

Time-Average Constraints in Stochastic Model Predictive Control

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Abstract—This paper presents two alternatives to using chance constraints in stochastic MPC, motivated by the observation that many stochastic constrained control algorithms aim to impose a bound on the time-average of constraint violations. We consider imposing a robust constraint on the time-average of constraint violations over a finite period. By allowing the controller to respond to the effects of past violations, two algorithms are presented that solve this problem, both requiring a single convex optimization after a preprocessing step. Stochastic MPC formulations that ‘remember’ previous violations and react accordingly were given previously in [1], [2], but in those works the focus was on asymptotic guarantees on the average number of violations. In contrast we give stronger robust bounds on the violation permissible in any time period of a specified length. The method is also applied to a bound on the sum of convex loss functions of the amount of constraint violation, thus allowing controllers to place greater importance on avoiding large violations.

I. INTRODUCTION

It is usually difficult to find closed-form solutions to infinite horizon optimal control problems for systems with inequality constraints on states and inputs. Model Predictive Control (MPC) gives an approximate solution by solving a finite horizon problem online using numerical methods, and applying only the first element in the computed sequence of control inputs to the plant. By repeating this process at regular intervals, taking into account the updated state each time, an effective feedback controller is produced.

MPC theory is well-developed for plants with linear dynamics and convex constraints, for which there exist controllers requiring a modest level of computational effort with guarantees of stability and constraint satisfaction [3]. Controllers with similar guarantees of constraint satisfaction exist for problems where the dynamics include an additive disturbance [4], [5] or parametric uncertainty [6], [7] and constraints must be satisfied for all possible outcomes. Some conservatism with respect to the constraints must usually be tolerated in these latter cases, and the different techniques available typically give a compromise between optimality and ease of computation.

Although the robust approach is effective in providing guarantees of constraint satisfaction, there exist applications in which it is not essential that constraints are satisfied for all possible uncertainty realisations or at all times. Examples include constraints intended to limit fatigue damage in wind turbine control [8], or constraints that ensure occupant

comfort in building climate control [9]. In these applications, ensuring that constraints hold at all times hinders controller performance. Nonetheless, we would like to limit the ‘amount’ of constraint violation to some acceptable level. To this end, several chance-constrained formulations of MPC have been proposed that limit the probability of a constraint violation [10], [11], [12].

These methods give constraint satisfaction guarantees, but they have a major drawback in that chance constraints are nonconvex or difficult to evaluate in many cases, making it difficult to develop efficient implementations without introducing conservatism. This manifests as a difference between the intended probability of constraint violation and the observed frequency of violations in closed loop operation. This led the authors of [1] to suggest an algorithm with constraint tightening parameters that adapt to the observed violation frequency. Similarly, [2] presents a framework for controlling the average constraint violation, and proves asymptotic bounds on this average.

In this paper, we propose a scheme that, similarly to [1], [2], modifies the inequality constraints present in the online optimization based on the history of constraint violations. But instead of yielding asymptotic bounds on the average violation, our scheme leads to stronger robust bounds over finite time periods. This is made possible by a constraint relaxation policy that ensures recursive feasibility of the optimization. Two MPC algorithms are presented, each requiring a single convex optimization at each sampling instant. These are shown to be mean square stable and to satisfy specified bounds on the rate of constraint violation in closed loop.

Notation: A sequence u_t, u_{t+1}, \dots predicted at time t is denoted $u_{t|t}, u_{t+1|t}, \dots$. The vector formed by concatenating the elements of a predicted sequence $u_{t|t}, \dots, u_{t+N-1|t}$ is denoted $\underline{u}_t = (u_{t|t}, \dots, u_{t+N-1|t})$. The vector of ones and the identity matrix are denoted $\underline{1}$ and I respectively, and the convex hull is $\text{Conv}\{\cdot\}$. The probability measure is \mathbb{P} , expectation is \mathbb{E} and conditional expectation at time t is \mathbb{E}_t .

II. PROBLEM FORMULATION

We consider a plant with an i.i.d. stochastic uncertainty,

$$x_{t+1} = A(\omega_t)x_t + B(\omega_t)u_t + w(\omega_t), \quad (1)$$

where ω_t is an uncertain parameter bounded within a polytope, so that the plant state x_t satisfies for all $t = 0, 1, \dots$

$$x_{t+1} \in \text{Conv}\{A^{(i)}x_t + B^{(i)}u_t + w^{(i)}, \quad i = 1, \dots, p\}. \quad (2)$$

The control objective is to regulate the state x_t to the origin while respecting a state constraint

$$Fx_t \leq \underline{1} \quad (3)$$

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at a ‘sufficiently large’ number of times t . We assume that it is possible to choose a linear feedback law, $u_t = Kx_t$, such that (2) admits a robustly invariant polyhedron satisfying the constraint (3). As shown in [13], this requires that the joint spectral radius of $A^{(i)} + B^{(i)}K$, $i = 1, \dots, p$ is less than 1.

