be filled with any proper rooted irreducible quadrangulation with combination parameter $(k_1, \ldots, k_\ell)$ with boundary quadrilaterals removed, while each part corresponding to a set of common diagonals (shown in blue) is either a face with contribution tx if its total boundary length is 4 or $[t^{a+1}](1 + x^{-1})(Q(t, x) - t^2x^2)$ if its boundary length is $2(a + 1) > 4$ (the factor $(1 + x^{-1})$ accounts for the possibility that the face corresponding to the diagonal can either be present in the map or not).

$$\begin{aligned}
&= [t_1^{\pi_1} \cdots t_\ell^{\pi_\ell}] \sum_{\underline{1}_\ell \leq \underline{\tau} \leq \underline{\pi}} q(\underline{\tau}; x) \{x^{-1}\Psi(t_1)\}^{\tau_1} \cdots \{x^{-1}\Psi(t_\ell)\}^{\tau_\ell} \\
&\overset{*}{=} [t_1^{\pi_1} \cdots t_\ell^{\pi_\ell}] \sum_{\underline{1}_\ell \leq \underline{\tau}} q(\underline{\tau}; x) \{x^{-1}\Psi(t_1)\}^{\tau_1} \cdots \{x^{-1}\Psi(t_\ell)\}^{\tau_\ell} \\
&= [t_1^{\pi_1} \cdots t_\ell^{\pi_\ell}] Q_\ell(x^{-1}\Psi(t_1, x), \ldots, x^{-1}\Psi(t_\ell, x))
\end{aligned}$$

where equality $(*)$ follows from the observation $x^{-1}\Psi(t, x) = t + \mathcal{O}(t^2)$ meaning that only terms with composition parameter $\leq \underline{\pi}$ contribute to the coefficient of $t_1^{\pi_1} \cdots t_\ell^{\pi_\ell}$. Multiplying both sides by z^p, using $p = \pi_1 + \cdots + \pi_\ell$ and summing over all $(p, \underline{\pi})$ gives the identity (3.64). □

Lemma 3.4.3 *The (formal) power series $x^{-1}\Psi(t, x) = t^{-1}x^{-1}(1 + x^{-1})Q(t, x) - tx$ has a unique compositional inverse $\psi(t, x) \in \mathbb{Z}[\![t, x]\!]$ given explicitly as follows*

$$\begin{aligned}
\psi(t, x) = -\frac{1 + x}{2x(1 + xt)(1 + u)^3} \Big\{ & ux^2t^2 - xt(1 + 3u + 4u^2) - u(1 + u - u^2) \\
& - (xt - u)\sqrt{u^2x^2t^2 - 2(1 + 3u + 3u^2)uxt + (1 + u - u^2)^2} \Big\}.
\end{aligned} \tag{3.67}$$

The inverse $\psi(t, x)$ has a unique singularity of minimal modulus at $x = x_0$ for each $t \in 2\mathbb{D}$ and admits a singular expansion of the form

$$\psi(t, x) = -20t^{-1}\alpha_0(t/4) + \frac{t(t + 84)}{16(t + 4)} + \beta(t)X^2 - 20t^{-1}\alpha_3(t/4)X^3 + \mathcal{O}(X^4).$$

Proof. We have already remarked that the first few terms of the expansion about the origin of the power series are $x^{-1}\Psi(t,x) = t + \mathcal{O}(t^2)$ so a compositional inverse $\psi(t,x)$ (with respect to t) exists and is unique. Using the quadratic equation (3.9) derived in theorem (3.3.1) and fundamental relation (3.13) we can first find a quadratic equation satisfied by $Q(t)$ and then deduce a quadratic equation satisfied by $x^{-1}\Psi(t)$. Making a substitution $t \to \psi(t)$ yields a quadratic for the compositional inverse $\psi(t)$ with two solutions of which only one is of order $t + \mathcal{O}(t^2)$ and given by (3.67). We leave the details to the reader.

The final part of the lemma concerning the singularity structure of $\psi(t,x)$ follows from lemma (3.3.7), keeping in mind that the polynomial appearing under the square-root is $\sigma(xt,x)$ so the critical t-value t_1 is now $x_0 t_1 = t_0$, while the singular expansion can be derived using `Mathematica`. □

An astute reader will have already noticed the remarkable similarity between the compositional inverse $\psi(t,x)$ and the generating function $\Phi_0(xt,x)$ of all rooted irreducible quadrangulations. Indeed, by direct inspection we have

$$\psi(t,x) = \frac{t(1-x^2t)}{1+xt} - t(1+x)\Phi_0(xt,x). \tag{3.68}$$

An interesting corollary of theorem (3.4.1) and lemma (3.4.3) is that the multivariate generating functions Q_ℓ are completely symmetric in the variables $(t_1,\ldots,t_\ell)$ marking the valency of black boundary vertices. Consequently,

Corollary 3.4.1 *Let* $\mathbf{k}$ *be an* r*-neighbourhood, then the number* $c_{\mathbf{k}}(N)$ *of* $\mathbf{k}$*-admissible quadrangulations with* N *faces depends only on the partition parameter* $\underline{\lambda}$ *of* $\mathbf{k}$*.*

We can now write explicitly a formula for the generating function $Q_\ell(\underline{t})$ by noting (according to the definition of ψ) that $Q(\psi(t)) = x^2(1+x)^{-1}(x\psi(t)+t)\psi(t)$, so

$$\begin{aligned} Q_\ell(t_1,\ldots,t_\ell;x) &= \frac{x^2}{1+x}\sum_{i=1}^{\ell}\psi(t_i)(x\psi(t_i)+t_i)\prod_{j=1,j\neq i}^{\ell}\frac{\psi(t_j)}{\psi(t_i)-\psi(t_j)} \\ &= \frac{x^2}{1+x}\sum_{i=1}^{\ell}t_i\psi(t_i)\prod_{j=1,j\neq i}^{\ell}\frac{\psi(t_j)}{\psi(t_i)-\psi(t_j)} \end{aligned} \tag{3.69}$$

It is possible to show that on an appropriate slit domain (which corresponds to taking the principal value of the square-root in the definition of $\psi(t)$) the function $\psi(t)$ is schlicht for each fixed $x \in (0,x_0)$. However, we can avoid the unnecessary calculation involved in checking univalence by using the following simpler argument. The formula (3.69) makes sense for any $(t_1,\ldots,t_\ell)$ with pairwise distinct coordinates

and can be made precise even for arguments with some coordinates equal by a limiting process. Hence, apart from singularities of $\psi(t,x)$, the only possible source are zeros of denominators. We will show that there exists $\epsilon > 0$ and $\phi \in (0, \pi/2)$ such that

$$\frac{\psi(t) - \psi(s)}{t - s} \neq 0$$

for any $x \in \Delta(\epsilon, \phi)$ and $t, s \in 2\mathbb{D}$. Indeed, using relation (3.68) we can write the difference quotient in the form

$$\frac{\psi(t) - \psi(s)}{t - s} = \frac{1 - x^2(s + t + xst)}{(1 + sx)(1 + tx)} \left(1 - \frac{(1 + x)(1 + sx)(1 + tx)}{1 - x^2(s + t + xst)} \frac{t\Phi_0(xt) - s\Phi_0(xt)}{t - s}\right).$$

Note that trivially (using say triangle inequality) the initial factor is well-defined and non-zero on $(t, s, x) \in 2\mathbb{D} \times 2\mathbb{D} \times \Delta(\epsilon, \phi)$ provided $\epsilon > 0$ is small enough. On the other hand, the subtrahend of the second factor can be expanded in a power series about the origin with respect to (t, s, x) and has non-negative coefficients. This is clearly true for the difference quotient of $t\Phi_0(xt)$ since its coefficients are non-negative (using their combinatorial interpretation) and it is easily checked by the binomial theorem that the same is true for the factor $(1+sx)(1+tx)(1-x^2(s+t+xst))^{-1}$. Furthermore, both series are absolutely convergent on $2\mathbb{D} \times 2\mathbb{D} \times x_0\overline{\mathbb{D}}$. It now follows easily using lemma (3.1.2) that the maximal value of the subtrahend on $2\overline{\mathbb{D}} \times 2\overline{\mathbb{D}} \times x_0\overline{\mathbb{D}}$ is achieved at the point $(2, 2, 1/4)$ and is equal to $13/33$. It follows (by continuity) that there exists some $\epsilon > 0$ (sufficiently small) and $\phi \in (0, \pi/2)$ such that the difference quotient is non-zero for $|t|, |s| < 2$ and $x \in \Delta(\phi, \epsilon)$.

As such on the ℓ-torus $\mathbb{D}(t_0)^\ell$ the unique singularity of minimal modulus occurs at x_0 and we can easily deduce the singular expansion about x_0 to be of the form

$$\alpha_1(\underline{t}) + \alpha_2(\underline{t})(1 - 4x) + \alpha_3(\underline{t})(1 - 4x)^{3/2} + \mathcal{O}((1 - 4x)^2)$$

The first few terms in an expansion of the dominant term about the origin are

$$\begin{aligned}
\alpha_3(\underline{t}) &= \frac{1}{250} e_{(1_3)} + \frac{17}{3125} e_{(2,1_2)} + \frac{2651}{625000} e_{(3,1_2)} + \frac{3803}{625000} e_{(2_2,1)} + \cdots \\
\alpha_4(\underline{t}) &= \frac{1}{125} e_{(1_4)} + \frac{991}{125000} e_{(2,1_3)} + \frac{17923}{3125000} e_{(3,1_3)} + \frac{20739}{3125000} e_{(2_2,1_2)} + \cdots \\
\alpha_5(\underline{t}) &= \frac{11}{1000} e_{(1_5)} + \frac{5783}{625000} e_{(2,1_4)} + \frac{395821}{62500000} e_{(3,1_4)} + \frac{434093}{62500000} e_{(2_2,1_3)} + \cdots
\end{aligned}$$

where we have used the standard notation for elementary symmetric polynomial $e_{\underline{\lambda}}(t_1, \ldots, t_r)$ of a partition $\underline{\lambda}$ with r parts. The formulas clearly reflect the observation of corollary (3.4.1). We are precisely in a situation where the transfer theorem may be applied to deduce an asymptotic expansion for the coefficients of the generating function $Q_\ell(\underline{t}; x)$. We summarize our findings in the following lemma:

Lemma 3.4.4 *Let $\ell \geq 3$ be a fixed positive integer and $t_1, \ldots, t_\ell \in [0, t_0]$ then the coefficients $[x^n]Q_\ell(\underline{t}; x)$ admit the following asymptotic expansion*

$$[x^n]Q_\ell(\underline{t}; x) = \frac{3}{4\sqrt{\pi}}\alpha_3(\underline{t})n^{-5/2}x_0^{-n}\left(1 + \mathcal{O}(n^{-1})\right) \text{ as } n \to \infty. \tag{3.70}$$

In particular, if $\underline{\lambda}$ denotes a (fixed) partition with ℓ parts and we denote by $c(\underline{\lambda})$ the coefficient of $e_{\underline{\lambda}}$ in $\alpha_3(\underline{t})$, then the coefficients $[x^n]q(\underline{\lambda}; x)$ admit an asymptotic expansion of the form

$$[x^n]q(\underline{\lambda}; x) = \frac{3}{4\sqrt{\pi}}c(\underline{\lambda})n^{-5/2}x_0^{-n}\left(1 + \mathcal{O}(n^{-1})\right) \text{ as } n \to \infty.$$

For future reference we will also denote by $a(\underline{\lambda})$ and $b(\underline{\lambda})$ the coefficient of $e_{\underline{\lambda}}$ in $\alpha_1(\underline{t})$ and $\alpha_2(\underline{t})$, respectively. Hence the generating function $q(\underline{\lambda}; x)$ of $\underline{\lambda}$-admissible quadrangulations admits a singular expansion of the form

$$q(\underline{\lambda}; x) = a(\underline{\lambda}) + b(\underline{\lambda})X^2 + c(\underline{\lambda})X^3 + \mathcal{O}(X^4).$$

3.4.2 Generating function $N_2(t, z, x)$

We introduce the final generating function of rooted irreducible quadrangulations that will be needed in the sequel. Let $N_2(t, z, x)$ denote the multivariate generating function of rooted irreducible quadrangulations with a simple boundary of arbitrary length $2p$ ($p \geq 2$) satisfying property (3.3.1) with a weight x per inner face, a weight $\sqrt{z}$ per boundary edge and weight t per inner quadrilateral adjacent to a black boundary vertex (in other words, the exponent of t in $N_2(t, z, x)$ is the total degree of all black boundary vertices).

Alternatively, we can also view $N_2(t, xtz, x)$ as the multivariate generating function of all rooted irreducible quadrangulations with a weight x per face, a weight z per face in the 1-neighbourhood of the root and a weight t per face in the 2-neighbourhood of the root. The following theorem provides an exact expression for the generating function:

Theorem 3.4.2. *The multivariate generating function $N_2(t, z, x)$ counting rooted irreducible quadrangulations according to the number of faces and the volume of the* 1- *and* 2*-neighbourhood of the root vertex is given by*

$$\begin{aligned} N_2(t, z, x) &= \frac{z}{1+z}Q((1+z)\psi(t, x), x) - z(1-z)Q(\psi(t, x), x) - z^2\psi(t)Q_t(\psi(t, x), x) \\ &= z^3(t^3x^3 + 3t^4x^5 + (t^3 + 3t^5)x^6 + (6t^4 + 9t^5)x^7 + \cdots) \\ &\quad + z^4(t^4x^4 + 2t^4x^5 + 8t^5x^6 + (4t^4 + 12t^5 + 6t^6)x^7 + \cdots) \\ &\quad + z^5(t^5x^5 + 5t^5x^6 + (5t^5 + 15t^6)x^7 + \cdots) + \cdots \end{aligned}$$

Proof. Using lemma (3.4.2), the generating function $N_2(t,z,x)$ is given by

$$N_2(t,z,x) = \sum_{\ell\geq 3} z^\ell Q_\ell(t,\ldots,t;x) = \sum_{\ell\geq 3} \frac{z^\ell \psi^\ell(t)}{(\ell-1)!} \left\{ \frac{\partial^{\ell-1}}{\partial \zeta^{\ell-1}} \zeta^{-1} Q(\zeta;x)\Big|_{\zeta=\psi(t)} \right\}.$$

Re-arranging and using Taylor's theorem gives the result. □

The generating function $N_2(t,z,x)$ is a composition of two algebraic generating functions whose dominant singularity (with respect to x) is $x_0 = 1/4$. Referring to the composition scheme, we are in the third case where confluence of singularities occurs and a singular expansion of the composition is obtained by a combination of the singular expansions of each factor.

We already know, using the lemma on minimal singularities (3.3.7) of $\Phi(t)$, that for $t \in 2\mathbb{D}$ the unique singularity of minimal modulus of $\psi(t,x)$ occurs at x_0. On the other hand, the generating function $Q(t,x)$ is expressible in terms of ψ via the following relation

$$Q(t,x) = t^2x^2 - 1 + \frac{(1-u)(1+2u)^2}{(1+x)^2} t(1+x-xt) - \frac{x(1-u)(1+u)^3}{(1+x)^2}\psi(s)$$

where $x(1+x)s = (1-u)(1+u)^3t-(1+x)$. We will show that for every $(t,z) \in \mathbb{D}\times t_0\mathbb{D}$ the generating function $N_2(t,z,x)$ has a unique singularity of minimal modulus at the point $x = x_0$ (recall that $t_0 = 1/2$).

We will first find a more convenient representation for the generating function. It is intuitively clear that the main source of singularities will be the zeros of quadratic polynomials appearing under the square root. The lemma (3.3.7) on minimal singularities of $\Phi(t,x)$ classifies the singularities of minimal modulus of $\psi(t)$. We need to consider the second source of singularities arising from the composition $\psi(s)$ with $x(1+x)s = (1-u)(1+u)^3(1+z)\psi(t) - (1+x)$. Clearly, only zeros of the quadratic polynomial $\sigma(s,x)$ need to be considered. We can factor the polynomial as $\sigma(s,x) = \sigma_1^2(t,x)\sigma_2(z,t,x)$ where

$$\begin{aligned} f_1(u) &= \frac{u(1-u)}{1+u-u^2}(1+3u+2\sqrt{u(1+2u)}) \\ f_2(u) &= \frac{u(1-u)}{1+u-u^2}(1+3u-2\sqrt{u(1+2u)}) \\ \sigma_2(z,t,x) &= \left(1 - z\frac{f_1(u)\psi(t)}{1-f_1(u)\psi(t)}\right)\left(1 - z\frac{f_2(u)\psi(t)}{1-f_2(u)\psi(t)}\right). \end{aligned}$$

and the initial factor is $\sigma_1^2(t,x) = (1-f_1(u)\psi(t))(1-f_2(u)\psi(t))$, which can further be simplified (after a lot of tedious algebraic calculations) as

$$\sigma_1^2(t,x) = \left[\frac{x^2(1+u)^3\psi^2(t) - (1-u)(1+x)(1+3u+4u^2-2uxt)\psi(t) + (1+x)^2}{(1+x)(u(1+x)-x(1+u)^2\psi(t))}\right]^2.$$

It thus does not introduce any additional singularities in $N_2(t,z,x)$. To simplify the representation even further we now introduce the following parametrization

$$xt = \alpha \frac{1+x-(1+u)\alpha}{1+u-\alpha}.$$

The auxiliary function $\alpha = \alpha(t,x)$ admits a bivariate expansion in (t,x) about the origin with positive coefficients (see proof of lemma (4.1.3) of chapter 3). In terms of the new parameters (u,α) the generating function N_2 takes the following form

$$\begin{aligned}
N_2 &= -\frac{z}{2(1+z)} + \frac{z\alpha}{2u(1+u)(u+\alpha)^2}\left\{P_1(\alpha) - \frac{z\alpha}{u+u^2-2u\alpha-\alpha^2}P_2(\alpha)\right\} \\
&\quad + \frac{z(u+u^2-2u\alpha-\alpha^2)(u^2-\alpha+\alpha^2-\alpha(1+u-\alpha)z)}{2u(1+u)(u+\alpha)^2(1+z)}\sqrt{(1-zr_1)\,(1-zr_2)} \\
P_1(\alpha) &= u+3u+4u^3-u(2-u)\alpha-(1-2u)\alpha^2+\alpha^3 \\
P_2(\alpha) &= u(1+5u+5u^2-u^3)-4u^2(2+3u)\alpha+(1+u)(1-4u)\alpha^2-2\alpha^2+\alpha^4.
\end{aligned}$$

The reciprocal roots $r_i(t,z)$ can be nicely expressed in terms of the parameter

$$w = x^{1/2} = \frac{q(1-q^2+q^4)^{1/2}}{1+q^4} \text{ and } u = \frac{q^2}{1+q^4}.$$

It is easily shown that the new parameter q satisfies $q = u(u(x)^2)^{1/2}u(x)^{-1/2}$ and has a power series expansion in $w = x^{1/2}$ with positive coefficients. We have

$$\begin{aligned}
r_1(t,x) &= \frac{(1+q^4)\alpha(1+q^2+q^4-(1+q^4)\alpha)}{(q-q^2+q^3-(1+q^4)\alpha)^2} \\
r_2(t,x) &= \frac{(1+q^4)\alpha(1+q^2+q^4-(1+q^4)\alpha)}{(q+q^2+q^3+(1+q^4)\alpha)^2}
\end{aligned}$$

Note that the form above also suggests a way of introducing a new parameter $\beta = \beta(t,z,x)$, which would yield a rational parametrization of the generating function. The next lemma settles the question of singularities of minimal modulus of N_2:

Lemma 3.4.5 *Let $r_i(t,x)$ denote the reciprocal zeros of the polynomial $\sigma_2(z)$ and $b \in (0,2)$, then there exist $\epsilon > 0$ (sufficiently small) and $\phi \in (0,\pi/2)$ such that*

$$\begin{aligned}
|r_2(t,x)| &\leq |r_1(t,x)| \\
|r_1(t,x)| &\leq r_1(b,x_0).
\end{aligned}$$

for all $(t,x) \in b\mathbb{D} \times \Delta(\epsilon,\phi)$.

It follows that for any $b \in (0,2)$, the unique singularity of minimal modulus of the square-root occurs at $x = x_0$ for all $(t,z) \in b\mathbb{D} \times c\mathbb{D}$, where the critical value $c = c(b)$ of parameter z is given explicitly as

$$c = \frac{2-b}{36b}(10 + b + \sqrt{(2-b)(50-b)}).$$

In particular, for $b = 1$ the critical value is $c(1) = 1/2$. Before we proceed with the proof of lemma (3.4.5), we will prove the following auxiliary lemma that will be needed to establish positivity of coefficients in power series expansions of various functions (alternatively, we could have again used the theory of positive definite matrices to establish the required inequalities; in particular, note that the succeeding lemma can be used to simplify the proof of lemma (3.3.7) since it implies at once that $t_-^{-1}(x)$ has a positive power series expansion):

Lemma 3.4.6 *Let the function $u = u(x)$ be implicitly defined by $u(1-u) = x$ and the condition $u(0) = 0$, then the function*

$$r(x) = (1+x)^{-1}\sqrt{u(1+2u)}$$

admits a power series expansion in $w = x^{1/2}$ with non-negative coefficients.

Proof. We introduce a new change of variables as $u = x(1+v)$, then the function $v = v(x)$ is implicitly defined by the equation $v = x(1+v)^2$ and condition $v(0) = 0$. It again admits a power series expansion in x with non-negative coefficients and in terms of v the function takes the form $r(x) = x^{1/2}(1+x)^{-1}\sqrt{1+3v}$. Unfortunately, using Lagrange's inversion to determine its coefficients does not make it clear that the coefficients are positive since the final sum representation is alternating. We will use the following trick to prove the lemma; we re-write the function $r(x)$ as

$$w^{-1}r(x) = \frac{\sqrt{1+3v}}{1+x} = \frac{\sqrt{1+v}}{1+x}\sqrt{1+\frac{2v}{1+v}} = \frac{1}{1+x}\sqrt{\frac{1-\sqrt{1-4x}}{2x}}\sqrt{2-\sqrt{1-4x}}.$$

The coefficients a_k of $\sqrt{1+v}$ can be explicitly evaluated (see [39]) and the expansion has the form

$$\sqrt{\frac{1-\sqrt{1-4x}}{2x}} = \sum_{k=0}^{\infty}\binom{4k}{2k}\frac{x^k}{4^k(2k+1)} = 1 + \frac{1}{2}x + \frac{7}{8}x^2 + \frac{33}{16}x^3 + \mathcal{O}(x^4).$$

Using elementary properties of binomial coefficients it is not difficult to show that the sequence of coefficients is unimodal at a_1. Moreover, straightforward computation (as

in the proof of lemma (3.3.7)) reveals that $(1+x)^{-1}\sqrt{1+v}$ has positive coefficients b_k, except the linear term $b_1 = -1/2$, which is negative, viz.

$$\frac{1}{1+x}\sqrt{\frac{1-\sqrt{1-4x}}{2x}} = 1 - \frac{1}{2}x + \frac{11}{8}x^2 + \frac{11}{16}x^3 + \mathcal{O}(x^4).$$

On the other hand, the sequence $\{c_k\}_{k\geq 0}$ of coefficients of $(2-\sqrt{1-4x})^{1/2}$ satisfies the linear difference equation (as can be checked with say `Mathematica`):

$$c_{k+2} = \frac{4(2k+1)}{3(k+2)}c_{k+1} + \frac{(4k-1)(4k+1)}{3(k+1)(k+2)}c_k$$

with initial conditions $c_0 = c_1 = 1$. It is not difficult to show that $c_2 = 1/2$ and the sequence is positive and increasing for $k \geq 2$. Taking the product we find that the coefficients r_n are positive for every $n \geq 0$,

$$r_n = \sum_{k=0}^{n} b_k c_{n-k} = c_n - \frac{1}{2}c_{n-1} + \sum_{k=2}^{n} b_k c_{n-k} > 0. \quad \square$$

Proof. Let us first check that both roots are well-defined and analytic on $2\mathbb{D}\times\Delta(\epsilon',\phi')$ for a sufficiently small $\epsilon' > 0$ and some $\phi' \in (0,\pi/2)$. We have found it easier to work with the normalized function α_0 related to α through the simple substitution

$$\alpha(t,x) = \frac{xu(1+u)}{1+x}\alpha_0(t,x).$$

Note that $\alpha_0(t,x) = t + \mathcal{O}(t^2)$ and it is easily checked using the ideas introduced in the proof of lemma (4.1.3) that the normalized function still has non-negative coefficients in a power series expansion about the origin. Furthermore, there exists some $\epsilon'' > 0$ and $\phi'' \in (0,\pi/2)$ such that $\alpha_0(t,x)$ is analytic on $2\mathbb{D}\times\Delta(\epsilon',\phi')$. The denominators of the roots r_i can be written as

$$1 \pm \frac{xu(1+q^4)(1+u)}{q(1\pm q+q^2)(1+x)}\alpha_0(t,x) = 1 - \frac{x}{1+x}\left(u \mp \sqrt{u(1+2u)}\right)\alpha_0(t,x).$$

As shown in the proof of lemma (4.1.3), the auxiliary map satisfies the inequality $|\alpha_0(t,x)| \leq \alpha_0(b,x_0)$ for $b \in (0,2)$ and every $(t,x) \in b\mathbb{D}\times\Delta(\epsilon'',\phi'')$ with equality if and only if $(t,x) = (b,x_0)$. On the other hand since $\Re[u^{-1}\sqrt{u(1+2u)}] \geq 0$ we also trivially have the inequality

$$\left|\frac{x}{1+x}\left(u - \sqrt{u(1+2u)}\right)\right| \leq \left|\frac{x}{1+x}\left(u + \sqrt{u(1+2u)}\right)\right|$$

for all $x \in \Delta(\epsilon'',\phi'')$. Finally, note that the root on the right-hand side admits a power series expansion in $w = x^{1/2}$ with positive coefficients (using the auxiliary lemma)

and admits a singular expansion about the dominant singular point at $x = x_0$ of the form $3/10 - X/4 + \mathcal{O}(X^2)$, so by lemma 4 in [29] there exist $\epsilon''' > 0$ and $\phi''' \in (0, \pi/2)$ such that

$$\left| \frac{x}{1+x} \left(u + \sqrt{u(1+2u)} \right) \right| \leq \frac{3}{10}$$

for every $x \in \Delta(\epsilon''', \phi''')$ with equality if and only if $x = x_0$. It follows that the denominator is bounded below on an appropriate Δ-domain and for $t \in b\mathbb{D}$ by

$$1 - \frac{3}{10}\alpha_0(b, x_0) = \frac{1}{12}(2 - b + \sqrt{(2-b)(25-b)}) > 0$$

provided $b \in (0, 2)$. Hence both roots are analytic on $2\mathbb{D} \times \Delta(\epsilon', \phi')$ for a sufficiently small $\epsilon' > 0$ and some $\phi' \in (0, \pi/2)$ (possibly depending on b). The estimates above also imply that the power series expansions (using the binomial theorem) are absolutely convergent on the same domain and take form

$$r_1(t, x) = \sum_{k \geq 1} \frac{(k + q + kq^2)(1+q^4)^k x^k u^k (1+u)^k}{q^{k+1}(1 - q + q^2)^k (1+x)^k} \alpha_0^k(t, x)$$
$$r_2(t, x) = \sum_{k \geq 1} \frac{(k - q + kq^2)(1+q^4)^k x^k u^k (1+u)^k}{q^{k+1}(1 + q + q^2)^k (1+x)^k} \alpha_0^k(t, x)$$

Since $\Re q \geq 0$, using triangle inequality it follows at once that the coefficients of r_1 are $\geq$ in absolute value than the coefficients of r_2, which implies $|r_2(t, x)| \leq |r_1(t, x)|$. It remains to prove the final inequality for $|r_1(t, x)|$. In lieu of techniques used so far, it will suffice to show that $r_1(t, x)$ admits a power series expansion in (t, x) with non-negative coefficients. According to lemma (4.1.3) this is true for α_0 and hence for α_0^k for any $k \geq 0$. We will prove that the coefficients, say $c_k(q)$, of α_0^k in the expansion of r_1 also admit a power series expansion in $w = x^{1/2}$ with non-negative coefficients. We can write

$$c_k(q) = [a(q)k + b(q)]\, b(q)^{k-1} \text{ for } k \geq 1.$$

It suffices to show that $a(q)$ and $b(q)$ both have a power series expansion in $w = x^{1/2}$ with non-negative coefficients, where

$$a(q) = \frac{q^2(1+q+q^2)(1+q^6)}{(1+q^4)(1+q^2+q^4+q^6+q^8)} = \frac{x}{1+x}\left(1 + 2u + \sqrt{u(1+2u)}\right),$$
$$b(q) = \frac{q^3(1+q+q^2)(1-q^2+q^4)}{(1+q^4)(1+q^2+q^4+q^6+q^8)} = \frac{x}{1+x}\left(u + \sqrt{u(1+2u)}\right).$$

Clearly, both $a(q)$ and $b(q)$ have non-negative power series expansion in $w = x^{1/2}$ according to lemma (3.4.6), which completes the proof. $\square$

We are now in a positive to apply the transfer theorem. Using `Mathematica` we can check that $N_2(t,z,x)$ admits a singular expansion of the form

$$N_2(t,z,x) = \alpha_0(t,z) + \alpha_2(t,z)X^2 + \alpha_3(t,z)X^3 + \mathcal{O}(X^4).$$

The coefficients are certain fairly complicated algebraic functions. However, as $\alpha_3(t,z)$ will play a pivotal role in the next chapter, we will write it out explicitly. In order to simplify the representation, we introduce the following change of variables

$$t = \frac{2(\omega^2-25)}{\omega^2-1} \text{ and } \zeta = \frac{1}{16}(\omega-5)(\omega+1)z \text{ for } \omega \in [3,7].$$

A tedious yet straightforward computation allows us to express the coefficient $\alpha_3(t,z)$ in terms of $w(t,z)$, as follows

$$\begin{aligned} w(t,z) &= \sqrt{(1-2\zeta)\left((\omega-2)^2-2\zeta\right)} \\ \alpha_3(t,z) &= \zeta\frac{a_0\zeta^3 - a_1\zeta^2 - a_2\zeta + a_3 - (b_0\zeta - b_1)(2\zeta-1)\,w(t,z)}{81\omega(\omega-1)^2(\omega-2)^2(\omega-3)^2(2\zeta-1)\,w(t,z)} \end{aligned} \tag{3.71}$$

where the coefficients $a_i = a_i(\omega)$ and $b_j = b_j(\omega)$ are polynomials in ω:

$$\begin{aligned} a_0(\omega) &= 64(\omega-2)^2(105-42\omega^2+\omega^4) \\ a_1(\omega) &= 24(\omega-2)^2(735-468\omega-125\omega^2+152\omega^3-35\omega^4-4\omega^5+\omega^6) \\ a_2(\omega) &= 24(\omega-2)^2(-525+504\omega-55\omega^2-64\omega^3+5\omega^4+8\omega^5-\omega^6) \\ a_3(\omega) &= (\omega-2)^3(1365-924\omega+21\omega^2+24\omega^3+23\omega^4+4\omega^5-\omega^6) \\ b_0(\omega) &= 7455-8400\omega-720\omega^2+4464\omega^3-1850\omega^4-112\omega^5+232\omega^6-48\omega^7+3\omega^8 \\ b_1(\omega) &= (\omega-2)^2(1365-924\omega+21\omega^2+24\omega^3+23\omega^4+4\omega^5-\omega^6) \end{aligned}$$

Finally, we can summarize our finding in the lemma below:

Lemma 3.4.7 *Let $t \in [0,1]$ and $z \in [0,1/2]$, then the coefficients $[x^n]N_2(t,z,x)$ admit the following asymptotic expansion*

$$[x^n]N_2(t,z,x) = \frac{3}{4\sqrt{\pi}}\alpha_3(t,z)n^{-5/2}x_0^{-n}\left(1+\mathcal{O}(n^{-1})\right) \text{ as } n\to\infty. \tag{3.72}$$

We will also write $c(p,\ell) = [z^p t^\ell]\alpha(t,z)$ for the coefficients of the dominant term.

Chapter 4

Random Quadrangulations

4.1 Random Quadrangulations

4.1.1 Uniform probabilities

Let us briefly recall our notation of various classes of rooted irreducible quadrangulations; let $n \in \mathbb{N}$ be a positive integer

$$\begin{aligned}
\mathbf{Q}_n &= \{\mathbf{q} \text{ rooted irreducible quadrangulations with } n \text{ faces}\} \\
\mathbf{Q}_f &= \cup_{n\geq 1}\mathbf{Q}_n = \{\mathbf{q} \text{ rooted irreducible quadrangulations with finitely many faces}\} \\
\mathbf{Q}_\infty &= \mathbf{Q}\backslash\mathbf{Q}_\infty = \{\mathbf{q} \text{ infinite rooted irreducible quadrangulations}\},
\end{aligned}$$

where $(\mathbf{Q}, d_{\mathbf{Q}})$ is the metric completion of the space $(\mathbf{Q}_f, d_{\mathbf{Q}})$ of finite quadrangulations. The space $\mathbf{Q}$ equipped with its Borel σ-algebra is a *standard Borel space* since, according to lemma (2.2.2), the completion is a complete separable metric space. We define μ_n to be the *uniform probability measure* supported on $\mathbf{Q}_n$. We are interested in the distribution of infinite quadrangulations, which we will define as the weak limit of the measure μ_n as $n \to \infty$.

To that end we must first establish that the weak limit of the uniform probability measures μ_n of rooted irreducible quadrangulations with n faces as $n \to \infty$ exists:

Definition 22. A measure μ on $\mathbf{Q}$ is the *weak limit* of μ_n if for every bounded continuous function $f : \mathbf{Q} \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu.$$

We have shown that for every $r \geq 1$ and rooted proper irreducible quadrangulation $\mathbf{k}$ the characteristic function of the event $\{B_r = \mathbf{k}\}$ is continuous with respect to the distance metric on the space of quadrangulations $(\mathbf{Q}, d_{\mathbf{Q}})$, so weak convergence

$\mu_n \to \mu$ implies convergence with respect to root neighbourhoods. In other words, for any $r \geq 1$ and $\mathbf{k}$,

$$\mu_n(B_r = \mathbf{k}) \to \mu(B_r = \mathbf{k}).$$

However, since the completion $\mathbf{Q}$ is not a compact metric space, the sole existence of the limits $\lim_{n\to\infty} \mu_n(B_r = \mathbf{k})$ for every $r \geq 1$ and $\mathbf{k}$ does not (in general) suffice to establish weak convergence of the measures. The following example is taken from O. Angel [5]: suppose τ_n are probability distributions on quadrangulations such that the root vertex degree in τ_n is a.s. n, then the probability of observing any given quadrangulation as the ball of radius $r \geq 1$ tends to 0 (so all the aforementioned limits exists), but the weak limit of τ_n does not exist.

Definition 23. Let (M, d) be a metric space equipped with its Borel σ-algebra $\mathscr{B}(M)$. Let $\mathscr{P}(M)$ denote the collection of all probability measures on the space $(M, \mathscr{B}(M))$. Given $\varepsilon > 0$ and a subset $A \subseteq M$ we define the *ε-neighbourhood* of A by

$$A^\varepsilon = \{p \in M | \exists q \in A \text{ such that } d(p, q) < \varepsilon\}.$$

The *Lévy-Prokhorov metric* $\pi : \mathscr{P}(M) \times \mathscr{P}(M) \to [0, +\infty)$ is defined by setting the distance between two probability measures μ and ν to be

$$\pi(\mu, \nu) = \inf\{\varepsilon > 0 | \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathscr{B}(M)\}.$$

Note in particular that for probability measures we clearly have $\pi(\mu, \nu) \leq 1$.

The following theorem relates weak convergence with the Lévy-Prokhorov metric for a separable (complete) metric space (M, d):

Theorem 4.1.1 (Prokhorov's theorem)**.** *Let (M, d) be a separable metric space, then the space $\mathscr{P}(M)$ of probability measures on $(M, \mathscr{B}(M))$ is metrizable and weak convergence is equivalent to convergence of measures in the Lévy-Prokhorov metric.*

As we have already remarked the existence of the limit $\lim_n \mu_n(B_r = \mathbf{k})$ is in itself not sufficient to ensure the existence of the weak limit of μ_n. However, Prokhorov's theorem (4.1.1) and the easily established observation that the events $\{B_r = \mathbf{k}\}$ generate the σ-algebra, may be used to show that if we know *a priori* there exists a probability measure μ on $\mathbf{Q}$ satisfying $\lim_n \mu_n(B_r = \mathbf{k}) = \mu(B_r = \mathbf{k})$ for every $r \geq 1$ and $\mathbf{k}$ then $\mu_n \to \mu$ in the sense of weak convergence.

We will prove in this section that for every $r \geq 1$ and $\mathbf{k}$ the limit $\lim_n \mu_n(B_r = \mathbf{k})$ exists and defines a measure μ on the space $\mathbf{Q}$ of quadrangulations. We begin by considering the simplest case when the combinatorial type $\mathbf{k}$ of the root neighbourhood has a single boundary component.

Let $\mathbf{k}$ be rooted proper irreducible quadrangulation with n faces and boundary of length $2p$ (for some $p \geq 3$) such that $B_r(\mathbf{q}) = \mathbf{k}$ for some rooted irreducible quadrangulation $\mathbf{q}$. The set of all $\mathbf{k}$-admissible quadrangulations (i.e. the set of all rooted proper irreducible quadrangulations $\mathbf{s}$ whose connected sum $\mathbf{k}\#\mathbf{s}$ is again an irreducible quadrangulation) depends only on the partition parameter of $\mathbf{k}$. Let's denote the parameter by $\underline{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$, where $\sum_i \lambda_i = p$. Accordingly, the probability of a uniformly distributed rooted irreducible quadrangulation $\mathbf{q}$ with N faces having a root neighbourhood of radius r isomorphic with $\mathbf{k}$ is given by

$$\mathbb{P}(B_r(\mu_N) = \mathbf{k}) = \frac{c_{\mathbf{k}}(N-n+p)}{c(N)} = \frac{c_{\underline{\lambda}}(N-n+p)}{c(N)} \sim \frac{729}{8} c(\underline{\lambda}) x_0^{n-p} \text{ as } N \to \infty.$$

The final estimate was obtained by using the asymptotics of coefficients of the generating function $Q_\ell(\underline{t}; x)$ about the dominant singularity at $x = x_0$. Collecting the results we have proven the following lemma:

Lemma 4.1.1 *Let $\mathbf{k}$ be a rooted proper irreducible quadrangulation with partition parameter $\underline{\lambda}$ with n faces and a simple boundary of length $2p$ (with $p \geq 3$) such that $B_r(\mathbf{q}) = \mathbf{k}$ for some rooted irreducible quadrangulation $\mathbf{q}$ and $r \geq 1$, then*

$$\lim_{N\to\infty} \mathbb{P}(B_r(\mu_N) = \mathbf{k}) = \frac{729}{8} c(\underline{\lambda}) x_0^{n-p} \tag{4.1}$$

where μ_N denotes the uniform measure on the set of all rooted irreducible quadrangulations with N faces.

Exactly the same reasoning also implies

Lemma 4.1.2 *Let $r \geq 1$ and denote by $A_{n,p,\ell}(\mu_N)$ the event that the r-ball of an irreducible quadrangulation selected uniformly by μ_N has a simple boundary of total length 2ℓ ($\ell \geq 3$), whose associated composition parameter has p parts ($p \geq 3$), and n faces, then*

$$\lim_{N\to\infty} \mathbb{P}(A_{n,p,\ell}(\mu_N)) = \frac{729}{8} c(p,\ell) x_0^{n-\ell} \tag{4.2}$$

where μ_N denotes the uniform measure on the set of all rooted irreducible quadrangulations with N faces.

We have so far restricted our consideration to the simplest case where the root neighbourhood has a single boundary component. In the general case, the boundary of an r-ball $B_r(\mathbf{q})$ will have multiple edge-disjoint components. We can assign a partition parameter to each hole as for the case of one boundary component. Since the holes are independent, following the same reasoning as before, and assuming that

$\mathbf{k}$ has r boundary components ("holes") with partition parameters $\underline{\lambda}_1, \ldots, \underline{\lambda}_r$ having total boundary length $2p_i$ for $1 \leq i \leq r$, respectively, and n faces, we obtain the following formula

$$\mathbb{P}(B_r(\mathbf{q}) = \mathbf{k}) = \frac{1}{c(N)} \sum_{N_1+\ldots+N_r=N-n} c_{\underline{\lambda}_1}(N_1+p_1)\cdots c_{\underline{\lambda}_r}(N_r+p_r).$$

We can now state two theorems concerning the limiting measure $\mu_\infty = \lim_{N\to\infty} \mu_N$.

