

Diophantine geometry and non-abelian reciprocity laws I

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Abstract

We use non-abelian fundamental groups to define a sequence of higher reciprocity maps on the adelic points of a variety over a number field satisfying certain conditions in Galois cohomology. The non-abelian reciprocity law then states that the global points are contained in the kernel of all the reciprocity maps.

Dedicated to John Coates on the occasion of his 70th birthday.

*I dive down into the depth of the ocean of forms,
hoping to gain the perfect pearl of the formless...*

–Tagore

1 Refined Hasse principles and reciprocity laws

Consider the Hasse-Minkowski theorem [11] for affine conics like

$$X : \quad ax^2 + by^2 = c \tag{1.1}$$

stating that X has a rational point in a number field F if and only if it has a point in F_v for all places v . In spite of its great elegance, even undergraduate students are normally left with a somewhat unsatisfactory sense of the statement, having essentially to do with the fact that the theorem says nothing about the locus of

$$X(F) \subset X(\mathbb{A}_F). \tag{1.2}$$

There are various attempts to rectify the situation, the most successful of which might be the theory of the Brauer-Manin obstruction [8].

The point of view of this paper is that one should consider such problems, even for more general varieties, as that of defining a good reciprocity map. That is, let's simplify for a moment and assume $X \simeq \mathbb{G}_m$ (so that existence of a rational point is not the issue). Then we are just asking about the locus of F^\times in the ideles \mathbb{A}_F^\times of F . In this regard, a description of sorts is provided by Abelian class field theory [1], which gives us a map

$$\mathrm{rec}^{\mathrm{ab}} : \mathbb{A}_F^\times \longrightarrow G_F^{\mathrm{ab}}, \tag{1.3}$$

with the property that

$$\mathrm{rec}^{\mathrm{ab}}(F^\times) = 0. \tag{1.4}$$

So one could well view the reciprocity map as providing a ‘defining equation’ for $\mathbb{G}_m(F)$ in $\mathbb{G}_m(\mathbb{A}_F)$, except for the unusual fact that the equation takes values in a group. Because F is a number field, there is also the usual complication that the kernel of $\mathrm{rec}^{\mathrm{ab}}$ is not exactly equal to $\mathbb{G}_m(F)$. But the interpretation of the reciprocity law as a refined statement of Diophantine geometry is reasonable enough.

In this paper, we obtain a generalization of Artin reciprocity to an *iterative non-abelian reciprocity law* with coefficients in smooth varieties whose étale fundamental groups satisfy rather mild cohomological conditions [Coh], to be described near the end of this section. They are satisfied, for example, by any smooth curve. Given a smooth variety X equipped with a rational point $b \in X(F)$ satisfying [Coh], we define a sequence of subsets

$$X(\mathbb{A}_F) = X(\mathbb{A}_F)_1 \supset X(\mathbb{A}_F)_1^2 \supset X(\mathbb{A}_F)_2 \supset X(\mathbb{A}_F)_2^3 \supset X(\mathbb{A}_F)_3 \supset X(\mathbb{A}_F)_3^4 \supset \cdots \quad (1.5)$$

and a sequence of maps

$$\text{rec}_n : X(\mathbb{A}_F)_n \longrightarrow \mathfrak{G}_n(X) \quad (1.6)$$

$$\text{rec}_n^{n+1} : X(\mathbb{A}_F)_n^{n+1} \longrightarrow \mathfrak{G}_n^{n+1}(X) \quad (1.7)$$

to a sequence $\mathfrak{G}_n(X), \mathfrak{G}_n^{n+1}(X)$ of profinite abelian groups in such a way that

$$X(\mathbb{A}_F)_n^{n+1} = \text{rec}_n^{-1}(0)$$

and

$$X(\mathbb{A}_F)_{n+1} = (\text{rec}_n^{n+1})^{-1}(0). \quad (1.8)$$

We visualize this as a diagram:

$$\begin{array}{ccccccc} \cdots X(\mathbb{A}_F)_2^3 = \text{rec}_2^{-1}(0) & \subset & X(\mathbb{A}_F)_2 = (\text{rec}_1^2)^{-1}(0) & \subset & X(\mathbb{A}_F)_1^2 = \text{rec}_1^{-1}(0) & \subset & X(\mathbb{A}_F)_1 \\ \downarrow \text{rec}_2^3 & & \downarrow \text{rec}_2 & & \downarrow \text{rec}_1^2 & & \downarrow \text{rec}_1 \\ \cdots & & \mathfrak{G}_2^3(X) & & \mathfrak{G}_2(X) & & \mathfrak{G}_1^2(X) & & \mathfrak{G}_1(X) \end{array} \quad (1.9)$$

in which each reciprocity map is defined not on all of $X(\mathbb{A}_F)$, but only on the kernel (the inverse image of 0) of all the previous maps. We put

$$X(\mathbb{A}_F)_\infty = \bigcap_{i=1}^\infty X(\mathbb{A}_F)_i. \quad (1.10)$$

The non-abelian reciprocity law then states

Theorem 1.1.

$$X(F) \subset X(\mathbb{A}_F)_\infty. \quad (1.11)$$

We give now a brief description of the groups \mathfrak{G}_n and \mathfrak{G}_n^{n+1} . Let $\Delta = \pi_1(\bar{X}, b)^{(2)}$ be the profinite prime-to-2 étale fundamental group¹ [3] of $\bar{X} = X \times_{\text{Spec}(F)} \text{Spec}(\bar{F})$, and let $\Delta^{[n]}$ be its lower central series defined as

$$\Delta^{[1]} = \Delta \quad (1.12)$$

and

$$\Delta^{[n+1]} := \overline{[\Delta, \Delta^{[n]}]}, \quad (1.13)$$

where the overline refers to the topological closure. We denote

$$\Delta_n := \Delta / \Delta^{[n+1]} \quad (1.14)$$

and

$$T_n := \Delta^{[n]} / \Delta^{[n+1]}. \quad (1.15)$$

¹The referee has asked for an explanation for leaving out the prime 2 in the fundamental groups. To include the full profinite π_1 , we would need to consider localization to Archimedean places in Poitou-Tate duality, which would then require us to include Archimedean places in the definition of restricted direct products. But then, because of non-trivial H^0 at Archimedean places, various long exact sequences in non-abelian cohomology would become problematic (cf. Lemma 4.2). We note in this regard that if the base field F had no real places, the full fundamental group could have been used in the entire paper.

Thus, we have an exact sequence

$$1 \longrightarrow T_n \longrightarrow \Delta_n \longrightarrow \Delta_{n-1} \longrightarrow 1 \quad (1.16)$$

for each n , turning Δ_n into a central extension of Δ_{n-1} .