Assumption 1: There exists g such that the set

$$\{x \in \mathbb{R}^n : Vx \leq g\} \subseteq \{x \in \mathbb{R}^n : Fx \leq 1\}$$

is robustly invariant for (2) under the control $u_t = Kx_t$.

The linear feedback gain K and the corresponding robustly invariant set should be chosen by a suitable design procedure. This could be achieved by designing K to give a desired robustly invariant set using the methods of [13]. Alternatively, LMI design techniques using quadratic candidate Lyapunov functions could be used to design K [14] and V and g could be found corresponding to the maximal [15], minimal [16], or some other robustly invariant polyhedron.

We introduce an indicator variable M_t equal to 1 if (3) is violated and equal to 0 if (3) is satisfied, and require

$$\frac{1}{T} \sum_{k=t}^{t+T} M_k \leq \varepsilon \quad (4a)$$

for all t and a given integer T . In words, the average number of constraint violations must not exceed ε over any time interval of length T . We note that for small ε , T must be large to obtain a good averaging, but this will present no issues for implementation as T may be chosen independently of the MPC prediction horizon.

Alternatively, if it is desirable to penalise large constraint violations more strongly, we may introduce a convex loss function $l : \mathbb{R}^q \rightarrow \mathbb{R}$ and require

$$\frac{1}{T} \sum_{k=t}^{t+T} l(Fx_k - 1) \leq \bar{l} \quad (4b)$$

for all t and a given integer T . In other words, the average loss over any time interval of length T must not exceed \bar{l} .

Remark 1: The time-average constraints (4a) and (4b) can be contrasted with chance constraints such as

$$\mathbb{P}\{Fx_t \not\leq 1\} \leq \varepsilon$$

and expected-value constraints like

$$\mathbb{E}[l(Fx_t - 1)] \leq \bar{l}$$

both of which have been considered in the MPC literature. The constraints (4a) and (4b) are analogs of these stochastic constraints, but with averaging carried out over a time period of length T rather than over the statistical ensemble.

Remark 2: A similar constraint to (4a) can be found in Section III.B of [17], which considers robustly bounding

$$\frac{1}{t} \sum_{k=0}^t M_k \leq \varepsilon,$$

that is, the average number of constraint violations from the initial time to the present. For times $t \geq T$, this condition is implied by (4a).

III. MPC FOR PARAMETRIC UNCERTAINTY

To handle parametric uncertainty in the dynamics of the plant, we adapt the approach of [7], which accounts for uncertainty by including variables in the online optimization to describe a ‘tube’ of predictions. This allows for parametric uncertainty while yielding an optimization problem with a number of variables and constraints that grows only linearly with the length N of the prediction horizon.

A. Input and tube parametrisations

We parameterise the predicted control input as

$$u_{k|t} = Kx_{k|t} + c_{k|t} \quad (5)$$

where K is the state feedback gain of Assumption 1. Predicted state trajectories then evolve according to

$$x_{k+1|t} \in \text{Conv}\{\Phi^{(i)}x_{k|t} + B^{(i)}c_{k|t} + w^{(i)}, \quad i = 1 \dots p\} \quad (6)$$

where $\Phi^{(i)} = A^{(i)} + B^{(i)}K$ for each i . The control parameters $c_{k|t}$ are to be determined online by solving an optimization problem. We set $c_{k|t} = 0$ for $k \geq t + N$, so the implied optimization is finite-dimensional, and we refer to the integer N as the *prediction horizon* of the MPC strategy.

To apply the constraint (3) to the uncertain state predictions, we bound the predicted states in a *state tube* which consists of a sequence of polyhedra of fixed complexity. We define the *cross sections* of this tube as the polyhedra

$$\mathcal{T}_{k|t} = \{x \in \mathbb{R}^n : Vx \leq \alpha_{k|t}\} \quad (7)$$

where $\alpha_{k|t}$ is a parameter that modifies the shape of the tube cross section at prediction time k . These parameters will appear in the online optimization as variables, therefore $\mathcal{T}_{k|t}$ is not fixed but rather determined online.