Theorem 4.1.2. *Let $\mathbf{k}$ be a rooted proper irreducible quadrangulation with partition parameter $\underline{\lambda}_0$ and simple boundary of total length $2p$ with r edge-disjoint "holes" with partition parameters $\underline{\lambda}_1, \ldots, \underline{\lambda}_r$ and simple boundaries of total length $2p_i$ for $1 \leq i \leq r$ having n faces. Then the following limit exists*

$$\lim_{N\to\infty} \mathbb{P}(\mathbf{k} \subseteq \mu_N(\underline{\lambda}_0)) = \frac{1}{c(\underline{\lambda}_0)} x_0^{-p_1} a(\underline{\lambda}_1)\cdots x_0^{-p_r} a(\underline{\lambda}_r) \sum_{i=1}^{r} \frac{c(\underline{\lambda}_i)}{a(\underline{\lambda}_i)} x_0^n \tag{4.3}$$

Proof. The proof is a straightforward application of singularity analysis. Since $\mathbf{k}$ is a 3-connected map, distinct completions of holes give distinct irreducible quadrangulations. Consequently,

$$\begin{aligned}\mathbb{P}(\mathbf{k} \subseteq \mathbf{q}) &= \frac{1}{c_{\underline{\lambda}_0}(N)} \sum_{N_1+\ldots+N_r=N-n} c_{\underline{\lambda}_1}(N_1+p_1)\cdots c_{\underline{\lambda}_r}(N_r+p_r) \\ &= \frac{[x^{N-n}]x^{-p_1}q(\underline{\lambda}_1;x)\cdots x^{-p_r}q(\underline{\lambda}_r;x)}{[x^N]q(\underline{\lambda}_0;x)}.\end{aligned}$$

The behaviour of the product near the dominant singularity at $x = x_0$ is governed by the term X^3 of the singular expansion. Recall that the generating function of $\underline{\lambda}$-admissible quadrangulations admits a singular expansion of the form $a(\underline{\lambda}) + b(\underline{\lambda})X^2 + c(\underline{\lambda})X^3 + \mathcal{O}(X^4)$. Plugging it in the product and rearranging gives

$$x^{-p_1}q(\underline{\lambda}_1;x)\cdots x^{-p_r}q(\underline{\lambda}_r;x) = \prod_{i=1}^{r} x_0^{-p_i} a(\underline{\lambda}_i) \left[1 + X\sum_{i=1}^{r} \frac{\tilde{b}(\underline{\lambda}_i)}{a(\underline{\lambda}_i)} + X^3 \sum_{i=1}^{r} \frac{c(\underline{\lambda}_i)}{a(\underline{\lambda}_i)} + \mathcal{O}(X^4)\right]$$

where $\tilde{b}(\underline{\lambda}) = b(\underline{\lambda}) + pa(\underline{\lambda})$. It now follows at once from the transfer theorem that the coefficients are asymptotically, as $N \to \infty$, of order

$$\begin{aligned}[x^{N-n}]\prod_i x^{-p_i}q(\underline{\lambda}_i;x) &= \prod_{i=1}^{r} x_0^{-p_i} a(\underline{\lambda}_i) \sum_{i=1}^{r} \frac{c(\underline{\lambda}_i)}{a(\underline{\lambda}_i)} [x^{N-n}](1-4x)^{3/2}\left(1+\mathcal{O}(N^{-1})\right) \\ &= \prod_{i=1}^{r} x_0^{-p_i} a(\underline{\lambda}_i) \sum_{i=1}^{r} \frac{c(\underline{\lambda}_i)}{a(\underline{\lambda}_i)} x_0^{n-N} \frac{N^{-5/2}}{\Gamma(-3/2)}\left(1+\mathcal{O}(N^{-1})\right).\end{aligned}$$

On the other hand, the asymptotics of $[x^N]q(\underline{\lambda}_0;x)$ as $N \to \infty$ is already known. Taking the quotient and simplifying gives exactly the limiting probability (4.3). □

Theorem 4.1.3. *Let* $\mathbf{k}$ *be a rooted proper irreducible quadrangulation with partition parameter* $\underline{\lambda}_0$ *and simple boundary of total length* $2p$ *with* r *edge-disjoint "holes" with partition parameters* $\underline{\lambda}_1, \ldots, \underline{\lambda}_r$ *and simple boundaries of total length* $2p_i$ *for* $1 \leq i \leq r$ *having* n *faces. Let* $\mathbf{q}_N(\underline{\lambda}_0)$ *be a random irreducible quadrangulation under condition* $\mathbf{k} \subseteq \mathbf{q}_N(\underline{\lambda}_0)$ *and write* $\mathbf{q}_N(\underline{\lambda}_0) = \mathbf{k}\#(\mathbf{s}_1, \ldots, \mathbf{s}_r)$. *Suppose* $N_i = N(\mathbf{s}_i)$ *is the number of inner faces of the* i*th component of the connected sum. Taking the limit* $N \to \infty$*, a.s. exactly one of the* N_i*'s is infinite while the others are finite, with probability*

$$\lim_{N\to\infty} \mathbb{P}(N_i = \max_{j=1,\ldots,r} N_j) = \frac{c(\underline{\lambda}_i)}{a(\underline{\lambda}_i)} \left(\sum_{j=1}^{r} \frac{c(\underline{\lambda}_j)}{a(\underline{\lambda}_j)} \right)^{-1}. \tag{4.4}$$

Moreover, the limiting conditional distributions $\mathbb{P}(N_j = n_j, j \neq i | N_i = \max N_j)$ *exist and the random variables* N_j *for* $j \neq i$ *are asymptotically independent.*

Proof. We follow closely the proof of the analogous statement for triangulations in [41]. We may assume without loss of generality that $i = 1$, then the probability is

$$p_1(N) = \mathbb{P}(N_1 = \max_{j=1,\ldots,r} N_j) = \frac{\sum_{\substack{N_1,\ldots,N_r \\ N_1 \geq N_2,\ldots,N_k}} c_{\underline{\lambda}_1}(N_1+p_1)\cdots c_{\underline{\lambda}_r}(N_r+p_r)}{\sum_{N_1,\ldots,N_r} c_{\underline{\lambda}_1}(N_1+p_1)\cdots c_{\underline{\lambda}_r}(N_r+p_r)}.$$

Let $M > 0$ be a fixed positive integer, then

$$\begin{aligned}
\lim_{N\to\infty} p_1(N) &\geq \lim_{N\to\infty} \mathbb{P}(N_2 + \cdots + N_r \leq M) \\
&= \sum_{N_2+\cdots+N_r\leq M} \prod_{i=2}^{r} c_{\underline{\lambda}_i}(N_i+p_i) \lim_{N\to\infty} \frac{[x^{N-n-(N_2+\cdots+N_k)}] x^{-p_1} q(\underline{\lambda}_1; x)}{[x^{N-n}] \prod_i x^{-p_i} q(\underline{\lambda}_i; x)} \\
&= \sum_{N_2+\cdots+N_r\leq M} \prod_{i=2}^{r} c_{\underline{\lambda}_i}(N_i+p_i) \frac{x_0^{-p_1} c(\underline{\lambda}_1) x_0^{N_2+\cdots+N_r}}{\prod_{i=1}^{r} x_0^{-p_i} a(\underline{\lambda}_i) \sum_{i=1}^{r} \frac{c(\underline{\lambda}_i)}{a(\underline{\lambda}_i)}} \\
&= \frac{c(\underline{\lambda}_1)}{a(\underline{\lambda}_2)} \left(\sum_{j=1}^{r} \frac{c(\underline{\lambda}_j)}{a(\underline{\lambda}_j)} \right)^{-1} \sum_{N_2+\cdots+N_r\leq M} \prod_{i=2}^{r} \frac{x_0^{N_i+p_i} c_{\underline{\lambda}_i}(N_i+p_i)}{a(\underline{\lambda}_i)}.
\end{aligned} \tag{4.5}$$

Taking the limit as $M \to \infty$, the final sum is just the product of generating functions $q(\underline{\lambda}_i; x)$ evaluated at $x = x_0$ (see also theorem 3.3.8), so the right-hand side becomes equal to (4.4). Clearly, the same lower bound holds for every $i = 1, \ldots, r$. However, as the sum of the right-hand sides of (4.4) for $i = 1, \ldots, r$ equals 1 each inequality can be replaced with equality. This proves the first part of the theorem.

Using relation (4.5) we have also deduced that

$$\lim_{N\to\infty} \mathbb{P}(N_1 = \max_{j=1,\ldots,r} N_j) = \lim_{M\to\infty} \lim_{N\to\infty} \mathbb{P}(N_2 + \cdots + N_r \leq M)$$

and the limiting conditional distribution of $(N_2, \ldots, N_k | N_1 = \max)$ has probability generating function

$$\lim_{N\to\infty} \mathbb{E}(t_2^{N_2} \cdots t_r^{N_2} | N_1 = \max_{j=1,\ldots,r} N_j) = \prod_{i=2}^{r} \frac{t_i^{-p_i} q(\underline{\lambda}_i; x_0 t_i)}{q(\underline{\lambda}_i; x_0)}.$$

Hence the limiting conditional distribution exists and the random variables $N_2, \ldots, N_r$ are asymptotically independent. □

4.1.2 Maximum degree concentration results

The following section will be devoted to proving a strong concentration result for the maximum vertex degree of a rooted irreducible quadrangulation with n faces. More precisely, we will prove

Theorem 4.1.4. *Let Δ_n denote the maximum vertex degree of a random rooted irreducible quadrangulation with n faces, then*

$$\mathbb{E}(\Delta_n) = \frac{\log n - \frac{1}{2}\log\log n}{\log 2} + \mathcal{O}(1), \qquad \mathbb{V}(\Delta_n) = \mathcal{O}(\log n). \tag{4.6}$$

Furthermore, if $\Omega_n \to \infty$ denotes any function then

$$\mathbb{P}\left(\left|\Delta_n - \frac{\log n - \frac{1}{2}\log\log n}{\log 2}\right| \le \Omega_n\right) = 1 - \mathcal{O}(1/\log n + (1/2)^{\Omega_n}). \tag{4.7}$$

Theorem 4.1.5 ([28])**.** *Let $T = (1 - t/2)^{1/2}$ and define (for $k > 2$):*

$$b_k = [t^{k-1}] \frac{\left(5T - \sqrt{24+T^2}\right)^2 \left(T + \sqrt{24+T^2}\right)^3 \left(7T + \sqrt{24+T^2}\right)}{2^8 3^6 T^3 \sqrt{24+T^2}} \text{ and } \mu_k = \frac{3^6}{2k} b_k.$$

Let ζ_k denote the number of vertices of degree k of a random rooted irreducible quadrangulation with n faces, then uniformly for all $k = \mathcal{O}(\log n)$ we have

$$\mathbb{E}(\zeta_k) = n\mu_k \left(1 + \mathcal{O}\left(\frac{\log^{10} n}{n}\right)\right), \text{ and } \mathbb{V}(\zeta_k) = n\mu_k + (n\mu_k)^2 \mathcal{O}\left(\frac{\log^{10} n}{n}\right).$$

Furthermore, we have

$$\mathbb{P}(\zeta_k = n\mu_k(1 + o(1))) = 1 - o(1)$$

uniformly for $k < (\log n - (1/2)\log\log n)/\log 2 - \Omega_n$ and

$$\mathbb{P}(\zeta_k = 0) = 1 - o(1)$$

uniformly for $k > (\log n - (1/2)\log\log n)/\log 2 + \Omega_n$.

The proof is based on a careful asymptotic analysis of the coefficients of the trivariate generating function $G(t,s)$ of pointed rooted irreducible quadrangulations. Note the similarity of our result with known results on the maximum vertex degree distribution of random rooted planar maps according to the number of edges (see [29] and [28]). Our proof will follow closely the ideas used in [29] and [28].

We will be using the analytic and asymptotic results from [29]. We first set up our notation. We will use the letter ϵ to denote a sufficiently small positive number and ϕ will denote a constant angle in the interval $\phi \in (0, \pi/2)$. The definition of a Δ-domain can naturally be extended to the multivariate case, following [29], as

$$\begin{aligned}
\Delta_x(\epsilon,\phi) &= \{x \in \mathbb{C} || x| \leq 1+\epsilon,\, x \neq 1,\, |\arg(x-1)| \geq \phi\} \\
\Delta_j(\epsilon,\phi) &= \{y_j \in \mathbb{C} || y_j| \leq 1+\epsilon,\, y_j \neq 1,\, |\arg(y_j-1)| \geq \phi\} \\
\mathcal{R}(\epsilon,\phi) &= \{(x,\underline{y}) \in \mathbb{C} \times \mathbb{C}^d |\, |y_j| < 1 \text{ for all } j = 1,\ldots,d,\, x \in \Delta_x(\epsilon,\phi)\}.
\end{aligned}$$

Let $\beta_j > 0$ for $j = 1,\ldots,d$ and α be any real number. We introduce the notation:

Definition 24.

$$f(x,\underline{y}) = \tilde{\Theta}\left((1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}\right)$$

if there exists some $\epsilon > 0$ and $\phi \in (0,\pi/2)$ such that in $\mathcal{R}(\epsilon,\phi)$:

(i) $f(x,\underline{y})$ is analytic and satisfies

$$f(x,\underline{y}) = \Theta\left(|1-x|^{-\alpha}\prod_{j=1}^{d}(1-|y_j|)^{-\beta_j}\right)$$

as $(1-x)(1-y_j)^{-p} \to 0$ for $j = 1,\ldots,d$ and some $p \geq 0$.

(ii) There exists some $r \geq 0$ and a real number α' such that

$$f(x,\underline{y}) = \Theta\left(|1-x|^{-\alpha}\prod_{j=1}^{d}(1-|y_j|)^{-r}\right).$$

Definition 25.

$$f(x,\underline{y}) \approx c(1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}$$

if $f(x,\underline{y})$ can be written in the form

$$f(x,\underline{y}) = c(\underline{y})(1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j} + \sum_{j=0}^{d} C_j(x,\underline{y}) + E(x,\underline{y})$$

such that the following conditions hold:

(i) $C_0(x,\underline{y})$ is a polynomial in x, and for $1 \le j \le d$, $C_j(x,\underline{y})$ is a polynomial in y_j.

(ii) There exist some $\alpha' < \alpha$ and $\beta'_j \ge 0$ for $j = 1,\dots,d$ such that

$$E(x,\underline{y}) = \tilde{\mathcal{O}}\left((1-x)^{-\alpha'}\prod_{j=1}^{d}(1-y_j)^{-\beta'_j}\right),$$

(iii) $c(\underline{y}) = c+\mathcal{O}(\sum_{j=1}^{d}|1-y_j|)$ is analytic in $\Delta_1(\epsilon,\phi)\times\cdots\times\Delta_d(\epsilon,\phi)$ and $c(\underline{1}) = c \neq 0$.

The next technical lemmas describe the necessary estimates needed to obtain (sharp) concentration results on the maximum degree of a rooted irreducible quadrangulation. Since all the generating functions are expressible in terms of u and α, we will first analyse their properties. Let

$$X = (1-4x)^{1/2}, \quad T = (1-t/2)^{1/2} \text{ and } S = (1-s/2)^{1/2}.$$

We can clearly write $u = (1-X)/2$ and using the well-known expansion of the generating function of shifted Catalan numbers, $u(x)$ admits a power series expansion about the origin with positive coefficients so for $x \in x_0\mathbb{D}\backslash\{x_0\}$ it satisfies the inequality $|u(x)| \le u(|x|) < u(x_0) = 1/2$. The next lemma establishes the analogous properties for $\alpha(t,x)$. For convenience sake we write

$$\mathscr{R}'(\epsilon,\phi) = \{(x,\underline{y}) : |y_i| < 2,\ 1 \le j \le d, x \in x_0\Delta_x(\epsilon,\phi)\}.$$

Lemma 4.1.3 (Estimates) *Let $\alpha = \alpha(t,x)$ be determined by equations (3.62), then*

(i) The function $\alpha = \alpha(t,x)$ is analytic on $\mathscr{R}'(\epsilon,\phi)$ for sufficiently small $\epsilon > 0$ and $\phi \in (0,\pi/2)$ and for each fixed $t \in 2\mathbb{D}_t$ it has a unique singularity of minimum modulus at $x = 1/4$.

(ii) $\alpha(t,x)$ admits a bivariate power series expansion in (t,x) about the origin with non-negative coefficients and there exist $\epsilon' \le \epsilon$ (sufficiently small), $\phi \le \phi' < \pi/2$ such that $|\alpha(t,x)| < |u(x)| \le 1/2$ for every $(t,x) \in \mathscr{R}'(\epsilon',\phi')$.

(iii) For $\varepsilon > 0$ and $\phi \in (0,\pi/2)$ sufficiently small,

$$\begin{aligned}\alpha(t,x) = {} & \frac{1}{12}\left\{6 - T^2 - T\sqrt{24+T^2}\right\} \\ & + \frac{1}{36}\left\{T(3+2T^2) - (3-2T^2)\sqrt{24+T^2}\right\}T^{-1}X \\ & - \frac{1}{108}\left\{T^3(15-11T^2) - \frac{108-216T^2+117T^4+11T^6}{\sqrt{24+T^2}}\right\}T^{-3}X^2 \\ & + \frac{1}{81}\left\{T^5(3-4T^2) - \frac{81-216T^2+81T^4+45T^6+4T^8}{\sqrt{24+T^2}}\right\}T^{-5}X^3 \\ & + \tilde{\mathcal{O}}(X^4T^{-8}) \end{aligned} \tag{4.8}$$

Proof. We can solve the quadratic equation (3.62) determining $\alpha(t,x)$ explicitly and using the condition $\alpha(0,x) \equiv 0$ the correct solution is:

$$\alpha(t,x) = \frac{1+x+uxt - \sqrt{(1+u-u^2)^2 - 2(1+3u+3u^2)uxt + u^2x^2t^2}}{2(1+u)}.$$

Apart from the simple poles at $x=-2$ and the fixed branch point at $x=1/4$, the only source of singularities of $\alpha(t,x)$ are the roots of the quadratic polynomial under the square-root that has already been investigated in the proof of lemma (3.3.7). We have shown that the polynomial factors as $(1+x)^2(1-tf_1(x))(1-tf_2(x))$ where f_1, f_2 are also the eigenvalues of the matrix (the quadratic polynomial is now $\sigma(xt,x)$):

$$N(x) = \begin{bmatrix} xA(x) & x \\ xB(x) & xA(x) \end{bmatrix}, A(x) = \frac{u(1+3u+3u^2)}{(1+u-u^2)^2} \text{ and } B(x) = \frac{4u^3(1+u)^2(1+2u)}{(1+u-u^2)^4}.$$

and the coefficients of the matrix $N(x)$ all admit power series expansions in x about the origin with non-negative coefficients. Using lemma 4 in [29], it follows that there exists some $\epsilon > 0$ and $\phi \in (0, \pi/2)$ such that $|A(x)| \leq A(x_0)$ and $|B(x)| \leq B(x_0)$ for $x \in x_0\overline{\Delta}_x(\epsilon,\phi)$ with equality if and only if $x = x_0$. It follows that $|N(x)| \leq N(x_0)$ for all $x \in x_0\overline{\Delta}_x(\epsilon,\phi)$. The matrix $N(x_0)$ is positive with unique maximal eigenvalue $1/2$, so (by the Perron-Frobenius theorem) the eigenvalues of $N(x)$ are dominated in absolute value by $1/2$ with equality if and only if $|N(x)| = N(x_0)$. In other words, $|f_i(x)| \leq 1/2$ for all $x \in x_0\overline{\Delta}_x(\epsilon,\phi)$ with equality if and only if $x = x_0$. It follows that $\alpha(t,x)$ is analytic in $\mathscr{R}'(\epsilon,\phi)$ and has a unique singularity of minimal modulus at $x = x_0$ for every $t \in 2\mathbb{D}$.

Recall that the implicit equation defining α is

$$\alpha = t\frac{xu(1+u-\alpha)}{1+x-(1+u)\alpha}.$$

The quadratic equation determines $\alpha(t,x)$ uniquely by the condition $\alpha(0,x) \equiv 0$ and is exactly of the form needed to apply Lagrange's theorem (3.1.2). We can thus determine the coefficients of $\alpha(t)$ in a power series expansion about $t=0$ as functions of (u,x). Let $k \geq 0$, then

$$\begin{aligned}
[t^{k+1}]\alpha(t,x) &= \frac{1}{k+1}[\lambda^k]\left(\frac{xu(1+u-\lambda)}{1+x-(1+u)\lambda}\right)^{k+1} \\
&= \frac{x^{k+1}u^{k+1}(1+u)}{(1+u-u^2)^{k+1}} \sum_{i=0}^{k} \frac{(-1)^{k-i}}{k+1}\binom{k+1}{i+1}\binom{k+i}{k}\frac{(1+u)^{2i}}{(1+x)^i} \\
&= \frac{x^{k+1}u^{k+1}(1+u)}{(1+u-u^2)^{k+1}} \sum_{i=0}^{k} \frac{(-1)^{k-i}}{i+1}\binom{k}{i}\binom{k+i}{k}\left(\frac{(1+u)^2}{1+u-u^2}\right)^i
\end{aligned}$$

$$= \frac{x^{k+1}u^{k+2}(1+2u)}{(1+x)^{k+1}(1+u)} \sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{k+i}{k}\left(\frac{u(1+2u)}{(1+x)}\right)^{i} \text{ for } k \geq 1.$$

The final expression is obtained by an application of the following identity

$$\sum_{i=0}^{k} \frac{(-1)^{k-i}}{i+1}\binom{k}{i}\binom{k+i}{k} z^i = z^{-1}\left\{\sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{k+i}{k}(z-1)^{i+1} + \delta_{k,0}\right\}. \quad (4.9)$$

The initial coefficient is simply $[t]\alpha(t,x) = xu(1+u)(1+x)^{-1}$. Appealing to the proof of lemma (3.3.7) it now easily follows that the initial coefficient and every term of the finite sum admits a power series expansion in ascending powers of x with non-negative coefficients, which completes the proof of the first part of (ii). Moreover, the same ideas also prove that both $u(x)^{-1}\alpha(t,x)$ and the normalized map $\alpha_0(t,x)$ (introduced in the proof of lemma (3.4.5)) admit a bivariate expansion in (t,x) with non-negative coefficients.

We will now prove the inequality (which will be needed later to establish the various generating functions connected with $G(t,s)$ are analytic in the extended region $\mathscr{R}'(\epsilon,\phi)$). We will show the stronger inequality $|u^{-1}\alpha| < 1$ for $(t,x) \in \mathscr{R}'(\epsilon',\phi')$ with $0 < \epsilon' \leq \epsilon$ and $\phi \leq \phi' < \pi/2$. As already shown above, the map $\alpha(t,x)$ is analytic on $\mathscr{R}'(\epsilon,\phi)$ and has a unique singularity of minimal modulus at $x = x_0$ for any $t \in 2\mathbb{D}$. Let $x \in x_0\Delta_x(\epsilon,\phi)$ be fixed then as $\alpha(t)$ is analytic on $2\mathbb{D}$ its power series expansion with respect to t about the origin is convergent on $2\mathbb{D}$ with coefficients say $c_k(x)$. We will prove there exist $0 < \epsilon' \leq \epsilon$ and $\phi \leq \phi' < \pi/2$ such that $|c_k(x)| \leq c_k(x_0)$ for any $x \in x_0\overline{\Delta}_x(\epsilon',\phi')$ and $k \geq 1$ with equality if and only if $x = x_0$. Supposing for the moment this is true we have (by triangle inequality for absolutely convergent series):

$$|u^{-1}\alpha| \leq \sum_k |c_k(x)||t|^k \leq \sum_k c_k(x_0)|t|^k = 2\alpha(x_0,|t|) \text{ for } t \in 2\mathbb{D}.$$

However, trivially $2\alpha(x_0,|t|) < 1$ for $|t| < 2$ which establishes the desired inequality. It remains to investigate the coefficients $c_k(x)$. We can write $c_1(x) = x(1+u)(1+x)^{-1}$ and

$$c_{k+1}(x) = \frac{1+2u}{1+u}\left(\frac{xu}{1+x}\right)^{k+1} \sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{k+i}{k}\left(\frac{u(1+2u)}{(1+x)}\right)^{i} \text{ for } k \geq 1.$$

Using again lemma 4 in [29], we deduce the existence of $\epsilon'' > 0$ and $\phi'' \in (0,\pi/2)$ such that for all $x \in x_0\Delta_x(\epsilon'',\phi'')$ we have

$$\left|\frac{x(1+u)}{1+x}\right| < \frac{3}{10}, \left|\frac{1+2u}{1+u}\right| < \frac{4}{3}, \left|\frac{xu}{1+x}\right| < \frac{1}{10} \text{ and } \left|\frac{u(1+2u)}{1+x}\right| < \frac{4}{5}.$$

Since all the coefficients of the finite sum representation of $c_k(x)$ are positive, it follows by triangle inequality that $|c_k(x)| < c_k(x_0)$ for $x \in x_0\Delta_x(\epsilon'', \phi'')$. Taking the intersection $x_0\Delta_x(\epsilon, \phi) \cap x_0\Delta_x(\epsilon'', \phi'') = x_0\Delta_x(\epsilon', \phi')$ establishes the required result.

Finally, to deduce the singular expansion of $\alpha(t, x)$ note that part (i) implies $\alpha(t, x)$ is analytic in the larger region $\mathscr{R}'(\epsilon, \phi)$ for sufficiently small $\epsilon > 0$ and $\phi \in (0, \pi/2)$. We will first verify condition (i) in definition (24). Let's again denote the reciprocal zeros of the quadratic polynomial appearing under the square-root by $f_1(x), f_2(x)$ and $x = (1 - X^2)/4$, $t = 2(1 - T^2)$, then we can write

$$f_1(x) = \frac{ux\left(1 + 3u + 3u^2 + 2(1+u)\sqrt{u(1+2u)}\right)}{(1+u-u^2)^2} = \frac{1}{2} - X + \frac{5X^2}{16} + \frac{31X^3}{64} + \mathcal{O}(X^4)$$

$$f_2(x) = \frac{ux\left(1 + 3u + 3u^2 - 2(1+u)\sqrt{u(1+2u)}\right)}{(1+u-u^2)^2} = \frac{1}{50} - \frac{49X^2}{2000} - \frac{7X^3}{1600} + \mathcal{O}(X^4)$$

and the function $\alpha(t, x)$ becomes

$$\begin{aligned}
\alpha(t,x) &= \frac{1 + x + uxt - (1+x)\sqrt{(1 - tf_1(x))(1 - tf_2(x))}}{2(1+u)} \\
&= \frac{6 - T^2}{12} + \frac{3 + 2T^2}{36}X - \frac{15 - 11T^2}{108}X^2 + \frac{3 - 4T^2}{81}X^3 + \mathcal{O}(X^4T^{-8}) \\
&\quad - \frac{5 - X^2}{4(3 - X)}T\left(1 + \left(2X - \frac{5}{8}X^2 - \frac{31}{32}X^3 + \mathcal{O}(X^4)\right)(1 - T^2)T^{-2}\right)^{1/2} \\
&\qquad \times \left(1 - (1 - T^2)\left(\frac{1}{25} - \frac{49}{1000}X^2 - \frac{7}{8000}X^3 + \mathcal{O}(X^4)\right)\right)^{1/2}.
\end{aligned}$$

When $XT^{-2} \to 0$ in $\mathscr{R}'(\epsilon, \phi)$ we can expand both square roots using the binomial theorem (with the help of say `Mathematica`) to obtain exactly the singular expansion (4.8) with an error term $E(t, x)$ of order $\mathcal{O}(X^4T^{-8})$. Clearly, the remainder $E(t, x)$ is analytic in $\mathscr{R}'(\epsilon, \phi)$ and satisfies $\mathcal{O}(|X|^4(1 - |t/2|)^{-8})$ as $XT^{-2} \to 0$, which verifies condition (i). On other hand, it is clear that $E(t, x) = \mathcal{O}((1 - |t/2|)^{-5})$ in $\mathscr{R}'(\epsilon, \phi)$ which verifies the second condition of definition (24) and hence gives the singular expansion (4.8). □

Recall that the generating function $G(t, s)$ is comprised of four parts. We will analyse each contribution separately to obtain the asymptotically dominant terms of $G(t, s)$ about the singularities of minimal modulus. For the remainder of the section $\epsilon > 0$ and $\phi \in (0, \pi/2)$ will denote sufficiently small positive real numbers, while $p(x, T, S)$ and $c(T, S)$ will stand, respectively, for a linear polynomial in x and a function analytic on $(t/2, s/2) \in \Delta_t(\epsilon, \phi) \times \Delta_s(\epsilon, \phi)$ such that $c(0, 0) \neq 0$, not necessarily the same at each occurrence.

Lemma 4.1.4 (Estimates for Φ and G) *For $\varepsilon > 0$ and $\phi \in (0, \pi/2)$ sufficiently small,*

$$x^2t^2\Phi(xt,x) = p(x,T) + c(T)T^{-3}X^3 + \tilde{\mathcal{O}}(X^4T^{-8}),$$
$$G(t,x) = p(x,T,S) - c(T,S)T^{-3}S^{-1}X + \tilde{\mathcal{O}}(X^3T^{-6}S^{-6})$$

where $p(x,T)$ and $p(x,T,S)$ denote an expression in x, T and S which is linear in x, $c(T)$ and $c(T,S)$ are analytic in $2\Delta_t(\epsilon,\phi)\times 2\Delta_s(\epsilon,\phi)$ with initial values $c(0) = \sqrt{6}/81$ and $c(0,0) = 1$. The coefficients $a_{nk} = [x^nt^k](xt)^2\Phi(xt,x)$ and $a_{nkl} = [x^nt^ks^l]G(t,s,x)$ satisfy the asymptotic estimates

$$a_{nk} = [x^nt^k]c(T)T^{-3}X^3 + \mathcal{O}(4^n2^{-k}n^{-5/2}\log^{10}n)$$
$$a_{nkk} = [x^nt^ks^k]c(T,S)T^{-3}S^{-1}X + \mathcal{O}(4^n2^{-2k}n^{-3/2}\log^{10}n)$$

uniformly for $2 < k = \mathcal{O}(\log n)$.

Proof. Both functions Φ and G are expressible as rational functions of u and α. The main technical assumption we need to verify is that each function is analytic in a region of the form $\mathscr{R}'(\epsilon,\phi)$ for some sufficiently small $\epsilon > 0$ and $\phi \in (0,\pi/2)$. Clearly, it suffices to check the denominators of the various rational expressions are non-zero on the given domain $\mathscr{R}'$. We use the estimates on α derived in lemma (4.1.3), e.g. in the denominator of Φ_0 we encounter the factor $1-\alpha$, which can be estimated from below as

$$|1-\alpha(t)| \geq 1 - |\alpha(t)| \geq 1 - |u(x)| \geq \frac{1}{2}.$$

We can similarly verify that any factor appearing in the denominator of any of the four contributions is bounded away from zero on $\mathscr{R}'$ as follows (we will for sake of brevity denote $\alpha = \alpha(t,x)$ and $\beta = \alpha(s,x)$):

$$|1-\alpha-\beta| \geq 1 - |\alpha+\beta| \geq 1 - 2|u| > 0 \text{ and } |1-u^{-2}\alpha\beta| \geq 1 - |u^{-1}\alpha||u^{-1}\beta| > 0$$
$$\left|1 - \frac{u}{1+u}\left(u^{-1}\alpha + u^{-1}\beta + u^{-2}\alpha\beta\right)\right| \geq 1 - 3\left|\frac{u}{1+u}\right| > 0$$

where in the last inequality we have used the easily proven observation that $u(1+u)^{-1}$ has a positive power series expansion in x, so lemma 4 in [29] can be used to find a region $\mathscr{R}'$ on which it is $< 1/3$. Furthermore, Φ_0 is clearly bounded on $\mathscr{R}'(\epsilon,\phi)$, while all four contributions of $G(t,s)$ satisfy $\mathcal{O}((1-2|t|)^{-1})$ for $(x,t,s) \in \mathscr{R}'(\epsilon,\phi)$.

We may now substitute the singular expansion of $\alpha(t,x)$ obtained in lemma (4.1.3) to obtain the expansion of $\Phi(xt,x)$ and the four contributions making up $G(t,s)$. Using `Mathematica`, for $(x,t,s) \in \mathscr{R}'(\epsilon,\phi)$ and $XT^{-2}, XS^{-2} \to 0$, we arrive exactly at the expansion for $x^2t^2\Phi(xt,x)$ above and

$$(xts)^2\Phi_\bullet(xt,s,x) = p(x,T,S) + \tilde{\mathcal{O}}(X^3T^{-8}S^{-6})$$

$$(xt)^2 s\Phi_\circ(xt,s,x) = p(x,T,S) + \tilde{\mathcal{O}}(X^3T^{-8}S^{-6})$$
$$(xt)^2\Phi_{i,\bullet}(xt,s,x) = \frac{1}{2}c(T,S)T^{-3}S^{-1}X + p(x,T,S) + \tilde{\mathcal{O}}(X^3T^{-9}S^{-8})$$
$$(xt)^2\Phi_{i,\circ}(xt,s,x) = \frac{1}{2}c(T,S)T^{-3}S^{-1}X + p(x,T,S) + \tilde{\mathcal{O}}(X^3T^{-9}S^{-8})$$

where $c(0,0) = -1$ in both cases. The function $c(T,S)$ admits the following closed-form expression

$$c(T) = \frac{(1-T^2)\left(5T - \sqrt{24+T^2}\right)^2\left(T + \sqrt{24+T^2}\right)^3\left(7T + \sqrt{24+T^2}\right)}{2^7 3^6\sqrt{24+T^2}}$$
$$c(T,S) = \frac{3}{256}c(T)\left(5S - \sqrt{24+S^2}\right)^3\left(S + \sqrt{24+S^2}\right)^2.$$

We will not write out explicitly all the terms of the aforementioned expansions. Let us merely remark that by introducing the substitution

$$\tau = \sqrt{\frac{50-t}{2-t}} = T^{-1}\sqrt{24+T^2} \text{ and } \zeta = \sqrt{\frac{50-s}{2-s}} = S^{-1}\sqrt{24+S^2}$$

the coefficients are all expressible as rational functions of (τ,ζ) and it is easily checked that each is analytic in the polydisk $2\mathbb{D}\times 2\mathbb{D}$ (the inequality $|\tau|>3$ on $2\overline{\mathbb{D}}$ is easily checked and suffices to show the denominators are always non-zero). □

Having obtained the necessary uniform estimates, we will investigate the first and second moment of the random variable counting the number of vertices of given degree in a random rooted irreducible quadrangulation. We will use the letters $\mathbb{P},\mathbb{E}$ and $\mathbb{V}$ to denote the probability, expectation and variance of a random variable, respectively. We will let $\zeta_k = \zeta_{k,n}$ denote the number of vertices of degree k in a random rooted irreducible quadrangulation with n faces (where as usual all probability distributions are always uniform over the family of rooted irreducible quadrangulations). According to Whitney's theorem, for irreducible quadrangulations $\zeta_{k,n} = 0$ whenever $k \leq 2$ or $n \leq 5$, so we assume $k \geq 3$ in the rest of the section. Let $[x]_k = x(x-1)\cdots(x-k+1)$ denote the *falling factorial*, then

Lemma 4.1.5 *For $k \geq 3$ we have*

$$\mathbb{E}([\zeta_k]_1) = \frac{4n}{k}\frac{[t^k x^n]t^2\Phi(xt,x)}{[x^n]x^2\Phi(x,x)}$$
$$\mathbb{E}([\zeta_k]_2) = \frac{4n}{k}\frac{[t^k s^k x^n]G(t,s)}{[x^n]x^2\Phi(x,x)}.$$

The proof is exactly the same as that of part (ii) of lemma 1 in [29] (upon noting that an irreducible quadrangulation with n faces has $2n$ edges). Using lemmas 2 and 3 in [29], we can deduce uniform asymptotic estimates for the first two moments (provided the range of parameter k is suitably restricted):

Lemma 4.1.6 *Let ϵ be any constant verifying $0 < \epsilon < 2$ and let $k \geq 3$. For a random rooted irreducible quadrangulation with n faces, we have*

$$\mathbb{E}(\zeta_k) = \mathcal{O}(n(2-\epsilon)^{-k}) \text{ uniformly for all } k,\ n.$$

and (where the coefficients b_k are defined as in the statement of theorem (4.1.5))

$$\mathbb{E}([\zeta_k]_1) = \frac{3^6}{2k} n b_k \left(1 + \mathcal{O}\left(\frac{\log^{10} n}{n}\right)\right) = \frac{9\sqrt{6}}{\sqrt{\pi}} n k^{-1/2} 2^{-k} \left(1 + \mathcal{O}(k^{-1})\right)$$

$$\mathbb{E}([\zeta_k]_2) = \frac{3^{12}}{4k^2} n^2 b_k^2 \left(1 + \mathcal{O}\left(\frac{\log^{10} n}{n}\right)\right) = \frac{486}{\pi} n^2 k^{-1} 2^{-2k} \left(1 + \mathcal{O}(k^{-1})\right)$$

uniformly for $k = \mathcal{O}(\log n)$.

Finally, theorem (4.1.4) follows at once from lemma 7 in [29], while theorem (4.1.5) follows from lemma 4 in [29] and an application of Chebyshev's inequality upon observing that $n\mu_k \to \infty$ when $2 < k < (\log n - (1/2)\log\log n)/\log 2 - \Omega_n$.

Since we have been unable to describe the generating function of multipointed rooted irreducible quadrangulations we could not determine the exact limiting distribution of the maximum vertex degree as that would require knowledge of the higher factorial moments of ζ_k. However, we believe the limiting distribution of $\Delta_n - \mathbb{E}(\Delta_n)$ will again be the (extreme value) Gumbel distribution with parameter $1/2$ because that is also the case for general maps as established in [29].

4.2 Hulls of Irreducible Quadrangulations

Recall that in theorem (4.1.3) we have shown that asymptotically (in the limit as $N \to \infty$) in a large quadrangulation $\mathbf{q}$ conditioned to have $B_r(\mathbf{q}) = \mathbf{k}$, where $\mathbf{k}$ is a rooted proper irreducible quadrangulation with (possibly several) disjoint boundary components, with high probability only one of the "holes" in $\mathbf{k}$ contains the major part of the quadrangulation. This has motivated the definition of a hull.

We will present the construction of the hull of rooted irreducible quadrangulations and its skeleton decomposition. The decomposition was first described by M. Krikun in [42] for general quadrangulations. We follow closely the presentation and definition of hulls in general quadrangulations as described by J. Le Gall and T. Lehéricy in [44] and for triangulation by Curien and J. Le Gall in [15] and L. Menard in [48].

4.2.1 Layer decomposition of hulls

We will define in this section the hull of a rooted pointed irreducible quadrangulation and its associated layer decomposition. Our description is analogous to the layer decomposition of hulls of triangulations of type II in [41] and quadrangulations in [42]. However, certain modifications of the construction are necessary since our limiting probabilities (4.1) do not depend solely on boundary length but also on the associated composition parameter.

Let's recall that an irreducible quadrangulation $\mathbf{q}$ with a distinguished face f_0 of degree $2p$ has property (3.3.1) if its boundary vertices can be bi-partitioned into two edge-disjoint subsets $b_\circ$ and $b_\bullet$ (in particular, two boundary vertices cannot be linked by an edge lying strictly inside the map) such that no two vertices of $b_\bullet$ have a common adjacent vertex lying strictly inside the map. With this convention, we will always assume that $\mathbf{q}$ is rooted and the root edge lies on the boundary of the distinguished face f_0, called the *root face*, is oriented clockwise and the root vertex belongs to the set $b_\bullet$. We canonically equip $\mathbf{q}$ with a composition parameter $\underline{\lambda} \vdash p$, which gives the degree sequence of vertices of $b_\bullet$ starting at the root vertex in clockwise direction. Notice that we can associate to $\mathbf{q}$ its *diagonal cycle* ∂_0, which consists of all the face diagonals of quadrilaterals incident with vertices of the set $b_\bullet$ and disjoint with the set $b_\bullet$ (in other words, given a vertex $v \in b_\bullet$ and a quadrilateral $f = vw_0v'w_1$ incident with v, we draw the diagonal w_0w_1). According to the definition ∂_0 is a simple cycle of length u.

Definition 26. Let $\mathbf{q}$ be a rooted proper irreducible quadrangulation satisfying property (3.3.1) and $v \in V_{\text{inn}}(\mathbf{q})$ an inner vertex of $\mathbf{q}$. We say v is of *height* h if the graph-distance of v to the diagonal cycle of $\mathbf{q}$ is equal to h.

Suppose then that $\mathbf{q}$ is a rooted proper irreducible quadrangulation satisfying property (3.3.1) with root face f_0. For every positive integer $r \geq 1$, the ball $B_r(\mathbf{q})$ of radius r is the planar map obtained by taking the union of the faces of $\mathbf{q}$ that have at least one vertex of height less than or equal to $r-1$ from the diagonal cycle of $\mathbf{q}$. We adopt the following convention: the ball $B_r(\mathbf{q})$ of radius $r > 0$ will be considered as an irreducible quadrangulation with root face f_0 and a number of edge-disjoint simple proper holes such that the vertices at distance $r+1$ from the diagonal cycle ∂_0 will not be considered as belonging to the original map $\mathbf{q}$.

Now let v be a distinguished vertex of $\mathbf{q}$. We will denote by $(\mathbf{q}, v)$ the pointed rooted irreducible quadrangulation satisfying property (3.3.1). Let h be the height of the distinguished vertex v in $(\mathbf{q}, v)$ (recall that the height is defined as the minimal

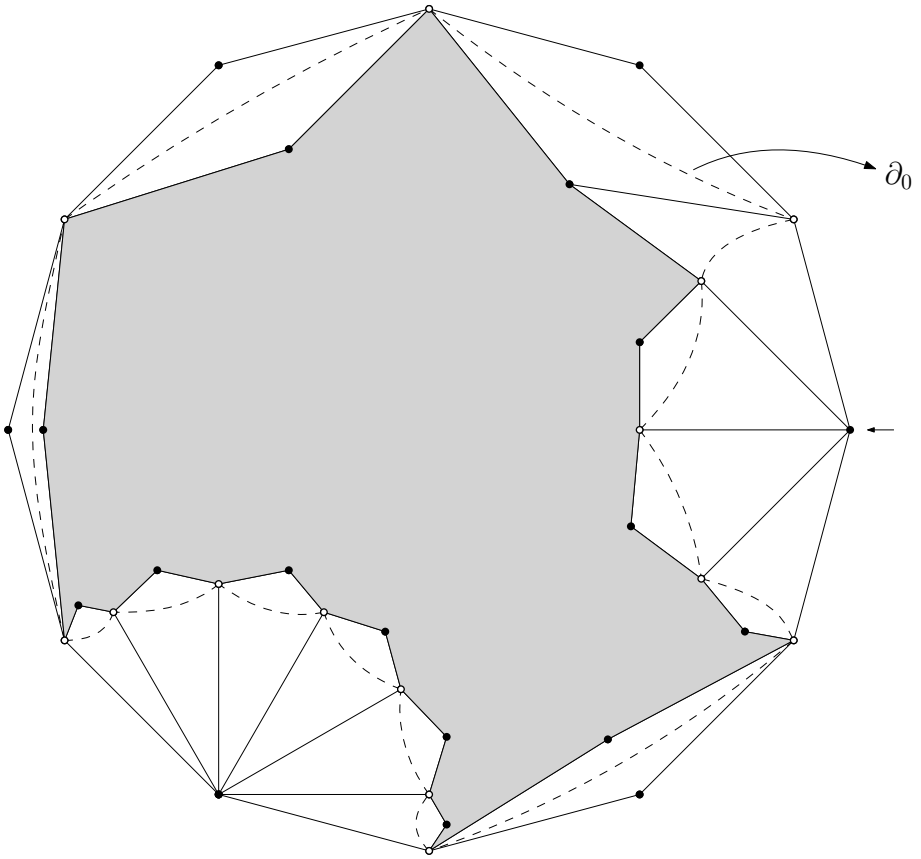


Figure 4.1: A possible boundary structure of an irreducible quadrangulation $\mathbf{q}$ with a distinguished unbounded face satisfying property (3.3.1) with its canonical bicolouration of boundary vertices and the associated diagonal cycle ∂_0 (shown dashed). The composition parameter is $\underline{\lambda} = (4, 1, 6, 1, 1, 1)$, starting with the root vertex (distinguished in the figure by the small arrow) in clockwise direction.

distance between v and a vertex of the diagonal cycle) and fix $0 < r < h$. In that case, the vertex v belongs to the complement of the ball $B_r(\mathbf{q}, v)$ in $\mathbf{q}$ (as per our convention) and we define *the r-hull* $B_r^\bullet(\mathbf{q}, v)$ of the pointed map $(\mathbf{q}, v)$ as the union of $B_r(\mathbf{q})$ and all the connected components of the complement of $B_r(\mathbf{q})$ in $\mathbf{q}$ that do not contain the distinguished vertex v. Then under the assumptions stated so far, it is easy to verify that the hull $B_r^\bullet(\mathbf{q}, v)$ is exactly an irreducible quadrangulation with root face f_0 and a distinguished face f_1, called the *exterior face*, whose boundary vertices form a proper simple cycle. If we canonically label the vertices of $B_r^\bullet(\mathbf{q}, v)$ by height from the diagonal cycle ∂_0, then according to the definition every quadrilateral incident with the exterior face is simple of type $(r, r-1, r, r+1)$. We define the boundary $\partial B_r^\bullet(\mathbf{q}, v)$ of the r-hull as the union of all the simple faces of type $(r, r-1, r, r+1)$ incident with the exterior face. A verbatim the same argument as in the proof of

lemma (3.4.1), shows that the vertices of the boundary of the exterior face have no non-trivial diagonals. We will describe below how to assign a composition parameter to the exterior boundary. For definiteness sake we set $B_r^\bullet(\mathbf{q}, v) = \dagger$ if $h \leq r$, where$\dagger$ is some cemetery point added to the set of all possible r-hulls.

Finally, note that as a consequence of our definitions, the hulls form an increasing sequence

$$B_1^\bullet(\mathbf{q}, v) \subseteq B_2^\bullet(\mathbf{q}, v) \subseteq \cdots \subseteq B_r^\bullet(\mathbf{q}, v).$$

We are now ready to define a layer of an irreducible quadrangulation.

Definition 27. Let $\mathbf{q}$ be an irreducible quadrangulation with simple boundary satisfying property (3.3.1) with a distinguished vertex v of height h. Let $0 < r < h$ be a fixed integer, then a *layer* $L_r^\bullet(\mathbf{q}, v)$ and a semi-layer $L_r^*(\mathbf{q}, v)$ of height r of $(\mathbf{q}, v)$ are defined as (where we adopt the convention $B_0^\bullet(\mathbf{q}, v) = \emptyset$)

$$L_r^\bullet(\mathbf{q}, v) = (B_r^\bullet(\mathbf{q}, v) \backslash B_{r-1}^\bullet(\mathbf{q}, v)) \cup \partial B_{r-1}^\bullet(\mathbf{q}, v) \tag{4.10}$$

$$L_r^*(\mathbf{q}, r) = L^\bullet(\mathbf{q}, v) \backslash \partial B_r^\bullet(\mathbf{q}, v). \tag{4.11}$$

Notice that this definition is incomplete, since we have not specified the root edge. We will see in a moment how to canonically root the layers of an irreducible quadrangulation $\mathbf{q}$. For now let us remark, that a layer is an irreducible quadrangulation whose lower boundary satisfies property (3.3.1) and whose upper boundary forms a proper simple cycle. We can associate to a layer its upper and lower diagonal cycle, which we denote by ∂_r and ∂_{r-1}, and are oriented in counterclockwise direction.

Let $\mathscr{L}_r^\bullet$, resp. $\mathscr{L}_r^*$ be a mapping on the set $\tilde{\mathbf{Q}}_f$ of pointed rooted irreducible with finitely many faces and a simple boundary satisfying property (3.3.1) that associates to each quadrangulation $(\mathbf{q}, v)$ its layer (resp. semi-layer) at height r, and denote by $\Lambda = \cup_r \mathscr{L}_r^\bullet(\tilde{\mathbf{Q}}_f)$ and $\Lambda' = \cup_r \mathscr{L}_r^*(\tilde{\mathbf{Q}}_f)$ the sets of all possible layers and semi-layers.

Let us examine some simple properties of a layer $L \in \Lambda$. We enumerate the vertices of the upper boundary of L as $(x_1, y_1, \ldots, x_p, y_p)$ starting from some arbitrarily chosen vertex and so that all x_i's are at height 2 from the lower diagonal cycle and all y_i's are at height 1 from the lower diagonal cycle (in particular, the face diagonals (y_{i-1}, y_i) form the upper diagonal cycle), then

(i) each face diagonal (y_{i-1}, y_i) of the upper boundary $\partial B_r^\bullet(\mathbf{q}, v)$ of the r-hull belongs to a unique quadrilateral of $L_r^\bullet(\mathbf{q}, v)$ that has a unique vertex on the lower boundary of the layer; we denote it by z_{i-1}. Such quadrilaterals will be called *down quadrilaterals of height* r.