All of the objects above are equipped with canonical actions of $G_F = \text{Gal}(\bar{F}/F)$. Given any topological abelian group A with continuous G_F -action, we have the continuous Galois dual

$$D(N) := \text{Hom}_{ct}(A, \mu_\infty), \quad (1.17)$$

and the Pontriagin dual

$$A^\vee = \text{Hom}_{ct}(A, \mathbb{Q}/\mathbb{Z}). \quad (1.18)$$

(See Appendix II for details.)

With this notation, we can define the targets of the reciprocity maps rec_n using continuous cohomology:

$$\mathfrak{G}_n(X) := H^1(G_F, D(T_n))^\vee. \quad (1.19)$$

Notice that when $X = \mathbb{G}_m$, we have $T_1 = \hat{\mathbb{Z}}^{(2)}(1)$ and $T_n = 0$ for $n > 1$. Thus, $D(T_1) = \oplus_{p \neq 2} \mathbb{Q}_p/\mathbb{Z}_p$ and

$$H^1(G_F, D(T_1)) = \oplus_{p \neq 2} \text{Hom}(G_F, \mathbb{Q}_p/\mathbb{Z}_p) = \oplus_{p \neq 2} \text{Hom}(G_F^{\text{ab}}, \mathbb{Q}_p/\mathbb{Z}_p). \quad (1.20)$$

Hence, by Pontrjagin duality, there is a canonical isomorphism

$$\mathfrak{G}_1((\mathbb{G}_m)_F) \simeq G_F^{\text{ab},(2)}, \quad (1.21)$$

and rec_1 will agree with the prime-to-2 part of the usual reciprocity map rec^{ab} .

For the $\mathfrak{G}_n^{n+1}(X)$, we need a little more notation. Let S be a finite set of places of F and $G_F^S = \text{Gal}(F_S/F)$ the Galois group of the maximal extension of F unramified outside of S . We denote by S^0 the set of non-Archimedean places in S . For a topological abelian group A with G_F^S -action, we have the kernel of localization

$$\text{III}_S^i(A) := \text{Ker}[H^i(G_F^S, A) \xrightarrow{\text{loc}_S} \prod_{v \in S^0} H^i(G_v, A)], \quad (1.22)$$

and what we might call the *strict* kernel,

$$s\text{III}_S^i(A) := \text{Ker}[H^i(G_F^S, A) \xrightarrow{\text{loc}} \prod_{v \in V_F^0} H^i(G_v, A)], \quad (1.23)$$

where the localization map now goes to *all* non-Archimedean places in F . Obviously,

$$s\text{III}_S^i(A) \subset \text{III}_S^i(A).$$

For the strict kernels, whenever $S \subset T$, the restriction maps on cohomology induce maps

$$s\text{III}_S^i(A) \longrightarrow s\text{III}_T^i(A), \quad (1.24)$$

For any finite set M of odd primes, denote by Δ^M the maximal pro- M quotient of Δ , together with corresponding notation $[\Delta^M]^{[n]}$, Δ_n^M , T_n^M . Given M , we consider sets of places S of F that contain all places lying above primes of M , all Archimedean places, and all places of bad reduction for (X, b) .

Then

$$\mathfrak{G}_n^{n+1}(X) := \varprojlim_M \varinjlim_S s\text{III}_S^2(T_{n+1}^M). \quad (1.25)$$

We will see in section 3 how to define the reciprocity maps from level of the filtration on $X(\mathbb{A}_F)$ to all these groups.

The conditions [Coh] are the following.

[Coh 1]: For each finite set M of odd primes, T_n^M is torsion-free.

[Coh 2]: For each finite set M of odd primes and non-Archimedean place v , $H^0(G_v, T_n^M) = 0$.

They are used in suitable injectivity statements for localization in cohomology, which, in turn, feed into the inductive definition of the reciprocity maps. Also, [Coh 2] is necessary for the long exact sequence (2.28) in the restricted direct product of local cohomology.

If we pick a place v , then the projection $X(\mathbb{A}_F) \longrightarrow X(F_v)$ induces the image filtration

$$X(F_v) = X(F_v)_1 \supset X(F_v)_1^2 \supset X(F_v)_2 \supset X(F_v)_2^3 \supset X(F_v)_3 \supset X(F_v)_3^4 \supset \dots \quad (1.26)$$

and of course,

$$X(F) \subset X(F_v)_\infty := \cap_n X(F_v)_n. \quad (1.27)$$

Conjecture 1.2. *When X is a proper smooth curve and v is an odd prime of good reduction, we have*

$$X(F) = X(F_v)_\infty.$$

This conjecture can be viewed as a refinement of the conjecture of Birch and Swinnerton-Dyer type made in [2]. By comparing the profinite reciprocity map here to a unipotent analogue, the computations of that paper can be viewed as evidence for (an affine analogue of) this conjecture as well. We will write more systematically about this connection and about *explicit* higher reciprocity laws in a forthcoming publication. In the meanwhile, we apologise a bit for the lack of examples in this paper, pointing out that it is possible to view the examples in [2] as illustrating the reciprocity laws here. That is, it is quite likely unfeasible to give explicit formulas for the full non-abelian reciprocity maps, in the same way such formulas are hard to come by in the abelian theory. It is rather that one should compose the reciprocity maps with natural projections or functions. Even though we do not spell it out in detail as yet, the computations in [2] arise exactly from such a process.

2 Pre-reciprocity

We will assume throughout that X is a smooth variety over F such that the conditions [Coh] are satisfied. We will denote by V_F the set of all places of F and by V_F^0 the set of non-Archimedean places.

The maps rec_n and rec_n^{n+1} will be constructed in general via non-abelian cohomology and an iterative application of Poitou-Tate duality. For this, it is important that the G_F -action on any fixed Δ^M , where M , as before, is a finite set of odd primes, factors through $G_F^S = \text{Gal}(F_S/F)$ for some finite set S of places of F . Here, F_S refers to the maximal algebraic extension of F unramified outside S . If $X \hookrightarrow X'$ is a smooth compactification with a normal crossing divisor D as complement, then it suffices to take S large enough to satisfy the conditions that

- X' has a smooth model over $\text{Spec}(\mathcal{O}_F[1/S])$;
 - D extends to a relative normal crossing divisor over $\text{Spec}(\mathcal{O}_F[1/S])$;
 - b extends to an S -integral point of the model of X , given as the complement of the closure of D in the smooth model of X' ;
 - S contains M and all Archimedean places of F .
- ([13], Theorem 2.1)

We will be using thereby the continuous cohomology sets and groups (Appendix I, and [5, 12])

$$H^1(G_F^S, \Delta_n^M), \quad H^i(G_F^S, T_n^M), \quad H^i(G_F^S, D(T_n^M)). \quad (2.1)$$

(Note that the first term is an H^1 , in anticipation of the possibility that Δ_n^M is non-abelian.) Whenever this notation is employed, we assume that the finite set S has been chosen large enough so that the G_F -action factors through G_F^S . Given any topological group U with continuous action of G_F , if this action factors through G_F^S for some set S , we will call S an *admissible set* of places. For any admissible set, we denote by S^0 the non-Archimedean places in S .