For the controller to be implementable, only a finite number of the parameters can be retained as free variables. Hence we choose $\alpha_{k|t} = g$ for $k \geq t + N$, where V and g correspond to the robustly invariant polyhedron of Assumption 1. We then have

$$\mathcal{T}_{N|t} = \mathcal{T}_{N+1|t} = \mathcal{T}_{N+2|t} = \dots$$

so, implicitly, there is an infinite sequence of tube cross-sections bounding the future state. All points within this set should satisfy the state constraint (3), ensuring that this constraint is satisfied beyond the prediction horizon. This is analogous to the terminal set often used in MPC, and to enlarge the feasible region of the controller we recommend that V and g are chosen so that $\mathcal{T}_{N|t}$ has a large volume.

B. Tube constraints

To bound the predicted states within the tube, we require that the current state is a member of the initial cross section

$$x_{t|t} \in \mathcal{T}_{t|t} \quad (8)$$

and that any state in $\mathcal{T}_{k|t}$ at prediction time k must evolve, under the uncertain dynamics, to a state in $\mathcal{T}_{k+1|t}$, i.e.

$$\mathcal{T}_{k|t} \subseteq \left\{x \in \mathbb{R}^n : \Phi^{(i)}x + B^{(i)}c_{k|t} + w^{(i)} \in \mathcal{T}_{k+1|t}, \quad \forall i\right\}. \quad (9)$$

Noting that (9) is equivalent to

$$x_{k|t} \in \mathcal{T}_{k|t} \implies x_{k+1|t} \in \mathcal{T}_{k+1|t}, \quad \forall k \geq t$$

it is immediate that (8) and (9) imply

$$x_{k|t} \in \mathcal{T}_{k|t}, \quad \forall k \geq t$$

so that the state tube contains the predicted states forever.

For any prediction time k , the state constraint (3) can be applied to all points in the corresponding tube cross-section by enforcing the subset relation

$$\mathcal{T}_{k|t} \subseteq \{x \in \mathbb{R}^n : Fx \leq \underline{1}\}. \quad (10)$$

If (8) and (9) also hold for all k , then the prediction $x_{k|t}$ will satisfy (3) because it is contained in $\mathcal{T}_{k|t}$.

To apply these set-theoretic conditions on the tube cross sections $\mathcal{T}_{k|t}$, we must translate them to inequality constraints on the parameters $\alpha_{k|t}$. For the initial condition (8), this is straightforward as we can apply the linear constraint

$$Vx_t \leq \alpha_{t|t} \quad (11)$$

in the online optimization. For the subset relations (9) and (10), we make use of a lemma from [15].

Lemma 1: Let $\mathcal{P}_i = \{x : F_i x \leq b_i\}$, $i = 1, 2$, then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if

$$Hb_1 \leq b_2$$

for some nonnegative matrix $H \geq 0$ satisfying $HF_1 = F_2$.

Suppose that we choose nonnegative matrices $\Gamma^{(i)} \geq 0$ for $i = 1, \dots, p$, and a matrix Δ satisfying the equalities

$$\Gamma^{(i)} V = V \Phi^{(i)} \quad (12a)$$

$$\Delta V = F \quad (12b)$$

then Lemma 1 immediately yields polyhedral constraints on the parameters $\alpha_{k|t}$ for each subset relation.

Proposition 1: The tube cross sections $\mathcal{T}_{k|t}$ and $\mathcal{T}_{k+1|t}$ satisfy (9) if

$$\Gamma^{(i)} \alpha_{k|t} + VB^{(i)} c_{k|t} + Vw^{(i)} \leq \alpha_{k+1|t} \quad (13)$$

for $\Gamma^{(i)}$ satisfying (12a).

Proposition 2: The tube cross section \mathcal{T}^k satisfies (10) if

$$\Delta \alpha_{k|t} \leq \underline{1} \quad (14)$$

for Δ satisfying (12b).

Constraints (13) and (14) will appear in the online optimisation. To relax these constraints, each row of $\Gamma^{(i)}$ and Δ may be chosen by minimising its 1-norm:

$$\Gamma_j^{(i)} = \arg \min_y \{ \underline{1}^T y : y^T V = V_j \Phi^{(i)}, \quad y \geq 0 \}$$

$$\Delta_j = \arg \min_y \{ \underline{1}^T y : y^T V = F_j, \quad y \geq 0 \}$$

When chosen in this way, these matrices are sparse with at most n nonzero entries in each row.