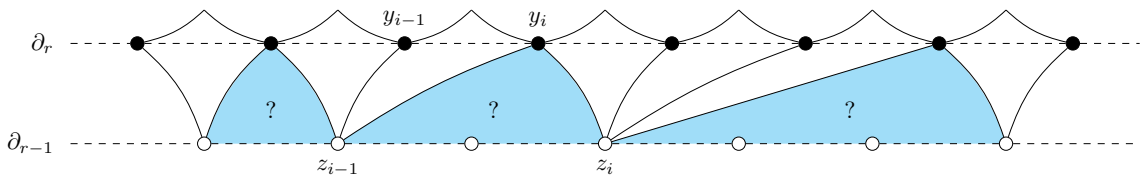


Figure 4.2: The layer decomposition of a hull of an irreducible quadrangulation.

(ii) The points z_i are cyclically ordered around ∂_{r-1} and there are $\ell_i \geq 0$ edges of ∂_{r-1} between z_i and z_{i+1} (note that we might have $\ell_i = 0$ if z_i and z_{i+1} refer to the same vertex). The part of the layer filling in the area (z_{i-1}, y_i, z_i) is called a *slot* or *interstice* associated with the down quadrilateral of (y_i, y_{i+1}) and rooted at its unique boundary edge belonging to the down quadrilateral, the orientation chosen so that the interior of the slot is one the left-hand side of the root.

The internal structure of an interstice is not as simple, since it can contain arbitrarily large sub-quadrangulations, which can have vertices of arbitrary height. This is where the reattached components of $B_r^\bullet(\mathbf{q}, v) \backslash B_r(\mathbf{q})$ go. Notice that a slot with $\ell_i = 0$ simply reduce to an edge by irreducibility.

Now let $\mathbf{q}$ be an irreducible quadrangulation with simple boundary satisfying property (3.3.1) with a distinguished vertex v of height h. Fix some $0 < r < h$ and construct the r-hull $\Delta = B_r^\bullet(\mathbf{q}, v)$. According to the definition it defines a growing sequence of hulls,

$$B_1^\bullet(\mathbf{q}, v) \subseteq B_2^\bullet(\mathbf{q}, v) \subseteq \cdots \subseteq B_r^\bullet(\mathbf{q}, v),$$

which is decomposed into layers $L_j^\bullet(\mathbf{q}, v)$ for $1 \leq j \leq r$. Let $\mathcal{N}(\Delta)$ be the collection of all face diagonals of Δ that belong to one of the cycles ∂_i for some $0 \leq i \leq r$. We can define a genealogy on $\mathcal{N}(\Delta)$ by saying that for any $1 \leq i \leq r$, an edge $e \in \partial_i$ is the parent of an edge $e' \in \partial_{i-1}$ if the first vertex encountered when turning around the oriented cycle ∂_{i-1} and starting at the end vertex of the oriented edge e' belongs to the down quadrilateral associated to e.

The relation introduced above defines a forest F of rooted trees, its vertices being in one-to-one correspondence with the edges of $\mathcal{N}(\Delta)$, which inherits its planar structure from that of Δ making it a planar rooted forest. In addition, we can order canonically the trees, starting from the one containing the root edge of Δ and following the orientation of $\partial_r \Delta$. Notice that the forest F consists of $|\partial_r|$ trees, such that every tree is of height at most r and the whole forest has exactly $|\partial_0|$ vertices of height r. The forest F is called the *skeleton* of Δ - we will complete the description

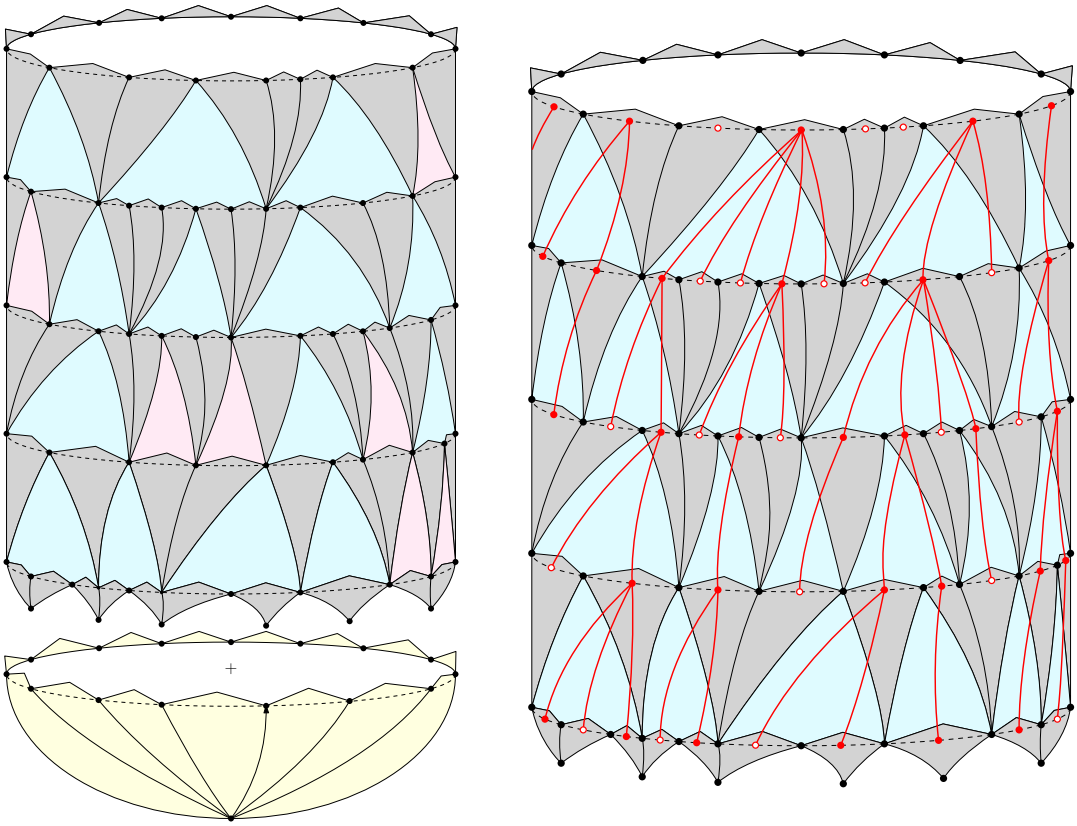


Figure 4.3: A schematic representation of the structure of hulls in rooted irreducible quadrangulations. On the left, the hull is obtained (as is standard) by first removing the star neighbourhood of the root vertex (shown in yellow) and has a canonical layered structure, where down quadrilaterals are grey and the non-trivial interstices are blue, while the trivial interstices (that are necessarily reduced to a face due to irreducibility) are red. On the right is presented the construction of the associated skeleton of a hull pictured in bold red. We have drawn the nodes without ancestors void, while the nodes with children have been drawn solid.

of hulls in terms of the skeleton below. As mentioned earlier this decomposition allows is to canonically root the upper and lower boundary of layers of an irreducible quadrangulation, by rooting each cycle ∂_i at the (unique) ancestor of the root edge of the quadrangulation in its skeleton. With these conventions we can now define the upper and lower composition parameters of a layer.

Definition 28. Let $\Delta = L_r^\bullet(\mathbf{q}, v)$ be a layer at height r of a pointed rooted irreducible quadrangulation $(\mathbf{q}, v)$ with simple boundary satisfying property (3.3.1) whose distinguished vertex is of height $h > r$. We assume the upper and lower boundaries of Δ are rooted as explained above, then we can associate to Δ its *upper* and *lower composition parameters* $\underline{\lambda}$ and $\underline{\pi} = (\underline{\pi}_1, \dots, \underline{\pi}_\ell)$ parameters as follows:

1. The upper boundary ∂_r of Δ forms a proper simple cycle with no non-trivial diagonals. We can thus associate to ∂_r a partition parameter $\underline{\lambda}'$ describing the structure of its non-trivial diagonals. However, as the upper boundary is oriented and rooted at an initial face of a block of the parameter $\underline{\lambda}'$, we can canonically order the partition starting with the root block in counterclockwise direction around the cycle ∂_r. This defines the upper composition parameter $\underline{\lambda}$.

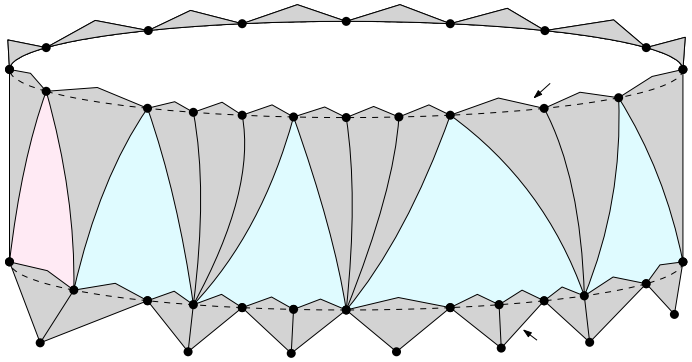


Figure 4.4: Assignment of upper and lower composition parameters to a layer, we have used the arrows to indicate the position of the root - the upper composition parameter will be a cyclic permutation of $(1, 1, 3, 3, 2, \mathbf{1}, \ldots)$ such that the bold term corresponding to the root is at the initial term. The slots have composition parameters (from left-to right) $(1), (1, 1), (1, 2), (1, 2, 1)$ and $(1, 1)$.

2. The lower boundary Δ is rooted and satisfies property (3.3.1), so we may canonically equip it with a composition parameter $\underline{\mu}$ which encodes the degree sequence of its $b_\bullet$ vertices. Suppose now that the upper composition parameter has k parts (note that by irreducibility we necessarily have $k \geq 3$), then the lower boundary cycle ∂_{r-1} is partitioned into k parts, which naturally induces a partition of its composition parameter into parts $\underline{\pi}_i$ (up to cyclic permutation). Hence every (non-trivial) slot is assigned a composition parameter $\underline{\pi}_i$, and we adopt the convention that the parameters are ordered in counterclockwise direction around the cycle ∂_{r-1} starting from the slot containing the root edge, and within each slot the parts of parameter $\underline{\pi}_i$ are ordered in counterclockwise direction starting with first child of the unique ancestor of the slot.

Note that the upper and lower composition parameters specify neither the lower boundary length nor the root of a layer Δ!

We have mentioned that the internal structure of an interstice can be fairly complicated. However, with the assignment of the lower composition parameter to a layer Δ, each slot of boundary length ≥ 4 (that does not reduce to an edge) is equipped

with a composition parameter $\underline{\pi}$. This additional piece of information allows us to describe a bijection between interstices and a class of rooted $\underline{\pi}$-admissible irreducible quadrangulations. The interstice is either reduced to a face if parameter $\underline{\pi}$ consist of single part; otherwise it is converted to a $\underline{\pi}$-admissible quadrangulation via the procedure illustrated below. In the first case, the interstice is called *trivial* and *non-trivial* otherwise. Clearly, the procedure is reversible.

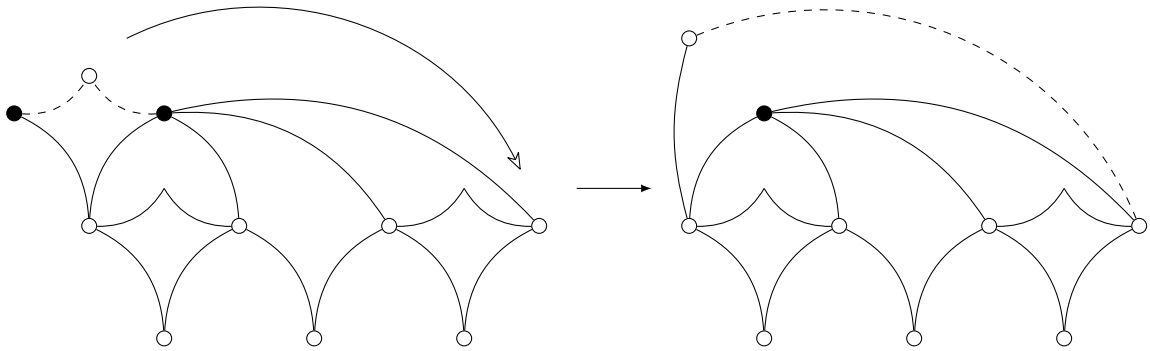


Figure 4.5: The enumeration procedure for admissible irreducible quadrangulations filling the slots in the skeleton decomposition of an irreducible quadrangulation (the figure has been take from [42]).

We can now similarly complete the definition of a semi-layer $L_r^*(\mathbf{q}, v)$ of height r. We assume it inherits its lower boundary root and composition parameter form the layer $L^\bullet(\mathbf{q}, v)$ of height r, while the upper boundary is rooted at the unique slot incident with the down quadrilateral of cycle ∂_r containing the root, so that the interior of the slot appears on the left-hand side of the root. With these convention set up, let us summarize the structure of the hull $\Delta = B_r^\bullet(\mathbf{q}, v)$ of a pointed rooted irreducible quadrangulation $(\mathbf{q}, v)$ with a simple boundary satisfying property (3.3.1):

- The upper boundary of Δ forms a proper simple cycle with no non-trivial diagonals, whose structure is described by the upper composition parameter $\underline{\lambda}$.
- The remainder $\Delta \backslash \partial B_r^\bullet(\mathbf{q}, v)$ of the hull decomposes uniquely into a sequence $L_1^*(\mathbf{q}, v), \ldots, L_r^*(\mathbf{q}, v)$ of semi-layers.
- Each semi-layer is equipped with a lower composition parameter $(\underline{\pi}_1, \ldots, \underline{\pi}_\ell)$, position of the root on both boundaries and the size of the lower boundary.
- Each slot has an associated composition parameter $\underline{\pi}$ with $k \geq 1$ parts and is filled with an irreducible quadrangulation, which is a 4-cycle if $k = 1$, or else a $(1, \underline{\pi})$-admissible rooted irreducible quadrangulation whose root face either belongs to the map or not (see figure above).

Clearly, every semi-layer uniquely specifies its lower composition parameter and the quadrangulations filling in the slots with appropriate admissible irreducible quadrangulations. Conversely, almost every set of admissible quadrangulations can be used to fill in the slots and obtain a semi-layer of an irreducible quadrangulation. There are, however, two exceptions.

Lemma 4.2.1 *Let $\underline{\pi} = (\underline{\pi}_1, \ldots, \underline{\pi}_\ell)$ be a sequence of composition parameters with $\ell \geq 3$ and $\mathbf{q}_1, \ldots, \mathbf{q}_\ell$ rooted irreducible quadrangulations such that $\mathbf{q}_i$ is $(1, \underline{\pi}_i)$-admissible. Let L be the result of gluing quadrangulations $\mathbf{q}_i$ (via the operation of connected sum) into the slots associated with parameters $\underline{\pi}_i$, then L is a semi-layer belonging to Λ', except in the following two exceptional cases*

(E_1) *there is at most one non-trivial interstice and all the trivial interstices are incident to a common vertex;*

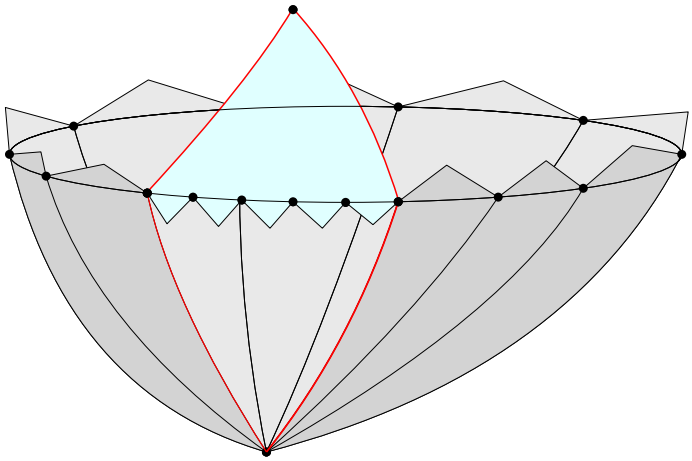


Figure 4.6: A semi-layer of exception (E_1) with a separating 4-cycle is shown in red.

(E_2) *there are at most two non-trivial interstices with composition parameters of length 2 such that the lower boundary quadrilaterals are all incident to exactly two vertices.*

Proof. We need only to check that L is strongly 4-irreducible, so every 4-cycle must bound an inner face. Since quadrangulations $\mathbf{q}_i$ are admissible for each slot, a separating 4-cycle cannot occur within a slot, so a 4-cycle can only occur if there exist two vertices of the lower boundary cycle incident to two common vertices and not reduced to a face. Clearly, not both common vertices can belong to the upper boundary since the upper boundary is of length $2\ell \geq 6$. Suppose first there exists a separating 4-cycle consisting of an upper boundary vertex and two vertices of the lower boundary

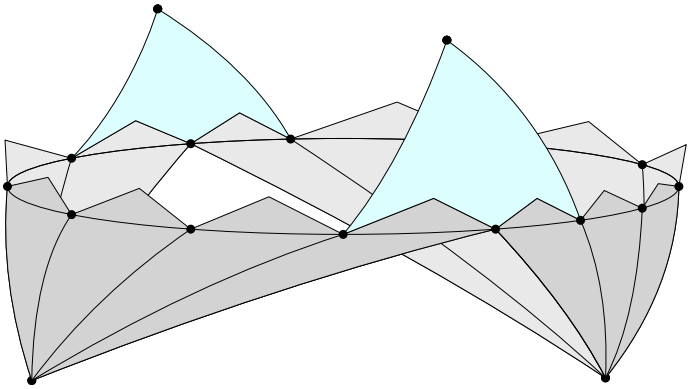


Figure 4.7: A semi-layer of exception (E_2) with lower boundary of length 4.

cycle, then according to the structure of the semi-layer the vertices must delimit a non-trivial slot. Hence the only possible way for the two lower boundary vertices to have a second common vertex that does not belong to the slot is if the other slots are all trivial and share a common vertex. This is exactly exception (E_1).

On the other hand if there exists a separating 4-cycle consisting of two vertices on the lower boundary, but otherwise disjoint with the upper boundary, then the lower boundary must have total length 4. Clearly, that implies at once there are at most two non-trivial slots with composition parameters of length 2. Furthermore, all the trivial slots and the quadrilaterals of the possibly non-trivial slots must be joined to exactly two vertices. We have described exactly exception (E_2). □

We can now also prove that any semi-layer $L \in \Lambda'$ can appear at any height r within some finite rooted irreducible quadrangulation $\mathbf{q}$ in $\mathbf{Q}_f$.

Lemma 4.2.2 *Let $L_1, \ldots, L_r$ be a sequence of semi-layers, such that the lower boundary length of L_{i+1} is equal to the upper boundary length of L_i for each i, then by identifying the upper boundary of L_i with the lower boundary of L_{i+1} for $i = 1, \ldots, r-1$, we obtain a valid irreducible quadrangulation. Consequently, the image of $\mathscr{L}_r^*$ is the same for all r:*

$$\mathscr{L}_1^*(\mathbf{Q}_f) = \ldots = \mathscr{L}_r^*(\mathbf{Q}_f) = \Lambda'.$$

Proof. When identifying the upper and lower boundaries of two semi-layers L and L' we only need to make sure that the resulting map contains no separating 4-cycles. Just as in the proof of lemma (3.3.4) a separating 4-cycle can only occur if only if there exists two boundary vertices b, b' on the upper, resp. lower boundary of L and L', with a common inner vertex in both semi-layers. However, such a 4-cycle cannot exist since the lower boundary of L' satisfies property (3.3.1). □

Finally, we will also need to know the exact generating function of the possible quadrangulations filling the slots of the skeleton decomposition. Recall that the admissible quadrangulations are the 4-cycle if the total boundary length of the slot is 4 or a rooted irreducible quadrangulation $\mathbf{q}$ with partition parameter of the form $(1, \underline{\lambda})$ whose root face (i.e. the inner face lying to the left of the root edge) either belongs to $\mathbf{q}$ or is removed thus forming a new map, say $\mathbf{q}'$. Let $Z(z, x)$ denote the generating function of all rooted admissible quadrangulations with a weight x per each inner face and a weight $\sqrt{z}$ per each non-root boundary face, then we have

Lemma 4.2.3 *The generating function $Z(z, x)$ of admissible quadrangulations filling in the slots of a semi-layer $L \in \Lambda$ with a weight x per each inner face and a weight $\sqrt{z}$ per each non-root boundary face is given by*

$$Z(z,x) = \frac{-z}{2u(1+z)^2}\Big\{uz^2 - (1+u+4u^2)z - u(-1+u-u^2) \\ +(z-u)\sqrt{u^2z^2 - 2(1+3u+3u^2)uz + (1+u-u^2)^2}\Big\} \tag{4.12}$$

Proof. Indeed, by definition the generating function can be expressed as

$$\begin{aligned} Z(z,x) &= xz + (1+x^{-1})\sum_{\ell\geq 2} z^\ell \sum_{\underline{\lambda}} q((1,\underline{\lambda});x) = xz + \sum_{\ell\geq 3} z^\ell \sum_{\underline{\lambda}} [t\underline{t}^{\underline{\lambda}}] Q_{\ell+1}(t,\underline{t};x) \\ &= xz + (1+x^{-1})\sum_{\ell\geq 2} z^\ell \left.\frac{\partial}{\partial t} Q_{\ell+1}(t,t_1,\ldots,t_{\ell+1})\right|_{t=0,t_i=1} \\ &= xz + (1+x^{-1})\sum_{\ell\geq 2} z^\ell \frac{\psi^\ell(1,x)}{(\ell-1)!}\left.\frac{d^{\ell-1}}{d\zeta^{\ell-1}}\zeta^{-2}Q(\zeta;x)\right|_{\zeta=\psi(1,x)} \\ &= xz + \frac{z(1+x)}{x\psi(1,x)}\left\{\frac{1}{(1+z)^2}Q((1+z)\psi(1,x)) - Q(\psi(1,z))\right\}. \end{aligned} \tag{4.13}$$

It is easily shown using lemma (3.4.3) that $\psi(1,x) = u(1+x)x^{-1}(1+u)^{-3}$. Plugging the value in (4.13) gives, after some calculation, exactly formula (4.12). □

We can now complete the skeleton decomposition of hulls. Recall that we have associated to an r-hull $\Delta = B^\bullet(\mathbf{q}, v)$ of a pointed rooted irreducible quadrangulation $(\mathbf{q}, v)$ with a simple boundary satisfying property (3.3.1) and a distinguished vertex of height $h > r$, an ordered rooted planar forest F, called the skeleton of Δ. Let $\underline{\lambda}$ be the upper composition parameter of Δ, $\underline{\nu}$ its canonical lower composition parameter encoding the degree sequence of the lower boundary vertices and p its lower boundary length. We will first augment the forest F to also encode $\underline{\nu}$ (which at the moment cannot be read of the skeleton) as follows:

• add a child to each left-most node of height r of every part of the parameter $\underline{\nu}$ and re-root at the child of the root vertex (which is necessarily a left-most child of the root part of $\underline{\nu}$).

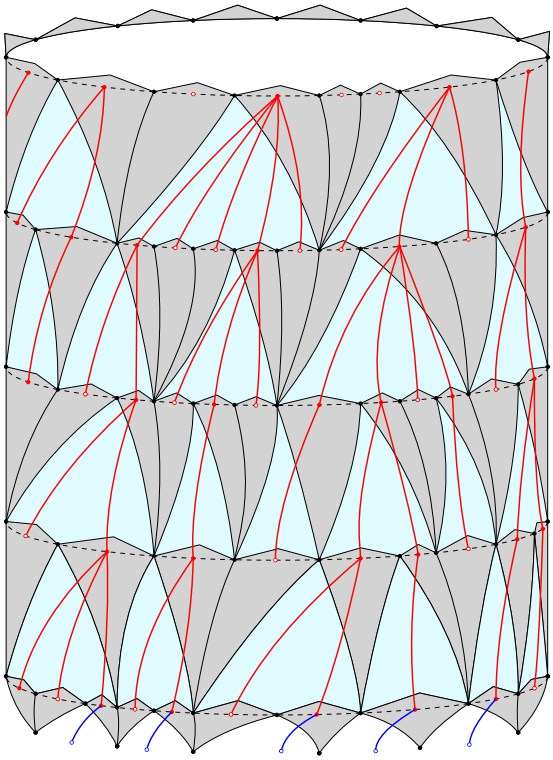


Figure 4.8: The construction of the augmented forest of the skeleton of a hull pictured in bold red with the newly added vertices and edges in blue. We have drawn the nodes without ancestors void, while the nodes with children have been drawn solid.

We let F_0 denote the augmented forest and we denote by $|\underline{\lambda}|_0$ the number of parts and by $|\underline{\lambda}|_1$ the sum of all parts of a composition parameter. Clearly, not every pointed linear ordered planar forest F_0 can arise as the skeleton of an irreducible hull. We will characterize the possible skeletons of irreducible hulls below (which will amount to translating exceptions (E_1) and (E_2) into tree-theoretic terms). It will be convenient to call a node of a forest *sterile* if it has no children. Let us say a pointed forest is $(r, \underline{\lambda}, p)$-*admissible* if

1. It is composed of an ordered sequence of $|\underline{\lambda}|_1$ rooted plane trees of height $r+1$, of which at most $|\underline{\lambda}|_0$ are non-trivial (i.e. not reduced to a single sterile node).
2. It has exactly p vertices at height $r+1$.
3. The distinguished vertex has height $r+1$ and belong to the first tree.

and the following exceptions do not occur

(E_0) A node has at least two children, all sterile expect perhaps the first one.

(E_1) At height i every node except possibly one has only sterile children.

(E_2) At height i there are at most two non-sterile nodes.

Exception (E_0) results from the irreducibility assumption. It does not explicitly arise in the construction of semi-layers, since we have already assumed that an admissible quadrangulation of a slot with composition parameter consisting of single part, must be a face. Note that the composition parameters associated to a given slot in the skeleton decomposition can be determined from the forest F; indeed, the composition parameter of a slot associated to a given node v in F of height strictly less than r is determined by the genealogy of children of node v in F as follows: order the children of v from left-to-right as $(v_0, v_1, \ldots, v_c)$. Let $s_1, \ldots, s_{k-1}$ be the positive indices of the non-sterile children arranged in increasing order and set $s_0 = 0$, $s_k = c + 1$, then the composition parameter is $\underline{\pi} = (s_1 - s_0, \ldots, s_k - s_{k-1})$.

We denote by $\mathfrak{F}(r, \underline{\lambda}, p)$ the set of all $(r, \underline{\lambda}, p)$-admissible forests and for any $F \in \mathfrak{F}(r, \underline{\lambda}, p)$ we denote by F^* the set of vertices of F at height strictly smaller than r. We can then summarize the skeleton decomposition as

Lemma 4.2.4 *The skeleton decomposition is a bijection between irreducible hulls of radius r, upper composition parameter $\underline{\lambda}$ and lower boundary cycle of length p and pairs consisting of an $(r, \underline{\lambda}, p)$-admissible forest F and a collection $\{(M_v, \underline{\pi}_v)\}_{v \in F^*}$, where for each node $v \in F^*$, $\underline{\pi}_v$ is the composition parameter describing the genealogy structure of the node v in F and M_v is an admissible irreducible quadrangulation with parameter $(1, \underline{\pi}_v)$.*

The skeleton F of an irreducible hulls Δ will be denoted by $\mathrm{Skel}(\Delta)$.

4.2.2 Hulls of irreducible quadrangulations

We can now describe the hulls of irreducible quadrangulations. Let $\mathbf{q}$ be a rooted irreducible quadrangulation and denote by $\mathbf{q}_0$ the irreducible quadrangulation with a simple boundary obtain upon removing the star-neighbourhood of the root vertex in $\mathbf{q}$, and re-rooted at the unique edge incident to the target of the original root. We have already shown that the resulting map satisfies property (3.3.1), so we could apply the construction of the hull of the preceding section to $\mathbf{q}_0$ if we also had a distinguished vertex. Note that working with $\mathbf{q}_0$ means no loss of information, since the original irreducible quadrangulation can be uniquely reconstructed from $\mathbf{q}_0$ by gluing back the quadrangular star.

Although the definition of the hull depends on a reference vertex v, it can be made precise also in a rooted irreducible quadrangulation $\mathbf{q}_0$ without a pointed vertex. Indeed, according to theorem (4.1.3), for any fixed $r \geq 1$, with high probability precisely one hole of the r-ball $B_r(\mathbf{q}_0)$ contains the major part of the quadrangulation. We may define the r-hull as the rooted proper irreducible quadrangulation obtained from $B_r(\mathbf{q}_0)$ by filling all the holes, except the maximal one. If there are multiple holes with maximal size, we chose one of them in some deterministic manner (the details are not important to the construction since such situation arises with small probability as the number faces $N \to \infty$).

Consider a rooted proper irreducible quadrangulation Δ with simple upper boundary and lower boundary satisfying property (3.3.1) such that $\Delta = B_r^\bullet(\mathbf{q}_0)$. If $\mathbf{q}_0$ is large enough, e.g. if the number of faces in $\mathbf{q}$ is at least twice that of Δ, then

$$B_r^\bullet(\mathbf{q}_0) \supseteq B_{r-1}^\bullet(\mathbf{q}_0) \supseteq \cdots \supseteq B_1^\bullet(\mathbf{q}_0). \tag{4.14}$$

Note that this would not be true for a general finite irreducible quadrangulation. Furthermore, provided $\mathbf{q}_0$ was chosen large enough the sequence (4.14) does not depend on $\mathbf{q}_0$ and we have $B_j^\bullet(\Delta) = B_j^\bullet(\mathbf{q}_0)$[1]. Note that in the construction of the skeleton decomposition, the distinguished vertex was needed only to guarantee that property (4.14) held, so provided $\mathbf{q}_0$ is large enough we can deduce the existence of a skeleton decomposition of Δ just as in the preceding section. The restriction that $\mathbf{q}_0$ is large enough is inconsequential since we are ultimately interested only in the limit as $N \to \infty$.

We also have an analogous result for limiting probabilities of hulls:

Lemma 4.2.5 (Uniform probabilities of hulls) *Let $\mathbf{k}$ be a rooted proper irreducible quadrangulation with partition parameter $\underline{\lambda}$ with n faces and a simple boundary of length $2p$ (with $p \geq 3$) such that $B_r^\bullet(\mathbf{q}) = \mathbf{k}$ for some rooted irreducible quadrangulation $\mathbf{q}$ and $r \geq 1$, then*

$$\lim_{N\to\infty} \mathbb{P}(B_r^\bullet(\mu_N) = \mathbf{k}) = \frac{729}{8} c(\underline{\lambda}) x_0^{n-p} \tag{4.15}$$

where μ_N denotes the uniform measure on the set of all rooted irreducible quadrangulations with N faces.

[1]We are being somewhat imprecise here since we have not formally defined the hull of an irreducible quadrangulation with two boundary components; we leave to the reader the simple task of extending the definition (the role of the pointed vertex is now played by the exterior face).

Proof. The proof is essentially the same as for lemma (4.1) for a ball with a single boundary component. Indeed, given $\mathbf{k}$, the connected sum $\mathbf{k}\#\mathbf{s}$ with a $\underline{\lambda}$-admissible irreducible quadrangulation $\mathbf{s}$ will have the same r-hull $B_r^\bullet(\mathbf{k}\#\mathbf{s}) = \mathbf{k}$, provided the number of faces of $\mathbf{s}$ is large enough (say larger than n). Thus for $N > 2n$, we again have

$$\mathbb{P}(B_r^\bullet(\mu_N) = \mathbf{k}) = \frac{c_{\underline{\lambda}}(N - n + p)}{c(N)}.$$

The limit as $N \to \infty$ follows as before. □

4.2.3 The hull operator

The skeleton decomposition would in principle allow us to compute the probability that an r-hull $B_r^\bullet$ of an infinite random irreducible quadrangulation $\mathbf{q}$ has a particular skeleton F. However, in the construction the exceptions (E_i) must be taken into account, which is not a simple task for a generic skeleton. The construction of the skeleton suggests we could instead define an operator that computes a statistical sum over all semi-layers in Λ' with respect to the lower boundary length and the number of faces as was done in [41] for triangulations of type II. We will prove the following result.

Lemma 4.2.6 *Let $p \geq 3$ denote the length of the lower boundary of a semi-layer L and let $n = n(L)$ denote the number of inner quadrilateral faces in L. Suppose we take a function $G(z)$ that allows expansion $G(z) = \sum_{k\geq 3} c_k z^{k-1}$, then we can write the sum over all semi-layers $L \in \Lambda'$ as*

$$\sum_L c_p x^n z^{p-1} = (\boldsymbol{A}(x)G)(z)$$

where $\boldsymbol{A}(x)$ is a linear operator defined as follows

$$\begin{aligned}(\boldsymbol{A}(x)G)(z) = R'(z,x)G(R(z,x)) - (R(z,x) - x + (1+z)R'(z,x))G(x) \\ - zR'(0,x)(R'(0,x)G'(x) - 2G(x)),\end{aligned} \tag{4.16}$$

in which prime denotes differentiation with respect to z and we have made the following notational convention $R(z,x) = (1+z)z^{-1}Z(z,x)$.

Proof. Let P be a semi-layer pattern with upper boundary of length ℓ and lower boundary of length p; we assume $\ell, p \geq 3$. Let us enumerate the slots, starting from the one that contains the root of the lower boundary. Let the slots have composition parameters $\underline{\lambda}_1, \ldots, \underline{\lambda}_\ell$. Given the parameters $\underline{\lambda}_1, \ldots, \underline{\lambda}_\ell$ the root can be placed in any

of $\ell_1 := |\underline{\lambda}_1|_0$ positions, so in order to define a semi-layer pattern such a position should be specified along with the composition parameters $\underline{\lambda}_1, \ldots, \underline{\lambda}_\ell$. Finally, let us write $\ell_i = |\underline{\lambda}_i|_0$, $r = \ell_1 + \cdots + \ell_\ell$ and $F(z) = \sum_{k\geq 3} c_k z^k$.

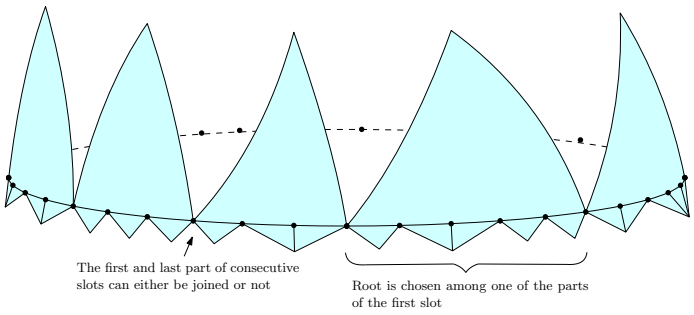


Figure 4.9: A semi-layer with specified upper boundary length and lower composition parameter. Note that the ambiguity of whether the first and last quadrilateral of two consecutive slots are joined or not corresponds on the level of generating functions to multiplication by the factor $(1+z)$. On the other hand, the root can be specified among any of the ℓ_1 parts of the first slot with the convention that if the first part is chosen then it cannot be joined with the quadrilateral of the last slot.

We have at first (without taking into account exceptions E_i)

$$\begin{aligned}\sum_L c_\ell x^n z^p &= \sum_{\ell\geq 3} \sum_{\underline{\lambda}_1,\ldots,\underline{\lambda}_\ell} c_\ell \left\{ \prod_i (1+x^{-1})[t_1 \underline{t}^{\underline{\lambda}}] Q(\underline{t}) \right\}^* z^{r-\ell}(1+z)^{\ell-1}(\ell_1(1+z)-1) \\ &= \sum_{\ell\geq 3} c_\ell (1+z^{-1})^{\ell-1} Z^{\ell-1}(z,x)((1+z)Z'(z,x) - z^{-1}Z(z,x)) \\ &= zu'(z,x)G(u(z,x)). \end{aligned} \tag{4.17}$$

where the last factor of the first line corresponds to the fact that the parameters $\underline{\lambda}_1, \ldots, \underline{\lambda}_\ell$ do not uniquely determine the length of the lower boundary and we have introduced the notation $G(z) = z^{-1}F(z)$ and $u(z,x) = (1+z)z^{-1}Z(z,x)$. The starred bracket is somewhat imprecise, since it does not include the term corresponding to a trivial slot with composition parameter $\underline{\lambda} = (1)$; if we augment the generating function $Q(\underline{t}; x)$ (determined in chapter 2) by adding a term $x(1+x^{-1})^{-1}t_1 t_2$ (and no other), then the bracketed term is correct. We will assume this modification has been made to Q throughout the proof to simplify the discussion.

Formula (4.17) is not yet correct due to the exceptional cases. Note that we have already taken into account exception (E_0) since $[z]Z(z,x) = x$. However, the term (4.17) includes some layers that should not be counted due to exceptions (E_1) and (E_2), so we have to subtract these exceptional layers from (4.17) to obtain a valid expression. We have two cases to consider: either the semi-layer pattern P

has exactly one non-trivial slot (i.e. a slot not reduced to a face) and all the trivial slots are joined or there are exactly two non-trivial slots. We refer the reader to the schematic representation of the two exceptions in figures (4.6) and (4.7). Note that $x = (d/dz)Z(z,x)|_{z=0}$ and $Z_2(x) = [z^2]Z(z,x) = (d/dz)z^{-1}Z(z,x)|_{z=0}$.

The invalid layers counted in the first case are expressed as the sum

$$\begin{aligned}\sum_{(E_1)} c_\ell x^n z^p &= \sum_{\ell\geq 3,\underline{\lambda}_1} c_\ell x^{\ell-1}(1+x^{-1})[t_1\underline{t}^{\underline{\lambda}_1}]Q(\underline{t})z^{\ell_1-1}((1+\ell_1)z^2+2\ell_1 z+\ell_1-1-\delta_{1,\ell_1}z)\\ &= \sum_{\ell\geq 3} c_\ell x^{\ell-1}((1+z)^2Z'(z,x)-z^{-1}(1-z^2)Z(z,x))-zF(x)\\ &= z\frac{d}{dz}[(1+z)u(z,x)]G(x)-zxG(x).\end{aligned}$$

The last term has been subtracted to avoid double counting. On the other hand, the invalid layers counted in the second case contribute

$$\begin{aligned}\sum_{(E_2)} c_\ell x^n z^\ell &= z^2\sum_{\ell\geq 3} c_\ell x^{\ell-2}\left\{(\ell-1)Z_2^2(x)+2(\ell-2)xZ_2(x)+(\ell-3)x^2\right\}\\ &= z^2(x+Z_2(x))^2G'(x)-2z^2(x+Z_2(x))G(x)\\ &= z^2R'(0,x)^2G'(x)-2z^2R'(0,x)G(x).\end{aligned}$$

Putting together all the contributions and re-writing the result in terms of G and u gives exactly expression (4.16) for the operator $\boldsymbol{A}$. □

We note for future reference the following explicit formula for the function $R(z,x)$ introduced above (which is an easy consequence of formula (4.12)):

$$\begin{aligned}R(z) = \frac{-1}{2u(1+z)}\Big\{&uz^2-(1+u+4u^2)z-u(-1+u-u^2)\\ &+(z-u)\sqrt{u^2z^2-2(1+3u+3u^2)uz+(1+u-u^2)^2}\Big\}\end{aligned}$$

The following result is an immediate corollary of lemmas (4.2.6), (4.2.1) and (4.2.2):

Lemma 4.2.7 *Let $p\geq 3$ denote the length of the lower boundary of a truncated hull $\bar{B}_r = B_r^\bullet\backslash\partial B_r^\bullet$ and let n denote the number of inner quadrilateral faces in $\bar{B}_r$. Suppose we take a function $G(t)$ that allows expansion $G(t)=\sum_{k\geq 3}c_k z^{k-1}$, then the sum over all truncated r-hulls $\bar{B}_r$ is expressible in terms of the operator $\boldsymbol{A}(x)$ as*

$$\sum_{\bar{B}_r} c_p x^n z^{p-1} = (\boldsymbol{A}^r(x)G)(z). \tag{4.18}$$

We will frequently write the operator $\boldsymbol{A}(x)$ simply as $\boldsymbol{A}_x$ (or $\boldsymbol{A}_u$, respectively $\boldsymbol{A}_q$ where $u=u(x)$ and $q=q(x)$ are certain auxiliary parameters that will be recalled in the next section).

4.3 The Hull Operator $\boldsymbol{A}_q$

We have introduced in the previous section the hull operator $\boldsymbol{A}_q$ as a convenient tool for studying the asymptotics of the volume and perimeter of hulls of irreducible quadrangulations. The operator has been defined only formally without specifying its domain and codomain. However, the properties of the operator will depend crucially on the analytic properties of the map $R(z, x)$. We will examine more closely the map $R(z, x)$ in the following section.

Before we start, we briefly recall the standard notational convention that have been used throughout the thesis.

Definition 29. We denote by $u \equiv u(x)$ the unique solution of the quadratic equation

$$u(x)[1 - u(x)] = x \tag{4.19}$$

satisfying $u(0) = 0$, The function $u(x)$ is simply the shifted generating function of Catalan numbers and is given explicitly by the following formula

$$u(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \sum_n \frac{1}{n+1}\binom{2n}{n} x^{n+1} = x + x^2 + 2x^3 + 5x^4 + \cdots \quad |x| \leq 1/4.$$

The radius of convergence of the power series expansion about the origin is $x_0 = 1/4$ using Stirling's formula. The point x_0 will be called the *critical value* of parameter x and we will always assume from now on, unless explicitly stated otherwise, that the values of the parameter x are restricted to the interval $[0, x_0]$. Since the map $u = u(x)$ establishes a one-to-one correspondence

$$u(x) : [0, x_0] \xrightarrow{\sim} [0, u_0]$$

where $u_0 = 1/2$ is the *critical value* of the parameter u, we will treat parameters x and u as equivalent, connected by the relation (4.19), and will use them interchangeably as appropriate.

Definition 30. We also define parameter $q = q(x)$ as the unique positive solution of the quartic equation

$$u(x)[1 + q^4(x)] = q^2(x) \tag{4.20}$$

satisfying $q(0) = 0$. Equivalently, parameter q can be expressed solely in terms of x by using relation (4.19) to eliminate the indeterminate u. The functions $q = q(x)$ and $q = q(u)$ are expressible in terms of radical as follows

$$q(u) = \sqrt{\frac{1 - \sqrt{1 - 4u^2}}{2u}} \text{ and } q(x) = \sqrt{\frac{1 - \sqrt{2\sqrt{1 - 4x} - (1 - 4x)}}{1 - \sqrt{1 - 4x}}}.$$

The radius of convergence of the power series expansion about the origin is $x_0 = 1/4$ and $u_0 = 1/2$ and it is once again easily checked that the map $q = q(u)$ establishes a one-to-one correspondence

$$q(u) : [0, u_0] \xrightarrow{\sim} [0, q_0],$$

where $q_0 = 1$ is the *critical value* of parameter q. As such we will freely switch between all three parameters (x, u, q). Finally, let us remark that the substitution $u \to u(q)$ will always yield an expression in q that is invariant under the action of the Klein viergruppe V of transformations since the same is true for the map $u(q)$.

Having set up our parameters, we will first fix the meaning of the function $R(z, x)$. Since the function is expressible in terms of multivalued radicals it will be important to make it precise which branch and branch-cut of the square-root is being used. As such we will first define the map $R \equiv R(z, x)$ as a solution of a certain irreducible quadratic equation.

Definition 31. Let $u \in [0, u_0]$ be fixed. The function $R(z, u) \equiv R_u(z)$ is an algebraic function of z of degree 2 defined as the unique solution of the quadratic equation $F_u(z, w) = 0$ given by

$$u(1+z)w^2+\{uz^2-(1+u+4u^2)z+u(1-u+u^2)\}w+u(z-u(1-u))(1+z) = 0 \quad (4.21)$$

such that $R_u(0) = u(1 - u)$. Provided $u \neq 0$, the conjugate root exists and will be denoted by $\bar{R}(z, u) \equiv \bar{R}_u(z)$. Note that the definition of R_u also makes sense for $u = 0$, where we set R_u identically equal to zero (whereas the conjugate root at $u = 0$ would be identically equal to ∞).

Although not immediately obvious from (4.21), the equation $F_u(z, w)$ is symmetric in the variables z and w, that is $F_u(z, w) = F_u(w, z)$. This is easily seen by writing it in the following more symmetric form

$$F_u(z, w) = u(1+z)w^2-(1+u+4u^2)zw+u(1+w)z^2+u(1-u+u^2)(z+w)-u^2(1-u).$$

As a result we will see that the conjugate root $\bar{R}_u$ can be interpreted as a (left) inverse of R_u (with respect to the variable z) over a suitable domain (this will be made more precise below).