For each non-Archimedean place v of F , let $G_v = \text{Gal}(\bar{F}_v/F_v)$. In the following, U denotes a topologically finitely-generated profinite group that is prime to 2, in the sense that it is the inverse limit of finite groups of order prime to 2. When U has a continuous G_F -action, define

$$\prod_{v \in S^0}^S H^i(G_v, U) := \prod_{v \in S^0} H^i(G_v, U) \times \prod_{v \in V_F^0 \setminus S^0} H_{un}^i(G_v, U) \quad (2.2)$$

where

$$H_{un}^i(G_v, U) := H^i(G_v/I_v, U^{I_v})$$

and $I_v \subset G_v$ is the inertia subgroup. Here, as in the following, if U is non-abelian, we only allow $i = 1$. In any case, we have a natural map

$$H_{un}^i(G_v, U) \longrightarrow H^i(G_v, U),$$

and hence, if $T \supset S$, a natural map

$$\prod_{v \in S^0}^S H^i(G_v, U) \longrightarrow \prod_{v \in T^0}^T H^i(G_v, U).$$

Define the ‘restricted direct product’ as a direct limit

$$\prod_{v \in S^0}^{\prime} H^i(G_v, U) := \varinjlim_S \prod_{v \in S^0}^S H^i(G_v, U). \quad (2.3)$$

For $i = 1$, the maps in the limit will be injective, but not in general for $i = 2$. We will also use the notation

$$\prod_S H^i(G_v, U) = \prod_{v \in S^0} H^i(G_v, U) \quad (2.4)$$

and

$$\prod_T^S H^i(G_v, U) = \prod_{v \in S^0} H^i(G_v, U) \times \prod_{v \in T^0 \setminus S^0} H_{un}^1(G_v, U) \quad (2.5)$$

for $T \supset S$, so that

$$\prod_{v \in S^0}^S H^i(G_v, U) = \varprojlim_T \prod_T^S H^i(G_v, U). \quad (2.6)$$

For each $n \geq 2$, we have an exact sequence

$$1 \longrightarrow T_n^M \longrightarrow \Delta_n^M \longrightarrow \Delta_{n-1}^M \longrightarrow 1. \quad (2.7)$$

of topological groups. By Appendix I, Lemma 4.4, the surjection $\Delta_n^M \longrightarrow \Delta_{n-1}^M$ is equipped with a continuous section, so that we get a long exact sequence of continuous cohomology

$$0 \longrightarrow H^1(G_F^S, T_n^M) \longrightarrow H^1(G_F^S, \Delta_n^M) \longrightarrow H^1(G_F^S, \Delta_{n-1}^M) \xrightarrow{\delta_{n-1}^g} H^2(G_F^S, T_n^M). \quad (2.8)$$

Here, the superscript in ‘ δ_{n-1}^g ’ refers to ‘global’. As explained in the Appendix I, Lemma 4.2 and Lemma 4.3, the meaning of exactness here is as follows. The group $H^1(G_F^S, T_n^M)$ acts freely on the space $H^1(G_F^S, \Delta_n^M)$. and the projection

$$p_{n-1} : H^1(G_F^S, \Delta_n^M) \longrightarrow H^1(G_F^S, \Delta_{n-1}^M) \quad (2.9)$$

identifies the orbit space with the kernel of the boundary map δ_{n-1}^g . To check that the conditions of Appendix I are satisfied, note that twisting the Galois action by a cocycle for a class $c \in H^1(G_F^S, \Delta_{n-1}^M)$ will not change the action on the graded pieces T_i^M , so that the condition [Coh 2] implies that Δ_{n-1}^M has no G_F^S -invariants. This is because the twisted action is an inner twist, which will not affect the subquotients of the lower central series.

Similarly, for each non-Archimedean local Galois group, we have exact sequences

$$0 \longrightarrow H^1(G_v, T_n^M) \longrightarrow H^1(G_v, \Delta_n^M) \longrightarrow H^1(G_v, \Delta_{n-1}^M) \xrightarrow{\delta_{n-1}} H^2(G_v, T_n^M). \quad (2.10)$$

For each n , there is a surjection

$$(T_1^M)^{\otimes n} \longrightarrow T_n^M. \quad (2.11)$$

Thus, T_n^M has strictly negative weights between $-2n$ and $-n$ as a Galois representation. By [4], Theorem 3(b), we see that the localization

$$H^1(G_F^S, T_n^M) \longrightarrow \prod^S H^1(G_v, T_n^M) \subset \prod' H^1(G_v, T_n^M) \quad (2.12)$$

is injective. In order to use [4], we need to make a few remarks. Firstly, there is the simple fact that

$$T_n^M = \prod_{l \in M} T_n^l, \quad (2.13)$$

so it suffices to consider l -adic representations for a fixed prime l . Next, we note that [4] proves the injectivity for the Galois representations $H^i(\bar{V}, \mathbb{Z}_l(n))/(tor)$ and $i \neq 2n$ where V is a smooth projective variety. But an examination of the proof shows that it only uses the fact that this is torsion-free, finitely-generated, and of non-zero weight. That is to say, it is shown that

$$H^1(G_F^S, N) \longrightarrow \prod^S H^1(G_v, N) \subset \prod' H^1(G_v, N) \quad (2.14)$$

is injective for any torsion-free finitely-generated \mathbb{Z}_l -module of non-zero Galois weight. Note that all the T_n^l are torsion-free by condition [Coh 1].

Now, by using the exact sequences (2.8) and (2.10) and an induction over n , we get injectivity of localization

$$H^1(G_F^S, \Delta_n^M) \longrightarrow \prod^S H^1(G_v, \Delta_n^M) \subset \prod' H^1(G_v, \Delta_n^M) \quad (2.15)$$

for every n .

Of course, we can repeat the discussion with any admissible $T \supset S$. Using these natural localization maps, we will regard global cohomology simply as subsets of the \prod^S or of \prod' .