C. Cost function

We introduce a cost function designed to penalise deviations from a specified target set. Hence we introduce a matrix W and vector h such that the target polyhedron

$$\Omega = \{x \in \mathbb{R}^n : Wx \leq h\} \subseteq \{x \in \mathbb{R}^n : Fx \leq \underline{1}\}$$

is robustly positively invariant. In closed loop, the system state will converge to this polyhedron, which may, for instance, be the minimal robustly invariant set.

We now define a cost function that is quadratic in the predicted input u^k and in $x_{k|t} - s_{k|t}$ where $s_{k|t}$ lies on a possible trajectory of the plant within Ω . Hence let

$$\mathcal{V}_t = \min_{s_t \in \Omega} \left\{ \sum_{k=t}^{\infty} \mathbb{E}_t (\|x_{k|t} - s_{k|t}\|_Q^2 + \|u_{k|t}\|_R^2) \right\}$$

where the sequence $s_{k|t}$ is generated by

$$s_{k+1|t} = A(\omega_{k|t})s_{k|t} + B(\omega_{k|t})u_{k|t} + w(\omega_{k|t})$$

and $u_{k|t}$, $x_{k|t}$ are given by (5) and (6). We suppose that R is chosen to be positive definite and Q such that $(Q^{\frac{1}{2}}, A)$ is an observable pair, so this cost is positive definite in $x_t - s_t$ and hence penalizes any deviation of the state from Ω .

To use \mathcal{V}_t as the cost function of an online optimization, it must be expressed in terms of x_t , s_t and c_t . We therefore construct a system that generates predictions of $x_{k|t} - s_{k|t}$ when its state is projected onto the first n dimensions. Let

$$\begin{bmatrix} z_{k+1|t} \\ c_{k+1|t} \end{bmatrix} = \begin{bmatrix} A(\omega) + B(\omega)K & B(\omega)E \\ 0 & T \end{bmatrix} \begin{bmatrix} z_{k|t} \\ c_{k|t} \end{bmatrix}$$

where $E = [I \ 0 \ \dots \ 0]$ and T is a shift matrix, so that $c_{k|t} = E c_{k|t}$ and $c_{k+1|t} = T c_{k|t}$. We express these dynamics more succinctly as $\phi_{k+1|t} = \Psi(\omega)\phi_{k|t}$ with $\phi_{k|t} = (z_{k|t}, c_{k|t})$. Hence we can rewrite the cost function as

$$\mathcal{V}_t = \min_{s_t \in \Omega} \sum_{k=t}^{\infty} \mathbb{E}_t \|\phi_{k|t}\|_{\bar{Q}}^2, \quad \bar{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix}$$

where we specify the initial condition $z_{t|t} = x_t - s_t$ and generate predictions of $\phi_{k|t}$ using the given dynamics. It is immediate that \mathcal{V}_t can be expressed in terms of a quadratic form in $\phi_t = (x_t - s_t, c_t)$.

Proposition 3: The cost \mathcal{V}_t may be calculated as

$$\mathcal{V}_t = \min_{s_t \in \Omega} \mathcal{V}(x_t - s_t, c_t) = \min_{s_t \in \Omega} \|\phi_t\|_P^2$$

where P satisfies the stochastic Lyapunov equation:

$$P - \mathbb{E}[\Psi^T(\omega)P\Psi(\omega)] = \bar{Q} \quad (15)$$

In practice, the matrix P can be found by expressing the uncertain parameters $A(\omega)$, $B(\omega)$, $w(\omega)$ in terms of the convex hull vertices as

$$(A(\omega), B(\omega), w(\omega)) = \sum_{i=1}^p (A^{(i)}, B^{(i)}, w^{(i)}) q_i(\omega)$$

so a substitution in the Lyapunov equation gives

$$P - \sum_{i,j} (\Psi^{(i)})^T P \Psi^{(j)} \mathbb{E}[q_i(\omega) q_j(\omega)] = \bar{Q}$$

and P can be calculated using standard Lyapunov equation solvers if the second moments of q are available.

IV. HANDLING TIME-AVERAGE CONSTRAINTS

We present two similar but distinct methods corresponding to the time-average constraints (4a) and (4b). In both cases, the algorithm consists of a preprocessing step in which the time history of previous constraint violations is processed, followed by the solution of a single quadratic program.

A. Bounding the average number of violations

To satisfy the bound (4a) on closed-loop constraint violations, the controller must be aware of the number of previous constraint violations and when they occurred. Therefore we suppose that the controller stores up to the last T values of M_t . We introduce a relaxed version of the constraint (3)

$$Fx_{k|t} \leq \underline{1} + b_{k|t}r \quad (16)$$

where $r > 0$ is an application-specific design parameter that limits the maximum violation and the binary variable $b_{k|t}$ selects either the original constraint or the relaxed version.