Analytic properties of R_u: We will assume from now that $u \in (0, u_0]$. In chapter 3 we studied the properties of function related to $R_u(z)$ by treating its defining equation as an affine algebraic plane curve. We will follow a different approach in this section by analysing instead the compact Riemann surface defined by equation $F_u(z, w) = 0$.

The quadratic polynomial $F_u(z,w)=0$ is clearly irreducible, so equation $F_u=0$ determines, for each fixed $u\in(0,u_0]$, a compact Riemann surface $\mathcal{R}_u$ and a (complete) meromorphic algebraic function $\pi_u:\mathcal{R}_u\to\mathbb{P}^1$. The *critical points* (i.e. points at which equation $F_u(z,w)=0$ does not admit two distinct solutions or else becomes meaningless) are exactly the zeros of the principal coefficient, zeros of the discriminant and (possibly) the point ∞. We will denote by $\mathcal{S}$ the set of all critical points. After some straightforward algebraic calculations we find that the discriminant of the equation is

$$\mathrm{Discr}(z)=(z-u)^2(uz^2-2u(1+3u+3u^2)z+(1+u-u^2)^2). \tag{4.22}$$

Consequently, the set of critical point is a subset of $\{-1,u,\infty,z_\pm\}$, where $z_\pm$ are the zeros of the quadratic factor of $\mathrm{Discr}(z)$ given by

$$z_\pm=\frac{1+3u+3u^2}{u}\pm 2(1+u)\frac{1+2u}{u}=\frac{(1\pm q+q^2\pm q^3+q^4)^2}{q^2(1+q^4)}. \tag{4.23}$$

To simplify notation we will also write $z_0=-1, z_1=u$ and $z_\infty=\infty$. It is easily checked that for $u\in(0,u_0)$ all the points are distinct. However, at the critical value $u=u_0$ the points z_1 and z_- coincide, so that case will have to be treated separately. The corresponding branching values are $w_0=0, w_1=u, w_\infty=-1$ and

$$w_\pm=\mp\frac{1\pm q+q^2\pm q^3+q^4\pm q^5+q^6}{q(1+q^4)}. \tag{4.24}$$

As is well-known critical points of $F_u(z,w)$ are at most algebraic singularities of R_u and $\bar{R}_u$, all other points being regular for both branches. Furthermore, zeros of the discriminant are necessarily ordinary algebraic branch points. We will determine next the ramification index and local behaviour of the branches R_u and $\bar{R}_u$ about each of the critical points using Newton's diagrams. This will also allow us to compute the genus of the Riemann surface $\mathcal{R}_u$ and find a convenient parametrization.

$\boldsymbol{z_0}$ Let's introduce the local change of coordinates $(z,w)\to(x-1,y)$ and re-write F_u in terms of (x,y):

$$\underbrace{x(-u(1+u-u^2)+ux)}_{(0,1)}+\underbrace{\{(1+u)^3-(1+3u+4u^2)x+ux^2\}y}_{(1,0)}+\underbrace{uxy^2}_{(2,1)}.$$

Since the principal coefficient of each term is clearly non-zero on $(0,u_0]$, we conclude form the diagram that the point z_0 is always unramified with index $e_0=1$ and in a neighbourhood of z_0 the equation admits two meromorphic solutions locally of the form

$$w_1(z)=\frac{u(1+u-u^2)}{(1+u)^3}(z+1)+o[(z+1)]\text{ as }z\to-1$$

$$w_2(z) = -\frac{(1+u)^3}{u}(z+1)^{-1} + o[(z+1)^{-1}] \text{ as } z \to -1.$$

$\boldsymbol{z_\infty}$ Let's introduce the local change of coordinates $(z, w) \to (x^{-1}, y-1)$ and re-write (the numerator of) F_u in terms of (x, y):

$$\underbrace{x(1+u)^3}_{(0,1)} + \underbrace{\{u - (1+3u+4u^2)x - u(1+u-u^2)x^2\}y}_{(1,0)} + \underbrace{x(u+ux)y^2}_{(2,1)}.$$

The diagram is the same as for the point z_0. Since the principal coefficients are again all non-zero in the interval $(0, u_0]$, it follows that the point z_∞ is also unramified with index $e_\infty = 1$ and in a neighbourhood of z_∞ the equation admits two meromorphic solutions locally of the form

$$w_1(z) = -1 - \frac{(1+u)^3}{u}z^{-1} + o[z^{-1}] \text{ as } z \to \infty$$
$$w_2(z) = -z + o(z) \text{ as } z \to \infty.$$

$\boldsymbol{z_1}$ Let's introduce the local change of coordinates $(z, w) \to (x+u, y+u)$ and re-write F_u in terms of (x, y):

$$\underbrace{u(1+u)x^2}_{(0,2)} + \underbrace{x\{-1-u+ux\}y}_{(1,1)} + \underbrace{(u(1+u)+ux)y^2}_{(2,0)}.$$

The principal coefficients are non-zero on $(0, u_0]$. However, its Newton diagram is a straight line segment, so the test is inconclusive. The point z_1 can either be unramified with index $e_1 = 1$ or ramified with index $e_1 = 2$. We will consider first the case $u \neq u_0$. Let $\mathbb{C}[\![x]\!]$ denote the ring of formal power series in x and $\mathbb{C}[\![x]\!][y]$ the ring of polynomials in y with coefficients in $\mathbb{C}[\![x]\!]$. Recall that $\mathbb{C}[\![x]\!]$ is a complete valuation field with a unique maximal ideal $\mathfrak{m} = (x)$. Since $F_u \equiv y^2 \mod \mathfrak{m}$, it is clear that $F_u(x, y)$ is reducible in $\mathbb{C}[\![x]\!][y]$ if and only if $F_u(x, xy)$ is reducible. Note that we can write $F_u(x, xy) = x^2 F_1(x, y)$ where F_1 is primitive and satisfies $F_1 \equiv u(1+u)(y-\alpha_1)(y-\alpha_2) \mod \mathfrak{m}$ (where α_i are the solutions of the quadratic equation $u - \alpha + u\alpha^2 = 0$). Since we assumed $u \neq u_0$, the roots α_i are distinct, so by Hensel's lemma F_1 is reducible in $\mathbb{C}[\![x]\!][y]$. It follows that the point x_1 is unramified with index $e_1 = 1$ and in a neighbourhood of z_1 the equation admits two holomorphic solutions locally of the form

$$w_1(z) = u + \frac{1-\sqrt{1-4u^2}}{2u}(z-u) + o[(z-u)] \text{ as } z \to u$$
$$w_2(z) = u - \frac{1-\sqrt{1-4u^2}}{2u}(z-u) + o[(z-u)] \text{ as } z \to u.$$

On the other hand, the argument above breaks down at $u = u_0$. The equation F_{u_0} in local coordinates (x, y) becomes

$$\tfrac{3}{4}x^2 + \tfrac{1}{2}x(-3+x)y + \tfrac{1}{4}(3+2x)y^2.$$

The first terms of the Puiseux expansion of the two branches obtained in the case $u \neq u_0$ are still valid at $u = u_0$ and suggest making the substitution $y \mapsto xy+x$ in the equation. Since translations preserve irreducibility and $F_u \equiv y^2 \mod \mathfrak{m}$ it is easily seen that the new equation will be irreducible in $\mathbb{C}[\![x]\!][y]$ if and only if $F_u(x, xy+x$ is. After the substitution the equation becomes $F_u(x, xy+x) = \frac{1}{4}x^2(4x+6xy+(3+2x)y^2)$ and is clearly irreducible in $\mathbb{C}[\![x]\!][y]$ by Eisenstein's criterion. Hence the point z_1 is ramified in this case with index $e_1 = 2$. Furthermore, by computing the Newton diagram of the transformed equation it is easily seen that in a neighbourhood of z_1 the solutions are locally of the form

$$w_1(z) = \tfrac{1}{2} + (z-u_0) + (z-u_0)^{3/2} \text{ as } z \to u_0$$
$$w_2(z) = \tfrac{1}{2} + (z-u_0) - (z-u_0)^{3/2} \text{ as } z \to u_0.$$

$\boldsymbol{z_-}$ Let's introduce the local change of coordinates $(z, w) \to (x + z_-, y + w_-)$ and re-write F_u in terms of (x, y), then we get a constant multiple of the polynomial:

$$\underbrace{x\left\{\frac{q(1-q)^4(1+q^2)^3}{\beta(1+q^4)^2} + \alpha x\right\}}_{(0,1)} + \underbrace{x\left\{\frac{P(q)}{1+q^4} + \beta x\right\}y}_{(1,1)} + \underbrace{\left\{\frac{(1-q+q^2)^3}{1+q^4} + \beta x\right\}y^2}_{(2,0)},$$

where $P(q) = 1-3q+5q^2-8q^3+5q^4-3q^5+q^6$ and $\beta = q/(1+q+q^2)$. We can exclude the critical point $q_0 = 1$, since we have already determined the index of $z_- = z_1$ above. On the interval $(0, q_0)$ the polynomial $P(q)$ has a unique zero $q_1 \approx 0.49$ but $\beta(q_1) \neq 0$. Hence the middle term will either represent the point $(1, 1)$ or $(1, 2)$ in the Newton diagram.

$$u_1 = u(q_1) = \frac{1}{3}\left[1 + \sqrt[3]{-\tfrac{50}{4} + \tfrac{3}{4}\sqrt{354}} - \frac{7}{\sqrt[3]{-100+6\sqrt{354}}}\right].$$

The coefficients α and β are non-zero on $(0, 1)$, so we conclude from the diagram that the point z_- is ramified with index $e_- = 2$ and in a neighbourhood of z_- the equation admits two solutions whose Puiseux expansions are locally of the form

$$w_1(z) = w_- - i\frac{(1-q)^2(1+q^2)}{q(1-q+q^2)}\sqrt{\frac{(1+q^2)(1+q^2+q^4)}{q(1+q^4)}}\sqrt{z-z_-} \text{ as } z \to z_-$$

$$w_2(z) = w_- + i\frac{(1-q)^2(1+q^2)}{q(1-q+q^2)}\sqrt{\frac{(1+q^2)(1+q^2+q^4)}{q(1+q^4)}}\sqrt{z-z_-} \text{ as } z \to z_-.$$

$\boldsymbol{z_+}$ We could again introduce a local change of coordinates as above and investigate the corresponding Newton's diagram. However, note that the transformation $q \mapsto -q$ exchanges the roots $z_\pm$. It follows that the equation and expansions for z_+ can be obtained from those of z_- by the substitution $q \to -q$. In particular, the analysis is also valid at $q = q_0$ since the initial term is non-zero at q_0 and there is no ambiguity with the middle term since the polynomial $P(-q)$ has no positive real roots. It follows that z_+ is ramified with index $e_+ = 2$ and in a neighbourhood of z_+ the equation admits two solutions whose Puiseux expansions are locally of the form

$$w_1(z) = w_- - \frac{(1+q)^2(1+q^2)}{q(1+q+q^2)}\sqrt{\frac{(1+q^2)(1+q^2+q^4)}{q(1+q^4)}}\sqrt{z-z_+} \text{ as } z \to z_+$$
$$w_2(z) = w_- + \frac{(1+q)^2(1+q^2)}{q(1+q+q^2)}\sqrt{\frac{(1+q^2)(1+q^2+q^4)}{q(1+q^4)}}\sqrt{z-z_+} \text{ as } z \to z_+.$$

Let's denote by $g(\mathscr{R}_u)$ the genus of the compact Riemann surface $\mathscr{R}_u$ determined by equation $F_u(z,w) = 0$. To compute the genus of $\mathscr{R}_u$ we apply the Riemann-Hurwitz formula to the map $\pi_u : \mathscr{R}_u \to \mathbb{P}^1$. It is well-known that π_u is covering map of degree 2 since it is determined by a quadratic equation. Furthermore, the genus of the Riemann sphere $\mathbb{P}^1$ is 0. It follows that

$$2g - 2 = -4 + \sum_{P \in S}(e_P - 1) = -2 \Rightarrow \underline{\text{genus}\,\mathscr{R}_u = 0}.$$

Consequently, $\mathscr{R}_u$ is analytically isomorphic to the Riemann sphere $\mathbb{P}^1$ and the equation $F_u(z,w) = 0$ admits uniformization by rational functions (which will be explicitly determined below). The two branches of the solution are given by

$$R_u(z) = \frac{-1}{2u(1+z)}\{uz^2 - (1+u+4u^2)z - u(-1+u-u^2) + (z-u)\sigma_u(z)\} \quad (4.25)$$
$$\overline{R}_u(z) = \frac{-1}{2u(1+z)}\{uz^2 - (1+u+4u^2)z - u(-1+u-u^2) - (z-u)\sigma_u(z)\} \quad (4.26)$$
$$\sigma_u(z)^2 = u^2z^2 - 2u(1+3u+3u^2)z + (1+u-u^2)^2. \quad (4.27)$$

When writing out the local expansions about the critical points we have taken care to ensure that the expansion of the first branch $w_1(z)$ always corresponded to R_u and the that of the second branch to the conjugate root. The points $z_\pm$ are thus the only branch points of $R_u, \overline{R}_u$ with ramification index 2, so by choosing $[z_-, z_+] \subseteq \mathbb{R}$

as a branch cut, both roots can be meromorphically extended to the slit complex plane $\mathscr{D}_u = \hat{\mathbb{C}}\backslash[z_-, z_+]$. When talking about square-roots we will always assume that the principal branch is being used with a branch cut along the negative real axis. It follows that the correct determination of R_u and $\overline{R}_u$ (analytic in $\mathscr{D}_u$) with respect to the established branch cut convention will be obtained by setting

$$\sigma_q(z) = \frac{q^2}{1+q^4}(z_+ - z)\sqrt{\frac{z_- - z}{z_+ - z}} = \frac{q^2}{1+q^4}\sqrt{z_- - z}\sqrt{z_+ - z}, \tag{4.28}$$

where $\arg(z - z_-) = \arg(z - z_+) = 0$ for $z > z_\pm$. Note that the second expression in (4.28) is formally defined only on the slit complex plane $\mathbb{C}\backslash[z_-, \infty)$ since the principal square root has a branch cut along the negative real axis. However, it is possible to prove that the product $\sqrt{z_- - z}\sqrt{z_+ - z}$ admits an analytic continuation to (z_+, ∞), so is well-defined on $\mathscr{D}_u$. The main properties of $R_u(z)$ are collected in the following theorem:

Theorem 4.3.1 (Analytic properties of R_q and $\bar{R}_q$)**.** *Let $q \in (0, 1]$ and denote by $\mathbb{D} = \{z \in \mathbb{C} || z| < 1\}$ the open unit disk. The function $z = z(\lambda) : \mathbb{D} \to \mathscr{D}_q$ is an analytic isomorphism such that $z(q) = q$ and $z'(q) > 0$, where*

$$z(\lambda) = \frac{(1+q^2+q^4 - q(1+q^2)\lambda)((1+q^2+q^4)\lambda - q(1+q^2)}{q^2(1+q^4)\lambda} \tag{4.29}$$

$$R_q(\lambda) = \frac{(1+q^2+q^4+q^6 - q(1+q^2+q^4)\lambda)((1+q^2)\lambda - q)}{q(1+q^4)(1+q^2-q\lambda)} \tag{4.30}$$

$$\overline{R}_q(\lambda) = \frac{(1+q^2-q\lambda)((1+q^2+q^4+q^6)\lambda - q(1+q^2+q^4)}{q(1+q^4)\lambda((1+q^2)\lambda - q)}. \tag{4.31}$$

The pair $(z(\lambda), R_q(\lambda))$ constitutes a rational parametrization of the surface $\mathscr{R}_q$ and has the following properties: $z(\lambda^{-1}) = z(\lambda)$ and $\overline{R}_q(\lambda) = R_q(\lambda^{-1})$ (when considered as rational functions over $\mathbb{P}^1$). Finally, $z(\pm 1) = z_\pm$ and each arc $\mathbb{S}^1_\pm$ is mapped bijectively onto the branch cut $[z_-, z_+]$.

Proof. It is easily checked that $F_q(z(\lambda), R_q(\lambda)) \equiv 0$ for all $\lambda \in \mathbb{P}^1$, so the pair $(z(\lambda), R_q(\lambda))$ gives a rational parametrization of the compact Riemann surface $\mathscr{R}_q$. Let us check that $z(\lambda) : \mathbb{D} \to \mathscr{D}_q$ is indeed an analytic isomorphism. We can write $z(\lambda)$ as a composition $z(\lambda) = \mu \circ b \circ c$, where $c : \mathbb{D} \xrightarrow{\sim} \mathscr{H}_+ = \{z \in \mathbb{C} | \Re z > 0\}$ is the Cayley map, which maps the open unit disk $\mathbb{D}$ biholomorphically onto the open right-half plane $\mathscr{H}_+$, $b : \mathscr{H}_+ \xrightarrow{\sim} \mathbb{C}\backslash(-\infty, 0]$ is the square map $b(z) = z^2$, which maps the open right-half plane biholomorphically onto the slit complex plane $\mathbb{C}\backslash(-\infty, 0]$ and $\mu : \mathbb{C}\backslash(-\infty, 0] \xrightarrow{\sim} \mathscr{D}_q$ is the unique Möbius transformation such that $\mu(0) = z_-, \mu(\infty) = z_+$ and $\mu(-1) = \frac{1}{2}(z_- + z_+)$, which maps the slit domain $\mathbb{C}\backslash(-\infty, 0]$ biholomorphically onto $\mathscr{D}_q$. □

We record for later the following easily provable observation

$$R_q(\lambda) = z\left(\frac{q}{1+q^2-q\lambda}\right) \text{ and } \bar{R}_q(\lambda) = z\left(\frac{q\lambda}{(1+q^2)\lambda - q}\right). \tag{4.32}$$

We have already observed that (at least formally) the roots R_u and $\overline{R}_u$ seem to be mutually inverse functions (with respect to the variable z). However, due to multi-valued square-roots the identities $R \circ_z \overline{R} = \mathrm{id}$ and $\overline{R} \circ_z R = \mathrm{id}$ (where $\circ_z$ denotes composition with respect to z) are not true in general over $\mathfrak{D}_u$. In order to determine their range of validity, we begin by observing the following identities on $\mathfrak{D}_u$:

$$\sigma_u^2(R(z,u),u) = \left[\frac{(z-u)(1+3u+u^2+uz)+(z+u+2)\sigma_u(z)}{2(1+z)}\right]^2 \tag{4.33}$$

$$\sigma_u^2(\overline{R}(z,u),u) = \left[\frac{(z-u)(1+3u+u^2+uz)-(z+u+2)\sigma_u(z)}{2(1+z)}\right]^2. \tag{4.34}$$

The next theorem determines precisely the values of the various compositions of R_q and $\bar{R}_q$ depending on the domain of definition.

Theorem 4.3.2 (Composition Identities)**.** *Let $q \in (0,1]$ and denote by $\Gamma_q = R_q z_q(S^1)$ the bicircular rational plane algebraic curve of degree 4 given in polar coordinates for $\vartheta \in [-\pi,\pi]$ as*

$$\rho^2 = \frac{(1+2q^2+q^4+q^6)(1+q^2+2q^4+q^6)}{q^2(1+q^4)^2} - \frac{2(1+q^2)(1+q^2+q^4)}{q(1+q^4)}\cos\vartheta.$$

Provided $q \in (0,1)$, Γ_q is a nested family of smooth simple closed curves of genus 0. Let I_q and E_q denote the interior and exterior of Γ_q, respectively, then

$$R_q : \mathfrak{D}_q \xrightarrow{\sim} I_q \text{ is an analytic isomorphism}$$
$$\overline{R}_q : E_q \xrightarrow{\sim} E_q \text{ is an analytic isomorphism}$$

and the following compositional identities are true

(i) $\overline{R}_q \circ_z R_q = \mathrm{id}$ *on* $\mathfrak{D}_q$.

(ii) $R_z \circ_z \overline{R}_q = \mathrm{id}$ *on* I_q.

(iii) $\overline{R}_q \circ_z \overline{R}_q = \mathrm{id}$ *on* E_q.

(iv) $R_q \circ_z R_q = z^{-1}(R_q(z) - R_q(0))$ *on* $\mathfrak{D}_q$.

Remark 3 The value q_1 of the parameter q, which arose as the unique positive zero of polynomial $P(q)$ for the Newton diagram of z_-, has an interesting geometric interpretation. It is possible to show that the curve Γ_q is convex if and only if $q \in (0, q_1)$.

Proof. We will only sketch the proof of the theorem. An elementary, albeit involved, computation reveals that the polar parametrization of the curve Γ_q is related to the map $R_q(\lambda)$ through the relation $\rho^2 = (\Re R_q(e^{i\vartheta}))^2 + (\Im R_q(e^{i\vartheta}))^2$, which immediately implies that I_q is exactly the image of the open unit disk $\mathbb{D}$ by $R_q(\lambda)$ (and similarly for the conjugate root). Relations (4.32) then imply that the restriction of R_q to the slit domain $\mathscr{D}_q$ is an analytic isomorphism onto I_q and the restriction of $\bar{R}_q$ to E_q is an analytic isomorphism onto E_q. The compositional identities follow from (4.33) and (4.34) by carefully considering the the value of the square root of their left-hand sides, depending on the domain of definition. □

Mapping properties of R_q: Mapping and dynamical properties of the map $R_q(z)$ will be important in the sequel when studying the properties of the hull operator. Let's denote the Möbius transformation appearing on the right-hand side of equation (4.32) with $\mu_q(z)$, so

$$\mu_q(z) = \frac{q}{1 + q^2 - qz}. \tag{4.35}$$

It is an automorphism of the extended complex plane $\hat{\mathbb{C}}$ with determinant $\det \mu_q(z) = q^2$ and (normalized) trace $\operatorname{tr} \mu_q = q + q^{-1} \geq 2$. The transformation is *hyperbolic* if $q \neq q_0$ and *parabolic* otherwise. It has two fixed points, the attractive fixed point q and the repelling fixed point q^{-1}. Its characteristic constant is therefore $\lambda_q = \mu_q'(q) = q^2$.

Let $k \geq 0$ be a non-negative integer. We will denote by $\mu_{q,k}$ the k-fold iterate of μ_q with the convention that $\mu_{q,0} = \mathrm{id}$ and $\mu_{q,1} = \mu_q$. Writing the transformation μ_q in *normal form* as

$$\frac{\mu_q - q}{\mu_q - q^{-1}} = q^2 \frac{z - q}{z - q^{-1}}$$

it is easily shown by induction on k that the kth iterate is given as follows:

$$\mu_{q,k}(z) = \frac{\mathfrak{m}_{k-1}(q)z - \mathfrak{m}_k(q)}{\mathfrak{m}_k(q)z - \mathfrak{m}_{k+1}(q)}, \text{ where } \mathfrak{m}_k(q) = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{4.36}$$

The coefficients $\mathfrak{m}_k(q)$ are reciprocal forms in q closely related to Chebyshev polynomials of the second kind $U_k(s)$. Indeed, if we introduce a new variable s by the relation $s = \frac{1}{2} \operatorname{tr} \mu_q$ then $\mathfrak{m}_k(q) = U_{k-1}(s)$. It follows that the forms $\mathfrak{m}_k(q)$ satisfy the recurrence relation $\mathfrak{m}_{k+1}(q) = (q + q^{-1})\mathfrak{m}_k(q) - \mathfrak{m}_{k-1}(q)$ with initial conditions $\mathfrak{m}_0(q) = 0$ and $\mathfrak{m}_1(q) = 1$. Furthermore, equation (4.36) also remains valid for negative integers k provided $\mu_{q,-k}(z)$ is interpreted as the kth iterate of the inverse map $\mu_q^{-1}(z)$ for $k > 0$.

The following classical theorem is well-known in complex analysis:

Theorem 4.3.3 (Mapping Properties of μ_q)**.** *Let $q \in (0, 1)$, then $\lim_{k \to \pm\infty} \mu_{q,k} = q^{\pm 1}$ uniformly on compact subsets of $\hat{\mathbb{C}} \backslash \{q^{\mp}\}$.*

4.3.1 Resolvent of the hull operator

We will first transform the hull operator $\boldsymbol{A}_q$ by an appropriate change of variables to a more manageable form. Let us recall that the hull operator $\boldsymbol{A}_q$ is defined as

$$(\boldsymbol{A}_q f)(z) = R_q'(z) f(R_q(z)) - \left\{R_q(z) - R_q(0) - 2zR_q'(0) + (1+z)R_q'(z)\right\} f(R_q(0)) \\ - z(R_q')^2(0) f'(R_q(0))$$

We have proven in the previous section that the symbol $R_q(z)$ admits a rational uniformization. We will use a conformal map $z : \mathbb{D} \to \mathscr{D}_q$ to perform a "change of variables" that will simplify the operator:

$$\begin{array}{ccc} \mathrm{Hol}(\mathscr{D}_q^*) & \xrightarrow{\boldsymbol{A}_q} & \mathrm{Hol}(\mathscr{D}_q^*) \\ \Big\downarrow P & & \Big\downarrow P \\ \mathrm{Hol}(\mathbb{D}^*) & \xrightarrow{\boldsymbol{B}_q} & \mathrm{Hol}(\mathbb{D}^*) \end{array}$$

where $P : f \mapsto f \circ z$ is the composition operator with symbol z. We introduce the following change of variables

$$z = \frac{(1+q^2+q^4 - q(1+q^2)w)((1+q^2+q^4)w - q(1+q^2))}{q^2(1+q^4)w} \tag{4.37}$$

$$w = \frac{\sqrt{z_+(q) - z} - \sqrt{z_-(q) - z}}{\sqrt{z_+(q) - z} + \sqrt{z_-(q) - z}}. \tag{4.38}$$

According to theorem (4.3.1), $z(w) : \mathbb{D} \to \mathscr{D}_q$ is the unique analytic isomorphism such that $z(q) = q$ and $z'(q) > 0$ with inverse $w(z)$ (where as always we take the principal branch of the square-roots). We have shown that the action of conjugation $\phi \mapsto z^{-1} \circ \phi \circ z$ on the composition map transforms it to the Möbius transformation $\mu_q(w)$ (see equation (4.32)). When switching to the new coordinate system a function $f(z)$ turns into a function $g(w)$ defined by the equation $f(z) = g(w)$ and the operator $\boldsymbol{A}_q$ turns into the operator $\boldsymbol{B}_q$ such that for all f and z we have $(\boldsymbol{A}_q f)(z) = (\boldsymbol{B}_q g)(w)$. The operator $\boldsymbol{B}_q$ is (formally) obtained from $\boldsymbol{A}_q$ by conjugation with the composition operator with symbol $z(w)$ and acts on g as follows

$$(\boldsymbol{B}_q g)(w) = \frac{w^2}{1-w^2}(1-\mu_q(w)^2) g(\mu_q(w)) + \varphi_q(w) g(\mu_q(w_0)) + \psi_q(w) g'(\mu_q(w_0)), \tag{4.39}$$

where we have introduced the following notation for the coefficients of the finite-rank part of the operator

$$\varphi_q(w) = -\frac{a_q - b_q w + c_q w^3 - d_q w^4}{q^2(1+q^4)^3(1+q^2+q^4+q^6+q^8)w(1-w^2)} \tag{4.40}$$

$$\psi_q(w) = \frac{q^5(1-q^6)^3(1-q^{14})(1-w_0 w)(w_0-w)}{(1-q^2)^4(1+q^2)(1+q^4)^4(1+q^2+q^4+q^6+q^8)^2 w} \tag{4.41}$$

and the coefficients of the quartic polynomial in the numerator of φ_q are given in the table below

$$\begin{aligned}
a_q &= 2q^3(1+q^2)(1+q^2+q^4)(1+q^2+q^4+q^6+q^8+q^{10}+q^{12}) \\
b_q &= q^2(3+7q^2+9q^4+7q^6+3q^8)(1+q^2+2q^4+2q^8+q^{10}+q^{12}) \\
c_q &= 1+7q^2+20q^4+41q^6+59q^8+71q^{10}+74q^{12} \\
&\quad +71q^{14}+59q^{16}+41q^{18}+20q^{20}+7q^{22}+q^{24} \\
d_q &= 2q(1+q^2)(1+q^2+q^4)^4(1-q^2+q^4).
\end{aligned}$$

We can decompose the hull operator into two parts $\boldsymbol{B}_q = T_1 + P$, where T_1 represents the weighted composition operator given by the first term in the representation (4.39) and P is the finite-rank remainder. Simple induction on $k \geq 1$ reveals both operator are *iterable*. Indeed, the Möbius transformation $\mu_q(w) = \alpha_q(q^2\alpha_q(w))$ is a *conformal dilatation*, where $\alpha_q \in \operatorname{Aut}\mathbb{D}$ is a disk automorphism given by

$$\alpha_q(w) = \frac{q-w}{1-qw}.$$

It is easily checked that α_q is an involution and it follows by induction on $k \geq 0$ that the kth iterate of the weighted composition operator T_1 is given by

$$(T_1^k g)(w) = \frac{q^{2k}(1-q^{4k}a_q^2(w))(1-q^{-1}a_q(w))^2(1-qa_q(w))^2}{(1-a_q^2(w))(1-q^{2k-1}a_q(w))^2(1-q^{2k+1}a_q(w))^2} g(a_q(q^{2k}a_q(w))). \tag{4.42}$$

A reason for the existence of such a simple formula for the iterates is that the multiplier $m(w)$ forms a standard coboundary cocycle, i.e. it has the form $m = q^2\omega/(\omega\circ\mu_q)$ for some function ω (see lemma below). More precisely, if we denote $\mu_t = \alpha_q q^{2t}\alpha_q$ for $t \in \mathbb{R}^+$ and $m_t = q^2\omega/(\omega\circ\mu_t)$, we can prove that the operator $(T_t g)(w) = m_t(w)(g\circ\mu_t)(w)$ is well-defined on (any) Hardy space H_r^p of the open disc $r\mathbb{D}$ for any $r \in (q,1)$[2] and has the following properties:

[2]We cannot define the operator T_t on the Hardy space of the whole open unit disk since its multiplier has simple poles at ± 1. On the other hand, it is easily checked that the condition $\mu_q(r\mathbb{D}) \subset r\mathbb{D}$ is equivalent to $r \in (q,1)$.

Lemma 4.3.1 (Weighted composition operator T_t) *Let $p > 1$, then $T_t : H_r^p \to H_r^p$ defines a bounded linear operator for all $t \in \mathbb{R}^+$ and the family $\{T_t\}$ forms a uniformly bounded strongly continuous one-parameter semigroup of operators on H_r^p.*

Let $t \in \mathbb{R}^+$, then $T_t : H_r^2 \to H_r^2$ is a one-to-one operator that belongs to the p-Schatten class $\mathcal{S}_p$ for every $p > 0$ (in particular, T_t is a trace class operator) with point spectrum $\sigma_p(T_t) = q^{2t\mathbb{Z}_+}$ and continuous spectrum $\sigma_c(T_t) = \{0\}$. The eigenfunction corresponding to the eigenvalue q^{2tk} is $\alpha_q^{k-1}(w)\omega(w)$ and the normalized eigenfunctions form a complete system in H_r^2, where

$$\omega(w) = \frac{(1-q^2)^3 w^2}{q^2(1-qw)^2(1-w^2)}.$$

The trace $\operatorname{tr} T_t$ and the Fredholm determinant $\det(I - zT_t)$ exist and admit the following closed-form expression

$$\operatorname{tr} T_t = \frac{q^{2t}}{1-q^{2t}} \text{ and } \det(I - zT_t) = (q^{2t}, q^2 z)_\infty.$$

We will not go into further details regarding the properties of the hull operator on Hardy spaces and will continue working in a purely formal setting (although our results can always be put in the context of Hardy spaces of an appropriately chosen domain). We introduce next the resolvent of a linear operator. Although we define resolvents for operators on Banach spaces, the definition also works in a purely formal context.

Definition 32. Let A be a linear operator defined on some Banach space. The *resolvent* of the operator A is defined as

$$R(z; A) = (zI - A)^{-1} = \sum_{k=0}^{\infty} z^{-k-1} A^k, \text{ for } |z| > \|A\| \tag{4.43}$$

where I denotes the identity operator and $z \in \rho(A)$ belongs to the resolvent set of the operator A. Let A and B be operators defined on a Banach space, then the following identities hold (called respectively, the *first resolvent identity* and the *second resolvent identity*):

$$R(z; A) - R(w; A) = (z - w)R(z; A)R(w; A), \text{ for all } z, w \in \rho(A) \tag{4.44}$$

$$R(z; A) - R(z; B) = R(z; A)(B - A)R(z; B), \text{ for all } z, w \in \rho(A) \cap \rho(B). \tag{4.45}$$

Note that (formally) the resolvent $z^{-1}R(z^{-1}, \boldsymbol{B}_q)$ is simply the generating function of the iterates of the hull operator. Since we will be interested in the asymptotic

behaviour of the iterates applied to a fixed map $g(w)$, we would like to determine a convenient expression for the resolvent that is computable and amenable to singularity analysis. Let us denote the resolvent of T_1 by $R_1(z,w) = z^{-1}R(z^{-1},T_1)$ and write $\Phi_q(z,w) = R_1(z,w)\varphi, \Psi_q(z,w) = R_1(z,w)\psi$. Note that R_1 is readily computable due to expression (4.42) for the kth iterate of T_1. Finally, let us also set $S(z,w) = z^{-1}R(z^{-1},\boldsymbol{B}_q)$ for the resolvent of the hull operator. The following lemma supplies the sought representation:

Lemma 4.3.2 (Decomposition of the resolvent $R(z,\boldsymbol{B}_q)$) *Let z be sufficiently small, then the following identity holds (prime denotes differentiation with respect to w):*

$$S(z,w) = z^{-1}R(z^{-1},\boldsymbol{B}_q) = R_1(z,w) + \vartheta_0(z,w)R_1(z,w_1) + \vartheta_1(z,w)R_1'(z,w_1), \quad (4.46)$$

where the coefficients are given as follows

$$\vartheta_0(z,w) = \frac{z\Phi_q(z,w)(1-z\Psi_q'(z,w_1)) + z^2\Psi_q(z,w)\Phi_q'(z,w_1)}{(1-z\Phi_q(z,w_1)(1-z\Psi_q'(z,w_1)) - z^2\Phi_q'(z,w_1)\Psi_q(z,w_1)}$$
$$\vartheta_1(z,w) = \frac{z^2\Phi_q(z,w)\Psi_q(z,w_1) + z\Psi_q(z,w)(1-z\Phi_q(z,w_1))}{(1-z\Phi_q(z,w_1)(1-z\Psi_q'(z,w_1)) - z^2\Phi_q'(z,w_1)\Psi_q(z,w_1)}.$$

Proof. We apply the second resolvent identity (4.45) to the decomposition $\boldsymbol{B}_q = T_1 + P$ of the hull operator. Recall that $S(z,w) = z^{-1}R(z^{-1},\boldsymbol{B}_q)$, so we obtain

$$\begin{aligned} S(z,w) &= R_1(z,w) + zR_1(z,w)PS(z,w) \\ &= R_1(z,w) + z\Phi_q(z,w)S(z,w_1) + z\Psi_q(z,w)S'(z,w_1) \end{aligned} \quad (4.47)$$

where prime indicates differentiation with respect to w and $w_1 = \mu(w_0)$. Let's evaluate the identity on an arbitrary function $g = g(w)$. If we take the derivative with respect to w of both sides and substitute $w = w_1$, we get a system of equations:

$$\begin{aligned} \{1-z\Phi_q(z,w_1)\}\, S(z,w_1)g - z\Psi_q(z,w_1)S'(z,w_1)g &= R_1(z,w_1)g \\ -z\Phi_q'(z,w_1)S(z,w_1)g + \left\{1-z\Psi_q'(z,w_1)\right\} S'(z,w_1)g &= R_1'(z,w_1)g. \end{aligned}$$

The determinant Δ of the system is easily evaluated to be $(1-z\Phi_q(z,w_1)(1-z\Psi_q'(z,w_1)) - z^2\Phi_q'(z,w_1)\Psi_q(z,w_1)$ and is equal to 1 at $z=0$. We will prove (see theorem 4.3.4) that the determinant is analytic about the origin, so if we take $\epsilon > 0$ sufficiently small, $\Delta \neq 0$ for $|z| < \epsilon$. Hence for values of z sufficiently close to the origin, the system has a unique solution given by

$$S(z,w_1)g = \frac{\{1-z\Psi_q'(z,w_1)\}R_1(z,w_1)g + z\Psi_q(z,w_1)R_1'(z,w_1)g}{(1-z\Phi_q(z,w_1)(1-z\Psi_q'(z,w_1)) - z^2\Phi_q'(z,w_1)\Psi_q(z,w_1)}$$

$$S(z,w_1)'g = \frac{z\Phi_q'(z,w_1)R_1(z,w_1)g + \{1 - z\Phi_q(z,w_1)\}\, R_1'(z,w_1)g}{(1 - z\Phi_q(z,w_1)(1 - z\Psi_q'(z,w_1)) - z^2\Phi_q'(z,w_1)\Psi_q(z,w_1)}.$$

Substituting these expression back in formula (4.47) gives the representation

$$S(z,w)g = R_1(z,w)g + \vartheta_0(z,w)R_1(z,w_1)g + \vartheta_1(z,w)R_1'(z,w_1)g.$$

Since g was arbitrary identity (4.46) follows. □

We will often denote the specialized values of the resolvent maps of φ and ψ (and their derivatives) at $w = w_1$ by suppressing the second variable, writing simply $\Phi_q(z) \equiv \Phi_q(z,w_1)$ etc. Note that in terms of the original hull operator, the substitution $w = w_1$ corresponds to $t = x$ and the decomposition (4.3.2) somewhat simplifies at $w = w_1$:

Corollary 4.3.1 *Let z be sufficiently small then the following identity holds*

$$S(z,w_1) = \frac{\left\{1 - z\Psi_q'(z)\right\} R_1(z,w_1) + z\Psi_q(z)R_1'(z,w_1)}{(1 - z\Phi_q(z)(1 - z\Psi_q'(z)) - z^2\Phi_q'(z)\Psi_q(z)} \tag{4.48}$$

Once again lemma (4.3.2) has been deduced in a purely formal setting. However, it is possible to derive the same result also in the context of Hilbert spaces (more specifically Hardy spaces) if all the operators involved had been defined on an appropriate Hardy space. Since a lot of our work has been motivated by the corresponding theory of operators on Hilbert spaces, we give another quick recap of Fredholm determinants.

Let A be a compact operator with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ (where $N = 1, 2, \ldots, \infty$) arranged in descending order of magnitude, counted up to algebraic multiplicity. The trace of the operator A is defined by $\operatorname{tr} A = \sum_n \langle \phi_n, A\phi_n \rangle$ for any orthonormal basis $\{\phi_n\}$ (the value of the series is independent of the choice of basis). The space $\mathcal{S}_1$ of *trace class operators* consists of all operators A such that $\operatorname{tr} A < +\infty$. The *Fredholm determinant* of a trace-class operator is defined by

$$\det(1 - zA) = \prod_{i=1}^{N}(1 - z\lambda_i)$$

and is an entire function of genus 0. Let A and B be bounded linear operators with $A - B \in \mathcal{S}_1$, then the operator $(I - zB)(I - zA)^{-1}$ is trace-class and its Fredholm determinant $\det[(I - zB)(I - zA)^{-1}]$ is meaningful and called the *perturbation determinant* of the operator A by the operator $T = B - A$. Although the hull operator has been defined only formally, the preceding discussion motivates the following definition:

Definition 33. The map

$$\Delta_q(z) = (1 - z\Phi_q(z)(1 - z\Psi_q'(z)) - z^2\Phi_q'(z)\Psi_q(z)$$

will be called the *perturbation determinant* of the hull operator.

Let us introduce next some terminology that will be useful when analyzing properties of the resolvent map $R_1(z,w)\phi$ applied to a suitable function ϕ. Recall that the weighted composition operator T_1 is defined as follows

$$(T_1\phi)(w) = \frac{w^2(1+q+q^2-qw)(1-q+q^2-qw)}{(1-w^2)(1+q^2-qw)^2}\phi\left(\frac{q}{1+q^2-qw}\right).$$

When defined on the Hardy space H_r^2 of the open disc $\mathbb{D}_r$, the operator is trace class with spectrum $q^{2\mathbb{Z}_+}$, so the resolvent $(zI - T_1)^{-1}$ will be analytic on the residual set $\mathbb{C}\backslash\sigma(T_1)$ with simple poles at the eigenvalues of T_1 and an essential singularity at the origin. We therefore expect the map $R_1(z,w)\phi = \sum_{k\geq 0} z^k T_1^k\phi$ to be meromorphic with simple poles contained in the set $\mathfrak{D}_q^* = q^{2\mathbb{Z}_{<0}}$ for all suitably chosen maps ϕ. We have already remarked above that the iterates of the composition operator T_1 can be evaluated in closed form, which we can also write as

$$G_q(s;\zeta) = \frac{(1-q^{-1}\zeta)^2(1-q\zeta)^2}{\zeta(1-\zeta^2)}\frac{s\zeta(1-s^2\zeta^2)}{(1-q^{-1}s\zeta)^2(1-qs\zeta)^2}\phi(a_q(s\zeta)), \tag{4.49}$$

where we have denoted $\zeta = a_q(w)$. We will also write $\exp_q(s) = q^{2s}$, then the coefficients of the map $R_1(z,w)$ are given by $\alpha_k(w;q) = G_q(\exp_q(k);\zeta)$ for $k \geq 0$. Note that the initial multiplier does not affect the convergence and analytic properties of the map $R_1(z,w)\phi$ (with respect to the variable z), while the rest of the iterate can be written as a function $g_q(z)$ in the variable $s\zeta$. More precisely we define the map

$$g_q(z) = \frac{z(1-z^2)}{(1-q^{-1}z)^2(1-qz)^2}\phi(a_q(z)) \tag{4.50}$$

called the *interpolating map* of ϕ. We will show that (under reasonable technical assumptions) the analytic character of the resolvent map (with respect to the variable z) is entirely determined by the behaviour of the interpolating map at the origin. This should come as no surprise since theorem (4.3.3) implies that the iterates of the Möbius transformation μ_q converge uniformly on compact subsets to the attractive fixed point q, which corresponds exactly to the origin in coordinate ζ.