For any U with continuous G_F^S -action such that the localization map

$$H^1(G_F^T, U) \longrightarrow \prod^T H^1(G_v, U) \subset \prod' H^1(G_v, U) \quad (2.16)$$

is injective for all admissible T , define

$$E(U) := \varinjlim_T \text{loc}(H^1(G_F^T, U)) = \bigcup_T \text{loc}(H^1(G_F^T, U)). \quad (2.17)$$

For admissible T , there is also the partial localization

$$H^1(G_F^T, U) \xrightarrow{\text{loc}_T} \prod_T H^1(G_v, U). \quad (2.18)$$

When U is topologically finitely-generated abelian profinite group with all finite quotients prime to 2, we have the duality isomorphism (local Tate duality, [9], Chapter VII.2)

$$D : \prod_T H^1(G_v, U) \simeq \prod_T H^1(G_v, D(U))^\vee \quad (2.19)$$

that can be composed with

$$\prod_T H^1(G_v, D(U))^\vee \xrightarrow{\text{loc}_T^*} H^1(G_F^T, D(U))^\vee \quad (2.20)$$

to yield a map

$$\text{loc}_T^* \circ D : \prod_{v \in T} H^1(G_v, U) \longrightarrow H^1(G_F^T, D(U))^\vee \quad (2.21)$$

such that

$$\text{Ker}(\text{loc}_T^* \circ D) = \text{loc}_T(H^1(G_F^T, U)) \quad (2.22)$$

(Poitou-Tate duality, [9], Chapter VIII.6). We denote also by $\text{loc}_T^* \circ D$ the map

$$\prod_T H^1(G_v, U) \longrightarrow H^1(G_F^T, D(U))^\vee \quad (2.23)$$

obtained by projecting the components in $\prod_{v \in V_F^0 \setminus T^0} H_{un}^1(G_v, U)$ to zero.

When U is abelian and $T' \supset T$, these maps fit into commutative diagrams as follows:

$$\begin{array}{ccc} \prod_T H^1(G_v, U) & \hookrightarrow & \prod_{T'} H^1(G_v, U) \\ \text{loc}_T^* \circ D \downarrow & & \downarrow \text{loc}_{T'}^* \circ D \\ H^1(G_F^T, D(U))^\vee & \xleftarrow{\text{Inf}^*} & H^1(G_F^{T'}, D(U))^\vee \end{array} \quad (2.24)$$

where the lower arrow is the dual to inflation. The commutativity follows from the fact that $H_{un}^1(G_v, U)$ and $H_{un}^1(G_v, D(U))$ annihilate each other under duality, so that the sum of the local pairings between $\prod_T H^1(G_v, U)$ and $H^1(G_F^T, D(U))$ will be independent of the contribution from $T' \setminus T$. Hence, we get a compatible family of maps

$$\prod_T H^1(G_v, U) \longrightarrow \varprojlim_{T'} H^1(G_F^{T'}, D(U))^\vee = H^1(G_F, D(U))^\vee \quad (2.25)$$

Taking the union over T , we then get

$$\text{prec}(U) : \prod_T H^1(G_v, U) \longrightarrow \varprojlim_T H^1(G_F^T, D(U))^\vee = H^1(G_F, D(U))^\vee \quad (2.26)$$

(The notation prec for ‘pre-reciprocity’ will be placed in context below.) According to Appendix II (5.13),

Proposition 2.1. *When U is a topologically finitely-generated abelian pro-finite group with all finite quotients prime to 2, then*

$$\text{Ker}(\text{prec}(U)) = E(U). \quad (2.27)$$

(Recall $E(U)$ defined in (2.17).)

One distinction from the appendix is that our product runs only over non-Archimedean places. However, because we are only considering prime-to-2 coefficients, the local H^1 vanishes at all Archimedean places. The goal of this section, by and large, is to generalise this result to the coefficients Δ_n^M , which are non-abelian.

In addition to the exact sequences (2.8) and (2.10), we have exact sequences with restricted direct products

$$0 \longrightarrow \prod' H^1(G_v, T_n^M) \longrightarrow \prod' H^1(G_v, \Delta_n^M) \xrightarrow{p_{n-1}} \prod' H^1(G_v, \Delta_{n-1}^M) \xrightarrow{\delta_{n-1}} \prod' H^2(G_v, T_n^M) \quad (2.28)$$

making the second term of the first line a $\prod' H^1(G_v, T_n^M)$ -torsor over the kernel of δ . To see this, let S be an admissible set of primes. Then the G_v -action for $v \notin S$ factors through G_v/I_v , so that we have an exact sequence

$$0 \longrightarrow H^1(G_v/I_v, T_n^M) \longrightarrow H^1(G_v/I_v, \Delta_n^M) \longrightarrow H^1(G_v/I_v, \Delta_{n-1}^M) \xrightarrow{\delta_{n-1}} H^2(G_v/I_v, T_n^M) \quad (2.29)$$

and hence, an exact sequence

$$\begin{aligned} 0 \rightarrow \prod_{v \in S^0} H^1(G_v, T_n^M) \times \prod_{v \in V_F^0 \setminus S^0} H_{un}^1(G_v, T_n^M) &\longrightarrow \prod_{v \in S} H^1(G_v, \Delta_n^M) \times \prod_{v \in V_F^0 \setminus S^0} H_{un}^1(G_v, \Delta_n^M) \\ \rightarrow \prod_{v \in S} H^1(G_v, \Delta_{n-1}^M) \times \prod_{v \in V_F^0 \setminus S^0} H_{un}^1(G_v, \Delta_{n-1}^M) &\xrightarrow{\delta_{n-1}} \prod_{v \in S} H^2(G_v, T_n^M) \times \prod_{v \in V_F^0 \setminus S^0} H_{un}^2(G_v, T_n^M) \end{aligned} \quad (2.30)$$

Taking the direct limit over S gives us the exact sequence with restricted direct products.

In the following, various local, global, and product boundary maps will occur. In the notation, we will just distinguish the level and the global boundary map, since the domain should be mostly clear from the context.

We go on to define a sequence of pre-reciprocity maps as follows. First, we let

$$\text{prec}_1 := \varprojlim_M \text{prec}(\Delta_1^M) : \varprojlim_M \prod' H^1(G_v, \Delta_1^M) \longrightarrow \varprojlim_M H^1(G_F, D(\Delta_1^M))^\vee = H^1(G_F, D(\Delta_1))^\vee \quad (2.31)$$

as above. The kernel of prec_1 is exactly $E_1 := \varprojlim_M E(\Delta_1^M)$. For $x \in E_1$, define

$$\text{prec}_1^2(x) := \delta_1^g(x) \in \varprojlim_M \varinjlim_T H^2(G_F^T, T_2^M) \quad (2.32)$$

(where we identify global cohomology with its image under the injective localisation in order to apply the boundary map to elements of E_1) and

$$E_1^2 := \text{Ker}(\text{prec}_1^2). \quad (2.33)$$

Given $x \in E_1^2$ we will denote by x_M the projection to

$$[E_1^2]_M := \text{Ker}[\delta_1^g|E(\Delta_1^M)]. \quad (2.34)$$

We will be considering various inverse limits over M below, and using superscripts M in a consistent fashion.