For each prediction time k , the controller must choose b^k based on the sum of constraint violations in the preceding T sample times and the possible future constraint violations over the prediction horizon. Hence we would like to ensure

$$\frac{1}{T} \sum_{l=k-T+1}^t M^l + \frac{1}{T} \sum_{l=t+1}^k b^l \leq \varepsilon$$

for all prediction times $k = t, \dots, t + N$. This suggests a procedure for choosing $b_{k|t}$: Evaluate the first term of the above expression. If the result is less than ε then the ‘worst case’ average number of violations up to time k is less than ε , so we can choose $b_{k|t} = 1$. Otherwise, choose $b_{k|t} = 0$.

We may simplify this procedure if we introduce a variable $\underline{v}_{k|t}$ that keeps track of the ‘worst-case’ number of possible constraint violations up to prediction time k . If we set

$$\underline{v}_{k|t} = \frac{1}{T} \sum_{l=k-T+1}^t M_{l|t} + \frac{1}{T} \sum_{l=t+1}^{k-1} b_{l|t}$$

then subsequent values of $\underline{v}_{k|t}$ satisfy:

$$\bar{v}_{k|t} = \bar{v}_{k-1|t} - \frac{1}{T} M_{k-T|t} + \frac{1}{T} b_{k|t}$$

This suggests an algorithm (Algorithm 1) where the values of $b_{k|t}$ are found in a pre-computation step and then an optimization solved using the strict and relaxed constraints.

Algorithm 1 requires a computation time approximately equal to a corresponding robust MPC problem, as steps 1 and 2 only require simple arithmetic and comparison operations and can typically be carried out very quickly. A crucial fact about the policy in step 2 for choosing $b_{k|t}$ is that in closed loop the value of $b_{k|t}$ corresponding to a fixed prediction time k can only increase, and not decrease.

Lemma 2: In closed loop, $\bar{v}_{k|t}$ satisfies $\bar{v}_{k|t+1} \leq \bar{v}_{k|t}$ for all k , for any t at which the optimization in step 3 is feasible, and hence $b_{k|t} = 1 \implies b_{k|t+1} = 1$.

Proof: Subtracting $\bar{v}_{k|t}$ from $\bar{v}_{k|t+1}$ gives

$$\bar{v}_{k|t+1} - \bar{v}_{k|t} = \frac{1}{T} (M_{t+1} - b_{t+1|t}) + \frac{1}{T} \sum_{l=t+2}^{k-1} (b_{l|t+1} - b_{l|t})$$

Algorithm 1: MPC with bounded number of violations

- 1) Calculate $\bar{v}_{t|t} = \frac{1}{T} \sum_{l=t-T+1}^t M_{l|t}$
- 2) For each $k = t, \dots, t + N - 1$:
 - a) Calculate $\bar{v}_{k|t} = \bar{v}_{k-1|t} - \frac{1}{T} M_{k-T|t} + \frac{1}{T} b_{k|t}$
 - b) If $\bar{v}_{k|t} + \frac{1}{T} \leq \varepsilon$, set $b_{k+1|t} = 1$. Else $b_{k+1|t} = 0$.
- 3) Solve the quadratic program:

$$\begin{aligned} & \underset{\underline{\alpha}_t, s_t, \underline{c}_t}{\text{minimise}} && \mathcal{V}(x_t - s_t, \underline{c}_t) \\ & \text{subject to} && Vx_t \leq g, \quad Ws_t \leq h \\ & && \Delta\alpha_{k|t} \leq \underline{1} + b^k r \\ & && \Gamma^{(i)} \alpha_{k|t} + VB^{(i)} c_{k|t} + Vw^{(i)} \leq \alpha_{k+1|t} \\ & && (\text{for } k = t \dots t + N - 1, \alpha_{t+N|t} = g) \end{aligned}$$