Theorem 4.3.4. *Let the interpolating map $g_q(z)$ of ϕ admit a Newton-Pusieux series expansion about the origin of the form*

$$g_q(z) = \sum_{k \geq p} a_k z^{k/d}, \text{ for } z \in \mathbb{D}_r^*$$

that is absolutely and uniformly convergent in a punctured disc of radius $r > 0$, where $d \geq 1$ is a positive integer, $p \in \mathbb{Z}$ is an integer and the initial coefficient $a_p \neq 0$ is non-zero, then the resolvent map $R_1(z, \zeta)\phi$ admits a meromorphic continuation to $\mathbb{C} \times \mathbb{D}_r^*$ with simple poles at the points of the set

$$\mathfrak{D}_\phi = \{q^{-2k/d} | k \geq p \text{ such that } a_k \neq 0\}.$$

Proof. Let $\zeta_0 \in \mathbb{D}_r^*$ be fixed (note in particular that $\zeta_0 \neq 0$ by assumption) and choose an integer $N \geq 0$ sufficiently large so that $q^{2k}|\zeta_0| < r$ for all $k \geq N$, then we can write the resolvent $R_1(z, \zeta_0)\phi$ as

$$\begin{aligned} R_1(z, \zeta_0)\phi &= \omega_q(\zeta_0) \left\{ \sum_{k=0}^{N-1} g_q(q^{2k}\zeta_0) z^k + \sum_{k \geq N} \sum_{l \geq p} a_l q^{2kl/d} \zeta_0^{l/d} z^k \right\} \\ &= \omega_q(\zeta_0) \left\{ \sum_{k=0}^{N-1} g_q(q^{2k}\zeta_0) z^k + z^N \sum_{l \geq p} a_l \zeta_0^{l/d} \frac{q^{2Nl/d}}{1 - q^{2l/d} z} \right\} \end{aligned} \tag{4.51}$$

where in the final representation it was permissible to switch the order of summation since the inner sum is absolutely and uniformly convergent by our choice of N. Notice that (4.51) is uniformly convergent on compact subsets of the punctured plane $\mathbb{C} \backslash \mathfrak{D}_\phi$ (again by our choice of N), so the expression defines a meromorphic function on the entire complex plane with simple poles at the points of the set $\mathfrak{D}_\phi$. Furthermore, the representation (4.51) remains valid in a small neighbourhood of ζ_0, since we can pick $\rho > 0$ sufficiently small so that $q^{2k}|\zeta| < r$ for every $k \geq N$ and $\zeta \in \mathbb{D}(\zeta_0, \rho)$. Since the series is again absolutely and uniformly convergent for any fixed $z_0 \in \mathbb{C} \backslash \mathfrak{D}_\phi$ it defines an analytic function on $\mathbb{D}(\zeta_0, \rho)$. Consequently, by Hartogs' theorem the resolvent map (as a bivariate function of (z, ζ)) is analytic on $\mathbb{C} \backslash \mathfrak{D}_\phi \times \mathbb{D}_r^*$. □

4.3.2 The hull boundary operator

The hull boundary operator will be denoted simply by $\boldsymbol{B}$ and corresponds to the critical value of the parameter $q = 1$ that has so far been excluded from consideration. Plugging in the critical value of the parameter in the representation (4.39) of the hull operator gives the following formula for its boundary value

$$(\boldsymbol{B}g)(w) = \frac{w^2(3-w)}{(1+w)(2-w)^2} g\left(\frac{1}{2-w}\right) - \frac{21 - 37w - 37w^2 + 81w^3}{10w(1+w)} g\left(\frac{3}{4}\right)$$

$$+\frac{21(3-2w)(2-3w)}{800w}g'\left(\frac{3}{4}\right). \tag{4.52}$$

Furthermore, the iterates of the weighted composition operator T_1 are expressible as

$$(T_1^k g)(w)=\frac{w^2}{(1-w)^3(1+w)}G\left(\frac{k-(k-1)w}{k+1-kw}\right) \text{ with } G(w)=\frac{(1-w)^3(1+w)g(w)}{w^2}.$$

A simple computation allows us to express the functions $\Phi_1(z,w_1)$, $\Psi_1(z,w_1)$ and their derivatives with respect to w evaluated at $w=w_1$ in terms of the polylogarithmic functions. We let $\mathrm{Li}_{n,r}(z)$ denote the *truncated polylogarithm of order* r given by

$$\mathrm{Li}_{n,r}(z)=\sum_{k>r}\frac{z^{k-r-1}}{k^n}=\frac{1}{z^{r+1}}\left\{\mathrm{Li}_n(z)-z-\frac{z^2}{2^n}-\cdots-\frac{z^r}{r^n}\right\}.$$

Clearly, the truncated polylogarithm has a removable singularity at the origin and admits an analytic continuation to the slit complex plane $\mathbb{C}\backslash[1,\infty)$. We have

Lemma 4.3.3 *The following formulas holds for the resolvent maps of the coefficients functions φ_1 and ψ_1 of the finite-rank operator P,*

$$\begin{aligned}
\Phi_1(z)&=\frac{2664}{35}\,\mathrm{Li}_{2,2}(z)-\frac{216}{5}\,\mathrm{Li}_{3,2}(z)-\frac{2664}{35}\,\mathrm{Li}_{2,3}(z)-\frac{5832}{35}\,\mathrm{Li}_{3,3}(z)\\
\Psi_1(z)&=-\frac{189}{50}\,\mathrm{Li}_{2,2}(z)+\frac{81}{25}\,\mathrm{Li}_{3,2}(z)+\frac{189}{50}\,\mathrm{Li}_{2,3}(z)+\frac{81}{25}\,\mathrm{Li}_{3,3}(z)\\
\Phi_1'(z)&=\frac{262848}{245}\,\mathrm{Li}_{2,2}(z)-\frac{21312}{7}\,\mathrm{Li}_{3,2}(z)+\frac{10368}{5}\,\mathrm{Li}_{4,2}(z)\\
&\quad-\frac{262848}{245}\,\mathrm{Li}_{2,3}+\frac{21312}{245}\,\mathrm{Li}_{3,3}(z)+\frac{10368}{5}\,\mathrm{Li}_{4,3}(z)\\
\Psi_1'(z)&=-\frac{1332}{25}\,\mathrm{Li}_{2,2}(z)+\frac{5832}{35}\,\mathrm{Li}_{3,2}(z)-\frac{3888}{25}\,\mathrm{Li}_{4,2}(z)\\
&\quad+\frac{1332}{25}\,\mathrm{Li}_{2,3}(z)-\frac{13176}{175}\,\mathrm{Li}_{3,3}(z)-\frac{3888}{25}\,\mathrm{Li}_{4,3}(z)
\end{aligned}$$

All the functions admit an analytic continuation to the slit complex plane $\mathbb{C}\backslash[1,\infty)$ with logarithmic branch-point at $z=1$ and have finite non-zero limits at $z=1$.

Explicitly the values of the resolvent maps at $z=1$ are:

$$\Phi_1(1)=\frac{8774}{35}-\frac{7344}{35}\zeta(3)\approx-1.54017 \tag{4.53}$$

$$\Psi_1(1)=-\frac{783}{100}+\frac{162}{25}\zeta(3)\approx-0.040671 \tag{4.54}$$

$$\Phi_1'(1)=-\frac{360480}{49}-\frac{724608}{245}\zeta(3)+\frac{352512}{35}\zeta(4)\approx-11.0063 \tag{4.55}$$

$$\Psi_1'(1)=\frac{7928}{35}+\frac{15984}{175}\zeta(3)-\frac{7776}{25}\zeta(4)\approx-0.33909 \tag{4.56}$$

An immediate consequence of lemma (4.3.3) and the representation of the perturbation determinant is that the determinant $\Delta_1(z)$ likewise admits an analytic continuation to the entire slit complex plane $\mathbb{C}\backslash[1,\infty)$ and has a finite (non-zero) limit as $z \to 1$. We will prove in the next section that it has no zeros on the entire closed unit disk $\overline{\mathbb{D}}$, which will imply that the coefficients $\vartheta_0(z, w_1)$ in the representation (4.3.8) have a unique dominant singularity at $z = 1$. Since we will ultimately be interested in the behaviour of the coefficients of $[z^r]S(z, w_1)g$ as $r \to \infty$ for some particular function g, the nature and location of dominant singularities will be crucial to our investigation. The following theorem will be of paramount importance:

Theorem 4.3.5 (Eneström-Kakeya theorem, [35])**.** *Suppose $p_0 \geq p_1 \geq p_2 \geq \cdots \geq 0$ is a non-increasing sequence of positive real numbers, then the power series*

$$F(z) = \sum_{n\geq 0} p_n z^n$$

is absolutely and uniformly convergent in the open unit disk $\mathbb{D}$, Abel summable on the unit circle, except possibly at $z = 1$, and has no zeros in $\overline{\mathbb{D}}\backslash\{1\}$ except if $\lim_n p_n = 0$ and there exists an integer $d \geq 2$ such that $p_n = p_{n-1}$ whenever $d \nmid n$.

Remark 4 The statements about Abel summability and the exceptional case are not included in the article by S. Kakeya [35] and have been noted independently by A. Peyerimhoff [51] and B. Kuttner [43].

4.3.3 Asymptotic analysis of the perturbation determinant

We will assume throughout this section, unless explicitly stated otherwise, that $q \in (0, 1)$. The resolvent and perturbation determinant are expressible in terms of the q-polylogarithmic functions $\mathrm{Li}_q[z]$, so let us briefly recall the relevant definitions.

Definition 34. The *q-polylogarithm* $\mathrm{Li}_{n,q}[z]$ is defined for $|z| < 1$ by the following power series

$$\mathrm{Li}_{n,q}[z] := \sum_{k\geq 1} \frac{z^k}{[k]_q^n} = \sum_{k\geq 1} \frac{(1-q)^n z^k}{(1-q^k)^n}, \tag{4.57}$$

where we have denoted $[k]_q = 1 + q + \cdots + q^{k-1}$.

The q-polylogarithm $\mathrm{Li}_{n,q}[z]$ is clearly analytic in the unit disk $\mathbb{D}$, since the inequality $[k]_q^n \geq 1$ implies it is dominated by the geometric series on $\mathbb{D}$. It can be analytically continued to a meromorphic function on $\mathbb{C}\backslash\mathfrak{D}_q$, where $\mathfrak{D}_q = \{q^{-k}\}_{k\geq 0}$. Its analytic continuation admits a Mittag-Leffler expansion of the form:

$$\mathrm{Li}_{n,q}[z] = (1-q)^n \sum_{r\geq 0} \binom{n+r-1}{r} \frac{q^r z}{1-q^r z} \tag{4.58}$$

$$\operatorname{Res}_{z=q^{-r}} \operatorname{Li}_{n,q}[z] = -(1-q)^n \binom{n+r-1}{r}.$$

At points of $\mathfrak{D}_q$ the q-polylogarithm has simple poles. Using (4.58) it is possible to show that $\lim_{q\to 1^-} \operatorname{Li}_{n,q}[z] = \operatorname{Li}_n(z)$ uniformly on compact subsets of the slit complex plane $\mathbb{C}\backslash[1,\infty)$. More generally, for $k \geq 0$ we will denoted by $\operatorname{Li}_{n,k,q}[z]$ the *truncation* of the q-polylogarithm defined as follows:

$$\operatorname{Li}_{n,k,q}[z] = \sum_{i>k} \frac{z^{i-k-1}}{[i]_q^n} = (1-q)^n \sum_{r\geq 0} \binom{n+r-1}{r} \frac{q^{r(k+1)}}{1-q^r z}.$$

It is an easy consequence of theorem 1 in [37] applied to the truncated q-polylogarithm that $\operatorname{Li}_{n,k,q}[z]$ converges uniformly on compact subsets of the slit plane $\mathbb{C}\backslash[1,+\infty)$ to the ordinary truncated polylogarithm.

The next two theorems will describe the main analytic properties of the resolvent maps appearing in the expression of the perturbation determinant. We will be particularly interested in the asymptotic behaviour of their coefficients in a series expansion about the origin and uniform estimate (with respect to the parameter q). Since the coefficients will be expressible in terms of polylogarithmic sequences of the form $\operatorname{Li}_{-n}(q^k)$, it will be convenient to also introduce the following Lambert series:

Definition 35. The Lambert series of the polylogarithmic sequence $\operatorname{Li}_{-n}(z^k)$ will be denoted by $\mathscr{L}_n(z,t)$ and is given by

$$\mathscr{L}_n(z,t) = \sum_{k\geq 0} \operatorname{Li}_{-n}(tz^k) = \sum_{k\geq 1} \frac{k^n t^k}{1-z^k}, \text{ for } z \in \mathbb{D}.$$

It is closely related to the Lambert series of the divisor sum function $\sigma_x(n) = \sum_{d|n} d^x$ given by $\mathscr{L}_n(z,z)$.

Theorem 4.3.6 (Analytic properties of Φ_q and Ψ_q)**.** *The resolvent maps $\Phi_q(z)$ and $\Psi_q(z)$ of $\varphi_q(w)$ and $\psi_q(w)$, respectively, have the following properties:*

(i) $\Phi_q(z)$ and $\Psi_q(z)$ can be analytically continued to the punctured plane $\mathbb{C}\backslash\mathfrak{D}_{q^2}^*$ and the points $z_k = q^{-2k}, k > 0$ are at most simple poles. We can express both resolvents as linear combinations of truncated q-polylogarithms and expand them in a Mittag-Leffler series:

$$\Phi_q(z) = \sum_{k=1}^{3} \frac{(-1)^k a_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,2,q^2}[z] + \sum_{k=3}^{3} \frac{(-1)^k b_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,3,q^2}[z] = \sum_{r=1}^{\infty} \frac{A_r(q)}{z-q^{-2r}}$$

$$\Psi_q(z) = \sum_{k=1}^{3} \frac{(-1)^k a_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,2,q^2}[z] + \sum_{k=1}^{3} \frac{(-1)^k b_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,3,q^2}[z] = \sum_{r=1}^{\infty} \frac{B_r(q)}{z-q^{-2r}}$$

where the constants appearing in the expansions are determined in the proof.

(ii) Let $x_k(q)$ denote the coefficient of z^k in a power series expansion about the origin of either $\Phi_q(z)$ or $\Psi_q(z)$, then the following properties are true:

- *The sequence $\{x_k(q)\}_{k\geq 0}$ is negative and increasing for each $q \in [0,1]$.*
- *The function $x_k(q) \in \mathscr{C}([0,1])$ is negative and decreasing for each $k \geq 0$.*
- *The following limits exist uniformly on their respective domains*

$$\lim_{k\to\infty} x_k(q) = 0 \text{ for } q \in [0,1],$$
$$\lim_{q\to 1^-} x_k(q) = x_k \text{ for } k \geq 0.$$

- *The bound $x_k(q) = \mathcal{O}((1-q^2)^3 \operatorname{Li}_{-2}(q^{2k}))$ holds uniformly on $\mathbb{N}_0 \times [0,1]$.*

Proof. According to theorem (4.3.4) the analytic character of the resolvent map of ϕ is completely determined by the behaviour of the interpolating map $g_q(\zeta)$ of ϕ (see (4.50)). We will thus examine first the interpolating maps of the coefficients $\varphi_q(w)$ and $\psi_q(w)$. It turns out that the resolvents Φ_q and Ψ_q can be treated simultaneously, if we introduce the following coefficients:

$$a(q) = \frac{(1+q^2)^2(1+q^2+q^4)^3(1-q^4)^2}{q^6(1-q^8)(1-q^{10})} \text{ for } \varphi_q \quad -\frac{(1+q^2+q^4)^4(1-q^4)^4}{2q(1-q^8)^2(1-q^{10})^2} \text{ for } \psi_q,$$
$$b(q) = \frac{(1+q^2+q^4)^6(1+q^6)(1-q^4)^3}{q^8(1-q^8)(1-q^{10})(1-q^{14})} \text{ for } \varphi_q \quad -\frac{(1+q^2+q^4)^4(1-q^4)^4}{2q(1-q^8)^2(1-q^{10})^2} \text{ for } \psi_q.$$

We also introduce the coefficient $c(q)$, to simplify the subsequent calculations, as

$$c(q) = -\frac{(1-q^2)^3 a(q)\operatorname{Li}_{-2}(q^4) + (1-q^2)^3 b(q)\operatorname{Li}_{-2}(q^6) - 1}{(1-q^2)^2(\operatorname{Li}_{-1}(q^4) - \operatorname{Li}_{-1}(q^6))} \text{ for } \varphi_q$$
$$c(q) = -\frac{(1-q^2)\left[a(q)\operatorname{Li}_{-2}(q^4) + b(q)\operatorname{Li}_{-2}(q^6)\right]}{\operatorname{Li}_{-1}(q^4) - \operatorname{Li}_{-1}(q^6)} \text{ for } \psi_q.$$

Recall that the coefficients of the resolvent map can be written as $G_q(\zeta) = f_q(\zeta)g_q(s\zeta)$, where $f_q(\zeta)$ is a constant factor. We are only interested in the value of the resolvent at the critical point $w_1 = \mu_q(w_0)$, which in the coordinate system of ζ corresponds to the value $\zeta_1 = q^7$. We will thus evaluate the constant factor at ζ_1 and work with the constant multiple $f_q(q^7)g_q(\zeta)$ of the interpolating map, which we will continue to denote by g_q. A straightforward calculation using formulas (4.40) and (4.41) reveals that the interpolating map of both φ_q and ψ_q can be expressed in the form

$$g_q(\zeta) = -(1-q^2)^3\left\{a(q)\operatorname{Li}_{-2}(q^{-1}\zeta) + b(q)\operatorname{Li}_{-2}(q\zeta) + c(q)\frac{\operatorname{Li}_{-1}(q^{-1}\zeta) - \operatorname{Li}_{-1}(q\zeta)}{1-q^2}\right\}$$

$$= -(1-q^2)^3\zeta\left\{\frac{a(q)(1+q^{-1}\zeta)}{q(1-q^{-1}\zeta)^3}+\frac{qb(q)(1+q\zeta)}{(1-q\zeta)^3}+\frac{c(q)(1-\zeta^2)}{q(1-q^{-1}\zeta)^2(1-q\zeta)^2}\right\}.$$

Clearly, the map admits an expansion about the origin of the form $\mathcal{O}(\zeta)$, which by theorem (4.3.4) implies each resolvent map $\Phi_q(z)$ and $\Psi_q(\zeta)$ admits a meromorphic continuation to the entire complex plane with simple poles at the points of the set

$$\{q^{-2k}|k\in\mathbb{Z}_{>0} \text{ such that } [\zeta^k]g_q(\zeta)\neq 0\}.$$

We could deduce the Mittag-Leffler expansion of the resolvent map using the same argument as in the proof of theorem (4.3.4). However, since our interpolating map $g_q(\zeta)$ is rational, it is easier to perform instead a partial fraction decomposition and express each resolvent in terms of the truncated q^2-polylogarithms as:

$$\sum_{k=1}^{3}(-1)^k a_k(q)\operatorname{Li}_{k,2,q^2}[z]+\sum_{k=1}^{3}(-1)^k b_k(q)\operatorname{Li}_{k,3,q^2}(z)=\sum_{r=1}^{\infty}\frac{A_r(q)}{z-q^{-2r}},$$

where the coefficients a_i and b_j are expressible in terms of the constants a,b,c as

$$a_1(q)=(1-q^2)^2a(q)-(1-q^2)c(q),\quad a_2(q)=3(1-q^2)a(q)-c(q),\quad a_3(q)=2a(q)$$
$$b_1(q)=(1-q^2)^2b(q)+(1-q^2)c(q),\quad b_2(q)=3(1-q^2)b(q)+c(q),\quad b_3(q)=2b(q).$$

The terms of the Mittag-Leffler expansion (i.e. the residues of the resolvent at the points of the set $\mathfrak{D}^*_{q^2}$) are given by:

$$A_r(q)=rq^{4r}(1-q^2)^3\left\{c(q)\frac{1-q^{2r}}{1-q^2}+ra(q)+rq^{2r}b(q)\right\}.$$

Note that $A_r=A_r(q)$ is a quasi-exponential polynomial in q, so in particular for each fixed value of $r\geq 1$, it has at most finitely many zeros. It follows that there exists an at most countable set E of points in $(0,1)$ such that for every $q\in(0,1)\backslash E$, the set of poles of both resolvent maps is exactly $\mathfrak{D}^*_{q^2}$. In particular, since $\operatorname{dist}(0,\mathfrak{D}^*_{q^2})\geq q^{-2}>1$ each map is analytic at $z=1$.

We are left with proving the statements about the behaviour of the coefficients of the maps. Recall that the coefficient sequence of the resolvent map with interpolating map $g_q(\zeta)$ for some constant (a,b,c) is given by $x_k(q)=g_q(q^{2k+7})$ for $k\geq 0$. We wish to prove that

(i) The sequence $\{x_k(q)\}_{k\geq 0}$ is negative and increasing for each $q\in(0,1)$.

(ii) The function $x_k(q)$ is negative and decreasing on $[0,1]_q$ for each $k\geq 0$.

(iii) The limit $\lim_k x_k(q)=0$ is uniform in $q\in[0,1]$.

(iv) The bound $x_k(q) = \mathcal{O}((1-q^2)^3 \operatorname{Li}_{-2}(q^{2k}))$ holds uniformly on $(k,q) \in \mathbb{N}_0 \times [0,1]$.

Notice that the interpolating map satisfies $g_q(0) = 0$. We will show that on the unit square $(q,s) \in [0,1]^2$ its derivative satisfies $g_q'(q^7 s) < 0$ (so g_q is decreasing). Since the map $s(q) = q^{2k}$ is monotonically decreasing in k, this will imply that the sequence is negative and increasing, and similarly it will imply that the functions $x_k(q)$ are negative and decreasing on the unit interval $[0,1]_q$. If we calculate the derivative and substitute $\zeta = q^7 s$ we obtain a bivariate rational function of (q,s). Its denominator is a positive multiple of $(1-q^6 s)^4 (1-q^8 s)^4$, so is clearly positive, whereas the numerator is a negative multiple of a certain bivariate polynomial $n(s,q)$ of degree 6 in s.

Since it is difficult to show directly that the numerator is of constant sign, we will use the following well-known theorem:

Theorem 4.3.7 (Positivity in multivariate Bernstein basis)**.** *Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a bivariate polynomial of separate degrees $n = \deg_x p(x,y)$ and $m = \deg_y p(x,y)$ such that $p(x,y) > 0$ for every $(x,y) \in [0,1]^2$, then there exists positive integers $u \geq n$, $v \geq m$ and positive constants c_{ij} for every $i = 1, \ldots, u$, $j = 1, \ldots, v$ such that*

$$p(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} x^i y^j = \sum_{i=0}^{u} \sum_{j=0}^{v} \binom{u}{i} \binom{v}{j} c_{ij} x^i (1-x)^{u-i} y^j (1-y)^{v-j},$$

where the coefficients are connected by the formula

$$c_{ij} = \sum_{r=0}^{n} \sum_{s=0}^{m} a_{rs} \frac{\binom{i}{r}\binom{j}{s}}{\binom{u}{r}\binom{v}{s}}.$$

Firstly, it is important to point out that it is not always possible to take $u = n$ and $v = m$ (although it is possible to constructively derive lower bounds for u and v). Secondly, the representation with positive coefficients in the bivariate Bernstein basis does not extend to the class of non-negative polynomials $p(x,y)$ on the unit square.

Despite the failure of the theorem for general non-negative bivariate polynomials, we are particularly lucky. As it turns out, the numerator $n(s,q)$ of both resolvent maps $\Phi_q(z)$ and $\Psi_q(z)$ can be written in the bivariate Bernstein basis with $u = \deg_s n(s,q)$ and $v = \deg_q n(s,q)$ and all non-negative coefficients. Indeed, either by direct calculation of coefficients using the formula of the theorem or by solving the resulting system of linear equation, it is easily checked using say `Mathematica` that the representation exists and has only non-negative coefficients (we will not write the coefficients explicitly). This proves parts (i) and (ii).

It is easily shown that the limit $\lim_{q \to 1^-} x_k(q)$ exists (pointwise) for every $k \geq 0$, so each function $x_k(q)$ admits a continuous extension to the entire closed unit interval

$[0,1]_q$. Furthermore, for each fixed value of $q \in [0,1]$ we have $\lim_{k\to\infty} x_k(q) = 0$, so by property (i) and Dini's theorem we conclude that the limit $\lim_k x_k(q) = 0$ exists uniformly in q. Furthermore, by the Moore-Osgood theorem (or rather its addendum) it also follows that the limit $\lim_{q\to 1^-} x_k(q) = x_k$ exists uniformly in $k \geq 0$. We are left with proving the uniform bound. We can write

$$\begin{aligned}\frac{x_k(q)}{(1-q^2)^3 \operatorname{Li}_{-2}(q^{2k})} = -q^6 a(q)\frac{(1+q^{2k+6})(1-q^{2k})^3}{(1+q^{2k})(1-q^{2k+6})^3} - q^8 b(q)\frac{(1+q^{2k+8})(1-q^{2k})^3}{(1+q^{2k})(1-q^{2k+8})^3} \\ - q^6 c(q)\frac{(1+q^{2k+7})(1-q^{2k+7})(1-q^{2k})^3}{(1+q^{2k})(1-q^{2k+6})^2(1-q^{2k+8})^2}. \qquad (4.59)\end{aligned}$$

Notice that all the quotients in (4.59) are balanced, so lemma (A.1.4) implies that the limit as $k \to \infty$ exists uniformly in $q \in [0,1]$ with value $-q^6 a(q) - q^8 b(q) - q^6 c(q)$. This completes the proof of the theorem. □

Theorem 4.3.8 (Analytic properties of Φ'_q and Ψ'_q)**.** *The derivatives (with respect to w) $\Phi'_q(z)$ and $\Psi'_q(z)$ evaluated at w_1 have the following properties:*

(i) $\Phi'_q(z)$ and $\Psi'_q(z)$ can be analytically continued to the punctured plane $\mathbb{C}\backslash\mathfrak{D}^*_{q^2}$ and the points $\lambda_k = q^{-2k}, k > 0$ are at most simple poles. We can express both resolvents as linear combinations of truncated q-polylogarithms and expand them in a Mittag-Leffler series:

$$\Phi_q(z) = \sum_{k=1}^{4} \frac{(-1)^k a'_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,2,q^2}[z] + \sum_{k=1}^{4} \frac{(-1)^k b'_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,3,q^2}[z] = \sum_{r=1}^{\infty} \frac{A'_r(q)}{z - q^{-2r}}$$
$$\Psi_q(z) = \sum_{k=1}^{4} \frac{(-1)^k a'_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,2,q^2}[z] + \sum_{k=1}^{4} \frac{(-1)^k b'_k(q)}{(1-q^2)^k} \operatorname{Li}_{k,3,q^2}[z] = \sum_{r=1}^{\infty} \frac{B'_r(q)}{z - q^{-2r}}$$

where the constants appearing in the expansions are determined in the proof.

(ii) Let $y_k(q)$ denote the coefficient of z^k in a power series expansion about the origin of either $\Phi'_q(z)$ or $\Psi'_q(z)$, then the following properties are true:

- *The sequence $\{y_k(q)\}_{k\geq 0}$ is negative and increasing for each $q \in [0,1]$.*
- *The function $y_k(q) \in \mathscr{C}([0,1])$ is negative and decreasing for each $k \geq 0$.*
- *The following limits exist uniformly on their respective domains*

$$\lim_{k\to\infty} y_k(q) = 0 \text{ for } q \in [0,1],$$
$$\lim_{q\to 1^-} y_k(q) = y_k \text{ for } k \geq 0.$$

- *The bound $y_k(q) = \mathcal{O}((1-q^2)^3 \operatorname{Li}_{-2}(q^{2k}))$ holds uniformly on $\mathbb{N}_0 \times [0,1]$.*

Proof. We follow closely the proof of theorem (4.3.6). Recall that in the proof of (4.3.6) we introduced a certain modified interpolating map $g_q(z)$, which was obtained as a constant multiple of the interpolating map (4.50), multiplied by the constant term $f_1(\zeta_1)$ evaluated at the critical point $\zeta_1 = q^7$. The map $G_q(s;\zeta)$ (as defined in (4.49)) can then be expressed in terms of the modified $g_q(\zeta)$ as

$$G_q(s;\zeta) = \frac{f_1(\zeta)}{f_1(\zeta_1)} g_q(s\zeta), \text{ where } f_1(\zeta) = \frac{(1-q^{-1}\zeta)^2(1-q\zeta)^2}{\zeta(1-\zeta^2)}.$$

To determine the coefficients of the differentiated resolvents $\Phi'_q(z)$ and $\Psi'_q(q)$, we need to evaluate the derivative of $G_q(s;q)$ with respect to w at the point w_1. Recall that the variables (w,ζ) are related via the disk automorphism $w = \alpha_q(\zeta)$ and $s = q^{2k}$ is a constant independent of w, so if we denote the modified interpolating map of the differentiated resolvent map by $h_q(\zeta)$ we have

$$h_q(\zeta) = \frac{q^{-7}}{\alpha_q(\zeta_1)} \left\{ q^7 \frac{f_1'(\zeta_1)}{f_1(\zeta_1)} g_q(\zeta) + \zeta g_q'(\zeta) \right\} = \alpha(q)\beta(q) g_q(\zeta) - \alpha(q)(1-q^2)\zeta g_q'(\zeta)$$

where the coefficients α and β are

$$\alpha(q) = q^{-7}(1+q^2)^2(1+q^4)^2 \text{ and } \beta(q) = -q^7(1-q^2)\frac{f_1'(\zeta_1)}{f_1(\zeta_1)}$$

and we have again evaluated the constant terms at the critical point ζ_1. Since $\zeta d_\zeta \operatorname{Li}_{-n}(a\zeta) = \operatorname{Li}_{-(n+1)}(a\zeta)$, it follows immediately that the interpolating map $h_q(\zeta)$ admits an expansion of the form $\mathcal{O}(\zeta)$. Hence the resolvent maps $\Phi'_q(z)$ and $\Psi'_q(\zeta)$ admit a meromorphic continuation to the entire complex plane with simple poles contained in the set $\mathfrak{D}^*_{q^2}$. Performing partial fraction decomposition of $h_q(\zeta)$ we can express each resolvent as a linear combination of truncated q^2-polylogarithms as

$$\sum_{k=1}^{4} (-1)^k a_k'(q) \operatorname{Li}_{k,2,q^2}[z] + \sum_{k=1}^{4} (-1)^k b_k'(q) \operatorname{Li}_{k,3,q^2}[z] = \sum_{r=1}^{\infty} \frac{A_r'}{z - q^{-2r}},$$

where the coefficients a_i' and b_j' are related to the coefficients of proof (4.3.6) by:

$$a_k'(q) = \alpha(q)\left\{k(1-q^2) + \beta(q)\right\} a_k(q) + \alpha(q)(k-1)a_{k-1}(q)$$
$$b_k'(q) = \alpha(q)\left\{k(1-q^2) + \beta(q)\right\} b_k(q) + \alpha(q)(k-1)b_{k-1}(q)$$

and the terms of the Mittag-Leffler expansion (i.e. the residues of the resolvents at the points of the set $\mathfrak{D}^*_{q^2}$) are given by

$$A_r'(q) = \alpha(q)(\beta(q) - (1-q^2)r)A_r(q) = m_r(q)A_r(q).$$

Once again there exists an at most countable set F of points in $(0,1)$ such that for every $q \in (0,1)\backslash F$, the set of poles of both resolvent maps is exactly $\mathfrak{D}^*_{q^2}$. Moreover, as $\operatorname{dist}(0, \mathfrak{D}^*_{q^2}) \geq q^{-2} > 1$, both maps are analytic at $z = 1$.

Let's again denote the coefficients sequence of the differentiated resolvent maps with interpolating function $h_q(\zeta)$ and constants (a, b, c) by $y_k(q) = h_q(q^{2k+7})$ for $k \geq 0$. We wish to show that

(i) The sequence $\{y_k(q)\}_{k\geq 0}$ is negative and increasing for each $q \in (0,1)$.

(ii) The function $y_k(q)$ is negative and decreasing on $[0,1]_q$ for each $k \geq 0$.

(iii) The limit $\lim_k y_k(q) = 0$ is uniform in $q \in [0,1]$.

(iv) The bound $y_k(q) = \mathcal{O}((1-q^2)^3 \operatorname{Li}_{-2}(q^{2k}))$ holds uniformly on $(k,q) \in \mathbb{N}_0 \times [0,1]$.

The first three assertions are proven verbatim the same as in (4.3.6). Once again we are lucky to find it possible to write the numerator of the differentiated resolvent $h'_q(q^7 s)$ in the bivariate Bernstein basis with non-negative coefficients (as can easily be checked by `Mathematica`). It remains to show the uniform bound. Note that the coefficients $y_k(q)$ are expressible as a linear combination $\alpha(q)\beta(q)x_k(q) - \alpha(q)(1-q^2)\tilde{x}_k(q)$, where both coefficients $\alpha(q)$ and $\beta(q)$ are continuous and bounded on $[0,1]$. We have already shown $x_k(q) = \mathcal{O}((1-q^2)^3 \operatorname{Li}_{-2}(q^{2k}))$ q-uniformly, so it remains to consider the second term. We can write

$$\begin{aligned}
\frac{\tilde{x}_k(q)}{(1-q^2)^2 \operatorname{Li}_{-2}(q^{2k})} &= -q^6 a(q) \frac{(1+4q^{2k+6}+q^{4k+12})(1-q^2)(1-q^{2k})^3}{(1+q^{2k})(1-q^{2k+6})^4} \\
&- q^8 b(q) \frac{(1+4q^{2k+8}+q^{4k+16})(1-q^2)(1-q^{2k})^3}{(1+q^{2k})(1-q^{2k+8})^4} - \frac{q^6 c(q)(1-q^2)(1-q^{2k})}{(1+q^{2k})(1-q^{2k+6})(1-q^{2k+8})} \\
&- q^6 c(q) \left\{ \frac{q^{2k+6}(1-q^2)(1-q^{2k})^3}{(1+q^{2k})(1-q^{2k+8})(1-q^{2k+6})^3} + \frac{q^{2k+8}(1-q^2)(1-q^{2k})^3}{(1+q^{2k})(1-q^{2k+6})(1-q^{2k+8})^3} \right\}.
\end{aligned}$$

It is not difficult to check that the limit as $k \to \infty$ of every term exists and is uniform in $q \in [0,1]$, so the term $\tilde{x}_k(q)$ is again q-uniformly of order $\mathcal{O}((1-q^2)^2 \operatorname{Li}_{-2}(q^{2k}))$. $\square$

The perturbation determinant $\Delta_q(z)$ is expressed in terms of the resolvent maps Φ^*_q, Ψ^*_q as follows:

$$\begin{aligned}
\Delta_q(z) &= (1 - z\Phi_q(z))(1 - z\Psi'_q(z)) - z^2 \Phi'_q(z)\Psi_q(z) \\
&= 1 - z\left[\Phi_q(z) + \Psi'_q(z)\right] + z^2\left[\Phi_q(z)\Psi'_q(z) - \Phi'_q(z)\Psi_q(z)\right].
\end{aligned} \tag{4.60}$$

Consequently, $\Delta_q(z)$ is a meromorphic function of z on the entire complex plane $\mathbb{C}$ with poles at the points of the set $\mathfrak{D}^*_{q^2}$. Furthermore, each point $z_k = q^{-2k}$ is at

most a double pole of the determinant. Plugging in the expressions for the resolvent maps in terms of q^2-polylogarithmic functions and simplifying, it follows that the determinant map can be expressed as a linear combination of products of truncated q^2-polylogarithms. Similarly, plugging in the Mittag-Leffler expansions and simplifying gives a partial fraction expansion of the perturbation determinant as follows. Direct evaluation using the explicit formulas for the residues gives $\sum_r m_r A_r = 0 = \sum_r B_r$ and $\sum_r A_r = 1 = \sum_r m_r B_r$, which allows us to express the residues as

$$\begin{aligned} C_r(q) &= -\operatorname{Res}_r \left[\left(\sum_s \frac{z_s A_s}{z - z_s} \right) \left(\sum_r \frac{z_r m_r B_r}{z - z_r} \right) - \left(\sum_s \frac{z_s m_s A_s}{z - z_s} \right) \left(\sum_r \frac{z_r B_r}{z - z_r} \right) \right] \\ &= \sum_{s \neq r} \frac{m_s - m_r}{z_s^{-1} - z_r^{-1}} [A_s B_r - A_r B_s] = \sum_{s \neq r} q^{4(r+s)} \frac{rs(s-r)}{q^{2r} - q^{2s}} (b_{rs}(q) - b_{sr}(q)). \end{aligned}$$

To deduce the final expression we have used the formulas for the residues of the resolvent maps Φ_q and Ψ_q, resp., which are of the form $A_r = rq^{4r}[c(1-q^{2r}) + r(a+q^{2r}b)]$ and $B_r = rq^{4r}[\gamma(1-q^{2r}) + r(\alpha+q^{2r}\beta)]$ for some triples (a, b, c) and (α, β, γ) of constants depending only on q. Since $m_s - m_r = m(s-r)$ for some constant m, upon writing $e_0 = m(a\beta - \alpha b), e_1 = m(b\gamma - \beta c)$ and $e_2 = m(c\alpha - \gamma a)$, after a straightforward algebraic calculation we obtain the expression above with

$$b_{rs}(q) = r(1 - q^{2s})(e_2 - q^{2r} e_1 + s e_0)$$

where e_i are constants depending only on q and given explicitly below:

$$\begin{aligned} e_0 &= \frac{(1-q^2)^7 (1+q^2)^7 (1+q^2+q^4)^7 (1+q^4)}{2q^{16}(1+q^2+q^4+q^6+q^8)^2 (1+q^2+q^4+q^6+q^8+q^{10}+q^{12})} \\ e_1 &= \frac{(1-q^2)^6 (1+q^2)^6 (1+q^2+q^4)^6 (1+2q^2+6q^4+5q^6+6q^8+2q^{10}+q^{12})}{2q^{16}(1+q^2+q^4+q^6+q^8)^2 (1+q^2+q^4+q^6+q^8+q^{10}+q^{12})} \\ e_2 &= \frac{(1-q^2)^6 (1+q^2)^6 (1+q^2+q^4)^6}{2q^{14}(1+q^2+q^4+q^6+q^8)(1+q^2+q^4+q^6+q^8+q^{10}+q^{12})}. \end{aligned}$$

Furthermore, the following simple relation is easily verified $e_0(q) \operatorname{Li}_{-3}(q^4) + [e_2(q) - e_1(q)] \operatorname{Li}_{-2}(q^4) = 0$. We should remark that a similar computation as above, easily reveals that the perturbation determinant has at most simple poles (the quadratic terms cancel), so the following corollary is now immediate

Corollary 4.3.2 *Let $q \in (0,1)$, then the perturbation determinant $\Delta_q(z)$ is meromorphic in the entire complex plane $\mathbb{C}$ with simple poles at the points of the set $\mathfrak{D}^*_{q^2}$. It admits a Mittag-Leffler expansion of the form*

$$\Delta_q(z) = \sum_{r=1}^{\infty} \frac{C_r(q)}{z_r - z}$$

where the residues $C_r(q)$ are given by the infinite series above.

We will devote the remainder of this section to proving the following result:

Theorem 4.3.9. *The perturbation determinant $\Delta_q(z)$ has no zeros in the closed unit disk $\bar{\mathbb{D}}$ for $q \in [0,1]$.*

In order to prove this theorem, we turn our attention to the coefficients $\{c_k(q)\}_{k\geq 0}$ of the perturbation determinant $\Delta_q(z)$. We will derive below an explicit formula for the coefficients (expressible in terms of generalized q^2-harmonic numbers) and prove the terms are positive and monotonically decreasing. According to the Eneström-Kakeya theorem (4.3.5) we will thus be able to conclude that the perturbation determinant has no zeros in the closed unit disc $\bar{\mathbb{D}}$. The ambiguity at the point $z = 1$ in the theorem is not relevant in the present case, since the power series expansion of the perturbation determinant is absolutely convergent on the closed unit disc $\overline{\mathbb{D}}$.

We will express the coefficients $c_k(q)$, in an expansion about the origin, of the perturbation determinant $\Delta_q(z)$ explicitly as follows:

$$\begin{aligned} c_k(q) = &-e_2\Delta_k\beta_{k+2}\sum_{i=0}^{k}\beta_i + e_1\Delta_k\beta_{k+3}\sum_{i=0}^{k}\beta_{i+1} - (e_1-e_2)\beta_{k+3}(\beta_0-\beta_{k+1}) \\ &- e_2\Delta_k\alpha_{k+2}\sum_{i=0}^{k}\gamma_i + e_1\Delta_k\alpha_{k+3}\sum_{i=0}^{k}\gamma_{i+1} - (e_1-e_2)\gamma_{k+3}(\alpha_0-\alpha_{k+1}) \\ &+ e_0(\gamma_0\beta_{k+1} - \beta_0\gamma_{k+1}) \end{aligned} \tag{4.61}$$

where we have introduced the shift operator $\Delta_k c_k = c_k - c_{k+1}$ and the following sequences $\alpha_k(q) = \mathrm{Li}_{-1}(q^{2(k+2)}), \beta_k(q) = \mathrm{Li}_{-2}(q^{2(k+2)})$ and $\gamma_k(q) = \mathrm{Li}_{-3}(q^{2(k+2)})$. Our aim is to show that the coefficients $\{c_k(q)\}_{k\geq 0}$ are positive and monotonically decreasing for every $q \in (0,1]$. Notice that we could have expressed the coefficients in terms of the generalized q^2-harmonic numbers by partial fraction decomposition of the polylogarithms β_i and γ_i. However, representation (4.61) turns out to be more convenient for our purposes. Formula (4.61) can be derived in the following manner. We can re-write the difference $b_{rs}(q) - b_{sr}(q)$ as follows:

$$(r-s)(e_2 + q^{2(r+s)}e_1) + (se_2 - re_1)q^{2r} - (re_2 - se_1)q^{2s} + rs(q^{2r} - q^{2s})e_0,$$

where $e_i = e_i(q)$ are fixed functions of q that do not depend on (r,s). Note that each of the three terms is anti-symmetric in (r,s); we will analyse each contribution separately. Let's denote them by $J_i(z)$ for $i = 0,1,2$, respectively. We will repeatedly be using the following simple identity

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\alpha_{rs}z^r = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\alpha_{r(s+r)}z^r(1-z^s),$$

which is true, provided the double sequence $\{\alpha_{rs}\}_{r,s\geq 1}$ is anti-symmetric in (r,s). We have:

$$\begin{aligned}
J_0(z) &= -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{4(r+s)}\frac{rs(s-r)^2}{q^{2r}-q^{2s}}(e_2+q^{2(r+s)}e_1)z^r \\
&= -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{6r+4s}\frac{rs^2(r+s)}{1-q^{2s}}(e_2+q^{4r+2s}e_1)z^r(1-z^s) \\
&= -e_2\,\mathrm{Li}_{-2}(q^6z)\left\{\mathscr{L}_2(q^4)-\mathscr{L}_2(q^4z)\right\} - e_1\,\mathrm{Li}_{-2}(q^{10}z)\left\{\mathscr{L}_2(q^6)-\mathscr{L}_2(q^6z)\right\} \\
&\quad - e_2\,\mathrm{Li}_{-1}(q^6z)\left\{\mathscr{L}_3(q^4)-\mathscr{L}_3(q^4z)\right\} - e_1\,\mathrm{Li}_{-1}(q^{10}z)\left\{\mathscr{L}_3(q^6)-\mathscr{L}_3(q^6z)\right\} \\
J_1(z) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{4(r+s)}\frac{rs(s-r)}{q^{2r}-q^{2s}}((se_2-re_1)q^{2r}-(re_2-se_1)q^{2s})z^r \\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{8r+4s}\frac{rs(r+s)}{1-q^{2s}}(r(1-q^{2s})(e_2-e_1)+s(e_2+q^{2s}e_1))z^r(1-z^s) \\
&= -(e_1-e_2)\,\mathrm{Li}_{-2}(q^8z)\left\{\mathrm{Li}_{-2}(q^4)-\mathrm{Li}_{-2}(q^4z)\right\} \\
&\quad - (e_1-e_2)\,\mathrm{Li}_{-3}(q^8z)\left\{\mathrm{Li}_{-1}(q^4)-\mathrm{Li}_{-1}(q^4z)\right\} \\
&\quad + \mathrm{Li}_{-2}(q^8z)\left\{e_2(\mathscr{L}_2(q^4)-\mathscr{L}_2(q^4z))+e_1(\mathscr{L}_2(q^6)-\mathscr{L}_2(q^6z))\right\} \\
&\quad + \mathrm{Li}_{-1}(q^8z)\left\{e_2(\mathscr{L}_3(q^4)-\mathscr{L}_3(q^4z))+e_1(\mathscr{L}_3(q^6)-\mathscr{L}_3(q^6z))\right\} \\
J_2(z) &= e_0\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{4(r+s)}r^2s^2(s-r)z^r = e_0\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{8r+4s}r^2s(r+s)^2z^r(1-z^s) \\
&= e_0\left\{\mathrm{Li}_{-2}(q^4z)\,\mathrm{Li}_{-3}(q^4)-\mathrm{Li}_{-3}(q^4z)\,\mathrm{Li}_{-2}(q^4)\right\}
\end{aligned}$$

Summing the three contributions and simplifying gives the generating function $J(z)$ of the residue sequence $\{C_r(q)\}_{q\geq 1}$. However, the coefficients of the perturbation determinant are expressible in terms of the generating function J as $c_k(q)=J(q^{2(k+1)})$. Plugging in the value $q^{2(k+1)}$ and simplifying gives exactly representation (4.61) above.