Now define

$$W(\Delta_2^M) \subset \prod' H^1(G_v, \Delta_2^M) \quad (2.35)$$

to be the inverse image of $[E_1^2]_M$ under the projection map

$$p_1 : \prod' H^1(G_v, \Delta_2^M) \longrightarrow \prod' H^1(G_v, \Delta_1^M), \quad (2.36)$$

which is, therefore, a $\prod' H^1(G_v, T_2^M)$ -torsor over $[E_1^2]_M$. (By (2.28) an element of $[E_1^2]_M$ is liftable to $\prod' H^1(G_v, \Delta_2^M)$.)

Consider the following diagram:

$$\begin{array}{ccc}
E(T_2^M) \hookrightarrow \prod' H^1(G_v, T_2^M) & & \\
\downarrow & & \downarrow \\
E(\Delta_2^M) \hookrightarrow W(\Delta_2^M) & & \\
\downarrow & & \downarrow \\
[E_1^2]_M & = & [E_1^2]_M
\end{array} \tag{2.37}$$

We see with this that $E(\Delta_2^M)$ provides a reduction of structure group for $W(\Delta_2^M)$ from $\prod' H^1(G_v, T_2^M)$ to $E(T_2^M)$. That is, $W(\Delta_2^M)$ is the torsor pushout of $E(\Delta_2^M)$ with respect to the map

$$E(T_2^M) \longrightarrow \prod' H^1(G_v, T_2^M).$$

Choose a set-theoretic splitting

$$s_1 : [E_1^2]_M \longrightarrow E(\Delta_2^M) \tag{2.38}$$

of the torsor in the left column. We then use this ‘global’ splitting to define

$$\text{prec}_2^M : W(\Delta_2^M) \longrightarrow H^1(G_F, D(T_2^M))^\vee \tag{2.39}$$

by the formula

$$\text{prec}_2^M(x) = \text{prec}(T_2^M)(x - s_1(p_1(x))) \tag{2.40}$$

Here, we denote by $x - s_1(p_1(x))$ the unique element $z \in \prod' H^1(G_v, T_2^M)$ such that $x = s_1(p_1(x)) + z$. (We are using additive notation because the context is the action of a vector group on a torsor.)

Because $E(T_2^M)$ is killed by $\text{prec}(T_2^M)$, it is easy to see that

Proposition 2.2. *prec_2^M is independent of the splitting s_1 .*

Now define

$$W_2 := \varprojlim_M W_2(\Delta_2^M) \tag{2.41}$$

and

$$\text{prec}_2 := \varprojlim_M \text{prec}_2^M : W_2 \longrightarrow \varprojlim_M H^1(G_F, D(T_2^M))^\vee = H^1(G_F, D(T_2))^\vee. \tag{2.42}$$

In general, define

$$E_n := \varprojlim_M E(\Delta_n^M) \tag{2.43}$$

and

$$\text{prec}_n^{n+1} := \delta_n^g : E_n \longrightarrow \varprojlim_M \varinjlim_T H^2(G_F^T, T_{n+1}^M). \tag{2.44}$$

Then define

$$E_n^{n+1} = \text{Ker}(\delta_n^g) \subset E_n, \tag{2.45}$$

and

$$W(\Delta_{n+1}^M) = p_n^{-1}([E_n^{n+1}]_M), \quad (2.46)$$

where $[E_n^{n+1}]_M = \text{Ker}(\delta^g|E(\Delta_n^M))$. As when $n = 1$, (2.28) implies that $W(\Delta_{n+1}^M)$ is a $\prod' H^1(G_v, T_{n+1}^M)$ torsor over $[E_n^{n+1}]_M$. Use a splitting s_n of

$$E(\Delta_{n+1}^M) \rightarrow [E_n^{n+1}]_M \quad (2.47)$$

to define

$$\text{prec}_{n+1}^M : W(\Delta_{n+1}^M) \longrightarrow H^1(G_F, D(T_{n+1}^M))^\vee \quad (2.48)$$

via the formula

$$\text{prec}_{n+1}^M(x) = \text{prec}(T_{n+1}^M)(x - s_n(p_n(x))). \quad (2.49)$$

Once again, because $E(T_{n+1}^M)$ is killed by $\text{prec}(T_{n+1}^M)$, we get

Proposition 2.3. *prec_n^M is independent of the splitting s_n .*

Finally, define

$$W_{n+1} := \varprojlim_M W(\Delta_{n+1}^M) \quad (2.50)$$

and

$$\text{prec}_{n+1} = \varprojlim_M \text{prec}_{n+1}^M : W_{n+1} \longrightarrow \varprojlim_M H^1(G_F, D(T_{n+1}^M))^\vee = H^1(G_F, D(T_{n+1}))^\vee. \quad (2.51)$$

Then we finally have the following generalisation of Proposition 2.1.

Proposition 2.4.

$$\text{Ker}(\text{prec}_{n+1}) = E_{n+1} \quad (2.52)$$

Proof. We have seen this already for $n = 1$. Let $x \in \text{Ker}(\text{prec}_{n+1})$ and x_M the projection to $\text{Ker}(\text{prec}_{n+1}^M)$. It is clear from the definition that $E(\Delta_{n+1}^M) \subset \text{Ker}(\text{prec}_{n+1}^M)$. On the other hand, if $\text{prec}_{n+1}^M(x_M) = 0$, then $y_M = x_M - s_n(p_n(x_M)) \in E(T_{n+1}^M)$, by Proposition 2.1. Hence, $x_M = y_M + s_n(p_n(x_M)) \in E(\Delta_{n+1}^M)$. Since this is true for all M , $x \in E_{n+1} = \varprojlim_M E(\Delta_{n+1}^M)$. (The assertion is a kind of ‘left exactness’ of the inverse limit for pointed sets, although we are giving a direct argument.) □

3 Reciprocity

Recall the product of the local period maps

$$j_n^M : X(\mathbb{A}_F) \longrightarrow \prod' H^1(G_v, \Delta_n^M), \quad (3.1)$$

$$x \mapsto (\pi_1^{et}(\bar{X}; b, x_v)_n^M)_v. \quad (3.2)$$

Here,

$$\pi_1^{et}(\bar{X}; b, x_v)_n^M := \pi_1^{et}(\bar{X}; b, x_v) \times_{\pi_1^{et}(\bar{X}, b)} \Delta_n^M = [\pi_1^{et}(\bar{X}; b, x_v) \times \Delta_n^M] / \pi_1^{et}(\bar{X}, b),$$