- 4) Apply $u_t = Kx_t + c_t$ to the system.
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after rearranging terms. We consider the outcome of step 2 of the algorithm at time $t + 1$. If $M_{t+1} = 1$, then (from feasibility of the optimization) step 2 chose $b_{t+1|t} = 1$ at time t and hence $\bar{v}_{t+1|t+1} = \bar{v}_{t+1|t}$. Hence $b_{k|t+2} = b_{k|t+1}$ for all $k \geq t + 1$ and the second term in the above expression cancels, so that $\bar{v}_{k|t+1} = \bar{v}_{k|t}$. Alternatively if $M_{t+1} = 0$, then either $b_{t+1|t} = 0$, in which case we repeat the above reasoning to find $\bar{v}_{k|t+1} = \bar{v}_{k|t}$ again, or $b_{t+1|t} = 1$, so that $\bar{v}_{t+1|t+1} = \bar{v}_{t+1|t} - \frac{1}{T}$. Considering the iteration in step 2, if we let $\mu = \min\{k \in \mathbb{Z} : k > t, b_{k|t} = 0\}$, we will find $\bar{v}_{k|t+1} = \bar{v}_{k|t} - \frac{1}{T}$ for all $k \leq \mu$ and $\bar{v}_{k|t+1} = \bar{v}_{k|t}$ thereafter. ■

An immediate consequence of Lemma 2 is that the constraint for a given k will only be relaxed, and not tightened, in closed loop operation. This implies that the QP in step 3 remains feasible for all time if it is feasible initially.

Theorem 1: Under the control of Algorithm 1 we have

$$\frac{1}{T} \sum_{k=t}^{t+T-1} M_k \leq \varepsilon$$

for all $t = 0, \dots, \infty$ if step 3 is feasible at $t = 0$.

Proof: From Lemma 2, we may verify that if

$$(\alpha_{t|t}, \alpha_{t+1|t}, \dots, \alpha_{N-1|t}), \underline{c}_t, s_t$$

is feasible in step 3 at time t , then

$$(\alpha_{t+1|t}, \alpha_{t+2|t}, \dots, g), T\underline{c}_t, s^{t+1}$$

is feasible at time $t + 1$, where $s_{t+1|t} = A(\omega_t)s_t + B(\omega_t)u_t + w(\omega_t)$. By induction, step 3 is then feasible at all times $t \geq 0$ if it is feasible at $t = 0$. Step 2 of the algorithm chooses $b_{t+1|t}$ such that

$$\bar{v}_{t|t} + \frac{1}{T} b_{t+1|t} \leq \varepsilon.$$

From the feasibility properties, $M_t = 1$ can only occur if $b_{t|t-1} = 1$, so this immediately gives the claimed bound. ■

B. Bounding the average loss

If we wish to satisfy the bound on average loss (4b) we can use an algorithm with a similar preprocessing step. Here, we relax the constraints by introducing a variable that bounds the allowable loss, $\lambda_{k|t}$, for each prediction time k .

$$l(\Delta\alpha_{k|t} - \underline{1}) \leq \lambda_{k|t} \quad (17)$$

Analogously to $\bar{v}_{k|t}$, we introduce scalars $\bar{L}_{k|t}$ that give the contribution to the average loss for the time period from $k - T$ to t . As the loss function is convex we may include constraints bounding the average loss directly in the optimisation, leading to Algorithm 2.

Algorithm 2: MPC with bounded average loss

- 1) For each $k = t, \dots, t + N - 1$:
 - a) Calculate $L_{k|t} = \frac{1}{T} \sum_{l=k-T+1}^t l(Fx_{k|t} - \underline{1})$
 - 2) Solve the convex optimization:

$$\begin{aligned} & \underset{\alpha_t, s_t, \underline{c}_t, \lambda_t}{\text{minimise}} && \mathcal{V}(x_t - s_t, \underline{c}_t) \\ & \text{subject to} && Vx_t \leq g, \quad Ws_t \leq h \\ & && L_{k|t} + \frac{1}{T} \sum_{l=t+1}^k \lambda_{l|t} \leq \bar{l} \\ & && l(\Delta\alpha_{k|t} - \underline{1}) \leq \lambda_{k|t} \\ & && \Gamma^{(i)}\alpha_{k|t} + VB^{(i)}c_{k|t} + Vw^{(i)} \leq \alpha_{k+1|t} \\ & && (\text{for } k = t \dots t + N - 1, \alpha_{t+N|t} = g) \end{aligned}$$
 - 3) Apply $u_t = Kx_t + c_t$ to the system.
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Theorem 2: Under the control of Algorithm 2 we have

$$\frac{1}{T} \sum_{k=t}^{t+T} l(Fx - \underline{1}) \leq \bar{l}$$

for all $t = 0, \dots, \infty$ in closed loop.