We introduce next the approximation sequences $\{c_k(q;N,0)\}$ and $\{c_k(q;N,1)\}$ defined as follows:

$$\begin{aligned}
c_k(q;N,0) &= -e_2\Delta_k\beta_{k+2}\left\{\sum_{i=0}^{N+1}\beta_i-\beta_{k+1}\right\} + e_1\Delta_k\beta_{k+3}\left\{\sum_{i=0}^{N+1}\beta_i-\beta_0\right\} \\
&\quad - e_2\Delta_k\alpha_{k+2}\left\{\sum_{i=0}^{N+1}\gamma_i-\gamma_{k+1}\right\} + e_1\Delta_k\alpha_{k+3}\left\{\sum_{i=0}^{N+1}\gamma_i-\gamma_0\right\} \\
&\quad - (e_1-e_2)\beta_{k+3}(\beta_0-\beta_{k+1}) - (e_1-e_2)\gamma_{k+3}(\alpha_0-\alpha_{k+1}) \\
&\quad + e_0(\gamma_0\beta_{k+1}-\beta_0\gamma_{k+1}) \\
c_k(q;N,1) &= -e_2\Delta_k^2\beta_{k+2}\left\{\sum_{i=0}^{N+1}\beta_i-\beta_{k+2}\right\} + e_1\Delta_k^2\beta_{k+3}\left\{\sum_{i=0}^{N+1}\beta_i-\beta_0\right\} \\
&\quad - e_2\Delta_k^2\alpha_{k+2}\left\{\sum_{i=0}^{N+1}\gamma_i-\gamma_{k+2}\right\} + e_1\Delta_k^2\alpha_{k+3}\left\{\sum_{i=0}^{N+1}\gamma_i-\gamma_0\right\}
\end{aligned}$$

$$+ e_2\beta_{k+1}\Delta_k\beta_{k+2} - e_1\beta_{k+2}\Delta_k\beta_{k+3} + e_2\gamma_{k+1}\Delta_k\alpha_{k+2} - e_1\gamma_{k+2}\Delta_k\alpha_{k+3}$$
$$- (e_1 - e_2)\Delta_k\beta_{k+3}(\beta_0 - \beta_{k+2}) - (e_1 - e_2)\Delta_k\gamma_{k+3}(\alpha_0 - \alpha_{k+2})$$
$$+ (e_1 - e_2)(\beta_{k+3}\Delta_k\beta_{k+1} + \gamma_{k+3}\Delta_k\alpha_{k+1}) + e_0(\gamma_0\Delta_k\beta_{k+1} - \beta_0\Delta_k\gamma_{k+1})$$

Then the sequence of coefficients $\{c_k(q)\}$ of the perturbation determinant and its first differences $\{\Delta_k c_k(q)\}$ are expressible as $c_k(q) = c_k(q; k, 0)$ and $\Delta_k c_k(q) = c_k(q; k+1, 1)$. However, the approximation sequences are not related through the difference operator, i.e. $c_k(q; N, 1) \neq \Delta_k c_k(q; N, 0)$. The following technical lemma will allows us to prove positivity and monotonicity of the sequence $\{c_k(q)\}$; more precisely, we prove:

Lemma 4.3.4 *Let $q \in [0, 1]$, then the sequences $\{c_k(q; N, 0)\}$ and $\{c_k(q; N, 1)\}$ have the following properties:*

(i) The sequence $c_k(q; N, s) < c_k(q; N + 1, s)$ is monotonically increasing for every $k, N \geq 0$ and $s \in \{0, 1\}$.

(ii) The sequence $c_k(q; s + 2, s) \geq 0$ is non-negative for every $k \geq 0$ and $s \in \{0, 1\}$.

Proof. The major advantage of a sequence $c_k(q; N, s)$ with fixed N (independent of k) over the sequence of coefficients $c_k(q)$ (or its differences) is that the interpolating map of the term $c_k(q; N, s)$ is a bivariate rational function of (q^2, q^{2k}) of fixed degree that is independent of k (where the degree of a rational map is defined as the number of its poles, counting multiplicities). We can establish its positivity in a manner similar to the proof of theorem (4.3.6) and (4.3.8), by writing the numerator in the bivariate Bernstein basis and checking the coefficients are non-negative.

Let us prove part (i) first. Note that the difference $c_k(q; N + 1, s) - c_k(q; N, s)$ is simply expressible as

$$\beta_{N+2}(e_1\Delta_k^{s+1}\beta_{k+3} - e_2\Delta_k^{s+1}\beta_{k+2}) + \gamma_{N+2}(e_1\Delta_k^{s+1}\alpha_{k+3} - e_2\Delta_k^{s+1}\alpha_{k+2}),$$

so it suffices to prove both brackets are positive. Evaluating the limit of each bracket as $q \to 1^-$, it is easily shown that each is a rational function of k with positive coefficients so we may assume without loss of generality that $q \in [0, 1)$. Recall that the sequences $\{\beta_k\}$ and $\{\alpha_k\}$ are both completely monotone, meaning $\Delta_k^r x_n > 0$ for all $r, n \geq 0$ (see lemma (A.1.1), and the sequence $\{\Delta_k^r \beta_{k+n}\}_{k\geq 1}$ is logarithmically convex (see lemma (A.1.2), so it suffices to prove both brackets are positive for $k = 0$. This is easily checked for both brackets by direct computation.

We have already described how to prove part (ii). We again dispose first of the boundary case $q = 1$, by explicitly evaluating each approximation sequence at $q = 1$.

We obtain a rational function of k with positive coefficients in both cases, so part (ii) is true for $q = 1$. Hence we may assume $q \in [0, 1)$ and write the terms of each approximation sequence $c_k(q; s + 2, s)$ as a bivariate rational function of $(q^2, q^{2k}) = (q^2, z)$. It is easily checked that the denominators are always non-negative on the unit square $[0, 1]^2$, while the numerators can both be written in the bivariate Bernstein basis with non-negative coefficients (using say `Mathematica`). It follows that both sequences are non-negative as asserted. □

According to lemma (4.3.4) we have $c_2(q) > c_3(q) > \cdots > c_k(q) > c_{k+1}(q) > \cdots > 0$. It remains to check the appropriate inequalities for the initial coefficients. We can explicitly compute $c_0(q) = 1$, $c_1(q)$ and $c_2(q)$ and check each (and their respective differences) is expressible as a q-rational map with positive coefficients. We will not write down the expressions for the sake of brevity.

4.4 Limiting Distributions

4.4.1 One-endedness of the UIIPQ

The following is the main theorem of the present section

Theorem 4.4.1 (L. Addario-Berry, [1])**.** *For every $n \geq 1$, let μ_n denote the uniform probability measure on the space of finite rooted irreducible quadrangulations with n faces, then the sequence $\{\mu_n\}_{n\geq 1}$ converges to a probability measure μ, in the sense of weak convergence, in the space of probability measures on $(\mathbf{Q}, d_{\mathbf{Q}})$. Moreover, μ is supported on the set of infinite rooted irreducible quadrangulations of the plane.*

The probability measure μ defines a certain random object in $(\mathbf{Q}, d_{\mathbf{Q}})$ called the *uniform infinite irreducible planar quadrangulation* and we will devote the subsequent sections to the investigation of local properties of this object.

We will prove of the existence of the uniform infinite irreducible planar quadrangulation by exploiting essentially the same ideas as used by M. Krikun in [42] to show the existence of the uniform infinite planar quadrangulation (UIPQ). We recall first a few definitions and results from probability theory.

Let $(I, \preccurlyeq)$ be a *directed set* (in other words, we assume the relation $\preccurlyeq$ is reflexive and transitive such that any pair of elements of I has an upper bound). Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces and suppose for every pair $i, j \in I$ that satisfies $i \preccurlyeq j$ we are given a continuous map

$$\mathrm{pr}_{ij} : X_j \to X_i$$

such that $\mathrm{pr}_{ii} = \mathrm{id}_{X_i}$ for every i, and $\mathrm{pr}_{ij} \circ \mathrm{pr}_{jk} = \mathrm{pr}_{ik}$ for all $i \preccurlyeq j \preccurlyeq k$.

The system $((X_i, \mathcal{T}_i)_{i\in I}, (\mathrm{pr}_{ij})_{i\preccurlyeq j})$ is called a *projective system* of topological spaces.

Definition 36. The *inverse limit* $\varprojlim X_i$ of the sets X_i with respect to the directed system $(X_i, \{\mathrm{pr}_{ij}\})$ is defined as the following set:

$$\varprojlim X_i = \left\{ (x_i)_{i\in I} \in \prod_{j\in I} X_j |\ \mathrm{pr}_{ij}(x_j) = x_i \text{ whenever } i \preccurlyeq j \right\}$$

equipped with canonical projection $\mathrm{pr}_i : \varprojlim X_i \to X_i$ simply defined by $\mathrm{pr}_i(x_j)_{j\in J} = x_i$. The *inverse limit topology* on $\varprojlim X_i$ is the weak topology generated by the maps pr_i for $i \in I$.

Inverse limit construction is a generalization of a product. However, unlike the product of non-empty space (which is non-empty by the axiom of choice), the inverse limit of non-empty spaces can be empty even if the projections pr_{ts} are surjective. The following lemma is well-known and gives a sufficient condition for the inverse limit to be non-trivial and well-behaved.

Lemma 4.4.1 *Let $\varprojlim X_i$ be the inverse limit over a countable directed set T.*

(i) If every map pr_{ij} is surjective, the inverse limit is non-empty and each canonical projection pr_i is surjective.

(ii) If additionally each space X_i is Polish, the inverse limit equipped with the inverse limit topology is also a Polish space.

Before continuing with the properties of inverse limits, let us take a closer look at the completion space $\mathbf{Q}$ of the space of finite irreducible quadrangulations with respect to the local topology. A peculiar property of the local metric is the easily proven observation that it satisfies the ultrametric inequality

$$d_{\mathbf{Q}}(\mathbf{q}, \mathbf{q}'') \leq \max\{d_{\mathbf{Q}}(\mathbf{q}, \mathbf{q}'), d_{\mathbf{Q}}(\mathbf{q}', \mathbf{q}'')\}.$$

It is now not difficult to show that the completion is in fact naturally isomorphic with a certain inverse limit. Indeed, if we denote by $\mathbb{B}_r$ the at most countable set of all possible r-balls of irreducible quadrangulations, equipped with the discrete topology and define the natural projections $\mathrm{pr}_{rr'} : \mathbb{B}_r \to \mathbb{B}_{r'}$ by $\mathbf{q} \mapsto B_{r'}(\mathbf{q})$ for every $r' \leq r$, then it is easily shown that we have an isomorphism of topological spaces[3]

$$\mathbf{Q} \cong \varprojlim \mathbb{B}_r.$$

[3]The result is not surprising since a metric space is ultrametric if and only if it admits a representation as an (at most countable) inverse limit of discrete spaces.

Let $\mathbf{P}$ denote the subset of all infinite planar irreducible quadrangulations. Note that there is a natural way of extending the definition of an r-hull to the set of planar infinite quadrangulations - we may simply define $B_r^\infty(\mathbf{q})$ to be the rooted proper irreducible quadrangulation obtained from $B_r(\mathbf{q})$ by filling in all the finite holes. Let's denote by $\mathbb{C}_r$ the at most countable set of all possible r-hull of infinite planar irreducible quadrangulations, then we can write $\mathbf{P}$ as a countable intersection

$$\mathbf{P} = \bigcap_{r=1}^{\infty} \bigcup_{\Delta \in \mathbb{C}_r} \{\mathbf{q} |\, \Delta \subseteq \mathbf{q}\}, \tag{4.62}$$

where notation $\Delta \subseteq \mathbf{q}$ is to be interpreted as saying $\Delta \subseteq B_{r'}(\mathbf{q})$ for some sufficiently large $r' \geq r$. Notice that each set of the union is open in the local topology generated by the combinatorial balls, so representation (4.62) implies $\mathbf{P}$ is a G_δ subset of $\mathbf{Q}$. According to Mazurkiewicz's theorem it is therefore topologically complete, meaning there exists a topologically equivalent metric d such that $(\mathbf{P}, d)$ is complete. Unfortunately, the induced metric on $\mathbf{P}$ is not complete (indeed, the example of figure 2.7 can easily be adapted to produce a Cauchy sequence of infinite planar quadrangulations whose limit is not planar). However, as we will show now, if we replace in the definition of the local metric combinatorial balls with hulls the resulting metric space $\mathbf{P}$ is complete. More precisely, we will prove the following

Lemma 4.4.2 *For each fixed $r \geq 1$, let $\mathbb{C}_r$ denote the (countable) discrete space of all possible r-hulls of irreducible quadrangulations and define for each $r \geq r' \geq 1$ the projection operator $\mathrm{pr}_{rr'} : \mathbb{C}_r \to \mathbb{C}_{r'}$ by $\mathrm{pr}^\bullet_{rr'}(\Delta) = B^\bullet_{r'}(\Delta)$, then $(\mathbb{C}_r, \mathrm{pr}^\bullet_{rr'})$ is an inverse system and we have an isomorphism of topological spaces*

$$(\mathbf{P}, d_{\mathbf{Q}}) \cong \varprojlim \mathbb{C}_r.$$

Proof. The underlying set of the inverse limit $\varprojlim \mathbb{C}_r$ is naturally identified with the set of all compatible sequences $(\mathbf{q}_1, \mathbf{q}_2, \ldots)$ of finite irreducible quadrangulations with a unique hole such that $B_r^\bullet(\mathbf{q}_{r+1}) = \mathbf{q}_r$. We can define functions

$$f : \mathbf{P} \to \varprojlim \mathbb{C}_r, \quad \mathbf{q} \mapsto (B_r^\infty(\mathbf{q}))$$
$$g : \varprojlim \mathbb{C}_r \to \mathbf{P}, \quad \mathbf{q} \mapsto (B_r(\mathbf{q})).$$

Using the definition of combinatorial balls and hulls, it is clear that the mappings f and g are mutually inverse bijections. It therefore suffices to show each is continuous. Notice that both spaces can naturally be identified with a subspace of a product space (namely, $\prod_r \mathbb{B}_r$ or $\prod_r \mathbb{C}_r$) with the induced topology. According to the universal

property of product topology it therefore suffices to prove that the compositions $\mathrm{pr}_r^\bullet \circ f$ and $\mathrm{pr}_r \circ g$ are continuous for every $r \geq 1$. Let $r \geq 1$ and choose $\Delta_0 \in \mathbb{B}_r$ and $\Delta_1 \in \mathbb{C}_r$. Since Δ_1 is a finite (irreducible) quadrangulation there exists some $r' \geq r$ such that $B_{r'}(\Delta_1) = \Delta_1$ (in other words, Δ_1 has a finite radius) so we may write

$$(\mathrm{pr}_r^\bullet \circ f)^{-1}(\Delta_1) = \{\mathbf{q} \in \mathbf{P} | B_r^\infty(\mathbf{q}) = \Delta_1\} = \bigcap_{s=1}^{r'} \{\mathbf{q} \in \mathbf{P} | B_s(\mathbf{q}) \supseteq B_s(\Delta_1)\}$$
$$(\mathrm{pr}_r \circ g)^{-1}(\Delta_0) = \{\mathbf{q} \in \varprojlim \mathbb{C}_r | B_r(\mathbf{q}) = \Delta_0\} = \bigcup_{\Delta_0 \subseteq \Delta \in \mathbb{C}_r} \{\mathbf{q} \in \varprojlim \mathbb{C}_r | B_r^\bullet(\mathbf{q}) = \Delta\}.$$

Clearly, both sets are open in their respective local topologies. It follows that f and g are mutually inverse homeomorphisms. □

Suppose now each space X_i comes equipped with a probability measure $\mathbb{P}_i$ defined on the Borel σ-algebra of X_i. Since the canonical projection pr_i is continuous for every $i \in I$, it is measurable. We say that a family $\{\mathbb{P}_i\}_{i \in I}$ of probability measures is *consistent* if the following condition holds

$$\mathrm{pr}_{ij}(\mathbb{P}_j) = \mathbb{P}_i \text{ whenever } i \preccurlyeq j.$$

Under suitable restrictions, a consistent family of probability measure defines a unique probability measure $\mathbb{P}$ on the inverse limit set $\varprojlim X_i$ equipped with its Borel σ-algebra induced by the inverse limit topology, called the *inverse limit Borel σ-algebra.* Alternatively, the inverse limit Borel σ-algebra could have also been defined as the smallest σ-algebra on the projective limit that makes all the canonical projection pr_i measurable. The following result is well-known:

Theorem 4.4.2 (Kolmogorov's extension theorem)**.** *Let $(I, \preccurlyeq)$ be a countable directed set and $\{X_i\}_{i \in I}$ a family of topological spaces equipped with a collection pr_{ij} (for $i \preccurlyeq j$) of continuous surjective maps. Suppose $\{\mathbb{P}_i\}_{i \in I}$ is a consistent family of probability measures on X_i, then there exists a unique probability measure $\mathbb{P}$ on the inverse limit σ-algebra satisfying $\mathrm{pr}_i(\mathbb{P}) = \mathbb{P}_i$ for all $i \in I$.*

We are now ready to prove theorem (4.4.1). Let's write $\alpha^*(z) = z^{-1}\alpha_3(z)$ (where α_3 is the coefficient of the dominant term in the singular expansion of $\Phi(z, x)$ about the dominant singular point at $x = x_0$ - see 3.29). If we re-write α^* in terms of the new coordinates $w = w(z)$, a simple calculation yields

$$\alpha^*(w) = \frac{2(4-3w)(2-3w)^2}{243w(1+w)(1-w)^3}.$$

We can now prove

Lemma 4.4.3 *Let $r \geq 1$ be a fixed positive integer, then*

$$\sum_{\Delta \in \mathbb{C}_r} \lim_{N\to\infty} \mathbb{P}(B_r^\bullet(\mu_N) = \Delta) = 1.$$

Proof. We need to treat the case $r = 1$ separately, so let us first show that the result is true for 1-neighbourhoods. We know that $B_1^\bullet(\mathbf{q}) = B_1(\mathbf{q}) = \mathbf{s}_p$ where p is the degree of the root vertex in $\mathbf{q}$. According to formula (4.1) the limiting probabilities $e_p = \lim_{N\to\infty}(\deg(\mu_n) = p)$ exist and their generating function is simply $p(z) = c_0\alpha_3(x_0 z)$. It is easily checked that $\alpha_3(x_0) = c_0^{-1}$, which immediately implies p is a probability generating function as asserted (here and below $c_0 = 729/8$).

Let now $r > 1$ be fixed. We can evaluate the sum of probabilities over all r-hulls using the boundary hull operator as follows

$$\begin{aligned}\sum_{\Delta \in \mathbb{C}_r} \lim_{N\to\infty} \mathbb{P}(B_r^\bullet(\mu_N) = \Delta) &= \sum_{\bar{B}_r} c_0 \sum_{|\underline{\lambda}|_0 = p} c(\underline{\lambda}) x_0^{n(B_r^\bullet) - |\underline{\lambda}|_1} \\ &= \sum_{\bar{B}_r} c_0 x_0^{n(\bar{B}_r)+p} c_p = x_0 c_0 (\boldsymbol{A}(x_0)^{r-2}\alpha^*)(x_0)\end{aligned}$$

where $c_p = [t^p]\alpha^*(t)$ is the (limiting) probability that a reduced r-hull $\bar{B}_r$ of a random irreducible quadrangulations has upper boundary length p. We wish to prove that the right-hand side is equal to 1, which follows from the equality $\boldsymbol{B}^{r-2}\alpha^* = \alpha^*$ and the fact that $\alpha^*(x_0) = x_0^{-1}c_0^{-1}$. This in turn follows from $\boldsymbol{B}\alpha^* = \alpha^*$ (in other words α^* is an eigenfunction of the boundary hull operator with eigenvalue 1). To check the last equation one simply plugs the expression for α^* found above in (4.52) and performs a trivial expression transformation. We omit it. □

Lemma (4.4.3) implies that for each fixed $r \geq 1$, the limits (4.15) define a unique probability measure $\mathbb{P}_r$ on the space $\mathbb{C}_r$ of r-hulls of irreducible quadrangulations.

Notice that for each $1 \leq r < r'$ there is a canonical projection $\mathrm{pr}_{rr'} : \mathbb{C}_{r'} \to \mathbb{C}_r$, which maps $\Delta \in \mathbb{C}_{r'}$ to $B_r^\bullet(\Delta)$ and for any $1 \leq r < r' < r''$ we have $\mathrm{pr}_{rr'} \circ \mathrm{pr}_{r'r''} = \mathrm{pr}_{rr''}$. Furthermore, the limits (4.15) imply the family $\{\mathbb{P}_r\}_{r\geq 1}$ is a consistent family of probability measures in the sense that $\mathbb{P}_r = \mathbb{P}_{r'} \circ \mathrm{pr}_{rr'}^{-1}$ for every $1 \leq r < r'$. We are exactly in a position to apply Kolmogorov's extension theorem to deduce the existence of a unique probability measure $\mathbb{P}_\infty$ on the projective limit set of the projective system $(\mathbb{C}_r, \mathrm{pr}_{rr'})$. However, we have shown that the projective limit of the latter system is exactly the space of all infinite irreducible planar quadrangulations. It follows that we can define a random infinite irreducible planar quadrangulation, which we denote by $\mathbf{q}_\infty$, such that $\mathbf{q}_\infty$ has a unique end and the law of $B_r^\bullet(\mathbf{q}_\infty)$ is equal to $\mathbb{P}_r$ for every positive integer $r \geq 1$. Note that here infinity plays the role of

the distinguished point, so the hull $B_r^\bullet(\mathbf{q}_\infty)$ is obtained by "filling in" the finite holes in the ball $B_r(\mathbf{q}_\infty)$, which is itself the union of all the faces that are incident with a vertex of distance strictly less than r from the root.

Finally, since we have already shown that for every irreducible quadrangulation $\mathbf{k}$ with a simple boundary

$$\lim_{N\to\infty} \mu_N(\mathbf{q}|B_r^\bullet(\mathbf{q}) = \mathbf{k}) = \mathbb{P}_\infty(\mathbf{q}|B_r^\bullet(\mathbf{q}) = \mathbf{k}).$$

It follows by Prokhorov's theorem (4.1.1) that the probability measure $\mathbb{P}_\infty$ is exactly the weak limit of the uniform probability measure μ_n as $n \to \infty$. This completes the proof of theorem (4.4.1).

4.4.2 Hull boundary length limiting distribution

We have shown that the resolvent of the boundary operator $\boldsymbol{B}$ is given by

$$S(z, w_1)g = \frac{1 - z\Psi_1'(z)}{\Delta_1(z)} R_1(z, w_1)g + \frac{z\Psi_1(z)}{\Delta_1(z)} R_1'(z, w_1)g$$

where $\Delta_1(z)$ is the boundary determinant function. According to theorem (4.3.9), the boundary determinant map admits a unique analytic continuation to the slit complex plane $\mathbb{C}\backslash[1, \infty)$ and is expressible in terms of polylogarithmic functions. We have also shown that the limit $\lim_{z\to 1} \Delta_1(z)$ exists and the boundary determinant map is non-zero on the closed unit disc $\overline{\mathbb{D}}$. It follows that the coefficients in the resolvent formula are analytic on the closed unit disk $\overline{\mathbb{D}}\backslash\{1\}$. We will show that the main contributions $R_1(z, w_1)g$ and $R_1'(z, w_1)g$ also have a pole-like singularity at $z = 1$, so the dominant singularity of the boundary resolvent map occurs at $z = 1$.

Recall that the higher moments of the random variable ℓ giving the length of the upper boundary of an r-hull $B_r^\bullet$ for $r > 1$ are expressible in terms of the boundary hull operator as (where we have denoted $\boldsymbol{A} = \boldsymbol{A}(x_0)$):

$$\begin{aligned}\mathbb{E}\ell^j(B_r^\bullet) &= \sum l^j \mathbb{P}(\ell(B_r^\bullet) = l) = \sum_{\bar{B}_r} \sum_{|\underline{\lambda}|_0 = p} l^j c_0 c(p, l) x_0^{n(B_r^\bullet) - l} \\ &= c_0 \sum_{\bar{B}_r} \left([z^p]\ \theta_t^j \alpha^*(t, z)\Big|_{t=1} \right) x_0^{n(\bar{B}_r)+p} = x_0 c_0 (\boldsymbol{A}^{r-2}\ \theta_t^j \alpha^*(t, z)\Big|_{t=1})(x_0)\end{aligned}$$

where $\alpha^*(t, z) = z^{-1}\alpha_3(t, z)$ is the dominant singular term of $N_2(t, z)$ about the nearest singularity at $x = x_0$ and we have introduced the differential operator $\theta_t = t\frac{d}{dt}$. The volume generating function $\alpha^*(t, z)$ is quite complicated, so in order to the simplify the expression we introduce the following change of variables.

$$t = \frac{2(25 - \omega^2)}{1 - \omega^2} \text{ and } z = \frac{2(\omega - 1)^2}{(\omega - 5)(\omega + 1)} \left(1 - \frac{\omega - 3}{\omega - 1} w\right) \left(1 - \frac{\omega - 3}{\omega - 1} w^{-1}\right),$$

where it is assumed that $(\omega, w) \in [3,7] \times [0,1]$. It is easily checked that upon substitution $\omega = 7$ (which corresponds to $t = 1$), $w(7,t)$ is exactly the coordinate transformation between the boundary operators $\boldsymbol{A}$ and $\boldsymbol{B}$. Note that with respect to the transformation introduced above, the fixed point occurs at $w = 1$. However, for the purposes of our calculations it is easier to work with a transformation whose fixed point is simply the origin $w = 0$. We can achieve this by further introducing the transformation $w = c(W)$, where c is the Cayley map (recall that the Cayley map is an involution). We can write t and the function $\alpha^*(t,z)$ in terms of the variables (ω, W) as follows:

$$t = g(\omega, W(\omega,t)) = \frac{8(1-(\omega-2)^2W^2)}{(\omega-5)(\omega+1)(1-W^2)},$$
$$\alpha^*(\omega,W) = \frac{(\omega+1)(\omega-5)(1-(\omega-2)W)^2 p_0(\omega,W)}{2^5 3^4(\omega-3)(\omega-2)^2(\omega-1)\omega W^3(1-W^2)}$$

where $p_0(\omega, W)$ is a bivariate polynomial given explicitly as follows

$$\begin{aligned} p_0(\omega,W) &= 2\omega(\omega-2)^2 + 4\omega(\omega-2)^3W + 6\omega(\omega-2)^2(\omega^2-4\omega-13)W^2 \\ &\quad + (3\omega^6 - 36\omega^5 + 81\omega^4 + 320\omega^3 - 875\omega^2 - 388\omega + 1155)W^3 \\ &\quad + 2(\omega-2)(2\omega^3 + 4\omega^2 - 76\omega - 105)W^4. \end{aligned}$$

It is easily checked that upon substitution $\omega = 7$ (which corresponds to $t = 1$) and $W = c(w)$ the above formula simplifies to the familiar expression for $\alpha^*(w)$ determined in the previous section. The differential operator θ can easily be re-written in terms of ω; we have

$$\vartheta = \frac{(1-\omega^2)(25-\omega^2)}{48\omega}\frac{d}{d\omega} \text{ and } w_\omega = -\frac{g_\omega(\omega,W)}{g_W(\omega,W)} = \frac{(\omega-2)(1-W^2)(1-9W^2)}{(1-\omega^2)(\omega-5)(\omega-3)W}.$$

if we denote by α^*_x and α^*_y the partial derivatives of α^* with respect to the first and second variable, then the effect of the operator ϑ on α^* is

$$\theta_z\alpha^* = \frac{(1-\omega^2)(25-\omega^2)}{48\omega}\alpha^*_x(\omega,W) - \frac{(\omega+5)(\omega-5)(1-W^2)(1-9W^2)}{48\omega(\omega-3)W}\alpha^*_y(\omega,W).$$

The next result describes the structure of the iterates $\alpha_j(\omega,W) = \theta_t^j\alpha^*(\omega,W)$ about the point $\omega_0 = 7$. We describe the structure of the iterates $\alpha_j(\omega,W)$ with respect to the new variable W; however, the original map can then be obtain by substituting $W = c(w)$ back into the expression.

Lemma 4.4.4

$$\alpha_j^*(\omega, W) = \frac{(2j+1)!!(\omega+1)(\omega-5)}{2^{4(j+1)}3^{j+4}(\omega-3)(\omega-1)}\left(\frac{(\omega+5)(\omega-2)}{\omega(\omega-3)}\right)^j \frac{P_j(\omega, W)}{W^{2j+3}(1-W^2)}$$
$$\alpha_j^*(\omega_0, W) = \frac{(2j+1)!!}{2^{4j+3}3^5}\left(\frac{5}{7}\right)^j \frac{P_j(\omega_0, W)}{W^{2j+3}(1-W^2)}$$

where $P_j(\omega, W) \in \mathbb{Z}(\omega)[W]$ *is a polynomial in* W *with rational coefficients in* ω *such that* $P_j(\omega, 0) = 1$, $P_j(\omega, W) - 1 = \mathcal{O}(W^2)$ *and* $\deg_W P_j = 4j + 6$.

Proof. It clearly suffices to prove the first identity. The proof will be by induction on j. Firstly, $\alpha_0^*(\omega, W) = \alpha^*(\omega, W)$ clearly satisfies the *ansatz*. The sequence $\{\alpha_j^*(\omega, W)\}_{j\geq 0}$ by definition satisfies the following differential relation:

$$\alpha_{j+1}^* = \frac{(1-\omega^2)(25-\omega^2)}{48\omega}\alpha_{j,x}^*(\omega, W) - \frac{(\omega+5)(\omega-2)(1-W^2)(1-9W^2)}{48\omega(\omega-3)W}\alpha_{j,y}^*(\omega, W).$$

Plugging in the *ansatz* gives, after some tedious algebra,

$$\alpha_{j+1}^*(\omega, W) = \frac{(2j+3)!!(\omega+1)(\omega-5)}{2^{4j+7}3^{j+5}(\omega-3)(\omega-1)}\left(\frac{(\omega+5)(\omega-2)}{\omega(\omega-3)}\right)^{j+1} \frac{P_{j+1}(\omega, W)}{W^{2j+5}(1-W^2)},$$

where the polynomials $\{P_j(\omega, W)\}_{j\geq 0}$ satisfy the following partial differential recurrence relation

$$P_{j+1}(\omega, W) = \left\{1 - 2\frac{(8\omega(\omega-2)^2(\omega+5) + 2jp(\omega)}{(3+2j)\omega(\omega-2)^2(\omega+5)}W^2 + 9\frac{5+3j}{3+2j}W^4\right\}P_j(\omega, W)$$
$$- \frac{1}{3+2j}W(1-W^2)(1-9W^2)P_{j,y}(\omega, W) + \frac{(\omega-5)(\omega-3)(\omega^2-1)}{(3+2j)(\omega-2)}W^2P_{j,x}(\omega, W),$$

where $p(\omega) = 25+45\omega-70\omega^2+24\omega^3-5\omega^4+\omega^5$, and $P_j(\omega, 0) = 1$. It now readily follows that $P_{j+1}(\omega, W)$ is a polynomial of degree $4(j+1)+6$ in W satisfying $P_{j+1}(\omega, 0) = 1$ and $P_{j+1}(\omega, W) - 1 = \mathcal{O}(W^2)$. □

We are now ready to determine the dominant singular term of the resolvent map $R_1(z, w_1)\theta_t^j\alpha^*$ in a singular expansion about the dominant singularity at $z = 1$. More precisely, we prove first the following auxiliary lemma:

Lemma 4.4.5 (Singular expansion) *Let* $g_j(w) = W^{-(j+3)}(1-W^2)^{-1}$ *for* $j \in \mathbb{Z}$, *then*

$$S(z, w_1)g_j = \frac{2^{j+8}\Gamma(j+1)}{3(1-z)^{j+1}} + \mathcal{O}\left(\frac{1}{(1-z)^j}\right) \text{ if } j \geq 0 \tag{4.63}$$

and $S(z, w_1)g_j = o(1-z)^{-1/2}$ *otherwise.*

Proof. We need to establish first that $z = 1$ is indeed the dominant singularity of the resolvent map $S(z, w_1)g_j$ for each $j \in \mathbb{Z}$. We have already shown that the coefficients $\vartheta_i(z, w_1)$ have a dominant singularity at $z = 1$. We will prove first the same is true for $R_1(z, w_1)g_j$ and $R_1'(z, w_1)g_j$. The singular expansion (4.63) is then obtained by a combination of the singular expansion of each term as described in chapter 3. According to formula (4.42), we can evaluate the iterates $(T_1^k g_j)(w_1)$ by introducing first the auxiliary map $G_j(w)$ defined as

$$G_j(w) = \frac{(1-w)^3(1+w)}{w^2} g_j(w) = \frac{16}{W^j(1-W^2)^3}.$$

The iterates (which are also the coefficients of the resolvent map $R_1(z, w_1)g_j$) are then given by (note that the critical value of the new variable W is $W_1 = c(w_1) = 1/7$)

$$\begin{aligned}(T_1^k g_j)(w_1) &= \frac{36}{7}\frac{(2k+7)^{j+6}}{(k+3)^3(k+4)^3} = \frac{36}{7}a_{kj}\\ (T_1^k g_j)'(w_1) &= \frac{3552}{49}a_{kj} + \frac{1152}{7}(j+6)a_{k(j-1)} - \frac{1728}{7}\underbrace{\frac{(2k+7)a_{kj}}{(k+3)(k+4)}}_{b_{kj}}.\end{aligned}$$

Let us introduce the functions $s_j(z) = \sum_{k\geq 0} a_{kj} z^k$ and $t_j(z) = \sum_{k\geq 0} b_{kj} z^k$, then we can express each resolvent in terms of s_j and $t_j(z)$ as follows

$$R_1(z, w_1)g_j = \frac{36}{7}s_j(z), \quad R_1'(z, w_1)g_j = \frac{3552}{49}s_j(z) + \frac{1152}{7}(j+6)s_{j-1}(z) - \frac{1728}{7}t_j(z).$$

We will prove that both s_j and t_j are expressible in terms of the Lerch transcendent, so admit an analytic continuation to the slit complex plane $\mathbb{C}\backslash[1, \infty)$ with a unique dominant singularity at $z = 1$. We will have to treat separately the cases $j \geq 0$ and $j < 0$. Suppose first $j \geq 0$ and perform a partial fraction decomposition of a_{kj}; it is easily shown by induction on j that we obtain

$$\begin{aligned}a_{kj} &= \frac{(2k+7)^{j+6}}{(k+3)^3(k+4)^3} = \left\{\frac{1}{k+3} + \frac{1}{k+4}\right\}^3 \sum_{i=0}^{j+3} \binom{j+3}{i}(k+3)^i(k+4)^{j+3-i}\\ &= \frac{1}{(k+3)^3} - \frac{1}{(k+4)^3} + \frac{2j+9}{(k+3)^2} - \frac{2j+9}{(k+4)^2} + \frac{2(j+3)(j+5)}{(k+3)(k+4)} + P_j(k),\end{aligned}$$

where $P_j(x) \in \mathbb{Z}[x]$ is a polynomial of degree j. Inductively it is possible to prove the polynomials satisfy the following difference equation $P_{j+1}(x) = (2x+7)^2 P_j(x) + 32(j+2)(j+3)$ with initial condition $q_0(x) = 2^6$, which allows us to determine them explicitly as

$$P_j(x) = 2^6 \sum_{n=0}^{j} \binom{j+2-n}{2}(2x+7)^n = 2^6 \sum_{n=0}^{j} \left(\sum_{m=n}^{j} \binom{j+2-m}{2}\binom{m}{n}7^{m-n}\right) 2^n x^n$$

On the other hand, if $j < 0$, we write $j = -i$. It is clear that the terms a_{kj} admit a partial fraction decomposition into terms of the form $(k+3)^{-n}, (k+4)^{-m}$ and $(2k+7)^{-l}$, where $1 \leq n, m \leq 3$ and $1 \leq l \leq i-6$. We will not write down the explicit formula, merely the dominant terms which are of the form (for some constants α_j and β_j)

$$a_{kj} = \frac{(2k+7)^{6-i}}{(k+3)^3(k+4)^3} = \frac{\alpha_j}{(k+3)(k+4)} + \frac{\beta_j}{2k+7} + \mathcal{O}(k^{-2}).$$

It is clear that in both cases the function $s_j(z)$ is expressible in terms of polylogarithms and the Lerch transcendent (which is again analytically continuable to the slit complex plane $\mathbb{C}\backslash[1,\infty)$ with logarithmic branch-points at $z = 1, \infty$). It follows that $s_j(z)$ admits a singular expansion of the form

$$s_j(z) = \begin{cases} 2^{j+6}\Gamma(j+1)(1-z)^{-(j+1)} + \mathcal{O}(1-z)^{-j} & \text{for } j \geq 0 \\ o(1-z)^{-1/2} & \text{for } j < 0. \end{cases}$$

A similar analysis is possible for the terms b_{kj} - this time we would need to treat separately the cases $j \geq 1$ and $j < 1$. We will not write out again the explicit forms of the partial fraction decompositions. Again the generating function $t_j(z)$ admits an analytic continuation to the slit complex plane $\mathbb{C}\backslash[1,\infty)$ and its singular expansion satisfies $t_j(z) = o(s_j(z))$ for $j \geq 0$ and $o(1-z)^{-1/2}$ otherwise. Hence for $j \geq 0$ it does not contribute to the dominant singular term at all. To deduce the final expression, suppose first $j \geq 0$ then the preceding analysis shows the singular expansion of the resolvents $R_1(z, w_1)g_j$ and $R_1'(z, w_1)g_j$ about the dominant singular point $z = 1$ is

$$R_1(z, w_1)g_j = \frac{36}{7}\frac{2^{j+6}\Gamma(j+1)}{(1-z)^{j+1}} + \mathcal{O}((1-z)^{-2j})$$
$$R_1'(z, w_1)g_j = \frac{3552}{49}\frac{2^{j+6}\Gamma(j+1)}{(1-z)^{j+1}} + \mathcal{O}((1-z)^{-2j})$$

On the other hand, the coefficients in the decomposition satisfy $\vartheta_i(z, w_1) = \vartheta_i(1, w_1) + o(1)$ as $z \to 1$ (in some Δ-domain). Combining the singular expansions yields:

$$S(z, w_1)g_j = \frac{36}{7}\left(\vartheta_0(1, w_1) + \frac{296}{21}\vartheta_1(1, w_1)\right)\frac{2^{j+6}\Gamma(j+1)}{(1-z)^{j+1}} + \mathcal{O}((1-z)^{-2j}).$$

Using the values (4.53)-(4.56) of the resolvents $\Phi_1(z), \Psi_1(z)$ and their derivatives at $z = 1$, we can easily show that the initial factor simplifies to $4/3$, which gives exactly formula (4.63) above. The proof of case $j < 0$ is similar, we omit it. $\square$

Theorem 4.4.3 (Limiting distribution of the hull boundary length)**.** *Let $|\gamma_r|$ denote the length of the r-hull under distribution μ, then the random variable $14/5|\gamma_r|r^{-2}$ converges in distribution to the $\Gamma(3/2)$ law.*

Proof. A combination of lemmas (4.4.4) and (4.4.5) gives

$$\begin{aligned} S(z,w_1)\theta_t^j\alpha^* &= \frac{(2j+1)!!}{2^{4j+3}3^5}\left(\frac{5}{7}\right)^j\frac{2^{2j+8}(2j)!}{3(1-z)^{2j+1}} + \mathcal{O}((1-z)^{-(2j-1)}) \\ &= \frac{(2j+1)!!}{2^{2j-5}3^6}\left(\frac{5}{7}\right)^j\frac{(2j)!}{(1-z)^{2j+1}} + \mathcal{O}((1-z)^{-(2j-1)}) \end{aligned} \tag{4.64}$$

valid in a Δ-domain $\Delta(\epsilon,\phi)$ for sufficiently small $\epsilon>0$ and some $\phi\in(0,\pi/2)$. We may thus apply the transfer theorem to deduce, as $r\to+\infty$,

$$\mathbb{E}^j\ell(B_r^\bullet) = x_0c_0[z^r]S(z,w_1)\theta_t^j\alpha^* = \frac{(2j+1)!!}{2^j}\left(\frac{5}{14}r^2\right)^j\left(1+\mathcal{O}(r^{-2})\right).$$

Consequently, the re-scaled limit exists

$$\lim_{r\to\infty}\mathbb{E}^j\left(\frac{14}{5}r^{-2}\ell(B_r^\bullet)\right) = \mathbb{E}^j\xi = \frac{(2j+1)!!}{2^j}.$$

The limiting random variable ξ has moment generating function $\mathbb{E}e^{-s\xi}=(1+s)^{-3/2}$. Applying the inverse Laplace transform gives the density of the $\Gamma(3/2)$-law. □

Let's denote by $\ell_0(B_r^\bullet)$ the number of parts of the composition parameter of the boundary of the r-hull $B_r^\bullet(\mathbf{q})$. We can similarly analyse the limiting probabilities of $\ell_0(B_r^\bullet)$ using the boundary hull operator. We can again write for $r>1$,

$$\begin{aligned} \mathbb{E}^j(\ell_0(B_r^\bullet)) &= \sum p^j\mathbb{P}(\ell_0(B_r^\bullet)=p) = \sum_{\bar{B}_r}\sum_{|\underline{\lambda}|_0=p} p^jc_0c(p,l)x_0^{n(B_r^\bullet)-l} \\ &= c_0\sum_{\bar{B}_r}p^jc_px_0^{n(\bar{B}_r)+p} = x_0c_0(\boldsymbol{A}^{r-2}z^{-1}\theta_z^j\alpha_3)(x_0). \end{aligned}$$

Introducing again the change of variables $z=\frac{1}{2}(3-2w)(3-2w^{-1})$ and $W=c(w)$, the map $\alpha_3(z)$ and the differential operator $\theta_z=z\frac{d}{dz}$ now take the form

$$\alpha_3(W) = \frac{(1+W)(1-5W)^3(1+5W)(1+7W)}{3888(1-W)^2W^3}, \quad \theta_z = -\frac{(1-W^2)(1-25W^2)}{48W}\frac{d}{dW}.$$

Using induction it is possible to prove the iterates have the following form

Lemma 4.4.6

$$z^{-1}\theta_z^j\alpha_3(W) = \frac{(2j+1)!!}{2^{4j+3}3^{j+5}}\frac{P_j(W)}{W^{2j+3}(1-W^2)},$$

where $P_j(W)\in\mathbb{Z}[W]$ is a polynomial in W of degree $4j+6$ such that $P_j(0)=1$ and $P_j(W)-1=\mathcal{O}(W^2)$.

The proof is very similar to the proof of lemma (4.4.4), we omit it. The following result is now immediate:

Theorem 4.4.4 (Limiting distribution of ℓ_0)**.** *Let $|\gamma_r|_0$ denote the composition length of the r-hull under distribution μ, then the random variable $6|\gamma_r|_0 r^{-2}$ converges in distribution to the $\Gamma(3/2)$ law.*

Proof. A combination of lemmas (4.4.6) and (4.4.5) gives

$$\begin{aligned} S(z,w_1)z^{-1}\theta_z^j\alpha_3 &= \frac{(2j+1)!!}{2^{4j+3}3^{j+5}}\frac{2^{2j+8}(2j)!}{3(1-z)^{2j+1}} + \mathcal{O}((1-z)^{-(2j-1)}) \\ &= \frac{(2j+1)!!}{2^{2j-5}3^6}\left(\frac{1}{3}\right)^j \frac{(2j)!}{(1-z)^{2j+1}} + \mathcal{O}((1-z)^{-(2j-1)}) \end{aligned}$$

valid in some Δ domain at $z=1$. Applying the transfer theorem, we obtain the asymptotically dominant term as $r \to +\infty$,

$$\mathbb{E}\ell_0^j(B_r^\bullet) = x_0 c_0[z^r]S(z,w_1)z^{-1}\theta_z^j\alpha_3 = \frac{(2j+1)!!}{2^j}\left(\frac{1}{6}r^2\right)^j\left(1+\mathcal{O}(r^{-2})\right).$$

Consequently, using the exact same reasoning as in the proof of theorem (4.4.3) we deduce the re-scaled length $6r^{-2}\ell_0(B_r^\bullet)$ converges towards the $\Gamma(3/2)$-law. □

4.4.3 Hull volume limiting distribution

Having investigated the limiting distribution of normalized boundary length of r-hulls in random irreducible quadrangulations, we turn our attention next to the limiting distribution of volumes of r-hulls. The approach used to examine the limiting distribution of hull perimeters unfortunately will not work for volumes. Indeed, by universality we expect that the limiting distribution of properly normalized volume will be of the form

$$G_c(\Lambda) = 3^{3/2}\frac{\cosh(c\Lambda^{1/4})}{(\cosh^2(c\Lambda^{1/4})+2)^{3/2}} \tag{4.65}$$

for some constant $c > 0$. However, we have been unable to compute the moments of the distribution given by $G_c(\Lambda)$ explicitly, so it would be impossible to identify the limiting distribution of the hull volumes for irreducible quadrangulations from its moments. We will instead compute the probability generating function using the hull operator and examine its asymptotics as $R \to \infty$. As a side note, the function $G_c(\Lambda)$ is called the *two-point function* of $2D$ quantum gravity in the physics literature (see e.g [2]). It describes how distances and volumes are related in the continuum limit and corresponds (in the framework of probability theory) to the Laplace transform

of the law of volume of a "Brownian sphere with marked points at distance 1" when sampled according to the excursion measure (see [17]).