(where the $\pi_1^{et}(\bar{X}, b)$ -action at the end is the diagonal one giving the pushout torsor) are torsors for Δ_n^M with compatible actions of G_v , and hence, define classes in $H^1(G_v, \Delta_n^M)$. When $v \notin S$ for S admissible and $x_v \in X(\mathcal{O}_{F_v})$, then this class belongs to $H_{un}^1(G_v, \Delta_n^M)$ ([13], Prop. 2.3). Therefore, $(\pi_1^{et}(\bar{X}; b, x_v)_n^M)_v$ defines a class in $\prod' H^1(G_v, \Delta_n^M)$. (This discussion is exactly parallel to the unipotent case [6, 7].) Clearly, we can then take the limit over M , to get the period map

$$j_n : X(\mathbb{A}_F) \longrightarrow \varprojlim_M \prod' H^1(G_v, \Delta_n^M). \quad (3.3)$$

The reciprocity maps will be defined by

$$\text{rec}_n(x) = \text{prec}_n(j_n(x)), \quad (3.4)$$

and

$$\text{rec}_n^{n+1}(x) = \text{prec}_n^{n+1}(j_n(x)). \quad (3.5)$$

Of course, these maps will not be defined on all of $X(\mathbb{A}_F)$. As in the introduction, define

$$X(\mathbb{A}_F)_1^2 = \text{Ker}(\text{rec}_1). \quad (3.6)$$

Then for $x \in X(\mathbb{A}_F)_1^2$, $j_1(x) \in E(\Delta_1)$, and hence, prec_1^2 is defined on $j_1(x)$. Thus, rec_1^2 is defined on $X(\mathbb{A}_F)_1^2$. Now define

$$X(\mathbb{A}_F)_2 := \text{Ker}(\text{rec}_1^2). \quad (3.7)$$

Then for $x \in X(\mathbb{A}_F)_2$, $j_1(x) \in E_1^2$, so that $j_2(x) \in W_2$. Hence, prec_2 is defined on $j_2(x)$, and rec_2 is defined on $X(\mathbb{A}_F)_2$.

In general, the following proposition is now clear.

Proposition 3.1. *Assume we have defined*

$$X(\mathbb{A}_F)_1^2 \supset X(\mathbb{A}_F)_2 \supset \dots \supset X(\mathbb{A}_F)_{n-1}^n \supset X(\mathbb{A}_F)_n \quad (3.8)$$

as the iterative kernels of $\text{rec}_1, \text{rec}_1^2, \dots, \text{rec}_{n-1}, \text{rec}_{n-1}^n$. Then, $j_n(x) \in W_n$ for $x \in X(\mathbb{A}_F)_n$ so that $\text{rec}_n = \text{prec}_n \circ j_n$ is defined on $X(\mathbb{A}_F)_n$ and rec_n^{n+1} is defined on $\text{Ker}(\text{rec}_n)$.

Note that prec_{n-1}^n takes values in $\varprojlim_M \varinjlim_T H^2(G_F^T, T_n^M)$. However, $j_{n-1}(x)$ lifts to $j_n(x) \in \varprojlim_M \prod' H^1(G_v, \Delta_n^M)$, and hence, is clearly in the kernel of δ_n . Therefore,

$$\text{prec}_{n-1}^n(j_{n-1}(x)) \in \varprojlim_M \varinjlim_T s\text{III}_T^2(T_n^M) =: \mathfrak{G}_{n-1}^n(X) \quad (3.9)$$

for all $x \in X(\mathbb{A}_F)_{n-1}$.

The global reciprocity law of theorem (1.1),

$$X(F) \subset X(\mathbb{A}_F)_\infty,$$

now follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} X(F) & \hookrightarrow & X(\mathbb{A}_F) \\ \downarrow & & \downarrow j_n \\ E(\Delta_n^M) = \varinjlim_T H^1(G_F^T, \Delta_n^M) & \hookrightarrow & \prod' H^1(G_v, \Delta_n^M) \end{array} \quad (3.10)$$

for each M .

To check compatibility with the usual reciprocity map for $X = \mathbb{G}_m$ note that the map

$$F_v^* \xrightarrow{\kappa} H^1(G_v, \hat{\mathbb{Z}}(1)^{(2)}) \xrightarrow{D} H^1(G_v, \oplus_{p \neq 2} \mathbb{Q}_p / \mathbb{Z}_p)^\vee = G_v^{\text{ab}, (2)} \quad (3.11)$$

is the local reciprocity map ([9], corollary (7.2.13), with the natural modification for the prime-to-2 part). Here, κ is the map given by Kummer theory, while D is local duality as before. Furthermore, the localization

$$H^1(G_F, \oplus_{p \neq 2} \mathbb{Q}_p / \mathbb{Z}_p) \xrightarrow{\text{loc}_v} H^1(G_v, \oplus_{p \neq 2} \mathbb{Q}_p / \mathbb{Z}_p) \quad (3.12)$$

is dual to the map,

$$G_v^{(2)} \longrightarrow G_F^{(2)} \quad (3.13)$$

induced by $\bar{F} \hookrightarrow \bar{F}_v$, so that the dual of localization

$$G_v^{\text{ab},(2)} = H^1(G_v, \oplus_{p \neq 2} \mathbb{Q}_p / \mathbb{Z}_p)^\vee \xrightarrow{\text{loc}_v^*} H^1(G_F, \oplus_{p \neq 2} \mathbb{Q}_p / \mathbb{Z}_p)^\vee = G_F^{\text{ab},(2)} \quad (3.14)$$

is simply the natural map we started out with. Since the global reciprocity map is the sum of local reciprocity maps followed by the inclusion of decomposition groups, we are done.

4 Appendix I: A few lemmas on non-abelian cohomology

We include here some basic facts for the convenience of the reader.

Given a continuous action

$$\rho : G \times U \longrightarrow U \quad (4.1)$$

of topological group G on a topological group U , we will only need $H^0(G, U)$ and $H^1(G, U)$ in general. Of course $H^0(G, U) = U^\rho \subset U$ is the subgroup of G -invariant elements. (We will put the homomorphism ρ into the notation or not depending upon the needs of the situation.) Meanwhile, we define

$$H^1(G, U) = U \setminus Z^1(G, U). \quad (4.2)$$

Here, $Z^1(G, U)$ consists of the 1-cocycles, that is, continuous maps $c : G \longrightarrow U$ such that

$$c(gh) = c(g)gc(h), \quad (4.3)$$

while the U action on it is given by

$$(uc)(g) := uc(g)g(u^{-1}). \quad (4.4)$$

We also need $H^2(G, A)$ for A abelian defined in the usual way as the 2-cocycles, that is, continuous functions $c : G \times G \longrightarrow A$ such that

$$gc(h, k) - c(gh, k) + c(g, hk) - c(g, h) = 0, \quad (4.5)$$

modulo the subgroup of elements of the form

$$df(g, h) = f(gh) - f(g) - gf(h) \quad (4.6)$$

for $f : G \longrightarrow A$ continuous. Any $H^i(G, U)$ defined in this way is pointed by the class of the constant map $G^n \longrightarrow e \in U$, even though it is a group in general only for U abelian. We denote by $[c]$ the equivalence class of a cocycle c .