Proof: By construction, if

$$(\alpha_{t|t}, \alpha_{t+1|t}, \dots, \alpha_{N-1|t}), \quad \underline{c}_t, \quad s_t, \quad (\lambda_{t+1|t}, \dots, \lambda_{N-1|t})$$

is feasible in step 3 at time t , then

$$(\alpha_{t+1|t}, \alpha_{t+2|t}, \dots, g), \quad T\underline{c}_t, \quad s_{t+1|t}, \quad (\lambda_{t+2|t}, \dots, \lambda_{N-1|t})$$

is feasible at time $t + 1$. Hence by induction, feasibility at $t = 0$ implies feasibility for all $t \geq 0$. Noting that $\lambda_{t+1|t}$ is an upper bound for the loss at time $t + 1$, the claimed bound now follows directly from the constraint

$$L_{t|t} + \frac{1}{T} \lambda_{t+1|t} \leq \bar{l}$$

which is present in the optimization in step 2. \blacksquare

C. Closed-loop stability

We now present a closed loop stability result that applies equally to both algorithms by virtue of their cost functions being identical. As the cost penalises the expectation of a

quadratic function of the deviation of the state from the target set, we expect convergence to this set in mean square.

Theorem 3: Under the control of Algorithm 1 or Algorithm 2, the expected cost satisfies the bound

$$\mathbb{E}\|\phi_t\|_P^2 \leq \delta^{\frac{t}{n}-1} \mathbb{E}\|\phi_0\|_P^2$$

so that $\mathbb{E}[x_t - s_t] \rightarrow 0$ and $\mathbb{E}[\underline{c}_t] \rightarrow 0$ exponentially in mean square in closed loop operation.

Proof: Pre- and post-multiplying (15) by ϕ_t^T , ϕ_t gives

$$\mathbb{E}_t \|\Psi(\omega)\phi_t\|_P^2 - \|\phi_t\|_P^2 \leq -\|\phi_t\|_Q^2.$$

For either algorithm, the solution at time t is feasible at time $t + 1$, so that $\mathbb{E}_t \|\phi_{t+1}\|_P^2 \leq \mathbb{E}_t \|\Psi(\omega)\phi_t\|_S^2$ and therefore

$$\mathbb{E}_t \|\phi_{t+1}\|_P^2 - \|\phi_t\|_P^2 \leq -\|\phi_t\|_Q^2.$$

Take expectations and sum over $k = t, \dots, t + n - 1$ to find

$$\mathbb{E}\|\phi_{t+n}\|_P^2 - \mathbb{E}\|\phi_t\|_P^2 \leq - \sum_{k=t}^{t+n-1} \mathbb{E}\|\phi_k\|_Q^2$$

and since $(Q^{\frac{1}{2}}, A)$ was assumed observable and R positive-definite, the right-hand-side is a negative-definite function of ϕ_t . Hence choose positive scalars $\sigma_1 < \sigma_2$ such that

$$\sigma_1 \|\phi_t\|^2 \leq \mathbb{E}\|\phi_t\|_P^2 \leq \sigma_2 \|\phi_t\|^2, \quad \sigma_1 \|\phi_t\|^2 \leq \sum_{k=t}^{t+n-1} \mathbb{E}\|\phi_k\|_Q^2$$

from which we deduce, after substitution and rearrangement,

$$\mathbb{E}\|\phi_{t+n}\|_P^2 \leq \left(1 - \frac{\sigma_1}{\sigma_2}\right) \mathbb{E}\|\phi_t\|_P^2$$

and finally, as this holds for all t , we conclude

$$\mathbb{E}\|\phi_t\|_P^2 \leq \left(1 - \frac{\sigma_1}{\sigma_2}\right)^{\frac{t}{n}-1} \mathbb{E}\|\phi_0\|_P^2$$

which is the claimed bound. \blacksquare

V. SIMULATION EXAMPLE

The example system has nominal matrices

$$A_0 = \begin{bmatrix} -1.9 & -1.4 \\ 0.7 & 0.5 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ -0.25 \end{bmatrix}.$$

where $A(\omega)$, $B(\omega)$ and $w(\omega)$ are defined by

$$A(\omega) = A_0 + \Delta_A^{(1)}\omega_1 + \Delta_A^{(2)}\omega_2 + \Delta_A^{(3)}\omega_3$$

$$B(\omega) = B_0 + \Delta_B^{(1)}\omega_4 + \Delta_B^{(2)}\omega_5$$

$$w(\omega) = w^{(1)}\omega_6 + w^{(2)}\omega_7$$

for $\omega_1, \dots, \omega_7$ i.i.d. and uniformly distributed on $[0, 1]$, and

$$\Delta_A^{(1)} = \begin{bmatrix} 0.01 & 0.05 \\ -0.05 & -0.01 \end{bmatrix}, \quad \Delta_A^{(2)} = \begin{bmatrix} -0.01 & -0.05 \\ 0 & -0.01 \end{bmatrix}$$

$$\Delta_A^{(3)} = \begin{bmatrix} 0 & 0 \\ 0.05 & 0.02 \end{bmatrix}, \quad \Delta_B^{(1)} = \begin{bmatrix} 0.03 \\ -0.02 \end{bmatrix}$$

$$\Delta_B^{(2)} = \begin{bmatrix} -0.03 \\ 0.02 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