As a warm-up, let us determine first the probability generating function of the volume of 2-hulls $B_2^\bullet$ (or equivalently, 2-balls since a combinatorial ball of radius 2 in an irreducible quadrangulation has a unique boundary component). We will make use of the fact that the joint probability $\mathbb{P}(\mathrm{Vol}(B_2^\bullet) = n, \mathrm{Vol}(B_1^\bullet) = p)$ depends only on the number of parts of the partition parameter of the 2-hull, but not on the upper boundary length ℓ (as follows easily from lemma (4.1) upon noting that the volume of a 2-hull is simply $p + \ell$). Consequently, we can write

$$\begin{aligned}\mathbb{E}(s^{\mathrm{Vol}(B_2^\bullet)}) &= \sum_{n\geq 6} s^n \mathbb{P}(\mathrm{Vol}(B_2^\bullet) = n) = \sum_{p\geq 3}\sum_{n\geq 2p} s^n \mathbb{P}(\mathrm{Vol}(B_2^\bullet) = n, \mathrm{Vol}(B_1^\bullet) = p) \\ &= \sum_{p\geq 3}\sum_{n\geq 2p} s^n \sum_{\underline{\lambda}\vdash n-p} c_0 c(\underline{\lambda}) x_0^{p+1} = x_0 c_0 \sum_{p\geq 3} (sx_0)^p \sum_{\ell\geq p} s^\ell \sum_{\underline{\lambda}\vdash q} c(\underline{\lambda}) \\ &= x_0 c_0 \sum_{p,\ell} c(p,\ell) s^\ell (sx_0)^p = x_0 c_0 s\alpha^*(s, sx_0)\end{aligned}$$

In general, the probability generating function of the volume of an r-hull $B_r^\bullet$ for $r \geq 2$ can then be expressed in terms of the bivariate operator as follows:

$$\begin{aligned}\mathbb{E}(s^{\mathrm{Vol}(B_r^\bullet)}) &= \sum_{n\geq 0} s^n \mathbb{P}(\mathrm{Vol}(B_r^\bullet) = n) = c_0 \sum_{\bar{B}_r} \sum_{|\underline{\lambda}|_0 = p} c(p,q) x_0^{n(\bar{B}_r)+p} s^{n(\bar{B}_r)+q+p} \\ &= x_0 c_0 s(\boldsymbol{A}^{r-2}(sx_0)\alpha^*(s,z))(sx_0). \qquad (4.66)\end{aligned}$$

Ultimately we will set $s = e^{-\lambda/r^4}$ and take the limit $r \to +\infty$ (note that for $\lambda \geq 0$ the expected value is guaranteed to exist by the trivial inequality $e^{-\lambda} \leq 1$). Indeed, it is well-known that for various other combinatorial classes of planar maps (e.g. planar triangulations and quadrangulations) the volume of the r-hull is asymptotically of order $\mathcal{O}(r^4)$. Hence we expect a similar result to be true also for the class of irreducible quadrangulations. If we again introduce the change of variables $z = z(w)$, which transforms the hull operator $\boldsymbol{A}_q$ to $\boldsymbol{B}_q$, then the probability generating function of the volume of r-hulls is expressible as (we abbreviate $\alpha_s^*(w) = \alpha^*(s,w)$)

$$\mathbb{E}(s^{\mathrm{Vol}(B_r^\bullet)}) = x_0 c_0 (\boldsymbol{B}_q^{r-2} \alpha_s^*(w))(w_1),$$

where parameter q is related to the variable s via the relation

$$sx_0 = \frac{q^2(1-q^2+q^4)}{(1+q^4)^2} \text{ and } w_1(q) = \frac{q(1+q^2+q^4)}{1+q^2+q^4+q^6}.$$

Due to the substitution $s = e^{-\lambda/r^4}$ we cannot use singularity analysis to determine the asymptotics of the probability generating function $\mathbb{E}(s^{\mathrm{Vol}(B_r^\bullet)})$ as $r \to \infty$. The

following auxiliary lemma will be used instead to obtain the asymptotics of the iterates in equation (4.66). Notice that the lemma is essentially a re-wording of Tannery's theorem (A.1.3). However, for convenience sake we write it out in the following form:

Lemma 4.4.7 (Diagonally dominant transformation) *Let $\{x_{n,k}\}_{k,n\geq 0}$ be a double sequence and suppose we are given a sequence of coefficients $\{a_{k,n}\}_{k,n\geq 0}$ that satisfy the following conditions:*

(x_1) $\lim_{n\to\infty} x_{n-k,n} = x$ *(independent of k) for each fixed $k \in \mathbb{Z}$.*

(x_2) *The sequence $x_{k,n}$ is uniformly bounded.*

(a_1) *The limit $\lim_{n\to\infty} a_{k,n} = a_k$ exists for each fixed $k \geq 0$.*

(a_2) *The sequence $a_{k,n}$ has a convergent majorant.*

Then the convolution sequence $s_n = a_{n,n}x_{0,n} + a_{n-1,n}x_{1,n} + \cdots + a_{0,n}x_{n,n}$ is convergent with limit $\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} a_k x$.

Let a_k, b_k denote the coefficients of z^k in $\vartheta_i(z, w_0)$ for $i = 1, 2$, respectively. Similarly, denote with x_k and y_k the coefficient of z^k of the resolvent maps $T_1(z, w_1)(\alpha^*)$ and $T_1(z, w_1)(\alpha^*)$, respectively. Then according to Cauchy's multiplication theorem the probability generating function of the volume of an n-hull is (up to a constant) given by the following convolution:

$$\sum_{k=0}^{n} a_k(q)x_{n-k}(q) + \sum_{k=0}^{n} b_k(q)y_{n-k}(q).$$

We will devote the rest of the section to proving the convolution sequences $s_n(q) = (a * x)_n(q)$ and $t_n(q) = (b * y)_n(q)$ satisfy the conditions of the lemma.

We will verify first conditions (x_1) and (x_2) for the sequence of coefficients of $R_1(z, w_0)\alpha^*$ and $R_1'(z, w_0)\alpha^*$. Recall that the map $\alpha^*(s, z)$ has the form (see 3.71)

$$\alpha^*(s,z) = (\omega-5)(\omega+1)\frac{a_0\zeta^3 - a_1\zeta^2 - a_2\zeta + a_3 - (b_0\zeta - b_1)\,(2\zeta-1)\,W(t,z)}{2^4 3^4 \omega(\omega-1)^2(\omega-2)^2(\omega-3)^2\,(2\zeta-1)\,W(t,z)}$$

where $\alpha_i = \alpha_i(\omega)$ and $\beta_j = \beta_j(\omega)$ are certain polynomials in ω, and the transformation formulas are $\zeta = (\omega-5)(\omega+1)z/16$,

$$\omega(s) = \sqrt{\frac{50-s}{2-s}} = \sqrt{\frac{25-2q^2+52q^4-2q^6+25q^8}{1-2q^2+4q^4-2q^6+q^8}},$$
$$W(t,z) = \sqrt{(1-2\zeta)\,((\omega-2)^2-2\zeta)}.$$

We are interested in the behaviour of the iterates $(T_1^{r-k}\alpha_s^*(w))(w_1)$ as $r \to \infty$, when $s = e^{-\lambda/r^4}$ for some $\lambda > 0$ and $k \in \mathbb{Z}$ is a fixed integer. We prefer to work for the time

being with parameter q as opposed to (λ, r) since the hull operator and its iterates are naturally expressed in terms of q. We will write $\alpha_s^*(z) = \beta_s^*(w)$ for the map in the w-coordinate system, obtained upon substitution of $z = z(w)$. Using the explicit formula for the iterates (4.42) we have (where we have suppressed the dependence of s on q in the notation):

$$\begin{aligned}
x_n(q) &= (T_1^n \alpha_s^*(w))(w_1) = m_0(q) \frac{q^{2n}(1-q^2)^3(1-q^{4n+14})}{(1-q^{2n+6})^2(1-q^{2n+8})^2} \beta_s^* \left(\frac{q(1-q^{2n+6})}{1-q^{2n+8}} \right) \\
y_n(q) &= (T_1^n \alpha_s^*(w))'(w_1) = m_1(q) \frac{q^{2n-1}(1-q^2)^3(1-q^{4n+14})}{(1-q^{2n+6})^2(1-q^{2n+8})^2} \beta_s^* \left(\frac{q(1-q^{2n+6})}{1-q^{2n+8}} \right) \\
&\quad - 2m_0(q) \frac{q^{4n-1}(1-q^2)^4(1+q^2-3q^{2n+8}+q^{6n+22})}{(1-q^{2n+6})^3(1-q^{2n+8})^3} \beta_s^* \left(\frac{q(1-q^{2n+6})}{1-q^{2n+8}} \right) \\
&\quad + (1+q^2+q^4+q^6)^2 m_0(q) \frac{q^{4n}(1-q^2)^5(1-q^{4n+14})}{(1-q^{2n+6})^2(1-q^{2n+8})^4} \beta_{s,w}^* \left(\frac{q(1-q^{2n+6})}{1-q^{2n+8}} \right)
\end{aligned}$$

where $\beta_{s,w}^*$ denote the partial derivative of β_s^* with respect to w and the multipliers are certain continuous bounded functions on $[0,1]$ given as

$$\begin{aligned}
m_0(q) &= \frac{(1+q^2+q^4)^2(1+q^2+q^4+q^6)^2}{(1+q^2+q^4+q^6+q^8+q^{10}+q^{12})} \\
m_1(q) &= \frac{2m_0(q)^2(1+3q^2+5q^4+7q^6+6q^8+5q^{10}+4q^{12}+3q^{14}+2q^{16}+q^{18})}{(1+q^2+q^4+q^6)(1+q^2+q^4)^3}
\end{aligned}$$

We should point out that neither sequence $x_n(q)$ or $y_n(q)$ is uniformly convergent on $q \in [0,1]$ and it is exactly the failure of uniform convergence that will be exploited later to establish property (x_1) (note that for fixed each $q \in [0,1]$, the pointwise limit of both sequences is 0). However, both sequences are nevertheless q-uniformly bounded as we will prove next. This is not *a priori* clear since in the limit $(n,q) \to (\infty, 1)$ the variables (s,z) tend to $(1,t_0)$ and the function $\alpha^*(s,z)$ has a singularity at $(1,t_0)$.

Lemma 4.4.8 *Let $q \in [0,1]$, then the coefficient sequences $\{x_n(q)\}_{n\geq 0}$ and $\{y_n(q)\}_{n\geq 0}$ are q-uniformly bounded.*

Proof. Recall that the maps $\alpha_s^*(z)$ and $\beta_s^*(w)$ are related via the transformation $z = z(w)$, so in particular,

$$\begin{aligned}
\beta_s^* \left(\frac{q(1-q^{2n+6})}{1-q^{2n+8}} \right) &= \alpha_s^* \left(\frac{q^2}{1+q^4} \frac{(1-q^{2n+2})(1-q^{2n+12})}{(1-q^{2n+6})(1-q^{2n+8})} \right) \\
\beta_{s,w}^* \left(\frac{q(1-q^{2n+6})}{1-q^{2n+8}} \right) &= m_2 \frac{(1-q^2)(1-q^{4n+14})}{q^3(1-q^{2n+6})^2} \alpha_{s,z}^* \left(\frac{q^2}{1+q^4} \frac{(1-q^{2n+2})(1-q^{2n+12})}{(1-q^{2n+6})(1-q^{2n+8})} \right) \\
m_2(q) &= \frac{(1+q^2)(1+q^2+q^4)}{(1+q^4)}
\end{aligned}$$

Recall that in lemma (3.4.5) we have shown that on $(s,z) \in \overline{\mathbb{D}} \times t_0\overline{\mathbb{D}}$ the map α^* has a unique dominant singularity at the point $(s,t) = (1,t_0)$. From the form of the map above, we conclude that $f(s,z) = (1-2\zeta)^{3/2}\alpha^*(s,z)$ and $g(s,z) = (1-2\zeta)^{5/2}\alpha^*_z(s,z)$ are continuous on the closed bidisk $\overline{\mathbb{D}} \times t_0\overline{\mathbb{D}}$. Hence there exists some $M > 0$ such that $|f(s,z)| \leq M$ and $|g(s,z)| \leq M$ for each $(s,z) \in \overline{\mathbb{D}} \times t_0\overline{\mathbb{D}}$. Note that for $q \in [0,1]$ and $n \geq 0$, the point $(s(q), z_n(q))$ where

$$s(q) = \frac{4q^2(1-q^2+q^4)}{(1+q^4)^2} \text{ and } z_n(q) = \frac{q^2}{1+q^4}\frac{(1-q^{2n+2})(1-q^{2n+12})}{(1-q^{2n+6})(1-q^{2n+8})}$$

belongs to the closed bidisk. Similarly, multipliers $m_0(q)$, $m_1(q)$ and $m_2(q)$ are continuous on $[0,1]$ so there exists some $m > 0$ such that $|m_i(q)| \leq m$ on $[0,1]$ for all i. Suppose $n \geq 1$ (since the point $(s(q), z_n(q))$ approaches the boundary of the bidisk only in the limit $n \to \infty$ and the map α^* is continuous on the interior of the bidisk, the initial coefficients are all continuous and bounded on $[0,1]$, so it suffices to prove some tail of the sequence is q-uniformly bounded), then the coefficients $|x_n(q)|$ and $|y_n(q)|$ are bounded above by sums of terms of the form

$$\begin{aligned}
\text{term}_1 &= mM\frac{q^{2n-1}(1-q^{2n+8})(1-q^{4n+14})}{(1-q^{2n+6})^2}\left\{\frac{(1-q^2)^2}{(1-q^{2n+8})^2(1-2\zeta_n)}\right\}^{3/2} \\
\text{term}_2 &= 2mM\frac{q^{4n-1}(1-q^2)(1+q^2-3q^{2n+8}+q^{6n+22})}{(1-q^{2n+6})^3}\left\{\frac{(1-q^2)^2}{(1-q^{2n+8})^2(1-2\zeta_n)}\right\}^{3/2} \\
\text{term}_3 &= mM\frac{q^{4n-3}(1-q^2)(1-q^{2n+8})(1-q^{4n+14})^2}{(1-q^{2n+6})^4}\left\{\frac{(1-q^2)^2}{(1-q^{2n+8})^2(1-2\zeta_n)}\right\}^{5/2}.
\end{aligned}$$

Since we assumed $n \geq 1$, the initial factor of each term is well-defined and continuous on $[0,1]$. Furthermore, the initial factors of the first and last term are clearly q-uniformly convergent balanced quotients. On the other hand, the initial factor of the second term can be re-written as

$$\frac{(1-q^2)(1+q^2-3q^{2n+8}+q^{6n+22})}{(1-q^{2n+6})^3} = \frac{(1-q)^2(1-q^{2n+8})^2}{(1-q^{2n+6})^3} + \frac{q^2(1-q^2)(1-q^{2n+14})}{(1-q^{2n+6})^2},$$

which again clearly demonstrates it is q-uniformly convergent.

It remains to consider the parenthesized expression. It is easily shown that the map $\omega = \omega(q)$ is monotonically increasing on $[0,1]$, so the final factor is bounded above by its value at $\omega(1) = 7$, viz.

$$\frac{(1-q^2)^2}{(1-q^{2n+8})^2(1-2\zeta_n)} \leq \frac{(1+q^4)(1-q^{2n+6})}{(1-q^{2n+8})(1+q^4(1+q^2)(2+q^2+2q^4)q^{2n}+q^{4n+14})}.$$

However, the right-hand side is again clearly q-uniformly bounded. □

The limit relation (x_1) is now easily verified and is the content of our next lemma:

Lemma 4.4.9 (Diagonal limits) *Let $k \in \mathbb{Z}$ be a fixed integer, then*

$$\lim_{n\to\infty} x_{n-k}(q_n) = \frac{3}{4}x_0^{-1}c_0^{-1}\cdot\frac{36}{7}G_1\left(x_0\lambda\right) \tag{4.67}$$

$$\lim_{n\to\infty} y_{n-k}(q_n) = \frac{3}{4}x_0^{-1}c_0^{-1}\cdot\frac{3552}{49}G_1\left(x_0\lambda\right), \tag{4.68}$$

where it is assumed n is taken sufficiently large, so that both coefficients are defined.

Proof. We will use the representation of the coefficients $x_n(q)$ and $y_n(q)$ established in the proof of lemma (4.4.8). Clearly, $\lim_n q_n = 1$ and $\lim_n s(q_n) = 1$. Since the functions $f(s,z)$ and $g(s,z)$ are continuous on the compact set $(s,z) \in [0,1]\times[0,t_0]$, they are uniformly continuous. Hence each sequence $f(s(q), z_n(q))$ and $g(s(q), z_n(q))$ is q-uniformly convergent, which immediately implies that

$$\lim_{n\to\infty} f(s(q_n), z_n(q_n)) = \lim_{(s,z)\to(1,t_0)} f(s,z) = \frac{2\sqrt{6}}{81}$$

$$\lim_{n\to\infty} g(s(q_n), z_n(q_n)) = \lim_{(s,z)\to(1,t_0)} g(s,z) = \frac{2\sqrt{6}}{27}.$$

Similarly, we deduce for the multipliers the limit $\lim_n m_i(q_n) = m_i(1)$. Using the binomial theorem (or `Mathematica`) we can easily prove that $\lim_n q_n^{n-k} = \exp\{-(x_0\lambda)^{1/4}\}$ (note that k does not affect the value, since it is a fixed constant), which allows us to evaluate the limit of the initial factor of each term$_i$, by repeatedly replacing q-uniformly convergent factors with their values at $q = 1$. We obtain (note that second and third term tend to zero as $n \to \infty$):

$$\lim_n x_{n-k}(q_n) = \frac{64\sqrt{6}}{63}e^{-3(x_0\lambda)^{1/4}}\cosh(x_0\lambda)^{1/4}\lim_n\left\{\frac{(1-q_n^2)^2}{(1-q_n^{2n+8})^2(1-2\zeta_n)}\right\}^{3/2}$$

$$\lim_n y_{n-k}(q_n) = \frac{18944\sqrt{6}}{1323}e^{-3(x_0\lambda)^{1/4}}\cosh(x_0\lambda)^{1/4}\lim_n\left\{\frac{(1-q_n^2)^2}{(1-q_n^{2n+8})^2(1-2\zeta_n)}\right\}^{3/2}.$$

Finally, to evaluate the limit of the last factor we expand the parameters q_n and ζ_n in ascending powers of n^{-1}, which yields

$$\frac{(1-q_n^2)^2}{(1-q_n^{2n+8})^2} = \frac{\sqrt{\lambda}e^{2(x_0\lambda)^{1/4}}}{2\sinh^2(x_0\lambda)^{1/4}}n^{-2} + \mathcal{O}(n^{-3})$$

$$1-2\zeta_n = \frac{\sqrt{\lambda}(2+\cosh^2(x_0\lambda)^{1/4})}{\sinh^2(x_0\lambda)^{1/4}}n^{-2} + \mathcal{O}(n^{-3})$$

Plugging in the expansions and evaluating the limit gives after simplifications exactly the right-hand side of (4.67) and (4.68). □

A combination of the preceding results and an application of lemma (4.4.7) yields

Theorem 4.4.5 (Limiting distribution of the volume of hulls)**.** *Let $\lambda > 0$, then*

$$\lim_{r\to\infty} \mathbb{E}\left(e^{-\lambda \operatorname{Vol}(B_r^\bullet)/r^4}\right) = 3^{3/2}\frac{\cosh(2^{-1/2}\lambda^{1/4})}{(\cosh^2(2^{-1/2}\lambda^{1/4})+2)^{3/2}}. \tag{4.69}$$

Proof. We have already verified conditions (x_1) and (x_2) for coefficient sequences of both resolvents $R_1(z,w_0)\alpha^*$ and $R_1'(z,w_0)\alpha^*$. It remains to check conditions (a_1) and (a_2) for the coefficient sequence of $\vartheta_i(z,w_1)$. Recall that

$$\vartheta_0(z,w_1) = \frac{1-z\Psi_q'(z)}{\Delta_q(z)} \text{ and } \vartheta_q(z,w_1) = \frac{z\Psi_q(z)}{\Delta_q(z)},$$

where the perturbation determinant $\Delta_q(z)$ is

$$\Delta_q(z) = (1-z\Phi_q(z))(1-z\Psi_1'(z)) - z^2\Phi_q'(z)\Psi_q(z).$$

According to theorems (4.3.6) and (4.3.8) the coefficient sequences of all the resolvents $\Phi_q(z), \Psi_q(z), \Phi_q'(z)$ and $\Psi_q'(z)$ consist of q-continuous functions that are $\mathcal{O}((1-q^2)^3 \operatorname{Li}_{-2}(q^{2k}))$ q-uniformly. It follows from lemma (A.1.3) that each belongs to the Banach algebra $\mathfrak{R}_+(\mathcal{O}(k^{-3})$. Hence according to lemma A.2.1 the perturbation determinant $\Delta_q(z)$ also belongs to $\mathfrak{R}_+(\mathcal{O}(k^{-3})$ and its coefficients belong to $\mathcal{C}([0,1])$.

We have shown in theorem (4.3.9) that the perturbation determinant has no zeros in the closed unit disk $\bar{\mathbb{D}}$ for every $q \in [0,1]$. Since the terms of $\Delta_q(z)$ in a series expansion about the origin are continuous functions of (z,q) and the expansion is absolutely and uniformly convergent on $\overline{\mathbb{D}}\times[0,1]$, the determinant is jointly continuous in both variables. Hence it is q-uniformly bounded away from zero. Using theorem (A.2.3) it follows that $1/\Delta_q(z) \in \mathfrak{R}_+(\mathcal{O}(k^{-3})$. A final application of theorem (A.2.1) then gives

$$\vartheta_i(z,w_1) \in \mathfrak{R}_+(\mathcal{O}(k^{-3})).$$

Hence condition (a_2) is true since $\sum_k k^{-3}$ is clearly convergent. Moreover, the coefficients of $\vartheta_i(z,w_1)$ are still continuous on $[0,1]$, which implies the limit of its coefficients as $q \to 1$ (or equivalently $n \to \infty$) exists. It follows that all the conditions of lemma (4.4.7) are satisfied, so

$$\lim_{r\to\infty} \mathbb{E}\left(e^{-\lambda \operatorname{Vol}(B_r^\bullet)/r^4}\right) = x_0c_0(\vartheta_0(1,w_1)\lim_{n\to\infty} x_n(q) + \vartheta_1(1,w_1)\lim_{n\to\infty} y_n(q)) = G_1(x_0\lambda).$$

This completes the proof of the theorem. □

We can similarly investigate the scaling limit of the volume of hulls conditionally on their perimeter. Let $r > 1$, then using the hull boundary operator we can again write the probability generating function as

$$\begin{aligned}
\mathbb{E}\left(s^{\mathrm{Vol}\, B_r^\bullet}\middle|\, |\partial B_r^\bullet|_1 = \ell\right) &= \sum_n s^n \mathbb{P}\left(\mathrm{Vol}(B_r^\bullet) = n \middle|\, |\partial B_r^\bullet|_1 = \ell\right) \\
&= \sum_n s^n \frac{\mathbb{P}\left(\mathrm{Vol}(B_r^\bullet) = n, |\partial B_r^\bullet|_1 = \ell\right)}{\mathbb{P}\left(|\partial B_r^\bullet|_1 = \ell\right)} \\
&= \frac{c_0}{\mathbb{P}\left(|\partial B_r^\bullet|_1 = \ell\right)} \sum_{\bar{B}_r} \sum_{\substack{|\underline{\lambda}|_0 = p \\ |\underline{\lambda}|_1 = \ell}} c(p,\ell) x_0^{n(\bar{B}_r)+p} s^{n(\bar{B}_r)+\ell+p} \\
&= \frac{x_0 c_0 s^{\ell+1} (\boldsymbol{A}^{r-2}(sx_0)\alpha_\ell^*(z))(sx_0)}{x_0 c_0 (\boldsymbol{A}^{r-2}(x_0)\alpha_\ell^*(z))(x_0)}
\end{aligned}$$

where $\alpha_\ell^*(z) = [t^\ell]\alpha^*(t,z)$ (note that α_ℓ^* is a polynomial in z of degree $\ell - 1$ and of order $\mathcal{O}(z^2)$ as follows easily from its combinatorial interpretation). We will need a few preliminary results before we can use again lemma (4.4.7) to calculate the limiting distribution.

We need to determine first the asymptotic expansion of the coefficients $\alpha_\ell^*(z)$ as $\ell \to +\infty$. Notice that upon application of the weighted composition operator T_1^n and taking the limit $n \to +\infty$, the argument of the map will tend uniformly to $t_0 = 1/2$. Hence to be able to describe accurately the diagonal limits, the asymptotic estimate of α_ℓ^* needs to be valid uniformly in z in some neighbourhood of the critical point t_0. Although singularity analysis (based on the transfer results of Flajolet and Odlyzko) is traditionally applicable only to univariate analytic functions, when applied to $\alpha^*(t,z)$ with respect to the variable t it will nevertheless return a uniform asymptotic estimate due to the particular nature of the singularity structure of the map. The following lemma gives the precise result

Lemma 4.4.10 *The following asymptotic expansion holds for coefficients $\alpha_n^*(z) = [t^n]\alpha^*(t,z)$ as $n \to +\infty$ uniformly for $z \in \delta\mathbb{D}(t_0)$ for some sufficiently small $\delta > 0$,*

$$\alpha_n^*(z) \sim \frac{\sqrt{1+z}}{9\sqrt{3/2}} \left(\frac{8+5z}{8+9z+(6+5z)\sqrt{z(8+9z)}}\right)^{3/2} \frac{\rho(z)^{-n} n^{1/2}}{\Gamma(3/2)} \tag{4.70}$$

$$\alpha_{n,z}^*(z) \sim \frac{\sqrt{1+z}}{\sqrt{6}} \left(\frac{8+5z}{8+9z+(6+5z)\sqrt{z(8+9z)}}\right)^{5/2} \rho(z)^{-n} \frac{n^{3/2}}{\Gamma(5/2)} \tag{4.71}$$

where $\rho(z)$ is the dominant singularity of $\alpha^*(t)$ for $z \in (t_0 - \delta, t_0 + \delta)$ given as

$$\rho(z) = \frac{2 - 4z - 9z^2 + 3z\sqrt{z(9z+8)}}{1+z}. \tag{4.72}$$

Proof. We can write the map $\alpha^*(t,z)$ in the form

$$\alpha^*(t,z) = A(t,z) + B(t,z)(1-2\zeta)^{-3/2}.$$

The initial term $A(t,z)$ is a linear polynomial in z whose coefficients are rational functions of $\omega = \omega(t)$, so is analytic for $(t,z) \in \overline{\mathbb{D}} \times \mathbb{C}$ (moreover, since the map α^* is of order $\mathcal{O}(z^2)$ it does not contribute at all to the coefficients α_n^*). The multiplier $B(t,z)$ is of the form

$$B(t,z) = \frac{(\omega-5)(\omega+1)(a_0\zeta^3 - a_1\zeta^2 - a_2\zeta + a_3)}{2^4 3^4 \omega(\omega-1)^2(\omega-2)^2(\omega-3)^2\sqrt{((\omega-2)^2-2\zeta)}}.$$

It is easily shown that $|\omega| \geq \sqrt{13}$ for $t \in 2\overline{\mathbb{D}}$, so the only other source of singularities of the multiplier $B(t,z)$, apart from the minimal singularity at 2 of the square-root $\omega(t)$, is the square-root appearing in the denominator. However, as has been shown in lemma (3.4.5) the minimal root of $W(t,z)$ for $t \in b\overline{\mathbb{D}}$ occurs uniquely at the conjugate root $c = c(b)$ for $b \in (0,2)$ (and given explicitly in the paragraph following the statement of the lemma). We conclude that the multiplier is likewise analytic on $\overline{\mathbb{D}} \times r\overline{\mathbb{D}}$ for some sufficiently small $r \in (t_0, 2)$ and $B(1,t_0) = 2\sqrt{6}/81$. Notice that inverting the equation for $c = c(t)$ gives the unique root $\rho = \rho(z)$ of $1 - 2\zeta = 0$ considered as an equation in t; it is a matter of simple algebraic calculations to show that it is given exactly by the equation (4.72) above. We also note for future reference the following is always true (provided all the square-roots are chosen on the principal branch)

$$\omega(\rho(z)) = 2 + \sqrt{\frac{8+9z}{z}} \text{ and } B(\rho(z),z) = \frac{\sqrt{2(1+z)}}{81z^{3/2}}.$$

Finally, it is clear from the combinatorial interpretation of the map $\alpha^*(t,z)$ that it admits a positive bivariate power series expansion in (t,z) about the origin. We can now deduce an asymptotic expansion using the transfer theorem. Indeed, as already remarked the initial term $A(t,z)$ can be disregarded. On the other hand, the multiplier $B(t,z)$ and $C(t,z) = 1-2\zeta$ can both be expanded in a power series about the dominant singularity $t = \rho(z)$, which will be absolutely and uniformly convergent in a neighbourhood of $\rho(z)$ for all $z \in \delta\mathbb{D}(t_0)$ for sufficiently small $\delta > 0$. We find

$$\omega(t) = \omega(\rho) + \frac{(2+3z+\sqrt{z(8+9z)})^2(\sqrt{z(8+9z)}-2z)}{6z^2(8+9z)}(t-\rho) + \mathcal{O}(t-\rho)^2$$

$$B(t,z) = \frac{\sqrt{2(1+z)}}{81z^{3/2}} + \frac{\sqrt{(1+z)(8+9z)}(24+113z)}{2592\sqrt{2}z(1+z)}(\omega - \omega(\rho)) + \mathcal{O}\left(\omega - \omega(\rho)\right)^2$$

$$C(t,z) = -\frac{1}{4}\sqrt{z(8+9z)}(\omega - \omega(\rho)) + \mathcal{O}\left(\omega - \omega(\rho)\right)^2.$$

Furthermore, $\rho(z)^{-3/2}$ and $(-C_t(\rho(z),z))^{-3/2}$ are both well-defined by their principal values since their specializations at $z = t_0$ are positive. Consequently, we can write

$$\alpha^*(t,z) = B(\rho(z),z)(-C_t(\rho(z),z)\rho(z))^{-3/2}\left(1-\frac{t}{\rho(z)}\right)^{-3/2}\left\{1+\mathcal{O}\left(1-\frac{t}{\rho(z)}\right)\right\}$$

where the $\mathcal{O}$-term is uniform in $z \in \delta\mathbb{D}(t_0)$ for some $\delta > 0$. We can now apply the transfer theorem upon noting that the uniform $\mathcal{O}$-estimate can be uniformly bounded of order $\mathcal{O}(n^{-1})$ (see the Uniformity lemma IX.2 in [25] for the precise statement and proof). This completes the proof of the asymptotic estimate (4.70). The proof of (4.71) is completely analogous since we can again write

$$\alpha^*_{n,z} = A_z(t,z) + \left\{B_z(t,z)(1-2\zeta) + \frac{3(\omega+1)(\omega-5)}{16}B(t,z)\right\}(1-2\zeta)^{-5/2}. \quad \square$$

We can now determine all the required diagonal limits. Before stating the result we introduce the following notational abbreviation

$$E_c(\lambda) = \lambda^{3/4}\frac{\cosh\lambda^{1/4}}{\sinh^3\lambda^{1/4}}\exp\left(-c\ell\left(\lambda^{1/2}\left(\coth^2\lambda^{1/4}-\frac{2}{3}\right)-1\right)\right).$$

The diagonal limits can then be expressed as:

Lemma 4.4.11 *Let $k \in \mathbb{Z}$ be a fixed integer, then*

$$\lim_{n\to\infty} x_{n-k}(q_n) = \frac{3}{4}x_0^{-1}c_0^{-1}\cdot\frac{36}{7}E_{14/5}\left(x_0\lambda\right) \tag{4.73}$$

$$\lim_{n\to\infty} y_{n-k}(q_n) = \frac{3}{4}x_0^{-1}c_0^{-1}\cdot\frac{3552}{49}E_{14/5}\left(x_0\lambda\right) \tag{4.74}$$

where it is assumed n is taken sufficiently large, so that all the coefficients are defined. Furthermore, all the coefficients are q-uniformly bounded for all $n \geq 0$.

Proof. We again use the representation

$$\tilde{x}_n(q) = m_0(q)\frac{q^{2n}(1-q^2)^3(1-q^{4n+14})}{(1-q^{2n+6})^2(1-q^{2n+8})^2}\alpha^*_{[\ell n^2]}\left(\frac{q^2}{1+q^4}\frac{(1-q^{2n+2})(1-q^{2n+12})}{(1-q^{2n+6})(1-q^{2n+8})}\right)$$

$$\begin{aligned}\tilde{y}_n(q) = {} & m_1(q)\frac{q^{2n-1}(1-q^2)^3(1-q^{4n+14})}{(1-q^{2n+6})^2(1-q^{2n+8})^2}\alpha^*_{[\ell n^2]}\left(\frac{q^2}{1+q^4}\frac{(1-q^{2n+2})(1-q^{2n+12})}{(1-q^{2n+6})(1-q^{2n+8})}\right)\\ & - 2m_0(q)\frac{q^{4n-1}(1-q^2)^4(1+q^2-3q^{2n+8}+q^{6n+22})}{(1-q^{2n+6})^3(1-q^{2n+8})^3}\times\\ & \qquad \alpha^*_{[\ell n^2]}\left(\frac{q^2}{1+q^4}\frac{(1-q^{2n+2})(1-q^{2n+12})}{(1-q^{2n+6})(1-q^{2n+8})}\right)\end{aligned}$$

$$+\tilde{m}_2(q)\frac{q^{4n-3}(1-q^2)^6(1-q^{4n+14})^2}{(1-q^{2n+6})^4(1-q^{2n+8})^4}\alpha^*_{[\ell n^2],z}\left(\frac{q^2}{1+q^4}\frac{(1-q^{2n+2})(1-q^{2n+12})}{(1-q^{2n+6})(1-q^{2n+8})}\right)$$

where $\tilde{m}_2(q) = (1+q^2+q^4+q^6)^2 m_0(q)m_2(q)$ (in the notation introduced). Recall that we need to set $q = q_n$ and that the following limits hold (for any fixed $k \in \mathbb{Z}$), $\lim_n q_n = 1$ and $\lim_n q_n^{n-k} = \exp\{-(x_0\lambda)^{1/4}\}$. We are allowed to replace every occurrence of $\alpha^*_{[\ell n^2]}$ and $\alpha^*_{[\ell n^2],z}$ with their asymptotically dominant terms (in the limits as $n \to +\infty$) since the expansion are uniform around the critical point t_0 and the expression within the parenthesis converges (q-uniformly according to lemma (A.1.4)) to $1/2$ as $n \to \infty$. Moreover, taking first the limit as $q \to 1^-$, the resulting formulas also apply to the study of the asymptotics of $\mathbb{P}(|\partial B_n^\bullet| = [\ell n^2])$ as $n \to \infty$.

We will consider first the limits of the α^* factors (both in the bivariate and boundary case). Since the initial factors of (4.70) and (4.71) are all (uniformly) continuous on a neighbourhood of $1/2$, the limit are simply their values at $1/2$. On the other hand, since $\rho(1/2) = 1$ we need to determine the next asymptotic term of $\rho(q_n)$ as $n \to +\infty$ to calculate the limit. We find without difficulty (using say `Mathematica`)

$$\rho(q_n) = 1 + \frac{14}{5n^2}(x_0\lambda)^{1/2}\left(\coth^2(x_0\lambda)^{1/4} - \frac{2}{3}\right) + \mathcal{O}(n^{-3}) \text{ and } \rho_n = 1 + \frac{14}{5n^2} + \mathcal{O}(n^{-3}).$$

It follows from the exponential formula that the limits $\lim_n \rho(q_n)^{[\ell n^2]}$ and $\lim_n \rho_n^{[\ell n^2]}$ exist and are finite. Accordingly, in both cases the limit is of (exact) order n^2. Similarly, the derivatives are of exact order n^3. Next, the limits of factors $m_i(q)$ are by uniform continuity simply their values at 1, while the terms term_i satisfy

$$\text{term}_1(q_n) = \frac{2}{n^3}(x_0\lambda)^{3/4}\frac{\cosh(x_0\lambda)^{1/4}}{\sinh^3(x_0\lambda)^{1/4}} + \mathcal{O}(n^{-4})$$
$$\text{term}_2(q_n) = \mathcal{O}(n^{-4}) \text{ and } \text{term}_3(q_n) = \mathcal{O}(n^{-6}).$$

Moreover, the same expansion holds in the limit $\lambda \to 0^+$ (which corresponds to the boundary case $q = 1$). Therefore, asymptotically the probability $\mathbb{P}(|\partial B_n^\bullet|)$ satisfies[4]

$$\mathbb{P}(|\partial B_n^\bullet| = [\ell n^2]) \sim \left(\frac{14}{5}\right)^{3/2}\frac{e^{-14\ell/5}\ell^{1/2}}{\Gamma(3/2)}n^{-2}$$

upon using lemma (4.4.7) (more precisely, we apply the lemma to the sequence rescaled by a factor n^2 to ensure uniform boundedness).

[4]The result can be used to given a second independent proof of theorem (4.4.3); let $x > 0$ then

$$\mathbb{P}\left(\frac{14}{5n^2}|\gamma_n| \in [0,x]\right) = \sum_{\ell=0}^{[5xn^2/14]}\left(\frac{14}{5}\right)^{3/2}\frac{e^{-14\ell/5n^2}(\ell/n^2)^{1/2}}{n^2\Gamma(3/2)} \xrightarrow{n\to\infty} \left(\frac{14}{5}\right)^{3/2}\int_0^{5x/14}\frac{\sqrt{x}e^{-14x/5}}{\Gamma(3/2)}dx$$

which after a simple change variables becomes exactly the cumulative distribution of the $\Gamma(3/2)$-law.

Putting all the parts together we obtain exactly the diagonal limits (4.73) and (4.74). Finally, it can easily be shown both sequences are q-uniformly bounded, where we may replace the probability $\mathbb{P}(|\partial B_n^\bullet| = [\ell n^2])$ with the uniform estimate found above, using similar ideas as in the proof of lemma (4.4.8) and repeated use of results in appendix A. □

The following result is now immediate.

Theorem 4.4.6 (Limiting conditional distribution of the volume of hulls)**.** *Let* $\lambda, \ell > 0$*, then*

$$\lim_{r\to\infty} \mathbb{E}\left(e^{-\lambda \operatorname{Vol}(B_r^\bullet)/r^4} \middle| |\partial B_r^\bullet|_1 = [\ell r^2]\right) = (x_0\lambda)^{3/4} \frac{\cosh(x_0\lambda)^{1/4}}{\sinh^3(x_0\lambda)^{1/4}} \exp\left(-\frac{14}{5}\ell\left((x_0\lambda)^{1/2}\left(\coth^2(x_0\lambda)^{1/4} - \frac{2}{3}\right) - 1\right)\right).$$

Finally, we can also investigate the limiting probability of volumes of hull conditioned on the number of parts of the perimeter. Expressed in terms of the hull operator we have

$$\begin{aligned}
\mathbb{E}\left(s^{\operatorname{Vol} B_r^\bullet} \middle| |\partial B_r^\bullet|_0 = p\right) &= \sum_n s^n \mathbb{P}\left(\operatorname{Vol}(B_r^\bullet) = n \middle| |\partial B_r^\bullet|_0 = p\right) \\
&= \sum_n s^n \frac{\mathbb{P}\left(\operatorname{Vol}(B_r^\bullet) = n, |\partial B_r^\bullet|_0 = p\right)}{\mathbb{P}\left(|\partial B_r^\bullet|_0 = p\right)} \\
&= \frac{c_0}{\mathbb{P}\left(|\partial B_r^\bullet|_0 = p\right)} \sum_{\bar{B}_r} \sum_{|\underline{\lambda}|_1 = \ell} c(p,\ell) x_0^{n(\bar{B}_r)+p} s^{n(\bar{B}_r)+\ell+p} \\
&= \frac{x_0 c_0 s \alpha_p^*(s) (\boldsymbol{A}^{r-2}(sx_0) z^{p-1})(sx_0)}{x_0 c_0 \alpha_p^*(1) (\boldsymbol{A}^{r-2}(x_0) z^{p-1})(x_0)}
\end{aligned}$$

where $\alpha_p^*(t) = [z^{p-1}]\alpha^*(t,z)$. We again need to derive first uniform asymptotic estimates for the coefficients of $\alpha^*(t,z)$ with respect to the variable z in a manner similar to the proof of lemma (4.4.10). The following lemma summarizes our results:

Lemma 4.4.12 *The following asymptotic expansion holds for coefficients* $\alpha_n^*(t) = [z^{n-1}]\alpha^*(t,z)$ *as* $n \to +\infty$ *uniformly for* $t \in \delta\mathbb{D}(1)$ *for some sufficiently small* $\delta > 0$*,*

$$\alpha_n^*(t) \sim \frac{(\omega-5)(\omega-3)(\omega^2-1)}{1296\sqrt{(\omega-5)(\omega+1)}} \frac{\varrho(t)^{1-n} n^{1/2}}{\Gamma(3/2)} \tag{4.75}$$

where $\varrho(t)$ *is the dominant singularity of* $\alpha^*(z)$ *for* $t \in (1-\delta, 1+\delta)$ *given as*

$$\varrho(t) = \frac{2-t}{36t}\left(10 + t + \sqrt{(2-t)(50-t)}\right). \tag{4.76}$$

Proof. We will only give a brief sketch of the proof (in particular, we leave to the reader the simple task of verifying all the technical constraints). Recall that the map $\alpha^*(t,z)$ can be written in the form

$$\alpha^*(t,z) = A(t,z) + B(t,z)(1-2\zeta)^{-3/2}.$$

The initial term $A(t,z)$ is a linear polynomial in z and both $A(t,z)$ and $B(t,z)$ are analytic on $r\overline{\mathbb{D}} \times t_0\overline{\mathbb{D}}$ for some sufficiently small $r \in (1,2)$ with $B(1,t_0) \neq 0$. The unique singularity (of minimal modulus) of $1-2\zeta = 1 - z/\varrho(t) = 0$ (with respect to the variable z) has been determined in lemma (3.4.5) and is given exactly by (4.76). As before we can now expand the multiplier $B(t,z)$ in a power series about the dominant singularity $z = \varrho(t)$, which is absolutely and uniformly convergent in a neighbourhood of $\varrho(t)$ for all $t \in \delta\mathbb{D}$ for a sufficiently small $\delta > 0$. We find

$$B(t,z) = \frac{(\omega-5)(\omega-3)(\omega^2-1)}{1296\sqrt{(\omega-5)(\omega+1)}} + \frac{97(\omega-5)^2(\omega+1)^2}{20736\sqrt{(\omega-5)(\omega+1)}}(z-\varrho(t)) + \mathcal{O}\left(z-\varrho(t)\right)^2.$$

We can therefore write

$$\alpha^*(t,z) = B(t,\varrho(t))\left(1-\frac{z}{\varrho(t)}\right)^{-3/2}\left\{1+\mathcal{O}\left(1-\frac{z}{\varrho(t)}\right)\right\}$$

where the $\mathcal{O}$-term is uniform in $t \in \delta\mathbb{D}$ for some $\delta > 0$. The rest of the proof is now a simple application of the (uniform) transfer theorem. □

Lemma 4.4.13 *Let $k \in \mathbb{Z}$ be a fixed integer, then*

$$\lim_{n\to\infty} x_{n-k}(q_n) = \frac{3}{4}x_0^{-1}c_0^{-1}\cdot\frac{36}{7}E_6\left(x_0\lambda\right) \tag{4.77}$$

$$\lim_{n\to\infty} y_{n-k}(q_n) = \frac{3}{4}x_0^{-1}c_0^{-1}\cdot\frac{3552}{49}E_6\left(x_0\lambda\right) \tag{4.78}$$

where it is assumed n is taken sufficiently large, so that all the coefficients are defined. Furthermore, all the coefficients are q-uniformly bounded for all $n \geq 0$.

Proof. We will only provide a sketch of the proof since we are merely recycling the same ideas that have already been used in the proofs of lemmas (4.4.8), (4.4.9) and (4.4.11). Upon substitution $s_n = e^{-\lambda/n^4}$ and $p \to [pn^2]$, we deduce using the uniform estimate (4.75) that the initial factor $\alpha^*_{[pn^2]}(s_n)$ is asymptotic to

$$\alpha^*_{[pn^2]}(s_n) \sim \alpha^*_{[pn^2]}(1) \sim \frac{\sqrt{6}}{81}\frac{2^{[pn^2]}p^{1/2}n}{\Gamma(3/2)} \text{ as } n \to \infty.$$

On the other hand, the asymptotics of the coefficients $T_1^{n-k} z(w)^{p-1}$ of the resolvent are easily estimated using the explicit formula for the iterates. We find without difficulties the following asymptotic estimates (and similarly for the boundary case)

$$\tilde{x}_{n-k}(q_n) \sim 486 e^{-6p} n^{-3} 2^{-[pn^2]} \frac{3}{4} x_0^{-1} c_0^{-1} \cdot \frac{36}{7} E_6(x_0 \lambda)$$
$$\tilde{y}_{n-k}(q_n) \sim 486 e^{-6p} n^{-3} 2^{-[pn^2]} \frac{3}{4} x_0^{-1} c_0^{-1} \cdot \frac{3552}{49} E_6(x_0 \lambda).$$

Combining the asymptotic estimates gives the result. □

The limiting conditional distribution again follows immediately

Theorem 4.4.7 (Limiting conditional distribution of the volume of hulls)**.** *Let* $\lambda, p > 0$*, then*

$$\lim_{r\to\infty} \mathbb{E}\left(e^{-\lambda \operatorname{Vol}(B_r^\bullet)/r^4} \middle| |\partial B_r^\bullet|_0 = [pr^2]\right)$$
$$= (x_0\lambda)^{3/4} \frac{\cosh(x_0\lambda)^{1/4}}{\sinh^3(x_0\lambda)^{1/4}} \exp\left(-6p\left((x_0\lambda)^{1/2}\left(\coth^2(x_0\lambda)^{1/4} - \frac{2}{3}\right) - 1\right)\right).$$

Moreover, we believe the techniques developed so far could also be used to evaluate the scaling limit $(r^{-2}|\partial B_{[xr]}^\bullet|_1, r^{-4} \operatorname{Vol} B_{[xr]}^\bullet)$ of the hull process to a stochastic process.