Given a 1-cocycle $c \in Z^1(G, U)$, we can define the twisted action

$$\rho_c : G \longrightarrow \text{Aut}(U) \quad (4.7)$$

as

$$\rho_c(g)u = c(g)\rho(g)(u)c(g)^{-1}. \quad (4.8)$$

The isomorphism class of this action depends only on the equivalence class $[c]$.

Given an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0 \quad (4.9)$$

of topological groups with G action such that the last map admits a continuous splitting (not necessarily a homomorphism) and A is central in B , we get the exact sequence

$$0 \longrightarrow H^0(G, A) \longrightarrow H^0(G, B) \longrightarrow H^0(G, C) \longrightarrow \quad (4.10)$$

$$\longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \longrightarrow H^2(G, A) \quad (4.11)$$

of pointed sets, in the sense that the image of one map is exactly the inverse image of the base-point for the next map ([12], Appendix to Chapter VII).

But there are several bits more of structure. Consider the fibers of the map

$$i_* : H^1(G, A) \longrightarrow H^1(G, B). \quad (4.12)$$

The group $H^0(G, C)$ will act on $H^1(G, A)$ as follows. For $\gamma \in H^0(G, C)$, choose a lift to $b \in B$. For $x \in Z^1(G, A)$, let $(bx)(g) = bx(g)g(b^{-1})$. Because γ is G -invariant, this takes values in A , and defines a cocycle. Also, a different choice of b will result in an equivalent cocycle, so that the action on $H^1(G, A)$ is well-defined. From the definition, the $H^0(G, C)$ -action preserves the fibers of i_* . Conversely, if $[x]$ and $[x']$ map to the same element of $H^1(G, B)$, then there is a $b \in B$ such that $x'(g) = bx(g)g(b^{-1})$ for all $g \in G$. But then by applying q , we get $1 = q(b)g(q(b)^{-1})$, that is, $q(b) \in H^0(G, C)$. We have shown:

Lemma 4.1. *The fibers of i_* are exactly the $H^0(G, C)$ -orbits of $H^1(G, A)$.*

We can say more. Given $x \in Z^1(G, A)$ and $y \in Z^1(G, B)$, consider the map

$$(xy)(g) := x(g)y(g). \quad (4.13)$$

This is easily seen to be in $Z^1(G, B)$ and defines an action of $H^1(G, A)$ on $H^1(G, B)$.

Lemma 4.2. *Suppose $C^{\rho_z} = 1$ for all $[z] \in H^1(G, C)$. Then $H^1(G, A)$ acts freely on $H^1(G, B)$.*

Proof. Fix an element $[y] \in H^1(G, B)$. We work out its stabilizer. We have $[x][y] = [xy] = [y]$ if and only if there is a $b \in B$ such that $x(g)y(g) = by(g)g(b^{-1})$. By composing with q , we get

$$qy(g) = q(b)qy(g)g(q(b)^{-1}) \quad (4.14)$$

or

$$qy(g)g(q(b))qy(g)^{-1} = q(b). \quad (4.15)$$

This says that $q(b)$ is invariant under the G -action ρ_{qy} given by

$$c \mapsto qy(g)g(c)qy(g)^{-1}. \quad (4.16)$$

Hence, by assumption, $q(b) = 1$, and hence, $b \in A$. But then, $x(g)y(g) = bg(b^{-1})y(g)$ for all g , from which we deduce that $x(g) = bg(b^{-1})$ for all g , so that $[x] = 0$. \square

On the other hand,

Lemma 4.3. *The action of $H^1(G, A)$ is transitive on the fibers of $q_* : H^1(G, B) \longrightarrow H^1(G, C)$.*

Proof. The action clearly preserves the fiber. Now suppose $[qy] = [qy'] \in H^1(G, C)$. Then there is a $c \in C$ such that

$$qy'(g) = cqy(g)g(c^{-1}) \quad (4.17)$$

for all g . We can lift c to $b \in B$, from which we get

$$y'(g) = x(g)by(g)g(b^{-1}) \quad (4.18)$$

for some $x(g) \in A$. Since y, y' and $g \mapsto by(g)g(b^{-1})$ are all cocycles and A is central, this equality implies that

$$x : G \longrightarrow A \quad (4.19)$$

is a cocycle, and $[y'] = [x][y]$. \square

The existence of the continuous splitting of exact sequences that we need for applying the results above always holds in the profinite case.

Lemma 4.4. *Suppose we have an exact sequence of profinite groups*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 0 \quad (4.20)$$

where all maps are continuous. Suppose $B = \varprojlim_j B_j$, where the j run over natural numbers. Then there is a continuous section to the map $B \longrightarrow C$.

Proof. If $B = \varprojlim B_j$, by replacing each B_j with the image of B if necessary, we can assume all the maps in the inverse system are surjective. Furthermore, if A_j is the image of A in B_j , and $C_j = B_j/A_j$, one gets $A = \varprojlim A_j$ (since A is closed in B) and $C = \varprojlim C_j$. That is, the exact sequence of profinite groups can be constructed as an inverse limit of exact sequences

$$0 \longrightarrow A_j \longrightarrow B_j \xrightarrow{p_j} C_j \longrightarrow 0 \quad (4.21)$$

indexed by the same category in such a way that all the transition maps

$$A_i \longrightarrow A_j, B_i \longrightarrow B_j, C_i \longrightarrow C_j \quad (4.22)$$

are surjective. From the commutative diagram

$$\begin{array}{ccc} B_i & \longrightarrow & C_i \\ f \downarrow & & \downarrow g \\ B_j & \longrightarrow & C_j \end{array} \quad (4.23)$$

we get the commutative diagram

$$\begin{array}{ccc} B_i & \xrightarrow{h} & B_j \times_{C_j} C_i \\ & \searrow & \downarrow \\ & & C_i \end{array} \quad (4.24)$$

We claim that the map h is surjective. To see this, let $c_j \in C_j$ and $b_j \in B_j$ map to c_j . We need to check that $f^{-1}(b_j)$ surjects onto $g^{-1}(c_j)$. Let $c_i \in g^{-1}(c_j)$. Choose $b'_i \in B_i$ mapping to c_i and let $b'_j = f(b'_i)$. Since b'_j and b_j both map to c_j , there is an $a_j \in A_j$ such that $b'_j = b_j + a_j$. Now choose a_i mapping to a_j and put $b_i = b'_i - a_i$. Then $b_i \in f^{-1}(b_j)$ and it still maps to c_i . This proves the claim.