We choose $Q = \text{diag}(1, 1)$ and $R = 1$ for the cost function, with K as a stabilising state feedback and Ω

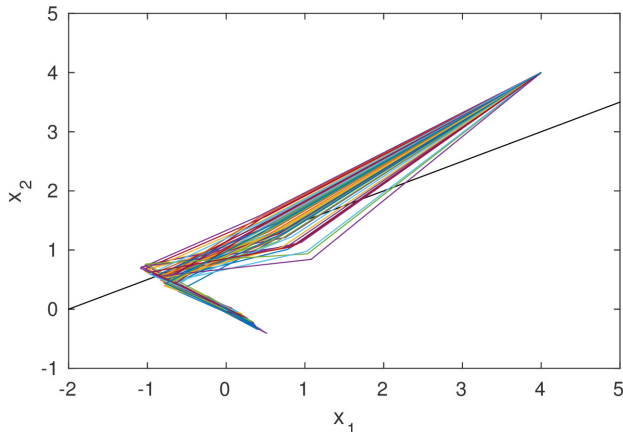


Fig. 1: Sampled chance constraints

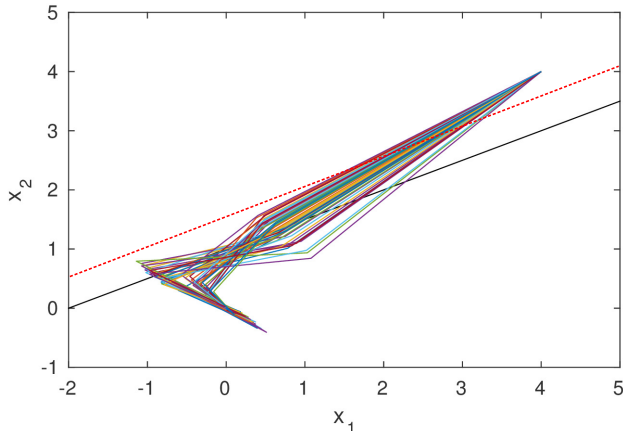


Fig. 2: Time-average constraints

as the corresponding minimal robustly invariant set. For implementation of Algorithm 1 we consider the constraint

$$\begin{bmatrix} -0.5 & 1 \end{bmatrix} x_t \leq 1$$

and a corresponding ‘relaxed’ version with $r = 0.8$. We choose $T = N = 3$, to approximately match the length of the system’s transients, and $\varepsilon = 1/3$ so that the constraint

$$\sum_{k=t}^{t+2} M_k \leq 2 \quad (18)$$

will be satisfied at all times t under Algorithm 1.

For comparison, we also consider running the sample-based MPC algorithm of [18] with the chance constraint:

$$\mathbb{P}_t \left\{ \begin{bmatrix} -0.5 & 1 \end{bmatrix} x_{t+1|t} \leq 1 \right\} \geq 2/3$$

In both cases, the resulting optimizations were solved by Gurobi running on a laptop with a mobile Core i5 processor.

The state trajectories resulting from the two approaches are shown in Figures 1 and 2, along with the original and relaxed constraints. The behaviour of the two controllers differs in closed-loop operation due to the robust bound (18) that must be satisfied in the latter case. From Figure 2, it is clear that every trajectory violating the constraint at $t = 0$ and $t = 1$ is forced to satisfy it at $t = 2$, which is the expected behaviour.

TABLE I: Comparison of MPC optimisations

	Algorithm of [18]	Algorithm 1
% Constraints satisfied, at $t=2$	68	66
Average solver time (ms)	132	69
Closed-loop cost	198.3	195.2

Some quantities of interest are listed in Table I for the two MPC algorithms. The most interesting result is that the average cost for the two algorithms is comparable, despite Algorithm 1 using a single robust optimisation and the algorithm of [18] requiring several. This suggests that this algorithm might be a favourable alternative to MPC controllers involving chance constraints.

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