Appendix A

q-Uniform Estimates

We have collected in the first part of this appendix some basic results from the theory of uniform convergence and the most frequently used results. In the second part, we have provided a detailed proof of a uniform version of the celebrated Levi-Wiener theorem since we have been unable to find it anywhere in literature, and have extended B. A. Rogozin's [56, 57] coefficient estimates to their q-uniform counterparts.

We will assume throughout the appendix (unless explicitly stated otherwise) that parameter q is constrained to the closed unit interval, which we denote by $I = [0, 1]$.

A.1 q-Uniform convergence

Let $s(q) = \{x_n(q)\}_{n\geq 0}$ be a sequence of complex-valued functions. We say the sequence $s(q)$ is *pointwise convergent* on I if for every fixed $q_0 \in I$, the sequence $s(q_0)$ is convergent. We say the sequence $s(q)$ is *uniformly convergent* with limit $x(q)$ if for every $\epsilon > 0$ there exists some N such that for any $n \geq N$ and $q \in I$, $|x_n(q) - x(q)| < \epsilon$. We will use the notation $x_n(q) \rightrightarrows x(q)$ to indicate the convergence is uniform. It is well-known that uniform convergence can equivalently be characterized in terms of the uniform norm as

$$x_n(q) \rightrightarrows x(q) \iff \lim_{n\to\infty} \sup_{q\in[0,1]} |x_n(q) - x(q)| = 0.$$

We collect below some properties of uniformly convergent sequences that have implicitly been used throughout the thesis. We assume that $s = \{x_n(q)\}$ and $t = \{y_n(q)\}$ are uniformly convergent sequences on I with limits $x(q)$ and $y(q)$ (all operations on s and t are performed pointwise):

(i) If $x_n(q)$ is bounded for every n, then s is q-uniformly bounded.

(ii) If $x_n(q)$ is continuous for every n, then $x(q)$ is continuous.

(iii) Let $\lambda, \mu \in \mathbb{C}$ be arbitrary constants then $\lambda s + \mu t$ is uniformly convergent.

(iv) If s and t are q-uniformly bounded, then the sequence $s \cdot t$ is uniformly convergent.

(v) Let y a complex-valued function whose range is contained in I, then the sequence $s \circ y$ is uniformly convergent.

(vi) Let y a complex-valued function that is uniformly continuous on a set containing the image $x_n(I)$ of every term, then the sequence $y \circ s$ is uniformly convergent.

It can be difficult in practice to determine whether a sequence is uniformly convergent or not. We provide below a simple sufficient condition

Theorem A.1.1 (Dini's theorem)**.** *Suppose $\{x_n(q)\}$ is a monotone sequence of continuous real-valued functions, which pointwise converges to a continuous function x, then the sequence is uniformly convergent.*

We have frequently encountered situations of iterated limits. It is well-known that given two limiting operators (such as limits, series, integrals, derivatives etc.) the result will in general depend on the order of operations. An important result that provides sufficient conditions for commutativity of limits is the Moore-Osgood theorem:

Theorem A.1.2 (Moore-Osgood theorem, [34])**.** *Let M be a complete metric space, A, B arbitrary subsets of a metric spaces M_1 and M_2, respectively and $f : A \times B \to M$ a bivariate mapping. Let $x_0 \in A' \backslash A$ and $y_0 \in B' \backslash B$ be two fixed limit points of the sets A and B. If*

1. $\lim_{x \to x_0} f(x, y) = \psi(y)$ *exists for each* $y \in B$.
2. $\lim_{y \to y_0} f(x, y) = \varphi(x)$ *exists uniformly in* $x \in A$,

then the limits $\lim_{x \to x_0} \lim_{y \to y_0} f(x, y)$, $\lim_{y \to y_0} \lim_{x \to x_0} f(x, y)$ and $\lim_{(x_0, y_0)} f(x, y)$ exist and are equal. Furthermore, if $B = \mathbb{N}$ (with the limit point ∞) then the conditions also imply that $\lim_{x \to x_0} f(x, y) = \psi(y)$ exists uniformly in $y \in B$.

A somewhat lesser known criterion sutured especially to series is Tannery's theorem. It is reminiscent of Weierstrass' criterion for uniform convergence of series.

Theorem A.1.3 (Tannery's theorem)**.** *Let $\{a_{k,n}\}_{k \geq 1}$ be a double sequence such that the following conditions hold:*

(i) The series $s_n = \sum_{k=1}^{\infty} a_{k,n}$ is convergent for every $n \geq 1$.

(ii) The limit $\lim_{n\to\infty} a_{k,n} = a_k$ *exists for all* $k \geq 1$.

(iii) There exists constants M_k *(independent of* n*) such that* $|a_{k,n}| \leq M_k$ *for every* $n, k \geq 1$ *and the series* $\sum_{k=1}^{\infty} M_k$ *is convergent.*

Let p_n *denote any strictly increasing sequence of positive integers, then*

$$\lim_{n\to\infty} \sum_{k=1}^{p_n} a_{k,n} = \sum_{k=1}^{\infty} \lim_{n\to\infty} a_{k,n}.$$

The terms of several sequences have been expressed in terms of polylogarithmic sequences of the form $\mathrm{Li}_{-n}(q^k)$. It will be convenient to collect in one place some of their properties. We introduce for convenience sake the forward and backward shift operators, $\nabla_k c_k = c_{k+1} - c_k$ and $\Delta_k c_k = c_k - c_{k+1}$. Assuming the parameter q belongs to the half-closed unit interval $[0, 1)$, we define the double sequence $x_q(n, k) = \mathrm{Li}_{-n}(q^k)$ for any positive integers $n, k \geq 1$. We have:

Lemma A.1.1 (Monotonicity properties of $x_q(n, k)$)

(i) [Complete monotonicity] $\Delta_k^r x_q(n, k) > 0$ *for every* $n, k, r \geq 0$.

(ii) [Absolute monotonicity] $\nabla_n^r x_q(n, k) > 0$ *for every* $n, k, r \geq 0$.

Proof. Monotonicity properties of the array are an easy consequence of the series representation of the polylogarithms; we have

$$\Delta_k^r x_q(n, k) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x_q(n, k+i) = \sum_{j=1}^{\infty} j^n q^{kj} (1 - q^j)^r \geq 0$$
$$\nabla_n^r x_q(n, k) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} x_q(n+i, k) = \sum_{j=1}^{\infty} j^n (j-1)^r q^{kj} \geq 0$$

and both series are absolutely convergent since we have assumed $q \in [0, 1)$. □

Lemma A.1.2 (Properties of polylogarithm sequences)

(i) [Total positivity] *The following matrix is totally positive (in other words, all the minors of the matrix are positive) for any* $k \geq 1$ *and* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$

$$\begin{bmatrix} x_q(1,\lambda_1) & x_q(2,\lambda_1) & x_q(3,\lambda_1) & \cdots & x_q(k,\lambda_1) \\ x_q(1,\lambda_2) & x_q(2,\lambda_2) & x_q(3,\lambda_2) & \cdots & x_q(k,\lambda_2) \\ x_q(1,\lambda_3) & x_q(2,\lambda_3) & x_q(3,\lambda_3) & \cdots & x_q(k,\lambda_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_q(1,\lambda_k) & x_q(2,\lambda_k) & x_q(3,\lambda_k) & \cdots & x_q(k,\lambda_k) \end{bmatrix}$$

(ii) [Total positivity] *The following matrix is totally positive (in other words, all the minors of the matrix are positive) for any* $n, k \geq 1$ *and* $\lambda_k > \lambda_{k-1} > \cdots > \lambda_1 \geq 1$

$$\begin{bmatrix} x_q(n,\lambda_1) & x_q(n,\lambda_1+1) & x_q(n,\lambda_1+2) & \cdots & x_q(n,\lambda_1+k-1) \\ x_q(n,\lambda_2) & x_q(n,\lambda_2+1) & x_q(n,\lambda_2+2) & \cdots & x_q(n,\lambda_2+k-1) \\ x_q(n,\lambda_3) & x_q(n,\lambda_3+1) & x_q(n,\lambda_3+2) & \cdots & x_q(n,\lambda_3+k-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_q(n,\lambda_k) & x_q(n,\lambda_k+1) & x_q(n,\lambda_k+2) & \cdots & x_q(n,\lambda_k+k-1) \end{bmatrix}$$

Moreover, the matrix remains totally positive if the sequence $\{x_q(n,k)\}_{k\geq 1}$ *is replaced by the sequence of iterated differences* $\{\Delta_k^r x_q(n,k+m)\}_{k\geq 1}$ *for any* $n \geq 1$ *and* $r, m \geq 0$.

Proof. Let's write $q^{\lambda_i} = e^{y_i}$, where $y_1 < y_2 < \cdots < y_k$, then by induction and problem **99** (generalized mean value theorem) in [52, p. 52], the matrix will be totally positive if the Wronskian $W(\mathrm{Li}_{-1}(e^y), \ldots \mathrm{Li}_{-k}(e^y))$ is totally positive for all y. Note that the Wronskian determinant of polylogarithms can be written as $e^{ky}(1-e^y)^{-(k+1)k}H_k$, where H_k is the shifted Hankel determinant of Eulerian polynomials. However, it is well-known that the Hankel matrix of a sequence is totally positive if and only if the principal minors of the Hankel matrix and the shifted Hankel matrix are positive. These minors are known explicitly for the case of Eulerian polynomials with values

$$W(\mathrm{Li}_{-1}(e^y), \ldots, \mathrm{Li}_{-k}(e^y)) = \frac{k! e^{\frac{1}{2}(k+1)ky}}{(1-e^y)^{(k+1)k}} \prod_{i=0}^{k-1} i!^2.$$

Part (i) follows. Total positivity of the second matrix follows by an easy application of problems **85** and **86** in [52, p. 49] to the power series

$$F(z) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \mathrm{Li}_{-n}(q^i z) = \sum_{k\geq 1} k^n (1-q^k)^r z^k$$

with $\alpha_i = q^{\lambda_i}$, $\beta_j = q^{\mu_j}$ where $1 \leq \lambda_1 < \cdots < \lambda_k$ and $0 \leq \mu_1 < \cdots < \mu_k$. □

We will also need a simple q-uniform estimate of the polylogarithmic sequence. The following lemma gives the desired bound

Lemma A.1.3 *Let* $n \geq 1$ *be a fixed positive integer, then the terms* $(1-q)^{n+1}x_q(n,k)$ *are well-defined and continuous on the entire closed unit interval* $[0,1]$ *and the following bound holds uniformly in* $(k,q) \in \mathbb{N} \times [0,1]$ *for each fixed* $n \geq 1$,

$$(1-q)^{n+1} x_q(n,k) = \mathcal{O}(k^{-(n+1)}).$$

Proof. Let $X_q(n,k) = k^{n+1}(1-q)^{n+1}x_q(n,k)$ and $\alpha = (n+1)^{-1}$, then the terms can be estimated as

$$X_q(n,k) = \frac{k^{n+1}q^k(1-q)^{n+1}}{(1-q^k)^{n+1}}\mathrm{Eu}_n(q^k) \le n!\left[\frac{f_\alpha(-k\operatorname{Log} q)}{f_0(-\operatorname{Log} q)}\right]^{n+1} \le n!C_\alpha^{n+1}, \qquad \text{(A.1)}$$

where have used the trivial inequality $\mathrm{Eu}_n(q^k) \le \mathrm{Eu}_n(1) = n!$ and have introduced, for $\alpha \in [0,1/2]$, the function

$$f_\alpha(z) = \frac{ze^{-\alpha z}}{1-e^{-z}} \text{ and } f'_\alpha(z) = \frac{e^{(1-\alpha)z}}{e^z-1}\left\{1+(1-\alpha)z - f_0(z)\right\}.$$

Using the well-known hyperbolic Cusa–Huygens inequality it is easily shown that f_0 is positive, strictly increasing and convex. Hence $f'_\alpha(z)$ has a unique zero, say z_α, and by examining the signs of limits at $\pm\infty$, it follow that the zero is positive, so the mapping f_α has a unique global maximum at z_α. We have thus shown there exists a constant $C_\alpha > 0$ such that $f_\alpha(z) \le C_\alpha$ on $\mathbb{R}$ and $f_0(z) \ge 1$ for $z \ge 0$, which proves (A.1). □

Lastly, the next two lemmas give sufficient conditions for uniform convergence of certain of quotients of quasi-polynomials, which we call *balanced quotients*:

Lemma A.1.4 *A balanced q-quotient is an expression of the form*

$$e_k(q) = \frac{(1-q^{\alpha_1 k+a_1})\cdots(1-q^{\alpha_n k+a_n})(1+q^{\beta_1 k+c_1})\cdots(1+q^{\beta_m k+c_m})}{(1-q^{\alpha_1 k+b_1})\cdots(1-q^{\alpha_n k+b_n})(1+q^{\beta_1 k+d_1})\cdots(1+q^{\beta_m k+d_m})}$$

where all the constants are arbitrary real numbers, and additionally $\alpha_i, \beta_j > 0$. Then the limit $\lim_{k\to\infty} e_k(q) = 1$ exists uniformly in $q \in [0,1]$.

Proof. Notice that each factor of the quotient is bounded on $[0,1]$, so it suffices to prove that convergence is uniform in the special cases $(n,m) = (1,0)$ and $(0,1)$. We can write

$$\frac{1 \pm q^{\alpha k+a}}{1 \pm q^{\alpha k+b}} - \frac{1 \pm q^{\alpha k+\alpha+a}}{1 \pm q^{\alpha k+\alpha+b}} = \pm\frac{q^{\alpha k}(1-q^\alpha)(q^a-q^b)}{(1 \pm q^{\alpha k+b})(1 \pm q^{\alpha k+\alpha+b})} \qquad \text{(A.2)}$$

The right-hand side of (A.2) is of constant sign $\pm\operatorname{sign}(b-a)$. Hence the sequence is monotonic and its pointwise limit is continuous on $[0,1]$. It follows by Dini's theorem that the sequence is uniformly convergent with respect to q. □

Lemma A.1.5 *Let $h_n \in \mathscr{C}([0,1])$ be a uniformly convergent sequence with limit h_∞ such that $h_\infty(1) = 0$, then the sequence $q^n h_n(q) \rightrightarrows 0$ is uniformly convergent on $[0,1]$.*

Proof. We estimate the supremum as follows $\sup_q |q^n h_n(q)| \leq \sup_q |h_n(q) - h_\infty(q)| + \sup_q |q^n h_\infty(q)|$. Note that the initial term converges to 0 as $n \to \infty$ since the sequence h_n was assumed uniformly convergent. On the other hand, the sequence $q^n h_\infty(q)$ is continuous, monotonically decreasing and has a continuous pointwise limit identically equal to 0 (note that h_∞ is continuous and bounded on I). Hence by Dini's theorem it is uniformly convergent, so the limit of the second term likewise converges to 0 as $n \to \infty$. It follows that the sequence $q^n h_n(q)$ is uniformly convergent. □

Similarly, we can prove a quotient of the form $x_k(q) = [k]_q^{-1}$ is uniformly convergent with limit $1 - q$ on $[0, 1]$. Indeed, the difference is positive

$$x_k(q) - x_{k+1}(q) = \frac{(1-q)^2 q^k}{(1-q^k)(1-q^{k+1})} > 0.$$

Another application of Dini's theorem again yields q-uniform convergence.

A.2 q-Uniform Wiener Algebra

Let $\mathfrak{R}_+$ denote the *Wiener algebra* of absolutely convergent power series $x(t) = \sum_{k=0}^{\infty} x_k t^k$ on the closed unit disc $\overline{\mathbb{D}}$ with norm $\|x\| = \sum_{k=1}^{\infty} |x_k| < +\infty$. Recall the following celebrated theorem, connected with the names of Levi and Wiener, due to their investigation of properties of Banach algebras,

Theorem A.2.1 (Levi-Wiener theorem)**.** *Let $x(t) \in \mathfrak{R}_+$ and $\Lambda(z)$ be an analytic function on simply connected bounded domain D. If the curve $\{x(\vartheta) | \vartheta \in [-\pi, \pi]\}$ lies entirely in D, then $\Lambda(x(t)) \in \mathfrak{R}_+$.*

We define the *q-Wiener algebra* $\mathfrak{R}'_+$ as the class of all absolutely and q-normally convergent power series $x(t, q) = \sum_{k=0}^{\infty} x_k(q) t^k$ on the closed unit disc $\overline{\mathbb{D}}$ with coefficients in the Banach algebra $\mathfrak{B}(I)$ of all bounded complex-valued functions on the interval I with the supremum norm equipped with the norm $\|x\|'_1 = \sum_{k=0}^{\infty} \sup_q |x_k(q)| < +\infty$. It is easily checked that $\mathfrak{R}'_+$ is again a Banach algebra with unit element. Indeed, due to absolute convergence we can use Cauchy's multiplication to deduce

$$\begin{aligned}\|xy\|'_1 &= \sum_{k=0}^{\infty} \sup_q \left| \sum_{l=0}^{k} x_{k-l}(q) y_l(q) \right| \leq \sum_{k=0}^{\infty} \sup_q \sum_{l=0}^{k} |x_{k-l}(q)||y_l(q)| \\ &\leq \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sup_q |x_{k-l}(q)| \sup_q |y_l(q)| = \sum_{k=0}^{\infty} \sup_q |x_k(q)| \sum_{l=0}^{\infty} \sup_q |y_l(q)| = \|x\|'_1 \|y\|'_1.\end{aligned}$$

Triangle inequality and homogeneity of the norm $\|\cdot\|_1'$ are clear. What remains is to prove the space $\mathfrak{R}_+'$ is complete with respect to the norm $\|\cdot\|_1'$. Suppose $x^n(t,q)$ is a Cauchy sequence in $\mathfrak{R}_+'$; that is, $\lim_{n\to\infty}\sup_{m\geq 0}\|x^n - x^{n+m}\|_1' = 0$. Let's write

$$x^n(t,q) = \sum_{k=0}^{\infty} x_k^n(q)t^k.$$

Clearly, the assumption implies the sequence $x_k^n(q)$ is Cauchy in $\mathfrak{B}(I)$ for each $k \geq 0$. It is well-known that the space $\mathfrak{B}(I)$ of bounded functions equipped with the supremum norm is complete, so there exists $x_k(q) \in \mathfrak{B}(I)$ such that $\lim_n x_k^n(q) = x_k(q)$ for each $k \geq 0$. What remains is to check that the series $x(t,q) = \sum_k x_k(q)t^k$ has finite $\|\cdot\|_1'$-norm and is the limit of the sequence $x^n(t,q)$ in $\mathfrak{R}_+'$. We can write

$$\begin{aligned}
\sum_{k=n}^{m}\sup_q|x_k(q)| &\leq \sum_{k=n}^{m}\sup_q|x_k(q)-x_k^N(q)| + \sum_{k=n}^{m}\sup_q|x_k^N(q)| \\
&= \sum_{k=n}^{m}\sup_q\lim_{M\to\infty}|x_k^{N+M}(q)-x_k^N(q)| + \sum_{k=n}^{\infty}\sup_q|x_k^N(q)| \\
&\leq \sum_{k=n}^{m}\sup_q\sup_M|x_k^{N+M}(q)-x_k^N(q)| + \sum_{k=n}^{\infty}\sup_q|x_k^N(q)| \\
&\leq \sup_M\sum_{k=n}^{\infty}\sup_q|x_k^{N+M}(q)-x_k^N(q)| + \sum_{k=n}^{\infty}\sup_q|x_k^N(q)|.
\end{aligned}$$

Both terms in the final inequality tend to 0 as $n, m \to \infty$, which proves $x_q \in \mathfrak{R}_+'$. Furthermore, looking only at the first term of the right-hand side with $n = 0$ and letting $m, N \to \infty$ establishes $x^n \to x$ in $\mathfrak{R}_+'$, which proves the space is complete.

We will now prove a q-version of the Wiener's theorem.[1] In what follows we will say a function $f(t,q)$ defined on a set D and depending on parameter $q \in I$ is q-uniformly bounded if $\sup_{q\in I}\sup_{t\in D}|f(t,q)| < +\infty$ and q-uniformly bounded away from zero if $\inf_{q\in I}\inf_{t\in D}|f(t,q)| > 0$.

Theorem A.2.2 (q-Wiener theorem)**.** *Let $x(t,q) \in \mathfrak{R}_+'$ and suppose $x(t,q)$ is q-uniformly bounded away from zero on the closed unit disc $\overline{\mathbb{D}}$, then $1/x(t,q) \in \mathfrak{R}_+'$.*

Proof. Firstly, note that the initial coefficients $x_0(q)$ is uniformly bounded away from zero on I. Hence its (multiplicative) inverse $x_0^{-1}(q)$ is bounded on I. Consequently,

[1] It is essential to require that the series are q-uniformly convergent since it is well-known that in the Wiener algebra $\mathfrak{R}_+$ there does not exist a constant $N(\delta, \|x\|_1)$ depending only on the norm of an element $x(t)$ and $\delta = \inf|x(t)|$ such that $\|x^{-1}\|_1 \leq N(\delta, \|x\|)$ for all $x \in \mathfrak{R}_+$ with $\delta > 0$. Moreover, it has been shown that a uniform estimate exists if and only if $\delta > 1/2$; in which case we may take $N(\delta) = (2\delta - 1)^{-1}$.

the multiplicative inverse of $x(t,q)$ (formally) exists as a power series for any $q \in I$ and using Faa di Bruno's formula we deduce its coefficients

$$[t^k]x^{-1}(t,q) = \sum_{i=1}^{k} x_0(q)^{-i} \sum_{\underline{\lambda} \in \mathscr{C}(k,i)} x_{\lambda_1}(q) \cdots x_{\lambda_i}(q).$$

are again bounded functions of parameter q. It remains to check the series defines a function in the uniform Wiener algebra $\mathfrak{R}'_+$. We follow D. J. Newman's elegant proof of the classical Wiener theorem (see [50]). We only need to verify the following inequality

$$\|f\|'_1 \leq \sup_{t,q} |f(t)| + 2 \sup_{t,q} |f''(t)|, \text{ for } f \in \mathscr{C}^2_q(\mathbb{T}), \tag{A.3}$$

where $\mathscr{C}^2_q(\mathbb{T})$ is the space of q-uniformly bounded twice continuously differentiable Fourier series with q-uniformly bounded second derivative. Let $f(t) = \sum_n a_n(q) e^{i2\pi t}$, then $|a_0(q)| \leq \sup_{t,q} |f(t)|$ and for $n \neq 0$ integrating by parts twice yields $|a_n(q)| \leq n^{-2} \sup_{t,q} |f''(t)|$. Taking the supremum with respect to q in both inequalities and using $\sum_n n^{-2} = \pi^2/6 < 2$ the result follows.

The proof is now almost verbatim the same as long as we replace every occurrence of $|\cdot|$ with the supremum norm $\|\cdot\|_{\infty,q}$ and $\|\cdot\|_1$ with the modified norm $\|\cdot\|'_1$ defined above. We apply inequality (A.3) to estimate the $\|\cdot\|'_1$-norm of $1/P^k$ (using the same notation as in [50]), where $k \geq 0$ is a positive integer and $P(t)$ is a finite trigonometric polynomial with q-bounded coefficients such that $\inf_{t,q} |P| \geq 2/3$. We can estimate the second derivative as

$$\max_{t,q} |(1/P^k)''| = \max_{t,q} |(k(k+1)(P')^2 - kP'' \cdot P)/P^{k+2}| \leq Ak^2(3/2)^{k+2}$$

for some sufficiently large constant A, since trivially P, P' and P'' are all q-uniformly bounded on the closed unit disk. The rest of the proof goes through unchanged. □

We will introduce the class $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ of absolutely convergent power series, which will be a subalgebra of the Banach algebra $\mathfrak{R}_+$ when equipped with the norm

$$\|x(t,q)\|'_{1,\alpha} = 2^\alpha \left\{ \sup_{k \geq 0} k^{\alpha-1} \|x_k\|_\infty + \|x\|'_1 \right\}. \tag{A.4}$$

We let $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ denote the class of all power series $x(t,q) = \sum_{k=0}^{\infty} x_k(q) t^k$ with coefficients in $\mathfrak{B}(I)$, such that $|x_k(q)| k^\alpha$ is q-uniformly bounded for all $k \geq 0$. Since the bound holds q-uniformly and the series $\sum_k k^{-\alpha}$ is convergent for $\alpha > 1$, it follows that class is a subset of the q-uniform Wiener algebra $\mathfrak{R}'_+$. We will prove first that when equipped with the norm (A.4) it forms a Banach algebra.

Lemma A.2.1 *Suppose $\alpha > 1$, then $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ equipped with the norm $\|\cdot\|'_{1,\alpha}$ is (unital) Banach algebra.*

Proof. Linearity and homogeneity of $\|\cdot\|'_{1,\alpha}$ are clear. Write $P_\alpha(x) = \sup_k k^\alpha \|x_k\|_\infty$. Let $x, y \in \mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$. Since series x and y are absolutely convergent, by Cauchy's multiplication theorem the coefficients of the product series xy are given by the convolution sequence $(x * y)_k$. Recall also the following well-known inequality $|k+l|^\alpha \leq 2^{\alpha-1}(|k|^\alpha + |l|^\alpha)$ valid for any $\alpha \geq 1$ and $l, k \in \mathbb{R}$. We can estimate $P_\alpha(xy)$ as follows

$$\begin{aligned}
|(x*y)_k(q)|k^\alpha = \left|\sum_{l=0}^{k} x_l(q) y_{k-l}(q) k^\alpha\right| &\leq \sum_{l=0}^{k} |x_l(q)||y_{k-l}(q)|k^\alpha \\
&\leq 2^{\alpha-1}\left(\sum_{l=0}^{k} |x_l(q)|l^\alpha |y_{k-l}(q)| + \sum_{l=0}^{l} |x_{k-l}(q)||y_l(q)|l^\alpha\right) \\
&\leq 2^{\alpha-1}\left(P_\alpha(x)\sum_{l=0}^{k}\sup_q |y_l(q)| + P_\alpha(y)\sum_{l=0}^{k}\sup_q |x_l(q)|\right) \\
&\leq 2^{\alpha-1}(P_\alpha(x)\|y\|'_1 + P_\alpha(y)\|x\|'_1.
\end{aligned}$$

Taking the supremum with respect to k, q on both sides yields the inequality

$$P_\alpha(xy) \leq 2^{\alpha-1}(P_\alpha(x)\|y\|'_1 + P_\alpha(y)\|x\|'_1). \tag{A.5}$$

Since $\alpha > 1$, we trivially have the inequality $K_\alpha = 2^{\alpha-1} > 1$ which gives

$$\begin{aligned}
\|xy\|'_{1,\alpha} = K_\alpha(P_\alpha(xy) + \|xy\|'_1) &\leq K_\alpha^2(P_\alpha(x)\|y\|'_1 + K_\alpha(y)\|x\|'_1) + K_\alpha\|x\|'_1\|y\|'_1 \\
&\leq K_\alpha^2(P_\alpha(x)\|y\|'_1 + K_\alpha(y)\|x\|'_1) + K_\alpha^2\|x\|'_1\|y\|'_1 \leq \|x\|'_{1,\alpha}\|y\|'_{1,\alpha}.
\end{aligned}$$

It remains to show the space $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ is complete in the $\|\cdot\|'_{1,\alpha}$-norm. Suppose x^n is a Cauchy sequence in $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$, then $\lim_{n\to\infty}\sup_{m\geq 0}\|x^n - x^{n+m}\|'_{1,\alpha} = 0$. Since $\|x\|'_1 \leq \|x\|'_{1,\alpha}$ for any $x \in \mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ it follows that x^n is Cauchy in $\mathfrak{R}'_+$, so there exists a limit $x \in \mathfrak{R}'_+$ of the sequence x^n considered as a sequence in $\mathfrak{R}'_+$. We will prove $\|x^n - x\|'_{1,\alpha} \to 0$ as $n \to \infty$. It is clearly sufficient to prove $P_\alpha(x^n - x) \to 0$ as $n \to \infty$. Let's write

$$x^n(t,q) = \sum_{k=0}^{\infty} x_k^n(q)t^k \text{ and } x(t,q) = \sum_{k=0}^{\infty} x_k(q)t^k.$$

Since $\lim_{n\to\infty}\|x^n - x\|'_1 = 0$, it follows that $\lim_{n\to\infty}\sup_k \|x_k^n - x_k\|_\infty = 0$ meaning we have $\lim_{n\to\infty} x_k^n(q) = x_k(q)$ uniformly in q for every $k \geq 0$. Hence we can estimate

$$\begin{aligned}
P_\alpha(x^n - x) = \sup_k \|x_k^n - x_k\|_\infty k^\alpha &= \sup_k \lim_{m\to\infty}\|x_k^n - x_k^{n+m}\|_\infty k^\alpha \\
&\leq \sup_k \sup_m \|x_k^n - x_k^{n+m}\|_\infty k^\alpha = \sup_m P_\alpha(x^n - x^{n+m}) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Consequently, $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ is a (commutative) Banach algebra with unit 1. □

The following definition is an abstraction of the Levi-Wiener theorem.

Definition 37. Let $\mathscr{A} \subseteq \mathscr{B}$ be two Banach algebras with a common identity. Then $\mathscr{A}$ is called *inverse-closed* in $\mathscr{B}$ if

$$a \in \mathscr{A} \text{ and } a^{-1} \in \mathscr{B} \implies a^{-1} \in \mathscr{A}.$$

Notice that the q-uniform theorem can be restated as saying the uniform Wiener algebra $\mathfrak{R}'_+$ is inverse-closed in $\mathscr{C}_q(\mathbb{T})$ (the algebra of q-uniformly bounded continuous functions on the unit circle). The following result will be needed in the proof of the q-uniform Wiener theorem for the algebra $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$.

Lemma A.2.2 (Brandenburg's trick [11]) *Let $\mathscr{A} \subseteq \mathscr{B}$ be two Banach algebras with a common unit element. Suppose for every $a \in \mathscr{A}$ there exists a sequence $c_n = c_n(a) > 0$ such that $\lim_{n\to\infty} c_n^{1/n} = 1$ and*

$$\|a^{2n}\|_{\mathscr{A}} \leq c_n \|a^n\|_{\mathscr{A}} \|a^n\|_{\mathscr{B}},$$

then $\mathscr{A}$ is inverse-closed in $\mathscr{B}$.

The next result describes the closure properties of the class - it is a harmless q-uniform extension of the results in [56, 57], albeit with a different proof.

Lemma A.2.3 (B. A. Rogozin, [56, 57]) *Suppose $\alpha > 1$ and $x(t,q) \in \mathfrak{R}_+$ is q-uniformly bounded away from zero on the closed unit disc $\overline{\mathbb{D}}$, then*

$$x(t,q) \in \mathfrak{R}_+(\mathcal{O}(k^{-\alpha})) \Rightarrow 1/x(t,q) \in \mathfrak{R}_+(\mathcal{O}(k^{-\alpha})).$$

Proof. Using inequality (A.5), given any $x \in \mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ we have

$$\|x^2\|'_{1,\alpha} \leq 2K_\alpha^2 P_\alpha(x)\|x\|'_1 + K_\alpha \|x^2\|'_1 \leq 2K_\alpha \|x\|'_1 \|x\|'_{1,\alpha},$$

where $K_\alpha = 2^{\alpha-1}$. It follows that $\|x^{2n}\|'_{1,\alpha} \leq 2^\alpha \|x^n\|'_1 \|x^n\|'_{1,\alpha}$ for any $x \in \mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$. Consequently, by Brandenburger's trick the algebra $\mathfrak{R}_+(\mathcal{O}(k^{-\alpha}))$ is inverse closed in $\mathfrak{R}'_+$. The conclusion follows from the uniform Wiener theorem (A.2.2). □

Bibliography

[1] L. Addario-Berry. "Growing random 3-connected maps". In: *Electron. Commun. Probab.* 19 (2014), no. 54, 12. ISSN: 1083-589X.

[2] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry*. Cambridge Monographs on Mathematical Physics. A statistical field theory approach. Cambridge University Press, Cambridge, 1997, pp. xiv+363. ISBN: 0-521-46167-7.

[3] E. M. Andreev. "Convex polyhedra in Lobačevskiĭ spaces". In: *Mat. Sb. (N.S.)* 81 (123) (1970), pp. 445–478.

[4] E. M. Andreev. "Convex polyhedra of finite volume in Lobačevskiĭ space". In: *Mat. Sb. (N.S.)* 83 (125) (1970), pp. 256–260.

[5] O. Angel and O. Schramm. "Uniform infinite planar triangulations". In: *Comm. Math. Phys.* 241.2-3 (2003), pp. 191–213. ISSN: 0010-3616.

[6] E. A. Bender and L. B. Richmond. "Central and local limit theorems applied to asymptotic enumeration. II. Multivariate generating functions". In: *J. Combin. Theory Ser. A* 34.3 (1983), pp. 255–265. ISSN: 0097-3165.

[7] E. A. Bender and N. C. Wormald. "The number of rooted convex polyhedra". In: *Canad. Math. Bull.* 31.1 (1988), pp. 99–102. ISSN: 0008-4395.

[8] M. Bodirsky et al. "A direct decomposition of 3-connected planar graphs". In: *Sém. Lothar. Combin.* 54A (2005), Art. B54Ak, 15. ISSN: 1286-4889.

[9] M. Bousquet-Mélou and A. Jehanne. "Polynomial equations with one catalytic variable, algebraic series and map enumeration". In: *J. Combin. Theory Ser. B* 96.5 (2006), pp. 623–672. ISSN: 0095-8956.

[10] J. Bouttier and E. Guitter. "On irreducible maps and slices". In: *Combin. Probab. Comput.* 23.6 (2014), pp. 914–972. ISSN: 0963-5483.

[11] L. H. Brandenburg. "On identifying the maximal ideals in Banach algebras". In: *J. Math. Anal. Appl.* 50 (1975), pp. 489–510. ISSN: 0022-247X.

[12] W. G. Brown. "Enumeration of quadrangular dissections of the disk". In: *Canad. J. Math.* 17 (1965), pp. 302–317. ISSN: 0008-414X.

[13] W. G. Brown. "On the existence of square roots in certain rings of power series". In: *Math. Ann.* 158 (1965), pp. 82–89. ISSN: 0025-5831.

[16] N. Curien and J.-F. Le Gall. "Scaling limits for the peeling process on random maps". In: *Ann. Inst. Henri Poincaré Probab. Stat.* 53.1 (2017), pp. 322–357. ISSN: 0246-0203.

[17] N. Curien and J.-F. Le Gall. “The hull process of the Brownian plane”. In: *Probab. Theory Related Fields* 166.1-2 (2016), pp. 187–231. ISSN: 0178-8051.

[14] N. Curien, L. Ménard, and G. Miermont. “A view from infinity of the uniform infinite planar quadrangulation”. In: *ALEA Lat. Am. J. Probab. Math. Stat.* 10.1 (2013), pp. 45–88. ISSN: 1980-0436.

[18] M. Drmota, O. Giménez, and M. Noy. “Degree distribution in random planar graphs”. In: *J. Combin. Theory Ser. A* 118.7 (2011), pp. 2102–2130. ISSN: 0097-3165.

[19] M. Drmota and K. Panagiotou. “A central limit theorem for the number of degree-k vertices in random maps”. In: *Algorithmica* 66.4 (2013), pp. 741–761. ISSN: 0178-4617.

[20] S. Eger. “Restricted weighted integer compositions and extended binomial coefficients”. In: *J. Integer Seq.* 16.1 (2013), Article 13.1.3, 25. ISSN: 1530-7638.

[21] J. A. Fill, P. Flajolet, and N. Kapur. “Singularity analysis, Hadamard products, and tree recurrences”. In: *J. Comput. Appl. Math.* 174.2 (2005), pp. 271–313. ISSN: 0377-0427.

[22] P. Flajolet. “Singularity analysis and asymptotics of Bernoulli sums”. In: *Theoret. Comput. Sci.* 215.1-2 (1999), pp. 371–381. ISSN: 0304-3975.

[23] P. Flajolet, S. Gerhold, and B. Salvy. “Lindelöf representations and (non-)holonomic sequences”. In: *Electron. J. Combin.* 17.1 (2010), Research Paper 3, 28. ISSN: 1077-8926.

[24] P. Flajolet and A. Odlyzko. “Singularity analysis of generating functions”. In: *SIAM J. Discrete Math.* 3.2 (1990), pp. 216–240. ISSN: 0895-4801.

[25] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009, pp. xiv+810. ISBN: 978-0-521-89806-5.

[26] W. B. Ford. *Studies on divergent series and summability & The asymptotic developments of functions defined by Maclaurin series*. Chelsea Publishing Co., New York, 1960, pp. x+342.

[27] É. Fusy, G. Schaeffer, and D. Poulalhon. “Dissections, orientations, and trees with applications to optimal mesh encoding and random sampling”. In: *ACM Trans. Algorithms* 4.2 (2008), Art. 19, 48. ISSN: 1549-6325.

[28] Z. Gao and N. C. Wormald. “Sharp concentration of the number of submaps in random planar triangulations”. In: *Combinatorica* 23.3 (2003), pp. 467–486. ISSN: 0209-9683.

[29] Z. Gao and N. C. Wormald. “The distribution of the maximum vertex degree in random planar maps”. In: *J. Combin. Theory Ser. A* 89.2 (2000), pp. 201–230. ISSN: 0097-3165.

[30] Z. Gao and N. C. Wormald. “The size of the largest components in random planar maps”. In: *SIAM J. Discrete Math.* 12.2 (1999), pp. 217–228. ISSN: 0895-4801.

[31] I. P. Goulden and D. M. Jackson. *Combinatorial enumeration*. Foreword by Gian-Carlo Rota, Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Inc., New York, 1983, pp. xxiv+569. ISBN: 0-471-86654-7.

[32] E. Guitter. "The distance-dependent two-point function of quadrangulations: a new derivation by direct recursion". In: *Ann. Inst. Henri Poincaré D* 4.2 (2017), pp. 213–244. ISSN: 2308-5827.

[33] F. Harary and W. T. Tutte. "On the order of the group of a planar map". In: *J. Combinatorial Theory* 1 (1966), pp. 394–395.

[34] T. H. Hildebrandt. *Introduction to the theory of integration*. Pure and Applied Mathematics, Vol. XIII. Academic Press, New York-London, 1963, pp. ix+385.

[35] S. Kakeya. "On the zero points of a power series with positive coefficients." English. In: *Tohoku Math. J.* 3 (1913), pp. 23–34. ISSN: 0040-8735.

[36] M. Kaneko, N. Kurokawa, and M. Wakayama. "A variation of Euler's approach to values of the Riemann zeta function". In: *Kyushu J. Math.* 57.1 (2003), pp. 175–192. ISSN: 1340-6116.

[37] M. Katsurada. "Asymptotic expansions for certain multiple q-integrals and q-differentials of Thomae-Jackson type". In: *Diophantine analysis and related fields 2010*. Vol. 1264. AIP Conf. Proc. Amer. Inst. Phys., Melville, NY, 2010, pp. 100–113.

[38] S. Kitaev, A. de Mier, and M. Noy. "On the number of self-dual rooted maps". In: *European J. Combin.* 35 (2014), pp. 377–387. ISSN: 0195-6698.

[39] W. Koepf. "Power series in computer algebra". In: *J. Symbolic Comput.* 13.6 (1992), pp. 581–603. ISSN: 0747-7171.

[40] A. Kolpakov. "On the optimality of the ideal right-angled 24-cell". In: *Algebr. Geom. Topol.* 12.4 (2012), pp. 1941–1960. ISSN: 1472-2747.

[41] M. Krikun. "A uniformly distributed infinite planar triangulation and a related branching process". In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 307.Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 10 (2004), pp. 141–174, 282–283. ISSN: 0373-2703.

[42] M. Krikun. "Local structure of random quadrangulations". 23 pages, 10 figures. Oct. 2006.

[43] B. Kuttner. "On discontinuous Riesz means of type n". In: *J. London Math. Soc.* 37 (1962), pp. 354–364. ISSN: 0024-6107.

[44] J.-F. Le Gall and T. Lehéricy. "Separating cycles and isoperimetric inequalities in the uniform infinite planar quadrangulation". In: arXiv:1710.02990 (Oct. 2017), arXiv:1710.02990. arXiv: `1710.02990 [math.PR]`.

[45] M. Luczak and P. Winkler. "Building uniformly random subtrees". In: *Random Structures Algorithms* 24.4 (2004), pp. 420–443. ISSN: 1042-9832.

[46] P. Mani. "Automorphismen von polyedrischen Graphen". In: *Math. Ann.* 192 (1971), pp. 279–303. ISSN: 0025-5831.

[47] M. Marcus and H. Minc. *A survey of matrix theory and matrix inequalities*. Reprint of the 1969 edition. Dover Publications, Inc., New York, 1992. ISBN: 0-486-67102-X.

[48] L. Ménard. "Volumes in the uniform infinite planar triangulation: from skeletons to generating functions". In: *Combin. Probab. Comput.* 27.6 (2018), pp. 946–973. ISSN: 0963-5483.

[49] R. C. Mullin and P. J. Schellenberg. "The enumeration of c-nets via quadrangulations". In: *J. Combinatorial Theory* 4 (1968), pp. 259–276.

[50] D. J. Newman. "A simple proof of Wiener's $1/f$ theorem". In: *Proc. Amer. Math. Soc.* 48 (1975), pp. 264–265. ISSN: 0002-9939.

[15] C. Nicolas and J. Le Gall. "First-passage percolation and local modifications of distances in random triangulations". In: 2015.

[51] A. Peyerimhoff. "On the modulus of power series of a certain type". In: *J. London Math. Soc.* 40 (1965), pp. 260–261. ISSN: 0024-6107.

[52] G. Pólya and G. Szegő. *Problems and theorems in analysis. II.* Classics in Mathematics. Theory of functions, zeros, polynomials, determinants, number theory, geometry, Translated from the German by C. E. Billigheimer, Reprint of the 1976 English translation. Springer-Verlag, Berlin, 1998, pp. xii+392. ISBN: 3-540-63686-2.

[53] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*. Second. Vol. 149. Graduate Texts in Mathematics. Springer, New York, 2006, pp. xii+779. ISBN: 978-0387-33197-3.

[54] I. Richards. "On the classification of noncompact surfaces". In: *Trans. Amer. Math. Soc.* 106 (1963), pp. 259–269. ISSN: 0002-9947.

[55] L. B. Richmond and N. C. Wormald. "The asymptotic number of convex polyhedra". In: *Trans. Amer. Math. Soc.* 273.2 (1982), pp. 721–735. ISSN: 0002-9947.

[56] B. A. Rogozin. "Asymptotic behavior of the coefficients in Levi-Wiener theorems on absolutely converging trigonometric series". In: *Sibirsk. Mat. Ž.* 14 (1973), pp. 1304–1312, 1367. ISSN: 0037-4474.

[57] B. A. Rogozin. "Asymptotic behavior of the coefficients of functions of power series and Fourier series". In: *Sibirsk. Mat. Ž.* 17.3 (1976), pp. 640–647, 717. ISSN: 0037-4474.

[58] K. V. Safonov. "On power series of algebraic and rational functions in $\mathbf{C}^n$". In: *J. Math. Anal. Appl.* 243.2 (2000), pp. 261–277. ISSN: 0022-247X.

[59] E. Steinitz. "Polyeder und Raumeinteilungen". In: *Encyk der Math Wiss* 12 (1922), pp. 38–43, 94–101.

[60] W. T. Tutte. "A census of planar maps". In: *Canad. J. Math.* 15 (1963), pp. 249–271. ISSN: 0008-414X.

[61] H. Whitney. "Congruent Graphs and the Connectivity of Graphs". In: *Amer. J. Math.* 54.1 (1932), pp. 150–168. ISSN: 0002-9327.

[62] D. Zeilberger. “The umbral transfer-matrix method. I. Foundations”. In: *J. Combin. Theory Ser. A* 91.1-2 (2000). In memory of Gian-Carlo Rota, pp. 451–463. ISSN: 0097-3165.