For any fixed j , suppose we've chosen a section s_j of $B_j \longrightarrow C_j$. Then

$$s_j \circ g : C_i \longrightarrow B_j \quad (4.25)$$

defines a section of

$$B_j \times_{C_j} C_i \longrightarrow C_i. \quad (4.26)$$

This section can then be lifted to a section s_i of $B_i \longrightarrow C_i$. Thereby, we have constructed a diagram of sections

$$\begin{array}{ccc} C_i & \xrightarrow{s_i} & B_i \\ \downarrow & & \downarrow \\ C_j & \xrightarrow{s_j} & B_j \end{array} \quad (4.27)$$

By composing s_j with the projection $g_j : C \longrightarrow C_j$, we have a compatible sequence of maps

$$C \xrightarrow{f_j = s_j \circ g_j} B_j \quad (4.28)$$

such that $p_j \circ f_j = g_j$. Thus, we get a continuous map $f : C \longrightarrow B$ such that $p \circ f = Id$. \square

It has been pointed out by the referee that a more general statement can be found in [10], proposition 2.2.2. However, we will retain the proof above for the convenience of the reader. That is, a continuous section exists in circumstances more general than countably ordered inverse limits, but we have just recalled this case since it is all we will need. This applies for example when B is the pro- M completion of a finitely-generated group: For every n , we can let $B(n) \subset B$ be the intersection of open subgroups of index $\leq n$. This is a characteristic subgroup, and still open. So the quotients defining the inverse limit can be taken as $B/B(n)$.

5 Appendix II: Some complements on duality for Galois cohomology

When A is topological abelian group, A^\vee denotes the continuous homomorphisms to the discrete group \mathbb{Q}/\mathbb{Z} . Thus, in the profinite case of $A = \varprojlim A/H$, where the H run over any defining system of open normal subgroups of finite index,

$$A^\vee = \varinjlim_H \text{Hom}(A/H, \mathbb{Q}/\mathbb{Z}) \quad (5.1)$$

with the discrete topology. If $A = \varinjlim_m A[m]$ is a torsion abelian group with the discrete topology, then

$$A^\vee = \varprojlim \text{Hom}(A[m], \mathbb{Q}/\mathbb{Z}) \quad (5.2)$$

with the projective limit topology. Meanwhile, if A has a continuous action of the Galois group of a local or a global field, then $D(A)$ denotes the continuous homomorphisms to the discrete group

$$\mu_\infty = \varinjlim_m \mu_m \quad (5.3)$$

with Galois action. As far as the topological group structure is concerned, $D(A)$ is of course the same as A^\vee .

We let F be a number field and T a finite set of places of F including the Archimedean places. We denote by G_F the Galois group $\text{Gal}(\bar{F}/F)$ and by $G_F^T = \text{Gal}(F_T/F)$ the Galois group of the maximal extension F_T of F unramified outside T . Let v be a place of F , and equip $G_v = \text{Gal}(\bar{F}_v/F_v)$ with a choice of homomorphism $G_v \longrightarrow G_F \longrightarrow G_F^T$ given by the choice of an embedding $\bar{F} \hookrightarrow \bar{F}_v$.

In the following A (with or without Galois action) will be in the abelian subcategory of all abelian groups generated by topologically finitely-generated profinite abelian groups and torsion groups A such that A^\vee is topologically finitely-generated. We choose this category to give a discussion of duality.

We have local Tate duality

$$H^i(G_v, A) \simeq^D H^{2-i}(G_v, D(A))^\vee. \quad (5.4)$$

We also use the same letter D to denote the product isomorphisms

$$\prod_{v \in T'} H^i(G_v, A) \simeq^D \prod_{v \in T'} H^{2-i}(G_v, D(A))^\vee \quad (5.5)$$

for any indexing set T' .

Let $\text{III}_T^i(A)$ be the kernel of the localization map

$$\text{III}_T^i(A) := \text{Ker}[H^i(G_F^T, A) \xrightarrow{\text{loc}_T} \prod_{v \in T} H^i(G_v, A)] \quad (5.6)$$

and $\text{Im}_T^i(A)$, the image of the localization map

$$\text{Im}_T^i(A) := \text{Im}[H^i(G_F^T, A) \xrightarrow{\text{loc}_T} \prod_{v \in T} H^i(G_v, A)]. \quad (5.7)$$

Assume now that $A = \varprojlim A_n$, where T contains all the places lying above primes dividing the order of any A_n . According to Poitou-Tate duality, we have an isomorphism

$$\text{III}_T^i(A) \simeq \text{III}_T^{2-i}(D(A))^\vee, \quad (5.8)$$

and an exact sequence

$$H^i(G_F^T, A) \longrightarrow \prod_{v \in T} H^i(G_v, A) \xrightarrow{\text{loc}_T^* \circ D} H^{2-i}(G_F^T, D(A))^\vee. \quad (5.9)$$

Note that this is usually stated for finite coefficients². But since all the groups in the exact sequence

$$H^i(G_F^T, A_n) \longrightarrow \prod_{v \in T} H^i(G_v, A_n) \xrightarrow{\text{loc}_T^* \circ D} H^{2-i}(G_F^T, D(A_n))^\vee \quad (5.10)$$

are finite, we can take an inverse limit to get the exact sequence above (since the inverse limit is exact on inverse systems of finite groups).

If $T' \supset T$, since all the inertia subgroups $I_v \subset G_v$ for $v \notin T$ act trivially on A , we have

$$\text{Im}_{T'}^1(A) \cap [\prod_{v \in T} H^1(G_v, A) \times \prod_{v \in T' \setminus T} H^1(G_v/I_v, A)] = \text{Im}_T^1(A). \quad (5.11)$$

In particular, we have an exact sequence

$$H^1(G_F^T, A) \xrightarrow{\text{loc}_{T'}} \prod_{v \in T} H^1(G_v, A) \times \prod_{v \in T' \setminus T} H^1(G_v/I_v, A) \xrightarrow{\text{loc}_{T'}^* \circ D} H^1(G_F^{T'}, D(A))^\vee. \quad (5.12)$$

Taking an inverse limit over T' , we get an exact sequence

$$H^1(G_F^T, A) \xrightarrow{\text{loc}} \prod_{v \in T} H^1(G_v, A) \times \prod_{v \notin T} H^1(G_v/I_v, A) \xrightarrow{\text{loc}^* \circ D} H^1(G_F, D(A))^\vee. \quad (5.13)$$

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²It has been pointed out by the referee that this general case is well-known, for example, in work of Nekovar or Schneider.

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